Linearized stability in case of state-dependent delay: a simple test example

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Abstract

In the theory of delay functional (Volterra) equations, as well as in ordinary or delay differential equations, the principle of linearized stability is an efficient tool for studying local asymptotic behaviour of non-linear systems near a steady state.

The aim of this thesis is to prove that the principle of linearized stability for Volterra equations can be used even in some cases when the model does not satisfy assumptions of the standard formulation of the principle (see [5]).

We consider a scalar system derived from a Daphnia population model, as an example of a relatively simple model that does not satisfy the assumptions of the standard formulation of the principle of linearized stability for Volterra equations and show that the actual principle applies anyway.

Contents

1 Introduction

The principle of linearized stability is an efficient tool to decide whether a steady state of a non-linear system is locally asymptotically stable or unstable, that is whether or not orbits starting close to the steady state converge to it as time goes to infinity. The reason why the word "linearized" occurs in the name of the principle (or theorem) is that we first linearize our original system around the steady state of interest, draw conclusions about stability or instability of the linearized system and finally prove that the asymptotic behaviour of the non-linear system is qualitatively the same.

In the case of non-linear Volterra functional equations, in particular renewal equations, the approach is based on spectral theory of linear operators. First we find a (non-linear) solution semigroup for the non-linear problem, differentiate it, prove that the derivative is actually the solution semigroup for the linear problem and then decide on its stability or instability using the functional analytic approach. The last and most important part of the proof justifies the already mentioned claim that orbits of the two systems that start close to the corresponding steady states behave qualitatively the same as time goes to infinity.

Roughly speaking, the standard formulation of the principle of linearized stability used for example in [5] requires the right hand side of the system to be differentiable (for details see Section 1.5) from which the differentiability of the corresponding solution semigroup is deduced. The aim of this thesis is to show that even if the right hand side of the system is not differentiable, the corresponding solution semigroup may be differentiable and thus the principle of linearized stability can be applied. We prove this statement for a simplified Daphnia population model.

1.1 Model

In this section I am going to introduce the mathematical model that will be used throughout the whole text. It is motivated by a Daphnia population model from [6], but we can imagine any other population that satisfies the set of assumptions from this section.

Regarding the essentially mathematical nature of the problem, we want to consider the simplest possible model that doesn't satisfy the differentiability condition from the principle of linearized stability. It is convenient to introduce a scaled model which makes the analysis simpler in terms of notation and thus clearer. The full model and the way how we simplify it are described in Appendix A.

Assumptions:

(i) An individual starts consuming food (substrate) at the age of $\epsilon > 0$, not directly after birth.

(ii) An individual cannot reproduce immediately after it is born, but only after it matures. The probability per unit of time of giving birth is zero before maturation and constant after maturation. This constant is denoted by $\beta > 0$.

(iii) An individual matures when it reaches a certain size. In the scaled case an individual matures after it grows by one unit, i.e. the difference of size at birth and at maturation is 1. The size, and therefore the age of maturation, depends on environmental conditions after the individual was born, concretely on the substrate concentration $S(t) \geq 0$.

(iv) The probability $\mathcal{F}(a)$ that an individual survives till age a depends only on a.

(v) There is balance between inflow of the substrate, its degradation and its consumption by Daphnias. We denote S_0 the rate of inflow and $S(t)$ the substrate concentration at time t.

Notation:

ϵ	age at which individuals start consuming food,
τ	maturation age,
$b(t)$	population birth rate at time t,
$S(t)$	food (substrate) concentration at time t,
S_0	rate of substrate supply,
β	(constant) reproduction rate after maturation,
b_t	$b_t(\theta) := b(t+\theta) \quad \text{for } \theta \leq 0 \text{ and } t \geq 0.$

Simple bookkeeping yields the renewal equation with an initial condition,

$$b(t) = \beta \int_{\tau(b_t)}^{\infty} b(t-a)\mathcal{F}(a)da, \quad t \geq 0. \tag{RE}$$

We provide the equation (RE) with the initial condition

$$b(\theta) = \varphi(\theta), \quad \theta < 0. \tag{IC}$$

The symbol $\tau(b_t)$ denotes the maturation age of individuals that mature at time t given the food history, which according to the formula 1.1 below, we can express in terms of the history of b itself. Therefore we use the notation $\tau = \tau(b_t)$. Indeed, we define $S(t)$ by

$$S(t) := \frac{S_0}{1 + \int_{\epsilon}^{\infty} b(t-a)\mathcal{F}(a)da}, \tag{1.1}$$

and then define the age at maturation τ for an individual that matures at time t by requiring that

$$\int_{t-\tau}^{t} S(\sigma)d\sigma = 1. \tag{1.2}$$

Remark. In fact, (1.1) is only an approximation of S derived using the so called quasi-steady-state approximation from a differential equation for S. For the derivation of (1.1) see Appendix A.

We assume that

(i) $\mathcal{F} : [0, \infty) \to [0, 1]$,

(ii) $\mathcal{F}$ is non-increasing,

(iii) $\mathcal{F}$ is continuous on $[0, \infty)$,

(iv) assumption (RoC) from Definition 1.1 below holds.

In the rest of this section I am going to make the definition of the variables and equations more mathematically rigorous.

1.2 State Space

Definition 1.1 *Consider the linear space*

$$X := L^1_\xi(-\infty, 0)$$

of Lebesgue measurable functions such that

$$\|\varphi\|_{1,\xi} := \int_{-\infty}^{0} |\varphi(\theta)| e^{\xi\theta} d\theta < \infty,$$

where $\xi > 0$ is such that

$$\int_0^\infty \mathcal{F}(a) e^{\xi a} da < \infty. \tag{RoC}$$

The abbreviation 'RoC' stands for 'rate of convergence'.

Remark. Assumption (RoC) together with continuity of the map $a \mapsto \mathcal{F}(a)e^{\xi a}$ on $[0, \infty)$ imply that

$$\sup_{a \geq 0} \mathcal{F}(a) e^{\xi a} < \infty. \tag{1.3}$$

I will very often refer to (RoC), but actually use (1.3).

Remark. We have to use the weighted L^1-norm in order to include constants (steady states) into the state space.

Moreover, we need that for any $\varphi \in X$

$$\int_0^\infty \varphi(-a) \mathcal{F}(a) da < \infty,$$

which is guaranteed by (RoC). Indeed, if

$$\sup_{a \geq 0} \mathcal{F}(a) e^{\xi a} = C,$$

then

$$\int_0^\infty \varphi(-a)\mathcal{F}(a)da = \int_0^\infty \varphi(-a)e^{-\xi a}\mathcal{F}(a)e^{\xi a}da \le C\|\varphi\|_{1,\xi} < \infty.$$

Integrability of the function $a \mapsto \mathcal{F}(a)e^{\xi a}$, $a \ge 0$ will be needed in Section 4.1 and beyond.

Remark. The reason why we choose L^1 instead of the presumably more natural continuous functions will be clear later (see the remark on page 18). Roughly speaking, we need the state space to be invariant under translation with extension by a constant value.

Proposition 1.2 *X is a Banach space.*

Proof. X is obviously a normed linear space, so we only need to check its completeness. Let $\{\phi_n\}$, $n = 1, 2, \ldots$ be a Cauchy sequence in X and denote $f_n(\theta) := \phi_n(\theta)e^{\xi\theta}$ for $\theta \le 0$. Then for any n, $m \in \mathbb{N}$

$$\begin{aligned}\|\phi_n - \phi_m\|_{1,\xi} &= \int_{-\infty}^0 |\phi_n(\theta) - \phi_m(\theta)|e^{\xi\theta}d\theta \\ &= \int_{-\infty}^0 |f_n(\theta) - f_m(\theta)|d\theta \\ &= \|f_n - f_m\|_{L^1(-\infty,0)}.\end{aligned}$$

Consequently, $\{f_n\}$ is a Cauchy sequence in $L^1(-\infty, 0)$ which is a complete space, so the sequence converges to f in $L^1(-\infty, 0)$. Let's define a function ϕ by

$$\phi(\theta) := f(\theta)e^{-\xi\theta}, \quad \text{for } \theta \le 0.$$

Then

$$\|\phi\|_{1,\xi} = \int_{-\infty}^0 |\phi(\theta)|e^{\xi\theta}d\theta = \int_{-\infty}^0 |f(\theta)|d\theta = \|f\|_{L^1(-\infty,0)} < \infty,$$

so $\phi \in X$. Moreover,

$$\|\phi_n - \phi\|_{1,\xi} = \int_{-\infty}^0 |\phi_n(\theta) - \phi(\theta)|e^{\xi\theta}d\theta = \|f_n - f\|_{L^1(-\infty,0)} \to 0,$$

and hence ϕ_n converges to ϕ in X. Consequently, the space X is complete.

□

1.3 Maturation Age

The relation between the age τ of individuals that mature at time $t = 0$ and the "history" φ is specified by the equation

$$\int_{-\tau}^{0} \frac{S_0}{1 + \int_{\epsilon}^{\infty} \varphi(s-a)\mathcal{F}(a)da} ds = 1, \tag{1.4}$$

with $\epsilon > 0$ being the age at which individuals start consuming the substrate.

However, this equation in general does not have a unique solution τ for given φ, and therefore some conditions on φ must be imposed. It is convenient to denote

$$F(\tau, \varphi) := \int_{-\tau}^{0} \frac{S_0}{1 + \int_{\epsilon}^{\infty} \varphi(s-a)\mathcal{F}(a)da} ds - 1 \tag{1.5}$$

for $\tau \geq 0$ and $\varphi \in X$. As a consequence, (1.4) takes the form $F(\tau, \varphi) = 0$.

Biological interpretation suggests that φ should be non-negative. Let's consider the subset X_+ of X such that all $\varphi \in X_+$ have non-negative values. This assumption guarantees that $F(\cdot, \varphi)$ is for every φ an increasing function, and thus it crosses 0 at most once.

Existence of τ is a more subtle problem than its uniqueness. We know that $F(0, \varphi) = -1$ for any $\varphi \in X_+$ and that $\mathcal{F}(\cdot, \varphi)$ is an increasing absolutely continuous (and therefore continuous) function, but it is not guaranteed that the integral

$$\int_{-\infty}^{0} \frac{S_0}{1 + \int_{\epsilon}^{\infty} \varphi(s-a)\mathcal{F}(a)da} ds$$

converges to a value greater or equal to 1. If the function

$$s \mapsto \int_{\epsilon}^{\infty} \varphi(s-a)\mathcal{F}(a)da$$

grows so fast with $s \to -\infty$ that

$$\int_{-\infty}^{0} \frac{S_0}{1 + \int_{\epsilon}^{\infty} \varphi(s-a)\mathcal{F}(a)da} ds < 1, \tag{1.6}$$

then the equation $F(\tau, \varphi) = 0$ has no solution. This means that the environmental conditions make it impossible to mature.

Another situation with a similar effect is when $F(\tau, \varphi)$ converges to 1 for $\tau \to \infty$. In this case individuals theoretically 'reach maturation' when time goes to infinity, which in practice leads to the same situation as in the first case.

To summarize, the equation $F(\tau, \varphi) = 0$ has in principle either

a) no solution, neither finite nor infinite, or

b) a solution $\tau = \infty$, or

c) a solution $0 < \tau < \infty$.

In the first two cases no individuals mature, so we can set $b(t) = 0$ for $t \geq 0$. To make sure that τ is defined for all $\varphi \in X_+$, we define $\tau = \infty$ as the solution of the equation $F(\tau, \varphi) = 0$ in the cases a) and b). As opposed to the generic third case, this situation is rather unrealistic, since in real situations we expect the population birth rate to be always finite. If φ is essentially bounded, then the integrand in (1.4) is bounded away from zero and thus we can take τ so large that the value of the integral reaches one. Formulated in mathematical language, if $\varphi \in X_+$ is such that

$$\operatorname*{ess\,sup}_{\theta \leq 0} \varphi(\theta) = c_1 > 0,$$

and

$$\sup_{a \geq 0} \mathcal{F}(a) e^{\xi a} = c_2 > 0,$$

then

$$\begin{aligned}
\int_{-\tau}^{0} \frac{S_0}{1 + \int_\epsilon^\infty \varphi(s-a)\mathcal{F}(a)da} ds &= \int_{-\tau}^{0} \frac{S_0}{1 + \int_\epsilon^\infty \varphi(s-a)e^{-\xi a}\mathcal{F}(a)e^{\xi a}da} ds \\
&\geq \frac{S_0 \tau}{1 + c_1 c_2 \int_\epsilon^\infty e^{-\xi a} da} \\
&= \frac{S_0 \tau}{1 + \frac{c_1 c_2}{\xi} e^{-\xi \epsilon}}. \qquad (1.7)
\end{aligned}$$

For $\tau = (1 + \frac{c_1 c_2}{\xi} e^{-\xi\epsilon})/S_0$ we have (1.7) $= 1$ and consequently $F(\tau, \varphi) \geq 0$, so $\tau(\varphi)$ exists finite (here we use continuity of $F(\cdot, \varphi)$, the fact that it is an increasing function and that $F(0, \varphi) = -1$ for any $\varphi \in X_+$).

To conclude, for any non-negative $\varphi \in X_+$ there exists a solution τ of the equation (1.4), denoted $\tau(\varphi)$. This solution is unique.

Finally if $t \geq 0$, then the equality

$$1 = \int_{t-\tau}^{t} \frac{S_0}{1 + \int_\epsilon^\infty b(\sigma - a)\mathcal{F}(a)da} d\sigma = \int_{-\tau}^{0} \frac{S_0}{1 + \int_\epsilon^\infty b_t(\sigma - a)\mathcal{F}(a)da} d\sigma$$

shows that $\tau(b_t)$ is well-defined and its interpretation is indeed the maturation age of individuals that mature at time t (given the initial condition φ).

Remarks.

1. There is an inconsistency in the case when the equation (1.4) does not have a finite solution. The initial condition $\varphi \in X_+$ is non-zero, hence some individuals were born in the past. But then there must exist some mature individuals that gave birth to them. Moreover, due to generally unbounded support of the survival probability $\mathcal{F}$ and the fact that reproduction rate is constant after maturation, these individuals are still able and even expected to give birth at present, which contradicts the definition $b(t) = 0$ for $t \geq 0$.

This problem is caused by the quasi-steady-state approximation that we used to derive the simplified model introduced in Section 1.1 (see Appendix A). Since the case when environmental conditions make it impossible to mature is neither interesting, nor realistic, we accept this inconsistency.

2. Non-negativity of φ is not necessary for the equation (1.4) to have a unique solution, but it seems to be the most reasonable assumption in view of the biological interpretation.

   From the mathematical point of view it is sufficient to assume for instance

$$\int_0^\infty \varphi(s-a)\mathcal{F}(a)da \geq -\frac{1}{2} \tag{1.8}$$

   for all $s \leq 0$. Under this assumption the denominator in the definition of F is bounded away from zero, thus F is well-defined, $F(\cdot, \varphi)$ is an increasing function, so the proof of existence and uniqueness of solution τ to the equation $F(\tau, \varphi) = 0$ for any $\varphi \in X$ such that (1.8) holds can be performed the same way as above. Assumption (1.8) is obviously satisfied for any non-negative φ.

In the rest of this section we prove, using the Implicit Function Theorem, that if τ is defined (and finite) for a given $\varphi_+ \in X_+$, then there exists a neighborhood of φ_+ in X where τ is defined (and finite).

Lemma 1.3 *Let $\psi \in X$ and $t \in [0, T]$. Then*

$$\|\psi_t^o\|_{1,\xi} = e^{-\xi t}\|\psi\|_{1,\xi},$$

where ψ_t^o is defined according to the convention (2.2).

Proof.

$$\begin{aligned} \|\psi_t^o\|_{1,\xi} &= \int_{-\infty}^0 |\psi_t^o(\theta)|e^{\xi\theta}d\theta \\ &= \int_{-\infty}^{-t} |\psi_t^o(\theta)|e^{\xi\theta}dy + \int_{-t}^0 |\psi_t^o(\theta)|e^{\xi\theta}d\theta \end{aligned} \tag{1.9}$$

Since $\psi_t^o(\theta) = 0$ for $t + \theta \in [0, T]$ (see (2.2)), the second integral in (1.9) is zero. Hence

$$\begin{aligned} \|\psi_t^o\|_{1,\xi} &= \int_{-\infty}^{-t} |\psi_t(\theta)|e^{\xi\theta}dy \\ &= e^{-\xi t}\int_{-\infty}^0 |\psi(y)|e^{\xi y}dy \\ &= e^{-\xi t}\|\psi\|_{1,\xi}. \end{aligned}$$

□

Let's consider any $\varphi_+ \in X_+$ for which $\tau(\varphi_+)$ exists finite and verify the assumptions of the Implicit Function Theorem.

Assumption 1

Let's extend the function F so that it is defined on $\mathbb{R} \times X$,

$$F(\tau,\varphi) := \begin{cases} \int_{-\tau}^{0} \frac{S_0}{1+\int_\epsilon^\infty \varphi(s-a)\mathcal{F}(a)da} ds - 1 & \text{if } \tau \geq 0 \\ -1 & \text{if } \tau < 0 \end{cases} \quad \text{for } \varphi \in X. \tag{1.10}$$

Then $F : \mathbb{R} \times X \to \mathbb{R}$, where $\mathbb{R}$ and X are Banach spaces.

Assumption 2

Next we need to check that there exists a neighbourhood V of φ_+ in X such that $F : (0,\infty) \times V \to \mathbb{R}$ is continuously differentiable. Let V be a neighbourhood of φ_+ in X and suppose that

$$\int_\epsilon^\infty \varphi(s-a)\mathcal{F}(a)da \geq -\frac{1}{2} \quad \text{for all } \varphi \in V \text{ and } s \leq 0.$$

Then for any $\varphi \in V$ and $\tau > 0$ the derivative

$$D_1F(\tau,\varphi)\sigma = \frac{S_0\,\sigma}{1+\int_\epsilon^\infty \varphi(-\tau-a)\mathcal{F}(a)da}, \quad \sigma \in \mathbb{R}, \tag{1.11}$$

is well-defined, because the denominator cannot become zero (see the remark on page 11). Assumption (RoC) allows us to denote

$$\sup_{a\geq 0} \mathcal{F}(a)e^{\xi a} =: C, \quad C \in (0,\infty).$$

For $\varphi \in V$ and $\tau > 0$ we have

$$\begin{aligned}
&\lim_{(\tilde\tau,\tilde\varphi)\to(\tau,\varphi)} \|D_1F(\tau,\varphi) - D_1F(\tilde\tau,\tilde\varphi)\| = \\
&= \lim_{(\tilde\tau,\tilde\varphi)\to(\tau,\varphi)} \sup_{0<\sigma\leq 1} |D_1F(\tau,\varphi)\sigma - D_1F(\tilde\tau,\tilde\varphi)\sigma| \\
&= \lim_{(\tilde\tau,\tilde\varphi)\to(\tau,\varphi)} \sup_{0<\sigma\leq 1} \sigma \left| \frac{S_0}{1+\int_\epsilon^\infty \varphi(-\tau-a)\mathcal{F}(a)da} - \frac{S_0}{1+\int_\epsilon^\infty \tilde\varphi(-\tilde\tau-a)\mathcal{F}(a)da} \right| \\
&= \lim_{(\tilde\tau,\tilde\varphi)\to(\tau,\varphi)} S_0 \left| \frac{\int_\epsilon^\infty \big(\varphi(\tau-a) - \tilde\varphi(-\tilde\tau-a)\big)\mathcal{F}(a)da}{(1+\int_\epsilon^\infty \tilde\varphi(-\tilde\tau-a)\mathcal{F}(a)da)(1+\int_\epsilon^\infty \varphi(-\tau-a)\mathcal{F}(a)da)} \right| \\
&= \lim_{(\tilde\tau,\tilde\varphi)\to(\tau,\varphi)} S_0 \left| \frac{\int_\epsilon^\infty \big(\varphi(\tau-a) - \tilde\varphi(-\tilde\tau-a)\big)e^{-\xi a}\mathcal{F}(a)e^{\xi a}da}{(1+\int_\epsilon^\infty \tilde\varphi(-\tilde\tau-a)\mathcal{F}(a)da)(1+\int_\epsilon^\infty \varphi(-\tau-a)\mathcal{F}(a)da)} \right| \\
&\leq \lim_{(\tilde\tau,\tilde\varphi)\to(\tau,\varphi)} \frac{S_0 C\,\|\varphi_{-\tau} - \tilde\varphi_{-\tilde\tau}\|_{1,\xi}}{(1+\int_\epsilon^\infty \tilde\varphi(-\tilde\tau-a)\mathcal{F}(a)da)(1+\int_\epsilon^\infty \varphi(-\tau-a)\mathcal{F}(a)da)}.
\end{aligned} \tag{1.12}$$

Since $\varphi \in V$ and $\tilde{\varphi} \to \varphi$ (so we can suppose that $\tilde{\varphi}$ satisfies the defining inequality for V as well), the denominator in (1.12) is always positive. Moreover, Lemma 2.3 yields continuity of the map $\tau \mapsto \varphi^o_\tau$, from which we conclude

$$\begin{aligned}(1.12) &= \lim_{(\tilde{\tau},\tilde{\varphi})\to(\tau,\varphi)} S_0 C \frac{\|\varphi_{-\tau} - \varphi_{-\tilde{\tau}} + \varphi_{-\tilde{\tau}} - \tilde{\varphi}_{-\tilde{\tau}}\|_{1,\xi}}{(1+\int_\epsilon^\infty \tilde{\varphi}(-\tilde{\tau}-a)\mathcal{F}(a)da)(1+\int_\epsilon^\infty \varphi(-\tau-a)\mathcal{F}(a)da)} \\ &\leq \lim_{(\tilde{\tau},\tilde{\varphi})\to(\tau,\varphi)} S_0 C \frac{\|\varphi_{-\tau} - \varphi_{-\tilde{\tau}}\|_{1,\xi} + \|\varphi_{-\tilde{\tau}} - \tilde{\varphi}_{-\tilde{\tau}}\|_{1,\xi}}{(1+\int_\epsilon^\infty \tilde{\varphi}(-\tilde{\tau}-a)\mathcal{F}(a)da)(1+\int_\epsilon^\infty \varphi(-\tau-a)\mathcal{F}(a)da)}\end{aligned}$$

Lemma 2.3 implies $\|\varphi_{-\tau} - \varphi_{-\tilde{\tau}}\|_{1,\xi} \to 0$ as $(\tilde{\tau},\tilde{\varphi}) \to (\tau,\varphi)$ and Lemma 1.3 yields $\|\varphi_{-\tilde{\tau}} - \tilde{\varphi}_{-\tilde{\tau}}\|_{1,\xi} \to 0$ as $(\tilde{\tau},\tilde{\varphi}) \to (\tau,\varphi)$. Altogether

$$\lim_{(\tilde{\tau},\tilde{\varphi})\to(\tau,\varphi)} \|D_1 F(\tau,\varphi) - D_1(\tilde{\tau},\tilde{\varphi})\| = 0$$

and hence $(\tau,\varphi) \mapsto D_1 F(\tau,\varphi)$ is continuous on $(0,\infty) \times V$.

Let's turn to the Fréchet derivative of $F(\tau,\cdot)$. For $\tau > 0$ and $\varphi \in V$ we get

$$D_2 F(\tau,\varphi)\psi = -S_0 \int_{-\tau}^0 \frac{\int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da}{(1+\int_\epsilon^\infty \varphi(s-a)\mathcal{F}(a)da)^2} ds, \quad \psi \in X. \tag{1.13}$$

We observe that $D_2 F(\cdot,\varphi)\psi$ is absolutely continuous and therefore continuous on $(0,\infty)$, and $D_2 F(\tau,\cdot)\psi$ is due to positivity of the denominator continuous on V. Now we need to check that $D_2 F$ is continuous on $(0,\infty) \times V$. Let's first assume that $\tilde{\tau}$ approaches τ from above, which I denote by $\tilde{\tau} \to \tau_+$ (here the subscript + means something different than in φ_+).

$$\begin{aligned}
&\lim_{(\tilde\tau,\tilde\varphi)\to(\tau_+,\varphi)} \|D_2F(\tau,\varphi)-D_2F(\tilde\tau,\tilde\varphi)\| = \\
&= \lim_{(\tilde\tau,\tilde\varphi)\to(\tau_+,\varphi)} \sup_{\|\psi\|_{1,\xi}\le 1} |D_2F(\tau,\varphi)\psi - D_2F(\tilde\tau,\tilde\varphi)\psi| \\
&= S_0 \lim_{(\tilde\tau,\tilde\varphi)\to(\tau_+,\varphi)} \sup_{\|\psi\|_{1,\xi}\le 1} \left| \int_{-\tau}^{0} \frac{\int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da}{\left(1+\int_\epsilon^\infty \varphi(s-a)\mathcal{F}(a)da\right)^2} ds \right. \\
&\qquad\qquad\qquad \left. - \int_{-\tilde\tau}^{0} \frac{\int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da}{\left(1+\int_\epsilon^\infty \tilde\varphi(s-a)\mathcal{F}(a)da\right)^2} ds \right| \\
&\le S_0 \lim_{(\tilde\tau,\tilde\varphi)\to(\tau_+,\varphi)} \sup_{\|\psi\|_{1,\xi}\le 1} \left| \int_{-\tau}^{0} \frac{\int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da}{\left(1+\int_\epsilon^\infty \varphi(s-a)\mathcal{F}(a)da\right)^2} - \frac{\int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da}{\left(1+\int_\epsilon^\infty \tilde\varphi(s-a)\mathcal{F}(a)da\right)^2} ds \right| \\
&\quad + S_0 \lim_{(\tilde\tau,\tilde\varphi)\to(\tau_+,\varphi)} \sup_{\|\psi\|_{1,\xi}\le 1} \left| \int_{-\tilde\tau}^{-\tau} \frac{\int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da}{\left(1+\int_\epsilon^\infty \tilde\varphi(s-a)\mathcal{F}(a)da\right)^2} ds \right| \\
&= S_0 \lim_{(\tilde\tau,\tilde\varphi)\to(\tau_+,\varphi)} \sup_{\|\psi\|_{1,\xi}\le 1} \left| \int_{-\tau}^{0} \int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da \left(\frac{1}{\left(1+\int_\epsilon^\infty \varphi(s-a)\mathcal{F}(a)da\right)^2} \right.\right. \\
&\qquad\qquad\qquad \left.\left. - \frac{1}{\left(1+\int_\epsilon^\infty \tilde\varphi(s-a)\mathcal{F}(a)da\right)^2} \right) ds \right| \\
&\quad + S_0 \lim_{(\tilde\tau,\tilde\varphi)\to(\tau_+,\varphi)} \sup_{\|\psi\|_{1,\xi}\le 1} \left| \int_{-\tilde\tau}^{-\tau} \frac{\int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da}{\left(1+\int_\epsilon^\infty \tilde\varphi(s-a)\mathcal{F}(a)da\right)^2} ds \right| \qquad (1.14)
\end{aligned}$$

If again $C = \sup_{a\ge 0} \mathcal{F}(a)e^{\xi a}$ we obtain

$$\begin{aligned}
\left| \int_\epsilon^\infty \psi(s-a)\mathcal{F}(a)da \right| &= \left| e^{-\xi s} \int_\epsilon^\infty \psi(s-a) e^{\xi(s-a)} \mathcal{F}(a) e^{\xi a} da \right| \\
&\le Ce^{-\xi s} \int_\epsilon^\infty |\psi(s-a)|\, e^{\xi(s-a)} da \\
&= Ce^{-\xi s} \int_{-\infty}^{s-\epsilon} |\psi(\theta)|\, e^{\xi\theta} d\theta \\
&\le Ce^{-\xi s} \|\psi\|_{1,\xi},
\end{aligned}$$

for any $s \leq 0$ and thus

$$
\begin{aligned}
(1.14) \leq S_0 C \lim_{(\tilde{\tau},\tilde{\varphi})\to(\tau_+,\varphi)} \sup_{\|\psi\|_{1,\xi}\leq 1} \|\psi\|_{1,\xi} \Bigg| \int_{-\tau}^{0} & \frac{e^{-\xi s}}{\left(1+\int_{\epsilon}^{\infty}\varphi(s-a)\mathcal{F}(a)da\right)^2} \\
& - \frac{e^{-\xi s}}{\left(1+\int_{\epsilon}^{\infty}\tilde{\varphi}(s-a)\mathcal{F}(a)da\right)^2} ds \Bigg| \\
+ S_0 C \lim_{(\tilde{\tau},\tilde{\varphi})\to(\tau_+,\varphi)} \sup_{\|\psi\|_{1,\xi}\leq 1} \|\psi\|_{1,\xi} & \left| \int_{-\tilde{\tau}}^{-\tau} \frac{e^{-\xi s}}{\left(1+\int_{\epsilon}^{\infty}\tilde{\varphi}(s-a)\mathcal{F}(a)da\right)^2} ds \right| \\
= S_0 C \lim_{(\tilde{\tau},\tilde{\varphi})\to(\tau_+,\varphi)} & \left| \int_{-\tau}^{0} \frac{e^{-\xi s}}{\left(1+\int_{\epsilon}^{\infty}\varphi(s-a)\mathcal{F}(a)da\right)^2} - \frac{e^{-\xi s}}{\left(1+\int_{\epsilon}^{\infty}\tilde{\varphi}(s-a)\mathcal{F}(a)da\right)^2} ds \right| \\
+ S_0 C \lim_{(\tilde{\tau},\tilde{\varphi})\to(\tau_+,\varphi)} & \left| \int_{-\tilde{\tau}}^{-\tau} \frac{e^{-\xi s}}{\left(1+\int_{\epsilon}^{\infty}\tilde{\varphi}(s-a)\mathcal{F}(a)da\right)^2} ds \right| \qquad (1.15)
\end{aligned}
$$

The first limit in (1.15) is zero due to continuity of $D_2F(\tau,\cdot)$ in V and continuity of $D_2F(\cdot,\varphi)$ on $(0,\infty)$ ensures that the second limit in (1.15) is zero as well. Hence, D_2F is continuous on $(0,\infty)\times V$.

Assumption 3

Last assumption requires

$$\sigma \mapsto D_1F(\tau,\varphi)\sigma \quad \text{for}(\tau,\varphi)\in(0,\infty)\times V \text{ and } \sigma\in\mathbb{R} \qquad (1.16)$$

to be a Banach space isomorphism of $\mathbb{R}$ onto $\mathbb{R}$.

This assumption is also satisfied, because

$$D_1F(\tau,\varphi)\sigma = \frac{S_0\,\sigma}{1+\int_{\epsilon}^{\infty}\varphi(-\tau-a)\mathcal{F}(a)da},$$

so $D_1F(\tau,\varphi)$ is a linear map of the form $\sigma\mapsto C\sigma$ for $\sigma\in R$ and some constant $C\in\mathbb{R}$, $C\neq 0$ which is obviously an isomorphism.

To conclude, the Implicit Function Theorem implies that τ is defined in a neighbourhood of φ_+ in X.

1.4 Renewal Equation

For clarity I repeat the renewal equation with corresponding initial condition that we are going to use throughout the text and add some necessary mathematical assumptions to its definition.

$$b(t) = \beta \int_{\tau(b_t)}^{\infty} b(t-a)\mathcal{F}(a)da, \quad t\geq 0, \qquad \text{(RE)}$$

$$b(\theta) = \varphi(\theta) \quad \text{for } \theta<0, \text{ and } \varphi\in X_+, \qquad \text{(IC)}$$

so the identity is satisfied in the sense of L^1, i.e. b and φ are actually equivalence classes containing L^1_ξ-integrable functions that coincide in all points except a set of (Lebesgue) measure zero.

I add an expression for the basic reproduction number R_0,

$$R_0 = \beta \int_\tau^\infty \mathcal{F}(a)da,$$

where τ is the unique solution of the equation (1.2) with b being identically zero, i.e. of the equation

$$\int_{-\tau}^0 S_0 ds = 1.$$

Since τ can be easily expressed explicitly as $\tau = 1/S_0$, we obtain

$$R_0 = \beta \int_{\frac{1}{S_0}}^\infty \mathcal{F}(a)da.$$

Let's assume that $R_0 > 1$.

1.5 Problem Description

Consider a renewal equation with an initial condition

$$x(t) = F(x_t), \quad t > 0, \tag{1.17}$$

$$x(\theta) = \varphi(\theta), \quad \theta \leq 0. \tag{1.18}$$

Suppose that $\varphi \in X$ and that $F : X \to \mathbb{R}$ is continuously Fréchet differentiable. Then we can use the principle of linearized stability, here presented as Theorem 1.4, that in generic cases[1] tell us whether a steady state $\bar{x}$ of (1.17) is locally exponentially stable or unstable.

Theorem 1.4 *[5] Assume that $F : X \to \mathbb{R}$ is continuously Fréchet differentiable. Let $\bar{x}$ be a steady state of (1.17) and let $k \in L^1_\xi(0,\infty) \cap L^\infty_\xi(0,\infty)$ represent $DF(\bar{x})$:*

$$DF(\bar{x})\varphi = \int_0^\infty k(s)\varphi(-s)ds.$$

(a) If all the roots of the characteristic equation

$$\int_0^\infty k(a)e^{-\lambda a}da = 1 \tag{1.19}$$

have negative real part, then the steady state $\bar{x}$ is exponentially stable.

(b) If there exists at least one root λ of (1.19) with positive real part, then the steady state $\bar{x}$ is unstable.

[1] If the characteristic equation (1.19) has no roots with positive real part but some root with real part zero, then Theorem 1.4 does not provide any information.

For the model introduced in Section 1.1 we obtain

$$F(\varphi) = \beta \int_{\tau(\varphi)}^{\infty} \varphi(-a)\mathcal{F}(a)da,$$

with the derivative $DF(\varphi)$ for $\varphi \in X$ formally expressed as

$$DF(\varphi)\psi = -\beta\varphi(-\tau(\varphi))\mathcal{F}(\tau(\varphi))[D\tau(\varphi)]\psi + \beta \int_{\tau(\varphi)}^{\infty} \psi(-a)\mathcal{F}(a)da.$$

However, the expression on the right hand-side is not defined, because φ is in fact an equivalence class of functions that coincide everywhere except in a set of measure zero. Hence it is not possible to consider a value of φ in a single point, as this value by definition is not known. In other words, F is not differentiable and thus does not satisfy the assumptions of Theorem 1.4.

Remark. Later on we will be interested in a derivative of F in a steady state (i.e. a constant function) $\bar{\varphi}$. Note that in that case we know even now what $DF(\bar{\varphi})$ means, because a constant function is continuous and thus the value $\varphi(-\tau(\varphi))$ is known (we consider the continuous representant of the L^1 equivalence class).

The following sections will be devoted to proving an analogous result to Theorem 1.4 for the equation (RE), (IC). Concretely we prove that even though the map F is not differentiable, the solution semigroup corresponding to the system is differentiable and the principle of linearized stability can be applied.

2 Solution Operator and Its Derivative

2.1 Construction of the Solution

For a given $\varphi \in X_+$ I am going to construct a solution of (RE) using the method of steps and thus prove existence and uniqueness of solutions to (RE), (IC).

Let's first observe that $\tau(\cdot)$ is bounded away from zero. Using non-negativity of φ, the definition by the equation (1.4) yields

$$1 = \int_{-\tau}^{0} \frac{S_0}{1 + \int_{\epsilon}^{\infty} \varphi(s-a)\mathcal{F}(a)da} ds \leq \int_{-\tau}^{0} S_0 ds = S_0\tau$$

and consequently,

$$\frac{1}{S_0} \leq \tau. \tag{2.1}$$

Let $T < \min\{\epsilon, 1/S_0\}$ and $t \in [0,T]$. In order for φ_t to belong to X_+, we extend the function φ by zero on $[0,t]$. The concrete value of the extension is actually irrelevant for the solution itself (this follows from the discussion below), but later it becomes convenient that the value is zero. So we define φ_t^o for $t \in [0, T]$ by

$$\varphi_t^o(\theta) := \begin{cases} \varphi(t+\theta) & t+\theta < 0, \\ 0 & t+\theta \in [0,t]. \end{cases} \tag{2.2}$$

This convention applies whenever the superscript ^{o} is used.

Remark. Note that $(\varphi + \psi)_t^o = \varphi_t^o + \psi_t^o$ for all φ, $\psi \in X$ and $t \geq 0$.

Remark. Now it becomes obvious why the space of continuous functions on $(-\infty, 0]$ would not work as a state space. If we shift and extend $\varphi \in C(-\infty, 0]$ as in (2.2), the resulting function φ_t^o is in general not continuous. Hence, if the state space was $C(-\infty, 0]$, φ_t^o would not belong to the state space.

Recall the definition of $\tau(b_t)$,

$$\int_{-\tau}^{0} \frac{S_0}{1 + \int_\epsilon^\infty b_t(s-a)\mathcal{F}(a)da} ds = 1. \tag{2.3}$$

Step 1. Recall that $T < \min\{\epsilon, 1/S_0\}$.

For $t \in [0, T]$, $s \leq 0$ and $a \geq \epsilon$ we have

$$t + s - a \leq t - a \leq T - \epsilon < 0. \tag{2.4}$$

Since $b(\theta) = \varphi(\theta)$ for $\theta < 0$, (2.3) and (2.4) imply that $\tau(b_t) = \tau(\varphi_t^o)$.

Moreover, in (RE)

$$b(t) = \beta \int_{\tau(b_t)}^{\infty} b(t-a)\mathcal{F}(a)da, \quad t \geq 0,$$

we have

$$t - a \leq T - \tau(b_t) = T - \tau(\varphi_t^o) \leq T - \frac{1}{S_0} < 0, \tag{2.5}$$

so for the same reason as in (2.4) only φ (or more precisely φ_t^o) comes into play.

Altogether, for $t \in [0, T]$ the equation (RE) becomes

$$b(t) = \beta \int_{\tau(\varphi_t^o)}^{\infty} \varphi(t-a)\mathcal{F}(a)da, \tag{2.6}$$

and hence on $[0, T]$ the solution of (RE), (IC) is given explicitly by (2.6).

Consider a formally defined solution operator $\Sigma(t, \cdot) : X_+ \to X_+$ for $t \in [0, T]$,

$$\Sigma(t, \varphi) := b_t,$$

i.e. for $\theta \leq 0$ and $t \in [0, T]$ we have

$$\Sigma(t,\varphi)(\theta) := \begin{cases} \varphi(t+\theta) & t+\theta < 0, \\ \beta \int_{\tau(\varphi_{t+\theta}^o)}^{\infty} \varphi(t+\theta-a)\mathcal{F}(a)da & t+\theta \in [0,T]. \end{cases} \tag{2.7}$$

In the L^1 class we can safely construct the solution on $(-\infty, T]$ by attaching the just obtained piece b on $[0, T]$ to the initial condition φ on $(-\infty, 0)$ and iterate the process to acquire the solution on $[T, 2T]$ as follows.

Step 2. We define the solution operator $\Sigma(t, \cdot) : X_+ \to X_+$ for $t \in [T, 2T]$ by

$$\Sigma(t, \varphi) = \Sigma(t - T, \Sigma(T, \varphi)), \quad \varphi \in X_+.$$

In other words, for $t \in (T, 2T]$ the equation (RE) becomes

$$b(t) = \beta \int_{\tau(b_t^T)}^{\infty} b^T(t-a)\mathcal{F}(a)da, \tag{2.8}$$

where

$$b^T(t) := \begin{cases} \varphi(t) & \text{for } t \in (-\infty, 0), \\ b(t) & \text{given by (2.6) for } t \in [0, T]. \end{cases}$$

The same way as before we extend the solution to $(-\infty, 2T]$ by attaching the new piece of b on $(T, 2T]$ to b^T and denote this extended solution b^{2T}. Then b^{2T} obviously satisfies the equation (RE) for $t \in (-\infty, 2T]$.

This process can be iterated infinitely many times to gradually build the whole solution b on $\mathbb{R}$ and define the corresponding non-linear solution semigroup $\Sigma(t, \cdot) : X_+ \to X_+$ for $t \geq 0$. $\{\Sigma(t, \cdot)\}_{t \geq 0}$ is indeed a semigroup, because for the solution b of (RE) with an initial condition $\varphi \in X$ and any t, $s \geq 0$ we can write

$$\Sigma(t, \Sigma(s, \varphi)) = \Sigma(t, b_s) = b_{t+s} = \Sigma(t + s, \varphi).$$

2.2 Continuity of the Solution on $[0, \infty)$

In this section I am going to prove that the solution b of (RE), (IC) is continuous on $[0, \infty)$ (the corresponding L^1 class contains a continuous function).

First I recall the following two well-known results – density of continuous functions with compact support in L^1 and the Moore-Osgood theorem on exchanging limits. Then I present the proofs of two auxiliary (also known) results that are in principle continuity of translation in L^1 (Lemma 2.3) and continuity of the function that arises by convolution of an L^1- with a bounded measurable function (Theorem 2.4).

Lemma 2.1 *[18] For any $f \in L^1(\mathbb{R})$ there exists a sequence $f^n \in C_c(\mathbb{R})$ (compactly supported continuous functions) such that*

$$\|f^n - f\|_{L^1(\mathbb{R})} \to 0 \quad \text{for } n \to \infty.$$

Theorem 2.2 (Moore-Osgood) *Let X be a metric space and let the functions $f^n : X \to \mathbb{R}$ and $f : X \to \mathbb{R}$ be defined on a neighbourhood $\mathcal{U}$ of x_0 in X. Suppose that*

1. $f^n(x)$ converges to $f(x)$ uniformly on $\mathcal{U}$,

2. the limit $\lim_{x\to x_0} f^n(x)$ exists and is finite for any $n \in \mathbb{N}$.

Then $\lim_{x\to x_0} f(x)$ exists and

$$\lim_{n\to\infty} \lim_{x\to x_0} f^n(x) = \lim_{x\to x_0} f(x).$$

Lemma 2.3 *Consider $f \in L^1(\mathbb{R})$. Then the function*

$$t \mapsto (a \mapsto f(t-a))$$

acting from $\mathbb{R}$ to $L^1(0,\infty)$ is continuous, i.e.

$$\lim_{\eta\to 0} \int_0^\infty |f(t+\eta-a) - f(t-a)| da = 0$$

for any $t \in \mathbb{R}$.

Proof. Let $f \in L^1(\mathbb{R})$. Lemma 2.1 yields a sequence of continuous and compactly supported functions $\{f^n\}_{n=1}^\infty$ on $\mathbb{R}$ such that

$$\lim_{n\to\infty} \|f^n - f\|_{L^1(\mathbb{R})} = 0.$$

Continuity and compact support of f^n for any $n \in \mathbb{N}$ imply

$$\begin{aligned}\lim_{\eta\to 0} \int_0^\infty |f^n(t+\eta-a) - f^n(t-a)| da &= \int_0^\infty \lim_{\eta\to 0} |f^n(t+\eta-a) - f^n(t-a)| da \\ &= 0. \qquad (2.9)\end{aligned}$$

Moreover, we have

$$\begin{aligned}&\lim_{n\to\infty} \int_0^\infty |f^n(t+\eta-a) - f^n(t-a)| da \\ &= \lim_{n\to\infty} \int_0^\infty |f^n(t+\eta-a) - f(t+\eta-a) + f(t-a) - f^n(t-a) + f(t+\eta-a) - f(t-a)| da \\ &\le \lim_{n\to\infty} \int_0^\infty |f^n(t+\eta-a) - f(t+\eta-a)| da + \lim_{n\to\infty} \int_0^\infty |f(t-a) - f^n(t-a)| da \\ &\qquad + \int_0^\infty |f(t+\eta-a) - f(t-a)| da \\ &= \lim_{n\to\infty} \int_{-\infty}^{t+\eta} |f^n(x) - f(x)| dx + \lim_{n\to\infty} \int_{-\infty}^{t} |f(x) - f^n(x)| dx \\ &\qquad + \int_0^\infty |f(t+\eta-a) - f(t-a)| da \\ &\le \lim_{n\to\infty} \|f^n - f\|_{L^1(\mathbb{R})} + \lim_{n\to\infty} \|f - f^n\|_{L^1(\mathbb{R})} + \int_0^\infty |f(t+\eta-a) - f(t-a)| da \\ &= \int_0^\infty |f(t+\eta-a) - f(t-a)| da \qquad (2.10)\end{aligned}$$

and the convergence in (2.10) is obviously uniform on the t-interval $[0, \infty)$.

Using (2.9) and (2.10) with the convergence uniformity, we apply the Moore-Osgood theorem (Theorem 2.2) that allows us to interchange the limits and write

$$\begin{aligned}\lim_{\eta\to 0}\int_0^\infty |f(t+\eta-a)-f(t-a)|da &= \lim_{\eta\to 0}\lim_{n\to\infty}\int_0^\infty |f^n(t+\eta-a)-f^n(t-a)|da\\ &= \lim_{n\to\infty}\lim_{\eta\to 0}\int_0^\infty |f^n(t+\eta-a)-f^n(t-a)|da\\ &= 0.\end{aligned}$$

□

Theorem 2.4 *If $f \in L^1(\mathbb{R})$ and $g \in L^\infty(0,\infty)$, then their convolution product*

$$t \mapsto \int_0^\infty f(t-a)g(a)da, \quad t \in \mathbb{R} \tag{2.11}$$

is continuous.

Proof. Let $f \in L^1(\mathbb{R})$ and $g \in L^\infty(0,\infty)$. Then

$$\begin{aligned}&\lim_{\eta\to 0}\left|\int_0^\infty f(t+\eta-a)g(a)da - \int_0^\infty f(t-a)g(a)da\right|\\ &\le \lim_{\eta\to 0}\int_0^\infty |f(t+\eta-a)-f(t-a)|\,|g(a)|da\\ &\le \|g\|_{L^\infty(0,\infty)}\lim_{\eta\to 0}\int_0^\infty |f(t+\eta-a)-f(t-a)|\,da.\end{aligned} \tag{2.12}$$

Lemma 2.3 yields that the limit in (2.12) is zero and therefore the convolution defined in (2.11) is continuous.

□

Corollary 2.5 *Consider $\varphi \in X$, τ defined by (1.4) and $\mathcal{F}$ bounded and measurable satisfying (RoC). Then the solution b of (RE), (IC) is continuous on $[0,\infty)$.*

Proof. First I recall the notation used in Section 2.1 when constructing the solution by method of steps. We denote

$$b^T(x) := \begin{cases}\varphi(x) & x<0\\ \beta\int_{\tau(\varphi_x)}^\infty \varphi(x-a)\mathcal{F}(a)da & x\in[0,T]\end{cases}$$

and then inductively

$$b^{nT}(x) := \begin{cases}b^{(n-1)T}(x) & x\in(-\infty,(n-1)T]\\ \beta\int_{\tau(b_x^{((n-1)T)(o)})}^\infty b^{(n-1)T}(x-a)\mathcal{F}(a)da & x\in[(n-1)T,nT]\end{cases}$$

for $n = 2, 3, \ldots$.

Let $\varphi \in X$ and consider the functions $f^n : \mathbb{R} \to \mathbb{R}$ defined by

$$f^n(x) := \begin{cases} b^{nT}(x)e^{\xi x} & -\infty < x \le nT, \\ 0, & nT < x, \end{cases} \qquad x \in \mathbb{R}.$$

and $g : [0,\infty) \times [T,\infty) \to \mathbb{R}$ defined by

$$g(a,c) := \begin{cases} 0, & a \le c, \\ \mathcal{F}(a)e^{\xi a}, & c < a, \end{cases} \qquad a \ge 0,\ c \ge T.$$

Then obviously $f \in L^1(\mathbb{R})$ and $g(\cdot, c)$ is due to assumption (RoC) bounded and measurable for any $c \ge T$. Now we apply Theorem 2.4 that yields continuity of the map

$$t \mapsto \int_0^\infty f^n(t-a)g(a,c)da = \beta \int_c^\infty b^{nT}(t-a)e^{\xi(t-a)}\mathcal{F}(a)e^{\xi a}da \qquad (2.13)$$
$$= \beta e^{\xi t} \int_c^\infty b^{nT}(t-a)\mathcal{F}(a)da$$

for any $c \ge T$ and $t \in [0, (n+1)T]$.

Continuity of the exponential function implies that

$$t \mapsto \beta e^{-\xi t} e^{\xi t} \int_c^\infty b^{nT}(t-a)\mathcal{F}(a)da = \beta \int_c^\infty b^{nT}(t-a)\mathcal{F}(a)da$$

is continuous for all $t \in [0, (n+t)T]$ and $c \ge T$. Hence the function $u : [T,\infty) \times \mathbb{R}_+ \to \mathbb{R}$ defined by

$$u(c,t) := \beta \int_c^\infty b^{nT}(t-a)\mathcal{F}(a)da$$

is continuous with respect to both c and t (continuity with respect to c follows from absolute continuity of Lebesgue integral as a function of lower limit).

Moreover, τ is continuous (even continuously differentiable, see Section 2.3) and Lemma 2.3 implies that the map $t \mapsto b_t^{nT}$ is continuous too. Then their composition $t \mapsto \tau(b_t^{(nT)(o)})$ is continuous and since $\tau(b_t^{(nT)(o)}) \ge T$, we conclude that also

$$t \mapsto u(\tau(b_t^{(nT)(o)}), t) = \beta \int_{\tau(b_t^{(nT)(o)})}^\infty b^{nT}(t-a)\mathcal{F}(a)da$$

is a continuous function. Indeed, for $t \in [0, (n+t)T]$ and $s \to t$ we have

$$\begin{aligned} &|u(\tau(b_t^{(nT)(o)}), t) - u(\tau(b_s^{(nT)(o)}), s)| \\ &= |u(\tau(b_t^{(nT)(o)}), t) - u(\tau(b_s^{(nT)(o)}), t) + u(\tau(b_s^{(nT)(o)}), t) - u(\tau(b_s^{(nT)(o)}), s)| \\ &\le \underbrace{|u(\tau(b_t^{(nT)(o)}), t) - u(\tau(b_s^{(nT)(o)}), t)|}_{\to 0} + \underbrace{|u(\tau(b_s^{(nT)(o)}), t) - u(\tau(b_s^{(nT)(o)}), s)|}_{\to 0}. \end{aligned}$$

Finally, arbitrariness of n implies continuity of b on $[0,\infty)$.

□

2.3 Fréchet Derivative of $\tau(\cdot)$

Before differentiating $\Sigma(t,\cdot)$, let's find the Fréchet derivative of τ that will be needed later.

Recall from Section 1.3 that we have verified assumptions of the Implicit Function Theorem for $F : (0,\infty) \times V \to \mathbb{R}$, where V is an X-neighbourhood of such $\varphi \in X_+$ that $\tau(\varphi)$ exists finite.

If $\varphi \in X_+$ is such that $\tau(\varphi)$ exists finite (for instance if φ is essentially bounded), then the Implicit Function Theorem yields differentiability of $\tau(\cdot)$ at φ (see Section 1.3). Thus, we can use the chain rule and obtain the equality

$$D_1F(\tau(\varphi),\varphi) \circ D\tau(\varphi)\psi + D_2F(\tau(\varphi,\varphi))\psi = 0. \tag{2.14}$$

for $\psi \in X$. Since

$$D_1F(\tau(\varphi),\varphi) \circ D\tau(\varphi)\psi = \frac{\partial}{\partial\tau}F(\tau(\varphi),\varphi) \cdot D\tau(\varphi)\psi,$$

we can rearrange (2.14) into

$$D\tau(\varphi)\psi = -\frac{D_2F(\tau(\varphi),\varphi)\psi}{\frac{\partial}{\partial\tau}F(\tau(\varphi),\varphi)}.$$

Concretely,

$$D\tau(\varphi)\psi := \left(1+\int_\epsilon^\infty \varphi(-\tau(\varphi)-a)\mathcal{F}(a)da\right)\int_{-\tau(\varphi)}^0 \frac{\int_\epsilon^\infty \psi(\sigma-a)\mathcal{F}(a)da}{(1+\int_\epsilon^\infty \varphi(\sigma-a)\mathcal{F}(a)da)^2}d\sigma. \tag{2.15}$$

2.4 Fréchet Derivative of $\Sigma(t,\cdot)$

Let $\varphi \in X_+$ and $\psi \in X$. We can formally express the candidate for the Fréchet derivative of $\Sigma(t,\cdot)$ for $t \in [0,T]$ at φ applied to ψ by

$$[D\Sigma(t,\varphi)\psi](\theta) = \begin{cases} \psi(t+\theta) & t+\theta<0, \\ -\beta\varphi(t+\theta-\tau(\varphi^o_{t+\theta}))\mathcal{F}(\tau(\varphi^o_{t+\theta}))[D\tau(\varphi^o_{t+\theta})]\psi^o_{t+\theta} \\ \quad +\beta\int_{\tau(\varphi^o_{t+\theta})}^\infty \psi(t+\theta-a)\mathcal{F}(a)da & 0 \le t+\theta \le T, \end{cases} \tag{2.16}$$

with $D\tau(\varphi^o_{t+\theta})$ following from (2.15),

$$\begin{aligned}&\left[D\tau(\varphi^o_{t+\theta})\right]\psi^o_{t+\theta} = \\ &= \left(1+\int_\epsilon^\infty \varphi(t+\theta-\tau(\varphi^o_{t+\theta})-a)\mathcal{F}(a)da\right)\int_{-\tau(\varphi^o_{t+\theta})}^0 \frac{\int_\epsilon^\infty \psi(t+\theta+\sigma-a)\mathcal{F}(a)da}{(1+\int_\epsilon^\infty \varphi(t+\theta+\sigma-a)\mathcal{F}(a)da)^2}d\sigma.\end{aligned} \tag{2.17}$$

$D\Sigma(t,\varphi)\psi$ from (2.16) is a well-defined element of X if and only if

$$\theta \mapsto \varphi(t+\theta-\tau(\varphi^o_{t+\theta})) \tag{2.18}$$

is a well-defined function on $[-t, 0]$. Since $\varphi \in X_+$ and $t + \theta - \tau(\varphi^o_{t+\theta}) \leq 0$, it is sufficient for $\theta \mapsto \varphi(t + \theta - \tau(\varphi^o_{t+\theta}))$ to be a well-defined function if the argument $t + \theta - \tau(\varphi^o_{t+\theta})$ satisfies

$$\frac{d}{d\theta}(t + \theta - \tau(\varphi^o_{t+\theta})) \neq 0$$

for all $\theta \in (-t, 0)$. Since t is for the time being considered to be fixed and the expression above depends only on $t + \theta$, it is sufficient to prove that

$$\frac{d}{dx}(x - \tau(\varphi^o_x)) \neq 0$$

for all $x \in (0, T)$. Let's denote

$$g(\tau, x) := \int_{-\tau}^{0} S(x + \sigma) d\sigma - 1,$$

where

$$S(x + \sigma) = \frac{S_0}{1 + \int_{\epsilon}^{\infty} \varphi(x + \sigma - a)\mathcal{F}(a)da} d\sigma.$$

Then the defining equation for $\tau(\varphi^o_x)$ is of the form $g(\tau, x) = 0$. As $S(x) \neq 0$ for all $x \in [0, T]$ and Theorem 2.4 implies that the function

$$x \mapsto S(x),$$

defined for $x \in [0, T]$ is continuous, the Implicit Function Theorem yields

$$\frac{d\tau}{dx}(x) = -\frac{\frac{dg}{dx}(\tau, x)}{\frac{dg}{d\tau}(\tau, x)} = -\frac{\int_{-\tau}^{0} \frac{d}{dx} S(x + \sigma) d\sigma}{S(x - \tau)}$$

$$= -\frac{\int_{-\tau}^{0} \frac{d}{d\sigma} S(x + \sigma) d\sigma}{S(x - \tau)} = -\frac{S(x) - S(x - \tau)}{S(x - \tau)} =$$

$$= 1 - \frac{S(x)}{S(x - \tau)} < 1.$$

Then

$$\frac{d}{dx}(x - \tau(\varphi^o_x)) = 1 - \frac{d}{dx}\tau(\varphi^o_x) > 0$$

for all $x \in (0, T)$ and thus the function in (2.18) is well-defined.

Remark. The fact that the function $x \mapsto x - \tau(\varphi^o_x)$ is increasing can also be inferred from the biological interpretation. If it was decreasing, then some individuals would mature later than other individuals that were born after them, which is under our assumptions clearly impossible. Similarly, individuals that were born at the same time must also mature at the same time, so the function $x \mapsto x - \tau(\varphi^o_x)$ cannot be constant.

Remark. Due to appearance of φ in the first term of (2.16), $D\Sigma(t,\varphi)\psi$ for $t \in [0,T]$ is not in general continuous on $-t \le \theta \le 0$. However, continuity of the constant function $\bar{\varphi}$ implies continuity of the actual object of interest $D\Sigma(t,\bar{\varphi})\psi$ on $[-t,0]$ for any $\psi \in X$. Details of this implication are presented in Section 4.1.

Let's now prove that (2.16) with (2.17) is indeed the Fréchet derivative of $\Sigma(t,\cdot)$. For any $\varphi \in X_+$ the operator $D\Sigma(t,\varphi)$ is clearly linear. Moreover, we know from Section 2.3 that $D\tau(\varphi_{t+\theta})$ is a bounded linear functional from X to $\mathbb{R}$, which together with assumption (RoC) yields boundedness of $D\Sigma(t,\varphi)$. Hence, $D\Sigma(t,\varphi)$ is for any $\varphi \in X_+$ and any $t \in [0,T]$ a well-defined bounded linear operator from X to X.

Theorem 2.6 *Suppose that $\mathcal{F} : [0,\infty) \to [0,1]$ is continuous and satisfies (RoC). For $\varphi \in X_+$ and $t \in [0,T]$ we have*

$$\lim_{\psi \to 0} \frac{\|\Sigma(t,\varphi+\psi) - \Sigma(t,\varphi) - D\Sigma(t,\varphi)\psi\|_{1,\xi}}{\|\psi\|_{1,\xi}} = 0, \tag{2.19}$$

where $\psi \in X$ and $D\Sigma(t,\varphi)\psi$ is defined by (2.16).

Proof. For the numerator from (2.19) we get

$$\|\Sigma(t,\varphi+\psi) - \Sigma(t,\varphi) - D\Sigma(t,\varphi)\psi\|_{1,\xi} \tag{2.20}$$

$$= \int_{-\infty}^{0} |\Sigma(t,\varphi+\psi)(\theta) - \Sigma(t,\varphi)(\theta) - [\Sigma(t,\varphi)\psi](\theta)|\, e^{\xi\theta} d\theta \tag{2.21}$$

$$\begin{aligned}
\le & \int_{-\infty}^{-t} |(\varphi+\psi)(t+\theta) - \varphi(t+\theta) - \psi(t+\theta)|\, e^{\xi\theta} d\theta \\
& + \int_{-t}^{0} \left| \beta \int_{\tau(\varphi^o_{t+\theta}+\psi^o_{t+\theta})}^{\infty} \psi(t+\theta-a)\mathcal{F}(a)da - \beta \int_{\tau(\varphi^o_{t+\theta})}^{\infty} \psi(t+\theta-a)\mathcal{F}(a)da \right| e^{\xi\theta} d\theta \\
& + \beta \int_{-t}^{0} \left| \int_{\tau(\varphi^o_{t+\theta}+\psi^o_{t+\theta})}^{\infty} \varphi(t+\theta-a)\mathcal{F}(a)da - \int_{\tau(\varphi^o_{t+\theta})}^{\infty} \varphi(t+\theta-a)\mathcal{F}(a)da \right. \\
& \qquad \left. + \varphi(t+\theta-\tau(\varphi^o_{t+\theta}))\mathcal{F}(\tau(\varphi^o_{t+\theta}))D\tau(\varphi^o_{t+\theta})\psi^o_{t+\theta} \right| e^{\xi\theta} d\theta
\end{aligned} \tag{2.22}$$

Let's denote the three additive terms I_1, I_2 and I_3 respectively. Then I_1 is obviously identically zero. Using the fact that the function $t \mapsto \tau(\varphi^o_t)$ is continuous (see Section 2.2) and that the interval $[-t, 0]$ is compact, we obtain

$$
\begin{aligned}
I_2 &= \beta \int_{-t}^{0} \left| \int_{\tau(\varphi^o_{t+\theta}+\psi^o_{t+\theta})}^{\tau(\varphi^o_{t+\theta})} \psi(t+\theta-a)\mathcal{F}(a)da \right| e^{\xi\theta} d\theta \\
&\leq \max_{s\in[-t,0]} \beta \int_{-t}^{0} \left| \int_{\tau(\varphi^o_{t+s}+\psi^o_{t+s})}^{\tau(\varphi^o_{t+s})} \psi(t+\theta-a)\mathcal{F}(a)da \right| e^{\xi\theta} d\theta \\
&\leq \max_{s\in[-t,0]} \beta \left| \int_{\tau(\varphi^o_{t+s}+\psi^o_{t+s})}^{\tau(\varphi^o_{t+s})} \int_{-t}^{0} |\psi(t+\theta-a)|\, e^{\xi\theta} d\theta \mathcal{F}(a)da \right| \\
&= \max_{s\in[-t,0]} \beta \left| \int_{\tau(\varphi^o_{t+s}+\psi^o_{t+s})}^{\tau(\varphi^o_{t+s})} \int_{-a}^{t-a} |\psi(y)|\, e^{\xi y} dy\, e^{\xi(a-t)} \mathcal{F}(a)da \right| \\
&\leq \max_{s\in[-t,0]} \beta \|\psi\|_{1,\xi} \left| \int_{\tau(\varphi^o_{t+s}+\psi^o_{t+s})}^{\tau(\varphi^o_{t+s})} e^{\xi(a-t)} \mathcal{F}(a)da \right| \\
&\leq \beta \|\psi\|_{1,\xi} e^{-\xi t} \max_{s\in[-t,0]} \left| \int_{\tau(\varphi^o_{t+s}+\psi^o_{t+s})}^{\tau(\varphi^o_{t+s})} e^{\xi a} da \right| \\
&= \beta \|\psi\|_{1,\xi}\, e^{-\xi t} \max_{s\in[-t,0]} \frac{\left| e^{\xi\tau(\varphi^o_{t+s}+\psi^o_{t+s})} - e^{\xi\tau(\varphi^o_{t+s})} \right|}{\xi}.
\end{aligned}
$$

We get

$$
\lim_{\psi\to 0} \frac{I_2}{\|\psi\|_{1,\xi}} = \frac{\beta\, e^{-\xi t}}{\xi} \lim_{\psi\to 0} \max_{s\in[-t,0]} \left| e^{\xi\tau(\varphi^o_{t+s}+\psi^o_{t+s})} - e^{\xi\tau(\varphi^o_{t+s})} \right|. \tag{2.23}
$$

Let's denote $s_m \in [-t, 0]$ the point where the maximum on the right hand-side of (2.23) is attained, i.e.

$$
(2.23) = \frac{\beta\, e^{-\xi t}}{\xi} \lim_{\psi\to 0} \left| e^{\xi\tau(\varphi^o_{t+s_m}+\psi^o_{t+s_m})} - e^{\xi\tau(\varphi^o_{t+s_m})} \right|
$$

It follows from Lemma 1.3 that $\|\psi_{t+s_m}\|_{1,\xi} \leq \|\psi\|_{1,\xi}$ for all $t \in [0, T]$, so $\|\psi\|_{1,\xi} \to 0$ implies $\|\psi_{t+s_m}\|_{1,\xi} \to 0$. Together with continuity of τ it yields

$$
\lim_{\psi\to 0} \frac{I_2}{\|\psi\|_{1,\xi}} = \frac{\beta\, e^{-\xi t}}{\xi} \lim_{\psi_{t+s_m}\to 0} \left| e^{\xi\tau(\varphi^o_{t+s_m}+\psi^o_{t+s_m})} - e^{\xi\tau(\varphi^o_{t+s_m})} \right| = 0. \tag{2.24}
$$

Now it only remains to prove that

$$
\lim_{\psi\to 0} \frac{I_3}{\|\psi\|_{1,\xi}} = 0.
$$

Let's first substitute s for $t+\theta$ in I_3,

$$I_3 = \beta \int_{-t}^{0} \bigg| \int_{\tau(\varphi^o_{t+\theta}+\psi^o_{t+\theta})}^{\infty} \varphi(t+\theta-a)\mathcal{F}(a)da - \int_{\tau(\varphi^o_{t+\theta})}^{\infty} \varphi(t+\theta-a)\mathcal{F}(a)da$$
$$+ \varphi(t+\theta-\tau(\varphi^o_{t+\theta}))\mathcal{F}(\tau(\varphi^o_{t+\theta}))D\tau(\varphi^o_{t+\theta})\psi^o_{t+\theta}\bigg| e^{\xi\theta}d\theta \tag{2.25}$$

$$= \beta \int_{0}^{t} \bigg| \int_{\tau(\varphi^o_{s}+\psi^o_{s})}^{\infty} \varphi(s-a)\mathcal{F}(a)da - \int_{\tau(\varphi^o_{s})}^{\infty} \varphi(s-a)\mathcal{F}(a)da$$
$$+ \varphi(s-\tau(\varphi^o_{s}))\mathcal{F}(\tau(\varphi^o_{s}))D\tau(\varphi^o_{s})\psi^o_{s}\bigg| e^{\xi(s-t)}ds. \tag{2.26}$$

If the integrand in (2.26) converges with $\psi \to 0$, then it converges uniformly on $[-t.0]$. Supposing it converges, we can exchange the limit and the integral and write

$$\lim_{\psi\to 0} \frac{I_3}{\|\psi\|_{1,\xi}} = \beta \int_{0}^{t} \lim_{\psi\to 0} \frac{1}{\|\psi\|_{1,\xi}} \bigg| \int_{\tau(\varphi^o_{s}+\psi^o_{s})}^{\infty} \varphi(s-a)\mathcal{F}(a)da - \int_{\tau(\varphi^o_{s})}^{\infty} \varphi(s-a)\mathcal{F}(a)da$$
$$+ \varphi(s-\tau(\varphi^o_{s}))\mathcal{F}(\tau(\varphi^o_{s}))D\tau(\varphi^o_{s})\psi^o_{s}\bigg| e^{\xi(s-t)}ds. \tag{2.27}$$

At the end of this proof we find that the sequence indeed converges and therefore this step was legal.

From differentiability of τ (see Section 2.3) and Lemma 1.3 we have

$$\lim_{\psi\to 0} \frac{|\tau(\varphi^o_s+\psi^o_s) - \tau(\varphi^o_s) - D\tau(\varphi^o_s)\psi^o_s|}{\|\psi\|_{1,\xi}} = 0 \quad \text{for all } s\in[0,t]. \tag{2.28}$$

Moreover, we use the Riemann summation method to write

$$\lim_{\tau\downarrow\tau_0} \frac{1}{\tau-\tau_0}\left(\int_{\tau}^{\infty} \varphi(s-a)\mathcal{F}(a)da - \int_{\tau_0}^{\infty} \varphi(s-a)\mathcal{F}(a)da \right)$$
$$= \lim_{\tau\downarrow\tau_0} \frac{-1}{\tau-\tau_0} \int_{\tau_0}^{\tau} \varphi(s-a)\mathcal{F}(a)da$$
$$= -\lim_{\tau\downarrow\tau_0} \frac{1}{\tau-\tau_0} \lim_{m\to\infty} \sum_{k=0}^{m-1} \varphi\left(s-\tau_0-\frac{k}{m}(\tau-\tau_0)\right)\mathcal{F}\left(\tau_0+\frac{k}{m}(\tau-\tau_0)\right)\frac{\tau-\tau_0}{m}$$
$$= -\lim_{\tau\downarrow\tau_0}\lim_{m\to\infty} \frac{1}{m}\sum_{k=0}^{m-1} \varphi\left(s-\tau_0-\frac{k}{m}(\tau-\tau_0)\right)\mathcal{F}\left(\tau_0+\frac{k}{m}(\tau-\tau_0)\right), \tag{2.29}$$

where $s\in[0,t]$ is a variable. Lemma 2.1 now yields a sequence $\{\varphi^n\}$ for $n = 1, 2, \ldots$ of compactly supported continuous functions such that $\varphi^n \to \varphi$ for $n\to\infty$ in X. Indeed, if we define

$$f(x) := \begin{cases} 0 & \text{for } x>0, \\ \varphi(x)e^{\xi x} & \text{for } x\le 0, \end{cases}$$

then Lemma 2.1 yields a sequence $\{f^n\}$ for $n = 1, 2, \ldots$ of compactly supported continuous functions such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f^n(x) - f(x)| dx = 0.$$

If we for all n define $\varphi^n(x) := f^n(x)e^{-\xi x}$ for $x \leq 0$, then all the functions φ^n are continuous and compactly supported and

$$\begin{aligned} \|\varphi^n - \varphi\|_{1,\xi} &= \int_{-\infty}^{0} |\varphi^n(x)e^{\xi x} - \varphi(x)e^{\xi x}| dx \\ &= \int_{-\infty}^{0} |f^n(x) - f(x)| dx \to 0. \end{aligned}$$

Then (2.29) is equal to

$$- \lim_{\tau\downarrow\tau_0} \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \lim_{n\to\infty} \varphi_n \left(s - \tau_0 - \frac{k}{m}(\tau - \tau_0) \right) \mathcal{F} \left(\tau_0 + \frac{k}{m}(\tau - \tau_0) \right) \tag{2.30}$$

almost everywhere in $[0, t]$. Since for $n \to \infty$ as well as for $m \to \infty$ the sequence in (2.30) converges uniformly with respect to τ (at least on some neighbourhood of τ_0), we can exchange the order of the limits (see Theorem 2.2) and use continuity of φ_n and $\mathcal{F}$ to write

$$\begin{aligned} (2.30) &= - \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \lim_{n\to\infty} \lim_{\tau\downarrow\tau_0} \varphi_n \left(s - \tau_0 - \frac{k}{m}(\tau - \tau_0) \right) \mathcal{F} \left(\tau_0 + \frac{k}{m}(\tau - \tau_0) \right) \\ &= - \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \lim_{n\to\infty} \varphi_n(s - \tau_0)\mathcal{F}(\tau_0) \\ &= - \lim_{m\to\infty} \frac{1}{m} \sum_{k=0}^{m-1} \varphi(s - \tau_0)\mathcal{F}(\tau_0) \\ &= -\varphi(s - \tau_0)\mathcal{F}(\tau_0), \end{aligned} \tag{2.31}$$

while the equality holds in $L^1(0, t)$. Proceeding analogously for τ approaching τ_0 from below we obtain

$$\begin{aligned} &\lim_{\tau\uparrow\tau_0} \frac{1}{\tau - \tau_0} \left(\int_{\tau}^{\infty} \varphi(s - a)\mathcal{F}(a)da - \int_{\tau_0}^{\infty} \varphi(s - a)\mathcal{F}(a)da \right) \\ &\quad = \lim_{\tau\uparrow\tau_0} \frac{1}{\tau - \tau_0} \int_{\tau}^{\tau_0} \varphi(s - a)\mathcal{F}(a)da \\ &\quad = -\varphi(s - \tau_0)\mathcal{F}(\tau_0) \end{aligned} \tag{2.32}$$

for almost all $s \in [0, t]$. Equations (2.31) and (2.32) together yield

$$\lim_{\tau\to\tau_0} \left| \frac{1}{\tau - \tau_0} \left(\int_{\tau}^{\infty} \varphi(s - a)\mathcal{F}(a)da - \int_{\tau_0}^{\infty} \varphi(s - a)\mathcal{F}(a)da \right) + \varphi(s - \tau_0)\mathcal{F}(\tau_0) \right| = 0 \tag{2.33}$$

almost everywhere in $[0,t]$.

Finally we use (2.27), (2.28) and (2.33) as follows.

$$
\begin{aligned}
&\lim_{\psi\to 0}\frac{I_3}{\|\psi\|_{1,\xi}} = \\
&= \beta\int_0^t \lim_{\psi\to 0}\frac{1}{\|\psi\|_{1,\xi}}\bigg|\int_{\tau(\varphi_s^o+\psi_s^o)}^{\infty}\varphi(s-a)\mathcal{F}(a)da - \int_{\tau(\varphi_s^o)}^{\infty}\varphi(s-a)\mathcal{F}(a)da \\
&\qquad + \varphi(s-\tau(\varphi_s^o))\mathcal{F}(\tau(\varphi_s^o))(\tau(\varphi_s^o+\psi_s^o)-\tau(\varphi_s^o)) \\
&\qquad - \varphi(s-\tau(\varphi_s^o))\mathcal{F}(\tau(\varphi_s^o))(\tau(\varphi_s^o+\psi_s^o)-\tau(\varphi_s^o)-D\tau(\varphi_s^o)\psi_s^o)\bigg| e^{\xi(s-t)}ds \\
&\le \beta\int_0^t \lim_{\psi\to 0}\frac{1}{\|\psi\|_{1,\xi}}\bigg|\int_{\tau(\varphi_s^o+\psi_s^o)}^{\infty}\varphi(s-a)\mathcal{F}(a)da - \int_{\tau(\varphi_s^o)}^{\infty}\varphi(s-a)\mathcal{F}(a)da \\
&\qquad + \varphi(s-\tau(\varphi_s^o))\mathcal{F}(\tau(\varphi_s^o))(\tau(\varphi_s^o+\psi_s^o)-\tau(\varphi_s^o))\bigg| e^{\xi(s-t)}ds \\
&\qquad + \beta\int_0^t\bigg|\varphi(s-\tau(\varphi_s^o))\mathcal{F}(\tau(\varphi_s^o))\lim_{\psi\to 0}\frac{|\tau(\varphi_s^o+\psi_s^o)-\tau(\varphi_s^o)-D\tau(\varphi_s^o)\psi_s^o)|}{\|\psi\|_{1,\xi}}\bigg| e^{\xi(s-t)}ds \\
&= \beta\int_0^t \lim_{\psi\to 0}\bigg|\frac{\int_{\tau(\varphi_s^o+\psi_s^o)}^{\infty}\varphi(s-a)\mathcal{F}(a)da - \int_{\tau(\varphi_s^o)}^{\infty}\varphi(s-a)\mathcal{F}(a)da}{\tau(\varphi_s^o+\psi_s^o)-\tau(\varphi_s^o)} + \varphi(s-\tau(\varphi_s^o))\mathcal{F}(\tau(\varphi_s^o))\bigg| \\
&\qquad \cdot\lim_{\psi\to 0}\frac{|\tau(\varphi_s^o+\psi_s^o)-\tau(\varphi_s^o)|}{\|\psi\|_{1,\xi}}e^{\xi(s-t)}ds
\end{aligned}
$$

The second limit is due to differentiability of τ at φ_s^o for all $s\in[0,t]$ uniformly bounded (the map $s\mapsto\tau(\varphi_s^o)$ is continuous and we consider s from a compact interval $[0,t]$), and thus we can write

$$
\begin{aligned}
&\lim_{\psi\to 0}\frac{I_3}{\|\psi\|_{1,\xi}} \le \\
&\le \beta K\int_0^t \lim_{\psi\to 0}\bigg|\frac{\int_{\tau(\varphi_s^o+\psi_s^o)}^{\infty}\varphi(s-a)\mathcal{F}(a)da - \int_{\tau(\varphi_s^o)}^{\infty}\varphi(s-a)\mathcal{F}(a)da}{|\tau(\varphi_s^o+\psi_s^o)-\tau(\varphi_s^o)|} \\
&\qquad + \varphi(s-\tau(\varphi_s^o))\mathcal{F}(\tau(\varphi_s^o))\bigg| e^{\xi(s-t)}ds
\end{aligned} \tag{2.34}
$$

for some constant $K>0$.

Now we would like to use (2.33) to deduce that the limit inside the integral in (2.34) is zero almost everywhere with respect to $s\in[0,t]$. We still need to justify this step, since τ_0 currently depends on s, whereas in (2.33) we proved that for any fixed τ_0 the limit is zero almost everywhere in $[0,t]$.

As φ is an element of L^1, it is necessary that $s-\tau(\varphi_s^o)$ is a function of s, i.e. that $\tau(\varphi_s^o)$ is not equal to some constant plus s. However, at the beginning of this section (see the page 24) we checked that

$$\frac{d}{ds}\left(s - \tau(\varphi_s^o)\right) > 0$$

for all $s \in [0, T]$, so this cannot happen.

Thus, we conclude from Lemma 1.3, continuity of τ and (2.33) that the limit inside the integral in (2.34) is zero almost everywhere for $s \in [0, t]$. As a consequence, the integral is zero and thus

$$\lim_{\psi \to 0} \frac{I_3}{\|\psi\|_{1\xi}} = 0,$$

which completes the proof.

□

To summarize, we have obtained a linearization of the semigroup $\Sigma(t, \cdot)$ for $t \in [0, T]$.

Differentiability of $\Sigma(t, \cdot)$ for $t \in [0, T]$ allows us to use the chain rule to derive $D\Sigma(t, \varphi)$ for any $t \geq 0$ as follows. Using the semigroup property of Σ for $t \in (T, 2T]$ we get

$$D\Sigma(t, \varphi) = D\big(\Sigma(t - T, \cdot) \circ \Sigma(T, \varphi)\big) = D\Sigma(t - T, \Sigma(T, \varphi))D\Sigma(T, \varphi).$$

The object on the right hand-side is already known, since $0 \leq t - T, T \leq T$. By iterating this process we obtain $D\Sigma(t, \varphi)$ for all $t \geq 0$.

3 Steady States

3.1 Trivial steady state

The renewal equation (RE) obviously has a zero steady state. Our previous assumption on R_0 implies instability of the trivial steady state, which can be seen as follows. Recall from Section 1.4 that

$$R_0 = \beta \int_{\frac{1}{S_0}}^{\infty} \mathcal{F}(a) da$$

and that we assume $R_0 > 1$. Consider the constant initial condition

$$\varphi(\theta) = \delta, \quad \theta \leq 0,$$

where $\delta > 0$ is small. Then

$$\|\varphi\|_{1,\xi} = \delta \int_{-\infty}^{0} e^{\xi\theta} d\theta = \frac{\delta}{\xi}.$$

We know from the defining equation (1.4) for τ that $\tau(0) = 1/S_0$ and $\tau(\delta) > \tau(0)$. Then for $t \in [0,T]$ we have

$$\begin{aligned} b(t) &= \beta\delta \int_{\tau(\delta)}^{\infty} \mathcal{F}(a)da \\ &= \beta\delta \int_{\tau(0)}^{\infty} \mathcal{F}(a)da - \beta\delta \int_{\tau(0)}^{\tau(\delta)} \mathcal{F}(a)da \\ &= \delta \left(R_0 - \beta \int_{\tau(0)}^{\tau(\delta)} \mathcal{F}(a)da \right) \end{aligned}$$

and

$$\Sigma(t,\varphi)(\theta) = \begin{cases} \delta, & \text{if } t+\theta < 0, \\ \delta \left(R_0 - \beta \int_{\tau(0)}^{\tau(\delta)} \mathcal{F}(a)da \right), & \text{if } t+\theta \in [0,T]. \end{cases}$$

For $t \in [0,T]$ we obtain

$$\begin{aligned} \|\Sigma(t,\varphi)\|_{1,\xi} &= \delta \int_{-\infty}^{-t} e^{\xi\theta} d\theta + \delta \left(R_0 - \beta \int_{\tau(0)}^{\tau(\delta)} \mathcal{F}(a)da \right) \int_{-t}^{0} e^{\xi\theta} d\theta \\ &= \|\varphi\|_{1,\xi} e^{-\xi t} + \|\varphi\|_{1,\xi} \left(R_0 - \beta \int_{\tau(0)}^{\tau(\delta)} \mathcal{F}(a)da \right) (1 - e^{-\xi t}) \\ &= \|\varphi\|_{1,\xi} \left(e^{-\xi t} + (1 - e^{-\xi t}) \left(R_0 - \beta \int_{\tau(0)}^{\tau(\delta)} \mathcal{F}(a)da \right) \right). \end{aligned}$$

We can suppose that $\delta > 0$ is so small that $\left(R_0 - \beta \int_{\tau(0)}^{\tau(\delta)} \mathcal{F}(a)da \right) > 1$ and consequently

$$e^{-\xi t} + (1 - e^{-\xi t}) \left(R_0 - \beta \int_{\tau(0)}^{\tau(\delta)} \mathcal{F}(a)da \right) > 1.$$

For any $t > 0$ we can find $k \in \mathbb{N}$ and $\tau \in [0,T)$ such that $t = kT + \tau$ and use the semigroup property of Σ to write

$$
\begin{aligned}
\|\Sigma(t,\varphi)\|_{1,\xi} &= \|\Sigma(kT+\tau,\varphi)\|_{1,\xi} \\
&= \|\Sigma((k-1)T+\tau,\varphi)\|_{1,\xi}\left(e^{-\xi T}+(1-e^{-\xi T})\left(R_0-\beta\int_{\tau(0)}^{\tau(\delta)}\mathcal{F}(a)da\right)\right) \\
&\ \vdots \\
&= \|\Sigma(\tau,\varphi)\|_{1,\xi}\left(e^{-\xi T}+(1-e^{-\xi T})\left(R_0-\beta\int_{\tau(0)}^{\tau(\delta)}\mathcal{F}(a)da\right)\right)^k \\
&= \|\varphi\|_{1,\xi}\left(e^{-\xi T}+(1-e^{-\xi T})\left(R_0-\beta\int_{\tau(0)}^{\tau(\delta)}\mathcal{F}(a)da\right)\right)^k \\
&\qquad\cdot\left(e^{-\xi \tau}+(1-e^{-\xi \tau})\left(R_0-\beta\int_{\tau(0)}^{\tau(\delta)}\mathcal{F}(a)da\right)\right),
\end{aligned}
$$

which implies instability of the trivial steady state.

3.2 Non-trivial steady state

Now we would like to find a non-trivial steady state, give conditions for its existence and uniqueness and examine its stability. From now on we are going to be concerned only with the non-trivial steady state.

First of all, if we plug a (yet unknown) constant $\bar{b}$ into (RE), we obtain an equation for $\tau(\bar{b})$

$$1=\beta\int_{\tau(\bar{b})}^{\infty}\mathcal{F}(a)da. \tag{3.1}$$

Using $\tau(\bar{b})$, $\bar{b}$ can be obtained from the definition of $\tau(\bar{b})$ (see (1.4)),

$$\int_{-\tau(\bar{b})}^{0}\frac{S_0}{1+\bar{b}\int_{\epsilon}^{\infty}\mathcal{F}(a)da}ds=1. \tag{3.2}$$

Since the integrand in (3.2) is constant with respect to the integration variable, we obtain

$$\frac{S_0\tau(\bar{b})}{1+\bar{b}\int_{\epsilon}^{\infty}\mathcal{F}(a)da}=1,$$

or equivalently,

$$\bar{b}=\frac{S_0\tau(\bar{b})-1}{\int_{\epsilon}^{\infty}\mathcal{F}(a)da}. \tag{3.3}$$

Remark on notation. In the following text the symbol $\bar{\varphi}$ stands for both a constant function (an element of X) and its value (an element of $\mathbb{R}$), i.e. $\bar{\varphi}(\theta)=\bar{\varphi}$ for $\theta<0$. Similarly, $\bar{b}$ means both a constant function on $\mathbb{R}$ and its value $\bar{b}=\bar{\varphi}\in\mathbb{R}_+$. The precise meaning should always be clear from context.

Theorem 3.1 *The steady state $\bar{b}$ of (RE) exists positive if and only if $R_0 > 1$. The equation (RE) has at most one non-trivial steady state.*

Proof.

- Existence (positivity)

  The defining equation for $\tau(\bar{b})$ (3.1) shows that τ restricted on constant functions is one-to-one.

  Since $\tau(0) = 1/S_0$, (RE) for $\bar{b} = 0$ yields

$$1 = \beta \int_{\tau(\bar{b})}^{\infty} \mathcal{F}(a)da = \beta \int_{\frac{1}{S_0}}^{\infty} \mathcal{F}(a)da = R_0,$$

  so $R_0 = 1$ if and only if $\bar{b} = 0$.

  Moreover, for $\bar{b} > 0$ we have the inequality $1/S_0 < \tau(\bar{\varphi})$ and thus

$$1 = \beta \int_{\tau(\bar{b})}^{\infty} \mathcal{F}(a)da < \beta \int_{\frac{1}{S_0}}^{\infty} \mathcal{F}(a)da = R_0. \tag{3.4}$$

  In other words, $R_0 > 1$ if and only if $\bar{b} > 0$.

  In Section 1.4 we assumed $R_0 > 1$, which now implies $\bar{b} > 0$.

- Uniqueness

  Recall the equation (3.1) for $\tau(\bar{b})$,

$$1 = \beta \int_{\tau(\bar{b})}^{\infty} \mathcal{F}(a)da.$$

  Non-negativity of $\mathcal{F}$ implies that the integral on the right hand-side of (3.1) as a function of $\tau(\bar{b})$ is a non-increasing function. As a consequence, its value can cross 1 at most once and therefore the equation (3.1) has at most one solution. In other words, $\tau(\bar{b})$ is unique. Uniqueness of $\bar{b}$ then follows at once from (3.3).

□

4 Linearized Model and Characteristic Equation

The aim of this section is to introduce the linearization of (RE) around the non-trivial steady state and derive the corresponding characteristic equation.

4.1 Linearized Renewal Equation

This section is devoted to linearizing the equation (RE), (IC) around the non-trivial steady state $\bar{\varphi}$. As I have mentioned in Section 3, we assume $\bar{\varphi} = \bar{b} > 0$.

Recall the definition of the nonlinear strongly continuous semigroup $\Sigma(t) : X_+ \to X_+$, $t \geq 0$,

$$\Sigma(t, \cdot) : \varphi \mapsto b_t,$$

where b is a solution of (RE) with initial condition φ. We can now define a strongly continuous semigroup $\Lambda(t) : X \to X$, $t \geq 0$ by

$$\Lambda(t) := D\Sigma(t, \bar{\varphi}).$$

For $t \in [0, T]$ and $\psi \in X$ we have

$$\begin{aligned} y_t(\theta) &= \left[\Lambda(t)\psi\right](\theta) \\ &= \begin{cases} \psi(t+\theta) & t+\theta < 0, \\ -\beta\bar{\varphi}\mathcal{F}(\tau(\bar{\varphi}))[D\tau(\bar{\varphi})]\psi^o_{t+\theta} & \\ \qquad +\beta\int_{\tau(\bar{\varphi})}^{\infty}\psi(t+\theta-a)\mathcal{F}(a)da & 0 \leq t+\theta \leq T, \end{cases} \end{aligned} \tag{4.1}$$

Then for $t \in (T, 2T]$ we get

$$y_t(\theta) = \begin{cases} \psi(t+\theta) & t+\theta < 0, \\ -\beta\bar{\varphi}\mathcal{F}(\tau(\bar{\varphi}))[D\tau(\bar{\varphi})]\psi^o_{t+\theta} & \\ \qquad +\beta\int_{\tau(\bar{\varphi})}^{\infty}\psi(t+\theta-a)\mathcal{F}(a)da & 0 \leq t+\theta \leq T, \\ -\beta\bar{\varphi}\mathcal{F}(\tau(\bar{\varphi}))[D\tau(\bar{\varphi})]y^T_{t+\theta} & \\ \qquad +\beta\int_{\tau(\bar{\varphi})}^{\infty}y^T(t+\theta-a)\mathcal{F}(a)da & T \leq t+\theta \leq 2T. \end{cases}$$

where y^T is given by (4.1). By iterating this procedure one finds y_t for all $t \geq 0$ and if we plug in the expression for $D\tau(\bar{\varphi})$, it follows that y_t satisfies the linearized renewal equation

$$\begin{aligned} y(t) = &-\frac{\beta\,\bar{\varphi}\,\mathcal{F}(\tau(\bar{\varphi}))}{1+\bar{\varphi}\int_{\epsilon}^{\infty}\mathcal{F}(a)da}\int_{-\tau(\bar{\varphi})}^{0}\int_{\epsilon}^{\infty}y_t(\sigma-a)\mathcal{F}(a)da\,d\sigma \\ &+\beta\int_{\tau(\bar{\varphi})}^{\infty}y(t-a)\mathcal{F}(a)da, \quad \text{for } t \geq 0, \end{aligned} \tag{LRE}$$

$$y(\theta) = \psi(\theta) \quad \text{for } \theta \leq 0, \tag{IC}$$

where $\psi \in X$ is the given initial condition. Again, the equation (LRE) is intended to approximate local behaviour of (RE) around $\bar{\varphi}$, so $\|\psi\|_{1,\xi}$ is expected to be small.

Riesz representation theorem implies that (LRE) can be transformed into the convenient convolution form

$$y(t) = \int_0^\infty k(a)y(t-a)da, \quad k \in L^\infty_\xi(0,\infty). \tag{4.2}$$

The right integral in (LRE) is already in the desired form, so we only need to transform the double integral. We substitute α for $a-\sigma$ and obtain

$$\begin{aligned}
&\int_{-\tau(\bar\varphi)}^0 \int_\epsilon^\infty y_t(\sigma-a)\mathcal{F}(a)da\,d\sigma = \\
&\quad = \int_{-\tau(\bar\varphi)}^0 \int_{\epsilon-\sigma}^\infty y(t-\alpha)\mathcal{F}(\sigma+\alpha)d\alpha\,d\sigma \\
&\quad = \int_\epsilon^{\tau(\bar\varphi)+\epsilon} \int_{\epsilon-\alpha}^0 y(t-\alpha)\mathcal{F}(\sigma+\alpha)d\sigma\,d\alpha \\
&\qquad\qquad + \int_{\epsilon+\tau(\bar\varphi)}^\infty \int_{-\tau(\bar\varphi)}^0 y(t-\alpha)\mathcal{F}(\sigma+\alpha)d\sigma\,d\alpha \\
&\quad = \int_\epsilon^{\tau(\bar\varphi)+\epsilon} y(t-\alpha)\left(\int_{\epsilon-\alpha}^0 \mathcal{F}(\sigma+\alpha)d\sigma\right)d\alpha \\
&\qquad\qquad + \int_{\epsilon+\tau(\bar\varphi)}^\infty y(t-\alpha)\left(\int_{-\tau(\bar\varphi)}^0 \mathcal{F}(\sigma+\alpha)d\sigma\right)d\alpha.
\end{aligned}$$

Then (LRE) becomes

$$\begin{aligned}
y(t) = -\frac{\beta\,\bar\varphi\,\mathcal{F}(\tau(\bar\varphi))}{1+\bar\varphi\int_\epsilon^\infty \mathcal{F}(a)da}&\Bigg[\int_\epsilon^{\tau(\bar\varphi)+\epsilon} y(t-\alpha)\left(\int_{\epsilon-\alpha}^0 \mathcal{F}(\sigma+\alpha)d\sigma\right)d\alpha \\
&\quad + \int_{\epsilon+\tau(\bar\varphi)}^\infty y(t-\alpha)\left(\int_{-\tau(\bar\varphi)}^0 \mathcal{F}(\sigma+\alpha)d\sigma\right)d\alpha\Bigg] \\
&+ \beta\int_{\tau(\bar\varphi)}^\infty y(t-a)\mathcal{F}(a)da,
\end{aligned}$$

and k is of the form

$$\begin{aligned}
k(a) = \frac{-\beta\bar\varphi\,\mathcal{F}(\tau(\bar\varphi))}{1+\bar\varphi\int_\epsilon^\infty \mathcal{F}(a)da}&\left(\chi_{(\epsilon,\tau(\bar\varphi)+\epsilon)}(a)\int_{\epsilon-a}^0 \mathcal{F}(\sigma+a)d\sigma + \chi_{(\tau(\bar\varphi)+\epsilon,\infty)}(a)\int_{-\tau(\bar\varphi)}^0 \mathcal{F}(\sigma+a)d\sigma\right) \\
&+ \beta\chi_{(\tau(\bar\varphi),\infty)}(a)\mathcal{F}(a).
\end{aligned} \tag{4.3}$$

Proposition 4.1 *The kernel $k \in L^\infty_\xi(0,\infty)$ defined in (4.3) is an element of $L^1_\xi(0,\infty)$.*

Proof. Let's denote

$$A := -\frac{\beta\bar\varphi\,\mathcal{F}(\tau(\bar\varphi))}{1+\bar\varphi\int_\epsilon^\infty \mathcal{F}(a)da}.$$

The assumption (RoC) guarantees that $\mathcal{F} \in L^1_\xi(0,\infty) \cap L^\infty_\xi(0,\infty)$ and therefore we can denote

$$C := \operatorname*{ess\,sup}_{a \geq 0} \mathcal{F}(a) e^{\xi a}.$$

Using the expression (4.3) and (RoC) we obtain

$$\begin{aligned}
\|k\|_{1,\xi} &= \int_0^\infty |k(a)| e^{\xi a} da \\
&= A \left(\int_\epsilon^{\tau(\bar\varphi)+\epsilon} \left| \int_{\epsilon - a}^0 \mathcal{F}(\sigma + a) d\sigma \right| e^{\xi a} da + \int_{\tau(\bar\varphi)+\epsilon}^\infty \left| \int_{-\tau(\bar\varphi)}^0 \mathcal{F}(\sigma + a) d\sigma \right| e^{\xi a} da \right) \\
&\quad + \beta \int_{\tau(\bar\varphi)}^\infty \mathcal{F}(a) e^{\xi a} da \\
&= A \int_\epsilon^{\tau(\bar\varphi)+\epsilon} \int_{\epsilon - a}^0 \mathcal{F}(\sigma + a) e^{\xi(\sigma + a)} e^{-\xi \sigma} d\sigma \, da \\
&\quad + A \int_{-\tau(\bar\varphi)}^0 \int_{\tau(\bar\varphi)+\epsilon}^\infty \mathcal{F}(\sigma + a) e^{\xi(\sigma + a)} da \, e^{-\xi\sigma} d\sigma + \beta \int_{\tau(\bar\varphi)}^\infty \mathcal{F}(a) e^{\xi a} da \\
&\leq AC \int_\epsilon^{\tau(\bar\varphi)+\epsilon} \int_{\epsilon - a}^0 e^{-\xi \sigma} d\sigma \, da + A \|\mathcal{F}\|_{1,\xi} \int_{-\tau(\bar\varphi)}^0 e^{-\xi\sigma} d\sigma + \beta \|\mathcal{F}\|_{1,\xi} \\
&= \frac{AC}{\xi} \left(\frac{e^{-\xi \tau(\bar\varphi)}}{\xi} - \frac{1}{\xi} - \tau(\bar\varphi) \right) + \frac{A}{\xi} \|\mathcal{F}\|_{1,\xi} (e^{\xi \tau(\bar\varphi)} - 1) + \beta \|\mathcal{F}\|_{1,\xi} < \infty.
\end{aligned}$$

□

Proposition 4.2 *Solution y of (LRE), (IC) is continuous on $[0,\infty)$, i.e. the map*

$$t \mapsto \int_0^\infty y(t-a) k(a) da, \quad t \geq 0,$$

is continuous.

Proof. Let $\psi \in X$. For $t \in [0,T]$ we have

$$y(t) = \int_0^\infty k(a) \psi(t-a) da.$$

Let's denote

$$y^T(x) := \begin{cases} \psi(x) & \text{if } x \leq 0, \\ \int_0^\infty k(a)\psi(x-a) da & \text{if } x \in [0,T], \end{cases}$$

and by induction

$$y^{(n+1)T}(x) := \begin{cases} y^{nT}(x) & \text{if } x \leq n\,T, \\ \int_0^\infty k(a) y^{nT}(x-a) da & \text{if } x \in (n\,T, (n+1)T]. \end{cases}$$

for $n = 1, 2\ldots$. The function

$$x \mapsto \begin{cases} y^{nT}(x)e^{\xi x} & x \leq n\,T, \\ 0 & n\,T < x \end{cases}$$

obviously belongs to $L^1(\mathbb{R})$ and since $k \in L^\infty_\xi(0,\infty)$, the map $a \mapsto k(a)e^{\xi a}$ for $a \geq 0$ is bounded. Theorem 2.4 yields continuity of

$$t \mapsto \int_0^\infty y^{nT}(t-a)e^{\xi(t-a)}k(a)e^{\xi a}da, \quad t \in [0,(n+1)T].$$

We conclude from continuity of the exponential function that also

$$\begin{aligned} y(t) &= e^{-\xi t}\int_0^\infty \psi(t-a)e^{\xi(t-a)}k(a)e^{\xi a}da \\ &= \int_0^\infty \psi(t-a)k(a)da, \quad t \in [0,(n+1)T]. \end{aligned}$$

is continuous.

Finally, arbitrariness of n implies continuity of y on $[0,\infty)$.

□

4.2 Characteristic Equation

Characteristic equation can now be obtained by plugging $y(t) = y_0 e^{\lambda t}$ into (4.2),

$$y_0 e^{\lambda t} = \int_0^\infty k(a)\, y_0\, e^{\lambda(t-a)}da,$$

which after rearranging becomes

$$1 = \int_0^\infty k(a)e^{-\lambda a}da. \tag{4.4}$$

If we denote $\widehat{k}$ the Laplace transform

$$\widehat{k}(\lambda) := \int_0^\infty k(a)e^{-\lambda a}da, \tag{4.5}$$

we can write the characteristic equation (4.4) as

$$1 = \widehat{k}(\lambda). \tag{4.6}$$

Remark. The Principle of Linearized Stability, that I present in the next section, gives conditions on roots of the characteristic equation (4.4) that in generic cases either ensure asymptotic stability or prove instability of the steady state of the non-linear equation (RE). Unfortunately, I do not have time to provide analysis of the particular characteristic equation (4.4) too.

5 Linearized Stability

This section is devoted to the proof the Principle of Linearized Stability for the equation (RE) with the initial condition (IC). I first prove the general result presented in Corollary 5.13 and thereafter apply it to our problem.

In this section the symbol X can refer to a general Banach space with an associated norm denoted by $\|\cdot\|_X$, but also the state space defined in Definition 1.1, in which case the norm $\|\cdot\|_{1,\xi}$ is used. The precise meaning should always be clear from the context.

5.1 Complexification

The principle of linearized stability employs spectral theory, for which complex vector spaces are needed. The space X is a real vector space, so it requires embedding into some complex vector space. The complexification is explained in detail in Section III.7 of [9], so I only present the method.

Definition 5.1 *[9] Let X be a real vector space. The Cartesian product $X_{\mathbb{C}} := X \times X$ can be given the structure of a complex vector space by defining*

- *addition,*
$$(x, y) + (u, v) := (x + u, y + v),$$
- *and multiplication by a complex number,*
$$(\alpha + i\beta)(x, y) := (\alpha x - \beta y, \alpha y + \beta x).$$

Given a norm $\|\cdot\|_X$ on X, one can construct an admissible norm on $X_{\mathbb{C}}$, but this norm is not uniquely determined. An example of such an admissible norm on $X_{\mathbb{C}}$ is

$$\|x + iy\|_{X_{\mathbb{C}}} := \sup_{-\pi \le \varphi \le \pi} \sqrt{\|x \cos\varphi - y \sin\varphi\|_X^2 + \|y \cos\varphi + x \sin\varphi\|_X^2}. \tag{5.1}$$

If X is an inner product space, this norm reduces to

$$\|x + iy\|_{X_{\mathbb{C}}} := \sqrt{\|x\|_X^2 + \|y\|_X^2}.$$

5.2 Properties of $\Lambda(t)$ and Its Infinitesimal Generator

Let A denote the infinitesimal generator of the strongly continuous semigroup (C_0-semigroup) $\{\Lambda(t)\}_{t\ge 0}$ of linear operators, i.e.

$$A\phi := \lim_{t \searrow 0} \frac{\Lambda(t)\phi - \phi}{t}, \quad \phi \in \mathcal{D}(A),$$

where $\mathcal{D}(A)$ is the domain of definition of A,

$$\mathcal{D}(A) := \left\{ \phi \in X : \lim_{t \searrow 0} \frac{\Lambda(t)\phi - \phi}{t} \text{ exists} \right\}.$$

Proposition 5.2 *[11] Assume that $\phi \in \mathcal{D}(A)$. Then*

(i) $\Lambda(t)\phi \in \mathcal{D}(A)$ *for each* $t \geq 0$,

(ii) $\Lambda(t)A\phi = A\,\Lambda(t)\phi$ *for each* $t > 0$,

(iii) the mapping $t \mapsto \Lambda(t)\phi$ *is differentiable for each* $t > 0$,

(iv) $\frac{d}{dt}\Lambda(t)\phi = A\Lambda(t)\phi$ *for each* $t > 0$.

Proof. Let $\phi \in X$.

(i) By the semigroup property and linearity of $\Lambda(t)$,

$$\lim_{s \searrow 0} \frac{\Lambda(s)\Lambda(t)\phi - \Lambda(t)\phi}{s} = \lim_{s \searrow 0} \frac{\Lambda(t)\Lambda(s)\phi - \Lambda(t)\phi}{s} \tag{5.2}$$

$$= \Lambda(t) \lim_{s \searrow 0} \frac{\Lambda(s)\phi - \phi}{s} \tag{5.3}$$

$$= \Lambda(t)A\phi. \tag{5.4}$$

The limit in (5.3) exists, thus also the limit in (5.2) exists and $\Lambda(s)\phi \in \mathcal{D}(a)$.

(ii) If the limit in (5.2) exists, it can be written as $A\Lambda(t)\phi$, which together with (5.4) yields

$$A\Lambda(t)\phi = \Lambda(t)A\phi.$$

(iii) + (iv)

$$\lim_{h \searrow 0} \frac{\Lambda(t)\phi - \Lambda(t-h)\phi}{h} - \Lambda(t)A\phi =$$

$$= \lim_{h \searrow 0} \Lambda(t-h)\frac{\Lambda(h)\phi - \phi}{h} - \Lambda(t)A\phi$$

$$= \lim_{h \searrow 0} \Lambda(t-h)\left(\frac{\Lambda(h)\phi - \phi}{h} - A\phi\right) + \left(\Lambda(t-h) - \Lambda(t)\right)A\phi. \tag{5.5}$$

The second term of (5.5) goes to zero due to continuity of Λ and the first term goes to zero since

$$\left\|\Lambda(t-h)\left(\frac{\Lambda(h)\phi - \phi}{h} - A\phi\right)\right\|_X$$

$$\leq \|\Lambda(t-h)\| \cdot \left\|\frac{\Lambda(h)\phi - \phi}{h} - A\phi\right\|_X$$

$$\leq C \underbrace{\left\|\frac{\Lambda(h)\phi - \phi}{h} - A\phi\right\|_X}_{\to 0} \to 0, \quad C > 0.$$

Altogether, (5.5) = 0. If we follow the same procedure for the right limit

$$\lim_{h \searrow 0} \frac{\Lambda(t+h)\phi - \Lambda(t)\phi}{h},$$

we get that this limit also exists and is zero, thus

$$\frac{d}{dt}\Lambda(t)\phi = A\Lambda(t)\phi$$

Hence assertions (iii) and (iv) are proven.

□

Proposition 5.3 *[11] A is a closed operator.*

Proof. We need to show that if $\phi_n \to \phi$ in X with $\phi_n \in \mathcal{D}(A)$ for all $n = 1, 2, \ldots$, and $A\phi_n \to \tilde{\phi}$ in X, then $\phi \in \mathcal{D}(A)$ and $A\phi = \tilde{\phi}$.

Let $\phi_n \in \mathcal{D}(A)$ for $n = 1, 2, \ldots$ be such that assumptions from the previous sentence are satisfied. According to Proposition 5.2,

$$\begin{array}{ccc} \int_0^t \frac{d}{ds}\Lambda(s)\phi ds & = & \Lambda(t)\phi - \phi \\ \| & & \\ \int_0^t A\Lambda(s)\phi ds & = & \int_0^t \Lambda(s)A\phi ds. \end{array}$$

Hence

$$\Lambda(t)\phi_n - \phi_n = \int_0^t \Lambda(s)A\phi_n ds.$$

For $n \to \infty$ we obtain

$$\Lambda(t)\phi - \phi = \int_0^t \Lambda(s)\tilde{\phi} ds$$

and consequently,

$$\lim_{t \searrow 0} \frac{\Lambda(t)\phi - \phi}{t} = \lim_{t \searrow 0} \frac{1}{t} \int_0^t \Lambda(s)\tilde{\phi} ds = \tilde{\phi}.$$

Then by definition $\phi \in \mathcal{D}(A)$ and $A\phi = \tilde{\phi}$. □

Lemma 5.4 *There exists $a > 0$ and $M \geq 1$ such that*

$$\|\Lambda(t)\| \leq M \quad \textit{for } 0 \leq t \leq a.$$

Proof. We know from Proposition 1.2 that X is a Banach space. Let $\Lambda(t)\phi = y_t$. It follows from Lemma 2.3 that the function $t \mapsto y_t$ is continuous, which implies existence and boundedness of

$$\max_{t\in[0,a]} \|\Lambda(t)\phi\|_X = \max_{t\in[0,a]} \|y_t\| < \infty,$$

where $0 < a < \infty$, so that $[0, a]$ is a compact subset of $\mathbb{R}$. Now Banach-Steinhaus Theorem (Uniform Boundedness Principle) yields

$$\sup_{t\in[0,a]} \|\Lambda(t)\| < \infty.$$

Hence there exists a constant $M \geq 1$ such that

$$\|\Lambda(t)\| \leq M \quad \text{for } 0 \leq t \leq a.$$

□

Proposition 5.5 *There exists $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|\Lambda(t)\| \leq Me^{\omega t} \quad \textit{for } t \geq 0.$$

Proof. Lemma 5.4 yields constants $a > 0$ and $M \geq 1$ such that

$$\|\Lambda(t)\| \leq M \quad \text{for } 0 \leq t \leq a.$$

Then there exists a sequence of real numbers $t_0, t_1, t_2, \ldots, t_n$ such that $0 < t_i < a$ for all $i = 0, 1, \ldots, n$ and

$$t = \sum_{i=0}^{n} t_i$$

It holds that

$$n \min_i t_i \leq t.$$

Using the semigroup property of Λ and the results above we get

$$\begin{aligned}
\|\Lambda(t)\| &= \|\Lambda(t_0 + t_1 + \ldots + t_n)\| \\
&= \|\Lambda(t_0) \cdot \Lambda(t_1) \cdot \ldots \cdot \Lambda(t_n)\| \\
&= \|\Lambda(t_0)\| \cdot \|\Lambda(t_1)\| \cdot \ldots \cdot \|\Lambda(t_n)\| \\
&\leq M \cdot M^n = Me^{n \ln M} \\
&\leq Me^{\frac{t}{\min_i t_t} \ln M}
\end{aligned}$$

Denoting

$$\omega := \frac{\ln M}{\min_i t_i}$$

we obtain

$$\|\Lambda(t)\| \leq Me^{\omega t}.$$

□

5.3 Principle of Linearized Stability

The approach and most of the proofs presented in this section come from [5], but some details and proofs skipped in [5] are provided here.

5.3.1 General Setting

Definition 5.6 *Let A be the infinitesimal generator of the C_0-semigroup $\{\Lambda(t)\}_{t\geq 0}$. Then the resolvent set $\rho(A)$ of A is a set of all $\lambda \in \mathbb{C}$ such that $(\lambda I - A)^{-1}$ exists, i.e. such that*

(i) $\lambda I - A$ *is one-to-one,*

(ii) $\lambda I - A$ *is onto.*

Remark. Definition 5.6 silently uses Proposition 5.3, that is closedness of the operator A. When defining a resolvent set for a general (non-closed) linear operator L, we would have to require $(\lambda I - A)^{-1}$ to be bounded. If we consider a closed operator A, the closed graph theorem implies that $(\lambda I - A)^{-1}$ is continuous.

Definition 5.7 *If $\lambda \in \rho(A)$, then the resolvent operator R_λ is defined by*

$$R_\lambda \varphi := (\lambda I - A)^{-1}\varphi.$$

Propositions 5.8, 5.9 and 5.12 together lead to a general version of the principle of linearized stability (Corollary 5.13) which serves as a tool for proving the particular form of the principle of linearized stability for our renewal equation (Theorem 5.29).

Proposition 5.8 *[9] Assume that $\gamma > 0$ and $M \geq 1$ exist such that for $t \geq 0$*

$$\|\Lambda(t)\| \leq Me^{-\gamma t}.$$

Then $\bar{\varphi}$ is locally exponentially stable as a stationary point of Σ.

Proof. Choose $t_0 > 0$ such that

$$Me^{-\gamma t_0} \leq \frac{1}{4}.$$

From Section 2.4 we know that

$$\begin{aligned} 0 &= \lim_{\|\varphi-\bar{\varphi}\|_X \to 0} \frac{\|\Sigma(t,\varphi) - \Sigma(t,\bar{\varphi}) - \Lambda(t)(\varphi - \bar{\varphi})\|_X}{\|\varphi - \bar{\varphi}\|_X} \\ &= \lim_{\|\varphi-\bar{\varphi}\|_X \to 0} \frac{\|\Sigma(t,\varphi) - \bar{\varphi} - \Lambda(t)(\varphi - \bar{\varphi})\|_X}{\|\varphi - \bar{\varphi}\|_X} \end{aligned}$$

and consequently $\delta > 0$ can be chosen such that for all φ satisfying $\|\varphi - \bar{\varphi}\| < \delta$ and all $t \in [0, t_0]$ we have

$$\|\Sigma(t,\varphi) - \bar{\varphi} - \Lambda(t)(\varphi - \bar{\varphi})\|_X \le \frac{1}{4}\|\varphi - \bar{\varphi}\|_X.$$

Then for such φ and t we get

$$\begin{aligned}\|\Sigma(t,\varphi) - \bar{\varphi}\|_X &\le \|\Sigma(t,\varphi) - \bar{\varphi} - \Lambda(t)(\varphi - \bar{\varphi})\|_X + \|\Lambda(t)(\varphi - \bar{\varphi})\|_X \\ &\le \frac{1}{4}\|\varphi - \bar{\varphi}\|_X + Me^{-\gamma t}\|\varphi - \bar{\varphi}\|_X \\ &\le \left(\frac{1}{4} + M\right)\|\varphi - \bar{\varphi}\|_X,\end{aligned}$$

and for $t = t_0$ we get the estimate

$$\|\Sigma(t_0,\varphi) - \bar{\varphi}\|_X \le \frac{1}{2}\|\varphi - \bar{\varphi}\|_X < \delta.$$

If we iterate this process, we obtain

$$\|\Sigma(kt_0,\varphi) - \bar{\varphi}\|_X \le \left(\frac{1}{2}\right)^k \|\varphi - \bar{\varphi}\|_X, \quad k \in \mathbb{N},\, k \ge 1.$$

For any $t > 0$ we now introduce $\tau \in [0, t_0)$ and an integer k by the relation

$$t = kt_0 + \tau.$$

Then for any $\varphi \in X$ such that

$$\|\varphi - \bar{\varphi}\|_X \le \frac{\delta}{\frac{1}{4} + M}$$

we have

$$\|\Sigma(\tau,\varphi) - \bar{\varphi}\|_X \le \left(\frac{1}{4} + M\right)\|\varphi - \bar{\varphi}\|_X \le \delta$$

and so we are allowed to write

$$\begin{aligned}\|\Sigma(t,\varphi) - \bar{\varphi}\|_X &= \|\Sigma(kt_0, \Sigma(\tau,\varphi)) - \bar{\varphi}\|_X \\ &\le \left(\frac{1}{2}\right)^k \|\Sigma(\tau,\varphi) - \bar{\varphi}\|_X \\ &\le \left(\frac{1}{2}\right)^k \left(\frac{1}{4} + M\right)\|\varphi - \bar{\varphi}\|_X \\ &= e^{-k\ln 2}\left(\frac{1}{4} + M\right)\|\varphi - \bar{\varphi}\|_X \\ &= e^{-\frac{(t-\tau)\ln 2}{t_0}}\left(\frac{1}{4} + M\right)\|\varphi - \bar{\varphi}\|_X \\ &\le e^{-\frac{(t-t_0)\ln 2}{t_0}}\left(\frac{1}{4} + M\right)\|\varphi - \bar{\varphi}\|_X \\ &= 2e^{-\frac{t\ln 2}{t_0}}\left(\frac{1}{4} + M\right)\|\varphi - \bar{\varphi}\|_X,\end{aligned}$$

which is the required exponential stability estimate.

□

Proposition 5.9 *[9] Let $\bar{\varphi}$ be a fixed point of a map $F : X \to X$. Assume that F is differentiable at $\bar{\varphi}$ with derivative L and that X admits a decomposition*

$$X = X_- \oplus X_+$$

into closed L-invariant subspaces such that

(i) $\|L\varphi\|_X \leq \eta \|\varphi\|_X$ *for all* $\varphi \in X_-$,

(ii) $\|L\varphi\|_X \geq (\eta + \delta)\|\varphi\|_X$ *for all* $\varphi \in X_+$,

where $\eta \geq 1$ and $\delta \geq 0$. Then $\bar{\varphi}$ is unstable as a fixed point of the iteration scheme

$$\varphi_{n+1} = F(\varphi_n).$$

Proof. Let P denote the projection on X_+ along X_- and define

$$|||\varphi||| = \|P\varphi\|_X + \|(I - P)\varphi\|_X.$$

Then $||| \cdot |||$ is an equivalent norm on X.

Choose $\epsilon > 0$ such that for $|||\varphi - \bar{\varphi}||| < \epsilon$

$$|||F(\varphi) - \bar{\varphi} - L(\varphi - \bar{\varphi})||| \leq \frac{1}{4}\delta |||\varphi - \bar{\varphi}|||.$$

If $\|(I - P)\varphi\|_X \leq \|P\varphi\|_X$, then

$$|||\varphi||| = \|P\varphi\|_X + \|(I - P)\varphi\|_X \leq 2\|P\varphi\|_X. \tag{5.6}$$

Then for φ such that $|||\varphi - \bar{\varphi}||| < \epsilon$ and $\|(I - P)(\varphi - \bar{\varphi})\|_X \leq \|P(\varphi - \bar{\varphi})\|_X$ we have

$$\begin{aligned}
\|P(F(\varphi) - \bar{\varphi})\|_X &= \|PL(\varphi - \bar{\varphi}) + P(F(\varphi) - \bar{\varphi}) - PL(\varphi - \bar{\varphi})\|_X \\
&\geq \|PL(\varphi - \bar{\varphi})\|_X - \|P(F(\varphi) - \bar{\varphi} - L(\varphi - \bar{\varphi}))\|_X \\
&\geq \|PL(\varphi - \bar{\varphi})\|_X - |||F(\varphi) - \bar{\varphi} - L(\varphi - \bar{\varphi})||| \\
&\geq (\eta + \delta)\|P(\varphi - \bar{\varphi})\|_X - \frac{1}{4}\delta|||\varphi - \bar{\varphi}||| \\
&\geq (\eta + \delta)\|P(\varphi - \bar{\varphi})\|_X - \frac{1}{2}\delta\|P(\varphi - \bar{\varphi})\|_X \\
&= \left(\eta + \frac{\delta}{2}\right)\|P(\varphi - \bar{\varphi})\|_X,
\end{aligned} \tag{5.7}$$

where the last inequality uses (5.6).

Similarly,

$$
\begin{aligned}
\|(I-P)(F(\varphi)-\bar{\varphi})\|_X &= \|(I-P)L(\varphi-\bar{\varphi})+(I-P)(F(\varphi)-\bar{\varphi})-(I-P)L(\varphi-\bar{\varphi})\|_X \\
&\le \|(I-P)L(\varphi-\bar{\varphi})\|_X + \|(I-P)(F(\varphi)-\bar{\varphi}-L(\varphi-\bar{\varphi}))\|_X \\
&\le \eta\|(I-P)(\varphi-\bar{\varphi})\|_X + |||F(\varphi)-\bar{\varphi}-L(\varphi-\bar{\varphi})||| \\
&\le \eta\|P(\varphi-\bar{\varphi})\|_X + \frac{1}{4}\delta|||\varphi-\bar{\varphi}||| \\
&\le \eta\|P(\varphi-\bar{\varphi})\|_X + \frac{1}{2}\delta\|P(\varphi-\bar{\varphi})\|_X \\
&= \left(\eta+\frac{\delta}{2}\right)\|P(\varphi-\bar{\varphi})\|_X. \qquad (5.8)
\end{aligned}
$$

Inequalities (5.7) and (5.8) together imply that $F(\varphi)$ satisfies the same estimate as φ, i.e.

$$\|(I-P)(F(\varphi)-\bar{\varphi})\|_X \le \left(\eta+\frac{\delta}{2}\right)\|P(\varphi-\bar{\varphi})\|_X \le \|P(F(\varphi)-\bar{\varphi})\|.$$

Suppose that $\|F^{(n)}(\varphi)-\bar{\varphi}\|_X \le \epsilon$ for all n and $\|\varphi-\bar{\varphi}\|_X$ sufficiently small. Then (5.7) yields

$$\|P(F^{(n)}(\varphi)-\bar{\varphi})\|_X \ge \left(\eta+\frac{\delta}{2}\right)^n \|P(\varphi-\bar{\varphi})\|_X,$$

which, since $\eta+\delta/2>1$, tends to infinity as $n\to 0$ whenever $\|P\varphi\|_X>0$. This contradicts our assumption that $\|F^{(n)}(\varphi)-\bar{\varphi}\|_X \le \epsilon$ for all n. Hence any orbit corresponding to an initial condition φ that belongs to the set

$$\{\varphi : \|P(\varphi-\bar{\varphi})\|_X \ge \|(I-P)(\varphi-\bar{\varphi})\|_X \,\&\, |||\varphi-\bar{\varphi}||| \le \epsilon\}$$

must leave the ϵ-ball.

□

Definition 5.10 *The growth bound ω_0 of $\Lambda(t)$ is defined by*

$$\omega_0 = \lim_{t\to\infty} \frac{\log\|\Lambda(t)\|}{t}.$$

Remark. In other words, ω_0 is the infimum of all $\omega \in \mathbb{R}$ such that there exists a constant $M \ge 1$ with

$$\|T(t)\| \le Me^{\omega t}$$

for all $t \ge 0$.

Theorem 5.11 *[9] Let $A : \mathcal{D}(A) \to X$ be the generator of a C_0-semigroup $\Lambda(t)$. If $\Lambda(t)$ is eventually compact, then the growth bound of $\Lambda(t)$ equals the spectral bound of A, i.e.*

$$\omega_0 = \sup\{Re\lambda \,|\, \lambda \in \sigma(A)\}.$$

Proposition 5.12 *[9] Assume that X admits a decomposition*

$$X = X_- \oplus X_+$$

into $\Lambda(t)$-invariant subspaces such that

(i) X_+ *is finite dimensional,*

(ii) $\|\Lambda(t)|_{X_-}\| \leq Me^{\omega t}$,

(iii) *Re* $\lambda > \omega$ *for all* $\lambda \in \sigma(A|_{X_+})$,

where $\omega \geq 0$ and $M \geq 1$. Then $\bar{\varphi}$ is unstable as a stationary point of Σ.

Proof. Since $X|_+$ is finite dimensional, $\Lambda(t)|_{X_+}$ can be extended to a group of operators on $X|_+$ for $t < 0$. Consider the semigroup $S(t) := \Lambda(-t)|_{X_+}$ with $t \geq 0$ and denote $\tilde{A}$ its infinitesimal generator.

Moreover, finite-dimensionality of X_+ implies existence of $\epsilon > 0$ such that assumption (iii) holds for $\omega + \epsilon$ and it follows from continuity of $t \mapsto \Lambda(t)$ for all $t \in \mathbb{R}$ that $\text{Re}\lambda < -(\omega + \epsilon)$ for all $\lambda \in \sigma(\tilde{A})$. Then Definition 5.10 of ω_0 (or in fact the remark below the definition) and Theorem 5.11 applied to $S(t)$ now yield constants $N \geq 1$ and $0 < \alpha < \omega + \epsilon$ such that

$$\|S(t)\| \leq Ne^{-\alpha t}. \tag{5.9}$$

In particular, (5.9) holds for some $\gamma \in (\omega, \omega + \epsilon)$. We have

$$\begin{aligned} \|x_+\|_X &= \|\Lambda(-t)\Lambda(t)x_+\|_X \\ &\leq \|\Lambda(-t)\|\|\Lambda(t)x_+\|_X \\ &= \|S(t)\|\|\Lambda(t)x_+\|_X \\ &\leq Ne^{-\gamma t}\|\Lambda(t)x_+\|_X. \end{aligned}$$

In other words, there exists $q := 1/N \in (0, 1]$ and $\omega < \gamma < \omega + \epsilon$ such that

$$\|\Lambda(t)x_+\|_X \geq qe^{\gamma t}\|x_+\|_X$$

for all $x_+ \in X|_+$.
Now choose s so large that

$$qe^{\gamma s} > Me^{\omega s}$$

and apply Proposition 5.9 with $L = \Lambda(t)$, $F(\varphi) = \Sigma(s, \varphi)$, $\eta = Me^{\omega s}$ and $\delta = qe^{\gamma s} - Me^{\omega s}$.

□

Corollary 5.13 *[9] Let Σ be a strongly continuous nonlinear semigroup. Let $\bar{\varphi}$ be a steady state of Σ and assume that for each $t \geq 0$, $\Sigma(t, \varphi)$ has a (uniform) Fréchet derivative $\Lambda(t)$ at $\bar{\varphi}$. Let A be the infinitesimal generator of Λ. Assume further that X admits a decomposition*

$$X = X_- \oplus X_+$$

into two $\Lambda(t)$-invariant subspaces X_- and X_+ such that *(i)* X_+ *is finite dimensional,*

(ii) the restriction of $\Lambda(t)$ *to* X_- *converges exponentially to* 0 *as* $t \to \infty$.

Then $\bar{\varphi}$ *is*

(a) (locally) exponentially stable if $Re\,\lambda < 0$ *for all* $\lambda \in \sigma(A|_{X_+})$,

(b) unstable if there exists a $\lambda \in \sigma(A|_{X_+})$ *with* $Re\,\lambda > 0$.

Proof. If $\mathrm{Re}\,\lambda < 0$ for all $\lambda \in \sigma(A|_{X_+})$, then Theorem 5.11 and assumptions (i), (ii) imply existence of constants $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)\| \leq Me^{-\omega t}$$

for all $t \geq 0$. Assertion (a) can now be concluded from Proposition 5.8.

On the other hand, if there exists a $\lambda \in \sigma(A|_{X_+})$ such that $\mathrm{Re}\,\lambda > 0$, we can decompose X_+ in two linear subspaces,

$$X_+ = X_{++} \oplus X_{+-},$$

such that every $\lambda \in \sigma(A|_{X_{++}})$ has positive real part and every $\lambda \in \sigma(A|_{X_{+-}})$ has negative or zero real part. Since X_+ is finite dimensional, $\sigma(A|_{X_+})$ coincides with the point spectrum $\sigma_p(A|_{X_+})$ and therefore the subspaces X_{++} and X_{+-} are generated by eigenfunctions corresponding to positive, respectively negative eigenvalues. As a consequence, the linear subspaces X_{++}, X_{+-} are again $\Lambda(t)$-invariant and thus also the subspace $X_- \oplus X_{+-}$ is $\Lambda(t)$-invariant.

Now Theorem 5.11 together with assumption (ii) imply existence of $M \geq 1$ and $\omega \geq 0$ such that $\|\Lambda(t)|_{X_- \oplus X_{+-}}\| \leq Me^{\omega t}$. Moreover, due to finite-dimensionality of X_+ (and thus also X_{++}) $\lambda \in \sigma(A|_{X_{++}})$ satisfies $\mathrm{Re}\lambda > \omega$ for some $\omega > 0$, so we can apply Proposition 5.12 that proves assertion (b).

□

5.3.2 Specific Case

In order to apply Theorem 5.13 to our problem, we need to find a direct sum decomposition $X = X_+ \oplus X_-$, such that $\Lambda(t)$ converges exponentially to 0 on X_- and that X_+ has finite dimension.

The following theorem says that the resolvent operator is in fact the Laplace transform of the semigroup $\Lambda(t)$. The proof is essentially taken from [11], but modified for the current situation.

Theorem 5.14 *[11] Let* $\lambda \in \rho(A)$ *have positive real part. Then*

$$R_\lambda \varphi = \int_0^\infty e^{-\lambda t} \Lambda(t) \varphi \, dt. \tag{5.10}$$

Proof. Let's first check that the integral on the right hand-side is a well-defined bounded linear operator form X to X. We have

$$\begin{aligned}\left\|\int_0^\infty e^{-\lambda t}\Lambda(t)\varphi\,dt\right\|_{1,\xi} &= \int_0^\infty \left|\int_0^\infty e^{-\lambda t}\Lambda(t)\varphi(-a)\,dt\right| e^{-\xi a}da \\ &\le \int_0^\infty \int_0^\infty \left|e^{-\lambda t}\Lambda(t)\varphi(-a)\right| dt\, e^{-\xi a}da \\ &= \int_0^\infty \int_0^\infty e^{-Re\lambda\, t}\left|\Lambda(t)\varphi(-a)\right| e^{-\xi a}dt\, da \\ &= \int_0^\infty e^{-Re\lambda\, t}\int_0^\infty \left|\Lambda(t)\varphi(-a)\right| e^{-\xi a}da dt \\ &= \int_0^\infty e^{-Re\lambda t}\|\Lambda(t)\varphi\|_{1,\xi}dt \\ &\le \int_0^\infty e^{-Re\lambda t}\|\Lambda(t)\|\|\varphi\|_{1,\xi}dt. \end{aligned} \tag{5.11}$$

The fact that $Re\lambda > 0$ together with Proposition 5.5 yields

$$\begin{aligned}(5.11) &\le M\|\varphi\|_{1,\xi}\int_0^\infty e^{-Re\lambda t}e^{-\omega t}dt \\ &= \frac{M\|\varphi\|_{1,\xi}}{Re\lambda+\omega} < \infty. \end{aligned} \tag{5.12}$$

Linearity of the integral on the right hand-side of (5.10) is obvious and (5.12) implies that it is a bounded operator from X to X. Let's denote

$$\tilde{R}_\lambda\phi := \int_0^\infty e^{-\lambda t}\Lambda(t)\varphi\,dt$$

and prove that $R_\lambda = \tilde{R}_\lambda$. For $h > 0$ and $\phi \in X$ we have

$$\begin{aligned}\frac{\Lambda(h)\tilde{R}_\lambda\phi - \tilde{R}_\lambda\phi}{h} &= \frac{1}{h}\int_0^\infty e^{-\lambda t}[\Lambda(t+h)\phi - \Lambda(t)\phi]dt \\ &= \frac{1}{h}\int_h^\infty e^{-\lambda(t-h)}\Lambda(t)\phi\,dt - \frac{1}{h}\int_0^\infty e^{-\lambda t}\Lambda(t)\phi\,dt \\ &= -\frac{1}{h}\int_0^h e^{\lambda(t-h)}\Lambda(t)\phi\,dt + \frac{1}{h}\int_0^\infty (e^{-\lambda(t-h)} - e^{-\lambda t})\Lambda(t)\phi\,dt \\ &= \frac{-e^{-\lambda h}}{h}\int_0^h e^{\lambda t}\Lambda(t)\phi\,dt + \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}\Lambda(t)\phi\,dt. \end{aligned}$$

Hence

$$\lim_{h\searrow 0}\frac{T(h)\tilde{R}_\lambda\phi - \tilde{R}_\lambda\phi}{h} = -\phi + \lambda\tilde{R}_\lambda\phi,$$

in other words $A\tilde{R}_\lambda\phi = -\phi + \lambda\tilde{R}_\lambda\phi$ or

$$(\lambda I - A)\tilde{R}_\lambda\phi = \phi, \quad \phi \in X. \tag{5.13}$$

On the other hand if $\phi \in \mathcal{D}(A)$, we can use closedness of A (see Proposition 5.3) to put A inside the integral and write

$$A\tilde{R}_\lambda \phi = A \int_0^\infty e^{-\lambda t} \Lambda(t)\phi dt = \int_0^\infty e^{-\lambda t} A\, \Lambda(t)\phi dt$$
$$= \int_0^\infty e^{-\lambda t} \Lambda(t) A\phi dt = \tilde{R}_\lambda A\phi.$$

Thus

$$\tilde{R}_\lambda(\lambda I - A)\phi = \phi, \quad \phi \in \mathcal{D}(A). \tag{5.14}$$

Formulas (5.13) and (5.14) imply that $\tilde{R}_\lambda$ is one-to-one and onto. Consequently, for $\lambda \in \rho(A)$

$$\tilde{R}_\lambda = (\lambda I - A)^{-1} = R_\lambda.$$

□

Recall from Section 4.1 that

$$y(t) = \begin{cases} \psi(t) & \text{for } -\infty < t < 0, \\ \int_0^\infty k(a)y(t-a)da & \text{for } t \geq 0. \end{cases} \tag{5.15}$$

From Theorem 5.14 we get

$$R_\lambda \psi = (\widehat{\Lambda(\cdot)\psi})(\lambda) = \int_0^\infty e^{-\lambda t} \Lambda(t)\psi\, dt$$

and consequently

$$\begin{aligned}(R_\lambda \psi)(\theta) &= \int_0^\infty e^{-\lambda t} y_t(\theta) dt \\ &= \int_0^\infty e^{-\lambda t} y(t+\theta) dt \\ &= \int_\theta^\infty e^{-\lambda(\sigma-\theta)} y(\sigma) d\sigma \\ &= e^{\lambda\theta} \int_\theta^\infty e^{-\lambda\sigma} y(\sigma) d\sigma \\ &= e^{\lambda\theta}\left(\widehat{y}(\lambda) + \int_\theta^0 e^{-\lambda\sigma}\psi(\sigma)d\sigma\right).\end{aligned} \tag{5.16}$$

By (5.15) (LRE) can be written as

$$\begin{aligned}y(t) &= \int_0^t k(a)y(t-a)da + \int_t^\infty k(a)y(t-a)da \\ &= \int_0^t k(a)y(t-a)da + \int_{-\infty}^0 k(t-s)y(s)ds \\ &= \int_0^t k(a)y(t-a)da + \int_{-\infty}^0 k(t-s)\psi(s)ds.\end{aligned} \tag{5.17}$$

Taking Laplace transform of (5.17) we obtain

$$\widehat{y}(\lambda) = \widehat{k}(\lambda)\widehat{y}(\lambda) + Q(\lambda)\psi, \tag{5.18}$$

where

$$Q(\lambda)\psi = \int_0^\infty e^{-\lambda t} \int_{-\infty}^0 k(t-s)\psi(s)ds\, dt. \tag{5.19}$$

The equation (5.18) implies

$$\widehat{y}(\lambda) = \frac{Q(\lambda)\psi}{1-\widehat{k}(\lambda)}. \tag{5.20}$$

By substituting expression (5.20) for $\widehat{y}$ into (5.16) we get

$$(R_\lambda \psi)(\theta) = e^{\lambda\theta} \left[\frac{Q(\lambda)\psi}{1-\widehat{k}(\lambda)} + \int_\theta^0 e^{-\lambda s}\psi(s)ds \right]. \tag{5.21}$$

It follows that the resolvent R_λ exists as a bounded linear operator from X to X provided the right hand-side of (5.21) is, as a function of θ, a well-defined element of X and depends continuously on ψ. Let's denote

$$(H(\lambda)\psi)(\theta) := e^{\lambda\theta} \int_\theta^0 e^{-\lambda s}\psi(s)ds.$$

Then

$$\begin{aligned}
\|H(\lambda)\psi\|_{1,\xi} &\leq \int_{-\infty}^0 e^{\xi\theta} e^{\mathrm{Re}\lambda\theta} \int_\theta^0 e^{-\mathrm{Re}\lambda s}|\psi(s)|ds\, d\theta \\
&= \int_{-\infty}^0 \int_{-\infty}^s e^{(\xi+\mathrm{Re}\lambda)\theta} d\theta\, e^{-\mathrm{Re}\lambda s}|\psi(s)|ds \\
&= \int_{-\infty}^0 \frac{1}{\xi+\mathrm{Re}\lambda} e^{(\xi+\mathrm{Re}\lambda)s} e^{-\mathrm{Re}\lambda s}|\psi(s)|ds \\
&= \frac{1}{\xi+\mathrm{Re}\lambda}\|\psi\|_{1,\xi}.
\end{aligned}$$

Thus $H(\lambda)$ is a bounded linear operator from X to X provided

$$0 < \frac{1}{\xi+\mathrm{Re}\lambda} < \infty,$$

i.e. provided

$$\mathrm{Re}\lambda > -\xi.$$

The first term on the right hand-side of (5.21) is not defined if $1-\widehat{k}(\lambda) = 0$, i.e. exactly when λ is a root of the characteristic equation (4.6),

$$\widehat{k}(\lambda) = 1.$$

The following proposition states a well-knows result from complex analysis.

Proposition 5.15 *[18] Suppose that Ω is a region and f is a holomorphic function on Ω and denote*

$$Z(f) := \{a \in \Omega : f(a) = 0\}.$$

Then either $Z(f) = \Omega$, or $Z(f)$ has no limit point in Ω.

In our case we have $\Omega = \mathbb{C}$ and the holomorphic function $\widehat{k}(\cdot) - 1$. As the case $Z(f) = \Omega$ can be obviously excluded, we conclude that roots of the characteristic equation (4.6) are isolated points in $\mathbb{C}$.

We know from Section 4.2 that the function $y(t) = y_0 e^{\lambda t}$ is a solution of (LRE) if and only if λ is a root of the characteristic equation (4.6). Let λ be a root of (4.6). If we define the initial condition $\psi(\theta) = y_0 e^{\lambda\theta}$, we have

$$y_0 e^{\lambda t} = \int_0^\infty k(a) y_0 e^{\lambda(t-a)} da,$$

and consequently

$$[\Lambda(t)\psi](\theta) = y_t(\theta) = y_0 e^{\lambda t} e^{\lambda\theta} = e^{\lambda t}\psi(\theta) \tag{5.22}$$

for all $t \geq 0$. By simple manipulation of (5.22) and using the definition of A we get

$$\lim_{t\searrow 0} \frac{\Lambda(t)\psi - \psi}{t} = \lambda \lim_{t\searrow 0} \frac{e^{\lambda t}\psi - \psi}{\lambda t},$$
$$A\psi = \lambda\psi.$$

Hence, λ is and eigenvalue of A. We conclude that the set of roots of the characteristic equation (4.6) in the half-plane $\mathrm{Re}\lambda > -\xi$ is a subset of the (point) spectrum of A. The next task is to show that in the half-plane $\mathrm{Re}\lambda > -\xi$ there are no other spectral values.

If λ is not a root of the characteristic equation (4.6), then the first term on the right hand-side of (5.21), defines a bounded linear operator

$$\psi \mapsto \left(\theta \mapsto e^{\lambda\theta} \frac{Q(\lambda)\psi}{1 - \widehat{k}(\lambda)}\right)$$

from X to X if and only if

$$\psi \mapsto \left(\theta \mapsto e^{\lambda\theta} Q(\lambda)\psi\right) \tag{5.23}$$

defines a bounded linear operator from X to X. Since for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\lambda > -\xi$ we have

$$\|e^{\lambda\theta} Q(\lambda)\psi\|_{1,\xi} \leq \int_0^\infty e^{\mathrm{Re}\lambda\,\theta} e^{\xi\theta} |Q(\lambda)\psi| d\theta = \frac{1}{\xi + \mathrm{Re}\lambda} |Q(\lambda)\psi|,$$

the operator defined in (5.23) is bounded and linear if and only if $Q(\lambda)$ is a bounded linear functional from X to $\mathbb{R}$.

Proposition 5.16 *[5] Let $k \in L^1_\xi(0,\infty)$. Then $Q(\lambda)$ is a bounded linear operator from X into $\mathbb{R}$ for all λ with $Re\lambda > -\xi$.*

Proof. Recall the definition (5.19) of $Q(\lambda)$,

$$Q(\lambda)\psi = \int_0^\infty e^{-\lambda t}\int_{-\infty}^0 k(t-s)\psi(s)ds\,dt.$$

Linearity of $Q(\lambda)$ is obvious. From definition of $Q(\lambda)$ we get

$$\begin{aligned}
|Q(\lambda)\psi| &\le \int_0^\infty e^{-\mathrm{Re}\lambda\, t}\int_{-\infty}^0 |k(t-\sigma)||\psi(\sigma)|d\sigma\,dt \\
&= \int_0^\infty e^{-(\mathrm{Re}\lambda+\xi)t}\int_{-\infty}^0 e^{\xi(t-\sigma)}|k(t-\sigma)|e^{\xi\sigma}|\psi(\sigma)|d\sigma\,dt \\
&= \int_{-\infty}^0 e^{\xi\sigma}|\psi(\sigma)|\int_0^\infty e^{-(\mathrm{Re}\lambda+\xi)(\sigma+\tau)}e^{\xi\tau)}|k(\tau)|d\tau\,d\sigma \\
&\le \|k\|_{1,\xi}\|\psi\|_{1,\xi}
\end{aligned}$$

for all λ with $\mathrm{Re}\lambda > -\xi$. Thus $Q(\lambda)$ is a bounded linear operator form X into X.

□

Now I present the well-known Riemann-Lebesgue lemma for Laplace transform that will be used to prove the following theorem, which mostly summarizes the already known facts.

Lemma 5.17 (Riemann-Lebesgue lemma for Laplace transform) *Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable function. If f is L^1-integrable and supported on $(0,\infty)$, then*

$$\widehat{f}(z) \to 0$$

as $z \to \infty$ within the half-plane $\{z \in \mathbb{C},\ Im\, z \ge 0\}$.

Theorem 5.18 *[5] Let $k \in L^1_\xi(0,\infty)$. A point $\lambda \in \mathbb{C}$ with $Re\lambda > -\xi$ belongs to the spectrum of A if and only if it is a root of the characteristic equation*

$$\widehat{k}(\lambda) = 1. \tag{5.24}$$

Every root of the characteristic equation in the right half-plane

$$\Pi_+^\xi := \{\lambda \in \mathbb{C} : Re\lambda > -\xi\}$$

is an eigenvalue of A. There are at most finitely many roots of (5.24) in Π_+^ξ.

The mapping $\lambda \mapsto R_\lambda$ is holomorphic in Π_+^ξ except at the roots λ_n of (5.24), where it has a pole of order at most equal to the multiplicity of λ_n as a root of (5.24).

Proof. It only remains to prove the last two statements, everything else has already been proven.

Since the function $\lambda \mapsto \widehat{k}(\lambda) - 1$ is not identically zero, Proposition 5.15 implies that (5.24) has at most finitely many solutions in any compact subset of $\mathbb{C}$. It follows from Lemma 5.17 that for any $\epsilon > 0$, say for $\epsilon = 1/2$, there exists an $x > 0$ such that

$$|\widehat{k}(\lambda)| < \epsilon \quad \text{for all } z \text{ with } |z| > x.$$

Thus, there exists a compact set $C \subset \mathbb{C}$ such that all roots λ of the characteristic equation $\widehat{k}(\lambda) = 1$ belong to the interior of C, denoted C^o. These to facts together yield that there is at most a finite number of eigenvalues (roots of $\widehat{k}(\lambda) = 1$) in the right half-plane $\Pi_+^\xi = \{z \in \mathbb{C}, \text{Re} z > -\xi\}$.

Recall the representation (5.21) of R_λ,

$$(R_\lambda\psi)(\theta) = e^{\lambda\theta}\left[\frac{Q(\lambda)\psi}{1-\widehat{k}(\lambda)} + \int_\theta^0 e^{-\lambda s}\psi(a)ds\right].$$

Since the map $\lambda \mapsto Q(\lambda)\psi$ is for any $\psi \in X$ holomorphic, the only potentially problematic term is

$$\frac{1}{1-\widehat{k}(\lambda)},$$

and the problem occurs when $1 - \widehat{k}(\lambda) = 0$, i.e. precisely when λ is a root of (5.24). Hence the function

$$\lambda \mapsto \frac{1}{1-\widehat{k}(\lambda)}$$

is analytic except at the roots of (5.24) and the same holds for $\lambda \mapsto R_\lambda$.

□

It remains to prove that on the subspace of X spanned by eigenfunctions corresponding to eigenvalues $\lambda \in \mathbb{C}$ with $\text{Re}\lambda < -\xi$ the semigroup $\Lambda(t)$ for $t \to \infty$ exponentially decays. If $\Lambda(t)$ were a compact operator, exponential decay of T would follow from Proposition 5.5. That is, however, not the case, so one has to use some different technique. First I prove that exponential decay of solution of (LRE), (IC) implies exponential decay of $\Lambda(t)$ and then give conditions for exponential decay of solution to (LRE), (IC).

Theorem 5.19 *[17] Let $\Lambda(t)$ be a C_0-semigroup. If*

$$\int_0^\infty \|\Lambda(t)\psi\|_{1,\xi}dt < \infty \quad \textit{for every } \psi \in X,$$

then there are constants $M \geq 1$ and $\epsilon > 0$ such that $\|\Lambda(t)\|_{1,\xi} \leq Me^{-\epsilon t}$.

Lemma 5.20 *[5] Assume that the solution y of (LRE), (IC) satisfies*

$$\int_0^\infty |y(\sigma)|e^{\xi\sigma}d\sigma < \infty, \tag{5.25}$$

for all initial conditions $\psi \in X$. *Then the solution semigroup* Λ *is uniformly exponentially stable, that is, there exists* $\epsilon > 0$ *and* $\widetilde{M} \geq 1$ *such that*

$$\|\Lambda(t)\|_{1,\xi} \leq \widetilde{M}e^{-\epsilon t}, \quad t > 0. \tag{5.26}$$

If, instead of (5.25), the stronger condition

$$\int_0^\infty |y(\sigma)|e^{\xi\sigma}d\sigma \leq M\|\psi\|_{1,\xi} \tag{5.27}$$

holds for all $\psi \in X$, *then (5.26) holds with* $\epsilon = \xi$.

Proof. Recall that $\Lambda(t)\psi = y_t$. Therefore

$$\begin{aligned}
\|\Lambda(t)\psi\|_{1,\xi} &= \|y_t\|_{1,\xi} \\
&= \int_{-\infty}^0 |y(t+\theta)|e^{\xi\theta}d\theta \\
&= \int_{-\infty}^t |y(\sigma)|e^{\xi(\sigma-t)}d\sigma \\
&= e^{-\xi t}\int_{-\infty}^t |y(\sigma)|e^{\xi\sigma}d\sigma \\
&= e^{-\xi t}\left(\|\psi\|_{1,\xi} + \int_0^t |y(\sigma)|e^{\xi\sigma}d\sigma\right).
\end{aligned} \tag{5.28}$$

If (5.27) holds, then we set $\widetilde{M} := \max\{1, \|\psi\|_{1,\xi}(1+M)\}$ and (5.26) is proven for $\epsilon = \xi$.

Under the weaker assumption (5.25) we still have

$$\int_0^\infty \|\Lambda(t)\psi\|_{1,\xi}dt < \infty$$

for all ψ, so Theorem 5.19 implies that (5.26) holds for some $\epsilon > 0$.

□

Using (IC) we can write (LRE) as

$$\begin{aligned}
y(t) &= \int_0^\infty k(a)y(t-a)da \\
&= \int_0^t k(a)y(t-a)da + \int_t^\infty k(a)y(t-a)da \\
&= \int_0^t k(a)y(t-a)da + \int_t^\infty k(a)\psi(t-a)da \\
&= \int_0^t k(a)y(t-a)da + \int_{-\infty}^0 k(t-\theta)\psi(\theta)d\theta
\end{aligned}$$

Definition 5.21 *Let $k \in L^1_\xi(0,\infty)$ and $y : \mathbb{R} \to \mathbb{R}$ be such that $y_t \in X$ for all $t \geq 0$. By the convolution product of k and y we understand the function*

$$(k * y)(t) := \int_0^t k(a)y(t-a)da, \quad t \geq 0.$$

Adopting the notation notation introduced in Definition 5.21 we obtain the equation (LRE) in the form

$$y = k * y + f, \tag{5.29}$$

where

$$f(t) := \int_{-\infty}^0 k(t-\theta)\psi(\theta)d\theta.$$

Theorems 5.22 and 5.24 taken from the book [12] together yield a variation of constants formula for the equation (5.29).

Theorem 5.22 *[12] Let $k \in L^1_{loc}(0,\infty)$. Then there is a unique solution $r \in L^1_{loc}(0,\infty)$ of the equation*

$$r = k + k * r \ (= k + r * k). \tag{5.30}$$

This solution r depends continuously on k in the topology of $L^1_{loc}(0,\infty)$.

Definition 5.23 *[12] Function r from Theorem 5.22 is called the resolvent of k.*

Theorem 5.24 *[12] Let $k \in L^1_{loc}(0,\infty)$. Then for every $f \in L^1_{loc}(0,\infty)$ there is a unique solution $y \in L^1_{loc}(0,\infty)$ of (5.29). This solution is given by the variation of constants formula*

$$y(t) = f(t) + (r * f)(t), \quad t \geq 0, \tag{5.31}$$

where r is the resolvent of k.

The Paley-Wiener theorem presented below gives a condition for the resolvent r of k to belong to $L^1(0,\infty)$. This result will be used in the proof of Lemma 5.26.

Theorem 5.25 (Paley-Wiener) *[12] Let $k \in L^1(0,\infty)$. Then the resolvent r of k satisfies*

$$r \in L^1(0,\infty)$$

if and only if

$$\widehat{k}(z) \neq 1$$

for all $z \in \mathbb{C}$ such that $Re(z) \geq 0$.

Lemma 5.26 *[5] Let $k \in L^1_\xi(0,\infty)$. If the characteristic equation*

$$\widehat{k}(\lambda) = 1$$

has no roots with $Re\lambda \geq -\xi$, then the solution y of (LRE), (IC) satisfies the estimate

$$\int_0^\infty e^{\xi\sigma}|y(\sigma)|d\sigma \leq M\|\psi\|_{1,\xi}$$

for some $M \geq 1$.

Proof. Recall that we can write (LRE) in the form as in (5.29), i.e.

$$y(t) = \int_0^t k(s)y(t-s)ds + \int_{-\infty}^0 k(t-s)\psi(s)ds. \tag{5.32}$$

If we multiply both sides of (5.32) by $e^{\xi t}$, we get

$$e^{\xi t}y(t) = \int_0^t e^{\xi s}k(s)e^{\xi(t-s)}y(t-s)ds + \int_{-\infty}^0 e^{\xi(t-s)}k(t-s)e^{\xi s}\psi(s)ds. \tag{5.33}$$

Denoting

$$\begin{aligned} y^\xi &= e^{\xi t}y(t), \\ k^\xi(t) &= e^{\xi t}k(t), \\ f^\xi(t) &= \int_{-\infty}^0 k^\xi(t-s)e^{\xi s}\psi(s)ds \end{aligned}$$

for $t \geq 0$ we can rewrite (5.33) into

$$y^\xi(t) = \int_0^t k^\xi(s)y^\xi(t-s)ds + f^\xi(t), \tag{5.34}$$

Because $k \in L^1_\xi(0,\infty)$, we have $k^\xi \in L^1(0,\infty)$. We assume that the characteristic equation $\widehat{k}(\lambda) = 1$ has no roots with $\text{Re}\lambda \geq -\xi$. Since

$$\widehat{k^\xi}(\lambda) = \int_0^\infty e^{\xi a}e^{-\lambda a}k(a)da = \widehat{k}(\lambda - \xi),$$

the equation

$$\widehat{k^\xi}(\lambda) = 1$$

has no roots $\lambda \in \mathbb{C}$ with $\text{Re}\lambda \geq 0$. Now we can infer from the Paley-Wiener theorem (Theorem 5.25) that the resolvent r^ξ of k^ξ belongs to $L^1(0,\infty)$.

Recall that r the resolvent kernel of k, i.e.

$$r = k + r * k. \tag{5.35}$$

Multiplying both sides of (5.35) by $e^{\xi t}$ we obtain

$$r^\xi(t) = e^{\xi t} r(t),$$

and consequently

$$\|r^\xi\|_1 = \|r\|_{1,\xi}.$$

The formula (5.31) applied to the equation (5.34) yields

$$|y^\xi(t)| \leq |f^\xi(t)| + \int_0^t |r^\xi(t-s)| \cdot |f^\xi(s)| ds. \tag{5.36}$$

Moreover,

$$\begin{aligned}
\int_0^\infty |f^\xi(t)| dt &= \int_0^\infty \left| e^{\xi t} \int_{-\infty}^0 k(t-\theta)\psi(\theta) d\theta \right| dt \\
&\leq \int_0^\infty \int_{-\infty}^0 |k(t-\theta)| e^{\xi(t-\theta)} |\psi(\theta)| e^{\xi\theta} d\theta dt \\
&= \int_{-\infty}^0 \int_0^\infty |k(t-\theta)| e^{\xi(t-\theta)} dt |\psi(\theta)| e^{\xi\theta} d\theta \\
&\leq \int_{-\infty}^0 \|k\|_{1,\xi} |\psi(\theta)| e^{\xi\theta} d\theta \\
&= \|k\|_{1,\xi} \|\psi\|_{1,\xi},
\end{aligned} \tag{5.37}$$

and consequently

$$\begin{aligned}
\int_0^\infty \int_0^t |r^\xi(t-s)| \cdot |f^\xi(s)| ds dt &= \int_0^\infty \int_s^\infty |r^\xi(t-s)| dt |f^\xi(s)| ds \\
&= \int_0^\infty \int_0^\infty |r^\xi(t)| dt |f^\xi(s)| ds \\
&= \|r\|_{1,\xi} \int_0^\infty |f^\xi(s)| ds \\
&\leq \|r\|_{1,\xi} \|k\|_{1,\xi} \|\psi\|_{1,\xi}.
\end{aligned} \tag{5.38}$$

Now (5.36), (5.37) and (5.38) yield

$$\begin{aligned}
\int_0^\infty e^{\xi t} |y(t)| dt &= \int_0^\infty |y^\xi(t)| dt \\
&\leq \int_0^\infty \left(|f^\xi(t)| + \int_0^t |r^\xi(t-s)| \cdot |f^\xi(s)| ds \right) dt \\
&= \int_0^\infty |f^\xi(t)| dt + \int_0^\infty \int_0^t |r^\xi(t-s)| \cdot |f^\xi(s)| ds dt \\
&\leq \|k\|_{1,\xi} \|\psi\|_{1,\xi} + \|r\|_{1,\xi} \|k\|_{1,\xi} \|\psi\|_{1,\xi} \\
&= \|k\|_{1,\xi} \left(1 + \|r\|_{1,\xi}\right) \|\psi\|_{1,\xi} \\
&\leq M \|\psi\|_{1,\xi}
\end{aligned}$$

with $M := \max\{\|k\|_{1,\xi}, 1\} \cdot (1 + \|r\|_{1,\xi}) \geq 1$.

□

The following theorem combines Lemma 5.26 with Lemma 5.20 in order to summarize the relevant information.

Theorem 5.27 *[5] Let $k \in L^1_\xi(0, \infty)$. If the characteristic equation $\widehat{k}(\lambda) = 1$ has no roots with $Re\lambda \geq -\xi$, then the solution semigroup Λ of the linearized problem satisfies*

$$\|\Lambda(t)\| \leq \widetilde{M} e^{-\xi t}, \quad t \geq 0$$

for some $\widetilde{M} \geq 1$.

Proof. It follows from Lemma 5.26 that

$$\int_0^\infty e^{\xi\sigma} |y(\sigma)| d\sigma \leq M \|\psi\|_{1,\xi}$$

for some $M \geq 1$ and consequently Lemma 5.20 yields the estimate

$$\|\Lambda(t)\| \leq \widetilde{M} e^{-\xi t}, \quad t \geq 0$$

for some $\widetilde{M} \geq 1$.

□

Now we use some abstract functional analytic results to obtain the direct sum decomposition

$$X = X_+ \oplus X_-.$$

It follows from [14] (Sections III.6.4-5) and [20] (Section VII.8) that the operator P_+ defined by

$$P_+ = \frac{1}{2\pi i} \int_{\Gamma_+} R_\lambda d\lambda, \tag{5.39}$$

where Γ_+ is a counter clock-wise contour in the open right half-plane $\Pi^\xi_+ = \{\lambda \in \mathbb{C}, \mathrm{Re}\lambda > -\xi\}$ surrounding all the eigenvalues of A in Π^ξ_+, is a projection onto a finite dimensional subspace X_+ and that Λ leaves X_+ and X_- invariant. The subspace X_- is defined by

$$X_- := P_- X,$$

with

$$P_- := I - P_+.$$

If we substitute expression (5.21) for the resolvent R_λ into (5.39), we obtain

$$(P_+\psi)(\theta) = \frac{1}{2\pi i} \int_{\Gamma_+} e^{\lambda\theta} \frac{Q(\lambda)\psi}{1 - \widehat{k}(\lambda)} d\lambda, \quad \theta < 0, \tag{5.40}$$

as the last term on the right hand-side of (5.21) is holomorphic and therefore vanishes when integrated over the closed curve Γ_+.

Lemma 5.28 *[5] Let $k \in L^1_\xi(0,\infty)$. Then the restriction of the solution semigroup to X_- is uniformly exponentially stable, i.e. there exist $M \geq 1$ and $\epsilon > 0$ such that for all $\psi \in X_-$*

$$\|\Lambda(t)\psi_-\|_{1,\xi} \leq Me^{-\epsilon t}\|\psi_-\|_{1,\xi}, \quad t \geq 0. \tag{5.41}$$

Proof. By the obvious part of Lemma 5.20, the result follows once we have shown that the solution y^- of (LRE) with the initial condition

$$y^-(\theta) = P_-\psi(\theta), \quad \theta \leq 0,$$

is an element of $L^1_\xi(0,\infty)$ for all $\psi \in X$. Thanks to linearity of (LRE), the solution y of (LRE), (IC) has the unique decomposition

$$y(t) = y^+(t) + y^-(t), \quad t \in \mathbb{R}, \tag{5.42}$$

where y^+ is the solution of (LRE) with the initial condition

$$y^+(\theta) = P_+\psi(\theta), \quad \theta \leq 0.$$

An application of the residue theorem shows that there are constants $\alpha_{jk}(\psi)$, depending on ψ, such that

$$\frac{1}{2\pi i}\int_{\Gamma_+} e^{\lambda t}\frac{Q(\lambda)\psi}{1-\widehat{k}(\lambda)}d\lambda = \sum_{j=1}^{n}\sum_{k=0}^{p_j-1}\alpha_{jk}(\psi)t^k e^{\lambda_j t}$$

for all $t \in \mathbb{R}$. In particular,

$$(P_+\psi)(\theta) = \sum_{j=1}^{n}\sum_{k=0}^{p_j-1}\alpha_{jk}(\psi)\theta^k e^{\lambda_j \theta}, \quad \theta \leq 0. \tag{5.43}$$

Since P_+ is a projection, (5.43) implies that

$$\alpha_{jk}(\psi) = \alpha_{jk}(P_+\psi) \tag{5.44}$$

for all $\psi \in X$. On the other hand, Theorem 2.5 on page 197 of [12] states that the solution $y(t)$ of the Volterra equation (5.17),

$$y(t) = \int_0^t k(a)y(t-a)da + \int_{-\infty}^0 k(t-s)\psi(s)ds$$

(which, as we know, is equivalent to (LRE), (IC)) has the form

$$y(t) = \sum_{j=1}^{n}\sum_{k=0}^{p_j-1}\alpha_{jk}(\psi)t^k e^{\lambda_j t} + \rho(\psi,t), \quad t > 0, \tag{5.45}$$

where $\rho(\psi, \cdot) \in L^1_\xi(0, \infty)$ and the coefficients $\alpha_{jk}(\psi)$ have been obtained by exactly the same application of the residue theorem as above. Using (5.44) we get in particular

$$y^+(t) = \sum_{j=1}^{n} \sum_{k=0}^{p_j-1} \alpha_{jk}(\psi) t^k e^{\lambda_j t} + \rho(P_+\psi, t), \quad t > 0. \tag{5.46}$$

Combining (5.42), (5.45) and (5.46), we conclude that

$$y^-(t) = y(t) - y^+(t) = \rho(\psi, t) - \rho(P_+\psi, t), \quad t > 0,$$

and hence that y^- is an element of $L^1_\xi(0, \infty)$, which completes the proof.

□

The following theorem states the final result – the principle of linearized stability for equation (RE), (IC).

Theorem 5.29 (Principle of linearized stability) *[5] Let $\bar{b}$ be a steady state of (RE).*

1. *If all the roots of the characteristic equation (4.4),*

$$\widehat{k}(\lambda) = 1,$$

   have negative real part, then the steady state $\bar{b}$ is exponentially stable.

2. *If there exists at least one root of (4.4) with positive real part, then the steady state $\bar{b}$ is unstable.*

Proof.

1. Theorem 5.18 shows that (4.4) has at most finitely many roots in any right half-plane. If there are no roots with non-negative real part, we can choose $\bar{\xi} \in (0, \xi)$ such that (4.4) has no roots λ with Re$\lambda \geq -\bar{\xi}$.

   Theorem 5.27 yields that the semigroup Λ corresponding to the linearized problem satisfies

$$\|\Lambda(t)\| \leq M e^{-\bar{\xi} t}, \quad t \geq 0,$$

   for some $M \geq 1$. Then the conditions (i) and (ii) of Corollary 5.13 are satisfied with X_+ equal to the trivial subspace $\{0\}$. Hence $\bar{y}$ is exponentially stable, because $\sigma(A|_{X_+})$ is empty.

2. Assume that the characteristic equation (4.4) has n roots $\lambda_1, \ldots, \lambda_n$ in the right half-plane $\Pi^\xi_+ = \{\lambda \in \mathbb{C}, \text{Re}\lambda > -\xi\}$ and consider the decomposition $X = X_+ \oplus X_-$ as above, where X_+ is finite dimensional. By Lemma 5.28, the restricted semigroup $\Lambda|_{X_-}$ is uniformly exponentially stable. Thus the decomposition $X = X_+ \oplus X_-$ satisfies conditions (i) and (ii) of Corollary 5.13. If there exists at least one characteristic root with positive real part, then this root is an eigenvalue of $A|_{X_+}$ and (b) follows from Corollary 5.13 (b).

□

A Non-scaled Model

As I have mentioned at the very beginning of Section 1.1, I have been using a scaled version of the original model throughout the whole text. Now I am going to introduce the full model with some remarks.

We are dealing with a stage structured model of Daphnia population. The phrase "stage structured" means that we distinguish two stages of development – the juvenile period (before maturation) and the adult period (after maturation). The reproduction rate is constant within the stages, but the constant is different for each stage.

As before, $b(t)$ denotes the population birth rate at time t, $S(t)$ the substrate concentration at time t, S_0 rate of substrate supply and β is the constant reproduction rate of mature individuals. The survival probability till age a is again determined by $\mathcal{F}$ and the age of maturation by τ. This is all well-known from previous sections.

Let's now denote ξ_b the size at birth, ξ_A the size at maturation and γ the constant consumption rate of an individual that has reached age ϵ. Then we can write down the renewal equation with an initial condition,

$$b(t) = \beta \int_{\tau(b_t)}^{\infty} b_t(-a)\mathcal{F}(a)da, \quad t \geq 0, \tag{A.1}$$

$$b(\theta) = \varphi(\theta), \quad \theta < 0. \tag{A.2}$$

The original system consists of the renewal equation (A.1) and a differential equation for S,

$$\frac{dS}{dt}(t) = cS_0 - cS(t) - c\gamma S(t) \int_{\epsilon}^{\infty} b(t-a)\mathcal{F}(a)da. \tag{A.3}$$

To simplify the model, we use the quasi-steady-state approximation and reduce the system to the scalar renewal equation (A.1). The idea is that the food consumption happens at a much faster time scale compared to the length of life (and thus the time scale of the b-dynamics). Therefore we first consider the dynamics of S alone, see that S converges to a steady state and then study the renewal equation (A.1) while supposing that S is in the steady state. The steady state is given by

$$S(t) := \frac{S_0}{1 + \gamma \int_{\epsilon}^{\infty} b(t-a)\mathcal{F}(a)da}, \tag{A.4}$$

and one can easily check that it is stable. Thus, we consider (A.4) as a "definition" of $S(t)$.

Remark. As I mentioned in Section 1.3, the quasi-steady-state approximation causes an inconsistency when defining the maturation age (see the remark on page 10). That is, however, predictable, since we consider only an approximation for one of the two mutually dependent variables.

We also prescribe the dependence of body size on substrate concentration,

$$\xi(t) = \xi_b + \int_{t_b}^{t} S(\sigma)d\sigma, \tag{A.5}$$

where t_b is the time of birth. For $t = t_b + \tau$ we now get

$$\xi(t) = \xi_A, \tag{A.6}$$

so an individual matures when it reaches size ξ_A. Equations (A.5) and (A.6) together yield a defining equation for τ,

$$\int_{t-\tau}^{t} S(\sigma)d\sigma = \xi_A - \xi_b. \tag{A.7}$$

The basic reproduction number R_0 corresponding to the non-scaled model can be expressed as

$$R_0 = \beta \int_{\frac{\xi_A - \xi_b}{S_0}}^{\infty} \mathcal{F}(a)da.$$

Finally, we scale the parameters to make the bookkeeping easier. First we divide the equation (A.7) with (A.4) plugged in by $\xi_A - \xi_b$ and obtain

$$\int_{t-\tau}^{t} \frac{S_0}{\xi_A - \xi_b} \cdot \frac{1}{1 + \gamma \int_{\epsilon}^{\infty} b(\sigma - a)\mathcal{F}(a)da} d\sigma = 1.$$

Let's denote

$$\tilde{S}_0 := \frac{S_0}{\xi_A - \xi_b} \quad \text{and} \quad \tilde{b} := \gamma b.$$

Then $\tau(\tilde{b_t})$is defined by

$$\int_{-\tau}^{0} \frac{\tilde{S}_0}{1 + \int_{\epsilon}^{\infty} \tilde{b}(s - a)\mathcal{F}(a)da} ds = 1.$$

Moreover, we multiply the renewal equation (A.1) by γ yielding

$$\gamma\, b(t) = \beta \int_{\tau(\tilde{b}_t)}^{\infty} \gamma\, b_t(-a)\mathcal{F}(a)da,$$

and hence

$$\tilde{b}(t) = \beta \int_{\tau(\tilde{b}_t)}^{\infty} \tilde{b}_t(-a)\mathcal{F}(a)da.$$

As a final step we recycle the symbols b and S_0, i.e. write b instead of $\tilde{b}$ and S_0 instead of $\tilde{S}_0$.

To conclude, I remark that the model is rather simple with not much dependence among variables (e.g. the probability per unit of time β or consumption rate γ may depend on food density S). We also ignored any changes in the environment like seasonality.

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