Master's Thesis

Types of von Neumann Algebras

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ABSTRACT

In this thesis we examine the classification theory regarding von Neumann algebras. We will focus on the classification of von Neumann algebras into types, namely type I, II and III. Furthermore the decomposition of von Neumann algebra into a direct integral of factors is included. The classification process starts with abelian von Neumann algebras. After the abelian von Neumann algebras are classified we examine the general structure of type I and type II von Neumann algebras. We then proceed to provide explicit constructions of factors of the possible types. The focus lies with the general structure theory of von Neumann algebras. Various constructions, such as, the crossed product construction are included, as it provides us with a tool to construct factors of type II_1, type II_∞ and type III.

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Preface

Introduction

In this thesis we examine the general structure of von Neumann algebras. The aim is to get acquainted with the types of von Neumann algebras and some of their properties. There are three different types of von Neumann algebras, in increasing complexity, they are denoted by type I, II, III. All these types are then further decomposable, these further decompositions are also investigated and examined. The structure theory for type I and II algebras is covered in chapter one, where we mostly focus on type I algebras.

Besides the type decomposition there also exists a decomposition of von Neumann algebras into factors. A factor is a von Neumann algebra $\mathcal{A}$ in which the center is trivial, i.e. $\mathcal{A} \cap \mathcal{A}' = \mathbb{C}$. The only weakness of the factor decomposition theorem is that it only holds on separable Hilbert spaces. Combining these two decompositions one obtains that the study of von Neumann algebras reduces to study of factors of the possible types.

In chapter two we will present the crossed product construction. This constructions assigns to an abelian von Neumann algebra $\mathcal{A}$ and a countable discrete group G a new von Neumann algebra $\mathcal{R}(\mathcal{A}, G)$. This construction can be extended to include von Neumann algebras that are not abelian, however that construction is not included. The algebra $\mathcal{R}(\mathcal{A}, G)$ can be thought of as the von Neumann algebra that incorporates the group structure on G with the algebra structure on $\mathcal{A}$. The crossed product construction yields examples of von Neumann algebras that are not of type I, we will give examples of type II algebras constructed with the crossed product.

Chapter three is devoted to traces of von Neumann algebras. The theory of traces is developed in some detail in order to construct a type III algebra using the crossed product construction. We will mainly focus how traces on von Neumann algebras interact with the crossed product construction.

The last chapter is devoted to the classification of type III von Neumann algebras, it turns out that there are many non-isomorphic type III factors on a separable Hilbert space, namely uncountably many. We will not provide explicit constructions of each type III factor however the decomposition theorem is covered in detail. We start of with the Tomita-Takesaki theorem, which states that any von Neumann algebra $\mathcal{A}$ can be represented isomorphically on a Hilbert space $\mathcal{H}$ such that $\mathcal{A}$ is anti isomorphic to its own commutant. In this process we will construct a group of automorphisms of $\mathcal{A}$ which in fact forms the basis for the decomposition theorem of type III von Neumann algebras.

For the sake of self containment we also included an appendix covering some basic theory regarding unbounded operators, the spectral theorem, anti-linear operators and integrals of operators.

Authors declaration and acknowledgements

I hereby declare that I am the sole author of this thesis. This thesis is based on the work of many mathematicians and therefor I am in debt to all who have contributed to the theory of von Neumann algebras. Especially the works of M.Takesaki on the matter have been a great guide for me. Though I took great care in writing this, errors are bound to remain, any such error is my responsibility.

The main inspiration comes from the nature and structure of algebras of operators on Hilbert space. I find the interplay between the topological and algebraic data in conjunction with the infinite dimensionality of von Neumann algebras fascinating. In some cases topological data of von Neumann algebras can be expressed in algebraic terms, the best known case is ofcourse the double commutant theorem. This shows that the topological structure of a von Neumann algebra is linked with its algebraic structure.

My foremost thanks go out to Dr. Bram Mesland who has been my guide. He provided me with support and useful discussions through the entire process, in particular his input regarding the Tomita Takesaki theorem was essential. Furthermore I am in debt to Dr. André Henriques who has provided me with an excellent exposition regarding the factor decomposition theorem, which, for some reason, is not covered in detail in most modern works on the subject.

I would also like to thank my family for giving me support when I felt stuck and providing me with a stable environment to work in. Furthermore I would like to thank Msc. Frank Tramper for his input regarding stylistic matters and the many interesting discussions we had.

Preliminaries and notation

We start of by fixing the notation. All Hilbert spaces are denoted by $\mathcal{H}$ and are over the complex numbers $\mathbb{C}$, if more then one Hilbert space is under consideration we will denote them by $\mathcal{H}_i$, with $i \in I$ some index set. Subspaces of Hilbert spaces are generally denoted by $\mathcal{V}$. Elements of Hilbert spaces will generally be denoted by h, k, g etc. The letters a, b, x and y are generally reserved for bounded operators acting on some Hilbert space, occasionally we will use capital letters T or A to denote some bounded (or unbounded) operator. For projections we usually reserve the letters e and f, if some projection is central then we usually assign the letter z to it.

Von Neumann algebras are almost exclusively denoted by $\mathcal{A}$ or $\mathcal{M}$, usually we use the symbol $\mathcal{A}$. If we want to stress the Hilbert space on which $\mathcal{A}$ is represented then we will denote this by $\{\mathcal{A}, \mathcal{H}\}$. We also use the symbol $\mathcal{A}$ for C^*-algebras, however they are seldom under consideration and when they are we will explicitly mention whether $\mathcal{A}$ is a C^*-algebra or a von Neumann algebra. We will use $\mathcal{Z}$ or $\mathcal{Z}(\mathcal{A})$ to indicate the center of some von Neumann algebra $\mathcal{A}$, i.e. $\mathcal{Z}(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$, here $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. Note that the commutant of $\mathcal{A}$ depends on the Hilbert space on which $\mathcal{A}$ is represented. The von Neumann algebra of all bounded operators on $\mathcal{H}$ is denoted by $\mathcal{B}(\mathcal{H})$. If we wish to consider a von Neumann algebra generated by some set $X \subset \mathcal{B}(\mathcal{H})$ then we denote the resulting von Neumann algebra by $W^*(X)$, C^*-algebras generated by some set are denoted by $C^*(X)$. Objects which are of special interest are usually denoted in some other font then the one we are using now, example: $\mathfrak{h}$ will denote a vector in $\mathcal{H}$ with very interesting properties relative to $\{\mathcal{A}, \mathcal{H}\}$. Ideals are usually denoted by $\mathcal{I}$ or $\mathcal{J}$, index sets are usually denoted with I, J or $\mathbb{N}$ when countable.

Now we will cover the preliminaries. We will assume that the reader is acquainted with functional analysis and a good deal of measure theory at the level of [1] and [6]. We assume that the reader is familiar with the Gelfand-Naimark-Segal construction, henceforth GNS construction. This construction assigns to a C^*-algebra $\mathcal{A}$ a Hilbert space on which $\mathcal{A}$ can be represented faithfully, for properties see [1] or [8]. We will also assume a familiarity with the different operator topologies on $\mathcal{B}(\mathcal{H})$, though this subject deserves a proper introduction it is not included. For a detailed discussion regarding operator topologies we refer to [8].

Let $S \subset \mathcal{B}(\mathcal{H})$, we denote by S' the set of elements in $x \in \mathcal{B}(\mathcal{H})$ such that $xs = sx$ for all $s \in S$. The set S' is called the commutant of S. Note that by definition $S \subset S''$. Consider S''', obviously $S' \subset S'''$ on the other hand we have that $S \subset S''$ thus it follows that $S''' \subset S'$. From this it follows that the operation of taking the commutant stabilizes, we find

$$S \subset S'' = S'''' = \cdots ,$$
$$S' = S''' = S''''' \cdots .$$

We define von Neumann algebras in the following way.

Definition 0.0.0.1. *A von Neumann algebra is a star closed algebra of operators $\mathcal{A} \subset \mathcal{B}(\mathcal{H})$ that is equal to its double commutant. In formula form:*

$$\mathcal{A} = \mathcal{A}''.$$

Note that by our last argument it follows that if S is any star closed subset of $\mathcal{B}(\mathcal{H})$ then $\mathcal{A} := S'$ is a von Neumann algebra.

No exposition regarding von Neumann algebras is complete without the double commutant theorem, for proof we refer to [8].

Theorem 0.0.0.2 (Double commutant theorem due to John von Neumann)**.** *Suppose that $\mathcal{A}$ is a star closed algebra of operators on $\mathcal{H}$ and suppose that $\mathcal{A}$ is strongly closed, then the following statements hold.*

1. *There exists a greatest projection $z \in \mathcal{A}$, this projection acts as the identity for $\mathcal{A}$ and satisfies $z := [\mathcal{A}\mathcal{H}]$, that is, z is the projection onto the closed linear span of $\mathcal{A}\mathcal{H}$.*

2. $\mathcal{A}'' = \{x + \lambda 1 \; ; \; x \in \mathcal{A}, \; \lambda \in \mathbb{C}\}$. *In particular if $1 \in \mathcal{A}$ then $\mathcal{A} = \mathcal{A}''$.*

□

This theorem links the algebraic data of being equal to its double commutant to the topological data of being strong closed. In particular it states that if $\mathcal{A}$ is unital then saying that $\mathcal{A}$ is closed in the strong operator topology is the same as saying that it is equal to its double commutant. We call a von Neumann algebra $\mathcal{A}$ a factor when $\mathcal{A} \cap \mathcal{A}' = \mathbb{C}$. Note that in order to talk about strong topology or commutants we need a Hilbert space on which $\mathcal{A}$ is faithfully represented. It could be that $\mathcal{A}$ represented on $\mathcal{H}_0$ is a factor but $\mathcal{A}$ represented on $\mathcal{H}_1$ is not.

Von Neumann algebras can also be characterized abstractly as C^*-algebras with a predual. This means that there exists some Banach space $\mathcal{F}$ such that $\mathcal{F}^* = \mathcal{A}$ as Banach spaces. If such a predual exists then it is necessarily unique (op to isomorphism). Note that having a predual induces a weak topology on $\mathcal{A}$, this topology is called the ultraweak topology. Consequently the unit ball of $\mathcal{A}$ is ultraweakly compact. This predual can be embedded naturally in its double dual (any Banach space can) and therefor consists of linear functionals of $\mathcal{A}$. The linear functionals on $\mathcal{A}$ that come from $\mathcal{F}$ are called normal or ultraweakly continuous. The characterization of these normal linear functionals is that they respect increasing bounded nets. Suppose that $\{a_i\}_{i \in I}$ is a bounded increasing net of operators i.e. $a_i \geq a_j$ when $i \geq j$, we say that $\phi \in \mathcal{A}^*$ is normal when

$$\phi(\sup_i a_i) = \sup_i \phi(a_i).$$

Suppose that ϕ is a normal state on $\mathcal{A}$ then its GNS representation $\{\pi_\phi, \mathcal{H}_\phi\}$ represents $\mathcal{A}$ as a von Neumann subalgebra of $\mathcal{B}(\mathcal{H}_\phi)$. These statements are all nontrivial (and in all honesty, presenting it in such a manner does not give enough right to the subject) but we will take them for granted. For proof we refer to [8]. On the other hand is π is an isomorphism of von Neumann algebras it is necessarily continuous in this ultraweak topology, this statement is also nontrivial. In the coming chapters we will not always mention in which topology we regard convergence (usually we pick the strong convergence). This is justified in some sense because any von Neumann algebra on $\mathcal{H}$ is closed in the weak topology, which is the weakest of the interesting topologies on $\mathcal{A}$. So if $a \in \mathcal{A}$ then we can a find a net converging to a in any of the weak, strong, norm, etc topologies associated to $\mathcal{A}$. When confronted with a sum $a = \sum_{i \in I} a_i$ of elements coming from $\mathcal{A}$ then this sum converges in the *strong* topology. Any such sum should be regarded in the strong topology.

We leave the preliminaries for what they are and start in chapter one with the basic definitions and constructions. The following chapters are all reasonably self contained, however we could not include all proofs and therefor if a proof is not included there will be a reference pointing to the proof of the corresponding statement.

Contents

Chapter 1

The General Structure of von Neumann Algebras

In this chapter we will cover the basic theory describing abelian von Neumann algebras and the decomposition of type I and type II von Neumann algebras. We start out by investigating the projection lattice of a von Neumann algebra. We conclude the first part of this chapter with a prelude to the decomposition into factors using direct integrals. After doing so we proceed with classifying abelian von Neumann algebras. After classifying abelian von Neumann we will finish off the factor decomposition. The last part of this chapter is devoted to decompose type I and type II von Neumann algebras into a direct sum of tensor products.

1.1 Projections in a von Neumann algebra

We start of by examining the set of all projections (defined below) in a von Neumann algebra. The general structure decomposition is then given in terms of projections the von Neumann algebra may or may not have. Most of this section is based on [8]

Definition 1.1.0.3. *A projection in a von Neumann algebra $\mathcal{A}$ is an element $e \in \mathcal{A}$ enjoying the properties:*

1. $e^2 = e$;
2. $e^* = e$.

We call a projection central when it is an element of $\mathcal{A} \cap \mathcal{A}'$. We call two projections e and f orthogonal when $ef = 0$.

The zero element is an example of a projection. If $e \in \mathcal{A}$ is a nonzero projection then $\|e\| = 1$, hence all projections are in the unit sphere of $\mathcal{A}$. We denote the set of all projections in $\mathcal{A}$ as $\mathcal{A}_p$. Note that any projection e is automatically positive since $e = e^2 = e^*e$.

Let be e any projection and suppose that $h \in \ker(e)^\perp$. Then it follows that $h \in \overline{\operatorname{ran}(e^*)} = \overline{\operatorname{ran}(e)}$, as such, there are h_i in $\mathcal{H}$ such that $\lim_i eh_i = h$. Using the continuity of e we find that $eh = \lim_i e^2 h_i = \lim_i eh_i = h$. We conclude that $\operatorname{ran}(e) = \ker(e)^\perp$ (in particular $\operatorname{ran}(e)$ is closed) and furthermore for all $h \in \ker(e)^\perp$ it holds that $eh = h$.

Suppose that e is a projection in $\mathcal{A}$ and that $a \in \mathcal{A}$ commutes with e. Then it follows that $a : \operatorname{ran}(e) \longrightarrow \operatorname{ran}(e)$ and $a : \ker(e) \longrightarrow \ker(e)$. On the other hand if $a : \ker(e) \longrightarrow \ker(e)$ and $a : \operatorname{ran}(e) \longrightarrow \operatorname{ran}(e)$ then it is not hard to see that a commutes with e. Consequently if z is a central projection then $\ker(z)$ and $\operatorname{ran}(z)$ are both invariant for every element of $\mathcal{A}$.

Lets investigate the projection space $\mathcal{A}_p$. First of all, since every projection is positive we have that $\mathcal{A}_p$ inherits the partial order from $\mathcal{A}_+$. It turns out that $\mathcal{A}_p$ is a complete lattice (A partial order is called a complete lattice when every subset has a least upper bound and a greatest lower bound).

Theorem 1.1.0.4. *If $\mathcal{A}$ is a von Neumann algebra then $\mathcal{A}_p$ is a complete lattice.*

Proof:

Let $P \subset \mathcal{A}_p$ and let $\mathcal{H}$ denote the Hilbert space on which $\mathcal{A}$ acts. Define

$$p_0\mathcal{H} := \bigcap_{p \in P} p\mathcal{H}.$$

It follows that p_0 is the projection on $\bigcap_{p\in P} p\mathcal{H}$. Our aim is to show that $p_0 \in \mathcal{A}$. To do this we will show that p_0 commutes with every unitary in $\mathcal{A}'$ (recall that every element in a C^*-algebra can be expressed as a linear combination of four unitary elements). If p_0 commutes with every unitary in $\mathcal{A}'$, then $p_0 \in \mathcal{A}'' = \mathcal{A}$. Let $u \in \mathcal{A}'$ be unitary. By definition u commutes with every $p \in P$ hence $p\mathcal{H}$ is invariant for u for all $p \in P$. We find that

$$\begin{aligned} u\left(\bigcap_{p\in P} p\mathcal{H}\right) &= \bigcap_{p\in P} u\,(p\mathcal{H}) \\ &= \bigcap_{p\in P} pu\mathcal{H} \\ &\subset \bigcap_{p\in P} p\mathcal{H}. \end{aligned}$$

it follows that $\bigcap_{p\in P} p\mathcal{H}$ is invariant for u hence p_0 commutes with u. We conclude that $p_0 \in \mathcal{A}_p$ and clearly p_0 is the greatest lower bound for P. Consider the projection $p_1 := \bigvee_i p_i\mathcal{H}$, via the same argument p_1 commutes with ever unitary in $\mathcal{A}'$, clearly p_1 is the greatest upper bound for P. We conclude that $\mathcal{A}_p$ is a complete lattice. □

This theorem allows us to consider subsets of $\mathcal{A}_p$ and take upper or lower bounds on that set. The point is that the resulting projection is guaranteed to be in $\mathcal{A}$. Suppose that I is any index set and that $\{e_i\}_{i\in I}$ is a family of orthogonal projections in $\mathcal{A}$, orthogonal in the sense that $e_ie_j = 0$ when $i \neq j$. Then by the previous theorem we can make sense of the sum $\sum_{i\in I} e_i$, by defining

$$\sum_{i\in I} e_i := \sup\left\{\sum_{i\in F\subset I} e_i \ ; \ F \text{ is a finite subset of } I\right\}.$$

The previous theorem justifies the following definition.

Definition 1.1.0.5. *For any projection e we define the central carrier of e, denoted by $z(e)$, to be the smallest central projection z such that $e \leq z$. We call two projections e and f centrally orthogonal when $z(e)z(f) = 0$.*

We now proceed to define an equivalence relation on $\mathcal{A}_p$.

Definition 1.1.0.6. *On the set $\mathcal{A}_p$ we define a equivalence relation by saying that $e, f \in \mathcal{A}_p$ are equivalent, denoted by $e \sim f$, if there is an element $u \in \mathcal{A}$ such that the following statements hold*

1. $u^*u = e$;

2. $uu^* = f$.

We write $e \preceq f$ when there is an $f_1 \in \mathcal{A}_p$ such that $e \sim f_1$ and $f_1 \leq f$. The symbol $\succeq$ is defined similarly.

We will show that this indeed is a equivalence relation. It is obvious that $e \sim e$, pick $u = e$. If u sets up the equivalence between two projections e and f, meaning that $u^*u = e$ and $uu^* = f$. Then $k := u^*$ sets up the equivalence between f and e, namely $k^*k = uu^* = f$ and $kk^* = u^*u = e$. Suppose that $e \sim f \sim g$, then we can find u and w such that

$$\begin{aligned} u^*u &= e, \qquad & w^*w &= f, \\ uu^* &= f, \qquad & ww^* &= g. \end{aligned}$$

Let $k = wu$, then it follows that $k^*k = u^*w^*wu = u^*fu = u^*uu^*u = e^2 = e$, furthermore $kk^* = wuu^*w^* = wfw^* = ww^*ww^* = g^2 = g$. We conclude that $\sim$ defines an equivalence relation.

Proposition 1.1.0.7. *If given two equivalent projections e and f, then the element u such that $u^*u = e$ and $uu^* = f$ is a partial isometry. Furthermore $e = u^*u$ is the projection onto $\ker(u)^\perp$ and $f = uu^*$ is the projection onto ran(u).*

Proof:

Let $h \in \ker(e)^{\perp}$, it follows that $\|h\|^2 = \|eh\|^2 = \langle eh, eh \rangle = \langle eh, h \rangle = \langle uh, uh \rangle = \|uh\|^2$. If $h \in \ker(e)$ then $0 = \|eh\|^2 = \|uh\|^2$. We conclude that $\ker(u)^{\perp} = \ker(e)^{\perp}$ and that $\|uh\| = \|h\|$ for all $h \in \ker(u)$, that is, u is a partial isometry. Also we obtained that e is the projection onto $\ker(u)^{\perp}$. Now note that $uh = ueh = uu^*uh = fuh$, we conclude $fu = u$. Similarly we conclude that $\ker(u^*) = \ker(f)$ and that f is the projection onto $\operatorname{ran}(u)$. □

There is a converse to this statement, namely, partial isometries in $\mathcal{A}$ give rise to equivalent projections.

Proposition 1.1.0.8. *Suppose that u is a partial isometry in $\mathcal{A}$. Then the projection $e = u^*u$ onto $\ker(u)^{\perp}$ is an element of $\mathcal{A}$, also the projection $f = uu^*$ onto the range of u is also in $\mathcal{A}$.*

Proof:

That e and f are both elements of $\mathcal{A}$ follows directly from the fact that $\mathcal{A}$ is a star closed ring. We must show that u^*u defines the projection onto $\ker(u)^{\perp}$ and that uu^* defines the projection onto $\operatorname{ran}(u)$. Let $e = u^*u$, it follows that $\ker(u) \subset \ker(e)$, on the other hand if $h \in \ker(e)$ then $0 = \langle u^*uh, h \rangle = \|u(h)\|^2$. We conclude that $\ker(e) = \ker(u)$. Let $h \in \ker(u)^{\perp}$ it follows that $\|h\|^2 = \|uh\|^2 = \langle u^*uh, h \rangle$. Since $\langle u^*uh, h \rangle > 0$, we find that u^*uh is of the form $u^*uh = h_1 \oplus h_{\perp}$ with $h_1 \in \operatorname{span}\{h\}$ and $h_{\perp} \in [\operatorname{span}\{h\}]^{\perp}$. It follows that $\|h\|^2 = \langle u^*uh, h \rangle = \langle h_1, h \rangle$. We conclude that $h_1 = \pm h$. Since -1 is not an element of the spectrum of u we conclude that $h_1 = h$. Since $\|u^*u\| = \|u\|^2 = 1$, it follows that $h_{\perp} = 0$. Using similar arguments we find that f is the projection onto $\operatorname{ran}(u)$. □

We conclude that any partial isometry u in $\mathcal{A}$ gives rise to two equivalent projections in $\mathcal{A}$. The question becomes if these partial isometries are abundant in any given von Neumann algebra. Though the word abundant should be interpreted informally, the answer is still yes.

Theorem 1.1.0.9. *If $\mathcal{A}$ is a von Neumann algebra then any element $a \in \mathcal{A}$ gives rise to a partial isometry u_a which belongs to $\mathcal{A}$. The partial isometry u_a maps $\ker(a)^{\perp}$ onto the closure of ran(a). In particular $\mathcal{A}$ contains the projections onto $\ker(a)^{\perp}, \ker(a), \overline{ran(a)}$ and $ran(a)^{\perp}$ for any $a \in \mathcal{A}$. Furthermore the projections onto $\ker(a)^{\perp}$ and $\overline{ran(a)}$ are equivalent.*

Proof:

Let $a \in \mathcal{A}$ and let $u_a \cdot |a|$ be the polar decomposition associated to a, so $a = u_a \cdot |a|$. By the functional calculus for C^*-algebras, $|a|$ belongs to $\mathcal{A}$. By the polar decomposition theorem, u_a is a partial isometry which maps $\ker(a)^{\perp}$ onto $\overline{\operatorname{ran}(a)}$. A priory it is not clear that $u_a \in \mathcal{A}$. We will show that u_a commutes with $\mathcal{A}'$. It suffices to show that u_a commutes with every unitary in $\mathcal{A}'$. Consider a unitary $w \in \mathcal{A}'$, since $|a| \in \mathcal{A}$ we have that $w\,|a| = |a|\,w$. We will first show that $\ker(|a|)$ and $\ker(|a|)^{\perp}$ are invariant subspaces for w. Suppose that $h \in \ker(|a|)$ then $0 = w\,|a|\,h = |a|\,wh$, we find that $\ker(|a|)$ is invariant for w. On the other hand if $w(h) \in \ker(|a|)$ for some $h \in \mathcal{H}$, then we find that $0 = |a|\,wh = w\,|a|\,h$ thus $|a|\,h \in \ker(w) = \{0\}$, as such, $h \in \ker(|a|)$. We conclude that $\ker(|a|)$ and $\ker(|a|)^{\perp}$ are both invariant for w. Note furthermore that $\ker(|a|) = \ker(a)$. Suppose that $h \in \ker(|a|)$ then $w(h) \in \ker(|a|)$, as such, $u_a w(h) = w u_a(h) = 0$. We conclude that w and u_a commute on $\ker(|a|)$. Suppose that $h \in \ker(|a|)^{\perp}$ and consider $u_a w(h)$. Since $w(h) : \ker(|a|)^{\perp} \longrightarrow \ker(|a|)^{\perp} = \overline{\operatorname{ran}(|a|)}$, there exists a net $\{h_i\}_{i \in I}$ such that $\lim_i |a|\,h_i = w(h)$. It follows that

$$u_a w(h) = \lim_i u_a\,|a|\,h_i = \lim_i a h_i := f.$$

Consider w^*f, we find that

$$\begin{aligned} w^*f &= w^*\left(\lim_i a h_i\right) \\ &= \lim_i w^* a h_i \\ &= \lim_i a w^* h_i \\ &= \lim_i u_a\,|a|\,w^* h_i = \lim_i u_a w^*\,|a|\,h_i = u_a w^* w h = u_a h. \end{aligned}$$

We conclude that $w^*(f) = w^* u_a w(h) = u_a(h)$, that is, u_a commutes with w. Using that $\mathcal{A} = \mathcal{A}''$, we conclude that $u_a \in \mathcal{A}$. It follows that $u_a^* u_a \in \mathcal{A}$ defines the projection onto $\ker(a)^\perp$ and $u_a u_a^* \in \mathcal{A}$ defines the projection onto $\overline{\mathrm{ran}(a)}$ and by definition they are equivalent. The projections onto $\ker(a)$ and $\mathrm{ran}(a)^\perp$ are obtained by considering $1 - u_a^* u_a$ and $1 - u_a u_a^*$ respectively. Note that these last two projections are not necessarily equivalent. □

Note that the previous theorem cannot be extended to C^*-algebras. For example, consider the interval $[0,1] \subset \mathbb{R}$ and let $\mathcal{C}$ be its C^*-algebra of continuous functions. Since $[0,1]$ is path-connected, it follows that there are no nontrivial projections in $\mathcal{C}$, as such, there cannot be any non unitary partial isometries. On the other hand we established that the von Neumann algebra $L^\infty([0,1], \lambda)$, with λ denoting the Lebesgue measure, comes with an abundance of both projections and partial isometries. This is an example of the difference between C^*-algebras and von Neumman algebras and it illustrates the power of the double commutant theorem.

Definition 1.1.0.10. *Given $a \in \mathcal{A}$, we define the left support of a, denoted by $s_l(a)$, as the smallest projection $e \in \mathcal{A}$ such that $ea = a$. We define the right support of a, denoted by $s_r(a)$, to be the smallest projection $f \in \mathcal{A}$ such that $af = a$.*

It is easy to see that for any $a \in \mathcal{B}(\mathcal{H})$ it holds that $s_l(a)$ is the projection onto the closure of $\mathrm{ran}(a)$. Similarly for any $a \in \mathcal{B}(\mathcal{H})$ it holds that $s_r(a)$ is the projection onto $\ker(a)^\perp$. Using theorem 1.1.0.9 we conclude that $s_l(a) \sim s_r(a)$ and that for any von Neumann algebra $\mathcal{A}$ it holds that $s_l(a), s_r(a) \in \mathcal{A}$ for all $a \in \mathcal{A}$.

Proposition 1.1.0.11. *If $f, e \in \mathcal{A}$ are equivalent projections then $z(f) = z(e)$ (see for definition* 1.1.0.5*).*

Proof:

Let u be the partial isometry such that $u^*u = e$ and $uu^* = f$. It follows that $s_l(u) = f$ and that $s_r(u) = e$. Consider $z(e)$, since $z(e) \geq e$ it follows that $z(e)u = u$. Since $z(e)$ is central it follows that $u = z(e)u = uz(e)$ thus $z(e) \geq f$. Note that $z(e)$ is a central projection dominating f, as such, $z(e) \geq z(f)$. Consider $z(f)$, we find that $uz(f) = u$. Since $z(f)$ is central it follows that $u = uz(f) = z(f)u$. We find that $z(f)$ is a central projection dominating e, as such, $z(f) \geq z(e)$. We conclude that $z(e) = z(f)$, as desired. □

It follows that if $z(e) \neq z(f)$, then f and e cannot be equivalent.

Proposition 1.1.0.12. *Suppose that $\{e_i\}_{i \in I}$ and $\{f_i\}_{i \in I}$ are two families of orthogonal projections such that $e_i \sim f_i$. Then it follows that*

$$e = \sum_{i \in I} e_i \sim \sum_{i \in I} f_i = f.$$

Proof:

Let u_i be the partial isometry such that $e = u_i^* u_i$ and $f = u_i u_i^*$. Consider u_i and u_j for $i \neq j$, since e_i and e_j are orthogonal it follows that $\ker(u_i) \perp \ker(u_j)$. Since $\mathrm{ran}(u_i) \perp \mathrm{ran}(u_j)$ are orthogonal (because $f_i \perp f_j$) it follows that $u_i + u_j$ is again a partial isometry. We find that $(u_i + u_j)$ is a partial isometry with $\ker(u_i + u_j)^\perp = \ker(u_i)^\perp \vee \ker(u_j)^\perp$ and $\mathrm{ran}(u_i + u_j) = \mathrm{ran}(u_i) \vee \mathrm{ran}(u_j)$. We conclude that $e_i + e_j \sim f_i + f_j$. We find that

$$u := \sum_i u_i,$$

sets up the equivalence between e and f. □

Note that in the last proposition the element u is a well defined partial isometry because $\{f_i\}$ and $\{e_i\}$ are orthogonal families.

Definition 1.1.0.13. *Given a von Neumann algebra $\mathcal{A}$ and a projection $e \in \mathcal{A}$, we define $\mathcal{A}_e$ as $\mathcal{A}_e := e\mathcal{A}e$. The set $\mathcal{A}_e$ is often called the corner algebra associated to e.*

By construction we have that $\mathcal{A}_e$ is a subset of $\mathcal{A}$. It is easy to see that elements in $\mathcal{A}_e$ are exactly those elements in $\mathcal{A}$ that map $\mathrm{ran}(e)$ into itself and are zero on $\ker(e)$. It turns out that $\mathcal{A}_e$ is a von Neumann algebra on the Hilbert space $\mathcal{H}_e := \mathrm{ran}(e)$.

Proposition 1.1.0.14. *Let $e \in \mathcal{A}$ be a projection and set $\mathcal{H}_e := ran(e)$, then $\mathcal{A}_e$ is a von Neumann algebra on $\mathcal{H}_e$.*

Proof:

That $\mathcal{A}_e$ is a star algebra of operators on $\mathcal{B}(\mathcal{H}_e)$ is obvious, also $e = 1 \in \mathcal{A}_e$ thus $\mathcal{A}_e$ has the identity. What remains is to show that $\mathcal{A}_e$ is strongly closed.

Let $M_l(e) : \mathcal{A} \longrightarrow \mathcal{A}$ be defined as $M_l(e)(a) = ea$, define $M_r(e) : \mathcal{A} \longrightarrow \mathcal{A}$ as $M_r(e)(a) = ae$. Since left and right multiplication by some element are strongly continuous maps we find that $M_l(e), M_r(e), M_l(1-e)$ and $M_r(1-e)$ are strongly continuous. If we can show that $\mathcal{A}_e = F^{-1}(\{0\})$ for some strongly continuous map F, then it follows that $\mathcal{A}_e$ is strongly closed. Consider $M_r(e)(\mathcal{A})$, the claim is that $M_r(e)(\mathcal{A}) = M_r(1-e)^{-1}(\{0\})$. Obviously if $a \in \mathcal{A}$ is of the form $a = xe$, then $M_r(1-e)(a) = M_r(1-e)(xe) = xe(1-e) = 0$ so $M_r(e)(\mathcal{A}) \subset M_r(1-e)^{-1}(\{0\})$. Suppose that $x \in \mathcal{A}$ satisfies $M_r(1-e)(x) = x(1-e) = 0$, then it follows that $\mathrm{ran}(1-e) \subset \ker(x)$, as such, $\ker(x)^{\perp} \subset \mathrm{ran}(1-e)^{\perp} = \ker(e)^{\perp}$. It follows that $s_r(x) \leq e$ thus $x = xe$, we conclude that $x \in M_r(e)(\mathcal{A})$. We find that $M_r(e)(\mathcal{A}) = M_r(1-e)^{-1}(\{0\})$. Via a similar argument we conclude that $M_l(e)(\mathcal{A}) = M_l(1-e)^{-1}(\{0\})$. In total we find that

$$\mathcal{A}_e = M_l(1-e)^{-1}\left[M_r(1-e)^{-1}(\{0\})\right],$$

as desired. □

We now present a handy tool in the study of projections.

Theorem 1.1.0.15. *For any two projections e and f in a von Neumann algebra $\mathcal{A}$ the following statements are equivalent.*

1. *$z(e)$ and $z(f)$ are not orthogonal.*
2. *$e\mathcal{A}f \neq \{0\}$.*
3. *There exists nonzero projections $e_1 \leq e$ and $f_1 \leq f$ such that $e_1 \sim f_1$.*

Proof.

We will prove $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 1$.

$1 \Longrightarrow 2$.

If e and f are not orthogonal, then the statement is trivial. Suppose that e and f are orthogonal, by considering the algebra $\mathcal{A}_{e+f}$ we may assume that $e + f = 1$. If $\mathcal{A}$ acts on $\mathcal{H}$, then, by decomposing $\mathcal{H}$ as $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_f$, we find that an element $a \in \mathcal{A}$ can be represented by a 2×2 matrix. It follows that $a \in \mathcal{A}$ is of the following form:

$$a = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

An element $h \in \mathcal{H}$ is represented by a vector $(h_e \oplus h_f)$ and a acts on $\mathcal{H} = \mathcal{H}_e \oplus \mathcal{H}_f$ by matrix multiplication. In this setting e and f take on the following form:

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad f = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

A matrix calculation reveals that for any $a \in \mathcal{A}$ it holds that eaf is of the form:

$$eaf = \begin{pmatrix} 0 & a_{12} \\ 0 & 0 \end{pmatrix}.$$

Suppose that $e\mathcal{A}f = \{0\}$, then for all $a \in \mathcal{A}$ it must follow that $a_{12} = 0$. Considering a^* we also conclude that $a_{21}^* = 0$ for all $a \in \mathcal{A}$, that is, $a_{21} = 0$ for all $a \in \mathcal{A}$ (because $\mathcal{A}$ is star closed).
We conclude that an element $a \in \mathcal{A}$ takes on the form

$$a = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}.$$

Now note that if this is indeed the case then $ea = ae$ for all $a \in \mathcal{A}$, also $af = fa$ for all $a \in \mathcal{A}$. This implies that e and f are central, since they are also orthogonal, they are centrally orthogonal. This contradicts the assumption *1.*

2 $\Longrightarrow$ *3.*

By assumption there exists some nonzero $x \in e\mathcal{A}f$. For $x \in e\mathcal{A}f$ we find that $exf = x$, it follows that $s_l(x) \leq e$ and $s_r(x) \leq f$, since x is nonzero it follows that $s_l(x) \neq 0$ and $s_r(x) \neq 0$. By the remark following definition 1.1.0.10 we find that $s_l(x) \sim s_r(x)$.

3 $\Longrightarrow$ *1.*

By proposition 1.1.0.11 we find that $z(e_1) = z(f_1)$. We conclude that $z(e)z(f) \geq z(e_1)z(f_1) = z(e_1) \neq 0$, as desired. □

We can now prove the comparability theorem, which is particularly powerful if the von Neumann algebra under consideration is a factor.

Theorem 1.1.0.16 (Comparability theorem)**.** *For any two projections e and f in a von Neumann algebra $\mathcal{A}$ there is a central projection z such that $zf \preceq ze$ and $(1-z)\,f \succeq (1-z)\,e$.*

Proof:

Pick maximal families of orthogonal projections $\{e_i\}$ and $\{f_i\}$ with $e_i \leq e$ and $f_i \leq f$ and $e_i \sim f_i$ for all i and define

$$e_1 := \sum_i e_i, \qquad \text{and} \qquad f_1 := \sum_i f_i.$$

By definition there are no projections $e_0 \leq e - e_1$ and $f_0 \leq f - f_0$ with $e_0 \sim f_0$. Using theorem 1.1.0.15 we find that $e - e_1$ and $f - f_1$ are centrally orthogonal. Let $z := z\,(e - e_1)$, then $e - e_1 \leq z$ and $f - f_1 \leq 1 - z$. It follows that $(f - f_1)\,z = 0$ this implies that $fz = f_1 z$, we find that $fz = f_1 z \sim e_1 z \leq ez$. Similarly we find that $(e - e_1)(1 - z) = 0$, which implies that $e\,(1-z) = e_1\,(1-z) \sim f_1\,(1-z) \leq f\,(1-z)$. □

Note that if $\mathcal{A}$ is a factor then $\mathcal{A}_p/\sim$ is a total order, that is, any two projections can be compared.

The reason why we consider the equivalence relation $\sim$ is that it provides us with an isomorphism of corner algebras.

Proposition 1.1.0.17. *If $e \sim f$ then $\mathcal{A}_e := e\mathcal{A}e$ and $\mathcal{A}_f := f\mathcal{A}f$ are isomorphic.*

Proof:

Let u be the partial isometry such that $u^*u = e$ and $uu^* = f$, consider the map

$$U : \mathcal{A}_e \longrightarrow \mathcal{A}_f,$$

defined by

$$U\,(x) := uxu^*.$$

It follows that

$$\begin{aligned}
&U\,(e) = u\,(u^*u)\,u^* = uu^*uu^* = f,\\
&U\,(ab) = U\,(aeb) = uau^*ubu^* = U\,(a)\,U\,(b)\,,\\
&U\,(a^*) = ua^*u^* = (uau^*)^* = U\,(a)^*\,,\\
&U\,(a+b) = U\,(a) + u\,(b)\,.
\end{aligned}$$

The last thing to check is that $\|U(a)\|^2 = \|a\|^2$. This follows from the following identities:

$$\begin{aligned}\|uau^*\|^2 &= \|au^*\|^2 \\ &= \|au^*ua^*\| \\ &= \|aa^*\| = \|a^*a\| = \|a\|^2 .\end{aligned}$$

We conclude that U is indeed an isometry and hence it is a isomorphism, its inverse is given by $U^*(y) := u^*yu$. □

These isomorphisms of corner algebras is further examined in the tensor decomposition of type I and type II von Neumann algebras. We now proceed to introduce the possible types of von Neumann algebras.

1.1.1 Types of von Neumann algebras

Here we introduce the possible types of von Neumann algebras and prove that every von Neumann algebra is decomposable into a direct sum of von Neumann algebras of those types. Hence the study of von Neumann algebras reduces to the study of the possible types.

Definition 1.1.1.1. *A projection $e \in \mathcal{A}_p$ is called finite when $e \sim f \leq e$ implies $e = f$. If e is not finite it will be called infinite. A projection e will be called purely infinite when there is no nonzero finite projection $f \in \mathcal{A}_p$ such that $f \leq e$. A projection e will be called properly infinite when fe is infinite for every central projection f such that $fe \neq 0$. A projection e will be called abelian when $e\mathcal{A}e$ is an abelian von Neumann algebra. A projection e will be called minimal when $\mathcal{A}_e := e\mathcal{A}e$=$\mathbb{C}e$. A projection is called σ-finite when $\mathcal{A}_f$ is a σ-finite von Neumann algebra, meaning that there are at most countably many orthogonal projections in $\mathcal{A}_f$.*

A von Neumann algebra is called finite, σ-finite, infinite, properly infinite or purely infinite according to the properties of the identity projection 1.

An example of an infinite projection is the following: let $\mathcal{H}$=$l^2(\mathbb{N})$ and let $\mathcal{A} = \mathcal{B}(\mathcal{H})$. Let $u : \mathcal{H} \longmapsto \mathcal{H}$ be defined as

$$u(x_1, x_2, x_3, ...) = (0, x_1, x_2, x_3, ...) .$$

The adjoint u^* behaves as follows:

$$u^*(x_1, x_2, x_3, ...) = (x_2, x_3, ...) .$$

It follows that $u^*u = 1$ and uu^* is a projection that leaves everything fixed except the first coordinate, which is mapped to 0. Let $e = u^*u = 1$ and $f = uu^*$ then $e \sim f \leq e$ but $f \neq e$ hence e is infinite.

Definition 1.1.1.2. *Let $\mathcal{A}$ be a von Neumann algebra, we say that*

1. *Type I: $\mathcal{A}$ is of type I when every nonzero central projection dominates a nonzero abelian projection in $\mathcal{A}$.*
2. *Type II: $\mathcal{A}$ is of type II when there exists no nonzero abelian projection but every nonzero central projection dominates some nonzero finite projection in $\mathcal{A}$.*
3. *Type III: $\mathcal{A}$ is of type III when there exists no nonzero finite projection in $\mathcal{A}$.*

An example of a type I von Neumann algebra is $\mathcal{B}(\mathcal{H})$ for any Hilbert space $\mathcal{H}$. Let $\{f_i\}_{i\in I}$ be a complete orthonormal system for $\mathcal{H}$ and fix $j \in I$. Let $e_j \in \mathcal{B}(\mathcal{H})$ be the projection on $\mathbb{C}f_j$. We find that $e_j\mathcal{B}(\mathcal{H})e_j = \mathbb{C}e_j$, as such, e_j is a minimal projection and therefor abelian. Since $\mathcal{B}(\mathcal{H})$ is a factor we conclude that 1 is the only nonzero central projection and obviously $e_j \leq 1$. We conclude that $\mathcal{B}(\mathcal{H})$ is of type I.

Definition 1.1.1.3. *We further decompose the type II von Neumann algebras as follows*

Type II_1: $\mathcal{A}$ is of type II_1 when $\mathcal{A}$ is finite and of type II.

Type II_∞: $\mathcal{A}$ is of type II_∞ when $\mathcal{A}$ has no nonzero central finite projection and $\mathcal{A}$ is of type II.

It turns out that every von Neumann algebra $\mathcal{A}$ is decomposable into a direct sum of von Neumann algebras belonging to one of the types described above and that this decomposition is unique. The way to prove this is by constructing projections in $\mathcal{A}$ with the desired properties.

Lemma 1.1.1.4. *Suppose that $\{e_i\}_{i\in I} \subset \mathcal{A}_p$ is a collection of centrally orthogonal abelian (resp. finite) projections. Then*

$$\sum_{i\in I} e_i,$$

is abelian (resp. finite).

Proof:

Consider the collection $\{e_i\}_{i\in I}$ and suppose that e_i is abelian for all $i \in I$. Note that the following holds:

$$\begin{aligned} z(e_i)\, e &= z(e_i) \sum_{j\in I} e_j \\ &= \sum_{j\in I} z(e_i)\, e_j \\ &= z(e_i)\, e_i = e_i. \end{aligned}$$

Consider the direct sum of the corner algebras $\mathcal{A}_{e_i}$, we find

$$\begin{aligned} \bigoplus_{i\in I} e_i \mathcal{A} e_i &= \bigoplus z(e_i)\, e\mathcal{A} z(e_i)\, e \\ &= \bigoplus z(e_i)\, e\mathcal{A}e \\ &= \left(\sum_{i\in I} z(e_i)\right) e\mathcal{A}e \\ &= e\mathcal{A}e. \end{aligned}$$

Since $\bigoplus_{i\in I} e_i\mathcal{A}e_i$ is abelian we conclude that $e\mathcal{A}e$ is abelian.

Suppose now that the collection $\{e_i\}_{i\in I}$ consists of finite projections, let e be as above. Consider any projection f such that $e \sim f \leq e$. Since $e \sim f \leq e$ we conclude that for all i it holds that $z(e_i)e \sim z(e_i)f \leq e_i$. Since e_i is finite it holds that $z(e_i)f = e_i$. It follows that $f = z(e)f = \sum_i z(e_i)f = \sum_i z(e_i)e_i = e$, we conclude that e is finite. $\square$

Theorem 1.1.1.5 (Type decomposition)**.** *Every von Neumann algebra $\mathcal{A}$ is uniquely decomposable into a direct sum of von Neumann algebras of type I, type II_1, type II_∞ and type III.*

Proof.

Let $\mathcal{A}$ be a von Neumann algebra and let $\mathcal{H}$ denote the Hilbert space on which it acts. Pick a maximal family of centrally orthogonal abelian projections $\{e_i\}_{i\in I} \subset \mathcal{A}_p$ and define

$$e := \sum_{i\in I} e_i.$$

If there are no nonzero abelian projections then $e = 0$ and $\mathcal{A}$ has no summand of type I. Lets assume that such a family $\{e_i\}_{i\in I}$ exists. By 1.1.1.4 it follows that e is abelian. Define $z_I := z(e)$, it follows that z_I is a nonzero central projection dominating e with the property that for any central projection $g \leq z_I$ we have that $e \not\leq z_I - g$. Suppose now that f is a nonzero central projection dominated by z_I then fe is an abelian projection (because f commutes with e) and $fe \leq f$. So what we have done is given a nonzero central projection f we constructed an abelian projection fe such that $fe \leq f$. We still need to conclude that $fe \neq 0$. Note that $z_I - e$ and $z_I - f$ commute (because z_I and f are both central projections) thus $0 \leq (z_I - e)(z_I - f)$. Suppose now that $fe = 0$ then it holds that

$$0 \leq (z_I - e)(z_I - f) = z_I - f - e.$$

By definition of z_I we have that $e \not\leq z_I - f$ hence we have reached a contradiction and we conclude that $fe \neq 0$. What we have found is that in $\mathcal{A}_{z_I}$ every nonzero central projection dominates a nonzero abelian projection, as such, $\mathcal{A}_{z_I}$ is of type I.

We claim now that there is no nonzero abelian projection in $\mathcal{A}(1 - z_I)$. Why is that? Pick an abelian projection g and write

$$g = gz_I + (1 - z_I)\, g.$$

Note that $(1 - z_I)\, g$ is an abelian projection and suppose that $(1 - z_I)\, g \neq 0$. By definition of e we have that e and $(1 - z_I)\, g$ are centrally orthogonal. If $g(1 - z_I)$ is nonzero then it would contradict the maximality of the family e_i, we conclude that there is no nonzero abelian projection in $\mathcal{A}_{1-z_I}$. So far we decomposed $\mathcal{A}$ into $\mathcal{A}_{z_I} \oplus \mathcal{A}_{1-z_I}$, where $\mathcal{A}_{z_I}$ is of type I and $\mathcal{A}_{1-z_I}$ contains no summand of type I.

Now we look at $\mathcal{A}_{1-z_I}$. Let $\{f_j\}_{j\in J}$ be a maximal family of centrally orthogonal finite projections in $\mathcal{A}_{1-z_I}$ and define

$$f := \sum_{j\in J} f_j.$$

By 1.1.1.4 we have that f is finite. Define $z_{II} := z\,(f)$, so z_{II} is the smallest central projection in $\mathcal{A}_{1-z_I}$ dominating f. Let p be a nonzero central projection in $\mathcal{A}_{1-z_I}$ dominated by z_{II}. Then fp is a nonzero finite projection dominated by p. Why is fp finite? Consider

$$f = fp + (1 - p)\, f,$$

and suppose that there is an α such that $fp \sim \alpha \leq fp$. Note that $f = fp + (1 - p)\, f \sim \alpha + f\,(1 - p) \leq f$ hence $\alpha + f\,(1 - p) = f \implies \alpha = fp$ so fp is finite. Note that fp is nonzero because if $fp = 0$ then $0 \leq (z_{II} - f)\,(z_{II} - p) = z_{II} - p - f$ but this is not possible by definition of z_{II}. We conclude that $\mathcal{A}_{z_{II}}$ has no abelian projections and that every nonzero central projection dominates a nonzero finite projection. It follows that $\mathcal{A}_{z_{II}}$ is of type II.

We now have that $\mathcal{A} = \mathcal{A}_{z_I} \oplus \mathcal{A}_{z_{II}} \oplus \mathcal{A}_{1-(z_I+z_{II})}$ where $\mathcal{A}_{z_I}$is of type I and $\mathcal{A}_{z_{II}}$ is of type II. Note that if g is a finite projection in $\mathcal{A}_{1-z_I}$ then $g \leq z_{II}$. Define now $z_{III} := 1 - z_I - z_{II}$ then there is no nonzero finite projection in $\mathcal{A}_{z_{III}}$ and thus $\mathcal{A}_{z_{III}}$ is of type III.

In $\mathcal{A}_{z_{II}}$ pick a maximal family $\{z_k\}_{k\in K}$ of orthogonal central finite projections. Then

$$z_{II_1} := \sum_{k\in K} z_k,$$

is finite. Note that in $\mathcal{A}_{z_{II_1}}$ there is no nonzero abelian projection. Since z_{II_1} is finite it follows that $\mathcal{A}_{z_{II_1}}$ is of type II_1. Define

$$z_{II_\infty} := z_{II} - z_{II_1}.$$

There is no nonzero central finite projection in $\mathcal{A}_{z_{II_\infty}}$, it follows that $\mathcal{A}_{z_{II_\infty}}$ is of type II_∞.

So far we decomposed $\mathcal{A}$ as

$$\mathcal{A} = \mathcal{A}_{z_I} \oplus \mathcal{A}_{z_{II_1}} \oplus \mathcal{A}_{z_{II_\infty}} \oplus \mathcal{A}_{z_{III}}.$$

What remains is to show that this decomposition is unique. Suppose that $a_I + a_{II_1} + a_{II_\infty} + a_{III} = 1$, is another orthogonal decomposition of 1 with the same properties. Consider $(1 - z_I)$ in $\mathcal{A}_{a_I}$, since $1 - z_I$ is a nonzero central projection it must follow that $1 - z_I$ dominates some nonzero abelian projection in $\mathcal{A}_{a_I}$. By construction $1 - z_I$ dominates no nonzero abelian projection, hence $1 - z_I = 0 \in \mathcal{A}_{a_I}$ it follows that $(1 - z_I)\, a_I = 0$, as such, $a_I \leq z_I$. Reversing the role of a_I and z_I we conclude that $z_I \leq a_I$ and hence $z_I = a_I$. We repeat this argument to see that $1 - z_{II_1} \in \mathcal{A}_{a_{II_1}}$ must dominate some nonzero finite projection. But $1 - z_{II_1}$ dominates no nonzero finite projection and hence $a_{II_1}\,(1 - z_{II_1}) = 0$, as such, $a_{II_1} \leq z_{II_1}$. Similarly we conclude $z_{II_1} \leq a_{II_1}$ thus $a_{II_1} = z_{II_1}$. Using similar arguments we conclude that $z_{II_\infty} = a_{II_\infty}$ and $z_{III} = a_{III}$. We conclude that the decomposition is unique. □

A direct consequence is that a factor is either of type I, II_1, II_∞ or type III.

We continue with the study of the corner algebra associated to an abelian projection e.

Proposition 1.1.1.6. *Suppose that $\mathcal{A}$ is a von Neumann algebra and $e \in \mathcal{A}$ is an abelian projection, then $\mathcal{A}_e = \mathcal{Z}(\mathcal{A})e$.*

Proof:

Consider $x \in \mathcal{A}_e$ and suppose that $y \in \mathcal{A}$ commutes with e. Then we find that

$$xy = xey = xye = exeye = eyexe = eyx = yex = yx.$$

It follows that $\mathcal{A}_e \subset [\mathcal{A} \cap \{e\}']' = [\mathcal{A}' \cup \{e\}'']''$. Since $\mathcal{A}_e$ is invariant under multiplication with e it follows that $\mathcal{A}_e \subset e[(\mathcal{A}' \cup \{e\}'')]''e$. Note that the von Neumann algebra $\{e\}''$ is given by

$$\{e\}'' = \{\lambda e + \mu(1-e) \; ; \; \lambda, \mu \in \mathbb{C}\}.$$

Since $\mathcal{A}_e$ is zero on the space $(1-e)\mathcal{H}$ it follows that $\mathcal{A}_e \subset e\mathcal{A}'e$. As such, $\mathcal{A}_e \subset e(\mathcal{A} \cap \mathcal{A}')e$, we conclude that $\mathcal{A}_e \subset \mathcal{Z}(\mathcal{A})e$. On the other hand it is obvious that $\mathcal{Z}(\mathcal{A})e \subset \mathcal{A}_e$, we conclude that $\mathcal{A}_e = \mathcal{Z}(\mathcal{A})e$. □

It follows that if $\mathcal{A}$ is a factor then any abelian projection is automatically minimal.

Theorem 1.1.1.7. *Let $\mathcal{A}$ be a von Neumann algebra and suppose that $e \in \mathcal{A}$ is an abelian projection. Suppose that $f \in \mathcal{A}$ is any projection with $z(f) \geq e$, then $e \preceq f$.*

Proof:

Suppose that $e \succeq f$. Then there exists f_1 such that $f \sim f_1 \leq e$, by proposition 1.1.0.11 we have that $z(f) = z(f_1) \leq z(e) \leq z(f)$, as such, $z(f_1) = z(e)$. Because e is abelian we can use the previous proposition to find a central projection c such that $f_1 = ce$. We find that $z(f_1) = z(ce) = z(e)$, hence $c \geq e$. We conclude that $f_1 = e$ and therefor $f \sim e$. So if $e \succeq f$ then $f = e$, in particular $e \preceq f$.

Suppose now that $f \not\preceq e$. By theorem 1.1.0.16 there exists a central projection g such that $ge \succeq gf$ and $(1-g)e \preceq (1-g)f$. Obviously ge is abelian, as such, $gf = ge$. We find that $e = ge + (1-g)e \preceq fg + (1-g)f = f$, as desired. □

We now proceed to study direct integrals of von Neumann algebras. This will be the prelude to the factor decomposition theorem. When the theory is sufficiently developed we will focus our attention on abelian von Neumann algebras. We do this because we need the properties of abelian von Neumann algebras in order to decompose a general von Neumann algebra over its center.

1.1.2 Direct Integrals of von Neumann algebras

Our aim is to decompose any given von Neumann algebra, working on a separable Hilbert space, into factors. The study of direct integrals in conjunction with the classification of abelian von Neumann algebras provides us with the necessary tools to tackle this problem.

Definition 1.1.2.1. *Let $\{\Gamma, \Omega, \mu\}$ be a measure space with Γ the space, Ω the σ-ring of subsets and μ the measure on Γ. Suppose that $\{\mathcal{H}(\gamma)\}_{\gamma \in \Gamma}$ is a collection of Hilbert spaces indexed by Γ. We call the collection $\{\mathcal{H}(\gamma)\}$ a measurable field of Hilbert spaces when it comes with a subspace $\mathcal{V} \subset \prod_{\gamma \in \Gamma} \mathcal{H}(\gamma)$ such that*

1. *For $\xi \in \mathcal{V}$ the map $\gamma \longrightarrow \|\xi(\gamma)\|$ is μ-measurable.*
2. *For all $\eta \in \prod_{\gamma} \mathcal{H}(\gamma)$ if the map $\gamma \longrightarrow \langle \xi(\gamma), \eta(\gamma) \rangle$ is μ-measurable for all $\xi \in \mathcal{V}$ then $\eta \in \mathcal{V}$.*
3. *There exists a countable subset $\{\xi_n\}_{n \in \mathbb{N}} \subset \mathcal{V}$ such that for all $\gamma \in \Gamma$ the set $\{\xi_n(\gamma)\}_{n \in \mathbb{N}}$ is total in $\mathcal{H}(\gamma)$ (this means that the linear span of $\{\xi_n(\gamma)\}_{n \in \mathbb{N}}$ is dense in $\mathcal{H}(\gamma)$.*

The subspace $\mathcal{V}$ is called the collection of measurable vector fields.

The last condition assures that each Hilbert space $\mathcal{H}(\gamma)$ is separable. The question becomes if such a subspace $\mathcal{V}$ exists. The following proposition settles this question.

Proposition 1.1.2.2. *Suppose that* $\{\mathcal{H}(\gamma)\}_{\gamma\in\Gamma}$ *is collection of Hilbert spaces indexed by the measure space* $\{\Gamma,\Omega,\mu\}$*. If there exists a sequence* $\{\xi_n\}_{n\in\mathbb{N}}$ *in* $\prod_{\gamma\in\Gamma}\mathcal{H}(\gamma)$ *such that for all* $n,m\in\mathbb{N}$ *the map* $\gamma\longrightarrow\langle\xi_n(\gamma),\xi_m(\gamma)\rangle$ *is* μ*-measurable and* $\{\xi_n(\gamma)\}$ *is total in* $\mathcal{H}(\gamma)$*, then the collection*

$$\mathcal{V}:=\left\{\xi\in\prod_{\gamma\in\Gamma}\mathcal{H}(\gamma)\ ;\ \gamma\longrightarrow\langle\xi(\gamma),\xi_n(\gamma)\rangle\ \ is\ \mu-measurable\ for\ every\ n\in\mathbb{N}\right\},$$

satisfies the requirements of definition 1.1.2.1.

Proof:

First of all $\mathcal{V}$ is linear and $\mathcal{V}$ satisfies *1* and *2* of definition 1.1.2.1. We only need to show that $\xi\in\mathcal{V}$ satisfies property *1* of definition 1.1.2.1. Since the sequence $\{\xi_n(\gamma)\}_{n\in\mathbb{N}}$ is total in $\mathcal{H}(\gamma)$ for all $\gamma\in\Gamma$ we can, by considering linear combinations of $\xi_n(\gamma)$ with complex rational coefficients, assume that $\xi_n(\gamma)$ is dense in $\mathcal{H}(\gamma)$. Pick $\xi\in\mathcal{V}$ then we find that

$$\|\xi(\gamma)\|=\sup_{n\in\mathbb{N}}\frac{|\langle\xi(\gamma),\xi_n(\gamma)\rangle|}{\|\xi_n(\gamma)\|},$$

here we set $\frac{|\langle\xi(\gamma),\xi_n(\gamma)\rangle|}{\|\xi_n(\gamma)\|}=0$ when $\|\xi_n(\gamma)\|=0$. Note that the map $\gamma\longrightarrow\frac{|\langle\xi(\gamma),\xi_n(\gamma)\rangle|}{\|\xi_n(\gamma)\|}$ is measurable by construction. Taking the supremum over a countable collection of measurable maps results in a new measurable map, as such, $\gamma\longrightarrow\|\xi(\gamma)\|$ is measurable. $\square$

Now an important proposition regarding these measurable fields of Hilbert spaces.

Proposition 1.1.2.3. *Suppose that* $\{\mathcal{H}(\gamma)\}$ *is a measurable field of Hilbert spaces and suppose that* $\{\eta_k\}_{k\in\mathbb{N}}$ *is a collection satisfying property* 3 *of definition* 1.1.2.1*. Then we can find a sequence* $\{\xi_j\}_{j\in\mathbb{N}}\subset\mathcal{V}$ *satisfying the following two conditions:*

- *If* $\dim(\mathcal{H}(\gamma))=n(\gamma)$*, with* $n(\gamma)\in\mathbb{N}\cup\infty$ *then* $\{\xi_1(\gamma),\cdots,\xi_{n(\gamma)}(\gamma)\}$ *is a orthonormal basis for* $\mathcal{H}(\gamma)$ *for every* $\gamma\in\Gamma$*.*
- *If* $j>n(\gamma)$ *then* $\xi_j(\gamma)=0$*.*

In particular $\{\xi_j\}_{j\in\mathbb{N}}$ *also satisfies property* 3 *of definition* 1.1.2.1*. Also* $\gamma\longrightarrow\dim(\mathcal{H}(\gamma))$ *is measurable.*

Proof:

For each $\gamma\in\Gamma$ pick $j_1(\gamma)\in\mathbb{N}$ such that $\eta_{j_1(\gamma)}(\gamma)\neq 0$ and define

$$\xi_1(\gamma):=\frac{\eta_{j_1(\gamma)}(\gamma)}{\left\|\eta_{j_1(\gamma)}(\gamma)\right\|}.$$

If for some γ_0 we could not pick $\eta_{j_1(\gamma_0)}(\gamma_0)$ such that $\eta_{j_1(\gamma_0)}(\gamma_0)$ is nonzero (meaning that $\mathcal{H}(\gamma_0)=\{0\}$) then we set $\xi_1(\gamma_0)=0$. Note that by construction we have that $\xi_1\in\mathcal{V}$. For every $\gamma\in\Gamma$ we select some $j_2(\gamma)$ such that $\eta_{j_2(\gamma)}(\gamma)\perp\xi_1(\gamma)$ and $\eta_{j_2(\gamma)}(\gamma)$ is nonzero, if for some γ we have that $\mathrm{span}(\xi_1(\gamma))=\mathcal{H}(\gamma)$ then for that γ we pick $\eta_{j_2(\gamma)}(\gamma)=0$. We can select $\eta_{j_2(\gamma)}(\gamma)$ in this way because the set $\eta_k(\gamma)$ is total in $\mathcal{H}(\gamma)$ for all γ. We define

$$\xi_2(\gamma):=\frac{\eta_{j_2(\gamma)}(\gamma)}{\left\|\eta_{j_2(\gamma)}(\gamma)\right\|},$$

by construction the map $\gamma\longrightarrow\xi_2(\gamma)$ is measurable. Inductively we construct ξ_n as above. We find that $\{\xi_n\}_{n\in\mathbb{N}}$ satisfies the requirements. Note that the map $\gamma\longrightarrow\langle\xi_n(\gamma),\xi_m(\gamma)\rangle$ is nonzero if and only if $n=m$. Suppose that $n=m$ and set $F_n:\Gamma\longrightarrow\mathbb{C}$, $F_n(\gamma):=\langle\xi_n(\gamma),\xi_n(\gamma)\rangle$ then $F_n^{-1}(\{1\})=\{\gamma\ ;\ \dim(\mathcal{H}_\gamma)\geq n\}$, also $\left[F_n^{-1}(\{0\})\right]^c=F_n^{-1}(\{1\})$, and both these sets are measurable for all n. Pick $n_0\in\mathbb{N}\cup\{\infty\}$ maximal such that $\dim(\mathcal{H}(\gamma))\geq n_0$ for all $\gamma\in\Gamma$, it then follows that the set $E_{n_0}:=\{\gamma\ ;\ \dim(\mathcal{H}(\gamma))=n_0\}=F_{n_0+1}^{-1}\{0\}$ is measurable (with the convention that $\infty-1=\infty$). Via similar arguments we find that the set $E_n:=\{\gamma\ ;\ \dim(\mathcal{H}(\gamma))=n\}$ is measurable, as desired. $\square$

Note that this is basically an elaborate application of the standard orthogonalization procedure. The properties of definition 1.1.2.1 are such that the resulting collection $\{\xi_n\}_{n\in\mathbb{N}}$ is measurable. The benefit of this proposition is that we can split the space Γ into measurable subspaces Γ_n such that for all $\gamma \in \Gamma_n$ it holds that $\dim(\mathcal{H}(\gamma)) = n$, we will use this later on. We can now construct the so-called direct integral of the measurable field $\{\mathcal{H}(\gamma)\}_{\gamma\in\Gamma}$. From now on we assume that the measure space $\{\Gamma, \Omega, \mu\}$ is σ-finite.

Consider those $\xi \in \mathcal{V}$ such that

$$\|\xi\|^2 := \int_\Gamma \|\xi(\gamma)\|^2 \ d\mu < \infty.$$

and denote this space by $\mathcal{H}$. It is obvious that $\mathcal{H}$ forms a vector space. It comes equipped with an inner product

$$\langle \xi, \eta \rangle := \int_\Gamma \langle \xi(\gamma), \eta(\gamma) \rangle \ d\mu,$$

After identifying ξ_1 and ξ_2 when $\xi_1(\gamma) = \xi_2(\gamma)$ μ almost everywhere, we find that $\mathcal{H}$ becomes a Hilbert space.

Definition 1.1.2.4. *The Hilbert space $\mathcal{H}$ constructed above is called the direct integral of the collection $\{\mathcal{H}(\gamma)\}$ with respect to the measure μ. It is denoted as*

$$\int_\Gamma^\oplus \mathcal{H}(\gamma) \ d\mu := \mathcal{H}.$$

Note that by the last proposition we can decompose Γ as

$$\Gamma = \bigcup_{n\in\mathbb{N}\cup\{\infty\}} \Gamma_n,$$

such that for each $\gamma \in \Gamma_n$ it holds that $\dim(\mathcal{H}(\gamma)) = n$. It follows that

$$\int_\Gamma^\oplus \mathcal{H}(\gamma) \ d\mu = \bigoplus_{n\in\mathbb{N}\cup\{\infty\}} \int_{\Gamma_n}^\oplus \mathcal{H}(\gamma) \ d\mu.$$

With this decomposition it suffices to study measurable fields of Hilbert spaces $\{\mathcal{H}(\gamma)\}$ such that $\dim(\mathcal{H}(\gamma_1)) = \dim(\mathcal{H}(\gamma_2))$ for all $\gamma_1, \gamma_2 \in \Gamma$. Note that if the map $\gamma \longrightarrow \dim(\mathcal{H}(\gamma))$ is constant over Γ, which we can assume it to be, then there exists a Hilbert space $\mathcal{H}_0$ such that $\mathcal{H}(\gamma) \cong \mathcal{H}_0$ for all γ. In this situation the direct integral $\int_\Gamma^\oplus \mathcal{H}_\gamma \ d\mu$ is naturally isomorphic to the square integrable functions $f : \Gamma \longrightarrow \mathcal{H}_0$.
We now proceed to study particular kinds of operators on $\int_\Gamma^\oplus \mathcal{H}(\gamma) \ d\mu$.

Definition 1.1.2.5. *Consider a collection $\{x(\gamma)\}_{\gamma\in\Gamma}$ of bounded operators such that $x(\gamma) \in \mathcal{B}(\mathcal{H}(\gamma))$ for all γ and the map $\gamma \longrightarrow \|x(\gamma)\|$ defines an element in $L^\infty(\Gamma, \mu)$. Then we say that $\{x(\gamma)\}$ is a measurable collection of operators if the vector $\{x(\gamma)h(\gamma)\}_{\gamma\in\Gamma}$ is an element of $\int_\Gamma^\oplus \mathcal{H}(\gamma) \ d\mu$ for all $h \in \int_\Gamma^\oplus \mathcal{H}(\gamma) \ d\mu$.*

In this definition we require that the collection $\{x(\gamma)\}$ is essentially bounded. One can omit this requirement from the definition, however we want to construct a *bounded* operator x from the collection $\{x(\gamma)\}$, when doing so we will need the requirement that $\{x(\gamma)\}$ is essentially bounded. For that reason it is included in the definition. Also note that all the measurable requirements are contained in the condition that the collection $\{x(\gamma)h(\gamma)\}_{\gamma\in\Gamma}$ is again measurable.

Definition 1.1.2.6. *Given a collection $\{x(\gamma)\}_{\gamma\in\Gamma}$ satisfying the requirements of definition* 1.1.2.5*, then we define an operator x as follows:*

$$x : \int_\Gamma^\oplus \mathcal{H}_\gamma \ d\mu \longrightarrow \int_\Gamma^\oplus \mathcal{H}_\gamma \ d\mu, \qquad xh := \int_\gamma^\oplus x(\gamma)h(\gamma) \ d\mu.$$

Operators of this form are called decomposable operators. If for each $\gamma \in \Gamma$, $x(\gamma)$ is a scalar then we call x a diagonal operator. The collection of diagonal operators is called the diagonal algebra associated to $\int_\Gamma^\oplus \mathcal{H}_\gamma \ d\mu$. We denote the diagonal algebra associated to $\mathcal{H}$ by $\mathcal{D}(\mathcal{H})$, the decomposable operators are denoted with $\mathcal{M}(\mathcal{H})$.

Note that definition 1.1.2.5 was chosen in such a way that x is a bounded operator. Given decomposable operators x and y, a direct calculation reveals that

$$x^* = \int_\Gamma^\oplus x(\gamma)^* \, d\mu, \qquad xy = \int_\Gamma^\oplus x(\gamma)y(\gamma) \, d\mu.$$

By construction the product of two decomposable operators is again decomposable. We identify elements in $\mathcal{D}(\mathcal{H})$ and $\mathcal{M}(\mathcal{H})$ when they are equal μ almost everywhere. Suppose that $\{x_i\}_{i\in I} \subset \mathcal{M}(\mathcal{H})$ converges strongly to some element $x \in \mathcal{B}(\mathcal{H})$. Since $\{x_i\}_{i\in I}$ is strongly convergent we find that for μ almost all γ that $\{x_i(\gamma)\}$ converges strongly to some $x(\gamma)$. It follows that $x = \int_\Gamma^\oplus x(\gamma) \, d\mu$. We find that $\mathcal{M}(\mathcal{H})$ is a von Neumann algebra on $\mathcal{H}$. Similarly we find that $\mathcal{D}(\mathcal{H})$ is a von Neumann algebra on $\mathcal{H}$.

Proposition 1.1.2.7. *Let* $\mathcal{H} := \int_\Gamma^\oplus \mathcal{H}_\gamma \, d\mu$, *then* $\mathcal{M}(\mathcal{H}) = \mathcal{D}(\mathcal{H})'$.

Proof:

Obviously $\mathcal{M}(\mathcal{H}) \subset \mathcal{D}(\mathcal{H})'$. On the other hand suppose that $x \in \mathcal{D}(\mathcal{H})'$. We decompose $\mathcal{H}$ as

$$\mathcal{H} = \int_\Gamma^\oplus \mathcal{H}(\gamma) \, d\mu = \bigoplus_{n\in\mathbb{N}\cup\{\infty\}} \int_{\Gamma_n}^\oplus \mathcal{H}(\gamma) \, d\mu,$$

here for each $\gamma \in \Gamma_n$, $\dim(\mathcal{H}(\gamma)) = n$. Since x commutes with $\mathcal{D}(\mathcal{H})$ we conclude that

$$x : \int_{\Gamma_n}^\oplus \mathcal{H}(\gamma) \, d\mu \longrightarrow \int_{\Gamma_n}^\oplus \mathcal{H}(\gamma) \, d\mu,$$

for all $n \in \mathbb{N} \cup \{\infty\}$. As such we may assume that the field of Hilbert spaces $\{\mathcal{H}(\gamma)\}$ is constant, that is, there exists some Hilbert space $\mathcal{H}_0$ such that $\mathcal{H}(\gamma) = \mathcal{H}_0$ for all $\gamma \in \Gamma$. Furthermore since Γ is σ-finite we can decompose Γ into a countable family of disjoint measurable subsets K_n such that each K_n has finite measure. Using the assumption that x commutes with $\mathcal{D}(\mathcal{H})$ once more we find that

$$x : \int_{K_n}^\oplus \mathcal{H}(\gamma) \, d\mu \longrightarrow \int_{K_n}^\oplus \mathcal{H}(\gamma) \, d\mu,$$

for all $n \in \mathbb{N}$. Because of this we may assume that $\mu(\Gamma) < \infty$, consequently, for each $h \in \mathcal{H}_0$, the map $\widehat{h} : \Gamma \longrightarrow \mathcal{H}_0$ defined by $\widehat{h}(\gamma) = h$ is an element of $\int_\Gamma^\oplus \mathcal{H}(\gamma) \, d\mu$. If F is a measurable set of Γ we denote by 1_F the map that is 0 when $\gamma \notin F$ and 1 when $\gamma \in F$. Recall that simple functions, functions of the form:

$$s : \Gamma \longrightarrow \mathcal{H}_0, \qquad s(\gamma) = \sum_{i=1}^n \lambda_i 1_{F_i}(\gamma)\widehat{h}_i(\gamma),$$

where $\{F_i\}_{i=1}^n$ is a finite disjoint partition of Γ, are dense in the $\int_\Gamma^\oplus \mathcal{H}(\gamma) \, d\mu$. We will show that there exists a decomposable operator $\int_\Gamma^\oplus x(\gamma) \, d\mu$ that agrees with x on this dense subset. Then, by density, this identity extends to all of $\int_\Gamma^\oplus \mathcal{H}(\gamma) \, d\mu$. Consider a simple function s, since $x \in \mathcal{D}(\mathcal{H})'$ we find that $x(s)$ is of the form:

$$x(s) = \sum_{i=1}^n \lambda_i \cdot x\left(1_{F_i}\widehat{h_i}\right) = \sum_{i=1}^n \lambda_i 1_{F_i} \cdot x\left(\widehat{h_i}\right).$$

We conclude that if we can find an operator $\int_\Gamma^\oplus x(\gamma) \, d\mu$ that agrees with x on all functions of the form $\widehat{h}$, with $h \in \mathcal{H}_0$ then we are done. Since $\mathcal{H}_0$ is separable we only need to check equality on a countable dense subset of $\mathcal{H}_0$. Let $\{h_n\}_{n\in\mathbb{N}}$ be a countable dense subset of $\mathcal{H}_0$, for each h_n we define $x(\gamma)h_n := x\left(\widehat{h_n}\right)(\gamma)$. Note that for each h_n this is well defined up to a set of measure zero. Since countable unions of sets of measure zero are still of measure zero we conclude that $\{x(\gamma)\}_{\gamma\in\Gamma}$ is well defined, up to a set of measure zero, on the collection $\{h_n\}_{n\in\mathbb{N}}$. For $h_j \in \{h_n\}_{n\in\mathbb{N}}$ we find that

$$\begin{aligned} x\widehat{h_j} &= \int_\Gamma^\oplus x\left(\widehat{h_j}\right)(\gamma)\ d\mu \\ &= \int_\Gamma^\oplus x(\gamma)h_j\ d\mu \\ &= \int_\Gamma^\oplus x(\gamma)\widehat{h_j}(\gamma)\ d\mu, \end{aligned}$$

as desired. We have found a decomposable operator that agrees with x, that is, x is decomposable and we are done. □

Consider a collection of von Neumann algebras $\{\mathcal{A}(\gamma)\}_{\gamma\in\Gamma}$ indexed by a σ-finite measure space Γ. We assume that the collection $\{\mathcal{A}(\gamma)\}_{\gamma\in\Gamma}$ acts pointwise on a collection $\{\mathcal{H}(\gamma)\}$ of Hilbert spaces.

Definition 1.1.2.8. *We call the collection $\{\mathcal{A}(\gamma)\}$ measurable if there exists a countable family $\{x_n\}_{n\in\mathbb{N}}$ of measurable operator fields such that $\{x_n(\gamma)\}_{n\in\mathbb{N}}$ generates $\mathcal{A}(\gamma)$ for almost all $\gamma \in \Gamma$.*

We now proceed to define direct integrals of von Neumann algebras, proposition 1.1.2.7 makes this an easy job.

Definition 1.1.2.9. *Given a measurable collection $\{\mathcal{A}(\gamma)\}_{\gamma\in\Gamma}$ of von Neumann algebras acting pointwise on $\{\mathcal{H}(\gamma)\}_{\gamma\in\Gamma}$ we then define the von Neumann algebra $\mathcal{A}$, acting on $\int_\Gamma^\oplus \mathcal{H}(\gamma)\ d\mu$, as those decomposable operators x such that $x(\gamma) \in \mathcal{A}(\gamma)$. Here we identify two decomposable operators when they agree almost everywhere.*

It is a priori not immediate that $\mathcal{A}$ is indeed a von Neumann algebra. Since $\mathcal{A}$ contains $\mathcal{D}(\mathcal{H})$ it follows by 1.1.2.7 that the commutant of $\mathcal{A}$ consists of decomposable operators. We find that $\mathcal{A}'$ consist of those decomposable operators y such that $y(\gamma) \in \mathcal{A}(\gamma)'$. Since $\mathcal{A}(\gamma)$ is a von Neumann algebra for all $\gamma \in \Gamma$ it follows that $\mathcal{A}'' = \mathcal{A}$. We denote the von Neumann algebra $\mathcal{A}$ by

$$\int_\Gamma^\oplus \mathcal{A}(\gamma)\ d\mu,$$

and call it the direct integral of the collection $\{\mathcal{A}(\gamma)\}_{\gamma\in\Gamma}$ with respect to μ. By our last argument we immediately find that

$$\left(\int_\Gamma^\oplus \mathcal{A}(\gamma)\ d\mu\right)' = \int_\Gamma^\oplus \mathcal{A}(\gamma)'\ d\mu \qquad \text{and} \qquad \mathcal{Z}\left(\int_\Gamma^\oplus \mathcal{A}(\gamma)\ d\mu\right) = \int_\Gamma^\oplus \mathcal{Z}\left(\mathcal{A}(\gamma)\right)\ d\mu.$$

Our aim is to decompose any given von Neumann algebra $\mathcal{A}$ acting on $\mathcal{H}$ as a direct integral of factors. In order to do so we need to construct a measure space $\{\Gamma, \mu\}$ and collections $\{\mathcal{A}(\gamma),\ \mathcal{H}(\gamma)\}_{\gamma\in\Gamma}$ with $\mathcal{A}(\gamma)$ acting on $\mathcal{H}(\gamma)$ such that $\mathcal{A} \cong \int_\Gamma^\oplus \mathcal{A}(\gamma)\ d\mu$ and $\mathcal{H} \cong \int_\Gamma^\oplus \mathcal{H}(\gamma)\ d\mu$. In order to do so we first need to study abelian von Neumann algebras, this is because the center $\mathcal{Z}(\mathcal{A})$ of $\mathcal{A}$ will play a crucial role in the factor decomposition. Since $\mathcal{Z}(\mathcal{A})$ is an abelian algebra we first need to study how it behaves and classify it as far as we can.

1.2 Abelian von Neumann algebras

In this section we will cover the classification of maximal abelian von Neumann algebras, we will begin with the classification of abelian unital C^* algebras before proceeding to von Neumann algebras. Compared to abelian C^* algebras, the classification of abelian von Neumann algebras as an algebra of L^∞ functions on some compact space measure space is involved. The large part of this section is based on lecture notes and [1]. We start of by briefly recalling the classification result of unital abelian C^*-algebras, stating:

Theorem 1.2.0.10. *If $\mathcal{A}$ is a unital abelian C^*-algebra then there exists a compact Hausdorff space X such that*

$$\mathcal{A} \cong C(X).$$

Since every von Neumann algebra is also a C^*-algebra this also holds for von Neumann algebras.

Given an abelian C^*-algebra $\mathcal{A}$, consider the set of nonzero homomorphisms $h : \mathcal{A} \longrightarrow \mathbb{C}$ and denote it by $\Sigma(\mathcal{A})$. By a homomorphism h we mean that h must respect the ring structure, the vector space structure and also the *-structure on $\mathcal{A}$, so

$$\begin{aligned} &h(\lambda a + b) = \lambda h(a) + h(b); \\ &h(ab) = h(a) h(b); \\ &h(1) = 1; \\ &h(a^*) = \overline{h(a)}. \end{aligned}$$

By the following argument any homomorphism on a C^*-algebra is continuous. Pick $h \in \Sigma(\mathcal{A})$ and let $a \in \mathcal{A}$. Suppose that $|h(a)| > \|a\|$, denote $\lambda := h(a)$ and consider the element $1 - \frac{a}{\lambda}$. The element $1 - \frac{a}{\lambda}$ is invertible with inverse $b := \sum_{n=0}^{\infty} \left(\frac{a}{\lambda}\right)^n$. Note that $\sum_{n=0}^{\infty} \left(\frac{a}{h(a)}\right)^n$ is convergent because $\|a\| < |h(a)|$. As such, we find that

$$\begin{aligned} 1 &= \left(1 - \frac{a}{\lambda}\right) b \\ &= h\left(1 - \frac{a}{\lambda}\right) h(b) \\ &= \left(1 - \frac{h(a)}{\lambda}\right) h(b) = 0, \end{aligned}$$

clearly a contradiction. it follows that $|h(a)| \leq \|a\|$ and h is continuous (in fact it has norm equal to 1). A consequence is that $\Sigma(\mathcal{A})$ is a subset of $\mathcal{A}^*$ (the dual of $\mathcal{A}$).

Proposition 1.2.0.11. *If $\mathcal{A}$ is a unital C^*-algebra then there exists a canonical bijection between the space of all homomorphisms of $\mathcal{A}$ and the maximal ideals of $\mathcal{A}$.*

Proof:

Let $\mathcal{I}$ be a maximal ideal in $\mathcal{A}$, then the quotient algebra is a field and also a C^*-algebra. We conclude that the quotient algebra is isomorphic to $\mathbb{C}$. It follows that the canonical map $h : \mathcal{A} \longrightarrow \mathcal{A}/\mathcal{I}$ satisfies the following properties: it is linear and multiplicative, $h(1) = 1$ and it respects the *-structure, thus it is a homomorphism. On the other hand, given a nonzero homomorphism h, by continuity its kernel $\mathcal{I}_h$ is a closed ideal of $\mathcal{A}$. Since $\mathcal{A}/\mathcal{I}_h \cong \mathbb{C}$ we conclude that $\mathcal{I}_h$ is maximal. □

For $a \in \mathcal{A}$ we denote by $\sigma(a)$ its spectrum, that is, $\sigma(a) := \{\lambda \in \mathbb{C} \ ; \ a - \lambda \text{ is not invertible}\}$. For any $a \in \mathcal{A}$ the set $\sigma(a)$ is nonempty and it is compact. Note that if $h \in \Sigma(\mathcal{A})$ then $h(a) \in \sigma(a)$ because $a - h(a) \in \ker(h)$, which is a maximal ideal of $\mathcal{A}$, in particular $a - h(a)$ is not invertible. Conversely if $\lambda \in \sigma(a)$ then $a - \lambda$ is not invertible, as such, $a - \lambda$ is in some maximal ideal of $\mathcal{A}$ meaning that there exists a homomorphism h such that $h(a - \lambda) = 0$. We conclude that $h(a) = \lambda$ and $\sigma(a) = \Sigma(a) := \{h(a) \ ; \ h \in \Sigma(\mathcal{A})\}$.

Equip $\Sigma(\mathcal{A})$ with the weak* topology it inherits from $\mathcal{A}^*$, thus $h_i \longrightarrow h$ iff for all $a \in \mathcal{A}$ we have $h_i(a) \longrightarrow h(a)$. We define the map $\rho : \mathcal{A} \longrightarrow C(\Sigma(\mathcal{A}), \text{weak}^*)$ by setting

$$\rho(a)(h) := h(a).$$

This map ρ is called the Gelfand transform. Note that the Gelfand transform really sends a to a continuous function on $\Sigma(\mathcal{A})$ because if $h_i \longrightarrow h$ then for all $a \in \mathcal{A}$ we have $h_i(a) \longrightarrow h(a)$, but $h_i(a) = \rho(a)(h_i)$ so $\rho(a)(h_i) \longrightarrow \rho(a)(h)$ as desired.

Theorem 1.2.0.12. *If ρ denotes the Gelfand transform then ρ defines an isomorphism between $\mathcal{A}$ and $C(\Sigma(\mathcal{A}), \textit{weak}^*)$.*

Proof

Since $\mathcal{A}$ is abelian every element $a \in \mathcal{A}$ is normal, as such, we have, by the spectral radius formula, that $\|a\| = \sup\{|\lambda| \,;\, \lambda \in \sigma(a)\}$. By our previous arguments we can rewrite this as $\|a\| = \sup\{|h(a)| \,;\, h \in \Sigma(\mathcal{A})\}$. Thus if a is nonzero then there exists some nonzero element in $\sigma(a)$ we conclude that $\rho(a)$ is not the zero function, hence ρ is injective. By the spectral radius formula we have $\|a\| = \|\rho(a)\|$ thus ρ is an isometry. It is easy to see that ρ is linear and multiplicative, also $\rho(a^*)(h) := h(a^*) = \overline{h(a)} = \rho(a)^*(h)$ thus it is a *-homomorphism. The only thing that remains is to see that ρ is surjective. Note that because ρ is an isometry its range is closed. Also its range contains the scalar functions and is closed under the star operation (since ρ preserves the star operation). Note also that its range separates points in $\Sigma(\mathcal{A})$ because given two distinct homomorphisms h_1, h_2 in $\Sigma(\mathcal{A})$ they are distinct because there is an $a \in \mathcal{A}$ such that $h_1(a) \neq h_2(a)$. By the Stone-Weierstrass theorem [1, p.145] we have that its range is dense. Using the fact that its range is also closed we conclude $\rho(\mathcal{A}) = C(\Sigma(\mathcal{A}), \text{weak}^*)$ as desired. □

We have now classified (unital) C^*-algebras as algebras of functions on a compact space. For completeness we shall describe its space of homomorphisms further. Given a compact space X and let $\mathcal{A} = C(X)$ be its C^*-algebra of functions. By our previous arguments $\Sigma(\mathcal{A}) \cong X$ but how precisely?
Given a homomorphism h on $C(X)$ then in particular $h \in C(X)^*$ (recall that the dual of $C(X)$ is the space of all finite Radon measures on X) thus there exists some measure μ such that

$$h(f) = \int_X f d\mu.$$

Since $h(1) = 1$ we have

$$1 = h(1) = \int_X d\mu,$$

thus μ is a probability measure on X. Note that $h(f)$ makes sense for any bounded μ-measurable function f. Consider the following, pick any measurable set F then $1 = \mu(F) + \mu(F^c)$ and also $0 = \mu(F)\mu(F^c)$ thus either $\mu(F) = 1$ or $\mu(F) = 0$. Since singleton sets are measurable $\mu(\{x\}) = 1$ for some singleton set $\{x\}$. We conclude $h = \delta_x$. As such, all homomorphisms are of this form and the correspondence $X \cong \Sigma(\mathcal{A})$ becomes clear via the map $x \longrightarrow \delta_x$.

We now proceed to classify maximal abelian von Neumann algebras on a separable Hilbert space. We start of by examining how the von Neumann algebra generated by an abelian C^*-algebra looks like.

Consider a unital abelian C^*-algebra $\mathcal{A}$, it follows that $\mathcal{A} \cong C(X)$ for some compact Hausdorff space. Suppose that μ is some positive measure on X. Using μ we can represent $\mathcal{A}$ on $\mathcal{B}(L^2(X,\mu))$ via the map $\pi_\mu : \mathcal{A} \longrightarrow \mathcal{B}(L^2(X,\mu))$ defined by $\pi_\mu(f)g(x) := f(x)g(x)$. Thus π_μ represents $\mathcal{A}$ as the collection of all continuous functions on X acting on $L^2(X,\mu)$ by multiplication.

Definition 1.2.0.13. *We say that a vector $\xi \in L^2(X,\mu)$ is cyclic for $\pi_\mu(\mathcal{A})$ when*

$$\overline{\{\pi_\mu(A)\xi\}} = L^2(X,\mu).$$

We note that in our case we have

$$\overline{\{\pi_\mu(\mathcal{A}) 1_X\}} = L^2(X,\mu);$$
$$\mu(a) = \langle \pi_\mu(a) 1_X, 1_X \rangle.$$

These assertions are easily verified. This means that this representation is equivalent to the one obtained from μ via the GNS construction (see for more reading on this [1, 8]). So we now have represented $\mathcal{A}$ as a C^*-sub-algebra of $\mathcal{B}(L^2(X,\mu))$. Consider the von Neumann algebra generated by $\{\pi_\mu(\mathcal{A})\}$ in $\mathcal{B}(L^2(X,\mu))$. Denote this von Neumann algebra by $\mathcal{M}(\pi_\mu(\mathcal{A}))$.

Theorem 1.2.0.14. *The von Neumann algebra $\mathcal{M}(\pi_\mu(\mathcal{A}))$ is of the form $L^\infty(X,\mu)$.*

Proof:

Since $\mathcal{A}$ is abelian we have the following inclusion:

$$\pi_\mu(\mathcal{A}) \subset \pi_\mu(\mathcal{A})' = \mathcal{M}(\pi_\mu(\mathcal{A}))',$$

where the last equality follows from the double commutant theorem [8, p.74]. If $f \in L^\infty(X,\mu)$, then it follows that $\pi_\mu(f)$ commutes with $\pi_\mu(\mathcal{A})$, hence $\pi_\mu(L^\infty(\mu)) \subset \mathcal{M}(\pi_\mu(\mathcal{A}))'$. On the other hand if $a \in \mathcal{M}(\pi_\mu(\mathcal{A}))'$ is positive then a defines a linear functional $\widehat{a}$ on $\pi_\mu(\mathcal{A})$ via

$$\widehat{a}(\pi(g)) := \langle a\pi(g)1_X, 1_X\rangle.$$

If we can show that $\widehat{a}$ is bounded then we can, by density, extend $\widehat{a}$ to $L^1(X,\mu)$. Suppose for the moment that we have shown that $\widehat{a}$ is bounded, it then follows then its extension defines a bounded linear functional on $L^1(X,\mu)$. As such a defines an element in $L^1(X,\mu)^*$. It follows that $a = \pi_\mu(f)$ for some function f in $L^\infty(X,\mu)$. We have then showed that $\pi_\mu(L^\infty(X,\mu)) \subset \mathcal{M}(\pi_\mu(\mathcal{A}))' \subset \pi_\mu(L^\infty(\mu))$ and equality follows.

Now to show that $\widehat{a}$ is bounded. We assumed that a was positive but this really is no restriction since every operator is the sum of its real part and i times its imaginary part, which are both self-adjoint, and every self-adjoint operator is the difference between two positive operators (this follows from the functional calculus for C^*-algebras). Consider $\widehat{a}(\pi_\mu(g))$ and let $g = uh$ with u a partial isometry and h a positive element in $\mathcal{A}$ (h is of the following form $h = |g| := \sqrt{(g^*g)}$). Such a decomposition exists and is called the polar decomposition of g (see for details 6.3.0.18) . We find

$$\begin{aligned}
|\widehat{a}(\pi(g))| &= |\langle a\pi(g)1_X.1_X\rangle| \\
&= \left|\left\langle a^{1/2}u\pi_\mu(|g|)^{1/2}1_X, a^{1/2}\pi_\mu(|g|)^{1/2}1_X\right\rangle\right| \\
&\le \left\|a^{1/2}\pi_\mu(|g|)^{1/2}1_X\right\|^2 \\
&\le \|a\| \cdot \left\|\pi_\mu(|g|)^{1/2}1_X\right\|^2 \\
&= \|a\| \int_X |g|\, d\mu,
\end{aligned}$$

we conclude that $\widehat{a}$ is bounded with $\|\widehat{a}\| \le \|a\|$. It follows by our previous consideration that $\mathcal{M}(\pi_\mu(\mathcal{A}))' = \pi_\mu(L^\infty(\mu))$. A nice consequence of this is that $\mathcal{M}(\pi_\mu(\mathcal{A}))'$ is abelian, as such, $\mathcal{M}(\pi_\mu(\mathcal{A}))' \subset \mathcal{M}(\pi_\mu(\mathcal{A}))'' = \mathcal{M}(\pi_\mu(\mathcal{A}))$. On the other hand we have that $\mathcal{M}(\pi_\mu(\mathcal{A})) \subset \mathcal{M}(\pi_\mu(\mathcal{A}))'$, we conclude

$$\mathcal{M}(\pi_\mu(\mathcal{A})) = \mathcal{M}(\pi_\mu(\mathcal{A}))'.$$

It follows that

$$\mathcal{M}(\pi_\mu(\mathcal{A})) = \pi_\mu(L^\infty(\mu)),$$

as desired. □

The representation π_μ need not to be injective, this depends on the measure μ chosen to represent $\mathcal{A}$. If μ respects open sets in the sense that if U is open then $\mu(U) > 0$ then the representation is injective.

We will now proceed to classify abelian von Neumann algebras. First we introduce the notion of star-cyclic vectors, it will help us to decompose normal operators to a direct sum of normal operators acting on a separable Hilbert space.

Definition 1.2.0.15. *Let $A \in \mathcal{B}(\mathcal{H})$, a vector $\xi_0 \in \mathcal{H}$ is called star-cyclic when the smallest reducing subspace for A containing ξ_0 is the whole Hilbert space $\mathcal{H}$. A vector ξ_0 is called cyclic when the smallest invariant subspace for A containing ξ_0 is $\mathcal{H}$. An operator A will be called star-cyclic (resp. cyclic) when it has a star-cyclic (resp. cyclic) vector.*

Recall that a subspace $\mathcal{V}$ is called reducing for A when it is invariant for both A and A^*. Note that ξ_0 is star-cyclic for A if and only if ξ_0 is a cyclic vector for $C^*(A)$. If there exists a star-cyclic operator A with star-cyclic vector ξ_0 then $\mathcal{H}$ is separable. This follows because $\{p(A)\xi_0 \ ; \ p \text{ is a polynomial}\}$ is a countable dense set in $\mathcal{H}$.

Consider a compactly supported measure μ on $\mathbb{C}$ with support K and define an operator A_μ on $L^2(K,\mu)$ by setting $A_\mu f = zf$. The C^*-algebra generated by A_μ is isomorphic to $C(K)$ so $C^*(A_\mu)$ acts on $L^2(K,\mu)$ as multiplication by continuous functions. Since $C(K)$ is dense in $L^2(K,\mu)$ we conclude that χ_K is a cyclic vector for $C^*(A_\mu)$, as such, χ_K is star-cyclic for A_μ. This provides us with an example of a star-cyclic operator. The following theorem states that this is the only type of example (for normal operators).

Theorem 1.2.0.16. *A normal operator A is star-cyclic if and only if it is unitarily equivalent to A_μ for some compactly supported measure μ on $\mathbb{C}$. If ξ_0 is a star-cyclic vector for A then μ can be chosen such that there exists a unique isomorphism $U : L^2(\mu) \longrightarrow \mathcal{H}$ with $U(1) = \xi_0$ and $U^{-1}AU = A_\mu$.*

Proof:

Suppose that A is normal and star-cyclic with vector ξ_0. Let $A = \int z \, dE$ be the spectral decomposition of A. Define a measure μ on $\sigma(A)$ by setting $\mu(E) := \langle P(E)\xi_0, \xi_0\rangle$. It follows that μ is a positive measure which assigns measure 1 to $\sigma(A)$ (after scaling of ξ_0). Define $U : L^2(\mu) \longrightarrow \mathcal{H}$ by

$$U(\phi) := \phi(A)\xi_0 := \left(\int \phi \, dP\right)\xi_0.$$

The map U is obviously linear, consider $\langle U(\phi), U(\phi)\rangle$, we find

$$\begin{aligned} \langle U(\phi), U(\phi)\rangle &= \langle \phi(A)\xi_0, \phi(A)\xi_0\rangle \\ &= \left\langle |\phi|^2(A)\xi_0, \xi_0\right\rangle \\ &= \int |\phi|^2 \, d\mu = \|\phi\|^2, \end{aligned}$$

as such, U is an isometry. Now note that $C^*(A) \cong C(\sigma(A))$ is dense in $L^2(\mu)$. Using that ξ_0 is star-cyclic we find that

$$\begin{aligned} U\left(L^2(\mu)\right) &= U\left(\overline{C(\sigma(A))}\right) \\ &= \overline{\{f(A)\xi_0 \ ; \ f \in C(\sigma(A))\}} \\ &= \overline{\{B\xi_0 \ ; \ B \in C^*(A)\}} = \mathcal{H}. \end{aligned}$$

We conclude that U is onto. Note that $U(1) = \xi_0$ and that

$$U^{-1}AU(\phi) = UA\left(\int \phi \, dP\right)\xi_0 = U^{-1}\left(\int z\phi \, dP\right)\xi_0 = \int z\phi \, dP = A_\mu(\phi).$$

It remains to be shown that U is unique. Suppose that U satisfies $U(1) = \xi_0$ and $U^{-1}AU = A_\mu$. Consider $U(z)$, we find

$$U(z) = UA_\mu(1) = AU(1) = A(\xi_0).$$

By induction we find $U(z^n) = A^n\xi_0$. Also $U(\overline{z}) = A_\mu^*(1) = U\left(U^{-1}A^*U\right)(1) = A^*(\xi_0)$. By these arguments we conclude that if $B \in C^*(A)$ and ϕ is its image in $C(\sigma(A))$, then $U(\phi) = B(\xi_0)$. But now we can use that $C(\sigma(A))$ is dense in $L^2(\mu)$ to conclude that $U(1) = \xi_0$ and $U^{-1}AU = A_\mu$ completely determines U, so U is unique. □

Let A be normal in $\mathcal{B}(\mathcal{H})$, for a vector $\xi \in \mathcal{H}$ define $\mathcal{H}_\xi$ as

$$\mathcal{H}_\xi := \overline{\operatorname{span}\{A^{*n}A^m(\xi) \ ; \ n, m \in \mathbb{N}\}}.$$

It follows that $\mathcal{H}_\xi = \overline{C^*(A)\xi}$ so that ξ is cyclic for $\mathcal{H}_\xi$. It follows that ξ is star-cyclic for $A_{|\mathcal{H}_\xi}$. By Zorn's lemma it follows now that every normal operator A is a direct sum of star-cyclic operators. One simply picks a maximal set of vectors $\{\xi_i\}_{i\in I}$ with the property that $\xi_i \notin \mathcal{H}_{\xi_j}$ when $i \neq j$, then it follows that

$$\mathcal{H} = \bigoplus_{i\in I} \mathcal{H}_{\xi_i} \qquad \text{and} \qquad A = \bigoplus_{i\in I} A_{|\mathcal{H}_{\xi_i}}.$$

By combining these results we can now state and prove the following theorem.

Theorem 1.2.0.17. *If A is normal then there exists a measure space X, a measure μ and a function $\phi \in L^\infty(X,\Omega,\mu)$ such that N is unitarily equivalent to A_μ on $L^2(X,\Omega,\mu)$.*

Proof:

Let A be normal, there are reducing subspaces $\mathcal{H}_i$ such that $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ and $A_{|\mathcal{H}_i}$ is star-cyclic. Denote by A_i the restriction of A to $\mathcal{H}_i$, so $A_i = A_{|\mathcal{H}_i}$. By theorem 1.2.0.16 it follows that $A_i \cong A_{\mu_i}$. Also note that $\sigma(A_i) \subset \sigma(A)$ for all i. Let X_i denote the support of the measure μ_i and let X be the disjoint union of the X_i. Let Ω be the collection of subsets of X defined by $\{E \subset X \; ; \; E \cap X_i \text{ is a Borel set of } X_i\}$ and let $U_i : L^2(X_i,\mu_i) \longrightarrow \mathcal{H}_i$ be as in theorem 1.2.0.16. We conclude that $A = \bigoplus A_i \cong \bigoplus A_{\mu_i}$, set $\mu(E) := \sum_i \mu_i(E \cap X_i)$ for $E \in \Omega$. It follows that $L^2(X,\Omega,\mu) = \bigoplus_i L^2(X_i,\mu_i)$ and that $U := \bigoplus_i U_i : L^2(X,\Omega,\mu) \longrightarrow \mathcal{H}$ is a unitary equivalence between A and $A_\mu := \bigoplus_i A_{\mu_i}$. For each i set $\phi_i(z_i) = z_i$ and let $\phi = \bigoplus_i \phi_i$ then $A \cong \phi$. Note that ϕ is bounded because X_i is bounded for each i, as such, $\phi \in L^\infty(X,\Omega,\mu)$ and A is unitarily equivalent to multiplying with ϕ. $\square$

Now we are ready to classify maximal abelian von Neumann algebras. Let (X,Ω,μ) be a measure space let and $f \in L^\infty(X,\Omega,\mu)$, define for f an operator M_f on $L^2(X,\Omega,\mu)$ by setting

$$M_f(\phi) = f \cdot \phi.$$

Denote by $\mathcal{A}_\mu$ the algebra $\{M_f \; ; \; f \in L^\infty(X,\Omega,\mu)\} \subset B\left(L^2(X,\Omega,\mu)\right)$ then M_f is a maximal abelian von Neumann algebra (by the discussion at the beginning of this section). The following theorem states that this is the only type of maximal abelian von Neumann algebra.

Theorem 1.2.0.18. *Let $\mathcal{H}$ be separable and let $\mathcal{A}$ be an abelian C^*-sub algebra of $\mathcal{B}(\mathcal{H})$, then the following are equivalent:*

1. *$\mathcal{A}$ is a maximal abelian von Neumann algebra, that is, $\mathcal{A} = \mathcal{A}'$;*
2. *$\mathcal{A}$ has a cyclic vector, contains 1 and $\mathcal{A}$ is closed in the strong operator topology;*
3. *There exists a compact metric space X, a positive Borel measure μ with support X and an isomorphism $U : L^2(X,\mu) \longrightarrow \mathcal{H}$ such that $U\mathcal{A}_\mu U^{-1} = \mathcal{A}$.*

Proof:

We shall prove $1 \Longrightarrow 2 \Longrightarrow 3 \Longrightarrow 1$.

$1 \Longrightarrow 2$
That $\mathcal{A}$ contains 1 follows from the fact that $\mathcal{A} = \mathcal{A}'$, that it is closed in the strong operator topology follows from the definition of a von Neumann algebra. It remains to be shown that $\mathcal{A}$ has a cyclic vector. If $\mathcal{V}$ is reducing for $\mathcal{A}$ call $\mathcal{V}$ minimal reducing when there exists no non trivial subspace $\mathcal{N} \subset \mathcal{V}$ such that $\mathcal{N}$ is also reducing for $\mathcal{A}$. Pick a maximal family of minimal reducing subspaces $\mathcal{H}_i$. It follows that $\mathcal{H}_i$ are orthogonal subspaces, because if $\mathcal{V}$ and $\mathcal{F}$ are reducing then so is $\mathcal{V} \cap \mathcal{F}$ contradicting the minimality of $\mathcal{H}_i$. Let p_i denote the projection onto $\mathcal{H}_i$, since $\mathcal{H}_i$ is reducing for $\mathcal{A}$ we have that if $A \in \mathcal{A}$ and $x_i \in \mathcal{H}_i$ then $Ap_i(x) \in \mathcal{H}_i$ so $p_iAp_i = Ap_i$. On the other hand $p_iA = (A^*p_i)^* = (p_iA^*p_i)^* = p_iAp_i = Ap_i$, thus p_i commutes with $\mathcal{A}$, that is, $p_i \in \mathcal{A}' = \mathcal{A}$. Let $p := \sum_i p_i$ then $p = 1$ because if not then $1 - p$ is a projection commuting with $\mathcal{A}$ so $(1-p)\mathcal{H}$ is a reducing subspace, contradicting the maximality of our family $\mathcal{H}_i$. So we conclude the following: $\mathcal{H} = \bigoplus_i \mathcal{H}_i$ and $\mathcal{A} = \bigoplus \mathcal{A}p_i$. Set $\mathcal{A}_i := \mathcal{A}p_i$ Obviously $\mathcal{H}_i$ is reducing for $\mathcal{A}_i$. Note that if $x_i \in \mathcal{H}_i$ then $\overline{\mathcal{A}_i x_i}$ is a reducing subspace for $\mathcal{A}_i$, by the minimality of $\mathcal{H}_i$ we conclude that $\overline{\mathcal{A}_i x_i} = \mathcal{H}_i$. We find that any nonzero $x_i \in \mathcal{H}_i$ is cyclic for $\mathcal{A}_i$. Since $\mathcal{H}$ is separable it

follows that I is at most countable. For each $i \in I$ select some nonzero $x_i \in \mathcal{H}_i$ of norm 1 and define x as follows

$$x := \sum_{i \in I} \frac{x_i}{2^i}.$$

It follows that

$$\overline{\mathcal{A}x} = \overline{\bigoplus_i \mathcal{A}_i x_i} = \bigoplus_i \mathcal{H}_i = \mathcal{H},$$

as such $\mathcal{A}$ has a cyclic vector.

$2 \Longrightarrow 3$

Pick a countable weakly dense subset of the unit ball of $\mathcal{A}$ and let $\mathcal{A}_1$ be the C^*-algebra generated by this set. Since the weak closure and strong closure agree on convex sets it follows that $\mathcal{A}$ is the strong closure of $\mathcal{A}_1$. Let X be the maximal ideal space of $\mathcal{A}_1$, it follows that the inverse Gelfand transform $\rho : C(X) \longrightarrow \mathcal{A}_1$ is given by integration with respect to some spectral measure P see for details the the prelude to theorem 6.3.0.16. If ϕ is a bounded Borel measurable function on X then it follows that $\rho(\phi) \in \mathcal{A}$ because $\mathcal{A}$ is strong closed. Let x_0 be the cyclic vector for $\mathcal{A}$ and define $\mu(E) := \langle P(E) x_0, x_0 \rangle$ for every Borel measurable set $E \subset X$. Let $B(X)$ be the space of all bounded Borel measurable functions on X and identify the functions that agree μ almost everywhere, that is, $\phi \sim \psi \Longleftrightarrow \mu(\{\phi \neq \psi\}) = 0$. With this identification $B(X)$ becomes a dense subspace of $L^2(X, \mu)$. For $\phi \in B(X)$ we find that

$$\begin{aligned}
\left\| \left(\int \phi \; dP \right) x_0 \right\|^2 &= \left\langle \left(\int \phi \; dP \right)^* \left(\int \phi \; dP \right) e_0, e_0 \right\rangle \\
&= \left\langle \left(\int \overline{\phi} \phi \; dP \right) x_0, x_0 \right\rangle \\
&= \int |\phi|^2 d\mu = \|\phi\|^2 .
\end{aligned}$$

It follows that the map $U : B(X) \longrightarrow \mathcal{H}$, $U(\phi) = \left(\int \phi dP \right) x_0$ defines an isometry from $B(X)$ into $L^2(\mu)$ and its range is dense because x_0 is cyclic. Hence U extends to an isomorphism from $L^2(X, \mu)$ onto $\mathcal{H}$. Let $\mathcal{A}_\mu := \{M_\phi \; ; \; \phi \in L^\infty(X, \mu)\}$ where $M_\phi(f) = \phi \cdot f$ acts on $L^2(X, \mu)$. Considering $UM_\phi(f)$ with $\phi \in L^\infty(X, \mu)$ and $f \in L^2(X, \mu)$, we find

$$\begin{aligned}
UM_\phi f = U(\phi f) &= \left(\int \phi f \; dP \right) x_0 \\
&= \left(\int \phi \; dP \right) \left(\int f \; dP \right) x_0 \\
&= \left(\int \phi \; dP \right) U(f) .
\end{aligned}$$

It follows that $UM_\phi U^{-1} = \int \phi \; dP$, as such, $U\mathcal{A}_\mu U^{-1} \subset \mathcal{A}$. For the reverse inclusion we note that $\mathcal{A} = \overline{\mathcal{A}_1} \subset \overline{U\mathcal{A}_\mu U^{-1}} \subset \overline{\mathcal{A}} = \mathcal{A}$, here the closure is taken in the strong operator topology.

$3 \Longrightarrow 1$

We have our representation π_μ sending $\mathcal{A}$ to $B\left(L^2(X, \mu)\right)$. At the beginning of this section we concluded that $\pi_\mu(\mathcal{A})$ is maximal abelian but now we have $\mathcal{A} \cong \mathcal{A}_\mu$ because the representation π_μ is now injective. □

This concludes the characterization of abelian algebras on separable Hilbert space. Note that in the last theorem we assumed $\mathcal{H}$ to be separable, though the statement remains (largely) true if $\mathcal{H}$ is non separable, some properties are still lost, X is in this case no longer metrizable.

From 1.2.0.18 we obtain the following result. Suppose that $S \subset \mathcal{B}(\mathcal{H})$ is a subset such that for all $a, b \in S$ it holds that $ab = ba$ then there exists a separating vector for S. Indeed S is contained in a maximal abelian von Neumann algebra $\mathcal{A}$, consequently $\mathcal{A}$ is equipped with a cyclic vector ξ. Since ξ is cyclic for $\mathcal{A}$ it is separating or $\mathcal{A}' = \mathcal{A}$ (see 4.1.1.3), in particular ξ is separating for S.

Proposition 1.2.0.19. *If $\mathcal{A}$ is a von Neumann algebra then $\mathcal{A}$ is the norm closed linear span of its projections.*

Proof:

Let $A \in \mathcal{A}$, and suppose that A is selfadjoint. Consider the von Neumann algebra generated by the identity and A, denoted by $W^*(A)$. it could be that $W^*(A)$ is not maximal abelian, however there is always a subspace $\mathcal{V} \subset \mathcal{H}$ such that $W^*(A)$ is maximal abelian on $\mathcal{V}$. It follows that $W^*(A) \cong L^\infty(X, \mu)$. The image of A in $L^\infty(X, \mu)$ is the identity function on X. Since every $L^\infty(X, \mu)$ is the limit of simple functions it follows that there are projections $\{e_i\}$ in $W^*(A)$ such that A is an element of the norm closed linear span of the set $\{e_i\}$. If A is not selfadjoint then we decompose A into its real and imaginary part to obtain the result. □

Remark

Note that the proof of the previous proposition only works in the case that $\mathcal{A}$ acts on a separable Hilbert space. However, we can decompose any selfadjoint operator as a direct sum of star-cyclic operators, which all act on a separable Hilbert space. This allows us to generalize to von Neumann algebras acting on Hilbert spaces of arbitrary dimension.

We now proceed to investigate the generator sets of an abelian von Neumann algebra. As noted before the set of projections associated to any von Neumann algebra $\mathcal{A}$ generate $\mathcal{A}$. The goal is to show that if $\mathcal{A}$ is an abelian von Neumann algebra on a separable Hilbert space then $\mathcal{A}$ is generated by a single selfadjoint element.

Consider an abelian von Neumann algebra $\mathcal{A}$ on a separable Hilbert space and let $\{a_i\}_{i\in\mathbb{N}}$ be any countable set of generators for $\mathcal{A}$. Denote by $W^*(a_i)$ the abelian von Neumann algebra generated by a_i, since $W^*(a_i) \cong L^\infty(\sigma(a_i), \mu_i)$ we find that $W^*(a_i)$ is countably generated by its projections, consequently $\mathcal{A}$ is generated by a countable collection of commuting projections.

Theorem 1.2.0.20. *Suppose that $\mathcal{A}$ is generated by a countable collection of commuting projections then there exists a selfadjoint element A such that $\mathcal{A} = W^*(A)$. Consequently any abelian von Neumann algebra on a separable Hilbert space is generated by a single element.*

Proof:

Let $\{P_n\}_{n\in\mathbb{N}}$ be a countable collection of projections generating $\mathcal{A}$ and define $A := \sum_{n=1}^{\infty} 3^{-n} P_n$. Consider $P_1 A$, we find that

$$\frac{1}{3}P_1 \leq P_1 A \leq P_1 \left(\sum_{n=1}^{\infty} 3^{-n} \cdot 1 \right) = \frac{1}{2}P_1.$$

Consider $(1 - P_1)A$, we find that

$$0 \leq (1 - P_1)A = \sum_{n=2}^{\infty} 3^{-n} P_n (1 - P_1) \leq \sum_{n=2}^{\infty} 3^{-n} P_n.$$

An application of the functional calculus yields that $\sigma(A) \subset [0, \frac{1}{6}] \cup [\frac{1}{3}, \frac{1}{2}]$. Let E denote the spectral measure associated to A, we find that

$$\int_{\sigma(A)} 1_{[\frac{1}{3}, \frac{1}{2}]}(x) \; dE \in W^*(A).$$

It follows that $P_1 A = \int_{[\frac{1}{3}, \frac{1}{2}]} x \; dE$ and $P_1 = \int 1_{[\frac{1}{3} \cdot \frac{1}{2}]} \; dE$. We conclude that $P_1 \in W^*(A)$. We subtract $\frac{1}{3}P_1$ from A and apply the same argument to find that P_2 in $W^*(A)$. Inductively we find that $P_n \in W^*(A)$, as such, $W^*(A) = \mathcal{A}$. □

Let $\mathcal{A}$ be any abelian von Neumann algebra on a separable Hilbert space and let $a \in \mathcal{A}$ be such that $\mathcal{A} = W^*(a)$. Pick a separating vector ξ for $\mathcal{A}$ and let E denote the spectral measure associated to a, it follows that the measure μ defined on $\sigma(a)$ by $\mu(F) = \langle E(F)\xi, \xi \rangle$ implements an isomorphism between $\mathcal{A}$ and $L^\infty(\sigma(a), \mu)$.

In the next section we will finish off the factor decomposition theorem, using the structure theory of abelian von Neumann algebras developed in this section.

1.2.1 Continuation of the factor decomposition

In this section we finish of the factor decomposition using the properties of abelian von Neumann algebras. We concluded that any abelian von Neumann algebra $\mathcal{A}$ is generated by a single selfadjoint element $a \in \mathcal{A}$, this gave rise to the measure space $\{\sigma(a), \mu\}$ equipped with the measure derived from the spectral measure associated to a and a separating vector ξ. Note that this in fact finishes the factor decomposition theorem for abelian von Neumann algebras, namely, for $\gamma \in \sigma(a)$ let $\mathcal{H}(\gamma) = \mathbb{C}$ then it follows that $\mathcal{A}$ is isomorphic to the diagonal algebra of $\int_{\sigma(a)}^{\oplus} \mathcal{H}(\gamma)\, d\mu$. Moreover $\mathcal{A}(\gamma) \cong \mathbb{C}$, as such, $\mathcal{A}(\gamma)$ is a factor for every γ. To extend this to an arbitrary von Neumann algebra $\mathcal{A}$ on a separable Hilbert space $\mathcal{H}$ we will decompose $\mathcal{A}$ over its center, for which we now have a good description.

We will give a description of how the factor decomposition is achieved without proving all the details.

Suppose that we are given a von Neumann algebra $\mathcal{A}$ acting on a separable Hilbert space $\mathcal{A}$. Let $\mathcal{Z}$ denote its center, it follows that $\mathcal{Z} \cong L^\infty(X, \mu)$ for some compact Hausdorff space X. In fact we can choose X to be a compact subset of $\mathbb{R}$ but we will not need this. Pick any nonzero vector $\xi_1 \in \mathcal{H}$ and consider the space $[\mathcal{Z}\xi_1]$, that is, $[\mathcal{Z}\xi_1]$ is the closed linear span of $\mathcal{Z}\xi_1 \subset \mathcal{H}$. Let $f_1 \in \mathcal{B}(\mathcal{H})$ denote the projection onto $[\mathcal{Z}\xi_1]$ and define a projection $p_1 \in \mathcal{A}$ by setting

$$p_1 := \inf\{e \in \mathcal{A}\ ;\ \text{e is a projection and } e \geq f_1\}.$$

This is well defined because the projection space of $\mathcal{A}$ is a complete lattice. Set $z_1 := z(p_1)$, that is, z_1 is the central carrier of p_1, it follows that $z_1 \in \mathcal{Z}$, as such, z_1 gives rise to a measurable subset $Y_1 \subset X$.

Consider now the Hilbert space $(1 - f_1)\mathcal{H}$ and pick $\xi_2 \in [\mathcal{Z}\xi_1]^\perp$. Again we let f_2 be the projection onto the space $[\mathcal{Z}\xi_2]$ and we construct p_2 and z_2 as above. It follows that z_2 gives rise to a measurable subset Y_2 of X, note that Y_1 and Y_2 need not be disjoint, in fact they seldom are.

We go on and pick $\xi_3 \in [[\mathcal{Z}\xi_1] \oplus [\mathcal{Z}\xi_2]]^\perp$ and construct a measurable subset $Y_3 \subset X$. We repeat this procedure until we have exhausted $\mathcal{H}$. Since $1 \in \mathcal{Z}$, the space $[\mathcal{Z}\xi_i]$ is at least one dimensional, using that $\mathcal{H}$ is separable we do indeed exhaust $\mathcal{H}$. In this construction we pick ξ_i such that $\|\xi_i\| = 1$. Set $\mathcal{H}_i := [\mathcal{Z}\xi_i]$, we find that

$$\mathcal{H} = \bigoplus_{i \in \mathbb{N}} \mathcal{H}_i.$$

Pick $i \in \mathbb{N}$ and consider the vector ξ_i, since $\|\xi_i\| = 1$ it induces a state ϕ_i on $\mathcal{Z}$ by setting $\phi(f) = \langle f\xi_i, \xi_i \rangle$. Since every functional of this form is ultraweakly continuous it follows that the set $\{g \in \mathcal{Z}\ ;\ \phi_i(g) = 0\}$ forms a weakly closed ideal of $\mathcal{Z}$. We conclude that $\ker(\phi_i)$ is of the from $\mathcal{Z}z_0$, with z_0 a projection in $\mathcal{Z}$. Obviously z_0 is the largest projection of $\mathcal{Z}$ such that $z_0\xi_i = 0$ it follows that $(1 - z_0)\xi_i = \xi_i$. Since z_0 is the largest projection that annihilates ξ_i it follows that $1 - z_0$ is the smallest projection in $\mathcal{Z}$ that satisfies $z\xi_i = \xi_i$, as such, $1 - z_0 = z_i$. We conclude that $\mathcal{Z}/\ker(\phi)$ is faithfully represented on $\mathcal{H}_i$. It follows that $\mathcal{H}_i \cong \{[\mathcal{Z}/\ker(\phi_i)], \langle \cdot, \cdot \rangle_{\phi_i}\} = L^2(Y_i, \mu)$. This isomorphism is implemented by the map $\rho_i : (f\xi_i) = \widehat{f} 1_{Y_i}$ where we let $\widehat{f}$ be the element in $L^\infty(X, \mu)$ corresponding to f. We find that modulo unitary equivalence it holds that

$$\mathcal{H} = \bigoplus_{i \in \mathbb{N}} \mathcal{H}_i = \bigoplus_{i \in \mathbb{N}} L^2(Y_i, \mu) = L^2\left(\coprod_{i \in \mathbb{N}} (Y_i, \mu)\right).$$

Here the space $\coprod_{i\in\mathbb{N}}(Y_i,\mu)$ should be interpreted as the disjoint union of the measure spaces (Y_i,μ). Our next aim is to construct for each $x \in X$ a Hilbert space $\mathcal{H}_x$ such that

$$\int_X \mathcal{H}_x \, d\mu = L^2\left(\coprod_{i\in\mathbb{N}}(Y_i,\mu)\right).$$

Since pointwise operations are not well defined on equivalence classes of functions we will pick a concrete measurable set Y_i for each i instead of working with its equivalence class. Since this does not alter the isomorphism constructed above this is allowed. For each $x \in X$ we consider the set $I_x \subset \mathbb{N}$ defined as follows:

$$I_x := \{i \in \mathbb{N} \; ; \; x \in Y_i\}.$$

We define a map $D : X \longrightarrow \mathbb{N} \cup \{\infty\}$ as $D(x) := |I_x|$, that is, $D(x)$ is the cardinality of the set I_x. The map D will play the role of the dimension of the fiber Hilbert space above x. Note that the following holds:

$$L^2(\coprod_{i\in\mathbb{N}} Y_i,\mu) := \int^{\oplus}_{\coprod_i Y_i} \mathbb{C} \, d\mu = \int^{\oplus}_X L^2(I_x) \, d\mu.$$

So we now have a direct integral of Hilbert spaces isomorphic to our original Hilbert space. We now need to construct the von Neumann algebras associated to each $x \in X$. Note that in any case that if $f \in \mathcal{Z}$ then f acts as

$$f\left(\int^{\oplus}_X h(x) \, d\mu\right) = \int^{\oplus}_X \widehat{f}(x)h(x) \, d\mu,$$

with $f(x)$ the value of f at the point x, note that this is almost everywhere defined. We conclude that the representation of $\mathcal{Z}$ on $L^2(I_x)$ reduces to the scalars for all $x \in X$. In particular $\mathcal{Z}$ is simply the diagonal algebra associated to $\int^{\oplus}_X L^2(I_x) \, d\mu$. By 1.1.2.7 we find that every operator is decomposable. Suppose that $\{a_n\}_{n\in\mathbb{N}}$ defines a generating set for $\mathcal{A}$ then we may decompose each a_n as

$$a_n = \int^{\oplus}_X a_n(x) \, d\mu.$$

We define a von Neumann algebra $\mathcal{A}(x)$ on $L^2(I_x)$ by

$$\mathcal{A}(x) = W^*\left(\{a_n(x) \; ; \; n \in \mathbb{N}\}\right),$$

it follows that

$$\mathcal{A} = \int^{\oplus}_X \{\mathcal{A}(x), \; L^2(I_x)\} \, d\mu.$$

Since the center $\mathcal{Z}$ of $\mathcal{A}$ reduces to the scalars for all $x \in X$ we find that the decomposition above is indeed a factor decomposition, as desired. □

Note that this construction in fact reduces the study of von Neumann algebras on separable Hilbert space to the study of factors. In the next section we will reduce the study of type I algebras to the study of abelian algebras and $\mathcal{B}(\mathcal{H})$. Since we already have a good classification result regarding abelian algebras there is not much more left in the case type I.

1.3 Structure theory for type I and type II von Neumann algebras

In this section we investigate the tensor product of von Neumann algebras. After developing the basics we give a decomposition of type I von Neumann algebra into a direct sum of tensor products. We will give a similar decomposition of type II algebras. We will mostly follow [8], we assume that the construction of the algebraic tensor product between vector spaces is known.

1.3.1 Tensor products of von Neumann algebras

We will first consider tensor products of Hilbert spaces, after this we will define the tensor product of von Neumann algebras.

Consider Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote the algebraic tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$ as $\mathcal{H}_0$. A general element $h \in \mathcal{H}_0$ is of the form

$$h = \sum_{i=1}^{n} h_i^1 \otimes h_i^2,$$

with $h_i^1 \in \mathcal{H}_1$ and $h_i^2 \in \mathcal{H}_2$ for all i. On the space $\mathcal{H}_0$ we define an inner product $\langle \cdot, \cdot \rangle_0$ by

$$\left\langle \left(\sum_{i=1}^{n} h_i^1 \otimes h_i^2 \right), \left(\sum_{j=1}^{m} k_j^1 \otimes k_j^2 \right) \right\rangle_0 := \sum_{i=1}^{n} \sum_{j=1}^{m} \left\langle h_i^1, k_j^1 \right\rangle_1 \cdot \left\langle h_i^2, k_j^2 \right\rangle_2,$$

here $\langle \cdot, \cdot \rangle_1$ and $\langle \cdot, \cdot \rangle_2$ denote the inner products on $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively. We denote the completion of $\mathcal{H}_0$, with respect to this inner product, by $\mathcal{H}$ and call it the tensor product of $\mathcal{H}_1$ and $\mathcal{H}_2$. The space $\mathcal{H}$ is usually denoted by

$$\mathcal{H} := \mathcal{H}_1 \otimes \mathcal{H}_2.$$

Consider an orthonormal basis $\{f_i\}_{i \in I}$ of $\mathcal{H}_2$ and for each $i \in I$ define a map $U_i : \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$ by setting

$$U_i(h_1) := h_1 \otimes f_i.$$

We find that U_i defines a isometry from $\mathcal{H}_1$ into $\mathcal{H}$, if $i \neq j$ then U_i and U_j have orthogonal ranges, as such the map U defined as

$$U : \bigoplus_{i \in I} \mathcal{H}_1 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2, \qquad U\left(\bigoplus_{i \in I} h_i^1 \right) = \bigoplus_{i \in I} U_i(h_i^1) = \sum_{i \in I} h_i^1 \otimes f_i,$$

is an isometry from $\bigoplus_{i \in I} \mathcal{H}_1$ into $\mathcal{H}_1 \otimes \mathcal{H}_2$. Note that if $\{e_k\}_{k \in K}$ and $\{f_i\}_{i \in I}$ are orthonormal bases for $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively then $\{e_k \otimes f_i\}_{(k,i) \in K \times I}$ is a basis for $\mathcal{H}_1 \otimes \mathcal{H}_2$. It follows that the range of U (as defined above) contains this basis, as such, U is onto and therefor an isomorphism of Hilbert spaces. By this argument we can regard $\mathcal{H}_1 \otimes \mathcal{H}_2$ as the square summable functions $g : I \longrightarrow \mathcal{H}_1$. Using a similar argument we can show that $\mathcal{H}_1 \otimes \mathcal{H}_2$ is isomorphic to the space of square summable functions $h : K \longrightarrow \mathcal{H}_2$.

By these arguments we find that, for a fixed basis $\{f_i\}_{i \in I}$ of $\mathcal{H}_2$, an element in $h \in \mathcal{H}_1 \otimes \mathcal{H}_2$ can be uniquely represented as

$$h = \sum_{i \in I} h_i \otimes f_i.$$

If U_i is defined as above then

$$U_i^* \left(\sum_j h_j \otimes f_j \right) = h_i,$$

this follows from the construction of U_i. Consider an element $h \in \mathcal{H}_1 \otimes \mathcal{H}_2$, it follows that h is of the form

$$h = \sum_i h_i \otimes f_i.$$

If $x \in \mathcal{B}(\mathcal{H})_1$ then $x \otimes 1$ acts on $\mathcal{H}$ as

$$x \otimes 1(h) = \sum_i x(h_i) \otimes 1(f_i) = \sum_i x(h_i) \otimes f_i,$$

similarly for $y \in \mathcal{B}(\mathcal{H})_2$, $1 \otimes y$ acts on $\mathcal{H}$ as

$$1 \otimes y(h) = \sum_i h_i \otimes y(e_i).$$

The operator $x \otimes y = (x \otimes 1)(1 \otimes y)$ is therefor bounded when both $(x \otimes 1)$ and $(1 \otimes y)$ are bounded. For $x \otimes 1$ we find that

$$\begin{aligned}\|x \otimes 1(h)\|^2 = \left\| \sum_i x(h_i) \otimes f_i \right\|^2 &= \left\langle \sum_i x(h_i) \otimes f_i, \sum_i x(h_i) \otimes f_i \right\rangle \\ &= \sum_i \|x(h_i)\|^2 \leq \|x\|^2 \sum_i \|h_i\|^2 \\ &= \|x\|^2 \left\| \sum_i h_i \otimes f_i \right\|^2 .\end{aligned}$$

We conclude that $x \otimes 1$ is bounded, a similar argument shows that $1 \otimes y$ is bounded, consequently $x \otimes y$ is bounded for all $x \in \mathcal{B}(\mathcal{H})_1$ and $y \in \mathcal{B}(\mathcal{H})_2$. The operator $x \otimes y$ is called the tensor product of x and y. The argument above shows that $\|x \otimes y\| = \|x\| \cdot \|y\|$. The tensor product of operators satisfies the following properties:

$$\begin{aligned}&(\lambda_1 x_1 + \lambda_2 x_2) \otimes y = \lambda_1 x_1 \otimes y + \lambda_2 x_2 \otimes y = x_1 \otimes \lambda_1 y + x_2 \otimes \lambda_2 y = \lambda_1(x_1 \otimes y) + \lambda_2(x_2 \otimes y), \\ &x \otimes (\mu_1 y_1 + \mu_2 y_2) = \mu_1(x \otimes y_1) + \mu_2(x \otimes y_2) = \mu_1 x \otimes y_1 + \mu_2 x \otimes y_2 = x \otimes \mu_1 y + x \otimes \mu_2 y_2, \\ &(x_1 \otimes y_1)(x_2 \otimes y_2) = x_1 x_2 \otimes y_1 y_2, \\ &(x \otimes y)^* = x^* \otimes y^*, \\ &\|x \otimes y\| = \|x\| \cdot \|y\| .\end{aligned}$$

Definition 1.3.1.1. *Suppose that $\mathcal{A}_1$ is a von Neumann algebra on $\mathcal{H}_1$ and that $\mathcal{A}_2$ is a von Neumann algebra on $\mathcal{H}_2$. Then we define the von Neumann algebra on $\mathcal{H}_1 \otimes \mathcal{H}_2$ as*

$$\mathcal{A}_1 \otimes \mathcal{A}_2 := W^* \left(\{(x \otimes y) \ ; \ x \in \mathcal{A}_1, \ y \in \mathcal{A}_2\} \right).$$

Suppose that x defines an operator on $\mathcal{H}_1 \otimes \mathcal{H}_2$. We identify $\mathcal{H}_1 \otimes \mathcal{H}_2$ with the square summable functions $f : I \longrightarrow \mathcal{H}_1$, here I is an index set for the basis of $\mathcal{H}_2$. By our arguments above

$$\mathcal{H}_1 \otimes \mathcal{H}_2 \cong \bigoplus_{i \in I} \mathcal{H}_i,$$

here $\mathcal{H}_i$ is a copy of $\mathcal{H}_1$. Consider the operator x on $\mathcal{H}_1 \otimes \mathcal{H}_2$ and let $\{U_i\}_{i \in I}$ be defined as above. It follows that $U_j^* x U_i$ defines a bounded operator on $\mathcal{H}_1$, as such, we may regard $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ as the $I \times I$ matrices with entries from $\mathcal{B}(\mathcal{H}_1)$. Of course not all possible matrices can occur (except when $\mathcal{H}_2$ if finite dimensional) because of the boundedness of each element in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$. We identify the operator x with its $I \times I$ matrix $(x_{i,j})$, here $x_{i,j} = U_j^* x U_i$.

Suppose that an operator $x \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is of the form $x_1 \otimes 1$. Consider $U_j^* x U_i$, we find that

$$U_j^*(x_1 \otimes 1)U_i(h_1) = \begin{cases} x_1(h_1) & \text{when } i = j, \\ 0 & \text{when } i \neq j, \end{cases}$$

so $x_1 \otimes 1$ is the $I \times I$ matrix with x_1 on the diagonal and zero elsewhere. Consider the operator $U_i U_j^* : \mathcal{H}_1 \otimes \mathcal{H}_2 \longrightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$. If $x = x_1 \otimes 1$, with $x_1 \in \mathcal{B}(\mathcal{H}_1)$ then

$$\begin{aligned}&U_i U_j^* x(h) = U_i x_1(h_j) = x_1(h_j) \otimes e_i \\ &x U_i U_j^*(h) = x U_i(h_j) = x \otimes 1(h_j \otimes e_i) = x_1(h_j) \otimes e_i,\end{aligned}$$

we conclude that operators x of the form $x_1 \otimes 1$ commute with all operators of the form $U_i U_j^*$. Conversely suppose that x commutes with $U_i U_j^*$ for all i, j. We find that $U_i U_j^* x U_j = x U_i U_j^* U_j = x U_i$ thus $U_i^* x U_i = U_i^* U_i U_j^* x U_j = U_j^* x U_j$, as such, x is constant along the diagonal. Note furthermore that $U_j^* x U_i = U_j^* x U_i U_i^* U_i = U_j^* U_i U_i^* x U_i = 0$ because $U_j^* U_i = 0$. So x is constant along the diagonal and zero elsewhere, we conclude that $x = x_1 \otimes 1$ with $x_1 \in \mathcal{B}(\mathcal{H}_1)$. Consequently $\{U_i U_j^* \ ; \ i, j \in I\}' = \mathcal{B}(\mathcal{H}_1) \otimes \mathbb{C}$, here $\mathbb{C}$ should be understood as the von Neumann algebra $\mathbb{C}$ acting on the Hilbert space $\mathcal{H}_2$.

Consider the representation $\pi : \mathcal{B}(\mathcal{H}_1) \longrightarrow \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ defined by $\pi(x_1) := x_1 \otimes 1$. The properties of the tensor product show that this is indeed a representation, in fact it is an isomorphism.

Proposition 1.3.1.2. *If $S \subset \mathcal{B}(\mathcal{H}_1)$ is any subset then $\pi(S'') = \pi(S)''$.*

Proof:

Since $\pi(S) \subset \pi(\mathcal{B}(\mathcal{H}_1))$ we find that $U_i U_j^* \in \pi(S)'$ for all $i, j \in I$. It follows that $\pi(S)'' \subset \pi(\mathcal{B}(\mathcal{H}_1)$ thus it follows that $\pi(S)'' = \pi(S'')$. □

Using this we are now able to prove the following proposition.

Proposition 1.3.1.3. *Suppose that $\{\mathcal{A}, \mathcal{H}_1\}$ is a von Neumann algebra and $\mathcal{H}_2$ is a Hilbert space. Then the following statements hold:*

1. $(\mathcal{A} \otimes \mathbb{C})' = \mathcal{A}' \otimes \mathcal{B}(\mathcal{H}_2)$,
2. $(\mathcal{A} \otimes \mathcal{B}(\mathcal{H}_2))' = \mathcal{A}' \otimes \mathbb{C}$,
3. *$\mathcal{A} \otimes \mathcal{B}(\mathcal{H}_2)$ consists of the $I \times I$, with I an index set for the basis of $\mathcal{H}_2$, matrices with entries from $\mathcal{A}$.*

Proof:

3.
Suppose that $\{f_i\}_{i \in I}$ is an orthonormal basis for $\mathcal{H}_2$ then in we define $u_{i,j} : \mathcal{H}_2 \longrightarrow \mathcal{H}_2$ as $u_{i,j}(g) := \langle g, f_j \rangle e_i$. The operator $1 \otimes u_{i,j}$ acting on $\mathcal{H}_1 \otimes \mathcal{H}_2$ satisfies $1 \otimes u_{i,j} \left(\sum_{k \in I} h_k \otimes e_k \right) = h_j \otimes e_i$ thus $U_i U_j^* = 1 \otimes u_{i,j}$. For $x \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ we set $x_{i,j} := U_i^* x U_j$ as operator from $\mathcal{H}_1$ to itself. We conclude that

$$x = \sum_{i,j} x_{i,j} \otimes u_{i,j}.$$

in the strong operator topology. If we can show that $x_{i,j} \in \mathcal{A}$ for all i, j then it follows that $x \in \mathcal{A} \otimes \mathcal{B}(\mathcal{H}_2)$. For $y \in \mathcal{B}(\mathcal{H}_1)$ we set $\kappa_{i,j}(y) := y \otimes u_{i,j} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$, note that $\kappa_{i,j}$ is an isometry of $\mathcal{B}(\mathcal{H}_1)$ into $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ also if $y_k \longrightarrow y$ weakly then $\kappa_{i,j}(y_k) \longrightarrow \kappa_{i,j}(y)$ weakly, as such, it is weakly continuous. Note that for $x \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ it holds that

$$\begin{aligned}(1 \otimes u_{i,i}) x (1 \otimes u_{j,j}) &= U_i U_i^* x U_j U_j^* \\ &= U_i x_{i,j} U_j^* \\ &= x_{i,j} \otimes u_{i,j} \\ &= \kappa_{i,j}(x_{i,j}).\end{aligned}$$

It follows that

$$\kappa_{i,j}\left(\mathcal{B}(\mathcal{H}_1)\right) = (1 \otimes u_{i,i}) \mathcal{B}\left(\mathcal{H}_1 \otimes \mathcal{H}_2\right) (1 \otimes u_{j,j}).$$

Consequently we find that

$$\left\{ x \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \ ; \ \kappa_{i,j}^{-1}\left((1 \otimes u_{i,i}) x (1 \otimes u_{j,j}) \right) \in \mathcal{A} \right\} = \left\{ x \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \ ; \ x_{i,j} \in \mathcal{A} \right\}.$$

Using that $\kappa_{i,j}$ is weakly continuous for all i, j we conclude that $\{x \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \ ; \ x_{i,j} \in \mathcal{A}\}$ is weakly closed in $\mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ and it contains $\kappa_{i,j}(\mathcal{A})$. We now use that the maps $u_{i,j} \in \mathcal{B}(\mathcal{H}_2)$ generate $\mathcal{B}(\mathcal{H}_2)$ as a von Neumann algebra and the fact that $\{x \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \ ; \ x_{i,j} \in \mathcal{A}\}$ is weakly closed, to conclude that

$$\mathcal{A} \otimes \mathcal{B}(\mathcal{H}_2) = W * \left(\{ a \otimes u_{i,j} \ ; \ a \in \mathcal{A} \ i, j \in I \} \right) = \bigcap_{i,j} \left\{ x \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \ ; \ x_{i,j} \in \mathcal{A} \right\},$$

as desired.

1.
Each $x \in \mathcal{A} \otimes \mathbb{C}$ is a $I \times I$ matrix such that $a_{i,j} = 0$ when $i \neq j$ and $a_{i,i} = a_{j,j} = a \in \mathcal{A}$ for all i, j. It follows that the commutant of $\mathcal{A} \otimes \mathbb{C}$ consists of matrices of the form $(a'_{i,j})$ with $a_{\prime}^{i,j} \in \mathcal{A}'$, using *3* we conclude that $(\mathcal{A} \otimes \mathbb{C})' = \mathcal{A}' \otimes \mathcal{B}(\mathcal{H}_2)$, as desired.

2.
Using *1* we find that $(\mathcal{A} \otimes \mathcal{B}(\mathcal{H}_2))' = (\mathcal{A}' \otimes \mathbb{C})'' = \mathcal{A}' \otimes \mathbb{C}$, as desired. □

1.3.2 The decomposition of type I and type II von Neumann algebras

We start of with matrix units. These systems will play a key role in decomposing type I or II von Neumann algebras in tensor products of simpler algebras.

Definition 1.3.2.1. *A collection of elements $\{w_{i,j}, i,j \in I\}$ in a von Neumann algebra is called a matrix unit when*

1. $w_{i,j}^* = w_{j,i}$,
2. $w_{i,j}w_{k,l} = \delta_{j,k}w_{i,l}$,
3. $\sum_{i\in I} w_{i,i} = 1$.

By definition $w_{i,i}$ is a projection, if we denote by $\mathcal{H}_i$ the space $w_{i,i}\mathcal{H}$, then it is not hard to see that $w_{i,j}$ is an isometry from $\mathcal{H}_j$ to $\mathcal{H}_i$. The point of the matrix unit is that the set $\{w_{i,i}\}_{i\in I}$ is a set of mutually equivalent orthogonal projections in the sense of definition 1.1.0.6. Suppose that $\mathcal{A}$ is a von Neumann algebra allowing for such a matrix unit $\{w_{i,j}\ i,j \in I\}$. Pick a fixed i_0 and set $\mathcal{H}_0 := w_{i_0,i_0}\mathcal{H}$. For $h \in \mathcal{H}_0$ consider $w_{j,i_0}h$, we find that

$$w_{j,j}w_{j,i_0}h = w_{j,i_0}h,$$

as such $w_{j,i_0} \in \mathcal{H}_j = w_{j,j}\mathcal{H}$. Let $V_j : \mathcal{H}_0 \longrightarrow \mathcal{H}_j$ be defined as $V_j(h) := w_{j,i_0}h$, since $\sum_{i\in I} w_{i,i} = 1$ we find that

$$V : \bigoplus_{j\in I} \mathcal{H}_0 \longrightarrow \mathcal{H}, \qquad V\left(\sum_{j\in I} h_j\right) := \sum_j V_j(h_j),$$

defines an isomorphism of the space $\bigoplus_{j\in I}\mathcal{H}_0$ onto the space $\mathcal{H}$. Note however that

$$\bigoplus_{j\in I} \mathcal{H}_0 \cong \mathcal{H}_0 \otimes L^2(I),$$

we conclude that if $\mathcal{A}$ allows for a matrix unit then it representation Hilbert space decomposes as a tensor product between the Hilbert spaces $\mathcal{H}_0$ and $L^2(I)$. Note furthermore that equivalent projections f and e induce an isomorphism of the corner algebras $e\mathcal{A}e \cong f\mathcal{A}f$.

Theorem 1.3.2.2. *Suppose that $\{\mathcal{A}, \mathcal{H}\}$ allows for a matrix unit $\{w_{i,j}\ ;\ i,j \in I\}$. Set $\mathcal{A}_j := w_{j,j}\mathcal{A}w_{j,j}$ and set $\mathcal{H}_j := w_{j,j}\mathcal{H}$ then for any $j \in I$ we have that*

$$\{\mathcal{A}, \mathcal{H}\} \cong \{\mathcal{A}_j, \mathcal{H}_j\} \otimes \{\mathcal{B}(L^2(I)), L^2(I)\}.$$

Proof:

We must show that $\mathcal{A}$ can be represented as the $I \times I$ matrices with entries from $\mathcal{H}_j$. Pick any $i_0 \in I$ and let $\mathcal{A}_{i_0}$ and $\mathcal{H}_{i_0}$ be as in the theorem. Let $\{e_j\}_{j\in I}$ be a orthonormal basis for $L^2(I)$, for $j \in I$ set $w_j := w_{j,i_0}$, that is, w_j is the partial isometry from $\mathcal{H}_{i_0}$ onto $\mathcal{H}_j$. Define the map $U : \mathcal{H}_{i_0} \otimes L^2(I)$ by

$$U(\sum_{j\in I} h_j \otimes e_j) := \sum_{j\in I} w_j(h_j).$$

A direct calculation reveals that U is an isometry. The adjoint of U works as follows:

$$U^*(h) = \sum_j w_j^*(h) \otimes e_j.$$

Since the range of U contains all the spaces $w_{j,j}\mathcal{H}$ and $\sum_j w_{j,j} = 1$ we conclude that U is in fact an isomorphism between $\mathcal{H}$ and $\mathcal{H}_{i_0} \otimes L^2(I)$. Let $x \in \mathcal{A}_{i_0} \otimes \mathcal{B}(L^2(I))$, by 1.3.1.3 we find that $x = (x_{i,j})$ with $x_{i,j} \in \mathcal{A}_{i_0}$, consider $UxU^* : \mathcal{H} \longrightarrow \mathcal{H}$. We find that

$$\begin{aligned} UxU^*h &= Ux\left(\sum_j w_j^*h \otimes e_j\right) \\ &= U\sum_{i,j} x_{i,j}w_j^* \otimes e_i \\ &= \sum_{i,j} w_i x_{i,j} w_j^* h. \end{aligned}$$

Since $x_{i,j} \in \mathcal{A}_{i_0}$ and $w_i \in \mathcal{A}$ for all i it follows that $UxU^* = \sum_{i,j} w_i x_{i,j} w_j^*$, considered in the strong topology, defines an element of $\mathcal{A}$. We find that $\mathcal{A}_{i_0} \otimes \mathcal{B}(L^2(I))$ has an isomorphic embedding in $\mathcal{A}$. On the other hand pick $a \in \mathcal{A}$ and consider U^*aU. We find that

$$\begin{aligned} U^*aU\left(\sum_j h_j \otimes e_j\right) &= U^*a\left(\sum_j w_j h_j\right) \\ &= U^*\left(\sum_j a w_j h_j\right) \\ &= \sum_i \sum_j w_i^* a w_j (h_j) \otimes e_i \\ &= \sum_{i,j} w_i^* a w_j h_j \otimes e_i. \end{aligned}$$

Since $a \in \mathcal{A}$ we conclude that $a_{i,j} := w_i^* a w_j \in \mathcal{A}_{i_0}$ and $a = (a_{i,j})$, as desired. □

The gain here is that if we know that a von Neumann algebra $\mathcal{A}$ allows for a matrix unit then in order to understand $\mathcal{A}$ one only needs to know $\mathcal{A}$ on one of the corner algebras associated to the family of equivalent projections induced by the matrix unit. We will now begin constructing a matrix unit for type I von Neumann algebras.

Definition 1.3.2.3. *Pick a cardinal number α, suppose p is a central projection with the property that there is an orthogonal family $\{e_j\}_{j \in J_\alpha}$, with $|J_\alpha| = \alpha$, of abelian projection such that*

1. $\sum_{j \in J_\alpha} e_j = z$,
2. $z(e_j) = z$ *for all* $j \in J_\alpha$,

then z is called α- homogeneous.

Proposition 1.3.2.4. *For every cardinal number α there exists a maximal α-homogeneous projection p_α (possibly $p_\alpha = 0$).*

Proof: Pick a family of orthogonal central α-homogeneous projections $\{p_i\}_{i \in I}$. We find that

$$\begin{aligned} p_\alpha &:= \sum_{i \in I} p_i \\ &= \sum_{i \in I} \sum_{j \in J} e_{i,j} \\ &= \sum_{j \in J} \sum_{i \in I} e_{i,j}. \end{aligned}$$

Note that $\{e_{i,j}\}_{i \in I}$ is a family of centrally orthogonal abelian projections and as such $\sum_{i \in I} e_{i,j}$ is abelian. For two centrally orthogonal projections e, f we have that $z(e+f) = z(e) + z(f)$ thus $z\left(\sum_{i \in I} e_{i,j}\right) = \sum_{i \in I} z(e_{i,j}) = \sum_{i \in I} p_i = p$ hence p is α-homogeneous. Suppose that p is any α-homogeneous projection then $(1 - p_\alpha)p = \sum_{j \in J_\alpha} (1 - p_\alpha) e_j$ is α-homogeneous if p is not zero then this would contradict the maximality of the family $\{p_i\}_{i \in I}$. □

For α a cardinal number consider our maximal α-homogeneous projection p_α. If it is nonzero then there exists a orthogonal family of abelian projections $\{e_i\}_{i \in I}$ with $|I| = \alpha$. By definition of our α-homogeneous projection we have $z(e_i) = p_\alpha$ and thus $z(e_i) \geq e_j$ for each $j \in I$. An application of theorem 1.1.1.7 yields that for each $i, j \in I$ it holds that $e_j \succeq e_i$, by reversing the role of i and j we find that $e_i \sim e_j$. By proposition 1.1.0.7 we find a collection $u_{i,j}$ such that $u_{i,j}^* u_{i,j} = e_i$ and $u_{i,j} u_{i,j}^* = e_j$. Note that this collection $\{u_{i,j}\}_{i,j \in I}$ defines a matrix unit for the algebra $\mathcal{A}p_\alpha$. An application of theorem 1.3.2.2 yields that

$$\mathcal{A}_{p_\alpha} \cong \{\mathcal{A}_{e_1}, \mathcal{H}_1\} \otimes l^2(I).$$

with $\dim\left(l^2(I)\right) = \alpha$, furthermore since e_i is abelian we have that $\mathcal{A}_{e_i} \cong \mathcal{A}_{e_j}$ is an abelian von Neumann algebra, which we understand quite well.

Theorem 1.3.2.5. *If $\mathcal{M}$ is a von Neumann algebra of type I then there exists a unique orthogonal family of $\{z_\alpha\}$ of central projections indexed by cardinal numbers α such that $\sum_\alpha z_\alpha = 1$ and $\mathcal{M}_{z_\alpha} \cong \mathcal{A}_\alpha \overline{\otimes} B(\mathcal{H}_\alpha)$ with $\dim \mathcal{H}_\alpha = \alpha$. Therefor we have that*

$$\mathcal{M} = \bigoplus_\alpha \mathcal{M}_\alpha \cong \bigoplus_\alpha \mathcal{A}_\alpha \overline{\otimes} B(\mathcal{H}_\alpha).$$

Proof:

By our last arguments we only need to find a family of orthogonal projections $\{p_\alpha\}$ indexed by cardinal numbers α such that p_α is maximal α-homogeneous, $p_\alpha \perp p_\beta$ when $\beta \neq \alpha$ and $\sum_\alpha p_\alpha = 1$. If we can show that the orthogonality for different cardinals is correct then by maximality of each p_α and picking $z_\alpha = p_\alpha$ we find the uniqueness of z_α.

We begin by showing that for different cardinal numbers α and β that p_α and p_β are orthogonal. Suppose that $\alpha = \beta$, then by maximality of p_α and p_β we find that $p_\alpha = p_\beta$. Consider p_1, meaning that $\alpha = 1$, then p_1 is a maximal central projection with the property that p_1 is abelian.

Claim:
Suppose that z is any central abelian projection then z is not α-homogeneous for $\alpha \geq 2$.

We will prove this for $\alpha = 2$, the proof works the same for any other α. Suppose that z is 2-homogeneous then we can find abelian projections e_1 and e_2 with $e_1 \perp e_2$, $e_1 \sim e_2$, $z(e_1) = z(e_2) = z$ and $e_1 + e_2 = z$. Note that $e_1 = e_1 z \in \mathcal{M}z = \mathcal{Z}(\mathcal{M})z$ so there exists a central projection z_1 such that $e_1 = ze_1 = zz_1$ similarly we find z_2 such that $e_2 = e_2 z = z_2 z$ since e_1 and e_2 are both central and orthogonal, they are centrally orthogonal, contradiction. Suppose that p_1 is 1-homogeneous and that $p_1 p_\alpha \neq 0$ for some $\alpha > 1$ then in particular $p_1 p_\alpha$ is a central projection which is also α-homogeneous, but this cannot happen. We conclude that $p_1 \perp p_\alpha$ for all $\alpha > 2$, as desired.

Note that p_1 is just the projection on the abelian summand of $\mathcal{M}$. Set $\mathcal{M}_1 := \{\mathcal{M}p_1, p_1\mathcal{H}\} \otimes \{\mathbb{C}, \mathbb{C}\}$, obviously $\mathcal{M}p_1 \cong \{\mathcal{M}p_1, p_1\mathcal{H}\} \otimes \{\mathbb{C}, \mathbb{C}\}$. By removing $\mathcal{M}p_1$ we may assume that $\mathcal{M}$ has no abelian summand. This means that if e is an abelian projection in $\mathcal{M}$ then e has no nonzero central subprojections.

Consider p_α and p_β and suppose that $p_\alpha p_\beta \neq 0$. It is easy to see that $p_\alpha p_\beta$ is the largest projection that is both α and β homogeneous. By restricting to $p_\alpha p_\beta$ we may assume that $p_\alpha = p_\beta$. So suppose that $p_\alpha = p_\beta$ We conclude that $e_i \sim e_j \sim f_l \sim f_k$ for all $i, j \in I_\alpha$ and $k, l \in I_\beta$ and as such $n := \dim e_i\mathcal{H} = \dim f_l\mathcal{H}$ for all $i \in I_\alpha$ and $l \in I_\beta$. Suppose $m := \dim z_\alpha\mathcal{H} = \dim z_\beta\mathcal{H} < \infty$ then it follows that

$$m = \alpha n = \beta n.$$

We conclude that $\alpha = \beta$.

Suppose now that $\dim z_\alpha\mathcal{H} = \gamma \geq \aleph_0$ and also suppose that $\dim e_i\mathcal{H} < \gamma$ then it follows that $\alpha = \gamma = \beta$. So the remaining case is $\dim z_\alpha\mathcal{H} = \dim e_i\mathcal{H} = \gamma \geq \aleph_0$. Suppose that $n := \alpha < \infty$ consider $M_n(\mathcal{A})$, the $n \times n$ matrices with entries in $\mathcal{A}$. By considering $\mathcal{A}_{p_\alpha}$ we can assume $p_\alpha = 1$. Define the map $\Phi : M_n(\mathcal{A}) \longrightarrow \mathcal{A}$ by

$$\Phi(A) = \frac{1}{n}\sum_{i=1}^{n} a_{i,i}$$

We can identify $a \in \mathcal{A}$ with the matrix which has a on the diagonal and zero elsewhere. It follows that if $a \in \mathcal{A}$ then $\Phi(a) = a$. Furthermore, for each $x \in M_n(\mathcal{A})$ we have $\Phi(x^*x) = \Phi(xx^*)$. This follows from the following calculation, if $A = (a)_{i,j}$ and $B = (b)_{i,j}$ then $(AB)_{i,j} = \sum_{k=1}^{n} a_{i,k} b_{k,j}$ thus

$$\begin{aligned}\Phi(aa^*) = \Phi\left((aa^*)_{i,j}\right) &= \frac{1}{n}\sum_{i=1}^{n}\sum_{k=1}^{n} a_{i,k}a_{i,k}^* \\ &= \frac{1}{n}\sum_{k,i}^{n} a_{k,i}a_{k,i}^* \\ &= \frac{1}{n}\sum_{k,i}^{n} a_{i,k}^* a_{i,k} = \Phi(a^*a).\end{aligned}$$

Note that that in our case we have $\mathcal{A} = \mathcal{Z} \otimes M_n(\mathbb{C})$. So our algebra $\mathcal{A}$ looks like the $n \times n$ matrices with entries from its center. We can apply this to our situation by noting that $e_i \sim f_l$ gives us an element $u_{i,l}$ such that $u_{i,l}^* u_{i,l} = e_i$ and $u_{i,l}u_{i,l}^* = f_l$. Each $u_{i,l}$ is a $n \times n$ matrix with entries in $\mathcal{Z}$ as such $\Phi(e_i) = \Phi(e_j) = \Phi(f_l) = \Phi(f_k)$. Thus it follows that

$$\alpha\Phi(e_i) = n\Phi(e_i) = \sum_{i=1}^{n}\Phi(e_i) = \Phi(1) = 1 = \sum_{l\in I_\beta}\Phi(f_l) = \sum_{l\in I_\beta}\Phi(e_i) = \beta\Phi(e_i).$$

We conclude $\alpha = \beta$.

Suppose that $\alpha \geq \aleph_0$, By symmetry $\beta \geq \aleph_0$. Let ϕ be a normal state on $\mathcal{Z}$. Since $\mathcal{A}_{e_i} \cong \mathcal{Z}$ by the map $\sigma_i : \mathcal{Z} \longrightarrow \mathcal{A}_{e_i}$ defined as $\sigma_i(x) = xe_i$ we can define ϕ_i on $\mathcal{A}$ by setting

$$\phi_i(x) := \phi \circ \sigma_i^{-1}(e_i x e_i).$$

The support e_ϕ of a normal state is the infimum over all projections e such that $\phi(e) = 1$. If z is the support of ϕ then the support of ϕ_i equals ze_i. Define

$$J_i := \{j \in J, \phi_i(f_j) \neq 0\}.$$

Because $\{f_j\}_{j\in J}$ is orthogonal we have that J_i is countable. We have that $J = \bigcup_{i\in I} J_i$ thus $\beta \leq \aleph_0\alpha$ hence by symmetry $\alpha = \beta$.

This shows that if z is an α homogeneous projection and also a β homogeneous projection then $\alpha = \beta$ as such p_α and p_β are orthognal. In total we conclude that

$$\mathcal{M} = \bigoplus_\alpha \mathcal{M}_\alpha \cong \bigoplus_\alpha \mathcal{A}_\alpha \overline{\otimes} B(\mathcal{H}_\alpha).$$

Here $\mathcal{A}_\alpha$ is abelian. By maximality the p_α are unique. □

This concludes the decomposition of type I von Neumann algebras. Note in particular that if $\mathcal{A}$ is a type I factor then $\mathcal{A} \cong \mathcal{B}(\mathcal{H}_\alpha)$ for some cardinal number α. There is a similar statement concerning type II von Neumann algebra. Without proof we state:

Theorem 1.3.2.6. *If $\{\mathcal{M}, \mathcal{H}\}$ is a properly infinite but semifinite von Neumann algebra then we can find a unique family $\{z_\alpha\}$ of orthogonal central projections with the properties:*

$$\mathcal{M}z_\alpha = \{\mathcal{A}_\alpha, z_\alpha H\} \otimes \{\mathcal{B}(\mathcal{H}_\alpha), \mathcal{H}_\alpha\}, \qquad \sum_\alpha z_\alpha = 1.$$

Here each $\mathcal{A}_\alpha$ is a finite von Neumann algebra. Furthermore, the family $\{z_\alpha\}$ is unique while $\mathcal{A}_\alpha$ is not. If $\mathcal{H}$ is separable and f is a finite projection in $\mathcal{M}$ with $z(f) = 1$ then

$$\mathcal{M} \cong \mathcal{M}_f \otimes \mathcal{B}(f\mathcal{H}).$$

For a rigorous proof we refer to [8]. Note that these last two theorems reduce the study of type I, II_1 and II_∞ von Neumann algebras to the study of abelian von Neumann algebras and type II_1 von Neumann algebras and the study of $\mathcal{B}(\mathcal{H})$. Indeed if $\mathcal{A}$ is of type II then we can decompose it into a direct sum of types I, II_1 and II_∞, the last theorem asserts that II_∞ reduces to the study of type II_1. Theorem 1.3.2.5 tells us that in order to understand all type I algebras we only need to know abelian algebras and type I factors. The next chapter will be devoted to construct type II_1 factors using the crossed product construction. This construction will also be used to construct type II_∞ and type III factors.

Chapter 2

Crossed products

In this chapter we will consider the crossed product construction. We consider an abelian von Neumann algebra and a countably discrete group G. From these objects we will construct a new von Neumann algebra called the crossed product. This crossed product construction is useful as it gives us examples of type II factors. In the first section we give the basic construction, we conclude with the construction of a type II_1 factor. Most of this chapter is based on [8].

2.1 General construction and basic properties

Given an abelian von Neumann algebra $\mathcal{A}$ algebra and a group G, denote by $\mathrm{Aut}\,(\mathcal{A})$ the set of automorphisms of $\mathcal{A}$. A homomorphism $\alpha : G \longrightarrow \mathrm{Aut}\,(\mathcal{A})$ is called an action of G on $\mathcal{A}$. Suppose that $\mathcal{A}$ acts on $\mathcal{H}$ and suppose that G is countable and discrete. Consider $\mathcal{R} := \mathcal{H} \otimes l^2\,(G)$ as the square summable functions from G into $\mathcal{H}$.

Definition 2.1.0.7. *We define a representation of $\mathcal{A}$ on $\mathcal{B}\,(\mathcal{R})$ as follows: for $\xi \in \mathcal{R}$, $a \in \mathcal{A}$ and $g \in G$ we define*

$$\pi\,(a)\,\xi\,(g) := \alpha_g^{-1}\,(a)\,\xi\,(g)\,.$$

Also we define a representation $u : G \longrightarrow \mathcal{B}\,(\mathcal{R})$ by setting

$$u\,(g)\,\xi\,(h) := \xi\left(g^{-1}h\right).$$

Note here that if we are given an $\mathcal{H}$ valued function ξ on G then for any g the transformation $g^{\dagger}\,(\xi)\,(h) := \xi\left(g^{-1}h\right)$ sends g to a unitary in $\mathcal{B}\,(\mathcal{R})$. This can be seen as follows

$$\begin{aligned}\left\langle g^{\dagger}\,(\xi)\,, g^{\dagger}\,(\xi)\right\rangle &= \sum_{h\in G} \left\|\xi\left(g^{-1}h\right)\right\|^2 \\ &= \sum_{h\in G} \left\|\xi\,(h)\right\|^2 \\ &= \langle \xi, \xi\rangle\,.\end{aligned}$$

We conclude that G gets mapped into the unitary group of $\mathcal{B}(\mathcal{R})$. A direct computation reveals that

$$u\,(g)\,\pi\,(a)\,u\,(g)^*\,\xi\,(h) = \pi \circ \alpha_g\,(a)\,. \tag{2.1}$$

This last relation between u, α and π is called the *covariance* relation between π and u.

Definition 2.1.0.8. *We define $\mathcal{R}\,(\mathcal{A}, G, \alpha)$ as the von Neumann algebra on $\mathcal{R}$ generated by $\pi\,(\mathcal{A})$ and $u\,(G)$. It is called the crossed product of $\mathcal{A}$ by G with respect to α. When $\mathcal{A} = \mathbb{C}$ then we denote it by $\mathcal{R}\,(G)$ and call it the group von Neumann algebra of G.*

We fix now an abelian von Neumann algebra $\mathcal{A}$, a group G and an action α. For ease of notation we denote the crossed product of $\mathcal{A}$ by G with respect to α by $\mathcal{R}(\mathcal{A})$. Our aim this section is to deduce how the elements in $\mathcal{R}(\mathcal{A})$ behave. We will first represent elements of $\mathcal{R}(\mathcal{A})$ in terms of of $\pi\,(\mathcal{A})$ and $u(g)$, $g \in G$.

Proposition 2.1.0.9. *For every $x \in \mathcal{R}(\mathcal{A})$ we can find a unique $\mathcal{A}$ valued function on G, also denoted by x, such that*

$$x = \sum_{g\in G} \pi(x(g))u(g).$$

Proof:

Let $\mathcal{R}\left(\mathcal{A}\right)_0$ be defined as

$$\mathcal{R}(\mathcal{A})_0 := \left\{ \sum_{g\in G} \pi\left(x(g)\right) u(g)\ ;\ x : G \longrightarrow \mathcal{A} \text{ is a function with finite support} \right\}. \tag{2.2}$$

Suppose that $y \in \mathcal{R}(\mathcal{A})_0$ then y^* is of the form $\sum_{g\in G} u(g^{-1})\pi(x(g)^*)$, using (2.1) we conclude that

$$y^* = \sum_{g\in G} \pi\left(\alpha_g\left(y(g^{-1})\right)\right) u(g) \in \mathcal{R}(\mathcal{A})_0.$$

Similarly we find that if $y, z \in \mathcal{R}(\mathcal{A})_0$, then $yz \in \mathcal{R}(\mathcal{A})_0$. As such $\mathcal{R}(\mathcal{A})_0$ is a * algebra. Since $\mathcal{R}(\mathcal{A})$ is the von Neumann algebra generated by $\pi(\mathcal{A})$ and $u(G)$, it holds that $\mathcal{R}(\mathcal{A})_0 \subset \mathcal{R}(\mathcal{A})$. Using that $\mathcal{R}(\mathcal{A})_0$ contains the generator set of $\mathcal{R}(\mathcal{A})$, we conclude that $\mathcal{R}(\mathcal{A})_0$ is weakly dense in $\mathcal{R}(\mathcal{A})$. For $g \in G$ we define an operator $P_g : \mathcal{R} \longrightarrow \mathcal{H}$ by

$$P_g\left(\xi\right) := \xi\left(g^{-1}\right).$$

Its adjoint $P_g^* : \mathcal{H} \longrightarrow \mathcal{R}$ satisfies

$$P_g^*\left(h\right)\left(p\right) = \begin{cases} 0 \text{ when } p \neq g^{-1}, \\ h \text{ when } p = g^{-1} \qquad h \in \mathcal{H} \quad g, p \in G. \end{cases}$$

It follows that

$$\begin{aligned} P_g u\left(h\right) &= P_{gh}, \\ \left(P_g \pi\left(a\right)\right)\xi = \pi\left(a\right)\xi\left(g^{-1}\right) &= \alpha_{g^{-1}}^{-1}\xi\left(g^{-1}\right) = \left(\alpha_g\left(a\right)P_g\right)\xi. \end{aligned}$$

Note that $P_g P_g^*\left(h\right) = h$ for all $h \in \mathcal{H}$ and $g \in G$. As such it follows that

$$P_g \pi\left(a\right) P_g^* = \alpha_g\left(a\right) P_g P_g^* = \alpha_g\left(a\right).$$

Furthermore we have that

$$P_g^* P_g\left(\xi\right)\left(p\right) = \begin{cases} 0 \text{ when } p \neq g^{-1}, \\ \xi\left(g^{-1}\right) \text{ when } p = g^{-1} \qquad \xi \in \mathcal{R} \quad g, p \in G. \end{cases}$$

We conclude that $\left\{P_g^* P_g\right\}_{g\in G}$ is an orthogonal family of projections in $B\left(\mathcal{R}\right)$ with sum equal to 1. We find that

$$\sum_{g\in G} P_g^* \alpha_g\left(a\right) P_g = \sum_{g\in G} P_g^* P_g \pi\left(a\right) = \pi\left(a\right).$$

Suppose now that x is of the form (2.2), then

$$\begin{aligned} P_e x P_e^* &= \sum_{g\in G} P_e \pi\left(x\left(g\right)\right) u\left(g\right) P_e^* \\ &= \sum_{g\in G} \alpha_e\left(x\left(g\right)\right) P_e u\left(g\right) P_e^* \\ &= \sum_{g\in G} x\left(g\right) P_g P_e^* = x\left(e\right) \in \mathcal{A}, \end{aligned}$$

by density of $\mathcal{R}\left(\mathcal{A}\right)_0$ we conclude that for all $x \in \mathcal{R}\left(\mathcal{A}\right)$ we have $P_e x P_e^* \in \mathcal{A}$. For ease of notation we define $E(x) := P_e x P_e^*$. Now let again x be of the form (2.2) and consider $u(g)\, x u(g)^*$, we find

$$\begin{aligned} u(g)\, x u(g)^* \xi(p) &= \sum_{h\in G} u(g)\, \pi(x(h))\, u(h)\, u(g)^* \xi(p) \\ &= \sum_{p\in G} u(g)\, \pi(x(h))\, \xi\left(h^{-1}gp\right) \\ &= \sum_{h\in G} u(g)\, \alpha^{-1}_{h^{-1}gp}(x(h))\, \xi\left(h^{-1}gp\right) \\ &= \sum_{h\in G} \alpha^{-1}_{h^{-1}gp}(x(h))\, \xi\left(g^{-1}h^{-1}gp\right) \\ &= \sum_{h\in G} \alpha^{-1}_{h^{-1}gp}(x(h))\, u\left(ghg^{-1}\right) \xi(p) \\ &= \sum_{h\in G} \pi\circ\alpha_g(x(h))\, u\left(ghg^{-1}\right) \xi(p)\,. \end{aligned}$$

We conclude that that $E\left(u(g)\, x u(g)^*\right) = \alpha_g(x(e)) = \alpha_g(E(x))$ for all $x \in \mathcal{R}(\mathcal{A})_0$. By density this identity extends to $\mathcal{R}(\mathcal{A})$. For $x \in \mathcal{R}(\mathcal{A})$ set $x(g) := E\left(xu(g)^*\right)$, note that $x(g) \in \mathcal{A}$ for all $g \in G$. We see that x gives rise to a $\mathcal{A}$ valued function on G.

Consider now $P_g x P_h^*$, note first of all that $P_h^* = (P_e u(h))^*$. Since $h = gg^{-1}h$ we find that

$$P_h^* = (P_e u(h))^* = \left(P_e u(g)\, u\left(g^{-1}h\right)\right)^* = u\left(g^{-1}h\right)^* u(g)^* P_e^*.$$

We use this to deduce that

$$\begin{aligned} P_g x P_h^* &= P_e u(g)\, x u\left(g^{-1}h\right)^* u(g)^* P_e^* \\ &= E\left(u(g)\, x u\left(g^{-1}h\right)^* u(g)^*\right) \\ &= \alpha_g\left(E\left(xu\left(g^{-1}h\right)^*\right)\right) = \alpha_g\left(x\left(g^{-1}h\right)\right). \end{aligned}$$

We find the following equalities

$$\begin{aligned} x &= \sum_{g,h\in G} \left(P_g^* P_g\right) x \left(P_h^* P_h\right) \\ &= \sum_{g,h\in G} P_g^* \alpha_g\left(x\left(g^{-1}h\right)\right) P_h \\ &\overset{*}{=} \sum_{g,p\in G} P_g^* \alpha_g(x(p))\, P_{gp} \\ &= \sum_{g,p\in G} P_g^* \alpha_g(x(p))\, P_g u(p) \\ &= \sum_{p\in G} \left(\sum_{g\in G} P_g^* \alpha_g(x(p))\, P_g\right) u(h) \\ &= \sum_{p\in G} \pi(x(p))\, u(p)\,. \end{aligned}$$

At $(*)$ we switched to $p = g^{-1}h$. In total we conclude that every $x \in \mathcal{R}(\mathcal{A})$ is of the form

$$x = \sum_{g\in G} \pi(x(g)) u(g), \tag{2.3}$$

with $x(p) := E\left(xu(p)^*\right)$ as the $\mathcal{A}$ valued function on G. it is not difficult to see that this expression is indeed unique. □

We will now turn to the arithmetic in $\mathcal{R}(\mathcal{A})$. Using formula (2.3) we find that

$$xy(g) = \sum_{h\in G} x(h)\,\alpha_h\left(y\left(h^{-1}g\right)\right),$$
$$x^*(g) = \alpha_g\left(x\left(g^{-1}\right)^*\right).$$

This describes how the elements of $\mathcal{R}(\mathcal{A})$ act on $\mathcal{R}$.

2.2 The influence of the action α and the group G

In this section we will investigate how the action $\alpha : G \longrightarrow \mathrm{Aut}(\mathcal{A})$ influences the structure of $\mathcal{R}(\mathcal{A})$. It will turn out to be the case that if α is free and ergodic (defined below), then $\mathcal{R}(\mathcal{A})$ is a factor. To show this we will first investigate what the necessary and sufficient conditions for α are to be free and ergodic. We start by defining what we mean by free and ergodic.

Definition 2.2.0.10. *Let f be a projection and let $\phi \in Aut(\mathcal{A})$. We call f absolutely invariant for ϕ when $\phi(f) = f$, and the restriction of ϕ to $\mathcal{A}_f$ is trivial. An automorphism ϕ is called free when it has no nonzero absolutely invariant projections. Given a triple $(\mathcal{A}, G, \alpha)$ with $\alpha : G \longrightarrow Aut(\mathcal{A})$ an action. We say that the action α is free when $\alpha(g)$ is free for all $g \in G$, $(g \neq e)$. We say that the action α is ergodic when $\alpha(G)$ has no non-trivial invariant projections other then 1 and 0. In other words α is ergodic when there is no non trivial projection f such that $\alpha_g(f) = f$ for all $g \in G$.*

Since we are considering an abelian von Neumann algebra we can simply denote $\mathcal{A} = L^\infty(\Omega, \mu)$ for some Borel measure space Ω with measure μ. Any automorphism $\gamma \in \mathrm{Aut}(\mathcal{A})$ induces a homeomorphism T_γ of Ω via $T_\gamma(\omega)(x) = \omega(\gamma(x))$. Consider the set $\Gamma := \{\omega \in \Omega : T_\gamma(\omega) = \omega\}$. We pose the first necessary and sufficient condition for γ to be free.

Proposition 2.2.0.11. *Suppose that γ is an automorphism of $\mathcal{A}$. Denote by $T_\gamma : \Omega \longrightarrow \Omega$ the homeomorphism associated to γ and set $\Gamma := \{\omega \in \Omega\ ;\ T_\gamma(\omega) = \omega\}$. Then γ is free if and only if $\mu(\Gamma) = 0$.*

Proof:

Suppose that γ is free and let p be the projection onto Γ, that is, p is indicator function of Γ. For $\omega \in \Omega$ we find

$$p(\omega) = p(T_\gamma(\omega)) = T_\gamma(\omega)(p) = \omega(\gamma(p)) = \gamma(p)(\omega).$$

We conclude that p is invariant for γ. Suppose now that $a \in \mathcal{A}p$. If $p(\omega) = 0$, that is $\omega \in \Gamma^c$, then $\gamma(a)(\omega) = \gamma(ap)(\omega) = \gamma(a)p(\omega) = 0$. Consider now $\gamma(ap)(\omega)$ for $\omega \in \Gamma$. We find

$$\gamma(a)(\omega) = T_\gamma(\omega)(a) = \omega(a) = a(\omega).$$

We conclude that p is absolutely invariant. Because γ is free $p = 0$, hence $\mu(\Gamma) = 0$.

Suppose now that $\mu(\Gamma) = 0$, and that γ has an absolutely invariant projection p. Since p is a projection, it is, modulo a subset of measure zero, defined by a subset of $V \subset \Omega$. Consider $\omega \in V$. For all $a \in \mathcal{A}p$ we find $a(\omega) = \gamma(a)(\omega) = T_\gamma(\omega)(a) = a(T_\gamma(\omega))$. We conclude that $T_\gamma(\omega) = \omega$, as such $V \subset \Gamma$. But $\mu(\Gamma) = 0$, hence $p = 0$. □

This already gives a necessary and sufficient condition for γ to be free. Using this we shall now give another characterization for γ to be free.

Proposition 2.2.0.12. *An automorphism γ is free if and only if the condition $xa = a\gamma(x)$ for all $x \in \mathcal{A}$ implies $a = 0$.*

Proof:

Suppose that γ has some absolutely invariant projection e. It follows that $xe = \gamma(xe) = \gamma(x)e$, hence there exists some nonzero element e satisfying the condition. Suppose now that there exists some nonzero a such that $xa = a\gamma(x)$ for all $x \in \mathcal{A}$. Then for $\omega \in \Omega$ we have

$$\begin{aligned} xa(\omega) &= \omega(x)\,\omega(a) \\ &= \omega(a)\,\omega(\gamma(x)) \\ &= \omega(a)\,T_\gamma(\omega)(x)\,. \end{aligned}$$

Hence $x(\omega) = x(T_\gamma(\omega))$ for all ω with $a(\omega) \neq 0$. We conclude that $T_\gamma(\omega) = \omega$ when $a(\omega) \neq 0$. Note that $\{\omega \in \Omega \ : \ a(\omega) \neq 0\}$ is a measurable set in Ω. Since a is nonzero $(\mu(\{\omega \in \Omega \ : \ a(\omega) \neq 0\}) \neq 0$. We conclude that $\mu(\Gamma) \neq 0$ as desired. □

Consider now a triple $(\mathcal{A}, G, \alpha)$ with $\mathcal{A}$ an abelian von Neumann algebra acting on $\mathcal{H}$, G a countable discrete group and $\alpha : G \longrightarrow \mathrm{Aut}(\mathcal{A})$ a free action. As defined in 2.1.0.7 we have a representation $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H} \otimes L^2(G))$ and a unitary representation of G into $\mathcal{B}(\mathcal{H} \otimes L^2(G))$. We will show that the canonical image of $\mathcal{A}$ is maximal abelian when α is free.

Proposition 2.2.0.13. *The image of $\mathcal{A}$ under π is maximal abelian if and only if α is free.*

Proof.
Recall from 2.1.0.9 that every element in $x \in \mathcal{R}(\mathcal{A})$ can be written uniquely as

$$x = \sum_{g \in G} \pi(x(g))\,u(g)\,.$$

Now suppose that x commutes with $\pi(\mathcal{A})$. For all $a \in \mathcal{A}$ we find that

$$\begin{aligned} \pi(a)\,x &= \sum_{g \in G} \pi(ax(g))\,u(g) \\ x\pi(a) &= \sum_{g \in G} \pi(x(g))\,u(g)\,\pi(a)\,. \end{aligned}$$

Consider the term $u(g)\,\pi(a)$, we have by the covariance relation that $u(g)\,\pi(a)\,u(g)^* = \pi(\alpha_g(a))$. Thus $u(g)\,\pi(a) = \pi(\alpha_g(a))\,u(g)$, as such

$$\begin{aligned} x\pi(a) &= \sum_{g \in G} \pi(x(g))\,u(g)\,\pi(a) \\ &= \sum_{g \in G} \pi(x(g)\,\alpha_g(a))\,u(g)\,. \end{aligned}$$

Putting this together and using the uniqueness of 2.1.0.9, we that conclude $ax(g) = x(g)\,\alpha_g(a)$ for all g. But now we use that α is free to conclude that $x(g) = 0$ when $g \neq e$. It follows that $x = \pi(x(e)) \in \pi(\mathcal{A})$, that is, $\mathcal{A}$ is maximal abelian.

Suppose now that $\pi(\mathcal{A})$ is maximal abelian but α is not free. There exists some $g_0 \neq e$ such that α_{g_0} has an absolutely invariant projection p. Let $x = \pi(p)\,u(g_0)$, for $a \in \pi(\mathcal{A})$ we have

$$\begin{aligned} \pi(a)\,\pi(p)\,u(g_0) &= \pi(ap)\,u(g_0) \\ &= \pi(\alpha_{g_0}(ap))\,u(g_0) \\ &= \pi(p\alpha_{g_0}(a))\,u(g_0) \\ &= \pi(p)\,u(g_0)\,\pi(a)\,, \end{aligned}$$

thus x commutes with $\pi(\mathcal{A})$. But on the other hand $\pi(p)\,u(g_0) \notin \pi(\mathcal{A})$ this is a contradiction because $\pi(\mathcal{A})$ was assumed to be maximal abelian. □

We will now state the necessary and sufficient conditions under which $\mathcal{R}(\mathcal{A}, \alpha, G)$ forms a factor.

Theorem 2.2.0.14. *Suppose that α is a free action of G then the following statements are equivalent.*

1. *α is ergodic.*
2. *$\mathcal{R}(\mathcal{A}, \alpha, G)$ is a factor.*

Proof:

$1 \Longrightarrow 2$
Suppose that α is ergodic, because α is also free it follows that $\pi(\mathcal{A})$ is maximal abelian in $\mathcal{R}(\mathcal{A}, \alpha, G)$. It follows that $\mathcal{Z}(\mathcal{R}(\mathcal{A}, \alpha, G)) \subset \pi(\mathcal{A})$. So if $x \in \mathcal{R}(\mathcal{A}, \alpha, G)$ commutes with $\mathcal{R}(\mathcal{A}, \alpha, G)$ then $x \in \pi(\mathcal{A})$. If $x \in \mathcal{Z}(\mathcal{R}(\mathcal{A}, \alpha, G))$ then in particular x commutes with $u(g)$ for all $g \in G$. So the question becomes when does $\pi(a)$ commute with $u(g)$ for all g? Suppose that $\pi(a) u(g) = u(g) \pi(a)$. Then by the covariance relation (2.1) we conclude that $\pi(a) = \pi(\alpha_g(a))$. It follows that $\alpha_g(a) = a$ for all g. Consider now the set $\mathcal{M} := \{x \in \mathcal{A}\ ;\ \alpha_g(x) = x \text{ for all } g \in G\}$, $\mathcal{M}$ is obviously a *-sub algebra, it contains 1 and also if $x_i \longrightarrow x$ with $\{x_i\} \subset \mathcal{M}$ in the strong topology then $x \in \mathcal{M}$. We conclude that $\mathcal{M}$ is a von Neumann algebra. It follows that $\mathcal{M}$ is generated by its projections. In particular if $\mathcal{M}$ is not equal to $\mathbb{C}$ then $\mathcal{M}$ has nontrivial invariant projections. But we assumed that α was ergodic so $\mathcal{M} = \mathbb{C}$. As such $\mathcal{Z}(\mathcal{R}(\mathcal{A}, \alpha, G)) = \mathbb{C}$. We conclude that $\mathcal{R}(\mathcal{A}, \alpha, G)$ is a factor. Note here that $\mathcal{M} = \mathcal{Z}(\mathcal{R}(\mathcal{A}, \alpha, G))$.

$2 \Longrightarrow 1$
Suppose that $\mathcal{R}(\mathcal{A}, \alpha, G)$ is a factor. It follows that $\mathcal{M} = \mathbb{C}$ thus α has no nontrivial invariant projections, that is, α is ergodic. □

We will now simplify the situation, let $\mathcal{A} = \mathbb{C}$ and α_g is the identity automorphism for all g. We define $\mathcal{R}(G) := \mathcal{R}(\mathcal{A}, \alpha, G)$. We see that $\mathcal{R}(G)$ acts on the square summable functions $f : G \longrightarrow \mathbb{C}$. By definition α is an ergodic action (since the only projections in $\mathbb{C}$ are 0 and 1), however it is never free. Lets look at the canonical image of $\mathcal{A} = \mathbb{C}$ in $\mathcal{R}(G)$. Since $\alpha_g = 1$ for all g we conclude

$$\pi(a) \xi(g) = a \xi(g).$$

By (2.3) an element $x \in \mathcal{R}(G)$ is of the form $\sum_{g \in G} \pi(x(g)) u(g)$. Here $x(g) = P_e x u(g)^* P_e^*$. Since $\pi(a)$ can be identified with a, the formula reduces to $x = \sum_{g \in G} x(g) u(g)$. Consider the map $x : G \longrightarrow \mathbb{C}$, let $\xi_e \in L^2(G)$ be defined as

$$\xi_e(p) = \begin{cases} 1 & \text{when } p = e, \\ 0 & \text{when } p \neq e. \end{cases}$$

Consider now $x\xi_e$, we find

$$\begin{aligned} x\xi_e(p) &= \sum_{g \in G} x(g) u(g) \xi_e(p) \\ &= \sum_{g \in G} x(g) \xi_e\left(g^{-1} p\right) \\ &= x(p). \end{aligned}$$

Since $x\xi_e \in L^2(G)$, it follows that $\sum_{p \in G} |x\xi_e(p)|^2 < \infty$, as such $\sum_{p \in G} |x(p)|^2 < \infty$. We conclude that $x : G \longrightarrow \mathbb{C}$ is square summable. We find the following inequalities for the norm on $\mathcal{R}(G)$:

$$\begin{aligned} \|x\|_{L^2}^2 &\leq \|x\|_{\mathcal{R}(G)} \\ &\leq \sum_{g \in G} \|x(g) u(g)\| \\ &= \sum_{g \in G} |x(g)| = \|x\|_{L^1}. \end{aligned}$$

We conclude that every $L^1(G)$ function has a canonical image in $\mathcal{R}(G)$. Some $L^2(G)$ functions are also in $\mathcal{R}(G)$ but not all. It is instructive to classify self-adjoint elements in $\mathcal{R}(G)$, so suppose $x \in \mathcal{R}(G)$ is self-adjoint. It follows that

$$\begin{aligned}x^* &= \sum_{g\in G} \overline{\lambda_g} u(g)^* \\ &= \sum_{g\in G} \overline{\lambda_{g^{-1}}} u(g) \\ x &= \sum_{g\in G} \lambda_g u(g).\end{aligned}$$

Hence if $x = x^*$ if and only if $\overline{\lambda_g} = \lambda_{g^{-1}}$, in particular if λ_e is not real then x is not self-adjoint.
Since $\mathcal{R}(G)$ is completely described by the group structure on G, it is reasonable to assume that there are criteria on G that make $\mathcal{R}(G)$ into a factor. Luckily these are not that hard to find.

Proposition 2.2.0.15. *Let G be a countably infinite discrete group and let $\mathcal{R}(G)$ be the group von Neumann algebra of G. Then $\mathcal{R}(G)$ is a factor if and only if every conjugacy class*

$$C(g) := \left\{hgh^{-1} \ ; \ h \in G\right\} \subset G,$$

is infinite except when $g = e$.

Proof:

Suppose that every conjugacy class of G is infinite except when $g = e$. Since $\mathcal{R}(G)$ is generated by $\{u(h) \ ; \ h \in G\}$ it follows that x commutes with $\mathcal{R}(G)$ if and only if it commutes with $u(h)$ for all h. Suppose that $x \in \mathcal{R}(G)$ commutes with $\mathcal{R}(G)$, we find that $xu(h) = u(h)x$ so $u(h)xu(h)^* = x$ thus

$$\begin{aligned}x &= u(h)xu(h)^* \\ &= \sum_{g\in G} \pi \circ \alpha_h\left(x(g)u\left(hgh^{-1}\right)\right) \\ &= \sum_{g\in G} x(g)u\left(hgh^{-1}\right) \\ &= \sum_{g\in G} x\left(hgh^{-1}\right)u(g) \\ x &= \sum_{g\in G} x(g)u(g).\end{aligned}$$

We conclude that if x commutes with $\mathcal{R}(G)$ then $x(g) = x\left(hgh^{-1}\right)$ thus x must be constant on every conjugacy class of G. Suppose $x(g) \neq 0$ for some $g \neq e$ then $\sum_{g\in G} |x(g)|^2 \geq \sum_{h\in G} |x\left(hgh^{-1}\right)|^2 = \infty$, contradicting the square summability of x. It follows that $x = x(e) \in \mathbb{C}$ thus $\mathcal{R}(G)$ is a factor. On the other hand suppose that G has some finite conjugacy class $C(g_0)$ with $g_0 \neq e$ then let $x = \sum_{g\in C(g_0)} u(g)$ by our previous arguments x commutes with $\mathcal{R}(G)$. □

Groups with the property that they are countably infinite and only have infinite conjugacy classes are often abbreviated to ICC groups and there are many of them. Some examples:

- the free group on two or more generators,
- the group of finite permutations on a countably infinite set,
- the cartesian product of a finite number of ICC groups.

We will now consider projections in $\mathcal{R}(G)$ and deduce some constraints on the coëfficients. Let $f \in \mathcal{R}(G)$ be a projection, thus $f = f^* = f^2$. We have that f is of the form

$$f = \sum_{g\in G} f(g)u(g).$$

We haver already seen that $f\left(g^{-1}\right) = \overline{f(g)}$, consider now f^2. We find the following

$$\begin{aligned} f^2 &= \left(\sum_{g\in G} f(g)\,u(g)\right)\left(\sum_{h\in G} f(h)\,u(h)\right) \\ &= \sum_{g\in G}\sum_{h\in G} f(g)\,f(h)\,u(g)\,u(h) \\ &= \sum_{g\in G}\sum_{h\in G} f(g)\,f(h)\,u(hg) \\ &= \sum_{p\in G} f(p)\,u(p) = f. \end{aligned}$$

For each g let $h = pg^{-1}$, it follows that for all p we have

$$f(p) = \sum_{g\in G} f(g)\,f\left(pg^{-1}\right).$$

Since f is self adjoint it also follows that $\overline{f(p^{-1})} = f(p)$, consider now $\overline{f(p^{-1})}$ we find

$$\begin{aligned} \sum_{g\in G} f(g)\,f\left(pg^{-1}\right) &= f(p) \\ &= \overline{f(p^{-1})} \\ &= \sum_{g\in G} \overline{f(g)\,f(p^{-1}g^{-1})} \\ &= \sum_{g\in G} \overline{f(g^{-1})\,f(p^{-1}g)} \\ &= \sum_{g\in G} f(g)\,\overline{f(p^{-1}g)} \\ &= \sum_{g\in G} f(g)\,\overline{f\left((g^{-1}p)^{-1}\right)} \\ &= \sum_{g\in G} f(g)\,f\left(g^{-1}p\right). \end{aligned}$$

In total we conclude that for any projection f it must hold that

$$\sum_{g\in G} f(g)\left(f\left(pg^{-1}\right) - f\left(g^{-1}p\right)\right) = 0.$$

Remark.
Note here in particular that $f(e) = \sum_{g\in G} f(g)\,\overline{f(g)} \geq 0$ so that $f(e) > 0$ when f is not zero. Suppose that f and h are projections such that $f - h \geq 0$. Then $f - h$ is again a projection so $(f-h)(e) > 0$ and if $(f-h)(e) = 0$ then $h = f$. We will now deduce of what type $\mathcal{R}(G)$ is if G is an ICC group.

2.2.1 Construction of a type II_1 factor

Here we construct a factor of type II_1 using the crossed product construction.

Theorem 2.2.1.1. *If G is an ICC group then $\mathcal{R}(G)$ is a factor of type II_1.*

Proof:
We have already seen that $\mathcal{R}(G)$ is a factor, if we can show that $1 \in \mathcal{R}(G)$ is a finite projection then it must necessarily follow that $\mathcal{R}(G)$ is of type II_1. Suppose for the moment that we have shown that 1 is finite. Then the case type I is ruled out because finite type I factors are just $B(\mathbb{C}^n)$ for some $n \in \mathbb{N}$, hence finite dimensional. Clearly $\mathcal{R}(G)$ is of infinite dimension, thus if 1 is finite then $\mathcal{R}(G)$ is of type

II_1. What remains is to show that 1 is a finite projection. Let f be a projection such that $f \sim 1$ then there exists some $r \in \mathcal{R}(G)$ such that $r^*r = f$ and $rr^* = 1$. Consider rr^*, we find

$$\begin{aligned}
1 = u(e) &= rr^* \\
&= \left(\sum_{g\in G} r(g)\,u(g)\right)\left(\sum_{h\in G} \overline{r(h)}u\left(h^{-1}\right)\right) \\
&= \sum_{g\in G}\sum_{h\in G} r(g)\,\overline{r(h)}u\left(h^{-1}g\right).
\end{aligned}$$

For each g let $h = g$ then we conclude that $\sum_{g\in G} r(g)\,\overline{r(g)} = 1$. For each g let $h = gp^{-1}$ with $p \neq e$. then we find that

$$rr^*(p) = \sum_{g\in G} r(g)\,\overline{r(gp^{-1})} = 0.$$

Now consider $r^*r(e)$, we find

$$r^*r(e) = \sum_{g\in G}\sum_{h\in G} \overline{r(g)}r(g) = 1.$$

It follows that $r^*r(e) = rr^*(e) = 1$. Since $rr^* = 1$ it follows that $rr^* - r^*r = 1 - f \geq 0$. Using our last remark we conclude that $r^*r = rr^* = 1$, that is, $\mathcal{R}(G)$ is a fatcor of type II_1. □

Note that the argument above actually shows more, namely that all projections in $\mathcal{R}(G)$ are finite. If h and f are equivalent projections then $f(e) = h(e)$. Now we use the comparability theorem 1.1.0.16 to conclude the following: Suppose f and h are projections then either $f \preceq h$ or $h \preceq f$. To determine which one holds one only needs to consider $h(e)$ and $f(e)$ if $f(e) \leq h(e)$ then $f \preceq h$ and vice versa. Note also that $f \sim h$ if and only if $f(e) = h(e)$. It follows that the equivalence classes of projections in $\mathcal{R}$ are uniquely determined by their e coëfficient. If f is a projection then $f(e) \leq 1$ thus $f(e)$ takes values in $[0, 1]$ for any projection. The map $M : \mathcal{R}(G)_p \longrightarrow [0, 1]$ defined by

$$M(f) = f(e),$$

extends to a faithful finite normal trace on $\mathcal{R}(G)$ (defined in 3.1). Perhaps an odd consequence is that if f is a projection with $f(e) = \frac{1}{2}$ then $f \sim 1 - f$.

We now leave the study of crossed products but we will return to it later. The example of a type II_1 factor described above can be constructed without going far into the theory of traces. We will now go in to the theory of traces to gain a better understanding what the existence of a trace, with perhaps desirable properties, implies for the type of the von Neumann algebra it acts on.

Traces and the construction of type II_∞ and type III factors

3.1 Traces

In this chapter we will use the theory of traces to construct a type II_∞ and a type III factor. The first section is devoted to the basic theory of traces. We will see that the theory of traces only applies to von Neumann algebras which are not of type III. Later on we give an example of a type II_∞ von Neumann algebra and a type III von Neumann algebra. We conclude this section with the application of traces to the representation theory of a type I or II von Neumann algebra. The result of this application is the conclusion that a von Neumann algebra is anti-isomorphic to its own commutant. This in turn implies that $\mathcal{A}$ and $\mathcal{A}'$ are of the same type.

Definition 3.1.0.2. *A trace on a von Neumann algebra $\mathcal{A}$ is a map $\tau : \mathcal{A}_+ \longrightarrow [0,\infty]$ with the following properties*

1. $\tau(x+y) = \tau(x) + \tau(y)$ *with* $x, y \in \mathcal{A}_+$,
2. $\tau(\lambda x) = \lambda\tau(x)$ *for* $\lambda \geq 0$,
3. $\tau(x^*x) = \tau(xx^*)$ *for all* $x \in \mathcal{A}$.

Here $\mathcal{A}_+$ denotes the positive cone in $\mathcal{A}$. Note that from the definition it follows that if $a, b \in \mathcal{A}$ with $b \leq a$ then $\tau(b) \leq \tau(a)$. A trace τ is called *faithful* when $\tau(x) > 0$ when $x > 0$, *semifinite* when for all nonzero $x \in \mathcal{A}_+$ there exists a nonzero $y \in \mathcal{A}_+$ with $\tau(y) < \infty$ such that $y \leq x$, *finite* when $\tau(1) < \infty$, *normal* when $\sup \tau(x_i) = \tau(\sup x_i)$ for every bounded increasing net $\{x_i\}$ in $\mathcal{A}_+$. Note that a finite trace τ extends uniquely to a linear functional on $\mathcal{A}$ because $\mathcal{A}_+$ spans $\mathcal{A}$ linearly and $\tau(x) < \infty$ for all $x \in \mathcal{A}_+$. So a finite trace is just a positive linear functional enjoying property *3.* Also if τ is finite then we can, by rescaling, assume that $\tau(1) = 1$.

We start by deriving some general properties of traces.

Proposition 3.1.0.3. *If τ is a normal trace then there exists a unique central projection $z \in \mathcal{A}$ with the property that τ is faithful on $\mathcal{A}z$ and $\tau = 0$ on $\mathcal{A}(1-z)$.*

Proof:

Let $\mathcal{E} := \{f \in \mathcal{A}_p \ ; \ \tau(f) = 0\}$, $\mathcal{E}$ has a least upper bound $e = \bigvee_{f \in \mathcal{E}} f$. Denote $\mathcal{E} = \{f_i\}_{i \in I}$ and for $i \in I$ set

$$e_i := \bigvee_{j \leq i} f_j.$$

It follows that $\{e_i\}$ is a bounded increasing net in $\mathcal{A}_+$, as such

$$\tau(e) = \tau\left(\sup_i e_i\right) = \sup_i \tau(e_i) = 0,$$

by the normality of τ. Obviously e is the largest projection with the property that τ annihilates it. The claim is that e is central. Suppose not then e and $1-e$ are not centrally orthogonal, by (1.1.0.15) there are projections $p_1 \preceq e$ and $p_2 \preceq 1-e$ with the property that $p_1 \sim p_2$. There exists $r \in \mathcal{A}$ with $r^*r = p_1$ and $rr^* = p_2$, we use now the trace property to conclude that

$$0 = \tau(p_1) = \tau(r^*r) = \tau(rr^*) = \tau(p_2).$$

But then $p_2 \leq e$, contradiction. We conclude that e must be central. Set $z = 1 - e$ and let $x \in \mathcal{A}z_+$ be nonzero, Since $\mathcal{A}z$ is generated by its projections it follows that there exists a bounded increasing net of finite linear combinations of projections (with coëfficients > 0) converging to x. Using that for all nonzero projections $f \leq z$ it holds that $\tau(f) > 0$ and that τ is normal, we conclude that $\tau(x) > 0$. Obviously by the same argument τ is zero on $\mathcal{A}(1 - z)$, uniqueness follows by construction. □

The projection z in the argument above is called the support projection of τ and will be denoted by $z(\tau)$. Note that if $\mathcal{A}$ is a factor then for every normal trace τ, either τ is faithful or $\tau = 0$.

Proposition 3.1.0.4. *Given a family $\{\tau_i\}_{i\in I}$ of semifinite normal traces on $\mathcal{A}$ with orthogonal supports $z_i = z(\tau_i)$ then the map*

$$\tau := \sum_i \tau_i,$$

is also a semifinite normal trace.

Proof:

That τ satisfies properties $1, 2, 3$ of definition 3.1.0.2 is obvious. We only need to show normality and semifiniteness. Normality follows from the fact that supremum and summation commute when every term is nonnegative. For semifiniteness we note that if $\tau(x)$ is nonzero for some x then it is nonzero for some τ_i. We have that $\tau_i(x) = \tau(z_i x)$. Since τ_i is semifinite there exists some $y_i \leq z_i x \leq x$ such that $\tau(y) = \tau_i(y) < \infty$. we conclude that τ is semifinite. □

We find that we can add semifinite normal traces with orthogonal support to produce other semifinite traces. The next proposition is an analogy of 3.1.0.3.

Proposition 3.1.0.5. *Suppose τ is a normal trace on $\mathcal{A}$ then there exists a unique central projection projection z such that τ is semifinite on $\mathcal{A}z$ and $\tau(x) = \infty$ for all $x \in \mathcal{A}(1 - z)_+$.*

Proof:

Let $\mathcal{I} : \{x \in \mathcal{A} \; ; \; \tau(x^*x) < \infty\}$, we shall show that $\mathcal{I}$ is an ideal of $\mathcal{A}$. Let $x \in \mathcal{I}$ then $\tau(x^*x) = \tau(xx^*) < \infty$ so $x^* \in \mathcal{I}$. Also $\mathcal{I}$ is closed under scalar multiplication. Suppose that $x, y \in \mathcal{I}$, note the following inequality:

$$x^*y + y^*x \leq x^*x + y^*y.$$

This is true because

$$\begin{aligned} x^*x + y^*y - x^*y - y^*x &= x^*(x - y) + y^*(y - x) \\ &= x^*(x - y) - y^*(x - y) \\ &= (x^* - y^*)(x - y) = (x - y)^*(x - y) \geq 0. \end{aligned}$$

It follows that

$$\tau\left((x + y)^*(x + y)\right) \leq 2\tau(x^*x + y^*y) = 2\tau(x^*x) + 2\tau(y^*y) < \infty,$$

as such $x + y \in \mathcal{I}$. Suppose that $a \in \mathcal{A}$ and $x \in \mathcal{I}$. By the spectral radius formula we obtain $(ax)^*(ax) \leq \|a\|^2 x^*x$. As such $ax \in \mathcal{I}$, so $\mathcal{I}$ is an ideal of $\mathcal{A}$. Taking the strong closure of $\mathcal{I}$ (making $\mathcal{I}$ a von Neumann algebra) we find that there exists a unique central projection $z \in \mathcal{A}$ that acts as a unit for $\overline{\mathcal{I}}$. It follows that $\overline{\mathcal{I}} = \mathcal{A}z$. Pick now a strongly convergent increasing net e_i in $\mathcal{I}$ converging to z, then for any $x \in \mathcal{A}z$ we have

$$x = \lim_i xe_i$$

It follows that $\tau(xe_i) < \infty$ and $xe_i \leq x$ thus τ is semifinite on $\mathcal{A}z$. By construction $\tau(x) = \infty$ if $x \in \mathcal{A}(1 - z)_+$ is nonzero. □

To summarize: Given a normal trace τ. By 3.1.0.3 and 3.1.0.5 we can find three unique central projections z_1, z_2, z_3 with $\sum_{i=1}^3 z_i = 1$, such that τ is zero on $\mathcal{A}z_1$, semifinite and faithful on $\mathcal{A}z_2$ and τ restricted to $\mathcal{A}z_3$ is faithful but takes only the value ∞ for all nonzero $x \in \mathcal{A}(z_3)_+$.

It is instructive to return to our example of a type II_1 factor, by 2.2.0.15 an ICC group (infinite conjugacy class group) G gives rise to a type II_1 factor by crossing it with the complex numbers $\mathbb{C}$. By 2.1.0.9 every element $x \in \mathcal{R}(G)$ is uniquely expressed by a function $f_x : G \longrightarrow \mathbb{C}$ such that

$$x = \sum_{g \in G} f_x(g)\, u(g).$$

Proposition 3.1.0.6. *If G is an ICC group then its group von Neumann algebra $\mathcal{R}(\mathcal{G})$ allows for a faithful finite normal trace.*

Proof:

Consider the positive cone of $\mathcal{R}(G)$ and set $\tau : \mathcal{R}(G)_+ \longrightarrow [0, \infty]$, $\tau(x^*x) = f_{x^*x}(e)$ The claim is that τ is a finite faithful normal trace on $\mathcal{R}(G)$. Why is this the case? It is obvious that τ is finite since $\tau(1) = 1$. To show that τ is faithful we only need to show that if $x \in \mathcal{R}(G)$ is nonzero then $f_{x^*x}(e)$ is nonzero. We find

$$\begin{aligned} x^*x &= \left(\sum_{g \in G} \overline{f(g^{-1})} u(g) \right) \left(\sum_{h \in G} f(h)\, u(h) \right) \\ &= \sum_{g \in G} \sum_{h \in G} \overline{f(g^{-1})} f(h)\, u(hg) \\ f_{x^*x}(e) &= \sum_{g \in G} \overline{f(g^{-1})} f\left(g^{-1}\right) > 0 \qquad \text{when } x \neq 0, \end{aligned}$$

so τ is faithful. We also find that $\tau(x^*x) = \tau(xx^*)$ via the same argument. That τ respects multiplication and addition is immediate. To show that τ is normal is also immediate because if x_i is a bounded increasing net converging to x then f_{x_i} converges pointwise to f_x, in particular $\tau(x_i(e)) \longrightarrow \tau(x(e))$. $\square$

Remark: Note that one of the key features of a trace is that it is constant on equivalence classes. Consider an infinite dimensional Hilbert space $\mathcal{H}$ and let $\mathcal{A} := \mathcal{B}(\mathcal{H})$. Then by 3.1.0.3 any nonzero normal trace is faithful. If there is $x \in \mathcal{B}(\mathcal{H})$ with $\tau(x^*x) < \infty$ then by 3.1.0.5 we find that τ is semifinite. However τ can never be finite, because if so then pick any projection f with $1 \sim f < 1$. We find that $\tau(f) = \tau(1) = 1$, note that since τ is faithful we have that $\tau(1 - f) > 0$ so $1 = \tau(1) = \tau(f + (1 - f)) = \tau(f) + \tau(1 - f) = 1 + \tau(1 - f) > 1$, clearly a contradiction. We conclude that if τ is a nonzero normal trace on $\mathcal{B}(\mathcal{H})$ where $\mathcal{H}$ is infinite dimensional then either τ is semifinite or $\tau(x^*x) = \infty$ for any nonzero $x \in B(\mathcal{H})$. The point is that if we can construct a nonzero finite trace on some factor $\mathcal{A}$ then $\mathcal{A}$ is either abelian, of type II_1 or $\mathcal{A} = B(\mathcal{H})$ for some *finite* dimensional Hilbert space. Note that in the argument above, if f is a projection with $\tau(f) < \infty$ then f is a finite projection.

Now we will construct a semifinite faithful normal trace on $\mathcal{B}(\mathcal{H})$. Let $\mathcal{F}(\mathcal{H})$ be the finite rank operators on $\mathcal{H}$.

Proposition 3.1.0.7. *Let $\mathcal{H}$ be any Hilbert space. Then $B(\mathcal{H})$ allows for a semifinite faithful normal trace.*

Proof:

Claim 1. $\mathcal{F}(\mathcal{H})$ is strongly dense in $B(\mathcal{H})$.

We will prove **Claim 1** after constructing the trace, for the moment assume that **Claim 1** holds. Given a finite rank operator a and a complete orthonormal system $\{e_i\}_{i \in I}$ for $\mathcal{H}$. Define τ_0 on $\mathcal{F}(\mathcal{H})$ as follows

$$\tau_0(a) := \sum_{i \in I} \langle a e_i, e_i \rangle.$$

Note that $\tau_0(a^*) = \overline{\tau_0(a)}$ so if $a = a^*$ then $\tau(a) \in \mathbb{R}$. Also if $(a)_{i,j}$ is the matrix representation of a with respect to the basis $\{e_i\}_{i \in I}$ then $\tau_0(a) = \sum_i a_{i,i}$.

Claim 2. τ_0 is independent of the orthonormal system $\{e_i\}$.

Proof of Claim 2.
Given another orthonormal system $\{f_j\}_{j\in I}$. Then the following holds

$$\begin{aligned}\tau_0(a) &= \sum_{i\in I}\langle ae_i, e_i\rangle \\ &= \sum_{i\in I}\left\langle a\left(\sum_{j\in I}\langle e_i, f_j\rangle f_j\right), \sum_{k\in I}\langle e_i, f_k\rangle f_k\right\rangle \\ &= \sum_{i\in I}\sum_{j\in I}\langle\langle e_i, f_j\rangle af_j, \langle e_i, f_j\rangle f_j\rangle \\ &= \sum_{i\in I}\sum_{j\in I}|\langle e_i, f_j\rangle|^2\langle af_j, f_j\rangle \\ &= \sum_{j\in I}\langle af_j, f_j\rangle.\end{aligned}$$

so indeed τ_0 is independent of the orthonormal system. □

Set now $\tau_1(a) := \tau_0(|a|)$ with $|a| := \sqrt{a^*a}$, it follows that

$$\begin{aligned}\tau_1(a) = \tau_0(|a|) &= \sum_i\sum_k\sqrt{\overline{a_{k,i}}a_{k,i}} \\ &= \sum_i\sum_k\sqrt{a_{i,k}\overline{a_{i,k}}} = \tau_0(|a^*|) = \tau_1(a^*).\end{aligned}$$

Thus for all $a \in \mathcal{F}(\mathcal{H})$ we have that $\tau_1(a^*a) = \tau_0(a^*a) = \tau_0(aa^*) = \tau_1(aa^*)$. For $a \in \mathcal{F}(\mathcal{H})$ we define

$$\|a\|_1 := \tau_1(a) = \tau_0(|a|).$$

Taking the closure of $\mathcal{F}(\mathcal{H})$ in $\mathcal{B}(\mathcal{H})$ with respect to the norm $\|.\|_1$ gives us the so called trace class operators, denoted by $\mathcal{T}(\mathcal{H})$. For all $a \in \mathcal{B}(\mathcal{H})$ we have

$$\|a\| \leq \|a\|_1.$$

By the arguments in 3.1.0.5 we find that the trace class operators $\mathcal{T}(\mathcal{H})$ form an ideal inside $\mathcal{B}(\mathcal{H})$. If we denote by $C(\mathcal{H})$ the compact operators in $\mathcal{B}(\mathcal{H})$ then $\mathcal{T}(\mathcal{H}) \subset C(\mathcal{H})$. Let p_i be a increasing net of finite rank projections in $\mathcal{T}(\mathcal{H})$ converging to 1 and let a be positive, define

$$\tau(a) = \sup_i \tau_0(p_i a).$$

Note that this does not depend on the net $\{p_i\}$. Obviously it follows that τ is semifinite, that it is normal follows from the fact that it is normal on finite rank operators. To prove that **Claim 1** holds one only needs to note that the finite rank operators form a two sided ideal in $B(\mathcal{H})$. Since $B(\mathcal{H})$ is a factor, it follows that the strong closure of the finite rank operators is $\mathcal{B}(\mathcal{H})$. □

Definition 3.1.0.8. *If $\mathcal{A}$ is a von Neumann algebra we say that $\mathcal{A}$ is semifinite when $\mathcal{A}z_{III} = 0$.*

So $\mathcal{A}$ is semifinite when it has no summand of type III.

Proposition 3.1.0.9. *Suppose that $\mathcal{A}$ is a von Neumann algebra and $\mathcal{B}$ is a type I factor. If $\mathcal{A}$ admits a semifinite faithful normal trace τ then so does $\mathcal{M} := \mathcal{A}\overline{\otimes}\mathcal{B}$.*

Proof:

Since $\mathcal{B}$ is a type I factor, it is isomorphic to $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. As such there exists a matrix unit $\{p_{i,j}\}_{i,j\in I}$ in $\mathcal{B}$, here $|I| = \dim(\mathcal{H})$. The tensor product $\mathcal{A}\overline{\otimes}\mathcal{B}$ is represented by a matrix $(x_{i,j})_{i,j\in I}$ with entries from $\mathcal{A}$. For $x \in \mathcal{M}_+$ set

$$\widehat{\tau} := \sum_{i\in I} \tau(x_{i,i}).$$

It follows that for any $y \in \mathcal{M}$ we have $\widehat{\tau}(y^*y) = \widehat{\tau}(yy^*)$, additivity and homogeneity follow immediately. If $\widehat{\tau}(x^*x) = 0$ then

$$\begin{aligned} 0 = \widehat{\tau}(x^*x) &= \sum_i \tau\left(\sum_j x_{j,i}^* x_{j,i}\right) \\ &= \sum_i \sum_j \tau\left(x_{j,i}^* x_{j,i}\right). \end{aligned}$$

Where the last equality follows from the normality of τ. Since τ is faithful it follows that $x_{j,i}^* x_{j,i} = 0$ thus $x_{j,i} = 0$ for all i,j, hence $x = 0$. We conclude that $\widehat{\tau}$ is faithful. That $\widehat{\tau}$ is normal follows from the usual interchanging of summation and supremum. To show that $\widehat{\tau}$ is semifinite, let $x \in \mathcal{M}_+$. If $x = (x_{i,j})$ pick any $i_0 \in I$ such that $(x_{i_0,i_0}) \neq 0$ and set $(x_0)_{i,j} := x_{i_0,i_0}$ if $i = i_0$ and $j = i_0$ and zero otherwise. So we picked a single diagonal entry and replaced everything else by zero. It follows that $\tau(x_{i_0,i_0}) = \widehat{\tau}(x_0) \leq \widehat{\tau}(x)$. Since τ is semifinite we can pick $y_0 \leq x_{i_0,i_0}$ such that $\tau(y_0) < \infty$. Set $(y)_{i,j} = y_0$ when $i = i_0$ and $j = i_0$ and zero otherwise. It follows that $\widehat{\tau}(y) = \tau(y_0) < \infty$ and $y \leq x$, we conclude that $\widehat{\tau}$ is a semifinite faithful normal trace on $\mathcal{M}$. □

Proposition 3.1.0.10. *If there exists a faithful semifinite normal trace τ on $\mathcal{A}$ then $\mathcal{A}$ is semifinite.*

Proof:

Suppose that $\mathcal{A}$ has a faithful normal semifinite trace, we need to show that every central projection dominates some finite projection. Let e be any projection, since τ is semifinite there exists some positive $x \leq e$ with $\tau(x) < \infty$. By the spectral theorem we can pick a projection f and $r > 0$ such that $rf \leq x$. It follows that $\tau(f) \leq \frac{1}{r}\tau(x) < \infty$, so $\tau(f) < \infty$. Since τ is faithful it follows that f is finite. We conclude that if τ is a semifinite normal trace then every projection dominates some finite projection. In particular every central projection dominates some finite projection, as such $\mathcal{A}$ is semifinite. □

The converse to this statement is also, we will sketch the proof. Suppose that $\mathcal{A}$ is semifinite then by 1.3.2.6 and 3.1.0.9 we only need to construct traces on finite von Neumann algebras. We already have that $\mathcal{B}(\mathbb{C}^n)$ allows for a finite trace. Abelian von Neumann algebras allow for a finite trace if and only if they allow for a faithful positive linear functional. Each abelian von Neumann algebra allows for such a functional since each abelian von Neumann algebra allows for a separating and cyclic vector, see for details 4.1.1.4. it remains to be seen that all type II_1 algebras allow for a faithful finite trace, without proof we state that they do, for proof we refer to [8].

We can now construct a type II_∞ factor. We will do this using a type II_1 factor derived from an ICC group.

3.1.1 Construction of a type II_∞ factor

As an application of traces we will now construct a type II_∞ factor using some of the properties of traces.

Theorem 3.1.1.1. *Given an ICC group G and a separable Hilbert space $\mathcal{H}$ then $\mathcal{R}(\mathcal{G})\overline{\otimes}B(\mathcal{H})$ is a factor of type II_∞.*

Proof:

Recall that a countable ICC group $\mathcal{G}$ gives rise to a type II_1 factor on a separable Hilbert space by considering its group von Neumann algebra $\mathcal{R}(\mathcal{G})$ (see 2.2.0.15). We concluded that the map $\tau : \mathcal{R}(\mathcal{G}) \longrightarrow \mathbb{C}$ defined by

$$\tau(f) = f(e),$$

is a faithful finite normal trace on $\mathcal{R}(\mathcal{G})$ (see 3.1.0.6). Let now $\mathcal{H}$ be an infinite dimensional separable Hilbert space and consider $\mathcal{M} := \mathcal{R}(G)\overline{\otimes}B(\mathcal{H})$. After selecting a matrix unit in $\mathcal{B}(\mathcal{H})$ we can represent elements of $\mathcal{M}$ by matrices $x = (x_{i,j})$ with $x_{i,j} \in \mathcal{R}(\mathcal{G})$ and $i, j \in \mathbb{N}$. We now know that $\mathcal{M}$ admits a semifinite normal faithful trace $\widehat{\tau}$ derived from τ namely

$$\widehat{\tau}((x_{i,j})) = \sum_i \tau(x_{i,i}).$$

As such $\mathcal{M}$ has no summand of type III (since it allows for a semifinite normal faithful trace). Since $\mathcal{B}(\mathcal{H})$ is a factor it follows that $\mathcal{Z}(\mathcal{M}) \cong \mathcal{Z}(\mathcal{R}(\mathcal{G})) \cong \mathbb{C}$ so $\mathcal{M}$ is a factor. If $\mathcal{M}$ allows for a finite trace then in particular the identity would be a finite projection but it clearly is not. We find that $\mathcal{M}$ is not of type II_1. Also $\mathcal{M}$ does not contain all bounded operators on $L^2(G) \otimes \mathcal{H}$ because $\mathcal{R}(\mathcal{G})$ does not contain all bounded operators on $L^2(G)$. We conclude that $\mathcal{M}$ is not of type I. So far we have ruled out the cases type I, II_1 and III, it follows that $\mathcal{M}$ is of type II_∞. □

We conclude that an ICC group $\mathcal{G}$ gives rise to a type II_1 factor and by considering its tensor product with $B(\mathcal{H})$ for some separable infinite dimensional Hilbert space $\mathcal{H}$, it gives rise to a type II_∞ factor on a separable Hilbert space. Note that the separability is not necessary for $\mathcal{M}$ to be of type II_∞. We picked a separable space to construct a type II_∞ factor on a separable Hilbert space. This is to show that type II_∞ factors are not restricted to spaces of dimension $> \aleph_0$.

3.2 Trace and crossed products

In this section we return to the crossed product of an abelian von Neumann algebra $\mathcal{A}$ with a group $\mathcal{G}$ with respect to some free and ergodic action $\alpha : \mathcal{G} \longrightarrow \mathrm{Aut}(\mathcal{A})$. We will see that the type question of the crossed product $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ depends very much on which kind of traces $\mathcal{A}$ admits. Since $\mathcal{A}$ is abelian it allows for a faithful semifinite normal normal trace. We will see that the type of $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ depends on how this trace interacts with the action α. In this section we will assume that the action α is free and ergodic (see 2.2.0.10).

Consider an abelian von Neumann algebra $\mathcal{A}$ together with a group $\mathcal{G}$ and an action $\alpha : G \longrightarrow \mathrm{Aut}(\mathcal{A}$. We construct the crossed product of $\mathcal{A}$ with G as in 2.1.0.8, and denote it by $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$.

Proposition 3.2.0.2. *Suppose τ is a semifinite faithful normal trace on $\mathcal{A}$ which is invariant under α_g for all g. Then τ defines a semifinite faithful normal trace $\widehat{\tau}$ on $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ by setting*

$$\widehat{\tau}(x) := \tau(x(e)).$$

Proof:

Pick $x \in \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ then $x = y^*y$ for some $y \in \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$. Using that τ is invariant for α_g and that $\mathcal{A}$ is abelian we find

$$\begin{aligned}
\widehat{\tau}(x) = \tau(x(e)) = \tau(y^*y(e)) &= \tau\left(\sum_g \alpha_g\left(y\left(g^{-1}\right)^* y\left(g^{-1}\right)\right)\right) \\
&= \sum_g \tau\left(\alpha_g\left(y\left(g^{-1}\right)^* y\left(g^{-1}\right)\right)\right) \\
&= \sum_g \tau\left(y\left(g^{-1}\right)^* y\left(g^{-1}\right)\right) \\
&= \sum_g \tau\left(y^*(g)y(g)\right) \\
&= \sum_g \tau\left(y(g)y^*(g)\right) = \tau(yy^*(e)).
\end{aligned}$$

We conclude that $\tau(y^*y) = \tau(yy^*)$ for all $y \in \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$. That $\widehat{\tau}$ respects multiplication with elements from $\mathbb{R}_+$ and respects addition is immediate. If $x \in \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is positive and nonzero then $x(e) \in \mathcal{A}$

is positive and nonzero. As such, $\widehat{\tau}(x) > 0$, we conclude $\widehat{\tau}$ is faithful. Normality and semifiniteness of $\widehat{\tau}$ follows from the normality and semifiniteness of τ on $\mathcal{A}$. □

We see that a faithful semifinite normal trace invariant under α on $\mathcal{A}$ gives rise to a semifinite faithful normal trace on $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$. As such, if $\mathcal{A}$ allows for a semifinite faithful normal trace invariant under α then $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is not of type III. Note that the trace on $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is determined by the trace on $\mathcal{A}$. There is a converse to this statement namely:

Theorem 3.2.0.3. *Suppose that τ is a semifinite faithful normal trace on $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$. Then the restriction of τ to $\pi(\mathcal{A})$, denoted by $\tau_{\mathcal{A}}$, is a semifinite faithful normal trace on $\mathcal{A}$. Moreover $\tau_{\mathcal{A}}$ is invariant under the action of α.*

proof:

Given a semifinite faithful normal trace τ on $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$. Then $\tau_{\mathcal{A}}$ is a normal faithful trace on $\pi(\mathcal{A})$. We will now identify $\mathcal{A}$ with its image $\pi(\mathcal{A}) \subset \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$. What remains is to show that $\tau_{\mathcal{A}}$ is semifinite and invariant under α. Let $a = bb^* \in \mathcal{A}_+$ and consider

$$\tau_{\mathcal{A}}(\pi(\alpha_g(a))).$$

By the covariance relation, (2.1), we have $u(g)\pi(a)u(g)^* = \pi(\alpha_g(a))$. We find that

$$\begin{aligned}\tau_{\mathcal{A}}(\pi(\alpha_g(a))) &= \tau(\pi(\alpha_g(a))) \\ &= \tau\left(u(g)\pi(b)\pi(b)^* u(g)^*\right) \\ &= \tau\left(\pi(b)^*\pi(b)\right) \\ &= \tau\left(\pi(b)\pi(b)^*\right) = \tau(\pi(a)) = \tau_{\mathcal{A}}(\pi(a)).\end{aligned}$$

So indeed $\tau_{\mathcal{A}}$ is invariant under α. What remains is to show that $\tau_{\mathcal{A}}$ is semifinite. Let $a \in \mathcal{A}_+$, then we can find $y \in \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)_+$ with $y \le a$ and $\tau(y) < \infty$. Note that since $a - y \ge 0$ we have that $(a-y)(e) = a - y(e) \ge 0$ so the natural candidate is $y(e)$, however it is not a priori clear that $\tau(y(e)) \le \tau(y)$. We will show that this is indeed the case. We are given $y \in \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)_+$ with $y \le a$ and $\tau(y) < \infty$. Let $\mathcal{U}_{\mathcal{A}}$ denote the unitary group of $\mathcal{A}$ and set

$$C(y) := \operatorname{conv}\{uyu^* \ ; \ u \in \mathcal{U}_{\mathcal{A}}\}.$$

Let $K(y)$ be the weak closure of $C(y)$, so $K(y)$ is a weakly compact convex set. Note that $K(y) \subset \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)_+$. since $C(y)$ is convex we have that the strong closure of $C(y)$ equals the weak closure of $C(y)$. Note that each $u \in \mathcal{U}_{\mathcal{A}}$ acts as an affine transformation of $K(y)$ into $K(y)$, sending $x \in K(y)$ to $u(x) := uxu^* \in K(y)$. Since $\mathcal{U}_{\mathcal{A}}$ is an abelian family of such transformations and $K(y)$ is weakly compact. The Markov-Kakutani fixed point theorem (see [1] for details) applies to this situation. It states that there exists an $x_0 \in K(y)$ with $u(x_0) = x_0$ for all $u \in \mathcal{U}_{\mathcal{A}}$. This particular x_0 commutes with every unitary from $\mathcal{A}$, as such $x_0 \in \mathcal{A}'$. Using that the canonical image of $\mathcal{A}$ in $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is maximal abelian, we conclude that $x_0 \in \mathcal{A}$. Let $E : \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha) \longrightarrow \mathcal{A}$ be defined by $E(x) = x(e)$, that is, E is the restriction of $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ to $\mathcal{A}$. Consider $x \in C(y)$ and $u \in \mathcal{U}_{\mathcal{A}}$ we find that

$$\begin{aligned}uxu^*(e) = E(uxu^*) &= E\left(\sum_g \pi(u)\pi(x(g))u(g)\pi(u)^*\right) \\ &= E\left(\sum_g \pi(x(g)u\alpha_g(u^*))u(g)\right) \\ &= \pi(x(e)uu^*) = \pi(x(e)) = E(x).\end{aligned}$$

We conclude that E is constant on $C(y)$. It is not hard to see that E is weakly continuous, as such E is constant on $K(y)$. Using this we find that

$$\begin{aligned}K(y) \cap \mathcal{A} &= E(K(y) \cap \mathcal{A}) \\ &= E(x_0) = x_0.\end{aligned}$$

So x_0 is in fact unique.
Note that $\tau(uyu^*) = \tau(y)$ so that τ is constant on $C(y)$. Let e_j be an increasing net of finite projections with $\sup_j e_j = 1$, and let $x \in K(y)$. Note that the existence of such a net e_j is guaranteed by the semifiniteness of τ. There exists a net y_i in $C(y)$ converging weakly to x. We find the following expression for x and $\tau(x)$.

$$x = \sup_j \left(\lim_i e_j y_i e_j \right)$$
$$\tau(x) = \tau\left(\sup_j \left(\lim_i e_j y_i e_j \right) \right) = \sup_j \tau\left(\lim_i e_j y_i e_j \right).$$

Note now that for all j we have that $e_j \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha) e_j$ is a finite von Neumann algebra so that τ is restricted to $e_j \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha) e_j$ is a finite trace, hence continuous. We have thus that

$$\begin{aligned} \tau(x) &= \sup_j \lim_i \tau(e_j y_i e_j) \\ &\leq \lim_i \sup_j \tau(e_j y_i e_j) \\ &= \lim_i \tau(y_i) = \tau(y). \end{aligned}$$

In particular $\tau(x_0) \leq \tau(y)$, but $x_0 = E(y)$ so $\tau(E(y)) \leq \tau(y)$. In total we conclude that for $a \in \mathcal{A}_+$ we can find $y \in \mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)_+$ with $y \leq a$ and $\tau(y) < \infty$. Since $y \leq a$ we have that $y(e) \leq a$ but now $\tau(y) = \tau(y(e))$ thus $\tau(y(e)) < \infty$, and $y(e) \in \mathcal{A}_+$. We conclude that the restriction of τ to $\mathcal{A}$ is also a semifinite faithful normal trace and invariant under the action of α. □

Remark Note that the action α was chosen to be free and ergodic, as such $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is a factor by 2.2.0.14. Using 3.2.0.2 and 3.2.0.3 we conclude that $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ allows for a semifinite faithfull normal trace if and only if $\mathcal{A}$ allows for a semifinite normal trace invariant under $\alpha(\mathcal{G})$.
We are now ready to classify the type of a crossed product in terms of the trace and the action of α. Our last remark proves *4* in the following theorem.

Theorem 3.2.0.4. *Let $\mathcal{A}$ be an abelian von Neumann algebra and let $\alpha : G \longrightarrow Aut(\mathcal{A})$ be a free ergodic action of a countable discrete group G. Then $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is a factor and its type depends on α in the following way:*

1. *$\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is of type I if and only if $\mathcal{A}$ has a minimal projection p with $\sum_g \alpha_g(p) = 1$.*
2. *$\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is of type II_1 if and only if $\mathcal{A}$ admits a faithful finite normal trace invariant under α.*
3. *$\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is of type II_∞ if and only if $\mathcal{A}$ has no minimal projection and admits a semifinite, but infinite, faithful normal trace invariant under α.*
4. *$\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is of type III if and only if $\mathcal{A}$ does not admit a semifinite faithful normal trace invariant under α.*

Proof:

1
Suppose $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is of type I, then $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha) \cong \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Thus $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ admits a semifinite faithful normal trace, namely the canonical trace on $\mathcal{B}(\mathcal{H})$. By 3.2.0.3 this trace is also faithful, semifinite and normal on $\mathcal{A}$ thus $\mathcal{A}$ admits a finite rank projection. We conclude that there exists some finite set of nonzero measure in the spectrum of $\mathcal{A}$. As such we can select a point in the spectrum of $\mathcal{A}$ with nonzero measure, meaning that $\mathcal{A}$ has a minimal projection p. Note now that $p\alpha_g(p) = 0$ for all $g \neq e$ (if not then by the minimality of p we have that $\alpha_g(p) = p$ for some g and then p is absolutely invariant for α_g, but this cannot happen since α is free). Making the substitution $g = h^{-1}k$ with $k \neq h$ and letting α_h act both sides of the equation we find

$$\alpha_h(p)\,\alpha_k(p) = 0,$$

for all $h \neq k$. We conclude that the family $\{\alpha_g(p)\}_{g\in\mathcal{G}}$ is orthogonal. Consider now the projection

$$e = \sum_{g\in\mathcal{G}} \alpha_g(p),$$

and note that $\alpha_g(e) = e$ (by construction), thus e is invariant for all α_g. By the ergodic property of α we conclude that $e = 1$. So far we have proved the "if" part of *1*. Now the "and only if" part, suppose that $\mathcal{A}$ admits a minimal projection p with $\sum_{g\in\mathcal{G}} \alpha_g(p) = 1$. Since p is minimal in $\mathcal{A}$ we have that $p\mathcal{A}p = \mathcal{A}p = \mathbb{C}p$. For $x \in \mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ consider $\pi(p)\,x\pi(p) \in \mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$. We find

$$\begin{aligned}
\pi(p)\,x\pi(p) &= \sum_g \pi(px(g))\,u(g)\,\pi(p) \\
&= \sum_g \pi(px(g))\,\pi(\alpha_g(p))\,u(g) \\
&= \pi(px(e)\,p) \in \mathbb{C}p.
\end{aligned}$$

So $\pi(p)$ is also minimal in $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$, we conclude that $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ is of type I.

2
Suppose that $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ is of type II_1. Then by 3.2.0.3 its finite faithful normal trace is also finite faithful and normal on $\mathcal{A}$ and invariant under α. If $\mathcal{A}$ admits a finite faithful normal trace invariant under α then this trace can be extended to $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ by 3.2.0.2 and remains finite faithful and normal. Thus then $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ is of type II_1.

3
Suppose that $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ is of type II_∞ then its semifinite (but infinite) faithful normal trace is semifinite (but infinite), normal, faithful and invariant under α when restricted to $\mathcal{A}$. If $\mathcal{A}$ had a minimal projection then by the arguments above we would have concluded that $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ is of type I but this is not the case. As such $\mathcal{A}$ has no minimal projection. On the other hand, if $\mathcal{A}$ does not admit a minimal projection and has a semifinite (but infinite) faithful normal trace invariant under α then $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ is either of type I or of type II_∞. But note now that if $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ has a minimal projection p then $\mathcal{A}$ has one as well. So $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ must be of type II_∞. □

We will now explicitly construct a type III factor on a separable Hilbert space.

3.2.1 Construction of a type III factor

We need to find an abelian von Neumann algebra $\mathcal{A}$ and a countable discrete group $\mathcal{G}$ and an action $\alpha : \mathcal{G} \longrightarrow \mathrm{Aut}(\mathcal{A})$ which is free and ergodic. If we pick this action in such a way that any semifinite normal faithful trace on $\mathcal{A}$ is not invariant under α, then by 3.2.0.4 it follows that $\mathcal{R}(\mathcal{A},\mathcal{G},\alpha)$ is of type III. Consider $\mathbb{R}$ with the Lebesgue measure λ and set

$$\mathcal{A} := L^\infty(\mathbb{R},\lambda).$$

Pick a rational number $r \in \mathbb{Q}$ with $|r| \neq 1$, let $q \in \mathbb{Q}$ and let $n \in \mathbb{Z}$. For $x \in \mathbb{R}$ we set

$$\gamma_{r,q,n}(x) := r^n x + q.$$

Let $\mathcal{G}_r$ be the set of transformations $\left\{\gamma_{(r,q,n)}\ ;\ q, \in \mathbb{Q}\ n \in \mathbb{Z}\right\}$. For $\gamma_{(n,q)}$ and $\gamma_{(m,l)}$ we find the following identities:

$$\begin{aligned}
\gamma_{(m,l)} \circ \gamma_{(n,q)}(x) &= r^{m+n}x + r^m q + l = \gamma_{(m+n,r^m q+l)} \\
\gamma_{(n,q)} \circ \gamma_{(m,l)}(x) &= r^{m+n}x + r^n l + q = \gamma_{(m+n,r^n l+q)} \\
\gamma_{(-n,-r^{-n}q)} \circ \gamma_{n,q} &= \gamma_{0,r^{-n}q-r^{-n}q} = \gamma_{0,0} = e.
\end{aligned}$$

We conclude that $\mathcal{G}_r$ is a non abelian group and obviously it is countable. Since every element in $\mathcal{G}_r$ is an affine transformation of the real line, it gives rise to a self-map of $\mathcal{A} = L^\infty(\mathbb{R},\lambda)$ by setting

$$\alpha_{n,q} f(x) := f(\gamma_{n,q}(x)) = f(r^n x + q).$$

It is easy to check that the following equalities hold

$$\begin{aligned}
\|\alpha_{n,q} f\| &= \|f\|\,, \\
\alpha_{n,q}(\mu f) &= \mu \alpha_{n,q}(f)\,, \\
\alpha_{n,q}(f+g) &= \alpha_{n,q}(f) + \alpha_{n,q}(g)\,, \\
\alpha_{n,q}(f \cdot g) &= \alpha_{n,q}(f) \cdot \alpha_{n,q}(g), \\
\alpha_{n,q}(f^*) &= \alpha_{n,q}(f)^*\,, \\
\alpha_{n,q}(1) &= 1.
\end{aligned}$$

We see that $\alpha_{n,q} : \mathcal{A} \longrightarrow \mathcal{A}$ is an isomorphism. We define an action $\alpha : \mathcal{G} \longrightarrow \mathrm{Aut}(\mathcal{A})$ by setting

$$\alpha(\gamma_{n,q}) := \alpha_{n,q}.$$

So far we have constructed an abelian von Neumann algebra together with a group $\mathcal{G}$ and an action α. It remains to be shown that α is free and ergodic and that no faithful semifinite normal trace is invariant under α. First the ergodic property. If α is not ergodic then in particular there exists a projection e such that $\alpha_{0,q}(e) = e$ for all $q \in \mathbb{Q}$, that is $e(x) = e(x+q)$ for all $q \in \mathbb{Q}$ an $x \in \mathbb{R}$. In particular $e(x) = e(x+1)$, as such, by considering $\mathbb{R}/\mathbb{Z}$ we may assume that the action of α is given on the interval $[0,1]$. Note that for given $x \in [0,1]$ the orbit of x under $\{\alpha_{0,q}\ ;\ q \in \mathbb{Q}\}$ is dense in $[0,1]$. Consider the invariant projection e, it gives rise to a measurable subset $E \subset [0,1]$. Let $\epsilon > 0$ and pick a continuous function f with the property that

$$\int_{[0,1]} |f - e|\ d\lambda < \epsilon.$$

Note that for all $q \in \mathbb{Q}$ and $g \in L^\infty([0,1], \lambda)$ we have that

$$\int_{[0,1]} |\alpha_{0,q}(g)|\ d\lambda = \int_{[0,1]} |g|\ d\lambda,$$

this is because $\alpha_{0,q}$ shifts g over q, and therefor does not change its L^1 norm. We denote the L^1 norm with $\|\cdot\|_1$, using the invariance of e under $\alpha_{0,q}$ for all $q \in \mathbb{Q}$ we find the following inequalities:

$$\begin{aligned}
\|\alpha_{0,q}(f) - f\|_1 &= \|\alpha_{0,q}(f) - e + e - f\|_1 \\
&\leq \|\alpha_{0,q}(f) - e\|_1 + \|e - f\|_1 \\
&= \|\alpha_{0,q}(f - e)\|_1 + \|e - f\|_1 \\
&= 2\,\|f - e\|_1 \leq 2\epsilon.
\end{aligned}$$

Using the continuity of f and applying the Lebesgue dominated convergence theorem we find that for all $t \in \mathbb{R}$ the following holds:

$$\begin{aligned}
\int_{[0,1]} |f(x+t) - f(x)|\ dx &= \int_{[0,1]} \lim_{q \longrightarrow t} |f(x+q) - f(x)\ dx \\
&= \lim_{q \longrightarrow t} \int |f(x+q) - f(x)|\ dx \leq 2\epsilon.
\end{aligned}$$

Using Fubini's theorem and the fact that for all $x \in [0,1]$ we have that $\int_{[0,1]} f(t)\ dt = \int_{[0,1]} f(t+x)\ dt$, we find that

$$\begin{aligned}
\left\| f - \int_{[0,1]} f(t)\ dt \right\|_1 &= \int_{[0,1]} \left| f(x) - \int_{[0,1]} f(x+t)\ dt \right|\ dx \\
&\leq \int_{[0,1]} \int_{[0,1]} |f(x) - f(x+t)|\ dtdx \\
&= \int_{[0,1]} \int_{[0,1]} |f(x) - f(x+t)|\ dxdt \\
&\leq \int_{[0,1]} 2\epsilon\ dt = 2\epsilon.
\end{aligned}$$

Consider the constant function $\lambda(E) = \int_{[0,1]} e(x)\ dx$, we find that

$$\begin{aligned}
\|e - \lambda(E)\|_1 &= \|e - f + f - \lambda(E)\|_1 \\
&\leq \|e - f\|_1 + \|f - \lambda(E)\|_1 \\
&\leq \|e - f\|_1 + \left\| f - \int_{[0,1]} f(t)\ dt \right\|_1 + \left\| \int_{[0,1]} f(t)\ dt - \lambda(E) \right\|_1 \\
&\leq 3\epsilon + \left\| \int_{[0,1]} f(t)\ dt - \lambda(E) \right\|_1 \\
&\leq 3\epsilon + \int_{[0,1]} \|f - e\|_1\ dt \\
&= 4\epsilon.
\end{aligned}$$

We let ϵ tend to zero to find that $\|e - \lambda(E)\|_1 = 0$. This implies that either $e = 0$ or $e = 1$, translating this back to the original action on $\mathbb{R}$ we find that if e is an invariant projection then $e = 1$ or $e = 0$, that is, α is ergodic.

Now to show that $\alpha_{n,q}$ is free for all $n, q \in \mathbb{Z} \times \mathbb{Q} \setminus \{0, 0\}$. Suppose that $\alpha_{n,q}$ has an invariant projection e. Since e is a projection it gives rise to some set V that is invariant for $\gamma_{n,q}$ and $\lambda(V) \neq 0$. Set for simplicity $r^n = p$ so that $\gamma_{n,q}(x) = px + q$. Note that $\gamma_{n,q}$ has a unique fixed point $x_0 = \frac{q}{1-p}$, because r was chosen in to be of absolute value different then 1. Note now that if e was absolutely invariant for $\alpha_{n,q}$ then all its points are fixed. But $\alpha_{n,q}$ has only one fixed point so then V has measure zero. we conclude that $\alpha_{n,q}$ is free for all (n, q).

What remains is to show that if a semifinite normal faithful trace is given then it is not invariant under the action α of $\mathcal{G}_r$. Consider a semifinite normal faithful trace τ and pick an increasing sequence e_i of finite projections (finite in the sense that $\tau(e_i) < \infty$) converging strongly to 1. This can be done since $\mathcal{A}$ is separable. Set now $\tau_i(f) := \tau(e_i f e_i)$, by normality it follows that

$$\tau(f) = \sup_i \tau_i(f).$$

Note now that τ_i is a faithful finite normal trace on $e_i \mathcal{A} e_i$. As such we have that (again by normality) it is given by integration against some function ϕ_i in the predual of $\mathcal{A}e_i$. Since τ_i is faithful it follows that the set $\{\phi_i = 0\}$ has measure zero (note that here we pick some fixed ϕ_i in the equivalence class of ϕ_i and work with that function). Since e_i is a projection it gives rise to a set E_i such that $\mathcal{A}e_i = L^\infty(E_i, \lambda)$. so ϕ_i is an element of $L^1(E_i, \lambda)$. We find an increasing sequence of functions ϕ_i such that $\phi_i \in L^1(E_i, \lambda)$. We can assume that E_i is a bounded set so that ϕ_i is a bounded measurable function on some bounded set E_i for all i. Define now

$$\phi(x) := \sup_i \phi_i(x).$$

Note that this is well defined since for all $x \in \mathbb{R}$ we have that there is some i_x such that $x \in E_{i_x}$ and as such $\phi(x) = \phi_{i_x}(x)$. Note that the zero set of ϕ has measure zero because

$$\{\phi = 0\} = \{\phi_1 = 0\} \cup \left[\bigcup_{i=1} \{(\phi_{i+1} - \phi_i) = 0\} \cap \{E_{i+1} - E_i\} \right].$$

We conclude that

$$\lambda(\{\phi = 0\}) = \lambda(\{\phi_1 = 0\}) + \sum_{i=1} \lambda(\{(\phi_{i+1} - \phi_i) = 0\} \cap \{E_{i+1} - E_i\}) = 0.$$

Also via a similiar argument we conclude that $\lambda(\{\phi = \infty\}) = 0$. By picking a suitable representative in its equivalence class, ϕ can be regarded a positive function which is nowhere zero and is possibly unbounded as x tends to plus or minus infinity. Furthermore it follows that

$$\tau(f) = \sup_i \tau_i(f) = \sup_i \int_{\mathbb{R}} f(x) \phi_i(x)\ d\lambda(x) = \int_{\mathbb{R}} f(x) \phi(x)\ d\lambda(x).$$

Let now $\alpha_{n,0} \in \mathcal{G}$ and let $f = 1_{[0,1]}$ be the indicator function over the interval $[0, 1]$. Consider $\tau(\alpha_{n,0}(f))$, we find

$$\tau(f) = \int_0^1 \phi(x)\ d\lambda(x)$$
$$\tau(\alpha_{n,0}(f)) = \int_0^{r^n} \phi(x)\ d\lambda(x).$$

If we picked $r > 1$ and $n \geq 0$ then by the fact that $\phi > 0$, we conclude that

$$\tau(f) \neq \tau(\alpha_{n,0}(f)).$$

In total we conclude that τ is *not* invariant under the action α of $\mathcal{G}$, that is, $\mathcal{R}(\mathcal{A}, \mathcal{G}, \alpha)$ is of type III.

We have used the crossed product construction to show that there are indeed factors of all types on a separable Hilbert space. From the definition it was not clear at all that there are indeed factors of type III. The type III algebras are now the main focus of study. It turns out that most of the von Neumann algebras encountered in physics are of type III, emphasizing the importance of understanding them better. Sadly enough most of the techniques used to study type I and type II do not work in type III. We will now proceed to a classical result in the theory of von Neumann algebras.

3.3 Tomita-Takesaki theory (semifinite case)

The Tomita-Takesaki Theorem states that any given von Neumann algebra $\mathcal{A}$ can be represented on a Hilbert space $\mathcal{H}_{\mathcal{A}}$, constructed form $\mathcal{A}$, such that $\mathcal{A}$ is anti-isomorphic to its own commutant. We will state and prove the theorem for a semifinite von Neumann algebra and for an arbitrary von Neumann algebra. The reason for this is that in the semifinite case one can prove the theorem using the trace, which is guaranteed to exist. The proof for an arbitrary von Neumann algebra also works for the semifinite case, however we feel that it is instructive to see that the existence of a faithful semifinite normal trace can be used to construct the predual and a canonical representation on a Hilbert space derived from the algebra in question. When reviewing this particular representation one concludes that there is not much mystery left in the semifinite case. In some sense (which will be made clear later on) there is a striking resemblance between abelian von Neumann algebras and semifinite von Neumann algebras.

Theorem 3.3.0.1 (Tomita-Takesaki (semifinite case))**.** *Suppose that $\mathcal{A}$ allows for a semifinite faithful normal trace τ, then $\mathcal{A}$ is isomorphic to a von Neumann algabra $\pi(\mathcal{A})$ on a Hilbert space $\mathcal{H}_\tau$ and there is an anti-linear isometry $J : \mathcal{H}_\tau \longrightarrow \mathcal{H}_\tau$ such that $J\pi(\mathcal{A})J = \pi(\mathcal{A})'$.*

Before we can prove this theorem we need some preparation. First we examine what the existence of a faithful semifinite normal trace implies for the algebra it acts on. In particular we will construct a Hilbert space which allows for an involution and multiplication derived from τ. This Hilbert space will be the one considered in the Tomita-Takesaki theorem. Furthermore we will construct the predual of a von Neumann algebra $\mathcal{A}$ with a faithful semifinite normal trace.

Given $\mathcal{A}$ with semifinite faithful normal trace τ. set $\mathcal{N}_\tau : \{x \in \mathcal{A}\ ;\ \tau(x^*x) < \infty\}$. Since $\tau(x^*x) = \tau(xx^*)$ we have that if $x \in \mathcal{N}_\tau$ then $x^* \in \mathcal{N}_\tau$, so $\mathcal{N}_\tau$ is $*$ closed. If $m \in \mathcal{A}$ and $x \in \mathcal{N}_\tau$ then we find that

$$\begin{aligned}\tau\left((mx)^*(mx)\right) &= \tau(x^*m^*mx) \\ &\leq \|m\|^2\,\tau(x^*x).\end{aligned}$$

We conclude that $mx \in \mathcal{N}_\tau$. Since $(xm)^* = m^*x^* \in \mathcal{N}_\tau$ and $\mathcal{N}_\tau$ is $*$ closed, we conclude that $(xm)^{**} = xm \in \mathcal{N}_\tau$ for all $x \in \mathcal{N}_\tau$ and $m \in \mathcal{A}$. By the inequality $(x+y)^*(x+y) \leq 2(x^*x + y^*y)$ we conclude that $\mathcal{N}_\tau$ is closed under summation. We conclude that $\mathcal{N}_\tau$ is a proper two sided ideal of $\mathcal{A}$. For a more detailed description of these inequalities see 3.1.0.5. We will now show that τ defines an inner product on $\mathcal{N}_\tau$, however this is not a priori clear. We would like to define on $\mathcal{N}_\tau$

$$\langle x, y\rangle := \tau(y^*x),$$

however τ is only defined for positive elements. If in some way we could extend τ to include non positive elements, then it is not even a priori clear that $\tau(y^*x)$ is finite. We will show that this can in fact be done.

Proposition 3.3.0.2. *Suppose that τ is a semifinite faithful normal trace on a von Neumann algebra $\mathcal{A}$ and $\mathcal{N}_\tau = \{x \in \mathcal{A}\ ;\ \tau(x^*x) < \infty\}$. Then on the ideal $\mathcal{I}_\tau := \mathcal{N}_\tau \cdot \mathcal{N}_\tau$, we can extend τ to be a faithful positive normal linear functional also denoted by τ. The linear functional τ satisfies the following properties:*

$$\begin{aligned} \tau(f^*) &= \overline{\tau(f)} && \text{for all } f \in \mathcal{I}_\tau, \\ \tau(af) &= \tau(fa) && \text{for all } a \in \mathcal{A} \text{ and } f \in \mathcal{I}_\tau, \\ \tau(xy) &= \tau(yx) && \text{for all } x, y \in \mathcal{N}_\tau. \end{aligned}$$

Furthermore for all $x, y \in \mathcal{N}_\tau$, we have that $y^*x \in \mathcal{I}_\tau$, so that

$$\langle x, y\rangle_\tau := \tau(y^*x),$$

is a well defined inner product on $\mathcal{N}_\tau$.

Proof:

Since $\mathcal{N}_\tau$ is an ideal of $\mathcal{A}$ it follows that

$$\mathcal{I}_\tau := \mathcal{N}_\tau \cdot \mathcal{N}_\tau := \left\{ \sum_{n=0}^{m} x_n y_n\ ;\ x_n, y_n \in \mathcal{N}_\tau,\ m \in \mathbb{N} \right\},$$

is also an ideal of $\mathcal{A}$. Set $\mathcal{F}_\tau := \{x \in \mathcal{A}_+\ ;\ \tau(x) < \infty\}$ and consider an element $f \in \mathcal{I}_\tau$. By construction f is of the following form:

$$f = \sum_{n=0}^{m} y_n^* x_n.$$

The reason why we picked y_n^* in the formula above will be made clear now. Note the following identity for $y_n^* x_n$, we have

$$4 y_n^* x_n = \sum_{k=0}^{3} i^k \left(x_n + i^k y_n\right)^* \left(x_n + i^k y_n\right).$$

This formula is called the polarization identity for the product $y_n^* x_n$. In total we conclude that f can be written in the following form:

$$4f = \sum_{n=0}^{m} \sum_{k=0}^{3} i^k \left(x_n + i^k y_n\right)^* \left(x_n + i^k y_n\right).$$

We conclude that $\mathcal{I}_\tau$ is spanned linearly by $\mathcal{I}_\tau \cap \mathcal{A}_+$. If f happens to be self-adjoint then f can be written in the following form

$$4f = \sum_{n=0}^{m} (x_n + y_n)^* (x_n + y_n) - \sum_{n=0}^{m} (x_n - y_n)^* (x_n - y_n).$$

Note now that if f is positive then in particular we have that

$$4f \le \sum_{n=0}^{m} (x_n + y_n)^* (x_n + y_n).$$

As such, we find that $\tau(f) < \infty$ when $f \in \mathcal{I}_\tau \cap \mathcal{A}_+$. We conclude that $\mathcal{I}_\tau \cap \mathcal{A}_+ \subset \mathcal{F}_\tau$. On the other hand if $g \in \mathcal{F}_\tau$ then $\sqrt{g} \in \mathcal{N}_\tau$, thus $g \in \mathcal{I}_\tau \cap \mathcal{A}_+$, we conclude that $\mathcal{F}_\tau = \mathcal{I}_\tau \cap \mathcal{A}_+$.

In general if f is self adjoint then there are positive elements f_+ and f_- such that $f = f_+ - f_-$.

These elements can be obtained by considering the functions $t_+ : \mathbb{R} \longrightarrow \mathbb{R}$ and $t_- : \mathbb{R} \longrightarrow \mathbb{R}$ where t_+ is defined as

$$t_+(r) := \begin{cases} r & \text{when } r \geq 0, \\ 0 & \text{when } r < 0, \end{cases}$$

and t_- is defined as

$$t_-(r) := \begin{cases} 0 & \text{when } r \geq 0, \\ -r & \text{when } r < 0. \end{cases}$$

By the functional calculus $t_+(f)$ is well defined and is a positive element, the same holds for $t_-(f)$ and together they obey the relation $f = t_+(f) - t_-(f)$. Note that $t_+(f)\, t_-(f) = 0$ and that $|f| = t_+(f) + t_-(f)$. We define $f_+ := t_+(f)$ and $f_- = t_-(f)$. By the polar decomposition we can find a partial isometry $u \in \mathcal{A}$ such that $f = u|f|$. It follows that $u^* f = |f|$, since $\mathcal{I}_\tau$ is an ideal we can conclude that $|f| \in \mathcal{I}_\tau$ and thus $\tau(|f|) < \infty$. Using this we conclude that

$$\tau(|f|) = \tau(f_+ + f_-) = \tau(f_+) + \tau(f_-) < \infty,$$

in particular $\tau(f_+) < \infty$ and $\tau(f_-) < \infty$. We already concluded that $\mathcal{F}_\tau = \mathcal{I}_\tau \cap \mathcal{A}_+$ so it follows that $f_+, f_- \in \mathcal{I}_\tau$.

Now we can extend τ to a linear functional on $\mathcal{I}_\tau$ denoted for now by $\widehat{\tau}$. For a selfadjoint element $f \in \mathcal{I}_\tau$ and positive elements $h, v \in \mathcal{F}_\tau$ such that $f = h - v$, we define

$$\widehat{\tau}(f) := \tau(h) - \tau(v).$$

We first need to check if this is well defined before we can extend τ further to non selfadjoint elements in $\mathcal{I}_\tau$. Suppose that $f = h_1 - v_1 = h_2 - v_2$ are two different representations of f with $h_1, h_2, v_1, v_2 \in \mathcal{F}_\tau \subset \mathcal{I}_\tau$. Then it follows that $h_1 + v_2 = h_2 + v_1 \in \mathcal{F}_\tau$, thus

$$\begin{aligned} \tau(h_1 + v_2) &= \tau(h_2 + v_1) \\ \tau(h_1) + \tau(v_2) &= \tau(h_2) + \tau(v_1) \\ \tau(h_1) - \tau(v_1) &= \tau(h_2) - \tau(v_2) = \widehat{\tau}(f). \end{aligned}$$

We conclude that $\widehat{\tau}$ is well defined, so that we can choose $h = f_+$ and $v = f_-$. Note also that $\widehat{\tau}$ is $\mathbb{R}$ linear because τ is. If $f \in \mathcal{I}_\tau$ is selfadjoint then we set $\widehat{\tau}(i \cdot f) = i \cdot \widehat{\tau}(f)$ so that $\widehat{\tau}$ becomes $\mathbb{C}$ linear over the selfadjoint elements of $\mathcal{I}_\tau$. Given now any element $f \in \mathcal{I}_\tau$ then there are $h, v \in \mathcal{I}_\tau$ selfadjoint such that $f = h + iv$. We extend $\widehat{\tau}$ further by defining

$$\widehat{\tau}(f) := \widehat{\tau}(h) + i\widehat{\tau}(v).$$

Again this is well defined so that we can pick $h = \Re(f)$ and $v = \Im(f)$. In total we find that

$$\widehat{\tau}(f) = \tau\left(\Re(f)_+\right) - \tau\left(\Re(f)_-\right) + i\tau\left(\Im(f)_+\right) - i\tau\left(\Im(f)_-\right).$$

From now on we will not distinguish between $\widehat{\tau}$ and τ and we will denote $\widehat{\tau}$ simply by τ. Consider $f \in \mathcal{I}_\tau$ then $f = \Re(f) + i\Im(f)$ and $f^* = \Re(f) - i\Im(f)$ thus $\tau(f^*) = \tau(\Re(f)) - i\tau(\Im(f)) = \overline{\tau(f)}$. pick $a^* \in \mathcal{A}$ and let $f \in \mathcal{I}_\tau$ then by the polarization identity and the linearity of τ we find

$$\begin{aligned} \tau(4a^* f) &= \tau\left[\sum_{n=0}^{3} i^n (f + i^n a)^* (f + i^n a)\right] \\ &= \sum_{n=0}^{3} i^n \tau\left[(f + i^n a)^* (f + i^n a)\right] \\ &= \sum_{n=0}^{3} i^n \tau\left[(f + i^n a)(f + i^n a)^*\right] \\ &= \tau\left[\sum_{n=0}^{3} i^n (f + i^n a)(f + i^n a)^*\right] \\ &= \tau(4fa^*). \end{aligned}$$

We conclude that $\tau(af) = \tau(fa)$ whenever $fa \in \mathcal{I}_\tau$ (thus also when $f, a \in \mathcal{N}_\tau$).

On $\mathcal{N}_\tau$ we define $\langle \cdot, \cdot \rangle_\tau$ by

$$\langle x, y \rangle_\tau := \tau(y^* x).$$

By our previous considerations we see that for all $x, y, z \in \mathcal{N}_\tau$ and $\lambda \in \mathbb{C}$, the following identities hold

$$\begin{aligned}
\langle x, x \rangle_\tau &\geq 0, \\
\langle x, x \rangle_\tau &= 0 \iff x = 0, \\
\langle x, y \rangle_\tau &= \tau(y^* x) = \overline{\tau(x^* y)} = \overline{\langle y, x \rangle_\tau}, \\
\langle x + z, y \rangle_\tau &= \tau(y^*(x+z)) = \tau(y^* x + y^* z) = \tau(y^* x) + \tau(y^* z) = \langle x, y \rangle_\tau + \langle z, y \rangle_\tau, \\
\langle x, y + z \rangle_\tau &= \tau((y^* + z^*) x) = \tau(y^* x) + \tau(z^* x) = \langle x, y \rangle_\tau + \langle x, z \rangle_\tau, \\
\langle \lambda x, y \rangle_\tau &= \tau(y^*(\lambda x)) = \lambda \tau(y^* x) = \lambda \langle x, y \rangle_\tau, \\
\langle x, \lambda y \rangle_\tau &= \tau\left(\overline{\lambda} y^* x\right) = \overline{\lambda} \tau(y^* x) = \overline{\lambda} \langle x, y \rangle_\tau.
\end{aligned}$$

We see that $\langle \cdot, \cdot \rangle_\tau$ indeed defines an inner product on $\mathcal{N}_\tau$ as desired. □

We define now

$$\mathcal{H}_\tau := \overline{\{\mathcal{N}_\tau, \langle \cdot, \cdot \rangle_\tau\}}.$$

Remark 1.

The space $\mathcal{H}_\tau$ is often denoted as $L^2(\mathcal{A}, \tau)$, as it the Hilbert space associated to $\mathcal{A}$ and the trace τ. On the ideal $\mathcal{I}_\tau$ we can define a norm as $\|\cdot\|_1(x) := \tau(|x|)$. The completion of $\mathcal{I}_\tau$ is often denoted as $L^1(\mathcal{A}, \tau)$ for the same reasons. It is worth noting that each positive element $a \in \mathcal{I}_\tau$ defines a positive linear functional on $\mathcal{A}$ by setting $\phi_a(x) := \tau(ax)$. Suppose that $a \in \mathcal{I}_\tau$ is positive. Then for any bounded increasing net $\{x_i\} \subset \mathcal{A}$ with $\sup_i x_i = x$, we have

$$\phi_a(x) = \tau\left(a \sup_i x_i\right) = \tau\left(\sup_i a x_i\right) = \sup_i \tau(a x_i) = \sup_i \phi_a(x_i).$$

We conclude that ϕ_a defines a normal linear functional. Note now that $\mathcal{I}_\tau$ is spanned linearly by its positive elements, as such any element a defines a normal linear functional on $\mathcal{A}$, that is $\mathcal{I}_\tau$ is a subset of the predual of $\mathcal{A}$ (denoted by $\mathcal{A}_*$). It follows that $\mathcal{I}_\tau$ defines a total subset of the predual of $\mathcal{A}$ so that $\mathcal{I}_\tau$ is norm dense in $\mathcal{A}_*$, that is. $L^1(\mathcal{A}, \tau) \cong \mathcal{A}_*$. In general the space $\mathcal{A}_*$ is the space of all normal linear functionals on $\mathcal{A}$.

We return our attention to the trace τ. The trace τ satisfies a variety of inequalities, because $\langle \cdot, \cdot \rangle_\tau$ is an inner product, we have the Cauchy-Schwarz inequality stating:

$$|\tau(y^* x)| \leq \sqrt{\tau(x^* x) \tau(y^* y)}.$$

Let $x \in \mathcal{I}_\tau$, the polar decomposition $x = u|x|$, gives us that $|x|$ is in $\mathcal{I}_\tau$. Note that

$$\begin{aligned}
|\tau(x)|^2 = |\tau(u|x|)|^2 &= \left|\tau\left(|x|^{1/2} u |x|^{1/2}\right)\right| \\
&\leq \tau\left(|x|^{1/2} u^* u |x|^{1/2}\right) \tau(|x|) \\
&\leq \|u^* u\| \, \tau(|x|)^2 \\
&= \tau(|x|)^2,
\end{aligned}$$

we conclude that $|\tau(x)| \leq \tau(|x|)$. Suppose that $x \in \mathcal{I}_\tau$ and $y \in \mathcal{A}$, let $x = u|x|$ and $y = s|y|$ be their polar decompositions. Using 3.3.0.2 we find that the following holds:

$$\begin{aligned}|\tau(yx)|^2 = |\tau((s|y|)(u|x|))|^2 &= \left|\tau\left(|x|^{1/2}s|y|u|x|^{1/2}\right)\right|^2 \\ &= \left|\tau\left[\left(|x|^{1/2}s|y|^{1/2}\right)\left(|y|^{1/2}u|x|^{1/2}\right)\right]\right|^2 \\ &\leq \tau\left(|y|^{1/2}s^*|x|s|y|^{1/2}\right)\tau\left(|x|^{1/2}u^*|y|u|x|^{1/2}\right) \\ &= \tau(s|y|s^*|x|)\,\tau(u|x|u^*|y|) \\ &= \tau(|y^*||x|)\,\tau(|x^*||y|)\,.\end{aligned}$$

For $y \in \mathcal{A}_+$ and $x \in \mathcal{F}_\tau$, we have

$$\tau(yx) = \tau\left(x^{1/2}y^{1/2}y^{1/2}x^{1/2}\right) \leq \|y\|\,\tau(x)\,.$$

Using this we conclude that for any $y \in \mathcal{A}$ and $x \in \mathcal{I}_\tau$ we have

$$\begin{aligned}|\tau(yx)\,|^2 &\leq \tau(|y^*||x|)\,\tau(|x^*||y|) \\ &\leq \|y\|\,\tau(|x|)\,\|y\|\,\tau(|x^*|) \\ &= \|y\|^2\,\tau(|x|)\,\tau(u|x|u^*) \\ &= \|y\|^2\,\tau(|x|)^2\,.\end{aligned}$$

We now proceed to investigate the structure on $\mathcal{H}_\tau$. Since $\mathcal{H}_\tau$ is derived from a von Neumann algebra it has additional structure.

Definition 3.3.0.3. *For $m \in \mathcal{A}$ we define on the dense subspace $\mathcal{N}_\tau \subset \mathcal{H}_\tau$, operations*

$$\begin{aligned}\pi_l(m) : \mathcal{N}_\tau \longrightarrow \mathcal{N}_\tau, &\qquad \pi_l(m)(x) := mx \\ \pi_r(m) : \mathcal{N}_\tau \longrightarrow \mathcal{N}_\tau, &\qquad \pi_r(m)(x) := xm.\end{aligned}$$

Also we define an operation

$$J : \mathcal{N}_\tau \longrightarrow \mathcal{N}_\tau, \qquad J(x) := x^*.$$

We call $\pi_l(m)$, (resp. $\pi_r(m)$), the left representation, (resp. the right representation), of m. The map J is called the unitary involution of $\mathcal{H}_\tau$.

The language used in the definition above suggests that the maps π_l and π_r define representations of $\mathcal{A}$, this is indeed the case.

Proposition 3.3.0.4. *The map $\pi_l : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H})$ defines an isometric representation of $\mathcal{A}$. The map π_r defines an isometric anti representation in the sense that it reverses the order of multiplication. Furthermore, the unitary involution defines an anti-linear unitary operator on $\mathcal{H}_\tau$.*

Proof:

Consider the map J, for $x \in \mathcal{N}_\tau$ we have

$$\|J(x)\|_\tau^2 = \langle x^*, x^*\rangle_\tau = \tau(xx^*) = \tau(x^*x) = \langle x, x\rangle_\tau = \|x\|_\tau^2\,.$$

We conclude that that J defines an anti-linear isometry on the dense subset $\mathcal{N}_\tau$. As such it is continuous and extends to an anti-linear isometry on $\mathcal{H}_\tau$. Using the properties of τ we find the following equalities for J and J^*

$$\begin{aligned}\langle J^*(a), b\rangle_\tau := \langle J(b), a\rangle_\tau &= \tau(a^*b^*) \\ &= \tau(b^*a^*) \\ &= \langle a^*, b\rangle_\tau \\ &= \langle J(a), b\rangle\,.\end{aligned}$$

We conclude that $J = J^*$ and $J^2 = 1$. For $m \in \mathcal{A}$ consider $\pi_r(m)$, for $x \in \mathcal{N}_\tau$ we find

$$\begin{aligned}\|\pi_r(m)(x)\|_\tau^2 = \|xm\|_\tau^2 &= \|JJ(xm)\|_\tau^2\\ &= \|J(xm)\|_\tau^2\\ &= \|m^*x^*\|_\tau^2\\ &= \tau(xmm^*x^*) \leq \|m\|^2 \tau(xx^*) = \|m\|^2 \|x\|_\tau^2.\end{aligned}$$

We conclude that $\pi_r(m)$ is continuous for each $m \in \mathcal{A}$, as such we can extend $\pi_r(m)$ to a continuous linear operator on $\mathcal{H}_\tau$. For $m \in \mathcal{A}$ consider $\pi_l(m)$, for $x \in \mathcal{N}_\tau$ we find

$$\begin{aligned}\|\pi_l(m)(x)\|_\tau^2 &= \|mx\|_\tau^2\\ &= \tau(x^*m^*mx) \leq \|m\|^2 \tau(x^*x) = \|m\|^2 \|x\|_\tau^2.\end{aligned}$$

We conclude that $\pi_l(m)$ is continuous on the dense subspace $\mathcal{N}_\tau$, as such it can be extended to the whole of $\mathcal{H}_\tau$. The map π_l satisfies the following equalities for all $m, n \in \mathcal{A}$ and $\lambda \in \mathbb{C}$

$$\begin{aligned}&\pi_l(1) = 1,\\ &\pi_l(m+n) = \pi_l(m) + \pi_l(n),\\ &\pi_l(mn) = \pi_l(m)\pi_l(n),\\ &\pi_l(m^*) = \pi_l(m)^*,\\ &\pi_l(\lambda m) = \lambda\pi_l(m),\\ &\|\pi_l(m)\| \leq \|m\|.\end{aligned}$$

We conclude that π_l defines a representation of $\mathcal{A}$ on $\mathcal{H}_\tau$. The case of π_r is similar but with one subtlety, for all $m, n \in \mathcal{A}$ and $\lambda \in \mathbb{C}$, we find the following equalities:

$$\begin{aligned}&\pi_r(1) = 1,\\ &\pi_r(m+n) = \pi_r(m) + \pi_r(n),\\ &\pi_r(mn) = \pi_r(n)\pi_r(m),\\ &\pi_r(m^*) = \pi_r(m)^*,\\ &\pi_r(\lambda m) = \lambda\pi_r(m),\\ &\|\pi_r(m)\| \leq \|m\|.\end{aligned}$$

We see that π_r reverses the order of multiplication, as such, it is an anti-representation of $\mathcal{A}$ on $\mathcal{H}_\tau$. Our aim is now to show that π_l and π_r are injective so that they in fact define isometries.

Given a nonzero $a \in \mathcal{A}$. By the semifiniteness of τ there exists $b \in \mathcal{N}_\tau$ with the property that $bb^* \leq a^*a$ and $\tau(bb^*) < \infty$. Since $bb^* \leq a^*a$ we can write $a^*a = bb^* + c$, with c some positive element. Consider now $\langle \pi_l(a)b, \pi_l(a)b\rangle_\tau$, using the properties of τ we find that

$$\begin{aligned}\langle \pi_l(a)b, \pi_l(a)b\rangle_\tau &= \langle ab, ab\rangle_\tau\\ &= \tau(b^*a^*ab)\\ &= \tau(bb^*a^*a)\\ &= \tau(bb^*(bb^*+c))\\ &= \tau(bb^*bb^* + bb^*c) \geq \tau(bb^*bb^*).\end{aligned}$$

Since $bb^* > 0$, also $bb^*bb^* > 0$, thus $\langle \pi_l(a)b, \pi_l(a)b\rangle_\tau \geq \tau(bb^*bb^*) > 0$. As such, $\pi_l(a)$ is nonzero when a is nonzero, that is, π_l is injective. Via a similar argument we conclude that π_r is also injective so that both π_l an π_r are in fact isometries. □

Now we will investigate the commutant of $\pi_l(\mathcal{A}) \subset \mathcal{B}(\mathcal{H}_\tau)$. For $a, b \in \mathcal{A}$ and $h \in \mathcal{H}_\tau$ consider $\pi_l(a)\pi_r(b)h$. We find the following equality:

$$\pi_l(a)\pi_r(b)h = ahb = \pi_r(b)(ah) = \pi_r(b)\pi_l(a)h.$$

From this remarkable equality we draw the following conclusions:

$$\pi_r(\mathcal{A}) \subset \pi_l(\mathcal{A})'$$
$$\pi_l(\mathcal{A}) \subset \pi_r(\mathcal{A})'.$$

Note here that this bears some resemblance with abelian algebras. Suppose that $\mathcal{M}$ is abelian, then in particular we have $\mathcal{M} \subset \mathcal{M}'$, here we have something similar, namely there are two representations π_l and π_r such that $\pi_l(\mathcal{A}) \subset \pi_r(\mathcal{A})'$. This relation also holds when the roles of π_l and π_r are reversed. One should realize that a semifinite faithful normal trace τ influences a lot of the properties of $\mathcal{A}$, in other words it is a powerful tool. This is one of the reasons that the study of the type III algebra is quite different, it lacks such a trace.

We want to show that $\pi_l(\mathcal{A})' = \pi_r(\mathcal{A})$ but before doing so we first need to answer the following questions: Given $x \in \mathcal{H}_\tau$ when can we conclude that $x \in \mathcal{N}_\tau$? If $x \in \mathcal{A}$ when can we conclude that $x \in \mathcal{N}_\tau$ (besides using its τ evaluation)?

Lemma 3.3.0.5. *For any $x \in \mathcal{A}$ the following statements are equivalent:*

1. $x \in \mathcal{N}_\tau$,
2. $\sup\{|\tau(y^*x)| \; ; \; y \in \mathcal{I}_\tau, \; \tau(y^*y) \leq 1\} < \infty$.

Proof:
1 $\Longrightarrow$ 2
Note here that we are viewing $\mathcal{I}_\tau$ as a subspace of $\mathcal{H}_\tau$, with the norm it inherits from $\mathcal{N}_\tau$. If $x \in \mathcal{N}_\tau$ then for any $y \in \mathcal{I}_\tau$ with $\tau(y^*y) \leq 1$ we find

$$|\tau(y^*x)| \leq \sqrt{\tau(y^*y)\,\tau(x^*x)} < \infty.$$

2 $\Longrightarrow$ 1
Consider $\mathcal{I}_\tau$ as a subspace of $\mathcal{H}_\tau$, we can find a net of projections $\{e_i\} \subset \mathcal{F}_\tau$ converging strongly to $1 \in \mathcal{A}$. For any $m \in \mathcal{N}_\tau$ we have that $e_i m \in \mathcal{I}_\tau$ and

$$\|m - e_i m\|_\tau^2 = \tau(m^*(1-e_i)m) = \langle (1-e_i)m, m\rangle \longrightarrow 0.$$

We conclude that $\mathcal{N}_\tau \subset \overline{\mathcal{I}_\tau}$ thus $\mathcal{I}_\tau$ is dense in $\mathcal{H}_\tau$.
Since $\sup\{|\tau(y^*x)| \; ; \; y \in \mathcal{I}_\tau, \; \tau(y^*y) \leq 1\} < \infty$ holds, we have that x determines a densely defined *bounded* anti-linear functional on $\mathcal{H}_\tau$, as such it can be extended to the whole of $\mathcal{H}_\tau$. We denote this anti linear functional by ϕ_x and we denote its extension by $\widehat{\phi}_x$. So for $y \in \mathcal{I}_\tau$ we have

$$\phi_x(y) = \tau(y^*x).$$

The Riesz representation theorem provides us with an element $x_0 \in \mathcal{H}_\tau$, such that for all $h \in \mathcal{H}_\tau$ we have that

$$\widehat{\phi}_x(h) = \langle x_0, h\rangle_\tau.$$

So for $y \in \mathcal{I}_\tau$ we have that

$$\widehat{\phi}_x(y) = \langle x_0, y\rangle_\tau = \tau(y^*x) = \phi_x(y).$$

Using our net of projections $\{e_i\} \subset \mathcal{F}_\tau$ converging to $1 \in \mathcal{A}$, we conclude that for every $y \in \mathcal{I}_\tau$ the following holds

$$\begin{aligned}\langle e_i x_0, y\rangle_\tau &= \langle x_0, e_i y\rangle \\ &= \tau(y^* e_i x) \\ &= \langle e_i x, y\rangle_\tau.\end{aligned}$$

Since $\mathcal{I}_\tau$ is dense in $\mathcal{H}_\tau$ we conclude that $e_i x_0 = e_i x \in \mathcal{I}_\tau \subset \mathcal{H}_\tau$ for all e_i. We find now that

$$\tau(x^*x) = \sup \tau(x^* e_i e_i x) = \sup \|e_i x\|_\tau^2 = \sup \|e_i x_0\|_\tau^2 = \|x_0\|_\tau^2 < \infty,$$

As such $x \in \mathcal{N}_\tau$. □

Lemma 3.3.0.6. *For $x \in \mathcal{H}_\tau$ the following statements are equivalent*

1. $x \in \mathcal{N}_\tau$,
2. $\sup\{|\langle x, y\rangle_\tau| \ ; \ y \in \mathcal{I}_\tau, \ \tau(|y|) \leq 1\} < \infty$,
3. $\sup\{\|mx\|_\tau \ ; \ m \in \mathcal{N}_\tau, \ \|m\|_\tau \leq 1\} < \infty$.

Proof:
1 $\Longrightarrow$ 3
If $x \in \mathcal{N}_\tau$ then we can consider its image in $\mathcal{B}(\mathcal{H}_\tau)$. We find $\|mx\|_\tau = \|\pi_r(x)(m)\|_\tau$. Since $\mathcal{N}_\tau$ is dense we conclude that

$$\sup\{\|mx\|_\tau \ ; \ m \in \mathcal{N}_\tau, \ \|m\|_\tau \leq 1\} = \sup\{\|mx\|_\tau \ ; \ m \in \mathcal{H}_\tau, \ \|m\|_\tau \leq 1\} = \|\pi_r(x)\| = \|x\| < \infty.$$

3 $\Longrightarrow$ 2
Note that in this case x defines a bounded linear operator on $\mathcal{N}_\tau$, by extension it defines a bounded linear operator on $\mathcal{H}_\tau$. Pick $y \in \mathcal{I}_\tau$ with $\tau(|y|) \leq 1$ and let $y = |y^*|u$ be its right polar decomposition. Consider $|\langle x, y\rangle_\tau|$, we find

$$\begin{aligned}
|\langle x, y\rangle_\tau|^2 &= |\tau(y^*x)|^2 \\
&= |\tau(u^*|y^*|x)|^2 \\
&= \left|\tau\left(u^*|y^*|^{1/2}|y^*|^{1/2}x\right)\right|^2 \\
&= \left|\left\langle |y^*|^{1/2}x, |y^*|^{1/2}u\right\rangle_\tau\right|^2 \\
&\leq \left\langle |y^*|^{1/2}x, |y^*|^{1/2}x\right\rangle_\tau \left\langle |y^*|^{1/2}u, |y^*|^{1/2}u\right\rangle_\tau \\
&= \left\| |y^*|^{1/2}x\right\|_\tau^2 \left\| |y^*|^{1/2}\right\|_\tau^2 \\
&\leq \|x\|^2 \tau(|y^*|)^2 = \|x\|^2 \tau(|y|)^2 \leq \|x\|^2 < \infty.
\end{aligned}$$

2 $\Longrightarrow$ 1
Note that x defines a normal functional on $\mathcal{I}_\tau$, as such $x \in \mathcal{A}_*^* = \mathcal{A}$. Since $x \in \mathcal{H}_\tau$, it also defines a continuous anti-linear functional on $\mathcal{H}_\tau$. Applying lemma 3.3.0.5 we find that $x \in \mathcal{N}_\tau$. □

We are now finally able to prove the Tomita-Takesaki theorem in the semifinite case.

Proof of 3.3.0.1, Tomita-Takesaki theorem (semifinite case):

Let $\mathcal{A}$ be a von Neumann algebra equipped with a semifinite faithful trace τ. We construct $\mathcal{H}_\tau$ as above an embed $\mathcal{A}$ into $\mathcal{B}(\mathcal{H}_\tau)$ via the representation π_l and the anti-representation π_r. We concluded that the following statements hold:

$$\pi_l(\mathcal{A}) \subset \pi_r(\mathcal{A})',$$
$$\pi_r(\mathcal{A}) \subset \pi_l(\mathcal{A})'.$$

Consider $b \in \pi_l(\mathcal{A})'$, for $x, y \in \mathcal{N}_\tau$ we have that

$$\begin{aligned}
\|\pi_l(y)bx\|_\tau^2 &= \|yb(x)\|_\tau^2 \\
&= \|b(yx)\|_\tau^2 \\
&\leq \|b\|^2 \|\pi_l(y)x\|_\tau^2 \\
&= \|b\|^2 \tau(x^*y^*yx) \\
&\leq \|b\|^2 \|x\|^2 \|y\|_\tau^2.
\end{aligned}$$

For all $y \in \mathcal{N}_\tau$, we find the inequality $\|y(b(x))\|_\tau^2 \leq \|b\|^2 \|x\|^2 \|y\|_\tau^2$. Taking the supremum over $y \in \mathcal{N}_\tau$, with $\|y\|_\tau \leq 1$, we conclude

$$\sup\{\|yb(x)\|_\tau \ ; \ y \in \mathcal{N}_\tau, \ \|y\|_\tau \leq 1\} \leq \|b\| \|x\| < \infty.$$

Applying lemma 3.3.0.6, we conclude that $b(x) \in \mathcal{N}_\tau$ for all $x \in \mathcal{N}_\tau$. Note the following identities

$$\begin{aligned}\pi_r(bx)\,y = yb(x) &= \pi_l(y)\,b(x) \\ &= b(\pi_l(y)\,x) \\ &= b\pi_r(x)\,y.\end{aligned}$$

We conclude that $b\pi_r(x) = \pi_r(bx)$ for all $x \in \mathcal{N}_\tau$. It follows that

$$b\pi_r(\mathcal{N}_\tau) = \pi_r(b(\mathcal{N}_\tau)) \subset \pi_r(\mathcal{N}_\tau).$$

Consider the map $J : \mathcal{H}_\tau \longrightarrow \mathcal{H}_\tau$ defined by $J(h) = h^*$. For $a \in \pi_l(\mathcal{A})$ and $h \in \mathcal{H}_\tau$ we have

$$J\pi_l(a)\,J(h) = J(ah^*) = ha^* = \pi_r(a^*)\,h.$$

Note that $\pi_r(e_i) = \pi_r(e_i^*) = J\pi_l(e_i)\,J$. The space of all continuous normal linear functional on $\mathcal{A}$ is the predual of $\mathcal{A}$. As such, it defines the ultra-weak topology on $\mathcal{A}$, that is, $a_i \longrightarrow a$ ultraweak when for all $\phi \in \mathcal{A}_*$ we have $|\phi(a_i) - \phi(a)| \longrightarrow 0$. We can pick an increasing net of projections $\{e_i\} \in \mathcal{I}_\tau$ such that $e_i \longrightarrow 1$ ultra-weak. Consider now π_l and suppose that $a_i \longrightarrow a$ ultra weak. Then in particular $\tau(a_i x) \longrightarrow \tau(ax)$ for all $x \in \mathcal{I}_\tau$, by definition $\tau(a_i x) = \tau(\pi_l(a_i)\,x) \longrightarrow \tau(ax) = \tau(\pi_l(a)\,x)$ so that π_l is continuous in the ultra-weak topology. It follows that the range of π_l is ultra-weakly closed, that is, $\pi_l(\mathcal{A})$ determines a von Neumann algebra on $\mathcal{H}_\tau$. Since we can find an increasing net $\{e_i\}$ of projections converging to 1 in the ultra-weak topology on $\mathcal{A}$, we have that

$$b\pi_r(e_i) = \pi_r(be_i) \longrightarrow \pi_r(c) \in \pi_r(\mathcal{A}),$$

for some $c \in \mathcal{A}$. We conclude that if $b \in \pi_l(\mathcal{A})'$, then, $b = \pi_r(c)$ for some element $c \in \mathcal{A}$. It follows that

$$\pi_r(\mathcal{A}) = \pi_l(\mathcal{A})'.$$

Now we return to the map J. We found $J\pi_l(a)\,J = \pi_r(a^*)$, since $\pi_r(\mathcal{A}) = \pi_l(\mathcal{A})'$ we conclude that

$$\pi_l(\mathcal{A})' = J\pi(\mathcal{A})\,J.$$

It follows that $\pi_l(\mathcal{A})$ is anti-isomorphic to its own commutant as desired. □

We end the proof with a remark on the nature of this representation and the relation it shares with the abelian von Neumann algebras. Any abelian von Neumann $\mathcal{A}$ algebra can be represented faithfully as an algebra of L^∞ functions working on some Hilbert space by multiplication. It follows that $\mathcal{A} = \mathcal{A}'$ because one can choose this representation in such a manner that $\mathcal{A}$ is represented as a maximal abelian algebra. We compare this with the semifinite case, we found a Hilbert space with a natural involution and multiplication (such a Hilbert space is often called a Hilbert algebra) on which we can represent the von Neumann algebra simply by left multiplication. We cannot expect that $\mathcal{A} = \mathcal{A}'$ but instead we found that $\mathcal{A} = J\mathcal{A}'J$ where J is the star involution on the representation Hilbert space. It is (in my view) remarkable that such a relation holds. In my view, the approach to the Tomita- Takesaki theorem in the semifinite case is natural. The left and right representations of $\mathcal{A}$ on $\mathcal{H}_\tau$ are objects that are naturally associated to τ. The property that makes τ special is that cares not in which order multiplication occurs, i.e. $\tau(xy) = \tau(yx)$ as long as $xy \in \mathcal{I}_\tau$. Because of this property the right representation of $\mathcal{A}$ is indeed a representation of $\mathcal{A}$ as an algebra of bounded operators. If τ is not a trace then the right representation is, in general, not an algebra of bounded operators. Though the next statement is not precise, it captures the essence of what the existence of a trace τ implies for $\mathcal{A}$. The product $\mathcal{N}_\tau \times \mathcal{N}_\tau \longrightarrow \mathcal{I}_\tau$ is abelian for τ. It is ofcourse not truly abelian but for τ it matters not. So in this imprecise sense $\mathcal{A}$ is one step away from being an abelian algebra. Measuring how far $\mathcal{A}$ is off to being abelian is done in terms of J, the unitary involution and π_l, and π_r, the left and right representations.

Consequences of the Tomita-Takesaki theorem

There are several consequences of this theorem, we will briefly go over them without going far into how these consequences are proved.

If $\mathcal{A}$ is a semifinite von Neumann algebra with a semifinite faithful normal trace τ then we can consider its representation on $\mathcal{H}_\tau$ as above. Since $\pi_l(\mathcal{A})$ is anti-isomorphic to its commutant it follows that also $\pi_l(\mathcal{A})'$ is semifinite. We will not prove this but there exists a Hilbert space $\mathcal{H}_1$ and a projection $e_1 \in \pi_l(\mathcal{A})' \overline{\otimes} B(\mathcal{H}_1)$ such that $\mathcal{A}' \cong e_1(\pi_r(\mathcal{A}) \overline{\otimes} B(\mathcal{H}_1))$ as such $\mathcal{A}'$ is semifinite because $\pi_r(\mathcal{A}) \overline{\otimes} B(\mathcal{H}_1)$ is.

If one is willing to believe that this is true then we immediately get that $\mathcal{A}$ is of type $III \iff \mathcal{A}'$ is of type III. In fact one can show the following equivalences

- $\mathcal{A}$ is of type $I \iff \mathcal{A}'$ is of type I,
- $\mathcal{A}$ is of type $II \iff \mathcal{A}'$ is of type II,
- $\mathcal{A}$ is of type $III \iff \mathcal{A}'$ is of type III.

It is our aim to proceed to the case type III. In the type III scenario we do not have the trace at our disposal, however we are compensated in some sense. We will find representations of $\mathcal{A}$ derived from the GNS construction using separating and cyclic vectors. It turns out that the spectral theorem has a generalization to unbounded operators and we will make use of that fact. Also the polar decomposition is generalized to unbounded operators. The goal of the next section is to familiarize ourselves with unbounded operators, we will not go very far into the theory but instead we will scratch the surface to get a feel for it.

Chapter 4

Tomita-Takesaki theory and Classification of type III von Neumann algebras

In this chapter we will focus mainly on type III von Neumann algebras, we will study them using the Tomita-Takesaki theory introduced in the first section. All Hilbert spaces in this section are assumed to be separable so that if we consider a von Neumann algebra $\mathcal{A}$ then it is assumed to act on a separable space. The Tomita-Takesaki theory was succesfully used by A. Connes to further classify type III von Neumann algebras on separable Hilbert spaces.

4.1 Tomita-Takesaki Theorem (general case)

In this section we will state and prove the Tomita-Takesaki theorem for general von Neumann algebras on separable Hilbert space. Before going into the proof of the theorem we will cover the tools and preliminaries needed to succesfully prove the theorem. We will not prove the Tomita-Takesaki theorem in its most general form. There is a more general statement which concerns Hilbert algebras, for a full treatment we refer to [9]

Theorem 4.1.0.7 (Tomita-Takesaki)**.** *Let $\mathcal{A}$ be a separable von Neumann algebra, then there exists a representation $\pi(\mathcal{A})$ on a Hilbert space $\mathcal{H}_{\mathcal{A}}$ such that there exists an anti-unitary operator J and a collection $\{\Delta^{it} \; ; \; t \in \mathbb{R}\}$ of unitary operators in $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ with the properties:*

- *$J\pi(\mathcal{A})J = \pi(\mathcal{A})'$ and $J\pi(\mathcal{A})'J = \pi(\mathcal{A})$.*
- *$\{\Delta^{it} \; ; \; t \in \mathbb{R}\}$ defines a one parameter group of automorphisms for $\pi(\mathcal{A})$ and $\pi(\mathcal{A})'$.*

We start by constructing the Hilbert space in which the Tomita-Takesaki theorem holds. We will see that this Hilbert space is cannonically equipped with unbounded maps s and r which roughly act as a star operation. From these maps we will derive the maps J and Δ in theorem 4.1.0.7. In terms of J and Δ we will then derive a relation between $\pi(\mathcal{A})$ and its commutant $\pi(\mathcal{A})'$. It is then our aim to invert this relation using Fourier theory. When all the preliminaries are covered we will formally prove theorem 4.1.0.7.

4.1.1 Construction of $\mathcal{H}_{\mathcal{A}}$

Here we will construct $\mathcal{H}_{\mathcal{A}}$ using the GNS construction. We will first show that any von Neumann algebra acting on a separable Hilbert space allows for a faithful normal state. We then apply the GNS construction to this state.

Definition 4.1.1.1. *Given a von Neumann algebra $\mathcal{A}$, then a vector $\xi_0 \in \mathcal{H}$ is called cyclic and separating for $\mathcal{A}$ when the following conditions hold*

1. *$\{\mathcal{A}\xi_0\}$ is dense in $\mathcal{H}$. (cyclic)*
2. *If $x \in \mathcal{A}$ then $x(\xi_0) = 0$ if and only if $x = 0$. (separating).*

If given a subset $\mathcal{V} \subset \mathcal{H}$ then we say that $\mathcal{V}$ is separating for $\mathcal{A}$ when the only $x \in \mathcal{A}$ that annihilates $\mathcal{V}$ is the zero vector, thus $x(v) = 0$ for all $v \in \mathcal{V}$ implies $x = 0$.

Definition 4.1.1.2. *A von Neumann algebra $\mathcal{A}$ is called σ-finite if given a set of mutually orthogonal projections $\{e_i\}_{i \in I} \subset \mathcal{A}$ then I is at most countable.*

Since we assume our Hilbert space to be separable it follows that $\mathcal{A}$ is σ-finite.

Proposition 4.1.1.3. *If $\mathcal{A}$ is a von Neumann algebra then the following statements hold.*

1. *ξ_0 is cyclic for $\mathcal{A} \Longrightarrow \xi_0$ is separating for $\mathcal{A}'$.*
2. *ξ_0 is separating for $\mathcal{A} \longrightarrow \xi_0$ is cyclic for $\mathcal{A}'$.*

Proof:

1.
Suppose that ξ_0 is cyclic for $\mathcal{A}$, pick $a' \in \mathcal{A}'$ and consider $a'(\xi_0)$. Suppose that $a'(\xi_0) = 0$, then for all $x \in \mathcal{A}$ we have $a'x(\xi_0) = xa'(\xi_0) = 0$. The set $x(\xi_0)$ is dense in $\mathcal{H}$, as such, $a' = 0$. We conclude that ξ_0 is cyclic for $\mathcal{A} \Longrightarrow \xi_0$ is separating for $\mathcal{A}'$.

2.
Suppose that ξ_0 is separating for $\mathcal{A}$ and assume that ξ_0 is not cyclic for $\mathcal{A}'$. Define p as the projection onto $\overline{\mathcal{A}'\xi_0}^{\perp}$, since ξ_0 is not cyclic we have that $p \neq 0$. It is easy to see that $\overline{[\mathcal{A}'\xi_0]}$ is an invariant subspace of $\mathcal{A}'$. Since $\mathcal{A}'$ is $*$ closed it follows that $\overline{[\mathcal{A}'\xi_0]}$ is also reducing for $\mathcal{A}'$. It follows that p and $1-p$ commute with $\mathcal{A}'$, we find that $p, (1-p) \in \mathcal{A}$. Since $p\mathcal{A}'(\xi_0) = 0$ we find that $p(\xi_0) = 0$. But $p \in \mathcal{A}$ implies that $p(\xi_0) \neq 0$, contradiction. □

So if a vector ξ_0 is cyclic and separating for $\mathcal{A}$ then it is also cyclic and separating for $\mathcal{A}'$. Note that the same statements are true if we allow for a set that is cyclic and separating.

Theorem 4.1.1.4. *The following conditions are equivalent (even when $\mathcal{H}$ is not separable):*

1. *$\mathcal{A}$ is σ-finite.*
2. *There exists a countable subset $\{\xi_i\}_{i\in\mathbb{N}} \subset \mathcal{H}$ which is separating for $\mathcal{A}$.*
3. *$\mathcal{A}$ admits a faithful normal positive linear functional of norm 1.*
4. *$\mathcal{A}$ is isomorphic to a von Neumann algebra $\pi(\mathcal{A})$ which has a separating and cyclic vector.*

Proof:

1. $\Longrightarrow$ 2.
Pick $\xi_1 \in \mathcal{H}$ and let p_1 be the projection onto the space $\overline{\mathcal{A}'\xi_1}$. Pick $\xi_2 \in \overline{\mathcal{A}'\xi_1}^{\perp}$ and let p_2 be the projection onto $\overline{\mathcal{A}'\xi_2}$, note that p_1 and p_2 commute with $\mathcal{A}'$ so that $p_1, p_2 \in \mathcal{A}$. By Zorn's lemma we can find a maximal family of vectors $\{\xi_i\}_{i\in I} \subset \mathcal{H}$ and their associated projections $\{p_i\}_{i\in I} \subset \mathcal{A}$ constructed as above. Note that the projections $\{p_i\}_{i\in I}$ are mutually orthogonal. It follows that I is countable by the σ-finiteness of $\mathcal{A}$. Note furthermore that $\sum_{i\in I} p_i = 1$, by the maximality of the set $\{\xi_i\}_{i\in I}$. By definition the vector ξ_i is cyclic for $\overline{\mathcal{A}\xi_i}$. Note that

$$\mathcal{H} = \bigoplus_{i\in I} \overline{\mathcal{A}\xi_i}.$$

Since the set $\{\xi_i\}$ is cyclic for $\mathcal{A}'$, it is separating for $\mathcal{A}$.

2. $\Longrightarrow$ 3.
We can assume that $\sum_{n=1}^{\infty} \|\xi_n\| = 1$, define

$$\phi(x) := \sum_{n=1}^{\infty} \langle x(\xi_n), \xi_n \rangle .$$

It follows that if $x = a^*a$ then $\phi(x) = \sum_{n=1}^{\infty} \|a(\xi_n)\|^2 \geq 0$. Suppose that $\phi(x) = \phi(a^*a) = 0$, then $\|a(\xi_n)\| = 0$ for all n. By the separating property of $\{\xi_n\}_{n\in\mathbb{N}}$ we conclude that $a = 0$, as such, $x = 0$, meaning that ϕ is a positive faithful linear functional on $\mathcal{A}$. Note that $\phi(1) = 1$ and that if $\|a\| \leq 1$ then

$$|\phi(a)| \leq \sum_{n=1}^{\infty} |\langle a(\xi_n), \xi_n \rangle| \leq \|a\| \leq 1.$$

We find that ϕ has norm 1. That ϕ is normal (meaning ϕ is an element of the predual of $\mathcal{A}$) is immediate.

3. $\Longrightarrow$ *4.*
Note that since ϕ is faithful and positive it gives rise to an inner product on $\mathcal{A}$ by setting

$$\langle x, y\rangle := \phi\left(y^* x\right).$$

Let $\mathcal{H}_{\mathcal{A}}$ be the completion of $\mathcal{A}$ with respect to this inner product. For $a \in \mathcal{A}$ we define the map $\widehat{a} : \mathcal{A} \longrightarrow \mathcal{A}$ by left multiplication so

$$\widehat{a}\,(x) := ax.$$

We find that with respect to the norm defined by ϕ the following holds

$$\left\|\widehat{a}\,(x)\right\|_{\phi} \leq \|a\| \, \|x\|_{\phi},$$

so that $\|\widehat{a}\| \leq \|a\|$. In particular, for all $a \in \mathcal{A}$, $\widehat{a}$ is continuous with respect to $\|\cdot\|_{\phi}$ and thus $\widehat{a}$ can be extended to be defined on $\mathcal{H}_{\mathcal{A}}$. We find a map $\pi : \mathcal{A} \longrightarrow \mathcal{B}\left(\mathcal{H}_{\mathcal{A}}\right)$ defined by $\pi\,(a) := \widehat{a} \in \mathcal{B}\left(\mathcal{H}_{\mathcal{A}}\right)$. what remains is to show that π is an isomorphism and that $\pi\,(\mathcal{A})$ allows for a cyclic and separating vector $\xi \in \mathcal{H}_{\mathcal{A}}$.

Let $1 = \xi \in \mathcal{H}_{\mathcal{A}}$, then for any $a \in \mathcal{A}$ we have that $\pi\,(a)\,\xi = \widehat{a}\,(1) = a \in \mathcal{H}_{\mathcal{A}}$ so $\pi\,(a)\,\xi = 0$ if and only if $a = 0$. Furthermore $\pi\,(\mathcal{A})\,\xi = \mathcal{A} \subset \mathcal{H}_{\mathcal{A}}$ is dense by construction. We conclude that $\pi\,(\mathcal{A})$ allows for a cyclic and separating vector.

Now to show that $\pi : \mathcal{A} \longrightarrow \mathcal{B}\left(\mathcal{H}_{\mathcal{A}}\right)$ is an isomorphism onto its range. Linearity is immediate, also $\pi\,(1) = 1$ is immediate. Consider $\pi\,(a^*)$, for $x \in \mathcal{A} \subset \mathcal{H}_{\mathcal{A}}$ we find that $\pi\,(a^*)\,x = a^* x = \pi\,(a)^*\,x$ by density of $\mathcal{A}$ in $\mathcal{H}_{\mathcal{A}}$ this extends to all of $\mathcal{H}_{\mathcal{A}}$ we conclude that $\pi\,(a^*) = \pi\,(a)^*$. Note also that for $x \in \mathcal{A} \subset \mathcal{H}_{\mathcal{A}}$ we have that $\pi\,(ab)\,(x) = abx = \pi\,(a)\,\pi\,(b)\,x$, again by density of $\mathcal{A}$ this equality extend to all of $\mathcal{H}_{\mathcal{A}}$. We conclude that π is multiplicative. Consider $\|\pi\,(a)\|$, we find

$$\begin{aligned} \|\pi\,(a)\| &= \sup_{\|x\| \leq 1} \|\pi\,(a)\,x\| \\ &= \sup_{\|x\| \leq 1} \sqrt{\phi\left(x^* a^* a x\right)} \\ &\leq \sup_{\|x\| \leq 1} \|a\| \sqrt{\phi\left(x^* x\right)} = \|a\|, \end{aligned}$$

we conclude that $\|\pi\,(a)\| \leq \|a\|$. Note now that if $a \in \ker(\pi)$ then $a\xi = 0$, by the separating property of ξ we find that $a = 0$, hence π is injective. We conclude that π is an injective *-homomorphism, as such it is an isometry. We conclude that π is an isomorphism onto its range. using the fact that ϕ is normal we conclude that π is normal.

4. $\Longrightarrow$ *1.*
Let ξ denote the cyclic and separating vector and let $\{p_i\}_{i \in I}$ be a set of mutually orthogonal projections, denote $p := \sum_{i \in I} p_i$ then

$$0 \neq \|p\,(\xi)\| = \sum_{i \in I} \|p_i\,(\xi)\| \leq \|\xi\| < \infty.$$

By the separating property of ξ we conclude that $\|p_i\,(\xi)\| \neq 0$ for all i. As such there are only countably many p_i, that is, I is countable and thus $\mathcal{A}$ is σ-finite. □

Using 4.1.1.4 and the GNS construction, we construct the Hilbert space on which $\mathcal{A}$ allows for a cyclic and separating vector. We will now start investigating the structure of this Hilbert space, in particular we will investigate the canonical maps r and s.

4.1.2 The maps s, r, J and Δ

We now fix a von Neumann algebra $\mathcal{A}$ acting on a separable Hilbert space $\mathcal{H}$. As above we can construct a Hilbert space $\mathcal{H}_{\mathcal{A}}$, which contains $\mathcal{A}$ as a dense subspace, and an isomorphism $\pi : \mathcal{A} \longrightarrow \mathcal{B}(\mathcal{H}_{\mathcal{A}})$, such that $\pi(\mathcal{A})$ allows for a cyclic and separating vector $\mathfrak{h}$. We are also given a canonical embedding of $\mathcal{A}$

into $\mathcal{H}_{\mathcal{A}}$ denoted by $i_{\mathcal{A}}$ such that $i_{\mathcal{A}}(1) = \mathfrak{h}, i_{\mathcal{A}}(a) = \pi(a)\mathfrak{h}$ and $\pi(a)(i_{\mathcal{A}}(b)) := i_{\mathcal{A}}(ab)$. Furthermore if ϕ denotes the faithful normal state on $\mathcal{A}$ then the norm of $i(a) \in \mathcal{H}_{\mathcal{A}}$ is given by $\phi(a^*a)$.

Since $\mathfrak{h} = i_{\mathcal{A}}(1)$ is cyclic and separating for $\pi(\mathcal{A})$ it also cyclic and separating for $\pi(\mathcal{A})' \subset \mathcal{B}(\mathcal{H}_{\mathcal{A}})$. We conclude that the set $\pi(\mathcal{A})'$ is also dense and that we can define a map $i_{\pi(\mathcal{A})'} : \pi(\mathcal{A})' \longrightarrow \mathcal{H}_{\mathcal{A}}$ by setting $i_{\pi(\mathcal{A})'}(x) := x(\mathfrak{h})$. It is obvious that the maps $i_{\mathcal{A}}$ and $i_{\pi(\mathcal{A})'}$ are continuous with respect to the strong operator topology on $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$. We can use these embeddings to transfer algebraic operations from $\mathcal{A}$ and $\pi(\mathcal{A})'$ to operations on the dense subspaces $i_{\mathcal{A}}(\mathcal{A}) = \pi(\mathcal{A})\mathfrak{h}$ and $i_{\pi(\mathcal{A})'}(\pi(\mathcal{A}')) = \pi(\mathcal{A})'\mathfrak{h}$.

We define a product $\cdot$ on $i_{\mathcal{A}}(\mathcal{A}) \times i_{\mathcal{A}}(\mathcal{A})$ onto $i_{\mathcal{A}}(\mathcal{A})$ as

$$i_{\mathcal{A}}(a) \cdot i_{\mathcal{A}}(b) := i_{\mathcal{A}}(ab).$$

Note here that $i_{\mathcal{A}}(a) \cdot i_{\mathcal{A}}(b) = i_{\mathcal{A}}(ab) = \pi(a)(i_{\mathcal{A}}(b))$. We can use this formula to extend the definition domain of the product. We extend $\cdot$ to the space $i_{\mathcal{A}} \times \mathcal{H}_{\mathcal{A}}$ with image in $\mathcal{H}_{\mathcal{A}}$, by setting $i_{\mathcal{A}}(a) \cdot h := \pi(a)h$.

Similarly we can define a product on the space $i_{\pi(\mathcal{A})'}(\pi(\mathcal{A}')) = \pi(\mathcal{A})'\mathfrak{h}$, for $x, y \in \pi(\mathcal{A})'$, we define

$$x\mathfrak{h} \cdot y\mathfrak{h} = xy\mathfrak{h} = x((y\mathfrak{h})).$$

This product is extended in a similar way to the domain $\pi(\mathcal{A})'\mathfrak{h} \times \mathcal{H}_{\mathcal{A}}$ by setting $x\mathfrak{h} \cdot v := x(v)$. We see that $\mathcal{H}_{\mathcal{A}}$ comes naturally with a densely defined product. We now aim to equip $\mathcal{H}_{\mathcal{A}}$ with a densely defined star operation.

Definition 4.1.2.1. *We define maps s_0 and r_0 as follows:*

$$s_0 : \mathcal{A}\mathfrak{h} \longrightarrow \mathcal{A}\mathfrak{h} \subset \mathcal{H}_{\mathcal{A}}, \qquad s_0\,(a\mathfrak{h}) := a^*\mathfrak{h}$$
$$r_0 : \pi(\mathcal{A})'\mathfrak{h} \longrightarrow \pi(\mathcal{A})'\mathfrak{h} \subset \mathcal{H}_{\mathcal{A}}, \qquad r_0\,(x\mathfrak{h}) := x^*\mathfrak{h}.$$

We note that s_0 and r_0 are densely defined injective anti-linear maps on $\mathcal{H}$. For a discussion regarding anti-linear or unbounded maps we refer to the appendix.

Proposition 4.1.2.2. *The maps s_0 and r_0 are closable operators.*

Proof:

Consider the graph of s_0, we aim to show that its closure is again a graph. Define $\mathcal{V}$ as follows

$$\mathcal{V} := \overline{\{a\mathfrak{h} \oplus a^*\mathfrak{h}\ ;\ a \in \mathcal{A}\}}.$$

It is immediate that $\mathcal{V}$ is an anti-linear subspace of $\mathcal{H} \oplus \mathcal{H}$, to show that $\mathcal{V}$ defines a graph it suffices to show that if $0 \oplus h \in \mathcal{V}$ then $h = 0$. Let $0 \oplus h \in \mathcal{V}$, by construction there are $a_i \in \mathcal{A}$ such that $a_i\mathfrak{h} \oplus a_i^*\mathfrak{h} \longrightarrow 0 \oplus h$. For $x \in \pi(\mathcal{A})'$ consider $\langle x\mathfrak{h}, h\rangle$, we find that

$$\begin{aligned} |\langle x\mathfrak{h}, h\rangle| &= \lim_i |\langle x\mathfrak{h}, a_i^*\mathfrak{h}\rangle| \\ &= \lim_i |\langle a_i x\mathfrak{h}, \mathfrak{h}\rangle| \\ &= \lim_i |\langle x a_i\mathfrak{h}, \mathfrak{h}\rangle| \\ &= \lim_i |\langle a_i\mathfrak{h}, x^*\mathfrak{h}\rangle| \\ &\leq \lim_i \|a_i\mathfrak{h}\| \cdot \|x^*\mathfrak{h}\| = 0, \end{aligned}$$

so that $h \in [\pi(\mathcal{A})'\mathfrak{h}]^\perp$. We use now that $\pi(\mathcal{A})'\mathfrak{h}$ is dense in $\mathcal{H}_{\mathcal{A}}$ to conclude that $h = 0$. We conclude that $\mathcal{V}$ is a graph and therefor s_0 is closable. Note that this actually shows a relation between s_0 and r_0, we find that for $a \in \mathcal{A}$ and $x \in \pi(\mathcal{A})'$, the following holds

$$\begin{aligned} \langle s_0^*(x\mathfrak{h}), a\mathfrak{h}\rangle = \langle s_0(a\mathfrak{h}), x\mathfrak{h}\rangle &= \langle a^*\mathfrak{h}, x\mathfrak{h}\rangle \\ &= \langle \mathfrak{h}, ax\mathfrak{h}\rangle \\ &= \langle \mathfrak{h}, xa\mathfrak{h}\rangle \\ &= \langle x^*\mathfrak{h}, a\mathfrak{h}\rangle\,. \end{aligned}$$

We see here that s_0^* defines a closed extension of r_0, in particular r_0 is closable. By symmetry we have that r_0^* defines a closed extension of s_0, we set $s = s_0^{**}$ and $r = r_0^{**}$ so that r (resp. s) is the closure of r_0 (resp. s_0). □

We write $A \subset B$ to say that B defines a closed extension of A. It is easy to see that if $A \subset B$, then $B^* \subset A^*$. We find the following relations between s_0 and r_0

$$\begin{aligned} r_0 &\subset s_0^* \qquad & s &\subset r_0^* \\ s_0 &\subset r_0^* \qquad & r &\subset s_0^*. \end{aligned}$$

Proposition 4.1.2.3. *The maps r and s satisfy the identities $r^* = s$ and $s^* = r$*

Proof:

We denote by $\mathcal{D}(A)$ the domain of the operator A. Consider $r = r_0^{**}$, we have that $r \subset s_0^* = s^*$. On the other hand suppose that $k \in \mathcal{D}(r)$, then k defines a bounded linear functional on $\mathcal{D}(r_0^*)$ by setting $k(h) := \langle r_0^*(h), k\rangle$. Since r_0^* extends s_0 we conclude that k also defines a bounded linear functional on $\mathcal{D}(s_0)$, so that $k \in \mathcal{D}(s_0^*)$. We conclude that $s^* = s_0^* \subset r$ thus $r = s^*$ and also $s = r^*$. □

Since we are dealing with anti-linear maps we will consider the absolute value of inner products so that we can write $|\langle s_0(x), y\rangle| = |\langle x, s_0^*(y)\rangle|$. The benefit is that we do not have to conjugate the inner product every time we take an adjoint, we found that this can be confusing. We will now investigate the domains of r and s.

Proposition 4.1.2.4. *The maps r and s satisfy the following identities:*

$$\begin{aligned} \mathcal{D}(r^2) &= \mathcal{D}(r), \qquad & r^2(f) &= f, \\ \mathcal{D}(s^2) &= \mathcal{D}(s), \qquad & s^2(h) &= h. \end{aligned}$$

Proof:

Suppose that $k \in \mathcal{D}(s^*) = \mathcal{D}(r)$ and that $a\mathfrak{h} \in \mathcal{D}(s_0)$ then we find the following identities

$$\begin{aligned} |\langle s_0(a\mathfrak{h}), r(k)\rangle| &= |\langle s_0(a\mathfrak{h}), s^*(k)\rangle| \\ &= |\langle a^*\mathfrak{h}, s_0^*(k)\rangle| \\ &= |\langle a\mathfrak{h}, k\rangle| \,. \end{aligned}$$

Thus we conclude that for all $k \in \mathcal{D}(r) = \mathcal{D}(s^*)$, $r(k)$ defines a bounded linear functional on $\mathcal{D}(s_0)$, that is, $r(k) = s^*(k) \in \mathcal{D}(s^*)$. Thus $s^* : \mathcal{D}(s^*) \longrightarrow \mathcal{D}(s^*)$. We find that for $k \in \mathcal{D}(s^*)$ it holds that

$$\begin{aligned} |\langle s_0(a\mathfrak{h}), s^*s^*(k)\rangle| &= |\langle a^*\mathfrak{h}, s^*s^*(k)\rangle| \\ &= |\langle a\mathfrak{h}, s^*(k)\rangle| \\ &= |\langle a^*h, k\rangle| \,. \end{aligned}$$

We conclude that $r^2(k) = (s^*)^2(k) = k$, as such, $(s^*)^2 = r^2$ is the identity on $\mathcal{D}(r)$. We note that $\mathcal{D}(r) \subset \mathcal{D}(r^2)$, on the other hand $\mathcal{D}(r^2) = r^{-1}[\mathcal{D}(r)] \subset r^{-1}(\mathcal{H}) = \mathcal{D}(r)$. It follows that $\mathcal{D}(r) = \mathcal{D}(r^2)$ and $\operatorname{ran}(r) = \mathcal{D}(r)$.

Now we consider $s = r^*$, for $k \in \mathcal{D}(r^*)$ and $b\mathfrak{h} \in \mathcal{D}(r_0)$ we find that

$$\begin{aligned} |\langle r_0(b\mathfrak{h}), r^*(k)\rangle| &= |\langle b^*\mathfrak{h}, r^*(k)\rangle| \\ &= |\langle b\mathfrak{h}, k\rangle| \,. \end{aligned}$$

It follows that $r^* : \mathcal{D}(r^*) \longrightarrow \mathcal{D}(r^*)$, we conclude that $s^2 = (r^*)^2(k) = k$ so that (r^*) is the identity on its domain and furthermore we conclude that $\mathcal{D}(s^2) = \mathcal{D}(s)$ and $\operatorname{ran}(s) = \mathcal{D}(s)$. We conclude that $s = s^{-1}$ and $r = r^{-1}$. Note that s is not boundedly invertible. □

The polar decomposition justifies the following definition.

Definition 4.1.2.5. *Set $\Delta := s^*s$, by the polar decomposition we can find a partial anti-isometry J such that*

$$s = J\Delta^{1/2}$$

We call Δ the modular operator and J the modular conjugation associated to $\mathcal{A}$.

We will now investigate the relations between s, r, Δ and J.

Proposition 4.1.2.6. *The maps r, s, J and Δ satisfy the following relations:*

$$\begin{array}{lll} J^2 = 1, & J = J^*, & \Delta = rs, \\ r = \Delta^{1/2}J, & r = \left(s^{-1}\right)^* = J\Delta^{-1/2}, & sr = \left(\Delta^{-1/2}J\right)\left(J\Delta^{-1/2}\right) = \Delta^{-1}. \end{array}$$

Furthermore $J(\mathfrak{h}) = \mathfrak{h}, s(\mathfrak{h}) = \mathfrak{h}$, $r(\mathfrak{h}) = \mathfrak{h})$ and $\Delta(\mathfrak{h}) = \mathfrak{h}$.

Proof:
Since $\ker s = \{0\}$ and $\operatorname{ran}(s)$ is dense we conclude that J is anti-unitary. Using that $s = s^{-1}$ we find that $J\Delta^{1/2} = s = s^{-1} = \Delta^{-1/2}J^*$. It follows that $\Delta^{-1/2}\left(J^*\right)^2 = J\Delta^{1/2}J^* > 0$, as such $\Delta^{-1/2}\left(J^*\right)^2 = \left|\Delta^{-1/2}\left(J^*\right)^2\right| = \Delta^{-1/2}$. We conclude that $\left(J^*\right)^2 = 1$.

We find that the following identities hold for s, r, J and Δ.

$$\begin{array}{lll} J^2 = 1, & J = J^*, & \Delta = rs, \\ r = \Delta^{1/2}J, & r = \left(s^{-1}\right)^* = J\Delta^{-1/2}, & sr = \left(\Delta^{-1/2}J\right)\left(J\Delta^{-1/2}\right) = \Delta^{-1} \end{array}$$

Note that if $c \in \mathcal{Z}(\pi(\mathcal{A}))$ then $\Delta(c\mathfrak{h}) = c\mathfrak{h}$. As such Δ is the identity on the subspace $\mathcal{Z}(\pi(\mathcal{A}))\mathfrak{h} \subset \mathcal{H}_{\mathcal{A}}$, in particular for all $c \in \mathcal{Z}(\pi(\mathcal{A}))$ we have $\Delta(c\mathfrak{h}) = \Delta^{1/2}(c\mathfrak{h}) = c\mathfrak{h}$. This leads to the following conclusions: $J(\mathfrak{h}) = \mathfrak{h}, s(\mathfrak{h}) = \mathfrak{h}$, $r(\mathfrak{h}) = \mathfrak{h}$ and $\Delta(\mathfrak{h}) = \mathfrak{h}$. □

It is instructive to compare this with the semifinite case. Suppose that our Hilbert space was constructed using a trace τ. Then the map s_0 defined a star operation on all of $\mathcal{H}_{\mathcal{A}}$. For $a\mathfrak{h}, b\mathfrak{h} \in \mathcal{A}\mathfrak{h}$ we would have had that $\langle s_0(a\mathfrak{h}), b\mathfrak{h}\rangle = \langle a^*\mathfrak{h}, b\mathfrak{h}\rangle = \tau(b^*a^*) = \tau(a^*b^*) = \langle b^*\mathfrak{h}, a\mathfrak{h}\rangle = \langle s_0^*(b\mathfrak{h}), a\mathfrak{h}\rangle$, so that $s_0 \subset s_0^*$. Also $\|s_0(a\mathfrak{h})\|^2 = \langle a^*\mathfrak{h}, a^*\mathfrak{h}\rangle = \tau(aa^*) = \tau(a^*a) = \|a\mathfrak{h}\|^2$, so s_0 would be isometric, meaning that s would be a unitary. It would follow that $\Delta = 1$ and $J = s = r$, simplifying the situation.

We will now examine the range and domains of the operators r, s, J and $\Delta^{1/2}$ in greater detail. Consider $\mathcal{D}(s)$ and suppose that $h \in \mathcal{D}(s)$, pick a sequence $\{a_i\mathfrak{h}\} \subset [\mathcal{A}\mathfrak{h}]$ such that

$$\lim_i \|a_i\mathfrak{h} - h\| = 0.$$

We note that $h - a_i\mathfrak{h} \in \mathcal{D}(s)$ for all i, as such, $h - a_i\mathfrak{h} \oplus s(h - a_i\mathfrak{h}) \subset \operatorname{graph}(s)$. Since $h - a_i\mathfrak{h} \longrightarrow 0$ it must follow that $a_i^*\mathfrak{h} \longrightarrow s(h)$. On the other suppose that for $h \in \mathcal{H}_{\mathcal{A}}$ there exists a sequence $\{a_i\mathfrak{h}\}$ such that $\lim_i \|h - a_i\mathfrak{h}\| \longrightarrow 0$ and $\{a_i^*\mathfrak{h}\}$ is a Cauchy sequence in $\mathcal{H}$. Denote the limit of $a_i^*\mathfrak{h}$ by f, it follows that $a_i\mathfrak{h} \oplus a_i^*\mathfrak{h} \longrightarrow h \oplus f$, we use now that s is closed to conclude that $h \oplus f \in \operatorname{graph}(s)$ thus $h \in \mathcal{D}(s)$ and $f = s(h)$. We have characterized the domain of s as all those $h \in \mathcal{H}_{\mathcal{A}}$ such that there exists a sequence $\{a_i\mathfrak{h}\}$ with the property that

$$\lim a_i\mathfrak{h} = h \qquad \text{and} \qquad \{a_i^*\mathfrak{h}\} \text{ is a Cauchy sequence in } \mathcal{H}_{\mathcal{A}}.$$

Via similar arguments we conclude that the domain of r can be characterized as the set of all $h \in \mathcal{H}_{\mathcal{A}}$ such that there exists a sequence $b_i\mathfrak{h} \subset \pi(\mathcal{A})'\mathfrak{h}$ satisfying

$$\lim_i b_i\mathfrak{h} = h \qquad \text{and} \qquad b_i^*\mathfrak{h} \text{ is a Cauchy sequence.}$$

We also have the identities $r_0^* = r^* = s$ and $s_0^* = s^* = r$, so that

$$\mathcal{D}(s) = \mathcal{D}(r_0^*) := \{k \in \mathcal{H}_{\mathcal{A}} \; ; \; k : \pi(\mathcal{A})'\mathfrak{h} \longrightarrow \mathbb{C},\ k(b\mathfrak{h}) := \langle b^*\mathfrak{h}, k\rangle \text{ defines a bounded anti-linear functional}\},$$
$$\mathcal{D}(r) = \mathcal{D}(s_0^*) := \{h \in \mathcal{H}_{\mathcal{A}} \; ; \; h : \mathcal{A}\mathfrak{h} \longrightarrow \mathbb{C},\ h(b\mathfrak{h}) := \langle a^*\mathfrak{h}, k\rangle \text{ defines a bounded anti-linear functional}\}.$$

This characterizes the domain and range of the maps s and r. Now we would like to study the domain and range of the maps $\Delta^{1/2}$ and J.

By the identities $r = \Delta^{1/2} J$ and $r(\mathcal{D}(r)) = \mathcal{D}(r)$ we find that for $h \in \mathcal{D}(r)$ it holds that

$$r(h) = \Delta^{1/2} J(h) \in \mathcal{D}(r),$$

as such,

$$J : \mathcal{D}(r) \longrightarrow \mathcal{D}(s) \qquad \text{and} \qquad \Delta^{1/2} : \mathcal{D}(s) \longrightarrow \mathcal{D}(r).$$

We use now that $J^2 = 1$ to find that

$$J(\mathcal{D}(s)) = \mathcal{D}(r) \qquad \text{and} \qquad J(\mathcal{D}(r)) = \mathcal{D}(s).$$

Using this, and the identities $s(\mathcal{D}(s)) = \mathcal{D}(s)$ and $s = J\Delta^{1/2}$ we find that

$$\Delta^{1/2}(\mathcal{D}(s)) = \mathcal{D}(r).$$

It follows that $\mathcal{D}\left(\Delta^{-1/2}\right) = \mathcal{D}(r)$ and $\mathrm{ran}(\Delta^{-1/2}) = \mathcal{D}(s)$.

Unfortunately the domain of the map Δ, although being a subset of $\mathcal{D}(\Delta^{1/2})$, is hard to describe. We will use that $\Delta + \lambda$ is boundedly invertible for all $\lambda > 0$ to give a relation between the sets $\mathcal{A}\mathfrak{h}$ and $\pi(\mathcal{A})'\mathfrak{h}$. It turns out that we can associate to every element $x' \in \pi(\mathcal{A})'$ an element $x_\lambda \in \pi(\mathcal{A})$ using $\Delta + \lambda$.

Proposition 4.1.2.7. *For any $x' \in \pi(\mathcal{A})'$ and any $\lambda > 0$ we can find an operator $L_{x'} \in \pi(\mathcal{A})$ such that $L_{x'}\mathfrak{h} = (\Delta + \lambda)^{-1} x'\mathfrak{h}$.*

Pick $\lambda > 0$ then $(\Delta + \lambda)^{-1}$ is a well defined bounded operator from $\mathcal{H}_\mathcal{A}$ into $\mathcal{D}(\Delta) \subset \mathcal{D}(s)$. For given $x' \in \pi(\mathcal{A})'$ we can consider $(\Delta + \lambda)^{-1}(x'\mathfrak{h})$. We denote $f := (\Delta + \lambda)^{-1}(x'\mathfrak{h})$ and define a new operator $L : \pi(\mathcal{A})'\mathfrak{h} \longrightarrow \mathcal{H}_\mathcal{A}$ by setting $L(y\mathfrak{h}) := y(f)$. Consider $y\mathfrak{h}$ and $z\mathfrak{h}$ with $y, z \in \pi(\mathcal{A})'$, we find the following identities

$$\begin{aligned} \langle L(y\mathfrak{h}), z\mathfrak{h}\rangle &= \langle y(f), z\mathfrak{h}\rangle \\ &= \langle f, y^* z\mathfrak{h}\rangle \\ &= \langle f, r(z^* y\mathfrak{h})\rangle \\ &= \langle z^* y\mathfrak{h}, s(f)\rangle \\ &= \langle y\mathfrak{h}, zs(f)\rangle . \end{aligned}$$

It follows that L^* defines an extension of the map $\pi(\mathcal{A})' \longrightarrow \mathcal{H}_\mathcal{A}$, $y\mathfrak{h} \longrightarrow y(s(f)) = y(f^*)$ which is densely defined, as such L is closable (we denote its closure also by L). Note furthermore that if $z \in \pi(\mathcal{A})'$ and $b \in \pi(\mathcal{A})'$ then

$$L \circ z(b\mathfrak{h}) = zb(f) = z \circ L(b\mathfrak{h}),$$

so that L commutes with $\pi(\mathcal{A})'$. What we want to show is that L is in fact bounded, it then follows that $L \in \pi(\mathcal{A})$. We can decompose L as $L = UK$ with U a partial isometry and $K = \sqrt{L^* L}$. If E is the spectral measure associated with K then for any bounded subset $\gamma \subset \sigma(K)$ we have $E(\gamma) \in \pi(\mathcal{A})$. Suppose now that L is unbounded, then in particular we can find $c, d \in \mathbb{R}$ such that $\frac{\|x'\|}{2\sqrt{\lambda}} < c < d$ and $E((c,d))K \neq 0$. We pick c, d as above and let $E = E((c,d))$, it follows that $E \in \pi(\mathcal{A})$ and E commutes with K. We will now show that L is bounded. Consider $x' \in \pi(\mathcal{A})'$, the operator L was defined by the formula $L(b\mathfrak{h}) = b\left((\Delta + \lambda)^{-1} x'\mathfrak{h}\right) = b(f)$. By definition we have that if $z \in \pi(\mathcal{A})'$ then

$$zf^* = L^*(z\mathfrak{h}) = (UK)^*(z\mathfrak{h}) = KU^*(z\mathfrak{h}).$$

Consider now x', we find the following inequalities

$$\begin{aligned} \|x'\|^2 \cdot \|Ef^*\|^2 &\geq \|x'(Ef^*)\|^2 = \|x' E(UK)^*\mathfrak{h}\|^2 \\ &= \|x' EKU^*\mathfrak{h}\|^2 = \|EKU^*(x'\mathfrak{h})\|^2 . \end{aligned}$$

The last equality follows because x' commutes with E and L^*. Now we use the operator $\Delta + \lambda$, we have that $(\Delta+\lambda)^{-1}(x'\mathfrak{h}) = f$, thus $(\Delta+\lambda)f = x'\mathfrak{h}$.
Now we consider the term $\|EKU^*(x'\mathfrak{h})\|^2$ more closely, we find

$$\|EKU^*(x'\mathfrak{h})\|^2 = \|EKU^*(\Delta+\lambda)(f)\|^2 .$$

We denote for notational convenience $N = EKU^*$, the following inequality although trivial, makes no sense at first sight but the result will justify the approach,

$$\|EKU^*(\Delta+\lambda)f\|^2 = \|N(\Delta+\lambda)f\|^2 \geq \|N(\Delta+\lambda)f\|^2 - \|N(\Delta-\lambda)f\|^2 .$$

Note that the following holds

$$\begin{aligned}
\|N(\Delta+\lambda)f\|^2 &= \langle N(\Delta+\lambda)f, N(\Delta+\lambda)f\rangle \\
&= \langle N\Delta f, N\Delta f\rangle + \langle N\Delta f, \lambda N f\rangle + \langle \lambda N f, N\Delta f\rangle + \langle \lambda N f, \lambda N f\rangle , \\
\|N(\Delta-\lambda)f\|^2 &= \langle N(\Delta-\lambda)f, N(\Delta-\lambda)f\rangle \\
&= \langle N\Delta f, N\Delta f\rangle - \langle N\Delta f, \lambda N f\rangle - \langle \lambda N f, N\Delta f\rangle + \langle \lambda N f, \lambda N f\rangle .
\end{aligned}$$

It follows that

$$\begin{aligned}
\|N(\Delta+\lambda)f\|^2 - \|N(\Delta-\lambda)f\|^2 &= 2\left(\langle N\Delta f, \lambda N f\rangle + \langle \lambda N f, N\Delta f\rangle\right) \\
&= 2\lambda\left(\langle N\Delta f, N f\rangle + \langle N f, N\Delta f\rangle\right) \\
&= 4\lambda \Re \langle N\Delta f, N f\rangle .
\end{aligned}$$

Here $\Re\langle N\Delta f, Nf\rangle$ denotes the real part of $\langle N\Delta f, Nf\rangle$. We will now investigate the term $4\lambda\Re\langle N\Delta f, Nf\rangle$, we find that

$$\begin{aligned}
4\lambda\Re\langle N\Delta f, Nf\rangle &= 4\lambda\Re\langle EKU^*\Delta f, EKU^* f\rangle \\
&= 4\lambda\Re\left\langle \Delta f, UK^2EU^* f\right\rangle \\
&= 4\lambda\Re\left\langle rs(f), UK^2EU^* f\right\rangle .
\end{aligned}$$

We substitute $f = UK(\mathfrak{h})$ to find

$$\begin{aligned}
4\lambda\Re\left\langle rs(f), UK^2EU^* f\right\rangle &= 4\lambda\Re\left\langle rs(f), UK^3E\mathfrak{h}\right\rangle \\
&= 4\lambda\Re\left\langle s(f), EK^3U^*\mathfrak{h}\right\rangle \\
&= 4\lambda\Re\left\langle f^*, EK^2 f^*\right\rangle = 4\lambda\|KEf^*\|^2 .
\end{aligned}$$

In total we conclude that

$$\|x'\|^2 \cdot \|Ef^*\|^2 \geq 4\lambda\|KEf^*\|^2 .$$

Now we use that E is the spectral projection of the interval (c,d). For every $h \in \operatorname{ran}(E)$ we have $\|Kh\|^2 > c^2\|h\|^2$, applying this to the situation we find that if $Ef^* \neq 0$ then

$$\|x'\|^2 \cdot \|Ef^*\|^2 \geq 4\lambda\|KEf^*\|^2 \geq 4\lambda c^2\|Ef^*\|^2 > \|x'\|^2 \cdot \|Ef^*\|^2 ,$$

clearly a contradiction. We conclude that $Ef^* = 0$. Note now that $f^* = L^*(\mathfrak{h})$ and that for any $b \in \pi(\mathcal{A})'$ we have $bE = Eb$ (because $E \in \pi(\mathcal{A})$) and $bL^*(\mathfrak{h}) = L^*(b\mathfrak{h})$. It follows that for all $b \in \pi(\mathcal{A})'$

$$0 = bEf^* = Ebf^* = EL^*(b\mathfrak{h}).$$

We use that EL^* is bounded and that $\pi(\mathcal{A})'\mathfrak{h}$ is dense to conclude that $EL^* = 0$. But then $0 = EL^*U = EK \neq 0$, this is in contradiction with the assumption that K is unbounded. We conclude that L is bounded. □

To summarize this last discussion: We have succeeded to associate to every $x' \in \pi(\mathcal{A})'$ and $\lambda > 0$ an operator $L_{x'} : \pi(\mathcal{A})'\mathfrak{h} \longrightarrow \mathcal{H}_{\mathcal{A}}$ defined as $L_{x'}(b\mathfrak{h}) := b\left((\Delta+\lambda)^{-1}x'\mathfrak{h}\right)$. We concluded that $L_{x'}$ commutes with $\pi(\mathcal{A})'$ and that $L_{x'}$ is in fact bounded so that $L_{x'}$ can be extended to the whole of $\mathcal{H}_{\mathcal{A}}$ and

$L_{x'} \in \pi(\mathcal{A})$. Note that if $x' \neq y'$ then $L_{x'} \neq L_{y'}$ because the map $(\Delta+\lambda)^{-1}$ is injective. We denote the map $L_{x'}$ by x_λ. Furthermore we have that $x_\lambda(\mathfrak{h}) = L_{x'}(\mathfrak{h}) = (\Delta+\lambda)^{-1}x'\mathfrak{h} = f$. We can already see that there is a relation between $\pi(\mathcal{A})$ and its commutant. We can also apply the same procedure to some $\pi(a) \in \pi(\mathcal{A})$ and obtain an operator in $\widehat{L}(a) \in \pi(\mathcal{A})'$, the proof works the same. The question then becomes how $x' \longrightarrow L_{x'} \longrightarrow \widehat{L}\left(L_{x'}\right)$ behaves. After a small calculation we see that the vector associated to $\widehat{L}\left(L_{x'}\right)$ is the vector

$$f_{\widehat{L}(L_{x'})} = (\Delta+\lambda)^{-2}x'\mathfrak{h}.$$

In particular we conclude that the spaces $\mathcal{A}\mathfrak{h}$ and $\pi(\mathcal{A})'\mathfrak{h}$ are invariant for $(\Delta+\lambda)^{-2}$, it is however not true that L and $\widehat{L}$ "invert" each other. Note that the following holds

$$\begin{aligned} (\Delta+\lambda)^{-1} &: \mathcal{A}\mathfrak{h} \longrightarrow \pi(\mathcal{A})'\mathfrak{h}, \qquad & (\Delta+\lambda) &: \mathcal{D}(\Delta)\cap\mathcal{A}\mathfrak{h} \longrightarrow \pi(\mathcal{A})'\mathfrak{h},\\ (\Delta+\lambda)^{-1} &: \pi(\mathcal{A})'\mathfrak{h} \longrightarrow \mathcal{A}\mathfrak{h}, \qquad & (\Delta+\lambda) &: \mathcal{D}(\Delta)\cap\pi(\mathcal{A})'\mathfrak{h} \longrightarrow \mathcal{A}\mathfrak{h}. \end{aligned}$$

It follows that

$$\begin{aligned} (\Delta+\lambda)\left[\mathcal{D}(\Delta)\cap\mathcal{A}\mathfrak{h}\right] &= \pi\left(\mathcal{A}\right)'\mathfrak{h},\\ (\Delta+\lambda)\left[\mathcal{D}(\Delta)\cap\pi(\mathcal{A})'\mathfrak{h}\right] &= \mathcal{A}\mathfrak{h}. \end{aligned}$$

We will now investigate some of the properties of the map $L_{x'}$ for $x \in \pi(\mathcal{A})'$. Above we have seen that $L_{x'}$ defines an element in $\pi(\mathcal{A})$. If we denote $f_{x'} := L_{x'}(\mathfrak{h})$, then on the dense subspace $\pi(\mathcal{A})'\mathfrak{h}$ we have that $L_{x'}(b\mathfrak{h}) := b(f) = bL_{x'}(\mathfrak{h})$. Let $a\mathfrak{h} \in \mathcal{A}\mathfrak{h}$, if $\{b_i\mathfrak{h}\}$ is a sequence in $\pi(\mathcal{A})'\mathfrak{h}$ such that $\lim_i b_i\mathfrak{h} = a\mathfrak{h}$, then we have that

$$L_{x'}(a\mathfrak{h}) = \lim_i L_{x'}(b_i\mathfrak{h}) = \lim_i b_i f_{x'},$$

as expected. However it is almost *never* the case that $\lim_i b_i f_{x'} = \pi(a)f_{x'}$.

Suppose that $c \in \mathcal{Z}\left(\pi(\mathcal{A})\right)$ then $(\Delta+\lambda)c = \Delta\circ c + \lambda c = c + \lambda c = (1+\lambda)c$, as such,

$$f_c = L_c(\mathfrak{h}) = (\Delta+\lambda)^{-1}c\mathfrak{h} = \frac{c\mathfrak{h}}{1+\lambda}.$$

In this case we have that if $\{b_i\mathfrak{h}\} \subset \pi(\mathcal{A})'\mathfrak{h}$ converges to $a\mathfrak{h} \in \mathcal{A}\mathfrak{h}$, then

$$\begin{aligned} c_\lambda(a\mathfrak{h}) = L_c(a\mathfrak{h}) = \lim_i L_c(b_i\mathfrak{h}) &= \lim_i \frac{b_i c\mathfrak{h}}{1+\lambda} = \lim_i \frac{c b_i\mathfrak{h}}{1+\lambda}\\ &= \frac{ca\mathfrak{h}}{1+\lambda} = \frac{ac\mathfrak{h}}{1+\lambda} = \pi(a)f_c. \end{aligned}$$

We conclude that if $c \in \mathcal{Z}(\pi(\mathcal{A}))$, then the map $L_c = c_\lambda$ also commutes with $\pi(\mathcal{A})$, that is, L_c is central for $\pi(\mathcal{A})$. Moreover for all $h \in \mathcal{H}_{\mathcal{A}}$ we have

$$L_c(h) = \frac{c(h)}{1+\lambda}.$$

It follows that $\mathcal{Z}\left(\pi(\mathcal{A})\right) \ni c \longrightarrow L_c = c_\lambda \in \mathcal{Z}\left(\pi(\mathcal{A})\right)$ defines a bijection. Note that this implies in particular that that there are elements $a\mathfrak{h} \in \pi(\mathcal{A})\mathfrak{h}$ such that $a\mathfrak{h} \in \mathcal{D}(\Delta)$ but $a \notin \mathcal{Z}\left(\pi(\mathcal{A})\right)$. Similarly there are elements $b\mathfrak{h}$ with $b \in \pi(\mathcal{A})'$ such that $b\mathfrak{h} \in \mathcal{D}(\Delta)$ but $b \notin \mathcal{Z}\left(\pi(\mathcal{A})\right)$.

We return to the case $x' \in \pi(\mathcal{A})'$, we have that $L_{x'} \in \pi(\mathcal{A})$ and $L_{x'}(b'\mathfrak{h}) = b'\left(\Delta+\lambda\right)^{-1}x'\mathfrak{h}$. We would like to find an expression for $x_\lambda = L_{x'}$ in terms of x', Δ and J, in other words we want to find an expression G such that $x_\lambda(b'\mathfrak{h}) = G(x',\Delta,J)(b'\mathfrak{h})$ as maps. What we currently have is an expression of the form $x_\lambda(b'\mathfrak{h}) = b'\left(G(\Delta,x',\mathfrak{h})\right)$, here b' is working on the expression G, what we want is an expression G that is working on $b'\mathfrak{h}$. In order to succeed in our goals we will consider x' as a sesquilinear form.

Definition From now on we will denote the map $L_{x'}$, constructed from $x' \in \pi(\mathcal{A})'$ and $\lambda > 0$, by x_λ.

Proposition 4.1.2.8. *On $\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2})$ we have the following relation between x' and x_λ,*

$$Jx'J = \Delta^{-1/2}x_\lambda^*\Delta^{1/2} + \lambda \cdot \Delta^{1/2}x_\lambda^*\Delta^{-1/2}.$$

Proof:

For $a', b' \in \pi(\mathcal{A})'$ we can find $a^*, b^* \in \pi(\mathcal{A})$ such that $a^*\mathfrak{h} = (\Delta+1)^{-1}a'\mathfrak{h}$ and $b^*\mathfrak{h} = (\Delta+1)^{-1}b'\mathfrak{h}$, that is, $a = a_1^*$ and $b = b_1^*$. We can define a sesquilinear form $\mathfrak{s}_{x'}$ on $\pi(\mathcal{A})'\mathfrak{h}$ by setting

$$\mathfrak{s}_{x'}(b'\mathfrak{h}, a'\mathfrak{h}) := \langle x'\mathfrak{h}, b^*a\mathfrak{h}\rangle .$$

For $\mathfrak{s}_{x'}$ we find

$$\begin{aligned}
\langle x'\mathfrak{h}, b^*a\mathfrak{h}\rangle &= \langle x'b\mathfrak{h}, a\mathfrak{h}\rangle \\
&= \langle x's(b^*\mathfrak{h}), s(a^*\mathfrak{h})\rangle \\
&= \left\langle x's\left((\Delta+1)^{-1}b'\mathfrak{h}\right), s\left((\Delta+1)^{-1}a'\mathfrak{h}\right)\right\rangle \\
&= \left\langle (\Delta+1)^{-1}a'\mathfrak{h}, rx's\left((\Delta+1)^{-1}b'\mathfrak{h}\right)\right\rangle \\
&= \left\langle a'\mathfrak{h}, (\Delta+1)^{-1}\Delta^{1/2}Jx'J\Delta^{1/2}(\Delta+1)^{-1}\left(b'\mathfrak{h}\right)\right\rangle .
\end{aligned}$$

On the other hand we have that

$$\begin{aligned}
\langle x'\mathfrak{h}, b^*a\mathfrak{h}\rangle &= \langle (\Delta+\lambda)\, x_\lambda(\mathfrak{h}), b^*a\mathfrak{h}\rangle \\
&= \langle \Delta x_\lambda(\mathfrak{h}), b^*a\mathfrak{h}\rangle + \lambda \langle x_\lambda(\mathfrak{h}), b^*a\mathfrak{h}\rangle .
\end{aligned}$$

We will try to cast the expressions $\langle \Delta x_\lambda(\mathfrak{h}), b^*a\mathfrak{h}\rangle$ and $\lambda \langle x_\lambda(\mathfrak{h}), b^*a\mathfrak{h}\rangle$ in a similar form so that we can compare x_λ to x'. Consider the term:

$$\langle x_\lambda(\mathfrak{h}), b^*a\mathfrak{h}\rangle ,$$

we want to write this as $\langle a'\mathfrak{h}, G(x_\lambda, \Delta, J)b'\mathfrak{h}\rangle$ so that we can compare x_λ to x'. We find the following identities:

$$\begin{aligned}
\langle x_\lambda\mathfrak{h}, b^*a\mathfrak{h}\rangle &= \langle bx_\lambda\mathfrak{h}, a\mathfrak{h}\rangle \\
&= \langle s\left(x_\lambda^*b^*\mathfrak{h}\right), s\left(a^*\mathfrak{h}\right)\rangle \\
&= \left\langle \left(\Delta+1\right)^{-1} a'\mathfrak{h}, rs(x_\lambda^*b^*\mathfrak{h})\right\rangle \\
&= \left\langle a'\mathfrak{h}, (\Delta+1)^{-1}\Delta x_\lambda^*(\Delta+1)^{-1}b'\mathfrak{h}\right\rangle .
\end{aligned}$$

Now we consider the term $\langle \Delta x_\lambda\mathfrak{h}, b^*a\mathfrak{h}\rangle$. We find the following identities

$$\begin{aligned}
\langle \Delta x_\lambda\mathfrak{h}, b^*a\mathfrak{h}\rangle &= \langle rsx_\lambda\mathfrak{h}, b^*a\mathfrak{h}\rangle \\
&= \langle a^*b\mathfrak{h}, x_\lambda^*\mathfrak{h}\rangle \\
&= \langle b\mathfrak{h}, ax_\lambda^*\mathfrak{h}\rangle \\
&= \left\langle s(\Delta+1)^{-1}b'\mathfrak{h}, sx_\lambda a^*\mathfrak{h}\right\rangle \\
&= \left\langle \Delta x_\lambda(\Delta+1)^{-1}a'\mathfrak{h}, (\Delta+1)^{-1}b'\mathfrak{h}\right\rangle \\
&= \left\langle a'\mathfrak{h}, (\Delta+1)^{-1}x_\lambda^*\Delta(\Delta+1)^{-1}\right\rangle .
\end{aligned}$$

We draw the conclusion that for $b'\mathfrak{h}, a'\mathfrak{h} \in \pi(\mathcal{A})'\mathfrak{h}$ it holds that

$$\begin{aligned}
\mathfrak{s}_{x'}(b'\mathfrak{h}, a'\mathfrak{h}) &= \left\langle a'\mathfrak{h}, (\Delta+1)^{-1}\Delta^{1/2}Jx'J\Delta^{1/2}(\Delta+1)^{-1}\left(b'\mathfrak{h}\right)\right\rangle \\
&= \left\langle a'\mathfrak{h}, (\Delta+1)^{-1}x_\lambda^*\Delta(\Delta+1)^{-1}\right\rangle + \lambda \left\langle a'\mathfrak{h}, (\Delta+1)^{-1}\Delta x_\lambda^*(\Delta+1)^{-1}b'\mathfrak{h}\right\rangle .
\end{aligned}$$

Now we will show that $\mathfrak{s}_{x'}$ is a bounded sesquilinear form. We have the following identities

$$\begin{aligned}
|\mathfrak{s}_{x'}(b'\mathfrak{h}), a'\mathfrak{h}| = |\langle x'\mathfrak{h}, b^*a\mathfrak{h}\rangle| &= \left|\left\langle s\left(x'\right)^* b^*\mathfrak{h}, s(a^*\mathfrak{h})\right\rangle\right| \\
&= \left|\left\langle \Delta\left(x'\right)^* (\Delta+1)^{-1}b'\mathfrak{h}, (\Delta+1)^{-1}a'\mathfrak{h}\right\rangle\right| \\
&= \left|\left\langle (\Delta+1)^{-1}\Delta\left(x'\right)^* (\Delta+1)^{-1}b'\mathfrak{h}.a'\mathfrak{h}\right\rangle\right| \\
&\leq \left\|(\Delta+1)^{-1}\Delta\left(x'\right)^* (\Delta+1)^{-1} b'\mathfrak{h}\right\| \cdot \|a'\mathfrak{h}\| \\
&\leq \left\|(\Delta+1)^{-1}\Delta\right\| \cdot \left\|\left(x'\right)^*\right\| \cdot \left\|(\Delta+1)^{-1}\right\| \cdot \|b'\mathfrak{h}\| \cdot \|a'\mathfrak{h}\|.
\end{aligned}$$

Since $(\Delta+1)^{-1}$ is bounded it follows that $\mathfrak{s}_{x'}$ is bounded if $\Delta(\Delta+1)^{-1}$ is bounded. We have that $\Delta(\Delta+1)^{-1} = 1-(\Delta+1)^{-1}$ thus $\Delta(\Delta+1)^{-1}$ is bounded. We conclude that $\mathfrak{s}_{x'}$ is bounded. We use that $\pi(\mathcal{A})'\mathfrak{h}$ is dense in $\mathcal{H}_{\mathcal{A}}$ to conclude that the following holds

$$(\Delta+1)^{-1}\Delta^{1/2}Jx'J\Delta^{1/2}(\Delta+1)^{-1} = (\Delta+1)^{-1}x_\lambda^*\Delta(\Delta+1)^{-1} + \lambda(\Delta+1)^{-1}\Delta x_\lambda^*(\Delta+1)^{-1}.$$

We now multiply this identity from the left by $\Delta^{-1/2}(\Delta+1)$ and from the right by $(\Delta+1)\Delta^{-1/2}$ to find

$$Jx'J = \Delta^{-1/2}x_\lambda^*\Delta^{1/2} + \lambda\Delta^{1/2}x_\lambda^*\Delta^{-1/2},$$

which is valid on $\mathcal{D}\left((\Delta+1)\Delta^{-1/2}\right) = \mathcal{D}\left(\Delta^{-1/2}\right) \cap \mathcal{D}\left(\Delta^{1/2}\right)$. □

Definition 4.1.2.9. *We define, for notational ease, the operators $\mathfrak{D}^{-1/2}$ and $\mathfrak{D}^{1/2}$ as*

$$\begin{aligned}
\mathfrak{D}^{-1/2}: \mathcal{B}(\mathcal{H}_{\mathcal{A}}) \longrightarrow \mathcal{L}(\mathcal{H}_{\mathcal{A}}), &\qquad \mathfrak{D}^{-1/2}(B) = \Delta^{-1/2}B\Delta^{1/2}, \\
\mathfrak{D}^{1/2}: \mathcal{B}(\mathcal{H}_{\mathcal{A}}) \longrightarrow \mathcal{L}(\mathcal{H}_{\mathcal{A}}), &\qquad \mathfrak{D}^{1/2}(B) = \Delta^{1/2}B\Delta^{-1/2}.
\end{aligned}$$

Here $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$ is the collection of bounded linear operators and $\mathcal{L}(\mathcal{H}_{\mathcal{A}})$ is the collection of all linear operators on $\mathcal{H}_{\mathcal{A}}$.

In this notation we have the relation $Jx'J = \left(\mathfrak{D}^{-1/2} + \lambda\mathfrak{D}^{1/2}\right)(x_\lambda)$. Note that $\mathfrak{D}^{1/2}$ and $\mathfrak{D}^{-1/2}$ invert each other. We aim to show that $\left(\mathfrak{D}^{-1/2} + \lambda\mathfrak{D}^{1/2}\right)(x_\lambda) = Jx'J$ defines an invertible relation. In order to do so we will need to study the collection of operators $\left\{\Delta^{it}\ ;\ t \in \mathbb{R}\right\}$.

Proposition 4.1.2.10. *The collection $\left\{\Delta^{it}\ ;\ t \in \mathbb{R}\right\}$ defines a strongly continuous one parameter group of unitaries in $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$. This means that $\left\{\Delta^{it}\ ;\ t \in \mathbb{R}\right\}$ satisfies the following conditions*

- *The map $t \longrightarrow \Delta^{it}$ defines a group homomorphism from the additive group $\mathbb{R}$ into the unitary group of $\mathcal{B}(\mathcal{H}_{\mathcal{A}})$.*
- *For all $h \in \mathcal{H}_{\mathcal{A}}$ it holds that $\lim_{r\longrightarrow t}\Delta^{ir}(h) = \Delta^{it}h$.*

Proof:

By the spectral theorem we have that

$$\Delta^{it} = \int_{\sigma(\Delta)} e^{it\log(x)}\ dE_\Delta(x),$$

here E_Δ is the spectral measure associated with Δ. We use that fact that Δ is positive to conclude that

$$\begin{aligned}
\log(\Delta)^* &= \int_{\sigma(\Delta)} \overline{\log(x)}\ dE_\Delta(x) \\
&= \int_{\sigma(\Delta)} \log(x)\ dE_\Delta(x) = \log(\Delta),
\end{aligned}$$

so $\log(\Delta)$ is selfadjoint. If A is any selfdjoint operator then for any $t \in \mathbb{R}$ we have

$$\begin{aligned}\left(e^{itA}\right)^* &= \int_{\sigma(A)} \overline{e^{itx}}\ dE_A(x)\\ &= \int_{\sigma(A)} e^{-itx}\ dE_A(x)\\ &= \left(e^{itA}\right)^{-1}.\end{aligned}$$

Thus A selfadjoint $\implies$ e^{itA} is unitary. We conclude that Δ^{it} is unitary for all $t \in \mathbb{R}$. That $t \longrightarrow \Delta^{it}$ defines a group homomorphism follows from the fact that integration against a spectral measure is a multiplicative operation, for details regarding integration against spectral measure we refer to the appendix. It follows that every selfadjoint operator defines a group homomorphism from the additive group $\mathbb{R}$ into the unitary group of $\mathcal{B}(\mathcal{H}_\mathcal{A})$. For $h \in \mathcal{H}_\mathcal{A}$ consider the limit $\lim_{r\longrightarrow t} \Delta^{ir} h$. Using that $t \longrightarrow \Delta^{it}$ defines a group homomorphism we find that

$$\lim_{r\longrightarrow t} \Delta^{it} h = \lim_{t\longrightarrow r} \Delta^{i(r-t+t)} h = \Delta^{it} \lim_{r\longrightarrow 0} \Delta^{ir}(h).$$

Using this we conclude that in order to show that Δ^{it} defines a strongly continuous one parameter group of unitaries it suffices to show that Δ^{it} converges strongly to the identity operator as t goes to zero. For $h \in \mathcal{H}_\mathcal{A}$ we find

$$\lim_{t\longrightarrow 0} \left\| \left(\Delta^{it} - 1\right) h \right\| = \lim_{t\longrightarrow 0} \int_{\sigma(\Delta)} \left| e^{it\log(x)} - 1 \right|\ dE^{\Delta}_{h,h}.$$

Since $\left|e^{it\log(x)} - 1\right| \leq 2$ we can apply the Lebesgue dominated convergence theorem to bring the limit under the integral, we conclude

$$\lim_{t\longrightarrow 0} \left\| \left(\Delta^{it} - 1\right) h \right\| = \lim_{t\longrightarrow 0} \int_{\sigma(\Delta)} \left| e^{it\log(x)} - 1 \right|\ dE^{\Delta}_{h,h} = \int_{\sigma(\Delta)} \lim_{t\longrightarrow 0} \left| e^{it\log(x)} - 1 \right|\ dE^{\Delta}_{h,h} = 0.$$

It follows that $\Delta^{it} \longrightarrow 1$ strongly as $t \longrightarrow 0$. □

The collection $\{\Delta^{it}\ ;\ t \in \mathbb{R}\}$ gives rise to a collection of automorphisms of $\mathcal{B}(\mathcal{H}_\mathcal{A})$.

Definition 4.1.2.11. *For $A \in \mathcal{B}(\mathcal{H}_\mathcal{A})$ we define*

$$\mathfrak{D}^{it}(A) = \Delta^{it} A \Delta^{-it}.$$

We call $\mathfrak{D}^{it}$ the modular automorphism group associated to $\mathcal{A}$.

The collection $\{\mathfrak{D}^{it}\ ;\ t \in \mathbb{R}\}$ will play the role of the one parameter group of automorphisms considered in the Tomita-Takesaki theorem 4.1.0.7.

The next ingredient we need to invert the relation $Jx'J = \left(\mathfrak{D}^{-1/2} + \lambda \mathfrak{D}^{1/2}\right) x_\lambda$ is the integral

$$\int_{-\infty}^{\infty} \frac{b^{it}}{e^{\pi t} + e^{-\pi t}}\ dt \qquad (b > 0).$$

We will show that for $b > 0$ it holds that

$$\int_{-\infty}^{\infty} \frac{b^{it}}{e^{\pi t} + e^{-\pi t}}\ dt = \frac{1}{b^{1/2} + b^{-1/2}}.$$

Using the functional calculus we will then apply this integral to the operation $\mathfrak{D}^{i/2}$ in order to invert the operation $\left(\mathfrak{D}^{-1/2} + \lambda \mathfrak{D}^{1/2}\right)$. A direct approach to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{b^{it}}{e^{\pi t} + e^{-\pi t}}\ dt \qquad (b > 0),$$

using the residue formula fails because the function $\frac{b^{iz}}{e^{\pi z} + e^{-\pi z}}$ has poles at the set $\{(n + 1/2)i\ ;\ n \in \mathbb{Z}\}$. In particular it has infinitely many poles and there is no bounded curve enclosing them all. We will use a different approach. The trick is to consider a different function (also with infinitely many poles) and evaluate its integral over a curve enclosing just one of its poles.

Proposition 4.1.2.12. *For any positive $b \in \mathbb{R}$ the following holds*

$$\int_{-\infty}^{\infty} \frac{b^{it}}{e^{\pi t}+e^{-\pi t}}\ dt = \frac{1}{b^{1/2}+b^{-1/2}}.$$

Proof:

For $b>0$ consider the function

$$f(z) := \frac{b^{iz}}{e^{\pi z}-e^{-\pi z}}.$$

Suppose that $e^{\pi z}-e^{-\pi z}=0$, let $z=a+id$ then we find that $0=e^{\pi a}e^{i\pi d}-e^{-\pi a}e^{-i\pi d}=0$. It follows that $\left|e^{\pi a}e^{i\pi d}\right| = \left|e^{-\pi a}e^{-\pi d}\right|$. If this is to hold then $a=0$. What remains is to solve $e^{i\pi d}-e^{-i\pi d}=0$. We find $0=e^{i\pi d}-e^{-i\pi d}=\cos(\pi d)+i\sin(\pi d)-(\cos(\pi d)-i\sin(\pi d))=2i\sin(\pi d)$, so that $d\in\mathbb{Z}$. We conclude that the set $\{\pi n i\ ;\ n\in\mathbb{Z}\}$ form the poles of f, furthermore all the poles are simple. The residue of f at 0 is given by

$$\mathrm{Res}_0(f) = \lim_{z\longrightarrow 0} zf(z) = \lim_{t\longrightarrow 0}\frac{it}{2i\sin(\pi t)} = \frac{1}{2\pi}.$$

For $l>0$ consider the rectangle R in $\mathbb{C}$ given by the vertices $\{(l,i/2),(l,-i/2),(-l,-i/2),(-l,i/2)\}$ oriented counterclockwise. The only pole of f inside R is the pole it has at zero, by the residue formula we have

$$\int_R f(z)\ dz = i.$$

We decompose the rectangle in its 4 parts and consider the contribution of each part separately. We have

$$\int_R f = \left(\int_{(-l,i/2)}^{(-l,-i/2)} f\right)+\left(\int_{(-l,-i/2)}^{(l,-i/2)} f\right)+\left(\int_{(l,-i/2)}^{(l,i/2)} f\right)+\left(\int_{(l,i/2)}^{(-l,i/2)} f\right).$$

First we consider $\int_{(-l,i/2)}^{(-l,-i/2)} f$, we find

$$\begin{aligned}
\left|\int_{(-l,i/2)}^{(-l,-i/2)} f\ dt\right| = \left|\int_{1/2}^{-1/2} f(-l+it)\ dt\right| &= \left|\int_{1/2}^{-1/2} \frac{b^{i(-l+it)}}{e^{\pi(-l+it)}-e^{-\pi(-l+it)}}\ dt\right| \\
&\leq \int_{-1/2}^{1/2}\left|\frac{b^{-t}}{e^{-\pi l}e^{i\pi t}-e^{\pi l}e^{-i\pi t}}\right|\ dt \\
&\leq \int_{-1/2}^{1/2}\left|\frac{1}{e^{-\pi l}e^{i\pi t}}\right|\cdot\left|\frac{b^{-t}}{1-e^{2\pi l}e^{-2i\pi t}}\right|\ dt \\
&= \int_{-1/2}^{1/2}\left|\frac{e^{\pi l}b^{-t}}{1-e^{2\pi l}e^{-2i\pi t}}\right|\ dt \\
&\leq \int_{-1/2}^{1/2}\left|\frac{e^{\pi l}b^{-t}}{e^{2\pi l}-1}\right|\ dt.
\end{aligned}$$

Note that this expression tends to 0 as l tends to ∞. Via a similar calculation we conclude that the contribution of the integral

$$\int_{(l,-i/2)}^{(l,i/2)} f$$

also tends to 0 as l tends to ∞. We find that

$$\begin{aligned}
i = \int_R f = \lim_{l\longrightarrow\infty}\int_R f &= \lim_{l\longrightarrow\infty}\left[\left(\int_{(-l,-i/2)}^{(l,-i/2)} f\right)+\left(\int_{(l,i/2)}^{(-l,i/2)} f\right)\right] \\
&= \int_{-\infty}^{\infty} f(t-i/2)\ dt + \int_{\infty}^{-\infty} f(t+i/2)\ dt \\
&= \int_{-\infty}^{\infty} f(t-i/2)-f(t+i/2)\ dt.
\end{aligned}$$

using that $e^{\frac{\pm\pi i}{2}} = \pm i$ we can simplify the expression above as follows

$$\begin{aligned}
i &= \int_{-\infty}^{\infty} f(t-i/2) - f(t+i/2)\ dt \\
&= \int_{-\infty}^{\infty} \left[\left(\frac{b^{\frac{1}{2}} b^{it}}{e^{\pi t} e^{\frac{-\pi i}{2}} - e^{-\pi t} e^{\frac{\pi i}{2}}} \right) - \left(\frac{b^{\frac{-1}{2}} b^{it}}{e^{\pi t} e^{\frac{\pi i}{2}} - e^{-\pi t} e^{\frac{-\pi i}{2}}} \right) \right] dt \\
&= \int_{-\infty}^{\infty} \left[\left(\frac{b^{\frac{1}{2}} b^{it}}{-ie^{\pi t} - ie^{-\pi t}} \right) - \left(\frac{b^{\frac{-1}{2}} b^{it}}{ie^{\pi t} + ie^{-\pi t}} \right) \right] dt \\
&= \frac{-1}{i} \int_{-\infty}^{\infty} \left[\left(\frac{b^{\frac{1}{2}} b^{it}}{e^{\pi t} + e^{-\pi t}} \right) + \left(\frac{b^{\frac{-1}{2}} b^{it}}{e^{\pi t} + e^{-\pi t}} \right) \right] dt \\
&= i \left(b^{\frac{1}{2}} + b^{\frac{-1}{2}} \right) \int_{-\infty}^{\infty} \frac{b^{it}}{e^{\pi t} + e^{-\pi t}}\ dt.
\end{aligned}$$

we conclude that

$$\int_{-\infty}^{\infty} \frac{b^{it}}{e^{\pi t} + e^{-\pi t}}\ dt = \frac{1}{b^{\frac{1}{2}} + b^{\frac{-1}{2}}},$$

as desired. □

We will use this integral in combination with the unitary conjugation $\mathfrak{D}^{it}$ to construct the inverse of the map $(\mathfrak{D}^{-1/2} + \lambda \mathfrak{D}^{1/2})$.

We return now to $\left(\mathfrak{D}^{-1/2} + \lambda \mathfrak{D}^{1/2}\right)$, we aim to invert this map. Note that for $\lambda > 0$ the map

$$t \longrightarrow \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}},$$

defines an element of $L^1(\mathbb{R})$ (with the Lebesgue measure). For fixed $A \in \mathcal{B}(\mathcal{H}_{\mathcal{A}})$ we consider the collection $\left\{\mathfrak{D}^{it}(A)\ ;\ t \in \mathbb{R}\right\}$. Since $\mathfrak{D}^{it}(A) = \Delta^{it} A \Delta^{-it}$, with Δ^{it} being unitary, we conclude that $\left\|\mathfrak{D}^{it}(A)\right\| \leq \|A\|$, so the collection is uniformly bounded. For $h, v \in \mathcal{H}_{\mathcal{A}}$ consider the map $t \longrightarrow \left\langle \Delta^{it} A \Delta^{-it} h, v \right\rangle$. We find that

$$\begin{aligned}
\lim_{t \longrightarrow x} \left\langle \Delta^{it} A \Delta^{-it} h, v \right\rangle &= \lim_{t \longrightarrow x} \left\langle \Delta^{i(t-x)} \Delta^{ix} A \Delta^{-ix} \Delta^{-i(t-x)} h, v \right\rangle \\
&= \left\langle \Delta^{ix} A \Delta^{-ix} h, v \right\rangle,
\end{aligned}$$

hence the map $t \longrightarrow \left\langle \mathfrak{D}^{it}(A) h, v \right\rangle$ is continuous and in particular it is measurable.

Let μ denote the Lebesgue measure on $\mathbb{R}$ and let $\lambda > 0$, by our previous arguments the integral

$$\mathfrak{J}_\lambda(A) := \lambda^{-1/2} \int_{-\infty}^{\infty} \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \mathfrak{D}^{it}(A)\ d\mu,$$

is well defined. Note that by the properties of the integral it follows that on $\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2})$ the identity

$$\mathfrak{J}_\lambda \circ (\mathfrak{D}^{-1/2} + \lambda \mathfrak{D}^{1/2})(B) = (\mathfrak{D}^{-1/2} + \lambda \mathfrak{D}^{1/2}) \circ \mathfrak{J}_\lambda(B),$$

holds for all $B \in \mathcal{B}(\mathcal{H}_{\mathcal{A}})$. For a short description concerning integrals of operators we refer to the appendix.

Proposition 4.1.2.13. *Let $\mathfrak{J}_\lambda$ be defined as above. Then the following identity holds*

$$\mathfrak{J}_\lambda(J x' J) = \mathfrak{J}_\lambda \left[\left(\mathfrak{D}^{-1/2} + \lambda \mathfrak{D}^{1/2} \right) x_\lambda^* \right] = x_\lambda^*,$$

on $\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2})$.

Proof:

Pick $h, v \in \mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2})$ and consider

$$\left\langle \left[\left(\mathfrak{D}^{-1/2} \mathfrak{J}_\lambda(B) \right) + \lambda \left(\mathfrak{D}^{1/2} \mathfrak{J}_\lambda(B) \right) \right] h, v \right\rangle.$$

Let E_Δ denote the spectral measure associated to Δ, we find the following identities

$$\begin{aligned}
&\left\langle \left[\left(\mathfrak{D}^{-1/2} \mathfrak{J}_\lambda(B) \right) + \lambda \left(\mathfrak{D}^{1/2} \mathfrak{J}_\lambda(B) \right) \right] h, v \right\rangle \\
&= \left\langle \left(\int_{-\infty}^{\infty} \lambda^{-1/2} \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \left[\Delta^{-1/2+it} B \Delta^{1/2-it} + \lambda \Delta^{1/2+it} B \Delta^{-1/2-it} \right] \, dt \right) h, v \right\rangle \\
&= \int_{-\infty}^{\infty} \lambda^{-1/2} \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \left\langle \left[\Delta^{-1/2+it} B \Delta^{1/2-it} + \lambda \Delta^{1/2+it} B \Delta^{-1/2-it} \right] h, v \right\rangle \, dt.
\end{aligned}$$

We will consider the terms $\left\langle \Delta^{-1/2+it} B \Delta^{1/2-it} h, v \right\rangle$ and $\lambda \left\langle \Delta^{1/2+it} B \Delta^{-(1/2+it)} h, v \right\rangle$ separately. For $\left\langle \Delta^{-1/2+it} B \Delta^{1/2-it} h, v \right\rangle$ we find

$$\begin{aligned}
\left\langle \Delta^{-1/2+it} B \Delta^{1/2-it} h, v \right\rangle &= \left\langle B \Delta^{1/2-it} h, \Delta^{-1/2-it} v \right\rangle \\
&= \left\langle B \left(\int_{\sigma(\Delta)} x^{1/2-it} \, dE_\Delta(x) \right) h, \left(\int_{\sigma(\Delta)} y^{-1/2-it} \, dE_\Delta(y) \right) v \right\rangle \\
&= \left\langle \left(\int_{\sigma(\Delta)} x^{1/2-it} \, dBE_\Delta(x) \right) h, \left(\int_{\sigma(\Delta)} y^{-1/2-it} \, dE_\Delta(y) \right) v \right\rangle \\
&= \int_{\sigma(\Delta)} x^{1/2-it} \left\langle BE_\Delta(x) h, \left(\int_{\sigma(\Delta)} y^{-1/2-it} dE_\Delta(y) \right) v \right\rangle \, dx \\
&= \int_{\sigma(\Delta)} \int_{\sigma(\Delta)} x^{1/2-it} y^{-1/2+it} \left\langle BE_\Delta(x) h, E_\Delta(y) v \right\rangle \, dy dx.
\end{aligned}$$

Similarly we find for $\lambda \left\langle \Delta^{1/2+it} B \Delta^{-(1/2+it)} h, v \right\rangle$ the expression

$$\lambda \left\langle \Delta^{1/2+it} B \Delta^{-(1/2+it)} h, v \right\rangle = \lambda \int_{\sigma(\Delta)} \int_{\sigma(\Delta)} x^{-1/2-it} y^{1/2+it} \left\langle BE_\Delta(x) h, E_\Delta(y) v \right\rangle \, dy dx.$$

We combine now these expressions to find that

$$\begin{aligned}
&\left\langle \left[\left(\mathfrak{D}^{-1/2} \mathfrak{J}_\lambda(B) \right) + \lambda \left(\mathfrak{D}^{1/2} \mathfrak{J}_\lambda(B) \right) \right] h, v \right\rangle \\
&= \int_{-\infty}^{\infty} \lambda^{-1/2} \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \left\langle \left[\Delta^{-1/2+it} B \Delta^{1/2-it} + \lambda \Delta^{1/2+it} B \Delta^{-1/2-it} \right] h, v \right\rangle \, dt \\
&= \int_{-\infty}^{\infty} \int_{\sigma(\Delta)} \int_{\sigma(\Delta)} \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \left[\lambda^{-1/2} x^{1/2-it} y^{-1/2+it} + \lambda^{1/2} x^{-1/2-it} y^{1/2+it} \right] \left\langle BE_\Delta(x) h, E_\Delta(y) v \right\rangle \, dy \cdot dx \cdot dt \\
&= \int_{-\infty}^{\infty} \int_{\sigma(\Delta)} \int_{\sigma(\Delta)} \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \left[\left(\frac{y}{x} \right)^{it} \left[\left(\frac{x}{\lambda y} \right)^{1/2} + \left(\frac{x}{\lambda y} \right)^{-1/2} \right] \right] \left\langle BE_\Delta(x) h, E_\Delta(y) v \right\rangle \, dy \cdot dx \cdot dt.
\end{aligned}$$

Fubini's theorem allows us to interchange the order of integration. We first integrate against t to find

$$\begin{aligned}
&\int_{-\infty}^{\infty} \int_{\sigma(\Delta)} \int_{\sigma(\Delta)} \frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \left[\left(\frac{y}{x} \right)^{it} \left[\left(\frac{x}{\lambda y} \right)^{1/2} + \left(\frac{x}{\lambda y} \right)^{-1/2} \right] \right] \left\langle BE_\Delta(x) h, E_\Delta(y) v \right\rangle \, dy \cdot dx \cdot dt. \\
&= \int_{\sigma(\Delta)} \int_{\sigma(\Delta)} \left[\int_{-\infty}^{\infty} \left(\frac{\lambda^{it}}{e^{\pi t} + e^{-\pi t}} \left(\frac{y}{x} \right)^{it} \right) \right] \left[\left(\frac{x}{\lambda y} \right)^{1/2} + \left(\frac{x}{\lambda y} \right)^{-1/2} \right] \left\langle BE_\Delta(x) h, E_\Delta(y) v \right\rangle \, dt \cdot dy \cdot dx.
\end{aligned}$$

Note that

$$\left[\left(\frac{x}{\lambda y} \right)^{1/2} + \left(\frac{x}{\lambda y} \right)^{-1/2} \right] = \left[\left(\frac{\lambda y}{x} \right)^{1/2} + \left(\frac{\lambda y}{x} \right)^{-1/2} \right].$$

Using this together with the identity derived in proposition 4.1.2.12:

$$\int_{-\infty}^{\infty} \frac{\left(\lambda \frac{y}{x}\right)^{it}}{e^{\pi t}+e^{-\pi t}}\ dt = \frac{1}{\left(\frac{\lambda y}{x}\right)^{1/2} + \left(\frac{\lambda y}{x}\right)^{-1/2}},$$

we conclude that

$$\begin{aligned}\left\langle \left[\left(\mathfrak{D}^{-1/2}\mathfrak{J}_\lambda(B)\right) + \lambda\left(\mathfrak{D}^{1/2}\mathfrak{J}_\lambda(B)\right)\right] h, v\right\rangle &= \int_{\sigma(\Delta)}\int_{\sigma(\Delta)} \langle BE_\Delta(x)h, E_\Delta(y)v\rangle\ dy\cdot dx. \\ &= \left\langle B\left(\int_{\sigma(\Delta)} 1\ dE_\Delta\right)h, \left(\int_{\sigma(\Delta)} 1\ dE_\Delta\right)v\right\rangle \\ &= \langle Bh, v\rangle .\end{aligned}$$

Let $x' \in \pi(\mathcal{A})'$ and $\lambda > 0$. By proposition 4.1.2.8 we have on $\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta)^{-1/2}$, the identity:

$$Jx'J = \left[\mathfrak{D}^{-1/2} + \lambda\mathfrak{D}^{1/2}\right] x_\lambda^*.$$

It follows that on $\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2})$ the identity: $x_\lambda^* = \mathfrak{J}_\lambda(Jx'J)$ holds. □

We will now extend this identity to all of $\mathcal{B}(\mathcal{H})$. Since $\mathfrak{J}_\lambda : \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H})$ it suffices to show that $\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2})$ is dense in $\mathcal{H}$. By definition $\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2}) = \mathcal{D}(\Delta^{1/2} + \Delta^{-1/2})$. Using proposition 6.1.1.4, we find that

$$\begin{aligned}\left[\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2})\right]^\perp = \mathcal{D}\left(\Delta^{-1/2}(\Delta+1)\right)^\perp &= \operatorname{ran}\left((\Delta+1)^{-1}\Delta^{1/2}\right)^\perp \\ &= \ker\left(\Delta^{1/2}(\Delta+1)^{-1}\right) \\ &= \{0\},\end{aligned}$$

here the last equality follows from the fact that $\Delta^{1/2}(\Delta+1)^{-1}$ is invertible. We conclude that $\mathcal{D}(\Delta^{1/2}) \cap \mathcal{D}(\Delta^{-1/2})$ is dense in $\mathcal{H}_\mathcal{A}$.

Using all these considerations we are now able to prove the Tomita-Takesaki theorem.

Proof of theorem 4.1.0.7**:**

Let $\mathcal{H} = \mathcal{H}_\mathcal{A}$ and let J and Δ^{it} as above. Recall that for $x' \in \pi(\mathcal{A})'$ we have that $x_\lambda \in \pi(\mathcal{A})$, consider $x', y' \in \pi(\mathcal{A})'$. We find for $h, v \in \mathcal{H}_\mathcal{A}$ that

$$\langle y'x_\lambda^* h, v\rangle = \langle y'\mathfrak{J}_\lambda(Jx'J)h, v\rangle = \langle \mathfrak{J}_\lambda(Jx'J)y'h, v\rangle .$$

By the properties of the integral we find the following formulas for $\langle y'\mathfrak{J}_\lambda(Jx'J)h, v\rangle$ and $\langle \mathfrak{J}_\lambda(Jx'J)y'h, v\rangle$:

$$\begin{aligned}\langle y'\mathfrak{J}_\lambda(Jx'J)h, v\rangle &= \int_{-\infty}^{\infty} \frac{\lambda^{it}}{e^{\pi t}+e^{-pit}} \left\langle y'\Delta^{it}Jx'J\Delta^{-it}h, v\right\rangle, \\ \langle \mathfrak{J}_\lambda(Jx'J)y'h, v\rangle &= \int_{-\infty}^{\infty} \frac{\lambda^{it}}{e^{\pi t}+e^{-pit}} \left\langle \Delta^{it}Jx'J\Delta^{-it}y'h, v\right\rangle.\end{aligned}$$

Subtracting these terms and using that $\lambda > 0$ to write $\lambda^{it} = e^{i\xi t}$ with $\xi = \log\lambda$, we find

$$\int_{-\infty}^{\infty} \frac{e^{i\xi t}}{e^{\pi t}+e^{-\pi t}} \left\langle \left[y'\Delta^{it}Jx'J\Delta^{-it} - \Delta^{it}Jx'J\Delta^{-it}y'\right] h, v\right\rangle\ dt = 0.$$

Define the function $g : \mathbb{R} \longrightarrow \mathbb{C}$ as

$$g_{(h,v)}(t) := \frac{\left\langle \left[y'\Delta^{it}Jx'J\Delta^{-it} - \Delta^{it}Jx'J\Delta^{-it}y'\right] h, v\right\rangle}{e^{\pi t}+e^{-\pi t}},$$

if we denote by $\mathcal{F}$ the Fourier transform then we conclude that for all ξ it holds that

$$\mathcal{F}(g_{(h,v)})(\xi) = \int_{-\infty}^{\infty} \frac{e^{i\xi t}}{e^{\pi t} + e^{-\pi t}} \left\langle \left[y'\Delta^{it} Jx'J\Delta^{-it} - \Delta^{it} Jx'J\Delta^{-it} y' \right] h, v \right\rangle \, dt = 0.$$

Since the Fourier transform is invertible we conclude that $g(t) = 0$ for all $t \in \mathbb{R}$. This implies that $\Delta^{it} Jx'J\Delta^{it} \in \pi(\mathcal{A})$ for all $x' \in \pi(\mathcal{A})'$ and $t \in \mathbb{R}$. In particular for $t = 0$ we conclude that

$$J\pi(\mathcal{A})'J \subset \pi(\mathcal{A}).$$

Now suppose that we can show that

$$J\pi(\mathcal{A})J \subset \pi(\mathcal{A})',$$

we then find that

$$\pi(\mathcal{A})' = JJ\pi(\mathcal{A})'JJ \subset J\pi(\mathcal{A})J \subset \pi(\mathcal{A})'.$$

It then follows that

$$J\pi(\mathcal{A})J = \pi(\mathcal{A})' \qquad \text{and} \qquad J\pi(\mathcal{A})'J = \pi(\mathcal{A}).$$

Claim:

$$J\pi(\mathcal{A})'J \subset \pi(\mathcal{A}) \Longrightarrow J\pi(\mathcal{A})J \subset \pi(\mathcal{A})'.$$

Proof:

Let $a, b \in \pi(\mathcal{A})$ and $x \in \pi(\mathcal{A})'$, we aim to show that $aJb\mathfrak{h} = JbJa\mathfrak{h}$ as vectors in $\mathcal{H}_{\mathcal{A}}$. Using that we can show that $JaJb = bJaJ$ as operators. For $a, b \in \pi(\mathcal{A})$ and $x \in \pi(\mathcal{A})'$ we find that

$$\begin{aligned} \langle aJb\mathfrak{h}, x\mathfrak{h} \rangle &= \langle x^* Jb\mathfrak{h}, s(a\mathfrak{h}) \rangle \\ &= \langle JJx^* Jb\mathfrak{h}, s(a\mathfrak{h}) \rangle \\ &= \left\langle \Delta^{1/2}(a\mathfrak{h}), Jx^* Jb\mathfrak{h} \right\rangle. \end{aligned}$$

Since $Jx^*J \in \pi(\mathcal{A})$ we have that $Jx^*Jb\mathfrak{h} = s(b^*JxJ\mathfrak{h})$, using this together with the fact that $r\Delta^{1/2} = J$ we find that

$$\begin{aligned} \left\langle \Delta^{1/2}(a\mathfrak{h}), Jx^* Jb\mathfrak{h} \right\rangle &= \left\langle \Delta^{1/2}(a\mathfrak{h}), s(b^* JxJ\mathfrak{h}) \right\rangle \\ &= \left\langle b^* JxJ\mathfrak{h}, r\Delta^{1/2}(a\mathfrak{h}) \right\rangle \\ &= \langle b^* JxJ\mathfrak{h}, J(a\mathfrak{h}) \rangle \\ &= \langle Jx\mathfrak{h}, bJa\mathfrak{h} \rangle \\ \langle aJb\mathfrak{h}, x\mathfrak{h} \rangle &= \langle JbJa\mathfrak{h}, x\mathfrak{h} \rangle. \end{aligned}$$

Using the density of $\pi(\mathcal{A})'\mathfrak{h}$ we conclude that for all $a, b \in \pi(\mathcal{A})$ it holds that $aJb\mathfrak{h} = JbJa\mathfrak{h}$. Consider now $bJaJ$ as an operator, let $c \in \pi(\mathcal{A})'$ we find that

$$\begin{aligned} bJaJ(c\mathfrak{h}) &= (bJ)(aJc\mathfrak{h}) \\ &= (bJ)(JcJa\mathfrak{h}) \\ &= bcJa\mathfrak{h} \\ &= bJaJc\mathfrak{h}. \end{aligned}$$

We use the density of $\pi(\mathcal{A})\mathfrak{h}$ to conclude that JaJ commutes with b for all $a, b \in \pi(\mathcal{A})$, as such $JaJ \in \pi(\mathcal{A})'$. □

We continue with the proof of the Tomita-Takseaki theorem. As described above we can conclude now that

$$J\pi(\mathcal{A})J = \pi(\mathcal{A})' \qquad \text{and} \qquad J\pi(\mathcal{A})'J = \pi(\mathcal{A}).$$

We return to the function

$$g_{(h,v)}(t) := \frac{\left\langle \left[y'\Delta^{it}Jx'J\Delta^{-it} - \Delta^{it}Jx'J\Delta^{-it}y'\right]h, v\right\rangle}{e^{\pi t} + e^{-\pi t}},$$

we concluded that it is zero for all t. Since $Jx'J \in \pi(\mathcal{A})$ and y' commutes with $\Delta^{it}Jx'J\Delta^{-it}$ for all $x', y' \in \pi(\mathcal{A})'$ it must follow that $\Delta^{it}\pi(\mathcal{A})\Delta^{-it} \subset \pi(\mathcal{A})$, for all t. Note that

$$\pi(\mathcal{A}) = \Delta^{-it}\Delta^{it}\pi(\mathcal{A})\Delta^{-it}\Delta^{it} \subset \Delta^{-it}\pi(\mathcal{A})\Delta^{it} \subset \pi(\mathcal{A}),$$

so in fact we have the equality: $\pi(\mathcal{A}) = \Delta^{it}\pi(\mathcal{A})\Delta^{-it}$ for all t. It follows that conjugation by Δ^{it} defines a one parameter group of automorphisms of $\pi(\mathcal{A})$. We will now show that conjugation by Δ^{it} also defines a one parameter group of automorphisms for $\pi(\mathcal{A})'$. Pick $x' \in \pi(\mathcal{A})'$ and let $a, b \in \pi(\mathcal{A})$, consider the identities

$$\begin{aligned}
\left\langle \Delta^{it}x'\Delta^{-it}a\mathfrak{h}, b\mathfrak{h}\right\rangle &= \left\langle x'\Delta^{-it}a\mathfrak{h}, \Delta^{-it}b\Delta^{it}\Delta^{-it}\mathfrak{h}\right\rangle \\
&= \left\langle \Delta^{-it}a\mathfrak{h}, \Delta^{-it}b\Delta^{it}\left(x'\right)^{*}\Delta^{-it}\mathfrak{h}\right\rangle \\
&= \left\langle a\mathfrak{h}, b\Delta^{it}\left(x'\right)^{*}\Delta^{-it}\mathfrak{h}\right\rangle \\
&= \left\langle b^{*}a\mathfrak{h}, \Delta^{it}\left(x'\right)^{*}\Delta^{-it}\mathfrak{h}\right\rangle \\
&= \left\langle s(a^{*}b\mathfrak{h}), \Delta^{it}\left(x'\right)^{*}\Delta^{-it}\mathfrak{h}\right\rangle \\
&= \left\langle \Delta^{it}x'\Delta^{-it}\mathfrak{h}, a^{*}b\mathfrak{h}\right\rangle \\
&= \left\langle a\Delta^{it}x'\Delta^{-it}\mathfrak{h}, b\mathfrak{h}\right\rangle.
\end{aligned}$$

We conclude by density of $\pi(\mathcal{A})\mathfrak{h}$ that $\Delta^{it}x'\Delta^{-it}a\mathfrak{h} = a\Delta^{it}x'\Delta^{-it}\mathfrak{h}$. For $b \in \pi(\mathcal{A})$ consider $\Delta^{it}x'\Delta^{-it}a(b\mathfrak{h})$, we find

$$\begin{aligned}
\Delta^{it}x'\Delta^{-it}a(b\mathfrak{h}) &= \Delta^{it}x'\Delta^{-it}ab\mathfrak{h} \\
&= ab\Delta^{it}x'\Delta^{-it}\mathfrak{h} \\
&= a\Delta^{it}x'\Delta^{-it}b\mathfrak{h}.
\end{aligned}$$

Using the density of $\pi(\mathcal{A})\mathfrak{h}$ once more we conclude that $\Delta^{it}x'\Delta^{-it}$ commutes with $\pi(\mathcal{A})$ for all $x' \in \pi(\mathcal{A})'$, therefor $\Delta^{it}x'\Delta^{-it} \in \pi(\mathcal{A})'$, as desired. □

4.2 The Modular automorphism group $\mathfrak{D}$

In this section we will use the Tomita-Takesaki theorem to further classify type III von Neumann algebras. We will see that type III von Neumann algebras can be classified into types indexed by the interval $[0,1] \subset \mathbb{R}$. While proving the Tomita-Takesaki theorem we used a cyclic and separating vector $\mathfrak{h}$, this vector was derived from a faithful normal state which every separable von Neumann algebra has. However the vector $\mathfrak{h}$ depends on the faithful normal state chosen and so does the one parameter group of automorphisms. This is the case because the faithful normal state gave rise to an inner product on $\mathcal{A}$, changing the inner product leads to a change in the topology it defines, as such, the automorphism group varies with it. We will show that if we consider a different inner product, defined from a faithful normal state, then only the inner automorphisms change and the outer automorphisms (defined below) are stable under this perturbation. We will start by defining inner and outer automorphisms.

Definition 4.2.0.14. *An automorphism α of a von Neumann algebra $\mathcal{A}$ is called inner when there exists a unitary $u_\alpha \in \mathcal{A}$ such that for all $x \in \mathcal{A}$*

$$\alpha(x) = u_\alpha^{*}xu_\alpha.$$

An automorphism is called outer when it is not inner.

We denote by $\mathrm{Aut}(\mathcal{A})$ the group of all automorphisms, by $\mathrm{Inn}(\mathcal{A})$ we denote the group of all inner automorphisms.

Remark
note that the inner automorphism of $\mathcal{A}$ are precisely given by the unitaries in $\mathcal{A}$, namely if u is unitary then $U(x) := u^*xu$ defines an inner automorphism of $\mathcal{A}$. The group of all inner automorphisms defines a normal subgroup of $\mathrm{Aut}(\mathcal{A})$. Pick a unitary u and denote its image in $\mathrm{Inn}(\mathcal{A})$ by U, for any automorphism α we find that

$$\begin{aligned}\alpha \circ U \circ \alpha^{-1}(x) &= \alpha(u^*\alpha^{-1}(x)u) \\ &= \alpha(u^*)x\alpha(u).\end{aligned}$$

Since α is an automorphism we have that $\alpha(u)$ is unitary for each unitary u.

By this last remark we can consider the quotient of $\mathrm{Aut}(\mathcal{A})$ by $\mathrm{Inn}(\mathcal{A})$, we define

$$\mathrm{Out}(\mathcal{A}) := \frac{\mathrm{Aut}(\mathcal{A})}{\mathrm{Inn}(\mathcal{A})}.$$

For a faithful normal state ϕ we consider its GNS Hilbert space $\mathcal{H}_\phi$, recall that $\mathcal{H}_\phi$ is the completion of $\mathcal{A}$ with respect to the inner product $\langle \cdot, \cdot \rangle_\phi$ defined by ϕ. This inner product is given by the formula $\langle x, y \rangle_\phi = \phi(y^*x)$. The Tomita-Takesaki theorem gives us a one parameter group of automorphisms, acting on $\pi_\phi(\mathcal{A})$ and its commutant, constructed from the modular operator Δ_ϕ. From now on we will use the subscript ϕ to stress that Δ_ϕ depends on ϕ. This one parameter group was given by $\mathfrak{D}^{it}_\phi$ defined by $\mathfrak{D}^{it}_\phi(\pi_\phi(x)) = \Delta^{it}_\phi \pi_\phi(x)\Delta^{-it}_\phi$. The question becomes: if given faithful normal states ϕ and ψ how do $\mathfrak{D}^{it}_\phi$ and $\mathfrak{D}^{it}_\psi$ relate? We denote by $\mathfrak{h}_\phi$ the cyclic and separating vector associated to ϕ. We start by investigating how the spaces $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ relate.

Theorem 4.2.0.15. *Suppose that $\mathcal{A}$ is a σ-finite von Neumann algebra and suppose that ϕ and ψ faithful normal states on $\mathcal{A}$, then we can find a unitary $W : \mathcal{H}_\psi \longrightarrow \mathcal{H}_\phi$ such that:*

$$W\pi_\phi W^* = \pi_\psi.$$

Proof:

Denote by $\mathcal{M}_2(\mathbb{C})$ the von Neumann algebra of 2×2 matrices over $\mathbb{C}$ and consider $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$. Every element x in $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ is represented by a matrix,

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}.$$

Given two faithful normal states ϕ and ψ, we define a faithful normal state $(\phi \oplus \psi)$, defined on $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$, by setting

$$(\phi \oplus \psi)(x) = \phi(x_{11}) + \psi(x_{22}).$$

It is easy to see that $(\phi \oplus \psi)$ is indeed a faithful normal state on $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$. We construct the GNS Hilbert space associated to $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ and the faithful normal state $(\phi \oplus \psi)$. We define

$$\mathcal{H}_{(\phi\oplus\psi)} := \overline{\left\{\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C}) \ ; \ \langle \cdot, \cdot \rangle_{(\phi\oplus\psi)}\right\}}$$

With this construction we find a cyclic and separating vector $\mathfrak{h}_{(\phi\oplus\psi)} \in \mathcal{H}_{(\phi\oplus\psi)}$ associated to $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$. We define now matrices e_{11}, e_{12}, e_{21} and e_{22} as

$$e_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \ e_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ e_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ e_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Note that $x \in \mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ can be represented as

$$x = x_{11} \otimes e_{11} + x_{12} \otimes e_{12} + x_{21} \otimes e_{21} + x_{22} \otimes e_{22}.$$

We define closed subspaces $\mathcal{H}_{11}, \mathcal{H}_{12}, \mathcal{H}_{21}, \mathcal{H}_{22}$ of $\mathcal{H}_{(\phi\oplus\psi)}$ by setting

$$\mathcal{H}_{ij} := \overline{\left\{(a \otimes e_{ij})\mathfrak{h}_{(\phi\oplus\psi)} \ ; \ a \in \mathcal{A}\right\}},$$

more concretely:

$$\mathcal{H}_{11} = \overline{\left\{ \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix} \mathfrak{h}_{(\phi\oplus\psi)} \ ; \ a \in \mathcal{A} \right\}}.$$

Note that if $a, b \in \mathcal{A}$ then $\left\langle a \otimes e_{ij}\mathfrak{h}_{(\phi\oplus\psi)}, b \otimes e_{n,m}\mathfrak{h}_{(\phi\oplus\psi)} \right\rangle_{(\phi\oplus\psi)} = 0$ when $(n,m) \neq (i,j)$ (this is because $(\phi \oplus \psi)$ ignores the top right entry and the bottom left entry of the matrix). It follows that the Hilbert spaces $\mathcal{H}_{ij}$ form closed orthogonal subspaces, we find that

$$\mathcal{H}_{(\phi\oplus\psi)} = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{22}.$$

We relate the spaces $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$ (the Hilbert spaces derived from $\mathcal{A}$ by the GNS construction using ϕ and ψ) to the spaces $\mathcal{H}_{ij}$ by defining maps U^1_ϕ and U^2_ϕ as follows:

$$\begin{aligned} U^1_\phi(a\mathfrak{h}_\phi) &:= (a \otimes e_{11})\mathfrak{h}_{(\phi\oplus\psi)}, \\ U^2_\phi(a\mathfrak{h}_\phi) &:= (a \otimes e_{21})\mathfrak{h}_{(\phi\oplus\psi)}. \end{aligned}$$

We see that $U^1_\phi : \mathcal{A}\mathfrak{h}_\phi \longrightarrow \mathcal{H}_{11}$, note furthermore that

$$\begin{aligned} \left\| U^1_\phi(a\mathfrak{h}_\phi) \right\|^2 &= \left\langle U^1_\phi(a\mathfrak{h}_\phi), U^1_\phi(a\mathfrak{h}_\phi) \right\rangle_{(\phi\oplus\psi)} \\ &= (\phi \oplus \psi) \left(\begin{pmatrix} a^*a & 0 \\ 0 & 0 \end{pmatrix} \right) \\ &= \phi(a^*a) = \|a\mathfrak{h}_\phi\|^2, \end{aligned}$$

so that U^1_ϕ is in fact an isometry. We can extend U^1_ϕ to be defined on the whole of $\mathcal{H}_\phi$, we denote this extension also by U^1_ϕ. Obviously the range of U^1_ϕ is dense in $\mathcal{H}_{11}$ thus U^1_ϕ defines an isomorphism from $\mathcal{H}_\phi$ onto $\mathcal{H}_{11}$. Similarly we conclude that U^2_ϕ defines an isomorphism from $\mathcal{H}_\phi$ onto $\mathcal{H}_{21}$. We see that the structure of $\mathcal{H}_\phi$ is present in $\mathcal{H}_{(\phi\oplus\psi)}$.

We define maps U^1_ψ and U^2_ψ, as follows:

$$\begin{aligned} U^1_\psi &: \mathcal{A}\mathfrak{h}_\psi \subset \mathcal{H}_\psi \longrightarrow \mathcal{H}_{12}, \qquad U^1_\psi(a\mathfrak{h}_\psi) := (a \otimes e_{12})\mathfrak{h}_{(\phi\oplus\psi)}, \\ U^2_\psi &: \mathcal{A}\mathfrak{h}_\psi \subset \mathcal{H}_\psi \longrightarrow \mathcal{H}_{22}, \qquad U^2_\psi(a\mathfrak{h}_\psi) := (a \otimes e_{22})\mathfrak{h}_{(\phi\oplus\psi)}. \end{aligned}$$

Via similar arguments we conclude that U^1_ψ can be extended to define an isomorphism from $\mathcal{H}_\psi$ onto $\mathcal{H}_{12}$, also the extension of U^2_ψ defines an isomorphism from $\mathcal{H}_\psi$ onto $\mathcal{H}_{22}$. Consider the map

$$s_0^{(\phi\oplus\psi)} : [\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})]\,\mathfrak{h}_{(\phi\oplus\psi)} \longrightarrow [\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})]\,\mathfrak{h}_{(\phi\oplus\psi)}, \qquad s_0^{(\phi\oplus\psi)}(x\mathfrak{h}_{(\phi\oplus\psi)}) = x^*\mathfrak{h}_{(\phi\oplus\psi)}.$$

By construction we have that $s_0^{(\phi\oplus\psi)}$ is preclosed, we denote its closure by $s_{(\phi\oplus\psi)}$. If given $x \in [\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})]\,\mathfrak{h}_{(\phi\oplus\psi)}$ then $x^*\mathfrak{h}_{(\phi\oplus\psi)}$ satisfies

$$\begin{aligned} x\mathfrak{h}_{(\phi\oplus\psi)} &= [x_{11} \otimes e_{11} + x_{12} \otimes e_{12} + x_{21} \otimes e_{21} + x_{22} \otimes e_{22}]\,\mathfrak{h}_{(\phi\oplus\psi)}, \\ x^*\mathfrak{h}_{(\phi\oplus\psi)} &= [x^*_{11} \otimes e_{11} + x^*_{21} \otimes e_{12} + x^*_{12} \otimes e_{21} + x^*_{22} \otimes e_{22}]\,\mathfrak{h}_{(\phi\oplus\psi)}. \end{aligned}$$

Recall that $s^{(\phi\oplus\psi)} = s^{(\phi\oplus\psi)^{-1}}$ consequently we find the following identities for $s^{(\phi\oplus\psi)}$

$$\begin{aligned} s^{(\phi\oplus\psi)}\left(\mathcal{D}\left(s^{(\phi\oplus\psi)}\right) \cap \mathcal{H}_{11}\right) &= \mathcal{D}\left(s^{(\phi\oplus\psi)}\right) \cap \mathcal{H}_{11} \\ s^{(\phi\oplus\psi)}\left(\mathcal{D}\left(s^{(\phi\oplus\psi)}\right) \cap \mathcal{H}_{12}\right) &= \mathcal{D}\left(s^{(\phi\oplus\psi)}\right) \cap \mathcal{H}_{21} \\ s^{(\phi\oplus\psi)}\left(\mathcal{D}\left(s^{(\phi\oplus\psi)}\right) \cap \mathcal{H}_{21}\right) &= \mathcal{D}\left(s^{(\phi\oplus\psi)}\right) \cap \mathcal{H}_{12} \\ s^{(\phi\oplus\psi)}\left(\mathcal{D}\left(s^{(\phi\oplus\psi)}\right) \cap \mathcal{H}_{22}\right) &= \mathcal{D}\left(s^{(\phi\oplus\psi)}\right) \cap \mathcal{H}_{22}. \end{aligned}$$

For notational ease we denote

$$\begin{aligned}
&S := s^{(\phi\oplus\psi)}, \\
&\mathcal{H}_1 := \mathcal{H}_{11}, \qquad U_1 := U^1_\phi, \\
&\mathcal{H}_2 := \mathcal{H}_{21}, \qquad U_2 := U^2_\phi, \\
&\mathcal{H}_3 := \mathcal{H}_{12}, \qquad U_3 := U^1_\psi, \\
&\mathcal{H}_4 := \mathcal{H}_{22}, \qquad U_4 := U^2_\psi.
\end{aligned}$$

Suppose that we represent S by a matrix, then as a consequence of the equalities above we conclude that the matrix representation of S is of the form

$$S = \begin{pmatrix} S_{11} & 0 & 0 & 0 \\ 0 & 0 & S_{23} & 0 \\ 0 & S_{32} & 0 & 0 \\ 0 & 0 & 0 & S_{44} \end{pmatrix}.$$

For $a \in \mathcal{A}$ consider $(a \otimes e_{11})\mathfrak{h}_{(\phi\oplus\psi)} \in \mathcal{H}_1$, we find

$$\begin{aligned}
S_{11}\left((a \otimes e_{11})\mathfrak{h}_{(\phi\oplus\psi)}\right) &= (a^* \otimes e_{11})\mathfrak{h}_{(\phi\oplus\psi)} \\
&= U_1\left(s_\phi(a\mathfrak{h}_\phi)\right) \\
&= U_1 \circ s_\phi \circ U_1^*\left((a \otimes e_{11})\mathfrak{h}_{(\phi\oplus\psi)}\right).
\end{aligned}$$

Similarly we find that

$$S_{44} = U_4 \circ s_\psi \circ U_4^*.$$

We would like to find similar expressions for S_{23} and S_{32}. We define maps

$$\begin{aligned}
s_{\phi,\psi} &: \mathcal{A}\mathfrak{h}_\phi \subset \mathcal{H}_\phi \longrightarrow \mathcal{A}\mathfrak{h}_\psi \subset \mathcal{H}_\psi, \qquad s_{\phi,\psi}(a\mathfrak{h}_\phi) := a^*\mathfrak{h}_\psi \\
s_{\psi,\phi} &: \mathcal{A}\mathfrak{h}_\psi \subset \mathcal{H}_\psi \longrightarrow \mathcal{A}\mathfrak{h}_\phi \subset \mathcal{H}_\phi, \qquad s_{\psi,\phi}(a\mathfrak{h}_\psi) := a^*\mathfrak{h}_\phi.
\end{aligned}$$

Consider S_{23}, we find

$$\begin{aligned}
S_{23}\left((a \otimes e_{21})\mathfrak{h}_{(\phi\oplus\psi)}\right) &= (a^* \otimes e_{12})\mathfrak{h}_{(\phi\oplus\psi)} \\
&= U_3(a^*\mathfrak{h}_\psi) \\
&= U_3\left(s_{\phi,\psi}(a\mathfrak{h}_\phi)\right) \\
&= U_3 \circ s_{\phi,\psi} \circ U_2^*\left((a \otimes e_{21})\mathfrak{h}_{(\phi\oplus\psi)}\right).
\end{aligned}$$

Via a similar argument we find that

$$S_{32}\left((a \otimes e_{12})\mathfrak{h}_{(\phi\oplus\psi)}\right) = U_2 \circ s_{\psi,\phi} \circ U_3^*\left((a \otimes e_{12})\mathfrak{h}_{(\phi\oplus\psi)}\right).$$

Since S_{32} and S_{23} are closed we conclude that $s_{\phi,\psi}$ and $s_{\psi,\phi}$ are closable, we denote their extensions also by $s_{\phi,\psi}$ and $s_{\psi,\phi}$. Consider now $R = S^*$, we find that R is of the following form:

$$\begin{pmatrix} R_{11} & 0 & 0 & 0 \\ 0 & 0 & R_{23} & 0 \\ 0 & R_{32} & 0 & 0 \\ 0 & 0 & 0 & R_{44} \end{pmatrix},$$

with $R_{11} = S_{11}^*, R_{23} = S_{32}^*, R_{32} = S_{23}^*$ and $R_{44} = S_{44}^*$, consequently, $\Delta_{(\phi\oplus\psi)} = RS$ is of the form:

$$\begin{pmatrix} R_{11} & 0 & 0 & 0 \\ 0 & 0 & R_{23} & 0 \\ 0 & R_{32} & 0 & 0 \\ 0 & 0 & 0 & R_{44} \end{pmatrix} \cdot \begin{pmatrix} S_{11} & 0 & 0 & 0 \\ 0 & 0 & S_{23} & 0 \\ 0 & S_{32} & 0 & 0 \\ 0 & 0 & 0 & S_{44} \end{pmatrix} = \begin{pmatrix} \Delta_{11} & 0 & 0 & 0 \\ 0 & \Delta_{22} & 0 & 0 \\ 0 & 0 & \Delta_{33} & 0 \\ 0 & 0 & 0 & \Delta_{44} \end{pmatrix}.$$

Since $J_{(\phi\oplus\psi)}$ satisfies $J_{(\phi\oplus\psi)}\Delta^{1/2}_{(\phi\oplus\psi)} = S$, we find that $J_{(\phi\oplus\psi)}$ is of the form:

$$J_{(\phi\oplus\psi)} = \begin{pmatrix} J_{11} & 0 & 0 & 0 \\ 0 & 0 & J_{23} & 0 \\ 0 & J_{32} & 0 & 0 \\ 0 & 0 & 0 & J_{44} \end{pmatrix}.$$

Consider the representation $\pi_{(\phi\oplus\psi)} : \mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C}) \longrightarrow \mathcal{B}\left(\mathcal{H}_{(\phi\oplus\psi)}\right)$, we view $\mathcal{H}_{(\phi\oplus\psi)}$ as

$$\mathcal{H}_{(\phi\oplus\psi)} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4.$$

An exercise in matrix manipulation yields that $\pi_{(\phi\oplus\psi)}(x)$ is of the form:

$$\pi_{(\phi\oplus\psi)}(x) = \begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{12} & x_{22} & 0 & 0 \\ 0 & 0 & x_{11} & x_{12} \\ 0 & 0 & x_{21} & x_{22} \end{pmatrix}.$$

Using our isomorphisms U_i $i \in \{1,2,3,4\}$ we conclude that we can express this in terms of π_ϕ and π_ψ, we find

$$\pi_{(\phi\oplus\psi)}(x) = \begin{pmatrix} U_1\pi_\phi(x_{11})U_1^* & U_1\pi_\phi(x_{12})U_2^* & 0 & 0 \\ U_2\pi_\phi(x_{21})U_1^* & U_2\pi_\phi(x_{22})U_2^* & 0 & 0 \\ 0 & 0 & U_3\pi_\psi(x_{11})U_3^* & U_3\pi_\psi(x_{12})U_4^* \\ 0 & 0 & U_4\pi_\psi(x_{21})U_3^* & U_4\pi_\psi(x_{22})U_4^* \end{pmatrix}.$$

We pick

$$a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

and consider $J\pi_{(\phi\oplus\psi)}(a)J$, a straightforward calculation shows that

$$J\pi_{(\phi\oplus\psi)}(a)J = \begin{pmatrix} 0 & 0 & J_{11}U_1U_2^*J_{23} & 0 \\ 0 & 0 & 0 & J_{23}U_3U_4^*J_{44} \\ J_{32}U_2U_1^*J_{11} & 0 & 0 & 0 \\ 0 & J_{44}U_4U_3^*J_{32} & 0 & 0 \end{pmatrix}.$$

Note here that if we write $v_1 := J_{11}U_1U_2^*J_{23}$ and $v_2 := J_{23}U_3U_4^*J_{44}$ then $J\pi_{(\phi\oplus\psi)}(a)J$ reduces to

$$J\pi_{(\phi\oplus\psi)}(a)J = \begin{pmatrix} 0 & 0 & v_1 & 0 \\ 0 & 0 & 0 & v_2 \\ v_1^* & 0 & 0 & 0 \\ 0 & v_2^* & 0 & 0 \end{pmatrix}.$$

Note furthermore that $v_1^* = v_1^{-1}$ and $v_2^* = v_2^{-1}$, so that $J\pi_{(\phi\oplus\psi)}(a)J$ is a selfadjoint unitary that commutes with $\pi_{(\phi\oplus\psi)}\left(\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})\right)$. Using this commutation relation we calculate and compare $\pi_{(\phi\oplus\psi)}(x)\left(J\pi_{(\phi\oplus\psi)}(a)J\right)$ with $\left(J\pi_{(\phi\oplus\psi)}(a)J\right)\pi_{(\phi\oplus\psi)}(x)$, we find that

$$\pi_{(\phi\oplus\psi)}(x)\left(J\pi_{(\phi\oplus\psi)}(a)J\right) = \begin{pmatrix} 0 & 0 & U_1\pi_\phi(x_{11})U_1^*v_1 & U_1\pi_\phi(x_{12})U_2^*v_2 \\ 0 & 0 & U_2\pi_\phi(x_{21})U_1^*v_1 & U_2\pi_\phi(x_{22})U_2^*v_2 \\ U_3\pi_\psi(x_{11})U_3^*v_1^* & U_3\pi_\psi(x_{12})U_4^*v_2^* & 0 & 0 \\ U_4\pi_\psi(x_{21})U_3^*v_1^* & U_4\pi_\psi(x_{22})U_4^*v_2^* & 0 & 0 \end{pmatrix},$$

and

$$\left(J\pi_{(\phi\oplus\psi)}(a)J\right)\pi_{(\phi\oplus\psi)}(x) = \begin{pmatrix} 0 & 0 & v_1U_3\pi_\psi(x_{11})U_3^* & v_1U_3\pi_\psi(x_{12})U_4^* \\ 0 & 0 & v_2U_4\pi_\psi(x_{21})U_3^* & v_2U_4\pi_\psi(x_{22})U_4^* \\ v_1^*U_1\pi_\phi(x_{11})U_1^* & v_1^*U_1\pi_\phi(x_{12})U_2^* & 0 & 0 \\ v_2^*U_2\pi_\phi(x_{21})U_1^* & v_2^*U_2\pi_\phi(x_{22})U_2^* & 0 & 0 \end{pmatrix},$$

and they are equal because of the commutation relation. In particular we see that the following holds

$$U_1\pi_\phi(x_{11})U_1^*v_1 = v_1U_3\pi_\psi(x_{11})U_3^*,$$

as such,

$$\pi_\phi(x_{11}) = U_1^* v_1 U_3 \pi_\psi(x_{11}) U_3^* v_1^* U_1.$$

Set $W = U_1^* v_1 U_3$, we conclude that

$$\pi_\phi = W \pi_\psi W^*.$$

meaning that π_ϕ and π_ψ are unitarily equivalent. □

We will use this to prove the Cocycle Derivative Theorem, this theorem is important because the type classification for type III algebras is a consequence of it.

Theorem 4.2.0.16 (Cocycle Derivative Theorem)**.** *If ϕ and ψ are faithful normal states on $\mathcal{A}$ then we can find a unique SOT continuous one parameter group $\{u_t \ ; \ t \in \mathbb{R}\}$ of unitaries in $\mathcal{A}$ with the following properties:*

1. $u_{s+t} = u_s \Delta_\psi^{is} u_t \Delta_\psi^{-is}$.
2. $\mathfrak{D}_\phi^{it} = u_t \mathfrak{D}_\psi^{it} u_t^*$.

This is not the full statement of the cocycle derivative theorem, we omitted the rest as we will not need it, for a full treatment of the cocycle derivative theorem we refer to [9].

Proof:

Consider $\mathcal{H}_\phi$ the map η defined by

$$\eta : \mathcal{H}_\psi \longrightarrow \mathcal{H}_\phi, \qquad \eta(v) = W(v),$$

it satisfies the following properties:

$$\begin{aligned} \pi_\phi(x)\eta(v) &= \pi_\phi W(v) = W(\pi_\psi(v)) = \eta(\pi_\psi(v)) \qquad x \in \mathcal{A},\ v \in \mathcal{H}_\psi, \\ \psi(b^*a) &= \langle a\mathfrak{h}_\psi, b\mathfrak{h}_\psi \rangle_\psi = \langle W(a\mathfrak{h}_\psi), W(b\mathfrak{h}_\psi) \rangle_\phi = \langle \eta(a\mathfrak{h}_\psi), \eta(b\mathfrak{h}_\psi) \rangle_\phi \qquad a, b \in \mathcal{A}, \\ \mathcal{H}_\phi &= \overline{[\eta(\mathcal{A}\mathfrak{h}_\psi)]}. \end{aligned}$$

Since W is an isometry it follows that η is an isometry. We conclude that there exists a Hilbert space $\mathcal{H}$ and an action of $\mathcal{A}$ on $\mathcal{H}$ given by $\pi(\mathcal{A})$ such that there are isometries

$$\begin{aligned} \xi &: \mathcal{A}\mathfrak{h}_\phi \longrightarrow \mathcal{H}, \\ \eta &: \mathcal{A}\mathfrak{h}_\psi \longrightarrow \mathcal{H}, \end{aligned}$$

with the properties

$$\begin{aligned} \pi(a)(\xi(b\mathfrak{h}_\phi)) &= \xi(\pi_\phi(b)a\mathfrak{h}_\phi) \qquad a \in \mathcal{A}, \\ \pi(a)(\eta(b\mathfrak{h}_\psi)) &= \eta(\pi_\psi(a)b\mathfrak{h}_\psi) \qquad a \in \mathcal{A}, \\ \phi(b^*a) &= \langle \xi(a\mathfrak{h}_\phi), \xi(b\mathfrak{h}_\phi) \rangle \qquad a, b \in \mathcal{A}, \\ \psi(b^*a) &= \langle \eta(a\mathfrak{h}_\psi), \eta(b\mathfrak{h}_\psi) \rangle \qquad a, b \in \mathcal{A}, \\ \mathcal{H} &= \overline{[\xi(\mathcal{A}\mathfrak{h}_\phi)]} = \overline{[\eta(\mathcal{A}\mathfrak{h}_\psi)]}. \end{aligned}$$

For example we can pick $\mathcal{H} = \mathcal{H}_\phi$ and let $\xi = 1$ then $\mathcal{H}$ satisfies the properties above. Fix $\mathcal{H}$ as above, we note that ξ and η extend to isometries from $\mathcal{H}_\psi$ and $\mathcal{H}_\psi$ onto $\mathcal{H}$, note furthermore that for all $x \in \mathcal{A}$ it holds that

$$\begin{aligned} \pi(x) &= \xi \circ \pi_\phi(x) \circ \xi^*, \\ \pi(x) &= \eta \circ \pi_\psi(x) \circ \eta^*. \end{aligned}$$

Recall that the action of $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$ was given by

$$\pi_{(\phi\oplus\psi)}(x) = \begin{pmatrix} U_1\pi_\phi(x_{11})U_1^* & U_1\pi_\phi(x_{12})U_2^* & 0 & 0 \\ U_2\pi_\phi(x_{21})U_1^* & U_2\pi_\phi(x_{22})U_2^* & 0 & 0 \\ 0 & 0 & U_3\pi_\psi(x_{11})U_3^* & U_3\pi_\psi(x_{12})U_4^* \\ 0 & 0 & U_4\pi_\psi(x_{21})U_3^* & U_4\pi_\psi(x_{22})U_4^* \end{pmatrix}.$$

We use now ξ and η to deduce that the action of $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ on $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3 \oplus \mathcal{H}_4$ is given by

$$\pi_{(\phi\oplus\psi)}(x) = \begin{pmatrix} U_1\xi^*\pi(x_{11})\xi U_1^* & U_1\xi^*\pi(x_{12})\xi U_2^* & 0 & 0 \\ U_2\xi^*\pi(x_{21})\xi U_1^* & U_2\xi^*\pi(x_{22})\xi U_2^* & 0 & 0 \\ 0 & 0 & U_3\eta^*\pi(x_{11})\eta U_3^* & U_3\eta^*\pi(x_{12})\eta U_4^* \\ 0 & 0 & U_4\eta^*\pi(x_{21})\eta U_3^* & U_4\eta^*\pi(x_{22})\eta U_4^* \end{pmatrix}.$$

It follows that $\pi_{(\phi\oplus\psi)}$ is of the following form:

$$\pi_{(\phi\oplus\psi)}(x)$$

$$=$$

$$\begin{pmatrix} U_1\xi^* & 0 & 0 & 0 \\ 0 & U_2\xi^* & 0 & 0 \\ 0 & 0 & U_3\eta^* & 0 \\ 0 & 0 & 0 & U_4\eta^* \end{pmatrix} \begin{pmatrix} \pi(x_{11}) & \pi(x_{12}) & 0 & 0 \\ \pi(x_{21}) & \pi(x_{22}) & 0 & 0 \\ 0 & 0 & \pi(x_{11}) & \pi(x_{12}) \\ 0 & 0 & \pi(x_{21}) & \pi(x_{22}) \end{pmatrix} \begin{pmatrix} \xi U_1^* & 0 & 0 & 0 \\ 0 & \xi U_2^* & 0 & 0 \\ 0 & 0 & \eta U_3^* & 0 \\ 0 & 0 & 0 & \eta U_4^* \end{pmatrix}.$$

It follows that the action of $x \in \mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ on $\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}\oplus\mathcal{H}$ is given by $\pi(x)$. So far we have succeeded in showing that we may assume that $\mathcal{A}$ acts on a Hilbert space $\mathcal{H}$ which contains an isometric copy of $\mathcal{H}_\phi$ and $\mathcal{H}_\psi$. The action of $x \in \mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ is then given by

$$x = \begin{pmatrix} x_{11} & x_{12} & 0 & 0 \\ x_{21} & x_{22} & 0 & 0 \\ 0 & 0 & x_{11} & x_{12} \\ 0 & 0 & x_{21} & x_{22} \end{pmatrix}.$$

We will denote Δ associated to $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$ by

$$\Delta = \begin{pmatrix} \Delta_{11} & 0 & 0 & 0 \\ 0 & \Delta_{22} & 0 & 0 \\ 0 & 0 & \Delta_{33} & 0 \\ 0 & 0 & 0 & \Delta_{44} \end{pmatrix}.$$

Since Δ^{it} determines the modular automorphism group of $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ on $\mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H}$, we conclude that $\Delta^{it}x\Delta^{-it} = y$, for some $y \in \mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$. In particular $\Delta^{it}x\Delta^{-it} = y$ is of the form

$$y = \begin{pmatrix} y_{11} & y_{12} & 0 & 0 \\ y_{21} & y_{22} & 0 & 0 \\ 0 & 0 & y_{11} & y_{12} \\ 0 & 0 & y_{21} & y_{22} \end{pmatrix}.$$

From this last expression it follows that

$$\Delta_{11}^{it}x_{11}\Delta_{11}^{-it} = \Delta_{33}^{it}x_{11}\Delta_{33}^{-it}, \quad \Delta_{11}^{it}x_{12}\Delta_{22}^{-it} = \Delta_{33}^{it}x_{12}\Delta_{44}^{-it},$$

$$\Delta_{22}^{it}x_{21}\Delta_{11}^{-it} = \Delta_{44}^{it}x_{21}\Delta_{33}^{-it}, \quad \Delta_{22}^{it}x_{22}\Delta_{22}^{-it} = \Delta_{44}^{it}x_{22}\Delta_{44}^{-it}.$$

For notational easy we define

$$\left[\Delta_{11}^{it}, \Delta_{22}^{it}\right] := \begin{pmatrix} \Delta_{11}^{it} & 0 & 0 & 0 \\ 0 & \Delta_{22}^{it} & 0 & 0 \\ 0 & 0 & \Delta_{11}^{it} & 0 \\ 0 & 0 & 0 & \Delta_{22}^{it} \end{pmatrix}.$$

We conclude that the modular automorphism group of Δ denoted by $\mathfrak{D} := \{\mathfrak{D}^{it} \ ; \ t \in \mathbb{R}\}$ reduces to the following form:

$$\mathfrak{D}^{it}(x) = \left[\Delta_{11}^{it}, \Delta_{22}^{it}\right] \cdot x \cdot \left[\Delta_{11}^{it}, \Delta_{22}^{it}\right]^*.$$

Consider now the isomorphism

$$[\xi, \eta] : \mathcal{H}_\phi \oplus \mathcal{H}_\psi \oplus \mathcal{H}_\phi \oplus \mathcal{H}_\psi \longrightarrow \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H} \oplus \mathcal{H},$$

defined by

$$[\xi,\eta]:=\begin{pmatrix}\xi&0&0&0\\0&\eta&0&0\\0&0&\xi&0\\0&0&0&\eta\end{pmatrix}.$$

We denote the action of $x\in\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ on $\mathcal{H}_\phi\oplus\mathcal{H}_\psi\oplus\mathcal{H}_\phi\oplus\mathcal{H}_\psi$ by $\pi(x)$. The action of $\mathfrak{D}^{it}$ on $\mathcal{H}_\phi\oplus\mathcal{H}_\psi\oplus\mathcal{H}_\phi\oplus\mathcal{H}_\psi$ will be denoted by $\mathfrak{D}^{it}_\pi$. By construction the following identities hold:

$$\begin{aligned}\pi(x)&=[\xi,\eta]^*x[\xi,\eta],\\ \mathfrak{D}^{it}_\pi(\pi(x))&=\pi\left(\mathfrak{D}^{it}(x)\right),\\ &=[\xi,\eta]^*\left[\Delta^{it}_{11},\Delta^{it}_{22}\right]\cdot x\cdot\left[\Delta^{it}_{11},\Delta^{it}_{22}\right]^*[\xi,\eta],\\ &=[\xi,\eta]^*\left[\Delta^{it}_{11},\Delta^{it}_{22}\right]\cdot([\xi,\eta]\cdot[\xi,\eta]^*)\cdot x\,([\xi,\eta]\cdot[\xi,\eta]^*)\cdot\left[\Delta^{it}_{11},\Delta^{it}_{22}\right]^*[\xi,\eta],\\ &=\Big([\xi,\eta]^*\left[\Delta^{it}_{11},\Delta^{it}_{22}\right][\xi,\eta]\Big)\cdot\pi(x)\cdot\Big([\xi,\eta]^*\left[\Delta^{it}_{11},\Delta^{it}_{22}\right]^*[\xi,\eta]\Big).\end{aligned}$$

Note that $\pi(x)$ is of the form

$$\pi(x)=\begin{pmatrix}\pi_\phi(x_{11})&\xi^*x_{12}\eta&0&0\\ \eta^*x_{21}\xi&\pi_\psi(x_{22})&0&0\\0&0&\pi_\phi(x_{11})&\xi^*x_{12}\eta\\0&0&\eta^*x_{21}\xi&\pi_\psi(x_{22})\end{pmatrix}.$$

Note now that $\xi^*\Delta^{it}_{11}\xi=\Delta^{it}_\phi$ and similarly $\eta^*\Delta^{it}_{22}\eta=\Delta^{it}_\psi$. It follows that $\mathfrak{D}^{it}_\pi(\pi(x))$ is of the form

$$\mathfrak{D}^{it}_\pi(\pi(x))=\begin{pmatrix}\Delta^{it}_\phi\pi_\phi(x_{11})\Delta^{-it}_\phi&\Delta^{it}_\phi\xi^*(x_{12})\eta\Delta^{-it}_\psi&0&0\\ \Delta^{it}_\psi\eta^*(x_{21})\xi\Delta^{-it}_\phi&\Delta^{it}_\psi\pi_\psi(x_{22})\Delta^{-it}_\psi&0&0\\0&0&\Delta^{it}_\phi\pi_\phi(x_{11})\Delta^{-it}_\phi&\Delta^{it}_\phi\xi^*(x_{12})\eta\Delta^{-it}_\psi\\0&0&\Delta^{it}_\psi\eta^*(x_{21})\xi\Delta^{-it}_\phi&\Delta^{it}_\psi\pi_\psi(x_{22})\Delta^{-it}_\psi\end{pmatrix}.$$

We conclude that if $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ is represented on $\mathcal{H}_\phi\oplus\mathcal{H}_\psi\oplus\mathcal{H}_\phi\oplus\mathcal{H}_\psi$, then the modular operator Δ_π, and its corresponding one parameter group of unitaries Δ^{it}_π, are of the form:

$$\Delta_\pi=\begin{pmatrix}\Delta_\phi&0&0&0\\0&\Delta_\psi&0&0\\0&0&\Delta_\phi&0\\0&0&0&\Delta_\psi\end{pmatrix},\qquad \Delta^{it}_\pi=\begin{pmatrix}\Delta^{it}_\phi&0&0&0\\0&\Delta^{it}_\psi&0&0\\0&0&\Delta^{it}_\phi&0\\0&0&0&\Delta^{it}_\psi\end{pmatrix}.$$

It follows now that $x\in\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ can be represented on $\mathcal{H}_\phi\oplus\mathcal{H}_\psi\oplus\mathcal{H}_\phi\oplus\mathcal{H}_\psi$ by a matrix

$$\pi(x)=\begin{pmatrix}\pi_\phi(x_{11})&W\pi_\psi(x_{12})\\W^*\pi_\phi(x_{21})&\pi_\psi(x_{22})\end{pmatrix},$$

with $x_{ij}\in\mathcal{A}$ and $W:\mathcal{H}_\psi\longrightarrow\mathcal{H}_\phi$ the unitary equivalence between π_ψ and π_ϕ. Note that for $a\in\mathcal{A}$ we have that $W\pi_\psi(a)$ is just the representation of a as a bounded linear operator from $\mathcal{H}_\psi$ to $\mathcal{H}_\phi$. As such we can determine when a linear operator $B:\mathcal{H}_\psi\longrightarrow\mathcal{H}_\phi$ defines an element of $\mathcal{A}$. The modular automorphism group $\mathfrak{D}^{it}_\pi$ acts on $\pi(x)$ as follows:

$$\mathfrak{D}^{it}_\pi(x)=\begin{pmatrix}\Delta^{it}_\phi\pi_\phi(x_{11})\Delta^{-it}_\phi&\Delta^{it}_\phi W\pi_\psi(x_{12})\Delta^{-it}_\psi\\ \Delta^{it}_\psi W^*\pi_\phi(x_{21})\Delta^{-it}_\phi&\Delta^{it}_\phi\pi_\phi(x_{22})\Delta^{-it}_\phi\end{pmatrix}.$$

Because $\mathfrak{D}^{it}_\pi$ defines an automorphism for all $t\in\mathbb{R}$, we conclude that

$$\mathfrak{D}^{it}_\pi(\pi(x))=\begin{pmatrix}\Delta^{it}_\phi\pi_\phi(x_{11})\Delta^{-it}_\phi&\Delta^{it}_\phi W\pi_\psi(x_{12})\Delta^{-it}_\psi\\ \Delta^{it}_\psi W^*\pi_\phi(x_{21})\Delta^{-it}_\phi&\Delta^{it}_\phi\pi_\phi(x_{22})\Delta^{-it}_\phi\end{pmatrix}=\begin{pmatrix}\pi_\phi(y_{11})&W\pi_\psi(y_{12})\\W^*\pi_\phi(y_{21})&\pi_\psi(y_{22})\end{pmatrix}.$$

For all $t\in\mathbb{R}$ we define an automorphism $\mathfrak{D}^{it}$ of $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ by setting

$$\mathfrak{D}^{it}(x):=\pi^{-1}\left[\mathfrak{D}^{it}_\pi(\pi(x))\right].$$

Note that this is well defined because $\mathfrak{D}_{\pi}^{it}$ is an automorphism of $\pi\left(\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})\right)$. Similarly we identify the action of $\mathfrak{D}_{\phi}^{it}$ on $\pi_{\phi}(\mathcal{A})$ with its action on $\mathcal{A}$ via

$$a \xrightarrow{\pi_{\phi}} \pi_{\phi}(a) \xrightarrow{\mathfrak{D}_{\phi}^{it}} \mathfrak{D}_{\phi}^{it}(a) \xrightarrow{\pi_{\phi}^{-1}} b \in \mathcal{A}.$$

We also identify the action of $\mathfrak{D}_{\psi}^{it}$ on $\pi_{\psi}(\mathcal{A})$ with its action on $\mathcal{A}$ via

$$a \xrightarrow{\pi_{\psi}} \pi_{\psi}(a) \xrightarrow{\mathfrak{D}_{\psi}^{it}} \mathfrak{D}_{\psi}^{it}(a) \xrightarrow{\pi_{\psi}^{-1}} b \in \mathcal{A}.$$

We denote the action of $\mathfrak{D}^{it}$ on $\mathcal{A}\overline{\otimes}\mathcal{M}_2(\mathbb{C})$ by

$$\mathfrak{D}^{it}\left[\begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}\right] = \begin{pmatrix} \mathfrak{D}_{\phi}^{it}(x_{11}) & \mathfrak{D}_{\{\psi,\phi\}}^{it}(x_{12}) \\ \mathfrak{D}_{\{\phi,\psi\}}^{it}(x_{21}) & \mathfrak{D}_{\psi}^{it}(x_{22}) \end{pmatrix}.$$

We define the collection u_t as

$$u_t := \mathfrak{D}^{it}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] = \mathfrak{D}_{\{\psi,\phi\}}^{it}(1).$$

Note that u_t determines an element of $\mathcal{A}$. For $x, y \in \mathcal{A}$ consider $\mathfrak{D}_{(\psi,\phi)}^{it}(xy)$, we find that

$$\begin{aligned}
\mathfrak{D}_{\{\psi,\phi\}}^{it}(xy) &= \mathfrak{D}^{it}\left[\begin{pmatrix} 0 & xy \\ 0 & 0 \end{pmatrix}\right], \\
&= \mathfrak{D}^{it}\left[\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}\right] = \mathfrak{D}_{\phi}^{it}(x)\mathfrak{D}_{\{\psi,\phi\}}^{it}(y), \\
&= \mathfrak{D}^{it}\left[\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}\right] = \mathfrak{D}_{\{\psi,\phi\}}^{it}(x)\mathfrak{D}_{\psi}^{it}(y).
\end{aligned}$$

We will now show *1.* By the identity

$$\mathfrak{D}_{\phi}^{it}(x)\mathfrak{D}_{\{\psi,\phi\}}^{it}(y) = \mathfrak{D}_{\{\psi,\phi\}}^{it}(xy) = \mathfrak{D}_{\{\psi,\phi\}}^{it}(x)\mathfrak{D}_{\psi}^{it}(y),$$

we find

$$\begin{aligned}
u_{s+t} = \mathfrak{D}_{\{\psi,\phi\}}^{i(s+t)}(1) &= \mathfrak{D}_{\{\psi,\phi\}}^{is}\left(\mathfrak{D}_{\{\psi,\phi\}}^{it}(1)\right), \\
&= \mathfrak{D}_{\{\psi,\phi\}}^{is}\left(1 \cdot \mathfrak{D}_{\{\psi,\phi\}}^{it}(1)\right), \\
&= \mathfrak{D}_{\{\psi,\phi\}}^{is}(1) \cdot \mathfrak{D}_{\psi}^{is}\left(\mathfrak{D}_{\{\psi,\phi\}}^{it}(1)\right), \\
&= u_s \cdot \mathfrak{D}_{\psi}^{is}(u_t),
\end{aligned}$$

as desired. For *2* we note that $\mathfrak{D}_{\phi}^{it}(x)$ satisfies:

$$\begin{aligned}
\mathfrak{D}_{\phi}^{it}(x) &= \mathfrak{D}^{it}\left[\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}\right], \\
&= \mathfrak{D}^{it}\left[\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right], \\
&= u_t \mathfrak{D}_{\psi}^{it}(x) u_t^*,
\end{aligned}$$

as desired. □

We are now in the position to classify type *III* von Neumann algebras. Recall that the inner automorphisms of $\mathcal{A}$ form a normal subgroup of $\mathrm{Aut}(\mathcal{A})$. The group $\mathrm{Out}(\mathcal{A})$ is defined to be the quotient of $\mathrm{Aut}(\mathcal{A})$ by $\mathrm{Inn}(\mathcal{A})$. For any faithful normal state ϕ we consider its modular automorphism group $\mathfrak{D}_{\phi}^{it}$ as a subgroup of $\mathrm{Out}(\mathcal{A})$. Pick some other faithful normal state ψ, by the cocycle derivative theorem we have that there exists a collection u_t of unitaries such that

$$\mathfrak{D}_{\psi}^{it} = u_t \mathfrak{D}_{\phi}^{it} u_t^*.$$

We denote the equivalence class of $\mathfrak{D}_\phi^{it}$ in $\mathrm{Out}(\mathcal{A})$ by $\left[\mathfrak{D}_\phi^{it}\right]$. We conclude that for any two faithful normal states ϕ and ψ the following holds:

$$\left[\mathfrak{D}_\phi^{it}\right] = \left[\mathfrak{D}_\psi^{it}\right].$$

It follows that $\left[\mathfrak{D}_\phi^{it}\right]$ forms a subgroup of the real line which is an invariant of the algebra (i.e. invariant under the choice of the state ϕ). Denote by $V \subset \mathbb{R}$ the set of all $t \in \mathbb{R}$ such that $\Delta_\phi^{it} \in \mathcal{A}$, in other words V is the set of all $t \in \mathbb{R}$ such that $\mathfrak{D}_\phi^{it}$ is inner. Suppose that $t_n \subset V$ is a convergent sequence and let t_0 be its limit. Consider $\Delta_\phi^{it_0}$, since Δ_ϕ^{it} determines a strongly continuous one parameter group we conclude that $\Delta_\phi^{it_0} = \lim_{n \longrightarrow \infty} \Delta_\phi^{it_n}$, in the strong operator topology. Since $\Delta_\phi^{it_n} \subset \mathcal{A}$, it follows that $\Delta_\phi^{it_0} \in \mathcal{A}$. We conclude that V is closed, obviously V is a subgroup of $\mathbb{R}$ thus one and only one of the following possibilities can occur:

$$V = \begin{cases} \mathbb{R}, \\ r\mathbb{Z} \text{ for some } r > 0, \\ \{0\}. \end{cases}$$

Suppose that V is $\mathbb{R}$ then all automorphisms are inner. If $V = r\mathbb{Z}$ then we apply the group homomorphism $e : V \longrightarrow (0, \infty)$ defined by

$$e(rb) = e^{-rb}, \qquad b \in \mathbb{Z},$$

and set $\lambda := e(r)$, to find that the inner automorphisms form a multiplicative subgroup of $\{(0, \infty), \cdot\}$ of the form $\{\lambda^b \; ; \; b \in \mathbb{Z}\}$. Note that in this case $0 < \lambda < 1$. If $V = \{0\}$ then all automorphisms are outer. Because V is stable under the choice of faithful normal state, the following definition makes sense.

Definition 4.2.0.17. *If $\mathcal{A}$ is a type III factor, then we say that $\mathcal{A}$ is of type III_0 if $V = \mathbb{R}$, we say that $\mathcal{A}$ is of type III_λ if $V = \lambda^{\mathbb{Z}}$ and finally we say that $\mathcal{A}$ is of type III_1 when $V = \{0\}$.*

We conclude the following theorem.

Theorem 4.2.0.18. *If $\mathcal{A}$ is a type III factor, then there exists $\lambda \in [0, 1]$ such that $\mathcal{A}$ is of type III_λ.*$\square$

One could wonder why we insist that $\mathcal{A}$ is of type III. The point is that if $\mathcal{A}$ is not of type III then $V = \mathbb{R}$. It is not at all obvious that this is always the case. We will not show this but instead we will give a sketch of the argument en the steps required to obtain the result.

We insisted that $\mathcal{A}$ is σ-finite so that we could work with a faithful normal *state*, if we drop this assumption then we are forced to work with weights. A weight is a map from the positive cone $\mathcal{A}_+$ to the extended half-line $[0, \infty]$ which is linear. A weight is simply a noncontinuous positive linear functional. For any von Neumann algebra $\mathcal{A}$ there exists a faithful semifinite normal weight ω. Much like in the tracial setting one obtains a Hilbert space $\mathcal{H}_\omega$ derived from the weight ω via the GNS construction. We can then represent $\mathcal{A}$ on $\mathcal{H}_\omega$ and consider the modular automorphism group associated to this weight. One then proceeds to show that the image of this group in $\mathrm{Out}(\mathcal{A})$ is also an invariant of $\mathcal{A}$. In particular if $\mathcal{A}$ is of type II then it has a trace τ. The modular operator Δ_τ associated to τ is always equal to the identity, hence the modular automorphism group associated to τ is the trivial group, meaning that all automorphisms are inner. As such $V_\tau = V_\omega = \mathbb{R}$. For a detailed discussion regarding weights we refer to [9].

Chapter 5

Conclusions and summary

Here we look back at what has been achieved and what not. Furthermore we informally discuss some subjects not mentioned in this thesis.

We have seen that every von Neumann algebra $\mathcal{A}$ can be decomposed as a direct sum of von Neumann algebras of the possible types, that is the content of theorem 1.1.1.5. Furthermore the factor decomposition, as explained in section 1.2.1, allows us to further reduce the study of von Neumann algebras to factors of the possible types.

Using the crossed product construction we found that von Neumann algebras of all the possible types do indeed exist. The study of type semifinite von Neumann algebras is governed by traces and further decompositions into abelian von Neumann algebras and von Neumann algebras type I and II_1. Some type II_1 von Neumann algebras can be obtained from groups using the crossed product construction, as described in chapter two. As a side remark: it is currently unknown if the free group on two generators defines the same type II_1 factor as the free group of three generators, so the study of type II_1 von Neumann algebras is far from done.

However the most complications arise in the type III case. Note that the decomposition theorem, theorem 1.3.2.5, largely depends on the existence of a matrix unit. One could try this trick with the type III algebra but alas it will not work. This is because of the fact that all projections in a type III factor on a separable Hilbert space are equivalent, see for proof [8]. In this case it would not help to try to understand $\mathcal{A}$ on its corner algebras because trying to understand a corner algebra is the same as understanding the whole algebra. I myself usually compare this with fractals since fractals also posses a lot of self similarity, when zooming in.

The previous chapter was devoted to the classification of type III von Neumann algebras into those of type III_λ with $\lambda \in [0, 1]$, this gives us at least some grip on the type III algebras. Though no examples of these algebras are given, they can be constructed using the crossed product construction as described in chapter two, for details we refer to [9].

There are many more matters that we have not covered or even mentioned, for example the hyperfinite factors form an interesting subtype of von Neumann algebras. A hyperfinite von Neumann algebra is a von Neumann algebra that can be obtained from finite dimensional ones using a sort of limiting process. There exists a powerful classification result stating that all hyperfinite type II_1 factors are isomorphic. For most of these matters we refer to the treatment of Takesaki, namely [8],[9] and [10].

As a last word from myself I would like to say to anyone reading this that the subject of von Neumann algebras is highly challenging and, perhaps more importantly, fun. I find the level of abstraction and generalization a fascinating thing. Perhaps it has something to do with my fascination of anything of dimension $\geq \aleph_0$. Although I believe that I now posses a relatively good feel for the subject I am still relatively new to the subject. For me most of the theorems presented in this thesis were hard work, with emphasis on the Tomita-Takesaki theorem. But I think that this is true for most of us, I must say, the first time i took a read in [8] I was rather intimidated. For me I hope that my studies do not end here.

Chapter 6

Appendix

6.1 Basic theory for unbounded operators and anti-linear operators

Here we shall briefly cover the basic and necessary theory for unbounded operators. Throughout this section we will make distinctions between linear and anti-linear operators. If given some operator A we will specifically mention if it is linear or anti-linear. Later on we describe how we can view an anti-linear operator as a linear operator at the expense of a change in the Hilbert space structure it acts on.

Definition 6.1.0.19. *Let $\mathcal{H}$ be a Hilbert space. A linear operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ is a linear map defined on a linear subspace $\mathcal{D}(A) \subset \mathcal{H}$ (not necessarily closed) called the domain of A. A is called anti-linear when for all $\lambda \in \mathbb{C}$ and $h, k \in \mathcal{D}(A)$ we have $A(\lambda h + k) = \overline{\lambda} A(h) + A(k)$. If given two linear operators A, B then $A+B$ is defined on $\mathcal{D}(A+B) = \mathcal{D}(A) \cap \mathcal{D}(B)$, also AB is defined on $B^{-1}(\mathcal{D}(A))$, A is called bounded when its operator norm exists as an element of $\mathbb{R}$.*

Note that if given a linear operator A then if $\mathcal{D}(A)^{\perp} \neq \{0\}$ then we can extend A to include $\mathcal{D}(A)^{\perp}$ by setting $A = 0$ on $\mathcal{D}(A)^{\perp}$, so we can assume that $\mathcal{D}(A)$ is dense in $\mathcal{H}$. If A is bounded then by the density of its domain and the continuity of A on its domain, A extends to a bounded linear map on all of $\mathcal{H}$. Thus if A is indeed unbounded then it is defined on a proper dense subspace and does not allow for an extension to the whole Hilbert space. The notion of an extension should be interpreted as follows, if A is an operator then B is called an extension of A when $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Bh = Ah$ for all $h \in \mathcal{D}(A)$.

For a linear operator A on $\mathcal{H}$ we define its graph as

$$\operatorname{graph}(A) := \{h \oplus Ah \; ; \; h \in \mathcal{D}(A)\}.$$

An operator A is called closed when $\operatorname{graph}(A) \subset \mathcal{H} \oplus \mathcal{H}$ is a closed subspace. A is called closable if it allows for an extension $\widehat{A}$ such that $\widehat{A}$ is a closed operator.

The question becomes now which operators allow for a closed extension? To answer this one must first have a characterization of the subspaces in $\mathcal{H} \oplus \mathcal{H}$ which are graphs of some linear map. Obviously if A is some linear map from $\mathcal{H}$ to itself then if $0 \oplus h \in \operatorname{graph}(A)$ then $h = 0$. On the other hand suppose that $\mathcal{V}$ is some linear subspace of $\mathcal{H} \oplus \mathcal{H}$ with the property that $0 \oplus h \in \mathcal{V} \Longrightarrow h = 0$, then set $\mathcal{D} := \{h \in \mathcal{H} \; ; \text{ there is a } k \in \mathcal{H} \text{ with the property that } h \oplus k \in \mathcal{V}\}$. Suppose that for given $h \in \mathcal{D}$ there are k_1 and k_2 such that $h \oplus k_1 \in \mathcal{V}$ and $h \oplus k_2 \in \mathcal{V}$ then because $\mathcal{V}$ is linear we find that $0 \oplus k_1 - k_2 \in \mathcal{V}$ thus $k_1 = k_2$ hence if $h \in \mathcal{D}$ then there exists a unique $k \in \mathcal{H}$ with $h \oplus k \in \mathcal{V}$. Define now $Ah := k$ then for h_1 and h_2 we find k_1, k_2 with $h_1 \oplus k_1, h_2 \oplus k_2 \in \mathcal{V}$ also there is a unique k such that $h_1 + h_2 \oplus k \in \mathcal{V}$, since $\mathcal{V}$ is a linear space we find that $k = k_1 + k_2$ thus $A(h_1 + h_2) = k_1 + k_2 = A(h_1) + A(h_2)$. A similar argument shows that A respects scalar multiplication, that is, A is linear. We have found a one to one correspondence between linear subspaces $\mathcal{V}$ of $\mathcal{H} \oplus \mathcal{H}$ with the property that $0 \oplus k \in \mathcal{V} \Longrightarrow k = 0$ and linear operators on $\mathcal{H}$.

Say that a subspace $\mathcal{V} \subset \mathcal{H} \oplus \mathcal{H}$ is anti-linear when for $h_1 \oplus k_1$ and $h_2 \oplus k_2$ we have $(h_1 + h_2) \oplus (k_1 + k_2) \in \mathcal{V}$ and for $\lambda \in \mathbb{C}$ and $h \oplus k \in \mathcal{V}$ we have $\lambda h \oplus \overline{\lambda} k \in \mathcal{V}$. If $\mathcal{V}$ is anti-linear and has the property that $0 \oplus k \in \mathcal{V} \Longrightarrow k = 0$ then in a similar way we conclude that $\mathcal{V}$ is the graph of some anti-linear map A. So again we find a one to one correspondence between anti-linear subspaces $\mathcal{V}$ with the property that $0 \oplus k \in \mathcal{V} \Longrightarrow k = 0$ and graphs of anti-linear maps A. An anti-linear map is called closed if its graph is a closed anti-linear subspace and it is called closable when there is some anti-linear map B such that B is an extension of A and $\operatorname{graph}(B)$ is a closed anti-linear subspace of $\mathcal{H} \oplus \mathcal{H}$.

Lemma 6.1.0.20. *An (anti) linear operator A is closable if and only if $\overline{\operatorname{graph}(A)}$ is a graph.*

Proof:

Suppose that $\overline{\text{graph}(A)}$ is a graph, that is, there is some $B : \mathcal{H} \longrightarrow \mathcal{H}$ such that $\text{graph}(B) = \overline{\text{graph}(A)}$ then obviously we have $\text{graph}(A) \subset \text{graph}(B)$, for $h \in \mathcal{D}(A)$ we have that $h \oplus Ah \in \text{graph}(B)$ so that $h \oplus Ah = h \oplus Bh$ that is $Ah = Bh$, clearly $\mathcal{D}(A) \subset \mathcal{D}(B)$ so that B is a closed extension of A, hence A is closable. On the other hand suppose that A is closable then it allows for a closed operator B such that $\mathcal{D}(A) \subset \mathcal{D}(B)$ and $Ah = Bh$ for all $h \in \mathcal{D}(A)$, suppose that $0 \oplus k \in \overline{\text{graph}(A)}$ then $0 \oplus k \in \text{graph}(B)$ so then by our remarks $k = 0$, thus $\overline{\text{graph}(A)}$ is a graph. $\square$

6.1.1 The Adjoint

The notion of the adjoint generalizes naturally to unbounded linear operators. Suppose that A is a linear operator on $\mathcal{H}$ and that A is densely defined. Let $\mathcal{D}(A)$ be the domain of A and set

$$\mathcal{D}^*(A) := \{k \in \mathcal{H} \; ; \; k : \mathcal{D}(A) \longrightarrow \mathbb{C}, \; , k(h) := \langle Ah, k\rangle \text{ defines a bounded linear functional on } \mathcal{D}(A)\}.$$

Since k defines a linear functional on $\mathcal{D}(A)$ it is given by pairing with some vector $f \in \mathcal{H}$, that is $k(h) = \langle Ah, k\rangle = \langle h, f\rangle$. Suppose that f_1 and f_2 satisfy this relation, that is, $k(h) = \langle h, f_1\rangle = \langle h, f_2\rangle$. Then for all $h \in \mathcal{D}(A)$ we have $\langle h, f_1 - f_2\rangle = 0$, now we use that $\mathcal{D}(A)$ is dense in $\mathcal{H}$ and the boundedness of $k(h)$ to conclude that $f_1 = f_2$. So the correspondence $k \longrightarrow f$ such that $k(h) = \langle h, f\rangle$ is unique, we define A^* on $\mathcal{D}^*(A)$ by setting $A^*(k) = f$ so that the relation $\langle Ah, k\rangle = \langle h, f\rangle = \langle h, A^*k\rangle$ holds for all $h \in \mathcal{D}(A)$ and $k \in \mathcal{D}^*(A) = \mathcal{D}(A^*)$.

Now we will construct the adjoint for anti-linear operators.
Given an anti-linear bounded operator $A : \mathcal{H} \longrightarrow \mathcal{H}$ then for $k \in \mathcal{H}$ we define a map $\widehat{k} : \mathcal{H} \longrightarrow \mathbb{C}$ by setting

$$\widehat{k}(h) := \overline{\langle Ah, k\rangle}.$$

We see that the following equalities hold

$$\begin{aligned}\widehat{k}(h+v) &= \overline{\langle A(h+v), k\rangle} = \overline{\langle A(h) + A(v), k\rangle} = \overline{\langle A(h), k\rangle} + \overline{\langle A(v), k\rangle} = \widehat{k}(h) + \widehat{k}(v)\\ \widehat{k}(\lambda h) &= \overline{\langle A(\lambda h), k\rangle} = \overline{\left\langle \overline{\lambda} A(h), k\right\rangle} = \lambda \overline{\langle A(h), k\rangle}.\end{aligned}$$

It follows that $\widehat{k}$ is a linear functional on $\mathcal{H}$ so that $\widehat{k}(h)$ is given by paring h with some unique vector $f_k \in \mathcal{H}$, that is $\widehat{k}(h) = \langle h, f_k\rangle$. We define A^* by setting $A^*k := f_k$. For all h we have the following

$$\begin{aligned}A^*(\lambda k)(h) = f_{\lambda k}(h) = \widehat{\lambda k}(h) &= \overline{\langle Ah, \lambda k\rangle}\\ &= \lambda \langle h, A^*k\rangle\\ &= \left\langle h, \overline{\lambda} A^*k\right\rangle,\end{aligned}$$

hence $A^*(\lambda k) = \overline{\lambda} A^*(k)$. Via a similar argument we conclude that A^* is additive. In total we conclude that A and A^* are both anti-linear and satisfy the following relation for all $h, k \in \mathcal{H}$:

$$\overline{\langle Ah, k\rangle} = \langle h, A^*k\rangle.$$

We also note that $\|A^*\| = \|A\|$ and that $\|A^*A\| = \|A\|^2$. For unbounded anti-linear operators A we apply the same construction as with linear operators. So given an anti-linear operator A with dense domain $\mathcal{D}(A)$ then set

$$\mathcal{D}^*(A) := \left\{k \in \mathcal{H} \; ; \; \widehat{k} : \mathcal{D}(A) \longrightarrow \mathbb{C}, \; \widehat{k}(h) := \overline{\langle Ah, k\rangle} \text{ defines a bounded linear functional on } \mathcal{D}(A)\right\}.$$

We apply the same argument to find for each $k \in \mathcal{D}^*(A)$ some unique vector $f_k \in \mathcal{H}$ such that $\widehat{k}(h) = \langle h, f_k\rangle$ and define $A^*k = f_k$. It follows that A^* is anti-linear on its domain $\mathcal{D}(A^*) = \mathcal{D}^*(A)$.

Note that if we construct A^{**} as above then an examination of its domain reveals the following; Suppose that $k \in \mathcal{D}(A^*)$ then $k(h) := \langle h, A^*k\rangle$ defines a bounded linear functional on $\mathcal{D}(A)$, then naturally $h : \mathcal{D}(A^*) \longrightarrow \mathbb{C}$ defined by $h(k) = \langle Ah, k\rangle = \langle h, A^*k\rangle$ defines a bounded linear functional on $D(A^*)$. If one would just formally construct the double adjoint A^{**} of A then we would conclude that $\mathcal{D}(A) \subset \mathcal{D}(A^{**})$ so that A^{**} is an extension of A. However it is not necessarily true that $D(A^*)$ is dense in $\mathcal{H}$ so it could be that A^{**} is not well defined. We will partially solve this issue and give a sufficient criterion for A^{**} to be well defined.

Lemma 6.1.1.1. *given a densely defined linear operator A and let $I : \mathcal{H} \oplus \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}$ be the isomorphism defined by*

$$I(h \oplus k) := (-k) \oplus h.$$

Then it follows that

$$\mathit{graph}(A^*) = [I(\mathit{graph}(A))]^{\perp}.$$

Proof:

Pick $k \oplus A^*k$ in graph (A^*) and let $h \oplus Ah \in$ graph (A), then it follows that

$$\begin{aligned}\langle k \oplus A^*k, (-Ah) \oplus h\rangle &= -\langle k, Ah\rangle + \langle A^*k, h\rangle \\ &= -\langle k, Ah\rangle + \langle k, Ah\rangle = 0.\end{aligned}$$

As such graph $(A^*) \subset [I(\text{graph}(A))]^{\perp}$. On the other hand suppose that $k \oplus f \in [I(\text{graph}(A))]^{\perp}$ then for all $h \in \mathcal{D}(A)$ we have

$$0 = \langle k \oplus f, (-Ah) \oplus h\rangle = -\langle k, Ah\rangle + \langle f, h\rangle,$$

so that $\langle f, h\rangle = \langle k, Ah\rangle$ for all $h \in \mathcal{D}(A)$. We conclude that $f = A^*(k)$ hence $k \oplus f = k \oplus A^*k \in$ graph (A^*). In total we find

$$\text{graph}(A^*) = [I(\text{graph}(A))]^{\perp},$$

as desired. □

Proposition 6.1.1.2. *If A is a densely defined linear operator then the following statements hold*

1. *A^* is a closed operator.*
2. *A^* is densely defined if and only if A is closable.*
3. *If A is closable then its closure is given by $A^{**} := (A^*)^*$.*

Proof:

1
By 6.1.1.1 we have that graph $(A^*) = [I(\text{graph}(A))]^{\perp}$ in particular it is closed, so indeed A^* is a closed operator.

2
Suppose that A^* is densely defined then its adjoint $(A^*)^*$ is closed, also we have seen that $(A^*)^*$ defines an extension of A as such A is closable.

On the other hand, suppose that A is closable and let $k_0 \in \mathcal{D}(A^*)^{\perp}$. Then, by 6.1.1.1, we that $k_0 \oplus 0 \in [\text{graph}(A^*)]^{\perp} = [I(\text{graph}(A))]^{\perp\perp} = \overline{I(\text{graph}(A))}$. It follows that $I^*(k_0 \oplus 0) \in \overline{\text{graph}(A)}$. The map I^* is given by $I^*(h \oplus k) = k \oplus -h$ so then $0 \oplus -k_0 \in \overline{\text{graph}(A)}$. By the fact that A is closable it follows that $\overline{\text{graph}(A)}$ is a graph, thus $k_0 = 0$, we conclude that A^* is densely defined.

3
Suppose that A is closable then by *2* we find that A^* is densely defined, as such, A^{**} is well defined. Using 6.1.1.1 we find that graph $(A^{**}) = [I(\text{graph}(A^*))]^{\perp}$. Note that $I^2 = -1$ and $I^3 = I^*$, so for any subspace $\mathcal{V} \subset \mathcal{H} \oplus \mathcal{H}$ we have $I^2(\mathcal{V}) = \mathcal{V}$ thus $I(\mathcal{V}) = I^3(\mathcal{V}) = I^*(\mathcal{V})$. Using this we conclude that

$$\text{graph}(A^{**}) = [I^*\text{graph}(A^*)]^{\perp} = \left[I^*[I\text{graph}(A)]^{\perp}\right]^{\perp}.$$

For any isomorphism J and subspace $\mathcal{V}$ we have $J(\mathcal{V})^{\perp} = J\left(\mathcal{V}^{\perp}\right)$. We conclude that

$$\begin{aligned}\operatorname{graph}(A^{**}) &= \left[I^{*}\left[I\operatorname{graph}(A)\right]^{\perp}\right]^{\perp} \\ &= \left[I^{*}I\operatorname{graph}(A)\right]^{\perp\perp} \\ &= \left[\operatorname{graph}(A)\right]^{\perp\perp} = \overline{\operatorname{graph}(A)}.\end{aligned}$$

So indeed the closure of A is given by A^{**} when A is closable. Note here that if A is closed then we have the identity $A^{**} = A$. □

We would like a similar statement about anti linear operators. We will find a similar result but we will approach the solution in a different way. The question really is: how much different are anti-linear operators compared to ordinary linear operators? We have seen so far that a σ-finite von Neumann algebra $\mathcal{A}$ has a isomorphic copy on some Hilbert space constructed from A, on that Hilbert space we could define an involution in a natural way. So in that sense one cannot just dismiss anti-linear operators because they arise naturally. An other example is given by the Riesz representation theorem which states that any bounded linear functional ϕ on a Hilbert space $\mathcal{H}$ is uniquely determined by some element $k_{\phi} \in \mathcal{H}$. The evaluation of ϕ in some point $h \in \mathcal{H}$ is given by

$$\phi(h) = \langle h, k_{\phi}\rangle .$$

So we have a nice correspondence between elements k of $\mathcal{H}$ and elements ϕ_k in the dual $\mathcal{H}^{*}$ of $\mathcal{H}$. However the map $\Phi : \mathcal{H} \longrightarrow \mathcal{H}^{*}$ defined by

$$\Phi(k) := \phi_k,$$

is anti-linear because of the following equalities:

$$\Phi(\lambda k)(h) = \langle h, \lambda k\rangle = \overline{\lambda}\langle h, k\rangle = \overline{\lambda}\Phi(k)(h).$$

So the correspondence $\mathcal{H} \longleftrightarrow \mathcal{H}^{*}$ is anti-linear. The proof of the Riesz representation theorem also works for anti-linear bounded functionals on $\mathcal{H}$. One can prove that if ψ is a bounded anti-linear functional on $\mathcal{H}$ then there exists a unique k_{ψ} such that $\psi(h) = \langle k_{\psi}, h\rangle$ for all $h \in \mathcal{H}$. If we denote the space of anti-linear functionals on $\mathcal{H}$ by $\overline{\mathcal{H}^{*}}$ then the map $\Psi : \mathcal{H} \longrightarrow \overline{\mathcal{H}^{*}}$ defined by

$$\Psi(k)(h) := \langle k, h\rangle ,$$

is a linear correspondence between $\mathcal{H}$ and $\overline{\mathcal{H}^{*}}$. We see that there is some (anti) symmetry going on here.

Given any Hilbert space $\mathcal{H}$ we can define a new Hilbert space $\overline{\mathcal{H}}$ by defining a new scalar multiplication and inner product in the following manner

$$\begin{aligned}\lambda \cdot_{\overline{\mathcal{H}}} h &:= \overline{\lambda} \cdot_{\mathcal{H}} h \\ \langle h, v\rangle_{\overline{\mathcal{H}}} &:= \overline{\langle h, v\rangle}_{\mathcal{H}}.\end{aligned}$$

Here $\cdot_{\mathcal{H}}$ denotes the original scalar multiplication in $\mathcal{H}$ and $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ denotes the original inner product in $\mathcal{H}$, we leave the summation in $\mathcal{H}$ unchanged. If A is any anti-linear map from $\mathcal{H}$ into itself then we can consider A as a map from $\mathcal{H}$ to $\overline{\mathcal{H}}$. Since A is anti-linear it follows that A becomes linear viewed as a map to $\overline{\mathcal{H}}$. If we denote the space of anti-linear operators on $\mathcal{H}$ by $\mathcal{AL}(\mathcal{H})$ and denote the space of linear operators from $\mathcal{H}$ to $\overline{\mathcal{H}}$ by $\mathcal{L}\left(\mathcal{H}, \overline{\mathcal{H}}\right)$ then it follows that

$$\mathcal{AL}(\mathcal{H}) \cong \mathcal{L}\left(\mathcal{H}, \overline{\mathcal{H}}\right).$$

As such, to study anti linear operators it suffices to study linear operators but then one needs to make a change in the Hilbert space structure.

So lets now consider the adjoint of a anti-linear operator A, by our previous consideration A becomes linear when viewed as an element in $L\left(\mathcal{H}, \overline{\mathcal{H}}\right)$ and its graph becomes a subset in $\mathcal{H} \oplus \overline{\mathcal{H}}$. Naturally A^{*} is an element in $L\left(\overline{\mathcal{H}}, \mathcal{H}\right)$

Proposition 6.1.1.3. *If A is a densely defined anti-linear operator on $\mathcal{H}$, that is $A \in \mathcal{L}\left(\mathcal{H}, \overline{\mathcal{H}}\right)$, then the following statements hold*

1. *A^* defines a closed operator in $\mathcal{L}\left(\overline{\mathcal{H}}, \mathcal{H}\right)$.*
2. *A^* is densely defined if and only if A is closable.*
3. *If A is closable then its closure is given by $A^{**} := (A^*)^*$.*

Proof:
The same as the previous proposition but now we consider $\text{graph}\,(A^*)$ as a subset of $\overline{\mathcal{H}} \oplus \mathcal{H}$. □

The construction of the Hilbert space $\overline{\mathcal{H}}$ is covered in greater detail in section 6.2. It is there that we find that the structure of $\mathcal{AL}(\mathcal{H})$ is completely determined by the structure of $\mathcal{L}(\mathcal{H} \oplus \overline{\mathcal{H}})$. In this sense anti-linear operators are solved in terms of linear operators. Some constructions (such as the construction of the adjoint) do not require the full Hilbert space $\mathcal{H} \oplus \overline{\mathcal{H}}$ but can be done in $\mathcal{H}$, therefor these particular constructions are included here. However the notion of symmetry and selfadjoint operators in the antilinear case require a more rigorous approach, as such, they are included in the next section. While discussing symmetric and selfadjoint operators we will assume that they are linear. In section 6.2 we will make the generalization to antilinear maps.

Proposition 6.1.1.4. *Suppose that A is a linear operator, then*

$$[ran(A)]^{\perp} = \ker(A^*).$$

If A is a closed operator then also

$$[ran(A^*)]^{\perp} = \ker(A).$$

Proof:

Given $h \in [\text{ran}(A)]^{\perp}$, then $\langle Af, h\rangle = 0$ for all $f \in \mathcal{D}(A)$. It follows that A^* is defined on $[\text{ran}(A)]^{\perp}$, using the density of $\mathcal{D}(A)$ we find that $A^* = 0$ on $[\text{ran}(A)]^{\perp}$. We conclude that $[\text{ran}(A)]^{\perp} \subset \ker(A^*)$. Suppose that $h \in \ker(A^*)$, then for all $f \in \mathcal{D}(A)$ we find $0 = \langle f, A^*h\rangle = \langle Af, h\rangle$, thus $h \in [\text{ran}(A)]^{\perp}$.

Suppose now that A is also closed, by 6.1.1.2 we find that $A = A^{**}$. We use the first equality of this proposition to find the second. □

Now we will review the notion of a selfadjoint linear operator. When $A \in B(\mathcal{H})$ is a bounded linear operator then A is self adjoint when A is symmetric ($\langle Af, g\rangle = \langle f, Ag\rangle$). If we are in the situation that A is not bounded we need to consider the domain issues associated to the situation. Suppose that the equality $A^* = A$ holds for some unbounded linear operator A. Since A^* is closed (because it is the adjoint of A) it follows that A is closed and thus also we find $A = A^* = A^{**}$. The equality $A = A^*$ can only hold if $\mathcal{D}(A) = \mathcal{D}(A^*)$ so for $h, g \in \mathcal{D}(A)$ we find that $\langle Af, g\rangle = \langle f, Ag\rangle$ in particular for all $f \in \mathcal{D}(A)$ we have $\langle Af, f\rangle \in \mathbb{R}$.

6.1.2 Symmetric operators

Definition 6.1.2.1. *We say that a linear operator A is symmetric when for all $f, g \in \mathcal{D}(A)$ it holds that $\langle Af, g\rangle = \langle f, Ag\rangle$.*

Suppose that A is symmetric then for $f \in \mathcal{D}(A)$ we have $\langle Af, f\rangle \in \mathbb{R}$, this implies that A^* is a closed extension of A because $A^*f = Af$ for all $f \in \mathcal{D}(A)$. This in turn implies that A is symmetric so the following are equivalent

- A is symmetric
- $\langle Af, f\rangle \in \mathbb{R}$ for all $f \in \mathcal{D}(A)$
- A^* is a closed extension of A

In particular we note that every symmetric operator allows for a closed extension, furthermore we note that since A is closable its closure is given by A^{**}, so how do A^* and A^{**} relate? By definition we have that $\text{graph}(A^*)$ is closed and contains the graph of A, since $\text{graph}(A^{**})$ is the closure of $\text{graph}(A)$ we conclude that $\text{graph}(A^{**}) \subset \text{graph}(A^*)$. Note that since A^* is closed we have that $A^{***} = A^*$. Consider A^{**}, we find that $(A^{**})^* = A^{***} = A^*$ is a closed extension of A^{**} because $\text{graph}(A^{**}) \subset \text{graph}(A^*)$. As such, A^{**} is also symmetric. We conclude that every symmetric operator allows for a closed symmetric extension, therefor when considering symmetric operators we can assume they are closed.

Proposition 6.1.2.2. *Let A be a symmetric linear operator on $\mathcal{H}$. The the following statements hold*

1. *If $\text{ran}(A)$ is dense, then A is injective.*
2. *If $A = A^*$ and A is injective, then $\text{ran}(A)$ is dense and A^{-1} is selfadjoint.*
3. *If $\mathcal{D}(A) = \mathcal{H}$, then A is bounded and selfadjoint.*
4. *If $\text{ran}(A) = \mathcal{H}$, then $A = A^*$ and A^{-1} is bounded.*

Proof:

1.
By proposition 6.1.1.4 we have that $\{0\} = \text{ran}(A)^\perp = \ker(A^*)$. Since A is symmetric, we have that $\ker(A) \subset \ker(A^*)$. We conclude that A is injective.

2.
By 6.1.1.4 we have that $\text{ran}(A)^\perp = \ker(A^*) = \ker(A) = \{0\}$, we find that $\text{ran}(A)$ is dense. Set $B = A^{-1}$, we aim to show that $B = B^*$. By definition we have that $\mathcal{D}(B) = \text{ran}(A)$. Consider $x, f \in \mathcal{D}(B)$, it follows that $x = Ay$ and $f = Ah$ for some $y, h \in \mathcal{D}(A)$. We find that

$$\begin{aligned} \langle Bx, f\rangle &= \langle y, Ah\rangle \\ &= \langle Ay, h\rangle \\ &= \langle x, Bf\rangle , \end{aligned}$$

as such, B is symmetric. Suppose that $x \in \mathcal{D}(B^*)$, then the map $\langle Bf, x\rangle$ defines a bounded linear functional on $\mathcal{D}(B)$. We find the following identities:

$$\begin{aligned} \langle Bf, x\rangle &= \langle f, B^*x\rangle \\ &= \langle Ah, B^*x\rangle . \end{aligned}$$

This means that $B^*x \in \mathcal{D}(A^*) = \mathcal{D}(A)$, as such, $x \in \mathcal{D}(B)$. We conclude that $B = B^*$.

3.
Since A is symmetric it follows that $\mathcal{D}(A) \subset \mathcal{D}(A^*)$. Using that $\mathcal{D}(A) = \mathcal{H}$ we conclude that $\mathcal{D}(A^*) = \mathcal{H}$. We find that $A = A^*$, in particular A is closed. We use the closed graph theorem to conclude that A is continuous, hence bounded.

4.
We use *1* to find that A is injective and its inverse, denoted by B, is defined on the whole of $\mathcal{H}$. Consider $f, h \in \mathcal{H}$, it follows that $f = Ah$ and $x = Ay$ for some $h, y \in \mathcal{D}(A)$. We find that

$$\begin{aligned} \langle f, B^*x\rangle &= \langle Bf, x\rangle \\ &= \langle Bf, Ay\rangle \\ &= \langle f, y\rangle \\ &= \langle f, Bx\rangle , \end{aligned}$$

therefor B is symmetric. It follows from *3* that B is bounded and selfadjoint. We use *2* to conclude that $A = B^{-1}$ is selfadjoint. □

The last statement gives a criterion for a symmetric operator to be self adjoint, namely if A is symmetric and its range is the whole of $\mathcal{H}$ then it is selfadjoint and furthermore its inverse is bounded. Note here that the inverse of A is not invertible as an element of $\mathcal{B}(\mathcal{H})$.

We will now diagnose the spectrum of a symmetric operator, but before doing so we first need some observations.

Let A be a symmetric operator, consider $\lambda = a + bi \in \mathbb{C}$ and $h \in \mathcal{D}(A)$. We find that

$$\begin{aligned}\|(A-\lambda)h\|^2 &= \|(A-a)h - ibh\|^2 \\ &= \|(A-a)h\|^2 - \langle (A-a)h, ibh\rangle - \overline{\langle (A-a)h, ibh\rangle} + \|bh\|^2 \\ &= \|(A-a)h\|^2 + \|bh\|^2 .\end{aligned}$$

In particular if $b \neq 0$ then $A-\lambda$ is injective. Suppose that $b \neq 0$, by considering A^{**} we may assume that A is closed. Consider the range of $(A-\lambda)$ and suppose that $g \in \overline{\operatorname{ran}(A-\lambda)}$. Then there are $h_i \in \mathcal{D}(A)$ such that $(A-\lambda)h_i \longrightarrow g$ as $i \longrightarrow \infty$. In particular for all $\epsilon > 0$ we can find i_ϵ such that if $i, j \geq i_\epsilon$ then $\|(A-\lambda)h_i - (A-\lambda)h_j\|^2 < \epsilon$ thus $(A-\lambda)h_i$ forms a Cauchy net. Note now that $b^2\|h_i\|^2 \leq \|(A-\lambda)h_i\|^2$ as such the set $\{h_i\}$ forms a Cauchy net in $\mathcal{H}$, denote the limit of this net by h. We find now that $h_i \oplus (A-\lambda)h_i \longrightarrow h \oplus g \in \operatorname{graph}(A-\lambda)$, since we assumed A to be closed. it follows $g = (A-\lambda)h \in \operatorname{ran}(A-\lambda)$ thus the range of $A-\lambda$ is closed. We conclude that if A is a closed symmetric operator then for all $\lambda = a + ib \in \mathbb{C}$ with $b \neq 0$ we have that $A-\lambda$ is injective and $\operatorname{ran}(A-\lambda)$ is closed.

Definition 6.1.2.3. *If A is a linear operator then we say that A is boundedly invertible when there is a linear operator B such that $AB = 1$ and 1 is an extension of BA*

It follows that A is boundedly invertible if and only if $\operatorname{ran}(A) = \mathcal{H}$ and A is injective. As such the domain of B is the whole of $\mathcal{H}$ and therefor B is bounded. Also the bounded inverse B of A is unique if it exists.

Definition 6.1.2.4. *Given a linear operator, we denote by $\rho(A)$ the set of all $\lambda \in \mathbb{C}$ such that $A-\lambda$ is boundedly invertible. We define the spectrum of A to be the set $\sigma(A) := \mathbb{C} \setminus \rho(A)$.*

Note that if A is unbounded then so is its spectrum.

Let A be a closed symmetric operator and let $\lambda = a + bi \in \mathbb{C}$ with $b \neq 0$. Suppose that $\mu \in \mathbb{C}$ is such that $|\lambda - \mu| < |b|$, consider now

$$\mathcal{V} := \ker(A^* - \mu) \cap [\ker(A^* - \lambda)]^{\perp} .$$

If $\mathcal{V} \neq \{0\}$ then there exists $h \in \mathcal{V}$ with $\|h\| = 1$. Note that $[\ker(A^*-\lambda)]^{\perp} = \overline{\operatorname{ran}\left(A-\overline{\lambda}\right)} = \operatorname{ran}\left(A-\overline{\lambda}\right)$ because $\operatorname{ran}\left(A-\overline{\lambda}\right)$ is closed by our previous considerations. Let $h \in \mathcal{V}$ with $\|h\| = 1$ then there is g such that $\left(A-\overline{\lambda}\right)g = h$. Since $h \in \ker(A^*-\mu)$ we find the following identity:

$$\begin{aligned}0 = \langle (A^*-\mu)h, g\rangle &= \langle h, (A-\overline{\mu})g\rangle \\ &= \left\langle h, \left(A-\overline{\lambda}+\overline{\lambda}-\overline{\mu}\right)g\right\rangle \\ &= \|h\|^2 + (\lambda-\mu)\langle h, g\rangle .\end{aligned}$$

We conclude that

$$1 = |\lambda-\mu| \cdot |\langle h, g\rangle| \leq |\lambda-\mu| \cdot \|g\| < |b| \cdot \|g\| .$$

On the other hand we have that $\left\|\left(A-\overline{\lambda}\right)g\right\| \geq |b|\,\|g\|$, thus

$$1 = \left\|\left(A-\overline{\lambda}\right)g\right\| \geq |b| \cdot \|g\| .$$

In total we reach the conclusion that

$$|b| \cdot \|g\| \leq 1 < |b| \cdot \|g\| ,$$

which is clearly a contradiction. It follows that $\mathcal{V} = \ker(A^* - \mu) \cap [\ker(A^* - \lambda)]^\perp = \{0\}$, as such, $\ker(A^* - \mu) \subset \ker(A^* - \lambda)$ when $|\lambda - \mu| < |b| = |\Im(\lambda)|$. Suppose that $|\lambda - \mu| < 1/2 \cdot |\Im(\lambda)|$, then in particular

$$1/2|\Im(\lambda)| < |\Im(\mu)| < 3/2|\Im(\lambda)|.$$

It follows that $|\lambda - \mu| < |\Im(\mu)|$. We use now the same argument to show that if $|\lambda - \mu| < 1/2|\Im(\lambda)|$ then

$$\ker(A^* - \lambda) \cap [\ker(A^* - \mu)]^\perp = \{0\}.$$

so that $\ker(A^* - \lambda) = \ker(A^* - \mu)$. In particular the map $\lambda \longrightarrow \dim\ker(A^* - \lambda)$ is constant on space $\{\lambda \in \mathbb{C} \ ; \ \Im(\lambda) > 0\}$ and also constant on the space $\{\lambda \in \mathbb{C} \ ; \ \Im(\lambda) < 0\}$.

Why is this important? This statement allows us to classify how the spectrum of a symmetric operator looks like

Theorem 6.1.2.5. *Suppose that A is a closed symmetric operator, the only possibilities for $\sigma(A)$ are the following sets*

- $\sigma(A) = \mathbb{C}$.
- $\sigma(A) = \{\lambda \in \mathbb{C} \ ; \ \Im(\lambda) \geq 0\}$.
- $\sigma(A) = \{\lambda \in \mathbb{C} \ ; \ \Im(\lambda) \leq 0\}$.
- $\sigma(A) \subset \mathbb{R}$.

So if A is any closed symmetric operator then its spectrum is one and only one of the sets above.

Proof:
Suppose that A is closed and symmetric, then for any $\lambda = a + bi$ with $b \neq 0$ we have that $A - \lambda$ is injective. Suppose that $A - \lambda$ is not surjective and $b > 0$, then $\lambda \in \sigma(A)$ but if $A - \lambda$ is not surjective then because its range is closed we conclude that $\operatorname{ran}(A - \lambda)^\perp = \ker\left(A^* - \overline{\lambda}\right)$. Since $0 \neq \dim\ker(A^* - \lambda)$ is constant on $\{\mu \in \mathbb{C} \ ; \ \Im(\mu) < 0\}$, we conclude that $\{\overline{\mu} \in \mathbb{C} \ ; \ \Im(\mu) < 0\} \subset \sigma(A)$. If $b < 0$ then we would have that $\{\mu \in \mathbb{C} \ ; \ \Im(\mu) > 0\} \subset \sigma(A)$. Since $\sigma(A)$ is closed we have that if there is $a + ib \in \sigma(A)$ with $b > 0$, then $\{\lambda \in \mathbb{C} \ ; \ \Im(\lambda) \geq 0\} \subset \sigma(A)$ if there also exists $a + ib \in \sigma(A)$ with $b < 0$ then also $\{\lambda \in \mathbb{C} \ ; \ \Im(\lambda) \leq 0\} \subset \sigma(A)$ and together they imply that $\sigma(A) = \mathbb{C}$. So far we have $\sigma(A) = \mathbb{C} \Longleftrightarrow \{i, -i\} \subset \sigma(A)$. Suppose that $\sigma(A) \neq \mathbb{C}$ and suppose that there exists $a + bi \in \mathbb{C}$ with $b > 0$ then by our previous arguments $\sigma(A) = \{\mu \in \mathbb{C} \ ; \ \Im(\mu) \geq 0\}$. If $\sigma(A) \neq \mathbb{C}$ and $a + bi \in \sigma(A)$ with $b < 0$ then by our previous arguments $\sigma(A) = \{\mu \in \mathbb{C} \ ; \ \Im(\mu) \leq 0\}$. The last possibility is $\sigma(A) \subset \mathbb{R}$. □

We are now able to classify selfadjoint linear operators among symmetric operators

Theorem 6.1.2.6. *If A is a closed symmetric operator then the following statements are equivalent*

1. *A is selfadjoint.*
2. *$\sigma(A) \subset \mathbb{R}$.*
3. *$\ker(A^* - i) = \ker(A^* + i) = \{0\}$.*

Proof:

1 $\Longrightarrow$ 2
Suppose that A is selfadjoint and let $\lambda \in \sigma(A)$. If $\Im(\lambda) \neq 0$, then $A - \lambda$ is injective and its range is closed thus it must follow that $A - \lambda$ is not surjective. We have that $\{0\} \neq \operatorname{ran}(A - \lambda)^\perp = \ker\left(A^* - \overline{\lambda}\right) = \ker\left(A - \overline{\lambda}\right) = \{0\}$ clearly a contradiction, we conclude that $\Im(\lambda)$ cannot be nonzero, that is, $\lambda \in \mathbb{R}$.

2 $\Longrightarrow$ 3
Suppose that $\sigma(A) \subset \mathbb{R}$, then in particular $\{0\} = \mathcal{H}^\perp = [\operatorname{ran}(A \pm i)]^\perp = \ker(A^* \mp i)$.

3 $\Longrightarrow$ 1

We have that $\operatorname{ran}(A \pm i)$ closed, thus by assumption *3* we find $\operatorname{ran}(A \pm i) = \mathcal{H}$ so that $A \pm i$ is surjective. For $h \in \mathcal{D}(A^*)$ we find $f \in \mathcal{D}(A)$ such that $(A+i) f = (A^*+i) h$ (because $A+i$ is surjective). Since A is symmetric we have that A^* is an extension of A so that $(A+i) f = (A^*+i) f$, we find $(A^*+i) f = (A^*+i) h$. Using *3* again we conclude also that A^*+i is injective, as such, $h = f \in \mathcal{D}(A)$ thus A is an extension of A^*. We conclude $A = A^*$. □

We now covered symmetric and selfadjoint linear operators. Normal operators are defined in the way one suspects them to be, for completion the definition is included.

Definition 6.1.2.7. *A linear operator A is called normal when A is closed, $\mathcal{D}(A^*A) = \mathcal{D}(AA^*)$ and on $\mathcal{D}(A^*A) = \mathcal{D}(AA^*)$ the equality $AA^* = A^*A$ holds.*

What remains is to find the corresponding statements for anti-linear operators. But before doing so it is instructive to formally introduce the conjugate vector space to see in detail how linear and anti-linear maps can be interchanged at the cost of a change in the vector space structure.

6.2 Conjugate vector spaces

We will now describe the structure of the conjugate vector space associated to any complex vector space, we will do this for Hilbert spaces but the generalization is simple enough. This section is devoted to make precise the statement that an anti-linear operator can be viewed as a linear operator defined on the same Hilbert space but with values in a conjugated space. At the end of this section it will be clear why this is the case and how we can transfer statements about linear operators to anti-linear operators.

Given $\mathcal{H}$ a Hilbert space then we can formally associate to $\mathcal{H}$ the set $\overline{\mathcal{H}}$ by defining

$$\overline{\mathcal{H}} := \left\{\overline{h} \; ; \; h \in \mathcal{H}\right\}.$$

In order to make $\overline{\mathcal{H}}$ a Hilbert space we need to define addition and scalar multiplication on $\overline{\mathcal{H}}$ as well as an inner product. We define

$$\begin{aligned} \overline{h} + \overline{v} &:= \overline{h+v}, \\ \alpha \overline{h} &:= \overline{\overline{\alpha} h}, \\ \langle \overline{h}, \overline{v} \rangle_{\overline{\mathcal{H}}} &:= \overline{\langle h, v \rangle}_{\mathcal{H}}. \end{aligned}$$

With this structure $\overline{\mathcal{H}}$ becomes a Hilbert space. Since $\overline{\mathcal{H}}$ is again a Hilbert space we may apply the same procedure again and we find the space $\overline{\overline{\mathcal{H}}}$ but luckily, as the following identities show, $\mathcal{H}$ and $\overline{\overline{\mathcal{H}}}$ are linearly isomorphic. We find

$$\begin{aligned} \overline{\overline{h}} + \overline{\overline{v}} &= \overline{\overline{h} + \overline{v}} = \overline{\overline{h+v}}, \\ \alpha \overline{\overline{h}} &= \overline{\overline{\alpha} \overline{h}} = \overline{\overline{\alpha h}}, \\ \left\langle \overline{\overline{h}}, \overline{\overline{v}} \right\rangle_{\overline{\overline{\mathcal{H}}}} &= \overline{\langle \overline{h}, \overline{v} \rangle}_{\overline{\mathcal{H}}} = \overline{\overline{\langle h, v \rangle}}_{\mathcal{H}} = \langle h, v \rangle_{\mathcal{H}}, \end{aligned}$$

so that the map $i : \mathcal{H} \longrightarrow \overline{\overline{\mathcal{H}}}$ is a linear isometric isomorphism. We will henceforth identify $\overline{\overline{\mathcal{H}}}$ with $\mathcal{H}$. We define now a map $c : \mathcal{H} \longrightarrow \overline{\mathcal{H}}$ simply by setting

$$c(h) := \overline{h},$$

by our previous arguments c is also defined on $\overline{\mathcal{H}}$ and $c^2 = 1$. The map c satisfies the following identities

$$\begin{aligned} c(h+v) &= \overline{h+v} = \overline{h} + \overline{v} = c(h) + c(v), \\ c(\alpha h) &= \overline{\alpha h} = \overline{\alpha} \overline{h} = \overline{\alpha} c(h), \end{aligned}$$

we conclude that c is anti-linear, by construction c is an anti-unitary.

Definition 6.2.0.8. *The space $\overline{\mathcal{H}}$ is called the conjugate space associated to $\mathcal{H}$.*

We see that $\mathcal{H}$ and $\overline{\mathcal{H}}$ are connected by the anti-unitary c. The map c will be extended to give the connection between operators on $\mathcal{H}$ and $\overline{\mathcal{H}}$.

Given an anti-linear map $A : \mathcal{H} \longrightarrow \mathcal{H}$ then the map $\overline{A} := c \circ A : \mathcal{H} \longrightarrow \overline{\mathcal{H}}$ satisfies the identities:

$$\overline{A}(\alpha h + v) = c(\overline{\alpha} A(h) + A(v)) = \overline{\overline{\alpha} A(h)} + \overline{A(v)} = \alpha \overline{A}(h) + \overline{A}(v).$$

As such, $\overline{A}$ becomes a linear map. On the other hand if $A : \mathcal{H} \longrightarrow \mathcal{H}$ is anti-linear we can define a map $\widehat{A} : \overline{\mathcal{H}} \longrightarrow \mathcal{H}$ by setting $\widehat{A} := A \circ c$. It follows that

$$\widehat{A}\left(\alpha\overline{h} + \overline{v}\right) = A\left(c\left(\alpha\overline{h}\right) + c(\overline{v})\right) = A(\overline{\alpha} h + v) = \alpha A(h) + A(v) = \alpha \widehat{A}\left(\overline{h}\right) + \widehat{A}(\overline{v}),$$

we conclude that both $\overline{A}$ and $\widehat{A}$ define linear maps. Furthermore it follows that any anti-linear map $A : \mathcal{H} \longrightarrow \mathcal{H}$ induces an anti-linear map $\widetilde{A} : \overline{\mathcal{H}} \longrightarrow \overline{\mathcal{H}}$ by setting

$$\widetilde{A} = c \circ \widehat{A} = c \circ A \circ c = \overline{A} \circ c.$$

It follows directly that $c \circ \overline{A} \circ c = A \circ c = \widehat{A}$ so that the maps $\widehat{A}$ and $\overline{A}$ are anti-unitarily equivalent. It is easy to see that if $L : \mathcal{H} \longrightarrow \mathcal{H}$ is a linear map then $\widehat{L} := L \circ c : \overline{\mathcal{H}} \longrightarrow \mathcal{H}$ defines an anti-linear map, also $\overline{L} := c \circ L : \mathcal{H} \longrightarrow \overline{\mathcal{H}}$ is anti-linear and $\widetilde{L} := c \circ L \circ c : \overline{\mathcal{H}} \longrightarrow \overline{\mathcal{H}}$ becomes a linear map. Furthermore we note for A linear or anti-linear we have $\widetilde{A} = \overline{\widehat{A}} = \overline{A \circ c} = c \circ A \circ c = \widehat{\overline{A}}$ so that the operations "hat" and "bar" commute.

For vector spaces $\mathcal{V}$ and $\mathcal{W}$ we denote now

$$\mathcal{L}(\mathcal{V}, \mathcal{W}) := \{L : \mathcal{V} \longrightarrow \mathcal{W} \; ; \; L \text{ is a linear map}\}.$$
$$\mathcal{AL}(\mathcal{V}, \mathcal{W}) := \{L : \mathcal{V} \longrightarrow \mathcal{W} \; ; \; L \text{ is an anti-linear map}\}.$$

The map c extends in a canonical way to a map $\overline{C} : \mathcal{L}(\mathcal{H}, \mathcal{H}) \longrightarrow \mathcal{AL}\left(\mathcal{H}, \overline{\mathcal{H}}\right)$ by defining

$$\overline{C}(A) := \overline{A}.$$

For all $h \in \mathcal{H}$ we find that $\overline{C}$ satisfies the identities

$$\overline{C}(A + B)(h) = \overline{A(h) + B(h)} = \overline{C}(A)(h) + \overline{C}(B)(h)$$
$$\overline{C}(\alpha A)(h) = \overline{\alpha A(h)} = \overline{\alpha}\overline{C}(A)(h).$$

As such $\overline{C}(\alpha A + B) = \overline{\alpha}\overline{C}(A) + \overline{C}(B)$ and we conclude that $\overline{C}$ determines an *anti-isomorphism* between $\mathcal{L}(\mathcal{H}, \mathcal{H})$ and $\mathcal{AL}\left(\mathcal{H}, \overline{\mathcal{H}}\right)$.

We can also extend c to a map $\widehat{C} : \mathcal{L}(\mathcal{H}, \mathcal{H}) \longrightarrow \mathcal{AL}\left(\overline{\mathcal{H}}, \mathcal{H}\right)$ by defining

$$\widehat{C}(A) := \widehat{A}.$$

For all $\overline{h} \in \overline{\mathcal{H}}$ we find that

$$\widehat{C}(\alpha A)\left(\overline{h}\right) = \alpha A(h) = \alpha \widehat{A}\left(\overline{h}\right) = \alpha \widehat{C}(A)\left(\overline{h}\right),$$
$$\widehat{C}(A + B)\left(\overline{h}\right) = A(h) + B(h) = \widehat{A}\left(\overline{h}\right) + \widehat{B}\left(\overline{h}\right) = \widehat{C}(A)\left(\overline{h}\right) + \widehat{C}(B)\left(\overline{h}\right).$$

We conclude that $\widehat{C}(\alpha A + B) = \alpha\widehat{C}(A) + \widehat{C}(B)$ and therefor $\widehat{C}$ determines a *linear* isomorphism between $\mathcal{L}(\mathcal{H}, \mathcal{H})$ and $\mathcal{AL}\left(\overline{\mathcal{H}}, \mathcal{H}\right)$.

It is now immediate that we can extend c to a map $\widetilde{C} : \mathcal{L}(\mathcal{H}, \mathcal{H}) \longrightarrow \mathcal{L}\left(\overline{\mathcal{H}}, \overline{\mathcal{H}}\right)$ by defining $\widetilde{C}(A) := \widetilde{A}$, it follows that $\widetilde{C}$ defines an *anti-isomorphism*.

If we use the symbol $\cong$ to denote a *linear* isomorphism and the symbol $\approx$ to denote an *anti-linear* isomorphism, then by our previous considerations we find the following identities

$$\mathcal{L}(\mathcal{H}, \mathcal{H}) \cong \mathcal{AL}\left(\overline{\mathcal{H}}, \mathcal{H}\right) \approx \mathcal{AL}\left(\mathcal{H}, \overline{\mathcal{H}}\right) \cong \mathcal{L}\left(\overline{\mathcal{H}}, \overline{\mathcal{H}}\right) \approx \mathcal{L}(\mathcal{H}, \mathcal{H}).$$

Via similar arguments we conclude

- $\overline{C} : \mathcal{AL}(\mathcal{H},\mathcal{H}) \longrightarrow \mathcal{L}\left(\mathcal{H},\overline{\mathcal{H}}\right)$ determines an *anti-linear* isomorphism.
- $\widehat{C} : \mathcal{AL}(\mathcal{H},\mathcal{H}) \longrightarrow \mathcal{L}\left(\overline{\mathcal{H}},\mathcal{H}\right)$ determines a *linear* isomorphism.
- $\widetilde{C} : \mathcal{AL}(\mathcal{H},\mathcal{H}) \longrightarrow \mathcal{AL}\left(\overline{\mathcal{H}},\overline{\mathcal{H}}\right)$ determines an *anti-linear* isomorphism.

Summarizing this we find

$$\mathcal{AL}(\mathcal{H},\mathcal{H}) \cong \mathcal{L}\left(\overline{\mathcal{H}},\mathcal{H}\right) \approx \mathcal{L}\left(\mathcal{H},\overline{\mathcal{H}}\right) \cong \mathcal{AL}\left(\overline{\mathcal{H}},\overline{\mathcal{H}}\right) \approx \mathcal{AL}(\mathcal{H},\mathcal{H}).$$

It is worth noting that by construction the metric topologies on $\mathcal{H}$ and $\overline{\mathcal{H}}$ coincide so that $\widetilde{C}$ defines also an *anti-linear* isomorphism between $B(\mathcal{H})$ and $B\left(\overline{\mathcal{H}}\right)$. We have for $x, y \in B(\mathcal{H})$ that $\widetilde{C}(xy) = cxyc = cxccyc = \widetilde{C}(x)\widetilde{C}(y)$, as such $\widetilde{C}$ is multiplicative. By construction c is an anti unitary as such $c = c^* = c^{-1}$ thus we find $\widetilde{C}(x)^* = (cxc)^* = c^*x^*c^* = cx^*c = \widetilde{C}(x^*)$ meaning that $\widetilde{C}$ defines a faithful anti-linear representation of $B(\mathcal{H})$ onto $B\left(\overline{\mathcal{H}}\right)$. It is easy to see that $\widetilde{C}$ is strongly continuous, as such, every von Neumann algebra on $\mathcal{H}$ can be represented faithfully though anti-linear on $\overline{\mathcal{H}}$.

We concluded that $\widetilde{C}$ is a $*$ preserving map form $\mathcal{L}(\mathcal{H},\mathcal{H})$ onto $\mathcal{L}\left(\overline{\mathcal{H}},\overline{\mathcal{H}}\right)$. We now want to find out how the adjoint behaves in the other cases. If given $B \in \mathcal{AL}\left(\overline{\mathcal{H}},\mathcal{H}\right)$ then B^* defines an element in $\mathcal{AL}\left(\mathcal{H},\overline{\mathcal{H}}\right)$. So how can B^* be found from B? Because $\mathcal{AL}\left(\overline{\mathcal{H}},\mathcal{H}\right) \cong \mathcal{L}(\mathcal{H},\mathcal{H})$ we have that B is given by $\widehat{A}$ for some $A \in \mathcal{L}(\mathcal{H},\mathcal{H})$ thus $B = Ac$. In order not to be distracted by domain considerations we assume B to be bounded, from there on it is easy to generalize to the unbounded case. By definition of the adjoint the following equality should hold for all $\overline{h} \in \overline{\mathcal{H}}$ and $v \in \mathcal{H}$

$$\left\langle B\left(\overline{h}\right), v\right\rangle_{\mathcal{H}} = \left\langle \overline{h}, B^* v\right\rangle_{\overline{\mathcal{H}}}.$$

But now we use our formula for B to find that

$$\begin{aligned}
\left\langle B\left(\overline{h}\right), v\right\rangle_{\mathcal{H}} &= \left\langle Ac\left(\overline{h}\right), v\right\rangle_{\mathcal{H}} \\
&= \left\langle \overline{h}, (Ac)^* v\right\rangle_{\overline{\mathcal{H}}} \\
&= \left\langle \overline{h}, cA^* v\right\rangle_{\overline{\mathcal{H}}} \\
&= \left\langle \overline{h}, \overline{A^*} v\right\rangle_{\overline{\mathcal{H}}}.
\end{aligned}$$

We conclude that $\left(\widehat{A}\right)^* = \overline{(A^*)}$. It follows that if $A \in \mathcal{L}(\mathcal{H},\mathcal{H})$ then $\overline{C}(A^*) = \widehat{C}(A)^*$. On the other hand we would like to have a formula for $\left(\overline{A}\right)^*$ via a similar argument we conclude that $\left(\overline{A}\right)^* = \widehat{(A^*)}$ it follows that $\overline{C}(A)^* = \widehat{C}(A^*)$. To summarize: we concluded that for $A \in \mathcal{L}(\mathcal{H},\mathcal{H})$ it holds that

- $\widetilde{C}(A)^* = \widetilde{C}(A^*)$.
- $\widehat{C}(A)^* = \overline{C}(A^*)$.
- $\overline{C}(A)^* = \widehat{C}(A^*)$.

Since $\mathcal{L}(\mathcal{H},\mathcal{H})$ allows for a product and an involution, we can define a product and an involution in $\mathcal{AL}\left(\overline{\mathcal{H}},\mathcal{H}\right)$ and $\mathcal{AL}\left(\mathcal{H},\overline{\mathcal{H}}\right)$. In $\mathcal{AL}\left(\overline{\mathcal{H}},\mathcal{H}\right)$ we define a product $\cdot$ and a involution $\dagger$ as follows:

$$\begin{aligned}
\widehat{A} \cdot \widehat{B} &:= \widehat{AB}, \\
\left(\widehat{A}\right)^{\dagger} &:= \widehat{A^*},
\end{aligned}$$

similarly we define an involution $\dagger$ and a product $\cdot$ in $\mathcal{AL}\left(\mathcal{H},\overline{\mathcal{H}}\right)$ by setting

$$\begin{aligned}
\overline{A} \cdot \overline{B} &:= \overline{AB}, \\
\overline{A}^{\dagger} &:= \overline{A^*}.
\end{aligned}$$

We use the notation $\dagger$ to avoid confusion between the involution and the adjoint. This is because the adjoint has a meaning as a map, while $\dagger$ is just a formal construction (although it relates to the real adjoint but then in a different space). It is easy to see that the maps $\overline{C}, \widehat{C}$ and $\widetilde{C}$ are norm preserving maps on $\mathcal{L}(\mathcal{H},\mathcal{H})$ to their respective targets, as such, the above introduced products and involutions define C^*-algebra structure's on their respective target spaces. Note furthermore that since $*$ defines an anti-linear isomorphism of $\mathcal{L}(\mathcal{H},\mathcal{H})$ we can define transformations derived from $*$ and c as follows: we define

$$\begin{aligned} \widehat{C}^* &: \mathcal{L}(\mathcal{H},\mathcal{H}) \longrightarrow \mathcal{AL}\left(\overline{\mathcal{H}},\mathcal{H}\right), && \widehat{C}^* := \widehat{C}\circ *, \\ \overline{C}^* &: \mathcal{L}(\mathcal{H},\mathcal{H}) \longrightarrow \mathcal{AL}\left(\mathcal{H},\overline{\mathcal{H}}\right), && \overline{C}^* := \overline{C}\circ *, \\ \widetilde{C}^* &: \mathcal{L}(\mathcal{H},\mathcal{H}) \longrightarrow \mathcal{L}\left(\overline{\mathcal{H}},\overline{\mathcal{H}}\right), && \widetilde{C}^* := \widetilde{C}\circ *. \end{aligned}$$

It follows that

$$\begin{aligned} \widehat{C}^* &\text{ is } \textit{anti-linear} \text{ since } \widehat{C} \text{ is } \textit{linear}, \\ \overline{C}^* &\text{ is } \textit{linear} \text{ since } \overline{C} \text{ is } \textit{anti-linear}, \\ \widetilde{C}^* &\text{ is } \textit{linear} \text{ since } \widetilde{C} \text{ is } \textit{anti-linear}. \end{aligned}$$

We will compare $\widehat{C}$ with $\widehat{C}^*$ and leave the cases $\overline{C}, \overline{C}^*$ and $\widetilde{C}, \widetilde{C}^*$ to the reader. We found that that $\widehat{C}$ has the following properties:

$$\begin{aligned} \widehat{C}(\alpha A + B) &= \alpha \widehat{C}(A) + \widehat{C}(B), \\ \widehat{C}(AB) &= \widehat{AB} = \widehat{A}\widehat{B} = \widehat{C}(A)\widehat{C}(B), \\ \widehat{C}(A^*) &= \widehat{A^*} = \left(\widehat{A}\right)^{\dagger} = \widehat{C}(A)^{\dagger}. \end{aligned}$$

Easy computations show that $\widehat{C}^*$ satisfies the following identities:

$$\begin{aligned} \widehat{C}^*(\alpha A + B) &= \overline{\alpha}\widehat{C}^*(A) + \widehat{C}^*(B), \\ \widehat{C}^*(AB) &= \widehat{C}^*(B)\widehat{C}^*(A), \\ \widehat{C}^*(A^*) &= \widehat{C}^*(A)^{\dagger}. \end{aligned}$$

So far we have discussed the similarity between the spaces $\mathcal{L}(\mathcal{H},\mathcal{H}), \mathcal{AL}\left(\overline{\mathcal{H}},\mathcal{H}\right), \mathcal{AL}\left(\mathcal{H},\overline{\mathcal{H}}\right)$ and $\mathcal{L}\left(\overline{\mathcal{H}},\overline{\mathcal{H}}\right)$. However our main interest in this section is anti-linear maps, the space $\mathcal{AL}(\mathcal{H},\mathcal{H})$. By our previous arguments $\mathcal{AL}(\mathcal{H},\mathcal{H}) \approx \mathcal{L}\left(\mathcal{H},\overline{\mathcal{H}}\right)$, considering $A \in \mathcal{AL}(\mathcal{H},\mathcal{H})$ gives rise to $\widehat{A} \in \mathcal{L}\left(\overline{\mathcal{H}},\mathcal{H}\right)$ and $\overline{A} \in \mathcal{L}\left(\mathcal{H},\overline{\mathcal{H}}\right)$ and also $\widehat{A} = c\overline{A}c$. We would like to extend the notion of a symmetric/selfadjoint operator to anti-linear operators, we do this using the symmetry between the spaces $\mathcal{L}(\mathcal{H},\overline{\mathcal{H}})$ and $\mathcal{L}(\overline{\mathcal{H}},\mathcal{H})$. We note that there are two canonical maps between $\mathcal{L}(\mathcal{H},\overline{\mathcal{H}})$ and $\mathcal{L}(\overline{\mathcal{H}},\mathcal{H})$ namely

$$\begin{aligned} \mathcal{L}(\mathcal{H},\overline{\mathcal{H}}) &\longleftrightarrow \mathcal{L}(\overline{\mathcal{H}},\mathcal{H}) \\ A &\longleftrightarrow \widetilde{A} = cAc \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(\mathcal{H},\overline{\mathcal{H}}) &\longleftrightarrow \mathcal{L}(\overline{\mathcal{H}},\mathcal{H}) \\ A &\longrightarrow A^* \\ B^* &\longleftarrow B \end{aligned}$$

Definition 6.2.0.9. *We say that $A \in \mathcal{L}(\mathcal{H},\overline{\mathcal{H}})$ is symmetric when $\widetilde{A^*} = cA^*c = \left(\widetilde{A}\right)^*$ defines a closed extension of A, we say that A is selfadjoint when $A = cA^*c$.*

Given $A \in \mathcal{L}\left(\mathcal{H},\overline{\mathcal{H}}\right)$ and $h \in \mathcal{D}(A)$ and $\overline{f} \in \overline{\mathcal{D}(A)}$ and suppose that $A \in \mathcal{L}\left(\mathcal{H},\overline{\mathcal{H}}\right)$ is symmetric. Then it follows that

$$\begin{aligned} \langle Ah, \overline{f}\rangle_{\overline{\mathcal{H}}} &= \langle cA^*c(h), \overline{f}\rangle_{\overline{\mathcal{H}}} \\ &= \langle \overline{h}, Af\rangle_{\overline{\mathcal{H}}}, \end{aligned}$$

in particular $\langle Ah, \overline{h} \rangle_{\overline{\mathcal{H}}} \in \mathbb{R}$ for all $h \in \mathcal{D}(A)$. Note also that if A is symmetric then cAc is symmetric.

The notion of the spectrum carries over to anti-linear maps in the following way; If given an anti-linear map, then that map defines an element $A \in \mathcal{L}(\mathcal{H}, \overline{\mathcal{H}})$ and an element $\widetilde{A} = cAc \in \mathcal{L}(\overline{\mathcal{H}}, \mathcal{H})$. In total A defines a linear operator in $A_2 \in \mathcal{L}(\mathcal{H} \oplus \overline{\mathcal{H}}, \mathcal{H} \oplus \overline{\mathcal{H}})$ as follows

$$A_2 := \begin{pmatrix} 0 & cAc \\ A & 0 \end{pmatrix}.$$

We see that A_2 defines a linear map from $\mathcal{H} \oplus \overline{\mathcal{H}}$ into itself and as such we can consider its spectrum.

Definition 6.2.0.10. *If A is linear map form H into $\overline{H}$ then we define its spectrum $\sigma(A)$ as follows*

$$\sigma(A) := \sigma(A_2) := \sigma \begin{pmatrix} 0 & cAc \\ A & 0 \end{pmatrix}.$$

It is easy to see that A is symmetric $\iff A_2$ is symmetric. Suppose that $A \in \mathcal{L}(\mathcal{H}, \overline{\mathcal{H}})$ is bounded, then consider the map $A_2 \in \mathcal{L}(\mathcal{H} \oplus \overline{\mathcal{H}}, \mathcal{H} \oplus \overline{\mathcal{H}})$. Since A is bounded so is A_2, furthermore since $A_2 \in B(\mathcal{H} \oplus \overline{\mathcal{H}})$ it allows for a polar decomposition. Note that $|A_2| := \sqrt{A_2^* A_2}$ has the following form:

$$|A_2| = \begin{pmatrix} \sqrt{A^*A} & 0 \\ 0 & \sqrt{cA^*Ac} \end{pmatrix}.$$

By the polar decomposition, there exists some partial isometry $U \in B(\mathcal{H} \oplus \overline{\mathcal{H}})$ such that $U \cdot |A_2| = A_2$. A small calculation yields that U is of the following form:

$$U = \begin{pmatrix} 0 & u_{12} \\ u_{21} & 0 \end{pmatrix}.$$

We see that $u_{21}\sqrt{A^*A} = A$ and $u_{12}\sqrt{cA^*Ac} = cAc$. By definition we have that u_{21} defines a partial isometry from H to $\overline{H}$ and thus we found a polar decomposition for A and also one for cAc. Note also that $\sqrt{cA^*Ac} = c\sqrt{A^*A}c$ so that $u_{12} = cu_{21}c$. We conclude that any anti-linear operator A on $\mathcal{H}$ can be decomposed as $u|A| = A$ with u a partial anti-isometry.

The notion of a normal operator carries over to anti-linear operators in the following way: A is called normal when A is closed and the equality $A^*A = cAA^*c$ holds. We conclude this intermezzo by saying that the embedding $\mathcal{L}(\mathcal{H}, \overline{\mathcal{H}}) \ni A \longrightarrow A_2 \in \mathcal{L}(\mathcal{H} \oplus \overline{\mathcal{H}}, \mathcal{H} \oplus \overline{\mathcal{H}})$ as described above can be used to transfer known statements about linear operators to corresponding statements about anti-linear operators, basically solving anti-linear operators in terms of linear operators.

6.3 The polar decomposition and Spectral theorem for unbounded (anti-) linear operators

In this section we will cover the spectral theorem and the polar decomposition for linear operators. We will cover the spectral theorem for bounded normal operators, for the general case we refer to [1]. The generalization of the polar decomposition to antilinear operators then follows from the observations in made in 6.2. These theorems allow us to classify abelian von Neumann algebras. Also the Tomita Takesaki theorem 4.1.0.7 relies on the polar decomposition and the spectral theorem. We start out with definition of a spectral measure.

Definition 6.3.0.11. *Let (X, Ω) be a measurable space (with Ω a σ-algebra of subsets), a spectral measure on (X, Ω) is a map $P : \Omega \longrightarrow \mathcal{B}(\mathcal{H})$ such that the following holds*

1. *$P(E)$ is a projection for all $E \in \Omega$;*
2. *$P(\emptyset) = 0$ and $P(X) = 1$;*
3. *$P(E_1 \cap E_2) = P(E_1) P(E_2)$;*

4. *If* $\{E_n\}_{n=1}^{\infty}$ *are pairwise disjoint then* $P\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} P(E_n)$.

Note here that for such a spectral measure P we have that $P_{g,h}(E) := \langle P(E)g, h\rangle$ defines an ordinary complex valued measure on X. We will show that the this measure $P_{g,h}$ is of bounded total variation and use this to show how we can integrate with respect to a spectral measure.

Proposition 6.3.0.12. *If P is a spectral measure on (X, Ω) then for all $g, h \in \mathcal{H}$ the measure $P_{g,h}$ defined by*

$$P_{g,h}(E) := \langle P(E)g, h\rangle\,,$$

defines an ordinary complex measure with total variation $\|g\| \cdot \|h\|$.

Proof:

Consider a spectral measure P on (X, Ω) and let $g, h \in \mathcal{H}$. It is obvious that $P_{g,h}$ satisfies the requirements for a complex measure, so we will show that it is of bounded total variation. The total variation of a measure μ, denoted by $\|\mu\|$, is defined as the supremum of $\sum_{j=1}^{n} |\mu(E_j)|$ where $\{E_1, ..., E_n\}$ runs over the finite partitions of X, so

$$\|P_{g,h}\| := \sup\left\{\sum_{j=1}^{n} |P_{g,h}(E_j)|\ ;\ \{E_j\}_{j=1}^{n} \text{ is a finite partition of } X\right\}.$$

Consider a partition $\{E_1, ..., E_n\}$ of X and pick complex numbers α_j such that $|P_{g,h}(E_j)| = \alpha_j P_{g,h}(E_j) = \langle \alpha_j P(E_j)g, h\rangle$. Note that $|\alpha_j| = 1$. It follows that

$$\begin{aligned}
\sum_{j=1}^{n} |P_{g,h}(E_j)| &= \sum_{j=1}^{n} \alpha_j P_{g,h}(E_j) \\
&= \sum_{j=1}^{n} \alpha_j \langle P(E_j)g, h\rangle \\
&= \left\langle \sum_{j=1}^{n} P(E_j)\alpha_j g, h\right\rangle \\
&\leq \left\|\sum_{j=1}^{n} P(E_j)\alpha_j g\right\| \cdot \|h\| \\
&\leq \left\|\sum_{j=1}^{n} P(E_j)\right\| \|g\|\,\|h\|\,.
\end{aligned}$$

We use that $\{E_1, ..., E_n\}$ is a partition to conclude that $\sum_{j=1}^{n} P(E_j)$ is a projection. We find that

$$\left\|\sum_{j=1}^{n} P(E_j)\right\| = 1.$$

The conclusion is that $\|P_{g,h}\| \leq \|g\| \cdot \|h\|$, as desired. □

The next result shows how to integrate with respect to a spectral measure. Denote by $B(X, \Omega)$ the bounded measurable functions $f : X \longrightarrow \mathbb{C}$.

Theorem 6.3.0.13. *If P is a spectral measure on (X, Ω) and $f \in B(X, \Omega)$. Then there exists a unique operator $A \in \mathcal{B}(\mathcal{H})$ such that for any $\epsilon > 0$ and $\{E_1, ..., E_n\}$ a partition of X with* $\sup\{|f(x_1) - f(x_2)|\ ;\ x_1, x_2 \in E_j\} \leq \epsilon$ *for all $j \leq n$ the inequality:*

$$\left\|A - \sum_{j=1}^{n} f(x_j) P(E_j)\right\| \leq \epsilon,$$

holds for any $x_j \in E_j$.

Proof:

Let $B(g,h) := \int f\, dP_{g,h}$ then B defines a sesquilinear form on $\mathcal{H}$. As such B defines a unique operator $A \in \mathcal{B}(\mathcal{H})$ with the property that

$$\langle Ag, h\rangle = B(g,h).$$

Let $\epsilon > 0$ and let $\{E_j, ..., E_n\}$ be a partition such that $\sup\{|f(x_1) - f(x_2)| \; ; \; x_1, x_2 \in E_j\} \leq \epsilon$ for all $j \leq n$. Then it follows that

$$\begin{aligned}
\left|\langle Ag, h\rangle - \sum_{j=1}^{n} f(x_j)\langle P(E_j) g, h\rangle\right| &= \left|\int f(x)\, dP_{g,h} - \sum_{j=1}^{n} f(x_k)\langle P(E_j) g, h\rangle\right| \\
&= \left|\sum_{j=1}^{n}\int_{E_j} f(x) - f(x_j)\, d\langle P(E_j) g, h\rangle\right| \\
&\leq \sum_{j=1}^{n}\int_{E_j} |f(x) - f(x_j)|\, d|\langle P(E_j) g, h\rangle| \\
&\leq \epsilon\|g\| \cdot \|h\|.
\end{aligned}$$

Pick $h = Ag$ and take the supremum over $\|g\| \leq 1$ to conclude that this convergence is in norm. □

The operator A obtained in this way is the integral of f with respect to the spectral measure P, it is denoted as

$$A := \int f\ dP.$$

Note that $B(X, \Omega)$ with the supremum norm is a unital C^*-algebra, we define a map $\pi : B(X, \Omega) \longrightarrow \mathcal{B}(\mathcal{H})$ by setting

$$\pi(f) := \int f\ dP.$$

Proposition 6.3.0.14. *The map π is an isometric representation of $B(X, \Omega)$ into $\mathcal{B}(H)$.*

Proof:

If f is a simple function in $B(X, \Omega)$, meaning that f is of the form $\sum_{j=1}^{n} \lambda_j 1_{E_j}$ for some partition $\{E_1, ..., E_n\}$ of X, then $\|\pi(f)\| = \|f\|$. By density of the simple functions in $B(X, \Omega)$ we conclude that π is an isometry. It is not hard to see that π is linear and respects the star operation. However it is not immediate that π is multiplicative. To see this let $f, g \in B(X, \Omega)$, and consider a partition $\{E_1, ..., E_n\}$ with the property that for f, g and fg we have $\sup\{|r(x_1) - r(x_2)| \; ; \; x_1, x_2 \in E_j\} < \epsilon$, here r denotes f, g or fg. We find the following

$$\left\|\pi(fg) - \sum_{j=1}^{n} fg(x_j) P(E_j)\right\| = \left\|\pi(fg) - \left(\sum_{j=1}^{n} f(x_j) P(E_j)\right)\left(\sum_{i=1}^{n} g(x_i) P(E_i)\right)\right\| \leq \epsilon,$$

so π is multiplicative. □

We conclude that spectral measures give rise to representations, the converse to this is also true, namely representations give rise to spectral measures.

Theorem 6.3.0.15. *If $\pi : C(X) \longrightarrow \mathcal{B}(\mathcal{H})$ is a representation then there exists a unique spectral measure P, on the Borel σ-algebra of X, such that*

$$\pi(f) = \int f\ dP.$$

Proof

For $g, h \in \mathcal{H}$ define a linear functional $[g,h] : C(X) \longrightarrow \mathbb{C}$ by setting $[g,h](f) := \langle \pi(f) g, h \rangle$. There exists some element $\mu_{g,h} \in C(X)^*$ such that $[g,h](f) = \mu_{g,h}(f) = \int f \, d\mu_{g,h}$. Because $\mu_{g,h}$ also makes sense for $\phi \in B(X,\Omega)$ (with Ω being the σ-algebra generated by the topology) we have that $[g,h]$ extends to a linear functional on $B(X,\Omega)$. So for $\phi \in B(X,\Omega)$ we define a sesquilinear map on $\mathcal{H}$ as follows

$$[\phi](g,h) := \int \phi \, d\mu_{g,h}.$$

Note that $[\phi]$ defines a bounded operator A on $\mathcal{H}$ such that $[\phi](g,h) = \langle Ag, h \rangle$. Define $\widehat{\pi}(\phi) := A$. It is obvious that if $f \in C(X)$, then $\widehat{\pi}(f) = \pi(f)$. The claim is that $\widehat{\pi}$ is a representation. We first show that $\widehat{\pi}$ is multiplicative. Consider functions ϕ and ξ in $B(X,\Omega)$ and consider them as elements of $C(X)^{**}$. Thus if $\mu \in C(X)^*$ then $f(\mu) = \mu(f) = \int f \, d\mu$. Recall that the natural embedding $i : C(X) \longrightarrow C(X)^{**}$ maps the unit ball of $C(X)$ into a dense set of the unit ball of $C(X)^{**}$, dense in the $\sigma\left(C(X)^{**}, C(X)^*\right)$ topology (thus $l_i \longrightarrow l$ in $\sigma\left(C(X)^{**}, C(X)^*\right)$ when for all $\mu \in C(X)^*$ it holds that $l_i(\mu) \longrightarrow l(\mu)$). We conclude that there is a net $\{\phi_i\} \subset C(X)$ such that $\int \phi_i d\mu \longrightarrow \int \phi \, d\mu$ for all $\mu \in C(X)^*$. Note also that $\widehat{\pi}$ is weakly continuous because if $\phi_i \longrightarrow \phi$ then $\langle \widehat{\pi}(\phi_i) g, h \rangle = \int \phi_i d\mu_{g,h} \longrightarrow \int \phi \, d\mu_{g,h} = \langle \widehat{\pi}(\phi) g, h \rangle$. Now if $\xi \in B(X,\Omega)$ and $\mu \in C(X)^*$ then $\xi\mu \in C(X)^*$ simply because $\xi\mu(f) = \int f\xi \, d\mu$. We find that if $\phi_i \longrightarrow \phi$ then

$$\widehat{\pi}(\phi_i \xi) \longrightarrow \widehat{\pi}(\phi\xi),$$

in the weak operator topology. In particular if $\xi \in C(X)$ then

$$\widehat{\pi}(\phi\xi) = \lim_i \widehat{\pi}(\phi_i \xi) = \lim_i \pi(\phi_i \xi) = \lim_i \pi(\phi_i) \pi(\xi) = \widehat{\pi}(\phi) \widehat{\pi}(\xi).$$

Thus we conclude that if $\phi \in B(X,\Omega)$ and $\xi \in C(X)$ then $\widehat{\pi}(\phi\xi) = \widehat{\pi}(\phi)\widehat{\pi}(\xi)$. But now the result follows easily because if $\{\phi_i\}$ is a net in $C(X)$ converging to ϕ, and $\xi \in B(X,\Omega)$, then by our previous arguments we have

$$\widehat{\pi}(\phi\xi) = \lim_i \widehat{\pi}(\phi_i \xi) = \lim_i \widehat{\pi}(\phi_i) \widehat{\pi}(\xi) = \widehat{\pi}(\phi) \widehat{\pi}(\xi).$$

We conclude that $\widehat{\pi}$ is multiplicative. That it is linear follows immediately from the linearity of the integral. Now to show that $\widehat{\pi}$ is *-preserving, let $\phi \in B(X,\Omega)$ and let $\{\phi_i\}$ be a net in $C(X)$ converging to ϕ. Then it follows that $\phi_i^* \longrightarrow \phi_i^*$ because if μ is a measure then $\overline{\mu}$ is a measure, using this we find that

$$\begin{aligned} \phi_i^*(\mu) &= \int \overline{\phi_i} d\mu \\ &= \overline{\int \phi_i d\overline{\mu}} \longrightarrow \overline{\int \phi \, d\overline{\mu}} = \overline{\phi}(\mu) = \phi^*(\mu), \end{aligned}$$

so $\phi_i^* \longrightarrow \phi_i^*$. We find that $\widehat{\pi}(\phi_i^*) \longrightarrow \widehat{\pi}(\phi^*)$ (since $\widehat{\pi}$ is weakly continuous). We conclude that

$$\widehat{\pi}(\phi^*) = \lim_i \widehat{\pi}(\phi_i^*) = \lim_i \pi(\phi_i^*) = \lim_i \pi(\phi_i)^* = \lim_i \widehat{\pi}(\phi_i)^*.$$

Now we investigate the limit $\lim_i \widehat{\pi}(\phi_i)^*$, we have

$$\begin{aligned} \lim_i \left\langle \widehat{\pi}(\phi_i)^* g, h \right\rangle &= \lim_i \overline{\langle \widehat{\pi}(\phi_i) h, g \rangle} \\ &= \overline{\langle \widehat{\pi}(\phi) h, g \rangle} \\ &= \left\langle \widehat{\pi}(\phi)^* g, h \right\rangle. \end{aligned}$$

Combining these results we conclude

$$\widehat{\pi}(\phi^*) = \lim_i \widehat{\pi}(\phi_i^*) = \lim_i \widehat{\pi}(\phi_i)^* = \widehat{\pi}(\phi)^*,$$

so $\widehat{\pi}$ is *-preserving.
So far we have concluded that $\widehat{\pi} : B(X,\Omega) \longrightarrow B(\mathcal{H})$ is a representation. We will now construct the

spectral measure associated with $\widehat{\pi}$. For each measurable subset E define χ_E as the indicator function for E, define now

$$P(E) := \widehat{\pi}(\chi_E).$$

It follows that $P(E)$ is a projection for each measurable subset E. If E is the empty set then χ_E is the zero function so $P(E) = 0$, if $E = X$ then $\chi_E = 1$ so $P(E) = 1$. For $E, F \in \Omega$ we have $P(E \cap F) = \widehat{\pi}(\chi_{E\cap F}) = \widehat{\pi}(\chi_E \chi_F) = \widehat{\pi}(\chi_E)\widehat{\pi}(\chi_F) = P(E)P(F)$. Let $\{E_i\}_{i\in\mathbb{N}}$ be a countable pairwise disjoint collection of measurable sets of X, we aim to show that $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$. Set $F = \bigcup_{i=1}^{\infty} E_i$ and set $F_n = \bigcup_{i=1}^{n} E_i$, consider $\|P(F \setminus F_n)h\|^2$. We find

$$\begin{aligned} \|P(F\setminus F_n)h\|^2 &= \langle P(F\setminus F_n)h, P(F\setminus F_n)h\rangle \\ &= \langle P(F\setminus F_n)h, h\rangle \\ &= \int \chi_{F\setminus F_n} dP_{h,h} \\ &= \sum_{i=n+1}^{\infty} P_{h,h}(E_i). \end{aligned}$$

But $P_{h,h}$ is a countably additive measure so $\sum_{i=n+1}^{\infty} P_{h,h}(E_i) \longrightarrow 0$ as $n \longrightarrow \infty$. We conclude that $P(F)h = \sum_{i=1}^{\infty} P(E_i)h$, that is, $P(F) = \sum_{i=1}^{\infty} P(E_i)$, hence P is a spectral measure. It remains to be shown that $\widehat{\pi}(f) = \int f\, dP$ for all $f \in C(X)$, and that a spectral measure with this property is unique. We shall show this for all $\phi \in B(X,\Omega)$. Let $\phi \in B(X,\Omega)$ then for all $\epsilon > 0$ there exists a partition $\{E_1, ..., E_n\}$ of X such that $\sup\{|\phi(x) - \phi(x')|\ ;\ x, x' \in E_i\} < \epsilon$ for all $i \leq n$. It follows that $\|\phi - \sum_{i=1}^{n}\phi(x_i)\chi_{E_i}\| \leq \epsilon$ (here for all $x_i \in E_i$). Since $\widehat{\pi}$ is a *-homomorphism we have $\|\widehat{\pi}\| \leq 1$ thus it follows that

$$\left\|\widehat{\pi}\left(\phi - \sum_{i=1}^{n}\phi(x_i)\chi_{E_i}\right)\right\| = \left\|\widehat{\pi}(\phi) - \sum_{i=1}^{n}\phi(x_i)P(E_i)\right\| \leq \left\|\phi - \sum_{i=1}^{n}\phi(x_i)\chi_{E_i}\right\| \leq \epsilon.$$

In other words $\widehat{\pi}$ is the integral of ϕ against P. For uniqueness just note that if $\widehat{\pi}(\phi) = \int \phi\, dP = \int \phi\, dP_2$ then in particular for all $E \in \Omega$ we have $\widehat{\pi}(\chi_E) = P(E) = P_2(E)$ thus $P_2 = P$. □

Note here that any representation $\pi : C(X) \longrightarrow B(\mathcal{H})$ extends uniquely (by the previous argument) to a representation of $B(X,\Omega)$. Now we can prove one of the most powerful tools in functional analysis, the so called spectral theorem. It completely describes normal operators and as a consequence it describes abelian operator algebras.

Theorem 6.3.0.16 (The Spectral Theorem, bounded case)**.** *Let $A \in \mathcal{B}(\mathcal{H})$ be a normal operator. Then there exists a unique spectral measure P on the Borel subsets of $\sigma(A)$ such that the following statements hold*

1. $A = \int x\, dP(x)$*;*
2. *If $U \subset \sigma(A)$ is open, then $P(U) \neq 0$;*
3. *If $B \in B(\mathcal{H})$ then $AB = BA$ and $A^*B = BA^*$ if and only if $BP(E) = P(E)B$ for every Borel set E in $\sigma(A)$.*

Proof:

Since A is normal we conclude that $C^*(A)$, the unital C^*-algebra generated by A and 1, is abelian. By the Gelfand transform we have $C^*(A) \cong C(\sigma(A))$. The inverse Gelfand transform, ρ^{-1}, is therefor a representation of $C(\sigma(A))$ into $\mathcal{B}(\mathcal{H})$. By 6.3.0.15 every operator in $C^*(A)$ is given by integration against the unique spectral measure P associated to ρ^{-1}.

1.
The image of $A \in C^*(A)$ in $C(\sigma(A))$ is the identity function of $\sigma(A)$. We conclude $A = \int x\, dP$

2.
Suppose $U \subset \sigma(A)$ is open. There exists some nonzero continuous function f with the property that $f(x) \leq \chi_U$. Although χ_U cannot be assumed to be in $C^*(A)$, it is a Borel function. By 6.3.0.15 we have that $\rho^{-1}(\chi_U) = P(U)$ is well defined. Since ρ^{-1} is an isomorphism we find that $P(U) = \widehat{\rho}^{-1}(\chi_U) \geq \rho^{-1}(f) \neq 0$.

3.
Suppose that B commutes with A and A^* then $B \in C^*(A)'$ we will show that B in fact commutes with every operator of the form $\widehat{\rho}^{-1}(\phi)$ with ϕ a bounded Borel measurable function on $\sigma(A)$. Let $\{\phi_i\}$ be a net in $C(\sigma(A))$ converging to $\phi \in B(\sigma(A))$, we find

$$\widehat{\rho}^{-1}(\phi) B = \lim_i \widehat{\rho}^{-1}(\phi_i) B = \lim_i B\widehat{\rho}^{-1}(\phi_i) = B \lim_i \widehat{\rho}^{-1}(\phi_i) = B\widehat{\rho}^{-1}(\phi).$$

Note here that we can in fact take B out of the limit because multiplication with an operator is weakly continuous. In particular we find that B commutes with every element of the form $\widehat{\rho}^{-1}(\chi_E)$ but $\widehat{\rho}^{-1}(\chi_E) = P(E)$ thus $P(E)B = BP(E)$ for every measurable set. Suppose that $BP(E) = P(E)B$ for every Borel set E. Then B also commutes with the weak closure of span $\{P(E)\ ;\ E \text{ is a Borel measurable set}\}$. Note now that the weak closure of
span $\{P(E)\ ;\ E \text{ is a Borel measurable set}\}$ contains A and A^*. □

Now the most general version of the spectral theorem. For proof see [1].

Theorem 6.3.0.17 (The spectral theorem for linear operators, general case)**.** *Suppose that A is a normal linear operator, then we can find a spectral measure P defined on all Borel subsets of $\mathbb{C}$ satisfying the following properties:*

1. $A = \int z \, dP$.
2. $P(E) = 0$ *when* $E \cap \sigma(A) = \emptyset$.
3. *If* $U \subset \mathbb{C}$ *is open and* $U \cap \sigma(A) \neq \emptyset$ *then* $P(U) \neq 0$.
4. *If* $B \in B(\mathcal{H})$ *is bounded such that* AB *is an extension of* BA *and* AB^* *is an extension of* B^*A *then*

$$\left(\int \phi \, dP\right) B,$$

defines an extension of

$$B\left(\int \phi \, dP\right),$$

for all Borel functions ϕ *on* $\mathbb{C}$.

The integral appearing in the spectral theorem should be considered as the usual integral appearing in the bounded version of the spectral theorem.

We will now cover the polar decomposition for linear operators.

Theorem 6.3.0.18. *Suppose that A is a closed densely defined (anti) linear operator then we can decompose A as*

$$A = U\sqrt{A^*A}.$$

Here U is a partial isometry.

Proof:

For $h \in \mathcal{D}(A) := \text{dom}(A)$, we note that

$$\begin{aligned}\|Ah\|^2 = \langle Ah, Ah\rangle &= \langle A^*Ah, h\rangle \\ &= \left\langle |A|^2 h, h\right\rangle \\ &= \langle |A| h, |A\rangle h\rangle .\end{aligned}$$

We define a map $U_0 : \mathcal{H} \longrightarrow \mathcal{H}$ as follows: for $h \in \ker(|A|)$ we set $U_0(h) = 0$, for $|A|\, h \in \operatorname{ran}(|A|)$ we set $U_0(|A|\, h) = Ah$. Note that $\overline{\operatorname{ran}(|A|)} = \ker(|A|)^{\perp}$ by 6.1.1.4, it follows that U_0 is densely defined. For all g in $\operatorname{ran}(|A|)$ we have the identity $\|U_0(g)\| = \|g\|$, as such, U is a densely defined partial isometry. We extend U_0 to the whole of $\mathcal{H}$ and denote its extension by U. Consider $h \in \mathcal{D}(A)$ if $h \in \ker(A)$ then $0 = Ah = U|A|h$, so A and $U|A|$ agree on $\ker(A)$. Suppose that $h \in \ker(A)^{\perp}$ then $U|A|h = Ah$ by construction. It follows that $U|A| = A$ as desired. Note that we can also pick other partial isometries $\widehat{U}$, as long as $\widehat{U}$ agrees with U on $\operatorname{ran}|A|$. The partial isometry U constructed above is precisely the one that gives the isomorphism between the Hilbert spaces $\ker(A)^{\perp}$ and $\overline{\operatorname{ran}(A)}$. If A is anti-linear then we can linearize it using the construction explained in the section on conjugate vector spaces and obtain a similar decomposition. □

6.4 Integrals of operators

Given any Hilbert space $\mathcal{H}$ and a measure space (Γ, Σ, μ) with Γ the space, Σ the σ-ring of subsets and μ the measure, we can consider collections $\{A_\gamma \ ; \ \gamma \in \Gamma\}$ of operators in $\mathcal{B}(\mathcal{H})$ indexed by Γ. We call such a collection measurable if for all $h, v \in \mathcal{H}$ it holds that the map

$$\gamma \longrightarrow \langle A_\gamma h, v \rangle ,$$

is a $\mathbb{C}$ measurable map. Suppose that $\{A_\gamma\}$ defines such a measurable collection and suppose furthermore that

$$\sup \{ |\langle A_\gamma h, v \rangle| \ ; \ \gamma \in \Gamma, \|h\| \leq 1, \|v\| \leq 1 \} = M < \infty.$$

For any $g \in L^1(\Gamma, \mu)$ we can then consider the map

$$\int_\Gamma g(\gamma) \langle A_\gamma h, v \rangle \ d\mu.$$

By construction it follows that

$$\left| \int_\Gamma g(\gamma) \langle A_\gamma h, v \rangle \right| \ d\mu \leq M \cdot \|h\| \cdot \|v\| \cdot \int |g(\gamma)| \ d\mu < \infty.$$

Derived from the collection $\{A_\gamma \ ; \ \gamma \in \Gamma\}$ and the function g, we define now a new sesquilinear form on $\mathcal{H}$ as follows:

$$\left[\int_\Gamma g(\gamma) A_\gamma \ d\mu \right] (h, v) := \int_\Gamma g(\gamma) \langle A_\gamma h, v \rangle \ d\mu.$$

It is easy to see that $\left[\int_\Gamma g(\gamma) A_\gamma \ d\mu \right]$ is sesquilinear, as such it gives rise to a bounded operator on $\mathcal{H}$ which we will also denote as

$$\int_\Gamma g(\gamma) A_\gamma \ d\mu.$$

By construction we have that

$$\left\langle \left(\int_\Gamma g(\gamma) A_\gamma \ d\mu \right) h, v \right\rangle = \int_\Gamma g(\gamma) \langle A_\gamma h, v \rangle \ d\mu.$$

Consider, for fixed $T \in \mathcal{B}(\mathcal{H})$, the map

$$T \left(\int_\Gamma g(\gamma) A_\gamma \ d\mu \right).$$

For $h, v \in \mathcal{H}$ we find the following identities

$$\begin{aligned} \left\langle T \left(\int_\Gamma g(\gamma) A_\gamma \ d\mu \right) h, v \right\rangle &= \left\langle \left(\int_\Gamma g(\gamma) A_\gamma \ d\mu \right) h, T^* v \right\rangle \\ &= \int_\Gamma g(\gamma) \langle A_\gamma h, T^* v \rangle \ d\mu \\ &= \int_\Gamma g(\gamma) \langle T A_\gamma h, v \rangle \ d\mu. \end{aligned}$$

It follows that

$$T\left(\int_\Gamma g(\gamma)A_\gamma \ d\mu\right) = \int_\Gamma g(\gamma)TA_\gamma \ d\mu.$$

Consider

$$\left(\int_\Gamma g(\gamma)A_\gamma \ d\mu\right)^*,$$

we aim to find an expression for this in terms of g and A_γ. For $h, v \in \mathcal{H}$ we find

$$\begin{aligned}
\left\langle\left(\int_\Gamma g(\gamma)A_\gamma \ d\mu\right)^* h, v\right\rangle &= \overline{\left\langle\left(\int_\Gamma g(\gamma)A_\gamma \ d\mu\right) v, h\right\rangle} \\
&= \overline{\int_\Gamma g(\gamma)\left\langle A_\gamma v, h\right\rangle \ d\mu} \\
&= \int_\Gamma \overline{g(\gamma)} \cdot \overline{\left\langle A_\gamma v, h\right\rangle} \ d\mu \\
&= \int_\Gamma \overline{g(\gamma)} \left\langle A_\gamma^* h, v\right\rangle \ d\mu,
\end{aligned}$$

it follows that

$$\left(\int_\Gamma g(\gamma)A_\gamma \ d\mu\right)^* = \int_\Gamma \overline{g(\gamma)}A_\gamma^* \ d\mu.$$

Using this identity we derive that for fixed $T \in \mathcal{B}(\mathcal{H})$ it holds that

$$\left(\int_\Gamma g(\gamma)A_\gamma \ d\mu\right)T = \int_\Gamma g(\gamma)A_\gamma T \ d\mu.$$

We see that integration in this sense behaves well with respect to multiplication and the star operation. There is much to say about integrals of operators, for example we can now examine the average of a collection of operators as a linear operator, but we will not go into this as we only need the basic construction.

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