Tropical Geometry

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Abstract

The main topic of this thesis is the tropical semiring. We discuss tropical addition and multiplication and what the graphs of polynomials, defined using this type of addition and multiplication, look like. We will discuss the notion of a tropical amoeba and how Voronoi diagrams relate to this notion. We will also consider power diagrams, a type of generalization of the Voronoi diagram, and look at their relation to tropical amoebas. We will show that every tropical polynomial defines a power diagram and, even better, also a Voronoi diagram.

We will also approach tropical geometry from the view of algebraic geometry. We discuss the notion of an amoeba in algebraic geometry and consider the spine of this amoeba. Finally we will show how the spine of an amoeba relates to the tropical amoeba.

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1 Introduction

In modern mathematics a concept one encounters a lot and that is widely studied, is the concept of a function. Basically a function is just a prescription: to each number in a given set, it assigns a given value. One way of describing the prescription is to tabulate all the numbers together with their assigned values. For large sets of numbers this can be tedious, moreover if the set is infinite it is impossible to tabulate all the numbers. Therefore we wish to describe the function in another way. A large family of functions can be described using polynomials.

To define a polynomial one needs the concept of addition, a $+$ operation, and multiplication, a $\times$ operation. We are used to using the same kind of addition and multiplication as the ancient Greeks did. But nothing is keeping us from defining addition and multiplication a different way. Still we have to be cautious, we want most of the conditions that hold for the original addition and multiplication, to hold for the newly defined addition and multiplication. This is where the concept of a semiring comes in. It tells us precisely which conditions have to be satisfied if we want to redefine addition and multiplication.

In this thesis we start by examining different definitions of addition and multiplication. We check that we can indeed form a semiring with them. Also we look at how they relate to each other. Doing this we find one semi ring that relates to a lot of other semirings. This will be the tropical semiring.

Because the tropical semiring relates to a lot of other semirings, it is not a bad idea to investigate it a bit further. A way of representing a function is to draw the graph of it. We will look at the graphs of tropical polynomials and find that they are easy to draw.

After having considered the geometry of the tropical semiring, we wonder if there are any relations between tropical geometry and other ways of doing geometry? This will be the main question we will try to answer in this thesis.

To this end we first look at a type of geometry we already knew in high school mathematics, the geometry of the Voronoi diagram. How can we construct Voronoi diagrams and can we express Voronoi diagrams using functions? We will find that there is a relation between Voronoi diagrams and tropical polynomials. Using this relation, constructing the Voronoi diagram of a set of points becomes much easier. This relation becomes even better if instead of considering Voronoi diagrams we look at the more general power diagrams. For power diagrams we will also look at ways of constructing them and how we can represent them using functions. We will find that every power diagram defines a tropical polynomial and, moreover, every tropical polynomial defines a power diagram. We will use the relation between power diagrams and Voronoi diagrams to determine if every tropical polynomial also defines a Voronoi diagram.

Another way of doing geometry we will consider is algebraic geometry. We will explore how amoebas and tropical polynomials relate to each other. To this end we will define the spine of an amoeba using the Ronkin function. We will explicitly calculate parts of the Ronkin function for polynomials and find that there is a one to one relation between tropical polynomials and the spine of an amoeba.

Throughout this thesis we will try to make calculations as explicit as possible and we will give a number of examples to make sure the theory that is presented can also be understood by readers who are not familiar with the fields we are working with.

2 Semirings

We are used to doing mathematics in fields and, if we drop the requirement of the existence of a multiplicative inverse, in rings. We could also wonder what happens if we also do not require the existence of an inverse for addition. What kind of structure is left?

Definition 2.1. *A semiring is a triple $(R, \oplus, \otimes)$ consisting of a nonempty set R equipped with a $\oplus$ operation which we call addition and a $\otimes$ operation referred to as multiplication, for which the following conditions hold:*

1. $a \oplus (b \oplus c) = (a \oplus b) \oplus c \ \forall a, b, c \in R$
   $a \oplus b = b \oplus a \ \forall a, b \in R$
   $\exists 0 \in R : a \oplus 0 = 0 \oplus a = a \ \forall a \in R$

2. $a \otimes (b \otimes c) = (a \otimes b) \otimes c \ \forall a, b, c \in R$
   $\exists 1 \in R \backslash \{0\} : 1 \otimes a = a \otimes 1 = a \ \forall a \in R$

3. $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c \ \forall a, b, c \in R$
   $(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c \ \forall a, b, c \in R$

4. $a \otimes 0 = 0 \otimes a = 0 \ \forall a \in R$

We will only consider semirings where $R \subset \mathbb{R} \cup \{-\infty, \infty\}$.

Of course every ring is automatically a semiring, but there are also some interesting nontrivial examples of semirings which are not rings themselves:

- $(\mathbb{N} \cup \{\infty\}, \gcd, \times)$
- $(\mathbb{N} \cup \{\infty\}, \min, +)$
- $(\mathbb{R} \cup \{\infty\}, \min, +)$
- $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +)$
- $(\mathbb{R}_{\geq 0}, \min, \times)$
- $(\mathbb{I}, \max, \min)$ where $\mathbb{I} = [0, 1]$
- $\mathbb{R}_h := (\mathbb{R}_{\geq 0}, \oplus_h, \times)$ where

$$a \oplus_h b = \begin{cases} \max\{a, b\} & \text{if } h = 0; \\ (a^{\frac{1}{h}} + b^{\frac{1}{h}})^h & \text{if } h > 0. \end{cases}$$

- $\mathbb{R}_{[t]} := (\mathbb{R}_{\geq 0}, \oplus_t, +)$ where

$$a \oplus_t b = \begin{cases} \log_t(t^a + t^b) & \text{if } 0 < t < \infty; \\ \max\{a, b\} & \text{if } t = \infty. \end{cases}$$

- Let R be a semiring and A a non-empty set. Then the set R^A of maps from A to R is also a semiring, where we take the scalar addition and multiplication of R as $\oplus$ and $\otimes$ operators.

Here with $\mathbb{N}$ we denote the natural numbers including 0.

2.1 The semiring $(\mathbb{R} \cup \{\infty\}, \min, +)$

Let us take a closer look at some of these semirings and start by looking at $(\mathbb{R} \cup \{\infty\}, \min, +)$. For $a, b \in \mathbb{R} \cup \{\infty\}$ we have:

$$\begin{aligned} a \oplus b &= \min\{a, b\} \\ a \otimes b &= a + b \end{aligned}$$

This is indeed a semiring, as we will show. For all $a, b, c \in \mathbb{R} \cup \{\infty\}$ we have:

$$\begin{aligned} a \oplus (b \oplus c) &= a \oplus \min\{b, c\} = \min\{a, b, c\} = \min\{a, b\} \oplus c = (a \oplus b) \oplus c \\ a \oplus b &= \min\{a, b\} = \min\{b, a\} = b \oplus a \\ a \oplus \infty &= \min\{a, \infty\} = a \text{ hence in this case } \infty \text{ is the zero.} \end{aligned}$$

Thus part 1 of the definition holds. Part 2 holds because of the known rules for using the $+$, here 0 has the role of the 1. Now part 3 and 4 also hold since for all $a, b, c \in \mathbb{R} \cup \{\infty\}$:

$$\begin{aligned} a \otimes (b \oplus c) &= a + \min\{b, c\} = \min\{a + b, a + c\} = a \otimes b \oplus a \otimes c \\ (a \oplus b) \otimes c &= \min\{a, b\} + c = \min\{a + c, b + c\} = a \otimes c \oplus b \otimes c \\ a \otimes \infty &= a + \infty = \infty \end{aligned}$$

Therefore $(\mathbb{R} \cup \{\infty\}, \min, +)$ is a semiring. Notice that $a \oplus a = \min\{a, a\} = a$ for all $a \in \mathbb{R} \cup \{\infty\}$ and that we need ∞ since in $\mathbb{R}$ there is no largest element and therefore there would not be an element that could have the zero role.

Also notice that this is not a ring since there is no such thing as subtraction, i.e. an operation $\ominus$ such that $\forall a \in \mathbb{R}\ \exists \ominus a \in \mathbb{R} : a \oplus \ominus a = 0$. Since if there would be such an operation, then

$$(a \oplus a) \oplus \ominus a = a \oplus \ominus a = 0$$

but also

$$a \oplus (a \oplus \ominus a) = a \oplus 0 = a$$

and if $a \neq 0$ this contradicts $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

Because in $\mathbb{R}$ $-$ is the inverse of $+$, $(\mathbb{R} \cup \{\infty\}, \min, +)$ has a multiplicative inverse and hence is a semifield. Now if we take $(\mathbb{R}_{\geq 0} \cup \{\infty\}, \min, +)$ then there is no multiplicative inverse anymore. Hence this is an example of a semiring which is not a semifield.

Lemma 2.2. $(\mathbb{R} \cup \{\infty\}, \min, +) \cong (\mathbb{R} \cup \{-\infty\}, \max, +)$

Proof. Let $f : \mathbb{R} \cup \{\infty\} \longrightarrow \mathbb{R} \cup \{-\infty\}, \quad x \longmapsto -x$. Then f is a bijection, moreover f is its own inverse: $f(f(x)) = f(-x) = x\ \forall x \in \mathbb{R} \cup \{\infty\}$. Also for all $x, y \in \mathbb{R} \cup \{\infty\}$

$$\begin{aligned} f(x \oplus y) &= f(\min\{x, y\}) = \max\{-x, -y\} = f(x) \oplus f(y) \\ f(x \otimes y) &= f(x + y) = -(x + y) = -x - y = f(x) \otimes f(y). \end{aligned}$$

Hence f is an isomorphism. $\square$

Lemma 2.3. $(\mathbb{R} \cup \{-\infty\}, \max, +) \cong (\mathbb{R}_{\geq 0}, \max, \times)$

Proof. Let $g : \mathbb{R} \cup \{-\infty\} \longrightarrow \mathbb{R}_{\geq 0}, \quad x \longmapsto e^x$. Then g is a bijection with the log as its inverse. Also for all $x, y \in \mathbb{R} \cup \{-\infty\}$

$$\begin{aligned} g(x \oplus y) &= g(\max\{x,y\}) = e^{\max\{x,y\}} = \max\{e^x, e^y\} = g(x) \oplus g(y) \\ g(x \otimes y) &= g(x+y) = e^{x+y} = e^x \times e^y = g(x) \times g(y) \\ g(0) &= e^0 = 1 \\ g(-\infty) &= e^{-\infty} = 0. \end{aligned}$$

Hence g is an isomorphism. $\square$

2.2 The semiring $(\mathbb{I}, \max, \min)$

If instead of $\mathbb{R} \cup \{\infty\}$ we take $\mathbb{I} = [0,1]$ we can take max as addition and min as multiplication. In this semiring the zero and one are just 0 and 1. The only thing we still need to check is that $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$. Now

$$\begin{aligned} a \otimes (b \oplus c) &= \min\{a, \max\{b,c\}\} = \max\{\min\{a,b\}, \min\{a,c\}\} \\ &= a \otimes b \oplus a \otimes c. \end{aligned}$$

To show this we may assume without loss of generality that $b < c$ and must subsequently consider the cases

i. $a < b < c$,

ii. $b < a < c$,

iii. $b < c < a$.

If $a < b < c$ we have

$$\begin{aligned} \min\{a, \max\{b,c\}\} &= \min\{a,c\} = a \quad \text{and} \\ \max\{\min\{a,b\}, \min\{a,c\}\} &= \max\{a,a\} = a. \end{aligned}$$

If $b < a < c$

$$\begin{aligned} \min\{a, \max\{b,c\}\} &= \min\{a,c\} = a \quad \text{and} \\ \max\{\min\{a,b\}, \min\{a,c\}\} &= \max\{b,a\} = a. \end{aligned}$$

And finally if $b < c < a$ we have

$$\begin{aligned} \min\{a, \max\{b,c\}\} &= \min\{a,c\} = c \quad \text{and} \\ \max\{\min\{a,b\}, \min\{a,c\}\} &= \max\{b,c\} = c. \end{aligned}$$

Therefore we see that the distributive laws hold in this semiring.

2.3 The semiring $(\mathbb{N} \cup \{\infty\}, \gcd, \times)$

First we will check the conditions to see that $(\mathbb{N} \cup \{\infty\}, \gcd, \times)$ is indeed a semiring. For all $a, b, c \in \mathbb{N} \cup \{\infty\}$ we have:

1. - Suppose $a \oplus (b \oplus c) = \gcd(a, \gcd(b, c)) = p$ and $\gcd(b, c) = q$. Then $p|a$ and $p|q$, therefore $a = np$ and $q = mp$ with $\gcd(n, m) = 1$. Because $q = \gcd(b, c)$, $b = iq = imp$ and $c = jq = jmp$ with $\gcd(i, j) = 1$. Now $(a \oplus b) \oplus c = \gcd(\gcd(a, b), c) = \gcd(\gcd(np, imp), jmp)$. It might be that $\gcd(n, i) \neq 1$, so n and i have a factor r in common. Then we get $\gcd(\gcd(np, imp), jmp) = \gcd(rp, jmp)$. But since $\gcd(i, j) = 1$ and $\gcd(n, m) = 1$, $\gcd(r, j) = \gcd(r, m) = 1$, hence $\gcd(rp, jmp) = p$ and $a \oplus (b \oplus c) = (a \oplus b) \oplus c$.

   - $a \oplus b = \gcd(a, b) = \gcd(b, a) = b \oplus a$
   - $a \oplus \infty = \gcd(a, \infty) = a$ hence in this case ∞ is the zero.

2. Because we have the normal multiplication, this part holds automatically.

3. - Suppose $a \otimes (b \oplus c) = a \times \gcd(b, c) = ap$ where $p = \gcd(b, c)$. So $b = np$ and $c = mp$ with $\gcd(n, m) = 1$. Now $a \otimes b \oplus a \otimes c = \gcd(a \times b, a \times c) = \gcd(anp, amp) = ap$, hence $a \otimes (b \oplus c) = a \otimes b \oplus a \otimes c$.
   - Suppose $(a \oplus b) \otimes c) = \gcd(a, b) \times c = qc$ where $q = \gcd(a, b)$. So $a = iq$ and $b = jq$ with $\gcd(i, j) = 1$. Now $a \otimes c \oplus b \otimes c = \gcd(a \times c, b \times c) = \gcd(iqc, jqc) = qc$, hence $(a \oplus b) \otimes c = a \otimes c \oplus b \otimes c$.

4. $a \otimes \infty = a \times \infty = \infty$.

Now for small numbers $a, b \in \mathbb{N} \cup \{\infty\}$ we can calculate $a \oplus b = \gcd(a, b)$ quite easily. But if a and b become large, then $a \oplus b$ can be a challenging calculation. A way to calculate $a \oplus b$ is to first factor a and b into primes. So let

$$a = \prod_{p\ prime} p_i^{a_i} \text{ and } b = \prod_{p\ prime} p_i^{b_i},$$

then

$$a \oplus b = \gcd(a, b) = \prod_{p\ prime} p_i^{c_i}$$

where $c_i = \min(a_i, b_i)$. For this we still first have to factor a and b into primes and for large numbers this also is a time consuming calculation. But if we only want to know the factors of one single prime p, then we are done quite fast. So instead of $a = \prod_{p\ prime} p_i^{a_i}$ we calculate $a = rp^{v_p(a)}$ where $v_p(a)$ is called the p-adic valuation of a at p and $\gcd(r, p) = 1$. Now if $a = rp^{v_p(a)}$ and $b = sp^{v_p(b)}$, $a \oplus b = \gcd(rp^{v_p(a)}, sp^{v_p(b)}) = \gcd(r, s)p^{\min(v_p(a), v_p(b))}$. This still does not give us an easy way for calculating $a \oplus b$. But suppose we are only interested in the p-adic valuation of $a \oplus b$, then

$$v_p(a \oplus b) = v_p(\gcd(a, b)) = v_p(\gcd(r, s)p^{\min(v_p(a), v_p(b))}) = \min(v_p(a), v_p(b)).$$

Also we have

$$v_p(a \otimes b) = v_p(a \times b) = v_p(a) + v_p(b).$$

So through the p-adic valuation the semiring $(\mathbb{N} \cup \{\infty\}, \gcd, \times)$ turns into $(\mathbb{N} \cup \{\infty\}, \min, +)$.

Lemma 2.4. $v_p : (\mathbb{N} \cup \{\infty\}, \gcd, \times) \longrightarrow (\mathbb{N} \cup \{\infty\}, \min, +)$ *is a homomorphism between semirings.*

Proof. We already showed that

$$\begin{aligned} v_p(a \oplus b) &= \min(v_p(a), v_p(b)) = v_p(a) \oplus v_p(b) \quad \text{and} \\ v_p(a \otimes b) &= v_p(a) + v_p(b) = v_p(a) \otimes v_p(b). \end{aligned}$$

Because $v_p(1) = 0$ we find that v_p is a homomorphism. $\square$

If we denote by $R = (\mathbb{N} \cup \{\infty\}, \min, +)$, $R' = (\mathbb{N} \cup \{\infty\}, \gcd, \times)$ and $P = \{p \in \mathbb{N} \mid p \text{ prime}\}$, then R^P is a semiring. We can form the map $n \mapsto (p \mapsto v_p(n))$ from R' to R^P. Because of lemma 2.4 this map is a homomorphism. Moreover, because of the unique prime factorization in $\mathbb{N}$ this map is injective, but it is not surjective, therefore we do not have an isomorphism.

2.4 The semirings $\mathbb{R}_h$ and $\mathbb{R}_{[t]}$

For $\mathbb{R}_h$ and $\mathbb{R}_{[t]}$ we will not check all conditions, just the ones that do not follow immediately from the definitions of $\oplus$ and $\otimes$. For all $a, b, c \in \mathbb{R}_{\geq 0}$, $h \neq 0$, $t \neq \infty$ we have:

$$\begin{aligned} a \oplus_h (b \oplus_h c) &= a \oplus_h [b^{\frac{1}{h}} + c^{\frac{1}{h}}]^h = [a^{\frac{1}{h}} + b^{\frac{1}{h}} + c^{\frac{1}{h}}]^h \\ &= (a \oplus_h b) \oplus_h c \\ a \otimes (b \oplus_h c) &= [a^{\frac{1}{h}}(b^{\frac{1}{h}} + c^{\frac{1}{h}})]^h = [a^{\frac{1}{h}} b^{\frac{1}{h}} + a^{\frac{1}{h}} c^{\frac{1}{h}}]^h \\ &= a \otimes b \oplus_h a \otimes c \\ a \oplus_t (b \oplus_t c) &= a \oplus_t \log_t(t^b + t^c) = \log_t(t^a + t^{\log_t(t^b + t^c)}) = \log_t(t^a + t^b + t^c) \\ &= (a \oplus_t b) \oplus_t c \\ a \otimes (b \oplus_t c) &= a + \log_t(t^b + t^c) = \log_t(t^a) + \log_t(t^b + t^c) = \log_t(t^a(t^b + t^c)) \\ &= \log_t(t^{a+b} + t^{a+c}) = a \otimes b \oplus_t a \otimes c. \end{aligned}$$

So we find that indeed $\mathbb{R}_h$ and $\mathbb{R}_{[t]}$ are semirings.

Lemma 2.5. $\mathbb{R}_h \cong \mathbb{R}_{[t]}$

Proof. Consider $\log : \mathbb{R}_h \longrightarrow \mathbb{R}_{[t]}$, where $t = e^{\frac{1}{h}}$. This is a bijection and

$$\begin{aligned} \log(a) \oplus_t \log(b) &= \log_t(t^{\log(a)} + t^{\log(b)}) = \log_{e^{\frac{1}{h}}}(a^{\frac{1}{h}} + b^{\frac{1}{h}}) \\ &= \log([a^{\frac{1}{h}} + b^{\frac{1}{h}}]^h) = \log(a \oplus_h b) \\ \log(a \otimes b) &= \log(a \times b) = \log(a) + \log(b) = \log(a) \otimes \log(b). \end{aligned}$$

Hence we have an isomorphism. $\square$

Because of this lemma we will only need to focus on $\mathbb{R}_h$. It seems that $\mathbb{R}_h$ provides us with infinitely many different semirings. But this is not the case as the following lemma will show.

Lemma 2.6. *For all $h \neq 0$, $\mathbb{R}_1 \cong \mathbb{R}_h$*

Proof. Let $f : \mathbb{R}_1 \longrightarrow \mathbb{R}_h,\ x \mapsto x^h$. Then f is a bijection with $y \mapsto y^{\frac{1}{h}}$ as its inverse. Also for all $x, y \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} f(a \oplus_1 b) &= f(a+b) = (a+b)^h \\ f(a) \oplus_h f(b) &= a^h \oplus_h b^h = (a+b)^h \\ \text{hence } f(a \oplus_1 b) &= f(a) \oplus_h f(b), \\ \text{also } f(a \otimes b) &= f(a \times b) = (a \times b)^h = a^h \times b^h = f(a) \otimes f(b). \end{aligned}$$

Therefore f is an isomorphism. □

Now if $h = 0$ then there can not exist an isomorphism, since in $\mathbb{R}_0$ $a \oplus_0 a = \max(a, a) = a$ and this is not the case in $\mathbb{R}_1$: $a \oplus_1 a = a + a = 2a$. Therefore $\mathbb{R}_h$ only provides us with two different semirings: $\mathbb{R}_1$ and $\mathbb{R}_0 = (\mathbb{R}_{\geq 0}, \max, \times)$.

Note that $\mathbb{R}_1$ is just $(\mathbb{R}_{\geq 0}, +, \times)$, which we will denote by $\mathbb{R}_+$, and therefore by allowing negative numbers we can easily extend the semiring $\mathbb{R}_1$ to the ring $\mathbb{R}$. Hence for $h \neq 0$ we can extend $\mathbb{R}_h$ to a ring that is isomorphic to $\mathbb{R}$. But in the case $h = 0$ we can not do this because there is no such thing as $-$. So we find that if we let h go to 0, we have a family of semirings that we can extend to a ring, but in the limit case this is not possible any more. This process is often referred to as dequantization.

Now at first sight the choice for $\oplus_0 = \max$ might seem quite random, but the following lemma shows us that this actually is the only logical choice.

Lemma 2.7.

$$\lim_{h \to 0} [a^{\frac{1}{h}} + b^{\frac{1}{h}}]^h = \max(a, b)$$

Proof. Suppose $a > b$, then

$$\begin{aligned} \lim_{h \to 0} [a^{\frac{1}{h}} + b^{\frac{1}{h}}]^h &= \lim_{h \to 0} \left[a^{\frac{1}{h}} \left(1 + \left(\frac{b}{a} \right)^{\frac{1}{h}} \right) \right]^h = \lim_{h \to 0} a \left[1 + \left(\frac{b}{a} \right)^{\frac{1}{h}} \right]^h \\ &= a, \end{aligned}$$

since if $h < 1$ we have

$$1 \leq \left[1 + \left(\frac{a}{b} \right)^{\frac{1}{h}} \right]^h \leq 1 + \left(\frac{a}{b} \right)^{\frac{1}{h}}$$

and because $\lim_{h \to 0} 1 + \left(\frac{b}{a}\right)^{\frac{1}{h}} = 1$ we must have $\lim_{h \to 0} [1 + (\frac{b}{a})^{\frac{1}{h}}]^h = 1$. By symmetry we have $\lim_{h \to 0} [a^{\frac{1}{h}} + b^{\frac{1}{h}}]^h = b$ if $b > a$.

If $a = b$ we have

$$\begin{aligned}\lim_{h\to 0}[a^{\frac{1}{h}} + b^{\frac{1}{h}}]^h &= \lim_{h\to 0}[2a^{\frac{1}{h}}]^h \\ &= \lim_{h\to 0} 2^h a \\ &= a.\end{aligned}$$

Hence $\lim_{h\to 0}[a^{\frac{1}{h}} + b^{\frac{1}{h}}]^h = \max(a, b)$. □

We see that the semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$ also occurs in the discussion of the semirings $(\mathbb{N} \cup \{\infty\}, \gcd, \times)$, $\mathbb{R}_h$ and $\mathbb{R}_{[t]}$. Therefore it is not a bad idea to investigate this semiring a bit further. We will do this in the next section.

3 The Tropical Semifield

In the previous section we found that the semiring $(\mathbb{R} \cup \{-\infty\}, \max, +)$ was worth more investigation. Since in this semiring multiplication is the same as the usual addition and in $\mathbb{R}$ the inverse of addition is subtraction, this semiring has a multiplicative inverse. Hence $(\mathbb{R} \cup \{-\infty\}, \max, +)$ is not only a semiring, it is a semifield. We refer to it as the tropical semifield and denote it as $\mathbb{R}_{trop}$. It appeared in computer science and is called 'tropical' in honour of Imre Simon who resides in São Paolo, Brazil. The importance of this ring was also recognized by Russian scientists who referred to it as 'dequantization' or 'idempotent analysis'. Strictly speaking the semiring that appeared in computer science used the minimum for addition instead of the maximum. But as was proven in the previous section, these two types of addition are isomorphic to each other.

3.1 Tropical Polynomials

In $\mathbb{R}_{trop}$ we defined addition and multiplication. We also noticed that there is no inverse for addition, but there is a multiplicative inverse. Now we will consider what happens if we repeat addition and multiplication, so for instance, for $a \in \mathbb{R}_{trop}$, what is $a \oplus a$ and $a \otimes a$?

We already mentioned that $a \oplus a = \max(a, a) = a$, hence repeating addition does not have any effect. Here we notice an important difference between the normal arithmetic and tropical arithmetic: in the normal arithmetic repeating addition shows us the relation between the normal addition and multiplication, $a + a = 2 \times a$. In tropical arithmetic such a relation does not exist.

What about $a \otimes a$? For $a \in \mathbb{R}_{trop}$ $a \otimes a = a + a = 2a$. If we write $a^2 = a \otimes a$, then we find that the 2 of the tropical square just comes in front of the a if we express it in the usual arithmetics. In the same way we find that for higher powers $b \in \mathbb{N}$, $a^b = ba$, where on the lefthandside we mean tropical multiplication and on the righthandside we use normal multiplication. Hence we see that taking powers the tropical way is just multiplying the normal way. Since there is also a tropical multiplicative inverse, we are also allowed to take negative powers and hence we have $a^b = ba$ for $b \in \mathbb{Z}$ and $a \in \mathbb{R}_{trop}$.

Now we know how to take powers the tropical way, we can consider monomials. So what does $ax^b = a \otimes x^b$, for $a \in \mathbb{R}$, $x \in \mathbb{R}_{trop}$ and $b \in \mathbb{Z}$, look like? Using the definition of tropical multiplication and what we just found for taking powers the tropical way, we find

$$ax^b = a \otimes x^b = a + bx.$$

Hence a tropical monomial in one variable is just an affine linear function. We can use the same construction for monomials in multiple variables:

$$\begin{aligned} ax_1^{b_1} \dots x_j^{b_j} &= a \otimes x_1^{b_1} \otimes \cdots \otimes x_j^{b_j} \\ &= a + b_1 x_1 + \cdots + b_j x_j \\ &= a + \langle b, x \rangle \end{aligned}$$

where $b = (b_1, \ldots, b_j)$ and $x = (x_1, \ldots, x_j)$.

We are now ready to consider tropical polynomials. Let $x = (x_1, \ldots, x_n)$, $j = (j_1, \ldots, j_n)$, $x^j = x_1^{j_1} \ldots x_n^{j_n}$ and $a_j \in \mathbb{R}$, then

$$\sum_{j \in J} a_j x^j = \max_j (a_j + \langle j, x \rangle),$$

where J is a finite non-empty subset of $\mathbb{Z}^n$. We see that tropical polynomials are piecewise affine linear functions. For example, in $\mathbb{R}^2$ we have the polynomials

$$\sum_{i,j} a_{ij} x^i y^j = \max_{i,j} (a_{ij} + ix + jy).$$

All the $a_{ij} + ix + jy$ are two dimensional affine linear subspaces of $\mathbb{R}^3$.

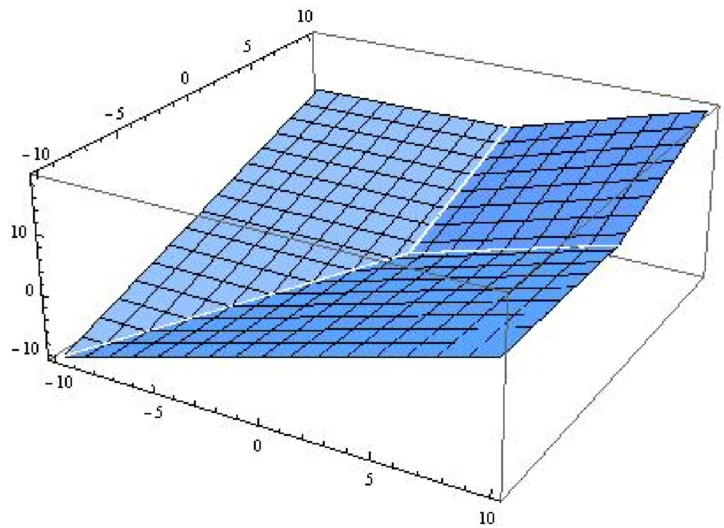

Figure 1: The graph of the tropical polynomial $-\frac{1}{2}x - \frac{1}{2}y - xy$

Notice that in this semiring there is a difference between $1x$ and x, since $1x = 1 + x$ instead of x. In the tropical polynomial we will use x and not $1x$.

To stress we are talking about polynomials in the tropical semiring, we will sometimes write them like

$$\bigoplus_{j=0}^{m} (a_j x^j) \quad \text{or} \quad \bigoplus_{j=0}^{m} (a_j + \langle j, x \rangle).$$

Also, we will usually denote tropical polynomials with capital letters.

3.2 Tropical Amoebas

Because a tropical polynomial is a piecewise affine linear function, around most points there exists a neighborhood at which the graph is given by an affine linear function. But at the points where the maximum is attained by two or more affine linear functions, such a neighborhood does not exist. Therefore, in a sense, these points can be considered special.

Definition 3.1. *Let $F : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial, i.e. F is of the form*

$$F(x) = \bigoplus_{j \in J} (a_j + \langle j, x \rangle)$$

where $j = (j_1, \ldots, j_n)$, $x = (x_1, \ldots, x_n)$ and J is a finite subset of $\mathbb{Z}^n$. Then the tropical amoeba $V_F \subset \mathbb{R}^n$ of F is the set of all points in $\mathbb{R}^n$ where F is not smooth.

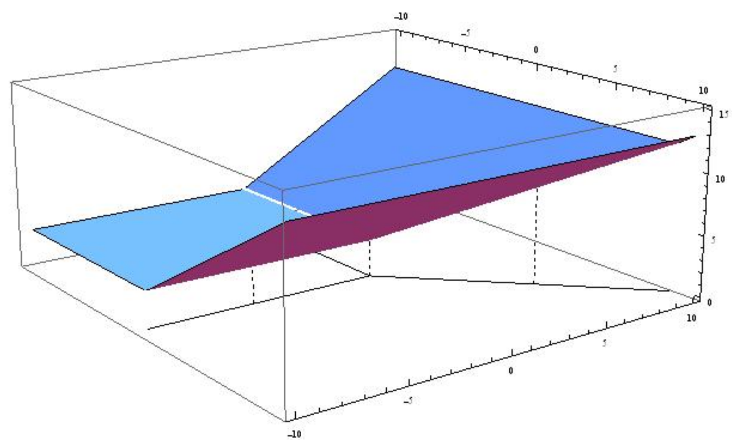

Figure 2: The graph of the tropical polynomial $3x + 2y + 3$ and its tropical amoeba

If $F(x, y) = \bigoplus_{j=1,\ldots,N} (a_j + b_j x + c_j y)$ we will call the tropical amoeba V_F of F a tropical curve.

Notice that if we add some number μ to all of the a_j, the graph of the tropical polynomial gets translated in the vertical direction, but the tropical amoeba remains the same. Since addition is the tropical multiplication, we see that $V_F = V_{\mu \otimes F}$. Hence as for null-sets of normal polynomials, the amoeba does not change if we multiply the polynomial with some number.

Let F be a tropical polynomial, then $F = \max_{i=1,\ldots,N}(f_i)$, where f_i, $i = 1, \ldots, N$, is an affine linear function. For a point x outside the amoeba V_F, the graph of F is given by the affine linear function f_j for which $f_j(x) > f_i(x)$ for all $i \neq j$. Hence at this point f_j dominates all the other affine linear functions of F.

For a point on V_F, there are at least two functions f_1 and f_2 that attain the maximum. So there are two functions which try to dominate. One can say that there is a conflict between these two functions at this point. In this view, V_F is given by all the points at which there is a conflict between two of the affine linear functions which F is composed of.

In classical algebraic geometry one is concerned with equations like $f(x) = 0$, where $f(x)$ is a polynomial. For this it is necessary to have the notion of an equation, or even better a system of equations. In tropical geometry we can formulate equations, but since there is no such thing as a tropical $\ominus$, we can not algebraically solve these equations as we are used to in algebraic geometry. This does not mean that there are no points satisfying the equation.

But in tropical geometry we encounter equations even without formulating them ourselves. Let $F = \max_{i \in I}\{f_i\}$, with $f_i,\ i \in I$ affine linear functions, be a tropical polynomial. To draw the graph of this polynomial, we need to find all the $x \in \mathbb{R} \cup \{-\infty\}$ such that $f_i(x) \geq f_j(x)$, for f_i a fixed. Or equivalently all the $x \in \mathbb{R} \cup \{-\infty\}$ such that for all $j \in I$

$$\max\{f_i(x), f_j(x)\} = f_i(x). \tag{1}$$

If we write this equation the tropical way, we get

$$f_i \oplus f_j = f_i \ \forall\ j \in I. \tag{2}$$

If there are two affine linear functions f_i and $f_{i'}$ which are equal to F at a point x, so we are on the tropical amoeba V_F, then we get the system of equations

$$\begin{aligned} f_i \oplus f_j &= f_i \ \forall\ j \in I \text{ and} \\ f_{i'} \oplus f_j &= f_{i'} \ \forall j \in I. \end{aligned}$$

If there are more affine linear functions equal to F at a point, in a similar way, we again get a system of equations. Therefore to draw the tropical polynomial, we need to solve several systems of equations.

To construct the amoeba V_F we want to find all the $x \in \mathbb{R} \cup \{-\infty\}$ such that x is a solution to two or more of these systems of equations. So we see that, like in classical algebraic geometry, tropical geometry is concerned with solving equations.

4 Voronoi diagrams and Tropical Amoebas

In this section we will mainly be concerned with points and functions in $\mathbb{R}^2$. But because most of the theorems and definitions hold for general $\mathbb{R}^n$, we will state and prove them as general as possible.

In the previous section we discussed the notion of a tropical amoeba and explained how we can look at it as the set of conflict points between the different affine functions. In classical geometry there is a similar notion, the Voronoi diagram. In a Voronoi diagram we consider the conflict points between a given set of points.

4.1 Voronoi Diagrams

In high-school mathematics one is familiar with distances in the Euclidean plane. One also encounters the notion of a (planar) Voronoi diagram.

Definition 4.1. *Let $A = \{A_1, \ldots, A_N\}$ be a nonempty finite subset of $\mathbb{R}^2$. The closed Voronoi cell $\bar{C}$ of the point A_i is defined as:*

$$\bar{C}(A_i) := \{(x, y) \in \mathbb{R}^2 \mid d((x, y), A_i) \leq d((x, y), A_j) \ \forall A_j \in A\}$$

The Voronoi diagram V of A is defined as:

$$V(A) := \{\bar{C}(A_1), \ldots, \bar{C}(A_N)\}.$$

Every Voronoi cell is a polyhedron, and the edges $e_{i1}, \ldots, e_{ir_i}$ of the boundary $\partial\bar{C}(A_i)$ are line segments, half lines or infinite lines. We will concern ourselves with the union of these edges

$$D(A) := \bigcup_{i=1}^{N} \left(\bigcup_{j=1}^{r_i} e_{ij} \right) = \bigcup_{i,j,\ i \neq j} \bar{C}(A_i) \cap \bar{C}(A_j).$$

Notice that $\mathbb{R}^2 \backslash (C(A_1) \cup \cdots \cup C(A_N))$, where $C(A_i)$ is the open Voronoi cell of A_i, is equal to $D(A)$. Therefore one could say that $D(A)$ divides $\mathbb{R}^2$ into different cells, each cell equal to one of the open Voronoi cells. Hence, in a sense, $D(A)$ defines the Voronoi diagram $V(A)$. The definition of $D(A)$ also relates more to our perception of a diagram than the definition of $V(A)$ does. Therefore we will sometimes use $D(A)$ instead of $V(A)$ when talking about the Voronoi diagram of A.

The definition of a Voronoi diagram in $\mathbb{R}^2$ easily extends to general $\mathbb{R}^n$.

Definition 4.2. *Let $A = \{A_1, \ldots, A_N\}$ with $A_i \in \mathbb{R}^n$ for $i = 1, \ldots, N$. The closed Voronoi cell $\bar{C}$ of the point A_i is defined as:*

$$\bar{C}(A_i) := \{x \in \mathbb{R}^n \mid \|x - A_i\| \leq \|x - A_j\| \ \forall A_j \in A\}$$

The Voronoi diagram V of A is defined as:

$$V(A) := \{\bar{C}(A_1), \ldots, \bar{C}(A_N)\}.$$

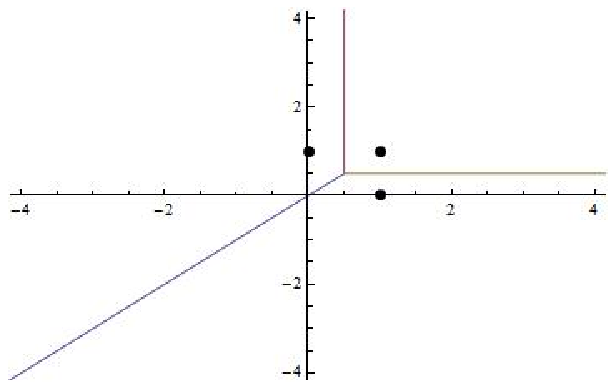

Figure 3: The Voronoi diagram $V(A)$ of $A = \{(1,0),(0,1),(1,1)\}$

Commonly a face is one of the two-dimensional polygons that bound a three-dimensional polytope. We can extend this to higher dimensions by saying that a face of a polytope P is the intersection of any supporting hyperplane of P and P itself, where a supporting hyperplane of P is a hyperplane such that P has at least one point in common with the hyperplane and P is entirely contained in one of the two closed halfspaces determined by the hyperplane. Therefore a face is any of the lower dimensional parts of the boundary of the polytope. Then the boundary of an m dimensional polyhedron in the Voronoi diagram consists of $m-1$-dimensional faces. An $m-1$-dimensional face is also called a facet. A two-dimensional face is just a face and we call a one-dimensional face an edge.

4.2 Construction of Voronoi Diagrams

Let us consider the Voronoi diagram given by the points $A = \{A_1, \ldots, A_N\}$ in $\mathbb{R}^2$, and write $A_i = (a_i, b_i)$. We can construct the Voronoi diagram $V(A)$ using the paraboloid T of revolution along the z-axis, $T := \{(x,y,z) \in \mathbb{R}^3 \mid z = x^2 + y^2\}$.

For $A_i \in A$ let us consider the lift-up A_i^* of A_i given by $A_i^* = (a_i, b_i, a_i^2 + b_i^2)$. Then $A_i^* \in T$. We denote the lift-up of all the points in A by A^*, thus $A^* = \{A_1^*, \ldots, A_N^*\}$.

Now we consider the tangent plane T_i to T at the point A_i^*. T_i is given by all the points $(x,y,z) \in \mathbb{R}^3$ such that

$$z = 2a_i x + 2b_i y - (a_i^2 + b_i^2).$$

Notice that $A_i^* \in T_i$ and that all the other points of A^* lie above T_i. Let L_{ij}^* be the line of intersection between the two tangent planes T_i and T_j, $i \neq j$. Denote by L_{ij} the orthogonal projection of L_{ij}^* onto $\mathbb{R}^2$. Then L_{ij} is given by

$$2(a_i - a_j)x + 2(b_i - b_j)y - (a_i^2 + b_i^2 - a_j^2 - b_j^2) = 0. \quad (3)$$

Later on we will derive this equation as the equation of the perpendicular bisector of A_i and A_j. Therefore we find that L_{ij} is the perpendicular bisector of A_i and A_j.

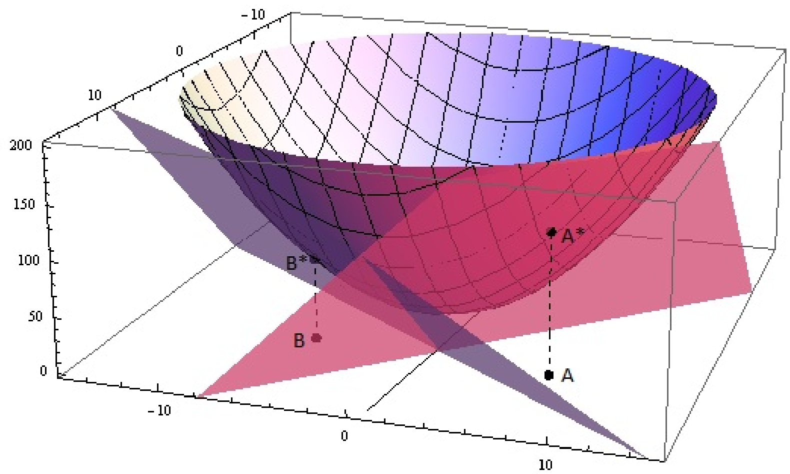

Figure 4: Construction of the perpendicular bisector of the points $A = (8, 8)$ and $B = (7, -5)$ using the tangent planes to the paraboloid

It is a theorem in standard Euclidean geometry that all the points on the perpendicular bisector have equal distance to A_i and A_j. Also the points on the side of the perpendicular bisector where A_i lies, are closer to A_i as to A_j. Therefore all these points lie in $V(A_i)$. Hence we see that the perpendicular bisector of two points defines the Voronoi diagram of these two points.

This construction is displayed in figure 4.

The tangent plane T_i divides $\mathbb{R}^3$ into two half spaces. Let

$$H_i := \{(x, y, z) \in \mathbb{R}^3 \mid z \geq 2a_i x + 2b_i y - (a_i^2 + b_i^2)\}$$

be the upper half space together with its boundary. Then $H_i \cap H_j$ is a polyhedron in $\mathbb{R}^3$ and the orthogonal projection of the edge of this polyhedron is the perpendicular bisector of A_i and A_j. If we extend this construction to all the points of A^*, we find the polyhedron $\bigcap_{i=1}^N H_i$. Then the orthogonal projection of all the facets of $\bigcap_{i=1}^N H_i$ onto $\mathbb{R}^2$ is equal to the Voronoi diagram $V(A)$.

In a similar way we can extend this construction to higher dimensions using the $n+1$ dimensional parabola $x_0 = \sum_{i=1}^n x_i^2$.

Using the construction described above, we find that in $\mathbb{R}^2$ the Voronoi diagram consists of parts of the perpendicular bisectors of the different points. In this construction we use a quadratic function to eventually construct lines. One may wonder if we can construct the Voronoi diagram using only affine functions.

In the case of a two dimensional Voronoi diagram, let us therefore explore $D(A)$ a bit further. As we noticed before, given two points $A_1, A_2 \in \mathbb{R}^2$, then $D(\{A_1, A_2\})$ is equal to the perpendicular bisector of A_1 and A_2. Suppose $A_1 = (a_1, b_1)$ and $A_2 = (a_2, b_2)$, then one can use the Pythagorean theorem to define the perpendicular bisector of A_1 and A_2: the perpendicular bisector of A_1 and A_2 consists of all points $(x, y) \in \mathbb{R}^2$ for which

$$\sqrt{(x - a_1)^2 + (y - b_1)^2} = \sqrt{(x - a_2)^2 + (y - b_2)^2}. \tag{4}$$

We can rewrite this equation to

$$\begin{aligned}
\frac{1}{2}((x-a_1)^2+(y-b_1)^2) &= \frac{1}{2}((x-a_2)^2+(y-b_2)^2)\\
\frac{1}{2}x^2-a_1x+\frac{1}{2}a_1^2+\frac{1}{2}y^2-b_1y+\frac{1}{2}b_1^2 &= \frac{1}{2}x^2-a_2x+\frac{1}{2}a_2^2+\frac{1}{2}y^2-b_2y+\frac{1}{2}b_2^2\\
(a_2-a_1)x+(b_2-b_1)y &= \frac{1}{2}(a_2^2-a_1^2+b_2^2-b_1^2) \qquad (5)
\end{aligned}$$

We can extend this construction to the case where A consists of more than two points. If $A = \{A_1, \ldots, A_N\}$ and $A_i = (a_i, b_i)$ for $i = 1, \ldots N$, then $(x, y) \in \bar{C}(A_i)$ if

$$\sqrt{(x-a_i)^2+(y-b_i)^2} \leq \sqrt{(x-a_j)^2+(y-b_j)^2} \quad \forall j = 1, \ldots, N. \qquad (6)$$

As before we can rewrite this to

$$-a_ix-b_iy+\frac{1}{2}(a_i^2+b_i^2) \leq -a_jx-b_jy+\frac{1}{2}(a_j^2+b_j^2). \qquad (7)$$

If we substitute $c_i = -\frac{1}{2}(a_i^2+b_i^2)$ for $i = 1, \ldots, N$ this equation becomes

$$a_ix+b_iy+c_i \geq a_jx+b_jy+c_j. \qquad (8)$$

Now let $F(x, y) = \max\{a_ix+b_iy+c_i \mid i = 1, \ldots, N\}$, then the graph of F is a polyhedron in $\mathbb{R}^3$. More over, $(x, y) \in \bar{C}(A_i)$ precisely if $F(x, y) = a_ix+b_iy+c_i$. Therefore we find that the Voronoi diagram $D(A)$ is given by the projection of the edges of the graph of F on $\mathbb{R}^2$ and we have thus proved the following theorem.

Theorem 4.3. *Let $A = \{A_1, \ldots, A_N\}$ with $A_i = (a_i, b_i)$ and*

$$F(x, y) = \bigoplus_{i=1}^{N} c_ix^{a_i}y^{b_i} = \max_{i=1,\ldots,N}(c_i+a_ix+b_iy)$$

with $c_i = -\frac{1}{2}(a_i^2+b_i^2)$. Then $D(A)$ is equal to the tropical amoeba V_F.

Notice that this is not the only tropical polynomial that defines $V(A)$. We could also have taken

$$G(x, y) = \bigoplus_{i=1}^{N} \bar{c}_ix^{2a_i}y^{2b_i} = \max_{i=1,\ldots,N}(\bar{c}_i+2a_ix+2b_iy),$$

with $\bar{c}_i = -(a_i^2+b_i^2)$. If we take our tropical polynomial this way, we see that the affine linear functions of G are just the hyperplanes in the construction of the Voronoi diagram using the parabola. However, the coordinates of the points of the Voronoi diagram are in direct correspondence with the exponents of the monomials of F. Also this allows us to look at tropical polynomials with odd exponents.

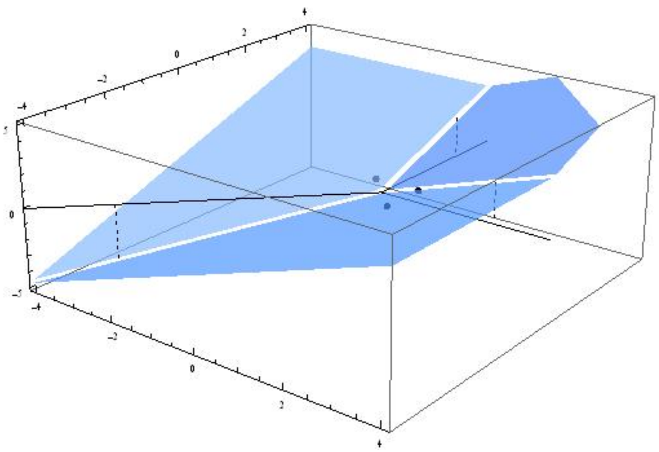

Figure 5: The graph of $F(x, y) = \max(x - \frac{1}{2}, y - \frac{1}{2}, x + y - 1)$ together with the Voronoi diagram of $A = \{(1, 0), (0, 1), (1, 1)\}$

Example 4.4. *Let* $A = \{(1, 0), (0, 1), (1, 1)\}$*, then the perpendicular bisectors are given by*

$$x - y = 0 \tag{9}$$

$$x = \frac{1}{2} \tag{10}$$

$$y = \frac{1}{2}. \tag{11}$$

We can use these to construct $D(A)$*, which is displayed in figure 3. By theorem 4.3* $D(A)$ *is equal to the tropical amoeba* V_F*, where* $F(x, y) = \max(x - \frac{1}{2}, y - \frac{1}{2}, x + y - 1)$*. The graph of* $F(x, y)$ *is given in figure 1. In figure 5 we see that the projection of the singular sides of* F *is indeed equal to the Voronoi diagram defined by* $D(A)$*.*

In theorem 4.3 the coefficients c_i are determined by the points of A. But what happens if we vary the c_i? Can we still find points in some $\mathbb{R}^n$ such that the Voronoi diagram in $\mathbb{R}^2$ of these points is equal to the tropical amoeba of F? We will try to answer this question in the remaining part of this section.

4.3 Power diagrams

Given a set of points in $\mathbb{R}^2$ we can construct a tropical polynomial, with coefficients depending on the set of points, such that its corner locus is equal to the Voronoi diagram of the set of points. Before we go any further, let us consider two 'actions' on these points and look at how they affect the coefficients of the tropical polynomial.

First let us consider lifting the points from the plane.

Suppose we have a point P in $\mathbb{R}^2$ and we lift it a distance d from the plane. We then get a point P' in $\mathbb{R}^3$. Now consider a point Q in the plane, not equal to P with distance q to P. The the distance between Q and P' is equal to

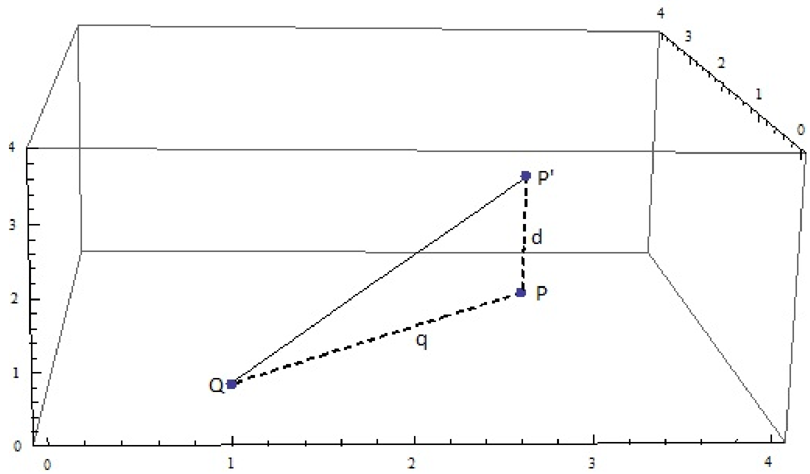

Figure 6: Changing the distance by lifting a point from the plane

$\sqrt{q^2 + d^2} > q$. Therefore by lifting P from the plane the distance between P' and the points on the plane becomes larger. Now what is the effect on the coefficients of our tropical polynomial?

Suppose $P = (p_1, p_2)$, then $P' = (p_1, p_2, d)$. For any point (x, y) in $\mathbb{R}^2$ the distance to P' is

$$\sqrt{(x - p_1)^2 + (y - p_2)^2 + d^2}.$$

Hence (x, y) lies in the 'Voronoi cell' of P if

$$\sqrt{(x - p_1)^2 + (y - p_2)^2 + d^2} < \sqrt{(x - a_j)^2 + (y - b_j)^2} \quad \forall\ j = 1, \dots, N,$$

where $A_j = (a_j, b_j)$ are the other points in our set of points. We can rewrite this to

$$-p_1 x - p_2 y + \frac{1}{2}(p_1^2 + p_2^2) + d^2 < -a_j x - b_j y + \frac{1}{2}(a_j^2 + b_j^2).$$

Therefore the new coefficient c_P becomes $-\frac{1}{2}(p_1^2+p_2^2)-d^2 < -\frac{1}{2}(p_1^2+p_2^2)$. Hence we see that we can make the coefficients smaller by lifting our points from the plane.

Notice that the 'Voronoi cell' of P is not a Voronoi cell in the usual sense of the word.

Now let us consider another way of defining the distance between points in the plane.

Let P be a given point in the plane and C be a circle of radius ϵ centered at P. For any point Q outside C we define the distance from Q to P as the length of the tangent between Q and C. If the usual distance between Q and P is q, then using the Pythagorean theorem, the new distance becomes

$$\sqrt{q^2 - \epsilon^2} < q.$$

Therefore we see that this new distance is smaller than the old distance. A way to look at this new distance is to consider P with a weight ϵ.

With this distance, the coefficients in our tropical polynomial become:

$$c_P = -\frac{1}{2}(p_1^2 + p_2^2) + \epsilon^2 > -\frac{1}{2}(p_1^2 + p_2^2).$$

Therefore by assigning a weight to a point, we can make the coefficients larger.

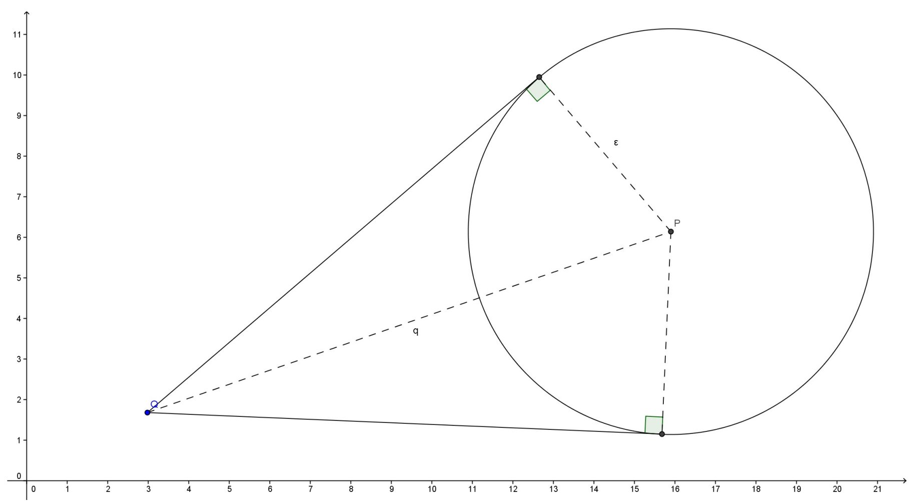

Figure 7: Changing the distance by assigning a weight to a point

As we have mentioned before, by lifting points from the plane or assigning a weight to them, the tropical curve that we get does not resemble a Voronoi diagram any more. Therefore we introduce the following notion.

Definition 4.5. *Consider a set of points* $P = \{P_1, \ldots, P_N\} \subset \mathbb{R}^n$*. To each point* P_i *we assign a weight* w_i*. Let* $g_i : \mathbb{R}^n \longrightarrow \mathbb{R}$ *be given by*

$$g_i(x) := \frac{1}{2}\|x - P_i\| - \frac{1}{2}w_i$$

and let

$$g(x) := \min_{1 \leq i \leq N} g_i(x)$$

We define $Pow(P_i)$ *as the closure of*

$$\{x \in \mathbb{R}^n \mid g_i(x) = g(x) \text{ and } g_j(x) > g(x) \ \forall \ P_j \neq P_i\}.$$

Then the power diagram $\mathcal{P}$ *of* P *is*

$$\mathcal{P} := \{Pow(P_1), \ldots, Pow(P_N)\}.$$

The separator $Sep(\{P_i, P_j\})$ *between* P_i *and* P_j *is the hyperplane* $\{x \in \mathbb{R}^n \mid g_i(x) = g_j(x)\}$.

In this definition, for convenience, we us the square of the Euclidean distance as norm, so $\|x\| = \sum_{i=1}^{n} x_i^2$. Also we do not require the weights to be positive.

Notice that if we add the same number to all of the weights, the functions g_i change but $\mathcal{P}$ remains the same. Hence the power diagram does not change at all. Therefore only the difference between the weights matters and we are free to choose all of the weights positive or negative if we like to.

If all the weights are equal to zero, we recover the definition of a Voronoi diagram.

4.4 Construction of Power Diagrams

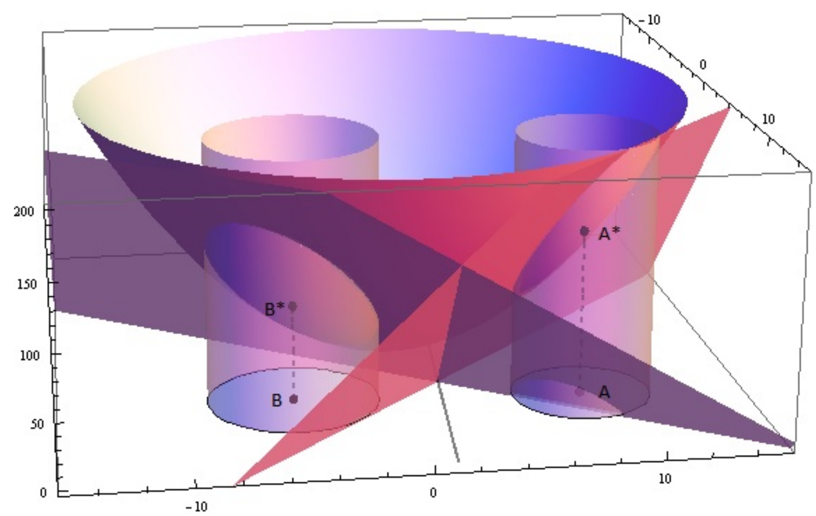

Figure 8: Construction of the power diagram of the two points $A = (8, 8)$ with weight $w_A = 10$ and $B = (7, -5)$ with weight $w_B = 15$.

To construct a power diagram we use a construction much like the construction of the Voronoi diagram. Let the power diagram be given by the points $P = \{P_1, \ldots, P_N\}$ with weights w_i. Consider the parabola $x_0 = \sum_{i=1}^{n} x_i^2$. We choose our weights in such a way that they are all positive such that we can take cylinders C_i of radius $\sqrt{w_i}$ around the lines (x_0, P_i) in $\mathbb{R}^{n+1}$. Then intersection of these cylinders with the parabola are of the form $(x_0, \ldots, x_n) \in \mathbb{R}^{n+1}$ with $x_0 = \sum_{i=1}^{n} x_i^2$ and $(x_1 - P_{i1})^2 + \cdots + (x_n - P_{in})^2 = w_i$. This last equation is equal to $x_1^2 - 2P_{i1}x_1 + P_{i1}^2 + \cdots + x_n^2 + 2P_{in}x_n + P_{in}^2 = w_i$ and from this it follows that $x_0 = 2\langle x, P_i \rangle - \|P_i\| + w_i$. Therefore the intersection of the cylinder C_i with the parabola lies in the hyperplane T_i given by

$$x_0 = 2\langle x, P_i \rangle - \|P_i\| + w_i.$$

If we now consider the orthogonal projection of the intersection of two hyperplanes T_i and T_j, we find that it is given by

$$\langle x, P_i \rangle - \frac{\|P_i\| - w_i}{2} = \langle x, P_j \rangle - \frac{\|P_j\| - w_j}{2}.$$

As we will see in a moment, these equations define the power diagram. Therefore, to construct the power diagram, we take the upper convex hull of all the $T_i,\ i = 1, \ldots, N$. The projection of the singular sides now defines the power diagram.

Notice that if $w_i = 0$, T_i becomes tangent to the parabola and hence we recover the construction of a Voronoi diagram.

We show how this construction works for three points in figure 9.

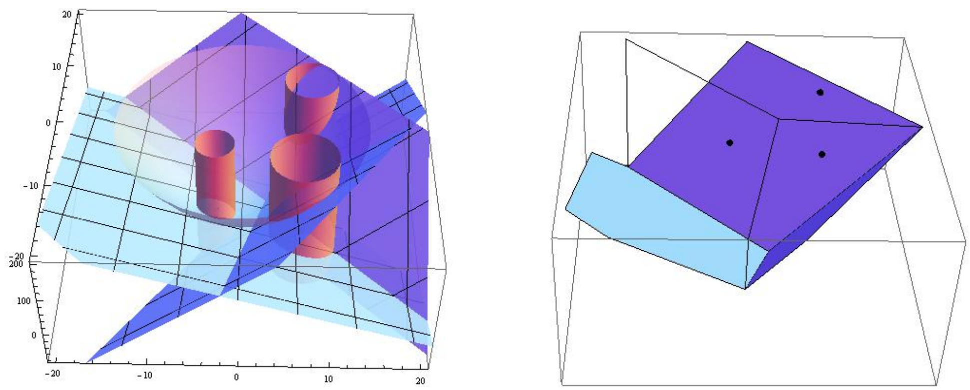

Figure 9: The construction of the powerdiagram of the points $(8, 8)$, $(7, -5)$ and $(-3, -3)$ with respectively weights 10, 15 and 5.

If we change the weights of the points, all the cylinders and hence also the hyperplanes change. But if we change all the weights the same way, by this we mean that we add or subtract the same constant to all of them, we see in the formula that the intersections of the hyperplanes do not change.

As we did for the Voronoi diagram, we will now try to express the power diagram by means of a tropical polynomial.

Theorem 4.6. ([SM05], prop. 1) *Let $P = \{P_1, \ldots, P_N\} \subset \mathbb{R}^n$ be a set of points and w_i the weight assigned to P_i. Then the power diagram $\mathcal{P}$ can be obtained from the tropical polynomial*

$$f(x) = \bigoplus_{i=1}^{N} c_i + \langle x, P_i \rangle$$

where

$$c_i = -\frac{\|P_i\| - w_i}{2}.$$

Proof. Notice that the power diagram can completely be determined by all the $x \in \mathbb{R}^n$ for which $g_i(x) = g_j(x)$ for some $i, j,\ i \neq j$. Now if $g_i(x) = g_j(x)$ for some $i, j,\ i \neq j$, we have

$$\begin{aligned} \frac{1}{2}\|x - P_i\| - \frac{1}{2}w_i &= \frac{1}{2}\|x - P_j\| - \frac{1}{2}w_j \\ \frac{1}{2}\sum_{k=1}^{n}(x_k - p_{ik})^2 - \frac{1}{2}w_i &= \frac{1}{2}\sum_{m=1}^{n}(x_m - p_{jm})^2 - \frac{1}{2}w_j \\ \sum_{k=1}^{n}\left(\frac{1}{2}x_k^2 - x_k p_{ik} + \frac{1}{2}p_{ik}^2\right) - \frac{1}{2}w_i &= \sum_{m=1}^{n}\left(\frac{1}{2}x_m^2 - x_m p_{jm} + \frac{1}{2}p_{jm}^2\right) - \frac{1}{2}w_j \\ -\langle x, P_i\rangle + \frac{\|P_i\| - w_i}{2} &= -\langle x, P_j\rangle + \frac{\|P_j\| - w_j}{2}. \end{aligned} \tag{12}$$

Let $f_i(x) = \langle x, P_i\rangle + c_i$, with $c_i = -\frac{\|P_i\| - w_i}{2}$ and

$$f(x) = \max_{i=1,\dots,N} f_i(x) = \bigoplus_{i=1}^{N} c_i + \langle x, P_i\rangle .$$

Then we find that $Pow(P_i)$ is the closure of

$$\{x \in \mathbb{R}^n \mid f_i(x) = f(x) \text{ and } f_j(x) < f(x)\ \forall\ j \neq i\}$$

and that

$$\{x \in \mathbb{R}^n \mid g_i(x) = g(x) \text{ and } g_j(x) = g(x) \text{ for some } i, j,\ i \neq j\}.$$

is equal to the tropical curve determined by $f(x)$. □

Notice that this theorem is just a generalization of what we already deduced for the Voronoi diagram in $\mathbb{R}^2$. In that case $w_i = 0$ for all $i = 1, \dots, N$, $\|P_i\| = (a_i^2 + b_i^2)$ and therefore $c_i = \frac{1}{2}(a_i^2 + b_i^2)$. Also, if we construct power diagrams using tropical polynomials, we can allow negative weights.

With the P_i fixed, we can choose any set of coordinates for our tropical polynomial and find a power diagram belonging to it by adjusting the weights w_i. Therefore we have the following result.

Theorem 4.7. *Every power diagram defines a tropical polynomial. Conversely every tropical polynomial $f = \bigoplus a_i + \langle P_i, x\rangle$ defines a power diagram. The relation between the coefficients of the polynomial and the power diagram are given by*

$$a_i = -\frac{\|P_i\| - w_i}{2}.$$

Proof. This is an immediate consequence of theorem 4.6. □

Now we can ask ourselves the following questions:

1. Every tropical polynomial defines a power diagram but does it define a Voronoi diagram?

2. In our original setting our points lie in $\mathbb{R}^2$. If we require our weights to be positive we need a power diagram in $\mathbb{R}^3$ and intersect it with the hyperplane $z = 0$ to recover the tropical line given by the tropical polynomial as constructed in theorem 4.6. But is this intersection still a power diagram?

We first address the second question:

Lemma 4.8. ([SM05], prop. 2) *The intersection of a hyperplane with a power diagram is again a power diagram.*

Proof. By theorem 4.6 we can define a power diagram solely using affine functions. Since we can describe the power diagram using affine functions, the intersection of a hyperplane and a power diagram can be described using affine functions. Because of theorem 4.7 these affine functions determine a power diagram. □

We can answer the first question using the following lemma.

Lemma 4.9. ([SM05], prop. 3) *Let $\mathcal{T}$ be a power diagram in $\mathbb{R}^n$. There is a Voronoi diagram $\mathcal{V}$ in $\mathbb{R}^{n+1}$ and a hyperplane $\mathcal{H}$ such that $\mathcal{H} \cap \mathcal{V} = \mathcal{T}$.*

Proof. Let $\{P_1, \ldots, P_N\}$ be the set of points that determine $\mathcal{T}$. Let $\mathcal{V}$ be the Voronoi diagram determined by the points $Q_i = (P_i, P_i')$, where the value of P_i' is still to be determined. We write $\bar{x} = (x, x')$. Then

$$\|\bar{x} - Q_i\| = \|x - P_i\| + (x' - P_i')^2.$$

If we intersect $\mathcal{V}$ with the hyperplane given by $x' = 0$, we get

$$\|\bar{x} - Q_i\| = \|x - P_i\| + P_i'^2.$$

Now let $\mathcal{T}$ be given by functions g_i as specified in definition 4.5. Notice that in this definition of a power diagram, only the difference between weights, $w_i - w_j$, matters. Therefore it is no restriction to assume that $w_i < 0$. Now if we choose $P_i' = \sqrt{-w_i}$, the the Voronoi diagram determined by the points Q_i, intersected with the hyperplane $x' = 0$ is equal to the power diagram $\mathcal{T}$. □

We see that in the construction of the Voronoi diagram we need to have negative weights, this in contrast to the construction of the power diagram where all the weights had to be positive.

Because of lemma 4.8 and 4.9 we now have the following theorem.

Theorem 4.10. *Every tropical polynomial $f(x, y) = \bigoplus a_i + b_i x + c_i y$ defines a Voronoi diagram in $\mathbb{R}^3$.*

Proof. As shown in the beginning of this subsection, $f(x, y)$ defines a power diagram $\mathcal{P}$ in $\mathbb{R}^3$ with positive weights. If we intersect this power diagram with the hyperplane $z = 0$, we get a power diagram $\mathcal{P}'$ in $\mathbb{R}^2$. By lemma 4.9 there exists a Voronoi diagram $\mathcal{V}$ in $\mathbb{R}^3$ such that $\mathcal{V}$ intersected with $\mathbb{R}^2$ is equal to $\mathcal{P}'$. Hence $f(x, y)$ determines a Voronoi diagram in $\mathbb{R}^3$. □

Remark: The weights of $\mathcal{P}$ are positive, but it might be that some of the weights of $\mathcal{P}'$ are negative. Therefore requiring the weights to be positive in the first power diagram only gives us one extra step in the proof of theorem 4.10. If we drop this requirement, we can start the proof with some $\mathcal{P}''$ in $\mathbb{R}^2$, whose weights are allowed to be negative, and immediately apply lemma 4.9.

Example 4.11. *Let us consider the tropical polynomial of example 4.4 again,*

$$F(x,y) = -\frac{1}{2}x \oplus -\frac{1}{2}y \oplus -1xy = \max(x-\frac{1}{2}, y-\frac{1}{2}, x+y-1).$$

Now we will change the coefficients and get

$$G(x,y) = -2x \oplus 2y \oplus -1xy = \max(x-2, y+2, x+y-1).$$

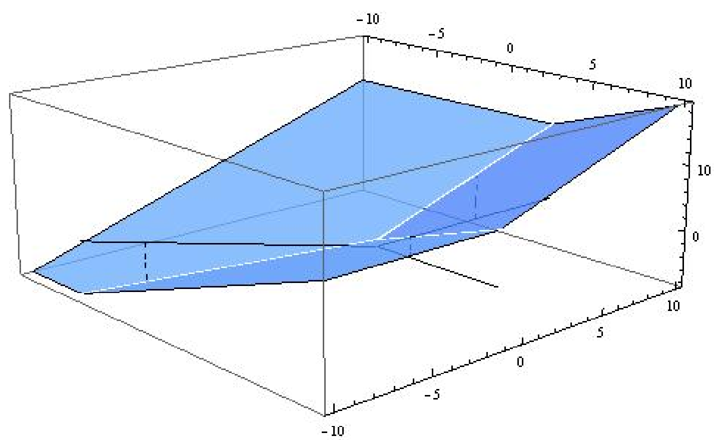

Figure 10: The graph and tropical amoeba of $G(x,y) = -2x \oplus 2y \oplus -1xy$

Then the tropical amoeba V_G is equal to the power diagram given by $P_1 = \{(1,0,\sqrt{\frac{3}{2}}),(0,1,0),(1,1,0)\}$ with weights $w_1 = 0$, $w_2 = 3$ and $w_3 = 0$, intersected with the hyperplane $z = 0$. This intersection is equal to the power diagram given by $P_2 = \{(1,0),(0,1),(1,1)\}$ with weights $w_1 = -3$, $w_2 = 3$ and $w_3 = 0$. Notice that this is the same power diagram as we would have if we had immediately allowed our weights to be negative.

Notice that this power diagram demonstrates one important property of power diagrams: although for all points the power cells belonging to them are not empty, not all of the points lie in their own cell, as can be seen in figure 11.

In order to construct the Voronoi diagram in $\mathbb{R}^3$, all the weights have to be negative. Therefore we subtract 3 from all the weights of $\mathcal{P}(P_2)$. So the weights become: $w_1 = -6$, $w_2 = 0$ and $w_3 = -3$. Since only the difference between the weights matters, this still defines the same power diagram. Using the proof of lemma 4.9 we find that the Voronoi diagram given by $A = \{(1,0,\sqrt{6}),(0,1,0),(1,1,\sqrt{3})\}$ intersected with the hyperplane $z = 0$ is equal to the tropical amoeba V_G. Hence G defines the Voronoi diagram $V(A)$ in $\mathbb{R}^3$.

Summarizing, we have seen that every Voronoi diagram defines a tropical polynomial and, if we extend our definition of a Voronoi diagram to a power

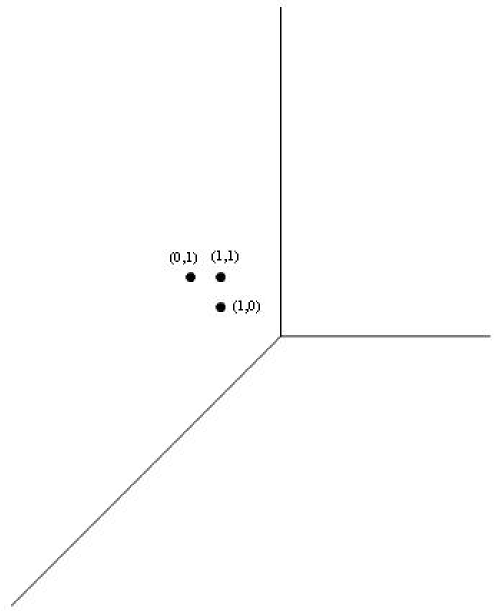

Figure 11: The power diagram of $P = \{(1,0),(0,1),(1,1)\}$ with weights $w_1 = -3, w_2 = 3$ and $w_3 = 0$.

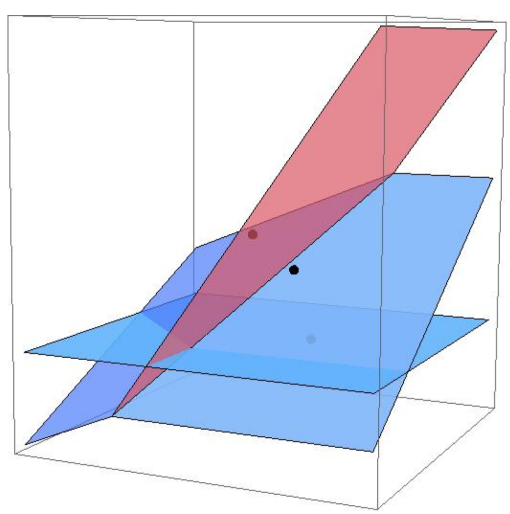

Figure 12: The Voronoi diagram of $P = \{(1,0,\sqrt{6}),(0,1,0),(1,1,\sqrt{3})\}$ intersected with the hyperplane $z = 0$.

diagram, every power diagram defines a tropical polynomial. Conversely every tropical polynomial defines a power diagram. Hence there is a one to one relation between power diagrams and tropical polynomials. If we use the relation between power diagrams and Voronoi diagrams of one dimension higher, we even find that every tropical polynomial defines a Voronoi diagram.

Notice that the exponents of our tropical polynomial are integers and therefore the points of the Voronoi and power diagrams need to have integer coordinates. We could loose this restriction just by saying that $a^b = ba$ for all $a \in \mathbb{R}_{trop}$ and $b \in \mathbb{R}$, but this would need some verification. Also we would not be talking about tropical polynomials any more, but about a more general group of tropical functions. The treatment of this is out of the scope of this thesis.

5 The Ronkin Function

In section 3 we already mentioned a relation between algebraic geometry and tropical geometry. Also, in section 2 we noticed that $\mathbb{R}_h$ is equal to $\mathbb{R}$ for all $h \neq 0$, on which we can do algebraic geometry, but for the limit case $h = 0$ we find $\mathbb{R}_{trop}$, on which we do tropical geometry. Therefore we could think of tropical geometry as a limit case of algebraic geometry. So for some notions in algebraic geometry, we might wonder what the tropical counterpart would be. In this section we will derive a relation between the notion of an amoeba in algebraic geometry and a tropical amoeba.

In algebraic geometry we can define varieties of hypersurfaces using polynomials. An example of such an algebraic variety is $\{x + y + 1 = 0 \mid x, y \in \mathbb{C}\}$.

5.1 Amoebas

Definition 5.1. *Let Log* : $(\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$ *be defined by* $Log(z_1, \ldots, z_n) \mapsto (\log|z_1|, \ldots, \log|z_n|)$ *and let* $V \subset (\mathbb{C}^*)^n$ *be an algebraic variety. The amoeba* $\mathcal{A}$ *of* V *is* $\mathcal{A} := Log(V) \subset \mathbb{R}^n$.

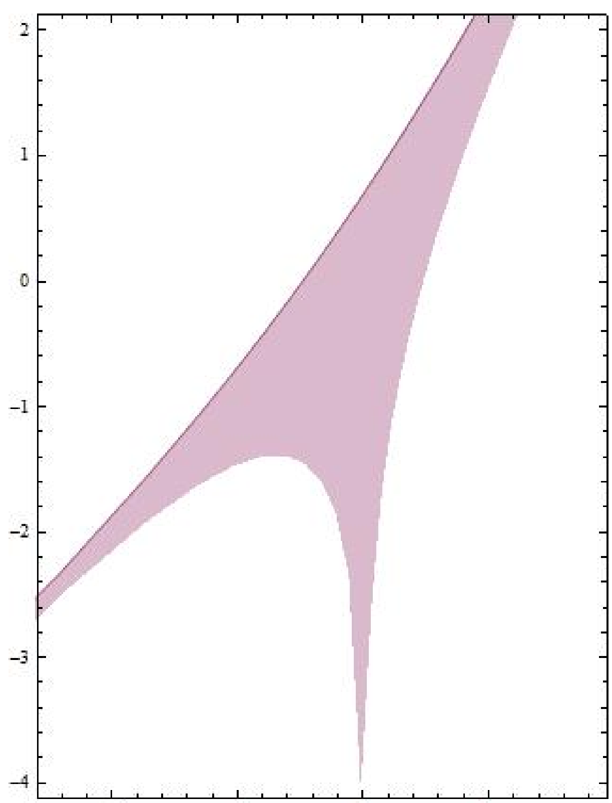

Figure 13: The amoeba of $x^2y + xy + y^2 = 0$

We will only consider varieties that are hypersurfaces. Thus $V = \{f = 0\}$ for some polynomial $f : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$. To stress the dependence of the amoeba

$\mathcal{A}$ of V on f, we will denote it by $\mathcal{A}_f$.

5.2 The Ronkin Function

In order to be able to understand the relation between these amoebas and tropical amoebas, we need to do some analysis.

First of all, since the product of n circles is a $n+1$-dimensional torus and $Log^{-1}(x_1, \dots, x_n)$ is the product of n circles of radius e^{x_i}, $Log^{-1}(x_1, \dots, x_n)$ is a $n+1$-dimensional torus.

Definition 5.2. *Let $N_f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined as the push-forward of* $\log |f|$ *under the previously defined Log. So*

$$N_f(x_1, \dots, x_n) = \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x_1,\dots,x_n)} \log |f(z_1, \dots, z_n)| \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

The function N_f is known as the Ronkin function and is well defined for $x \notin \mathcal{A}_f$, since for $x \notin \mathcal{A}_f$, $|f(z_1, \dots, z_n)| \neq 0$ for all $z = (z_1, \dots, z_n) \in Log^{-1}(x_1, \dots, x_n)$. Therefore $\log |f(z_1, \dots, z_n)|$ only takes finite values on $Log^{-1}(x_1, \dots, x_n)$ and we can integrate over this torus. In the literature we find that it is possible to prove that it takes real (finite) values, even over $\mathcal{A}_f$ where the integral is singular.

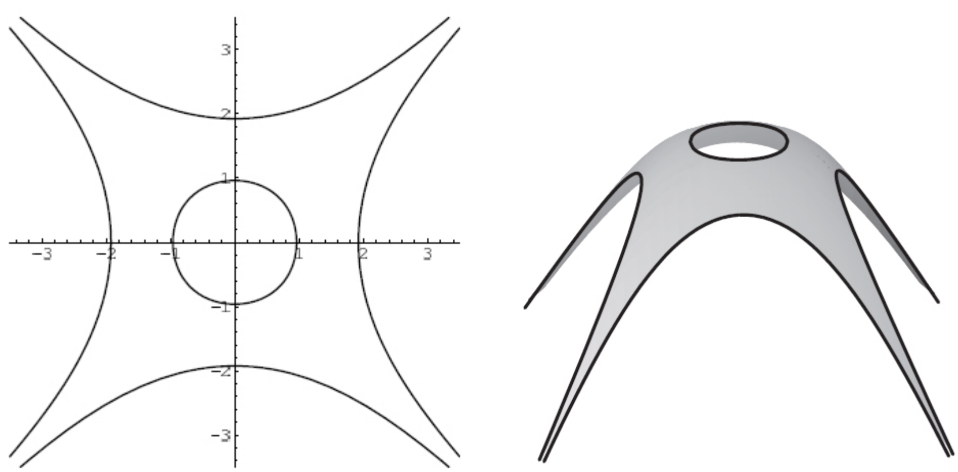

Figure 14: The amoeba and Ronkin function of $f(x,y) = x + x^{-1} + y + y^{-1} + 5$ [KOS06].

Lemma 5.3. ([PE03], prop. 1.3 & prop. 1.4, [RU03], thm. 1) *N_f is convex over $\mathbb{R}^n$, strictly convex over $\mathcal{A}$ and affine over $\mathbb{R}^n \backslash \mathcal{A}$*

Proof. First we prove that the derivative of N_f with respect to x_j is the real part of

$$v_j(x) = \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \frac{\partial f}{\partial z_j} \frac{z_j}{f(z)} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.$$

For this we use polar coordinates $z_k = e^{x_k+i\theta_k}$. If we keep x_k fixed, we get $\frac{dz_k}{z_k} = \frac{e^{x_k+i\theta_k} id\theta_k}{e^{x_k+i\theta_k}} = id\theta_k$. Therefore we have the equality

$$(2\pi i)^n N_f(x) = \int_0^{2\pi} \cdots \int_0^{2\pi} \log|f(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n})| i^n d\theta_1 \wedge \cdots \wedge d\theta_n.$$

Differentiating with respect to x_j, for $x \notin \mathcal{A}_f$, now gives

$$\begin{aligned}
(2\pi i)^n \frac{\partial N_f}{\partial x_j}(x) &= \int_0^{2\pi} \cdots \int_0^{2\pi} Re\left(\frac{e^{x_j+i\theta_j}}{f(z)} \frac{\partial f}{\partial z_j}\right) i^n d\theta_1 \wedge \cdots \wedge d\theta_n \\
&= \int_{Log^{-1}(x)} Re\left(\frac{\partial f}{\partial z_j} \frac{z_j}{f(z)}\right) \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}.
\end{aligned}$$

Hence $\frac{\partial N_f}{\partial x_j}(x) = Re(v_j(x))$.

Let E be a connected component of $\mathbb{R}^n \backslash \mathcal{A}$, then for $x \in E$, $v_j(x)$ is an integer. To see this, we consider

$$\frac{1}{2\pi i} \int_{|z_j| = e^{x_j}} \frac{\partial f}{\partial z_j} \frac{1}{f(z)} dz_j$$

for θ_k fixed for $k \neq j$. As a consequence of the residue formula this integral is equal to the number of zeros minus the number of poles of the function $z_j \mapsto f(z_1, \ldots, z_n)$ in the disk with boundary $|z_j| = e^{x_j}$. Hence it must be an integer. And since it depends continuously on the θ_k it must be independent of them. From this it follows that v_j is constant on E. We will demonstrate this for v_1.

$$\begin{aligned}
(2\pi i)^{n-1} v_1(x) &= \frac{1}{2\pi i} \int_{Log^{-1}(x)} \frac{\partial f}{\partial z_1} \frac{z_1}{f(z)} \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \\
&= \int_0^{2\pi} \cdots \int_0^{2\pi} \frac{1}{2\pi i} \int_0^{2\pi} \frac{\partial f}{\partial z_1} \frac{e^{x_1+i\theta_1}}{f(z)} i^n d\theta_1 \wedge \cdots \wedge d\theta_n \\
&= \int_0^{2\pi} \cdots \int_0^{2\pi} (\frac{1}{2\pi i} \int_{|z_1| = e^{x_1}} \frac{\partial f}{\partial z_1} \frac{1}{f(z)} dz_1) d\theta_2 \wedge \cdots \wedge d\theta_n
\end{aligned}$$

The calculations for other v_j are similar, the only difference is that we first have to change the order of integration.

Notice that we must have the condition $x \in E$. Since if this is not the case $z_j \mapsto f(z_1, \ldots, z_n)$ would have a zero on the boundary $|z_j| = e^{x_j}$ and the residue formula would not hold.

Because $v_j(x)$ is constant for $x \in E$ and E is connected, all the partial derivatives of N_f are constant and hence N_f is an affine function on E and it is given by

$$N_f|_E(x) = \langle (v_1, \ldots, v_n), (x_1, \ldots, x_n) \rangle + C$$

where C is some constant.

Because f is a holomorphic function, $\log|f| : (\mathbb{C}^*)^n \backslash \{f = 0\} \longrightarrow \mathbb{R}$ is a pluriharmonic function. If we set $\log(0) = -\infty$ the we have a plurisubharmonic function $\log|f| : (\mathbb{C}^*)^n \longrightarrow \mathbb{R} \cup \{-\infty\}$, which is strictly plurisubharmonic over {f=0}. The (strict) convexity of N_f now follows from $\log|f|$ being (strictly) plurisubharmonic. $\square$

In the proof of lemma 5.3 we found that the vector $v(x) = (v_1(x), \dots, v_n(x))$ was constant for $x \in E$ where E was a connected component of $\mathbb{R}^n \backslash \mathcal{A}_f$. This gives rise to the following definition.

Definition 5.4. *Let E be a connected component of $\mathbb{R}^n \backslash \mathcal{A}_f$. Then the order of E is the vector*

$$v = (v_1, \dots, v_n) \in \mathbb{Z}^n$$

where v_j is defined as in the proof of lemma 5.3.

From the proof of lemma 5.3 we immediately deduce that the gradient of $N_f(x)$ is equal to the order of the component containing x.

Definition 5.5. *Let $f(z) = \sum_j a_j z^j,\ a_j \in \mathbb{C}$. Then the Newton polyhedron Δ_f of f is defined as*

$$\Delta_f = \textit{Convex hull}\{j \mid a_j \neq 0\} \subset \mathbb{R}^n.$$

Let $A = \{\alpha \in \mathbb{Z} |\ \alpha$ is the order of a connected component of $\mathbb{R}^n \backslash \mathcal{A}_f\}$. Then it can be shown that the convex hull of A coincides with the Newton polyhedron of f. Moreover, there exists a map from $\mathbb{R}^n \backslash \mathcal{A}_f$ to $\Delta_f \cap \mathbb{Z}^n$ which maps different components of the complement of $\mathcal{A}_f$ to different lattice points of Δ_f. This map is given by $x \mapsto v$ where $x \in E$ for E a connected component of $\mathbb{R}^n \backslash \mathcal{A}_f$ and v is the order of E.

5.3 The Spine of an Amoeba

Because of lemma 5.3 we can define the function

$$N_f^\infty := \max_E N_E$$

where $N_E : \mathbb{R}^n \longrightarrow \mathbb{R}$ is obtained by extending $N_f|_E$ to $\mathbb{R}^n$ by linearity and E runs over the components of $\mathbb{R}^n \backslash \mathcal{A}_f$.

Definition 5.6. *The spine $\mathcal{S}_f$ of an amoeba $\mathcal{A}_f$ is the collection of all $x \in \mathbb{R}^n$ at which N_f^∞ is not locally linear.*

In topology the spine of a topological manifold M has to be a strong deformation retract of M. Therefore, for $\mathcal{S}_f$ to be a spine in the topological sense of the word, we have to verify that it is a strong deformation retract of $\mathcal{A}_f$. Recall that $\mathcal{S}_f$ is a strong deformation retract of $\mathcal{A}_f$ if there exists a homotopy $F : \mathcal{A}_f \times I \longrightarrow \mathcal{A}_f$, where $I = [0, 1]$, such that for all $a \in \mathcal{A}_f,\ s \in \mathcal{S}_f$ and $t \in I$:

i. $F(a, 0) = a$

ii. $F(a, 1) \in \mathcal{S}$

iii. $F(s, t) = s \ \ \forall t \in I,\ \forall s \in S$

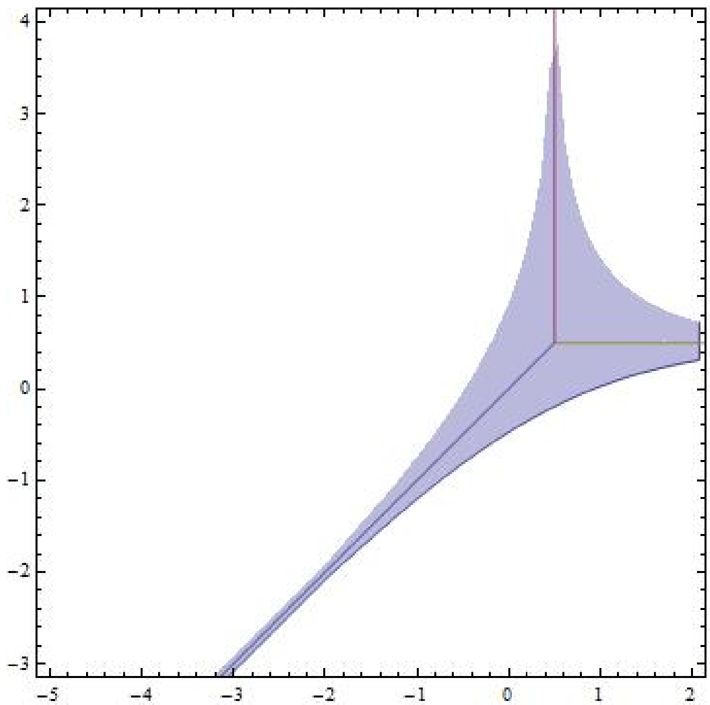

Figure 15: The amoeba of $\{f(x,y) = e^{-\frac{1}{2}}x + e^{-\frac{1}{2}}y + e^{-1}xy = 0\}$ together with its spine

Let E_α denote the connected component of $\mathbb{R}^n \backslash \mathcal{A}_f$ of order α, $c_\alpha = N_f(x) - \langle \alpha, x \rangle$ for $x \in E_\alpha$, $S(x) = \max_{\alpha \in A}(c_\alpha + \langle \alpha, x \rangle)$ and $F_\alpha = \{x \mid S(x) = c_\alpha + \langle \alpha, x \rangle\}$. Notice that the spine $\mathcal{S}_f$ of $\mathcal{A}_f$ is equal to the corner locus of $S(x)$ (i.e. the set of x where $S(x)$ is non smooth), in turn the corner locus of $S(x)$ is equal to the union of the boundaries of the F_α. Now $E_\alpha \subset F_\alpha$.

For every $\alpha \in A$ choose a $p_\alpha \in E_\alpha$ and consider the line segments from the p_α to the boundary of F_α. Then $\mathcal{A}_f$ is contained in the union of these line segments and hence we can retract $\mathcal{A}_f$ along these lines to $\mathcal{S}_f$, see figure 16.

In [PR04] Passare and Rullgard prove that $\mathcal{A}_f$ is contained in the union of these line segments, by showing that if a line segment does not intersect the boundary of any F_α, it does not intersect $\mathcal{A}_f$. The complete proof can be found in [PR04], theorem 1.

5.4 Tropical Amoebas and the Spine of an Amoeba

In the previous subsections we defined the Ronkin function N_f of a polynomial f and derived a piecewise linear function N_f^∞. We defined the spine of $\mathcal{A}_f$ as the collection of all $x \in \mathbb{R}^n$ at which N_f^∞ is not locally linear.

Now we notice that a tropical polynomial is a convex, locally linear function and conversely every convex, locally linear function can be viewed as a tropical polynomial. Thus we can view N_f^∞ as a tropical polynomial. Then the spine of $\mathcal{A}_f$ is exactly the tropical amoeba of N_f^∞. Therefore we find that for each

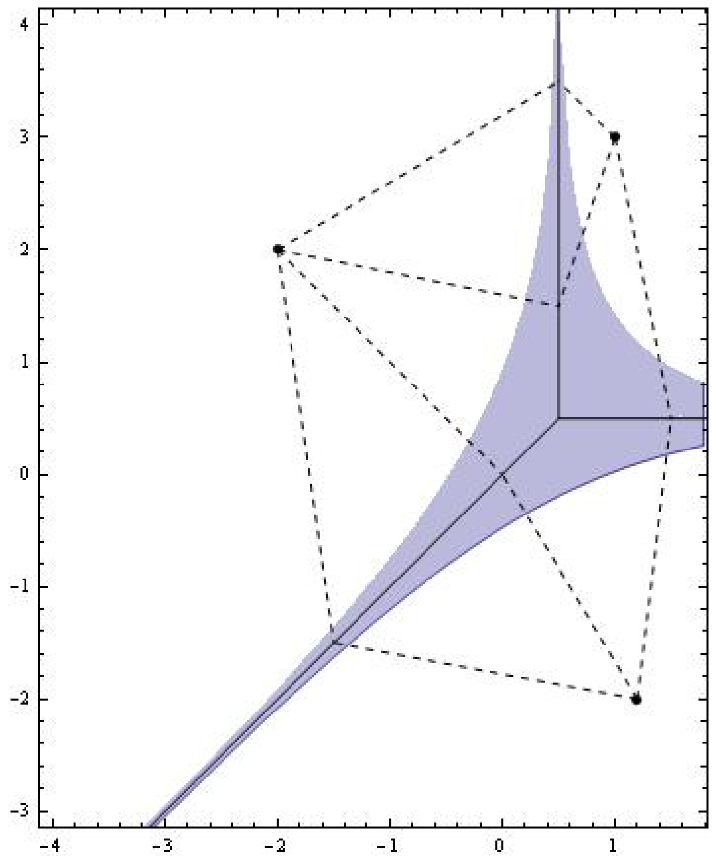

Figure 16: Retracting the amoeba of $e^{-\frac{1}{2}}x + e^{-\frac{1}{2}}y + e^{-1}xy = 0$ to its spine

polynomial $f : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$ we can find a tropical polynomial $F = N_f^\infty$ such that $\mathcal{S}_f$ and V_F coincide.

Can we also do this the other way around? Given a tropical polynomial F, does there exist a polynomial $f : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}$ such that V_F and $\mathcal{S}_f$ coincide? To answer this question, we will have to take a closer look at the Ronkin function.

Let $f(z) = \sum_{j \in J} a_j z^j$ and suppose that on a set $E \subset (\mathbb{C}^*)^n$, $a_0 z^{j_0}$ is a dominating term in the sense that $\left|a_0 z^{j_0}\right| > \left|\sum_{j \in J} \left(a_j z^j\right) - a_0 z^{j_0}\right|$ for all $z \in E$. We can rewrite $f(z)$ to

$$f(z) = a_0 z^{j_0} \left(1 + g(z)\right),$$

where

$$g(z) = \sum_{\substack{j \in J \\ j \neq j_0}} \frac{a_j}{a_0} z^{j - j_0}.$$

Let x be in the component where $a_0 z^{j_0}$ is the dominating term. Then

$$\begin{aligned} N_f(x) &= \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \log|f(z)| \frac{dz}{z} \\ &= \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \log|a_0 z^{j_0}| \frac{dz}{z} + \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \log|1+g(z)| \frac{dz}{z}. \end{aligned}$$

Here $\frac{dz}{z}$ is a short hand notation for $\frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n}$.

We consider the integral with $\log|a_0 z^{j_0}|$ first.

$$\begin{aligned} \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \log|a_0 z^{j_0}| \frac{dz}{z} &= \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \log|a_0| + \log|z^{j_0}| \frac{dz}{z} \\ &= \frac{1}{(2\pi i)^n} \log|a_0| \int_{Log^{-1}(x)} \frac{dz}{z} \\ &\quad + \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \langle j_0, \log|z| \rangle \frac{dz}{z}. \end{aligned}$$

Since $\log|z|$ is constantly equal to x on $Log^{-1}(x)$ and $\int_{Log^{-1}(x)} \frac{dz}{z} = (2\pi i)^n$, we find that

$$\frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \log|a_0 z^{j_0}| \frac{dz}{z} = log|a_0| + \langle j_0, x \rangle. \tag{13}$$

Now we consider the second integral. Because $\left|a_0 z^{j_0}\right| > \left| \sum_{j \in J} \left(a_j z^j\right) - a_0 z^{j_0} \right|$ we find that $|g(z)| < 1$. Hence $log|1+g(z)| = Re\left(\log(1+(g(z))\right)$. Now

$$\begin{aligned} \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \log|1+g(z)| \frac{dz}{z} &= \frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} Re\left(\log(1+g(z))\right) \frac{dz}{z} \\ &= \frac{1}{(2\pi)^n} \int_{Log^{-1}(x)} Re\left(\log(1+g(e^{x_1+i\theta_1}, \ldots, e^{x_n+i\theta_n}))\right) d\theta \\ &= \frac{1}{(2\pi)^n} Re \frac{1}{i^n} \int_{Log^{-1}(x)} \log(1+g(z)) \frac{dz}{z} \\ &= \frac{1}{(2\pi)^n} Re\left((2\pi)^n Res\left(\frac{\log(1+g(z))}{z}\right)\right) \\ &= Re\left(Res\left(\frac{\log(1+g(z))}{z}\right)\right). \end{aligned} \tag{14}$$

To determine the residue of $\frac{\log(1+g(z))}{z}$, we need to find the coefficient of $\frac{1}{z}$ in the expansion of $\frac{\log(1+g(z))}{z}$. But this is exactly the constant term of the expansion of $\log(1+g(z))$. Now

$$\log(1+g(z)) = \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (g(z))^n. \tag{15}$$

Since $g(z) = \sum_{j \neq j_0} \frac{a_j}{a_0} z^{j-j_0}$, we find that

$$
\begin{aligned}
(g(z))^n &= \sum \frac{n!}{\prod k_j!} \prod \left(\frac{a_j}{a_0} z^{j-j_0} \right)^{k_j} && (16) \\
&= \sum \frac{n!}{\prod k_j!} \frac{\prod \left(a_j z^j \right)^{k_j}}{a_0^n z^{n j_0}}, && (17)
\end{aligned}
$$

where we sum over $\{k_j \geq 0\}_{j \in J \backslash j_0}$ such that $\sum k_j = n$. For our logarithmic expansion we now get

$$
\sum_{n \geq 1} \frac{(-1)^{n-1}}{n} (g(z))^n = \sum_{\substack{n \geq 1 \\ \sum k_j = n}} (-1)^{n-1} \frac{(n-1)!}{\prod k_j!} \frac{\prod \left(a_j z^j \right)^{k_j}}{a_0^n z^{n j_0}}. \tag{18}
$$

If we substitute $k_0 = -n$ we get

$$
log(1 + g(z)) = \sum (-1)^{k_0 - 1} \frac{(-k_0 - 1)!}{\prod k_j!} a^k z^{\sum k_j j}, \tag{19}
$$

where $a^k = a_0^{k_0} \ldots a_r^{k_r}$ if $|J| = r+1$ and we sum over all the $k_0 < 0,\ k_j \geq 0$ and $\sum_{j=0}^r k_j = 0$. The constant terms of this expression are precisely those where $\sum k_j j = 0$. Hence we find

$$
\frac{1}{(2\pi i)^n} \int_{Log^{-1}(x)} \log |1 + g(z)| \frac{dz}{z} = Re \left(\sum (-1)^{k_0 - 1} \frac{(-k_0 - 1)!}{\prod k_j!} a^k \right), \tag{20}
$$

where we have to sum over all the $k_0 < 0,\ k_j \geq 0,\ \sum_{j=0}^r k_j = 0$ and $\sum k_j j = 0$.

Therefore if we combine equation 13 and equation 20, we find that

$$
N_f(x) = log|a_0| + \langle j_0, x \rangle + Re \left(\sum_K (-1)^{k_0 - 1} \frac{(-k_0 - 1)!}{\prod k_j!} a^k \right), \tag{21}
$$

with $K = \{k_0 < 0,\ k_j \geq 0,\ \sum_{j=0}^r k_j = 0,\ \sum k_j j = 0\}$.

As a consequence we find that for $f(z) = \sum_{j \in J} a_j z^j$ the amoeba $\mathcal{A}_f$ of f is equal to the tropical amoeba of the polynomial

$$
F = \bigoplus_{j \in J} (\bar{a}_j + \langle j, x \rangle)
$$

where

$$
\bar{a}_j = log|a_j| + Re \left(\sum_K (-1)^{k_0 - 1} \frac{(-k_0 - 1)!}{\prod k_j!} a^k \right).
$$

Notice that in this calculation we need to assume that there exists a dominating term.

We also notice that $e^{\bar{a}_j} = a_j e^{\sum_K (-1)^{k_0 - 1} \frac{(-k_0 - 1)!}{\prod k_j!} a^k}$ is invertible. Hence if we are given $\bar{a}_j$ of a tropical polynomial F, we can use this relation to calculate the a_j and find a polynomial f such that the tropical amoeba V_F and the amoeba $\mathcal{A}_f$ coincide. Now we would like to find that there is a one to one relation between tropical polynomials and the usual polynomials. But there is one problem, we made the assumption that there exists a dominating term. But if we pick a tropical polynomial with random coefficients, this assumption need not be true.

6 Conclusion

In this thesis we considered polynomials in the tropical semiring. We looked at what their graphs look like and found that since the polynomials are piecewise affine linear functions, the graphs are very easy to draw.

Our main concern in this thesis was how tropical geometry relates to other ways of doing geometry. To this end we considered Voronoi diagrams and amoebas in algebraic geometry.

Voronoi diagrams can be constructed using a parabola in one dimension higher than the dimension of the Voronoi diagram itself. But if we take a closer look at this construction and the functions concerned with it, we find that we can construct the Voronoi diagram solely using affine functions. Hence the Voronoi diagram can be constructed using a tropical polynomial. If we jump to power diagrams we even find a one to one relation between tropical polynomials and the diagrams.

In algebraic geometry we can construct the spine of an amoeba using the Ronkin function. For this we need to calculate a quite complicated integral. But using the relation between the function defining the amoeba and the tropical polynomial that defines the spine of the amoeba, the construction becomes more straight forward.

Therefore we find that tropical polynomials also pop up in other parts of geometry and can be considered very helpful there. But also, in this way tropical geometry functions as a bridge between different types of geometry. It might be worth further study to how tropical geometry functions as a bridge between Voronoi diagrams and algebraic geometry. What topics and concepts does it influence? For instance: in algebraic geometry one can be interested in the dual triangulation of the Newton polytope of a Laurent polynomial. In the theory of power diagrams one encounters the concept of a Delaunay triangulation. How do these concepts relate to each other and can tropical geometry help us understand this relation a bit better?

For now these questions remain unanswered. But the link has been made and further study has to show how valuable this link is.

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