Orthonormal bases in Inverse semigroups, a categorical approach.

Thesis submitted for the degree of Master of Science.

Pim van der Hoorn
Department of Mathematics, University of Utrecht

Abstract

The role the choices of orthogonal bases play in the structure of the category **Hilb**, remains a problem in categorical quantum mechanics. In this thesis we take a closer look at the finite dimensional Hilbert spaces. We will show that there is a link between these and certain inverse semigroups. Thus reducing this geometric problem to a problem in the field of combinatorics.

Using the work of Samson Abramsky *et al.*, we will show that we have an equivalence between the categories $\mathbb{F}\text{rob}(\mathbf{PInj})$ and $\mathbb{F}\text{rob}(\mathbf{Hilb})$ of Frobenius semigroups. Next, we will study symmetric inverse semigroups and construct the category **RepInv** of representable inverse semigroups. Using the above equivalence and the Wagner-Preston representation we prove that we have an adjunction between **RepInv** and $\mathbb{F}\text{rob}(\mathbf{Hilb})$. This shows that these inverse semigroups carry a structure similar to that of finite- dimensional Hilbert spaces.

Acknowledgments

This thesis could not have possible without the guidance of Chris Heunen en Jaap van Oosten. I would like to thank them for their suggestions, insights and comments. At times, when I was completely stuck, they helped me to get back on track.

Contents

Chapter 1

Introduction

Throughout the years the theory of quantum mechanics has always fascinated mathematicians and has been the basis for many parts of modern mathematics. Much work has been put into revealing the mathematical structure behind this physical theory. For some years now, this has been done on the level of category theory. Categorical quantum mechanics tries to build a general abstract theory for quantum systems, as well as understand the interactions of classical quantum systems.

Although the structure of the category **Hilb** of Hilbert spaces and bounded linear functions is almost fully understood. It is not known what role the choice of an orthogonal basis plays here. The existence of this problem could be explained by saying that the numerous properties of each individual choice of an orthogonal basis make it hard to distinct which actually depend on this choice.

In this thesis we try to explore this problem by looking at the relation between finite-dimensional Hilbert spaces and inverse semigroups. We arrive at inverse semigroups because of the relation between Hilbert spaces and the category **PInj** of sets and partial injective functions. From the latter we can form so called symmetric inverse semigroups in which every inverse semigoup can be represented by means of the Wagner-Preston representation theorem. We explore the structure of these symmetric inverse semigroups and construct a new, minimal, representation for certain finite inverse semigroups. This minimal representation will give rise to a functor and an adjunction between these inverse semigroups and Hilbert spaces.

Because we only consider finite-dimensional Hilbert space, the category **Hilb** only has finite-dimensional Hilbert spaces as objects. Similar, **PInj** has finite sets.

We have structured this thesis as follows. Chapter 1 will serve as an intro-

duction to the categorical theory of symmetric monoidal dagger categories, Frobenius algebras and Frobenius inverse semigroups. We also give an extension theorem for functors between monoidal categories to functors between Frobenius inverse semigroups. In chapter 2, we will give two examples of symmetric monoidal dagger categories **PInj** and **Hilb**. We show that the categories of their Frobenius inverse semigroups $\mathbb{F}$rob(**PInj**) and $\mathbb{F}$rob(**Hilb**) are equivalent. After this, we move in to inverse semigroups. We start with a brief introduction of inverse semigroups in chapter 3. Next, we investigate symmetric inverse semigroups and give a characterization of them. Chapter 4 introduces the definition of representable inverse semigroups. We show that for these inverse semigroups we can construct a minimal representation. The last chapter puts all results together. Here we prove several adjunctions between subcategories of inverse semigroups and Frobenius inverse semigroups of Hilbert spaces. At the end, we will shortly discuss the possible implications of these adjunctions, as well as give some ideas for future research.

Chapter 2

Categorical framework

In order to arrive at the results we need to recall some notions of category theory as well as introduce some new ones. In this chapter we will define a number of categorical properties such as Dagger Categories, Monoidal Categories and Frobenius inverse semigroup. We will also define the notion of a Monoidal and Frobenius functor and develop extension theorems for such functors. These will be used later to construct a functor between the categories $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ and $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$. More about this in chapter 2.4.3.

2.1 Dagger Categories

In quantum mechanics one usually deals with bounded linear operators A on some Hilbert space. These operators have an adjoint, which is again a linear operator and is denoted by $A^\dagger$. Taking an adjoint can therefore be viewed as an operation on the set of bounded linear operators. This idea is converted to the field of category theory in the form of *dagger categories*. For those who know some of the properties of the adjoint the following definition should look familiar.

Definition 2.1. *A* dagger category $\mathcal{C}$ *is a category together with an identity on objects functor* $\mathcal{C}^{op} \xrightarrow{\ \dagger\ } \mathcal{C}$ *such that for all morphisms* $a \xrightarrow{\ f\ } b$, $b \xrightarrow{\ g\ } c$

D1 $\dagger(\dagger f^{op})^{op} = f$

D2 $\dagger(f^{op} \circ g^{op}) = \dagger g^{op} \circ \dagger f^{op}$

We call the functor $\dagger$ *the dagger* and will refer to applying it as *taking the dagger of* or in more popular terms as *daggering*. To keep notation clear and readable we write $f^\dagger$ instead of $\dagger f^{op}$. With this the conditions from definition 2.1 become

D1 $f^{\dagger\dagger} = f$

D2 $(gf)^{\dagger} = f^{\dagger}g^{\dagger}$

In this sense taking the dagger of a morphism can be seen as reversing its direction. We will later see a concrete example of this as well as some dagger categories.

When dealing with multiple dagger categories we will distinguish between the daggers by means of subscripts. However when no confusion arises we will omit the subscript to improve readability.

In correspondence with the theory of Hilbert spaces we call a morphism f in a dagger category *isometric* if $f^{\dagger}f = 1$. If it is invertible and $f^{\dagger} = f^{-1}$ then we call f *unitary*, if $f^{\dagger} = f$ then it is *self-adjoint*.

As is common practice in category theory, when defining a category, we define the notion of a *dagger functor*.

Definition 2.2. *A* dagger functor F *is a functor* $\mathcal{C} \xrightarrow{F} \mathcal{D}$ *between two dagger categories such that* $\dagger_{\mathcal{D}}F = F\dagger_{\mathcal{C}}$.

In this thesis we will be working a lot with natural transformations. If the functors in question are dagger functors then we have the following

Lemma 2.3. *Let* $\mathcal{C}, \mathcal{D}$ *be dagger categories,* $\mathcal{C} \xrightarrow{S,T} \mathcal{D}$ *two dagger functors and* $\tau : S \Rightarrow T$ *a natural transformation. Then* $\tau^{\dagger} : T \Rightarrow S$

Proof. For each $c \xrightarrow{f} c' \in \mathcal{C}$ the following diagram commutes

$$
\begin{array}{ccc}
Sc' & \xrightarrow{\tau} & Tc' \\
\uparrow{\scriptstyle Sf^{\dagger}} & & \uparrow{\scriptstyle Tf^{\dagger}} \\
Sc & \xrightarrow{\tau} & Tc
\end{array}
$$

Therefore we have that

$$
\begin{array}{ccc}
Sc & \xleftarrow{\tau^{\dagger}} & Tc \\
\downarrow{\scriptstyle (Sf^{\dagger})^{\dagger}} & & \downarrow{\scriptstyle (Tf^{\dagger})^{\dagger}} \\
Sc' & \xleftarrow{\tau^{\dagger}} & Tc'
\end{array}
$$

commutes. Now because S, T are dagger functors this implies that

$$
\begin{array}{ccc}
Sc & \xleftarrow{\tau^{\dagger}} & Tc \\
\downarrow{\scriptstyle Sf} & & \downarrow{\scriptstyle Tf} \\
Sc' & \xrightarrow{} & Tc'
\end{array}
$$

commutes showing that $\tau^{\dagger} : T \Rightarrow S$. $\qquad\square$

Dagger functors will play an important role when we try to extend functors to the category of Frobenius semigroups. However, before we get there we need to know more about *monoidal categories*.

2.2 Monoidal categories

Every mathematician is familiar with the direct product of sets which takes two sets and turns it into an other set. Another example of a similar construction is the tensor product of rings. The similarity here is that we have a construction involving some 'associative' product , which takes two objects and by 'multiplication' turns them into a new object of the same type. This general concept is defined in category theory by means of *monoidal categories*.

Definition 2.4. *A* monoidal category *is a triple* $\langle \mathcal{C}, \square, e \rangle$ *where*
$C \times C \xrightarrow{\square} C$ *is a functor and e an object in $\mathcal{C}$, such that there are natural isomorphisms*

$$\alpha : \square(1 \times \square) \Rightarrow \square(\square \times 1)$$
$$\lambda : \square(e \times 1) \Rightarrow 1_{\mathcal{C}}$$
$$\rho : \square(1 \times e) \Rightarrow 1_{\mathcal{C}}$$

which make the following diagrams commute.

$$
\begin{array}{ccccc}
a\square(b\square(c\square d)) & \xrightarrow{\alpha} & (a\square b)\square(c\square d) & \xrightarrow{\alpha} & ((a\square b)\square c)\square d) \\
\downarrow{\scriptstyle 1\square\alpha} & & & & \uparrow{\scriptstyle \alpha\square 1} \\
a\square((b\square c)\square d) & & \xrightarrow{\quad\alpha\quad} & & (a\square(b\square c))\square d
\end{array}
$$

$$
\begin{array}{ccc}
a\square(e\square b) & \xrightarrow{\alpha} & (a\square e)\square b \\
\downarrow{\scriptstyle 1\square\lambda} & & \downarrow{\scriptstyle \rho\square 1} \\
a\square b & = & a\square b
\end{array}
$$

If all three isomorphisms reduce to identities then we call $\mathcal{C}$ a *strict monoidal category*. There is a theorem stating that every monoidal category is equivalent to a strict one, see [5]. This is why we will not really make a distinction between the two cases. A monoidal category is called *symmetric* if in addition we have an isomorphism $\gamma : a\square b \to b\square a$ for all $a, b \in \mathcal{C}$ making the following diagrams commute.

$$
\begin{array}{ccc}
a\square b \xrightarrow{\ \gamma\ } b\square a & \qquad & b\square e \xrightarrow{\ \gamma\ } e\square b \\
\searrow{\scriptstyle 1}\quad \downarrow{\scriptstyle \gamma} & & \searrow{\scriptstyle \rho}\quad \downarrow{\scriptstyle \lambda} \\
a\square b & & b
\end{array}
$$

$$
\begin{array}{ccc}
a\square(b\square c) \xrightarrow{\ \alpha\ } (a\square b)\square c \xrightarrow{\ \gamma\ } c\square(a\square b) \\
\downarrow{\scriptstyle 1\square\lambda} \hspace{5cm} \downarrow{\scriptstyle \alpha} \\
a\square(c\square b) \xrightarrow{\ \alpha\ } (a\square c)\square b \xrightarrow{\ \gamma\square 1\ } (c\square a)\square b
\end{array}
$$

We refer to α, λ, ρ and γ as the structure morphisms of the (symmetric) monoidal category.

One nice fact about monoidal categories is that they contain certain objects called *monoids*. These objects generalize several algebraic objects such as rings and ordinary monoids in **Set**.

Definition 2.5. *A* monoid *in a monoidal category* $\mathcal{C}$ *consists of a triple* $\langle a, \mu, \iota\rangle$ *where a is an object in $\mathcal{C}$ and* $a\square a \xrightarrow{\ \mu\ } a$ *,* $e \xrightarrow{\ \iota\ } a$ *are morphisms such that the following diagrams commute*

$$
\begin{array}{ccc}
a\square(a\square a) \xrightarrow{\quad\alpha\quad} (a\square a)\square a & \qquad & e\square a \xrightarrow{\ \iota\square 1\ } a\square a \xleftarrow{\ 1\square\iota\ } a\square e \\
\downarrow{\scriptstyle 1\square\mu} \hspace{3cm} \downarrow{\scriptstyle \mu\square a} & & \searrow{\scriptstyle \lambda}\quad\downarrow{\scriptstyle \mu}\quad\swarrow{\scriptstyle \rho} \\
a\square a \xrightarrow{\ \mu\ } a \xleftarrow{\ \mu\ } b\square a & & a
\end{array}
$$

We call μ the *monoid multiplication* on a. The first diagram resembles the associativity while the other says that ι is the identity with respect to μ. A *comonoid* in $\mathcal{C}$ is a monoid in $\mathcal{C}^{op}$.

We can turn the monoids of a monoidal category $\mathcal{C}$ into a category $\mathbf{Mon}_\mathcal{C}$ by defining a morphism $\langle a, \mu_a, \iota_a\rangle \xrightarrow{\ f\ } \langle b, \mu_b, \iota_b\rangle$ to be a morphism $a \xrightarrow{\ f\ } b$ in $\mathcal{C}$ such that the following diagrams commute:

$$
\begin{array}{ccc}
a\square a \xrightarrow{\ \mu_a\ } a & \qquad & e \xrightarrow{\ \iota_a\ } a \\
\downarrow{\scriptstyle f\square f} \quad \downarrow{\scriptstyle f} & & \searrow{\scriptstyle \iota_b}\quad\downarrow{\scriptstyle f} \\
b\square b \xrightarrow{\ \mu_b\ } b & & b
\end{array}
$$

The category of comonoids is defined in a similar way and denoted by $\mathbf{Mon}_\mathcal{C}^*$. There are functors between monoidal categories which deserve special attention. These are the functors which preserve the monoidal structure.

Definition 2.6. *A functor* $\mathcal{C} \xrightarrow{\ M\ } \mathcal{D}$ *between monoidial categories* $\langle\mathcal{C}, \square, e_\mathcal{C}\rangle$, $\langle\mathcal{D}, \Diamond, e_\mathcal{D}\rangle$ *is called* monoidal *if there exist a natural transformation*

$$\beta : \Diamond(M \times M) \Rightarrow M\square$$

and a morphism $e_{\mathcal{D}} \xrightarrow{\tau} M(e_{\mathcal{C}})$ *which make the following diagrams commute*

$$
\begin{array}{ccc}
M(a)\Diamond(M(b)\Diamond M(c)) & \xrightarrow{\alpha_{\mathcal{D}}} & (M(a)\Diamond M(b))\Diamond M(c) \\
\downarrow{\scriptstyle 1\Diamond\beta} & & \downarrow{\scriptstyle \beta\Diamond 1} \\
M(a)\Diamond M(b\square c) & & M(a\square b)\Diamond M(c) \\
\downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta} \\
M(a\square(b\square c)) & \xrightarrow{M(\alpha_{\mathcal{C}})} & M((a\square b)\square c)
\end{array}
$$

$$
\begin{array}{ccccccc}
M(b)\Diamond e_{\mathcal{D}} & \xrightarrow{\rho_{\mathcal{D}}} & M(b) & \qquad & e_{\mathcal{C}}\Diamond M(b) & \xrightarrow{\lambda_{\mathcal{D}}} & M(b) \\
\downarrow{\scriptstyle 1\Diamond\tau} & & \uparrow{\scriptstyle M(\rho_{\mathcal{C}})} & & \downarrow{\scriptstyle \tau\Diamond 1} & & \uparrow{\scriptstyle M(\lambda_{\mathcal{C}})} \\
M(b)\Diamond M(e_{\mathcal{C}}) & \xrightarrow{\beta} & M(b\square e_{\mathcal{C}}) & & M(e_{\mathcal{C}})\Diamond M(b) & \xrightarrow{\beta} & M(e_{\mathcal{C}}\square b)
\end{array}
$$

If $\mathcal{C}, \mathcal{D}$ are symmetric and

$$
\begin{array}{ccc}
M(a)\Diamond M(b) & \xrightarrow{\beta} & M(a\square b) \\
\downarrow{\scriptstyle M(\gamma_{\mathcal{C}})} & & \downarrow{\scriptstyle \gamma_{\mathcal{D}}} \\
M(b)\Diamond M(a) & \xrightarrow{\beta} & M(b\square a)
\end{array}
$$

commutes we call M a *commutative monoidal functor*. Using the definition of a monoidal functor we can state its comonoidal variant. We say that M is *comonoidal* if there is a natural transformation $\beta^* : M\square \Rightarrow \Diamond(M \times M)$ and a morphism $M(e_{\mathcal{C}}) \xrightarrow{\tau^*} e_{\mathcal{D}}$ which make the obvious coversions of the diagrams for definition 2.6 commute.

There is a relation between the monoidal and comonoidal functors which uses a dagger. We will come to this later.

The fact that monoidal functors preserve the structure of the monoidal categories suggest that they can be applied to monoids. This is actually the case. We can extend these functors to the category of monoids.

Proposition 2.7. *Let* $\langle \mathcal{C}, \square, e_{\mathcal{C}} \rangle, \langle \mathcal{D}, \Diamond, e_{\mathcal{D}} \rangle$ *be monoidal categories and* $\mathcal{C} \xrightarrow{M} \mathcal{D}$ *a monoidal functor. Then* M *extends to a functor* $\mathbf{Mon}_{\mathcal{C}} \xrightarrow{\mathfrak{M}} \mathbf{Mon}_{\mathcal{D}}$ *, such that the following diagram commutes.*

$$
\begin{array}{ccc}
\mathbf{Mon}_{\mathcal{C}} & \xrightarrow{\mathfrak{M}} & \mathbf{Mon}_{\mathcal{D}} \\
\downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\
\mathcal{C} & \xrightarrow{M} & \mathcal{D}
\end{array}
$$

where U is the obvious forgetful functor.

Proof. Because M is monoidal we have a natural transformation
$\beta : \Diamond(M \times M) \Rightarrow M\square$ and a morphism $\tau : e_{\mathcal{D}} \Rightarrow M(e_{\mathcal{C}})$
We define $\mathfrak{M}$ on objects first. Let $\langle a, \mu, \varepsilon \rangle \in \mathbf{Mon}_{\mathcal{C}}$. We will show that the following diagram commutes.

$$e_{\mathcal{D}} \Diamond M(c) \xrightarrow{M(\varepsilon)\tau \Diamond 1} M(c) \Diamond M(c) \qquad\qquad (2.1)$$

with $\lambda_{\mathcal{D}}$ and $M(\mu)\beta$ to $M(c)$.

If we write this out and fill in diagrams for the naturality of β and the corresponding one for the monoid a we get this diagram:

$$e_{\mathcal{D}} \Diamond M(a) \xrightarrow{\tau \Diamond 1} M(e_{\mathcal{C}}) \Diamond M(a) \xrightarrow{M(\varepsilon)\Diamond 1} M(a) \Diamond M(a)$$

with vertical maps $\lambda_{\mathcal{D}}$, β, β; middle row $M(e_{\mathcal{C}}\square a) \xrightarrow{M(\varepsilon\square 1)} M(a\square a)$; then $M(\lambda_{\mathcal{C}})$ and $M(\mu)$; bottom row $M(a) \quad = \quad M(a) \quad = \quad M(a)$.

The left diagram commutes because M is monoidal, the upper right one commutes by naturality of β and the bottom right diagram because a is a monoid. This proves that (2.1) commutes. The proof for the other two monoid diagrams is similar.

The above proves that $\langle M(a), M(\mu)\beta, M(\varepsilon)\tau \rangle$ is a monoid in $\mathcal{D}$ and hence we define $\mathfrak{M}$ on objects as $\langle a, \mu, \varepsilon \rangle \mapsto \langle M(a), M(\mu)\beta, M(\varepsilon)\tau \rangle$.

Let $\langle a, \mu, \varepsilon \rangle \xrightarrow{f} \langle a', \mu', \varepsilon' \rangle$. Then the following diagram commutes

$$M(c) \Diamond M(c) \xrightarrow{\beta} M(c\square c) \xrightarrow{M(\mu)} M(c)$$

with vertical maps $M(f)\Diamond M(f)$, $M(f\square f)$, $M(f)$; bottom row $M(c') \Diamond M(c') \xrightarrow{\beta} M(c'\square c') \xrightarrow{M(\mu')} M(c')$.

The left diagram commutes again by naturality of β while the right commutes because $f\mu = \mu'(f\square f)$ in $\mathbf{Mon}_{\mathcal{C}}$. This proves that $M(f)M(\mu)\beta = M(\mu')\beta(M(f)\Diamond M(f))$ in $\mathbf{Mon}_{\mathcal{D}}$. The proof that $M(f)M(\varepsilon)\tau = M(\varepsilon')\tau$ is trivial, so we get that $\langle M(a), M(\mu)\beta, M(\varepsilon)\tau \rangle \xrightarrow{M(f)} \langle M(a'), M(\mu')\beta, M(\varepsilon')\tau \rangle$ in $\mathbf{Mon}_{\mathcal{D}}$. Hence, we define $\mathfrak{M}$ on arrows as $\mathfrak{M}(f) = M(f)$.

This completes the proof that $\mathfrak{M}$ is a functor. Now, because the functor U sends $\langle a, \mu, \varepsilon \rangle$ to a, it is easy to see that the diagram in the proposition commutes. $\qquad \square$

Its is not hard to see that we can rewrite the above proof to prove the comonoidal equivalent of proposition 2.7. Now if we start with two monoidal categories $\langle \mathcal{C}, \square, e_{\mathcal{C}} \rangle$, $\langle \mathcal{D}, \Diamond, e_{\mathcal{D}} \rangle$ and a functor $\mathcal{C} \xrightarrow{F} \mathcal{D}$ and we want to define a functor on the monoids, it is enough to prove that F is monoidal. This technique is extended to a subclass of monoidal categories and then used in chapter 2 to extend the functor $\mathbf{PInj} \xrightarrow{\ell^2} \mathbf{Hilb}$ to $\mathbb{F}\mathrm{rob}(\mathbf{PInj}) \longrightarrow \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$.

2.3   Monoids, comonoids and daggers

When defining dagger categories it was mentioned that taking a dagger is like reversing the direction of the morphism. The definition of a comonoid is also 'reversed' with respect to that of the monoid. The question that arises is if $\langle c, \mu, \varepsilon \rangle$ is a monoid is it true that taking the dagger of the morphisms turns it into a comonoid. This is not true in general for it relies on the interaction of the dagger with the monoidal structure. We will therefore define a category to be *dagger monoidal* if it is both, but in such a way that the dagger 'preserves' the monoidal structure. This is specified in the following definition.

Definition 2.8. *A* dagger monoidal category *is a monoidal category $\mathcal{B}$ equipped with a dagger $\dagger : \mathcal{B}^{op} \to \mathcal{B}$ such that*

DM1 $\dagger\square = \square(\dagger \times \dagger)$

DM2 α, λ, ρ *are unitary.*

We call a category *symmetric dagger monoidal* or *SDM*, if it is also symmetric and γ is unitary. The functors between two dagger monoidal categories are those that are both monoidal and dagger and are conveniently called dagger monoidal functors. The unitariness of the structure morphisms implies that monoids can be turned into comonoids by the dagger and visa versa.

Lemma 2.9. *Let $\mathcal{C}$ be a dagger monoidal category. Then if $\langle c, \mu, \iota \rangle$ is a monoid in $\mathcal{C}$, $\langle c, \mu^{\dagger}, \iota^{\dagger} \rangle$ is a comonoid in $\mathcal{C}$ and vice versa.*

Proof. Let $\langle c, \mu, \iota \rangle$ be a monoid in $\mathcal{C}$. Then

$$e\Box c \xleftarrow{\iota^\dagger\Box 1} c\Box c \qquad\qquad e\Box c \xleftarrow{\iota^\dagger\Box 1} c\Box c$$

which commutes. The same is true for the right side of the unit diagram and the diagram for α. This proves that $\langle c, \mu^\dagger, \iota^\dagger \rangle$ is a comonoid. The reverse implication is similar. $\qquad\square$

The following lemma is a refinement of proposition 2.7 and relates the monoidal and comonoidal functors on monoidal dagger categories.

Lemma 2.10. *Let* $\mathcal{C}$*,* $\mathcal{D}$ *be dagger monoidal categories and* $\mathcal{C} \xrightarrow{M} \mathcal{D}$ *be a dagger monoidal functor. Then* M *is comonoidal and hence extends to a functor* $\mathbf{Mon}^*_{\mathcal{C}} \xrightarrow{\mathfrak{M}} \mathbf{Mon}^*_{\mathcal{D}}$ *, such that the following diagram commutes.*

$$
\begin{array}{ccc}
\mathbf{Mon}^*_{\mathcal{C}} & \xrightarrow{\ \mathfrak{M}\ } & \mathbf{Mon}^*_{\mathcal{D}} \\
\downarrow{\scriptstyle U} & & \downarrow{\scriptstyle U} \\
\mathcal{C} & \xrightarrow{\ M\ } & \mathcal{D}
\end{array}
$$

where U *is the obvious forgetful functor.*

Proof. The only thing needed to show is that M is comonoidal, the rest of the proof is then similar to that of Proposition 2.7. Because M is monoidal there exists a natural transformation β and a morphism τ. By definition of the dagger and because the categories and the functor are dagger monoidal, we have that $\beta^\dagger : M\Box \Rightarrow \Diamond(M \times M)$. Together with $M(e_{\mathcal{B}}) \xrightarrow{\tau^\dagger} e_{\mathcal{C}}$ this defines the structure needed to make M comonoidal. $\qquad\square$

2.4 Frobenius structures

The symmetric dagger monoidal categories form the basis for classical categorical quantum mechanics. At its center stand the Frobenius algebras, which are used to define classical quantum systems. In [2] it has even been proved there is a class of Frobenius algebras which in **Hilb** constitute an orthonormal basis. This is a very strong result, which justifies why these Frobenius algebras are said to be the categorical equivalent of orthonormal bases.

Frobenius algebras were already defined in algebra. The definition given here is the categorical translation of that definition. However, it is not very clear at first why this is a valid translation. Because we don't need this translation we can just see this as a purely categorical definition. For those who are interested in the link between the two definitions see [4].

2.4.1 Algebras, structures and semigroups

Definition 2.11. *A* Frobenius algebra *in a monoidal category* $\mathcal{C}$ *consists of a monoid* $\langle a, \mu, \iota \rangle$ *and comonoid* $\langle a, \mu^*, \iota^* \rangle$*, which satisfies the following diagram*

$$
\begin{array}{ccc}
a \square a & \xrightarrow{\mu^* \square 1} & a \square a \square a \\
\downarrow{\scriptstyle 1 \square \mu^*} & \searrow{\scriptstyle \mu} \quad a \quad \searrow{\scriptstyle \mu^*} & \downarrow{\scriptstyle 1 \square \mu} \\
a \square a \square a & \xrightarrow{\mu \square 1} & a \square a
\end{array}
\tag{2.2}
$$

We write $\langle a, \mu, \iota, \mu^*, \iota^* \rangle$ for the Frobenius algebra given by $\langle a, \mu, \iota \rangle$ and $\langle a, \mu^*, \iota^* \rangle$. The commutativity of diagram 2.2 is referred to as the *Frobenius property*. We call a Frobenius algebra *special* if $\mu\mu^* = 1$. These objects together with a dagger structure in the category **Hilb** are the categorical representations of quantum systems. This dagger structure, like in the case of dagger monoidal categories, is such that it respects the Frobenius structure.

Definition 2.12. *A special commutative †-Frobenius algebra in a symmetric dagger monoidal category is a special Frobenius algebra* $\langle a, \mu, \iota, \mu^*, \iota^* \rangle$ *such that* $\gamma\mu^* = \mu^*$ *and* $\mu = (\mu^*)^\dagger$*. We denote this algebra by either* $\langle a, \mu, \iota \rangle$ *or* $\langle a, \mu^*, \iota^* \rangle$*.*

The special commutative †-Frobenius algebras have a very rich structure. This can however turn into a disadvantage as will be the case for the category **PInj**. We will see that the only special commutative †-Frobenius algebra here is the trivial one. In order to remedy this we drop the conditions on the (co)-unit for Frobenius algebras. Because the new structures have multiplication but no unit we call them Frobenius semigroups.

Definition 2.13. *A* Frobenius structure *in a monoidal category* $\mathcal{C}$ *consists of an object* a*, together with a monoid multiplication* ∇ *and comonoid multiplication* Δ*, which together satisfy the Frobenius property.*

In correspondence with Frobenius algebras we call a Frobenius structure special if $\nabla\Delta = 1$. We can now define a *Frobenius semigroup* to be a special Frobenius structure $\langle a, \nabla, \Delta \rangle$ in a SDM such that $\gamma\Delta = \Delta$ and $\nabla = \Delta^\dagger$. We will denote these by either $\langle a, \nabla \rangle$ or $\langle a, \Delta \rangle$, depending on which operation is more important. In the rest of this thesis this will mostly be the latter.

2.4.2 Frobenius categories

Using the above definitions we can define, for a SDM category $\mathcal{C}$, three categories $\mathbb{F}\mathrm{rob}(\mathcal{C})$, $\mathbb{F}\mathrm{rob}_a(\mathcal{C})$ and $\mathbb{F}\mathrm{rob}_m(\mathcal{C})$. The category $\mathbb{F}\mathrm{rob}(\mathcal{C})$ is defined as follows. Objects are Frobenius semigroups $\langle c, \Delta \rangle$. A morphism $\langle c, \Delta \rangle \xrightarrow{f} \langle c', \Delta' \rangle$ is a morphism $c \xrightarrow{f} c'$ in $\mathcal{C}$, such that the following diagrams commute.

$$
\begin{array}{ccc}
c \xrightarrow{f} c' & \qquad & c \xrightarrow{\Delta} c\square c \\
\searrow^{1} \downarrow^{f^\dagger} & & \downarrow^{f} \qquad \downarrow^{f\square f} \\
c & & c' \xrightarrow{\Delta'} c'\square c'
\end{array}
$$

The category $\mathbb{F}\mathrm{rob}_m(\mathcal{C})$ is defined almost similar to $\mathbb{F}\mathrm{rob}(\mathcal{C})$ with the exception that instead of $f^\dagger f = 1$ we require that f is monic.

The category $\mathbb{F}\mathrm{rob}_a(\mathcal{C})$ has special commutative $\dagger$-Frobenius algebras as objects. The morphisms are morphisms in $\mathcal{C}$ which make the same diagrams commute as those for morphisms in $\mathbb{F}\mathrm{rob}(\mathcal{C})$, but with the extra requirement that $e \xrightarrow{\tau} c$ commutes.

$$
\begin{array}{c}
e \xrightarrow{\tau} c \\
\searrow^{\tau'} \downarrow^{f} \\
c'
\end{array}
$$

Some authors write $\mathbb{F}\mathrm{rob}(\mathcal{C})$ for $\mathbb{F}\mathrm{rob}_a(\mathcal{C})$. We have chosen this convention because the absence of (co)units will make sure that we can relate the Frobenius semigroups in **PInj** to those in **Hilb**.

2.4.3 Frobenius Functors

The next step is one which we have already done several times. We define special functors between SDMs which we can then extend to functors on the Frobenius semigroups. In section 3.3 we will see that the ℓ^2 functor is an example of such a functor.

Definition 2.14. *A* Frobenius functor[1] *is a functor* $\mathcal{M} \xrightarrow{F} \mathcal{N}$ *between symmetric dagger monoidal categories, which is both a dagger and commutative*

---

[1]We have chosen the term Frobenius functor because these functors preserve the Frobe-

monoidal functor, such that the natural transformations β and τ from definition 2.6 are componentwise unitary.

The fact that the β and τ morphisms are unitary is the strength of these functors and is almost the sole reason why they can be extended to Frobenius semigroups. We close this chapter with this extension theorem which forms the basis for the adjunction between $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ and $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$.

Theorem 2.15. *Let* $\mathcal{C}$, $\mathcal{D}$ *be symmetric dagger monoidal categories and* $\mathcal{C} \xrightarrow{F} \mathcal{D}$ *be a Frobenius functor. Then F lifts to a functor* $\mathbb{F}\mathrm{rob}(\mathcal{C}) \xrightarrow{\mathbb{F}} \mathbb{F}\mathrm{rob}(\mathcal{D})$.

Proof. Because F is a dagger monoidal functor by Proposition 2.7 and Lemma 2.10 we can transport (co)monoids from $\mathcal{C}$ to $\mathcal{D}$ via F. Starting with the structure $\langle c, \nabla, \Delta \rangle$ we get $\langle c, F(\nabla)\beta, \beta^\dagger F(\Delta) \rangle$. First, we check that this satisfies the Frobenius property in $\mathcal{D}$. Lets look at the left part of the diagram 2.2.

$$
\begin{array}{ccc}
F(c) \lozenge F(c) & \xrightarrow{\;\;F(\nabla)\beta\;\;} & F(c) \\[2pt]
{\scriptstyle (1\lozenge\beta^\dagger)(1\lozenge F(\Delta))}\Big\downarrow & & \Big\downarrow{\scriptstyle \beta^\dagger F(\Delta)} \\[2pt]
F(c)\lozenge F(c)\lozenge F(c) & \xrightarrow{(F(\nabla)\lozenge 1)(\beta\lozenge 1)} & F(c)\lozenge F(c)
\end{array}
$$

Expanding this and adding some extra morphisms we get the following.

$$
\begin{array}{ccccc}
F(c)\lozenge F(c) & \xrightarrow{\;\;\beta\;\;} & F(c\square c) & \xrightarrow{\;F(\nabla)\;} & F(c) \\[2pt]
{\scriptstyle 1\lozenge F(\Delta)}\Big\downarrow & & {\scriptstyle F(1\square\Delta)}\Big\downarrow & & \Big\downarrow{\scriptstyle F(\Delta)} \\[2pt]
F(c)\lozenge F(c\square c) & \xrightarrow{\;\;\beta\;\;} & F(c\square c\square c) & \xrightarrow{F(\nabla\square 1)} & F(c\square c) \\[2pt]
{\scriptstyle 1\lozenge\beta^\dagger}\Big\downarrow & & {\scriptstyle \beta^\dagger}\Big\downarrow & & \Big\downarrow{\scriptstyle \beta^\dagger} \\[2pt]
F(c)\lozenge F(c)\lozenge F(c) & \xrightarrow{\;\beta\lozenge 1\;} & F(c\square c)\lozenge F(c) & \xrightarrow{F(\nabla)\lozenge 1} & F(c)\lozenge F(c)
\end{array}
$$

The upper left and lower right diagram commute because of the naturality of β respectively $\beta^\dagger$, while the upper right diagram commutes due to the Frobenius property on c. The lower left diagram commutes because F is monoidal and β is unitary. The proof for the other part is similar. Now only the commutativity and speciality are left. The first follows immediately from

nius structure. However, there is also another definition of a Frobenius functor in category theory. A functor F is called Frobenius if there is a functor G such that $F \dashv G \dashv F$. In this thesis however they are just those functors that preserve the structure.

the fact that F is commutative monoidal and β is unitary, while the second follows directly from the unitarity of β. $\qquad\square$

In the proof of theorem 2.15 we have not used any of the properties of τ. This, because we are working with Frobenius semigroups instead of Frobenius algebras. If we consider special commutative †-Frobenius algebras, the (co)monoid diagrams for the unit ι commute due to the unitariness of τ. Hence, F lifts to the categories of special commutative †-Frobenius algebras. In this setting we have proven the following theorem.

Theorem 2.16. *Let* $\mathcal{C}$, $\mathcal{D}$ *be symmetric dagger monoidal categories and* $\mathcal{C} \xrightarrow{\ F\ } \mathcal{D}$ *be a Frobenius functor. Then F lifts to a functor* $\mathbb{F}\mathrm{rob}_a(\mathcal{C}) \xrightarrow{\ \mathbb{F}\ } \mathbb{F}\mathrm{rob}_a(\mathcal{D})$.

Chapter 3

Partial injections and Hilbert spaces

Now that we have some basic theory on Frobenius semigroups and functors we can turn to the world of categorical quantum mechanics. We are interested in the structure of the category of Hilbert spaces **Hilb**. This category will turn out to be a SDM and the Frobenius semigroups on it will categorize the notion of a orthonormal basis. However we start with the category of partial injections **PInj** which will also turn out to be a SDM. The structure of its Frobenius semigroups is related to the Frobenius semigroups of **Hilb**. We will construct an adjunction between $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ and $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ and prove that in the finite-dimensional case this is an equivalence of categories. This adjunction will later be used to relate $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ to certain Inverse semigroups.

3.1 The category PInj

The category of partial injections can be defined in several ways. We start with the definition from injective relations and then move to partial injective functions.

A injective relation F on $X \times Y$ is a subset $F \subseteq X \times Y$ such that for all $x_1, x_2 \in X$ and $y_1, y_2 \in Y$

P1 $(x_1, y_1), (x_1, y_2) \in F \Rightarrow y_1 = y_2$

P2 $(x_1, y_1), (x_2, y_1) \in F \Rightarrow x_1 = x_2$

If we have two injective relations $F \subseteq X \times Y$ and $G \subseteq Y \times Z$ we define their

composition $G \circ F$ as follows

$$G \circ F = \{(x, z) \in X \times Z : \exists y \in Y; (x, y) \in F, (y, z) \in G\}$$

If $(x_1, z_1), (x_1, z_2) \in G \circ F$, then $\exists y_1, y_2 \in Y$ such that $(x_1, y_1), (x_1, y_2) \in F$ and $(y_1, z_1), (y_2, z_2) \in G$. The former implies that $y_1 = y_1$ and hence we the latter implies $z_1 = z_2$. Similarly, we get that if $(x_1, z_1), (x_2, z_1) \in G \circ F$ then $x_1 = x_2$. This proves that $G \circ F \subseteq X \times Z$ is an injective relation.

Because we can compose injective relations, it is natural to consider them as morphisms i.e. $X \xrightarrow{F} Y$ is an injective relation $F \subseteq X \times Y$. This defines a category **PInj**, called the category of partial injections. We say that an injective relation $F \subseteq X \times Y$ is total if for all $x \in X$ there is a $y \in Y$, such that $(x, y) \in F$.

Given a injective relation $F \subseteq X \times Y$ we can define a partial function $X \xrightarrow{f} Y$ as follows:

- $\mathrm{dom}(f) = \{x \in X | \exists y \in Y; (x, y) \in F\}$

- $f(x) = y \iff (x, y) \in F$

This function is well defined because of P1 and injective by P2. We call these functions partial injections. From a partial injection $X \xrightarrow{f} Y$ we can define an injective relation $F \subseteq X \times Y$ by $(x, y) \in F \iff x \in \mathrm{dom}(f)$ and $f(x) = y$. These operations are inverse to each other and hence we have a 1-1 correspondence between partial injections and injective relations. We shall denote the morphisms in **PInj** as partial injections. The composition is defined by the composition of the corresponding relations. In correspondence with injective relations, we call a partial injection $X \xrightarrow{f} Y$ *total* if its domain is all of X.

Now that we have defined the category **PInj**, we can define the structure needed to turn it into a SDM. First notice that the properties P1 and P2 of an injective relation are symmetric. Hence, given such a relation $F \subseteq X \times Y$ we can define $F^\dagger \subseteq Y \times X$ by $(y, x) \in F^\dagger \iff (x, y) \in F$ which is again an injective relation. In terms of partial injections this operation translates to the following. Given a partial injection $X \xrightarrow{f} Y$ we define $Y \xrightarrow{f^\dagger} X$ to be the function with $\mathrm{dom}(f^\dagger) = \mathrm{im}(f)$ and $f^\dagger(y) = f^{-1}\{y\}$. Using relations we see that $(GF)^\dagger = F^\dagger G^\dagger$ and $(F^\dagger)^\dagger = F$. This shows that we have a dagger on **PInj**, which really reverses the morphisms.

The monoidal structure is defined using the Cartesian product which we denote by $\oplus$. Given two morphisms $X \xrightarrow{f} U$ and $Y \xrightarrow{g} V$ we define

their product $X \oplus Y \xrightarrow{f \oplus g} U \oplus V$ componentwise i.e. $f \oplus g(x \oplus y) = f(x) \oplus g(y)$. Composition is also defined component wise, hence well defined and therefore $\oplus$ is a functor $\mathbf{PInj} \times \mathbf{PInj} \xrightarrow{\oplus} \mathbf{PInj}$. This functor is clearly associative, so we have $\alpha : x \oplus (y \oplus z) \mapsto (x \oplus y) \oplus z$. The next step is to define the unit e of the monoidal structure. We need this unit to be such that we have the natural transformations λ and ρ. Well if we take e to be the set with one element $\{1\}$, then $\{1\} \oplus X \cong X$ by $1 \oplus x \mapsto x$. This is clearly a natural isomorphism, hence we have $\lambda : \oplus(e \times 1) \Rightarrow 1$. The definition of $\rho : \oplus(1 \times e) \Rightarrow 1$ is now trivial as is the commutativity of the diagrams from 2.4. To wrap things up we notice that $X \oplus Y \cong Y \oplus X$ in a trivial way, which gives us γ. We have now proven the following proposition.

Proposition 3.1. $\langle \mathbf{PInj}, \oplus, \{1\} \rangle$ *is a symmetric monoidal category.*

If we look closer at the proof of this proposition, we see we have also proven that $\langle \mathbf{Set}, \oplus, \{1\} \rangle$ is a symmetric monoidal category. The only difference is that $\mathbf{PInj}$ also has a dagger and we will see that this dagger preserves the monoidal structure, turning the category into a SDM.

Because the product $f \oplus g$ is defined component wise, it is easy to see that $f^\dagger \oplus g^\dagger = (f \oplus g)^\dagger$. Now clearly α, λ, ρ and γ are unitary , thus turning $\mathbf{PInj}$ into a SDM.

We have defined $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ as the category of *Frobenius semigroups* on $\mathbf{PInj}$ instead of Frobenius algebras, because the latter would be too restrictive. We will now see what that means. Suppose that $\langle X, \mu, \iota \rangle$ is a monoid. Then in particular the following diagram commutes.

$$\{1\} \oplus X \xrightarrow{\iota \oplus 1} X \oplus X \xleftarrow{1 \oplus \iota} X \oplus \{1\}$$

with λ, μ, ρ mapping to X.

Because λ is total, this means that $\mathrm{dom}(\iota \oplus 1) = \{1\} \oplus X$ and similar $\mathrm{dom}(1 \oplus \iota) = X \oplus \{1\}$. Now let $x \in X$ and denote $\iota(1) = e$ then $\mu(e \oplus x) = x = \mu(x \oplus e)$. Because μ is injective, it follows that $e \oplus x = x \oplus e$ which implies that $x = e$. This proves that only the set $\{1\}$ can be part of a monoid triple. We have remedied this by dropping the existence of the (co)unit in the definition of Frobenius structures.

Even though we have expanded the range of objects by using semigroups instead of algebras, the structure of $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ is still very simple. In order to see this, we define for any set X the function $X \xrightarrow{\Delta_X} X \oplus X$ by $x \mapsto x \oplus x$. With this we can prove the following lemma.

Lemma 3.2. *Let $X \in \mathbf{PInj}$ then $\langle X, \Delta_X \rangle \in \mathbb{F}\mathrm{rob}(\mathbf{PInj})$.*

Proof. It is easy to see that Δ_X is a comonoid multiplication. To prove that it satisfies the Frobenius property we prove the commutivity of the left side of the diagram 2.2.

$$
\begin{array}{ccc}
X \oplus X & \xrightarrow{\Delta_X^\dagger} & X \\
{\scriptstyle 1 \oplus \Delta_X^\dagger} \downarrow & & \downarrow {\scriptstyle \Delta_X} \\
X \oplus X \oplus X & \xrightarrow{\Delta_X \oplus 1} & X \oplus X
\end{array}
$$

First we have to check that the domains are equal. Clearly $\mathrm{dom}(\Delta_X^\dagger \Delta_X) = \{x \oplus x\}$ while

$$
\begin{aligned}
\mathrm{dom}((\Delta_X \oplus 1)(1 \oplus \Delta_X^\dagger)) &= \{x \oplus y : (1 \oplus \Delta_X^\dagger)(x \oplus y) \in \mathrm{dom}(\Delta_X \oplus 1)\} \\
&= \{x \oplus y : x \oplus y \oplus y \in \mathrm{dom}(\Delta_X \oplus 1)\} \\
&= \{x \oplus y : x = y\} = \{x \oplus x\}
\end{aligned}
$$

Now let $x \oplus x \in X \oplus X$, then

$$(\Delta_X \oplus 1)(1 \oplus \Delta_X^\dagger)(x \oplus x) = \Delta_X \oplus 1(x \oplus x \oplus x) = x \oplus x = \Delta_X(x) = \Delta_X \Delta_X^\dagger(x \oplus x)$$

Because $\mathrm{dom}\left((\Delta_X \oplus 1)(1 \oplus \Delta_X^\dagger)\right) = \{x \oplus x | x \in X\}$,it follows that the diagram commutes. The other side is done in a similar way. This shows that $\langle X, \Delta_X \rangle$ is a Frobenius structure. Because $\gamma(x \oplus x) = x \oplus x$ it is also commutative and clearly it is special hence a Frobenius semigroup. $\qquad \square$

These Frobenius semigroups are very simple and completely determined by the objects of $\mathbf{PInj}$. Moreover, they completely characterize the objects in $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$.

Proposition 3.3. *Let $\langle X, \Delta \rangle$ be a Frobenius semigroup in $\mathbf{PInj}$ then $\Delta = \Delta_X$.*

Proof. By speciality we get $\Delta^\dagger \Delta = 1$, thus $\mathrm{dom}(\Delta) = X$. Commutivity implies that $\Delta = \Delta_X f$ for some total morphism $f : X \to X$, hence the domain of $\Delta^\dagger$ is contained in the diagonal of $X \oplus X$. The only thing left to show is that $f = 1$. The frobenius property says that $(\Delta^\dagger \oplus 1)(1 \oplus \Delta) = \Delta\Delta^\dagger = (1 \oplus \Delta^\dagger)(\Delta \oplus 1)$ writing these functions as relations we get

$$(\Delta^\dagger \oplus 1)(1 \oplus \Delta) = \{x^4 | \exists y^3 : (x_1, x_2, y_1, y_2, y_3) \in 1 \oplus \Delta \tag{3.1}$$

$$(y_1, y_2, y_3, x_3, x_4) \in \Delta^\dagger \oplus 1\} \tag{3.2}$$

$$(1 \oplus \Delta^\dagger)(\Delta \oplus 1) = \{x^4 | \exists y^3 : (x_1, x_2, y_1, y_2, y_3) \in \Delta \oplus 1 \tag{3.3}$$

$$(y_1, y_2, y_3, x_3, x_4) \in 1 \oplus \Delta^\dagger\} \tag{3.4}$$

$$\Delta\Delta^\dagger = \{x^4 | \exists y : (x_1, x_2, y) \in \Delta^\dagger, (y, x_3, x_4) \in \Delta\} \tag{3.5}$$

Now (3.1) implies that $y_1 = x_1$ and $y_2 = y_3$, while (3.2) implies that $y_3 = x_4$ and $y_1 = y_2$, hence $x_1 = x_4$. In a similar manner (3.3) and (3.4) imply that $x_2 = x_3$ and by (3.5) we get $x_3 = x_4$. Using (3.1) we see that because $y_2 = y_3 = x_4 = x_1 = x_2$ it follows that $f = 1$. $\qquad\square$

We have now seen that **PInj** is a SDM and we have characterized its Frobenius semigroups. It was already mentioned these where completely determined by the objects of **PInj**. With this we can define the category $\mathbb{F}$rob(**PInj**) in a different way. To this end define the category **TInj** to have sets as objects and injective functions as morphisms. Then this category defines the former one.

Theorem 3.4. TInj $\cong \mathbb{F}$rob(**PInj**)

Proof. By lemma 3.2 $\langle X, \Delta_X \rangle \in \mathbb{F}$rob(**PInj**). So define $\textbf{TInj} \xrightarrow{\ F\ } \mathbb{F}$rob(**PInj**) on objects by $X \mapsto \langle X, \Delta_X \rangle$ and the identity on morphisms. To make this a valid functor we need to check that if $X \xrightarrow{\ f\ } Y \in \textbf{TInj}$, then $\Delta_Y f = f \oplus f \Delta_X$ and $f^\dagger f = 1$. The former is obvious while the latter is true because f is injective. The functor in the other direction is the forgetful functor U. This is well defined because all morphisms in $\mathbb{F}$rob(**PInj**) are injective functions on the underlying sets. Clearly $UF = 1$ and by Proposition 3.3 we get that $FU = 1$. $\qquad\square$

With this result we end this section on the category **PInj**. We have seen that it is a symmetric dagger monoidal category whose Frobenius semigroups are completely described by its objects. We remark that using the above proof, we can show that $\mathbb{F}$rob$_m$(**PInj**) $\cong$ **PInj**. This might seem like a better result. However, the fact that the morphisms in $\mathbb{F}$rob(**PInj**) are injective functions on sets, will make it possible to relate this category to the category of inverse semigroups.

3.2 The category Hilb

We will now investigate the structure of the category of Hilbert spaces and continuous linear function **Hilb**. Hilbert spaces have a very rich structure. As we mentioned in the introduction this is so rich, that it is hard to the impact of the choise of basis. Therefore it would be nice if **Hilb** could be related to a category which is better understood.

We will show that **Hilb** is also a SDM and later we will see, we have a relation between $\mathbb{F}$rob(**Hilb**) and $\mathbb{F}$rob(**PInj**). In order to define the monoidal structure on **Hilb** we need some basic results from functional analysis. For this we used the book by Conway [3].

3.2.1 Hilbert tensor product

Let us start with the monoidal multiplication on **Hilb**. There are two ways to do this. The first is to use the direct sum of Hilbert spaces. This is a simple construction and one can show that this defines a monoidal structure on **Hilb**. We will, however, focus on the second construction which involves the tensor product $\otimes$. This, because this product is used to compose quantum systems. The construction however, is harder then was the case with **PInj**. Here it was clear that the Cartesian product of two sets is again a set. In **Hilb** we start with the algebraic tensor product of complex vector spaces. The problem here is that the vector tensor of two Hilbert spaces is not necessarily complete. Hence, not a Hilbert space. This problem is solved by making it complete.

Definition 3.5. *Let $\mathcal{H}$, $\mathcal{K}$ be two Hilbert spaces. We define the tensor product $\mathcal{H} \otimes \mathcal{K}$ as the completion of the complex vector space tensor product with respect to the linear extension of the following inner product*

$$\langle h \otimes k, h' \otimes k' \rangle_{\mathcal{H} \otimes \mathcal{K}} = \langle h, h' \rangle_{\mathcal{H}} \langle k, k' \rangle_{\mathcal{K}}$$

From here on we will omit the subscripts of the inner products we use. This, because most of the time it is completely clear in what space we take this inner product.

Because we complete the tensor product with respect to the inner product on the tensor, it follows that $\mathcal{H} \otimes \mathcal{K} \in$ **Hilb**. However, in the finite case the completion of the tensor is not necessary.

Proposition 3.6. *Let $\mathcal{H}, \mathcal{K}$ be two finite-dimensional Hilbert spaces. Then the vector tensor product $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{K}$ is complete*

Proof. Let $N = \dim(\mathcal{H})$ and $M = \dim(\mathcal{K})$. Denote by $e(i) \otimes f(i)$ the i-th orthonormal basis vector of $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{K}$ and let $\{\sum_i a_n(i) e(i) \otimes f(i)\}_{n \in \mathbb{N}}$ be a Cauchy sequence. Then, by the Parsevals Theorem, we have

$$\| \sum_i a_n(i) e(i) \otimes f(i) - \sum_j a_m(j) e(j) \otimes f(j) \|^2 = \sum_i |a_n(i) - a_m(i)|^2$$

This shows that $\{a_n(i)\}_{n \in \mathbb{N}}$ is a Cauchy sequence for each $1 \leq i \leq NM$. Hence, for each i we have $a(i) := \lim_{n \to \infty} a_n(i)$. Now because of the finiteness and the linearity of the limit we have

$$\lim_{n \to \infty} \sum_i a_n(i) e(i) \otimes f(i) = \sum_i \left(\lim_{n \to \infty} a_n(i) \right) e(i) \otimes f(i) = \sum_i a(i) e(i) \otimes f(i)$$

Thus $\mathcal{H} \otimes_{\mathbb{C}} \mathcal{K}$ is complete $\qquad \square$

This proposition makes working with the tensor a lot more simple. This is because elements in $\mathcal{H} \otimes \mathcal{K}$ are in general converging sequences of elements in the vector tensor product. This means we are dealing with objects of the form $\{\sum_i h_{i,n} \otimes k_{i,n}\}_{n \in N}$.

This is not the case when we are dealing with finite-dimensional Hilbert spaces. Here we only have to deal with elements of the tensor product $\sum_n h_n \otimes k_n$.

In order to keep notations clear and simple we will write elements from the tensor as $h \otimes k$. We use this shorthand notation because all our functions need to be linear and therefore to define them on $\mathcal{H} \otimes \mathcal{K}$ it is enough to define them on the simple elements $h \otimes k$ and then use linear extension. We will do this without specification throughout this chapter. In some cases one has to check that the functions are well defined with respect to the relations on the tensor. However this is clear most of the time.

We will now prove that the tensor product from definition 3.5 defines a functor. Given two morphisms $\mathcal{H} \xrightarrow{f} \mathcal{H}'$ and $\mathcal{K} \xrightarrow{g} \mathcal{K}'$ in **Hilb** we define their tensor component wise i.e. $f \otimes g(h \otimes k) = f(h) \otimes g(k)$. This is well defined, linear and bounded hence a morphism in **Hilb**. Composition is also defined component wise and therefore we have a functor $\mathbf{Hilb} \times \mathbf{Hilb} \xrightarrow{\otimes} \mathbf{Hilb}$. This completes the multiplication part of the monoidal structure on **Hilb**. To complete the structure we need a unit. Because we used the complex tensor, it will come as no surprise that $\mathbb{C}$ satisfies the necessary properties. The remaining details on the natural isomorphisms are given in the following proposition.

Proposition 3.7. $\langle \mathbf{Hilb}, \otimes, \mathbb{C} \rangle$ *is symmetric monoidal.*

Proof. We have already seen that $\mathbf{Hilb} \times \mathbf{Hilb} \xrightarrow{\otimes} \mathbf{Hilb}$ is a functor. Thus we only need to prove the existence of the four natural isomorphisms. Let $\mathcal{H}, \mathcal{K}, \mathcal{L} \in \mathbf{Hilb}$. The morphism $\mathcal{H} \otimes (\mathcal{K} \otimes \mathcal{L}) \xrightarrow{\alpha} (\mathcal{H} \otimes \mathcal{K}) \otimes \mathcal{L}$ is defined by $\alpha(h \otimes (k \otimes l)) = (h \otimes k) \otimes l$. It is easy to see that this is well defined and linear. By definition of the inner product we have

$$\|\alpha(h \otimes (k \otimes l))\|^2 = \langle (h \otimes k) \otimes l, (h \otimes k) \otimes l \rangle^2 = \langle h \otimes k, h \otimes k \rangle \langle l, l \rangle$$
$$= \langle h, h \rangle \langle k, k \rangle \langle l, l \rangle = \|h\|^2 \|k\|^2 \|l\|^2$$

Therefore α is bounded and hence a morphism in **Hilb**. Moreover because $\otimes$ is defined component wise on functions it is natural. The inverse is obvious so α is a natural isomorphism.

Next we define the transformation $\lambda : \otimes(\mathbb{C} \times 1) \Rightarrow 1_{\mathbf{Hilb}}$ by $\lambda(c \otimes h) = ch$. This is also well defined, linear, bounded and natural. It has an inverse

defined by $\lambda^{-1}(h) = 1 \otimes h$, for

$$\lambda^{-1}\lambda(c \otimes h) = \lambda^{-1}(ch) = 1 \otimes ch = c \otimes h$$

$$\lambda\lambda^{-1}(h) = \lambda(1 \otimes h) = h$$

Hence, λ is a natural isomorphism. The definition of the transformation ρ is given by $h \otimes c \mapsto ch$. While the commutative transformation γ has the obvious definition $h \otimes k \mapsto k \otimes h$. The commutivity of the diagrams is straight forward. $\qquad\square$

3.2.2   A dagger on Hilb

Being monoidal is a start but we need **Hilb** to be SDM. This means we have to define a dagger. As was the case with the monoidal structure, this is more involved than the dagger on **PInj**. We use Riesz Representation Theorem which states that for each bounded linear functional $\mathcal{H} \xrightarrow{f} \mathbb{C}$, there exists a unique vector $h_0 \in \mathcal{H}$ such that $f(h) = \langle h, h_0 \rangle$ for all $h \in \mathcal{H}$ and $\|h_0\| = \|f\|$. Now let $\mathcal{H} \xrightarrow{f} \mathcal{K}$ and fix $k \in \mathcal{K}$. Consider the function $\mathcal{H} \xrightarrow{F_k} \mathbb{C}$ defined by $F_k(h) = \langle f(h), k \rangle$. This is clearly linear and bounded for

$$\|F_k(h)\|^2 = |\langle f(h), k \rangle|^2 \leq \langle f(h), f(h) \rangle \langle k, k \rangle \leq \|f(h)\|^2 \|k\|^2 \leq \|f\|^2 \|k\|^2 \|h\|^2$$

Hence by the Riesz Representation Theorem there is a unique h_k such that $\langle f(h), k \rangle = \langle h, h_k \rangle$. Now define $\mathcal{K} \xrightarrow{f^\dagger} \mathcal{H}$ by $f^\dagger(k) = h_k$, then we have the following

Lemma 3.8. *Let* $\mathcal{H} \xrightarrow{f} \mathcal{K} \in$ **Hilb** *then*

1. $f^\dagger$ *is linear*

2. $f^\dagger$ *is bounded*

Proof.   1. Let $k_1, k_2 \in \mathcal{K}$, $h \in \mathcal{H}$ and $c_1, c_1 \in \mathbb{C}$ then

$$\begin{aligned}
\langle h, c_1 f^\dagger(k_1) + c_2 f^\dagger(k_2) \rangle &= \langle \overline{c_1} h, f^\dagger(k_1) \rangle + \langle \overline{c_2} h, f^\dagger(k_2) \rangle \\
&= \langle f(\overline{c_1} h), k_1 \rangle + \langle f(\overline{c_2} h), k_2 \rangle \\
&= \langle f(h), c_1 k_1 + c_2 k_2 \rangle
\end{aligned}$$

Hence by uniqueness $f^\dagger(c_1 k_1 + c_2 k_2) = c_1 f^\dagger(k_1) + c_2 f^\dagger(k_2)$.

2. Let $k \in \mathcal{K}$, $h \in \mathcal{H}$. Then by the calculation we did earlier $\|F_k\| \leq \|f\|\|k\|$ and hence by Riesz Representation Theorem $\|f^\dagger(k)\| = \|F_k\| \leq \|f\|\|k\|$ so $f^\dagger$ is bounded.

$\square$

This lemma proves that $f^\dagger$ is a morphism in **Hilb**. Next we show that it satisfies the conditions of a dagger.

Lemma 3.9. *Let* $\mathcal{H} \xrightarrow{\ f\ } \mathcal{K}$ *and* $\mathcal{K} \xrightarrow{\ g\ } \mathcal{L}$ *be morphisms in* **Hilb** *then*

D1 $1^\dagger = 1$

D2 $(f^\dagger)^\dagger = f$

D3 $(gf)^\dagger = f^\dagger g^\dagger$

Proof. Recall that by lemma 3.8 $f^\dagger$ and $g^\dagger$ are morphisms in **Hilb**.

D1 This is trivial.

D2 Take $h \in \mathcal{H}$, $k \in \mathcal{K}$ then

$$\langle k, f(h) \rangle = \overline{\langle f(h), k \rangle} = \overline{\langle h, f^\dagger(k) \rangle} = \langle f^\dagger(k), h \rangle$$

hence by uniqueness $(f^\dagger)^\dagger = f$

D3 Take $h \in \mathcal{H}$, $l \in \mathcal{L}$ then

$$\langle h, f^\dagger g^\dagger(l) \rangle = \langle f(h), g^\dagger(l) \rangle = \langle gf(h), l \rangle$$

so again by uniqueness $(gf)^\dagger = f^\dagger g^\dagger$

$\square$

The above considerations and lemma's are summarized in the following definition.

Definition 3.10. *Define the functor* **Hilb**$^{op} \xrightarrow{\ \dagger\ }$ **Hilb** *as the identity on objects and on morphisms* $\mathcal{H} \xrightarrow{\ f\ } \mathcal{K}$ *, by defining $f^\dagger(k)$ to be the unique element in $\mathcal{H}$, such that $\langle h, f^\dagger(k) \rangle = \langle f(h), k \rangle$ for all $h \in \mathcal{H}$.*

Lemma 3.8 together with lemma 3.9 say that $\dagger$ is well defined and a dagger on **Hilb**. What remains is to show that it preserves the monoidal structure. The proof is given in proposition 3.12 which summarizes the results in this section. First a lemma, showing that unitary maps preserve the inner product.

Lemma 3.11. *Let* $\mathcal{H} \xrightarrow{f} \mathcal{K} \in \mathbf{Hilb}$ *then*

1. $f^\dagger f = 1 \iff \langle h, h' \rangle = \langle f(h), f(h') \rangle$ *for all* $h, h' \in \mathcal{H}$

2. $ff^\dagger = 1 \iff \langle k, k' \rangle = \langle f^\dagger(k), f^\dagger(k') \rangle$ *for all* $k, k' \in \mathcal{K}$

Proof.   1. Suppose $f^\dagger f = 1$ then $\langle h, h' \rangle = \langle h, f^\dagger f(h') \rangle = \langle f(h), f(h') \rangle$. Now suppose that $\langle h, h' \rangle = \langle f(h), f(h') \rangle$ for all $h, h' \in \mathcal{H}$. Then $\langle h, h' \rangle = \langle f(h), f(h') \rangle = \langle h, f^\dagger f(h') \rangle$ and because of uniqueness it follows that $f^\dagger f(h') = h'$ for all $h' \in \mathcal{H}$.

2. Suppose $ff^\dagger = 1$ then $\langle k, k' \rangle = \langle k, ff^\dagger(k') \rangle = \langle f^\dagger(k), f^\dagger(k') \rangle$. Now suppose that $\langle k, k' \rangle = \langle f^\dagger(k), f^\dagger(k') \rangle$ for all $k, k' \in \mathcal{K}$. Then $\langle k, k' \rangle = \langle f^\dagger(k), f^\dagger(k') \rangle = \langle k, ff^\dagger(k') \rangle$ and because of uniqueness it follows that $ff^\dagger(k') = k'$ for all $k' \in \mathcal{K}$.

$\square$

Proposition 3.12. Hilb *is a SDM.*

Proof. We have already shown that **Hilb** is a symmetric monoidal category and that is has a dagger. What remains to prove is that the dagger and the tensor commute and the four structure morphisms α, λ, ρ and γ are unitary. We start with the former. Therefore $\mathcal{H} \xrightarrow{f} \mathcal{H}'$ and $\mathcal{K} \xrightarrow{g} \mathcal{K}'$ be morphisms in **Hilb**. Then

$$\langle h \otimes k, f^\dagger \otimes g^\dagger(h' \otimes k') \rangle = \langle h, f^\dagger(h') \rangle \langle k, g^\dagger(k') \rangle$$
$$= \langle f(h), h' \rangle \langle g(k), k' \rangle$$
$$= \langle f \otimes g(h \otimes k), h' \otimes k' \rangle$$

so by uniqueness $(f \otimes g)^\dagger = f^\dagger \otimes g^\dagger$ which proves that $\dagger \otimes = \otimes \dagger$.
We will prove that λ is unitary. The prove for the other transformations is similar.

$$\langle c \otimes h, \lambda^{-1}(h') \rangle = \langle c \otimes h, 1 \otimes h' \rangle = c\langle h, h' \rangle = \langle ch, h' \rangle = \langle \lambda(c \otimes h), h' \rangle$$

hence $\lambda^{-1} = \lambda^\dagger$. $\square$

3.3   A functor between PInj and Hilb

In the previous sections we have seen that both **PInj** and **Hilb** are SDMs. Our aim in this chapter was to establish an adjunction between their categories of Frobenius semigroups. To do this, we need a functor between

$\mathbb{F}$rob($\mathbf{PInj}$) and $\mathbb{F}$rob($\mathbf{Hilb}$). This is where the extension theorem 2.15 comes in. In this section we will give a functor from $\mathbf{PInj}$ to $\mathbf{Hilb}$. We will show this functor is Frobenius and then use 2.15 to extend it to the Frobenius semigroups. The functor in question is the ℓ^2 functor and follows from the ℓ^2 construction on sets. We recall that we are working with finite-dimensional Hilbert spaces and finite sets.

3.3.1 ℓ^2 spaces

Definition 3.13. *Let X be a set then we define the space $\ell^2(X)$ by*

$$\ell^2(X) = \left\{ X \xrightarrow{\phi} \mathbb{C} : \sum_{x \in X} |\phi(x)|^2 < \infty \right\}$$

The space $\ell^2(X)$ is given a linear structure by defining $(\phi + \psi)(x) = \phi(x) + \psi(x)$ for $\phi, \psi \in \ell^2(X)$, $x \in X$. In order to make $\ell^2(X)$ into a Hilbert space we need an inner product. For this we use the inner product on $\mathbb{C}$, which is defined by $\langle c_1, c_2 \rangle = c_1 \bar{c}_2$ and the fact that elements of $\ell^2(X)$ map to $\mathbb{C}$.

Lemma 3.14. *Let X be a set, then the map $\langle \cdot, \cdot \rangle : \ell^2(X) \times \ell^2(X) \to \mathbb{C}$ defined by*

$$\langle \phi, \psi \rangle := \sum_{x \in X} \phi(x) \overline{\psi(x)}$$

is an inner product.

Proof. The map is clearly linear in the first and conjugate linear in the second variable. Now observe that $\langle \phi, \phi \rangle = \sum_{x \in X} |\phi(x)|^2$, hence the map is positive. It is also positive definite, for if $\langle \phi, \phi \rangle = 0$ it must be that $\phi(x) = 0$ for each $x \in X$ hence $\phi = 0$ as a function. That $\langle \phi, \psi \rangle = \overline{\langle \psi, \phi \rangle}$ is evident. $\qquad \square$

The next result completes the ℓ^2 construction by showing that it turns a set X into a Hilbert space.

Lemma 3.15. *Let X be a set then $\ell^2(X)$ with the inner product from 3.14 is a Hilbert space.*

Proof. We have already shown that $\ell^2(X)$ is an inner product space. Thus we need to show it is complete. Let $\{\phi_n\}_{n \in \mathbb{N}}$ be a Cauchy sequence in $\ell^2(X)$ then

$$\sum_{x \in X} |\phi_n(x) - \phi_m(x)|^2 = \langle \phi_n - \phi_m, \phi_n - \phi_m \rangle = \|\phi_n - \phi_m\|^2$$

This implies that $\{\phi_n(x)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for each $x \in X$ and because this is complete $\lim_{n\to\infty} \phi_n(x)$ exists for all $x \in X$. We define $\phi(x) = \lim_{n\to\infty} \phi_n(x)$. Then $\lim_{n\to\infty} \phi_n = \phi$ and hence using dominated convergence and the fact that $\left\{\sum_{x\in X} |\phi_n(x)|^2\right\}_{n\in\mathbb{N}}$ is Cauchy we get.

$$\sum_{x\in X} |\phi(x)|^2 = \sum_{x\in X} |\lim_{n\to\infty} \phi_n(x)|^2$$
$$= \sum_{x\in X} \lim_{n\to\infty} |\phi_n(x)|^2$$
$$= \lim_{n\to\infty} \sum_{x\in X} |\phi_n(x)|^2 < \infty$$

This shows that $\phi \in \ell^2(X)$ and so $\ell^2(X)$ is complete. $\qquad\square$

Now we know $\ell^2(X)$ is a Hilbert space we can take a closer look at its structure. We are especially interested in a basis for this space. It turns out this basis is given by the set X. To show this, we need a class of characteristic functions $X \xrightarrow{\chi_x} \mathbb{C}$ for each $x \in X$. Recall that a characteristic function χ_U is defined for any $U \subseteq X$ by

$$\chi_U(x) = \begin{cases} 1 & \text{if } x \in U \\ 0 & \text{else} \end{cases}$$

We denote $\chi_{\{x\}}$ by χ_x.

Lemma 3.16. *Let X be a set. Then $\{\chi_x\}_{x\in X}$ form an orthonormal basis of $\ell^2(X)$.*

Proof. Let $x, y \in X$ then

$$\langle \chi_x, \chi_y \rangle = \sum_{z\in X} \chi_x(z)\overline{\chi_y(z)} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{else} \end{cases}$$

hence χ_x, χ_y are orthonormal and clearly they are linearly independent. Now let $\phi \in \ell^2(X)$ then because supp_ϕ is finite $\phi = \sum_{x\in\text{supp}_\phi} \phi(x)\chi_x$, which shows that the $\{\chi_x\}_{x\in X}$ span $\ell^2(X)$. $\qquad\square$

This lemma as well as the functions χ_x will play an important role in the remainder of this chapter. They will be used to establish an adjunction between $\mathbb{F}\text{rob}(\mathbf{PInj})$ and $\mathbb{F}\text{rob}(\mathbf{Hilb})$.

3.3.2 The ℓ^2 functor

That $\ell^2(X)$ is a Hilbert space is a well known statement. The natural question is whether this construction is functorial and if so, which starting category should we take? In order to answer this question we need to define what the ℓ^2 construction on morphisms is.

Given two sets X, Y, a function $X \xrightarrow{f} Y$ and $\phi \in \ell^2(X)$, we want $\ell^2(\phi) \in \ell^2(Y)$. The first thing we could try is to reverse via f from Y to X and apply ϕ on the result. However, to do this we need f to be injective. Now if $f^{-1}\{y\} = \emptyset$ then we could define $\ell^2(\phi)(y) = 0$. This shows that actually f does not need to be a total function on X. Now we have arrived at the category **PInj**. Here the construction discussed above works and therefore this could be the right candidate for the starting category of the functor ℓ^2. The next proposition formalizes all the considerations above. We use the convention that sums over the empty set are 0.

Proposition 3.17. *The operation ℓ^2 defines a functor* $\mathbf{PInj} \xrightarrow{\ell^2} \mathbf{Hilb}$

Proof. Define ℓ^2 on objects just by $X \mapsto \ell^2(X)$, which by lemma 3.15 is in **Hilb**. Now let $X \xrightarrow{f} Y$ be an arrow in **PInj**, then define the morphism $\ell^2(X) \xrightarrow{\ell^2(f)} \ell^2(Y)$ by

$$\ell^2(f)(\phi)(y) = \sum_{f^{-1}(y)} \phi(x)$$

where $\phi \in \ell^2(X)$. Composition is defined by

$$\ell^2(g)\ell^2(f)\phi(z) := \sum_{f^{-1}(g^{-1}(z))} \phi(x)$$

This is well defined for $f^{-1}\{g^{-1}\{z\}\} = (gf)^{-1}\{z\}$ so $\ell^2(gf) = \ell^2(g)\ell^2(f)$. It needs to be shown that $\ell^2(f)(\phi) \in \ell^2(Y)$ for all $\phi \in \ell^2(X)$. First observe that because f is injective there can be at most one $x \in X$ with $f(x) = y$, if such an x does not exist then $\ell^2(f)(\phi)(y) = 0$. For the finiteness of the square sum, we observe that $f^{-1}\{Y\} \subseteq X$. Hence

$$\sum_{y \in Y} |\ell^2(f)(\phi)(y)|^2 = \sum_{y \in Y} |\sum_{f^{-1}(y)} \phi(x)|^2$$
$$\leq \sum_{x \in X} |\phi(x)|^2 < \infty$$

proving that indeed $\ell^2(f)(\phi) \in \ell^2(Y)$. $\square$

The functor ℓ^2 is the first step in constructing the adjunction between the Frobenius semigroups of **PInj** and **Hilb**. It turns out it has a very rich structure, which will eventually make it a Frobenius functor. The rest of this section will therefore be devoted to proving this result. The first step is proving that ℓ^2 is monoidal. The hardest thing here is to construct the natural transformation β. Let $X, Y \in$ **PInj**. Then for each $\phi \otimes \psi \in \ell^2(X) \otimes \ell^2(Y)$ we need $X \oplus Y \xrightarrow{\beta(\phi \otimes \psi)} \mathbb{C}$. To this end we define

$$\beta(\phi \otimes \psi)(x, y) = \phi(x)\psi(y)$$

It is an easy exercise to show that β is well defined. For the other transformation we have the following.

Lemma 3.18. $\ell^2(\{1\}) \cong \mathbb{C}$

Proof. Define $\mathbb{C} \xrightarrow{\tau} \ell^2(\{1\})$ by $\tau(c)(1) = c$. This is clearly linear and it is bounded because

$$\|\tau(c)\|^2 = \tau(c)(1)\overline{\tau(c)(1)} = \|c\|^2$$

Now suppose $\tau(c_1) = \tau(c_2)$. Then $c_1 = \tau(c_1)(1) = \tau(c_2)(1) = c_2$ and hence τ is injective. For surjectivity let $\phi \in \ell^2(\{1\})$. Then $\text{dom}(\phi) = \{1\}$ and so $\phi = \tau(\phi(1))$. $\qquad\square$

Due to this result we will not distinguish between $\mathbb{C}$ and $\ell^2(\{1\})$. When we write $c \otimes \phi$ it is clear that we mean c as a function. The isomorphism τ will be used as the other transformation for the monoidal structure of ℓ^2. There is a generalization of this lemma, which we will give later in this chapter. Now that we have our transformations, we are ready to prove the following.

Theorem 3.19. *The functor ℓ^2 is a commutative monoidal functor.*

Proof. We need to proof that the transformations β and τ are natural. It is clear from the definition that τ is natural so we do β. Let $X \xrightarrow{f} X'$ and $Y \xrightarrow{g} Y'$ be arrows in **PInj**, $\phi \otimes \psi \in \ell^2(X) \otimes \ell^2(Y)$. Then in order to prove that β is natural we must show that $\ell^2(f \oplus g)\beta(\phi \otimes \psi) = \beta(\ell^2(f)\phi \otimes \ell^2(g)\psi)$ as functions from $X' \otimes Y'$ to $\mathbb{C}$. So take $x' \in X'$ and $y' \in Y'$ then

$$\ell^2(f \oplus g)\beta(\phi \otimes \psi)(x', y') = \sum_{(f \oplus g)^{-1}(x', y')} \phi(x)\psi(y)$$
$$= \sum_{f^{-1}(x')} \phi(x) \sum_{g^{-1}(y')} \psi(y)$$
$$= \beta(\ell^2(f)\phi \otimes \ell^2(g)\psi)(x', y')$$

Now that β is natural we need to prove the diagrams from definition 2.6 commute. We will prove that the following diagram commutes

$$
\begin{array}{ccc}
\mathbb{C} \otimes \ell^2(X) & \xrightarrow{\;\lambda_{\mathbf{Hilb}}\;} & \ell^2(X) \\
{\scriptstyle \tau \otimes 1}\big\downarrow & & \big\uparrow{\scriptstyle \ell^2(\lambda_{\mathbf{PInj}})} \\
\ell^2(\{1\}) \otimes \ell^2(X) & \xrightarrow{\;\beta\;} & \ell^2(\{1\} \oplus X)
\end{array}
$$

the proof for the other two diagrams is similar.
Let $c \otimes \phi \in \mathbb{C} \otimes \ell^2(X)$ and $x \in X$ then

$$
\begin{aligned}
\ell^2(\lambda_{\mathbf{PInj}})\beta(\tau \otimes 1)(c \otimes \phi)(x) &= \ell^2(\lambda_{\mathbf{PInj}})\beta(c \otimes \phi)(x) \\
&= \ell^2(\lambda_{\mathbf{PInj}})(c\phi)(x) \\
&= \sum_{\lambda_{\mathbf{PInj}}^{-1}(x)} c\phi(1 \oplus x') \\
&= c\phi(x) \\
&= \lambda_{\mathbf{Hilb}}(c \otimes \phi)(x)
\end{aligned}
$$

$\square$

Now that the ℓ^2 is monoidal we take it a step further. To prove it is Frobenius we need β and τ to be unitary. The latter is not hard for

$$
\langle c, \tau^{-1}(c') \rangle = c\overline{c'} = \langle \tau(c), c' \rangle
$$

which shows that $\tau^{-1} = \tau^{\dagger}$.
The proof for β is a bit harder because we do not have an inverse to test. To proof that $\beta\beta^{\dagger} = 1$ we take the inner product of $\beta^{\dagger}(\phi)$ with a basis element $\chi_x \otimes \chi_y$.

Lemma 3.20. *The natural transformation* $\beta : \otimes(\ell^2 \times \ell^2) \Rightarrow \ell^2 \oplus$ *is unitary.*

Proof. Let $X, Y \in \mathbf{PInj}$ and $\phi \otimes \psi, \phi' \otimes \psi' \in \ell^2(X) \otimes \ell^2(Y)$ then

$$
\begin{aligned}
\langle \phi \otimes \psi, \phi' \otimes \psi' \rangle &= \langle \phi, \phi' \rangle \langle \psi, \psi' \rangle \\
&= \sum_{x \in X} \phi(x)\overline{\phi'(x)} \sum_{y \in Y} \psi(y)\overline{\psi'(y)} \\
&= \sum_{(x,y) \in X \oplus Y} \phi(x)\psi(y)\overline{\phi'(x)\psi'(y)} \\
&= \langle \beta(\phi \otimes \psi), \beta(\phi' \otimes \psi') \rangle
\end{aligned}
$$

By lemma 3.11 $\beta^\dagger\beta = 1$. Now for the other identity let $\phi \in \ell^2(X \oplus Y)$ and write $\beta^\dagger\phi = \phi_1 \otimes \phi_2$. Then if we take $x \in X$, $y \in Y$ we have

$$
\begin{aligned}
\phi(x,y) &= \sum_{X \oplus Y} \chi_x(x')\chi_y(y')\phi(x',y') \\
&= \sum_{X \oplus Y} \overline{\beta(\chi_x \otimes \chi_y)}(x',y')\phi(x',y') \\
&= \overline{\langle \beta(\chi_x \otimes \chi_y), \phi \rangle} \\
&= \overline{\langle \chi_x \otimes \chi_y, \beta^\dagger(\phi) \rangle} \\
&= \overline{\langle \chi_x \otimes \chi_y, \phi_1 \otimes \phi_2 \rangle} \\
&= \overline{\langle \chi_x, \phi_1 \rangle\langle \chi_y, \phi_2 \rangle} = \phi_1(x)\phi_2(y)
\end{aligned}
$$

Therefore $\phi(x,y) = \phi_1(x)\phi_2(y) = \beta(\phi_1 \otimes \phi_2)(x,y) = \beta\beta^\dagger(\phi)(x,y)$, which shows that $\beta\beta^\dagger = 1$. □

We have now proven that the functor ℓ^2 is not only symmetric monoidal but in addition its structure morphisms are unitary. The only thing left, is to show that ℓ^2 is a dagger symmetric monoidal functor.

Theorem 3.21. *The functor ℓ^2 is Frobenius.*

Proof. By theorem 3.19 ℓ^2 is symmetric monoidal. We shall now prove that it is also symmetric dagger monoidal. Let $X \xrightarrow{f} Y$ in **PInj** and $\psi \in \ell^2(Y)$. Then

$$
\ell^2(f^\dagger)\psi(x) = \sum_{f^\dagger(y)=x} \psi(y) = \sum_{f(x)=y} \psi(y)
$$

and hence

$$\begin{aligned}
\langle \phi, \ell^2(f)^\dagger \psi \rangle &= \langle \ell^2(f)\phi, \psi \rangle \\
&= \sum_{y \in Y} \ell^2(f)\phi(y)\overline{\psi(y)} \\
&= \sum_{y \in Y} \left(\sum_{f(x)=y} \phi(x) \right) \overline{\psi(y)} \\
&= \sum_{f(x)=y} \phi(x)\overline{\psi(y)} \\
&= \sum_{x \in X} \phi(x) \left(\sum_{f(x)=y} \overline{\psi(y)} \right) \\
&= \sum_{x \in X} \phi(x)\overline{\ell^2(f^\dagger)\psi(x)} \\
&= \langle \phi, \ell^2(f^\dagger)\psi \rangle
\end{aligned}$$

for all $\phi \in \ell^2(X)$, which proves that $\ell^2(f)^\dagger = \ell^2(f^\dagger)$. The proof that the transformations α, λ, ρ and γ are unitary are very similar hence we do only λ. Let $h, h' \in \mathcal{H}$ and $c \in \mathbb{C}$. Then

$$\langle c \otimes h, \lambda^{-1}(h') \rangle = \langle c \otimes h, 1 \otimes h' \rangle = c\langle h, h' \rangle = \langle ch, h' \rangle = \langle \lambda(c \otimes h), h' \rangle$$

and hence $\lambda^{-1} = \lambda^\dagger$. This proves that ℓ^2 is symmetric dagger monoidal. We have already seen that τ is unitary and by lemma 3.20 so is β, which concludes the proof that ℓ^2 is Frobenius. $\square$

All the hard work is done. The result is that by theorem 2.15 we can extend ℓ^2 to a functor between $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ and $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$. Recall from Chapter 1 that if $\langle X, \Delta_X \rangle \in \mathbb{F}\mathrm{rob}(\mathbf{PInj})$, then ℓ^2 sends it to $\langle \ell^2(X), \beta^\dagger \ell^2(\Delta_X) \rangle \in \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$.

One might wonder why if we already had a functor between $\mathbf{PInj}$ and $\mathbf{Hilb}$ we had to go through all these details to get a Frobenius functor. The answer is very simple: there is no adjunction between $\mathbf{PInj}$ and $\mathbf{Hilb}$. For if we did have an adjunction $\mathbf{PInj} \overset{\ell^2}{\underset{\perp}{\rightleftarrows}} \mathbf{Hilb}$ it follows that all limits in $\mathbf{Hilb}$ are preserved. This would imply that $\mathbf{PInj}$ has all finite limits but this is not the case. For example $\mathbf{PInj}$ has no products. To see this suppose that we have a product $X \prod Y$, fix $x \in X$ and $y \in Y$ and take the injections

$x \xhookrightarrow{f} X$ and $y \xhookrightarrow{g} Y$. Then there is a unique $\{x, y\} \xrightarrow{h} X \prod Y$ such that $f = \pi_X h$ and $g = \pi_Y h$. This implies that h is total. However, $f = \pi_X h$ implies that $\mathrm{dom}(f) = \mathrm{dom}(h)$, which is a contradiction.

Although there can't be an adjunction between **PInj** and **Hilb** this could however be true for the Frobenius semigroups, because these categories have no finite limits. We will construct this adjunction in section 3.5. First we need to take a look at certain special elements within the Frobenius semigroups on **Hilb**.

3.4 Copyables

Definition 3.22. *Let $\langle \mathcal{H}, \delta \rangle \in \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$. A copyable $\mathfrak{c}$ is a element in $\mathcal{H}$ such that $\delta(\mathfrak{c}) = \mathfrak{c} \otimes \mathfrak{c}$.*

We denote by $\mathfrak{C}_\delta(\mathcal{H})$ the set of non zero copyable elements of $\langle \mathcal{H}, \delta \rangle$ where we omit the δ if no confusion arises.

Copyables can be defined in a general setting as a morphism $e \xrightarrow{\mathfrak{c}} c$ such that $(\mathfrak{c} \square \mathfrak{c})\mu = \mu \mathfrak{c}$, where $\langle c, \mu \rangle$ is a Frobenius semigroup in a SDM $\langle \mathcal{C}, \square, e \rangle$. However, we only need this specific case.

We have already mentioned that Frobenius semigroups in **Hilb** can be viewed as a categorical equivalent to an orthonormal basis. The key elements here are the copyables. It is proved in [2] that in the case of finite-dimensional Hilbert spaces the copyables form an orthonormal basis. Moreover starting with a Hilbert space and an orthonormal basis one can construct a Frobenius semigroup such that these two constructions are inverse to each other. The difficult part is to show that the copyables span the entire space because the following is true for any Frobenius semigroup.

Lemma 3.23. *Let $\langle \mathcal{H}, \delta \rangle \in \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ then*

1. *Elements from $\mathfrak{C}(\mathcal{H})$ are linearly independent*

2. *$\|\mathfrak{c}\| = 1$ for each $\mathfrak{c} \in \mathfrak{C}(\mathcal{H})$*

3. *Elements from $\mathfrak{C}(\mathcal{H})$ are pairwise orthonormal.*

Proof. The proof can be found in [1] pages [12-13]. $\square$

This means that $\mathfrak{C}(\mathcal{H})$ defines a Hilbert subspace in $\mathcal{H}$ which equals $\mathcal{H}$ when it is finite-dimensional.

We use the copyables to define a functor from $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ to $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$.

Definition 3.24. *Define the functor* $\mathbb{F}\mathrm{rob}(\mathbf{Hilb}) \xrightarrow{\mathfrak{C}} \mathbb{F}\mathrm{rob}(\mathbf{PInj})$ *as follows. Objects* $\langle \mathcal{H}, \delta \rangle$ *are sent to* $\langle \mathfrak{C}_\delta(\mathcal{H}), \Delta \rangle$*, while morphisms* $\langle \mathcal{H}, \delta_\mathcal{H} \rangle \xrightarrow{f} \langle \mathcal{K}, \delta_\mathcal{K} \rangle$ *are mapped to their restriction to the copyables.*

The functor $\mathfrak{C}$ is well defined because $\delta_\mathcal{K} f = f \otimes f \delta_\mathcal{H}$ guarantees that copyable elements are preserved while, $f^\dagger f = 1$ implies that f is injective.

3.5   The adjunction

Now that we have functors $\mathbb{F}\mathrm{rob}(\mathbf{PInj}) \underset{\mathfrak{C}}{\overset{\ell^2}{\rightleftarrows}} \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ it will come as no surprise that these define the long predicted adjunction. In this section we will prove this. Moreover we will show that this adjunction is actually an equivalence of categories.

To construct an adjunction we need a unit and co-unit which are natural transformations satisfying two triangle diagrams. We start with the unit $\eta : 1 \Rightarrow \mathfrak{C}\ell^2$. It turns out that this map is in fact an isomorphism. First let us prove that the copyables of an ℓ^2 space are the same as the base set.

Lemma 3.25. *Let X be a set then $\mathfrak{C}(\ell^2(X)) \cong X$ in* $\mathbf{Set}$*.*

Proof. Let $x \in X$ then $\ell^2(\Delta_X)\chi_x = \chi_{x \oplus x}$ for

$$\ell^2(\Delta_X)\chi_x(x_1 \oplus x_2) = \sum_{\Delta_X^{-1}(x_1,x_2)} \chi_x(x') = \begin{cases} 1 & x_1 = x_2 = x \\ 0 & \text{else} \end{cases}$$

hence

$$\begin{aligned}
&\langle \chi_x \otimes \chi_x - \beta^\dagger \ell^2(\Delta_X)\chi_x, \chi_x \otimes \chi_x - \beta^\dagger \ell^2(\Delta_X)\chi_x \rangle \\
&= \langle \chi_x \otimes \chi_x - \beta^\dagger \chi_{x \oplus x}, \chi_x \otimes \chi_x - \beta^\dagger \chi_{x \oplus x} \rangle \\
&= \langle \chi_x \otimes \chi_x, \chi_x \otimes \chi_x \rangle - \langle \chi_x \otimes \chi_x, \beta^\dagger \chi_{x \oplus x} \rangle \\
&\quad - \langle \beta^\dagger \chi_{x \oplus x}, \chi_x \otimes \chi_x \rangle + \langle \beta^\dagger \chi_{x \oplus x}, \beta^\dagger \chi_{x \oplus x} \rangle \\
&= \langle \chi_x \otimes \chi_x, \chi_x \otimes \chi_x \rangle - \langle \chi_x \otimes \chi_x, \beta^\dagger \chi_{x \oplus x} \rangle \\
&\quad - \langle \beta^\dagger \chi_{x \oplus x}, \chi_x \otimes \chi_x \rangle + \langle \chi_{x \oplus x}, \chi_{x \oplus x} \rangle \\
&= 0
\end{aligned}$$

From this it follows that $\beta^\dagger \ell^2(\Delta_X)\chi_x = \chi_x \oplus \chi_x$. Now let $\phi \in \mathfrak{C}(\ell^2(X))$ then

$\beta^\dagger \ell^2(\Delta_X)\phi = \phi \otimes \phi$ and therefore

$$\begin{aligned}
\phi(x)\phi(y) &= \langle \chi_x \otimes \chi_y, \phi \otimes \phi \rangle \\
&= \langle \chi_x \otimes \chi_y, \beta^\dagger \ell^2(\Delta_X)\phi \rangle \\
&= \langle \beta(\chi_x \otimes \chi_y), \ell^2(\Delta_X)\phi \rangle \\
&= \sum_{\Delta_X^{-1}(x' \oplus y')} \phi(x'')
\end{aligned}$$

From this we deduce that

$$\phi(x)\phi(y) = \begin{cases} \phi(x) & \text{if } x = y \\ 0 & \text{else} \end{cases}$$

Now suppose that $x, y \in X$ such that $x \neq y$, $\phi(x) \neq 0$ and $\phi(y) \neq 0$. Then $\phi(x)\phi(y) \neq 0$, which is a contradiction hence $\phi = \chi_x$ for some $x \in X$. $\square$

We now define the unit $X \xrightarrow{\eta} \mathfrak{C}(\ell^2(X))$ by $\eta(x) = \chi_x$. This is now well defined because of the above lemma. Next we show that it is a morphism in $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$. This follows immediately from the fact that η is injective and total. The last thing to check is that η is natural. For this let $X \xrightarrow{f} Y \in \mathbf{PInj}$ then

$$\eta_Y f(x)(y) = \chi_{f(x)}(y) = \sum_{f^{-1}(y)} \chi_x(x') = \ell^2(f)\chi_x(y) = \eta_X \ell^2(f)\chi_x(y)$$

hence

$$\begin{array}{ccc}
X & \xrightarrow{\eta} & \mathfrak{C}\ell^2(X) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathfrak{C}\ell^2(f)} \\
Y & \xrightarrow{\eta} & \mathfrak{C}\ell^2(Y)
\end{array}$$

commutes. Thus η is a natural transformation $\eta : 1 \Rightarrow \mathfrak{C}\ell^2$.

Now it is time to define the co-unit $\varepsilon : \ell^2 \mathfrak{C}(\mathcal{H}) \Rightarrow 1$. The crucial thing here is if $\phi \in \ell^2(X)$ then it has finite support. Let $\phi \in \ell^2(\mathfrak{C}(\mathcal{H}))$. Then we define

$$\varepsilon(\phi) = \sum_{\mathfrak{c} \in \mathrm{supp}_\phi} \phi(\mathfrak{c})\mathfrak{c}$$

Clearly $\varepsilon(\phi) \in \mathcal{H}$. To show that it is indeed a morphism in $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ and natural requires some work.

Proposition 3.26. *ε is a morphism in $\mathbb{F}$rob(**Hilb**)*

Proof. We need to show that ε is linear and bounded and that it satisfies the required commutivity rules.
Let $\phi, \psi \in \ell^2(\mathfrak{C}(\mathcal{H}))$. Then

$$
\begin{aligned}
\varepsilon(c\phi + c'\psi) &= \sum_{\text{supp}_{\phi+\psi}} (c\phi + c'\psi)(\mathfrak{c})\mathfrak{c} \\
&= \sum_{\text{supp}_{\phi+\psi}} c\phi(\mathfrak{c})\mathfrak{c} + c'\psi(\mathfrak{c})\mathfrak{c} \\
&= c\sum_{\text{supp}_\phi} \phi(\mathfrak{c})\mathfrak{c} + c'\sum_{\text{supp}_\psi} \psi(\mathfrak{c})\mathfrak{c} \\
&= c\varepsilon(\phi) + c'\varepsilon(\psi)
\end{aligned}
$$

which shows that ε is linear. For the boundedness we have

$$
\begin{aligned}
\|\varepsilon(\phi)\|^2 &= \langle \varepsilon(\phi), \varepsilon(\phi) \rangle \\
&= \langle \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\mathfrak{c}, \sum_{\text{supp}_\phi} \phi(\mathfrak{c}')\mathfrak{c}' \rangle \\
&= \sum_{\text{supp}_\phi} |\phi(\mathfrak{c})|^2 \langle \mathfrak{c}, \mathfrak{c} \rangle \\
&= \sum_{\text{supp}_\phi} |\phi(\mathfrak{c})|^2 \\
&= \langle \phi, \phi \rangle = \|\phi\|^2
\end{aligned}
$$

Now that ε is a morphism in **Hilb** we need to check the identities

$$
\begin{array}{ccc}
\ell^2(\mathfrak{C}(\mathcal{H})) \xrightarrow{\varepsilon} \mathcal{H} & \qquad & \ell^2(\mathfrak{C}(\mathcal{H})) \xrightarrow{\beta^\dagger \ell^2(\Delta_{\mathfrak{C}(\mathcal{H})})} \ell^2(\mathfrak{C}(\mathcal{H}))\Box \ell^2(\mathfrak{C}(\mathcal{H})) \\
\searrow^{1} \quad \downarrow{\varepsilon^\dagger} & & \downarrow{\varepsilon} \qquad\qquad \downarrow{\varepsilon\Box\varepsilon} \\
\ell^2(\mathfrak{C}(\mathcal{H})) & & \mathcal{H} \xrightarrow{\quad\delta\quad} \mathcal{H}\Box\mathcal{H}
\end{array}
$$

We start with the left.

$$
\begin{aligned}
\langle \varepsilon(\phi), \varepsilon(\psi) \rangle &= \Big\langle \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\mathfrak{c}, \sum_{\text{supp}_\psi} \psi(\mathfrak{c}')\mathfrak{c}' \Big\rangle = \sum_{\mathfrak{C}(\mathcal{H})} \phi(\mathfrak{c})\overline{\psi(\mathfrak{c})}\langle \mathfrak{c}, \mathfrak{c} \rangle \\
&= \sum_{\mathfrak{C}(\mathcal{H})} \phi(\mathfrak{c})\overline{\psi(\mathfrak{c})} = \langle \phi, \psi \rangle
\end{aligned}
$$

so $\varepsilon^\dagger\varepsilon = 1$. For the right identity we recall that $\phi = \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\chi_\mathfrak{c}$ for each $\phi \in \ell^2(\mathfrak{C}(\mathcal{H}))$, $\beta^\dagger\ell^2(\Delta_{\ell^2(\mathfrak{C}(\mathcal{H}))})\chi_\mathfrak{c} = \mathfrak{c} \otimes \mathfrak{c}$ and $\varepsilon\chi_\mathfrak{c} = \mathfrak{c}$. Using this we get

$$\varepsilon \otimes \varepsilon\beta^\dagger\ell^2(\Delta_{\ell^2(\mathfrak{C}(\mathcal{H}))})\phi = \varepsilon \otimes \varepsilon\beta^\dagger\ell^2(\Delta_{\ell^2(\mathfrak{C}(\mathcal{H}))}) \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\chi_\mathfrak{c}$$

$$= \varepsilon \otimes \varepsilon \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\beta^\dagger\ell^2(\Delta_{\ell^2(\mathfrak{C}(\mathcal{H}))})\chi_\mathfrak{c}$$

$$= \varepsilon \otimes \varepsilon \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\chi_\mathfrak{c} \otimes \chi_\mathfrak{c}$$

$$= \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\mathfrak{c} \otimes \mathfrak{c}$$

$$= \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\delta(\mathfrak{c})$$

$$= \delta\left(\sum_{\text{supp}_\phi} \phi(\mathfrak{c})\mathfrak{c}\right) = \delta\varepsilon\phi$$

This proves that $\varepsilon \in \mathbb{F}\text{rob}(\mathbf{Hilb})$. $\qquad\square$

Now that ε is a morphism in $\mathbb{F}\text{rob}(\mathbf{Hilb})$ there is one thing left.

Lemma 3.27. *The map ε is natural.*

Proof. Let $\langle\mathcal{H}, \delta_\mathcal{H}\rangle \xrightarrow{f} \langle\mathcal{K}, \delta_\mathcal{K} \in \mathbf{FdHilb}, \phi \in \ell^2\mathfrak{C}(\mathcal{H})$ and denote $\hat{\phi} = \ell^2(\mathfrak{C}(\mathcal{H}))f\phi$. Then because $\delta_\mathcal{K}f = f \otimes f\delta_\mathcal{H}$ and $f^\dagger f = 1$ it follows that $f(\mathfrak{C}(\mathcal{H})) \subseteq \mathfrak{C}(\mathcal{K})$ hence

$$\varepsilon\ell^2(\mathfrak{C}(\mathcal{H}))f\phi = \sum_{\text{supp}_{\hat\phi}} \ell^2(\mathfrak{C}(\mathcal{H}))f\phi(\mathfrak{c})\mathfrak{c}$$

$$= \sum_{\text{supp}_{\hat\phi}} \left(\sum_{f^{-1}(\mathfrak{c})} \phi(\mathfrak{c}')\right)\mathfrak{c}$$

$$= \sum_{f(\text{supp}_\phi)} \left(\sum_{f^{-1}(\mathfrak{c})} \phi(\mathfrak{c}')\right)\mathfrak{c}$$

$$= \sum_{\text{supp}_\phi} \phi(\mathfrak{c}')f(\mathfrak{c}')$$

$$= f\varepsilon\phi$$

We used here that $\text{supp}_{\hat\phi} = f(\text{supp}_\phi)$. It follows that the following diagram commutes.

$$
\begin{array}{ccc}
\ell^2 B\mathcal{H} & \xrightarrow{\ \varepsilon\ } & \mathcal{H} \\
{\scriptstyle \ell^2 Bf}\downarrow & & \downarrow{\scriptstyle f} \\
\ell^2 B\mathcal{K} & \xrightarrow{\ \varepsilon\ } & \mathcal{K}
\end{array}
$$

and so ε is natural. $\qquad\square$

Summing up everything we have a unit $\eta : 1 \Rightarrow \mathfrak{C}\ell^2$ defined by $\eta(x) = \chi_x$ and a co-unit $\varepsilon : \ell^2\mathfrak{C} \Rightarrow 1$ defined by $\varepsilon(\phi) = \sum_{\text{supp}_\phi} \phi(\mathfrak{c})\mathfrak{c}$.

Before we prove the adjunction, we will show that in the finite-dimensional case ε is an isomorphism. We will prove this by showing that every finite-dimensional Frobenius semigroup in **Hilb** is isomorphic to the ℓ^2 space of its copyables.

Lemma 3.28. *Let* $\langle \mathcal{H}, \delta\rangle \in \mathbb{F}\text{rob}(\mathbf{FdHilb})$ *then* $\mathcal{H} \cong \ell^2(\mathfrak{C}(\mathcal{H}))$

Proof. We use the map $\ell^2(\mathfrak{C}(\mathcal{H})) \xrightarrow{\ \varepsilon\ } \mathcal{H}$. From proposition 3.26 it follows that ε is an injective linear bounded function hence we only need to prove the surjectivity. Let $h \in \mathcal{H}$ then define $\phi_h \in \ell^2(\mathfrak{C}(\mathcal{H}))$ by $\phi_h(\mathfrak{c}) = \langle h, \mathfrak{c}\rangle$ then

$$
\varepsilon\phi_h = \sum_{\text{supp}_{\phi_h}} \phi_h(\mathfrak{c})\mathfrak{c} = \sum_{\mathfrak{C}(\mathcal{H})} \langle h, \mathfrak{c}\rangle\mathfrak{c} = h
$$

The last identity is due to the fact that the copyables form an orthonormal basis of $\mathcal{H}$. This proves that ε is an isomorphism, moreover because $\varepsilon^\dagger\varepsilon = 1$ it is an isometry. $\qquad\square$

Now we are ready to prove the main result of this chapter.

Theorem 3.29. $\mathbb{F}\text{rob}(\mathbf{PInj}) \underset{\mathfrak{C}}{\overset{\ell^2}{\underset{\perp}{\rightleftarrows}}} \mathbb{F}\text{rob}(\mathbf{Hilb})$

Proof. We use the unit η and co-unit ε which have been proven to be natural transformations. So all that needs to be done is to check the commutivity of the following diagrams

$$
\begin{array}{ccc}
\mathfrak{C}(\mathcal{H}) & \xrightarrow{\ \eta\ } & \mathfrak{C}\ell^2(\mathfrak{C}(\mathcal{H})) \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \mathfrak{C}\varepsilon} \\
& & \mathfrak{C}(\mathcal{H})
\end{array}
\qquad
\begin{array}{ccc}
\ell^2(X) & \xrightarrow{\ \ell^2(\eta)\ } & \ell^2(\mathfrak{C}\ell^2(X)) \\
& {\scriptstyle 1}\searrow & \downarrow{\scriptstyle \varepsilon} \\
& & \ell^2(X)
\end{array}
$$

For the left diagram let $\mathfrak{c} \in \mathfrak{C}(\mathcal{H})$. Then $\eta\mathfrak{c} = \chi_{\mathfrak{c}}$ and we have already observed that $\varepsilon\chi_{\mathfrak{c}} = \mathfrak{c}$. The right diagram is a bit more work. Let $\phi \in \ell^2(X)$ then $\ell^2(\eta)\phi : \mathfrak{C}(\ell^2(X)) \to \mathbb{C}$, with $\ell^2(\eta)\phi\chi_x = \sum_{\eta^{-1}(\chi_x)} \phi(x') = \phi(x)$. Observe that $\mathrm{supp}_{\ell^2(\eta)\phi} = \mathrm{supp}_\phi$. Therefore

$$\varepsilon\ell^2(\eta)\phi = \sum_{\mathrm{supp}_{\ell^2(\eta)\phi}} \ell^2(\eta)\phi(\chi_x)\chi_x = \sum_{\mathrm{supp}_{\ell^2(\eta)\phi}} \phi(x)\chi_x = \sum_{\mathrm{supp}_\phi} \phi(x)\chi_x = \phi$$

$\square$

The adjunction given above is actually an equivalence. The key in the proof is the result from [2] that in the finite-dimensional case the copyables form an orthonormal basis which is used in the form of lemma 3.28.

Corollary 3.30. $\mathbb{F}\mathrm{rob}(\mathbf{PInj}) \cong \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$

Proof. Applying theorem 3.29 to this setting we get an adjunction $\mathbb{F}\mathrm{rob}(\mathbf{PInj}) \underset{\mathfrak{C}}{\overset{\ell^2}{\underset{\perp}{\rightleftarrows}}} \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ with unit η and co-unit ε. Now η is a natural isomorphism because of lemma 3.25 while lemma 3.28 says that ε is a natural isomorphism. $\square$

Although we have restricted ourselves to the finite case, every result except for corollary 3.30 is also true in the infinite case. However, the proves are not completely the same. For instance, because here we would be working with converging sequences, we would have to check that the images of these sequences also converge. Still, this can be done proving that there is an adjunction $\mathbb{F}\mathrm{rob}(\mathbf{PInj}) \underset{\mathfrak{C}}{\overset{\ell^2}{\underset{\perp}{\rightleftarrows}}} \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ where $\mathbf{Hilb}$ has abitrary dimensional Hilbert spaces as objects and $\mathbf{PInj}$ arbitrary sets.

Chapter 4

Inverse semigroups

We have related the category $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ to $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$. Now we will relate the latter to the category of inverse semigroups, $\mathbf{Inv}$. We will see that $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ is equivalent to a subcategory of $\mathbf{Inv}$. This equivalence will give rise to adjunctions between $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ and several subcategories of $\mathbf{Inv}$. However, before we can do this we need to cover some theory on inverse semigroups. Although we refere to the category $\mathbf{PInj}$ several times in this chapter, all results presented here also hold for infinite and inverse semigroups sets.

4.1   The basics

The theory of inverse semigroups is a very rich one and is well covered in the book of Mark V. Lawson [6]. We will refer to this book several times for both basic, easy to check, properties as well as more complicated theorems. To start things off, let us recall that a semigroup consists of a set S together with an associative operation on it. This operation is often called multiplication and denoted by concatenation. An element $e \in S$ is called an *idempotent* if $e^2 = e$. We denote the set of idempotents of a semigroup S by $\mathcal{E}(S)$ omitting the S if it is clear which semigroup we use. With this reminder we are ready to give the definition of an inverse semigroup.

Definition 4.1. *An inverse semigroup S is a semigroup with the following properties*

S1 *S is* regular *i.e. for every $s \in S$ there is a $s^{-1} \in S$ such that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$.*

S2 *The idempotents in S commute.*

We refer to s^{-1} as the inverse of s. One might ask whether s^{-1} really does behave as the inverse we know from group theory. For instance, is it unique and do we have that $(st)^{-1} = t^{-1}s^{-1}$. The former is summarized in the following theorem.

Theorem 4.2. *Let S be a regular semigroup. Then the idempotents of S commute if and only if every $s \in S$ has a unique inverse.*

Proof. This is not straigthforeward but easy to follow, see [6] page 6-7. $\square$

Because the inverse is unique, inverse semigroups satisfy many properties that also hold for groups. We have summarized the most important ones in the next lemma.

Lemma 4.3. *Let S be an inverse semigroup and $s, t \in S$.*

1. $(s^{-1})^{-1} = s$

2. *If e is an idempotent then ses^{-1} is again an idempotent.*

3. $(st)^{-1} = t^{-1}s^{-1}$

Proof. 1. By definition we have that $ss^{-1}s = s$ and $s^{-1}ss^{-1} = s^{-1}$. Now by uniqueness of the inverses the result follows.

2. Notice that $s^{-1}s$ is an idempotent. Because these commute we have $ses^{-1}ses^{-1} = s(s^{-1}s)e^2s^{-1} = ses^{-1}$.

3. Because idempotents commute we have $t^{-1}s^{-1}stt^{-1}s^{-1} = t^{-1}tt^{-1}s^{-1}ss^{-1} = t^{-1}s^{-1}$. Similar $stt^{-1}s^{-1}st = ss^{-1}stt^{-1}t = st$ so the result follows again by uniqueness of the inverse.

$\square$

Now that we have seen that the inverses of inverse semigroups behave similar to those from group theory, it is time to consider the homomorphisms between inverse semigroups.

Definition 4.4. *Let S, T be inverse semigroups. A map $S \xrightarrow{\phi} T$ is called a homomorphism of inverse semigroups when*

$$\phi(ss') = \phi(s)\phi(s')$$

for all $s, s' \in S$.

Just as with the inverse elements in inverse semigroups the homomorphisms satisfy most of the properties we know from group theory. For instance $\phi(s) = \phi(s^{-1}ss^{-1}) = \phi(s^{-1})\phi(s)\phi(s)^{-1}$ and similar $\phi(s^{-1}) = \phi(ss^{-1}s) = \phi(s)\phi(s^{-1})\phi(s)$ showing that $\phi(s^{-1}) = \phi(s)^{-1}$.

The inverse semigroups and homomorphisms form a category **Inv**, the category of inverse semigroups. Later we will construct several subcategories of **Inv** and relate them to $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ and $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$. In order to do this we need to further investigate the structure of inverse semigroups. The idempotents play an important role in this. We will see that they contain a subset which generates a special class of inverse semigroups which relate **Inv** to $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ among other things. However the most important use of the idempotents is that we can define an order on inverse semigroups through them.

Let S be an inverse semigroup. Then we define the relation $\leq$ on S by

$$s \leq t \iff s = et$$

for some idempotent e.

This relation has numerous properties. The ones used in this thesis are summarized in the following lemma. The proof can be found in [6] pages 21-22.

Lemma 4.5. *Let S be an inverse semigroup. Then the following are equivalent:*

1. $s \leq t$

2. $s = tf$ *for some* $f \in \mathcal{E}$

3. $s^{-1} \leq t^{-1}$

4. $s = ss^{-1}t$

5. $s = ts^{-1}s$

The relation $\leq$ defines a partial order on S, called the *natural partial order*. It will play a leading role in the rest of this chapter.

An important quality of this partial order is that if $s \leq t$ and $u \leq v$, then $su \leq tv$. Also, if $u \leq st$ then by lemma 4.5 $ss^{-1}u = ss^{-1}stu^{-1}u = stu^{-1}u = u$ and similar $ut^{-1}t = u$. Moreover, it is preserved by the inverse semigroup homomorphism. To see this, let $S \xrightarrow{\phi} T \in \mathbf{Inv}$ and $s \leq t$. Then $s = et$ for some $e \in \mathcal{E}(S)$ and therefore $\phi(s) = \phi(et) = \phi(e)\phi(t)$. Because $\phi(e) \in \mathcal{E}(T)$ it follows that $\phi(s) \leq \phi(t)$. If ϕ is injective we have the following:

Lemma 4.6. *Let* $S \xrightarrow{\phi} T$ *be an injective homomorphism of inverse semigroups and* $s, t \in S$. *Then*

$$s \leq t \iff \phi(s) \leq \phi(t)$$

Proof. We have already covered one direction. So suppose that $\phi(s) \leq \phi(t)$. Then by lemma 4.5 $\phi(s) = \phi(s)\phi(s)^{-1}\phi(t) = \phi(ss^{-1}t)$. Because ϕ is injective it follows that $s = ss^{-1}t$, hence $s \leq t$. $\qquad\square$

Apart from the natural partial order we will also use an other relation. Define the relation $\lessdot$ on S by

$$s \lessdot t \iff s \leq t \text{ and } \forall u < t; s \leq u \Rightarrow s = u$$

It clear from the definition and lemma 4.5 that

$$s \lessdot t \iff s^{-1} \lessdot t^{-1}$$

This relations has a few important properties regarding products that we will use in chapter 4.

Lemma 4.7. *Suppose that* $u \lessdot st$ *then the following holds*

1. $s^{-1}u \lessdot s^{-1}st$

2. $ut^{-1} \lessdot stt^{-1}$

Proof. Because the proofs of 1 and 2 are similar we only prove 1. Suppose that $v \leq s^{-1}st$ and $s^{-1}u \leq v$. Then $sv \leq st$ and $u = ss^{-1}u \leq sv$ hence $u = sv$ and therefore $s^{-1}u = s^{-1}sv = v$. $\qquad\square$

From this lemma we deduce that if $s \lessdot t = t(t^{-1}t)$ then $t^{-1}s \lessdot t^{-1}t$ and similar $st^{-1} \lessdot tt^{-1}$.

4.2 Inverse semigroups with zero

Now that we have done the basics it is time to study inverse semigroups with a bit more structure. We are especially interested in inverse semigroups which have a zero. This will make it possible to define bottom elements with respect to the natural partial order. Some of these elements will turn out to generate a special class of inverse semigroups. These inverse semigroups will later be related to objects of the category $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$.
In this section we will introduce a lot of technical definitions, which at first will not have a very clear interpretation. These definitions however, come

from careful considerations regarding symmetric inverse semigroups, which
are covered in the next section. Although we could have given the definitions
there, they are applicable to any inverse semigroup with zero. Hence, because
it was more logical to consider inverse semigroups with zero before symmetric
inverse semigroups they are given in this section. The idea behind these
properties will be explained later, in more detail, in the section on symmetric
inverse semigroups.
Having said this, let us give the definition of an inverse semigroups with zero.

Definition 4.8. *Let S be a inverse semigroup. We say that S has a zero if
there is an $0 \in S$ such that $0s = 0 = s0$ for all $s \in S$*

For the rest of this section we will only consider inverse semigroups with a
zero. Hence if we say that S is an inverse semigroup we mean an inverse
semigroup with a zero.

4.2.1 Primitives

From the definition it is clear that $0 \in E$ hence $0 \leq s$ for all $s \in S$. The
elements we are interested in are those just above 0.

Definition 4.9. *Let S be an inverse semigroup. We say that a non-zero
element $p \in S$ is* primitive *if $s \leq p$ implies that $s = 0$ or $s = p$. If p is a
idempotent we say that p is an* atom.

We denote the sets of primitives and atoms of S by $\mathcal{P}(S)$ and $\mathcal{A}(S)$ respec-
tively. If no confusion arises then we omit the S as we did with the set of
idempotents.
The use of the term atom might look a bit off with respect to lattice theory.
However, $(S, \leq)$ is not a lattice while $(\mathcal{E}(S), \leq)$ is. This justifies the terminol-
ogy, for here the atoms from lattice theory are the atoms from definition 4.9.
In this chapter and the next one we will frequently consider whole subsets of
inverse semigroups. For this reason we will introduce some notation as well
as define what we mean by the product of subsets.
If $U \subset S$ and $s \in S$, then we define the following sets.

$$U^{-1} = \{s \in S : s^{-1} \in U\}$$
$$U_{\leq s} = \{u \in U : u \leq s\}$$
$$U_{\geq s} = \{u \in U : u \geq s\}$$

Also, given two subsets $U, V \subseteq S$ we define their product UV to be the set
of all products uv with $u \in U$ and $v \in V$. This multiplication is clearly
associative. For $s \in S$ and $U \subseteq S$, we write sU for $\{s\}U$.

We will use the convention that if $U, V \subseteq \mathcal{P}$, we exclude the zero from their product. Thus if the product is not empty, it will again be a subset of the primitives.

4.2.2   Properties of primitives

Primitives satisfy many properties. We start by summarizing the most basic ones.

Lemma 4.10. *Let S be an inverse semigroup, $p \in \mathcal{P}$, $a, b \in \mathcal{A}$, $e \in \mathcal{E}$ and $s \in S$ then*

1. $p^{-1} \in \mathcal{P}$

2. $ab \neq 0$ *if and only if $a = b$*

3. *If $ep \neq 0$ then $ep = p$*

4. *If $sp \neq 0$ then $sp \in \mathcal{P}$*

5. $pp^{-1} \in A$

6. *If $ap \neq 0$ then $a = pp^{-1}$*

Proof.   1. Suppose that $s \leq p^{-1}$. Then $s^{-1} \leq p$, hence $s^{-1} = 0$ or $s^{-1} = p$. It now follows that $s = 0$ or $s = p^{-1}$.

2. Suppose that $a \neq b$. Clearly $ab \leq a$ and $ab \leq b$ hence $ab = 0$ or $ab = b$ and $ab = a$. So if $ab \neq 0$ then $a = ab = b$. The other implication is trivial.

3. If $ep \neq 0$ then because $ep \leq p$ it follows that $ep = p$.

4. Suppose $sp \neq 0$ and $t \leq sp$ then $t = spt^{-1}t$. If $t \neq 0$ then $pt^{-1}t \neq 0$. Hence $pt^{-1}t = p$ and so $t = spt^{-1}t = sp$.

5. Clearly $pp^{-1} \in E$ and $pp^{-1} \neq 0$ hence by 4 $pp^{-1} \in \mathcal{A}$.

6. Suppose that $ap \neq 0$ then because $app^{-1}p = ap \neq 0$, $app^{-1} \neq 0$. Hence by 2 and 5 $a = p^{-1}p$

$\square$

From this lemma we deduce the following. Given a primitive p there exist $a, b \in \mathcal{A}$ such that $p = apb$. Just take $a = pp^{-1}$ and $b = p^{-1}p$. We could turn this around. Given two atoms a, b, does there exist a $p \in \mathcal{P}$ such that $p = apb$? This is trivially true if $a = b$. However, it is not always true for different a, b. Therefore we turn this into a definition.

Definition 4.11. *Let S be an inverse semigroup and $a, b \in \mathcal{A}$. We say that a and b are* connected *if there exists a $p \in P$ such that $p = apb$. The element p is called the* connector *of a and b, or equivalently we say that p connects a and b.*

Connecting atoms can be done in symmetric inverse semigroups. In section 4.3 we give a nice graphical interpretation of this.

Lemma 4.12. *Let $a, b, c \in \mathcal{A}$ and p be the connector of a and b. Then:*

1. *$a = pp^{-1}$ and $b = p^{-1}p$*

2. *p^{-1} connects b and a*

Proof.   1. Because p connects a and b, it follows that $ap \neq 0$. Hence $app^{-1} \neq 0$ and so by lemma 4.10 $a = pp^{-1}$. In a similar way it follows that $b = p^{-1}p$.

2. This is clear for $p^{-1} = (apb)^{-1} = bp^{-1}a$.

$\square$

Property 4 of lemma 4.10 raises another question. Given a $s \in S$, can we always find a $p \in \mathcal{P}$ with $sp \neq 0$? This is not true in general. However there is a class of inverse semigroups which do have this property.
An inverse semigroup S with zero is called *primitive* if for any non-zero $s \in S$ there is a $p \in \mathcal{P}$ such that $p \leq s$.

Lemma 4.13. *Let S be a primitive inverse semigroup and $s \in S$ non-zero. Then there exists a $p \in \mathcal{P}$ such that $sp \neq 0$.*

Proof. Because S is primitive, there is a $r \in \mathcal{P}$ with $r \leq s$. Then by lemma 4.5 $r = sr^{-1}r$. Therefore $sr^{-1} \neq 0$ and hence we can take $p = r^{-1}$. $\square$

Given $s, t \in S$ such that $st \neq 0$, there are $p, r \in \mathcal{P}$ with $p \leq s$ and $r \leq t$ such that $pr \neq 0$. Therefore $pr \leq st$. This raises the question if all primitives $p \leq st$ are products of primitives. The answer is yes.

Lemma 4.14. *Let $s, t \in S$ and $p \in \mathcal{P}_{\leq st}$. Then there are $p_s \in \mathcal{P}_{\leq s}$ and $p_t \in \mathcal{P}_{\leq t}$ such that $p = p_s p_t$*

Proof. Because $p \leq st$ it follows that $pp^{-1}s \neq 0$ and $tp^{-1}p \neq 0$ hence $pp^{-1}s \in \mathcal{P}_{\leq s}$ and $tp^{-1}p \in \mathcal{P}_{\leq t}$. Now take $p_s = pp^{-1}s$ and $p_t = tp^{-1}p$, then $p_s p_t = pp^{-1}stp^{-1}p = pp^{-1}p = p$. $\square$

It it not hard to see that we can extend this lemma for a product of any number of elements of S. However, we need to be careful. Although $p \leq sts$ implies that $p = p_s p_t p_s'$ it is not true in general that $p_s = p_s'$.

4.2.3 Orthogonality and completeness

From now on, when we say S is an inverse semigroup we mean that it is a primitive inverse semigroup with a zero.

We say that two primitives p, r are disjoint if $pp^{-1} \neq rr^{-1}$ or $p^{-1}p \neq r^{-1}r$. We call them completely disjoint if both inequalities hold. An inverse semigroup S is said to be (completely) disjoint if all its primitives are pairwise (completely) disjoint. In a disjoint inverse semigroup, for all $p, r \in \mathcal{P}$ such that $pp^{-1} = rr^{-1}$ and $p^{-1}p = r^{-1}r$ we have that $p = r$. Observe that if $p \leq ss^{-1}t$ in a disjoint inverse semigroup then, in contrast to the case where $p \leq sts$, it is true that $p = p_s p_s^{-1} p_t$.

Two elements $s, t \in S$ are said to be semi orthogonal if $st^{-1} = 0$ or $s^{-1}t = 0$. If both hold we call s and t orthogonal and denote this by $s \perp t$. We say that a set $A \subset S$ is (semi) orthogonal if all elements in it are pairwise (semi) orthogonal. There is a relation between the orthogonality and disjointness of primitives.

Lemma 4.15. *Let $p, r \in \mathcal{P}$. Then p, r are orthogonal if and only if they are completely disjoint.*

Proof. Suppose that $p \perp r$. Then $p^{-1}pr^{-1}r = 0$ and $pp^{-1}rr^{-1} = 0$. Hence by lemma 4.10 p and r are completely disjoint. Now if p and r are completely disjoint then again by lemma 4.10 $p^{-1}pr^{-1}r = 0 = pp^{-1}rr^{-1}$. Therefore $pr^{-1} = pp^{-1}pr^{-1}rr^{-1} = 0$ and similar $p^{-1}r = 0$. $\square$

Sets of orthogonal primitives can be used to construct new elements in the case of symmetric inverse semigroups. Because of this we introduce the notion of a complete inverse semigroup. We say that an inverse semigroup is *complete* if every set of orthogonal primitives has a least upper bound.

Take $s \in S$ and consider the set $\mathcal{P}_{\leq s}$. If $p, r \in \mathcal{P}_{\leq s}$ then $pp^{-1}s = p = sp^{-1}p$ and similar $rr^{-1}s = r = srr^{-1}$. From this we observe that if p and r are not orthogonal, then $p = r$. Hence $\mathcal{P}_{\leq s}$ is a set of orthogonal primitives. We now define S to be *semi complete* if for all $s \in S$ the join $\bigvee \mathcal{P}_{\leq s}$ exists. It is clear that every *complete* inverse semigroup is semi complete.

There is a result regarding orthogonal joins which is valid for any inverse semigroup.

Lemma 4.16. *Let $P \subseteq S$ be an orthogonal set of primitives such that $\bigvee P$ exists then*

1. $\bigvee P^{-1}P$ exists and $\bigvee P^{-1}P = (\bigvee P)^{-1}(\bigvee P)$.

2. $\bigvee PP^{-1}$ exists and $\bigvee PP^{-1} = (\bigvee P)(\bigvee P)^{-1}$.

3. If $\bigvee P^{-1}$ exists then $\bigvee P^{-1} = (\bigvee P)^{-1}$.

Proof. The proof of 1 and 2 can be found in [6] page 27. The proof of 3 which we give, is inspired by this proof.

Because for all $p \in P^{-1}$ we have that $p^{-1} \in P$ it follows that $p \leq (\bigvee P)^{-1}$ for all $p \in P^{-1}$. Therefore $\bigvee P^{-1} \leq (\bigvee P)^{-1}$. Now suppose that $p \leq s$ for all $p \in P^{-1}$. Then $p^{-1} = p^{-1}pp^{-1} \leq p^{-1}sp^{-1}$ from which it follows that $\bigvee P \leq (\bigvee P)s(\bigvee P)$. We now have that $(\bigvee P)^{-1} = (\bigvee P)^{-1}(\bigvee P)(\bigvee P)^{-1} \leq (\bigvee P)^{-1}(\bigvee P)s(\bigvee P)(\bigvee P)^{-1} \leq s$. Thus proving that $(\bigvee P)^{-1} \leq \bigvee P^{-1}$ and therefore $\bigvee P^{-1} = (\bigvee P)^{-1}$ □

Therefore, in semi complete inverse semigroups, we have that $\bigvee \mathcal{P}_{\leq s^{-1}} = \bigvee \mathcal{P}_{\leq s}^{-1} = (\bigvee \mathcal{P}_{\leq s})^{-1}$.

If every $s \in S$ is a supremum of primitives we call S *primitivistic*. It is clear that any primitivistic inverse semigroup is primitive. Moreover, every primitivistic inverse semigroup is semi complete for here $s = \bigvee \mathcal{P}_{\leq s}$.

Given two elements $s, t \in S$ we could consider the primitives that lie beneath them and ask when these completely define s and t. In other words when does $\mathcal{P}_{\leq s} = \mathcal{P}_{\leq t}$ imply that $s = t$. We say that S is *separated* by its primitives, or just *separated*, if it satisfies this property.

Let S be a semi complete inverse semigroup which is separated. Then for all $s \in S$ it follows that $s = \bigvee \mathcal{P}_{\leq s}$. To see this, observe that $\mathcal{P}_{\leq s} \subseteq \mathcal{P}_{\bigvee \mathcal{P}_{\leq s}}$ and $\bigvee \mathcal{P}_{\leq s} \leq s$. Hence $\mathcal{P}_{\leq s} = \mathcal{P}_{\bigvee \mathcal{P}_{\leq s}}$ and therefore $s = \bigvee \mathcal{P}_{\leq s}$. This proves that separated semi complete inverse semigroups are primitivistic. But more importantly it shows that for a complete inverse semigroup, being separated says that it is completely determined by its primitives. Hence we can see them as a sort of basis for all elements in the inverse semigroup. This observation gives a first glance at a possible relation between **Inv** and $\mathbb{F}$rob(**Hilb**). The idea of primitives as a basis will become more clear in the next section.

4.3 Symmetric inverse semigroups

In this section we take the first step in relating inverse semigroups to the category $\mathbb{F}$rob(**PInj**). The key objects here are the hom-sets **PInj**(X, X), which we denote by **PInj**(X). They turn out to have a very rich structure.

4.3.1 Partial injections as semigroup

Proposition 4.17. *Let X be a set then* **PInj**(X) *is an inverse semigroup*

Proof. We define multiplication by composition, which is clearly associative. Let $f \in \mathbf{PInj}(X)$ then $ff^{\dagger}f = f$ and $f^{\dagger}ff^{\dagger} = f^{\dagger}$ hence $f^{\dagger} = f^{-1}$. Now take $\epsilon \in \mathcal{E}$. Then $\epsilon^2 = \epsilon$ implies that $\mathrm{dom}(\epsilon) = \mathrm{im}(\epsilon)$. Suppose that there is a $x \in \mathrm{dom}(\epsilon)$ such that $\epsilon(x) \neq x$ then because ϵ is injective $\epsilon^2(x) \neq \epsilon(x)$. Therefore $\epsilon(x) = x$ for all $x \in \mathrm{dom}(\epsilon)$ so $\epsilon = 1_{\mathrm{dom}(\epsilon)}$. From this it follows that all idempotents commute. $\square$

We say that an inverse semigroup is *symmetric* if it is of the form $\mathbf{PInj}(X)$ for some set X.

Because we will be working a lot with restrictions of functions we introduce the following notation. If $X \xrightarrow{f} Y \in \mathbf{PInj}$ and $U \subset \mathrm{dom}(f)$. Then we denote by f_U the restriction of f to U. We write f_x for $f_{\{x\}}$.

The theory of inverse semigroups actually started with the discovery of this type of structure in the homsets $\mathbf{PInj}(X)$. For this reason a lot of work has been put in relating inverse semigroups to symmetric inverse semigroups. On of these results is the Wagner-Preston Representation Theorem. Which we will briefly discuss in chapter 4.4.3. There we will also construct a relation between representable inverse semigroups and symmetric inverse semigroups.

4.3.2 Graphical representation

A symmetric inverse semigroup $\mathbf{PInj}(X)$ can be represented in a graphical way by showing where each point of X is mapped to. We will show how this works. Let $X = \{1, 2, 3\}$, the set with three elements and take $f, g \in \mathbf{PInj}(X)$ defined by

$$\mathrm{dom}(f) = \{1, 2\} \qquad\qquad \mathrm{dom}(g) = \{1, 2\}$$
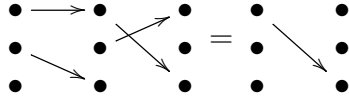$$f(1) = 1, f(2) = 3 \qquad\qquad g(1) = 3, g(2) = 1$$

Then we represent them by

While their inverses are represented by reversing the direction of the arrows

Using this representation we can calculate the composition gf by tracing the arrows

4.3.3 Structure of symmetric inverse semigroups

As we mentioned in the previous section, we used the symmetric inverse semigroups as models for inverse semigroups with zero. We will now see what some definitions translate to in the case of symmetric inverse semigroups, using the graphical representation.

Because the idempotents of symmetric inverse semigroups are the identity on their domain we see that $f \leq g$ if all arrows in f are also in g. For instance

This observation redefines the natural partial order on $\mathbf{PInj}(X)$.

Lemma 4.18. *Let $f, g \in \mathbf{PInj}(X)$ then $f \leq g$ if and only if $dom(f) \subseteq dom(g)$ and $f(x) = g(x)$ for all $x \in dom(f)$.*

Proof. Suppose $f \leq g$ then there is a $\epsilon \in E$ such that $f = g\epsilon$. Because $\epsilon = 1_{\mathrm{dom}(\epsilon)}$ it follows that $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$. Now let $x \in \mathrm{dom}(f)$. Then $f(x) = g\epsilon(x) = g(x)$ hence $f \subseteq g$.

Now suppose $\mathrm{dom}(f) \subseteq \mathrm{dom}(g)$ and $f(x) = g(x)$ for all $x \in \mathrm{dom}(f)$ and define $\epsilon = 1_{\mathrm{dom}(f)}$. Then $\epsilon \in E$, $\mathrm{dom}(g\epsilon) = \{x \in \mathrm{dom}(\epsilon) : \epsilon(x) \in \mathrm{dom}(g)\} = \mathrm{dom}(f)$ and $g\epsilon(x) = g(x) = f(x)$ for all $x \in \mathrm{dom}(f)$. Therefore $f = g\epsilon$ and so $f \leq g$. $\square$

This lemma gives a partial order $\subseteq$ on $\mathbf{PInj}(X)$, defined by

$$f \subseteq g \iff \mathrm{dom}(f) \subseteq \mathrm{dom}(g); f(x) = g(x)\forall x \in \mathrm{dom}(f)$$

which coincides with the natural partial order of inverse semigroups. Because $\subseteq$ is a more natural way to relate functions we will use this definition for the remainder of this section.

Using this new definition of the order we obtain the following for the primitives.

Lemma 4.19. *Let $S = \mathbf{PInj}(X)$ be a symmetric inverse semigroup. Then $\pi \in \mathcal{P}$ if and only if $|dom(\pi)| = 1$.*

Proof. Suppose $\pi \in \mathcal{P}$, let $x, y \in \mathrm{dom}(\pi)$ and suppose that $x \neq y$. Then $\pi_x \subseteq \pi$ hence $\pi_x = \pi$. However this is in contradiction with $x \neq y$ and therefore $\mathrm{dom}(\pi) = \{x\}$.

Now suppose that $|\mathrm{dom}(\pi)| = 1$ and $f \subseteq \pi$. Then, if $f \neq 0$, it follows that $\mathrm{dom}(f) = \mathrm{dom}(\pi)$ and hence $f = \pi$. $\qquad\square$

From this we deduce that in our graphical representation, primitives are represented by single arrows. Moreover, if $\pi \leq f$ then $\pi = f_{\mathrm{dom}(\pi)}$. We can also say something about the atoms $\pi^{-1}\pi$ and $\pi\pi^{-1}$ in this case.

Lemma 4.20. *Let $f \in \mathbf{PInj}(X)$, $\pi \leq f$ with $dom(\pi) = \{x\}$. Then $1_x = \pi^{-1}\pi$ and $\pi\pi^{-1} = 1_{f(x)}$.*

Proof. Because $\pi \leq f$ we have $\pi = f_x$ and so $\pi^{-1}\pi(x) = f_x^{-1} f_x(x) = x$. For the other equality we observe that $\mathrm{dom}(\pi^{-1}) = \{f(x)\}$ and $\pi^{-1} \leq f^{-1}$. Hence we can copy the proof of the first equality. $\qquad\square$

These observations are crucial in determining the properties of symmetric inverse semigroups.

We will now start investigating several properties. First we use the graphical representation to show the intuition behind them. Later we will give an exact proof of these statements.

We start by taking a look at the relation between the functions in $\mathbf{PInj}(X)$ and its primitives. For this, let $X = \{1, 2, 3\}$ and take the following $f \in \mathbf{PInj}(X)$
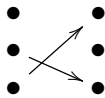

Then we see f is completely defined by the single arrows connecting the elements of X.
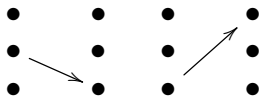

These however, are primitives and because $g \leq f$ if and only if all arrows in g are in f, it follows that f is the supremum of all primitives less or equal to f. Thus symmetric inverse semigroups are primitivistic.

Being primitivistic means that all functions can be constructed out of the primitives which lie beneath it. Lets try to go the other way. Given a set of
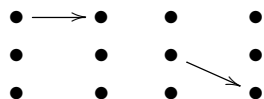
primitives
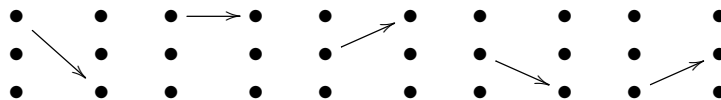
we could try to construct a function out of them by combining the arrows. However, we can only combine those primitives that do not have a common domain or image. For example
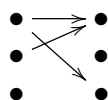
is not a valid partial function.

If we take two primitives without a common domain or image
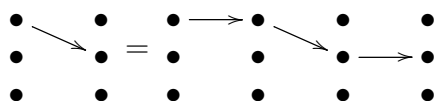

we see that

The left identity says that the images are disjoint while the right identity does the same for the domain. This is exactly what we defined as two primitives being completely disjoint which by lemma 4.15 is equivalent to being orthogonal. Hence we observe that symmetric inverse semigroups are complete.

We now observe that two primitives are disjoint if either their domain or codomain is.This observation, together with lemma 4.20 and 4.19 shows that **PInj**(X) is disjoint.

There is also a relation between primitives and atoms. Take two atoms in **PInj**(X)
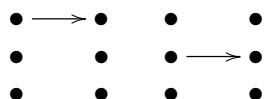
Then we can take the primitive, sending the domain of the first atom to the image of the second atom. This primitive then satisfies:

From this graphical representation it is clear that this primitive connects the atoms. This is where the definition of connecting atoms comes from. One can now see that in addition to being disjoint, primitivistic and complete, symmetric inverse semigroups also have the property that all atoms are connected.

These graphical intuitions are quite nice. However, we need a real proof of these statements. We will do this now, using the graphical intuition as guideline.

Proposition 4.21. *Let X be a set then:*

1. $\mathbf{PInj}(X)$ *is complete.*

2. $\mathbf{PInj}(X)$ *is primitivistic.*

3. *All atoms of* $\mathbf{PInj}(X)$ *are connected.*

Proof.   1. Let $P \subseteq \mathcal{P}$ be a set of disjoint primitives. Take $\pi \in P$ and $x \in \mathrm{dom}(\pi)$. Suppose now there is a $\rho \in P$ with $\rho \neq \pi$ such that $x \in \mathrm{dom}(\rho)$. Then $\mathrm{dom}(\pi\rho^{-1}) = \mathrm{im}(\rho)$ so $\pi\rho^{-1}(y) = \pi(x)$ and hence $\pi\rho^{-1} \neq 0$ which is a contradiction. Similar, there is no $\rho \neq \pi$ such that $\mathrm{im}(\pi) \cap \mathrm{im}(\rho) \neq \emptyset$. Now define $f \in \mathbf{PInj}(X)$ by $\mathrm{dom}(f) = \bigcup_{\pi \in P} \mathrm{dom}(\pi)$ and $f(x) = \pi(x)$ if $x \in \mathrm{dom}(\pi)$. Then f is well defined, injective and clearly $\pi \subseteq f$ for all $\pi \in P$. Now suppose that $g \in \mathbf{PInj}(X)$ such that $\pi \subseteq g$ for all $\pi \in P$. Then $\mathrm{dom}(f) = \bigcup_{\pi \in P} \mathrm{dom}(\pi) \subseteq \mathrm{dom}(g)$ and $f(x) = \pi(x) = g(x)$ for all $x \in \mathrm{dom}(f)$ hence $f \subseteq g$.

2. Let $f \in \mathbf{PInj}(X)$ and define $P_{\subseteq f}$ to be the set of $\pi \in \mathcal{P}$ such that $\pi \subseteq f$. Then $P_{\subseteq f}$ is a set of pairwise disjoint primitives for if $\pi, \rho \in P_{\subseteq f}$; $\pi \neq \rho$ then $\pi\rho^{-1} \neq 0$ implies that $\mathrm{dom}(\pi) \cap \mathrm{dom}(\rho) \neq \emptyset$. Therefore, if $x \in \mathrm{dom}(\pi) \cap \mathrm{dom}(\rho)$ it follows that $\rho(x) = f(x) = \pi(x)$, which implies that $\pi = \rho$. Similar, if $\pi^{-1}\rho \neq 0$ it follows that $\pi^{-1}(x) = f^{-1}(x) = \rho^{-1}(x)$. Now by (1) there is a least upper bound $\hat{f}$ of $P_{\subseteq f}$. Now if $x \in \mathrm{dom}(f)$, then lemma 4.19 implies that there is a $\pi \in P_{\subseteq f}$ such that $f(x) = \pi(x)$. Hence $\mathrm{dom}(f) = \bigcup_{\pi \in P_{\subseteq f}} = \mathrm{dom}(\hat{f})$ and $f(x) = \pi(x) = \hat{f}(x)$ from which it follows that $f = \hat{f}$.

3. Take $\alpha, \beta \in \mathcal{A}$ then by lemma 4.19 there are $x, y \in X$ such that $\alpha = 1_x$ and $\beta = 1_y$. Now define $\pi \in P$ by $\mathrm{dom}(\pi) = y$ and $\pi(y) = x$. Then $\mathrm{dom}(\alpha\pi\beta) = \mathrm{dom}(\pi)$ and $\alpha\pi\beta(x) = \alpha\pi(y) = \alpha(x) = x$ hence $\pi = \alpha\pi\beta$.

                                                         $\square$

4.4 Classification of symmetric inverse semigroups

It is clear from the previous section that symmetric inverse semigroups have a very rich structure which is easily deduced from the graphical representation of its elements. The question is: given an inverse semigroup when is it symmetric? This is what we will be investigating in this section.

4.4.1 Lifts

We have seen that, given a set X, we can construct a symmetric inverse semigroup $\mathbf{PInj}(X)$. Now given a function $X \xrightarrow{f} Y$, can we construct an inverse semigroup homomorphism between $\mathbf{PInj}(X)$ and $\mathbf{PInj}(Y)$? The key problem is, given a $X \xrightarrow{f} Y$, how do we turn $g \in \mathbf{PInj}(X)$ into a function in $\mathbf{PInj}(Y)$. This is done by 'lifting' g using f.

Definition 4.22. *Let* $X \xrightarrow{f} Y \in \mathbf{PInj}$ *and* $g \in \mathbf{PInj}(X)$*. Then we define the function* f^*g *by* $dom(f^*g) = f\,dom(g)$ *and* $f^*g(y) = fgf^\dagger(y)$*. We call* f^*g *the lift of* g *under* f*.*

It is clear that $f^*g \in \mathbf{PInj}$. The term lifting comes from the fact that the following diagram commutes in $\mathbf{PInj}$.

$$
\begin{array}{ccc}
Y & \xrightarrow{\ f^*g\ } & Y \\
\downarrow{\scriptstyle f^\dagger} & & \uparrow{\scriptstyle f} \\
X & \xrightarrow{\ g\ } & X
\end{array}
$$

Hence we can regard the f-lift f^* as a map $\mathbf{PInj}(X) \xrightarrow{f^*} \mathbf{PInj}(Y)$. If f is an injective function then $f^\dagger f = 1_X$ hence we have the following for $g, h \in \mathbf{PInj}(X)$

$$f^*(gh) = fghf^\dagger = fg1_Xhf^\dagger = fgf^\dagger fhf^\dagger = f^*(g)f^*(h)$$

Thus f^* is a homomorphism of inverse semigroups.

4.4.2 Atoms and sets

We will now address the following question: if S is a symmetric inverse semigroup, what is the relation between S and the set X such that $S = \mathbf{PInj}(X)$? The answer is atoms and it will follow from the following result.

Theorem 4.23. *Let X, Y be sets. Then* $\mathbf{PInj}(X) \cong \mathbf{PInj}(Y)$ *in* $\mathbf{Inv}$ *if and only if $X \cong Y$ in* $\mathbf{Set}$.

Proof. Suppose that $\mathbf{PInj}(X) \cong \mathbf{PInj}(Y)$ where we denote the isomorphism by ϕ. Then define $X \xrightarrow{\hat{\phi}} Y$ by $\hat{\phi}(x) = \text{dom}(\phi(1_x))$. This is well defined by lemma 4.19 and injective. For if $\hat{\phi}(x_1) = \hat{\phi}(x_2)$. Then $\text{dom}(\phi(1_{x_1})) = \text{dom}(\phi(1_{x_2}))$ and $\phi(1_{x_1})(y) = y = \phi(1_{x_2})(y)$ for $y \in \text{dom}(\phi(1_{x_1}))$. Thus $\phi(1_{x_1}) = \phi(1_{x_2})$ and hence $1_{x_1} = 1_{x_2}$ from which it follows that $x_1 = x_2$. For surjectivity let $y \in Y$. Then $1_y \in \mathbf{PInj}(Y)$ and therefore we have a $f \in \mathbf{PInj}(X)$ such that $\phi(f) = 1_y$. We claim that f is an atom for then $f = 1_x$ for some $x \in X$ and so $1_y = \phi(1_x)$, which implies that $y = \hat{\phi}(x)$. Because ϕ is injective and $\phi(f) = 1_y = 1_y^2 = \phi(f^2)$ it follows that $f \in \mathcal{E}(\mathbf{PInj}(X))$. Now let $g \in \mathbf{PInj}(X)$ such that $g \subseteq f$ then $\phi(g) \subseteq \phi(f) = 1_y$ and hence $\phi(g) = 0$ or $\phi(g) = 1_y$ from which we deduce that $g = 0$ or $g = f$ hence $f \in \mathcal{A}(\mathbf{PInj}(X))$.

Suppose now that $X \cong Y$ with isomorphism ψ. Then $\mathbf{PInj}(X) \xrightarrow{\psi^*} \mathbf{PInj}(Y)$ is an isomorphism. It is injective for $\psi^* f = \psi^* g$ implies that $\text{dom}(f) = \text{dom}(g)$ and $f(x) = \psi^{-1}\psi^* f(\psi(x)) = \psi^{-1}\psi^* g(\psi(x)) = g(x)$ hence because ψ is a bijection $f = g$. Now let $g \in \mathbf{PInj}(Y)$ and define $f \in \mathbf{PInj}(X)$ by $\text{dom}(f) = \psi^{-1}(\text{dom}(g))$ and $f(x) = \psi^{-1} g \psi(x)$ then clearly this is well defined and $\phi^* f = \psi \psi^{-1} g \psi \psi^{-1} = g$. $\square$

Corollary 4.24. *Let S be a symmetric inverse semigroup. Then* $S \cong \mathbf{PInj}(\mathcal{A}(S))$

Proof. Let $S = \mathbf{PInj}(X)$ for some set X. Then define $X \xrightarrow{\phi} \mathcal{A}(S)$ by $\phi(x) = 1_x$. This is clearly well defined and injective. Now let $a \in \mathcal{A}(S)$ then $a = 1_x$ for some $x \in X$ hence $a = \phi(x)$ so ϕ is surjective. Now by theorem 4.23 $S = \mathbf{PInj}(X) \cong \mathbf{PInj}(\mathcal{A}(S))$. $\square$

4.4.3 Inverse semigroups determined by atoms

The result from corollary 4.24 tells us that a symmetric inverse semigroup is completely determined by its atoms. Hence given an inverse semigroup. If it is symmetric it must be isomorphic to $\mathbf{PInj}(\mathcal{A})$. Hence, we need to define a homomorphism $S \xrightarrow{\phi} \mathbf{PInj}(\mathcal{A})$ and check when this is an isomorphism. We start with $\mathbf{PInj}(X)$ for some set X. It was proven that $\mathcal{A} \cong X$ in $\mathbf{Set}$ and hence $\mathbf{PInj}(X) \cong \mathbf{PInj}(\mathcal{A})$ in $\mathbf{Inv}$. We can write out this isomorphism ψ explicitly using the proof of theorem 4.23 and corollary 4.24.

Let $f \in \mathbf{PInj}(X)$ then $\mathrm{dom}(\psi(f)) = \{1_x : x \in \mathrm{dom}(f)\}$ and $\psi(f)(1_x) = 1_{f(x)}$. Now if $x \in \mathrm{dom}(f)$ then $f^{-1}f1_x \neq 0$. Conversely if $x \in X$ such that $f^{-1}f1_x \neq 0$ then $x \in \mathrm{dom}(f)$. From this we deduce that $\mathrm{dom}(\psi(f)) = \{1_x : f^{-1}f1_x \neq 0\} = f^{-1}f\mathcal{A}$. Given a $1_x \in f^{-1}f\mathcal{A}$ have $\psi(f)(1_x) = 1_{f(x)} = f1_xf^{-1}$. These observations lead to an alternative definition of the isomorphism from corollary 4.24. First we need to check if it is well defined.

Lemma 4.25. *Let S be a primitive inverse semigroup and $s \in S$. Then the function $s^{-1}s\mathcal{A} \xrightarrow{\theta(s)} \mathcal{A}$ defined by $\theta(s)(a) = sas^{-1}$ is injective and $\mathrm{im}(\theta(s)) = ss^{-1}\mathcal{A}$.*

Proof. First, because S is primitive, there is a $p \in \mathcal{P}$ such that $sp \neq 0$ hence $s^{-1}spp^{-1} \neq 0$ which proves that $s^{-1}s\mathcal{A} \neq \emptyset$. Now let $a, b \in s^{-1}s\mathcal{A}$ and suppose that $\theta(s)(a) = \theta(s)(b)$. Then $a = s^{-1}sa$, $b = s^{-1}sb$ and $sas^{-1} = sbs^{-1}$. Hence $b = s^{-1}sb = s^{-1}sbs^{-1}s = s^{-1}sas^{-1}s = s^{-1}sa = a$. For the image of $\theta(s)$ let $a \in s^{-1}s\mathcal{A}$. Then $a = s^{-1}sa$. and therefore $sas^{-1} = ss^{-1}sas^{-1} \in ss^{-1}\mathcal{A}$. Conversely let $a \in ss^{-1}\mathcal{A}$. Then $s^{-1}ss^{-1}as \in s^{-1}s\mathcal{A}$ and $\theta(s)(s^{-1}ss^{-1}as) = ss^{-1}ss^{-1}ass^{-1} = ss^{-1}ass^{-1} = ss^{-1}a = a$. $\square$

This lemma proves that if S is primitive, then for each $s \in S$, $\theta(s)$ is a bijection between $s^{-1}s\mathcal{A}$ and $ss^{-1}\mathcal{A}$. Thus θ_s is an element in $\mathbf{PInj}(\mathcal{A})$. The following is therefore well defined.

Lemma 4.26. *Let S be a primitive inverse semigroup. Then the map $S \xrightarrow{\theta_S} \mathbf{PInj}(\mathcal{A})$ defined by $\theta_S(s) = \theta(s)$ is a homomorphism of semigroups.*

Proof. Let $s, t \in S$ then we need to show that $\theta_s\phi_t = \theta_{st}$. We start with the domains. By definition of θ and lemma 4.10,

$$\mathrm{dom}\theta_s \cap \mathrm{im}\theta_t = s^{-1}s\mathcal{A} \cap tt^{-1}\mathcal{A} = s^{-1}stt^{-1}\mathcal{A}.$$

Hence by definition

$$\mathrm{dom}(\theta_s\theta_t) = \theta_t^{-1}(s^{-1}stt^{-1}\mathcal{A}) = t^{-1}s^{-1}stt^{-1}\mathcal{A}t$$

On the other hand $\mathrm{dom}(\theta_{st}) = t^{-1}s^{-1}st\mathcal{A}$. Lemma 4.10 now implies that $\mathrm{dom}(\theta_s\phi_t) \subseteq \mathrm{dom}(\theta_{st})$. For the other inclusion let $a \in \mathrm{dom}(\theta_{st})$. Then

$$a = t^{-1}s^{-1}sta = t^{-1}s^{-1}stt^{-1}ta = t^{-1}s^{-1}stat^{-1}t$$

hence by lemma 4.10 $a = t^{-1}bt$ with $b \in \mathcal{A}$ and therefore $a = t^{-1}s^{-1}stt^{-1}bt \in \mathrm{dom}(\theta_s\theta_t)$. If $a \in \mathrm{dom}(\theta_{st})$ then $\theta_{st}(a) = (st)a(st)^{-1} = stat^{-1}s^{-1} = \theta_s(tat^{-1})$ $= \theta_s\theta_t(a)$. $\square$

It now follows that for any primitive inverse semigroup S we have a homomorphism $S \xrightarrow{\theta_S} \mathbf{PInj}(\mathcal{A})$. We have already shown that if S is symmetric then θ_S is an isomorphism which corresponds to the isomorphism from corollary 4.24. What needs to be done, is to define those properties that make θ_S into an isomorphism. First we want θ_S to be injective. This depends on the orthogonality and separation of the inverse semigroup.

Proposition 4.27. *Let S be a primitive inverse semigroup. Then the homomorphism θ_S is injective if and only if S is disjoint and separated by primitives.*

Proof. First observe that θ is injective if and only if for all $s, t \in S$, $sas^{-1} = tat^{-1}$ for all $a \in \mathcal{A}$ implies that $s = t$.
Suppose that θ is injective. Let $p, r \in \mathcal{P}$ such that $pp^{-1} = rr^{-1}$ and $p^{-1}p = r^{-1}r$. Then for all $a \in \mathcal{A}$, $pap^{-1} \neq 0$ implies that $a = p^{-1}p$. Hence

$$pap^{-1} = pp^{-1}pp^{-1} = rr^{-1}rr^{-1} = rp^{-1}pr^{-1} = rar^{-1}$$

for all $a \in \mathcal{A}$ and therefore $p = r$. Thus if $p \neq r$ then they are disjoint. To prove that S is separated by primitives, let $s, t \in S$ such that $\mathcal{P}_{\leq s} = \mathcal{P}_{\leq t}$. Take $a \in \mathcal{A}$ such that $sas^{-1} \neq 0$ then $sa \neq 0$ hence $sa \in \mathcal{P}_{\leq s} = \mathcal{P}_{\leq t}$. Also, $as^{-1}s = a$ and therefore, because $sa \leq t$, $sa = tas^{-1}sa = ta$. Thus $sas^{-1} = tas^{-1} = t(sa)^{-1} = t(ta)^{-1} = tat^{-1}$ and so $s = t$.
Now suppose that S is disjoint and separated by primitives, let $s, t \in S$ such that $sas^{-1} = tat^{-1}$ for all $a \in \mathcal{A}$ and take $p \in \mathcal{P}_{\leq s}$. Then $p = pp^{-1}s = sp^{-1}p$, therefore $ps^{-1} \neq 0$ and $s^{-1}p \neq 0$. From this, it follows that $s^{-1}p \in \mathcal{A}$ and hence $s^{-1}p = p^{-1}p$. Consider the element $tp^{-1}p$. Then $tp^{-1}pt^{-1} = sp^{-1}ps^{-1} = ps^{-1} \neq 0$ and therefore $tp^{-1}p \neq 0$ so $tp^{-1}p \in \mathcal{P}_{\leq t}$. Furthermore $tp^{-1}p(tp^{-1}p)^{-1} = tp^{-1}pt^{-1} = sp^{-1}ps^{-1} = ps^{-1} = pp^{-1}$ and $(tp^{-1}p)^{-1}tp^{-1}p = p^{-1}pt^{-1}t = p^{-1}p$ for $tp^{-1}p \neq 0$. Because S is disjoint and $p = tp^{-1}p \in \mathcal{P}_{\leq t}$ it follows that $\mathcal{P}_{\leq s} \subseteq \mathcal{P}_{\leq t}$. If we interchange s and t we also get that $\mathcal{P}_{\leq s} \supseteq \mathcal{P}_{\leq t}$. Therefore $\mathcal{P}_{\leq s} = \mathcal{P}_{\leq t}$ and because S is separated we get that $s = t$. $\square$

We are now ready to state the main result of this section, a characterization of symmetric inverse semigroups.

Theorem 4.28. *An inverse semigroup is symmetric if and only if*

1. *It is primitive*

2. *It is disjoint*

3. *It is separated*

4. It is complete

5. All its atoms are connected

Proof. Proposition 4.21 proves one direction. Hence let S be inverse semigroup which satisfies the above properties. Then we use the map θ_S, which is well defined by lemma 4.25 and a homomorphism by lemma 4.26. Injectivity follows directly from proposition 4.27. The hard thing here is to prove surjectivity. We will do this in three parts. We will prove it first for primitives, then we show that θ preserves joins and finally we prove surjectivity for general elements.

Let $\pi \in \mathcal{P}(\mathbf{PInj}(\mathcal{A}(S)))$. We claim there is a $p \in \mathcal{P}$ such that $\pi = \theta_S(p)$. Take $a \in \mathrm{dom}(\pi)$ and define p_a to be the connector of $\pi(a)$ and a. Then $\mathrm{dom}(\theta_S(p_a)) = p_a^{-1} p_a \mathcal{A} = \{a\} = \mathrm{dom}(\pi)$ and

$$\theta_S(p_a)(a) = p_a a p_a^{-1} = p_a a (a p_a \pi(a)) = p_a a p_a \pi(a)$$

Thus $\theta_S(p_a)(a) \leq \pi(a)$ and because $\theta_S(p_a)(a) \neq 0$ it follows that $\theta_S(p_a) = \pi$. Take $P \subseteq \mathcal{P}(S)$ orthogonal and define $s = \bigvee P$. Then because θ_S is bijective on primitives $\mathcal{P}(\mathbf{PInj}(\mathcal{A}))_{\leq \theta_S(s)} = \theta_S(\mathcal{P}(S)_{\leq s})$. Therefore

$$\theta_S\left(\bigvee P\right) = \theta_S(s) = \bigvee \mathcal{P}(\mathbf{PInj}(\mathcal{A}))_{\leq \theta_S(s)} = \bigvee \theta_S(\mathcal{P}(S)_{\leq s}) = \bigvee \theta_S(P)$$

Now let $f \in \mathbf{PInj}(\mathcal{A})$ then for each $\pi \in \mathcal{P}(\mathbf{PInj}(\mathcal{A}))_{\leq f}$, $\pi = \theta_S(p)$ for some $p \in \mathcal{P}(S)$. Define $P = \theta_S^{-1} \mathcal{P}(\mathbf{PInj}(\mathcal{A}))_{\leq f}$, then

$$\theta_S\left(\bigvee P\right) = \bigvee \theta_S(P) = \bigvee \mathcal{P}(\mathbf{PInj}(\mathcal{A}))_{\leq f} = f$$

$\square$

Chapter 5

Representable inverse semigroups

In this chapter we will introduce a special class of inverse semigroups called representable inverse semigroups. These will form a category **RepInv** when we add so called representing homomorphisms. The reason for calling these inverse semigroups representable is because we can embed them into a symmetric inverse semigroup, such that the embedding satisfies a universal property.

The embedding is done via a construction that maps a finite disjoint inverse semigroup into a finite symmetric inverse semigroup. This construction is the most important step in attaining the adjunction between $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ and **RepInv**.

The construction is done in two steps. First we construct a disjoint separated inverse semigroup. Then we use the map θ from theorem 4.28 to embed this inverse semigroup into the symmetric inverse semigroup $\mathbf{PInj}(\mathcal{A})$.

Before we discuss the construction, we will say a few words about representations in general and especially the Wagner-Preston representation.

We say that a homomorphism $S \xrightarrow{\phi} T$ is a representation of S if T is a symmetric inverse semigroup and ϕ is injective.

A lot of work has been done in the field of representations. The most impressive result is the Wagner-Preston Representation theorem.

Theorem 5.1. *Let S be an inverse semigroup. Then there is an injective homomorphism $S \xrightarrow{\phi} \mathbf{PInj}(S)$ such that $s \leq t \iff \phi(s) \subseteq \phi(t)$.*

This theorem states that each inverse semigroup can be represented as a subsemigroup of the symmetric inverse semigroup $\mathbf{PInj}(S)$. We could try to

extend this representation into a functor from **Inv** to $\mathbb{F}$rob(**PInj**). However the Wagner-Preston representation is very large. Consider a symmetric inverse semigroup S. Then the Wagner-Preston representation embeds this into **PInj**(S). If S is not the trivial inverse semigroup then $\mathcal{A}(\textbf{PInj}(S)) \supsetneq \mathcal{A}(S)$, hence it is much larger. We are trying to give a minimal representation, which turns into a isomorphism when we consider symmetric inverse semigroups. This minimality is needed to define a functor between **RepInv** and $\mathbb{F}$rob(**Hilb**). Wagner-Preston does not satisfy this property.

Although Wagner-Preston does not give a minimal representation, it is still really powerful. Because every inverse semigroup can be embedded into a symmetric inverse semigroup we can restrict ourselves to inverse subsemigroups of symmetric inverse semigroups. We have already seen that the latter have a very rich structure. This will make the proofs more easy and insightful, because we can use our graphical representation. Furthermore, because we will consider injective homomorphisms, the ordering on the original inverse semigroup is completely determined by the ordering in the symmetric inverse semigroup.

5.1 Constructing a disjoint separated inverse semigroup part I

From now on we will only be working with finite inverse semigroups. Therefore we omit the word finite i.e. when we say that S is an inverse semigroup we mean that it is a finite inverse semigroup. Observe that these are always primitive.

The definition of representable inverse semigroups comes from careful considerations regarding a minimal construction of a disjoint separated inverse semigroup. Therefore before giving the definition we will first give a sketch of the construction and the problems we need to avoid.

5.1.1 A sketch of the construction

We have seen that symmetric inverse semigroups are completely determined by its atoms and via connection by its primitives. In order to get a separated inverse semigroup we need to carefully "add" primitives to the original inverse semigroup. Using our graphical representation we can easily see where this needs to be done.

Consider the inverse semigroup generated by the following three elements,

seen as a subsemigroup of $\mathbf{PInj}(\{1, 2, 3, 4\})$.
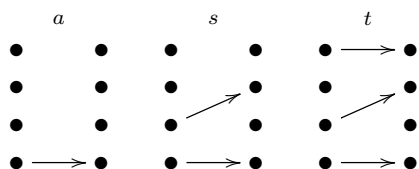
In addition to these and zero, the complete inverse semigroup contains the following elements.
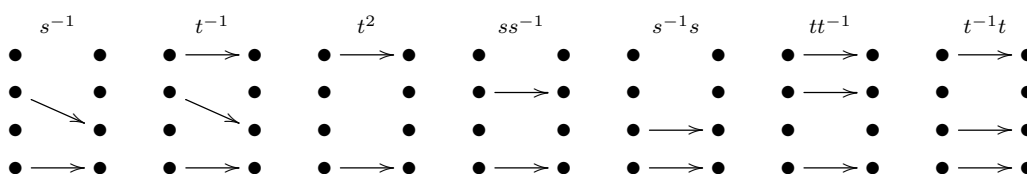
We want to embed this inverse semigroup into a disjoint separated inverse semigroup. This means that each of the elements has to be uniquely defined by the primitives that lie beneath it. Therefore, the first step is to separate the inverse semigroup into two disjoint pieces. Those elements that are already uniquely defined by their primitive and those that are not. In order to do this we take a look at the lattice of this inverse semigroup.
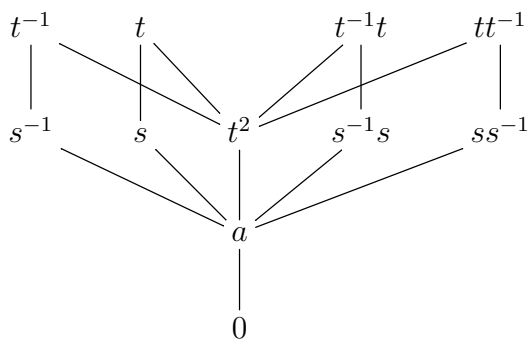
In our case, only a is completely defined by its primitives. Every element above a is not. Hence we can make a horizontal slice through the inverse

semigroup separating a from all elements above.
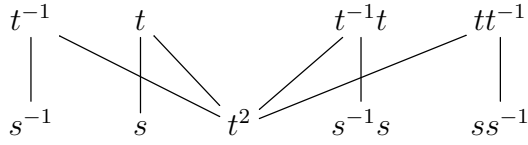
For all the top elements we need to add primitives in order to make the inverse semigroup separated by primitives. This would mean that we need nine primitives. However, from the graphical representation, it is immediately clear that we are only missing the following primitives.
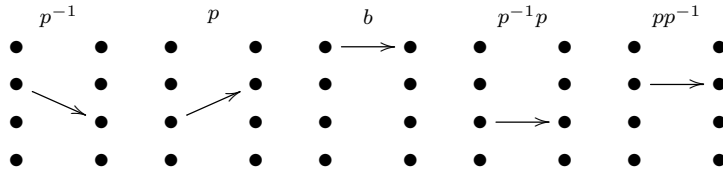

The key observation here is that some elements are uniquely linked to more then one element in the top below it. In our case, for instance, t is uniquely linked to s and t^2. We observe that p is needed to define s, while b is needed to define t^2. The element t however, is defined by both p and b. Hence, to let t be uniquely defined by its primitives we only need to add those primitives that uniquely define the elements below it. Therefore we need to make an other cut in the top of our inverse semigroup, where we eliminate those elements that are uniquely connected to more then one other element in the top. We are now left with the following top elements.

$$s^{-1} \qquad s \qquad t^2 \qquad s^{-1}s \qquad ss^{-1}$$

Each of these elements needs only one extra primitive for it to be uniquely defined by its primitives. Moreover, the primitive needed for s^{-1} is the inverse of the primitive needed for s. We also see that the primitive needed for ss^{-1} is the product of those needed for s and s^{-1} respectively.

These observations give a good idea, how the construction should go. However, we need to be careful. We attempt to construct a disjoint separated

inverse semigroup. Hence if we start with a disjoint inverse semigroup we need the new primitives to be pairwise disjoint. This requires some extra conditions on the original inverse semigroup. To see what can go wrong, consider the inverse semigroup containing the following elements.
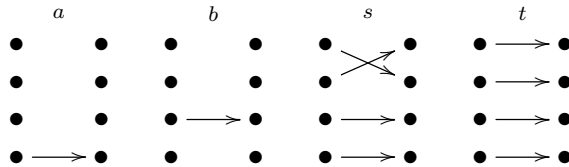
The top of this inverse semigroup is given by s and t. Both elements are connected to more the one element, hence our construction would end here. This can not be the case, for this inverse semigroup is clearly not separated. In our other example, it turns out that for all s the join of its primitives $\bigvee \mathcal{P}_{\leq s}$ exists. Therefore we should add the condition that the original inverse semigroup is semi complete. Now this last inverse semigroup is a non-example. However if we add $a \vee b$, it satisfies semi completeness but still we can not carry out our construction. This is because the selected top is still s and t, hence we have to add primitives for these two elements. What happens then is that we add the following elements.
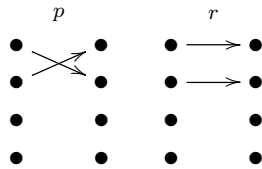
It is clear that these are indeed primitives. However they are not disjoint for $pp^{-1} = r = rr^{-1}$. The difference between this example and the former is that there, every selected element was completely defined by its domain and codomain. While in this last example that is not the case.

This means that we have to restrict ourselves to those inverse semigroups which are semi complete and whose selected top elements are completely determined by their domain and codomain.

5.1.2   Separation

We have seen a graphical representation of the construction. Now it is time to formalize this. One of the key steps was to separate the inverse semigroup into two parts, a bottom and a top. For this we give the following definition.

Definition 5.2. *Let S be a disjoint inverse semigroup. Then we define the bottom, top and defining part of S by*

$$\underline{S} = \{s \in S | \forall t \in S : \mathcal{P}_{\leq s} = \mathcal{P}_{\leq t} \Rightarrow s \leq t\}$$

$$\overline{S} = S \setminus \underline{S}$$

$$S^* = \{s \in \overline{S} | \forall t, u : t \lessdot s \wedge u \lessdot s \Rightarrow t = u\}$$

respectively.

Let us see if we did what we set out to do. First observe that $0 \in \underline{S}$. Now take $s, t \in \underline{S}$ such that $\mathcal{P}_{\leq s} = \mathcal{P}_{\leq t}$. Then $s \leq t$ and $t \leq s$ hence $s = t$. Thus all elements in $\underline{S}$ are uniquely defined by their primitives as was the idea. It turns out, however, that we have done even better.

Lemma 5.3. *Let $s \in \underline{S}$. Then $s = \bigvee \mathcal{P}_{\leq s}$.*

Proof. Clearly $p \leq s$ for all $p \in \mathcal{P}_{\leq s}$. Now, suppose that $t \in S$ such that $\mathcal{P}_{\leq s} \subseteq \mathcal{P}_{\leq t}$. Then $ss^{-1}t \neq 0$, so take $p \in \mathcal{P}_{\leq ss^{-1}t}$. It follows that $p = p_s p_s^{-1} p_t$ where $p_s \in \mathcal{P}_{\leq s}$ and $p_t \in \mathcal{P}_{\leq t}$. However, $p_s \in \mathcal{P}_{\leq t}$ and $p_s^{-1} p_t$ imply that $p_s = p_t$, hence $p \in \mathcal{P}_{\leq s}$. Because $p \in \mathcal{P}_{\leq s} \subseteq \mathcal{P}_{\leq t}$ implies that $p = pp^{-1}p \leq ss^{-1}t$ we have that $\mathcal{P}_{\leq s} = \mathcal{P}_{\leq ss^{-1}t}$. Therefore $s \leq ss^{-1}t \leq t$ which proves that $s = \bigvee \mathcal{P}_{\leq s}$ $\qquad \square$

This shows that $\underline{S}$ is already separated and semi complete. Therefore it is clear that no work needs to be done on this part of S.

We will now further investigate the separation $S = \underline{S} \bigsqcup \overline{S}$. It will turn out that both $\underline{S}$ and $\overline{S}$ can be seen as an inverse semigroup. Hence we can see $\underline{S} \bigsqcup \overline{S}$ as a splitting of S into two inverse semigroups where one is primitivistic.

As was said at the beginning of this chapter, we will consider inverse semigroups as subsemigroups of symmetric inverse semigroups. By Wagner-Preston this can always be done. We denote this by $S \lhd \mathbf{PInj}(X)$.

In order to keep notation clear we use the Greek letters π, ρ to denote the primitives of the symmetric inverse semigroup while for those of the original inverse semigroup we use p, r.

We can now restate, in terms of the primitives in the symmetric inverse semigroup, when $s \in \underline{S}$.

Lemma 5.4. *Let $S \lhd \mathbf{PInj}(X)$. Then $s \in \underline{S}$ if and only if for all $\pi \in \mathcal{P}(X)_{\leq s}$ there is a $p \in \mathcal{P}(S)_{\leq s}$ such that $\pi \leq p$.*

Proof. $\Rightarrow$ Let $\pi \in \mathcal{P}(X)_{\leq s}$. Then because $s \in \underline{S}$, $s = \bigvee \mathcal{P}(S)_{\leq s} = \bigvee \mathcal{P}(X)_{\leq s}$. Hence there is a $p \in \mathcal{P}(S)_{\leq s}$ such that $\pi \leq p$.

$\Leftarrow$ Let $t \in S$ with $\mathcal{P}(S)_{\leq t} = \mathcal{P}(S)_{\leq s}$. Then for all $\pi \in \mathcal{P}(X)_{\leq s}$ there is a $p \in \mathcal{P}(S)_{\leq s}$ such that $\pi \leq p$. Thus $\pi \leq t$ and hence $s \leq t$ which shows that $s \in \underline{S}$.

$\square$

This lemma make it possible to show that $\underline{S}$ is an ideal in S

Lemma 5.5. *Let $S \lhd \mathbf{PInj}(X)$, $s \in \underline{S}$ and $t \in S$. Then $st \in \underline{S}$.*

Proof. If $st = 0$ then $st \in \underline{S}$, so suppose that $st \neq 0$. Then $\mathcal{P}(S)_{\leq st} \neq \emptyset$. So take $\pi \in \mathcal{P}(X)_{\leq st}$, then $\pi = \pi_s \pi_t$ with $\pi_s \in \mathcal{P}(X)_{\leq s}$ and $\pi_t \in \mathcal{P}(X)_{\leq t}$. Suppose now that for all $r \in \mathcal{P}(S)_{\leq t}$ $\pi_t \not\leq r$. Then $\pi_t \pi_t^{-1} \rho \rho^{-1} = 0$ for all $\rho \in \mathcal{P}(X)_{\leq r}$. Now, because $st \neq 0$, there is a $p \in \mathcal{P}(S)_{\leq s}$ and a $r \in \mathcal{P}(S)_{\leq t}$ such that $\pi_s \leq p$ and $pr \neq 0$. Therefore $\pi_s^{-1} \pi_s = \rho \rho^{-1}$ for all $\rho \in \mathcal{P}(X)_{\leq r}$. However, then we have

$$\pi = \pi_s \pi_t = \pi_s \pi_s^{-1} \pi_s \pi_t \pi_t^{-1} \pi_t = \pi_s \rho \rho^{-1} \pi_t \pi_t^{-1} \pi_t = 0$$

which is a contradiction. Hence there is a $r \in \mathcal{P}(X)_{\leq t}$ such that $\pi_t \leq r$ and $pr \neq 0$. This proves that there exists a $q \in \mathcal{P}(S)_{\leq st}$ such that $\pi \leq q$. Thus by lemma 5.4 it follows that $st \in \underline{S}$.

$\square$

In a similar way one could prove that if $s \in \underline{S}$, $t \in S$ and $ts \neq 0$, then $ts \in \underline{S}$. Given $s, t \in S$, it follows that $st \in \overline{S}$ only if $s, t \in \overline{S}$.

Lemma 5.6. *If $s \in \overline{S}$ then also $s^{-1} \in \overline{S}$.*

Proof. Because $s \in \overline{S}$, there is a $t \in S$ with $\mathcal{P}(S)_{\leq t} = \mathcal{P}(S)_{\leq s}$ and $s \not\leq t$. Hence we have $s^{-1} \not\leq t^{-1}$ and because $\mathcal{P}(S)_{\leq t^{-1}} = \mathcal{P}(S)_{\leq s^{-1}}$ it follows that $s^{-1} \in \overline{S}$.

$\square$

We have to show that $\underline{S}$ is an inverse subsemigroup of S which is primitivistic. Because $\underline{S}$ and $\overline{S}$ are disjoint it follows that $\overline{S} \cup \{0\}$ is an inverse semigroup with zero if we define $st = 0$ if $st = 0 \in \overline{S}$ or $st \in \underline{S}$.

For S^* we have the following.

Lemma 5.7. *Let $s \in S^*$ then*

1. $s^{-1} \in S^*$.

2. $ss^{-1}, s^{-1}s \in S^*$.

Proof.   1. Let $u, v \lessdot s^{-1}$. Then $u^{-1}, v^{-1} \lessdot s$ hence $u^{-1} = v^{-1}$ from which it follows that $u = v$.

2. Let $u, v \lessdot ss^{-1}$. Then $us \neq 0$ and $vs \neq 0$. Then by lemma 4.7 $us \lessdot s$ and $vs \lessdot s$. Because $s \in S^*$ it follows that $us = vs$ and therefore we get $u = uss^{-1} = vss^{-1} = v$. The proof for $s^{-1}s$ is similar.

$\square$

This lemma is just the tip of the structure of S^*. However, before we can further investigate this we need to precisely define the inverse semigroups in which we will be working.

5.2 Representable inverse semigroups and homomorphisms

As was mentioned in 5.1.1, in order to construct a disjoint separated inverse semigroup, we need to restrict ourselves to a certain subclass of inverse semigroups. We are now ready to give a definition.

Definition 5.8. *Let $S \lhd \mathbf{PInj}(X)$ be a disjoint inverse semigroup. Then S is called representable if S is semi complete and the following holds for all $s \in S^*$ and $t, u \in S$.*

Rp1 *$s^{-1}s = t^{-1}t$ and $ss^{-1} = tt^{-1}$ imply that $s = t$.*

Rp2 *There exists a $\pi \in \mathcal{P}(X)_{\leq s}$ such that for all $t \in S$, if $\pi \leq t$ then $s \leq t$.*

Rp3 *If $\mathcal{P}_{\leq t} = \mathcal{P}_{\leq u}$ and $S^*_{\leq t} = S^*_{\leq u}$, then $t = u$.*

Observe that every disjoint separated inverse semigroup is representable. Hence every symmetric inverse semigroup is representable.
Apart from representable inverse semigroups, we also introduce the notion of a representing primitive.

Definition 5.9. *Let S, T be representable inverse semigroups, $S \xrightarrow{\phi} T$ an injective homomorphism and $p \in \mathcal{P}(T)_{\leq \phi(s)}$. Then p is said to be a representing primitive of s in T if $p \notin \phi(\mathcal{P}(S))$ and for all $s' \in S$ with $\mathcal{P}(S)_{\leq s'} = \mathcal{P}(S)_{\leq s}$ and $s' < s$, $p \not\leq \phi(s')$.*

The idea behind these representing primitives is the following. In S we are adding primitives for the defining elements S^*. It could be possible that some of these are already present T. The representable primitives should then correspond to these primitives.
We have already seen that if $p \leq st$ then $p = p_s p_t$. This is also the case for representing primitives.

Lemma 5.10. *Let S, T be representable inverse semigroups, $S \xrightarrow{\phi} T$ an injective homomorphism and $s, t \in S$ such that $st \neq 0$. If p is a representing primitive for st then $p = p_s p_t$ where p_s and p_t are representing primitives for s and t respectively.*

Proof. Because $p \in \mathcal{P}(T)_{\leq \phi(st)}$ it follows that $p = p_s p_t$ with $p_s \in \mathcal{P}(S)_{\leq s}$ and $p_t \in \mathcal{P}(S)_{\leq t}$. We will show that p_s is a representing primitive for s. The proof for p_t is similar. Suppose that $u < s$ and $\mathcal{P}_{\leq u} = \mathcal{P}_{\leq s}$. If $p_s \leq \phi(u)$ then $p \leq \phi(ut)$. Hence $\mathcal{P}(S)_{\leq ut} \neq \mathcal{P}(S)_{\leq st}$, which is a contradiction with $\mathcal{P}(S)_{\leq u} = \mathcal{P}(S)_{\leq s}$. Thus p_s is a representing primitive. $\square$

The representing primitives play an important role in the minimality of our construction, because they correspond to the primitives we are adding in S. We need our homomorphisms to respect these representing primitives, as well as the original primitives. To this end we define the notion of a representing homomorphism.

Definition 5.11. *Let S, T be representable inverse semigroups and $S \xrightarrow{\phi} T$ a homomorphism. We call ϕ representing if it is injective and the following holds for all $s \in S$.*

$$T^*_{\leq \phi(s)} = \phi(S^*_{\leq s}) \cap \overline{T}$$

$$|\mathcal{P}(T)_{\leq \phi(s)}| = |\mathcal{P}(S)_{\leq s}| + |\phi(S^*_{\leq s}) \cap \underline{T}|$$

This definition says that a representing homomorphism should respect the defining elements of S and should not add extra primitives beneath then the elements of S apart from the representing primitives. In a moment we will make this last claim more precise. First we will use representable inverse semigroups and representing homomorphisms to construct the category **RepInv**.

Given a representable inverse semigroup S it is clear that the identity on S is representing. Hence, in order to define the category **RepInv**, we need to prove that we can compose two representing homomorphisms.

Lemma 5.12. *Let S, T, U be representable inverse semigroups and $S \xrightarrow{\phi} T \xrightarrow{\psi} U$ be two representing homomorphisms. Then $S \xrightarrow{\psi\phi} U$ is a representing homomorphism.*

Proof.

$$
\begin{aligned}
|\mathcal{P}(U)_{\leq \psi\phi(s)}| &= |\mathcal{P}(T)_{\phi(s)}| + |\psi(T^*_{\leq \phi(s)}) \cap \underline{U}| \\
&= |\mathcal{P}(S)_{\leq s}| + |\phi(S^*_{\leq s}) \cap \underline{T}| + |\psi(T^*_{\leq \phi(s)}) \cap \underline{U}| \\
&= |\mathcal{P}(S)_{\leq s}| + |\psi(\phi(S^*_{\leq s}) \cap \underline{T}) \cap \underline{U}| + |\psi(\phi(S^*_{\leq s}) \cap \overline{T}) \cap \underline{U}| \\
&= |\mathcal{P}(S)_{\leq s}| + |\psi(\phi(S^*_{\leq s})) \cap \underline{U}|
\end{aligned}
$$

$\square$

Hence, we can now define the category **RepInv** to have representable inverse semigroups as objects and representing homomorphisms as arrows. The composition is just the ordinary composition. We define the categories **DSRpInv** and **SymRpInv** to have disjoint separated inverse semigroups and symmetric inverse semigroups as objects respectively and representing homomorphisms as arrows. In chapter 5.3.2 we will show that these three categories form a chain of adjunctions. At the beginning of this chapter we mentioned a universal property. Now that we have defined the category **RepInv** we can be more precise about this.

Definition 5.13. *Let* $S \xrightarrow{\phi} T$ *be a morphism in* **RepInv**. *Then we say that* ϕ *is a universal representing homomorphism if for all* $S \xrightarrow{\psi} U$ *in* **RepInv** *there exists a unique* $T \xrightarrow{\varphi} U$ *such that the following diagram commutes*

$$
\begin{array}{ccc}
S & \xrightarrow{\phi} & T \\
& \searrow{\psi} & \downarrow{\varphi} \\
& & U
\end{array}
$$

It is clear that we have a similar notion of universal representing homomorphisms in **DSRpInv** and **SymRpInv**.

It turns out that there always exists a representing primitive for those $s \in \overline{S}$ with $\phi(s) \in \underline{T}$. Moreover, if ϕ is a representing homomorphism and $s \in S^*$, then this representing primitive is unique. This proves our claim that representing primitives only add a minimal number of primitives.

Lemma 5.14. *Let* $S \xrightarrow{\phi} T$ *in* **RepInv** *and* $s \in \overline{S}$ *such that* $\phi(s) \in \underline{T}$. *Then there exists a representing primitive* $p \in \mathcal{P}(T)_{\leq \phi(s)}$. *Moreover, if* $s \in S^*$ *then this primitive is unique.*

Proof. We proof the existence by contradiction. Therefore suppose that no representing primitive exists. Then for all $p \in \mathcal{P}(T)_{\leq \phi(s)}$ and $s' \in S$ with $\mathcal{P}(S)_{\leq s'} = \mathcal{P}(S)_{\leq s}$ and $s' < s$ we have $p \leq \phi(s')$, which contradicts the assumption that $\phi(s) \in \underline{T}$.

Suppose now that $s \in S^*$. Then because $\phi(s) \in \underline{T}$, $|\phi(S^*_{\leq s}) \cap \underline{T}| = |S^*_{\leq s}|$. Therefore $|\mathcal{P}(T)_{\leq \phi(s)}| = |\mathcal{P}(S)_{\leq s}| + |S^*_{\leq s}|$. Hence, because we have a representing primitive for each $t \in S^*_{\leq t}$ and these are all distinct by definition, the representing primitive for s must be unique. $\qquad\square$

Because representing primitives of defining elements of $s \in S^*$ are unique, we denote these by $r_\phi(s)$. Using lemma 5.10 we deduce that if $s, u \in S^*$ such that $\phi(s), \phi(u) \in \underline{T}$ and $su \in S^*$, then $r(s)r(u) = r(su)$.

Now that we have given a definition it is time to give an example of a representing homomorphism. It turns out that we have already seen one, the map θ_S from theorem 4.28. To see this first observe that both inverse semigroups we considered where disjoint and separated, hence representable. Moreover because both inverse semigroups are separated $\overline{S} = \emptyset$. Because θ_S is bijective we only have to show that $|\mathcal{P}(\mathbf{PInj}(\mathcal{A}))_{\leq \theta_S(s)}| = |\mathcal{P}(S)_{\leq s}|$. However, we have already shown that $\mathcal{P}(\mathbf{PInj}(\mathcal{A}))_{\leq \theta_S(s)} = \theta_S(\mathcal{P}(S)_{\leq s})$. Hence it follows that θ_S is representing. We remark, that given a disjoint separated inverse semigroup θ_S is still a representing homomorphism. We will later show that θ_S satisfies the universal property.

In theorem 4.28 we have proven several properties for θ_S. One is that $\mathcal{P}(\mathbf{PInj}(\mathcal{A}))_{\leq \theta_S(s)} = \theta_S(\mathcal{P}(S)_{\leq s})$. An other was that θ_S preserved joins. These properties are not restricted to just θ_S.

Lemma 5.15. *Let S be a disjoint separated inverse semigroup and*

$S \overset{\phi}{\longrightarrow} \mathbf{PInj}(X)$ *a representing homomorphism. Then the following holds for all $s \in S$.*

1. *$\phi(\mathcal{P}(S)) \subseteq \mathcal{P}(\mathbf{PInj}(X))$*

2. *$\mathcal{P}(\mathbf{PInj}(X))_{\leq \phi(s)} = \phi(\mathcal{P}(S)_{\leq s})$*

3. *$\phi(\bigvee \mathcal{P}(S)_{\leq s}) = \bigvee \phi(\mathcal{P}(S)_{\leq s})$*

Proof.   1. Let $p \in \mathcal{P}(S)$. Then $1 = |\mathcal{P}(S)_{\leq p}| = |\mathcal{P}(\mathbf{PInj}(X))_{\leq \phi(p)}|$ and hence $\phi(p) \in \mathcal{P}(\mathbf{PInj}(X))$.

2. By 1 and the fact that ϕ is injective, it follows that $|\mathcal{P}(S)_{\leq s}| = |\phi(\mathcal{P}(S)_{\leq s})|$. Therefore we get that $|\phi(\mathcal{P}(S)_{\leq s})| = |\mathcal{P}(S)_{\leq s}| = |\mathcal{P}(\mathbf{PInj}(X))_{\leq \phi(s)}|$ and hence $\mathcal{P}(\mathbf{PInj}(X))_{\leq \phi(s)} = \phi(\mathcal{P}(S)_{\leq s})$.

3. By 2 $\phi(\bigvee \mathcal{P}(S)_{\leq s}) = \phi(s) = \bigvee \mathcal{P}(\mathbf{PInj}(X))_{\leq \phi(s)} = \bigvee \phi(\mathcal{P}(S)_{\leq s})$.
$\qquad\square$

5.3 Constructing a disjoint separated inverse semigroup part II

We return to the construction of our disjoint inverse semigroup. So far, we have separated the inverse semigroup into a bottom and top and have defined the set of defining elements in the top. We are now ready to define our primitives and prove that our embedding satisfies the universal property in **RepInv**.

5.3.1 Adding primitives

We need to define what our new primitives should be. As was mentioned several times, the elements in S^* should define these new elements. In order to define these we will make maximal use of the fact that we regard S as a subsemigroup of some symmetric $\mathbf{PInj}(X)$. For this makes it possible to define these new primitives as joins of certain primitives from the symmetric inverse semigroup.

Let $S \in \mathbf{RepInv}$ and $s \in S^*$. Then we define the following

$$\Pi_s = \{\pi \in \mathcal{P}(X)_{\leq s} | \forall t \in S; \pi \leq t \Rightarrow s \leq t\}$$

$$\pi(s) = \bigvee \Pi_s$$

Because $S \in \mathbf{RepInv}$, $\Pi_s \neq \emptyset$ for all $s \in S^*$ and hence $\pi(s) \neq 0$. The idea here is that $\pi(s)$ is the primitive uniquely defining s. First we show that these elements can indeed function as primitives.

Lemma 5.16. *Let $S \in \mathbf{RepInv}$, $s \in S^*$ and $t \in S$. Then the following holds:*

1. $t \leq \pi(s) \Rightarrow t = 0$ or $t = \pi(s)$.

2. *If $t \in S^*$ and $\pi(t) \leq \pi(s)$ then $\pi(t) = \pi(s)$*

Proof.   1. $t \neq 0$ implies that $\mathcal{P}(X)_{\leq t} \neq \emptyset$, so let $\pi \in \mathcal{P}(X)_{\leq t}$. Then $\pi \leq \pi(s)$, hence $\pi \in \Pi_s$ and therefore $s \leq t$. Now we have $\pi(s) \leq s \leq t$, thus $\pi(s) = t$.

2. Take $\pi \in \mathcal{P}(X)_{\leq \pi(t)}$. Then, because $\pi(t) \leq \pi(s)$, $\pi \in \Pi_s$. Together with $\pi \leq t$ we get that $s \leq t$. Similar, because $\pi \leq \pi(s) \leq s$ and $\pi \in \Pi_t$ it follows that $t \leq s$. Therefore $s = t$ and so $\pi(s) = \pi(t)$.

$\square$

The above lemma shows that every $\pi(s)$ indeed satisfies primitive like properties. Moreover, from the prove of the second part we deduce that if $\pi(s) = \pi(t)$ then $s = t$.

Loosely saying, the construction of our disjoint separated inverse semigroup is just adding these $\pi(s)$ to $S \subseteq \mathbf{PInj}(X)$. However it is not even clear that this is an inverse semigroup.

Let us be more formal and introduce the set $\Pi(S)$ defined by $\Pi(S) = S \sqcup \pi(S^*)$, where $\pi(S^*) := \{\pi(s)|s \in S^*\}$. Because $S, \pi(S^*) \subseteq \mathbf{PInj}(X)$ we can use the multiplication in $\mathbf{PInj}(X)$ to multiply elements in $\Pi(S)$. What we have to show is that this multiplication $\Pi(S)$ stays in $\Pi(S)$. Because S is already an inverse semigroup we need to focus on multiplications of the form $s\pi(t)$ and $\pi(s)\pi(t)$. The next lemma will be a first step in showing that multiplications of the latter type are well defined.

Lemma 5.17. *Let $S \in \mathbf{RepInv}$ and $t \in S^*$. Then*

1. $\pi(t^{-1}) = \pi(t)^{-1}$

2. $\pi(t^{-1}t) = \pi(t)^{-1}\pi(t)$

3. $\pi(tt^{-1}) = \pi(t)\pi(t)^{-1}$

Proof. Observe that by lemma 5.7 we have that $t^{-1}, tt^{-1}, t^{-1}t \in S^*$, hence everything is well defined. The proof now immediately follows from lemma 4.16. $\square$

We can use this lemma to prove there is a relation between those $s \in S$ and $t \in S^*$, for which $s\pi(t) \neq 0$.

In the following results we focus on the multiplication $s\pi(t)$. However, all of these results can also be proven for multiplications of the type $\pi(t)s$.

Lemma 5.18. *Let $S \in \mathbf{RepInv}$, $s \in S$ and $t \in S^*$ such that $s\pi(t) \neq 0$. Then $s^{-1}st = t$.*

Proof. By lemma 5.17 $\pi(t^{-1}t) = \pi(t)^{-1}\pi(t)$ and $\pi(tt^{-1}) = \pi(t)\pi(t)^{-1}$. Now $s\pi(t) \neq 0$ implies there exists a $\rho \in \mathcal{P}(X)_{\leq s}$ such that $\rho\pi(t) \neq 0$. Hence $\pi(t^{-1}t) = \pi(t)^{-1}\pi(t) = \pi(t)^{-1}\rho^{-1}\rho\pi(t) \leq ts^{-1}st$. From this it follows that $t^{-1}t \leq t^{-1}s^{-1}st$ and because $t^{-1}s^{-1}st \leq t^{-1}t$ we get $t^{-1}t = t^{-1}s^{-1}st$. On the other hand, $\pi(tt^{-1}) = \rho^{-1}\rho\pi(t)\pi(t)^{-1} \leq s^{-1}stt^{-1}$, which proves that $tt^{-1} \leq s^{-1}stt^{-1} \leq tt^{-1}$ and therefore $tt^{-1} = s^{-1}stt^{-1}$. From these results it follows that $s^{-1}stt^{-1}s^{-1}s = tt^{-1}$ and $t^{-1}s^{-1}ss^{-1}st = t^{-1}t$. Because $S \in \mathbf{RepInv}$ and $t \in S^*$ it now follows that $s^{-1}st = t$. $\square$

Lemma 5.19. *Let $S \in \mathbf{RepInv}$, $s \in S$ and $t \in S^*$. Then $s\pi(t) \neq 0 \Rightarrow st \in S^*$*

Proof. By lemma 5.18 $t = s^{-1}st$. Now suppose that $u, v \lessdot st$. Then $s^{-1}u \neq 0$ and $s^{-1}v \neq 0$. Therefore $s^{-1}u \lessdot s^{-1}st = t$ and $s^{-1}u \lessdot s^{-1}st = t$, hence $s^{-1}u = s^{-1}v$. From this it follows that $u = ss^{-1}u = ss^{-1}v = v$ and thus $st \in S^*$. □

We can now prove the following result.

Proposition 5.20. *Let $S \in$ **RepInv**. Then $\pi(S^*) \cup \{0\}$ is an inverse semigroup.*

Proof. First we need to show that the multiplication of $\pi(s)\pi(t)$ is again in $\pi(S^*) \cup \{0\}$. If $\pi(s)\pi(t) = 0$. then we are done. So suppose that $\pi(s)\pi(t) \neq 0$. Then $s\pi(t) \neq 0$ and hence by lemma 5.19 $st \in S^*$. We now claim that $\pi(s)\pi(t) = \pi(st)$.
Because $\pi(s)\pi(t) \neq 0$ we have $\pi(s) = \pi(s)\pi(t)\pi(t)^{-1} \leq stt^{-1}$. Hence $s \leq stt^{-1} \leq s$, thus $s = stt^{-1}$. In a similar way we can proof that $t = s^{-1}st$.
Take $\rho_s \in \Pi_s$ and $\rho_t \in \Pi_t$ such that $\rho_s\rho_t \neq 0$. Then $\rho_s\rho_t \leq st$. Now suppose that $\rho_s\rho_t \leq u$. Then $\rho_s = \rho_s\rho_t\rho_t^{-1} \leq u\rho_t^{-1} \leq ut^{-1}$, hence $s \leq ut^{-1}$ and so $st \leq ut^{-1}t \leq u$. This proves that $\rho_s\rho_t \in \Pi_{st}$, from which it follows that $\pi(s)\pi(t) \leq \pi(st)$.
Now let $\rho \in \Pi_{st}$. Then $\rho \leq st$ and hence $\rho = \rho_s\rho_t$. Suppose that $\rho_s \leq u$, then $\rho \leq u\rho_t \leq ut$ and so $st \leq ut$. Now using that $s = stt^{-1}$, we get that $s = stt^{-1} \leq utt^{-1} \leq u$. Thus proving that $\rho_s \in \Pi_s$. In a similar way, using that $t = s^{-1}st$, it follows that $\rho_t \in \Pi_t$ and therefore $\rho \in \Pi_s\Pi_t$. Thus $\pi(st) \leq \pi(s)\pi(t)$ and hence $\pi(s)\pi(t) = \pi(st)$.
Together with lemma 5.17 this proves that $\pi(S^*) \cup \{0\}$ is regular. Now let $\pi(s), \pi(t)$ be nonzero idempotents. Then $\pi(s)^2 = \pi(s)$, hence $\pi(s) = \pi(s^2)$ from which it follows that $s = s^2$. The same is true for t. Therefore we conclude that $s, t \in \mathcal{E}$. If $\pi(s)\pi(t) \neq 0$ then by the result above $\pi(s)\pi(t) = \pi(st) = \pi(ts) = \pi(t)\pi(s)$. Thus the idempotents of $\pi(S^*)$ commute. □

This proposition proves that the multiplications $\pi(s)\pi(t)$ are again in $\Pi(S)$. We are now left with the interacting multiplications $s\pi(t)$. These also behave well.

Proposition 5.21. *Let $s \in S$ and $t \in S^*$, such that $s\pi(t) \neq 0$. Then $s\pi(t) = \pi(st)$.*

Proof. By lemma 5.19 $st \in S^*$, so $\pi(st)$ exists. Take $\rho \in \mathcal{P}(X)_{\leq s}$ such that $\rho\pi(t) \neq 0$. Suppose now that $\rho\pi(t) \leq u$. Then $\pi(t) = \rho^{-1}\rho\pi(t) \leq \rho^{-1}u \leq s^{-1}u$. By definition we now have that $t \leq s^{-1}u$ hence $st \leq ss^{-1}u \leq u$. Thus, $\rho\pi(t) \in \Pi_{st}$ and so $s\pi(t) \leq \pi(st)$.

For the other inequality, let $\pi \in \Pi_{st}$. Then $\pi = \rho_s \rho_t$ with $\rho_s \in \mathcal{P}(X)_{\leq s}$ and $\rho_t \in \mathcal{P}(X)_{\leq t}$. Suppose that $\rho_s \leq u$. Then $\pi \leq \rho_s u \leq su$ hence $st \leq su$. By lemma 5.18 we now have that $t = s^{-1}st \leq s^{-1}su \leq u$ which proves that $\rho_t \in \Pi_t$. Thus, $\pi(st) \leq s\pi(t)$ and therefore $s\pi(t) = \pi(st)$ $\qquad\square$

5.3.2 Completing the construction

We can now combine all our results to finish the construction and prove its minimality.

Theorem 5.22. *Let S be a representable inverse semigroup. Then there exists an finite disjoint separated inverse semigroup $\Pi(S)$ and a universal representing homomorphism $S \xrightarrow{\Pi_S} \Pi(S)$.*

Proof. We recall that we defined $\Pi(S) = S \sqcup \pi(S^*)$. Proposition 5.20 and 5.21 prove that $\Pi(S)$ is an inverse semigroup. Hence we need to show that it is disjoint and separated. First observe that $\mathcal{P}(\Pi(S)) = \mathcal{P}(S) \cup \pi(S^*)$. Because S is disjoint and $\mathcal{P}(S) \perp \pi(S^*)$, we only need to prove that the elements of $\pi(S^*)$ are disjoint. Therefore, suppose that $\pi(s)\pi(s)^{-1} = \pi(t)\pi(t)^{-1}$ and $\pi(s)^{-1}\pi(s) = \pi(t)^{-1}\pi(t)$. Then it follows that $ss^{-1} = tt^{-1}$ and $s^{-1}s = t^{-1}t$ and because $S \in \mathbf{RepInv}$ it follows that $s = t$ so $\pi(s) = \pi(t)$.
To prove that $\Pi(S)$ is separated we only need to consider elements in $\overline{S}$. Suppose that $s, t \in \overline{S}$ such that $\mathcal{P}_{\leq s} = \mathcal{P}_{\leq t}$. Then $\mathcal{P}(S)_{\leq s} = \mathcal{P}(S)_{\leq t}$ and $S^*_{\leq s} = S^*_{\leq t}$. Hence, because S is representable, $s = t$.

We define $S \xrightarrow{\Pi_S} \Pi(S)$ by $\Pi_S(s) = s$. This is clearly an injective homomorphism. It is representing because $\mathcal{P}(\Pi(S))_{\leq \Pi_S(s)} = \mathcal{P}(S)_{\leq s} \cup S^*_{\leq s}$.

Now suppose that $S \xrightarrow{\phi} T$ is a representing homomorphism, where T is a disjoint separated inverse semigroup. Then we define $\Pi(S) \xrightarrow{\psi} T$ by $\psi(s) = \phi(s)$ and $\psi(\pi(s)) = r(s)$. Because $r(st) = r(s)r(t)$ and T is disjoint and separated, it follows that ψ is a homomorphism. It is clearly uniquely defined by ϕ and a representing homomorphism, because ϕ is. Furthermore, $\psi\Pi_S(s) = \psi(s) = \phi(s)$ hence Π_S is universal. $\qquad\square$

This theorem states that our construction is truly minimal in the category **RepInv**. The disjoint separated inverse semigroup $\Pi(S)$ is the smallest one containing S. We can obtain a similar result for disjoint separated inverse semigroups.

Theorem 5.23. *Let S be a disjoint separated inverse semigroup. Then there exists a finite symmetric inverse semigroup $\mathbf{PInj}(X)$ and a universal representing homomorphism $S \xrightarrow{\phi} \mathbf{PInj}(X)$.*

Proof. Because S is disjoint and separated, we have that the map $S \xrightarrow{\theta_S} \mathbf{PInj}(\mathcal{A}(S))$ is a representing homomorphism. To proof that θ_S is universal let $S \xrightarrow{\phi} \mathbf{PInj}(X)$ be an other representing homomorphism. Because of theorem 4.28, $\mathbf{PInj}(X) \cong \mathbf{PInj}(\mathcal{A}(\mathbf{PInj}(X)))$ with isomorphism $\theta_{\mathbf{PInj}(X)}$, we need to find a representing homomorphism $\mathbf{PInj}(\mathcal{A}(S)) \xrightarrow{\psi} \mathbf{PInj}(\mathcal{A}(\mathbf{PInj}(X)))$ such that the following diagram commutes.

$$
\begin{array}{ccc}
S & \xrightarrow{\ \theta_S\ } & \mathbf{PInj}(\mathcal{A}(S)) \\
& \theta_{\mathbf{PInj}(X)}\phi \searrow & \downarrow \psi \\
& & \mathbf{PInj}(\mathcal{A}(\mathbf{PInj}(X)))
\end{array}
$$

Because ϕ is injective we can consider $\phi_{\mathcal{A}(S)}$. This is a morphism in $\mathbf{PInj}$ and hence we can use lifts to define $\psi := \phi^*_{\mathcal{A}(S)}$. We will abuse notation a bit and write ϕ instead of $\phi_{\mathcal{A}(S)}$. Because ϕ is representing it follows that ϕ^* is also a representing homomorphism. Also, it is clear that ϕ^* is uniquely determined by ϕ. Hence all that we have to show is that the diagram commutes. For the domain we have

$$
\begin{aligned}
\mathrm{dom}(\phi^*\theta_S(s)) &= \phi(\mathrm{dom}(\theta_S(s))) \\
&= \phi(\mathcal{P}(S)^{-1}_{\leq s}\mathcal{P}(S)_{\leq s}) \\
&= \phi(\mathcal{P}(S)^{-1}_{\leq s})\phi(\mathcal{P}(S)_{\leq s}) \\
&= \mathcal{P}(\mathbf{PInj}(X))^{-1}_{\leq \phi(s)}\mathcal{P}(\mathbf{PInj}(X))_{\leq \phi(s)} \\
&= \mathrm{dom}(\theta_{\mathbf{PInj}(X)}(\phi(s)))
\end{aligned}
$$

while for all $\phi(p^{-1}p) \in \mathrm{dom}(\psi\theta_S(s))$,

$$
\begin{aligned}
\phi^*\theta_S(s)(\phi(p^{-1}p)) &= \phi\theta_S(s)(p^{-1}p) \\
&= \phi(sp^{-1}ps^{-1}) = \phi(pp^{-1}) \\
&= \phi(p)\phi(p^{-1}p)\phi(p)^{-1} \\
&= \theta_{\mathbf{PInj}(X)}(\phi(s))(\phi(p^{-1}p))
\end{aligned}
$$

$\square$

Combining theorems 5.22 and 5.23 we have proven the following.

Theorem 5.24. *Let S be a finite orthogonal inverse semigroup. Then there exists a finite symmetric inverse semigroup $\mathbf{PInj}(X)$ and a universal representing homomorphism $S \xrightarrow{\phi} \mathbf{PInj}(X)$.*

The symmetric inverse semigroup is $\mathbf{PInj}(\mathcal{A}(\Pi(S)))$, while the universal representing homomorphism is given by $\theta_{\Pi(S)}\Pi_S$. This completes our construction of a minimal representation.

The results of theorems 5.22, 5.23 and 5.24 can be restated as follows:

5.22: Free finite disjoint seperated inverse semigroups over representable inverse semigroups exist.

5.23: Free finite symmetric inverse semigroups over finite disjoint seperated inverse semigroups exist.

5.24: Free finite symmetric inverse semigroups over representable inverse semigroups exist.

As was mentioned at the start of this chapter, we needed this minimality to be able to define a functor to $\mathbb{F}\text{rob}(\mathbf{Hilb})$. These three theorems will actually turn out to produce a variety of categorical adjunctions, one of them being the adjunction between $\mathbf{RepInv}$ and $\mathbb{F}\text{rob}(\mathbf{Hilb})$.

Chapter 6

Hilbert spaces and Inverse semigroups

We have come to the final part of this thesis. Here we will combine all the main results to attain the adjunction between **RepInv** and $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$. As was already mentioned, the result from section 5.3.2 gives several adjunctions. Therefore we will construct our main adjunction in three steps.

6.1 SymRpInv and $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$

The first step is to establish a relation between the category **SymRpInv** and $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$. This starts with the observation that each $X \xrightarrow{f} Y$ in $\mathbb{F}\mathrm{rob}(\mathbf{PInj})$ is injective. Therefore its lift $\mathbf{PInj}(X) \xrightarrow{f^*} \mathbf{PInj}(Y)$ is a homomorphism of inverse semigroups. Composition of lifts is well defined. For, if $g \in \mathbb{F}\mathrm{rob}(\mathbf{PInj})(Y, Z)$ and $h \in \mathbf{PInj}(X)$, $g^* f^*(h) = g^*(fhf^\dagger) = gfhf^\dagger g^\dagger = (gf)h(gf)^\dagger = (gf)^*$.

Therefore we get a functor $\mathbb{F}\mathrm{rob}(\mathbf{PInj}) \xrightarrow{\mathbf{PInj}^*} \mathbf{SymRpInv}$ sending X to $\mathbf{PInj}(X)$ and f to f^*. Because representing homomorphisms preserve primitives and hence atoms we also have a functor $\mathbf{SymRpInv} \xrightarrow{\mathcal{A}} \mathbb{F}\mathrm{rob}(\mathbf{PInj})$ defined on objects by $\mathcal{A}_*(S) = \mathcal{A}(S)$ and on morphisms $S \xrightarrow{\phi} T$ by $\mathcal{A}_*(\phi) = \phi|_{\mathcal{A}(S)}$. Using these functors we obtain the following:

Theorem 6.1. SymRpInv $\cong \mathbb{F}\mathrm{rob}(\mathbf{PInj})$.

Proof. We prove that $\mathcal{A}_* \dashv \mathbf{PInj}^*$, where de unit and co-unit are isomorphisms. We define the unit $\eta : 1 \Rightarrow \mathbf{PInj}^* \mathcal{A}_*$ by the isomorphism

$S \xrightarrow{\theta_S} \mathbf{PInj}(\mathcal{A}(S))$. Naturality follows from theorem 5.23, together with the observation that $\mathbf{PInj}^* \mathcal{A}_*(\phi) = \phi^*$ for all $S \xrightarrow{\phi} T$ in $\mathbf{SymRpInv}$. The co-unit $\varepsilon : \mathcal{A}_* \mathbf{PInj}^* \Rightarrow 1$ is given by the isomorphism $\mathcal{A}(\mathbf{PInj}(X)) \xrightarrow{\phi} X$ from corollary 4.24. This is natural because $\mathrm{dom}(\phi_f(1_x)) = f(\mathrm{dom}(1_x)) = f(x) = f(\phi(1_x))$. The triangle diagrams

$$\mathbf{PInj}^* \xRightarrow{\eta * \mathbf{PInj}^*} \mathbf{PInj}^* \mathcal{A}_* \mathbf{PInj}^* \qquad \mathcal{A}_* \xRightarrow{\mathcal{A}_* \eta} \mathcal{A}_* \mathbf{PInj}^* \mathcal{A}_*$$

with $\Downarrow \mathbf{PInj}^* \varepsilon$ to $\mathbf{PInj}^*$ and $\Downarrow \varepsilon * \mathcal{A}_*$ to $\mathcal{A}_*$

commute because $\mathbf{PInj}^* \varepsilon = \eta^{-1}$ and $\mathcal{A}_* \eta = \varepsilon^{-1}$. Thus $\mathbf{SymRpInv} \cong \mathbb{F}\mathrm{rob}(\mathbf{PInj})$. $\qquad\square$

6.2 The adjunction between $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$ and RepInv

Because of theorem 6.1 and corollary 3.30 we have that $\mathbf{SymRpInv} \cong \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$. The next step is to construct an adjunction between $\mathbf{DSRpInv}$ and $\mathbf{SymRpInv}$. Finally we will show that we have an adjunction between $\mathbf{RepInv}$ and $\mathbf{DSRpInv}$.

The universal statement of theorem 5.23 gives us a functor $\mathbf{DSRpInv} \xrightarrow{\Theta} \mathbf{SymRpInv}$, defined on objects by $\Theta(S) = \mathbf{PInj}(\mathcal{A}(S))$ and on morphisms $S \xrightarrow{\phi} T$ by $\Theta(\phi) = \theta^{-1}_{\mathbf{PInj}(\mathcal{A}(T))} \phi^*$. The functor in the other direction is the inclusion functor I.

Theorem 6.2. $\mathbf{DSRpInv} \underset{I}{\overset{\Theta}{\rightleftarrows}} \mathbf{SymRpInv}$

Proof. The unit $\eta : 1 \Rightarrow I\Theta$ is given by $S \xrightarrow{\theta_S} \mathbf{PInj}(\mathcal{A}(S))$. To see that this is natural consider the following diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\theta_S} & \mathbf{PInj}(\mathcal{A}(S)) \\
\downarrow{\phi} & & \searrow{\phi^*} \\
T & \xrightarrow{\theta_T} & \mathbf{PInj}(\mathcal{A}(T)) \xleftarrow{\theta^{-1}_{\mathbf{PInj}(\mathcal{A}(T))}} \mathbf{PInj}(\mathcal{A}(\mathbf{PInj}(\mathcal{A}(T))))
\end{array}
$$

This diagram commutes because of the universal property of θ_S and the fact that $\theta_{\mathbf{PInj}(\mathcal{A}(T))}$ is an isomorphism. Now if we fill in the definition of $\Theta(\phi)$ we get the following.

$$
\begin{array}{ccc}
S & \xrightarrow{\ \theta_S\ } & \mathbf{PInj}(\mathcal{A}(S)) \\
\phi \downarrow & \theta^{-1}_{\mathbf{PInj}(\mathcal{A}(T))}\phi^* \downarrow & \searrow \phi^* \\
T & \xrightarrow{\ \theta_T\ } & \mathbf{PInj}(\mathcal{A}(T)) \xleftarrow{\ \theta^{-1}_{\mathbf{PInj}(\mathcal{A}(T))}\ } \mathbf{PInj}(\mathcal{A}(\mathbf{PInj}(\mathcal{A}(T))))
\end{array}
$$

The right side commutes by definition and hence the left side commutes proving that η is natural. Because I is the inclusion functor the co-unit $\varepsilon : \Theta I \Rightarrow 1$ is given by $\mathbf{PInj}(\mathcal{A}(\mathbf{PInj}(X))) \xrightarrow{\ \theta^{-1}_{\mathbf{PInj}(X)}\ } \mathbf{PInj}(X)$. Given a morphism $S \xrightarrow{\ \phi\ } T$ in $\mathbf{SymRpInv}$, the following diagram commutes.

$$
\begin{array}{ccc}
\mathbf{PInj}(\mathcal{A}(S)) & \xrightarrow{\ \theta_S^{-1}\ } & S \\
\theta_S^{-1} \downarrow & & \downarrow \phi \\
S & \xrightarrow{\ \phi\ } & T \\
\theta_S \downarrow & & \downarrow \theta_T \\
\mathbf{PInj}(\mathcal{A}(S)) & & \mathbf{PInj}(\mathcal{A}(T)) \\
& \searrow \phi^* \qquad \downarrow \theta_{\mathbf{PInj}(\mathcal{A}(T))} & \\
& \mathbf{PInj}(\mathcal{A}(\mathbf{PInj}(Y))) &
\end{array}
$$

Here the commutivity of the top is clear while the bottom part commutes by definition of ϕ^*. From this and because all instances of θ are isomorphisms, we deduce that ε is natural. For the triangles

$$
\begin{array}{ccc}
I \xRightarrow{\ \eta*I\ } I\Theta I & \qquad\qquad & \Theta \xRightarrow{\ \Theta\eta\ } \Theta I\Theta \\
\ \ \searrow \ \ \Big\Downarrow I\varepsilon & & \ \ \searrow \ \ \Big\Downarrow \varepsilon*\Theta \\
\quad I & & \quad \Theta
\end{array}
$$

we observe that the left triangle commutes because $\eta * I = \varepsilon^{-1}$. While the right one commutes because $\Theta(\theta_S) = \theta^{-1}_{\mathbf{PInj}(\mathcal{A}(S))}$. $\square$

The functor Θ was obtained from theorem 5.23. In a similar way we get a functor $\mathbf{RepInv} \xrightarrow{\Pi} \mathbf{DSRpInv}$ from theorem 5.22. This functor is defined on objects by $\Pi(S)$ and on morphisms by the universal property of the map Π_S. Again we have the inclusion functor $\mathbf{DSRpInv} \xrightarrow{I} \mathbf{RepInv}$. Observe that if $S \in \mathbf{RepInv}$ is disjoint and separated, $\Pi(S) = S$ and $\Pi_S = 1_S$. From this it follows that $\Pi^2 = \Pi$ so the co-unit is just the identity. Therefore, if take $S \xrightarrow{\Pi_S} \Pi(S)$ as our unit, which is natural because Π_S satisfies the universal property. Then the following is immediate.

Theorem 6.3. $\mathbf{RepInv} \;\underset{I}{\overset{\Pi}{\rightleftarrows}}\; \mathbf{DSRpInv}$

Combining these theorems and corollary 3.30 we have the following chain of adjunctions.

$$\mathbf{RepInv} \;\underset{I}{\overset{\Pi}{\rightleftarrows}}\; \mathbf{DSRpInv} \;\underset{I}{\overset{\Theta}{\rightleftarrows}}\; \mathbf{SymRpInv} \;\underset{\mathcal{A}_*}{\overset{\mathbf{PInj}^*}{\rightleftarrows}}\; \mathbb{F}\mathrm{rob}(\mathbf{PInj}) \;\underset{\mathfrak{C}}{\overset{\ell^2}{\rightleftarrows}}\; \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$$

The two most left adjunction are reflections while the other two are equivalences. From this chain we obtain the following, long awaited result.

Theorem 6.4. $\mathbf{RepInv} \;\overset{}{\underset{}{\rightleftarrows}}^{\perp}\; \mathbb{F}\mathrm{rob}(\mathbf{Hilb})$

Chapter 7

Conclusions and further work

In this thesis we have given a subclass of finite inverse semigroups and proven that they carry a structure which is similar to that of a basis of a finite-dimensional Hilbert space. The definition of these representable inverse semigroups contains a lot of technical requirements. However, some of these might be redundant. The idea behind this is that given $s \in S^*$ and $t \in S$ such that $st \neq 0$, it seems that $st^{-1}s \prec t$. If this is true one could prove, using the finiteness, that S is semi complete and that $\Pi_s \neq \emptyset$ for all $s \in S^*$. It also seems that if $s, t \in S$ such that $S^*_{\leq s} \cap S^*_{\leq t} \neq \emptyset$, then $st^{-1}s = ts^{-1}t \leq s, t$. This could be used to prove that $\overline{\Pi}(S)$ is separated. Hence, if both claims are true, representable inverse semigroups would just be finite disjoint inverse semigroups with property Rp1.

Although the results we have given in chapter 5.3.2 seem nice, we did not have time to investigate their precise implications. For instance, the adjunction from theorem 6.4 might be useful to get a grip on the interactions between different bases on the same Hilbert space. For it reduces this geometric problem to a problem in the field of inverse semigroups where computations are much more simple. More work on this is needed.

Also given a finite-dimensional Hilbert space, its lattice of orthonormal projections is an E-unitary inverse semigroup. The atoms of this inverse semigroup are exactly the basis elements of the Hilbert space. Hence $\mathbf{PInj}(\mathcal{A})$ is isomorphic to the original Hilbert space. It is also clear that this inverse semigroup is disjoint and separated. It might be possible to determine when a disjoint separated inverse semigroup corresponds to a lattice of orthonormal projections of some Hilbert space. The category of these inverse semigroups will then be equivalent to $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$. Hence, would give a different look at the structure of Hilbert spaces. For then we can deduce all the properties of

the Hilbert space by looking at its orthonormal projections.

In conclusion, we can say that there is indeed a strong link between finite inverse semigroups and finite-dimensional Hilbert spaces, which could be used to get a grip on the structure of the category $\mathbb{F}\mathrm{rob}(\mathbf{Hilb})$. We hope that this thesis will inspire others to further investigate these relations.

Bibliography

[1] Samson Abramsky and Chris Heunen, H^*-algebras and nonunital frobenius algebras: first step in infinite dimensional categorical quantum mechanics, Tech. report, Oxford University Computing Laboratory, 2010.

[2] Bob Coecke, Dusko Pavlovic, and Jamie Vicary, A new description of orthogonal bases, Tech. report, Oxford University Computing Laboratory, 2008, arXiv:0810:0812v1.

[3] John B. Conway, A course in functional analysis, Springer, 2007.

[4] Joachim Kock, Frobenius algebras and 2d topological quantum field theories, Cambridge University Press, 2003.

[5] Saunders Mac Lane, Categories for the working mathematician, Springer, 1997.

[6] Mark V Lawson, Inverse semigroups, the theory of partial symmetries, World Scientific, 1998.