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On the characterization of geometric logic

by

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To my parents

Contents

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Preface

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Dear reader,

The development of topos theory dates back to Alexander Grothendieck in the late 1950's who was one of the first that felt the need to put notions like coverings and sheaves in a more abstract context. He was motivated by the seemingly dual notions of subgroups of groups of deck transformations for a covering space and subgroups of the Galois group of a normal field extension. The parallel fails because covering spaces factor over open neighbourhoods (ie. monomorphisms) while fields factor over more general maps (not necessarily epimorphisms). Grothendieck figured out that the dualization works out fine if one replaces the notion of a neighbourhood $U \subset X$ by a more general map (continuous of course) $U \to X$. This allows for the theory to be developed even in the generality of category theory.

Though originally motivated by geometry, topos theory can equivalently well be developed from a logical point of view. Since this is the proper context for this thesis, we will combine the best of both worlds to develop basic topos theory in the first chapter. For a more elaborate version of these preliminaries we refer to [11] and the third volume of [1]. As our starting we point, we assume the reader is familiar with basic category theory and has followed some course in logic. If the reader feels uncertain about these foundations it might be helpful to consult the first two volumes of [1] or have a look at [10]. If the reader is familiar with model theory he or she might discover a connection between the general characterization theorem in chapter 2 of this thesis and the Łos-Tarski preservation theorem.

During the first year of my masters in Mathematical Sciences I came across many different topics in geometry which allow for a more general development in the context of category theory and topos theory. Motivated by the beauty of unification of these topics at an abstract level I visited dr. Jaap van Oosten and discussed the possibilities for me to write my master thesis about a topos theory related subject. After studying both category theory and topos theory I acquainted myself with the unlying theory of Caramellos characterization theorem with the aid of dr. van Oosten. The original plan to extend the result by Caramello turned out to be too ambitious for me to realize. Therefore, we decided to adapt the goal of this research: give a concise description of the characterization theorem and all its prerequisites, such that its beauty can be admired by a much wider audience.

Yours faithfully,

Ralph Langendam

Chapter 1

Prerequisites from Topos Theory

Throughout the following we assume that $\mathcal{C}$ is a small category.

1.1 The Notion of Site

Given a small category $\mathcal{C}$, we have the associated functor category $\widehat{\mathcal{C}} := \mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$ of contravariant functors $\mathcal{C} \to \mathrm{Set}$ (called *presheaves*) and the *Yoneda embedding* $y : \mathcal{C} \to \widehat{\mathcal{C}}$ given by $y_C : \mathcal{C}^{\mathrm{op}} \to \mathrm{Set} : D \mapsto \mathcal{C}(D, C)$ on objects $C, D \in \mathcal{C}_0$. The functors y_C are called *representable*, while subobjects of y_C are called *sieves* on C. As such, a sieve $S \leq y_C$ can be thought of as a family of morphisms to C which form a right ideal under composition, ie. $\forall f : f \in S \to f \circ g \in S$, whenever the composition is defined. Note that any arrow $h : D \to C$ allows us to transform S into a sieve on D, by selecting those arrows to D whose composition with h lies in S, ie. the sieve $h^*(S) \leq y_D$ is represented by the set $\{g : E \to D | h \circ g \in S\} \subset \bigcup_{E \in \mathcal{C}_0} \mathcal{C}(E, D)$.

Definition 1.1.1. *A* (Grothendieck) topology *on $\mathcal{C}$ is a function J on $\mathcal{C}_0$ which assigns to each object C a set of sieves $J(C)$ on C, such that the following three properties are satisfied*

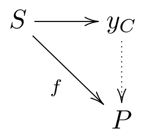
1. *For all objects C, the maximal sieve t_C of all morphisms to C is in $J(C)$.*
2. *For all sieves $S \in J(C)$ and all morphisms $h : D \to C$ we have $h^*(S) \in J(D)$.*
3. *Given $S \in J(C)$ and R any sieve on C with $\forall h : (h : D \to C) \in S \to h^*(R) \in J(D)$, then $R \in J(C)$.*

A pair $(\mathcal{C}, J)$ of a small category $\mathcal{C}$ and a Grothendieck topology J is called a (Grothendieck) site *and a sieve $S \in J(C)$ is said to J-cover C. We say that a sieve S on C is* closed *for J if for all morphisms $f : D \to C$ in $\mathcal{C}$ we have $f^*S \in J(D) \to f \in S$.*

1.2 The Notion of a Sheaf

Definition 1.2.1. *Given a site* $(\mathcal{C}, J)$ *and a presheaf* P *on* $\mathcal{C}$*, then a* matching family *for a sieve* $S \in J(C)$ *of elements of* P *is an association of* $x_f \in P(D)$ *to any* $(f : D \to C) \in S$ *such that* $x_f \cdot g := P(g)(x_f) = x_{fg}$ *for all* $g \in \mathcal{C}(E, D)$*. An* amalgamation *of such a matching family is a single* $x \in P(C)$ *such that for all* $f \in S : P(f)(x) = x_f$*. Compare this to the notion of matching family in classical sheaf theory for topological spaces, where the* x_f *are meant to agree on overlaps of their domain. As such, our presheaf* P *will be a* sheaf*, precisely if the "collation property" holds, ie. if every matching family for any cover of any object has a unique amalgamation.*

In view of our definition of a sieve S on C as a subfunctor of y_C we see that a matching family $f \to x_f$ for $f \in S$ is just a natural transformation $S \to P$ for any presheaf P. Hence, P is a sheaf precisely when for all $S \in J(C)$, all $f : S \to P$ extend uniquely to y_C, as in



The family of sheaves over the site $(\mathcal{C}, J)$ forms a full subcategory of $\widehat{\mathcal{C}}$ and is denoted $\mathrm{Sh}(\mathcal{C}, J)$. The inclusion $\mathrm{Sh}(\mathcal{C}, J) \to \widehat{\mathcal{C}}$ has a left adjoint (*the associated sheaf functor*) $a : \widehat{\mathcal{C}} \to \mathrm{Sh}(\mathcal{C}, J)$, which commutes with finite limits.

The process of sending a presheaf to its associated sheaf under a is called *sheafification*. More explicitly, sheafification can be formulated in terms of the *plus construction*. One may show, there is a well-defined functor $+ : \widehat{\mathcal{C}} \to \widehat{\mathcal{C}} : P \to P^+$, such that for any presheaf P and any $C \in \mathcal{C}_0$, $P^+(C)$ is the colimit over all S, J-covering sieves of C, of matching families for S of C. More explicitly, $P^+(C)$ is an equivalence class of matching families $\{x_f \in P(D) | f : D \to C \in S\}$ with for all $g : E \to D$ we have $x_f \cdot g = x_{fg}$. Two such $\{x_f | f \in S\}$ and $\{y_g | g \in S'\}$ are equivalent if there exists a common refinement $T \subset S \cap S'$ in $J(C)$ such that for all $f \in T$ we have $x_f = y_f$.

Given a presheaf P, the resulting presheaf P^+ is *separated*, ie. any matching family has at most one amalgamation. Finally, sheafification is the same as applying the plus functor twice. That is, for any presheaf P, $a(P) = (P^+)^+$.

1.3 Elementary topoi

Recall from category theory that a category $\mathcal{C}$ is said to be *cartesian closed* if it has a terminal object 1, any two objects of $\mathcal{C}$ have a product in $\mathcal{C}$ and an exponential in $\mathcal{C}$. Equivalently, any finite family of objects of $\mathcal{C}$ admits a product in $\mathcal{C}$ and the product functor $- \times Y$ has a right adjoint, denoted $-^Y$ for every object Y.
Furthermore, we recall that in a category $\mathcal{C}$ with finite limits, a *subobject classifier* is a monomorphism true : $1 \to \Omega$ such that for every monomorphism $A \to B$ there is a unique $\varphi : B \to \Omega$ such that the following is a pull-back square.

$$\begin{array}{ccc} A & \longrightarrow & 1 \\ \downarrow & & \downarrow{\scriptstyle \text{true}} \\ B & \underset{\varphi}{\longrightarrow} & \Omega \end{array}$$

Definition 1.3.1. *An* elementary topos *is a cartesian closed category $\mathcal{E}$ which has all finite limits and a subobject classifier* true : $1 \to \Omega$.

Example 1.3.2. *An (elementary) topos can be seen as a generalisation of the category* Set. *Henceforth,* Set *will be the prototype example of a topos. Its suboject classifier is the function* true : $\{*\} \to \{0,1\} : * \mapsto 1$. *Therefore, each monomorphism is classified by a characteristic function.*

Remark 1.3.3. *Since every topos $\mathcal{E}$ has a subobject classifier Ω and is a cartesian closed category it also has power objects. That is, in any category $\mathcal{C}$ with finite products, P is a power object of X if there is, for all Y, a natural 1-1 correspondence*

$$\mathcal{E}(Y, P) \cong \mathrm{Sub}_{\mathcal{C}}(Y \times X)$$

Power objects of X are unique up to isomorphism and in a topos $\mathcal{E}$ we let $\mathcal{P}(X) = \Omega^X$ be the power object of X. In that situation we have $\mathcal{P}(X)(E) = \mathrm{Sub}_{\mathcal{E}}(y_E \times X)$. As such, the power object functor $\mathcal{P} : \mathcal{E}^{\mathrm{op}} \to \mathcal{E}$ is right adjoint to the product functor.

1.4 Grothendieck topoi

The following we state without proof.

Theorem 1.4.1. *Given a site $(\mathcal{C}, J)$, the associated sheaf category is a topos, where its subobject classifier Ω is the sheaf defined by*

$$\Omega(C) = \{S \text{ sieve on } C | S \text{ is closed for } J\}$$

Definition 1.4.2. *A Grothendieck topos is a category which is equivalent to the category of sheaves over some site.*

Example 1.4.3. *From a geometrical point of view one could consider the category $\mathcal{O}(X)$ of opens of a topological space X, with inclusions as morphisms, to recover the usual notion of a presheaf as an object of the elementary topos $\widehat{\mathcal{O}(X)}$. To $\widehat{\mathcal{O}(X)}$, is associated a canonical Grothendieck topology J, consisting of those sieves S, such that $\bigcup S$ contains an open subset of X. One may verify that the topos of sheaves associated to this site, $\mathrm{Sh}(\widehat{\mathcal{O}(X)}, J)$, coincides with the usual sheaves on a topological space X.*

1.5 The Heyting algebra of Subobjects

Given a sheaf E in a Grothendieck topos $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$, then any subobject A of E can be uniquely represented by a functor $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$, which we deliberately denote by A too. As such, for all $C \in \mathcal{C}_0$ we have $A(C) \subset E(C)$ and for every $S \in J(C)$ and $e \in E(C)$ we have $e \in A(C)$ whenever $e \cdot f \in A(D)$ for all $f : D \to C$ in S. Finally, for all $C' \to C$ in $\mathcal{C}$, the restriction $A(C) \to A(C')$ agrees with that of E. Hence, each subobject A of E can be viewed as a *subsheaf* of E, ie. a subfunctor of E which is itself a sheaf. Still, the set of subobjects of E in $\mathcal{E}$ is denoted by $\mathrm{Sub}_{\mathcal{E}}(E)$ and it becomes a partial order by defining the relation, for $F, F' \in \mathrm{Sub}_{\mathcal{E}}(E)$

$$F \leq F' \Leftrightarrow \forall C \in \mathcal{C}_0 : FC \subset F'C$$

In this sense, E itself is seen to be the top element of $\mathrm{Sub}_{\mathcal{E}}(E)$.

Definition 1.5.1. *We define the pointwise meet and join as usual: given any set-indexed family $\{F_i \in \mathrm{Sub}_{\mathcal{E}}(E) | i \in I\}$ we define their infimum (or meet) resp. supremum (or* join*) by*

$$\left(\bigwedge_{i\in I} F_i\right)(C) = \bigcap_{i\in I} F_i(C)$$
$$\bigvee_{i\in I} F_i = \bigwedge\{G \in \mathrm{Sub}_{\mathcal{E}}(E) | \forall i \in I : F_i \subset G\}$$

As such, $\mathrm{Sub}_{\mathcal{E}}(E)$ is a complete lattice for every Grothendieck topos $\mathcal{E}$ and every sheaf $E \in \mathcal{E}_0$. Even stronger, we have the following proposition.

Proposition 1.5.2. $\mathrm{Sub}_{\mathcal{E}}(E)$ *is a complete Heyting algebra for every sheaf E in a Grothendieck topos $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$*

For this, it remains to verify the distributivity of the lattice.

Proof. Given a set of subobjects $\{A_i | i \in I\}$ and a subobject B we need to show that

$$B \wedge \bigvee_{i\in I} A_i = \bigvee_{i\in I} B \wedge A_i$$

The inclusion $\supset$ trivially holds, so we concentrate on $\subset$. Take $e \in E(C)$ for any $C \in \mathcal{C}_0$ and suppose that $e \in B(C)$ and $e \in \bigvee_{i\in I} A_i$. Then $S := \{f : D \to C | \exists i \in I : e \cdot f \in A_i(D)\} \in J(C)$. So, for $f \in S$ there exists an $i \in I$ for which $e \cdot f \in (B \wedge A_i)$ and hence $e \in \bigvee_{i\in I} B \wedge A_i$. □

Now, given any morphism of sheaves $\varphi \in \mathcal{E}(E, F)$ we may form the canonical inverse-image functor $\varphi^{-1} : \mathrm{Sub}_{\mathcal{E}}(F) \to \mathrm{Sub}_{\mathcal{E}}(E)$ by pull-back, ie. for G a subsheaf of F and $C \in \mathcal{C}_0$ we have

$$\varphi^{-1}(G)(C) = \{e \in FC | \varphi_C(e) \in GC\}$$

φ^{-1} has both a left and right adjoint, which will be introduced in the next section.

1.6 Some categorical logic

The first part of this chapter is largely based on chapter 4 of [16] and chapter 1 of [15].

Definition 1.6.1. *A first order* language $\mathcal{L}$ *will consist of a set of sorts* $\{S_1, S_2, \cdots\}$*; a denumerable collection of variables* $x_1^{S_i}, x_2^{S_i}, \cdots$ *for every sort* S_i *and a collection of function symbols* ($f : S_{i_1}, \cdots, S_{i_n} \to S$) *and relation symbols* ($R \subset S_{i_1}, \cdots, S_{i_n}$) *which come with an arity*[1] n*. The terms and formulas are defined in the usual inductive way.*

Definition 1.6.2. *A category* $\mathcal{C}$ *is called* regular *if it has all finite limits, regular epimorphisms are stable under pull-back and for all morphisms* f*, whenever*

$$\begin{array}{ccc} Z & \xrightarrow{\pi_0} & X \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle f} \\ X & \xrightarrow[f]{} & Y \end{array}$$

is a pull-back, then the coequalizer of $Z \underset{\pi_1}{\overset{\pi_0}{\rightrightarrows}} X$ *exists.*

One of the key properties of regular categories is that they allow for a unique factorisation of morphisms.

Proposition 1.6.3. *Every morphism* $f : X \to Y$ *in a regular category* $\mathcal{C}$ *can be uniquely factored as* $X \xrightarrow{e} E \xrightarrow{m} Y$*, where* e *is a regular epimorphism and* m *is a monomorphism. That is, given another such factorisation* $f = m'e'$*, there exists an isomorphism* $\sigma : E \to E'$ *such that* $\sigma e = e'$ *and* $m'\sigma = m$*.*

Since every topos is a regular category[2] this proposition applies in particular to topoi.

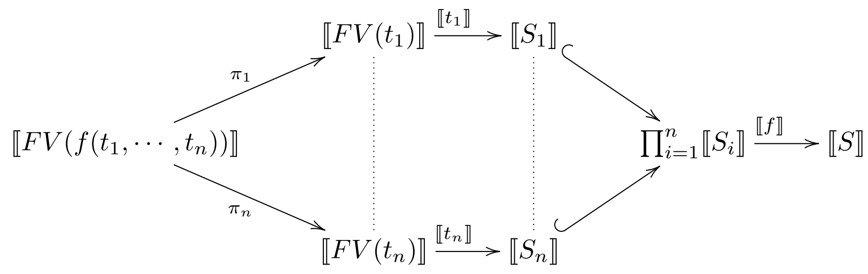
Definition 1.6.4. *An* interpretation $[\![\cdot]\!]$ *of a first order language* $\mathcal{L}$ *in a regular category* $\mathcal{C}$ *is given by choosing an object* $[\![S]\!]$ *for every sort* S*; an arrow* $[\![f]\!] : [\![S_{i_1}]\!] \times \cdots \times [\![S_{i_n}]\!] \to [\![S]\!]$ *for every function symbol* f *and a subobject* $[\![R]\!]$ *of* $[\![S_{i_1}]\!] \times \cdots \times [\![S_{i_n}]\!]$ *for every relation symbol* R*. Given terms* t *and formulas* φ *we denote by* $FV(t)$ *resp.* $FV(\varphi)$ *the collection of their free variables. These are interpreted as the products of interpretations of free variables; one occurrence of each sort for each free variable, for each occurrence of that free variable. We define the interpretation of terms and formulas inductively.*

[1] Any nonnegative integer. In this sense, constants are 0-ary function symbols, while atomic propositions are 0-ary relation symbols.

[2] For a proof of this fact we refer to [11].

terms *A term t of sort S is interpreted as a morphism $[\![t]\!] : [\![FV(t)]\!] \to [\![S]\!]$ and is defined by induction on the complexity of t:*

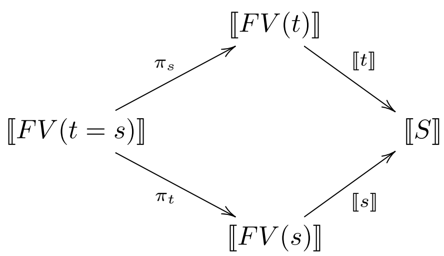
- $[\![x^S]\!] = \mathrm{id}_{[\![S]\!]}$ *for every variable x of sort S.*
- *Given interpretations $[\![t_i]\!] : [\![FV(t_i)]\!] \to [\![S_i]\!]$ for terms $t_1, \cdots, t_n$ and a function symbol $f : S_1, \cdots, S_n \to S$ we define $[\![f(t_1, \cdots, t_n)]\!]$ to be the map*



where the π_i are the obvious projections.

fomulas *A formula φ is interpreted as a subobject $[\![\varphi]\!]$ of $[\![FV(\varphi)]\!]$. Note that this definition is already fixed by the corresponding operations of the Heyting algebra of subobjects, except for quantifiers. In a topos, every inverse image arrow α^{-1} : $\mathrm{Sub}(E) \to \mathrm{Sub}(E')$ of an arrow $\alpha : E' \to E$ has both a left and right adjoint: $\exists_\alpha \dashv \alpha^{-1} \dashv \forall_\alpha$. Interpretation of quantifiers can be done through these adjoints as we will see explicitly in the case of universal quantification.*
Next, we write down the cases for equality, conjunction, implication and existential and universal quantification explicitly. Here it should be noted that the definitions for implication and universal quatification are to be understood in the context of topoi, because the subobject lattice of a topos is a complete Heyting algebra.

- $[\![\top]\!]$ *is the maximal subobject of* $[\![FV(\top)]\!] = [\![\emptyset]\!] = 1$.
- *Given terms t, s of sort S, then $[\![t = s]\!] \to [\![FV(t = s)]\!]$ is the equalizer of*



- *For $R \subset S_{i_1}, \cdots, S_{i_n}$ a relation symbol and $t_1, \cdots, t_n$ terms of sorts $S_{i_1}, \cdots, S_{i_n}$ respectively, we take $[\![R(t_1, \cdots, t_n)]\!] \to [\![FV(R(t_1, \cdots, t_n))]\!]$ to be the subobject defined by pulling back $[\![R]\!]$ along*

$$[\![FV(R(t_1, \cdots, t_n))]\!] \to \prod_{i=j}^{n} [\![FV(t_j)]\!] \overset{\prod_{j=1}^n [\![t_j]\!]}{\longrightarrow} \prod_{j=1}^{n} [\![S_{i_j}]\!]$$

- *Given interpretations of formulas* φ *and* ψ *and projections*

$$[\![FV(\varphi)]\!] \xleftarrow{\pi_\varphi} [\![FV(\varphi \wedge \psi)]\!] \xrightarrow{\pi_\psi} [\![FV(\psi)]\!]$$

  we define $[\![\varphi \wedge \psi]\!] \to [\![FV(\varphi \wedge \psi)]\!]$ *to be the greatest lower bound of the pull-backs of the interpreted formulas (as subobjects of* $[\![FV(\varphi \wedge \psi)]\!]$*) along their corresponding projections. Furthermore, we define* $[\![\varphi \to \psi]\!] \to FV(\varphi \to \psi)$ *to be the Heyting implication between the pull-backs of the interpreted formulas along their corresponding projection, again in* $\mathrm{Sub}([\![FV(\varphi \to \psi)]\!])$.

- *Let* $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ *be the interpretation of the formula* φ *and let* $\pi : [\![FV(\varphi)]\!] \to [\![FV(\exists x\varphi)]\!]$ *be a projection. Depending on whether or not* x *appears freely in* φ *we have different interpretations of* $\exists x\varphi$*. Therefore, consider the (possibly trivial) projection* $\pi' : [\![FV(\varphi) \cup \{x\}]\!] \to [\![FV(\varphi)]\!]$ *and define the subobject* $[\![\exists x\varphi]\!] \to [\![FV(\exists x\varphi)]\!]$ *to be the image of the composition*

$$(\pi')^*([\![\varphi]\!]) \to [\![FV(\varphi) \cup \{x\}]\!] \xrightarrow{\pi'} [\![FV(\varphi)]\!] \xrightarrow{\pi} [\![FV(\exists x\varphi)]\!]$$

- *Finally, given* $[\![\varphi]\!] \to [\![FV(\varphi)]\!]$ *and the corresponding projection* $\pi : [\![FV(\varphi)]\!] \to [\![FV(\forall x\varphi)]\!]$ *we have the projection* $\pi' : FV(\varphi \wedge x = x) \to [\![FV(\varphi)]\!]$ *which combines to define* $[\![\forall x\varphi]\!] := \forall_{\pi\circ\pi'}([\![\varphi]\!])$.

We now say that a formula φ *is true under this interpretation if its interpretation is the maximal subobject.*

Notation: We often write $[\![\cdot]\!]_{\mathcal{E}}$ or $[\![\cdot]\!]_{\mathcal{M}}$, instead of $[\![\cdot]\!]$, to emphasize that the interpretation lands in a topos $\mathcal{E}$ or belongs to an $\mathcal{L}$-structure $\mathcal{M}$.

1.7 Some First-Order Infinitary Geometric Logic

Definition 1.7.1. *For cadinals κ and λ we form the* infinitary *language $\mathcal{L}_{\kappa,\lambda}$ which is just the extension of the empty language in which we allow the formation of formulas of the form $\bigvee_{\varphi\in X}\varphi$ and $\bigwedge_{\varphi\in X}\varphi$ for any set of formulas X with cardinality less then κ and $\left|\bigcup_{\varphi\in X} FV(\varphi)\right| < \lambda$*

It should be noted that if κ is an infinite cardinal, the compactness theorem no longer holds for these languages. There exist however so called "weak compactness theorems" for $\mathcal{L}_{\kappa,\kappa}$ whenever κ is a weakly inaccessible cardinal. For a detailed discussion about this topic we refer to chapter 17 of [8].

Next, we focus our attention to the infinitary language $\mathcal{L}_{\infty,\omega}$, which is just $\mathcal{L}_{\kappa,\lambda}$, where we drop the κ-restriction. Instead of considering the language $\mathcal{L}_{\infty,\omega}$ as a whole, we restrict ourselves to the geometric fragment:

Definition 1.7.2. *A formula φ in a first-order (possibly infinitary) language $\mathcal{L}$ is called* geometric *if it is built up from atomic formulas by means of conjunction, arbitrary disjunction, existential quantification and the truth values $\top$ and $\bot$.*
Then, a geometric $\mathcal{L}$-theory T is a theory in which all axioms are (equivalent to formulas) of the form

$$\forall x_1 \cdots \forall x_n\, (\varphi(x_1, \cdots, x_n) \to \psi(x_1, \cdots, x_n))$$

where φ, ψ are geometric formulas with all free variables contained in $\{x_1, \cdots, x_n\}$.

In order to interpret a general geometric formula in a Grothendieck topos we need, in addition to the machinary of the previous section, also a method to interpret the arbitrary disjunctions in a consistent way. To this end we let $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ be a Grothendieck topos and $\Phi := \{\varphi_i | i \in I\}$ a set of geometric formulas together with interpretations $\{\pi_i : [\![\varphi_i]\!]_{\mathcal{E}} \to [\![FV(\varphi_i)]\!]_{\mathcal{E}}\}_{i\in I}$. We then define the interpretation $[\![\bigvee \Phi]\!]_{\mathcal{E}} \to [\![FV(\bigvee \Phi)]\!]_{\mathcal{E}}$ to be the least upper bound of the pull-backs of the interpreted formulas along their corresponding projections. Recall that this least upper bound is guaranteed to exist by virtue of the fact that $\mathrm{Sub}_{\mathcal{E}}(FV(\bigvee \Phi))$ is a complete Heyting algebra.

Now, let $\mathcal{M} = (M, [\![\cdot]\!]_{\mathcal{E}})$ be an $\mathcal{L}$-structure in $\mathcal{E}$, where M is an object of $\mathcal{E}$ and $[\![\cdot]\!]_{\mathcal{E}}$ is an interpretation of $\mathcal{L}$ in M, then a geometric formula $\forall x(\varphi(x) \to \psi(x))$ is true under this interpretation if $[\![\varphi(x)]\!]_{\mathcal{E}}$ is a subobject of $[\![\psi(x)]\!]_{\mathcal{E}}$.

1.8 Forcing over a site

Given a Grothendieck topos $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ we have the functors

$$\mathcal{C} \xrightarrow{y} \widehat{\mathcal{C}} \underset{a}{\overset{\iota}{\leftrightarrows}} \mathcal{E} \qquad (a \dashv \iota)$$

where ι is the inclusion. By the Yoneda lemma, we have for all sheaves X and objects C of $\mathcal{C}$ that $X(C) \cong \widehat{\mathcal{C}}(y_C, \iota X)$. Secondly, since ι has the left adjoint a we also obtain the equivalence $\widehat{\mathcal{C}}(y_C, \iota X) \cong \mathcal{E}(a \circ y_C, X)$ for all $C \in \mathcal{C}_0$ and $X \in \mathcal{E}_0$. Composing these two yields the equivalence $X(C) \cong \mathcal{E}(a \circ y_C, X)$ for all $C \in \mathcal{C}_0$ and $X \in \mathcal{E}_0$. This allows us to associate the $\alpha \in X(C)$ in a 1-1 fashion with the $\alpha' : y_C \to X$ and the $\alpha'' : a \circ y_C \to X$.

Definition 1.8.1. *Given a Grothendieck topos $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ and a first order formula $\varphi(x)$ with free variable x of sort $X \in \mathcal{E}_0$, then for any $C \in \mathcal{C}_0$ and $\alpha \in X(C)$ we define the* forcing relation $C \Vdash \varphi(\alpha)$ *to mean that $\alpha \in [\![\varphi(x)]\!]_{\mathcal{E}}(C)$.*

By virtue of the previous isomorphisms the forcing relation is equivalent to the following two statements

- $\alpha' : y_C \to X$ factors through $[\![\varphi]\!]_{\mathcal{E}} \to X$.
- $\alpha'' : a \circ y_C \to X$ factors through $[\![\varphi]\!]_{\mathcal{E}} \to X$.

The following theorem states the Kripke-Joyal semantics

Theorem 1.8.2. *Given a Grothendieck topos $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$; let $\varphi(x)$, $\psi(x)$ and $\chi(x, y)$ be formulas in the language of $\mathcal{E}$ with free variables x, y respectively of sorts $X, Y \in \mathcal{E}_0$ and suppose $\alpha \in X(C)$ for some $C \in \mathcal{C}_0$. Then*

1. *$C \Vdash \varphi(\alpha) \wedge \psi(\alpha)$ if and only if both $C \Vdash \varphi(\alpha)$ and $C \Vdash \psi(\alpha)$.*

2. *$C \Vdash \varphi(\alpha) \vee \psi(\alpha)$ if and only if there exists a set-indexed J-cover of C, say $\{f_i : C_i \to C | i \in I\}$, such that for each $i \in I$ either $C_i \Vdash \varphi(\alpha \circ f_i)$ or $C_i \Vdash \psi(\alpha \circ f_i)$.*

3. *$C \Vdash \varphi(\alpha) \to \psi(\alpha)$ if and only if for all morphisms $f : D \to C$ in $\mathcal{C}$, if $D \Vdash \varphi(\alpha \circ f)$, then $D \Vdash \psi(\alpha \circ f)$.*

4. *$C \Vdash \neg\varphi(\alpha)$ if and only if for all morphisms $f : D \to C$ in $\mathcal{C}$, if $D \Vdash \varphi(\alpha \circ f)$, then $\emptyset$ J-covers D.*

5. *$C \Vdash \exists y : \chi(\alpha, y)$ if and only if there exists a set-indexed J-cover of C, say $\{f_i : C_i \to C | i \in I\}$, and elements $\beta_i \in Y(C_i)$ for each $i \in I$, such that $C_i \Vdash \chi(\alpha \circ f_i, \beta_i)$ for all $i \in I$.*

6. *$C \Vdash \forall y : \chi(\alpha, y)$ if and only if for all morphisms $f : D \to C$ in $\mathcal{C}$ and all $\beta \in Y(D)$ we have $D \Vdash \chi(\alpha \circ f, \beta)$.*

1.9 Geometric Morphisms

When given a continuous map between topological spaces $f : X \to Y$ this induces an adjoint pair of functors on the sheaf topoi: $f^* : \mathrm{Sh}(X) \leftrightarrows \mathrm{Sh}(Y) : f_*$ with $f^* \dashv f_*$ where the direct image functor f_* is defined by composition with f^{-1} and the inverse image functor f^* is defined by pulling back the sheaf as an étale bundle along f. As such, f^* is left exact. We'll now generalise this construction to adapt to Grothendieck topoi.

Definition 1.9.1. *Let $\mathcal{E}$ and $\mathcal{F}$ be topoi. A* geometric morphism *$f : \mathcal{F} \to \mathcal{E}$ is a pair of adjoint functors*

$$\mathcal{F} \underset{f_*}{\overset{f^*}{\leftrightarrows}} \mathcal{E} \qquad (\perp)$$

such that f^* is left exact[3]*. f^* is called the* inverse image part *of f while f_* is called the* direct image part *of f.*
We say that f is a surjection[4] *if f^* is* conservative*, ie. f^* is faithful and reflects isomorphisms. f is called an* embedding *whenever f_* is fully faithful. Finally, the geometric morphism f is called* open *when for every object $E \in \mathcal{E}_0$ we have an adjunction $(f_E)_! \dashv f_E^*$ with*

$$(f_E)_! : \mathrm{Sub}_{\mathcal{F}}(f^*E) \leftrightarrows \mathrm{Sub}_{\mathcal{E}}(E) : f_E^*$$

The last definition is of course motivated by the notion of an open map between topological spaces. That is, given an open map $f : X \to Y$, which induces the inverse image functor $f^{-1} : \mathcal{O}(X) \to \mathcal{O}(Y)$, we have $V \subset f^{-1}(U) \Leftrightarrow f(V) \subset U$ forall open $U \subset X$ and $V \subset Y$. Hence, if f is open, the functor $\mathcal{O}(Y) \to \mathcal{O}(X) : V \mapsto f(V)$ has a left adjoint $f_!$.

Definition 1.9.2. *Given a site $(\mathcal{C}, J)$ and a topos $\mathcal{E}$, we say that $F : \mathcal{C} \to \mathcal{E}$ is* continuous *for J if it sends covering sieves to colimit diagrams. The category of continuous left exact functors is denoted* $\mathrm{ConLex}(\mathrm{Sh}(\mathcal{C}, J), \mathcal{E})$.

Definition 1.9.3. *Given any set-indexed family of geometric morphisms $\{f_i : \mathcal{E}_i \to \mathcal{E} | i \in I\}$ with common codomain. Then this set is said to be* jointly surjective *if for any two morphisms $\alpha, \beta : E \to E'$ in $\mathcal{E}$ there exists an $i \in I$, such that $f_i^*(\alpha) \neq f_i^*(\beta)$. In this case, we also say that the inverse image functors f_i^* are* jointly conservative
A geometric morphism $\mathrm{Set} \to \mathcal{E}$ *is called a* point *of the topos $\mathcal{E}$ and $\mathcal{E}$ is said to have* enough points *if the class of points of $\mathcal{E}$ is jointly surjective.*

Remark 1.9.4. *Given two topoi $\mathcal{E}, \mathcal{F}$ one may consider the family of geometric morphisms $\mathcal{E} \to \mathcal{F}$. These form a category, denoted* $\mathrm{Geom}(\mathcal{E}, \mathcal{F})$*, if we define morphisms between geometric morphisms $f, g : \mathcal{E} \rightrightarrows \mathcal{F}$ to be natural transformations $f^* \to g^*$. One may show that we could have chosen natural transformations $g_* \to f_*$ equivalently well, as these correspond bijectively with natural transformations $f^* \to g^*$.*

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[3] That is, f^* preserves finite limits. It already preserves colimits, because it has a right adjoint.

[4] In fact it is sufficient to require that f^* is faithful, because every faithful functor reflects epimorphisms and monomorphisms and, in topoi, an arrow is an isomorphism precisely if it is both a monomorphism and an epimorphism. Hence, every faithful functor between topoi reflects isomorphisms.

1.10 Classifying Topoi

Motivated by topology, the notion of classiying topos will generalise the notion of classifying space (for cohomology) in a special way. Given a space X and an abelian group G we may form the cohomology group $H^n(X, G)$ with respect to G, for any $n \in \mathbb{N}$. We now have the classifying space $K^n(G)$ which has the property that every n-dimensional cohomology class of X arises as the pull-back of a universal cohomology class along a unique map $X \to K^n(G)$ (up to homotopy). In that sense, $K^n(G)$ classifies $H^n(X, G)$.

In generalising to topoi we have the following in mind: suppose we have the notion of a "structure" such that for any topos $\mathcal{E}$ we have a category of such structures in $\mathcal{E}$. Intuitively one would like to say that a topos $\mathcal{B}$ is a classifying topos for these structures if there is an equivalence between the subcategory of $\mathcal{E}$ of such structures and the category of geometric morphisms $\mathcal{E} \to \mathcal{B}$.
Now, let these structures in a topos $\mathcal{E}$ be axiomatised in a theory T such that these structures form the set of T-models in $\mathcal{E}$, denoted $\mathrm{Mod}(T, \mathcal{E})$. This set becomes a category if one defines a morphism between two models, $\mu : \mathcal{M} \to \mathcal{N}$, to be a map $\mu : M \to N$ which preserves the structure of the language. That is, for every function symbol f and relation symbol R, we have $\mu([\![f]\!]_{\mathcal{M}}) = [\![f]\!]_{\mathcal{N}}$ and $\mu([\![R]\!]_{\mathcal{M}}) = [\![R]\!]_{\mathcal{N}}$.

Now, suppose that we have a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ whose inverse image part preserves T-models[5].

Definition 1.10.1. *In the above context, a* classifying topos *for T-models is a Grothendieck topos $\mathcal{B}(T)$ such that for every Grothendieck topos $\mathcal{E}$ we have a natural (in $\mathcal{E}$) equivalence*

$$c_{\mathcal{E}} : \mathrm{Mod}(\mathcal{E}, T) \to \mathrm{Hom}(\mathcal{E}, \mathcal{B}(T))$$

In case $\mathcal{E} = \mathcal{B}(T)$, the model corresponding to the identity on $\mathcal{B}(T)$ under $c_{\mathcal{E}}$ is called the universal T-model *and is denoted $U_T := c^{-1}_{\mathcal{B}(T)}\left(\mathrm{id}_{\mathcal{B}(T)}\right)$.*

Remark 1.10.2. *(Universal property for the classifying topos) The universal T-model has the property that for any topos $\mathcal{E}$ and T-model $\mathcal{M}$ there exists up to isomorphism a unique geometric morphism $f : \mathcal{E} \to \mathcal{B}(T)$ such that $\mathcal{M} \cong f^*(U_T)$.*

[5] We'll see later that this property holds for any geometric morphism, as long as T is a geometric theory.

1.11 Geometric Logic

In this section we'll show that every geometric theory has a classifying topos. To this end we first fix a first-order language $\mathcal{L}$ and an interpretation $[\![\cdot]\!]_{\mathcal{E}}$ of our language in a topos $\mathcal{E}$. Given a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ we have a canonical way of transporting our interpretation in $\mathcal{E}$ to an interpretation $[\![\cdot]\!]_{\mathcal{F}}$ in $\mathcal{F}$ using the inverse image functor f^*, because it is left exact.

Remark 1.11.1. *Given a left exact functor $F : \mathcal{E} \to \mathcal{F}$ we may define $[\![S]\!]_{\mathcal{F}} = F([\![S]\!]_{\mathcal{E}})$ on sorts S. Since F preserves products and monomorphisms it will also preserve interpretations of relation symbols: $[\![R]\!]_{\mathcal{F}} = F([\![R]\!]_{\mathcal{E}})$ and function symbols: $[\![f]\!]_{\mathcal{F}} = F([\![f]\!]_{\mathcal{E}})$. Although this gives rise to a functor between $\mathcal{L}$-interpretations, we don't expect it to restrict to a functor* $\mathrm{Mod}(T, \mathcal{F}) \to \mathrm{Mod}(T, \mathcal{E})$ *for any theory T, because that would imply that $M \vDash \varphi \Rightarrow F(M) \vDash \varphi$ for any $\mathcal{L}$-structure M and T-axiom φ. However, we shall prove that if T is a geometric theory, f^* does send T-models to T-models functorially.*

Theorem 1.11.2. *Given a geometric morphism $f : \mathcal{F} \to \mathcal{E}$ and an interpretation $[\![\cdot]\!]_{\mathcal{E}}$ of $\mathcal{L}$ in $\mathcal{E}$, then for any geometric formula $\varphi(x)$ we have $f^*([\![\varphi(x)]\!]_{\mathcal{E}}) = [\![\varphi(x)]\!]_{\mathcal{F}}$, where $[\![\cdot]\!]_{\mathcal{F}}$ is the induced interpretation as before.*

Proof. We prove this by induction on the construction of the formula φ.

atomic formulas Given a term $t(x_1, \cdots, x_n)$ of sort S and free variables among the x_i of sort S_i we have the interpretation $[\![t]\!]_{\mathcal{E}} : \prod_{i=1}^n [\![S_i]\!]_{\mathcal{E}} \to [\![S]\!]_{\mathcal{E}}$ of t in $\mathcal{E}$. Since f^* preserves products, an induction on the complexity of terms shows the commutativity of

$$\begin{array}{ccc} f^*\left(\prod_{i=1}^n [\![S_i]\!]_{\mathcal{E}}\right) & \xrightarrow{f^*([\![t]\!]_{\mathcal{E}})} & f^*([\![S]\!]_{\mathcal{E}}) \\ \downarrow \cong & & \| \\ \prod_{i=1}^n [\![S_i]\!]_{\mathcal{F}} & \xrightarrow[{[\![t]\!]_{\mathcal{F}}}]{} & [\![S]\!]_{\mathcal{F}} \end{array}$$

Hence, for atomic formulas $t = t'$ or $R(t_1, \cdots, t_k)$ with free variables among the x_i, we have $f^*[\![t = t']\!]_{\mathcal{E}} \cong [\![t = t']\!]_{\mathcal{F}}$ and $f^*[\![R(t_1, \cdots, t_k)]\!]_{\mathcal{E}} \cong [\![R(t_1, \cdots, t_k)]\!]_{\mathcal{F}}$. Finally, f^* preserves the top and bottom elements of the subobject lattice $\mathrm{Sub}_{\mathcal{E}}\left(\prod_{i=1}^n [\![S_i]\!]_{\mathcal{E}}\right)$ because these are finite limits. So, $f^*[\![\top]\!]_{\mathcal{E}} = [\![\top]\!]_{\mathcal{F}}$ and $f^*[\![\bot]\!]_{\mathcal{E}} \cong [\![\bot]\!]_{\mathcal{F}}$. Hence, the theorem holds for all atomic formulas.

conjunction Given formulas φ and ψ for which the theorem holds, then the theorem also holds for $\varphi \wedge \psi$, because f^* preserves the meets of the subobject lattices[6]

arbitrary disjunction Given a set-indexed family of geometric formulas $\{\varphi_i | i \in I\}$ we need to see why $f^*[\![\bigvee_{i \in I} \varphi_i]\!]_{\mathcal{E}} = [\![\bigvee_{i \in I} \varphi_i]\!]_{\mathcal{F}}$. To this end it suffices to prove that the join is a supremum of subobjects, because then it is preserved by f^*. However, it's already the top element of $\mathrm{Sub}_{\mathcal{E}}\left(FV\left(\bigvee_{i \in I} \varphi_i\right)\right)$.

existential quantification Let $\alpha : E \to E'$ be any arrow in $\mathcal{E}$ and let $A \to E$ be any subobject. Then $f^*(\exists_\alpha A) = \exists_{f^*\alpha} f^* A$, since $\exists_\alpha A$ is the image of $A \to E \to E'$ and f^* preserves images. In particular, this holds for any subobject which is the interpretation of a formula φ for which the theorem already holds. So the theorem holds for formulas $\exists x \varphi$ as well.

We conclude that the theorem holds for all geometric formulas, as they're built up out of these three elementary building blocks. □

Corollary 1.11.3. *For a geometric theory T, each geometric morphism $f : \mathcal{F} \to \mathcal{E}$ induces a functor $f^* : \mathrm{Mod}(T, \mathcal{E}) \to \mathrm{Mod}(T, \mathcal{F})$.*

Proof. Let $\mathcal{M} = (M, [\![\cdot]\!]_{\mathcal{E}})$ be an $\mathcal{L}$-structure in $\mathcal{E}$, such that all axioms of T are valid in $\mathcal{M}$. That is, if $T \vdash \alpha$, then $[\![\alpha]\!]_{\mathcal{M}}$ is a maximal subobject. To prove the corollary we need to show that the axioms of T are also valid in $f^*\mathcal{M}$. Since T is geometric, α is of the form $\forall x\,(\varphi(x) \to \psi(x))$. Now, assume that α is true in $\mathcal{M}$, ie. $[\![\varphi(x)]\!]_{\mathcal{M}} \leq [\![\psi(x)]\!]_{\mathcal{M}}$. Since, f^* is left exact we have $f^*[\![\varphi(x)]\!]_{\mathcal{M}} \leq f^*[\![\psi(x)]\!]_{\mathcal{M}}$. Hence, by the previous theorem $[\![\varphi(x)]\!]_{f^*\mathcal{M}} \leq [\![\psi(x)]\!]_{f^*\mathcal{M}}$, ie. α holds in $f^*\mathcal{M}$ and so, $f^*\mathcal{M}$ is a model of T in $\mathcal{F}$. □

[6] Meets are preserved by left exactness of f^*

1.12 Classifying Topoi for Geometric Theories

In this section we'll concentrate on the problem of finding a classifying topos for a (possibly infinitary) geometric theory T. Remarkably enough, this problem is completely solvable and below we will discuss how to explicitly construct the solution $\mathcal{B}(T)$. We start with the construction of a syntactic category $B(T)$, which comes equipped with a canonical Grothendieck topology $J(T)$. Then our classifying topos $\mathcal{B}(T)$ will be the Grothendieck topos of sheaves over the syntactic site.

The construction of the syntactic site will be a generalisation of the more intuitive and motivating idea of a category of definable objects. That is, given a Grothendieck topos $\mathcal{E}$ and a T-model $\mathcal{M}$ in $\mathcal{E}$, then an object A of $\mathcal{E}$ is called *definable in* $\mathcal{M}$ if there exists a geometric formula φ such that $A \cong [\![\varphi(x)]\!]_{\mathcal{E}}$. Now, let the free variables of φ (defining A) resp. ψ (defining B) be listed among $(x_1, \cdots, x_n)$ resp. $(y_1, \cdots, y_m)$ where each x_i is of sort S_i resp. each y_j is of sort S'_j and denote $\mathcal{S}_{\mathcal{M}} = \prod_{i=1}^{n} [\![S_i]\!]_{\mathcal{E}}$ resp. $\mathcal{S}'_{\mathcal{M}} = \prod_{j=1}^{m} [\![S'_j]\!]_{\mathcal{E}}$. Then an arrow between definable objects $(A, \mathcal{S}_{\mathcal{M}}) \to (B, \mathcal{S}'_{\mathcal{M}})$ is an arrow $A \to B$ in $\mathcal{E}$ whose graph (as a subobject of $\mathcal{S}_{\mathcal{M}} \times \mathcal{S}'_{\mathcal{M}}$) is definable. This forms the category $\mathrm{Def}(\mathcal{M})$ of definable objects in M. This category can be shown to inherit all finite limits from $\mathcal{E}$. It also inherits a basis for a Grothendieck topology, making all finite epimorphic families covering. More explicitly, a finite family of definable arrows $\{s_i : (A_i, \mathcal{S}^i_{\mathcal{M}}) \to (B, \mathcal{S}'_{\mathcal{M}})\}_{i=1}^{n}$ covers $(B, \mathcal{S}'_{\mathcal{M}})$ if it becomes an epimorphic family under the forgetful functor $\mathrm{Def}(\mathcal{M}) \to \mathcal{E}$, ie. if $\coprod_{i=1}^{n} A_i \to B$ is epimorphic in $\mathcal{E}$.

Somehow, we'd like to avoid choosing a particular model for our theory and instead treat all model of T in $\mathcal{E}$ simultaneously. It's precisely this generalisation which leads to the notion of a syntactic site.

Definition 1.12.1. *Given a geometric theory T, the* syntactic category $B(T)$ *of T is defined as follows:*

Objects *An object of $B(T)$ is a list of sorts $\mathfrak{S} = (S_1, \cdots, S_n)$, together with an equivalence class of geometric formulas $[\varphi(x_1, \cdots, x_n)]$ whose free variables are among the x_i, each of sort S_i. $\varphi(x_1, \cdots, x_n)$ is equivalent to $\psi(y_1, \cdots, y_n)$ (and hence defines the same object of $B(T)$) if the free variables of φ resp. ψ are among $x_1, \cdots, x_n$ resp. $y_1, \cdots, y_n$ from the same list of sorts $\mathfrak{S}$ and when for every T-model $\mathcal{M}$ we have $[\![\varphi(x)]\!]_{\mathcal{M}} \cong [\![\psi(y)]\!]_{\mathcal{M}}$.*

Arrows *An arrow between two objects $([\varphi(x)], \mathfrak{S}) \to ([\psi(y)], \mathfrak{S}')$ is an equivalence class of geometric formulas*[7] *$[\sigma(x, y)]$ whose interpretations as subobjects, are graphs of arrows $[\![\varphi]\!]_{\mathcal{M}} \to [\![\psi]\!]_{\mathcal{M}}$ for every T-model $\mathcal{M}$. Two such σ, σ' are equivalent if they define the graph of the same arrow in every model of T.*
More explicitly, the functionality of σ can be expressed as

$$
\begin{aligned}
&T \vdash \forall x \forall y (\sigma(x, y) \to \varphi(x) \wedge \psi(y)) \\
&T \vdash \forall x (\varphi(x) \to \exists y \sigma(x, y)) \\
&T \vdash \forall x \forall y \forall z (\sigma(x, y) \wedge \sigma(x, z) \to y = z)
\end{aligned}
$$

To see that this forms a category we observe that the identity arrow for an object $([\varphi], \mathfrak{S})$ is represented by the formula

$$\varphi(x) \wedge \varphi(x') \wedge \bigwedge_{i=1}^{n} x_i = x'_i$$

and that composition of two arrows $[\sigma] : ([\varphi(x)], \mathfrak{S}) \to ([\psi(y)], \mathfrak{S}')$ and $[\tau] : ([\psi(y')], \mathfrak{S}') \to ([\chi(z)], \mathfrak{S}'')$ is an arrow $[\tau \circ \sigma] : (\varphi(x), \mathfrak{S}) \to (\chi(z), \mathfrak{S}'')$ represented by

$$\exists y (\sigma(x, y) \wedge \tau(y, z))$$

where we note that the representant in the codomain of $[\sigma]$ can be chosen equal to the representant in the domain of $[\tau]$, because $\psi(y)$ and $\psi(y')$ define the same subobject of $\mathcal{S}'_{\mathcal{M}}$ for any T-model $\mathcal{M}$.

Remark 1.12.2. *Note that $B(T)$ might not be a small category. Semantically we see that it refers to all models in all topoi, while syntactically we observe that the class of (infinitary) formulas is not a set. From a foundational point of view there are two ways out of this: either we restrict the class of topoi, or we use a larger universe for our set theory. Although the first method can be applied without harming the theory we prefer the second method, because it gives us as much freedom as possible.*

The syntactic category defines, as a generalisation of the category of definable objects, a family of functors $\{F_{\mathcal{M}} : B(T) \to \mathrm{Def}(M) | \mathcal{M} \in \mathrm{Mod}(T, \mathcal{E})_0\}$ by $F_{\mathcal{M}}([\varphi(x)], \mathfrak{S}) = [\![\varphi(x)]\!]_{\mathcal{M}}$ on objects and $F_{\mathcal{M}}([\sigma] : ([\varphi(x)], \mathfrak{S}) \to ([\psi(y)], \mathfrak{S}'))$ is the arrow $[\![\varphi(x)]\!]_{\mathcal{M}} \to [\![\psi(y)]\!]_{\mathcal{M}}$ which has graph $[\![\sigma(x, y)]\!]_{\mathcal{M}}$.

[7] Each $\sigma(x, y)$ defines a subobject (the graph) $[\![\sigma(x, y)]\!]_{\mathcal{M}} \leq \mathcal{S}_{\mathcal{M}} \times \mathcal{S}'_{\mathcal{M}}$ for every T-model $\mathcal{M}$, as before.

We now define a basis for a Grothendieck topology $J(T)$ on $B(T)$.

Definition 1.12.3. *A finite family* $\{s_i : A_i \to B\}_{i=1}^n$ *of arrows in* $B(T)$ *is defined to cover* B *if for every* T*-model* $\mathcal{M}$ *this family is sent to a cover of* $F_{\mathcal{M}}(B)$ *by* $F_{\mathcal{M}}$*. The* syntactic site *is now defined to be the site* $(B(T), J(T))$*.*

Theorem 1.12.4. *Let* T *be a geometric* $\mathcal{L}$*-theory, then* $\mathcal{B}(T) := \mathrm{Sh}(B(T), J(T))$ *is the classifying topos for models of* T*.*

Remark 1.12.5. *Although we will not prove this theorem here, it's worthwile to observe how the equivalence* $\mathrm{Hom}(\mathcal{E}, \mathcal{B}(T)) \cong \mathrm{Mod}(T, \mathcal{E})$ *(natural in* $\mathcal{E}$*) comes about for every co-complete topos* $\mathcal{E}$*. By what we've seen about the syntactic category it's not difficult to see that every geometric morphism* $\mathcal{E} \to \mathcal{B}(T)$ *arises uniquely from a left-exact continuous functor* $\mathcal{B}(T) \to \mathcal{E}$*, such that we have an equivalence* $\mathrm{ConLex}(\mathcal{B}(T), \mathcal{E}) \cong \mathrm{Hom}(\mathcal{E}, \mathcal{B}(T))$*.*

As we anticipated, the first equivalence will be a generalisation of the notion of classifying space. That is, given a morphism $\mathcal{E} \to \mathcal{B}(T)$*, it will arise uniquely from a continuous left exact morphism* $\mathcal{B}(T) \to \mathcal{E}$ *which can be factored uniquely over* $\mathrm{Def}(\mathcal{M})$ *for a unique* T*-model* $\mathcal{M}$*. The equivalence in the theorem is precisely this correspondence between* T*-models* $\mathcal{M}$ *and geometric morphisms* $\mathcal{E} \to \mathcal{B}(T)$*.*

Remark 1.12.6. *In view of Remark 1.12.2 it is not immediately clear why* $\mathfrak{B}(T)$ *could be regarded as the sheaf category over a small site. To this end, we first observe that if* $\mathfrak{B}(T) = \mathrm{Sh}(\mathcal{C}, J)$ *is the classifying topos for a geometric theory* T *and* $\mathcal{E}$ *is a subtopos of* $\mathfrak{B}(T)$*, then we obtain a finer Grothendieck topology* $J \subset J_{\mathcal{E}}$ *on* $\mathcal{C}$*, corresponding to the models of* T *in* $\mathcal{E}$*. However, this corresepondence between subtopoi of* $\mathfrak{B}(T)$ *and finer Grothendieck topologies* $J_{\mathcal{E}}$ *is 1-1 by virtue of the previous remark. Hence, subtopoi of* $\mathfrak{B}(T)$ *can be regarded as sets, such that* $(\mathcal{C}, J)$ *can be assumed to be a small site.*

Chapter 2

Characterization of Geometric Logic

2.1 The Finitary Characterization Problem

Given a language $\mathcal{L}$ and a topos $\mathcal{E}$, we denote the $\mathcal{L}$-structures in $\mathcal{E}$ by $\mathcal{L}(\mathcal{E})$.

Definition 2.1.1. Coherent sequents *are sequents of the form $\varphi \vdash \psi$ where φ and ψ are coherent formulas, ie. formulas in coherent logic: the fragment of finitary first-order logic using only connectives and quantifiers. In full first order logic, coherent sequents are equivalent to geometric formulas $\forall x : \varphi \to \psi$. A theory T is called a* coherent theory *if it consists of coherent sequents.*

Suppose that $\mathcal{L}$ is a finitary first-order language. Then we may wonder whether an $\mathcal{L}$-theory T is equivalent to a coherent one. Having a precise answer to this question yields a characterization of coherent logic. I. Moerdijk was the first to tackle this problem in a letter to M. Makkai in 1989.

Theorem 2.1.2. *A finitary first-order $\mathcal{L}$-theory T can be axiomatized by coherent sequents over $\mathcal{L}$ precisely if*

1. *All inverse image parts of geometric morphisms between Grothendieck topoi preserve T-models, ie. for all geometric morphisms $f : \mathcal{F} \to \mathcal{E}$, if $\mathcal{M} \in \mathcal{L}(\mathcal{E})$ is a model of T, then $f^*\mathcal{M} \in \mathcal{L}(\mathcal{F})$ is a model of T.*

2. *All inverse image parts of surjective geometric morphisms between Grothendieck topoi reflect T-models, ie. for all surjective geometric morphisms $f : \mathcal{F} \to \mathcal{E}$ and all $\mathcal{M} \in \mathcal{L}(\mathcal{E})$, if $f^*\mathcal{M} \in \mathcal{L}(\mathcal{F})$ is a model of T, then $\mathcal{M}$ is a model of T.*

Moerdijk asked in his letter whether it was possible to generalise his result to infinitary logic. In 2009 Olivia Caramello published a paper [3] in which she adresses a more general problem of whether a class of structures is the class of models of a geometric theory inside a Grothendieck topos. In specializing to infinitary first-order theories that are geometric, in terms of their models in Grothendieck topoi, the question posed by Moerdijk, back in 1989, could now be answered by a decisive yes.

The orginal proof by Moerdijk uses the compactness theorem extensively. However, in the infinitary generalisation one cannot rely on this result anymore and so the proof fails. Nevertheless, for the infinitary case there do exist so called ”weak compactness theorems”, but they are beyond the scope of this paper. Detailed information about them can be found in chapter 17 of [8].

The remaining part of this chapter is devoted to the result by Olivia Caramello: the solution to the more general problem posed by Moerdijk.

2.2 Local Operators and Subtopoi

It turns out there are different but equivalent ways to express the data of a Grothendieck topology. One of those is the local operator or Lawverre-Tierney topology.

Definition 2.2.1. *A* local operator *on a topos $\mathcal{E}$ is a morphism $j : \Omega \to \Omega$ (where Ω is the subobject classfier of $\mathcal{E}$) which satisfies the following properties on the (possibly infinitary) logic of $\mathcal{E}$:*

1. $j(\top) = \top$
2. *For all formulas φ, $j \circ j(\varphi) = \varphi$.*
3. *For all formulas φ and ψ we have $j(\varphi \wedge \psi) = j(\varphi) \wedge j(\psi)$.*

Example 2.2.2. *In case $\mathcal{E} = \widehat{\mathcal{C}}$ for some category $\mathcal{C}$, the definition of a local operator $j : \Omega \to \Omega$ on $\mathcal{E}$ takes the usual form:*

1. *For every sieve S on $C \in \mathcal{C}_0$ we have $S \subset j_C(S)$.*
2. *For every sieve S on $C \in \mathcal{C}_0$ we have $j_C(j_C(S)) = j_C(S)$.*
3. *Given any two sieves S, S' on $C \in \mathcal{C}_0$ we have $j_C(S \cap S') = j_C(S) \cap j_C(S')$.*

In particular, a local operator can be thought of as a representant of a subobject. Hence, we have a canonical notion of comparability ($\leq$) between local operators on the same topos.

Proposition 2.2.3. *The following notions are equivalent. That is to say, that one determines the other uniquely.*

- *A Grothendieck topology J on a category $\mathcal{C}$.*
- *A local operator j on $\widehat{\mathcal{C}}$.*

Although we do not prove the proposition it might be helpful to see how j can be expressed in terms of J explicitly. Given any sieve S on $C \in \mathcal{C}_0$ we define

$$j_C(S) := \{g : C' \to C | g^*(S) \in J(C')\}$$

Definition 2.2.4. *Given a topos $\mathcal{E}$, a* subtopos *of $\mathcal{E}$ is a topos $\mathcal{E}'$ together with a geometric embedding $\mathcal{E}' \hookrightarrow \mathcal{E}$.*

Theorem 2.2.5. *Every local operator j on a topos $\mathcal{E}$ determines a unique subtopos $\mathcal{E}_j \hookrightarrow \mathcal{E}$. Conversely, every geometric embedding (inclusion) $\iota : \mathcal{E}' \hookrightarrow \mathcal{E}$ of topoi determines a unique local operator j on $\mathcal{E}$, such that $\mathcal{E}_j$ is equivalent to $\mathcal{E}'$.*

This theorem expresses in particular the bijective correspondence between subtopoi and Grothendieck topologies, which will be needed in the next section.

2.3 Preliminaries

Remark 2.3.1. *Two arrows, in a category with equalizers, are equal precisely if their equalizer is an isomorphism. Hence, an equalizer preserving functor between two categories with equalizers is conservative, precisely when it reflects isomorphisms. So, a family of geometric morphisms with common codomain is jointly surjective if and only if all the associated inverse image functors jointly reflect isomorphisms. That is, a set of geometric morphisms* $\{f_i : \mathcal{E}_i \to \mathcal{E} | i \in I\}$ *is jointly surjective precisely if for all morphisms* $g : E \to E'$ *in* $\mathcal{E}$ *we have that if* $f_i^*(g)$ *is an isomorphism for all* $i \in I$*, then so is* g *itself.*

Remark 2.3.2. *In [4] it is proved that the class of subtopoi of a Grothendieck topos* $\mathcal{E}$ *is a set. Hence, given an indexed family of subtopoi* $\{\mathcal{E}_i \hookrightarrow \mathcal{E} | i \in I\}$*, we can form the smallest subtopos* $\bigcup_{i \in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$ *containing all the* $\mathcal{E}_i$*.*[1]

Lemma 2.3.3. *Given an indexed family of geometric morphisms* $\{f_i : \mathcal{E}_i \to \mathcal{E} | i \in I\}$*, then it's jointly surjective, precisely if the surjection-embedding factorisations* $f_i : \mathcal{E}_i \overset{e_i}{\twoheadrightarrow} \mathcal{E}'_i \overset{m_i}{\hookrightarrow} \mathcal{E}$ *combine to yield an isomorphism* $\coprod_{i \in I} m_i : \bigcup_{i \in I} \mathcal{E}'_i \to \mathcal{E}$*.*

Proof. The joint surjectivity of the indexed family is equivalent to the joint surjectivity of the m_i. Secondly, to each m_i is associated a local operator j_i on $\mathcal{E}$ as seen in the previous section. Now, let $a_{j_i} : \mathcal{E} \to \mathcal{E}'_i$ be the sheaf functor associated to j_i.

"$\Rightarrow$" Let be given any local operator j on $\mathcal{E}$ which is smaller (as a subobject) then all the j_i, ie. for all $i \in I$ we have $j \leq j_i$. Then "$a_j(f)$ is an isomorphism $\Rightarrow a_{j_i}(f)$ is an isomorphism" for all morphisms f in $\mathcal{E}$ and $i \in I$. However, the latter implies that f is an isomorphism by the joint surjectivity of the f_i. Hence j must be the smallest local operator on $\mathcal{E}$ such that $\coprod_{i \in I} m_i$ is an isomorphism.

"$\Leftarrow$" Conversely, to see that the f_i are jointly surjective we need to show that any morphism f in $\mathcal{E}$ is an isomorphism whenever $a_{j_i}(f)$ is an isomorphism for all $i \in I$. To this end we consider the smallest local operator[2] k_f on $\mathcal{E}$ such that $a_{k_f}(f)$ is an isomorphism. Here, a_{k_f} is the sheaf functor associated to k_f. As subobjects, $k_f \leq j_i$ for all $i \in I$ and hence k_f is the smallest local operator. This implies that f must be an isomorphism, since $\coprod_{i \in I} m_i$ is an isomorphism by assumption.

□

From now on we assume our topoi to be Grothendieck, such that Remark 2.3.2 applies and we immediately obtain the following corollary.

Corollary 2.3.4. *Let be given a set-indexed family of geometric morphisms* $\{f_i : \mathcal{E}_i \to \mathcal{E} | i \in I\}$ *between Grothendieck topoi. By the previous lemma the coproduct Grothendieck topos* $\coprod_{i \in I} \mathcal{E}_i$ *exist, so we have a coproduct map* $f : \coprod_{i \in I} \mathcal{E}_i \to \mathcal{E}$*, whose surjection-embedding factorisation is* $f : \coprod_{i \in I} \mathcal{E}_i \twoheadrightarrow \bigcup_{i \in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$*. Moreover, the family is jointly surjective, precisely if* f *is surjective.*

[1] Even if $\mathcal{E}$ is not Grothendieck, it is still common to use $\bigcup_{i \in I} \mathcal{E}_i \hookrightarrow \mathcal{E}$ to denote the smallest subtopos of $\mathcal{E}$ containing all the $\mathcal{E}_i$, provided it exists.

[2] This exists by virtue of Example A 4.5.14 (c) in [9].

2.4 The Duality Theorem

To prepare for the characterization theorem in the next section, we now focus our attention to a duality between subtopoi of the classifying topos of a geometric $\mathcal{L}$-theory T and the closed geometric $\mathcal{L}$-theories which are "quotients of T". This is needed to ensure the preservation of the universal model under an associated sheaf functor. For an elaboration on the fact that every subtopos of a Grothendieck topos is a Grothendieck topos, we refer to chapter 3 of [4].

Definition 2.4.1. *Given a geometric $\mathcal{L}$-theory T and geometric $\mathcal{L}$-sequents σ, σ', then σ, σ' are said to be* T-equivalent *if $T \cup \{\sigma\} \vdash \sigma'$ and $T \cup \{\sigma'\} \vdash \sigma$. A* quotient *or an* extension *of T is a geometric $\mathcal{L}$-theory T', such that for all $\varphi \in T : T' \vdash \varphi$ and the set of such quotients of T is denoted by $Q(T)$. We say say that two geometric $\mathcal{L}$-theories T, T' are* syntactically equivalent *(and write $T \equiv_s T'$) if for any geometric $\mathcal{L}$-sequent σ we have $T \vdash \sigma \Leftrightarrow T' \vdash \sigma$. We say that T, T' are* Morita-equivalent *(and write $T \equiv_M T'$) if they have equivalent classifying topoi. Finally, T is called* closed *if every geometrical sequent provable from T is already an axiom of T.*

Remark 2.4.2. *Syntactic equivalence is clearly an equivalence relation on the geometric $\mathcal{L}$-theories and each equivalence class can be canonically represented by the theory of all geometric $\mathcal{L}$-sequents provable in any (and hence all) theories of the equivalence class. This canonical representative is of course the unique theory of the equivalence class which is closed. This motivates the following definition.*

Definition 2.4.3. *The* closure *of a geometric $\mathcal{L}$-theory T is the unique (closed) canonical representative of the syntactic equivalence class of T, denoted by $\overline{T}$.*

Definition 2.4.4. *Given a small category $\mathcal{C}$, a cocomplete category $\mathcal{E}$ (e.g. a topos) and a functor $F : \mathcal{C} \to \mathcal{E}$, then the Hom-functor $R : \mathcal{E} \to \widehat{C}$ with $R(E)(C) = \mathcal{E}(FC, E)$ has a left adjoint which we denote by $- \otimes_{\mathcal{C}} F$ (the* tensor product functor*). The functor F is called* flat *if the corresponding tensor product functor $- \otimes_{\mathcal{C}} F : \widehat{C} \to \mathcal{E}$ is left exact.*

Given a site $(\mathcal{C}, J)$, then the J-continuous flat functors $\mathcal{C} \to \mathcal{E}$ form a category which we denote by $\mathrm{Flat}_J(\mathcal{C}, \mathcal{E})$.

We need the following classical theorem as a lemma for the proof of the duality theorem.

Lemma 2.4.5. *(Diaconescu) Any presheaf topos is the classifying topos for flat functors on its site.*

Detailed proofs of a more general version of this theorem can be found at B 3.2.7 of [9] or in the original work by Diaconescu: [7].

Theorem 2.4.6. *(The Duality Theorem) Given a geometric $\mathcal{L}$-theory T, we may send its quotients T' to their classifying topoi $\mathcal{B}(T')$. This induces a bijection between $Q(T)/\equiv_s$ and the subtopoi of $\mathcal{B}(T)$.*

Instead of proving this theorem in full detail we shall give a sketch of it and indicate how the predicted bijection comes about.

"proof". By the soundness theorem for geometric logic, syntactically equivalent geometric theories have the same models in all Grothendieck topoi and are hence Morita equivalent. Assume $\mathcal{B}(T) = \mathrm{Sh}(B(T), J(T))$, where $(B(T), J(T))$ is the syntactic site of T. Given any Grothendieck topos $\mathcal{E}$ and any $\mathcal{M} \in \mathrm{Mod}(T, \mathcal{E})$, we have a flat functor $F_{\mathcal{M}} : B(T) \to \mathcal{E}$ assigning to every $\mathcal{L}$-formula $\varphi(x)$ the domain of $[\![\varphi(x)]\!]_{\mathcal{M}}$. This yields an equivalence $\mathrm{Mod}(T, \mathcal{E}) \cong \mathrm{Flat}_{J(T)}(B(T), \mathcal{E})$. It is shown in volume 2 part D of [9] that $B(T)$ is equivalent to a small category and hence all we now about small Grothendieck sites applies here.

Given a quotient T' of T, we construct $\mathcal{B}(T')$ like this: T' can be obtained from T by adding a set ΔT of axioms of the form $\varphi \vdash \psi$. One may show that given any Grothendieck topos $\mathcal{E}$ and $\mathcal{M} \in \mathrm{Mod}(T, \mathcal{E})$ the morphism associated to any $\varphi \vdash \psi$ in $B(T)$ is sent to an epimorphism by $F_{\mathcal{M}}$ precisely if $\varphi \vdash \psi$ holds in $\mathcal{M}$. Hence, the models of T' are classified by those $J(T)$-continuous flat functors which send the axioms of ΔT to epimorphisms.

If we let $J(T, T')$ be the smallest Grothendieck topology on $B(T)$ which makes all sieves of $J(T)$ and those corresponding to the axioms of ΔT covering then, by Diaconescu's theorem, $\mathcal{B}(T') = \mathrm{Sh}(B(T), J(T, T'))$. In particular, the inclusion $J(T) \subset J(T, T')$ yields the subtopos $\mathcal{B}(T') \hookrightarrow \mathcal{B}(T)$.

To see that this is well-defined on syntactic equivalence classes one needs to show that the above construction of the subtopos is independent of the choice of axioms in ΔT. We will omit this rather technical detail and concentrate on the reverse construction: given a subtopos $\mathcal{E}$ of $\mathcal{B}(T)$, then $\mathcal{E} = \mathrm{Sh}(B(T), J)$ for a unique J containing $J(T)$. Now, define the $\mathcal{L}$-theory T_J to consist of the axioms $\psi \vdash \exists x : \sigma$ where $[\sigma] \in B(T)_1$ is any monomorphism generating a sieve in J. One may now show that the equivalence $\mathrm{Mod}(T, \mathcal{E}) \cong \mathrm{Flat}_{J(T)}(B(T), \mathcal{E})$ restricts to an equivalence $\mathrm{Mod}(T_J, \mathcal{E}) \cong \mathrm{Flat}_J(B(T), \mathcal{E})$ and hence that $\mathcal{E} = \mathrm{Sh}(B(T), J)$ classifies T_J.

The last part of the proof amounts to showing that $J \mapsto T_J$ and $T' \mapsto J(T, T')$ are bijections inverse to each other. To this end we first observe that $J(T, T_J) = J$ by definition of the assignment $T' \mapsto J(T, T')$. Conversely, we need to show that every quotient T' of T is syntactically equivalent to $T_{J(T,T')}$. For this, one shows that the $\mathcal{L}$-structure $U(T, T') := \mathrm{image}\left(a_{J(T,T')} \circ y^T\right)$ is a universal model for both T' as well as $T_{J(T,T')}$. Here, y^T is of course the Yoneda embedding $B(T) \to \widehat{B(T)}$. So $U(T, T')$ is a conservative model of both T' and $T_{J(T,T')}$, yielding their syntactic equivalence. $\square$

2.5 The Characterization Theorem

Remark 2.5.1. *Given the (trivially geometric) empty theory $\mathbb{O}_{\mathcal{L}}$ in the language $\mathcal{L}$, consider its classifying topos $\mathcal{B}(\mathbb{O}_{\mathcal{L}})$ and observe that, since $\mathbb{O}_{\mathcal{L}}$ is empty, its models in any $\mathcal{E}$ are precisely the $\mathcal{L}$-structures in $\mathcal{E}$, $\mathcal{L}(\mathcal{E})$. Hence we have a correspondence between $\mathcal{M} \in \mathcal{L}(\mathcal{E})$ and geometric morphisms $f_{\mathcal{M}} : \mathcal{E} \to \mathcal{B}(\mathbb{O}_{\mathcal{L}})$.*

Theorem 2.5.2. *Given a class $\mathcal{S}$ of $\mathcal{L}$-structures in Grothendieck topoi which is closed under isomorphisms of $\mathcal{L}$-structures, then $\mathcal{S}$ is the class[3] of all models in all possible Grothendieck topoi of some geometric $\mathcal{L}$-theory $T_{\mathcal{S}}$ if and only if ("$\Leftrightarrow$") the following two conditions are satisfied:*

1. *Given any $\mathcal{L}$-structure $\mathcal{M} \in \mathcal{S}$ in $\mathcal{E}$ and any geometric morphism $f : \mathcal{F} \to \mathcal{E}$, then $f^*(M) \in \mathcal{S}$.*
2. *Given any set-indexed jointly surjective family of geometric morphisms $\{f_i : \mathcal{E}_i \to \mathcal{E} | i \in I\}$ and any $\mathcal{M} \in \mathcal{L}(\mathcal{E})$; if for all $i \in I : f_i^*(\mathcal{M}) \in \mathcal{S}$, then $\mathcal{M} \in \mathcal{S}$.*

Proof.

"$\Rightarrow$"

1. Given $f : \mathcal{F} \to \mathcal{E}$ and $\mathcal{M} \in \mathcal{L}(\mathcal{E})$ in $\mathcal{S}$, then $f^*(\mathcal{M})$ is also a model of $T_{\mathcal{S}}$. Since $\mathcal{S}$ is the class of all models in Grothendieck topoi of $T_{\mathcal{S}}$ we must have $f^*(\mathcal{M}) \in \mathcal{S}$ as well.
2. Given a set-indexed jointly surjective family of geometric morphisms $\{f_i : \mathcal{E}_i \to \mathcal{E} | i \in I\}$ we have $f : \coprod_{i \in I} \mathcal{E}_i \to \mathcal{E}$ surjective as indicated in Corollary 2.3.4. Let $\mathcal{M} \in \mathcal{L}(\mathcal{E})$ be given with $f_i^*(\mathcal{M}) \in \mathcal{S}$ for all $i \in I$, such that $f_i^*(\mathcal{M}) \in \mathrm{Mod}(T_{\mathcal{S}}, \mathcal{E}_i)$ for all $i \in I$, then $f^*(\mathcal{M})$ is a model of $T_{\mathcal{S}}$ in $\coprod_{i \in I} \mathcal{E}_i$. To see that $\mathcal{M}$ is also a model of $T_{\mathcal{S}}$ we assume $T_{\mathcal{S}} \vDash \varphi$ implying $f^*(\mathcal{M}) \vDash \varphi$. Hence, $[\![\varphi]\!]$ is the maximal subobject and so is reflected by f^* to yield $\mathcal{M} \vDash \varphi$. We conclude that $\mathcal{M}$ is a model of $T_{\mathcal{S}}$ and so $\mathcal{M} \in \mathcal{S}$.

[3] Given a class $\mathcal{S} = \{x | \varphi(x, p_1, \cdots, p_n)\}$ we deliberately write $y \in \mathcal{S}$ for $\varphi(y, p_1, \cdots, p_n)$.

"$\Leftarrow$" In view of our last remark, the $f_{\mathcal{M}}$ with $\mathcal{M} \in \mathcal{S}$ correspond to the set $\mathcal{L}(\mathcal{E})$, so the associated class of geometric morphisms is a set: $\{f_{\mathcal{M}} | \mathcal{M} \in S\}$. Now, given $\mathcal{M} \in \mathcal{S}$ in $\mathcal{E}$, we have the epi-mono factorisation of the associated geometric morphism $f_{\mathcal{M}} : \mathcal{E}_{\mathcal{M}} \overset{e_{\mathcal{M}}}{\twoheadrightarrow} \mathcal{E}'_{\mathcal{M}} \overset{m_{\mathcal{M}}}{\hookrightarrow} \mathcal{B}(\mathbb{O}_{\mathcal{L}})$. Using Remark 2.3.2 and the fact that $\mathcal{B}(\mathbb{O}_{\mathcal{L}})$ is Grothendieck, we obtain a subtopos[4] $\mathcal{E}_{\mathcal{S}} := \coprod_{\mathcal{M} \in S} \mathcal{E}'_{\mathcal{M}} \hookrightarrow \mathcal{B}(\mathbb{O}_{\mathcal{L}})$ represented by $m := \coprod_{\mathcal{M} \in S} m_{\mathcal{M}}$; we denote the inclusions by $\iota_{\mathcal{M}} : \mathcal{E}'_{\mathcal{M}} \to \mathcal{E}_{\mathcal{S}}$. This subtopos has an associated sheaf functor $a_{\mathcal{S}} : \mathcal{B}(\mathbb{O}_{\mathcal{L}}) \to \mathcal{E}_{\mathcal{S}}$ and by the Dualization Theorem (2.4.6) it corresponds to a unique (up to $\equiv_s$) geometric quotient $T_{\mathcal{S}}$ of $\mathbb{O}_{\mathcal{L}}$, such that if $U_{\mathbb{O}_{\mathcal{L}}}(\mathcal{E})$ is a universal model for $\mathbb{O}_{\mathcal{L}}$ in $\mathcal{E}$, then $U_{T_{\mathcal{S}}}(\mathcal{E}) := a\left(U_{\mathbb{O}_{\mathcal{L}}}(\mathcal{E})\right)$ is a universal model for $T_{\mathcal{S}}$ in $\mathcal{E}$. It remains to show that our $T_{\mathcal{S}}$ is properly chosen as to axiomatize the $\mathcal{L}$-structures in $\mathcal{S}$.

Define the family of geometric morphisms $h_{\mathcal{M}} := \iota_{\mathcal{M}} \circ e_{\mathcal{M}} : \mathcal{E}_{\mathcal{M}} \to \mathcal{E}_{\mathcal{S}}$ and observe that $f_{\mathcal{M}} = m \circ h_{\mathcal{M}}$. Since $\mathcal{M} \cong f^*_{\mathcal{M}}\left(U_{\mathbb{O}_{\mathcal{L}}}\right)$ we have $h^*_{\mathcal{M}}\left(U_{T_{\mathcal{S}}}\right) \cong \mathcal{M}$, such that $\mathcal{M}$ is a model of $T_{\mathcal{S}}$ and hence all structures in $\mathcal{S}$ are $T_{\mathcal{S}}$-models. Conversely, by Lemma 2.3.3 we see that the $h_{\mathcal{M}}$ are jointly surjective such that by (2), $U_{T_{\mathcal{S}}} \in \mathcal{S}$. By Remark 1.10.2 for any $\mathcal{F}$ Grothendieck and any $N \in \mathrm{Mod}(T_{\mathcal{S}}, \mathcal{F})$ we have $N = g^*(U_{T_{\mathcal{S}}})$ for some geometric morphism $g : \mathcal{F} \to \mathcal{E}_{\mathcal{S}}$ and so, by (1), any $T_{\mathcal{S}}$-model in a Grothendieck topos already lies in $\mathcal{S}$.

□

[4] In words: it is the union of the inclusion parts of the geometric morphisms.

2.6 Applications

We will now use the characterization theorem to prove the conjecture posed by Moerdijk.

Remark 2.6.1. *Similar to what we've done in the "only if"-part of the proof we can effectively replace property (2) in the characterization theorem by the following statements, by virtue of Corollary 2.3.4.*

(a) *Given any surjective geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ *and* $\mathcal{M} \in \mathcal{S}$ *we have* $f^*(\mathcal{M}) \in \mathcal{S} \Rightarrow \mathcal{M} \in \mathcal{S}$.

(b) *For any set-indexed family* $\{\mathcal{M}_i \in \mathcal{L}(\mathcal{E}_i) | i \in I\}$ *of* $\mathcal{L}$*-structures in* $\mathcal{S}$*, the associated* $\mathcal{L}$*-structure in* $\coprod_{i \in I} \mathcal{E}_i$ *also lies in* $\mathcal{S}$.

The following lemma is stated without proof.

Lemma 2.6.2. *Two infinitary first-order* $\mathcal{L}$*-theories are deductively equivalent if and only if they have the same models in all Grothendieck topoi.*

Theorem 2.6.3. *An infinitary first-order* $\mathcal{L}$*-theory* T *can be axiomatized by geometric* $\mathcal{L}$*-sequents if and only if*

1. *For any geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ *between Grothendieck topoi we have* $\mathcal{M} \in \mathrm{Mod}(T, \mathcal{E}) \Rightarrow f^*(\mathcal{M}) \in \mathrm{Mod}(T, \mathcal{F})$.

2. *For any surjective geometric morphism* $f : \mathcal{F} \to \mathcal{E}$ *between Grothendieck topoi we have* $f^*(\mathcal{M}) \in \mathrm{Mod}(T, \mathcal{F}) \Rightarrow \mathcal{M} \in \mathrm{Mod}(T, \mathcal{E})$.

Proof. Let $\mathcal{S}_T$ be the set of all models of T inside Grothendieck topoi, such that it automatically satisfies property (b) of the previous remark. Then T is axiomatizable by geometric $\mathcal{L}$-sequents if and only if property (a) and property (1) of the characterization theorem hold. So, the previous lemma finishes the proof. □

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List of Symbols and Abbreviations

Abbreviation	Description	Definition
$\cdot$	dot-operator for a presheaf	page 6
$[\![\cdot]\!], [\![\cdot]\!]_{\mathcal{E}}, [\![\cdot]\!]_{\mathcal{M}}$	interpretation (in $\mathcal{E}$) (of $\mathcal{M}$)	page 9
$\leq$	inequality of subobjects	page 23
$\Vdash$	forcing relation	page 13
$\equiv_s, \equiv_M$	syntactic and Morita equivalence	page 25
$0, 1$	initial resp. terminal object	page 10
a	sheafification functor	page 6
$\mathcal{B}(T)$	classifying topos for T-models	page 20
$\mathcal{C}$	category	page 5
$\widehat{\mathcal{C}}$	presheaf category $\mathrm{Set}^{\mathcal{C}^{\mathrm{op}}}$	page 5
$\mathrm{ConLex}(\mathrm{Sh}(\mathcal{C}, J), \mathcal{E})$	continuous left exact functors	page 14
$\mathcal{E}, \mathcal{F}, \cdots$	(elementary) topoi	page 7
f, f_*, f^*	geometric morphism	page 14
$f_!$	left adjoint to f^*	page 14
$\mathrm{Flat}(\mathcal{C})$	flat functors $\mathcal{C} \to \mathrm{Set}$	page 26
$\mathrm{Geom}(\mathcal{E}, \mathcal{F})$	category of geometric morphisms $\mathcal{E} \to \mathcal{F}$	page 14
$J, J(T)$	Grothendieck topology (associated with $B(T)$)	page 20
$\mathrm{Hom}(A, B)$	morphisms $A \to B$	page 15
$\mathcal{L}$	language	page 9
$\mathcal{L}_{\kappa,\lambda}, \mathcal{L}_{\infty,\omega}$	infinitary language	page 12
$\mathcal{L}(\mathcal{E})$	$\mathcal{L}$-structures in a topos $\mathcal{E}$	page 21
$\mathrm{Mod}(T, \mathcal{E})$	T-models in $\mathcal{E}$	page 15
$\mathbb{O}_{\mathcal{L}}$	empty $\mathcal{L}$-theory	page 27
$\mathcal{O}(X)$	category of opens of a topological space	page 7
$Q(T)$	quotients of T	page 25
$\mathrm{Sh}(\mathcal{C}, J)$	topos of sheaves on the site $(\mathcal{C}, J)$	page 6
$\mathrm{Sub}_{\mathcal{E}}(E)$	category of subobjects of $E \in \mathcal{E}_0$ in $\mathcal{E}$	page 8
T	(geometric) theory	page 12
$\overline{T}$	closure of T	page 25
U_T	universal T-model	page 15
x_f	matching family for a sieve	page 6
x	amalgamation of a matching family	page 6
y	Yoneda embedding	page 5

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