An Introduction to

FUNCTIONAL SPACES

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Master's Thesis Mathematical Sciences

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Preface

> *"Time is the best teacher, but unfortunately, it kills all of its students."*
>
> *– Robin Williams*

In mathematics there are many important spaces of distributions and functions. In this thesis, written under the supervision of Dr. Marius Crainic at Utrecht University as part of the Master's programme Mathematical Sciences and in partial fulfillment of the requirements for the degree of Master of Science, I will discuss a framework to work with such spaces: the theory of functional spaces.

Inspired by the book *Foundations of Global Non-Linear Analyis* by Richard Palais, the foundations for the theory of functional spaces as presented in this thesis were laid in a set of lecture notes by Prof. Dr. Erik van den Ban and Dr. Marius Crainic for a Master's course in mathematics about analysis on manifolds in the fall of 2009. My mission was to extend these foundations and to fill in the details, which required both original research and a careful study of the existing material. The result of this mission is an extensive and precise introduction to functional spaces that goes beyond the original foundations and hopefully is of added value. In addition to the lecture notes by Erik van den Ban and Marius Crainic and the book by Richard Palais, the book *Topological Vector Spaces, Distributions and Kernels* by François Trèves has been an important source of information.

The main text of the thesis consists of three parts, corresponding to three different 'settings'. Roughly speaking, the first part is about scalar-valued functions on open subsets of Euclidean space, the second part is about sections of vector bundles and the third part is about sections of fiber bundles. Apart from different settings, also the nature of the three parts is quite different. The first part focuses on developing a formal theory of functional spaces: various new concepts, including of course the concept of a functional space, are introduced in an axiomatic way and their relation is investigated. In the second part, this formal theory is generalized to the vector bundle setting and it is shown that sufficiently well-behaved functional spaces on Euclidean space can be used as a model for functional spaces on vector bundles. Finally, in the third part, a modeling procedure that enables an extension from vector bundles to fiber bundles is discussed.

The three parts of the main text are not equally accesible in terms of prerequisites and required mathematical experience. The first part should be readable for a wide spectrum of mathematics students and professionals. For the majority of this part the most advanced requirement is some knowledge about locally convex vector spaces and some, but not all, of this required knowledge is even provided in an appendix. In this way, everyone with a background in mathematics should be able to get an idea of the topic. The second and third part, however, are more technical and require some fluency in the language of differential geometry. The appendix about differential geometry is certainly not meant to learn this language; it only covers some concepts and results that are not standard enough to assume to be known.

A valuable lesson that I have learned while writing this thesis is that, especially when the setting becomes more technical, it is sometimes hard to decide how many details should be given. Writing down too many details might result in proofs that are hard to follow and is very time-consuming, but giving too few details might cause readers to get frustrated or put the validity of the statements in doubt. Just writing down statements without sufficient explanations and proofs is no mathematics, but neither is only writing down small trivial steps. This thesis might be a bit on the 'too many details' side of the balance, but I suppose that this reflects what I hope to be as a mathematician: precise and rigorous.

Another valuable lesson that I have learned is that mathematics is never finished. My ambition was to give a 'complete' introduction to functional spaces, but for each question that I solved another question came up. As a consequence, there are many questions about functional spaces that remain to be answered. On the one hand, I would have liked to answer these remaining questions in this thesis as well. But, on the other hand, there will always be remaining questions and I think that we should be very happy about that.

Marcel de Reus
Capelle aan den IJssel, August 2011

Notation and conventions

In this text, we use the following notation and conventions:

- The natural numbers include 0.
- $\mathbb{N}_\infty := \mathbb{N} \cup \{\infty\}$, $\mathbb{Z}_\infty := \mathbb{Z} \cup \{-\infty, \infty\}$ and we adopt the usual rules for 'calculating with infinity'.
- We use a slightly different typesetted 'n' for the dimension of the Euclidean space or the manifold we are working with, namely n instead of n. We do this because this dimensional 'n', which is assumed to be a natural number greater than or to 1 throughout the text, is around all the time and we want to be able to use the letter 'n' for other purposes as well.
- To avoid endless quantifications, we use the 'power of consistent notation'. That is, for symbols that are typically used for real numbers, such as ε, δ and C, it is understood that, for example, $\varepsilon > 0$ also implicitly means $\varepsilon \in \mathbb{R}$. Similarly, for symbols that are typically used for natural numbers, such as k, ℓ, m and n, it is understood that, for example, $0 \leq \ell \leq k$ also implicitly means ℓ, $k \in \mathbb{N}$.
- Unless the context suggests otherwise, the symbols α, β and γ denote multi-indices. The number of components of these multi-indices will be clear from the context and we write things like $\sum_{|\alpha| \leq k}$ without further quantification.
- Except for the fifth chapter, it does not matter whether we work with vector spaces and vector bundles over the real or complex numbers. Therefore, we let $\mathbb{K}$ be either $\mathbb{R}$ or $\mathbb{C}$ and we stipulate that the word 'scalar' refers to elements of $\mathbb{K}$.
- All integrals over subsets of $\mathbb{R}^{\mathrm{n}}$ are Lebesgue integrals and we denote the Lebesgue measure on $\mathbb{R}^{\mathrm{n}}$ by λ.
- We usually do not distinguish in notation between a (linear) map and its restriction (to some subspace); we constantly work with restrictions and explicitly writing the usual 'restriction bar' all the time is therefore simply too cumbersome.
- Statements that contain multiple words inside parentheses should usually be read as two statements: one with the words inside the parentheses and one without them.
- The collection of all compact subsets of a topological space X is conveniently denoted by $\mathscr{P}_{\mathrm{c}}(X)$.

- A net in a topological space X is usually denoted by $\{x_i\}_{i\in I}$ or $\{x_j\}_{j\in J}$ and it is understood that I and J are directed sets. If x is in addition an element of X, then we write '$x_i \to x$ in X' or $x = \lim_{i\to\infty} x_i$ for 'the net $\{x_i\}_{i\in I}$ converges to x in X'.

- The words 'stronger' and 'weaker' have their usual 'mathematical' meaning. That is, 'stronger' means 'strictly stronger or of equal strength' and 'weaker' means 'strictly weaker or of equal strength'.

- For topological spaces X and Y, $X \subseteq_{\mathrm{c}} Y$ means that X is a subset of Y with continuous inclusion map.

- When we consider an equivalence relation on a set X, the equivalence class of an element $x \in X$ is denoted by $[x]$.

- Locally convex vector spaces are not required to be Hausdorff.

- By the dual $\mathscr{X}^*$ of a locally convex vector space $\mathscr{X}$, we always mean the strong dual.

- If $\mathscr{X}$ and $\mathscr{Y}$ are locally convex vector spaces, $\mathscr{X} = \mathscr{Y}$ means, unless we explicitly indicate otherwise, that $\mathscr{X}$ and $\mathscr{Y}$ are equal as locally convex vector spaces.

- A linear topological isomorphism between two topological vector spaces is a linear isomorphism that is simultaneously a homeomorphism. Similarly, a linear topological embedding is a linear injective map between topological vector spaces that is a homeomorphism onto its image.

- The product of n copies of a locally convex vector space $\mathscr{X}$ is usually denoted by $\mathscr{X}^n$, but sometimes we write $\mathscr{X}^{\times n}$ instead to avoid confusion with the 'powers of functional spaces' that will be discussed.

- Unless explicitly stated otherwise, when working with notions from differential geometry, we are working in the smooth setting. That is, manifold means smooth manifold, vector bundle means smooth vector bundle, etc. Moreover, all manifolds are assumed to be second-countable.

- We only consider vector bundles of constant rank and we often denote vector bundles and fiber bundles just by their total space.

- Fiber bundle homomorphisms (including vector bundle homomorphisms) between fiber bundles over the same manifold are assumed to be the identity on the base manifold (i.e., a fiber bundle homomorphism $f\colon P \to Q$ between fiber bundles P and Q over the same base manifold sends P_x into Q_x for all points x of the base manifold).

- Because we do not want to be bothered by uninteresting exceptions, we always assume that the dimension of the total space of a fiber bundle is strictly larger than the dimension of the base manifold (for vector bundles this means that the rank of the vector bundle is greater than or equal to 1). For the same reason, all manifolds (including open subsets of $\mathbb{R}^{\mathrm{n}}$) are implicitly assumed to be nonempty, unless there is a good reason to include the trivial case.

- With 'differential operator' we mean 'linear partial differential operator'.
- When a partition of unity subordinate to an open cover $\{U_i\}_{i\in I}$ of a manifold M is again indexed by I, say $\{\eta_i\}_{i\in I}$, the word 'subordinate' means that $\mathrm{supp}(\eta_i) \subseteq U_i$ for every $i \in I$. However, if the partition is indexed by another index set, say $\{\eta_j\}_{j\in J}$, the word 'subordinate' means that there exists an $i_j \in I$ for every $j \in J$ such that $\mathrm{supp}(\eta_{i_j}) \subseteq U_{i_j}$. In both cases, it is taken to be part of the definition of a partition of unity that the supports of the functions of the partition form a locally finite family of subsets of M.

Contents

Introduction

One of the most successful ‘export products’ of mathematics is probably the partial differential equation. The importance of partial differential equations in modern science is hard to overestimate. The study of these equations is a vast research area that crosses the boundaries of mathematics into many other scientific disciplines such as physics, chemistry, biology, economics and engineering. Although in an implicit way, the introduction of functional spaces is strongly motivated by the relevance of partial differential equations.

When attempting to solve a partial differential equation, one often hopes to find a smooth solution. That is, one looks for a solution in the space of smooth functions. However, to successfully achieve the goal of finding a smooth solution, it is sometimes easier to focus first on finding *any* solution, without insisting on smoothness. In this case, not the space of smooth functions but a larger space is used as ‘solution space’. This larger space might be a space of functions which are less well-behaved than smooth functions, but it might also be a space of more general objects than functions, namely distributions. In fact, there are many spaces of functions and distributions that can be used as ‘solution space’, the most famous ones probably being the Sobolev spaces, and it depends on the context which is the most suitable one.

To study these ‘solution spaces’ of functions and distributions in an abstract fashion, we introduce functional spaces. The idea is that the notion of a functional space precisely captures the relevant properties that many familiar spaces of functions and distributions have in common, allowing us to develop a general framework that can be used to work with such spaces. Of course, this process of recognizing the important common properties of various objects and then abstracting this lies at the heart of mathematics; it is precisely this process that results in ‘mathematical theories’ like group theory, ring theory, category theory, etc. The succes of abstract theories, and hence of mathematics, has two major reasons: results only have to be derived once instead of for each individual example and, maybe more surprisingly but not less important, removing the ‘burden’ of having all kinds of case specific details around allows us to ‘see much clearer’.

Although functional spaces describe much more ‘specialized’ objects than for example groups, rings or categories, we believe that these objects are general enough to benefit from the two major advantages of abstraction and we hope to deliver a useful addition to mathematics by giving such an abstract treatment.

Part I

1

Distributions on $\mathbb{R}^n$

Before we can give a precise and detailed introduction to functional spaces, we have to treat some distribution theory. The focus hereby lies on giving the necessary definitions and results and establishing notation; we do not give examples and the usual clarifying and motivational discussions are scarce and short. Most sections even contain a subsection that lists, without futher ado, some 'relevant results'. Although we give proofs for the majority of the results, we refer to other sources if a proof is too lenghty or too much of a detour. Readers that are not too confident about their knowledge of locally convex vector spaces are advised to have a look at Appendix A first.

Throughout this chapter, Ω denotes an open subset of $\mathbb{R}^n$ and whenever $\Omega = \emptyset$ would cause difficulties or require changes, we implicitly assume that Ω is nonempty.

1.1 Test functions

Consider the linear space $C^\infty(\Omega)$ of scalar-valued smooth functions on Ω. We want to endow $C^\infty(\Omega)$ with a suitable locally convex topology. In order to do so, we define for every $K \in \mathscr{P}_c(\Omega)$ and $k \in \mathbb{N}$, a seminorm $\|\cdot\|_{K,k}$ on $C^\infty(\Omega)$ by

$$\|\varphi\|_{K,k} := \sum_{|\alpha|\leq k} \sup_{x\in K} |\partial^\alpha \varphi(x)|$$

and we endow $C^\infty(\Omega)$ with the topology induced by these seminorms. To emphasize the presence of this specifically chosen topology, the resulting locally convex vector space will be denoted by $\mathscr{E}(\Omega)$.

Next, we define $\mathscr{E}_K(\Omega)$, for $K \in \mathscr{P}_c(\Omega)$, to be the linear subspace of $\mathscr{E}(\Omega)$ consisting of the smooth functions with support inside K endowed with the subspace topology and we define $\mathscr{D}(\Omega)$ to be

$$\bigcup_{K\in\mathscr{P}_c(\Omega)} \mathscr{E}_K(\Omega)$$

endowed with the inductive limit topology (see Definition A.3.1). $\mathscr{D}(\Omega)$ consists of all compactly supported smooth functions on Ω and is often called the space of 'test functions'.

Relevant results

Proposition 1.1.1. $\mathscr{D}(\Omega) \subseteq_c \mathscr{E}(\Omega)$.

Proof: Because $\mathscr{E}_K(\Omega)$, with $K \in \mathscr{P}_c(\Omega)$, carries the topology induced from $\mathscr{E}(\Omega)$, we have that $\mathscr{E}_K(\Omega) \subseteq_c \mathscr{E}(\Omega)$ for every $K \in \mathscr{P}_c(\Omega)$. Applying Proposition A.3.2 to the inclusion map $\mathscr{D}(\Omega) \hookrightarrow \mathscr{E}(\Omega)$ finishes the proof. $\square$

Remark 1.1.2. $\mathscr{E}(\Omega)$ is Hausdorff because the inducing collection of seminorms

$$\{\|\cdot\|_{K,k} \mid K \in \mathscr{P}_c(\Omega) \text{ and } k \in \mathbb{N}\}$$

for $\mathscr{E}(\Omega)$ is clearly total (i.e., if $\|\varphi\|_{K,k} = 0$ for all $K \in \mathscr{P}_c(\Omega)$ and $k \in \mathbb{N}$, then $\varphi = 0$). The previous proposition subsequently implies that $\mathscr{D}(\Omega)$ is Hausdorff as well. ⊘

Lemma 1.1.3. *For every $K \in \mathscr{P}_c(\Omega)$, $\mathscr{E}_K(\Omega)$ is closed in $\mathscr{E}(\Omega)$.*

Proof: Let $\{\varphi_i\}_{i\in I}$ be a net in $\mathscr{E}_K(\Omega)$ and $\varphi \in \mathscr{E}(\Omega)$ such that $\varphi_i \to \varphi$ in $\mathscr{E}(\Omega)$. We should prove that $\operatorname{supp}(\varphi) \subseteq K$. That is, we should prove that for every $x \in \Omega \setminus K$ there exists a neighborhood of x on which φ vanishes.

So let $x_0 \in \Omega \setminus K$. Because $\Omega \setminus K$ is open in Ω and Ω is locally compact, we find a compact neighborhood K' of x_0 such that $K' \subseteq \Omega \setminus K$. Since $\varphi_i \to \varphi$ in $\mathscr{E}(\Omega)$, we in particular have that $\|\varphi_i\|_{K',0} \to \|\varphi\|_{K',0}$ in $\mathbb{R}$. But, for all $i \in I$, the fact that $\operatorname{supp}(\varphi_i) \subseteq K$ implies that φ_i equals 0 on $\Omega \setminus K$ and hence that $\|\varphi_i\|_{K',0} = 0$. Thus $\sup_{x\in K'} |\varphi(x)| = \|\varphi\|_{K',0} = \lim_{i\to\infty} \|\varphi_i\|_{K',0} = 0$, which shows that K' is a neighborhood of x_0 on which φ vanishes. $\square$

Remark 1.1.4. We will often use, without explicit mention, the following trivial observation: if $K, K' \in \mathscr{P}_c(\Omega)$ and $k, k' \in \mathbb{N}$ such that $K \subseteq K'$ and $k \le k'$, then $\|\cdot\|_{K,k} \le \|\cdot\|_{K',k'}$ on $\mathscr{E}(\Omega)$. ⊘

Lemma 1.1.5. *Let $\mathscr{Y}$ be a locally convex vector space, $\mathscr{Q}$ an inducing collection of seminorms for $\mathscr{Y}$ and $T\colon \mathscr{E}(\Omega) \to \mathscr{Y}$ a linear map. Then T is continuous if and only if for every $q \in \mathscr{Q}$ there exist $C \ge 0$, $K \in \mathscr{P}_c(\Omega)$ and $k \in \mathbb{N}$ such that*

$$q(T(\varphi)) \le C\|\varphi\|_{K,k}$$

for every $\varphi \in \mathscr{E}(\Omega)$.

Proof: Suppose that T is continuous. Because

$$\mathscr{P} := \{\|\cdot\|_{K,k} \mid K \in \mathscr{P}_c(\Omega) \text{ and } k \in \mathbb{N}\}$$

is, by definition, an inducing collection of seminorms for $\mathscr{E}(\Omega)$, Lemma A.1.2 tells us that for every $q \in \mathscr{Q}$ there exist $C \ge 0$, $K_0, \dots, K_n \in \mathscr{P}_c(\Omega)$ and $k_0, \dots, k_n \in \mathbb{N}$ such that

$$q(T(\varphi)) \le C \sum_{i=0}^{n} \|\varphi\|_{K_i,k_i}$$

for every $\varphi \in \mathscr{E}(\Omega)$. But then $K := \cup_{0=1}^{n} K_i \in \mathscr{P}_c(\Omega)$ and $k := \max_{0\le i\le n} k_i \in \mathbb{N}$ satisfy

$$q(T(\varphi)) \le C \sum_{i=0}^{n} \|\varphi\|_{K_i,k_i} \le C(n+1)\|\varphi\|_{K,k}$$

for every $\varphi \in \mathscr{E}(\Omega)$.

The converse implication is a direct consequence of Lemma A.1.2 and the fact that $\mathscr{P}$ is an inducing collection of seminorms for $\mathscr{E}(\Omega)$. $\square$

Lemma 1.1.6. *A seminorm $p\colon \mathscr{E}(\Omega) \to \mathbb{R}$ is continuous if and only if there exist $C \geq 0$, $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and $k \in \mathbb{N}$ such that*

$$p(\varphi) \leq C\|\varphi\|_{K,k}$$

for every $\varphi \in \mathscr{E}(\Omega)$.

Proof: The proof is analogous to the proof of the previous lemma, just use Lemma A.1.1 instead of Lemma A.1.2. □

Corollary 1.1.7. *$\mathscr{E}(\Omega)$ does not allow a continuous norm.*

Proof: Suppose for a contradiction that $\|\cdot\|\colon \mathscr{E}(\Omega) \to \mathbb{R}$ is a continuous norm on $\mathscr{E}(\Omega)$. On behalf of the previous lemma, we find $C \geq 0$, $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and $k \in \mathbb{N}$ such that

$$\|\varphi\| \leq C\|\varphi\|_{K,k}$$

for all $\varphi \in \mathscr{E}(\Omega)$. Now let $\varphi_0 \in \mathscr{E}(\Omega) \setminus \{0\}$ such that φ_0 equals 0 on an open neighborhood of K (since K is compact, such a function clearly exists). Then $\|\varphi_0\| \leq C\|\varphi_0\|_{K,k} = 0$, hence $\|\varphi_0\| = 0$. Since $\|\cdot\|$ is assumed to be a norm, this implies $\varphi_0 = 0$, contradicting our choice of φ_0. □

Lemma 1.1.8. *Let $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. Then $\mathscr{P} := \{\|\cdot\|_{K,k} \mid k \in \mathbb{N}\}$ is an inducing collection of (semi)norms for $\mathscr{E}_K(\Omega)$.*

Proof: Because $\mathscr{E}_K(\Omega)$ is endowed with the topology induced from $\mathscr{E}(\Omega)$, we already know that $\mathscr{P}' := \{\|\cdot\|_{K',k} \mid K' \in \mathscr{P}_{\mathrm{c}}(\Omega) \text{ and } k \in \mathbb{N}\}$ is an inducing collection of seminorms for $\mathscr{E}_K(\Omega)$ and clearly $\mathscr{P} \subseteq \mathscr{P}'$. Furthermore, since $\operatorname{supp}(\varphi) \subseteq K$ for every $\varphi \in \mathscr{E}_K(\Omega)$, we have for every $K' \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and $k \in \mathbb{N}$ that $\|\cdot\|_{K',k} \leq \|\cdot\|_{K,k}$ on $\mathscr{E}_K(\Omega)$. In other words, for every $p' \in \mathscr{P}'$ there is an $p \in \mathscr{P}$ such that $p' \leq p$ on $\mathscr{E}_K(\Omega)$. Applying Corollary A.1.6 then shows that $\mathscr{P}$ is an inducing collection of (semi)norms for $\mathscr{E}_K(\Omega)$. □

Remark 1.1.9. Formally speaking, we defined $\|\cdot\|_{K,k}$ on $\mathscr{E}(\Omega)$, so we should actually write $\|\cdot\|_{K,k}|_{\mathscr{E}_K(\Omega)}$ in the lemma above. However, we will be working with restricted mappings a lot and it is very cumbersome to constantly write down the restriction explicitly. Therefore, as announced in 'Notation and conventions', we will not write down the restriction symbol in cases where the context makes already clear that we consider the restriction of some map. ⊘

Lemma 1.1.10. *Let $\mathscr{Y}$ be a locally convex vector space, $\mathscr{Q}$ an inducing collection of seminorms for $\mathscr{Y}$, $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and $T\colon \mathscr{E}_K(\Omega) \to \mathscr{Y}$ a linear map. Then T is continuous if and only if for every $q \in \mathscr{Q}$ there exist $C \geq 0$ and $k \in \mathbb{N}$ such that*

$$q(T(\varphi)) \leq C\|\varphi\|_{K,k}$$

for every $\varphi \in \mathscr{E}_K(\Omega)$.

Proof: Suppose that T is continuous. Because the previous lemma tells us that

$$\mathscr{P} := \{\|\cdot\|_{K,k} \mid k \in \mathbb{N}\}$$

is an inducing collection of seminorms for $\mathscr{E}_K(\Omega)$, Lemma A.1.2 gives that for every $q \in \mathscr{Q}$ there exist $C \geq 0$ and $k_0, \ldots, k_n \in \mathbb{N}$ such that

$$q(T(\varphi)) \leq C \sum_{i=0}^{n} \|\varphi\|_{K,k_i}$$

for every $\varphi \in \mathscr{E}_K(\Omega)$. But then $k := \max_{0 \leq i \leq n} k_i \in \mathbb{N}$ satisfies

$$q(T(\varphi)) \leq C \sum_{i=0}^{n} \|\varphi\|_{K,k_i} \leq C(n+1)\|\varphi\|_{K,k}$$

for every $\varphi \in \mathscr{E}_K(\Omega)$.

The converse implication is a direct consequence of Lemma A.1.2 and the fact that $\mathscr{P}$ is an inducing collection of seminorms for $\mathscr{E}_K(\Omega)$. □

Lemma 1.1.11. *Let $\mathscr{Y}$ be a locally convex vector space, $\mathscr{Q}$ an inducing collection of seminorms for $\mathscr{Y}$ and $T\colon \mathscr{D}(\Omega) \to \mathscr{Y}$ a linear map. Then T is continuous if and only if for every $q \in \mathscr{Q}$ and $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ there exist $C \geq 0$ and $k \in \mathbb{N}$ such that*

$$q(T(\varphi)) \leq C\|\varphi\|_{K,k}$$

for every $\varphi \in \mathscr{E}_K(\Omega)$.

Proof: According to Proposition A.3.2, T is continuous if and only if

$$T|_{\mathscr{E}_K(\Omega)} : \mathscr{E}_K(\Omega) \to \mathscr{Y}$$

is continuous for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. Now apply the the previous lemma to get the desired result. □

Remark 1.1.12. If $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and U is an open subset of Ω such that $K \subseteq U$, then there exists an $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of K and $\mathrm{supp}(\varphi) \subseteq U$. This is a direct consequence of the existence of finite partitions of unity over K by compactly supported smooth functions subordinate to any open cover of K. For a proof of the existence of such partitions, we refer to [3, Theorem 2.15]. ⊘

Remark 1.1.13. It is important to know that $\mathscr{D}(\Omega)$ can also be realized as a *strict* inductive limit (see Definition A.3.3). To see this, let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion by compacts of Ω, i.e., a collection of compact subsets of Ω such that

$$\Omega = \bigcup_{i\in\mathbb{N}} K_i \quad \text{and} \quad K_i \subseteq \mathrm{int}(K_{i+1}) \text{ for all } i \in \mathbb{N}$$

(such an exhaustion always exists, see for example [8, Proposition 4.76]). Then for any $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, there must be an $i \in \mathbb{N}$ such that $K \subseteq K_i$. After all, we easily see that $\{\mathrm{int}(K_i)\}_{i\in\mathbb{N}}$ is an open cover of Ω, so for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ we find $i_0, \ldots, i_n \in \mathbb{N}$ such that $K \subseteq \cup_{j=0}^{n}\mathrm{int}(K_{i_j})$ and $\cup_{j=0}^{n}\mathrm{int}(K_{i_j}) \subseteq K_i$ for $i := \max_{0\leq j\leq n} i_j$. In combination with Proposition A.3.2, this implies that $\mathscr{D}(\Omega)$ is equal, as locally convex vector space, to $\cup_{i\in\mathbb{N}}\mathscr{E}_{K_i}(\Omega)$ endowed with the inductive limit topology and the latter is in fact a strict inductive limit.

Indeed, from Lemma 1.1.3 and the fact that the $\mathscr{E}_{K_i}(\Omega)$ are endowed with the restricted topology of $\mathscr{E}(\Omega)$, we deduce, for every $i \in \mathbb{N}$, that $\mathscr{E}_{K_i}(\Omega)$

is closed in $\mathscr{E}_{K_{i+1}}(\Omega)$ and that the inclusion map $\mathscr{E}_{K_i}(\Omega) \hookrightarrow \mathscr{E}_{K_{i+1}}(\Omega)$ is an embedding. Moreover, the previous remark makes clear that the inclusion $\mathscr{E}_{K_i}(\Omega) \subseteq_{\mathrm{c}} \mathscr{E}_{K_{i+1}}(\Omega)$ is strict (indeed, there exists a smooth function with support inside $\mathrm{int}(K_{i+1}) \subseteq K_{i+1}$ that equals 1 on an open neighborhood of K_i). ⊘

Lemma 1.1.14. *A subset B of $\mathscr{D}(\Omega)$ is bounded if and only if there exists an $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ such that B is a bounded subset of $\mathscr{E}_K(\Omega)$.*

Proof: Suppose that B is bounded in $\mathscr{D}(\Omega)$ and let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion by compacts of Ω. Using the previous remark and Proposition A.3.4, we see that there must be an $i \in \mathbb{N}$ such that B is a bounded subset of $\mathscr{E}_{K_i}(\Omega)$.

Conversely, if B is a bounded subset of $\mathscr{E}_K(\Omega)$ for some $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, then $\mathscr{E}_K(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}(\Omega)$ implies that B is bounded in $\mathscr{D}(\Omega)$ because continuous linear maps send bounded sets to bounded sets. □

Proposition 1.1.15. *$\mathscr{D}(\Omega)$ is sequentially dense in $\mathscr{E}(\Omega)$.*

Proof: Let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion by compacts of Ω and take, for every $i \in \mathbb{N}$, $\varphi_i \in \mathscr{D}(\Omega)$ such that φ_i equals 1 on an open neighborhood of K_i (see Remark 1.1.12). We claim that for every $\psi \in \mathscr{E}(\Omega)$ the sequence $\{\varphi_i\psi\}_{i\in\mathbb{N}}$, which is clearly a sequence in $\mathscr{D}(\Omega)$, converges to ψ in $\mathscr{E}(\Omega)$. For this, we should check that $\lim_{i\to\infty} \|\psi - \varphi_i\psi\|_{K,k} = 0$ for all $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and $k \in \mathbb{N}$. As explained in Remark 1.1.13, we find for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ an $i_0 \in \mathbb{N}$ such that $K \subseteq K_{i_0}$ and because the K_i are increasing, we in fact have $K \subseteq K_i$ for every $i \geq i_0$. As a consequence, ψ and $\varphi_i\psi$ coincide on an open neighborhood of K for every $i \geq i_0$ and this clearly implies $\lim_{i\to\infty} \|\psi - \varphi_i\psi\|_{K,k} = 0$. □

Lemma 1.1.16. *$\mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega)$ are both reflexive.*

Proof: This is a consequence of the fact that $\mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega)$ are Montel spaces (see [13, Proposition 34.4]) and the fact that Montel spaces are reflexive (see [13, Corollary of Proposition 36.9]). □

1.2 Distributions

We define the space of distributions on Ω as the strong dual of $\mathscr{D}(\Omega)$ and denote it by $\mathscr{D}'(\Omega)$. Similarly, we define the space of compactly supported distributions on Ω as the strong dual of $\mathscr{E}(\Omega)$ and denote it by $\mathscr{E}'(\Omega)$. It will become clear later on why the terminology 'compactly supported' is appropriate.

The following characterizations are direct consequences of Lemma 1.1.11 and Lemma 1.1.5.

Proposition 1.2.1. *A distribution on Ω is a linear map $u\colon \mathscr{D}(\Omega) \to \mathbb{K}$ with the following property: for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ there exist $C \geq 0$ and $k \in \mathbb{N}$ such that*

$$|u(\varphi)| \leq C\|\varphi\|_{K,k}$$

for every $\varphi \in \mathscr{E}_K(\Omega)$.

Proposition 1.2.2. *A compactly supported distribution on Ω is a linear map $u\colon \mathscr{E}(\Omega) \to \mathbb{K}$ with the following property: there exist $C \geq 0$, $K \in \mathscr{P}_c(\Omega)$ and $k \in \mathbb{N}$ such that*

$$|u(\varphi)| \leq C\|\varphi\|_{K,k}$$

for every $\varphi \in \mathscr{E}(\Omega)$.

Note that $\mathscr{D}'(\Omega)$ and $\mathscr{E}'(\Omega)$, as strong duals of locally convex vector spaces, are both Hausdorff (see Lemma A.4.2).

Lemma 1.2.3. *The map that sends $u \in \mathscr{E}'(\Omega)$ to $u|_{\mathscr{D}(\Omega)}$ is an injective continuous linear map from $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$.*

Proof: We already know that the inclusion map $\mathscr{D}(\Omega) \hookrightarrow \mathscr{E}(\Omega)$ is a continuous linear map (see Proposition 1.1.1). The described 'restriction' map is clearly the adjoint of this map and therfore a continuous linear map from $\mathscr{E}'(\Omega)$ into $\mathscr{D}'(\Omega)$. The injectivity follows from Proposition 1.1.15 and Lemma A.4.4. □

The previous lemma shows that we can identify $\mathscr{E}'(\Omega)$ as vector space with the linear subspace of $\mathscr{D}'(\Omega)$ consisting of all distributions that 'allow a continuous extension to $\mathscr{E}(\Omega)$'. But be careful, the topology of $\mathscr{E}'(\Omega)$ and the induced topology of this vector subspace of $\mathscr{D}'(\Omega)$ 'do not match'. However, the continuity of $u \mapsto u|_{\mathscr{D}(\Omega)}$ does tell us that the topology on $\mathscr{E}'(\Omega)$ is stronger than this induced topology. Having a linear subspace of $\mathscr{D}'(\Omega)$ with a stronger topology than the induced topology will play a central role in the theory of functional spaces.

1.3 Canonical identifications

The crucial point about distributions is that they 'generalize' ordinary, sufficiently well-behaved, functions. To see this, let f be a locally integrable function on Ω. Then for every $K \in \mathscr{P}_c(\Omega)$ and $\varphi \in \mathscr{E}_K(\Omega)$

$$\left|\int_\Omega f\varphi \, d\lambda\right| \leq \int_K |f\varphi| \, d\lambda \leq \|\varphi\|_{K,0} \int_K |f| \, d\lambda < \infty, \tag{1.1}$$

so we may define $u_f : \mathscr{D}(\Omega) \to \mathbb{K}$ by

$$u_f(\varphi) := \int_\Omega f\varphi \, d\lambda.$$

Looking at equation (1.1) and Proposition 1.2.1, we see that u_f is in fact a distribution on Ω. If g is another locally integrable function on Ω, it can be shown that $u_f = u_g$ if and only if f and g are equal almost everywhere (see for example [6, Proposition 2.2 on page 269]) and since it is customary to identify locally integrable functions that are almost everywhere the same, the assignment $f \mapsto u_f$ 'embeds' (in a purely set theoretic, non-topological fashion) the locally integrable functions in the space of distributions. This explains why distributions are sometimes called 'generalized functions'.

Because every smooth function is clearly locally integrable, we have in particular a map

$$\jmath\colon \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega)\colon \psi \mapsto u_\psi.$$

Using the linearity of the integral, we see that $\jmath$ is linear and since two continuous functions are the same almost everywhere if and only if they are the same everywhere, $\jmath$ is injective. We claim that $\jmath$ is even continuous.

Claim. $\jmath\colon \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega)$ is continuous.

Proof: Because $\jmath$ is a linear map between locally convex vector spaces, we can use Lemma A.1.2 to establish the claim. By definition

$$\mathscr{Q} := \{q_B \mid B \text{ a bounded subset of } \mathscr{D}(\Omega)\},$$

with $q_B : \mathscr{D}'(\Omega) \to \mathbb{R}\colon u \mapsto \sup_{\varphi\in B} |u(\varphi)|$, is an inducing collection of seminorms for $\mathscr{D}'(\Omega)$, while

$$\mathscr{P} := \{\|\cdot\|_{K,k} \mid K \in \mathscr{P}_{\mathrm{c}}(\Omega) \text{ and } k \in \mathbb{N}\}$$

is an inducing collection of seminorms for $\mathscr{E}(\Omega)$.

Fix a bounded subset B of $\mathscr{D}(\Omega)$. By Lemma 1.1.14, we find an $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ such that B is a bounded subset of $\mathscr{E}_K(\Omega)$. In other words, $B \subseteq \mathscr{E}_K(\Omega)$ and for every $k \in \mathbb{N}$ there exists an $r_k \geq 0$ such that $\|\varphi\|_{K,k} \leq r_k$ for all $\varphi \in B$. Hence, for every $\psi \in \mathscr{E}(\Omega)$ and $\varphi \in B$, we have

$$|u_\psi(\varphi)| = \left|\int_\Omega \psi\varphi\, d\lambda\right| \leq \int_K |\psi\varphi|\, d\lambda \leq \lambda(K)\|\varphi\|_{K,0}\|\psi\|_{K,0} \leq \lambda(K) r_0 \|\psi\|_{K,0}$$

(the Lebesgue measure $\lambda(K)$ of K if finite because K is compact). As a consequence, we get

$$q_B(\jmath(\psi)) = q_B(u_\psi) = \sup_{\varphi\in B} |u_\psi(\varphi)| \leq \lambda(K) r_0 \|\psi\|_{K,0},$$

which is of the desired form. □

If ψ is not only smooth, but also compactly supported, we can do even better. After all, if $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and $\psi \in \mathscr{E}_K(\Omega)$, then for every $\varphi \in \mathscr{E}(\Omega)$

$$\left|\int_\Omega \psi\varphi\, d\lambda\right| \leq \int_K |\psi\varphi|\, d\lambda \leq \lambda(K)\|\psi\|_{K,0}\|\varphi\|_{K,0} < \infty, \tag{1.2}$$

so we may define $\hat{u}_\psi\colon \mathscr{E}(\Omega) \to \mathbb{K}$ by

$$\hat{u}_\psi(\varphi) := \int_\Omega \psi\varphi\, d\lambda.$$

In view of Proposition 1.2.2, equation (1.2) shows that $\hat{u}_\psi$ is in fact a compactly supported distribution on Ω and we get a map $\hat{\jmath}\colon \mathscr{D}(\Omega) \to \mathscr{E}'(\Omega)\colon \psi \mapsto \hat{u}_\psi$. Again, we easily see that $\hat{\jmath}$ is injective and linear and we claim that $\hat{\jmath}$ is also continuous.

Claim. $\hat{\jmath}\colon \mathscr{D}(\Omega) \to \mathscr{E}'(\Omega)$ is continuous.

Proof: It suffices to prove that, for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, $\hat{\jmath}$ restricts to a continuous linear map from $\mathscr{E}_K(\Omega)$ into $\mathscr{E}'(\Omega)$ (see Proposition A.3.2). So fix $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$.

In order to prove that $\hat{\jmath}\colon \mathscr{E}_K(\Omega) \to \mathscr{E}'(\Omega)$ is continuous, we once more use Lemma A.1.2. By definition

$$\mathscr{Q} := \{q_B \mid B \text{ a bounded subset of } \mathscr{E}(\Omega)\},$$

with $q_B : \mathscr{E}'(\Omega) \to \mathbb{R} \colon u \mapsto \sup_{\varphi \in B} |u(\varphi)|$, is an inducing collection of seminorms for $\mathscr{E}'(\Omega)$, while

$$\mathscr{P} := \{\|\cdot\|_{K,k} \mid k \in \mathbb{N}\}$$

is an inducing collection of seminorms for $\mathscr{E}_K(\Omega)$.

Let B be a bounded subset of $\mathscr{E}(\Omega)$. Then we in particular find an $r_0 \geq 0$ such that $\|\varphi\|_{K,0} \leq r_0$ for all $\varphi \in B$. Together with equation (1.2), this shows that for all $\varphi \in B$ and $\psi \in \mathscr{E}_K(\Omega)$

$$|\hat{u}_\psi(\varphi)| \leq \lambda(K)\|\psi\|_{K,0}\|\varphi\|_{K,0} \leq \lambda(K) r_0 \|\psi\|_{K,0}.$$

As a consequence,

$$q_B(\hat{\jmath}(\psi)) = q_B(\hat{u}_\psi) = \sup_{\varphi \in B} |\hat{u}_\psi(\varphi)| \leq \lambda(K) r_0 \|\psi\|_{K,0},$$

which is an estimate of the desired form. □

So we have natural injective continuous linear maps $\jmath\colon \mathscr{E}(\Omega) \to \mathscr{D}'(\Omega)$ and $\hat{\jmath}\colon \mathscr{D}(\Omega) \to \mathscr{E}'(\Omega)$. If we add the inclusion $\imath\colon \mathscr{D}(\Omega) \hookrightarrow \mathscr{E}(\Omega)$ and its adjoint $\imath^* \colon \mathscr{E}'(\Omega) \to \mathscr{D}'(\Omega)$ to this list, we get a nice diagram with injective continuous linear maps as arrows:

$$\begin{array}{ccc} \mathscr{D}(\Omega) & \xrightarrow{\ \imath\ } & \mathscr{E}(\Omega) \\ {\scriptstyle\hat{\jmath}}\big\downarrow & & {\scriptstyle\jmath}\big\downarrow \\ \mathscr{E}'(\Omega) & \xrightarrow{\ \imath^*\ } & \mathscr{D}'(\Omega) \end{array}$$

(that $\imath$ and $\imath^*$ are injective continuous linear maps is something we have already seen; see Proposition 1.1.1 and Lemma 1.2.3). Because $\hat{u}_\psi|_{\mathscr{D}(\Omega)}$ clearly equals u_ψ for every $\psi \in \mathscr{D}(\Omega)$, this diagram is actually commutative. We will call the arrows from the diagram and their compositions *canonical identifications*. For example, $\jmath \circ \imath = \imath^* \circ \hat{\jmath}$ will be referred to as the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$ and $\hat{\jmath}$ will be referred to as the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{E}'(\Omega)$. The image of these canonical identifications is always assumed to be endowed with the unique topology that turns the identification into a linear topological isomorphism and we will usually make no distinction between the 'original' space and its image under a canonical identification. Under this convention, the arrows in the diagram become continuous inclusions and we write $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{E}(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$ and $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{E}'(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$.

Relevant results

According to Lemma 1.1.16, $\mathscr{D}(\Omega)$ is reflexive. Thus the usual 'evaluation in' map

$$\hat{\imath}\colon \mathscr{D}(\Omega) \to ((\mathscr{D}(\Omega))^*)^* = (\mathscr{D}'(\Omega))^*,$$

which is characterized by

$$\hat{\imath}(\varphi)\colon \mathscr{D}'(\Omega) \to \mathbb{K} \colon u \mapsto u(\varphi),$$

is a linear topological isomorphism. The following straightforward lemma and its corollary show that this linear topological isomorphism combines very naturally with the canonical identifications.

Lemma 1.3.1. $\jmath^* \circ \hat{\imath} = \hat{\jmath}$.

Proof: For every $\psi \in \mathscr{D}(\Omega)$ and $\varphi \in \mathscr{E}(\Omega)$

$$((\jmath^* \circ \hat{\imath})(\psi))(\varphi) = (\hat{\imath}(\psi))(\jmath(\varphi)) = (\jmath(\varphi))(\psi) = u_\varphi(\psi) = \hat{u}_\psi(\varphi) = (\hat{\jmath}(\psi))(\varphi).$$

□

Corollary 1.3.2. $(\jmath \circ \imath)^* \circ \hat{\imath} = \jmath \circ \imath$.

Proof:

$$(\jmath \circ \imath)^* \circ \hat{\imath} = (\imath^* \circ \jmath^*) \circ \hat{\imath} = \imath^* \circ (\jmath^* \circ \hat{\imath}) = \imath^* \circ \hat{\jmath} = \jmath \circ \imath.$$

□

In the next lemma we put our new convention into practice. Although formally $\mathscr{D}(\Omega)$ is a subset of neither $\mathscr{E}'(\Omega)$ nor $\mathscr{D}'(\Omega)$, the statement of this lemma still makes sense because we have agreed to identify $\mathscr{D}(\Omega)$ with $\hat{\jmath}(\mathscr{D}(\Omega))$ and $(\jmath \circ \imath)(\mathscr{D}(\Omega))$.

Lemma 1.3.3. *$\mathscr{D}(\Omega)$ is sequentially dense in $\mathscr{E}'(\Omega)$ and $\mathscr{D}'(\Omega)$.*

Proof: See [13, Corollary of Theorem 28.2]. □

1.4 Support of a distribution

Let U be an open subset of Ω. Then U is also an open subset of $\mathbb{R}^n$, so we can consider the space $\mathscr{D}(U)$ of compactly supported smooth functions on U and the space $\mathscr{D}'(U)$ of distributions on U. It is easy to see that we can extend every element φ of $\mathscr{D}(U)$ to an element $\tilde{\varphi}$ of $\mathscr{D}(\Omega)$ by putting

$$\tilde{\varphi}(x) := \begin{cases} \varphi(x) & \text{if } x \in U, \\ 0 & \text{if } x \notin U. \end{cases}$$

After all, because $\operatorname{supp}(\varphi)$ is compact, $\operatorname{supp}(\varphi)$ is closed in Ω, so for every $x \notin U$, $\Omega \setminus \operatorname{supp}(\varphi)$ is an open neighborhood of x in Ω on which $\tilde{\varphi}$ is smooth (a consequence of $\tilde{\varphi}|_{\Omega\setminus\operatorname{supp}(\varphi)} = 0$), while for every $x \in U$, U itself is an open neighborhood of x in Ω on which $\tilde{\varphi}$ is smooth (a consequence of $\tilde{\varphi}|_U = \varphi$). Thus $\tilde{\varphi}$ is a smooth function on Ω and clearly $\operatorname{supp}(\tilde{\varphi}) = \operatorname{supp}(\varphi)$.

We call $\tilde{\varphi}$ the 'extension by zero' of φ and we denote the associated map $\mathscr{D}(U) \to \mathscr{D}(\Omega)\colon \varphi \mapsto \tilde{\varphi}$ by $\operatorname{ext}_{U,\Omega}$. It is evident that $\operatorname{ext}_{U,\Omega}$ is linear and because $\operatorname{ext}_{U,\Omega}$ clearly restricts to a continuous linear map from $\mathscr{E}_K(U)$ into $\mathscr{E}_K(\Omega)$ for every $K \in \mathscr{P}_{\mathrm{c}}(U)$, $\operatorname{ext}_{U,\Omega}$ is actually a continuous linear map (see Proposition A.3.2). If V is another open subset of Ω and $V \subseteq U$, then the same arguments show that we have a continuous linear 'extension by zero' map $\operatorname{ext}_{V,U}\colon \mathscr{D}(V) \to \mathscr{D}(U)$. By taking the adjoint of this map, we get a continuous linear 'restriction' map

$$\operatorname{res}_{U,V} := (\operatorname{ext}_{V,U})^*\colon \mathscr{D}'(U) = (\mathscr{D}(U))^* \to (\mathscr{D}(V))^* = \mathscr{D}'(V),$$

which sends a distribution on U to its 'restriction' to $V \subseteq U$. In line with this terminology, we will often write $u|_V$ instead of $\text{res}_{U,V}(u)$. (Note that this has nothing to do with the 'ordinary' restriction of compactly supported distributions as linear functionals discussed in Lemma 1.2.3.)

If W is a third open of Ω such that $W \subseteq V \subseteq U$, we easily deduce that $\text{ext}_{V,U} \circ \text{ext}_{W,V} = \text{ext}_{W,U}$. Taking adjoints then gives $\text{res}_{V,W} \circ \text{res}_{U,V} = \text{res}_{U,W}$ and because in addition $\text{res}_{U,U}$ clearly equals $\text{id}_{\mathscr{D}'(U)}$, we see that the assignment $U \mapsto \mathscr{D}'(U)$, together with these restriction mappings, forms a presheaf over Ω. On behalf of the the following lemma, this presheaf is even a sheaf.

Lemma 1.4.1. *Suppose that* $\{U_i\}_{i\in I}$ *is a collection of open subsets of* Ω *such that* $\Omega = \cup_{i\in I} U_i$ *and that for every* $i \in I$, $u_i \in \mathscr{D}'(U_i)$. *If for all* $i, j \in I$

$$u_i|_{U_i \cap U_j} = u_j|_{U_i \cap U_j} ,$$

there exists a unique $u \in \mathscr{D}'(\Omega)$ *such that* $u|_{U_i} = u_i$ *for every* $i \in I$.

Proof: See [3, Theorem 7.6]. □

The following result is a consequence of this sheaf property.

Corollary 1.4.2. *For every* $u \in \mathscr{D}'(\Omega)$ *there exists a largest open subset* Ω_u *of* Ω *on which* u *vanishes (that is, such that* $u|_{\Omega_u} = 0$*).*

Proof: Let $\{U_i\}_{i\in I}$ be the collection of all open subsets of Ω on which u vanishes and take $\Omega_u := \cup_{i\in I} U_i$. Then, by application of the previous lemma to Ω_u, $\{U_i\}_{i\in I}$ and $u_i := u|_{U_i} = 0$, there must be a unique distribution u' on Ω_u such that $u'|_{U_i} = u_i$ for all $i \in I$. Because $\left(u|_{\Omega_u}\right)\big|_{U_i} = u|_{U_i}$, $u' = u|_{\Omega_u}$ satisfies this and since clearly also $u' = 0$ has this property, we conclude that $u|_{\Omega_u} = 0$. □

Thanks to this corollary, we can speak of the support of a distribution.

Definition 1.4.3. For a distribution u on Ω, we define its *support* by

$$\text{supp}(u) := \Omega \setminus \Omega_u,$$

where Ω_u is the largest open subset of Ω on which u vanishes. ⊘

Remark 1.4.4. We easily see that

$$\Omega_u = \{x \in \Omega \mid u|_U = 0 \text{ for some open neighborhood } U \subseteq \Omega \text{ of } x\},$$

which gives rise to the following alternative description of supp(u):

$$\text{supp}(u) = \{x \in \Omega \mid u|_U \neq 0 \text{ for any open neighborhood } U \subseteq \Omega \text{ of } x\}.$$ ⊘

Note that we also regard smooth functions as distributions and that we already had a notion of support for smooth functions. To remove any ambiguity, we should make sure that the 'old' and 'new' definition coincide. That is, we should check that, using the notation of the previous section,

$$\text{supp}(u_\psi) = \text{supp}(\psi)$$

for all $\psi \in \mathscr{E}(\Omega)$. For this, it suffices to show that for any open subset U of Ω,

$$(u_\psi)|_U = 0 \quad \text{if and only if} \quad \psi|_U = 0,$$

which is a consequence of the linearity and injectivity of

$$\mathscr{E}(U) \to \mathscr{D}'(U) \colon \psi' \mapsto u_{\psi'}$$

and the following claim.

Claim. $(u_\psi)|_U = u_{\left(\psi|_U\right)}$ for any open subset U of Ω and $\psi \in \mathscr{E}(\Omega)$.

Proof: Let $\varphi \in \mathscr{D}(U)$ and let $\tilde{\varphi} \in \mathscr{D}(\Omega)$ be its extension by zero. Then, because $\operatorname{supp}(\tilde{\varphi}) = \operatorname{supp}(\varphi) \subseteq U$,

$$(u_\psi)|_U(\varphi) = u_\psi(\tilde{\varphi}) = \int_\Omega \psi\tilde{\varphi}\, d\lambda = \int_U \psi\tilde{\varphi}\, d\lambda = \int_U \psi|_U\, \varphi\, d\lambda = u_{\left(\psi|_U\right)}(\varphi). \quad \square$$

Remark 1.4.5. For an arbitrary locally integrable function f, we in general do *not* have $\operatorname{supp}(u_f) = \operatorname{supp}(f)$ if we take $\operatorname{supp}(f)$ to be the complement of the largest open subset of Ω on which f vanishes (which is equivalent to taking the closure of $\{x \in \Omega \mid f(x) \neq 0\}$). Instead we get that $\operatorname{supp}(u_f)$ equals the complement of the largest open subset of Ω on which f vanishes *almost everywhere*. This is caused by the fact that u_f represents all locally integrable functions that are almost everywhere equal to f and not just f itself. Note however that we can always find a locally integrable function g that is almost everywhere equal to f which satisfies $\operatorname{supp}(u_g) = \operatorname{supp}(u_f) = \operatorname{supp}(g)$. $\oslash$

Remark 1.4.6. We can reformulate the claim that we just have proven as follows: for any open subset U of Ω, the restriction to $\mathscr{E}(\Omega)$ of the continuous linear map $\operatorname{res}_{\Omega,U} \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(U)$ coincides with the ordinary restriction of smooth functions. This ordinary restriction of smooth functions is obviously a continuous linear map from $\mathscr{E}(\Omega)$ into $\mathscr{E}(U)$ and we easily check that its adjoint, which is a continuous linear map from $\mathscr{E}'(U)$ into $\mathscr{E}'(\Omega)$, extends the 'extension by zero' map $\operatorname{ext}_{U,\Omega} \colon \mathscr{D}(U) \to \mathscr{D}(\Omega)$. Indeed, for all $\psi \in \mathscr{D}(U)$ and $\varphi \in \mathscr{E}(\Omega)$,

$$((\operatorname{res}^o_{\Omega,U})^* \hat{u}_\psi)(\varphi) = \hat{u}_\psi(\varphi|_U) = \int_U \psi\, \varphi|_U\, d\lambda = \int_\Omega \tilde{\psi}\varphi\, d\lambda = \hat{u}_{\tilde{\psi}}(\varphi),$$

where $\tilde{\psi} = \operatorname{ext}_{U,\Omega}(\psi)$ and $\operatorname{res}^o_{\Omega,U}$ denotes the ordinary restriction of smooth functions from Ω to U. This extended 'extension by zero' map is again denoted by $\operatorname{ext}_{U,\Omega}$, allowing us to state that $\operatorname{ext}_{U,\Omega}$ is a continuous linear map from $\mathscr{E}'(U)$ into $\mathscr{E}'(\Omega)$ that restricts to the ordinary continuous linear 'extension by zero' map from $\mathscr{D}(U)$ into $\mathscr{D}(\Omega)$, which we will temporarily denote by $\operatorname{ext}^o_{U,\Omega}$. Now observe that we clearly have

$$\operatorname{res}^o_{\Omega,U} \circ \operatorname{ext}^o_{U,\Omega} = \operatorname{id}_{\mathscr{D}(U)}.$$

Using this, a trivial mental computation shows that $\operatorname{res}_{\Omega,U} \circ \operatorname{ext}_{U,\Omega} = \operatorname{id}_{\mathscr{E}'(U)}$, so we have a similar identity for the extended maps and we therefore in particular have that $\operatorname{ext}_{U,\Omega}$ is injective. Of course, if V is an open subset of Ω such that $V \subseteq U$, the same arguments show that $\operatorname{ext}_{V,U} \colon \mathscr{D}(V) \to \mathscr{D}(U)$ extends to an injective continuous linear map from $\mathscr{E}'(V)$ into $\mathscr{E}'(U)$ which satisfies $\mathrm{res}_{U,V} \circ \mathrm{ext}_{V,U} = \mathrm{id}_{\mathscr{E}'(V)}$. If in addition W is an open subset of Ω such that $W \subseteq V$, taking the adjoint on both sides of the identity $\mathrm{res}^o_{U,W} = \mathrm{res}^o_{V,W} \circ \mathrm{res}^o_{U,V}$ shows that $\mathrm{ext}_{W,U} = \mathrm{ext}_{V,U} \circ \mathrm{ext}_{W,V}$. ⊘

Before we give a list of 'relevant results' related to the support of a distribution, let us introduce some additional terminology.

Definition 1.4.7. A family $\{u_i\}_{i\in I}$ of distributions on Ω is called *locally finite* if $\{\mathrm{supp}(u_i)\}_{i\in I}$ is a locally finite family of subsets of Ω. ⊘

Definition 1.4.8. Let T be a linear map from a linear subspace of $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$. We say that T is *local* if

$$\mathrm{supp}(Tu) \subseteq \mathrm{supp}(u)$$

for all $u \in \mathrm{dom}(T)$. ⊘

Relevant results

Lemma 1.4.9. *For every open subset U of Ω and every $u \in \mathscr{D}'(\Omega)$*

$$\mathrm{supp}(\,u|_U) \subseteq \mathrm{supp}(u)$$

and if $\mathrm{supp}(u) \subseteq U$*, then even*

$$\mathrm{supp}(\,u|_U) = \mathrm{supp}(u).$$

Proof: Let $x \in U$ and suppose that $x \notin \mathrm{supp}(u)$. By the characterization of the support given in Remark 1.4.4, we find an open neighborhood V of x in Ω such that $u|_V = 0$. But then $U \cap V$ is an open neighborhood of x in U such that

$$(\,u|_U)|_{U\cap V} = u|_{U\cap V} = (\,u|_V)|_{U\cap V} = 0|_{U\cap V} = 0,$$

so, again by the characterization from Remark 1.4.4, we get $x \notin \mathrm{supp}(\,u|_U)$.

Conversely, if $x \in U$ and $x \notin \mathrm{supp}(\,u|_U)$, then we find an open neighborhood V of x in U such that $(\,u|_U)|_V = 0$. Since U is open, V is then also an open neighborhood of x in Ω and

$$u|_V = (\,u|_U)|_V = 0,$$

thus $x \notin \mathrm{supp}(u)$.

So if $x \in U$, $x \in \mathrm{supp}(u)$ if and only if $x \in \mathrm{supp}(\,u|_U)$, which implies both statements of the lemma. □

Lemma 1.4.10. *Let U be an open subset of Ω and $u \in \mathscr{D}'(\Omega)$. Then u vanishes on U if and only if $u(\varphi) = 0$ for every $\varphi \in \mathscr{D}(\Omega)$ with* $\mathrm{supp}(\varphi) \subseteq U$.

Proof: First suppose that u vanishes on U. For every $\varphi \in \mathscr{D}(\Omega)$ that satisfies $\mathrm{supp}(\varphi) \subseteq U$, we have that $\varphi|_U$ is an element of $\mathscr{D}(U)$ with φ as extension by zero to Ω, hence

$$u(\varphi) = u|_U\,(\varphi|_U) = 0.$$

Now suppose that $u(\varphi) = 0$ for every $\varphi \in \mathscr{D}(\Omega)$ with $\mathrm{supp}(\varphi) \subseteq U$. To prove that u vanishes on U, let $\psi \in \mathscr{D}(U)$ and let $\tilde{\psi} \in \mathscr{D}(\Omega)$ be its extension by zero to Ω. Then $\mathrm{supp}(\tilde{\psi}) = \mathrm{supp}(\psi) \subseteq U$, hence

$$u|_U\,(\psi) = u(\tilde{\psi}) = 0.$$ □

Lemma 1.4.11. *For every closed subset A of Ω and every $u \in \mathscr{D}'(\Omega)$, we have* $\operatorname{supp}(u) \subseteq A$ *if and only if* $u(\varphi) = 0$ *for every* $\varphi \in \mathscr{D}(\Omega)$ *with* $\operatorname{supp}(\varphi) \subseteq \Omega \setminus A$.

Proof: Let Ω_u be the largest open subset of Ω on which u vanishes. By taking complements, we see that $\operatorname{supp}(u) = \Omega \setminus \Omega_u \subseteq A$ is equivalent to $\Omega \setminus A \subseteq \Omega_u$. But $\Omega \setminus A \subseteq \Omega_u$ is in turn equivalent to the statement that u vanishes on $\Omega \setminus A$. Indeed, because $\Omega \setminus A$ is open, we get from the definition of Ω_u that u vanishes on $\Omega \setminus A$ implies $\Omega \setminus A \subseteq \Omega_u$ and conversely, if $\Omega \setminus A \subseteq \Omega_u$, then

$$u|_{\Omega\setminus A} = \left(u|_{\Omega_u}\right)\big|_{\Omega\setminus A} = 0|_{\Omega\setminus A} = 0.$$

The result now follows by applying the previous lemma to $U := \Omega \setminus A$. □

Lemma 1.4.12. *Let $u, v \in \mathscr{D}'(\Omega)$ and $\mu \in \mathbb{K}$. Then*

1. $\operatorname{supp}(\mu u) = \emptyset$ *if* $\mu = 0$,
2. $\operatorname{supp}(\mu u) = \operatorname{supp}(u)$ *if* $\mu \neq 0$ *and*
3. $\operatorname{supp}(u + v) \subseteq \operatorname{supp}(u) \cup \operatorname{supp}(v)$.

Proof: The first statement is totally trivial and the second statement follows from the easy observation that for $\mu \neq 0$ and for every open subset U of Ω, $(\mu u)|_U = 0$ if and only if $u|_U = 0$.

For the third statement, we use the previous lemma. As union of two closed subsets, $\operatorname{supp}(u) \cup \operatorname{supp}(v)$ is a closed subset of Ω, so by the previous lemma it suffices to prove that $(u + v)(\varphi) = u(\varphi) + v(\varphi) = 0$ for every $\varphi \in \mathscr{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq \Omega \setminus (\operatorname{supp}(u) \cup \operatorname{supp}(v))$. But

$$\operatorname{supp}(\varphi) \subseteq \Omega \setminus (\operatorname{supp}(u) \cup \operatorname{supp}(v)) = (\Omega \setminus \operatorname{supp}(u)) \cap (\Omega \setminus \operatorname{supp}(v))$$

implies

$$\operatorname{supp}(\varphi) \subseteq \Omega \setminus \operatorname{supp}(u) \quad \text{and} \quad \operatorname{supp}(\varphi) \subseteq \Omega \setminus \operatorname{supp}(v),$$

so for such φ we have, again by the previous lemma, $u(\varphi) = v(\varphi) = 0$. □

Lemma 1.4.13. *If $u \in \mathscr{D}'(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$ such that* $\operatorname{supp}(\varphi) \cap \operatorname{supp}(u) = \emptyset$, *then* $u(\varphi) = 0$.

Proof: Observe that $\operatorname{supp}(\varphi) \cap \operatorname{supp}(u) = \emptyset$ implies $\operatorname{supp}(\varphi) \subseteq \Omega \setminus \operatorname{supp}(u)$ and take $A := \operatorname{supp}(u)$ in the direct implication of Lemma 1.4.11. □

Lemma 1.4.14. *If $u \in \mathscr{D}'(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$ vanishes on an open neighborhood of* $\operatorname{supp}(u)$, *then* $u(\varphi) = 0$.

Proof: Let U be an open neighborhood of $\operatorname{supp}(u)$ on which φ vanishes. Then $\operatorname{supp}(\varphi) \subseteq \Omega \setminus U$ and $\operatorname{supp}(u) \subseteq U$, so $\operatorname{supp}(\varphi) \cap \operatorname{supp}(u) = \emptyset$ and we can apply the previous lemma. □

Lemma 1.4.15. *If $u \in \mathscr{D}'(\Omega)$ and φ, $\psi \in \mathscr{D}(\Omega)$ such that φ and ψ coincide on an open neighborhood of* $\operatorname{supp}(u)$, *then* $u(\varphi) = u(\psi)$.

Proof: Apply the previous result to $\varphi - \psi$ and use the linearity of u. □

Lemma 1.4.16. *Let $u \in \mathscr{D}'(\Omega)$. If there exists an open subset U of Ω such that* $\operatorname{supp}(u) \subseteq U$ *and* $u|_U = 0$*, then* $u = 0$.

Proof: Let Ω_u be the largest open subset of Ω on which u vanishes. Because by assumption u vanishes on U, we have $U \subseteq \Omega_u$, while $\Omega \setminus \Omega_u = \operatorname{supp}(u) \subseteq U$ implies that also $\Omega \setminus U \subseteq \Omega_u$. As a consequence, $\Omega = U \cup (\Omega \setminus U) \subseteq \Omega_u$. Hence, $\Omega = \Omega_u$ and we conclude that u vanishes on Ω. ☐

Lemma 1.4.17. *Let $u, v \in \mathscr{D}'(\Omega)$. If there exists an open subset U of Ω such that* $\operatorname{supp}(u) \subseteq U$, $\operatorname{supp}(v) \subseteq U$ *and* $u|_U = v|_U$*, then* $u = v$.

Proof: Observe that

$$\operatorname{supp}(u - v) \subseteq \operatorname{supp}(u) \cup \operatorname{supp}(-v) = \operatorname{supp}(u) \cup \operatorname{supp}(v) \subseteq U$$

(see Lemma 1.4.12) and apply the previous lemma to $u - v$. ☐

Lemma 1.4.18. *Let U and V be open subsets of Ω and let $u \in \mathscr{D}'(U)$ and $v \in \mathscr{D}'(V)$ such that $u|_{U\cap V} = v|_{U\cap V}$. Then $u(\varphi|_U) = v(\varphi|_V)$ for all $\varphi \in \mathscr{D}(\Omega)$ with* $\operatorname{supp}(\varphi) \subseteq U \cap V$.

Proof: Because $\operatorname{supp}(\varphi) \subseteq U \cap V$, $\varphi|_{U\cap V}$ is an element of $\mathscr{D}(U \cap V)$ with extension by zero to U equal to $\varphi|_U$ and extension by zero to V equal to $\varphi|_V$, hence

$$u(\varphi|_U) = u|_{U\cap V}(\varphi|_{U\cap V}) = v|_{U\cap V}(\varphi|_{U\cap V}) = v(\varphi|_V).$$ ☐

Lemma 1.4.19. *Suppose that $\{u_i\}_{i\in I}$ is a locally finite family of distributions on Ω. Then*

$$\sum_{i\in I} u_i \colon \mathscr{D}(\Omega) \to \mathbb{K} \colon \varphi \mapsto \sum_{i\in I} u_i(\varphi)$$

is a well-defined distribution on Ω.

Proof: Using the fact that the family $\{\operatorname{supp}(u_i)\}_{i\in I}$ is locally finite, we readily check that for every $K \in \mathscr{P}_c(\Omega)$ there exists a *finite* subset I_K of I such that $K \cap \operatorname{supp}(u_i) \neq \emptyset$ if and only if $i \in I_K$. Looking at Lemma 1.4.13 we now see that for every $\varphi \in \mathscr{D}(\Omega)$ and every $K \in \mathscr{P}_c(\Omega)$ such that $\operatorname{supp}(\varphi) \subseteq K$, we have

$$\sum_{i\in I} u_i(\varphi) = \sum_{i\in I_K} u_i(\varphi) < \infty,$$

so $\sum_{i\in I} u_i$ is well-defined and a trivial mental computation shows that $\sum_{i\in I} u_i$ is linear. In order to prove that $\sum_{i\in I} u_i$ is continuous, it suffices to prove that $\sum_{i\in I} u_i$ is continuous on $\mathscr{E}_K(\Omega)$ for every $K \in \mathscr{P}_c(\Omega)$ (see Proposition A.3.2), which is the case because $\sum_{i\in I} u_i$ equals the distribution $\sum_{i\in I_K} u_i$ on $\mathscr{E}_K(\Omega)$. ☐

Lemma 1.4.20. *Let $\{u_i\}_{i\in I}$ be a net in $\mathscr{D}'(\Omega)$, $u \in \mathscr{D}'(\Omega)$ and $K \in \mathscr{P}_c(\Omega)$ such that $u_i \to u$ in $\mathscr{D}'(\Omega)$ and* $\operatorname{supp}(u_i) \subseteq K$ *for every $i \in I$. Then also* $\operatorname{supp}(u) \subseteq K$.

Proof: Let $\varphi \in \mathscr{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq \Omega \setminus K$. By Lemma 1.4.11 it suffices to prove that $u(\varphi) = 0$ and by the same lemma we get that $u_i(\varphi) = 0$ for all $i \in \mathbb{N}$. Because $\{\varphi\}$ is clearly a bounded subset of $\mathscr{D}(\Omega)$, the assumption that $u_i \to u$ in $\mathscr{D}'(\Omega)$ in particular implies that $|u(\varphi) - u_i(\varphi)| = \sup_{\psi\in\{\varphi\}} |(u - u_i)(\psi)| \to 0$ in $\mathbb{R}$ and we subsequently find that $u(\varphi) = \lim_{i\to\infty} u_i(\varphi) = \lim_{i\to\infty} 0 = 0$. ☐

1.5 Multiplication by smooth functions

For every $\varphi \in \mathscr{E}(\Omega)$, we can consider the associated 'multiplication by φ' map

$$m_\varphi\colon \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)\colon \psi \mapsto \varphi\psi.$$

It is evident that m_φ is linear and the computation

$$\begin{aligned}
\|m_\varphi \psi\|_{K,k} &= \sum_{|\alpha|\leq k} \|\partial^\alpha(\varphi\psi)\|_{K,0} = \sum_{|\alpha|\leq k} \|\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\partial^{\alpha-\beta}\varphi\partial^\beta\psi\|_{K,0} \\
&\leq \sum_{|\alpha|\leq k}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\|\partial^{\alpha-\beta}\varphi\|_{K,0}\|\partial^\beta\psi\|_{K,0} \qquad (1.3)\\
&\leq \sum_{|\alpha|\leq k}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\|\varphi\|_{K,k}\|\psi\|_{K,k} = \left(\sum_{|\alpha|\leq k}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\|\varphi\|_{K,k}\right)\|\psi\|_{K,k},
\end{aligned}$$

where $K \in \mathscr{P}_\mathrm{c}(\Omega)$, $k \in \mathbb{N}$ and the Leibniz rule for multi-indices is used, shows that m_φ is continuous (use Lemma A.1.2 and the 'standard' inducing collection of seminorms for $\mathscr{E}(\Omega)$, i.e., $\{\|\cdot\|_{K,k} \mid K \in \mathscr{P}_\mathrm{c}(\Omega) \text{ and } k \in \mathbb{N}\}$). Moreover, because

$$\operatorname{supp}(\varphi\psi) \subseteq \operatorname{supp}(\varphi) \cap \operatorname{supp}(\psi)$$

and $\mathscr{E}_K(\Omega)$, for $K \in \mathscr{P}_\mathrm{c}(\Omega)$, carries the topology induced from $\mathscr{E}(\Omega)$, we find that m_φ restricts to a continuous linear map from $\mathscr{E}_K(\Omega)$ into $\mathscr{E}_K(\Omega)$. Combining this with Proposition A.3.2 and the fact that $\mathscr{E}_K(\Omega) \subseteq_\mathrm{c} \mathscr{D}(\Omega)$ shows that m_φ also restricts to a continuous linear map from $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$.

Definition 1.5.1. Let $\varphi \in \mathscr{E}(\Omega)$, let $m_\varphi\colon \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$ be the continuous linear (restriction of the) 'multiplication by φ' map that we have just discussed and let $(m_\varphi)^*\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ be its adjoint. For every $u \in \mathscr{D}'(\Omega)$, we define

$$\varphi u := (m_\varphi)^* u. \qquad \oslash$$

Because the adjoint $(m_\varphi)^*$ in the definition above is automatically continuous, we see that for every $\varphi \in \mathscr{E}(\Omega)$

$$\mathscr{D}'(\Omega) \mapsto \mathscr{D}'(\Omega)\colon u \mapsto \varphi u$$

is a continuous linear map. We will denote this new 'multiplication by φ' map by m_φ as well. This is justified because on $\mathscr{E}(\Omega)$ this new 'multiplication by φ' map coincides with the old one. Indeed, for all φ, $\psi \in \mathscr{E}(\Omega)$ and $\chi \in \mathscr{D}(\Omega)$

$$(\varphi u_\psi)(\chi) = u_\psi(\varphi\chi) = \int_\Omega \psi(\varphi\chi)\,d\lambda = \int_\Omega (\varphi\psi)\chi\,d\lambda = u_{\varphi\psi}(\chi),$$

proving that $\varphi u_\psi = u_{\varphi\psi}$.

So, for every $\varphi \in \mathscr{E}(\Omega)$, the new continuous linear multiplication map $m_\varphi\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to the original continuous linear multiplication map from $\mathscr{E}(\Omega)$ into $\mathscr{E}(\Omega)$. Since $\mathscr{D}(\Omega)$, and therefore also $\mathscr{E}(\Omega)$, is dense in $\mathscr{D}'(\Omega)$ (see Lemma 1.3.3 and use that $\mathscr{D}(\Omega) \subseteq \mathscr{E}(\Omega)$), $m_\varphi\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is in fact the *unique* continuous extension of the original multiplication map

$m_\varphi\colon \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$, which shows that this extension is very natural and independent on any choices.

If $\varphi \in \mathscr{D}(\Omega)$, the situation is even better. After all, the fact that for every $\psi \in \mathscr{E}(\Omega)$, $\operatorname{supp}(\varphi\psi) \subseteq \operatorname{supp}(\varphi) \cap \operatorname{supp}(\psi)$, shows that the continuous linear map $m_\varphi\colon \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ can be viewed as a continuous linear map from $\mathscr{E}(\Omega)$ into $\mathscr{E}_{\operatorname{supp}(\varphi)}(\Omega)$ and hence as a continuous linear map from $\mathscr{E}(\Omega)$ into $\mathscr{D}(\Omega)$. The adjoint of m_φ as map from $\mathscr{E}(\Omega)$ into $\mathscr{D}(\Omega)$ is then a continuous linear map from $\mathscr{D}'(\Omega)$ into $\mathscr{E}'(\Omega)$ and it is clear that this adjoint coincides with the extended 'multiplication by φ' map under the canonical identification of $\mathscr{E}'(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$. In other words, if $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ can also be viewed as a continuous linear map from $\mathscr{D}'(\Omega)$ into $\mathscr{E}'(\Omega)$.

We can now explain why we call $\mathscr{E}'(\Omega)$ the space of compactly supported distributions.

Proposition 1.5.2. *The subspace of $\mathscr{D}'(\Omega)$ that we canonically identify with $\mathscr{E}'(\Omega)$ consists precisely of those $u \in \mathscr{D}'(\Omega)$ for which* $\operatorname{supp}(u)$ *is compact.*

Proof: As discussed in Section 1.2, the subspace of $\mathscr{D}'(\Omega)$ that we canonically identify with $\mathscr{E}'(\Omega)$ consists of those $u \in \mathscr{D}'(\Omega)$, $u\colon \mathscr{D}(\Omega) \to \mathbb{K}$, that allow a continuous linear extension to $\mathscr{E}(\Omega)$.

So let $u \in \mathscr{D}'(\Omega)$ that allows a continuous linear extension to $\mathscr{E}(\Omega)$ and denote this extension by $\hat{u}$. According to Proposition 1.2.2, there exist $C \geq 0$, $K \in \mathscr{P}_c(\Omega)$ and $k \in \mathbb{N}$ such that

$$|\hat{u}(\varphi)| \leq C\|\psi\|_{K,k}$$

for all $\psi \in \mathscr{E}(\Omega)$. Then $\Omega \setminus K$ is an open subset of Ω. Moreover, if $\varphi \in \mathscr{D}(\Omega \setminus K)$ and $\tilde{\varphi} \in \mathscr{D}(\Omega)$ is its extension by zero, then $\tilde{\varphi}$ vanishes on an open neighborhood of K (namely on $\Omega \setminus \operatorname{supp}(\varphi)$) and therefore

$$\left| u|_{\Omega\setminus K}(\varphi) \right| = |u(\tilde{\varphi})| = |\hat{u}(\tilde{\varphi})| \leq C\|\tilde{\varphi}\|_{K,k} = 0.$$

So $u|_{\Omega\setminus K} = 0$, which implies $\operatorname{supp}(u) \subseteq K$. Since a closed subset of a compact subset is again compact, we conclude that $\operatorname{supp}(u)$ is compact.

Next, let $u \in \mathscr{D}'(\Omega)$ such that $\operatorname{supp}(u)$ is compact and let $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of $\operatorname{supp}(u)$ (see Remark 1.1.12). Then $m_\varphi\colon \mathscr{E}(\Omega) \to \mathscr{D}(\Omega)\colon \psi \mapsto \varphi\psi$ is a continuous linear map (see the discussion above) and since the distribution u is also continuous, $\hat{u}\colon \mathscr{E}(\Omega) \to \mathbb{K}$ defined by $\hat{u} := u \circ m_\varphi$ is a continuous linear map as well. That $\hat{u}$ extends u is an easy consequence of Lemma 1.4.15. Indeed, for every $\psi \in \mathscr{D}(\Omega)$, $\varphi\psi$ and ψ coincide on an open neighborhood of $\operatorname{supp}(u)$, so application of this lemma shows that $\hat{u}(\psi) = u(\varphi\psi) = u(\psi)$. □

Relevant results

Lemma 1.5.3. *For every $u \in \mathscr{D}'(\Omega)$ and $\psi \in \mathscr{E}(\Omega)$*

$$\operatorname{supp}(\psi u) \subseteq \operatorname{supp}(\psi) \cap \operatorname{supp}(u).$$

Proof: Let Ω_u be the largest open subset of Ω on which u vanishes and define Ω_ψ and $\Omega_{\psi u}$ accordingly. If $\varphi \in \mathscr{D}(\Omega_\psi)$ and $\tilde{\varphi}$ is its extension by zero to Ω, then $\psi\tilde{\varphi} = 0$ (indeed, $\psi(x) = 0$ if $x \in \Omega_\psi$ and $\tilde{\varphi}(x) = 0$ if $x \notin \Omega_\psi$), hence

$$(\psi u)|_{\Omega_\psi}(\varphi) = (\psi u)(\tilde{\varphi}) = u(\psi\tilde{\varphi}) = 0.$$

This shows that ψu vanishes on Ω_ψ and, due to the definition of $\Omega_{\psi u}$, we find $\Omega_\psi \subseteq \Omega_{\psi u}$. Moreover, if $\varphi \in \mathscr{D}(\Omega_u)$ and $\tilde{\varphi}$ is its extension by zero to Ω, then $\psi\tilde{\varphi}$ equals the extension by zero of $\psi|_{\Omega_u}\varphi$ (indeed, because $\operatorname{supp}(\tilde{\varphi}) \subseteq \Omega_u$, $\psi\tilde{\varphi}$ equals 0 outside Ω_u) and therefore

$$(\psi u)|_{\Omega_u}(\varphi) = (\psi u)(\tilde{\varphi}) = u(\psi\tilde{\varphi}) = u|_{\Omega_u}(\psi|_{\Omega_u}\varphi) = 0$$

(this equals zero because u vanishes on Ω_u). That is, ψu vanishes on Ω_u as well, so we also have $\Omega_u \subseteq \Omega_{\psi u}$. Combining the two inclusions gives $\Omega_\psi \cup \Omega_u \subseteq \Omega_{\psi u}$ and by taking complements we conclude

$$\begin{aligned}\operatorname{supp}(\psi u) &= \Omega \setminus \Omega_{\psi u} \subseteq \Omega \setminus (\Omega_\psi \cup \Omega_u)\\ &= (\Omega \setminus \Omega_\psi) \cap (\Omega \setminus \Omega_u) = \operatorname{supp}(\psi) \cap \operatorname{supp}(u).\end{aligned}$$ □

Lemma 1.5.4. *Let $u \in \mathscr{D}'(\Omega)$, $\psi \in \mathscr{E}(\Omega)$ and U an open subset of Ω. Then*

$$(\psi u)|_U = \psi|_U\, u|_U\,.$$

Proof: Let $\varphi \in \mathscr{D}(U)$ and let $\tilde{\varphi}$ be its extension by zero to Ω. As in the proof of the previous lemma, $\psi\tilde{\varphi}$ equals the extension by zero of $\psi|_U\varphi$, so

$$(\psi u)|_U(\varphi) = (\psi u)(\tilde{\varphi}) = u(\psi\tilde{\varphi}) = u|_U(\psi|_U\varphi) = (\psi|_U\, u|_U)(\varphi).$$ □

Lemma 1.5.5. *Let $u \in \mathscr{D}'(\Omega)$ and $\psi \in \mathscr{E}(\Omega)$. If $x \in \operatorname{supp}(u)$ and ψ equals 1 on an open neighborhood $U \subseteq \Omega$ of x, then $x \in \operatorname{supp}(\psi u)$.*

Proof: We use the characterization of the support that we have discussed in Remark 1.4.4. Suppose for a contradiction that $x \notin \operatorname{supp}(\psi u)$, hence that there exists an open neighborhood V of x in Ω such that $(\psi u)|_V = 0$. Let $\varphi \in \mathscr{D}(U \cap V)$ and let $\tilde{\varphi}$ be its extension by zero to V. Then $\psi|_V\,\tilde{\varphi} = \tilde{\varphi}$ (indeed, $\operatorname{supp}(\tilde{\varphi}) \subseteq U$ and ψ equals 1 on U). Using the previous lemma, we obtain

$$\begin{aligned}u|_{U\cap V}(\varphi) &= (u|_V)|_{U\cap V}(\varphi) = u|_V(\tilde{\varphi}) = u|_V(\psi|_V\,\tilde{\varphi})\\ &= (\psi|_V\, u|_V)(\tilde{\varphi}) = (\psi u)|_V(\tilde{\varphi}) = 0.\end{aligned}$$

So $U \cap V$ is an open neighborhood of x with $u|_{U\cap V} = 0$, contradicting the assumption that $x \in \operatorname{supp}(u)$. □

Lemma 1.5.6. *If $u \in \mathscr{D}'(\Omega)$ and $\psi \in \mathscr{E}(\Omega)$ equals 1 on an open neighborhood of $\operatorname{supp}(u)$, then $\psi u = u$.*

Proof: Let $\varphi \in \mathscr{D}(\Omega)$. Then $\psi\varphi$ is a compactly supported smooth function that coincides with φ on an open neighborhood of $\operatorname{supp}(u)$, hence by Lemma 1.4.15,

$$(\psi u)(\varphi) = u(\psi\varphi) = u(\varphi).$$ □

Lemma 1.5.7. *If $u \in \mathscr{D}'(\Omega)$ and $\psi \in \mathscr{E}(\Omega)$ such that* $\operatorname{supp}(\psi) \cap \operatorname{supp}(u) = \emptyset$*, then $\psi u = 0$.*

Proof: Let $\varphi \in \mathscr{D}(\Omega)$. Then $\operatorname{supp}(\psi\varphi) \subseteq \operatorname{supp}(\psi) \cap \operatorname{supp}(\varphi) \subseteq \operatorname{supp}(\psi)$, hence $\operatorname{supp}(\psi\varphi) \cap \operatorname{supp}(u) = \emptyset$ and using Lemma 1.4.13 we get $(\psi u)(\varphi) = u(\psi\varphi) = 0$. □

Lemma 1.5.8. *If $\psi \in \mathscr{E}(\Omega)$, U is an open subset of Ω and $u \in \mathscr{E}'(U)$, then*

$$\psi(\operatorname{ext}_{U,\Omega} u) = \operatorname{ext}_{U,\Omega}(\psi|_U\, u).$$

Proof: Let $\varphi \in \mathscr{D}(\Omega)$. Then

$$\begin{aligned}(\psi(\operatorname{ext}_{U,\Omega} u))(\varphi) &= (\operatorname{ext}_{U,\Omega} u)(\psi\varphi) = u((\psi\varphi)|_U) \\ &= u(\psi|_U\, \varphi|_U) = \psi|_U\, u(\varphi|_U) = (\operatorname{ext}_{U,\Omega}(\psi|_U\, u))(\varphi). \qquad \square\end{aligned}$$

Lemma 1.5.9. *If $u \in \mathscr{E}'(\Omega)$ and U is an open subset of Ω with the property that* $\operatorname{supp}(u) \subseteq U$*, then $u|_U \in \mathscr{E}'(U)$ and*

$$\operatorname{ext}_{U,\Omega}(u|_U) = u.$$

Proof: Using Lemma 1.4.9, we find that $\operatorname{supp}(u|_U) = \operatorname{supp}(u)$ is compact, so $u|_U$ is indeed an element of $\mathscr{E}'(U)$. To derive the stated equality, let $\psi \in \mathscr{D}(\Omega)$ such that ψ equals 1 on an open neighborhood of $\operatorname{supp}(u)$ and $\operatorname{supp}(\psi) \subseteq U$. Then $\psi u = u$ by Lemma 1.5.6 and for every $\varphi \in \mathscr{D}(\Omega)$, $\psi\varphi$ is an element of $\mathscr{D}(\Omega)$ with support in U, so the extension by zero from U to Ω of $(\psi\varphi)|_U$ equals $\psi\varphi$. If we combine this with Lemma 1.5.4, we find that for every $\varphi \in \mathscr{D}(\Omega)$,

$$\begin{aligned}(\operatorname{ext}_{U,\Omega}(u|_U))(\varphi) &= (u|_U)(\varphi|_U) = ((\psi u)|_U)(\varphi|_U) \\ &= (\psi|_U\, u|_U)(\varphi|_U) = (u|_U)(\psi|_U\, \varphi|_U) \\ &= (u|_U)((\psi\varphi)|_U) = u(\psi\varphi) = (\psi u)(\varphi) = u(\varphi). \qquad \square\end{aligned}$$

Lemma 1.5.10. *For every open subset U of Ω and every $u \in \mathscr{E}'(U)$*

$$\operatorname{supp}(\operatorname{ext}_{U,\Omega} u) = \operatorname{supp}(u).$$

Proof: Because $(\operatorname{ext}_{U,\Omega} u)|_U = (\operatorname{res}_{\Omega,U} \circ \operatorname{ext}_{U,\Omega})(u) = u$, Lemma 1.4.9 directly tells us that

$$\operatorname{supp}(u) = \operatorname{supp}\big((\operatorname{ext}_{U,\Omega} u)|_U\big) \subseteq \operatorname{supp}(\operatorname{ext}_{U,\Omega} u).$$

For the converse inclusion, we have to do a bit more work. First of all, $\operatorname{supp}(u)$ is compact, hence a closed subset of Ω. Due to Lemma 1.4.11, it therefore suffices to prove that $(\operatorname{ext}_{U,\Omega} u)(\varphi) = 0$ for every $\varphi \in \mathscr{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq \Omega \setminus \operatorname{supp}(u)$. So fix $\varphi \in \mathscr{D}(\Omega)$ with $\operatorname{supp}(\varphi) \subseteq \Omega \setminus \operatorname{supp}(u)$ and let $\psi \in \mathscr{D}(U)$ such that ψ equals 1 on an open neighborhood of $\operatorname{supp}(u)$. Then $\psi u = u$ by Lemma 1.5.6 and $\psi\, \varphi|_U$ is an element of $\mathscr{D}(U)$ with

$$\operatorname{supp}(\psi\, \varphi|_U) \subseteq \operatorname{supp}(\psi) \cap \operatorname{supp}(\varphi) \subseteq U \cap (\Omega \setminus \operatorname{supp}(u)) = U \setminus \operatorname{supp}(u).$$

Hence,

$$(\operatorname{ext}_{U,\Omega} u)(\varphi) = u(\varphi|_U) = (\psi u)(\varphi|_U) = u(\psi\, \varphi|_U) = 0,$$

where we haved used Lemma 1.4.11 to establish the last equality. □

Lemma 1.5.11. *Let $\{u_i\}_{i\in I}$ be a net in $\mathscr{D}'(\Omega)$ and $u \in \mathscr{D}'(\Omega)$. Then $u_i \to u$ in $\mathscr{D}'(\Omega)$ if and only if $\varphi u_i \to \varphi u$ in $\mathscr{D}'(\Omega)$ for every $\varphi \in \mathscr{D}(\Omega)$.*

Proof: The implication from left to right is a direct consequence of the fact that multiplication by φ is a continuous linear map from $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ for every $\varphi \in \mathscr{D}(\Omega)$. For the converse implication, assume that $\varphi u_i \to \varphi u$ for every $\varphi \in \mathscr{D}(\Omega)$, let B be a bounded subset of $\mathscr{D}(\Omega)$ and let q_B be its associated seminorm on $\mathscr{D}'(\Omega)$. According to Lemma 1.1.14 we find a compact subset K of Ω such that $\operatorname{supp}(\psi) \subseteq K$ for all $\psi \in B$ and according to Remark 1.1.12 we find an $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of K. Now observe that

$$\begin{aligned} q_B(u-u_i) &= \sup_{\psi\in B} |u(\psi) - u_i(\psi)| = \sup_{\psi\in B} |u(\varphi\psi) - u_i(\varphi\psi)| \\ &= \sup_{\psi\in B} |(\varphi u)(\psi) - (\varphi u_i)(\psi)| = q_B(\varphi u - \varphi u_i) \to 0. \end{aligned}$$ □

1.6 Differentiation

Similar to the way in which we have introduced multiplication by smooth functions on $\mathscr{D}'(\Omega)$ we can introduce differentiation of distributions. To this end, let α be some multi-index and consider the map

$$\partial^\alpha \colon \mathscr{E}(\Omega) \to \mathscr{E}(\Omega).$$

Of course ∂^α is linear and the estimate

$$\|\partial^\alpha\psi\|_{K,k} = \sum_{|\beta|\le k} \|\partial^\beta\partial^\alpha\psi\|_{K,0} \le \left(\sum_{|\beta|\le k} 1\right) \|\psi\|_{K,k+|\alpha|},$$

with $\psi \in \mathscr{E}(\Omega)$, $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and $k \in \mathbb{N}$, shows that ∂^α is continuous. Since

$$\operatorname{supp}(\partial^\alpha\psi) \subseteq \operatorname{supp}(\psi),$$

we see that for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, ∂^α restricts to a continuous linear map from $\mathscr{E}_K(\Omega)$ into $\mathscr{E}_K(\Omega)$. Combining this with Proposition A.3.2 and the fact that $\mathscr{E}_K(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}(\Omega)$ shows that ∂^α also restricts to a continuous linear map from $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$. Because of the continuity of scalar multiplication, the same is true for $(-1)^{|\alpha|}\partial^\alpha$.

Definition 1.6.1. Let α be a multi-index and let

$$((-1)^{|\alpha|}\partial^\alpha)^* \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

be the adjoint of the continuous linear map $(-1)^{|\alpha|}\partial^\alpha \colon \mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$. For every $u \in \mathscr{D}'(\Omega)$, we define

$$\partial^\alpha u := ((-1)^{|\alpha|}\partial^\alpha)^* u.$$ ⊘

Remark 1.6.2. It is clear that with this definition the composition of partial derivatives works as expected: if α and β are multi-indices and $u \in \mathscr{D}'(\Omega)$, then $\partial^\alpha(\partial^\beta u) = \partial^{\alpha+\beta} u$. In other words, only defining $\partial_i u$ for $1 \leq i \leq \mathrm{n}$ and then introducing ∂^α in the usual way (as composition of single partial derivatives) results in the same definition. $\oslash$

Since the adjoint $((-1)^{|\alpha|}\partial^\alpha)^*$ in the definition above is automatically continuous, the map

$$\mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)\colon u \mapsto \partial^\alpha u$$

is a continuous linear map. We will denote this map by ∂^α. Of course, we should check that this new 'distributional derivative' coincides with the ordinary derivative if we apply it to a smooth function. As we will see in a moment, it is this requirement that causes the presence of the at first surprising and odd looking factor $(-1)^{|\alpha|}$.

Let $\psi \in \mathscr{E}(\Omega)$ and $\varphi \in \mathscr{D}(\Omega)$. Then

$$(\partial^\alpha u_\psi)(\varphi) = u_\psi((-1)^{|\alpha|}\partial^\alpha \varphi) = (-1)^{|\alpha|} \int_\Omega \psi \partial^\alpha \varphi \, d\lambda.$$

We now use integration by parts $|\alpha|$ times to get

$$(-1)^{|\alpha|} \int_\Omega \psi \partial^\alpha \varphi \, d\lambda = \int_\Omega \partial^\alpha \psi \varphi \, d\lambda = u_{\partial^\alpha \psi}(\varphi)$$

(note that due to the compact support of φ the boundary terms vanish). This shows that $\partial^\alpha u_\psi = u_{\partial^\alpha \psi}$, which means that the restriction to $\mathscr{E}(\Omega)$ of the new map $\partial^\alpha\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ indeed coincides with the usual differentiation of smooth functions. As with multiplication by smooth functions, because $\mathscr{E}(\Omega)$ is dense in $\mathscr{D}'(\Omega)$, the extension of $\partial^\alpha\colon \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ to a continuous linear map from $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$ is in fact unique.

It turns out that differentiation of distributions interacts very nicely with the multiplication of distributions by smooth functions: we have a Leibniz rule.

Lemma 1.6.3. *For every $u \in \mathscr{D}'(\Omega)$, $\psi \in \mathscr{E}(\Omega)$ and $1 \leq i \leq \mathrm{n}$, we have*

$$\partial_i(\psi u) = (\partial_i \psi) u + \psi(\partial_i u).$$

Proof: For every $\varphi \in \mathscr{D}(\Omega)$

$$\begin{aligned}(\partial_i(\psi u))(\varphi) &= -(\psi u)(\partial_i \varphi) = -u(\psi \partial_i \varphi) = -u(\partial_i(\psi\varphi) - \partial_i \psi \varphi) \\ &= -u(\partial_i(\psi\varphi)) + u(\partial_i \psi \varphi) = (\partial_i u)(\psi\varphi) + ((\partial_i \psi)u)(\varphi) \\ &= (\psi(\partial_i u))(\varphi) + ((\partial_i \psi)u)(\varphi).\end{aligned}$$ $\square$

As a consequence, we also have a Leibniz rule for multi-indices (which follows in the usual way from the ordinary Leibniz rule; it is just a matter of combinatorics).

Corollary 1.6.4. *For every $u \in \mathscr{D}'(\Omega)$, $\psi \in \mathscr{E}(\Omega)$ and multi-index α, we have*

$$\partial^\alpha(\psi u) = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \partial^{\alpha-\beta}\psi \partial^\beta u.$$

Also the interaction between differentiation and the support of distributions is as expected.

Lemma 1.6.5. *For every $u \in \mathscr{D}'(\Omega)$ and every multi-index α*

$$\operatorname{supp}(\partial^\alpha u) \subseteq \operatorname{supp}(u).$$

Proof: Let Ω_u be the largest open subset of Ω on which u vanishes and let $\Omega_{\partial^\alpha u}$ be the largest open subset of Ω on which $\partial^\alpha u$ vanishes. Then the statement of the lemma is equivalent to $\Omega_u \subseteq \Omega_{\partial^\alpha u}$ and because of the definition of $\Omega_{\partial^\alpha u}$ it suffices to prove that $\partial^\alpha u$ vanishes on Ω_u.

So let $\varphi \in \mathscr{D}(\Omega_u)$ and let $\tilde{\varphi}$ be its extension by zero to Ω. We easily see that $\partial^\alpha \tilde{\varphi}$ is the extension by zero of $\partial^\alpha \varphi$ (note that $\tilde{\varphi}$ equals 0 on the *open* subset $\Omega \setminus \operatorname{supp}(\varphi)$ of Ω), hence

$$(\partial^\alpha u)|_{\Omega_u}(\varphi) = (\partial^\alpha u)(\tilde{\varphi}) = (-1)^{|\alpha|} u(\partial^\alpha \tilde{\varphi}) = (-1)^{|\alpha|} \, u|_{\Omega_u}(\partial^\alpha \varphi) = 0. \qquad \square$$

1.7 Change of coordinates

In this section Ω' will also denote an open subset of $\mathbb{R}^{\mathrm{n}}$ and $\chi \colon \Omega \to \Omega'$ will be a diffeomorphism. Because the open subsets Ω and Ω' can then be 'identified' by χ, we also expect to find a way to 'identify' $\mathscr{D}'(\Omega)$ and $\mathscr{D}'(\Omega')$ using χ. Like we did in the sections on multiplication by smooth functions and differentiation, we start by looking at a natural map on the level of smooth functions.

So let

$$\chi^* \colon \mathscr{E}(\Omega') \to \mathscr{E}(\Omega) \colon \varphi \mapsto \varphi \circ \chi$$

be the so-called *pullback under* χ (beware, the * here does not have anything to do with adjoints; Ω and Ω' are not linear spaces and χ is not a linear map). It is clear that χ^* is a linear map and we claim that χ^* is also continuous.

Claim. $\chi^* \colon \mathscr{E}(\Omega') \to \mathscr{E}(\Omega)$ is continuous.

Proof: Let $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. We will prove by induction on $\mathbb{N}$ that for every $n \in \mathbb{N}$, the following statement holds: if α is a multi-index with $|\alpha| \leq n$, then there exists a $C_\alpha \geq 0$ (which is allowed to depend on χ) such that

$$\|\partial^\alpha(\varphi \circ \chi)\|_{K,0} \leq C_\alpha \|\varphi\|_{\chi(K),|\alpha|}$$

for all $\varphi \in \mathscr{E}(\Omega')$.

If $|\alpha| = 0$, then for every $\varphi \in \mathscr{E}(\Omega')$

$$\begin{aligned}\|\partial^\alpha(\varphi \circ \chi)\|_{K,0} &= \sup_{x \in K} |(\varphi \circ \chi)(x)| = \sup_{x \in K} |\varphi(\chi(x))| \\ &= \sup_{y \in \chi(K)} |\varphi(y)| = \|\varphi\|_{\chi(K),0} = \|\varphi\|_{\chi(K),|\alpha|},\end{aligned}$$

hence for $n = 0$ the statement holds.

Now suppose that the statement holds for $n = k$ with $k \in \mathbb{N}$. We want to prove that the statement also holds for $n = k + 1$. So let α be a multi-index with $0 < |\alpha| \leq k + 1$ (the case $|\alpha| = 0$ has just been covered). Then there is an $1 \leq i \leq \mathrm{n}$ and a multi-index β with $|\beta| = |\alpha| - 1 \leq k$ such that $\partial^\alpha = \partial^\beta \partial_i$. Using the chain rule, the Leibniz rule for multi-indices and the induction hypothesis, we find that for every $\varphi \in \mathscr{E}(\Omega')$

$$\begin{aligned}
\|\partial^\alpha(\varphi\circ\chi)\|_{K,0} &= \|\partial^\beta\partial_i(\varphi\circ\chi)\|_{K,0} = \|\partial^\beta\sum_{j=1}^{\mathrm{n}}(\partial_j\varphi\circ\chi)\partial_i\chi_j\|_{K,0}\\
&= \|\sum_{j=1}^{\mathrm{n}}\partial^\beta((\partial_j\varphi\circ\chi)\partial_i\chi_j)\|_{K,0}\\
&= \|\sum_{j=1}^{\mathrm{n}}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\partial^\gamma(\partial_j\varphi\circ\chi)\partial^{\beta-\gamma}\partial_i\chi_j\|_{K,0}\\
&\le \sum_{j=1}^{\mathrm{n}}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}\|\partial^\gamma(\partial_j\varphi\circ\chi)\|_{K,0}\|\partial^{\beta-\gamma}\partial_i\chi_j\|_{K,0}\\
&\le \sum_{j=1}^{\mathrm{n}}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}C_\gamma\|\partial_j\varphi\|_{\chi(K),|\gamma|}\|\partial^{\beta-\gamma}\partial_i\chi_j\|_{K,0}\\
&\le \sum_{j=1}^{\mathrm{n}}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}C_\gamma\|\varphi\|_{\chi(K),|\gamma|+1}\|\partial^{\beta-\gamma}\partial_i\chi_j\|_{K,0}\\
&\le \left(\sum_{j=1}^{\mathrm{n}}\sum_{\gamma\le\beta}\binom{\beta}{\gamma}C_\gamma\|\partial^{\beta-\gamma}\partial_i\chi_j\|_{K,0}\right)\|\varphi\|_{\chi(K),|\alpha|},
\end{aligned}$$

where the C_γ are the constants provided by the induction hypothesis. Because this estimate is of the desired form, we conclude that the statement indeed holds for $n = k + 1$.

By induction on $\mathbb{N}$ we are now allowed to conclude that the statement holds for every $n \in \mathbb{N}$, which implies that for *every* multi-index α there exists a $C_\alpha \ge 0$ such that

$$\|\partial^\alpha(\varphi\circ\chi)\|_{K,0} \le C_\alpha\|\varphi\|_{\chi(K),|\alpha|}$$

for all $\varphi \in \mathscr{E}(\Omega')$. As a consequence, we finally get our desired estimate for the continuity of χ^*. Indeed, for every every $k \in \mathbb{N}$ and $\varphi \in \mathscr{E}(\Omega')$

$$\begin{aligned}
\|\chi^*\varphi\|_{K,k} = \|\varphi\circ\chi\|_{K,k} &= \sum_{|\alpha|\le k}\|\partial^\alpha(\varphi\circ\chi)\|_{K,0}\\
&\le \sum_{|\alpha|\le k}C_\alpha\|\varphi\|_{\chi(K),|\alpha|} \le \left(\sum_{|\alpha|\le k}C_\alpha\right)\|\varphi\|_{\chi(K),k}. \qquad \square
\end{aligned}$$

The next step is to prove that χ^* restricts to a continuous linear map from $\mathscr{D}(\Omega')$ into $\mathscr{D}(\Omega)$. For this we need the following lemma.

Lemma 1.7.1. *For every* $\varphi \in \mathscr{E}(\Omega')$

$$\operatorname{supp}(\chi^*\varphi) = \chi^{-1}(\operatorname{supp}(\varphi)).$$

Proof: We first prove that

$$\operatorname{supp}(\chi^*\varphi) = \operatorname{supp}(\varphi\circ\chi) \subseteq \chi^{-1}(\operatorname{supp}(\varphi)). \tag{1.4}$$

Let $\Omega_{\varphi\circ\chi}$ be the largest open subset of Ω on which $\varphi\circ\chi$ vanishes and let Ω'_φ be the largest open subset of Ω' on which φ vanishes. Then the inclusion in equation (1.4) is equivalent to

$$\Omega\setminus\Omega_{\varphi\circ\chi}\subseteq\chi^{-1}(\Omega'\setminus\Omega'_\varphi)=\chi^{-1}(\Omega')\setminus\chi^{-1}(\Omega'_\varphi)=\Omega\setminus\chi^{-1}(\Omega'_\varphi),$$

which is in turn equivalent to $\chi^{-1}(\Omega'_\varphi)\subseteq\Omega_{\varphi\circ\chi}$. For the latter it suffices to prove that $\varphi\circ\chi$ vanishes on $\chi^{-1}(\Omega'_\varphi)$, which is clearly the case.

To prove the converse inclusion, we note that Ω, Ω' and χ are all arbitrarily chosen. Therefore, the inclusion of equation (1.4) also holds if we replace χ by χ^{-1} and φ by $\varphi\circ\chi$ (a smooth function on Ω). This results in

$$\operatorname{supp}(\varphi)=\operatorname{supp}((\varphi\circ\chi)\circ\chi^{-1})\subseteq\chi(\operatorname{supp}(\varphi\circ\chi))$$

and applying χ^{-1} to both sides then gives

$$\chi^{-1}(\operatorname{supp}(\varphi))\subseteq\operatorname{supp}(\varphi\circ\chi)=\operatorname{supp}(\chi^*\varphi).$$ □

Let $K'\in\mathscr{P}_{\mathrm{c}}(\Omega')$. Then $\chi^{-1}(K')\in\mathscr{P}_{\mathrm{c}}(\Omega)$ and the previous lemma shows that for all $\varphi\in\mathscr{E}_{K'}(\Omega')$, $\operatorname{supp}(\chi^*\varphi)=\chi^{-1}(\operatorname{supp}(\varphi))\subseteq\chi^{-1}(K')$. That is, χ^* maps $\mathscr{E}_{K'}(\Omega')$ into $\mathscr{E}_{\chi^{-1}(K')}(\Omega)$. By applying our usual trick (i.e., using Proposition A.3.2 and the fact that $\mathscr{E}_K(\Omega)\subseteq_{\mathrm{c}}\mathscr{D}(\Omega)$ for all $K\in\mathscr{P}_{\mathrm{c}}(\Omega)$) we now get that χ^* restricts to a continuous linear map from $\mathscr{D}(\Omega')$ into $\mathscr{D}(\Omega)$.

Definition 1.7.2. We call the adjoint $(\chi^*)^*\colon\mathscr{D}'(\Omega)\to\mathscr{D}'(\Omega')$ of the continuous linear map $\chi^*\colon\mathscr{D}(\Omega')\to\mathscr{D}(\Omega)$ the *pushforward under* χ and we denote it by χ_*. When u is a distribution on Ω, we say that χ_*u is the *pushforward of* u *under* χ. ⊘

As adjoint of a continuous linear map

$$\chi_*\colon\mathscr{D}'(\Omega)\to\mathscr{D}'(\Omega')$$

is itself a continuous linear map and because Ω, Ω' and χ were arbitrarily chosen, we also get a continuous linear pushforward $(\chi^{-1})_*\colon\mathscr{D}'(\Omega')\to\mathscr{D}'(\Omega)$. We easily check that χ_* and $(\chi^{-1})_*$ are each others inverse and we conclude that they are linear topological isomorphisms. So we have found the expected identification between $\mathscr{D}'(\Omega)$ and $\mathscr{D}'(\Omega')$. When studying functional spaces, we will only look at pushforwards under diffeomorphisms $\chi\colon\Omega\to\Omega'$ with $\Omega'=\Omega$. Such a diffeomorphism is often called a *change of coordinates*.

Lemma 1.7.3. *For every* $u\in\mathscr{D}'(\Omega)$

$$\operatorname{supp}(\chi_*u)=\chi(\operatorname{supp}(u)).$$

Proof: Let Ω_u be the largest open subset of Ω on which u vanishes and let Ω'_{χ_*u} be the largest open subset of Ω' on which χ_*u vanishes. In order to prove that $\operatorname{supp}(\chi_*u)\subseteq\chi(\operatorname{supp}(u))$ it suffices to prove that χ_*u vanishes on $\chi(\Omega_u)$ (the reasoning for this is similar to the reasoning in the proof of Lemma 1.7.1).

So let $\varphi\in\mathscr{D}(\chi(\Omega_u))$ and let $\tilde\varphi$ be its extension by zero to Ω'. Then, by Lemma 1.7.1,

$$\operatorname{supp}(\chi^*\tilde\varphi)=\chi^{-1}(\operatorname{supp}(\tilde\varphi))=\chi^{-1}(\operatorname{supp}(\varphi))\subseteq\chi^{-1}(\chi(\Omega_u))=\Omega_u,$$

so $\chi^*\tilde{\varphi}$ is the extension by zero to Ω of $(\chi^*\tilde{\varphi})|_{\Omega_u}$. Using this, we find

$$(\chi_* u)|_{\chi(\Omega_u)}(\varphi) = (\chi_* u)(\tilde{\varphi}) = u(\chi^*\tilde{\varphi}) = u|_{\Omega_u}((\chi^*\tilde{\varphi})|_{\Omega_u}) = 0,$$

which shows that $\chi_* u$ indeed vanishes on $\chi(\Omega_u)$.

The converse inclusion is obtained by replacing χ by χ^{-1} and u by $\chi_* u$ and then applying χ to both sides (again, this is similar to the approach in the proof of Lemma 1.7.1). □

In the case of multiplication by smooth functions and differentiation, the new maps on the space of distributions restricted to the familiar ones on the space of smooth functions. However, the familiar pushforward under the diffeomorphism χ on the space of smooth functions equals $(\chi^{-1})^* : \varphi \mapsto \varphi \circ \chi^{-1}$ (the pullback under χ^{-1}), while the following lemma shows that the restriction of $\chi_* : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ to $\mathscr{E}(\Omega)$ sends φ to $\frac{\varphi}{|\det D\chi|} \circ \chi^{-1}$.

Lemma 1.7.4. *Let f be a locally integrable function on Ω. Then*

$$\chi_* u_f = u_{\frac{f}{|\det D\chi|} \circ \chi^{-1}}.$$

Proof: Using the change of variables theorem, we find that if f is locally integrable on Ω, $\frac{f}{|\det D\chi|} \circ \chi^{-1}$ is locally integrable on Ω' and that for all $\varphi \in \mathscr{D}(\Omega')$,

$$\begin{aligned}
(\chi_* u_f)(\varphi) &= u_f(\chi^*\varphi) = u_f(\varphi \circ \chi) \\
&= \int_\Omega f(\varphi \circ \chi)\, d\lambda = \int_{\Omega'} (f \circ \chi^{-1})\varphi |\det D\chi^{-1}|\, d\lambda \\
&= \int_{\Omega'} \left(\frac{f}{|\det D\chi|} \circ \chi^{-1} \right) \varphi\, d\lambda = u_{\frac{f}{|\det D\chi|} \circ \chi^{-1}}(\varphi).
\end{aligned}$$ □

So the restriction of $\chi_* : \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ to $\mathscr{E}(\Omega)$ is only equal to the familiar pushforward for functions if $|\det D\chi| = 1$ (i.e., if χ is *volume preserving*). This is not a problem, but it is good to be aware of it. To avoid confusion, we consistently stick to the notation used so far: the maps χ_* and $(\chi^{-1})_*$ (that have $*$ as subscript) are the new distribution theoretic maps, while the maps χ^* and $(\chi^{-1})^*$ (that have $*$ as superscript) are the familiar maps on functions.

Relevant results

Lemma 1.7.5. *For every $\psi \in \mathscr{E}(\Omega')$, we have*

$$m_\psi \circ \chi_* = \chi_* \circ m_{\chi^*\psi}.$$

Proof: Let $u \in \mathscr{D}'(\Omega)$ and $\varphi \in \mathscr{D}(\Omega')$. Then

$$\begin{aligned}
((m_\psi \circ \chi_*)u)(\varphi) &= (m_\psi(\chi_* u))(\varphi) = (\chi_* u)(\psi\varphi) = u(\chi^*(\psi\varphi)) \\
&= u((\chi^*\psi)(\chi^*\varphi)) = (m_{\chi^*\psi} u)(\chi^*\varphi) \\
&= (\chi_*(m_{\chi^*\psi} u))(\varphi) = ((\chi_* \circ m_{\chi^*\psi})u)(\varphi).
\end{aligned}$$ □

Lemma 1.7.6. *For every $1 \le i \le \mathrm{n}$ and $u \in \mathscr{D}'(\Omega)$, we have*

$$\chi_* \partial_i u = \sum_{j=1}^{\mathrm{n}} \partial_j \chi_*((\partial_i \chi_j)u).$$

Proof: Let $\varphi \in \mathscr{D}(\Omega')$. Then

$$\begin{aligned}(\chi_* \partial_i u)(\varphi) &= (\partial_i u)(\varphi \circ \chi) = -u(\partial_i(\varphi \circ \chi)) \\ &= -\sum_{j=1}^{\mathrm{n}} u((\partial_j \varphi \circ \chi)\partial_i \chi_j) = -\sum_{j=1}^{\mathrm{n}} ((\partial_i \chi_j)u)(\partial_j \varphi \circ \chi) \\ &= -\sum_{j=1}^{\mathrm{n}} (\chi_*((\partial_i \chi_j)u))(\partial_j \varphi) = \sum_{j=1}^{\mathrm{n}} (\partial_j \chi_*((\partial_i \chi_j)u))(\varphi). \qquad \square\end{aligned}$$

Lemma 1.7.7. *For every $1 \le i \le \mathrm{n}$ and $u \in \mathscr{D}'(\Omega)$, we have*

$$\chi_* \partial_i u = \sum_{j=1}^{\mathrm{n}} (\partial_i \chi_j \circ \xi)\partial_j \chi_* u + \sum_{j=1}^{\mathrm{n}} \sum_{k=1}^{\mathrm{n}} (\partial_k \partial_i \chi_j \circ \xi)(\partial_j \xi_k)\chi_* u,$$

where ξ is used as convenient shorthand for χ^{-1}.

Proof: Using the previous two lemmas, we indeed get

$$\begin{aligned}\chi_* \partial_i u &= \sum_{j=1}^{\mathrm{n}} \partial_j \chi_*((\partial_i \chi_j)u) = \sum_{j=1}^{\mathrm{n}} \partial_j((\partial_i \chi_j \circ \xi)\chi_* u) \\ &= \sum_{j=1}^{\mathrm{n}} (\partial_i \chi_j \circ \xi)\partial_j \chi_* u + \sum_{j=1}^{\mathrm{n}} (\partial_j(\partial_i \chi_j \circ \xi))\chi_* u \\ &= \sum_{j=1}^{\mathrm{n}} (\partial_i \chi_j \circ \xi)\partial_j \chi_* u + \sum_{j=1}^{\mathrm{n}} \sum_{k=1}^{\mathrm{n}} (\partial_k \partial_i \chi_j \circ \xi)(\partial_j \xi_k)\chi_* u. \qquad \square\end{aligned}$$

2

Functional spaces on $\mathbb{R}^n$

Now that we have seen some distribution theory, it is time to introduce functional spaces. In this chapter we will restrict our attention to the convenient setting of 'scalar-valued' functional spaces on (open subsets of) $\mathbb{R}^n$. Those are a special case of the functional spaces on manifolds that 'take values in vector bundles' that we will introduce later. However, we are talking about a special case of significant importance: a lot of the more general functional spaces on manifolds will be 'modeled' after some functional space on $\mathbb{R}^n$. And, although the setting is simpler, many arguments for the more general case are the same as or very similar to the arguments in this specific setting.

Besides giving the definition of a functional space and a lot of examples, we will introduce properties that a functional space might (or might not) have and we will discuss constructions that turn a given functional space into another. For some important combinations of properties and constructions, we will also investigate whether or not the property is 'preserved' under the construction. Because many properties are related to a certain construction, such questions are often resolved by looking at the way in which two constructions 'interact'.

As in the previous chapter, Ω denotes an open subset of $\mathbb{R}^n$ and whenever $\Omega = \emptyset$ would cause difficulties or require changes, we implicitly assume that Ω is nonempty.

2.1 Definition and examples

Let us give the most important definition of this chapter straight away (recall that $\mathscr{D}(\Omega)$ is viewed as a subspace of $\mathscr{D}'(\Omega)$ whenever the context suggests so).

Definition 2.1.1. A *functional space* on Ω is a linear subspace $\mathscr{F}$ of $\mathscr{D}'(\Omega)$ that contains $\mathscr{D}(\Omega)$ and carries a locally convex topology such that:

1. $\mathscr{D}(\Omega) \subseteq_c \mathscr{F} \subseteq_c \mathscr{D}'(\Omega)$ and
2. for every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$. ⊘

The first condition in the definition above just says that the topology on $\mathscr{F}$ needs to be stronger than the topology induced from $\mathscr{D}'(\Omega)$ and that the topology that $\mathscr{F}$ induces on $\mathscr{D}(\Omega)$ needs to be weaker than the intrinsic topology on $\mathscr{D}(\Omega)$. The observation that the topology of a functional space on Ω is stronger than the topology induced from $\mathscr{D}'(\Omega)$ already leads to a first result: when combined with the fact that $\mathscr{D}'(\Omega)$ is Hausdorff, it shows that every functional space on Ω must be Hausdorff.

Lemma 2.1.2. *A functional space on Ω is always Hausdorff.*

Another simple observation that needs to be made is that a functional space on Ω is always dense in $\mathscr{D}'(\Omega)$. Indeed, on the strength of Lemma 1.3.3 we have that $\mathscr{D}(\Omega)$ is dense in $\mathscr{D}'(\Omega)$ and since a functional space $\mathscr{F}$ on Ω always contains $\mathscr{D}(\Omega)$, we see that $\mathscr{F}$ must be dense in $\mathscr{D}'(\Omega)$ as well.

Lemma 2.1.3. *Every functional space on Ω is a dense linear subspace of $\mathscr{D}'(\Omega)$.*

The framework of functional spaces makes it possible to give an abstract and unified treatment of important spaces of (equivalence classes of) functions, which are often grouped under the informal term ‘function spaces’, and important spaces of distributions.

Obviously, the space of all distributions is a functional space.

Example 2.1.4. $\mathscr{D}'(\Omega)$ is a functional space on Ω. ⊘

We have seen in the previous chapter that for every $\varphi \in \mathscr{E}(\Omega)$, the map $m_\varphi \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to the ordinary continuous linear ‘multiplication by φ’ map on $\mathscr{E}(\Omega)$ and that this map in turn restricts to the ordinary continuous linear ‘multiplication by φ’ map on $\mathscr{D}(\Omega)$, so we also have:

Example 2.1.5. $\mathscr{D}(\Omega)$ and $\mathscr{E}(\Omega)$ are functional spaces on Ω. ⊘

For the previous example, we have implicitly used our ‘indentification convention’. Making such identifications will often be necessary, so let us capture the underlying idea in a proposition.

Proposition 2.1.6. *Let $\mathscr{X}$ be a locally convex vector space and*

$$\imath \colon \mathscr{D}(\Omega) \to \mathscr{X} \quad \text{and} \quad \imath' \colon \mathscr{X} \to \mathscr{D}'(\Omega)$$

injective continuous linear maps such that:

1. *$\imath' \circ \imath \colon \mathscr{D}(\Omega) \to \mathscr{D}'(\Omega)$ equals the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$ and*
2. *for every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi(\imath'(\mathscr{X})) \subseteq \imath'(\mathscr{X})$ and $(\imath')^{-1} \circ m_\varphi \circ \imath' \colon \mathscr{X} \to \mathscr{X}$ is continuous.*

Then $\iota'(\mathscr{X})$, endowed with the topology that turns ι' into a linear topological isomorphism from $\mathscr{X}$ onto $\iota'(\mathscr{X})$, is a functional space on Ω.

That this proposition holds is immediately clear; it is nothing more than a simple translation of the definition of a functional space on Ω to the case where the locally convex vector space under consideration is not a subspace of the space of distributions. That is also precisely the point: a lot of our examples are usually not treated as subspaces of the space of distributions, but they can be identified with such a subspace in a canonical fashion. The proposition above then tells us whether their image under this indentification is a functional space. If this is the case, we will usually identify $\mathscr{X}$ with its image and say that $\mathscr{X}$ is a functional space on Ω.

So why did we not have to use this proposition when we stated that $\mathscr{D}(\Omega)$ and $\mathscr{E}(\Omega)$ are functional spaces? Well, we did, but implicitly. It is hidden in the statements from the previous chapter, where we already treated $\mathscr{D}(\Omega)$ and $\mathscr{E}(\Omega)$ as subspaces of the space of distributions, that we have used. For example, for $\mathscr{D}(\Omega)$ we implicitly took $\imath$ to be the identity on $\mathscr{D}(\Omega)$ and $\imath'$ to be the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$. Then the first condition of the previous proposition is clearly satisfied, while the second condition follows because $\varphi u_\psi = u_{\varphi\psi}$ and ordinary multiplication by a smooth function is continuous on $\mathscr{D}(\Omega)$.

The following should not come as a surprise.

Example 2.1.7. $\mathscr{E}'(\Omega)$ is a functional space on Ω. Formally speaking, we use Proposition 2.1.6 to achieve this. Indeed, let $\imath$ be the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{E}'(\Omega)$ and let $\imath'$ be the canonical identification of $\mathscr{E}'(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$. Then, by definition, $\imath' \circ \imath$ equals the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$ and we easily check that for every $\varphi \in \mathscr{E}(\Omega)$, $(\imath')^{-1} \circ m_\varphi \circ \imath'$ equals the adjoint of the continuous linear map $m_\varphi \colon \mathscr{E}(\Omega) \to \mathscr{E}(\Omega) \colon \psi \mapsto \varphi\psi$. ⊘

The next example, which shows that the well-known L^p spaces are functional spaces, is of great importance for the theory of functional spaces. These L^p spaces are namely the 'starting point' for the famous Sobolev spaces and the latter have been the leading example for important concepts in the theory.

Example 2.1.8. The spaces $L^p(\Omega)$ with $1 \le p \le \infty$ are functional spaces on Ω. Fix $1 \le p \le \infty$. Because every element of $L^p(\Omega)$ is an equivalence class of locally integrable functions that are equal almost everywhere (that every 'function' that belongs to $L^p(\Omega)$ is locally integrable follows by looking at the characteristic function of compact subsets and the Hölder inequality), the assignment $[f] \mapsto u_f$ gives an injective linear map $\imath'$ from $L^p(\Omega)$ into $\mathscr{D}'(\Omega)$ (see Section 1.3 of the previous chapter). This is already quite nice: every *equivalence class* of functions corresponds to *one* distribution, showing that the concept of a distribution is in fact more natural than the concept of a function when talking about $L^p(\Omega)$. Moreover, it is easy to see that the assignment $\varphi \mapsto [\varphi]$ gives an injective linear map $\imath$ from $\mathscr{D}(\Omega)$ into $L^p(\Omega)$. We claim that $\imath$ and $\imath'$ are also continuous.

Claim. $\imath \colon \mathscr{D}(\Omega) \to L^p(\Omega)$ is continuous.

Proof: It suffices to check that for every $K \in \mathscr{P}_c(\Omega)$, $\varphi \mapsto [\varphi]$ is a continuous linear map from $\mathscr{E}_K(\Omega)$ into $L^p(\Omega)$. So let $\|\cdot\|_p$ be the usual norm on $L^p(\Omega)$, fix $K \in \mathscr{P}_c(\Omega)$ and let $\varphi \in \mathscr{E}_K(\Omega)$. If $p = \infty$, we have

$$\|[\varphi]\|_p = \|[\varphi]\|_\infty = \|\varphi\|_{K,0}$$

and if $1 \le p < \infty$, we have

$$\|[\varphi]\|_p^p = \int_\Omega |\varphi|^p \, d\lambda = \int_K |\varphi|^p \, d\lambda \le \int_K (\|\varphi\|_{K,0})^p \, d\lambda = \lambda(K)(\|\varphi\|_{K,0})^p.$$

Hence, under the convention that $\frac{1}{\infty} = 0$,

$$\|[\varphi]\|_p \le \lambda(K)^{\frac{1}{p}} \|\varphi\|_{K,0}, \tag{2.1}$$

which is an estimate of the desired form. □

Claim. $\imath' \colon L^p(\Omega) \to \mathscr{D}'(\Omega)$ is continuous.

Proof: Let B be a bounded subset of $\mathscr{D}(\Omega)$ and let q_B be its associated seminorm on $\mathscr{D}'(\Omega)$. By Lemma 1.1.14 we find a compact subset K of Ω and a constant $r_0 \geq 0$ such that $\operatorname{supp}(\varphi) \subseteq K$ and $\|\varphi\|_{K,0} \leq r_0$ for all $\varphi \in B$. Using the Hölder inequality and equation (2.1), we find that for all $[f] \in L^p(\Omega)$ and $\varphi \in B$

$$|u_f(\varphi)| = \left|\int_\Omega f\varphi\, d\lambda\right| \leq \int_\Omega |f\varphi|\, d\lambda \leq \|[\varphi]\|_q \|[f]\|_p$$
$$\leq \lambda(K)^{\frac{1}{q}} \|\varphi\|_{K,0} \|[f]\|_p \leq \lambda(K)^{\frac{1}{q}} r_0 \|[f]\|_p,$$

where q is the Hölder conjugate of p. Therefore, for every $[f] \in L^p(\Omega)$

$$q_B(\imath'([f])) = \sup_{\varphi \in B} |u_f(\varphi)| \leq \lambda(K)^{\frac{1}{q}} r_0 \|[f]\|_p,$$

which is an estimate of the desired form. □

It remains to be shown that $\imath$ and $\imath'$ satisfy the conditions of Proposition 2.1.6. That $\imath' \circ \imath$ equals the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$ is clear. Moreover, it is easy to check that for $\varphi \in \mathscr{D}(\Omega)$, $[f] \mapsto [\varphi f]$ is a well-defined continuous linear map from $L^p(\Omega)$ into $L^p(\Omega)$ (use that $\|[\varphi f]\|_p \leq \|[\varphi]\|_\infty \|[f]\|_p$, which follows from an easy property of the essential supremum for $p = \infty$ and from

$$\int_\Omega |\varphi f|^p\, d\lambda \leq \int_\Omega |\varphi|^p |f|^p\, d\lambda \leq (\|[\varphi]\|_\infty)^p \int_\Omega |f|^p\, d\lambda = (\|[\varphi]\|_\infty)^p (\|[f]\|_p)^p$$

for $1 \leq p < \infty$). Since a trivial mental computation shows that $m_\varphi u_f = u_{\varphi f}$ holds for every $\varphi \in \mathscr{D}(\Omega)$ and $[f] \in L^p(\Omega)$, we deduce that the second condition of Proposition 2.1.6 is satisfied as well. Using this proposition, we may finally conclude that (the image under $\imath'$ of) $L^p(\Omega)$ is a functional space on Ω. ⊘

Example 2.1.9. Let $\mathscr{C}(\Omega)$ be the linear space of continuous functions on Ω and endow it with the topology induced by the seminorms $\{\|\cdot\|_{K,0} \mid K \in \mathscr{P}_{\mathrm{c}}(\Omega)\}$, with $\|\psi\|_{K,0} := \sup_{x \in K} |\psi(x)|$. Then by precisely the same arguments that we have used for $\mathscr{E}(\Omega)$ in Section 1.3 of the previous chapter

$$\imath' \colon \mathscr{C}(\Omega) \to \mathscr{D}'(\Omega) \colon \psi \mapsto u_\psi$$

is an injective continuous linear map. Furthermore, it is clear that we have $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{E}(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}(\Omega)$, so the inclusion map $\imath \colon \mathscr{D}(\Omega) \hookrightarrow \mathscr{C}(\Omega)$ is an injective continuous linear map as well. That $\imath' \circ \imath$ equals the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$ is evident and we easily check, using almost the same arguments as for $\mathscr{E}(\Omega)$, that for every $\varphi \in \mathscr{E}(\Omega)$, $\mathscr{C}(\Omega) \to \mathscr{C}(\Omega) \colon \psi \mapsto \varphi\psi$ is a continuous linear map that equals $(\imath')^{-1} \circ m_\varphi \circ \imath$. So by Proposition 2.1.6 (the image under $\imath'$ of) $\mathscr{C}(\Omega)$ is a functional space on Ω. ⊘

The following example shows that it is sometimes really convenient to use our 'identification convention' (which allows us to regard $\mathscr{C}(\Omega)$ as a subspace of $\mathscr{D}'(\Omega)$).

Example 2.1.10. Let $\mathscr{C}_b(\Omega)$ be the linear space of all bounded continuous functions on Ω. On this space the supremum norm $\|\cdot\|_\infty$ is well-defined and we endow it with the topology induced by this norm. We claim that $\mathscr{C}_b(\Omega)$ is a functional space on Ω.

First of all, we clearly have

$$\mathscr{C}_b(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega) \tag{2.2}$$

(note however that the topology on $\mathscr{C}_b(\Omega)$ is *strictly* larger than the topology that $\mathscr{C}(\Omega)$ induces on it). Moreover, using Proposition A.3.2, we easily check that $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_b(\Omega)$ (indeed, $\mathscr{E}_K(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_b(\Omega)$ because $\|\psi\|_\infty = \|\psi\|_{K,0}$ for every $\psi \in \mathscr{E}_K(\Omega)$). Finally, it is also easy to check that for every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \colon \mathscr{C}(\Omega) \to \mathscr{C}(\Omega) \colon \psi \mapsto \varphi\psi$ restricts to a continuous linear map from $\mathscr{C}_b(\Omega)$ into $\mathscr{C}_b(\Omega)$ (for the continuity, use that $\|\varphi\psi\|_\infty \leq \|\varphi\|_\infty \|\psi\|_\infty$). Since we already know that $m_\varphi \colon \mathscr{C}(\Omega) \to \mathscr{C}(\Omega)$ is the restriction of $m_\varphi \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$, it follows that $\mathscr{C}_b(\Omega)$ is a functional spaces on Ω. ⊘

Note that we have only implicitly told how we identify $\mathscr{C}_b(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$: by writing down the chain of inclusions in equation (2.2), we silently agreed to view $\mathscr{C}_b(\Omega)$ as a subspace of (the subspace of $\mathscr{D}'(\Omega)$ that corresponds to) $\mathscr{C}(\Omega)$. Hence, to view an element of $\mathscr{C}_b(\Omega)$ as distribution, we first view it as a continuous function and then use the canonical identification of $\mathscr{C}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$. In this case, this procedure is rather obvious (and results in $\mathscr{C}_b(\Omega) \to \mathscr{D}'(\Omega) \colon \psi \mapsto u_\psi$, as was to be expected) because an element of $\mathscr{C}_b(\Omega)$ really *is* a continuous function, but the concept of 'stacking identifications' also works if both identifications are nontrivial.

Example 2.1.11. Let $\mathscr{C}_0(\Omega)$ be the linear space of all continuous functions on Ω that 'vanish at infinity' (that is, all continuous functions φ on Ω such that $\{x \in \Omega \mid |\varphi(x)| \geq \varepsilon\}$ is compact for every $\varepsilon > 0$) and let $\mathscr{C}_s(\Omega)$ be the linear space of all continuous functions on Ω that 'become constant at infinity' (that is, all continuous functions φ on Ω for which there exists an $c \in \mathbb{K}$ such that $c_\Omega - \varphi$ vanishes at infinity). We easily deduce that $\mathscr{C}_0(\Omega) \subseteq \mathscr{C}_s(\Omega) \subseteq \mathscr{C}_b(\Omega)$ and we endow $\mathscr{C}_0(\Omega)$ and $\mathscr{C}_s(\Omega)$ with the induced topology from $\mathscr{C}_b(\Omega)$ (i.e., the topology induced by the supremum norm). We claim that $\mathscr{C}_0(\Omega)$ and $\mathscr{C}_s(\Omega)$ are functional spaces on Ω.

To prove this claim, we will exploit that $\mathscr{C}_b(\Omega)$ is a functional space on Ω and that the new spaces use the 'same' topology. To begin with, it is clear from the definition that $\mathscr{C}_0(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_s(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_b(\Omega)$ and we already know that $\mathscr{C}_b(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$. Furthermore, we certainly have $\mathscr{D}(\Omega) \subseteq \mathscr{C}_0(\Omega)$ (for every $\varphi \in \mathscr{D}(\Omega)$ and $\varepsilon > 0$, $\{x \in \Omega \mid |\varphi(x)| \geq \varepsilon\}$ is a closed subset of $\operatorname{supp}(\varphi)$) and combining this with $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_b(\Omega)$ and the fact that $\mathscr{C}_0(\Omega)$ carries the induced topology from $\mathscr{C}_b(\Omega)$ shows that $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_0(\Omega)$. Putting everything together, we get

$$\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_0(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_s(\Omega) \subseteq_{\mathrm{c}} \mathscr{C}_b(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega),$$

which gives the desired continuous inclusions for $\mathscr{C}_0(\Omega)$ and $\mathscr{C}_s(\Omega)$. The desired statement regarding m_φ for $\varphi \in \mathscr{D}(\Omega)$ follows from the continuity of $m_\varphi \colon \mathscr{C}_b(\Omega) \to \mathscr{C}_b(\Omega)$ and the simple observation that $m_\varphi(\mathscr{C}_s(\Omega)) \subseteq \mathscr{C}_0(\Omega)$ for every $\varphi \in \mathscr{D}(\Omega)$ (for every $\varphi \in \mathscr{D}(\Omega)$ and $\psi \in \mathscr{C}_s(\Omega)$, $\varphi\psi$ is a compactly supported continuous function). ⊘

We will come back to giving examples at various points in the chapter, often after the introduction of a 'construction' on functional spaces. These 'constructions' will actually be functors on the category of functional spaces on Ω, which we will now define.

Definition 2.1.12. The *category of functional spaces on* Ω has all functional spaces on Ω as objects and continuous inclusions as arrows. ⊘

It might seem strange that we do not allow more general continuous linear maps as arrows, but most of the 'constructions' that we will encounter can only be regarded as a functor on the category of functional spaces if the arrows satisfy some specific property and the most natural class of continuous linear maps that satisfies all these properties is the class of continuous inclusions.

2.2 Semi-functional spaces

Altough functional spaces are the most important objects in our theory, we will also encounter spaces that are 'almost' functional spaces.

Definition 2.2.1. A *semi-functional space* on Ω is a linear subspace $\mathscr{F}$ of $\mathscr{D}'(\Omega)$ carrying a locally convex topology such that:

1. $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$ and
2. for every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$.

Note that in comparison with the defintion of a functional space on Ω there is only one difference: for a semi-functional space $\mathscr{F}$ on Ω we do not require that $\mathscr{D}(\Omega)$ is a subspace of $\mathscr{F}$ (with continuous inclusion). This extra property that functional spaces are required to have turns out to be essential for only a limited, but important, part of the theory. Despite the fact that functional spaces are the main concept, the part of the theory for which this extra assumption is not required will be largely formulated in terms of semi-functional spaces. This is not because we want to be as general as possible; it is just very convenient for a smooth treatment of the theory.

It is clearly true that every functional space on Ω is a semi-functional space on Ω. Also, by the same argument that we have used for functional spaces, we see that every semi-functional space is Hausdorff. The fact that every functional space on Ω is a dense subspace of $\mathscr{D}'(\Omega)$ does *not* generalize to semi-functional spaces.

Example 2.2.2. $\mathscr{E}_K(\Omega)$, with $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, is a semi-functional space on Ω. Indeed, since $\mathscr{E}(\Omega)$ is a functional space on Ω and $\mathscr{E}_K(\Omega)$ carries the topology that is induced on it by $\mathscr{E}(\Omega)$, we have that $\mathscr{E}_K(\Omega) \subseteq_{\mathrm{c}} \mathscr{E}(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$. Moreover, for every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{E}(\Omega)$ into $\mathscr{E}(\Omega)$ and because $\mathrm{supp}(\varphi\psi) \subseteq \mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi) \subseteq K$ for all $\psi \in \mathscr{E}_K(\Omega)$, we find that m_φ also restricts to a continuous linear map from $\mathscr{E}_K(\Omega)$ into $\mathscr{E}_K(\Omega)$. ⊘

Example 2.2.3. Let A be an arbitrary subset of Ω. Then the linear space $\mathscr{C}(\Omega; A)$ of all continuous functions on Ω that vanish on A endowed with the induced topology from $\mathscr{C}(\Omega)$ is easily seen to be a semi-functional space on Ω: $\mathscr{C}(\Omega; A) \subseteq_{\mathrm{c}} \mathscr{C}(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$ gives the desired continuous inclusion and the simple observation that for every $\varphi \in \mathscr{D}(\Omega)$ the continuous linear map $m_\varphi \colon \mathscr{C}(\Omega) \to \mathscr{C}(\Omega)$ satisfies $m_\varphi(\mathscr{C}(\Omega; A)) \subseteq \mathscr{C}(\Omega; A)$ shows that m_φ restricts to a continuous linear map from $\mathscr{C}(\Omega; A)$ into $\mathscr{C}(\Omega; A)$.

So in particular $N := \mathscr{C}(\mathbb{R}^{\mathrm{n}}; \{0\})$ is a semi-functional space on $\mathbb{R}^{\mathrm{n}}$. This space plays a natural role when we look at distributions on $\mathbb{R}^{\mathrm{n}}$ with support at the origin, see [13, page 264]. ⊘

The discussion from the previous section about identifying spaces with a subspace of $\mathscr{D}'(\Omega)$ and the associated 'indentification convention' is also relevant in the context of semi-functional spaces. The following proposition is the analogue for semi-functional spaces of Proposition 2.1.6. Because it is nothing more than a simple 'translation' of the definition of a semi-functional space, we will not give a proof.

Proposition 2.2.4. *Let $\mathscr{X}$ be a locally convex vector space and*

$$\imath' \colon \mathscr{X} \to \mathscr{D}'(\Omega)$$

an injective continuous linear map such that for every $\varphi \in \mathscr{D}(\Omega)$,

$$m_\varphi(\imath'(\mathscr{X})) \subseteq \imath'(\mathscr{X}) \quad \text{and} \quad (\imath')^{-1} \circ m_\varphi \circ \imath' \colon \mathscr{X} \to \mathscr{X} \text{ is continuous.}$$

Then $\imath'(\mathscr{X})$, endowed with the topology that turns $\imath'$ into a linear topological isomorphism from $\mathscr{X}$ onto $\imath'(\mathscr{X})$, is a semi-functional space on Ω.

Example 2.2.5. Let U be an open subset of Ω. We have already seen that $\mathscr{E}'(U)$ is a functional space on U, but (surprisingly enough) $\mathscr{E}'(U)$ is also a semi-functional space on Ω in a natural way. To see this, we use the previous proposition with $\imath'$ equal to the injective continuous linear 'extension' map $\mathrm{ext}_{U,\Omega} \colon \mathscr{E}'(U) \to \mathscr{E}'(\Omega)$ from Remark 1.4.6 (which can also be viewed as a continuous map from $\mathscr{E}'(U)$ into $\mathscr{D}'(\Omega)$ because $\mathscr{E}'(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$). Of course, we should check that the condition formulated in the proposition above holds for this $\imath'$. So fix $\varphi \in \mathscr{D}(\Omega)$. Looking back at Example 2.1.7, we see that m_ψ restricts to a continuous linear map from $\mathscr{E}'(U)$ into $\mathscr{E}'(U)$ for all $\psi \in \mathscr{E}(U)$ (rather than only for $\psi \in \mathscr{D}(U)$), so in particular $m_{\varphi|_U}$ restricts to a continuous linear map from $\mathscr{E}'(U)$ into $\mathscr{E}'(U)$. Next, on behalf of Lemma 1.5.8,

$$m_\varphi \circ \mathrm{ext}_{U,\Omega} = \mathrm{ext}_{U,\Omega} \circ m_{\varphi|_U},$$

which clearly implies $m_\varphi(\mathrm{ext}_{U,\Omega}(\mathscr{E}'(U))) \subseteq \mathrm{ext}_{U,\Omega}(\mathscr{E}'(U))$. Moreover, using this identity, we also find that

$$\begin{aligned}(\mathrm{ext}_{U,\Omega})^{-1} \circ m_\varphi \circ \mathrm{ext}_{U,\Omega} &= \mathrm{res}_{\Omega,U} \circ m_\varphi \circ \mathrm{ext}_{U,\Omega} = \mathrm{res}_{\Omega,U} \circ \mathrm{ext}_{U,\Omega} \circ m_{\varphi|_U} \\ &= \mathrm{id}_{\mathscr{E}'(U)} \circ m_{\varphi|_U} = m_{\varphi|_U}\end{aligned}$$

is a continuous linear map from $\mathscr{E}'(U)$ into $\mathscr{E}'(U)$, so we are done. ⊘

Of course, we also have a category of semi-functional spaces on Ω.

Definition 2.2.6. The *category of semi-functional spaces on* Ω has all semi-functional spaces on Ω as objects and continuous inclusions as arrows. ⊘

2.3 Fixed compact support

We will now introduce the first construction on (semi-)functional spaces.

Definition 2.3.1. Let $\mathscr{F}$ be a semi-functional space on Ω. For all $K \in \mathscr{P}_c(\Omega)$, we define $\mathscr{F}_K$ to be

$$\{u \in \mathscr{F} \mid \operatorname{supp}(u) \subseteq K\} \subseteq \mathscr{F}$$

endowed with the subspace topology. ⊘

Proposition 2.3.2. *$\mathscr{F}_K$ is a semi-functional space on Ω and $\mathscr{F}_K \subseteq_c \mathscr{F}$.*

Proof: It trivially follows from Lemma 1.4.12 that $\mathscr{F}_K$ is a linear subspace of $\mathscr{F}$ and Lemma 1.5.3 shows that for $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \colon \mathscr{F} \to \mathscr{F}$ restricts to a map from $\mathscr{F}_K$ into $\mathscr{F}_K$. Everything else follows from the fact that $\mathscr{F}$ is a semi-functional space on Ω and the fact that $\mathscr{F}_K$ carries the subspace topology. □

So this is what we call a 'construction'. It is basically a recipe for transforming (semi-)functional spaces. However, the construction $\mathscr{F} \mapsto \mathscr{F}_K$, with $K \in \mathscr{P}_c(\Omega)$, is a little odd: it does not send functional spaces to functional spaces (after all, $\mathscr{D}(\Omega)$ can never be a subset of some $\mathscr{F}_K$), something that all other constructions that we are going to encounter will do.

Example 2.3.3. Let $K \in \mathscr{P}_c(\Omega)$. If we look back at the way in which we defined $\mathscr{E}_K(\Omega)$ from $\mathscr{E}(\Omega)$, we look at a blueprint of the construction that we have just introduced, so $\mathscr{E}_K(\Omega) = (\mathscr{E}(\Omega))_K$. ⊘

The equality sign in the previous example really stands for equality as topological vector spaces. The convention that an equality sign with topological vector spaces on both sides always has this interpretation (unless explicitly stated otherwise) was already announced in 'Notation and conventions', but we recall it here because it will be frequently used from now on.

Proposition 2.3.4. *For every semi-functional space $\mathscr{F}$ on Ω and $K \in \mathscr{P}_c(\Omega)$, $\mathscr{F}_K$ is closed in $\mathscr{F}$.*

Proof: Let $\{u_i\}_{i \in I}$ be a net in $\mathscr{F}_K$ and $u \in \mathscr{F}$ such that $u_i \to u$ in $\mathscr{F}$. Since $\mathscr{F} \subseteq_c \mathscr{D}'(\Omega)$, we get that $u_i \to u$ in $\mathscr{D}'(\Omega)$ as well and application of Lemma 1.4.20 subsequently shows that $\operatorname{supp}(u) \subseteq K$. □

Since $\mathscr{E}_K(\Omega) = (\mathscr{E}(\Omega))_K$ for every $K \in \mathscr{P}_c(\Omega)$, we see that the previous proposition is actually a generalization of Lemma 1.1.3.

Lemma 2.3.5. *If $\mathscr{F}$ and $\mathscr{G}$ are semi-functional spaces on Ω and $T \colon \mathscr{F} \to \mathscr{G}$ is a local continuous linear map, then T restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{G}_K$ for every $K \in \mathscr{P}_c(\Omega)$.*

Proof: Because T is local, $\operatorname{supp}(u) \subseteq K$ implies $\operatorname{supp}(Tu) \subseteq K$, so T maps $\mathscr{F}_K$ into $\mathscr{G}_K$ and because $\mathscr{F}_K$ and $\mathscr{G}_K$ carry the subspace topology, the lemma follows. □

As promised, the construction $\mathscr{F} \mapsto \mathscr{F}_K$ is going to be a functor. Because it always produces semi-functional spaces that are not a functional space, we should regard it as a functor from the category of semi-functional spaces on Ω to the category of semi-functional spaces on Ω. The previous lemma tells us that the natural restriction that the assignment $\mathscr{F} \mapsto \mathscr{F}_K$ puts on the class of arrows for the category of semi-functional spaces on Ω is that the arrows need to be *local* continuous linear maps. However, we have already decided to take only continuous inclusions as arrows, so this restriction is certainly met.

Proposition 2.3.6. *Let $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. The assignment $\mathscr{F} \mapsto \mathscr{F}_K$ is a functor from the category of semi-functional spaces on Ω to the category of semi-functional spaces on Ω.*

Proof: This follows straight from the previous lemma. After all, if $\mathscr{F}$ and $\mathscr{G}$ are semi-functional spaces on Ω such that $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$, then the inclusion map $\mathscr{F} \hookrightarrow \mathscr{G}$ is a local continuous linear map and application of the previous lemma gives $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{G}_K$. □

Note that there is something typical about the statement 'the assignment $\mathscr{F} \mapsto \mathscr{F}_K$ is a functor from ...'. Indeed, we only specify what it does on objects, so what happens to the arrows? Well, because we take only continuous inclusions as arrows in the category of (semi-)functional spaces on Ω, there is no choice: two semi-functional spaces $\mathscr{F}$ and $\mathscr{G}$ on Ω either satisfy $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$ or not, so there is at most *one* arrow from $\mathscr{F}$ to $\mathscr{G}$. Therefore the statement 'the assignment $\mathscr{F} \mapsto \mathscr{F}_K$ is a functor from ...' can only have one meaning: if $\mathscr{F}$ and $\mathscr{G}$ are semi-functional spaces on Ω such that $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$, then $\mathscr{F}_K$ and $\mathscr{G}_K$ are semi-functional spaces on Ω such that $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{G}_K$.

Lemma 2.3.7. *Let $\mathscr{F}$ be a semi-functional space on Ω and $K, K' \in \mathscr{P}_{\mathrm{c}}(\Omega)$. Then $(\mathscr{F}_K)_{K'} = \mathscr{F}_{K \cap K'} = (\mathscr{F}_{K'})_K$.*

Proof: By symmetry, it suffices to prove $(\mathscr{F}_K)_{K'} = \mathscr{F}_{K \cap K'}$. Equality as sets is trivial and observing that both spaces carry the restricted topology from $\mathscr{F}$ completes the argument. □

This lemma is a first example of a result about the 'interaction' of constructions. After all, it investigates how the constructions $\mathscr{F} \mapsto \mathscr{F}_K$ and $\mathscr{F} \mapsto \mathscr{F}_{K'}$ 'behave' when we apply them successively. The result of this investigation is that the functors $\mathscr{F} \mapsto \mathscr{F}_K$ and $\mathscr{F} \mapsto \mathscr{F}_{K'}$ commute and that their composition has a nice form. We will see that actually quite a few of the construction functors that we are going to encounter commute with each other and that this commutativity plays an important role in the 'preservation' of 'properties'.

Lemma 2.3.8. *Let $\mathscr{F}$ be a semi-functional space on Ω, $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and let $\{\eta_i\}_{i \in I}$ be a finite partition of unity over K by compactly supported smooth functions on Ω. Then*

$$\mathcal{I} \colon \mathscr{F}_K \to \prod_{i \in I} \mathscr{F}_{K_i} : u \mapsto \{\eta_i u\}_{i \in I},$$

with $K_i := \operatorname{supp}(\eta_i) \cap K$, is a linear topological embedding with closed image.

Proof: Since for every $i \in I$ multiplication by η_i restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$ and $\operatorname{supp}(\eta_i u) \subseteq \operatorname{supp}(\eta_i) \cap \operatorname{supp}(u) \subseteq K_i$ for all $u \in \mathscr{F}_K$, we see that $\mathcal{I}$ is a well-defined continuous linear map. Moreover, using the fact that $\operatorname{supp}(u+v) \subseteq \operatorname{supp}(u) \cup \operatorname{supp}(v)$ for all $u, v \in \mathscr{D}'(\Omega)$ and the fact that I is finite, it is follows that $\mathcal{P}\colon \prod_{i\in I} \mathscr{F}_{K_i} \to \mathscr{F}_K : \{u_i\}_{i\in I} \mapsto \sum_{i\in I} u_i$ is a well-defined continuous linear map as well (note that $\mathcal{P}$ is just a finite sum of continuous linear projection maps). Since clearly $\mathcal{P} \circ \mathcal{I} = \mathrm{id}_{\mathscr{F}_K}$, application of Lemma A.1.8 gives the desired result. □

2.4 Properties

In most mathematical theories there are many 'properties' that an instance of the main concept of the theory (e.g., a group, field, topological space, manifold, etc.) might or might not have. The theory of (semi-)functional spaces is no exception and in this section we introduce quite a few of such properties.

Because a semi-functional space is in particular a locally convex vector space, there already is a huge class of properties available. In our context it suffices to focus on a limited number of these 'inherited' properties and for each of them we have an associated *local* version.

Definition 2.4.1. Let $\mathscr{F}$ be a semi-functional space on Ω. We say that:

1. $\mathscr{F}$ is *metrizable* if $\mathscr{F}$ is metrizable as a locally convex vector space and

2. $\mathscr{F}$ is *locally metrizable* if, for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, $\mathscr{F}_K$ is metrizable.

Similarly, we talk about *(locally) normable*, *(locally) complete*, *(locally) Fréchet*, *(locally) Banach* and *(locally) Hilbert.* ⊘

It should be noted that the word 'locally' has a different meaning here than in topology and functional analysis where a space has some property locally if around every point there exists a neighborhood that satisfies this property. Indeed, usually $\mathscr{F}_K$, with $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, is a *proper* linear subspace of $\mathscr{F}$ and therefore has empty interior (see Proposition A.2.4) and there also might be elements of $\mathscr{F}$ that do not belong to any $\mathscr{F}_K$.

We should also mention that we say that a locally convex vector space is Banach if the space is complete and normable and that we say that a locally convex vector space is Hilbert if the space is complete and its topology is induced by some inner product. In other words, we do not assume that there is a designated norm or inner product on a Banach, respectively Hilbert, space; we only look at the topology. (As usual, a locally convex vector space is called Fréchet if it is metrizable and complete.)

Proposition 2.4.2. *Let $\mathscr{F}$ be a semi-functional space on Ω and let P be short for: metrizable, normable, complete, Fréchet, Banach or Hilbert. Then $\mathscr{F}$ is locally P if and only if for every $x \in \Omega$ there exists a compact neighborhood K_x of x in Ω such that $\mathscr{F}_{K_x}$ is P.*

Proof: The direct implication is a trival consequence of the fact that Ω is locally compact. For the converse implication, note that as a consequence of the assumption, we can find a collection $\{K_i\}_{i\in I}$ of compact subsets of Ω such that

$\mathscr{F}_{K_i}$ is P for every $i \in I$ and such that $\{\text{int}(K_i)\}_{i\in I}$ is an open cover of Ω. Now let K be an arbitrary compact subset of Ω and let $\{\eta_i\}_{i\in I}$ be a smooth partition of unity subordinate to $\{\text{int}(K_i)\}_{i\in I}$ (note that since $\text{supp}(\eta_i) \subseteq K_i$ and closed subsets of compact subsets are compact, the η_i are compactly supported). Because $\{\text{supp}(\eta_i)\}_{i\in I}$ is locally finite, we find a finite subset I_K of I such that $\text{supp}(\eta_i) \cap K \neq \emptyset$ if and only if $i \in I_K$. Then $\{\eta_i\}_{i\in I_K}$ is a finite partition of unity over K by compactly supported smooth functions on Ω, so on behalf of Lemma 2.3.8, $\mathscr{F}_K$ is linearly topologically isomorphic to a closed subspace of $\prod_{i\in I_K} \mathscr{F}_{K_i'}$, with $K_i' := \text{supp}(\eta_i) \cap K$. But, as a consequence of Proposition 2.3.4, $\mathscr{F}_{K_i'}$ is a closed subspace of $\mathscr{F}_{K_i}$ for every $i \in I_K$ and since the properties that P can resemble are all inherited by closed subspaces and preserved under finite products, it follows that $\mathscr{F}_K$ is P. □

The previous proposition shows that, despite the difference with topology and functional analysis, the terminology 'locally' is actually very appropriate here. The idea behind both usages of the word 'locally' is also pretty much the same, only instead of saying something about the existence of neighborhoods around points of $\mathscr{F}$, here the word 'locally' says something about the existence of neighborhoods around points of Ω.

Example 2.4.3. Let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion by compacts of Ω. Then we find for every $K \in \mathscr{P}_c(\Omega)$ an $i \in \mathbb{N}$ such that $K \subseteq K_i$ (see Remark 1.1.13), so by Corollay A.1.6, $\{\|\cdot\|_{K_i,0} \mid i \in \mathbb{N}\}$ is an inducing collection of seminorms for $\mathscr{C}(\Omega)$. Since this collection is countable, $\mathscr{C}(\Omega)$ is pseudometrizable and because we already know that $\mathscr{C}(\Omega)$ is Hausdorff (after all, it is a functional space on Ω), we see that $\mathscr{C}(\Omega)$ is metrizable. Furthermore, it is a well-known fact that $\mathscr{C}(\Omega)$ is complete (the idea is simple: if $\{\varphi_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathscr{C}(\Omega)$, then $\{\varphi_i(x)\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$ for every $x \in \Omega$ and by completeness of $\mathbb{K}$ we get an obvious candidate for the limit), so $\mathscr{C}(\Omega)$ is actually Fréchet. ⊘

Example 2.4.4. By definition $\mathscr{C}_b(\Omega)$ is normable and with an argument similar to the one for $\mathscr{C}(\Omega)$ we can prove that $\mathscr{C}_b(\Omega)$ is complete (see, e.g., [2, page 65]), so $\mathscr{C}_b(\Omega)$ is Banach.

Claim. $\mathscr{C}_s(\Omega)$ is a *closed* subspace of $\mathscr{C}_b(\Omega)$.

Proof: Let $\{\varphi_i\}_{i\in\mathbb{N}}$ be a sequence in $\mathscr{C}_s(\Omega)$ and $\varphi \in \mathscr{C}_b(\Omega)$ such that $\varphi_i \to \varphi$ in $\mathscr{C}_b(\Omega)$. We should show that φ 'becomes constant at infinity'. To this end, let for every $i \in \mathbb{N}$, $c_i \in \mathbb{K}$ be the 'limit at infinity' of φ_i. We are first going to prove that $\{c_i\}_{i\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{K}$.

So let $\varepsilon > 0$. Since $\{\varphi_i\}_{i\in\mathbb{N}}$ converges in $\mathscr{C}_b(\Omega)$, it is in particular a Cauchy sequence, hence we find an $N \in \mathbb{N}$ such that $\|\varphi_i - \varphi_j\|_\infty < \frac{\varepsilon}{3}$ for all $i, j \geq N$. Now fix $i, j \geq N$. By definition of c_i and c_j, we find $K_i, K_j \in \mathscr{P}_c(\Omega)$ such that

$$|c_i - \varphi_i(x)| < \frac{\varepsilon}{3} \quad \text{and} \quad |c_j - \varphi_j(x)| < \frac{\varepsilon}{3}$$

for all $x \in \Omega \setminus (K_i \cup K_j)$. Take any such x (since Ω is not compact, there always exists one), then

$$\begin{aligned} |c_i - c_j| &\leq |c_i - \varphi_i(x)| + |c_j - \varphi_j(x)| + |\varphi_i(x) - \varphi_j(x)| \\ &\leq |c_i - \varphi_i(x)| + |c_j - \varphi_j(x)| + \|\varphi_i - \varphi_j\|_\infty < \varepsilon. \end{aligned}$$

Thus $\{c_i\}_{i\in\mathbb{N}}$ is indeed a Cauchy sequence in $\mathbb{K}$ and by completeness of $\mathbb{K}$ we find an $c \in \mathbb{K}$ such that $c_i \to c$ in $\mathbb{K}$.

Of course, the next step is to show that c is the 'limit at infinity' of φ. So again let $\varepsilon > 0$. Because $\varphi_i \to \varphi$ in $\mathscr{C}_b(\Omega)$ and $c_i \to c$ in $\mathbb{K}$, we find an $N \in \mathbb{N}$ such that $\|\varphi - \varphi_i\|_\infty < \frac{\varepsilon}{3}$ and $|c - c_i| < \frac{\varepsilon}{3}$ for all $i \geq N$. As a consequence, if $x \in \Omega$ such that $|c_N - \varphi_N(x)| < \frac{\varepsilon}{3}$, then

$$\begin{aligned}|c - \varphi(x)| &\leq |c - c_N| + |c_N - \varphi_N(x)| + |\varphi(x) - \varphi_N(x)| \\ &\leq |c - c_N| + |c_N - \varphi_N(x)| + \|\varphi - \varphi_N\|_\infty < \varepsilon.\end{aligned}$$

In other words, $\{x \in \Omega \mid |c - \varphi(x)| \geq \varepsilon\} \subseteq \{x \in \Omega \mid |c_N - \varphi_N(x)| \geq \frac{\varepsilon}{3}\}$ and since closed subsets of compacts are compact, this finishes the proof. □

Note that this argument also shows that $\mathscr{C}_0(\Omega)$ is a closed subspace of $\mathscr{C}_b(\Omega)$ (and hence of $\mathscr{C}_s(\Omega)$): if all the c_i in the proof above equal 0, then also c equals 0. Since closed subspaces of a Banach space are again Banach, we conclude that $\mathscr{C}_0(\Omega)$ and $\mathscr{C}_s(\Omega)$ are Banach as well. ⊘

Example 2.4.5. It is well-known that $L^p(\Omega)$ is Banach for every $1 \leq p \leq \infty$ and that $L^2(\Omega)$ is Hilbert. The nontrivial ingredient is the completeness of $L^p(\Omega)$, which is often referred to as the Riesz-Fischer Theorem (see, e.g., [13, Theorem 11.2 and Theorem 11.4]). ⊘

In addition to the properties that we have as consequence of the fact that (semi-)functional spaces are locally convex vector spaces, we also have properties that are more intrinsic to the specific context we are working in.

Definition 2.4.6. Let $\mathscr{F}$ be a functional space on Ω. We say that:

1. $\mathscr{F}$ is *invariant* if for any diffeomorphism $\chi\colon \Omega \to \Omega$, $\chi_*\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a linear topological isomorphism from $\mathscr{F}$ onto $\mathscr{F}$ and

2. $\mathscr{F}$ is *locally invariant* if for any diffeomorphism $\chi\colon \Omega \to \Omega$ and any compact $K \subseteq \Omega$, $\chi_*\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a linear topological isomorphism from $\mathscr{F}_K$ onto $\mathscr{F}_{\chi(K)}$. ⊘

Definition 2.4.7. Let $\mathscr{F}$ be a functional space on Ω. We say that:

1. $\mathscr{F}$ is *normal* if $\mathscr{D}(\Omega)$ is dense in $\mathscr{F}$ and

2. $\mathscr{F}$ is *locally normal* if, for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, $\mathscr{F}_K$ is contained in the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}$. ⊘

There are two things that should be noted here. First, using the fact that every diffeomorphism $\chi\colon \Omega \to \Omega$ has an inverse $\chi^{-1}\colon \Omega \to \Omega$ which is also a diffeomorphism and the fact that $(\chi_*)^{-1} = (\chi^{-1})_*$, we see that in the definition of both invariance and locally invariance we can safely replace 'restricts to a linear topological isomorphism' by 'restricts to a continuous linear map'. And second, it is important to be aware of the fact that we restrict ourselves to 'true' functional spaces (as opposed to semi-functional spaces) for the concepts of normality and invariance.

Example 2.4.8. It is evident that $\mathscr{D}(\Omega)$ is normal and by Proposition 1.1.15 and Lemma 1.3.3 also $\mathscr{E}(\Omega)$, $\mathscr{E}'(\Omega)$ and $\mathscr{D}'(\Omega)$ are normal. Moreover, according to [13, Corollary 1 and Corollary 3 on page 159], $L^p(\Omega)$ is normal for $1 \leq p < \infty$ and $\mathscr{C}(\Omega)$ is normal. That we do not allow $p = \infty$ in the previous sentence has a good reason: $L^\infty(\Omega)$ is not normal. The assignment $\varphi \mapsto [\varphi]$ canonically *embeds* $\mathscr{C}_b(\Omega)$ into $L^\infty(\Omega)$ and because the uniform limit of continuous functions is continuous and not every element of $L^\infty(\Omega)$ is (the equivalence class of) a continuous function, we see that $\mathscr{C}_b(\Omega)$ is identified with a closed and proper subspace of $L^\infty(\Omega)$. Since $\mathscr{D}(\Omega) \subseteq \mathscr{C}_b(\Omega)$, this shows that $L^\infty(\Omega)$ cannot be normal. ⊘

Since $\mathscr{D}(\Omega) = \cup_K \mathscr{E}_K(\Omega)$, it is tempting to think that for a normal functional space $\mathscr{F}$ on Ω, $\mathscr{E}_K(\Omega)$ is dense in $\mathscr{F}_K$ for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. However, this is *not* the case. For example, $\mathscr{D}'(\mathbb{R}^{\mathrm{n}})$ is normal, but $\mathscr{E}_{\{0\}}(\mathbb{R}^{\mathrm{n}}) = \emptyset$ is certainly not dense in the nonempty $(\mathscr{D}'(\mathbb{R}^{\mathrm{n}}))_{\{0\}}$ (which contains for example the famous delta distribution). This also makes clear why it is not a good idea to define normality for a semi-functional space $\mathscr{G}$ on Ω by requiring that $\mathscr{D}(\Omega) \cap \mathscr{G}$ is dense in $\mathscr{G}$. After all, if a concept of normality for semi-functional spaces is available, we would expect to have that a functional space $\mathscr{F}$ on Ω is locally normal if and only if $\mathscr{F}_K$ is normal for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. But $\mathscr{D}'(\mathbb{R}^{\mathrm{n}})$ is clearly locally normal and we have just seen that $\mathscr{D}(\mathbb{R}^{\mathrm{n}}) \cap (\mathscr{D}'(\mathbb{R}^{\mathrm{n}}))_{\{0\}} = \mathscr{E}_{\{0\}}(\mathbb{R}^{\mathrm{n}})$ is not dense in $(\mathscr{D}'(\mathbb{R}^{\mathrm{n}}))_{\{0\}}$. For the same reason we do not talk about invariance of semi-functional spaces.

Example 2.4.9. Of course $\mathscr{D}'(\Omega)$ is invariant and from the material in Section 1.7 of the previous chapter we easily deduce that $\mathscr{D}(\Omega)$, $\mathscr{E}(\Omega)$ and $\mathscr{E}'(\Omega)$ are invariant as well. Indeed, if $\chi\colon \Omega \to \Omega$ is a diffeomorphism, then χ_* equals $m_{|\det D\chi^{-1}|} \circ (\chi^{-1})^*$ on $\mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega)$, which is a continuous linear map from $\mathscr{E}(\Omega)$ into $\mathscr{E}(\Omega)$ and from $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$, and χ_* equals the adjoint of the continuous linear map $\chi^*\colon \mathscr{E}(\Omega) \to \mathscr{E}(\Omega)$ on $\mathscr{E}'(\Omega)$. ⊘

Example 2.4.10. $L^1(\Omega)$ is invariant. To see this, let $\chi\colon \Omega \to \Omega$ be a diffeomorphism. By the change of variables theorem, we have for every $[f] \in L^1(\Omega)$

$$\int_\Omega \left| \frac{f}{|\det D\chi|} \circ \chi^{-1} \right| d\lambda = \int_\Omega |f \circ \chi^{-1}| |\det D\chi^{-1}| \, d\lambda = \int_\Omega |f| \, d\lambda.$$

In view of Lemma 1.7.4, this implies that χ_* maps $L^1(\Omega)$ into $L^1(\Omega)$ and it subsequently implies that for every $[f] \in L^1(\Omega)$

$$\|\chi_*[f]\|_1 = \int_\Omega \left| \frac{f}{|\det D\chi|} \circ \chi^{-1} \right| d\lambda = \int_\Omega |f| \, d\lambda = \|[f]\|_1,$$

which shows that χ_* is continuous as map from $L^1(\Omega)$ into $L^1(\Omega)$. ⊘

Example 2.4.11. It is not difficult to see that $L^2(\Omega)$ is not invariant in general. (Take, e.g., $\Omega = (0,1)$, $\chi\colon (0,1) \to (0,1)\colon x \mapsto x^2$ and $f\colon (0,1) \to \mathbb{K}\colon x \mapsto x^{-\frac{1}{3}}$. Then $[f] \in L^2((0,1))$, while $\chi_*[f] \notin L^2((0,1))$ because $\chi_* f = \frac{1}{2} x^{-\frac{2}{3}}$ is not square integrable on $(0,1)$.) However, we do have for every $1 \leq p \leq \infty$ that $L^p(\Omega)$ is locally invariant.

To verify this, let $\chi\colon \Omega \to \Omega$ be a diffeomorphism and let $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. We should check that χ_* restricts to a continuous linear map from $(L^p(\Omega))_K$

into $(L^p(\Omega))_{\chi(K)}$, but because χ_* always maps distributions with support in K to distributions with support in $\chi(K)$ (see Lemma 1.7.3), this boils down to checking that χ_* restricts to a continuous linear map from $(L^p(\Omega))_K$ into $L^p(\Omega)$. For $1 \leq p < \infty$ this follows from the observation that for every $[f] \in (L^p(\Omega))_K$

$$\begin{aligned}\|\chi_*[f]\|_p^p &= \int_\Omega \frac{|f \circ \chi^{-1}|^p}{|(\det D\chi) \circ \chi^{-1}|^{p-1}} |\det D\chi^{-1}|\, d\lambda = \int_\Omega \frac{|f|^p}{|\det D\chi|^{p-1}}\, d\lambda \\ &= \int_K |\det D\chi|^{1-p} |f|^p\, d\lambda \leq \|\det D\chi\|_{K,0}^{1-p} \|[f]\|_p^p,\end{aligned}$$

while for $p = \infty$ it follows from the observation that for $[f] \in (L^p(\Omega))_K$

$$\begin{aligned}\|\chi_*[f]\|_\infty &= \|[(|\det D\chi|^{-1} f) \circ \chi^{-1}]\|_\infty \\ &= \|[|\det D\chi|^{-1} f]\|_\infty \leq \|\det D\chi\|_{K,0}^{-1} \|[f]\|_\infty.\end{aligned}$$ ⊘

We end this section with an easy result that was already suggested by the terminology. At various points later in the chapter more properties will be introduced.

Lemma 2.4.12. *Let $\mathscr{F}$ be a (semi-)functional space on Ω and let P be short for: metrizable, normable, complete, Fréchet, Banach, Hilbert, invariant or normal. Then $\mathscr{F}$ is P implies $\mathscr{F}$ is locally P.*

Proof: For the properties that are familiar from the context of locally convex vector spaces (i.e., the first six properties from the list), this follows because $\mathscr{F}_K$ is a closed subspace of $\mathscr{F}$ for every $K \in \mathscr{P}_c(\Omega)$ and these properties are automatically 'inherited' by closed subspaces (note that $\mathscr{F}_K$ carries the subspace topology). Furthermore, for normality the statement is clear and for invariance it is a direct consequence of Lemma 1.7.3. □

Remark 2.4.13. In the previous lemma it is assumed to be understood that if P is short for either invariant or normal, $\mathscr{F}$ is assumed to be a functional space on Ω rather than just a semi-functional space. ⊘

2.5 Compact support

Each of the remaining sections of this chapter revolves around its own construction and the interaction of this construction with the previously defined constructions and properties. Just as the construction $\mathscr{F} \mapsto \mathscr{F}_K$ is a generalization of the way in which we obtain $\mathscr{E}_K(\Omega)$ from $\mathscr{E}(\Omega)$, the construction that we introduce in this section is a generalization of the way in which we obtain $\mathscr{D}(\Omega)$ from $\mathscr{E}(\Omega)$.

Definition 2.5.1. Let $\mathscr{F}$ be a semi-functional space on Ω. We define $\mathscr{F}_{\text{comp}}$ to be

$$\bigcup_{K \in \mathscr{P}_c(\Omega)} \mathscr{F}_K$$

endowed with the inductive limit topology. ⊘

Proposition 2.5.2. *$\mathscr{F}_{\text{comp}}$ is a semi-functional space on Ω and for every compact subset K of Ω, $\mathscr{F}_K \subseteq_c \mathscr{F}_{\text{comp}} \subseteq_c \mathscr{F}$. If $\mathscr{F}$ is a functional space on Ω, then so is $\mathscr{F}_{\text{comp}}$.*

Proof: It trivially follows from Lemma 1.4.12 that $\mathscr{F}_{\mathrm{comp}}$ is a linear subspace of $\mathscr{F}$, hence of $\mathscr{D}'(\Omega)$. On behalf of Proposition A.3.2, in order to check that the inclusion $\mathscr{F}_{\mathrm{comp}} \subseteq \mathscr{F}$ is continuous, it suffices to check that $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{F}$ for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. But we already know that this is the case (see Proposition 2.3.2). Hence $\mathscr{F}_{\mathrm{comp}} \subseteq_{\mathrm{c}} \mathscr{F} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$. That $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{comp}}$ for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ is a direct consequence of the definition of the inductive limit topology.

Let $\varphi \in \mathscr{D}(\Omega)$. Because of Proposition 2.3.2, we already know that m_φ restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{comp}}$ for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. Combining this with Proposition A.3.2 gives that m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{comp}}$ into $\mathscr{F}_{\mathrm{comp}}$. So $\mathscr{F}_{\mathrm{comp}}$ is indeed a semi-functional space on Ω.

If $\mathscr{F}$ is a functional space on Ω, we can say more. From $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}$ and the fact that every element of $\mathscr{D}(\Omega)$ has compact support, we get $\mathscr{D}(\Omega) \subseteq \mathscr{F}_{\mathrm{comp}}$. To check that this inclusion is continuous, it suffices to check that $\mathscr{E}_K(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{comp}}$ for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. But this is easy: from $\mathscr{E}_K(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}$ and the fact that $\mathscr{E}_K(\Omega) \subseteq \mathscr{F}_K$ we get $\mathscr{E}_K(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}_K$ and combining this with $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{comp}}$ gives the desired result. Hence $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{comp}}$ and we conclude that $\mathscr{F}_{\mathrm{comp}}$ is a functional space on Ω. □

Example 2.5.3. As announced, we clearly have that $\mathscr{D}(\Omega) = (\mathscr{E}(\Omega))_{\mathrm{comp}}$. ⊘

Example 2.5.4. We easily deduce that for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$

$$(\mathscr{C}_0(\Omega))_K = (\mathscr{C}_s(\Omega))_K = (\mathscr{C}_b(\Omega))_K = (\mathscr{C}(\Omega))_K,$$

hence

$$(\mathscr{C}_0(\Omega))_{\mathrm{comp}} = (\mathscr{C}_s(\Omega))_{\mathrm{comp}} = (\mathscr{C}_b(\Omega))_{\mathrm{comp}} = (\mathscr{C}(\Omega))_{\mathrm{comp}}.$$ ⊘

In the definition of a semi-functional space on Ω, we required that for such a semi-functional space $\mathscr{F}$ the continuous linear maps $m_\varphi \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ with $\varphi \in \mathscr{D}(\Omega)$ restrict to continuous linear maps from $\mathscr{F}$ into $\mathscr{F}$. Without any additional requirements, the situation turns out to be even better.

Lemma 2.5.5. *Let $\mathscr{F}$ be a semi-functional space on Ω. For every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\mathrm{comp}}$.*

Proof: Let $\varphi \in \mathscr{D}(\Omega)$ and let K be the support of φ. Since $\mathscr{F}$ is a semi-functional space, we already know that m_φ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$ and since $\operatorname{supp}(\varphi u) \subseteq \operatorname{supp}(\varphi) \cap \operatorname{supp}(u) \subseteq K$ for every $u \in \mathscr{F}$, we find that m_φ actually restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_K$. Using $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{comp}}$ completes the proof. □

Because $\mathscr{F}_{\mathrm{comp}}$ by definition carries the inductive limit topology, we already know that the topology of $\mathscr{F}_{\mathrm{comp}}$ can be characterized as the largest locally convex topology such that $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{comp}}$ for all $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. It is however possible to characterize the topology of $\mathscr{F}_{\mathrm{comp}}$ in a way that is more intrinsic to the specific setting of (semi-)functional spaces. The key to this is the following lemma, which is inspired by the just proven fact that for $\varphi \in \mathscr{D}(\Omega)$, m_φ can be regarded as a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\mathrm{comp}}$.

Lemma 2.5.6. *Let $\mathscr{F}$ be a semi-functional space on Ω, $\mathscr{Y}$ a locally convex vector space and $T\colon \mathscr{F}_{\mathrm{comp}} \to \mathscr{Y}$ a linear map. Then T is continuous if and only if for every $\varphi \in \mathscr{D}(\Omega)$*

$$T \circ m_\varphi \colon \mathscr{F} \to \mathscr{Y}$$

is continuous.

Proof: Since for every $\varphi \in \mathscr{D}(\Omega)$, m_φ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\mathrm{comp}}$, the direct implication is clear.

Now suppose that $T \circ m_\varphi \colon \mathscr{F} \to \mathscr{Y}$ is continuous for every $\varphi \in \mathscr{D}(\Omega)$. In order to prove that $T\colon \mathscr{F}_{\mathrm{comp}} \to \mathscr{Y}$ is continuous, it suffices to prove that $T|_{\mathscr{F}_K}$ is continuous for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ (see Proposition A.3.2). So fix $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. According to Remark 1.1.12, we find an $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of K. By assumption, we then have that $T \circ m_\varphi \colon \mathscr{F} \to \mathscr{Y}$ is continuous and hence that $(T \circ m_\varphi)|_{\mathscr{F}_K} \colon \mathscr{F}_K \to \mathscr{Y}$ is continuous. But by Lemma 1.5.6, $m_\varphi u = u$ for every $u \in \mathscr{F}_K$. Hence $(T \circ m_\varphi)|_{\mathscr{F}_K} = T|_{\mathscr{F}_K}$, which proves that $T|_{\mathscr{F}_K}$ is continuous. □

From this lemma we see that we can characterize the topology of $\mathscr{F}_{\mathrm{comp}}$ as the largest locally convex topology such that for every $\varphi \in \mathscr{D}(\Omega)$, m_φ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\mathrm{comp}}$. Indeed, if $\mathscr{Y} = \mathscr{F}_{\mathrm{comp}}$ as set and $\mathscr{Y}$ carries a locally convex topology such that m_φ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{Y}$ for all $\varphi \in \mathscr{D}(\Omega)$, we can apply the above lemma to $T = \mathrm{id}_{\mathscr{F}_{\mathrm{comp}}}$ to obtain that $\mathrm{id}_{\mathscr{F}_{\mathrm{comp}}} \colon \mathscr{F}_{\mathrm{comp}} \to \mathscr{Y}$ is continuous and this precisely means that the topology of $\mathscr{F}_{\mathrm{comp}}$ is larger than the topology of $\mathscr{Y}$.

The following lemma and proposition are very similar to Lemma 2.3.5 and Proposition 2.3.6. Nevertheless, there is also an important difference: in constrast to $\mathscr{F} \mapsto \mathscr{F}_K$, the construction $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{comp}}$ sends functional spaces to functional spaces. We express this in the proposition below by using our 'parentheses convention' (see 'Notation and conventions').

Lemma 2.5.7. *If $\mathscr{F}$ and $\mathscr{G}$ are semi-functional spaces on Ω and $T\colon \mathscr{F} \to \mathscr{G}$ is a local continuous linear map, then T restricts to a continuous linear map from $\mathscr{F}_{\mathrm{comp}}$ into $\mathscr{G}_{\mathrm{comp}}$.*

Proof: We already know that, for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, T restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{G}_K$ (see Lemma 2.3.5). Combining this with the fact that $\mathscr{G}_K \subseteq_{\mathrm{c}} \mathscr{G}_{\mathrm{comp}}$ shows that T restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{G}_{\mathrm{comp}}$. As a consequence, T restricts to a linear map from $\mathscr{F}_{\mathrm{comp}}$ into $\mathscr{G}_{\mathrm{comp}}$ and because of Proposition A.3.2 this restriction is continuous. □

Proposition 2.5.8. *The assignment $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{comp}}$ is a functor from the category of (semi-)functional spaces on Ω to the category of (semi-)functional spaces on Ω.*

Proof: This follows straight from the previous lemma. After all, if $\mathscr{F}$ and $\mathscr{G}$ are (semi-)functional spaces on Ω such that $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$, then the inclusion map $\mathscr{F} \hookrightarrow \mathscr{G}$ is a local continuous linear map and application of the previous lemma gives $\mathscr{F}_{\mathrm{comp}} \subseteq_{\mathrm{c}} \mathscr{G}_{\mathrm{comp}}$. □

The 'interaction' between the two construction functors that we have seen so far is simple.

Proposition 2.5.9. *Let $\mathscr{F}$ be a semi-functional space on Ω and $K \in \mathscr{P}_c(\Omega)$. Then $(\mathscr{F}_{\mathrm{comp}})_K = \mathscr{F}_K = (\mathscr{F}_K)_{\mathrm{comp}}$.*

Proof: Since the assignment $\mathscr{F} \mapsto \mathscr{F}_K$ is a functor, $\mathscr{F}_{\mathrm{comp}} \subseteq_c \mathscr{F}$ implies $(\mathscr{F}_{\mathrm{comp}})_K \subseteq_c \mathscr{F}_K$, while $\mathscr{F}_K \subseteq_c \mathscr{F}_{\mathrm{comp}}$ together with Lemma 2.3.7 implies

$$\mathscr{F}_K = \mathscr{F}_{K\cap K} = (\mathscr{F}_K)_K \subseteq_c (\mathscr{F}_{\mathrm{comp}})_K.$$

This proves the first equality. The second equality is also readily proven: because $\mathscr{G}_K \subseteq_c \mathscr{G}_{\mathrm{comp}}$ for every semi-functional space $\mathscr{G}$ on Ω, we get (by taking $\mathscr{G} = \mathscr{F}_K$)

$$\mathscr{F}_K = \mathscr{F}_{K\cap K} = (\mathscr{F}_K)_K \subseteq_c (\mathscr{F}_K)_{\mathrm{comp}}$$

and the fact that $(\mathscr{F}_K)_{K'} = \mathscr{F}_{K\cap K'} \subseteq_c \mathscr{F}_K$ for every $K' \in \mathscr{P}_c(\Omega)$ implies $(\mathscr{F}_K)_{\mathrm{comp}} \subseteq_c \mathscr{F}_K$ (use Proposition A.3.2). □

Since we will see quite a few constructions, we are certainly not going to discuss all possible 'interactions' between construction functors. However, at this point the number of interactions is still limited to three and we get the only interaction that we have not yet discussed without any effort.

Corollary 2.5.10. *For every semi-functional space $\mathscr{F}$ on Ω*

$$(\mathscr{F}_{\mathrm{comp}})_{\mathrm{comp}} = \mathscr{F}_{\mathrm{comp}}.$$

Proof: This follows immediately from the just proven fact that $(\mathscr{F}_{\mathrm{comp}})_K = \mathscr{F}_K$ for every $K \in \mathscr{P}_c(\Omega)$. □

Example 2.5.11. On behalf of the previous corollary, we have

$$(\mathscr{D}(\Omega))_{\mathrm{comp}} = ((\mathscr{E}(\Omega))_{\mathrm{comp}})_{\mathrm{comp}} = (\mathscr{E}(\Omega))_{\mathrm{comp}} = \mathscr{D}(\Omega). \qquad \oslash$$

Also the next result seems to be a direct consequence of Proposition 2.5.9 at first glance, but we should be careful. While most of the local versions of the properties for a (semi-)functional space $\mathscr{F}$ that we have introduced in the previous section only depend on the spaces $\mathscr{F}_K$ with $K \in \mathscr{P}_c(\Omega)$, local normality also depends on $\mathscr{F}$ itself (after all, it depends on the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}$).

Lemma 2.5.12. *Let $\mathscr{F}$ be a (semi-)functional space on Ω and let P be short for: metrizable, normable, complete, Fréchet, Banach, Hilbert, invariant or normal. Then $\mathscr{F}$ is locally P if and only if $\mathscr{F}_{\mathrm{comp}}$ is locally P.*

Proof: For everything but normality this is a direct consequence of the fact that $(\mathscr{F}_{\mathrm{comp}})_K = \mathscr{F}_K$ for every $K \in \mathscr{P}_c(\Omega)$. If P is short for normal, we can also use this fact, but we should do more.

Suppose that $\mathscr{F}_{\mathrm{comp}}$ is locally normal and fix $K \in \mathscr{P}_c(\Omega)$. If $u \in \mathscr{F}_K$, then $u \in (\mathscr{F}_{\mathrm{comp}})_K = \mathscr{F}_K$, so by local normality of $\mathscr{F}_{\mathrm{comp}}$ we find a net $\{\varphi_i\}_{i\in I}$ in $\mathscr{D}(\Omega)$ such that $\varphi_i \to u$ in $\mathscr{F}_{\mathrm{comp}}$. Since $\mathscr{F}_{\mathrm{comp}} \subseteq_c \mathscr{F}$, we subsequently get that $\varphi_i \to u$ in $\mathscr{F}$. Hence u lies in the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}$ and we conclude that $\mathscr{F}$ is locally normal.

Next, suppose that $\mathscr{F}$ is locally normal and again take $K \in \mathscr{P}_c(\Omega)$. If $u \in (\mathscr{F}_{\mathrm{comp}})_K = \mathscr{F}_K$, the local normality of $\mathscr{F}$ implies that we find a net

$\{\varphi_i\}_{i\in I}$ in $\mathscr{D}(\Omega)$ such that $\varphi_i \to u$ in $\mathscr{F}$. Now let $\psi \in \mathscr{D}(\Omega)$ such that ψ equals 1 on an open neighborhood of K (see Remark 1.1.12). Then m_ψ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\text{comp}}$ (Lemma 2.5.5) and $m_\psi u = u$ (Lemma 1.5.6). Hence $\psi\varphi_i = m_\psi\varphi_i \to m_\psi u = u$ in $\mathscr{F}_{\text{comp}}$ and since $\{\psi\varphi_i\}_{i\in I}$ is also a net in $\mathscr{D}(\Omega)$, this proves that u lies in the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}_{\text{comp}}$ and we conclude that $\mathscr{F}_{\text{comp}}$ is locally normal. □

The direct implication from the previous lemma can be seen as the first example of a result that describes the preservation of properties under a construction functor. After all, it tells us that if $\mathscr{F}$ has the property 'locally P', the result of $\mathscr{F}$ under the construction functor $\mathscr{F} \mapsto \mathscr{F}_{\text{comp}}$ still has the property 'locally P', i.e., the property 'locally P' is preserved. Because we readily see that local normality of $\mathscr{F}_{\text{comp}}$ implies normality of $\mathscr{F}_{\text{comp}}$ (if $\mathscr{F}_{\text{comp}}$ is locally normal, the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}_{\text{comp}}$ contains $\mathscr{F}_K$ for every $K \in \mathscr{P}_{\text{c}}(\Omega)$, hence it also contains $\cup_K \mathscr{F}_K = \mathscr{F}_{\text{comp}}$), we immediately arrive at our second result of this type.

Lemma 2.5.13. *Let $\mathscr{F}$ be a functional space on Ω. If $\mathscr{F}$ is normal, then $\mathscr{F}_{\text{comp}}$ is normal as well.*

Proof: According to Lemma 2.4.12, the assumption that $\mathscr{F}$ is normal implies that $\mathscr{F}$ is locally normal. Lemma 2.5.12 then tells us that $\mathscr{F}_{\text{comp}}$ is locally normal, which in turn implies that $\mathscr{F}_{\text{comp}}$ is normal. □

As with the interaction of construction functors, we do not have the ambition to investigate the preservation of every property under every construction; we restrict our attention to those questions about preservation that arise naturally when developing the theory or when considering examples.

Remark 2.5.14. Let $\mathscr{F}$ be a functional space on Ω. By a slight modification of the steps in Remark 1.1.13 to the more general context of funtional spaces, we deduce that if $\{K_i\}_{i\in\mathbb{N}}$ is an exhaustion by compacts of Ω, $\cup_{i\in\mathbb{N}}\mathscr{F}_{K_i}$ endowed with the inductive limit topology is a *strict* inductive limit which equals $\mathscr{F}_{\text{comp}}$ as locally convex vector space. (For the strictness of the inclusion $\mathscr{F}_{K_i} \subseteq \mathscr{F}_{K_{i+1}}$, we can again use a smooth function with support inside $\text{int}(K_{i+1}) \subseteq K_{i+1}$ that equals 1 on an open neighborhood of K_i because $\mathscr{D}(\Omega) \subseteq \mathscr{F}$.) ⊘

We end this section with a generalization of Lemma 1.1.14.

Lemma 2.5.15. *Let $\mathscr{F}$ be a functional space on Ω. A subset B of $\mathscr{F}_{\text{comp}}$ is bounded if and only if there exists an $K \in \mathscr{P}_{\text{c}}(\Omega)$ such that B is a bounded subset of $\mathscr{F}_K$.*

Proof: Suppose that B is bounded in $\mathscr{F}_{\text{comp}}$ and let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion by compacts of Ω. Using the previous remark and Proposition A.3.4, we find that there must be an $i \in \mathbb{N}$ such that B is a bounded subset of $\mathscr{F}_{K_i}$.

Conversely, if B is a bounded subset of $\mathscr{F}_K$ for some $K \in \mathscr{P}_{\text{c}}(\Omega)$, then $\mathscr{F}_K \subseteq_{\text{c}} \mathscr{F}_{\text{comp}}$ implies that B is bounded in $\mathscr{F}_{\text{comp}}$ because continuous linear maps send bounded sets to bounded sets. □

2.6 Locality

The next construction will play an important role in the fourth chapter where we make the transition to vector bundles over manifolds.

Definition 2.6.1. Let $\mathscr{F}$ be a semi-functional space on Ω and let $\mathscr{P}$ be an inducing collection of seminorms for $\mathscr{F}$. As a set, we define

$$\mathscr{F}_{\mathrm{loc}} := \{u \in \mathscr{D}'(\Omega) \mid \varphi u \in \mathscr{F} \text{ for all } \varphi \in \mathscr{D}(\Omega)\}.$$

Subsequently, we define for each $p \in \mathscr{P}$ and $\varphi \in \mathscr{D}(\Omega)$

$$q_{p,\varphi} \colon \mathscr{F}_{\mathrm{loc}} \to \mathbb{R} \colon u \mapsto p(\varphi u).$$

We easily see that $\mathscr{F}_{\mathrm{loc}}$ is a vector subspace of $\mathscr{D}'(\Omega)$ and that the $q_{p,\varphi}$, with $p \in \mathscr{P}$ and $\varphi \in \mathscr{D}(\Omega)$, are seminorms on $\mathscr{F}_{\mathrm{loc}}$. We endow $\mathscr{F}_{\mathrm{loc}}$ with the topology induced by these seminorms. ⊘

Like $\mathscr{F}_K$ and $\mathscr{F}_{\mathrm{comp}}$, $\mathscr{F}_{\mathrm{loc}}$ is again a semi-functional space on Ω, but this time it is more efficient to discuss some important properties of $\mathscr{F}_{\mathrm{loc}}$ before we prove this.

To begin with, we should note that the topology of $\mathscr{F}_{\mathrm{loc}}$ is independent of the chosen inducing collection of seminorms for $\mathscr{F}$ (an easy consequence of Corollary A.1.4; just observe that $p' \leq C\sum_{i=0}^{n} p_i$ implies $q_{p',\varphi} \leq C\sum_{i=0}^{n} q_{p_i,\varphi}$). Furthermore, it is clear from the definition of $\mathscr{F}_{\mathrm{loc}}$ as vector subspace of $\mathscr{D}'(\Omega)$ that for all $\varphi \in \mathscr{D}(\Omega)$, m_φ restricts to a linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$ and because for every $p \in \mathscr{P}$ and $u \in \mathscr{F}_{\mathrm{loc}}$

$$p(m_\varphi u) = p(\varphi u) = q_{p,\varphi}(u),$$

we even have that $m_\varphi \colon \mathscr{F}_{\mathrm{loc}} \to \mathscr{F}$ is continuous. Apart from a swap of the domain and codomain, this looks very similar to the continuity of the maps $m_\varphi \colon \mathscr{F} \to \mathscr{F}_{\mathrm{comp}}$, with $\varphi \in \mathscr{D}(\Omega)$, that we have seen in the previous section. The resemblance between the following lemma and Lemma 2.5.6 confirms this similarity.

Lemma 2.6.2. *Let $\mathscr{F}$ be a semi-functional space on Ω, $\mathscr{X}$ a locally convex vector space and $T \colon \mathscr{X} \to \mathscr{F}_{\mathrm{loc}}$ a linear map. Then T is continuous if and only if for every $\varphi \in \mathscr{D}(\Omega)$*

$$m_\varphi \circ T \colon \mathscr{X} \to \mathscr{F}$$

is continuous.

Proof: Because for every $\varphi \in \mathscr{D}(\Omega)$, m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$, the direct implication is clear.

Now suppose that $m_\varphi \circ T \colon \mathscr{X} \to \mathscr{F}$ is continuous for every $\varphi \in \mathscr{D}(\Omega)$, let $\mathscr{P}$ be an inducing collection of seminorms for $\mathscr{F}$ and let

$$\mathscr{Q} := \{q_{p,\varphi} \mid p \in \mathscr{P} \text{ and } \varphi \in \mathscr{D}(\Omega)\}$$

be the associated inducing collection of seminorms for $\mathscr{F}_{\mathrm{loc}}$. Because of the continuity of $m_\varphi \circ T$, we find for every $\varphi \in \mathscr{D}(\Omega)$ and $p \in \mathscr{P}$ a continuous seminorm r on $\mathscr{X}$ such that $p((m_\varphi \circ T)(x)) \leq r(x)$ for every $x \in \mathscr{X}$ (see Corollary A.1.3). But

$$p((m_\varphi \circ T)(x)) = p(\varphi(Tx)) = q_{p,\varphi}(Tx),$$

so we have actually found for every $q_{p,\varphi} \in \mathscr{Q}$ a continuous seminorm r on $\mathscr{X}$ such that $q_{p,\varphi}(Tx) \leq r(x)$ for all $x \in \mathscr{X}$ and this proves that $T\colon \mathscr{X} \to \mathscr{F}_{\mathrm{loc}}$ is continuous (again, see Corollary A.1.3). □

From this lemma we see that we can characterize the topology of $\mathscr{F}_{\mathrm{loc}}$ as the *smallest* locally convex topology such that $m_\varphi\colon \mathscr{F}_{\mathrm{loc}} \to \mathscr{F}$ is continuous for every $\varphi \in \mathscr{D}(\Omega)$. Indeed, if $\mathscr{X} = \mathscr{F}_{\mathrm{loc}}$ as set and $\mathscr{X}$ carries a locally convex topology such that $m_\varphi\colon \mathscr{X} \to \mathscr{F}$ is continuous for every $\varphi \in \mathscr{D}(\Omega)$, we can apply the above lemma to $T = \mathrm{id}_{\mathscr{X}}$ to obtain that $\mathrm{id}_{\mathscr{X}}\colon \mathscr{X} \to \mathscr{F}_{\mathrm{loc}}$ is continuous and this precisely means that the topology of $\mathscr{X}$ is stronger than the topology of $\mathscr{F}_{\mathrm{loc}}$. (Compare this to the discussion following Lemma 2.5.6.) In practice, the following corollary will be very convenient to work with.

Corollary 2.6.3. *Let $\mathscr{F}$ be a semi-functional space on Ω, $\mathscr{X}$ a locally convex vector space and $T\colon \mathscr{X} \to \mathscr{D}'(\Omega)$ a linear map. If for every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \circ T$ is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}$, then T is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}_{\mathrm{loc}}$.*

Proof: Since $\varphi(Tx) = (m_\varphi \circ T)(x) \in \mathscr{F}$ for all $\varphi \in \mathscr{D}(\Omega)$ and $x \in \mathscr{X}$, we see that $\mathrm{im}(T) \subseteq \mathscr{F}_{\mathrm{loc}}$. The continuity now follows from the previous lemma. □

Using this corollary, we easily prove that $\mathscr{F}_{\mathrm{loc}}$ is indeed a semi-functional space on Ω (and a bit more).

Proposition 2.6.4. *Let $\mathscr{F}$ be a semi-functional space on Ω. Then $\mathscr{F}_{\mathrm{loc}}$ is a semi-functional space on Ω as well and $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{loc}}$. If $\mathscr{F}$ is a functional space on Ω, then so is $\mathscr{F}_{\mathrm{loc}}$.*

Proof: We first want to show that the inclusion $\mathscr{F}_{\mathrm{loc}} \subseteq \mathscr{D}'(\Omega)$ is continuous. So let $\{u_i\}_{i\in I}$ be a net in $\mathscr{F}_{\mathrm{loc}}$ and $u \in \mathscr{F}_{\mathrm{loc}}$ such that $u_i \to u$ in $\mathscr{F}_{\mathrm{loc}}$. Because for every $\varphi \in \mathscr{D}(\Omega)$, m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$, this implies that $\varphi u_i \to \varphi u$ in $\mathscr{F}$ for every $\varphi \in \mathscr{D}(\Omega)$. Using $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$ then shows that $\varphi u_i \to \varphi u$ in $\mathscr{D}'(\Omega)$ for every $\varphi \in \mathscr{D}(\Omega)$ and on behalf of Lemma 1.5.11 we conclude that $u_i \to u \in \mathscr{D}'(\Omega)$. Hence $\mathscr{F}_{\mathrm{loc}} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$.

Applying Corollary 2.6.3 to the inclusion map $\mathscr{F} \hookrightarrow \mathscr{D}'(\Omega)$ directly gives $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{loc}}$. Combining this with the fact that m_φ, for $\varphi \in \mathscr{D}(\Omega)$, restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$, shows that m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}_{\mathrm{loc}}$. So $\mathscr{F}_{\mathrm{loc}}$ is indeed a semi-functional space on Ω. If $\mathscr{F}$ is a functional space, then $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{loc}}$ proves that $\mathscr{F}_{\mathrm{loc}}$ is a functional space as well. □

Example 2.6.5. We clearly have $(\mathscr{D}'(\Omega))_{\mathrm{loc}} = \mathscr{D}'(\Omega)$. Indeed, $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{loc}}$ gives $\mathscr{D}'(\Omega) \subseteq_{\mathrm{c}} (\mathscr{D}'(\Omega))_{\mathrm{loc}}$ and because $(\mathscr{D}'(\Omega))_{\mathrm{loc}}$ is a functional space on Ω, we also have $(\mathscr{D}'(\Omega))_{\mathrm{loc}} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$. ⊘

Using a partition of unity, it is possible to embed $\mathscr{F}_{\mathrm{loc}}$ in a product of 'local pieces' of $\mathscr{F}$.

Lemma 2.6.6. *Let $\mathscr{F}$ be a semi-functional space on Ω, let $\{\eta_i\}_{i\in I}$ be a partition of unity on Ω consisting of compactly supported smooth functions and let K_i denote the support of η_i. Then*

$$\mathcal{I}\colon \mathscr{F}_{\text{loc}} \to \prod_{i\in I} \mathscr{F}_{K_i} : u \mapsto \{\eta_i u\}_{i\in I}$$

is a linear topological embedding with closed image.

Proof: The idea is to use Lemma A.1.8. So we should prove that $\mathcal{I}$ is continuous and linear and we should find a continuous linear map $\mathcal{P}\colon \prod_{i\in I} \mathscr{F}_{K_i} \to \mathscr{F}_{\text{loc}}$ such that $\mathcal{P} \circ \mathcal{I} = \text{id}_{\mathscr{F}_{\text{loc}}}$.

For every $i \in I$, m_{η_i} restricts to a continuous linear map from $\mathscr{F}_{\text{loc}}$ into $\mathscr{F}$ and because $\text{supp}(\eta_i u) \subseteq \text{supp}(\eta_i) \cap \text{supp}(u) \subseteq K_i$ for every $u \in \mathscr{F}_{\text{loc}}$, we actually have that m_{η_i} restricts to a continuous linear map from $\mathscr{F}_{\text{loc}}$ into $\mathscr{F}_{K_i}$. This shows that every component of $\mathcal{I}$ is a continuous linear map and hence that $\mathcal{I}$ is a continuous linear map.

Furthermore, because a partition of unity is a locally finite family of functions, we have that $\{K_i\}_{i\in I} = \{\text{supp}(\eta_i)\}_{i\in I}$ is a locally finite family of subsets of Ω. As a consequence, every $\{u_i\}_{i\in I} \in \prod_{i\in I} \mathscr{F}_{K_i}$ is a locally finite family of distributions on Ω, so according to Lemma 1.4.19 we have a well-defined map

$$\mathcal{P}\colon \prod_{i\in I} \mathscr{F}_{K_i} \to \mathscr{D}'(\Omega)\colon \{u_i\}_{i\in I} \mapsto \sum_{i\in I} u_i.$$

It is evident that $\mathcal{P}$ is linear and since $\sum_{i\in I} \eta_i = 1$, we have that $\mathcal{P} \circ \mathcal{I} = \text{id}_{\mathscr{F}_{\text{loc}}}$.

In order to prove that $\mathcal{P}$ is actually a continuous linear map from $\prod_{i\in I} \mathscr{F}_{K_i}$ into $\mathscr{F}_{\text{loc}}$, it suffices to prove that $m_\varphi \circ \mathcal{P}$ is a continuous linear map from $\prod_{i\in I} \mathscr{F}_{K_i}$ into $\mathscr{F}$ for every $\varphi \in \mathscr{D}(\Omega)$ (see Corollary 2.6.3). So fix $\varphi \in \mathscr{D}(\Omega)$. Because $\{K_i\}_{i\in I}$ is locally finite and $\text{supp}(\varphi)$ is compact, we find a finite subset I_φ of I such that $K_i \cap \text{supp}(\varphi) \neq \emptyset$ if and only if $i \in I_\varphi$. On behalf of Lemma 1.5.7, we then have

$$(m_\varphi \circ \mathcal{P})(\{u_i\}_{i\in I}) = \sum_{i\in I} m_\varphi u_i = \sum_{i\in I_\varphi} m_\varphi u_i$$

for every $\{u_i\}_{i\in I} \in \prod_{i\in I} \mathscr{F}_{K_i}$, thus $m_\varphi \circ \mathcal{P}$ is a finite sum of continuous linear projections into $\mathscr{F}$ composed with continuous linear multiplications on $\mathscr{F}$, hence a continuous linear map into $\mathscr{F}$. □

Because there always exists a partition of unity on Ω that consists of compactly supported smooth functions, the *proof* of the previous lemma shows us that an element of $\mathscr{F}_{\text{loc}}$ can always be written as the sum of a locally finite family of elements of $\mathscr{F}_{\text{comp}}$.

Example 2.6.7. $(\mathscr{E}(\Omega))_{\text{loc}} = \mathscr{E}(\Omega)$. In order to prove this, we first prove that $(\mathscr{E}(\Omega))_{\text{loc}} \subseteq \mathscr{E}(\Omega)$. So let $u \in (\mathscr{E}(\Omega))_{\text{loc}}$. As we have just discussed, we can find a locally finite family $\{\varphi_i\}_{i\in I}$ of elements of $(\mathscr{E}(\Omega))_{\text{comp}} = \mathscr{D}(\Omega)$ such that u equals $\sum_{i\in I} \varphi_i$. To be precise, we actually find a locally finite family $\{u_{\varphi_i}\}_{i\in I}$ of elements of the subspace of $\mathscr{D}'(\Omega)$ that we canonically identify with $\mathscr{D}(\Omega)$ such that $u = \sum_{i\in I} u_{\varphi_i}$. However, since $\text{supp}(u_{\varphi_i}) = \text{supp}(\varphi_i)$, this indeed gives a locally finite family $\{\varphi_i\}_{i\in I}$ of elements of $\mathscr{D}(\Omega)$. Because the locally finite sum

$\varphi := \sum_{i\in I} \varphi_i$ is locally equal to a finite sum of smooth functions, φ itself is a smooth function and we easily check that $\sum_{i\in I} u_{\varphi_i} = u_\varphi$. Hence $u = u_\varphi$ and we conclude that $u \in \mathscr{E}(\Omega)$.

To prove that the inclusion $(\mathscr{E}(\Omega))_{\mathrm{loc}} \subseteq \mathscr{E}(\Omega)$ is continuous, fix $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and $k \in \mathbb{N}$ and let $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of K. Then we have for every $\psi \in (\mathscr{E}(\Omega))_{\mathrm{loc}} \subseteq \mathscr{E}(\Omega)$ that

$$\|\psi\|_{K,k} = \|\varphi\psi\|_{K,k}$$

and since $\|\cdot\|_{K,k}$ is an arbitrarily chosen seminorm from the 'standard' inducing collection of seminorms for $\mathscr{E}(\Omega)$ and $\psi \mapsto \|\varphi\psi\|_{K,k}$ is a seminorm from the associated inducing collection of seminorms for $(\mathscr{E}(\Omega))_{\mathrm{loc}}$, the continuity of the inclusion follows (see Lemma A.1.2). Because $\mathscr{E}(\Omega) \subseteq_{\mathrm{c}} (\mathscr{E}(\Omega))_{\mathrm{loc}}$ is automatic (it is of the form $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{loc}}$), we are done. ⊘

Example 2.6.8. The functional space $(L^1(\Omega))_{\mathrm{loc}}$ consists precisely of all 'almost everywhere' equivalence classes of locally integrable functions on Ω. One half of the proof is easy. If f is a locally integrable function on Ω, then φf is integrable for every $\varphi \in \mathscr{D}(\Omega)$ and we easily check that $m_\varphi u_{[f]} = u_{[\varphi f]}$. Hence $m_\varphi u_{[f]} \in L^1(\Omega)$ for all $\varphi \in \mathscr{D}(\Omega)$ and this precisely means that $u_{[f]} \in (L^1(\Omega))_{\mathrm{loc}}$.

For the other half we proceed in a similar fashion as in the previous example. Let $u \in (L^1(\Omega))_{\mathrm{loc}}$. Then we find a locally finite family of distributions $\{u_{[f_i]}\}_{i\in I}$ with $[f_i] \in L^1(\Omega)$ such that $u = \sum_{i\in I} u_{[f_i]}$. By replacing f_i by some integrable function that is almost everywhere the same if necessary, we may assume that $\mathrm{supp}(u_{[f_i]}) = \mathrm{supp}(f_i)$ (see Remark 1.4.5), hence we obtain a locally finite family $\{f_i\}_{i\in I}$ of integrable functions. We readily check that the locally finite sum $f := \sum_{i\in I} f_i$ is a locally integrable function and that $\sum_{i\in I} u_{[f_i]} = u_{[f]}$, so $u = u_{[f]}$ is indeed a distribution that represents an equivalence class of locally integrable functions. ⊘

The following lemma is an improvement of Lemma 2.5.5.

Lemma 2.6.9. *Let $\mathscr{F}$ be a semi-functional space on Ω. For every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}_{\mathrm{comp}}$.*

Proof: Fix $\varphi \in \mathscr{D}(\Omega)$ and let K be its support. Take $\psi \in \mathscr{D}(\Omega)$ such that ψ equals 1 on an open neighborhood of K. Then m_ψ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\mathrm{comp}}$ (because of Lemma 2.5.5) and m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$ (because of the properties of $\mathscr{F}_{\mathrm{loc}}$). As a consequence, $m_\varphi = m_{\psi\varphi} = m_\psi \circ m_\varphi$ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}_{\mathrm{comp}}$. □

The assignment $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{loc}}$ is again a functor, but because $\mathscr{F}_{\mathrm{loc}}$ is in general bigger than $\mathscr{F}$, the preparatory lemma looks a bit different.

Lemma 2.6.10. *If $\mathscr{F}$ and $\mathscr{G}$ are semi-functional spaces on Ω and*

$$T \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

is a linear map such that:

1. *T restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{G}$ and*

2. $m_\varphi \circ T = T \circ m_\varphi$ *for all* $\varphi \in \mathscr{D}(\Omega)$*,*

then T *restricts to a continuous linear map from* $\mathscr{F}_{\mathrm{loc}}$ *into* $\mathscr{G}_{\mathrm{loc}}$*.*

Proof: Consider T as linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{D}'(\Omega)$. For every $\varphi \in \mathscr{D}(\Omega)$, we have that $m_\varphi \circ T = T \circ m_\varphi$ as linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{D}'(\Omega)$. Since m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$ and T restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{G}$, we see that $m_\varphi \circ T = T \circ m_\varphi \colon \mathscr{F}_{\mathrm{loc}} \to \mathscr{D}'(\Omega)$ is actually a continuous map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{G}$. Applying Corollary 2.6.3 now gives the desired result. □

So the natural restriction that the assignment $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{loc}}$ puts on the class of arrows for the category of (semi-)functional spaces on Ω is that the arrows need to be continuous restrictions of linear maps on the space of distributions that commute with multiplication by compactly supported smooth functions. Since every continuous inclusion is a continuous restriction of the identity $\mathrm{id}_{\mathscr{D}'(\Omega)}$ on $\mathscr{D}'(\Omega)$, the class of arrows that we have chosen certainly satisfies this restriction.

Proposition 2.6.11. *The assignment* $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{loc}}$ *is a functor from the category of (semi-)functional spaces on* Ω *to the category of (semi-)functional spaces on* Ω*.*

Proof: This follows straight from the previous lemma. After all, if $\mathscr{F}$ and $\mathscr{G}$ are (semi-)functional spaces on Ω such that $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$, then the identity $\mathrm{id}_{\mathscr{D}'(\Omega)} \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ satisfies the conditions of the previous lemma, so $\mathrm{id}_{\mathscr{D}'(\Omega)}$ restricts to a continuous map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{G}_{\mathrm{loc}}$ and this precisely means that $\mathscr{F}_{\mathrm{loc}} \subseteq_{\mathrm{c}} \mathscr{G}_{\mathrm{loc}}$. □

The following result is the analogue of the first equality of Proposition 2.5.9.

Lemma 2.6.12. *Let* $\mathscr{F}$ *be a semi-functional space on* Ω *and* $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$*. Then*

$$(\mathscr{F}_{\mathrm{loc}})_K = \mathscr{F}_K.$$

Proof: Since $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{loc}}$ and the assignment $\mathscr{F} \mapsto \mathscr{F}_K$ is a functor, we obtain $\mathscr{F}_K \subseteq_{\mathrm{c}} (\mathscr{F}_{\mathrm{loc}})_K$.

For the converse inclusion, take $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of K. We know that m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$ and because m_φ is local, this implies that m_φ restricts to a continuous linear map from $(\mathscr{F}_{\mathrm{loc}})_K$ into $\mathscr{F}_K$ (see Lemma 2.3.5). But $m_\varphi u = u$ for all $u \in (\mathscr{F}_{\mathrm{loc}})_K$, so this actually proves that $(\mathscr{F}_{\mathrm{loc}})_K \subseteq_{\mathrm{c}} \mathscr{F}_K$. □

The analogue of the second equality of Proposition 2.5.9 also holds. However, it will have a special meaning in terms of a property that still has to be introduced and therefore we postpone its treatment for a moment. Instead, we present the analogue of Lemma 2.5.12.

Lemma 2.6.13. *Let* $\mathscr{F}$ *be a (semi-)functional space on* Ω *and let* P *be short for: metrizable, normable, complete, Fréchet, Banach, Hilbert, invariant or normal. Then* $\mathscr{F}$ *is locally* P *if and only if* $\mathscr{F}_{\mathrm{loc}}$ *is locally* P*.*

Proof: For everything but normality this is a direct consequence of the fact that $(\mathscr{F}_{\mathrm{loc}})_K = \mathscr{F}_K$ for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. If P is short for normal, we use an argument similar to the one that we have used in the proof of Lemma 2.5.12.

Suppose that $\mathscr{F}$ is locally normal and fix $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. If $u \in (\mathscr{F}_{\mathrm{loc}})_K = \mathscr{F}_K$, the local normality of $\mathscr{F}$ implies that we find a net $\{\varphi_i\}_{i\in I}$ in $\mathscr{D}(\Omega)$ such that $\varphi_i \to u$ in $\mathscr{F}$. Since $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{loc}}$, we subsequently get that $\varphi_i \to u$ in $\mathscr{F}_{\mathrm{loc}}$. Hence u lies in the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}_{\mathrm{loc}}$ and we conclude that $\mathscr{F}_{\mathrm{loc}}$ is locally normal.

Next, suppose that $\mathscr{F}_{\mathrm{loc}}$ is locally normal and again take $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$. If $u \in \mathscr{F}_K = (\mathscr{F}_{\mathrm{loc}})_K$, the local normality of $\mathscr{F}_{\mathrm{loc}}$ implies that we find a net $\{\varphi_i\}_{i\in I}$ in $\mathscr{D}(\Omega)$ such that $\varphi_i \to u$ in $\mathscr{F}_{\mathrm{loc}}$. Now let $\psi \in \mathscr{D}(\Omega)$ such that ψ equals 1 on an open neighborhood of K. Then m_ψ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$ and $m_\psi u = u$. Hence $\psi\varphi_i = m_\psi\varphi_i \to m_\psi u = u$ in $\mathscr{F}$ and since $\{\psi\varphi_i\}_{i\in I}$ is also a net in $\mathscr{D}(\Omega)$, this proves that u lies in the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}$ and we conclude that $\mathscr{F}$ is locally normal. □

We now introduce the promised property.

Definition 2.6.14. Let $\mathscr{F}$ be a semi-functional space on Ω. We say that $\mathscr{F}$ is *local* if $\mathscr{F}_{\mathrm{loc}} = \mathscr{F}$. ⊘

Example 2.6.15. We have already seen that $(\mathscr{D}'(\Omega))_{\mathrm{loc}} = \mathscr{D}'(\Omega)$ and also that $(\mathscr{E}(\Omega))_{\mathrm{loc}} = \mathscr{E}(\Omega)$, so both $\mathscr{D}'(\Omega)$ and $\mathscr{E}(\Omega)$ are local. ⊘

There are two things that should be noted here. First, since we always have $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{loc}}$, $\mathscr{F}_{\mathrm{loc}} = \mathscr{F}$ is equivalent to $\mathscr{F}_{\mathrm{loc}} \subseteq_{\mathrm{c}} \mathscr{F}$, hence a semi-functional space $\mathscr{F}$ is local if and only if $\mathscr{F}_{\mathrm{loc}} \subseteq_{\mathrm{c}} \mathscr{F}$. And second, contrary to the properties that we have seen so far, there is no notion of 'locally local' (which would have sounded absurd anyway). The most logical definition for 'locally local' would be to call a semi-functional space $\mathscr{F}$ on Ω 'locally local' if $\mathscr{F}_K$ is local for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, but it turns out that this is always the case.

Lemma 2.6.16. *For every semi-functional space $\mathscr{F}$ on Ω and all $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, $\mathscr{F}_K$ is local.*

Proof: We should show that $(\mathscr{F}_K)_{\mathrm{loc}} \subseteq_{\mathrm{c}} \mathscr{F}_K$. To this end, we first prove that $\mathrm{supp}(u) \subseteq K$ for every $u \in (\mathscr{F}_K)_{\mathrm{loc}}$. So let $u \in (\mathscr{F}_K)_{\mathrm{loc}}$. By definition of $(\mathscr{F}_K)_{\mathrm{loc}}$, we already know that for every $\varphi \in \mathscr{D}(\Omega)$, $\varphi u \in \mathscr{F}_K$, hence in particular $\mathrm{supp}(\varphi u) \subseteq K$. Now let $x \in \mathrm{supp}(u)$. Because $\{x\}$ is compact, we find an $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of x. By Lemma 1.5.5 we then get $x \in \mathrm{supp}(\varphi u) \subseteq K$, so we indeed have that $\mathrm{supp}(u) \subseteq K$.

Next, let $\psi \in \mathscr{D}(\Omega)$ such that ψ equals 1 on an open neighborhood of K. Then m_ψ restricts to a continuous linear map from $(\mathscr{F}_K)_{\mathrm{loc}}$ into $\mathscr{F}_K$ and because ψ equals 1 on an open neighborhood of the support of every $u \in (\mathscr{F}_K)_{\mathrm{loc}}$, this restriction is in fact a continuous linear inclusion. □

The previous lemma actually states that $\mathscr{F}_K = (\mathscr{F}_K)_{\mathrm{loc}}$ for all $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, so it is the analogue of the second equality of Proposition 2.5.9 in disguise. Also the next result, which is equivalent to $(\mathscr{F}_{\mathrm{loc}})_{\mathrm{loc}} = \mathscr{F}_{\mathrm{loc}}$, is really a statement about the interaction of construction functors.

Lemma 2.6.17. *For every semi-functional space $\mathscr{F}$ on Ω, $\mathscr{F}_{\mathrm{loc}}$ is local.*

Proof: We should show that $(\mathscr{F}_{\text{loc}})_{\text{loc}} \subseteq_{\text{c}} \mathscr{F}_{\text{loc}}$. In view of Corollary 2.6.3 it suffices to prove that for every $\varphi \in \mathscr{D}(\Omega)$, m_φ restricts to a continuous linear map from $(\mathscr{F}_{\text{loc}})_{\text{loc}}$ into $\mathscr{F}$. After all, if this is the case, we can apply Corollary 2.6.3 to the inclusion map $(\mathscr{F}_{\text{loc}})_{\text{loc}} \hookrightarrow \mathscr{D}'(\Omega)$ to obtain the desired result.

So fix $\varphi \in \mathscr{D}(\Omega)$. Let K be the support of φ and take $\psi \in \mathscr{D}(\Omega)$ such that ψ equals 1 on an open neighborhood of K. Then m_φ restricts to a continuous linear map from $(\mathscr{F}_{\text{loc}})_{\text{loc}}$ into $\mathscr{F}_{\text{loc}}$ and m_ψ restricts to a continuous linear map from $\mathscr{F}_{\text{loc}}$ into $\mathscr{F}$. As a consequence, $m_\psi \circ m_\varphi$ restricts to a continuous linear map from $(\mathscr{F}_{\text{loc}})_{\text{loc}}$ into $\mathscr{F}$. But because of the choice of ψ, $m_\psi \circ m_\varphi = m_{\psi\varphi} = m_\varphi$, so we are done. □

In general, if $\mathscr{F}$ is a semi-functional space on Ω, $\mathscr{F}_{\text{comp}}$ will be smaller than $\mathscr{F}$ and $\mathscr{F}_{\text{loc}}$ will be larger than $\mathscr{F}$. Nevertheless, the following two results show that $\mathscr{F}_{\text{comp}}$ and $\mathscr{F}_{\text{loc}}$ can be obtained from each other, so in some sense they contain 'the same information'.

Lemma 2.6.18. *For every semi-functional space $\mathscr{F}$ on Ω, we have*

$$(\mathscr{F}_{\text{loc}})_{\text{comp}} = \mathscr{F}_{\text{comp}}.$$

Proof: Since $\mathscr{F} \subseteq_{\text{c}} \mathscr{F}_{\text{loc}}$ and the assignment $\mathscr{F} \mapsto \mathscr{F}_{\text{comp}}$ is a functor, we directly obtain $\mathscr{F}_{\text{comp}} \subseteq_{\text{c}} (\mathscr{F}_{\text{loc}})_{\text{comp}}$.

For the converse inclusion, $(\mathscr{F}_{\text{loc}})_{\text{comp}} \subseteq_{\text{c}} \mathscr{F}_{\text{comp}}$, it suffices to prove that $(\mathscr{F}_{\text{loc}})_K \subseteq_{\text{c}} \mathscr{F}_{\text{comp}}$ for every $K \in \mathscr{P}_{\text{c}}(\Omega)$. But $(\mathscr{F}_{\text{loc}})_K = \mathscr{F}_K$ according to Lemma 2.6.12, so this is clear. □

Lemma 2.6.19. *For every semi-functional space $\mathscr{F}$ on Ω, we have*

$$(\mathscr{F}_{\text{comp}})_{\text{loc}} = \mathscr{F}_{\text{loc}}.$$

Proof: Since $\mathscr{F}_{\text{comp}} \subseteq_{\text{c}} \mathscr{F} \subseteq_{\text{c}} \mathscr{F}_{\text{loc}}$ and the assignment $\mathscr{F} \mapsto \mathscr{F}_{\text{loc}}$ is a functor, we directly obtain $(\mathscr{F}_{\text{comp}})_{\text{loc}} \subseteq_{\text{c}} (\mathscr{F}_{\text{loc}})_{\text{loc}} = \mathscr{F}_{\text{loc}}$.

The converse inclusion follows from applying Corollary 2.6.3 to the inclusion map $\imath\colon \mathscr{F}_{\text{loc}} \hookrightarrow \mathscr{D}'(\Omega)$. Indeed, by Lemma 2.6.9, $m_\varphi \circ \imath$ is a continuous linear map from $\mathscr{F}_{\text{loc}}$ into $\mathscr{F}_{\text{comp}}$ for every $\varphi \in \mathscr{D}(\Omega)$, so by Corollary 2.6.3, $\imath$ is a continuous linear map from $\mathscr{F}_{\text{loc}}$ into $(\mathscr{F}_{\text{comp}})_{\text{loc}}$. □

Apart from the 'philosophical significance', these lemmas also have interesting concrete implications.

Example 2.6.20. Using a similar argument as in Example 2.6.7, we find that $\mathscr{C}(\Omega)$ is local. If we subsequently combine Example 2.5.4 with the previous lemma, we get

$$(\mathscr{C}_0(\Omega))_{\text{loc}} = (\mathscr{C}_s(\Omega))_{\text{loc}} = (\mathscr{C}_b(\Omega))_{\text{loc}} = (\mathscr{C}(\Omega))_{\text{loc}} = \mathscr{C}(\Omega).$$

So in particular we see that $\mathscr{C}_0(\Omega)$, $\mathscr{C}_s(\Omega)$ and $\mathscr{C}_b(\Omega)$ are not local. ⊘

Example 2.6.21. Using the previous lemma and the already proven fact that $\mathscr{E}(\Omega)$ is local, we find

$$(\mathscr{D}(\Omega))_{\text{loc}} = ((\mathscr{E}(\Omega))_{\text{comp}})_{\text{loc}} = (\mathscr{E}(\Omega))_{\text{loc}} = \mathscr{E}(\Omega).$$

So in particular we see that $\mathscr{D}(\Omega)$ is not local. ⊘

The latter example has two important consequences, both based on the following observation: because $\mathscr{D}(\Omega) \subseteq_c \mathscr{F}$ for every functional space $\mathscr{F}$ on Ω, we have $\mathscr{E}(\Omega) = (\mathscr{D}(\Omega))_{\mathrm{loc}} \subseteq_c \mathscr{F}_{\mathrm{loc}}$ for every functional space $\mathscr{F}$ on Ω and hence $\mathscr{E}(\Omega) \subseteq_c \mathscr{F}$ for every local functional space $\mathscr{F}$ on Ω.

Lemma 2.6.22. *Let $\mathscr{F}$ be a functional space on Ω. Then $\mathscr{F}_{\mathrm{comp}}$ is not local and $\mathscr{F}_{\mathrm{comp}}$ is strictly smaller than $\mathscr{F}_{\mathrm{loc}}$.*

Proof: Clearly, $\mathscr{F}_{\mathrm{comp}}$ cannot contain $\mathscr{E}(\Omega)$, so in view of the discussion above it cannot be local. Moreover, since $\mathscr{F}_{\mathrm{loc}}$ does contain $\mathscr{E}(\Omega)$, $\mathscr{F}_{\mathrm{loc}}$ and $\mathscr{F}_{\mathrm{comp}}$ cannot be equal (actually, because of Lemma 2.6.19, this is equivalent to saying that $\mathscr{F}_{\mathrm{comp}}$ is not local), thus the inclusion $\mathscr{F}_{\mathrm{comp}} \subseteq \mathscr{F}_{\mathrm{loc}}$ must be strict. □

Lemma 2.6.23. *Let $\mathscr{F}$ be a local functional space on Ω. Then $\mathscr{F}$ does not allow a continuous norm.*

Proof: Suppose that it does. Then we get a continuous norm on $\mathscr{E}(\Omega)$ because $\mathscr{E}(\Omega) \subseteq_c \mathscr{F}$, which is in contradiction with Corollary 1.1.7. □

Note that as a consequence of the previous lemma, local functional spaces can certainly not be Banach.

Example 2.6.24. The functional spaces $L^p(\Omega)$ with $1 \leq p \leq \infty$ are Banach and therefore not local. ⊘

The following proposition generalizes Proposition 1.1.15 and supports the idea that $\mathscr{F}_{\mathrm{comp}}$ and $\mathscr{F}_{\mathrm{loc}}$ are not 'too far apart'. In the proof we use that for a net $\{u_i\}_{i\in I}$ in $\mathscr{F}_{\mathrm{loc}}$ and an element u of $\mathscr{F}_{\mathrm{loc}}$, $u_i \to u$ in $\mathscr{F}_{\mathrm{loc}}$ if and only if $\varphi u_i \to \varphi u$ in $\mathscr{F}$ for every $\varphi \in \mathscr{D}(\Omega)$, which is a direct consequence of the definition of $\mathscr{F}_{\mathrm{loc}}$.

Proposition 2.6.25. *Let $\mathscr{F}$ be a semi-functional space on Ω. Then $\mathscr{F}_{\mathrm{comp}}$ is sequentially dense in $\mathscr{F}_{\mathrm{loc}}$.*

Proof: Let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion by compacts of Ω and take, for every $i \in \mathbb{N}$, $\varphi_i \in \mathscr{D}(\Omega)$ such that φ_i equals 1 on an open neighborhood of K_i. We claim that for every $u \in \mathscr{F}_{\mathrm{loc}}$ the sequence $\{\varphi_i u\}_{i\in\mathbb{N}}$, which is a sequence in $\mathscr{F}_{\mathrm{comp}}$ because of Lemma 2.6.9, converges to u in $\mathscr{F}_{\mathrm{loc}}$.

In order to prove this claim we should, as we have just discussed, verify that $\psi\varphi_i u \to \psi u$ in $\mathscr{F}$ for every $\psi \in \mathscr{D}(\Omega)$. So take $\psi \in \mathscr{D}(\Omega)$ and let K be its support. As explained in Remark 1.1.13, we find an $i_0 \in \mathbb{N}$ such that $K \subseteq K_{i_0}$ and because the K_i are increasing, we in fact have $K \subseteq K_i$ for every $i \geq i_0$. As a consequence, φ_i equals 1 on an open neighborhood of the support of ψ for every $i \geq i_0$, hence $\psi\varphi_i = \psi$ for every $i \geq i_0$. But then certainly $\psi\varphi_i u = \psi u$ for all $i \geq i_0$, which proves that $\psi\varphi_i u \to \psi u$ in $\mathscr{F}$. □

Corollary 2.6.26. *Let $\mathscr{F}$ be a functional space on Ω. If $\mathscr{F}$ is locally normal, then $\mathscr{F}_{\mathrm{loc}}$ is normal.*

Proof: Looking at the proof of Lemma 2.5.13, we see that the assumption that $\mathscr{F}$ is locally normal implies that $\mathscr{F}_{\mathrm{comp}}$ is normal. So $\mathscr{D}(\Omega)$ is dense in $\mathscr{F}_{\mathrm{comp}}$, while $\mathscr{F}_{\mathrm{comp}}$ is in turn dense in $\mathscr{F}_{\mathrm{loc}}$ by the previous proposition. Because we also have the chain of continuous inclusions $\mathscr{D}(\Omega) \subseteq_c \mathscr{F}_{\mathrm{comp}} \subseteq_c \mathscr{F}_{\mathrm{loc}}$, we can use Lemma A.2.2 to conclude that $\mathscr{D}(\Omega)$ is dense in $\mathscr{F}_{\mathrm{loc}}$. □

Although weaker, it will be useful to capture the following result, which is of the 'preservation of a property' type, in a corollary too.

Corollary 2.6.27. *Let $\mathscr{F}$ be a functional space on Ω. If $\mathscr{F}$ is normal, then $\mathscr{F}_{\mathrm{loc}}$ is normal as well.*

Proof: This follows from the previous corollary and the fact that normality of $\mathscr{F}$ implies local normality of $\mathscr{F}$. □

We finish this section with two results similar to Corollary 2.6.26 that will be quite useful in the fourth chapter.

Proposition 2.6.28. *Let $\mathscr{F}$ be a semi-functional space on Ω. If $\mathscr{F}$ is locally Fréchet, then $\mathscr{F}_{\mathrm{loc}}$ is Fréchet.*

Proof: Let $\{K_i\}_{i\in\mathbb{N}}$ be an exhaustion by compacts of Ω, let $\{\eta_i\}_{i\in\mathbb{N}}$ be a smooth partition of unity on Ω subordinate to the open cover $\{\mathrm{int}(K_i)\}_{i\in\mathbb{N}}$ and let K_i' denote the support of η_i (note that $\mathrm{supp}(\eta_i)$ is a closed subset of K_i, hence compact). Because $\mathscr{F}$ is locally Fréchet, we then have that $\prod_{i\in\mathbb{N}}\mathscr{F}_{K_i'}$ is a countable product of Fréchet spaces, hence Fréchet and on the strength of Lemma 2.6.6, we have a linear topological isomorphism between $\mathscr{F}_{\mathrm{loc}}$ and a closed subspace of $\prod_{i\in\mathbb{N}}\mathscr{F}_{K_i'}$. Since closed subspaces of a Fréchet space are again Fréchet, the result follows. □

Proposition 2.6.29. *Let $\mathscr{F}$ be a functional space on Ω. If $\mathscr{F}$ is locally invariant, then $\mathscr{F}_{\mathrm{loc}}$ is invariant.*

Proof: Let $\chi\colon\Omega\to\Omega$ be a diffeomorphism. As discussed after the definition of invariance, to prove that $\mathscr{F}_{\mathrm{loc}}$ is invariant it suffices to prove that $\chi_*\colon\mathscr{D}'(\Omega)\to\mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}_{\mathrm{loc}}$. Since χ_* certainly restricts to a linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{D}'(\Omega)$, Corollary 2.6.3 subsequently tells us that it even suffices to prove that for every $\varphi\in\mathscr{D}(\Omega)$, $m_\varphi\circ\chi_*$ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$.

So let $\varphi\in\mathscr{D}(\Omega)$. Then also $\chi^*\varphi\in\mathscr{D}(\Omega)$ and $\mathrm{supp}(\chi^*\varphi)=\chi^{-1}(\mathrm{supp}(\varphi))$ (see Lemma 1.7.1), hence $m_{\chi^*\varphi}$ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}_{\chi^{-1}(\mathrm{supp}(\varphi))}$. Moreover, by local invariance of $\mathscr{F}$, χ_* restricts to a continuous linear map from $\mathscr{F}_{\chi^{-1}(\mathrm{supp}(\varphi))}$ into $\mathscr{F}_{\mathrm{supp}(\varphi)}\subseteq_{\mathrm{c}}\mathscr{F}$. As a consequence, $\chi_*\circ m_{\chi^*\varphi}$ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{loc}}$ into $\mathscr{F}$ and since $m_\varphi\circ\chi_*=\chi_*\circ m_{\chi^*\varphi}$ (see Lemma 1.7.5), we are done. □

2.7 Restrictions to opens

The key feature of local (semi-)functional spaces is that we can 'restrict' them to open subsets. Before we make this precise, note that if U is an open subset of Ω, $u\in\mathscr{D}'(U)$ and $\varphi\in\mathscr{D}(U)$, φu can be naturally viewed as an element of $\mathscr{D}'(\Omega)$. Indeed, clearly $\varphi u\in\mathscr{E}'(U)$ and since $\mathscr{E}'(U)$ can be viewed as a semi-functional space on Ω via $\mathrm{ext}_{U,\Omega}$ (see Example 2.2.5), we have $\varphi u\in\mathscr{E}'(U)\subseteq_{\mathrm{c}}\mathscr{D}'(\Omega)$.

Definition 2.7.1. Let $\mathscr{F}$ be a local semi-functional space on Ω, $\mathscr{P}$ an inducing collection of seminorms for $\mathscr{F}$ and U an open subset of Ω. As a set, we define

$$\mathscr{F}(U):=\{u\in\mathscr{D}'(U)\mid\varphi u\in\mathscr{F}\text{ for all }\varphi\in\mathscr{D}(U)\}$$

(as explained in the discussion above, φu is implicitly identified with its extension to Ω, so $\varphi u \in \mathscr{F}$ indeed makes sense and formally means $\operatorname{ext}_{U,\Omega}(\varphi u) \in \mathscr{F}$). Subsequently, we define for each $p \in \mathscr{P}$ and $\varphi \in \mathscr{D}(U)$

$$q_{p,\varphi} \colon \mathscr{F}(U) \to \mathbb{R} \colon u \mapsto p(\varphi u).$$

We easily see that $\mathscr{F}(U)$ is a vector subspace of $\mathscr{D}'(U)$ and that the $q_{p,\varphi}$, with $p \in \mathscr{P}$ and $\varphi \in \mathscr{D}(U)$, are seminorms on $\mathscr{F}(U)$. We endow $\mathscr{F}(U)$ with the topology induced by these seminorms. ⊘

Remark 2.7.2. It depends on the situation whether or not we explicitly write the 'identification map' $\operatorname{ext}_{U,\Omega}$ when we are working with $\mathscr{F}(U)$; results often look nicer if we hide the identifications, but in order to prove these results it is sometimes convenient to be a bit more verbose. ⊘

One should note that this definition is very similar to the definition of $\mathscr{F}_{\text{loc}}$. Indeed, taking $U = \Omega$ in fact gives the definition of $\mathscr{F}_{\text{loc}}$, so thanks to the assumption that $\mathscr{F}$ is local, we have $\mathscr{F} = \mathscr{F}_{\text{loc}} = \mathscr{F}(\Omega)$. Moreover, because of the similarity between the definition of $\mathscr{F}_{\text{loc}}$ and $\mathscr{F}(U)$, a trivial adaption of the arguments following the definition of $\mathscr{F}_{\text{loc}}$ shows that for every $\varphi \in \mathscr{D}(U)$, m_φ (which is a priori a map from $\mathscr{D}'(U)$ into $\mathscr{E}'(U)$, hence from $\mathscr{D}'(U)$ into $\mathscr{D}'(\Omega)$ when implicitly composed with $\operatorname{ext}_{U,\Omega}$) restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$ and that the topology of $\mathscr{F}(U)$ is the smallest locally convex topology with this property (so it is in particular independent of the chosen inducing collection of seminorms for $\mathscr{F}$). The following results are also obtained by a trivial adaption of the corresponding statements for $\mathscr{F}_{\text{loc}}$ (that is, Lemma 2.6.2 and Corollary 2.6.2).

Lemma 2.7.3. *Let $\mathscr{F}$ be a local semi-functional space on Ω, U an open subset of Ω, $\mathscr{X}$ a locally convex vector space and $T \colon \mathscr{X} \to \mathscr{F}(U)$ a linear map. Then T is continuous if and only if for every $\varphi \in \mathscr{D}(U)$*

$$m_\varphi \circ T \colon \mathscr{X} \to \mathscr{F}$$

is continuous.

Corollary 2.7.4. *Let $\mathscr{F}$ be a local semi-functional space on Ω, U an open subset of Ω, $\mathscr{X}$ a locally convex vector space and $T \colon \mathscr{X} \to \mathscr{D}'(U)$ a linear map. If for every $\varphi \in \mathscr{D}(U)$, $m_\varphi \circ T$ is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}$, then T is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}(U)$.*

With the help of the previous corollary, we easily prove that $\mathscr{F}(U)$ is a semi-functional space on U:

Proposition 2.7.5. *Let $\mathscr{F}$ be a local semi-functional space on Ω and U an open subset of Ω. Then $\mathscr{F}(U)$ is a semi-functional space on U. Moreover, if $\mathscr{F}$ is a functional space on Ω, then $\mathscr{F}(U)$ is a functional space on U.*

Proof: We first want to show that the inclusion $\mathscr{F}(U) \subseteq \mathscr{D}'(U)$ is continuous. So let $\{u_i\}_{i\in I}$ be a net in $\mathscr{F}(U)$ and $u \in \mathscr{F}(U)$ such that $u_i \to u$ in $\mathscr{F}(U)$. Furthermore, fix $\varphi \in \mathscr{D}(U)$. Then $\operatorname{ext}_{U,\Omega} \circ m_\varphi$ is a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$, so $\operatorname{ext}_{U,\Omega}(\varphi u_i) \to \operatorname{ext}_{U,\Omega}(\varphi u)$ in $\mathscr{F}$ and since $\mathscr{F} \subseteq_{\text{c}} \mathscr{D}'(\Omega)$,

$\mathrm{ext}_{U,\Omega}(\varphi u_i) \to \mathrm{ext}_{U,\Omega}(\varphi u)$ in $\mathscr{D}'(\Omega)$ as well. By applying the continuous linear restriction map $\mathrm{res}_{\Omega,U} : \mathscr{D}'(\Omega) \to \mathscr{D}'(U)$ and using the identity

$$\mathrm{res}_{\Omega,U} \circ \mathrm{ext}_{U,\Omega} = \mathrm{id}_{\mathscr{E}'(U)}$$

(see Remark 1.4.6), we subsequently get that $\varphi u_i \to \varphi u$ in $\mathscr{D}'(U)$. Because this holds for every $\varphi \in \mathscr{D}(U)$, Lemma 1.5.11 gives that $u_i \to u$ in $\mathscr{D}'(U)$, thus we indeed have $\mathscr{F}(U) \subseteq_{\mathrm{c}} \mathscr{D}'(U)$.

Next, we should show that for every $\varphi \in \mathscr{D}(U)$, m_φ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}(U)$. So fix $\varphi \in \mathscr{D}(U)$. Then m_φ is a linear map from $\mathscr{F}(U)$ into $\mathscr{D}'(U)$ such that for every $\psi \in \mathscr{D}(U)$, $m_\psi \circ m_\varphi = m_{\psi\varphi}$ is a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$, so on behalf of Corollary 2.7.4, m_φ is a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}(U)$.

Finally, suppose that $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}$ (i.e., that $\mathscr{F}$ is a functional space on Ω) and let $\imath$ denote the inclusion map $\mathscr{D}(U) \hookrightarrow \mathscr{D}'(U)$. Clearly, for every $\varphi \in \mathscr{D}(U)$, $m_\varphi \circ \imath$ is a continuous linear map from $\mathscr{D}(U)$ into $\mathscr{D}(U) \subseteq_{\mathrm{c}} \mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}$, hence by Corollary 2.7.4, $\imath$ is in fact a continuous linear map from $\mathscr{D}(U)$ into $\mathscr{F}(U)$ and this precisely means that $\mathscr{D}(U) \subseteq_{\mathrm{c}} \mathscr{F}(U)$. □

So if $\mathscr{F}$ is a local (semi-)functional space on Ω, we have for every open subset U of Ω a (semi-)functional space on U that is related to $\mathscr{F}$ in a natural way. The relevance of this becomes immediately clear if we think of Ω as a manifold and of U as a chart domain; it then morally means that we can restrict $\mathscr{F}$ to chart domains. It is therefore not hard to imagine that precisely this feature of local (semi-)functional spaces will be very useful to make the transition from $\mathbb{R}^n$ to the setting of vector bundles over manifolds in the fourth chapter.

Proposition 2.7.6. *For every open subset U of Ω the assignment $\mathscr{F} \mapsto \mathscr{F}(U)$ is a functor from the category of local (semi-)functional spaces on Ω to the category of (semi-)functional spaces on U.*

Proof: Suppose that $\mathscr{F}$ and $\mathscr{G}$ are (semi-)functional spaces on Ω with $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$. We already know that for every $\varphi \in \mathscr{D}(U)$, m_φ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$ and because $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$, we also have that m_φ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{G}$. Applying Corollary 2.7.4 to the inclusion map $\mathscr{F}(U) \hookrightarrow \mathscr{D}'(U)$ then shows that $\mathscr{F}(U) \subseteq_{\mathrm{c}} \mathscr{G}(U)$. □

The previous proposition shows that the assignment $\mathscr{F} \mapsto \mathscr{F}(U)$ is also a construction functor, although it is a bit different than the construction functors that we have seen so far. After all, it creates (semi-)functional spaces on U rather than on Ω and it is not defined on the entire category of (semi-)functional spaces on Ω, but only on a subcategory of (semi-)functional spaces with an additional property. Despite this difference, we have the usual questions about interaction and the preservation of properties.

Remark 2.7.7. If $\mathscr{F}$ is a semi-functional space on Ω, $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and U is an open subset of Ω that contains K, then $\mathscr{F}_K$ can be viewed as a semi-functional space on U via

$$\imath' : \mathscr{F}_K \to \mathscr{D}'(U) : u \mapsto u|_U .$$

Since $\imath'$ is the restriction of the continuous linear map $\mathrm{res}_{\Omega,U} : \mathscr{D}'(\Omega) \to \mathscr{D}'(U)$ to $\mathscr{F}_K$ and $\mathscr{F}_K \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$, $\imath'$ is a continuous linear map. Moreover, due to

Lemma 1.4.16, $u|_U = 0$ implies $u = 0$ for every $u \in \mathscr{F}_K$, so $\imath'$ is also injective. Now let $\varphi \in \mathscr{D}(U)$, let $\tilde{\varphi}$ be the extension by zero to Ω of φ and let $u \in \mathscr{F}_K$. Then $m_{\tilde{\varphi}}$ restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}_K$ and

$$(\imath' \circ m_{\tilde{\varphi}})(u) = (\tilde{\varphi}u)|_U = \tilde{\varphi}|_U \, u|_U = \varphi \, u|_U = (m_\varphi \circ \imath')(u)$$

(see Lemma 1.5.4). Hence,

$$\imath' \circ m_{\tilde{\varphi}} = m_\varphi \circ \imath' \tag{2.3}$$

on $\mathscr{F}_K$, which shows that $m_\varphi(\imath'(\mathscr{F}_K)) \subseteq \imath'(\mathscr{F}_K)$ and that $(\imath')^{-1} \circ m_\varphi \circ \imath' = m_{\tilde{\varphi}}$ is continuous. Putting everything together, we see that all requirements of Proposition 2.2.4 are met, thus $\mathscr{F}_K$ can indeed be identified with a semi-functional space on U via $\imath'$. ⊘

Lemma 2.7.8. *Let $\mathscr{F}$ be a local semi-functional space on Ω and U an open subset of Ω. For every $K \in \mathscr{P}_c(U)$, we have*

$$(\mathscr{F}(U))_K = \mathscr{F}_K.$$

Proof: Let $\varphi \in \mathscr{D}(U)$ such that φ equals 1 on an open neighborhood of K. Then m_φ is a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$ that sends $(\mathscr{F}(U))_K$ into $\mathscr{F}_K$ (see Lemma 1.5.10) and equals the identity (well, actually $\mathrm{ext}_{U,\Omega}$) on $(\mathscr{F}(U))_K$, so we obtain $(\mathscr{F}(U))_K \subseteq_c \mathscr{F}_K$.

Next, let $\imath' \colon \mathscr{F}_K \to \mathscr{D}'(U)$ be the 'inclusion' map described in the remark above, let $\varphi \in \mathscr{D}(U)$ and let $\tilde{\varphi}$ be its extension by zero to Ω. Using Lemma 1.5.9 and equation (2.3), we find that

$$\mathrm{ext}_{U,\Omega} \circ m_\varphi \circ \imath' = \mathrm{ext}_{U,\Omega} \circ \imath' \circ m_{\tilde{\varphi}} = m_{\tilde{\varphi}}$$

on $\mathscr{F}_K$, so $\mathrm{ext}_{U,\Omega} \circ m_\varphi \circ \imath'$ is a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}_K \subseteq_c \mathscr{F}$. According to Corollary 2.7.4 (where $\mathrm{ext}_{U,\Omega}$ is only implicitly present), this implies that $\imath'$ is a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}(U)$. Since Lemma 1.4.9 subsequently tells us that the 'inclusion' $\imath'$ actually maps $\mathscr{F}_K$ into $(\mathscr{F}(U))_K$, we conclude that $\mathscr{F}_K \subseteq_c (\mathscr{F}(U))_K$. □

Corollary 2.7.9. *Let $\mathscr{F}$ be a local semi-functional space on Ω and U an open subset of Ω. Then*

$$(\mathscr{F}(U))_{\mathrm{comp}} \subseteq_c \mathscr{F}_{\mathrm{comp}} \subseteq_c \mathscr{F}.$$

Proof: Due to Proposition A.3.2, it suffices to prove that $(\mathscr{F}(U))_K \subseteq_c \mathscr{F}_{\mathrm{comp}}$ for every $K \in \mathscr{P}_c(U)$, which is a direct consequence of the previous lemma and the fact that $\mathscr{F}_K \subseteq_c \mathscr{F}_{\mathrm{comp}}$ for every $K \in \mathscr{P}_c(\Omega)$. □

One of the properties that is preserved under $\mathscr{F} \mapsto \mathscr{F}(U)$ is locality. However, the locality of $\mathscr{F}$ is non-optional if we want to apply the construction functor $\mathscr{F} \mapsto \mathscr{F}(U)$, so we actually see that semi-functional spaces of the form $\mathscr{F}(U)$ are always local.

Lemma 2.7.10. *For every local semi-functional space $\mathscr{F}$ on Ω and every open subset U of Ω, $\mathscr{F}(U)$ is local.*

Proof: We should prove that $(\mathscr{F}(U))_{\mathrm{loc}} \subseteq_{\mathrm{c}} \mathscr{F}(U)$ and for this it suffices to prove that for every $\varphi \in \mathscr{D}(U)$, m_φ restricts to a continuous linear map from $(\mathscr{F}(U))_{\mathrm{loc}}$ into $\mathscr{F}$ (after all, if this is the case, we can apply Corollary 2.7.4 to the inclusion map $(\mathscr{F}(U))_{\mathrm{loc}} \hookrightarrow \mathscr{D}'(U)$). So let $\varphi \in \mathscr{D}(U)$. Then by Lemma 2.6.9, m_φ restricts to a continuous linear map from $(\mathscr{F}(U))_{\mathrm{loc}}$ into $(\mathscr{F}(U))_{\mathrm{comp}}$ and since the previous corollary tells us that $(\mathscr{F}(U))_{\mathrm{comp}} \subseteq_{\mathrm{c}} \mathscr{F}$, m_φ is also a continuous linear map from $(\mathscr{F}(U))_{\mathrm{loc}}$ into $\mathscr{F}$. □

Also (local) normality is preserved under $\mathscr{F} \mapsto \mathscr{F}(U)$:

Lemma 2.7.11. *Let $\mathscr{F}$ be a local functional space on Ω and U an open subset of Ω. If $\mathscr{F}$ is (locally) normal, then so is $\mathscr{F}(U)$.*

Proof: Both $\mathscr{F}$ and $\mathscr{F}(U)$ are local and since we know that for local functional spaces normality and local normality are equivalent, it suffices to prove the 'local' version of the statement (i.e., the statement that one obtains by removing the parentheses around the word 'locally').

Fix $K \in \mathscr{P}_{\mathrm{c}}(U)$ and let $u \in (\mathscr{F}(U))_K = \mathscr{F}_K$. Because $\mathscr{F}$ is assumed to be locally normal, we find a net $\{\varphi_i\}_{i\in I}$ in $\mathscr{D}(\Omega)$ such that $\varphi_i \to u$ in $\mathscr{F}$. Now take $\psi \in \mathscr{D}(\Omega)$ such that ψ equals 1 on an open neighborhood of K and $\mathrm{supp}(\psi) \subseteq U$. Then $\psi u = u$ and m_ψ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\mathrm{supp}(\psi)} = (\mathscr{F}(U))_{\mathrm{supp}(\psi)} \subseteq_{\mathrm{c}} \mathscr{F}(U)$. As a consequence, we obtain that $\psi\varphi_i \to \psi u = u$ in $\mathscr{F}_{\mathrm{supp}(\psi)}$ and that $(\psi\varphi_i)|_U \to u$ in $\mathscr{F}(U)$ (note that there is implicitly a restriction and an extension map around when we write $\mathscr{F}_{\mathrm{supp}(\psi)} = (\mathscr{F}(U))_{\mathrm{supp}(\psi)}$). Since $\mathrm{supp}(\psi) \subseteq U$ implies that $\{\,(\psi\varphi_i)|_U\}_{i\in I}$ is a net in $\mathscr{D}(U)$, this shows that $\mathscr{F}(U)$ is locally normal. □

Regarding invariance, we even have more than 'just' a preservation result; it turns out that invariance, locality and the concept of 'restricting $\mathscr{F}$ to U' work beautifully together.

Lemma 2.7.12. *Let $\mathscr{F}$ be a local functional space on Ω and let U and V be open subsets of Ω. If $\mathscr{F}$ is (locally) invariant and $\chi\colon U \to V$ is a diffeomorphism, then $\chi_*\colon \mathscr{D}'(U) \to \mathscr{D}'(V)$ restricts to a linear topological isomorphism from $\mathscr{F}(U)$ onto $\mathscr{F}(V)$.*

Proof: Fix $K \in \mathscr{P}_{\mathrm{c}}(U)$. We will first prove that χ_* restricts to a continuous linear map from $\mathscr{F}_K = (\mathscr{F}(U))_K$ into $(\mathscr{F}(V))_{\chi(K)} = \mathscr{F}_{\chi(K)}$. For this, we use the nontrivial fact that for any $x \in U$ there exists an open neighborhood U_x of x in U and a diffeomorphism $\chi_x\colon \Omega \to \Omega$ such that $\chi_x|_{U_x} = \chi|_{U_x}$ (see, for example, [10, Theorem 5.5]). Clearly, $\{U_x\}_{x\in U}$ is an open cover of K in Ω, so there exists a finite partition of unity $\{\eta_0, \dots, \eta_n\}$ over K subordinate to $\{U_x\}_{x\in U}$ of compactly supported smooth functions on Ω. Now pick for every $0 \le i \le n$ an $x \in U$ such that $\mathrm{supp}(\eta_i) \subseteq U_x$ and denote U_x by U_i and the associated χ_x by χ_i. Because $\mathscr{F}$ is invariant (note that since $\mathscr{F}$ is local, invariance and local invariance are equivalent for $\mathscr{F}$), $(\chi_i)_*$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$ for every $0 \le i \le n$ and because $\mathscr{F}$ is a functional space, we subsequently find that $\sum_{i=0}^n m_{\eta_i}(\chi_i)_*$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$.

We claim that χ_* (or actually $\mathrm{ext}_{V,\Omega} \circ \chi_* \circ \mathrm{res}_{\Omega,U}$) and $\sum_{i=0}^n m_{\eta_i}(\chi_i)_*$ coincide on $\mathscr{F}_K = (\mathscr{F}(U))_K$. To prove this, let $u \in \mathscr{F}_K$ and $\varphi \in \mathscr{D}(\Omega)$. Moreover, let

V be an open neighborhood of K such that $\sum_{i=0}^n \eta_i(x) = 1$ for every $x \in V$ (the existence of such an open neighborhood is one of the conditions of a finite partition of unity over K). After wrestling through some details and looking at Lemma 1.4.15, we see that to establish

$$(\chi_* u)(\varphi) = \sum_{i=0}^n (m_{\eta_i}(\chi_i)_* u)(\varphi)$$

it suffices to prove that $\varphi \circ \chi$ and $\sum_{i=0}^n \eta_i(\varphi \circ \chi_i)$ coincide on V (note that V is evidently a subset of U). Since $\chi_i|_{U_i} = \chi|_{U_i}$ and $\mathrm{supp}(\eta_i) \subseteq U_i$, we have that $\eta_i(x)\varphi(\chi_i(x)) = \eta_i(x)\varphi(\chi(x))$ for every $x \in U$ and $0 \leq i \leq n$. So we indeed have that

$$\begin{aligned}\left(\sum_{i=0}^n \eta_i(\varphi \circ \chi_i)\right)(x) &= \sum_{i=0}^n \eta_i(x)\varphi(\chi_i(x)) = \sum_{i=0}^n \eta_i(x)\varphi(\chi(x)) \\ &= \varphi(\chi(x)) \sum_{i=0}^n \eta_i(x) = \varphi(\chi(x)) = (\varphi \circ \chi)(x)\end{aligned}$$

for every $x \in V$. Hence, the restriction of χ_* to $\mathscr{F}_K = (\mathscr{F}(U))_K$ coincides with the restriction of the continuous linear map $\sum_{i=0}^n m_{\eta_i}(\chi_i)_* \colon \mathscr{F} \to \mathscr{F}$ to $\mathscr{F}_K$. As a result, χ_* restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}$ and because $\mathrm{supp}(\chi_* u) \subseteq \chi(\mathrm{supp}(u)) \subseteq \chi(K)$ for every $u \in \mathscr{F}_K$, we conclude that χ_* restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}_{\chi(K)}$.

The remainder of the proof is fairly easy. By symmetry, it suffices to prove that χ_* restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}(V)$ and for this it in turn suffices to prove that for every $\varphi \in \mathscr{D}(V)$, $m_\varphi \circ \chi_*$ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$ (see Corollary 2.7.4). So let $\varphi \in \mathscr{D}(V)$. Then $\chi^*\varphi \in \mathscr{D}(U)$, so $m_{\chi^*\varphi}$ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}_{\mathrm{supp}(\chi^*\varphi)} = \mathscr{F}_{\chi^{-1}(\mathrm{supp}(\varphi))}$. Moreover, $m_\varphi \circ \chi_* = \chi_* \circ m_{\chi^*\varphi}$ (see Lemma 1.7.5) and since we have just proven that χ_* restricts to a continuous linear map from $\mathscr{F}_{\chi^{-1}(\mathrm{supp}(\varphi))}$ into $\mathscr{F}_{\mathrm{supp}(\varphi)} \subseteq_{\mathrm{c}} \mathscr{F}$, we indeed get that $m_\varphi \circ \chi_*$ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$. □

Let us put all the 'preservation information' about $\mathscr{F} \mapsto \mathscr{F}(U)$ in a theorem:

Theorem 2.7.13. *Let $\mathscr{F}$ be a local (semi-)functional space on Ω, let U be an open subset of Ω and let P be short for: metrizable, normable, complete, Fréchet, Banach, Hilbert, invariant or normal. Then $\mathscr{F}$ is locally P implies $\mathscr{F}(U)$ is locally P.*

Proof: When P is short for: metrizable, normable, complete, Fréchet, Banach or Hilbert, this is a direct consequence of the fact that $(\mathscr{F}(U))_K = \mathscr{F}_K$ for every $K \in \mathscr{P}_{\mathrm{c}}(U)$ (that is, of Lemma 2.7.8). Furthermore, when P is short for invariant, the statement is a consequence of the previous lemma (note that invariance and local invariance are equivalent for the local spaces $\mathscr{F}$ and $\mathscr{F}(U)$) and the case where P is short for normal has already been dealt with in Lemma 2.7.11. □

Considering the fact that $\mathscr{F}(\Omega) = \mathscr{F}_{\mathrm{loc}}$, we see that the following lemma is a generalization of Lemma 2.6.6.

Lemma 2.7.14. *Let $\mathscr{F}$ be a local semi-functional space on Ω, let U be an open subset of Ω, let $\{\eta_i\}_{i\in I}$ be a partition of unity on U consisting of compactly supported smooth functions and let K_i denote the support of η_i. Then*

$$\mathcal{I}\colon \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}_{K_i} : u \mapsto \{\eta_i u\}_{i\in I}$$

is a linear topological embedding with closed image.

Proof: The proof is analogous to the proof of Lemma 2.6.6, only minor adjustments have to be made. Most importantly, the definition of $\mathcal{P}$ should be modified to $\prod_{i\in I} \mathscr{F}_{K_i} \to \mathscr{D}'(U)\colon \{u_i\}_{i\in I} \mapsto \sum_{i\in I} u_i|_U$. With this in mind, all other adjustments should be obvious. (Note that when we do not hide any identifications, $\mathcal{I}(u)$ should in fact be written as $\{\mathrm{ext}_{U,\Omega}(\eta_i u)\}_{i\in I}$.) □

Example 2.7.15. We have already seen that $\mathscr{E}(\Omega)$ is a local functional space on Ω, so if U is an open subset of Ω, we can consider $(\mathscr{E}(\Omega))(U)$. Of course, we would like to have that $(\mathscr{E}(\Omega))(U) = \mathscr{E}(U)$, especially because it is customary to abbreviate $\mathscr{E}(\mathbb{R}^{\mathrm{n}})$ by $\mathscr{E}$, so we need $(\mathscr{E}(\mathbb{R}^{\mathrm{n}}))(U) = \mathscr{E}(U)$ to prevent ambiguity.

Luckily for us, $(\mathscr{E}(\Omega))(U) = \mathscr{E}(U)$ indeed holds. In order to prove that $\mathscr{E}(U) \subseteq_{\mathrm{c}} (\mathscr{E}(\Omega))(U)$, it suffices on behalf of Corollary 2.7.4 to prove that $\mathrm{ext}_{U,\Omega} \circ m_\varphi$ restricts to a continuous linear map from $\mathscr{E}(U)$ into $\mathscr{E}(\Omega)$ for all $\varphi \in \mathscr{D}(U)$, which is indeed the case because the extension by zero of a compactly supported smooth function is a compactly supported smooth function with the same $\|\cdot\|_{K,k}$ norms (with $K \in \mathscr{P}_{\mathrm{c}}(U)$ and $k \in \mathbb{N}$). The proof of the converse inclusion, $(\mathscr{E}(\Omega))(U) \subseteq_{\mathrm{c}} \mathscr{E}(U)$, is similar to the proof of $(\mathscr{E}(\Omega))_{\mathrm{loc}} \subseteq_{\mathrm{c}} \mathscr{E}(\Omega)$ from Example 2.6.7. As a consequence of (the proof of) the previous lemma, every element of $(\mathscr{E}(\Omega))(U)$ can be written as a locally finite sum of restrictions to U of elements from $\mathscr{E}(\Omega)$, hence as a locally finite sum of smooth functions on U. Since locally finite sums of smooth functions are smooth, this proves that $(\mathscr{E}(\Omega))(U) \subseteq \mathscr{E}(U)$. Now if $K \in \mathscr{P}_{\mathrm{c}}(U)$ and $k \in \mathbb{N}$, then we can choose an $\varphi \in \mathscr{D}(U)$ such that φ equals 1 on an open neighborhood of K and the observation that

$$\|\psi\|^U_{K,k} = \|\varphi\psi\|^U_{K,k} = \|\mathrm{ext}_{U,\Omega}(\varphi\psi)\|^\Omega_{K,k}$$

for all $\psi \in (\mathscr{E}(\Omega))(U)$ subsequently shows that the inclusion $(\mathscr{E}(\Omega))(U) \subseteq \mathscr{E}(U)$ is continuous. ⊘

Example 2.7.16. Similar to the topic of the previous example, one might wonder whether or not $(\mathscr{D}'(\Omega))(U) = \mathscr{D}'(U)$ (note that we have already seen that $\mathscr{D}'(\Omega)$ is local). Since $(\mathscr{D}'(\Omega))(U)$ is a functional space on U, we directly get $(\mathscr{D}'(\Omega))(U) \subseteq_{\mathrm{c}} \mathscr{D}'(U)$, while $\mathscr{D}'(U) \subseteq_{\mathrm{c}} (\mathscr{D}'(\Omega))(U)$ is a consequence of Corollary 2.7.4, the fact that m_φ is a continuous linear map from $\mathscr{D}'(U)$ into $\mathscr{E}'(U)$ for every $\varphi \in \mathscr{D}(U)$ and the fact that $\mathrm{ext}_{U,\Omega}$ is a continuous linear map from $\mathscr{E}'(U)$ into $\mathscr{E}'(\Omega) \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$. ⊘

Instead of fixing an open subset U of Ω and talking about the assignment $\mathscr{F} \mapsto \mathscr{F}(U)$, we can also fix a local semi-functional space $\mathscr{F}$ on Ω and consider the assignment $U \mapsto \mathscr{F}(U)$, where U runs over the open subsets of Ω. The next two lemmas show that this assignment is a special type of (enriched) sheaf of distributions over Ω.

Lemma 2.7.17. *Let $\mathscr{F}$ be a local semi-functional space on Ω and let U and V be open subsets of Ω such that $V \subseteq U$. The map $\mathrm{res}_{U,V} \colon \mathscr{D}'(U) \to \mathscr{D}'(V)$ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}(V)$.*

Proof: According to Corollary 2.7.4, it suffices to prove that for every $\varphi \in \mathscr{D}(V)$, $\mathrm{ext}_{V,\Omega} \circ m_\varphi \circ \mathrm{res}_{U,V}$ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$. So fix $\varphi \in \mathscr{D}(V)$ and let $\tilde{\varphi}$ be its extension by zero to U. We need two identities. First, thanks to Lemma 1.5.4, $m_\varphi \circ \mathrm{res}_{U,V} = \mathrm{res}_{U,V} \circ m_{\tilde{\varphi}}$. And, second, Lemma 1.5.9 and the fact that $\mathrm{ext}_{V,\Omega} = \mathrm{ext}_{U,\Omega} \circ \mathrm{ext}_{V,U}$ (see Remark 1.4.6) together show that

$$(\mathrm{ext}_{V,\Omega} \circ \mathrm{res}_{U,V})(u) = (\mathrm{ext}_{U,\Omega} \circ \mathrm{ext}_{V,U} \circ \mathrm{res}_{U,V})(u) = (\mathrm{ext}_{U,\Omega})(u)$$

for all $u \in \mathscr{E}'(U)$ with the property that $\mathrm{supp}(u) \subseteq V$. As a consequence of these identities,

$$\mathrm{ext}_{V,\Omega} \circ m_\varphi \circ \mathrm{res}_{U,V} = \mathrm{ext}_{V,\Omega} \circ \mathrm{res}_{U,V} \circ m_{\tilde{\varphi}} = \mathrm{ext}_{U,\Omega} \circ m_{\tilde{\varphi}}$$

(note that for every $u \in \mathscr{D}'(U)$, $m_{\tilde{\varphi}} u$ is compactly supported and satisfies $\mathrm{supp}(m_{\tilde{\varphi}} u) \subseteq V$ because of Lemma 1.5.3) and since we know that $\mathrm{ext}_{U,\Omega} \circ m_{\tilde{\varphi}}$ restricts to a continuous linear map from $\mathscr{F}(U)$ into $\mathscr{F}$, the result follows. □

Lemma 2.7.18. *If $\mathscr{F}$ is a local semi-functional space on Ω, U is an open subset of Ω and $\{U_i\}_{i\in I}$ is a collection of open subsets of Ω such that $U = \cup_{i\in I} U_i$, then*

$$\mathcal{I} \colon \mathscr{F}(U) \to \prod_{i\in I} \mathscr{F}(U_i) \colon u \mapsto \{u|_{U_i}\}_{i\in I}$$

is a linear topological embedding with closed image and

$$\mathrm{im}(\mathcal{I}) = \{\{u_i\}_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i) \mid u_i|_{U_i \cap U_{i'}} = u_{i'}|_{U_i\cap U_{i'}} \textit{ for all } i, i' \in I\}.$$

Proof: It is a direct consequence of the previous lemma that $\mathcal{I}$ is well-defined and continuous. To prove that $\mathcal{I}$ is in fact a linear topological embedding with closed image, let $\{\eta_j\}_{j\in J}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$ consisting of compactly supported smooth functions on U and choose for every $j \in J$ an $i_j \in I$ with the property that $\mathrm{supp}(\eta_j) \subseteq U_{i_j}$. Using Lemma 1.5.10 and Lemma 1.4.19, we see that

$$\mathcal{P} \colon \prod_{i\in I} \mathscr{F}(U_i) \to \mathscr{D}'(U) \colon \{u_i\}_{i\in I} \mapsto \sum_{j\in J} \mathrm{ext}_{U_{i_j},U}(\eta_j|_{U_{i_j}}\, u_{i_j})$$

is a well-defined map and we readily verify that $\mathcal{P}$ is linear (just use that the locally finite sum becomes a finite sum when applied to a test function). Moreover, thanks to Lemma 1.5.4 and Lemma 1.5.9,

$$\begin{aligned}(\mathcal{P} \circ \mathcal{I})(u) &= \sum_{j\in J} \mathrm{ext}_{U_{i_j},U}(\eta_j|_{U_{i_j}}\, u|_{U_{i_j}}) = \sum_{j\in J} \mathrm{ext}_{U_{i_j},U}((\eta_j u)|_{U_{i_j}}) \\ &= \sum_{j\in J} \eta_j u = u\end{aligned}$$

for every $u \in \mathscr{F}(U)$, so if we can prove that $\mathcal{P}$ is actually a continuous linear map into $\mathscr{F}(U)$, we can invoke Lemma A.1.8 to conclude that $\mathcal{I}$ is an embedding with closed image.

In order to prove that $\mathcal{P}$ indeed has this property, it suffices to prove that $\mathrm{ext}_{U,\Omega} \circ m_\varphi \circ \mathcal{P}$ is a continuous linear map into $\mathscr{F}$ for all $\varphi \in \mathscr{D}(U)$ (see Corollary 2.7.4). So fix $\varphi \in \mathscr{D}(U)$. Because φ has compact support and $\{\mathrm{supp}(\eta_j)\}_{j\in J}$ is locally finite, there exists a finite subset J_φ of J such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\eta_j) \neq \emptyset$ if and only if $j \in J_\varphi$. Using Lemma 1.5.8 and the fact that $\mathrm{ext}_{U_{i_j},\Omega} = \mathrm{ext}_{U,\Omega} \circ \mathrm{ext}_{U_{i_j},U}$ (see Remark 1.4.6), we then deduce that

$$\begin{aligned}(\mathrm{ext}_{U,\Omega} \circ m_\varphi \circ \mathcal{P})(\{u_i\}_{i\in I}) &= \sum_{j\in J} (\mathrm{ext}_{U,\Omega} \circ m_\varphi)(\mathrm{ext}_{U_{i_j},U}(\eta_j|_{U_{i_j}} u_{i_j})) \\ &= \sum_{j\in J_\varphi} (\mathrm{ext}_{U,\Omega} \circ \mathrm{ext}_{U_{i_j},U})((\varphi\eta_j)|_{U_{i_j}} u_{i_j}) \\ &= \sum_{j\in J_\varphi} (\mathrm{ext}_{U_{i_j},\Omega} \circ m_{(\varphi\eta_j)|_{U_{i_j}}})(u_{i_j})\end{aligned}$$

for all $\{u_i\}_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i)$. In other words, denoting the projection from $\prod_{i\in I} \mathscr{F}(U_i)$ onto $\mathscr{F}(U_{i'})$ by $\pi_{i'}$,

$$\mathrm{ext}_{U,\Omega} \circ m_\varphi \circ \mathcal{P} = \sum_{j\in J_\varphi} \mathrm{ext}_{U_{i_j},\Omega} \circ m_{(\varphi\eta_j)|_{U_{i_j}}} \circ \pi_{i_j}$$

and since the right hand side is a finite sum of continuous linear maps into $\mathscr{F}$, we obtain that $\mathrm{ext}_{U,\Omega} \circ m_\varphi \circ \mathcal{P}$ is a continuous linear map into $\mathscr{F}$.

So $\mathcal{P}$ is indeed a continuous linear map into $\mathscr{F}(U)$ and as a consequence $\mathcal{I}$ is indeed a linear topological embedding with closed image. It remains to be shown that this closed image of $\mathcal{I}$ satisfies

$$\mathrm{im}(\mathcal{I}) = \{\{u_i\}_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i) \mid u_i|_{U_i\cap U_{i'}} = u_{i'}|_{U_i\cap U_{i'}} \text{ for all } i,\, i' \in I\}.$$

The inclusion '$\subseteq$' is easy; it is a direct consequence of the fact that if $u \in \mathscr{F}(U)$, then for all i, $i' \in I$,

$$(u|_{U_i})\big|_{U_i\cap U_{i'}} = u|_{U_i\cap U_{i'}} = (u|_{U_{i'}})\Big|_{U_i\cap U_{i'}}.$$

The inclusion '$\supseteq$' is a bit more work. Let $\{u_i\}_{i\in I} \in \prod_{i\in I} \mathscr{F}(U_i)$ such that $u_i|_{U_i\cap U_{i'}} = u_{i'}|_{U_i\cap U_{i'}}$ for all i, $i' \in I$. First observe that if we can prove that for every $i' \in I$

$$(\mathcal{P}\{u_i\}_{i\in I})|_{U_{i'}} = u_{i'}, \tag{2.4}$$

we are done. After all, $\{u_i\}_{i\in I} = \mathcal{I}(\mathcal{P}\{u_i\}_{i\in I})$ if this is the case. To establish equation (2.4), fix $i' \in I$ and $\varphi \in \mathscr{D}(U_{i'})$ and let J_φ be the finite subset of J such that $\mathrm{supp}(\varphi) \cap \mathrm{supp}(\eta_j) \neq \emptyset$ if and only if $j \in J_\varphi$. Moreover, let $\tilde{\varphi}$ be the extension by zero to Ω of φ. Using Lemma 1.4.18, we find

$$\begin{aligned}((\mathcal{P}\{u_i\}_{i\in I})|_{U_{i'}})(\varphi) = (\mathcal{P}\{u_i\}_{i\in I})(\tilde{\varphi}) &= \sum_{j\in J}(\operatorname{ext}_{U_{i_j},U}(\eta_j|_{U_{i_j}}\, u_{i_j}))(\tilde{\varphi}) \\ &= \sum_{j\in J}(\eta_j|_{U_{i_j}}\, u_{i_j})(\tilde{\varphi}|_{U_{i_j}}) = \sum_{j\in J_\varphi} u_{i_j}((\eta_j\tilde{\varphi})|_{U_{i_j}}) \\ &= \sum_{j\in J_\varphi} u_{i'}((\eta_j\tilde{\varphi})|_{U_{i'}}) = u_{i'}(\sum_{j\in J_\varphi}(\eta_j\tilde{\varphi})|_{U_{i'}}) \\ &= u_{i'}((\sum_{j\in J_\varphi}\eta_j\tilde{\varphi})|_{U_{i'}}) = u_{i'}(\tilde{\varphi}|_{U_{i'}}) = u_{i'}(\varphi). \qquad \square\end{aligned}$$

Remark 2.7.19. There is also a 'converse' to the previous two lemmas: if we have an assignment $U \mapsto \hat{\mathscr{F}}(U)$ that associates to every open subset U of Ω a (semi-)functional space $\hat{\mathscr{F}}(U)$ such that:

1. for all open subsets U and V of Ω with the property that $V \subseteq U$, the map $\operatorname{res}_{U,V} \colon \mathscr{D}'(U) \to \mathscr{D}'(V)$ restricts to a continuous linear map from $\hat{\mathscr{F}}(U)$ into $\hat{\mathscr{F}}(V)$ and

2. if U is an open subset of Ω and $\{U_i\}_{i\in I}$ is a collection of open subsets of Ω with the property that $U = \cup_{i\in I} U_i$, then

$$\mathcal{I} \colon \hat{\mathscr{F}}(U) \to \prod_{i\in I}\hat{\mathscr{F}}(U_i) \colon u \mapsto \{u|_{U_i}\}_{i\in I}$$

is a linear topological embedding with closed image and

$$\operatorname{im}(\mathcal{I}) = \{\{u_i\}_{i\in I} \in \prod_{i\in I}\hat{\mathscr{F}}(U_i) \mid u_i|_{U_i\cap U_{i'}} = u_{i'}|_{U_i\cap U_{i'}} \text{ for all } i,\, i' \in I\}$$

(i.e., if $U \mapsto \hat{\mathscr{F}}(U)$ is a special type of enriched sheaf of distributions over Ω), then $\mathscr{F} := \hat{\mathscr{F}}(\Omega)$ is a local (semi-)functional space on Ω and $\mathscr{F}(U) = \hat{\mathscr{F}}(U)$ for all open subsets U of Ω. In other words, there is a one-to-one correspondence between local (semi-)functional spaces on Ω and special types of enriched sheaves of distributions over Ω. The proof of this converse is not difficult, but because we do not really need it, we better move on. $\oslash$

2.8 Semi-locality

Although local functional spaces have some nice properties and are of significant importance, we have also seen that requiring a functional space to be local is quite restrictive: for example, local functional spaces do not allow a continuous norm. As a consequence, even one of the most elementary functional spaces, namely $\mathscr{D}(\Omega)$, does not belong to the class of local functional spaces. To remedy this, we introduce the concept of being *semi-local*; a property of (semi-)functional spaces that is less restrictive than being local, but that is still strong enough to single out a convenient class of (semi-)functional spaces.

Definition 2.8.1. Let $\mathscr{F}$ be a semi-functional space on Ω. We say that $\mathscr{F}$ is *semi-local* if for every $\varphi \in \mathscr{E}(\Omega)$, $m_\varphi\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$. ⊘

In other words, a semi-functional space is semi-local if the second requirement in Definition 2.2.1 holds for all smooth functions rather than only for the ones with compact support. This seems very natural to ask for and in some sense semi-locality is indeed a more natural notion than locality: it has this nice intrinsic formulation, it will naturally pop up in various places and we will see that, contrary to the class of local functional spaces, the class of semi-local functional spaces is closed under all construction functors that we encounter (i.e., semi-locality is always preserved under such functors).

But why do we call this property 'semi-local'? Well, while on first sight there is no apparent relation between semi-locality and locality, the properties are actually quite similar. This becomes clear by discussing the 'alternative approach' for introducing semi-locality, which starts with the introduction of another construction.

Definition 2.8.2. Let $\mathscr{F}$ be a semi-functional space on Ω and let $\mathscr{P}$ be an inducing collection of seminorms for $\mathscr{F}$. As a set, we define

$$\mathscr{F}_{\text{semi}} := \{u \in \mathscr{D}'(\Omega) \mid \varphi u \in \mathscr{F} \text{ for all } \varphi \in \mathscr{E}(\Omega)\}.$$

Subsequently, we define for each $p \in \mathscr{P}$ and $\varphi \in \mathscr{E}(\Omega)$

$$q_{p,\varphi}\colon \mathscr{F}_{\text{semi}} \to \mathbb{R}\colon u \mapsto p(\varphi u).$$

We easily see that $\mathscr{F}_{\text{semi}}$ is a vector subspace of $\mathscr{D}'(\Omega)$ and that the $q_{p,\varphi}$, with $p \in \mathscr{P}$ and $\varphi \in \mathscr{E}(\Omega)$, are seminorms on $\mathscr{F}_{\text{semi}}$. We endow $\mathscr{F}_{\text{semi}}$ with the topology induced by these seminorms. ⊘

If we compare this definition to the definition of $\mathscr{F}_{\text{loc}}$, we see that there is only one difference: $\mathscr{D}(\Omega)$ is replaced by $\mathscr{E}(\Omega)$. By replacing $\mathscr{D}(\Omega)$ by $\mathscr{E}(\Omega)$ in the arguments following the definition of $\mathscr{F}_{\text{loc}}$, we then obtain that for every $\varphi \in \mathscr{E}(\Omega)$, m_φ restricts to a continuous linear map from $\mathscr{F}_{\text{semi}}$ into $\mathscr{F}$ and that the topology of $\mathscr{F}_{\text{semi}}$ is the smallest locally convex topology with this property (hence it is in particular independent of the chosen inducing collection of seminorms for $\mathscr{F}$). In the same manner (that is, by replacing $\mathscr{D}(\Omega)$ by $\mathscr{E}(\Omega)$), we obtain the following results.

Lemma 2.8.3. *Let $\mathscr{F}$ be a semi-functional space on Ω, $\mathscr{X}$ a locally convex vector space and $T\colon \mathscr{X} \to \mathscr{F}_{\text{semi}}$ a linear map. Then T is continuous if and only if for every $\varphi \in \mathscr{E}(\Omega)$*

$$m_\varphi \circ T\colon \mathscr{X} \to \mathscr{F}$$

is continuous.

Corollary 2.8.4. *Let $\mathscr{F}$ be a semi-functional space on Ω, $\mathscr{X}$ a locally convex vector space and $T\colon \mathscr{X} \to \mathscr{D}'(\Omega)$ a linear map. If for every $\varphi \in \mathscr{E}(\Omega)$, $m_\varphi \circ T$ is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}$, then T is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}_{\text{semi}}$.*

However, not everything can be realized by just replacing $\mathscr{D}(\Omega)$ by $\mathscr{E}(\Omega)$.

Proposition 2.8.5. *Let $\mathscr{F}$ be a semi-functional space on Ω. Then $\mathscr{F}_{\mathrm{semi}}$ is a semi-functional space on Ω as well and $\mathscr{F}_{\mathrm{comp}} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{semi}} \subseteq_{\mathrm{c}} \mathscr{F}$. If $\mathscr{F}$ is a functional space on Ω, then so is $\mathscr{F}_{\mathrm{semi}}$.*

Proof: Because $1 \in \mathscr{E}(\Omega)$ (that is, the constant 1 function), m_1 restricts to a continuous linear map from $\mathscr{F}_{\mathrm{semi}}$ into $\mathscr{F}$ and since $m_1 u = u$ for all $u \in \mathscr{F}_{\mathrm{semi}}$, this restriction is in fact a continuous linear inclusion. Hence $\mathscr{F}_{\mathrm{semi}} \subseteq_{\mathrm{c}} \mathscr{F}$.

To prove that $\mathscr{F}_{\mathrm{comp}} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{semi}}$, it suffices to prove that for every $\psi \in \mathscr{E}(\Omega)$, m_ψ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{comp}}$ into $\mathscr{F}$. After all, if this is the case, we can apply Corollary 2.8.4 to the inclusion map $\mathscr{F}_{\mathrm{comp}} \hookrightarrow \mathscr{D}'(\Omega)$ to get the desired result. But according to Proposition A.3.2 it then suffices to prove that for every $\psi \in \mathscr{E}(\Omega)$ and $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, m_ψ restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}$. So fix $\psi \in \mathscr{E}(\Omega)$ and $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and let $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of K. Then $\varphi\psi \in \mathscr{D}(\Omega)$, so $m_{\varphi\psi}$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$ and because $m_{\varphi\psi}$ equals m_ψ on $\mathscr{F}_K$, we find that m_ψ indeed restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}$.

It now easily follows that $\mathscr{F}_{\mathrm{semi}}$ is a (semi-)functional space on Ω. First of all, combining $\mathscr{F}_{\mathrm{semi}} \subseteq_{\mathrm{c}} \mathscr{F}$ and $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$ gives $\mathscr{F}_{\mathrm{semi}} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$. Next, if $\varphi \in \mathscr{D}(\Omega)$, then m_φ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\mathrm{comp}}$ and using $\mathscr{F}_{\mathrm{semi}} \subseteq_{\mathrm{c}} \mathscr{F}$ and $\mathscr{F}_{\mathrm{comp}} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{semi}}$ then shows that m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{semi}}$ into $\mathscr{F}_{\mathrm{semi}}$. Finally, if $\mathscr{F}$ is a functional space, then $\mathscr{F}_{\mathrm{comp}}$ is a functional space, so $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{comp}} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{semi}}$ and we conclude that $\mathscr{F}_{\mathrm{semi}}$ is a functional space as well. □

Following the analogy with $\mathscr{F}_{\mathrm{loc}}$ and locality, in the 'alternative approach' for introducing semi-locality we declare a semi-functional space $\mathscr{F}$ on Ω to be semi-local if $\mathscr{F}_{\mathrm{semi}} = \mathscr{F}$. On behalf of the following proposition, the direct and alternative approach result in the same notion.

Proposition 2.8.6. *Let $\mathscr{F}$ be a semi-functional space on Ω. Then $\mathscr{F}$ is semi-local if and only if $\mathscr{F}_{\mathrm{semi}} = \mathscr{F}$.*

Proof: Suppose that $\mathscr{F}$ is semi-local, i.e., that m_φ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$ for every $\varphi \in \mathscr{E}(\Omega)$. By applying Corollary 2.8.4 to the inclusion map $\mathscr{F} \hookrightarrow \mathscr{D}'(\Omega)$, we then get $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}_{\mathrm{semi}}$ and since we always have $\mathscr{F}_{\mathrm{semi}} \subseteq_{\mathrm{c}} \mathscr{F}$, we obtain $\mathscr{F}_{\mathrm{semi}} = \mathscr{F}$.

Next, suppose that $\mathscr{F}_{\mathrm{semi}} = \mathscr{F}$ and let $\varphi \in \mathscr{E}(\Omega)$. As we have discussed, m_φ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{semi}}$ into $\mathscr{F}$, hence to a continuous linear map from $\mathscr{F} = \mathscr{F}_{\mathrm{semi}}$ into $\mathscr{F}$. So $\mathscr{F}$ is semi-local. □

So there is indeed a striking similarity between semi-locality and locality. However, the terminology does not just suggest that the properties are similar: the use of the prefix 'semi' indicates that semi-locality is a weaker property than locality, something which we also proclaimed in the first paragraph of this section. Before we verify this, let us observe that a trivial adaptation of (the proof of) Lemma 2.6.10 and Proposition 2.6.11 gives:

Lemma 2.8.7. *If $\mathscr{F}$ and $\mathscr{G}$ are semi-functional spaces on Ω and*

$$T\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

is a linear map such that:

1. *T restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{G}$ and*

2. *$m_\varphi \circ T = T \circ m_\varphi$ for all $\varphi \in \mathscr{E}(\Omega)$,*

then T restricts to a continuous linear map from $\mathscr{F}_{\text{semi}}$ into $\mathscr{G}_{\text{semi}}$.

Proposition 2.8.8. *The assignment $\mathscr{F} \mapsto \mathscr{F}_{\text{semi}}$ is a functor from the category of (semi-)functional spaces on Ω to the category of (semi-)functional spaces on Ω.*

The fact that locality is a stronger property than semi-locality is a consequence of the following result, which can now be seen as another example of a statement about the interaction of construction functors.

Lemma 2.8.9. *For every semi-functional space $\mathscr{F}$ on Ω, we have*

$$(\mathscr{F}_{\text{loc}})_{\text{semi}} = \mathscr{F}_{\text{loc}}.$$

Proof: The inclusion $(\mathscr{F}_{\text{loc}})_{\text{semi}} \subseteq_{\text{c}} \mathscr{F}_{\text{loc}}$ is of the form $\mathscr{G}_{\text{semi}} \subseteq_{\text{c}} \mathscr{G}$, hence automatic. So it suffices to prove $\mathscr{F}_{\text{loc}} \subseteq_{\text{c}} (\mathscr{F}_{\text{loc}})_{\text{semi}}$. To this end, observe that for every $\varphi \in \mathscr{D}(\Omega)$ and $\psi \in \mathscr{E}(\Omega)$, $m_\varphi \circ m_\psi = m_{\varphi\psi}$ restricts to a continuous linear map from $\mathscr{F}_{\text{loc}}$ into $\mathscr{F}$. By Corollary 2.6.3, this implies that for every $\psi \in \mathscr{E}(\Omega)$, m_ψ restricts to a continuous linear map from $\mathscr{F}_{\text{loc}}$ into $\mathscr{F}_{\text{loc}}$, which by Corollary 2.8.4 in turn implies that the inclusion $\mathscr{F}_{\text{loc}} \hookrightarrow \mathscr{D}'(\Omega)$ is actually a continuous linear map from $\mathscr{F}_{\text{loc}}$ into $(\mathscr{F}_{\text{loc}})_{\text{semi}}$. □

Proposition 2.8.10. *Let $\mathscr{F}$ be a semi-functional space on Ω. If $\mathscr{F}$ is local, then $\mathscr{F}$ is also semi-local.*

Proof: Because $\mathscr{F}$ is local, we have $\mathscr{F}_{\text{loc}} = \mathscr{F}$, hence using the previous lemma we get $\mathscr{F}_{\text{semi}} = (\mathscr{F}_{\text{loc}})_{\text{semi}} = \mathscr{F}_{\text{loc}} = \mathscr{F}$. □

Strictly speaking, the just proven fact that locality implies semi-locality does not yet prove that locality is 'stronger' in the normal linguistic sense of the word; after all, the properties still might be 'of equal strength'. Of course, this is not the case (otherwise we would not have bothered to introduce semi-locality) and this becomes clear if we consider $\mathscr{F}_{\text{comp}}$.

Lemma 2.8.11. *For every semi-functional space $\mathscr{F}$ on Ω, $\mathscr{F}_{\text{comp}}$ is semi-local.*

Proof: Let $\psi \in \mathscr{E}(\Omega)$. In the proof of Proposition 2.8.5, we have already seen that for every $K \in \mathscr{P}_{\text{c}}(\Omega)$, m_ψ restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}$. Because m_ψ maps $\mathscr{F}_K$ into $\mathscr{F}_K$ and $\mathscr{F}_K \subseteq_{\text{c}} \mathscr{F}_{\text{comp}}$, this implies that m_ψ restricts to a continuous linear map from $\mathscr{F}_K$ into $\mathscr{F}_{\text{comp}}$ for every $K \in \mathscr{P}_{\text{c}}(\Omega)$ and application of Proposition A.3.2 subsequently shows that m_ψ restricts to a continuous linear map from $\mathscr{F}_{\text{comp}}$ into $\mathscr{F}_{\text{comp}}$. □

This result is in sharp contrast with the situation for locality. After all, we have seen that a for a functional space $\mathscr{F}$ on Ω, $\mathscr{F}_{\text{comp}}$ is *never* local. So semi-locality is indeed less restrictive (i.e., really 'weaker') than locality (for example, $\mathscr{D}(\Omega) = (\mathscr{E}(\Omega))_{\text{comp}}$ is semi-local but not local) and we see that the assignment $\mathscr{F} \mapsto \mathscr{F}_{\text{comp}}$ is an example of a construction functor under which the class of semi-local functional spaces is closed while the class of local functional spaces is not. Furthermore, contrary to local functional spaces, semi-local functional spaces are not bothered by the restriction of not allowing continuous norms: $\mathscr{D}(\Omega)$ is a semi-local functional space on Ω and the continuous inclusions $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} L^p(\Omega)$ for $1 \leq p \leq \infty$ provide $\mathscr{D}(\Omega)$ with infinitely many different continuous norms.

Now that it is clear and proven that there is a very important difference between locality and semi-locality, we end with two more analogues of results from the previous section. The first one, which is analogous to Lemma 2.6.16, explains why we do not have a notion of 'locally semi-local' and the second one, which is analogous to Lemma 2.6.17, tells us that the construction functor $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{semi}}$ always produces semi-local spaces.

Lemma 2.8.12. *For every semi-functional space $\mathscr{F}$ on Ω and all $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, $\mathscr{F}_K$ is semi-local.*

Proof: By Lemma 2.6.16, $\mathscr{F}_K$ is local, so as a consequence of Proposition 2.8.10 it is certainly semi-local. □

Lemma 2.8.13. *For every semi-functional space $\mathscr{F}$ on Ω, $\mathscr{F}_{\mathrm{semi}}$ is semi-local.*

Proof: Let $\psi \in \mathscr{E}(\Omega)$. On behalf of Corollary 2.8.4, in order to prove that m_ψ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{semi}}$ into $\mathscr{F}_{\mathrm{semi}}$, it suffices to prove that for every $\varphi \in \mathscr{E}(\Omega)$, $m_\varphi \circ m_\psi = m_{\varphi\psi}$ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{semi}}$ into $\mathscr{F}$. But for every $\varphi \in \mathscr{E}(\Omega)$, $\varphi\psi \in \mathscr{E}(\Omega)$, so this is clear. □

2.9 Positive powers

A key example of functional spaces are the famous Sobolev spaces. These Sobolev spaces are usually defined 'in terms of' the spaces $L^p(\Omega)$ for $1 \leq p \leq \infty$ and the construction that we introduce in this section generalizes the procedure in which the Sobolev spaces of non-negative integer order are obtained from the spaces $L^p(\Omega)$ to arbitrary (semi-)functional spaces.

Definition 2.9.1. Let $\mathscr{F}$ be a semi-functional space on Ω, $\mathscr{P}$ an inducing collection of seminorms for $\mathscr{F}$ and $k \in \mathbb{N}$. As a set, we define

$$\mathscr{F}^k := \{u \in \mathscr{D}'(\Omega) \mid \partial^\alpha u \in \mathscr{F} \text{ for all } |\alpha| < k+1\}.$$

Subsequently, we define for each $p \in \mathscr{P}$

$$p_k \colon \mathscr{F}^k \to \mathbb{R} \colon u \mapsto \sum_{|\alpha|<k+1} p(\partial^\alpha u).$$

We easily see that $\mathscr{F}^k$ is a vector subspace of $\mathscr{D}'(\Omega)$ and that the p_k, with $p \in \mathscr{P}$, are seminorms on $\mathscr{F}^k$. We endow $\mathscr{F}^k$ with the topology induced by these seminorms. ⊘

Remark 2.9.2. It might seems strange to write $|\alpha| < k+1$ instead of $|\alpha| \leq k$. However, later on k will be allowed to equal ∞ and in that setting $|\alpha| < k+1$ will still have the right meaning, while $|\alpha| \leq k$ would not. ⊘

It should be noted that there is a clear similarity between the definition of $\mathscr{F}^k$ and the definitions of $\mathscr{F}_{\mathrm{loc}}$ and $\mathscr{F}_{\mathrm{semi}}$, which becomes even more clear if we observe that the collection $\{p_\alpha \mid |\alpha| < k+1\}$, with $p_\alpha \colon \mathscr{F}^k \to \mathbb{R} \colon u \mapsto p(\partial^\alpha u)$, is also an inducing collection of seminorms for $\mathscr{F}^k$ (an easy consequence of Corollary A.1.4). Therefore, by arguments completely analogous to the ones after the definition of $\mathscr{F}_{\mathrm{loc}}$, we deduce that that the topology of $\mathscr{F}^k$ is independent of the chosen inducing collection of seminorms for $\mathscr{F}$ and that for every $|\alpha| < k+1$, ∂^α restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$. Furthermore, we have analogues of Lemma 2.6.2/Lemma 2.8.3 and Corollary 2.6.3/Corollay 2.8.4.

Lemma 2.9.3. *Let $\mathscr{F}$ be a semi-functional space on Ω, $k \in \mathbb{N}$, $\mathscr{X}$ a locally convex vector space and $T \colon \mathscr{X} \to \mathscr{F}^k$ a linear map. Then T is continuous if and only if for every $|\alpha| < k+1$*

$$\partial^\alpha \circ T \colon \mathscr{X} \to \mathscr{F}$$

is continuous.

Proof: Because for every $|\alpha| < k+1$, ∂^α restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$, the direct implication is clear.

Now suppose that $\partial^\alpha \circ T \colon \mathscr{X} \to \mathscr{F}$ is continuous for every $|\alpha| < k+1$, let $\mathscr{P}$ be an inducing collection of seminorms for $\mathscr{F}$ and let

$$\mathscr{P}_k := \{p_k \mid p \in \mathscr{P}\}$$

be the associated inducing collection of seminorms for $\mathscr{F}^k$. Because of the continuity of $\partial^\alpha \circ T$, we find for every $|\alpha| < k+1$ and $p \in \mathscr{P}$ a continuous seminorm $r_{p,\alpha}$ on $\mathscr{X}$ such that $p((\partial^\alpha \circ T)(x)) \leq r_{p,\alpha}(x)$ for every $x \in \mathscr{X}$ (see Corollary A.1.3). But then $r_p := \sum_{|\alpha|<k+1} r_{p,\alpha}$ is also a continuous seminorm on $\mathscr{X}$ and

$$p_k(Tx) = \sum_{|\alpha|<k+1} p(\partial^\alpha(Tx)) = \sum_{|\alpha|<k+1} p((\partial^\alpha \circ T)(x)) \leq \sum_{|\alpha|<k+1} r_{p,\alpha}(x) = r_p(x).$$

So we have actually found for every $p_k \in \mathscr{P}_k$ a continuous seminorm r_p on $\mathscr{X}$ such that $p_k(Tx) \leq r_p(x)$ for all $x \in \mathscr{X}$ and this proves that $T \colon \mathscr{X} \to \mathscr{F}^k$ is continuous (again, see Corollary A.1.3). □

Corollary 2.9.4. *Let $\mathscr{F}$ be a semi-functional space on Ω, $k \in \mathbb{N}$, $\mathscr{X}$ a locally convex vector space and $T \colon \mathscr{X} \to \mathscr{D}'(\Omega)$ a linear map. If for every $|\alpha| < k+1$, $\partial^\alpha \circ T$ is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}$, then T is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}^k$.*

Proof: Since $\partial^\alpha(Tx) = (\partial^\alpha \circ T)(x) \in \mathscr{F}$ for all $|\alpha| < k+1$ and $x \in \mathscr{X}$, we see that $\mathrm{im}(T) \subseteq \mathscr{F}^k$. The continuity then follows from the previous lemma. □

Similar to what we have for $\mathscr{F}_{\mathrm{loc}}$ and $\mathscr{F}_{\mathrm{semi}}$, the topology of $\mathscr{F}^k$ is the smallest locally convex topology such that $\partial^\alpha \colon \mathscr{F}^k \to \mathscr{F}$ is continuous for every $|\alpha| < k+1$. Indeed, if $\mathscr{X} = \mathscr{F}^k$ as set and $\mathscr{X}$ carries a locally convex topology such that $\partial^\alpha \colon \mathscr{X} \to \mathscr{F}$ is continuous for every $|\alpha| < k+1$, we can apply the above lemma to $T = \mathrm{id}_{\mathscr{X}}$ to obtain that $\mathrm{id}_{\mathscr{X}} \colon \mathscr{X} \to \mathscr{F}^k$ is continuous and this precisely means that the topology of $\mathscr{X}$ is stronger than the topology of $\mathscr{F}^k$.

Proposition 2.9.5. *Let $\mathscr{F}$ be a semi-functional space on Ω. Then for every $k \in \mathbb{N}$, $\mathscr{F}^k$ is a semi-functional space on Ω as well. Furthermore, $\mathscr{F}^{k+1} \subseteq_c \mathscr{F}^k$, $\mathscr{F}^0 = \mathscr{F}$ and if $\mathscr{F}$ is a functional space on Ω, then so is $\mathscr{F}^k$.*

Proof: Application of Corollary 2.9.4 to the inclusion map $\mathscr{F}^k \hookrightarrow \mathscr{D}'(\Omega)$ immediately shows that $\mathscr{F}^k \subseteq_c \mathscr{F}^\ell$ for all $\ell \leq k$, hence in particular $\mathscr{F}^{k+1} \subseteq_c \mathscr{F}^k$ and $\mathscr{F}^k \subseteq_c \mathscr{F}^0$. Moreover, it is clear from the definition of $\mathscr{F}^k$ that $\mathscr{F}^0 = \mathscr{F}$ and using this, we obtain $\mathscr{F}^k \subseteq_c \mathscr{F}^0 = \mathscr{F} \subseteq_c \mathscr{D}'(\Omega)$.

Next, fix $\varphi \in \mathscr{D}(\Omega)$. To prove that m_φ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^k$, we consider m_φ as a linear map from $\mathscr{F}^k$ into $\mathscr{D}'(\Omega)$. We easily deduce that for every $|\alpha| < k+1$

$$\partial^\alpha \circ m_\varphi = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} m_{\partial^{\alpha-\beta}\varphi} \circ \partial^\beta$$

is a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$. After all, for every $\beta \leq \alpha$, we have $|\beta| \leq |\alpha| < k+1$, so ∂^β is a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$, while $m_{\partial^{\alpha-\beta}\varphi}$ is a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$ since $\partial^{\alpha-\beta}\varphi \in \mathscr{D}(\Omega)$. Application of Corollary 2.9.4 now gives that m_φ indeed restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^k$, so $\mathscr{F}^k$ is indeed a semi-functional space on Ω.

If $\mathscr{F}$ is a functional space on Ω, then $\mathscr{D}(\Omega) \subseteq_c \mathscr{F}$, so for all $|\alpha| < k+1$, $\partial^\alpha \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$, which a priori restricts to a continuous linear map from $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$, in fact also restricts to a continuous linear map from $\mathscr{D}(\Omega)$ into $\mathscr{F}$. Applying Corollary 2.9.4 to the inclusion map $\mathscr{D}(\Omega) \hookrightarrow \mathscr{D}'(\Omega)$ then shows that $\mathscr{D}(\Omega) \subseteq_c \mathscr{F}^k$ and we conclude that $\mathscr{F}^k$ is a functional space as well. □

Before we move on, let us look at some examples.

Example 2.9.6. For every $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, $(L^p(\Omega))^k$ equals

$$\{u \in \mathscr{D}'(\Omega) \mid \partial^\alpha u \in L^p(\Omega) \text{ for all } \alpha \leq k\}$$

and its topology is induced by the norm

$$(L^p(\Omega))^k \to \mathbb{R} \colon u \mapsto \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_p,$$

where $\|\cdot\|_p$ is the usual norm of $L^p(\Omega)$. Since this is one of the common definitions of the Sobolev space $W^{k,p}(\Omega)$, we see that $W^{k,p}(\Omega) = (L^p(\Omega))^k$ and in particular that $W^{k,p}(\Omega)$ is a functional space on Ω. Of course this is no surprise; like we have already said, we have actually based the definition of $\mathscr{F}^k$ on the way in which $W^{k,p}(\Omega)$ is defined from $L^p(\Omega)$. For $1 < p < \infty$

$$(L^p(\Omega))^k \to \mathbb{R} \colon u \mapsto \left(\sum_{|\alpha| \leq k} (\|\partial^\alpha u\|_p)^p \right)^{\frac{1}{p}}$$

is a different but equivalent norm on $W^{k,p}(\Omega)$ that is also commonly used. ⊘

Example 2.9.7. We easily deduce that for every $k \in \mathbb{N}$, $(\mathscr{D}(\Omega))^k = \mathscr{D}(\Omega)$, $(\mathscr{E}(\Omega))^k = \mathscr{E}(\Omega)$ and $(\mathscr{D}'(\Omega))^k = \mathscr{D}'(\Omega)$. ⊘

Example 2.9.8. Since for every $k \in \mathbb{N}_\infty$, a distribution whose partial derivatives with order strictly less than $k+1$ are equal to continuous functions is equal to a C^k function (see [4, Proposition 5.9.1]), we see that for every $k \in \mathbb{N}$, $(\mathscr{C}(\Omega))^k$ is equal to the vector space $\mathscr{C}^k(\Omega)$ of C^k functions on Ω. The latter does not carry a topology yet, but from now on we will assume that it is equipped with the topology of $(\mathscr{C}(\Omega))^k$, which allows us to write $\mathscr{C}^k(\Omega) = (\mathscr{C}(\Omega))^k$. ⊘

Example 2.9.9. Let $k \in \mathbb{N}$. Then $(\mathscr{C}_b(\Omega))^k$ coincides with the vector space of those C^k functions on Ω whose partial derivatives up to order k are bounded and we define $\mathscr{C}_b^k(\Omega) := (\mathscr{C}_b(\Omega))^k$. Similary, $(\mathscr{C}_0(\Omega))^k$ coincides with the vector space of C^k functions on Ω whose partial derivatives up to order k 'vanish at infinity' and we define $\mathscr{C}_0^k(\Omega) := (\mathscr{C}_0(\Omega))^k$, while $(\mathscr{C}_s(\Omega))^k$ coincides with the vector space of C^k functions on Ω whose partial derivatives up to order k 'become constant at infinity' and we define $\mathscr{C}_s^k(\Omega) := (\mathscr{C}_s(\Omega))^k$. If $\Omega = \mathbb{R}^{\mathrm{n}}$, we can say something more about $\mathscr{C}_s^k(\Omega)$: it easily follows that for an element of $\mathscr{C}_s^k(\mathbb{R}^{\mathrm{n}})$ the partial derivatives up to order k that have nonzero order actually vanish at infinity (i.e., 'their constants' must equal zero). ⊘

So far we have restricted ourselves to $k \in \mathbb{N}$ when speaking about $\mathscr{F}^k$. However, it is also possible to allow $k = \infty$.

Definition 2.9.10. Let $\mathscr{F}$ be a semi-functional space on Ω and let $\mathscr{P}$ be an inducing collection of seminorms for $\mathscr{F}$. As a set, we define

$$\mathscr{F}^\infty := \{u \in \mathscr{D}'(\Omega) \mid \partial^\alpha u \in \mathscr{F} \text{ for all } |\alpha| < \infty\} = \bigcap_{k=0}^{\infty} \mathscr{F}^k.$$

Subsequently, we define for each $p \in \mathscr{P}$ and $k \in \mathbb{N}$

$$p_k \colon \mathscr{F}^\infty \to \mathbb{R} \colon u \mapsto \sum_{|\alpha|<k+1} p(\partial^\alpha u).$$

We easily see that $\mathscr{F}^\infty$ is a vector subspace of $\mathscr{D}'(\Omega)$ and that the p_k, with $p \in \mathscr{P}$ and $k \in \mathbb{N}$, are seminorms on $\mathscr{F}^\infty$. We endow $\mathscr{F}^\infty$ with the topology induced by these seminorms. ⊘

In a similar way as before, we see that the topology on $\mathscr{F}^\infty$ is independent of the chosen inducing collection of seminorms for $\mathscr{F}$, that for all $|\alpha| < \infty$, $\partial^\alpha \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a linear map from $\mathscr{F}^\infty$ into $\mathscr{F}$ and that $\mathscr{F}^\infty$ is equipped with the smallest locally convex topology such that $\partial^\alpha \colon \mathscr{F}^\infty \to \mathscr{F}$ is continuous for every $|\alpha| < \infty$. In addition, when looking at the proof of Lemma 2.9.3, we see that only minor adjustments are necessary to show that this lemma also holds for $k = \infty$. As a consequence, we have the following improved version of Corollary 2.9.4.

Corollary 2.9.11. *Let $\mathscr{F}$ be a semi-functional space on Ω, $k \in \mathbb{N}_\infty$, $\mathscr{X}$ a locally convex vector space and $T \colon \mathscr{X} \to \mathscr{D}'(\Omega)$ a linear map. If for every $|\alpha| < k+1$, $\partial^\alpha \circ T$ is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}$, then T is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}^k$.*

An easy adaption of the proof of Proposition 2.9.5, using Corollary 2.9.11 instead of Corollary 2.9.4, subsequently gives:

Proposition 2.9.12. *Let $\mathscr{F}$ be a semi-functional space on Ω. Then $\mathscr{F}^\infty$ is a semi-functional space on Ω as well and $\mathscr{F}^\infty \subseteq_c \mathscr{F}^k$ for every $k \in \mathbb{N}$. If $\mathscr{F}$ is a functional space on Ω, then so is $\mathscr{F}^\infty$.*

So for every functional space $\mathscr{F}$ on Ω, we have a nice chain of continuous inclusions

$$\mathscr{D}(\Omega) \subseteq_c \mathscr{F}^\infty \subseteq_c \dots \subseteq_c \mathscr{F}^{k+1} \subseteq_c \mathscr{F}^k \subseteq_c \dots \subseteq_c \mathscr{F}^1 \subseteq_c \mathscr{F} \subseteq_c \mathscr{D}'(\Omega)$$

(if $\mathscr{F}$ is only semi-functional, we have to remove '$\mathscr{D}(\Omega) \subseteq_c$').

Example 2.9.13. We easily verify that $(\mathscr{C}(\Omega))^\infty = \mathscr{E}(\Omega)$. Moreover, we can extend the definitions of $\mathscr{C}_b^k(\Omega)$, $\mathscr{C}_0^k(\Omega)$ and $\mathscr{C}_s^k(\Omega)$ to $k = \infty$. Then $\mathscr{C}_b^\infty(\Omega)$ becomes the space of smooth function on Ω with bounded partial derivatives and $\mathscr{C}_0^\infty(\Omega)$, respectively $\mathscr{C}_s^\infty(\Omega)$, becomes the space of smooth functions on Ω whose partial derivatives 'vanish at infinity', respectively 'become constant at infinity'. ⊘

Of course, the assignment $\mathscr{F} \mapsto \mathscr{F}^k$ is a functor.

Lemma 2.9.14. *If $\mathscr{F}$ and $\mathscr{G}$ are semi-functional spaces on Ω and*

$$T \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$$

is a linear map such that:

1. *T restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{G}$ and*
2. *$\partial^\alpha \circ T = T \circ \partial^\alpha$ for all multi-indices α,*

then for all $k \in \mathbb{N}_\infty$, T restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{G}^k$.

Proof: Consider T as linear map from $\mathscr{F}^k$ into $\mathscr{D}'(\Omega)$. For every $|\alpha| < k+1$, we have that $\partial^\alpha \circ T = T \circ \partial^\alpha$ as linear map from $\mathscr{F}^k$ into $\mathscr{D}'(\Omega)$. Since ∂^α restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$ and T restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{G}$, we see that $\partial^\alpha \circ T = T \circ \partial^\alpha \colon \mathscr{F}^k \to \mathscr{D}'(\Omega)$ is actually a continuous linear map from $\mathscr{F}^k$ into $\mathscr{G}$. Applying Corollary 2.9.11 now gives the desired result. □

Proposition 2.9.15. *For every $k \in \mathbb{N}_\infty$, we have that the assignment $\mathscr{F} \mapsto \mathscr{F}^k$ is a functor from the category of (semi-)functional spaces on Ω to the category of (semi-)functional spaces on Ω.*

Proof: This follows straight from the previous lemma. After all, if $\mathscr{F}$ and $\mathscr{G}$ are (semi-)functional spaces on Ω such that $\mathscr{F} \subseteq_c \mathscr{G}$, then the identity $\mathrm{id}_{\mathscr{D}'(\Omega)} \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ satisfies the conditions of the previous lemma, so $\mathrm{id}_{\mathscr{D}'(\Omega)}$ restricts to a continuous map from $\mathscr{F}^k$ into $\mathscr{G}^k$ and this precisely means that $\mathscr{F}^k \subseteq_c \mathscr{G}^k$. □

The interaction of two construction functors of the type $\mathscr{F} \mapsto \mathscr{F}^k$ looks very pretty and natural.

Lemma 2.9.16. *For every semi-functional space $\mathscr{F}$ on Ω and all $k, \ell \in \mathbb{N}_\infty$, we have*

$$(\mathscr{F}^k)^\ell = \mathscr{F}^{k+\ell}.$$

Proof: For every $|\alpha| < k + \ell + 1$, we can find multi-indices β and γ such that $|\alpha| = |\beta| + |\gamma|$, $|\beta| < k + 1$, $|\gamma| < \ell + 1$ and $\partial^\alpha = \partial^\beta \circ \partial^\gamma$. Then ∂^γ restricts to a continuous linear map from $(\mathscr{F}^k)^\ell$ into $\mathscr{F}^k$ and ∂^β restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$, so we see that $\partial^\alpha = \partial^\beta \circ \partial^\gamma$ restricts to a continuous linear map from $(\mathscr{F}^k)^\ell$ into $\mathscr{F}$ for every $|\alpha| < k+\ell+1$. Applying Corollary 2.9.11 to the inclusion map $(\mathscr{F}^k)^\ell \hookrightarrow \mathscr{D}'(\Omega)$ now gives $(\mathscr{F}^k)^\ell \subseteq_c \mathscr{F}^{k+\ell}$.

To prove the converse inclusion, $\mathscr{F}^{k+\ell} \subseteq_c (\mathscr{F}^k)^\ell$, we will use Corollary 2.9.11 twice. First observe that, because of this corollary, it suffices to prove that ∂^γ restricts to a continuous linear map from $\mathscr{F}^{k+\ell}$ into $\mathscr{F}^k$ for every $|\gamma| < \ell + 1$. But, again thanks to Corollary 2.9.11, in order to prove that ∂^γ restricts to a continuous linear map from $\mathscr{F}^{k+\ell}$ into $\mathscr{F}^k$ it suffices to prove that $\partial^\beta \circ \partial^\gamma$ restricts to a continuous linear map from $\mathscr{F}^{k+\ell}$ into $\mathscr{F}$ for every $|\beta| < k + 1$. Since $\partial^\beta \circ \partial^\gamma$ equals ∂^α for some multi-index α with $|\alpha| = |\beta| + |\gamma| < k + \ell + 1$, this is clear. □

Also the interplay between the spaces $\mathscr{F}^k$ and the partial derivatives ∂^α is very natural, which is actually one of the main reasons for introducing $\mathscr{F}^k$.

Proposition 2.9.17. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}_\infty$. For all $|\alpha| < k+1$ and $|\alpha| \leq \ell < k+1$, $\partial^\alpha \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-\ell}$.*

Proof: By the previous lemma, $\mathscr{F}^k = \mathscr{F}^{(k-|\alpha|)+|\alpha|} = (\mathscr{F}^{k-|\alpha|})^{|\alpha|}$. So for every $|\beta| < |\alpha| + 1$, ∂^β restricts to a continuous linear map from $\mathscr{F}^k = (\mathscr{F}^{k-|\alpha|})^{|\alpha|}$ into $\mathscr{F}^{k-|\alpha|}$. Since clearly $|\alpha| < |\alpha| + 1$, we in particular have that ∂^α restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\alpha|}$ and because $\mathscr{F}^{k-|\alpha|} \subseteq_c \mathscr{F}^{k-\ell}$ (after all, $k - \ell \leq k - |\alpha|$), the result follows. □

As a direct consequence, every differential operator on Ω with constant coefficients and order $\ell \in \mathbb{N}$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-\ell}$ for every $k \in \mathbb{N}_\infty$ with $\ell \leq k$. Because of the 'multiplication axiom' of semi-functional spaces, the same is true for differential operators whose coefficients are compactly supported and if $\mathscr{F}$ is semi-local, the statement even holds for all differential operators on Ω.

Of course, it is very elegant that differential operators (of a special type) can be viewed as continuous linear maps between semi-functional spaces of the form $\mathscr{F}^k$. However, this would still be of limited value if we cannot guarantee that the spaces $\mathscr{F}^k$ are 'good enough' to work with. So we would like to have that for a 'nice' (semi-)functional space $\mathscr{F}$ on Ω, the $\mathscr{F}^k$ are also 'nice'. In other words, we would like to have some results about the preservation of properties under $\mathscr{F} \mapsto \mathscr{F}^k$.

Lemma 2.9.18. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}_\infty$. If $\mathscr{F}$ is metrizable, then $\mathscr{F}^k$ is metrizable as well.*

Proof: Because semi-functional spaces are always Hausdorff, a semi-functional space is metrizable if and only if there exists a countable inducing collection of seminorms. So suppose that $\mathscr{P}$ is a countable inducing collection of seminorms for $\mathscr{F}$ and let $\mathscr{P}_k$ be the associated inducing collection of seminorms for $\mathscr{F}^k$. If $k \neq \infty$, $\mathscr{P}_k = \{p_k \mid p \in \mathscr{P}\}$, thus $|\mathscr{P}_k| \leq |\mathscr{P}|$ and $\mathscr{P}_k$ is countable. If $k = \infty$, $\mathscr{P}_k = \{p_\ell \mid p \in \mathscr{P} \text{ and } \ell \in \mathbb{N}\} = \cup_{\ell\in\mathbb{N}}\{p_\ell \mid p \in \mathscr{P}\}$, which is a countable union of countable sets, hence countable. □

Lemma 2.9.19. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}$. If $\mathscr{F}$ is normable, then $\mathscr{F}^k$ is normable as well.*

Proof: If $\|\cdot\|$ is a norm on $\mathscr{F}$ which induces the topology of $\mathscr{F}$, then $\|\cdot\|_k$ is a norm on $\mathscr{F}^k$ which induces the topology of $\mathscr{F}^k$ (note that $\|u\|_k = 0$ in particular implies $\|u\| = 0$). □

Lemma 2.9.20. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}$. If the topology of $\mathscr{F}$ is induced by an inner product, then the topology of $\mathscr{F}^k$ is induced by an inner product as well.*

Proof: Let $\langle \cdot \mid \cdot \rangle$ be an inner product on $\mathscr{F}$ which induces the topology of $\mathscr{F}$. This means that the norm

$$\|\cdot\| \colon \mathscr{F} \to \mathbb{R} \colon u \mapsto \sqrt{\langle u \mid u \rangle}$$

induces the topology of $\mathscr{F}$. Hence, by definition of $\mathscr{F}^k$, the norm

$$\|\cdot\|_k \colon \mathscr{F}^k \to \mathbb{R} \colon u \mapsto \sum_{|\alpha|<k+1} \sqrt{\langle \partial^\alpha u \mid \partial^\alpha u \rangle}$$

induces the topology of $\mathscr{F}^k$ and we easily check that

$$\langle \cdot \mid \cdot \rangle_k \colon \mathscr{F}^k \times \mathscr{F}^k \to \mathbb{K} \colon (u, v) \mapsto \sum_{|\alpha|<k+1} \langle \partial^\alpha u \mid \partial^\alpha v \rangle$$

is an inner product on $\mathscr{F}^k$ whose associated norm is equivalent to $\|\cdot\|_k$ (for the latter statement, it is convenient to use the fact that on Euclidean space the Manhattan norm and Euclidean norm are equivalent). □

Remark 2.9.21. Let $k \in \mathbb{N}_\infty$. It is a trivial consequence of the definition of $\mathscr{F}^k$ that for a net $\{u_i\}_{i\in I}$ in $\mathscr{F}^k$ and an element u of $\mathscr{F}^k$, $u_i \to u$ in $\mathscr{F}^k$ if and only if $\partial^\alpha u_i \to \partial^\alpha u$ in $\mathscr{F}$ for every $|\alpha| < k + 1$. ⊘

Proposition 2.9.22. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}_\infty$. If $\mathscr{F}$ is complete, then $\mathscr{F}^k$ is complete as well.*

Proof: Let $\{u_i\}_{i\in I}$ be a Cauchy net in $\mathscr{F}^k$. Because for every $|\alpha| < k + 1$, ∂^α is a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$, $\{\partial^\alpha u_i\}_{i\in I}$ is a Cauchy net in $\mathscr{F}$ for every $|\alpha| < k + 1$ (see Lemma A.1.7). So using the completeness of $\mathscr{F}$, we find for every $|\alpha| < k + 1$ an $v_\alpha \in \mathscr{F}$ such that $\partial^\alpha u_i \to v_\alpha$ in $\mathscr{F}$ and since $\mathscr{F} \subseteq_c \mathscr{D}'(\Omega)$, we also have $\partial^\alpha u_i \to v_\alpha$ in $\mathscr{D}'(\Omega)$.

Now let $\bar{0}$ be the multi-index with all entries equal to zero and take $u := v_{\bar{0}}$. We then have that $u_i = \partial^{\bar{0}} u_i \to v_{\bar{0}} = u$ in $\mathscr{D}'(\Omega)$ and because for every multi-index α, $\partial^\alpha \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ is continuous, we find that $\partial^\alpha u_i \to \partial^\alpha u$ in $\mathscr{D}'(\Omega)$ for every $|\alpha| < k + 1$. But we already had that $\partial^\alpha u_i \to v_\alpha$ in $\mathscr{D}'(\Omega)$, so by Hausdorffness of $\mathscr{D}'(\Omega)$ we conclude that $\partial^\alpha u = v_\alpha \in \mathscr{F}$ for all $|\alpha| < k + 1$. This shows that $u \in \mathscr{F}^k$ and, when combined with the fact that $\partial^\alpha u_i \to v_\alpha$ in $\mathscr{F}$, that $\partial^\alpha u_i \to \partial^\alpha u$ in $\mathscr{F}$ for every $|\alpha| < k + 1$, which on behalf of the preceding remark means that $u_i \to u$ in $\mathscr{F}^k$. □

Corollary 2.9.23. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}_\infty$. If $\mathscr{F}$ is Fréchet, then $\mathscr{F}^k$ is Fréchet as well.*

Corollary 2.9.24. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}$. If $\mathscr{F}$ is Banach, then $\mathscr{F}^k$ is Banach as well.*

Corollary 2.9.25. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}$. If $\mathscr{F}$ is Hilbert, then $\mathscr{F}^k$ is Hilbert as well.*

So for the 'topological vector space properties' the results are very nice and we can use these results to effortlessly prove some properties of a few of our favorite examples.

Example 2.9.26. For every $k \in \mathbb{N}$, $\mathscr{C}^k(\Omega) = (\mathscr{C}(\Omega))^k$ is Fréchet and we also have that $\mathscr{E}(\Omega) = (\mathscr{C}(\Omega))^\infty$ is Fréchet (use Example 2.4.3). ⊘

Example 2.9.27. For every $k \in \mathbb{N}$, $\mathscr{C}_b^k(\Omega)$, $\mathscr{C}_s^k(\Omega)$ and $\mathscr{C}_0^k(\Omega)$ are Banach and $\mathscr{C}_b^\infty(\Omega)$, $\mathscr{C}_s^\infty(\Omega)$ and $\mathscr{C}_0^\infty(\Omega)$ are Fréchet (use Example 2.4.4). ⊘

Example 2.9.28. For every $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(\Omega)$ is Banach and $W^{k,2}(\Omega)$ is even Hilbert (use Example 2.4.5). Along with the special property of $W^{k,2}(\Omega)$ comes special notation: it is customary to denote $W^{k,2}(\Omega)$ by $H^k(\Omega)$. ⊘

The next four results show that also locality and semi-locality behave perfectly under $\mathscr{F} \mapsto \mathscr{F}^k$.

Lemma 2.9.29. *For every semi-functional space $\mathscr{F}$ on Ω and all $k \in \mathbb{N}_\infty$, we have*

$$(\mathscr{F}^k)_{\mathrm{loc}} = (\mathscr{F}_{\mathrm{loc}})^k.$$

Proof: Thanks to Corollary 2.6.3 and Corollary 2.9.11 it suffices to prove that for every $|\alpha| < k+1$ and $\varphi \in \mathscr{D}(\Omega)$

1. $\partial^\alpha \circ m_\varphi$ restricts to a continuous linear map from $(\mathscr{F}_{\mathrm{loc}})^k$ into $\mathscr{F}$ and
2. $m_\varphi \circ \partial^\alpha$ restricts to a continuous linear map from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}$.

The first statement is an easy consequence of

$$\partial^\alpha \circ m_\varphi = \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} m_{\partial^{\alpha-\beta}\varphi} \circ \partial^\beta,$$

but the second statement is a bit more work. We will prove it by induction on $\{0, \ldots, k\}$ from the following induction hypothesis (where $n \in \mathbb{N}$ with $n \leq k$): for every $|\alpha| < n+1$ and $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \circ \partial^\alpha$ restricts to a continuous linear map from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}^{k-|\alpha|}$.

For $n = 0$, the only multi-index α that satisfies $|\alpha| < n+1$ is the multi-index with all entries equal to zero. But then $m_\varphi \circ \partial^\alpha$ becomes m_φ, which restricts to a continuous linear map from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}^k$ and therefore, since $k - |\alpha| = k$, to a continuous linear map from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}^{k-|\alpha|}$. So for $n = 0$ the induction hypothesis holds.

Now suppose that the induction hypothesis holds for $n = m$ with $0 \leq m < k$. To prove that the induction hypothesis then also holds for $n = m+1$, let α be some multi-index with $0 < |\alpha| < m+2$ (we already have covered the $|\alpha| = 0$ case) and $\varphi \in \mathscr{D}(\Omega)$. Clearly, we have $\partial^\alpha = \partial_i \circ \partial^\beta$ for some $1 \leq i \leq$ n and some multi-index β with $|\beta| = |\alpha| - 1 < m + 1$. Furthermore, because of the Leibniz rule we then have

$$m_\varphi \circ \partial^\alpha = m_\varphi \circ \partial_i \circ \partial^\beta = \partial_i \circ m_\varphi \circ \partial^\beta - m_{\partial_i \varphi} \circ \partial^\beta.$$

Since the induction hypothesis by assumption holds for $n = m$, we find that $m_\varphi \circ \partial^\beta$ and $m_{\partial_i\varphi} \circ \partial^\beta$ restrict to continuous linear maps from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}^{k-|\beta|}$. Proposition 2.9.17 and the fact that $\mathscr{F}^{k-|\beta|} \subseteq_{\mathrm{c}} \mathscr{F}^{k-|\beta|-1}$ then give that $\partial_i \circ m_\varphi \circ \partial^\beta$ and $m_{\partial_i\varphi} \circ \partial^\beta$ restrict to continuous linear maps from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}^{k-|\beta|-1} = \mathscr{F}^{k-|\alpha|}$. As a consequence, also $m_\varphi \circ \partial^\alpha = \partial_i \circ m_\varphi \circ \partial^\beta - m_{\partial_i\varphi} \circ \partial^\beta$ restricts to a continuous linear map from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}^{k-|\alpha|}$. This shows that the induction hypothesis also holds for $n = m + 1$.

By induction on $\{0, \ldots, k\}$ we now deduce that the induction hypothesis holds for every $0 \leq n < k + 1$. Accordingly, we get: for every $|\alpha| < k + 1$ and $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi \circ \partial^\alpha$ restricts to a continuous linear map from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}^{k-|\alpha|} \subseteq_{\mathrm{c}} \mathscr{F}$. Since this is precisely what we needed to prove, we are done. ☐

Proposition 2.9.30. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}_\infty$. If $\mathscr{F}$ is local, then so is $\mathscr{F}^k$.*

Proof: This is a direct consequence of the previous lemma. Indeed, if $\mathscr{F}_{\mathrm{loc}} = \mathscr{F}$, then $(\mathscr{F}^k)_{\mathrm{loc}} = (\mathscr{F}_{\mathrm{loc}})^k = \mathscr{F}^k$. ☐

Lemma 2.9.31. *For every semi-functional space $\mathscr{F}$ on Ω and all $k \in \mathbb{N}_\infty$, we have*

$$(\mathscr{F}^k)_{\mathrm{semi}} = (\mathscr{F}_{\mathrm{semi}})^k.$$

Proof: The proof is completely analogous to the proof of Lemma 2.9.29, just use $\mathscr{E}(\Omega)$ instead of $\mathscr{D}(\Omega)$. ☐

Proposition 2.9.32. *Let $\mathscr{F}$ be a semi-functional space on Ω and $k \in \mathbb{N}_\infty$. If $\mathscr{F}$ is semi-local, then so is $\mathscr{F}^k$.*

Proof: If $\mathscr{F}_{\mathrm{semi}} = \mathscr{F}$, then $(\mathscr{F}^k)_{\mathrm{semi}} = (\mathscr{F}_{\mathrm{semi}})^k = \mathscr{F}^k$. ☐

Unfortunately, for normality and invariance the situation is not so ideal. For example, while $L^2(\Omega)$ is normal, it can be shown that $H^1(\Omega) = (L^2(\Omega))^1$ is not always normal (see [13, page 324]). So normality is in general not preserved under $\mathscr{F} \mapsto \mathscr{F}^k$ and since we have already seen that $L^1(\Omega)$ is invariant, the following claim shows that also invariance is not preserved.

Claim. $W^{1,1}((0,1)) = (L^1((0,1)))^1$ is not invariant.

Proof: Define $\chi\colon (0,1) \to (0,1)$ by $\chi(x) := x^2$ and $f\colon (0,1) \to \mathbb{K}$ by $f(x) := x^{\frac{1}{2}}$. Then χ is a diffeomorphism and $[f] \in W^{1,1}((0,1))$ (indeed, both $f = x^{\frac{1}{2}}$ and $\partial f = \frac{1}{2}x^{-\frac{1}{2}}$ are integrable on $(0,1)$). However, by Lemma 1.7.4,

$$\chi_* f = \frac{f}{|\det D\chi|} \circ \chi^{-1} = \frac{1}{2} x^{-\frac{1}{4}},$$

whose equivalence class is not an element of $W^{1,1}((0,1))$ because its derivative, $-\frac{1}{8}x^{-\frac{5}{4}}$, is not integrable on $(0,1)$. ☐

Of course, it is pitiful that invariance is not preserved. However, it turns out that for semi-local functional spaces we do have a nice preservation result for invariance and since we will be mainly interested in the *combination* of invariance and (semi-)locality, there is no reason to despair.

Lemma 2.9.33. *Let $\mathscr{F}$ be a semi-local functional space on Ω, $k \in \mathbb{N}_\infty$ and $\chi\colon \Omega \to \Omega$ a diffeomorphism. For every $|\alpha| < k+1$, $(\chi^{-1})_* \circ \partial^\alpha \circ \chi_*$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\alpha|}$.*

Proof: We will prove this by induction on $\{0,\dots,k\}$ from the following induction hypothesis (where $n \in \mathbb{N}$ with $n < k+1$): for every $|\alpha| < n+1$, $(\chi^{-1})_* \circ \partial^\alpha \circ \chi_*$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\alpha|}$.

For $n = 0$, the only multi-index α that satisfies $|\alpha| < n+1$ is the multi-index with all entries equal to zero. But then $(\chi^{-1})_* \circ \partial^\alpha \circ \chi_*$ equals $\mathrm{id}_{\mathscr{D}'(\Omega)}$, which clearly restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\alpha|} = \mathscr{F}^k$. So for $n = 0$ the induction hypothesis holds.

Now suppose that the induction hypothesis holds for $n = m$ with $0 \leq m < k$. To prove that the induction hypothesis then also holds for $n = m+1$, let α be some multi-index with $0 < |\alpha| < m+2$ (we already have covered the $|\alpha| = 0$ case). Clearly, we have $\partial^\alpha = \partial_i \partial^\beta$ for some $1 \leq i \leq \mathrm{n}$ and some multi-index β with $|\beta| = |\alpha| - 1 < m+1$. Furthermore, because of Lemma 1.7.7, we find that for every $u \in \mathscr{D}'(\Omega)$

$$(\chi^{-1})_* \partial^\alpha \chi_* u = (\chi^{-1})_* \partial_i \partial^\beta \chi_* u$$
$$= \sum_{j=1}^{\mathrm{n}} (\partial_i \xi_j \circ \chi) \partial_j (\chi^{-1})_* \partial^\beta \chi_* u + \sum_{j=1}^{\mathrm{n}} \sum_{\ell=1}^{\mathrm{n}} (\partial_\ell \partial_i \xi_j \circ \chi)(\partial_j \chi_\ell)(\chi^{-1})_* \partial^\beta \chi_* u,$$

where ξ is again short for χ^{-1}. Since the induction hypothesis by assumption holds for $n = m$, $(\chi^{-1})_* \partial^\beta \chi_*$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\beta|}$ and if we combine this with Proposition 2.9.17, $\mathscr{F}^{k-|\beta|} \subseteq_{\mathrm{c}} \mathscr{F}^{k-|\alpha|}$, the semi-locality of $\mathscr{F}^{k-|\alpha|}$ (use Proposition 2.9.32) and the expression above, we see that $(\chi^{-1})_* \partial^\alpha \chi_*$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\alpha|}$. Hence the induction hypothesis also holds for $n = m+1$.

By induction on $\{0,\dots,k\}$ we now deduce that the induction hypothesis holds for every $0 \leq n < k+1$ and the result follows. □

Proposition 2.9.34. *Let $\mathscr{F}$ be a semi-local functional space on Ω and $k \in \mathbb{N}_\infty$. If $\mathscr{F}$ is invariant, then $\mathscr{F}^k$ is invariant as well.*

Proof: Let $\chi\colon \Omega \to \Omega$ be a diffeomorphism. To prove that $\mathscr{F}^k$ is invariant, it suffices to prove that $\chi_*\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^k$ for which it in turn suffices to prove that $\partial^\alpha \circ \chi_*$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$ for all $|\alpha| < k+1$ (use Corollary 2.9.4).

So let $|\alpha| < k+1$. By the previous lemma, $(\chi^{-1})_* \circ \partial^\alpha \circ \chi_*$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\alpha|}$, hence in particular to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$. Moreover, because of the assumed invariance of $\mathscr{F}$, χ_* restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$, so we subsequently find that $\partial^\alpha \circ \chi_* = \chi_* \circ (\chi^{-1})_* \circ \partial^\alpha \circ \chi_*$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$. □

Now that we have discussed the preservation under $\mathscr{F} \mapsto \mathscr{F}^k$ of the 'main' version of all properties that we have encountered so far, it remains to discuss the preservation of some of the 'local' versions.

Lemma 2.9.35. *For every semi-functional space $\mathscr{F}$ on Ω, all $k \in \mathbb{N}_\infty$ and all $K \in \mathscr{P}_c(\Omega)$, we have*

$$(\mathscr{F}^k)_K = (\mathscr{F}_K)^k.$$

Proof: For every $|\alpha| < k+1$, ∂^α restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$ and because ∂^α is local, this implies that ∂^α restricts to a continuous linear map from $(\mathscr{F}^k)_K$ into $\mathscr{F}_K$. Applying Corollary 2.9.4 to the inclusion $(\mathscr{F}^k)_K \hookrightarrow \mathscr{D}'(\Omega)$ then gives $(\mathscr{F}^k)_K \subseteq_c (\mathscr{F}_K)^k$. For the converse inclusion, observe that $\mathscr{F}_K \subseteq_c \mathscr{F}$ and the fact that $\mathscr{F} \mapsto \mathscr{F}^k$ is a functor together imply $(\mathscr{F}_K)^k \subseteq_c \mathscr{F}^k$, while $(\mathscr{F}_K)^k \subseteq_c \mathscr{F}_K$ shows that the inclusion $(\mathscr{F}_K)^k \hookrightarrow \mathscr{F}^k$ lands inside $(\mathscr{F}^k)_K$. □

Lemma 2.9.36. *Let $\mathscr{F}$ be a semi-functional space on Ω, $k \in \mathbb{N}_\infty$ and let P be short for: metrizable, normable, complete or Fréchet. Then $\mathscr{F}$ is locally P implies $\mathscr{F}^k$ is locally P. If $k < \infty$, the same holds when P is short for Banach or Hilbert.*

Proof: Let $K \in \mathscr{P}_c(\Omega)$. Because $\mathscr{F}$ is locally P, $\mathscr{F}_K$ is P and since all the mentioned properties are preserved under $\mathscr{F} \mapsto \mathscr{F}^k$ (for Banach and Hilbert we use here that $k < \infty$), $(\mathscr{F}^k)_K = (\mathscr{F}_K)^k$ is P as well and this precisely means that $\mathscr{F}^k$ is locally P. □

In constrast with the preservation of invariance, it turns out that for the preservation of *local* invariance under $\mathscr{F} \mapsto \mathscr{F}^k$ we do not have to make any assumptions about (semi-)locality.

Proposition 2.9.37. *Let $\mathscr{F}$ be a functional space on Ω and $k \in \mathbb{N}_\infty$. If $\mathscr{F}$ is locally invariant, then so is $\mathscr{F}^k$.*

Proof: If $\mathscr{F}$ is locally invariant, $\mathscr{F}_{\mathrm{loc}}$ is invariant (Proposition 2.6.29) and because $\mathscr{F}_{\mathrm{loc}}$ is also semi-local, we obtain that $(\mathscr{F}^k)_{\mathrm{loc}} = (\mathscr{F}_{\mathrm{loc}})^k$ is invariant (Lemma 2.9.29 and Proposition 2.9.34). But then $(\mathscr{F}^k)_{\mathrm{loc}}$ is certainly locally invariant, which implies that $\mathscr{F}^k$ is locally invariant (Lemma 2.6.13). □

Example 2.9.38. By combining the previous proposition with Example 2.4.11, we deduce that for every $1 \leq p \leq \infty$ and $k \in \mathbb{N}$, the Sobolev space $W^{k,p}(\Omega)$ is locally invariant. ⊘

We end this section with a result about normality.

Lemma 2.9.39. *Let $\mathscr{F}$ be a semi-functional space on Ω. If T is a continuous linear functional on $\mathscr{F}^\infty$, then there exists an $k \in \mathbb{N}$ such that T extends to a continuous linear functional on $\mathscr{F}^k$.*

Proof: Let $\mathscr{P}$ be an arbitrary inducing collection of seminorms for $\mathscr{F}$. Because $T\colon \mathscr{F}^\infty \to \mathbb{K}$ is continuous, we find $C \geq 0$, $p_0, \ldots, p_n \in \mathscr{P}$ and $k_0, \ldots, k_n \in \mathbb{N}$ such that $|T(u)| \leq C \sum_{i=0}^n (p_i)_{k_i}(u)$ for all $u \in \mathscr{F}^\infty$. Take $k := \max_{0 \leq i \leq n} k_i$. Then $|T(u)| \leq C \sum_{i=0}^n (p_i)_k(u)$ for all $u \in \mathscr{F}^\infty$ and because $\mathscr{P}_k = \{p_k \mid p \in \mathscr{P}\}$ is an inducing collection of seminorms for $\mathscr{F}^k$, this shows that $T\colon \mathscr{F}^\infty \to \mathbb{K}$ is also continuous if we consider the restricted topology of $\mathscr{F}^k$ on $\mathscr{F}^\infty$. It now follows from the Hahn-Banach Theorem that T has a continuous linear extension to $\mathscr{F}^k$. □

Proposition 2.9.40. *Let $\mathscr{F}$ be a functional space on Ω. If $\mathscr{F}^k$ is normal for every $k \in \mathbb{N}$, then $\mathscr{F}^\infty$ is normal as well.*

Proof: Suppose that T is a continuous linear functional on $\mathscr{F}^\infty$ that vanishes on $\mathscr{D}(\Omega)$. By the previous lemma we find an $k \in \mathbb{N}$ such that T extends to a continuous linear functional $\hat{T}$ on $\mathscr{F}^k$. Clearly, $\hat{T}$ still vanishes on $\mathscr{D}(\Omega)$ and since $\mathscr{D}(\Omega)$ is dense in $\mathscr{F}^k$, this implies that $\hat{T} = 0$, which in turn implies that $T = 0$. On the strength of Lemma A.2.1, we conclude that $\mathscr{D}(\Omega)$ must be dense in $\mathscr{F}^\infty$. □

2.10 Duals

In this section we look at a construction that we already know from the theory of locally convex vector spaces: dualizing. (Recall that by the dual $\mathscr{X}^*$ of a locally convex vector space $\mathscr{X}$, we always mean the *strong* dual.) Well, to be more precise, we look at a *translation* of this procedure to the context of functional spaces. For this, we need to work with *normal* functional spaces and in order to give a precise definition, we will temporarily make the distinction between $\mathscr{D}(\Omega)$ and its canonical identification with a subspace of $\mathscr{D}'(\Omega)$ and explicitly use the canonical identification map $\jmath\colon \mathscr{D}(\Omega) \to \mathscr{D}'(\Omega)$.

Definition 2.10.1. Let $\mathscr{F}$ be a normal functional space on Ω and let

$$\imath\colon \jmath(\mathscr{D}(\Omega)) \to \mathscr{F} \quad \text{and} \quad \imath'\colon \mathscr{F} \to \mathscr{D}'(\Omega)$$

be the inclusion mappings. Because $\jmath(\mathscr{D}(\Omega))$ is dense in $\mathscr{F}$ (by normality of $\mathscr{F}$) and $\mathscr{F}$ is dense in $\mathscr{D}'(\Omega)$ (see Lemma 2.1.3), the adjoints of the continuous linear maps

$$\imath \circ \jmath\colon \mathscr{D}(\Omega) \to \mathscr{F} \quad \text{and} \quad \imath'\colon \mathscr{F} \to \mathscr{D}'(\Omega),$$

denoted by

$$(\imath \circ \jmath)^*\colon \mathscr{F}^* \to (\mathscr{D}(\Omega))^* = \mathscr{D}'(\Omega) \quad \text{and} \quad (\imath')^*\colon (\mathscr{D}'(\Omega))^* \to \mathscr{F}^*,$$

are injective continuous linear maps (see Lemma A.4.4). We define $\mathscr{F}'$ to be the vector subspace $(\imath \circ \jmath)^*(\mathscr{F}^*)$ of $\mathscr{D}'(\Omega)$ endowed with the topology that turns $(\imath \circ \jmath)^*$ into a linear topological isomorphism from $\mathscr{F}^*$ onto $\mathscr{F}'$. ⊘

Proposition 2.10.2. *$\mathscr{F}'$ is a functional space on Ω.*

Proof: Due to the reflexivity of $\mathscr{D}(\Omega)$ (see Lemma 1.1.16), the 'evaluation in' map $\hat{\imath}\colon \mathscr{D}(\Omega) \to (\mathscr{D}'(\Omega))^*$ is a linear topological isomorphism. So we have a locally convex vector space $\mathscr{F}^*$ together with two injective continuous linear maps $(\imath')^* \circ \hat{\imath}\colon \mathscr{D}(\Omega) \to \mathscr{F}^*$ and $(\imath \circ \jmath)^*\colon \mathscr{F}^* \to \mathscr{D}'(\Omega)$. On behalf of Proposition 2.1.6 the desired result follows if we can prove that

1. $(\imath \circ \jmath)^* \circ (\imath')^* \circ \hat{\imath}\colon \mathscr{D}(\Omega) \to \mathscr{D}'(\Omega)$ equals the canonical identification of $\mathscr{D}(\Omega)$ with a subspace of $\mathscr{D}'(\Omega)$ and

2. for every $\varphi \in \mathscr{D}(\Omega)$, $m_\varphi(\mathscr{F}') \subseteq \mathscr{F}'$ and $((\imath \circ \jmath)^*)^{-1} \circ m_\varphi \circ (\imath \circ \jmath)^* \colon \mathscr{F}^* \to \mathscr{F}^*$ is continuous.

The first statement is a direct consequence of Corollary 1.3.2, which states that $\jmath^* \circ \hat{\imath} = \jmath$ (note that in this corollary the symbol $\jmath$ has a different meaning!). Indeed, using this, we obtain

$$(\imath \circ \jmath)^* \circ (\imath')^* \circ \hat{\imath} = (\imath' \circ \imath \circ \jmath)^* \circ \hat{\imath} = \jmath^* \circ \hat{\imath} = \jmath.$$

For the second statement, note that a distribution $u \in \mathscr{D}'(\Omega)$ is an element of $\mathscr{F}'$ if and only if there exists an $\hat{u} \in \mathscr{F}^*$ (i.e., a continuous linear map $\hat{u} \colon \mathscr{F} \to \mathbb{K}$) such that $\hat{u}(u_\psi) = \hat{u}(\jmath(\psi)) = u(\psi)$ for every $\psi \in \mathscr{D}(\Omega)$. Now fix $\varphi \in \mathscr{D}(\Omega)$, let $u \in \mathscr{F}'$ and let $\hat{u}$ be the corresponding element of $\mathscr{F}^*$. Since $\mathscr{F}$ is a functional space, m_φ is a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$, hence its adjoint m_φ^* is a continuous linear map from $\mathscr{F}^*$ into $\mathscr{F}^*$. Therefore, $m_\varphi^* \hat{u}$ is also an element of $\mathscr{F}^*$ and for every $\psi \in \mathscr{D}(\Omega)$

$$(m_\varphi^* \hat{u})(u_\psi) = \hat{u}(m_\varphi u_\psi) = \hat{u}(u_{\varphi\psi}) = u(\varphi\psi) = (m_\varphi u)(\psi).$$

This shows that $m_\varphi u \in \mathscr{F}'$ and that $((\imath \circ \jmath)^*)^{-1} \circ m_\varphi \circ (\imath \circ \jmath)^* \colon \mathscr{F}^* \to \mathscr{F}^*$ coincides with the continuous linear map $m_\varphi^* \colon \mathscr{F}^* \to \mathscr{F}^*$, so we are done. □

So $\mathscr{F}'$ is just the natural identification of $\mathscr{F}^*$ with a subspace of $\mathscr{D}'(\Omega)$ (and therefore in particular linearly topologically isomorphic to $\mathscr{F}^*$) and if we again forget about the distinction between $\mathscr{D}(\Omega)$ and $\jmath(\mathscr{D}(\Omega))$, the elements of $\mathscr{F}'$ are precisely those distributions $u \colon \mathscr{D}(\Omega) \to \mathbb{K}$ that extend to a continuous linear map $\hat{u} \colon \mathscr{F} \to \mathbb{K}$.

Example 2.10.3. We easily verify that $(\mathscr{D}(\Omega))' = \mathscr{D}'(\Omega)$ and $(\mathscr{E}(\Omega))' = \mathscr{E}'(\Omega)$.⊘

Example 2.10.4. Let $1 \leq p < \infty$ and let $1 \leq q \leq \infty$ be its Hölder conjugate. Then $(L^p(\Omega))' = L^q(\Omega)$ (see [13, page 244-245]). ⊘

Now suppose that $\mathscr{F}$ and $\mathscr{G}$ are normal functional spaces on Ω and that $T \colon \mathscr{F} \to \mathscr{G}$ is a continuous linear map. Then the adjoint T^* of T is a continuous linear map from $\mathscr{G}^*$ into $\mathscr{F}^*$ and since $\mathscr{G}^*$ is linearly topologically isomorphic to $\mathscr{G}'$ and $\mathscr{F}^*$ is linearly topologically isomorphic to $\mathscr{F}'$, we also have an associated continuous linear map T' from $\mathscr{G}'$ into $\mathscr{F}'$. This map can be described as follows: an element of $\mathscr{G}'$ is first 'upgraded' to a continuous linear functional on $\mathscr{G}$, then turned into a continuous linear functional on $\mathscr{F}$ by prepending T and the result is subsequently 'downgraded' to an ordinary distribution.

Proposition 2.10.5. *The assignment $\mathscr{F} \mapsto \mathscr{F}'$ is a contravariant functor from the category of normal functional spaces on Ω to the category of functional spaces on Ω.*

Proof: Suppose that $\mathscr{F}$ and $\mathscr{G}$ are normal functional spaces on Ω such that $\mathscr{F} \subseteq_c \mathscr{G}$ and denote the inclusion map $\mathscr{F} \hookrightarrow \mathscr{G}$ by ς. Then, as we have just discussed, we have a continuous linear map ς' from $\mathscr{G}'$ into $\mathscr{F}'$ and we easily check that $\varsigma'(u) = u$ for every $u \in \mathscr{G}'$. So ς' is in fact an inclusion map and we conclude that $\mathscr{G}' \subseteq_c \mathscr{F}'$. □

The fact that the assignment $\mathscr{F} \mapsto \mathscr{F}'$ is not an ordinary (covariant) functor but a contravariant one is a noteworthy difference between the construction functor $\mathscr{F} \mapsto \mathscr{F}'$ and the construction functors that we have introduced earlier. Moreover, $\mathscr{F} \mapsto \mathscr{F}'$ is not defined on the full category of (semi-)functional spaces on Ω, but only on the category of normal functional spaces (which is, as the name suggests, the category with normal functional spaces on Ω as objects and continuous inclusions as arrows). Since $L^\infty(\Omega) = (L^1(\Omega))'$ and $L^\infty(\Omega)$ is not normal (see Example 2.4.8), we see that normality is in general not preserved under $\mathscr{F} \mapsto \mathscr{F}'$. However, we do have the following:

Lemma 2.10.6. *Let $\mathscr{F}$ be a normal functional space on Ω. If $\mathscr{F}$ is reflexive (as locally convex vector space), then $\mathscr{F}'$ is again normal.*

Proof: Sticking to the notation that we have used so far in this section, the 'inclusion map' from $\mathscr{D}(\Omega)$ into $\mathscr{F}'$ is given by $\jmath = (\imath \circ \jmath)^* \circ (\imath')^* \circ \hat{\imath}$. By Lemma A.4.6, the image of the adjoint $(\imath')^* \colon (\mathscr{D}'(\Omega))^* \to \mathscr{F}^*$ of $\imath' \colon \mathscr{F} \to \mathscr{D}'(\Omega)$ is dense in $\mathscr{F}^*$ and because $(\imath \circ \jmath)^*$ is a linear topological isomorphism from $\mathscr{F}^*$ onto $\mathscr{F}'$ and $\hat{\imath}$ is a linear topological isomorphism from $\mathscr{D}(\Omega)$ onto $(\mathscr{D}'(\Omega))^*$, this implies that the image of $(\imath \circ \jmath)^* \circ (\imath')^* \circ \hat{\imath}$ is dense in $\mathscr{F}'$. □

As a consequence of the previous lemma, for every normal reflexive functional space $\mathscr{F}$ on Ω, $(\mathscr{F}')'$ is a well-defined functional space and it turns out that the reflexivity of $\mathscr{F}$ beautifully translates into $(\mathscr{F}')' = \mathscr{F}$.

Lemma 2.10.7. *For every normal reflexive functional space $\mathscr{F}$ on Ω, we have*

$$(\mathscr{F}')' = \mathscr{F}.$$

Proof: Let ς be the natural linear topological isomorphism from $\mathscr{F}'$ onto $\mathscr{F}^*$, i.e., the map that extends a distribution $u \in \mathscr{F}'$ to a continuous linear map from $\mathscr{F}$ into $\mathbb{K}$ (note that these extensions are unique because ς is invertible with restriction as inverse). We want to find out how ς acts on $\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}'$. So fix $\varphi \in \mathscr{D}(\Omega)$. Then 'evaluation in φ', denoted by $\hat{\imath}_\varphi$, is a continuous linear map from $\mathscr{D}'(\Omega)$ into $\mathbb{K}$ and because $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)$, $\hat{\imath}_\varphi$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathbb{K}$. We claim that this restriction is the unique extension of φ (or more precisely, u_φ) to $\mathscr{F}$. Indeed, for every $\psi \in \mathscr{D}(\Omega)$,

$$\hat{\imath}_\varphi(\psi) = u_\psi(\varphi) = u_\varphi(\psi).$$

Hence, for every $\varphi \in \mathscr{D}(\Omega)$, $\varsigma(\varphi) = \hat{\imath}_\varphi$.

Next, let ϑ be the natural linear topological isomorphism from $(\mathscr{F}')'$ onto $(\mathscr{F}')^*$ and let ϱ be the natural linear topological isomorphism from $\mathscr{F}$ onto $(\mathscr{F}^*)^*$ (which is the usual 'evaluation in' map). Then $\vartheta^{-1} \circ \varsigma^* \circ \varrho$ is a linear topological isomorphism from $\mathscr{F}$ onto $(\mathscr{F}')'$ and for every $u \in \mathscr{F}$ and $\varphi \in \mathscr{D}(\Omega)$, we have

$$((\vartheta^{-1} \circ \varsigma^* \circ \varrho)u)(\varphi) = (\varrho u)(\varsigma\varphi) = (\varsigma\varphi)(u) = (\hat{\imath}_\varphi)(u) = u(\varphi).$$

This shows that the linear topological isomorphism $\vartheta^{-1} \circ \varsigma^* \circ \varrho \colon \mathscr{F} \to (\mathscr{F}')'$ is just the identity map, so $\mathscr{F}$ and $(\mathscr{F}')'$ must be equal. □

It is also possible to relate $\mathscr{F}_{\mathrm{comp}}$ and $\mathscr{F}_{\mathrm{loc}}$ using duals. Before we look at the relevant results, note that for a normal functional space $\mathscr{F}$ on Ω, Lemma 2.5.13 guarantees that $(\mathscr{F}_{\mathrm{comp}})'$ is defined.

Lemma 2.10.8. *For every normal functional space $\mathscr{F}$ on Ω, we have*

$$(\mathscr{F}_{\mathrm{comp}})' \subseteq_{\mathrm{c}} (\mathscr{F}')_{\mathrm{loc}}.$$

Proof: Thanks to Corollary 2.6.3, it suffices to prove that for every $\varphi \in \mathscr{D}(\Omega)$, m_φ restricts to a continuous linear map from $(\mathscr{F}_{\mathrm{comp}})'$ into $\mathscr{F}'$ (because if this is the case, application of Corollary 2.6.3 to the inclusion $(\mathscr{F}_{\mathrm{comp}})' \hookrightarrow \mathscr{D}'(\Omega)$ gives the desired result). So fix $\varphi \in \mathscr{D}(\Omega)$. According to Lemma 2.5.5, m_φ can be viewed as continuous linear map from $\mathscr{F}$ into $\mathscr{F}_{\mathrm{comp}}$ and as a consequence, we get a continuous linear map m'_φ from $(\mathscr{F}_{\mathrm{comp}})'$ into $\mathscr{F}'$. Now let $u \in (\mathscr{F}_{\mathrm{comp}})'$, $\psi \in \mathscr{D}(\Omega)$ and let $\hat{u}$ be the extension of u to $\mathscr{F}_{\mathrm{comp}}$. Then

$$(m'_\varphi u)(\psi) = (m^*_\varphi \hat{u})(\psi) = \hat{u}(\varphi\psi) = u(\varphi\psi) = (m_\varphi u)(\psi),$$

which shows that the restriction of m_φ to $(\mathscr{F}_{\mathrm{comp}})'$ coincides with the continuous linear map $m'_\varphi \colon (\mathscr{F}_{\mathrm{comp}})' \to \mathscr{F}'$ and hence that m_φ indeed restricts to a continuous linear map from $(\mathscr{F}_{\mathrm{comp}})'$ into $\mathscr{F}'$. $\square$

Lemma 2.10.9. *For every normal functional space $\mathscr{F}$ on Ω, we have*

$$(\mathscr{F}')_{\mathrm{loc}} \subseteq_{\mathrm{c}} (\mathscr{F}_{\mathrm{comp}})'.$$

Proof: Take $u \in (\mathscr{F}')_{\mathrm{loc}}$ and let ς be the natural linear topological isomorphism from $\mathscr{F}'$ onto $\mathscr{F}^*$. To prove that $u \in (\mathscr{F}_{\mathrm{comp}})'$, we need to find a continuous linear extension $\hat{u} \colon \mathscr{F}_{\mathrm{comp}} \to \mathbb{K}$. We define $\hat{u}$ as follows: for $v \in \mathscr{F}_{\mathrm{comp}}$ we pick an $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of $\mathrm{supp}(v)$ and we define $\hat{u}(v) := (\varsigma(\varphi u))(v)$ (note that $\varphi u \in \mathscr{F}'$ and $\varsigma(\varphi u) \in \mathscr{F}^*$).

First of all, we need to make sure that the given definition of $\hat{u}(v)$ does not depend on the choice of φ. So let $v \in \mathscr{F}_{\mathrm{comp}}$ and let φ and φ' be elements of $\mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood U_φ of $\mathrm{supp}(v)$ and φ' equals 1 on an open neighborhood $U_{\varphi'}$ of $\mathrm{supp}(v)$. Then also $U_\varphi \cap U_{\varphi'}$ is an open neighborhood of $\mathrm{supp}(v)$ and we can find an $\chi \in \mathscr{D}(\Omega)$ such that $\mathrm{supp}(\chi) \subseteq U_\varphi \cap U_{\varphi'}$ and χ equals 1 on an open neighborhood of $\mathrm{supp}(v)$ (see Remark 1.1.12). Moreover, because $\mathscr{F}_{\mathrm{comp}}$ is normal, we find a net $\{\psi_i\}_{i \in I}$ in $\mathscr{D}(\Omega)$ such that $\psi_i \to v$ in $\mathscr{F}_{\mathrm{comp}}$ and because m_χ restricts to a continuous linear map from $\mathscr{F}_{\mathrm{comp}}$ into $\mathscr{F}_{\mathrm{comp}}$ and $\mathscr{F}_{\mathrm{comp}} \subseteq_{\mathrm{c}} \mathscr{F}$, we also have that $\chi\psi_i \to \chi v = v$ in $\mathscr{F}$. Using the continuity of $\varsigma(\varphi u)$ and $\varsigma(\varphi' u)$, we now find

$$\begin{aligned}(\varsigma(\varphi u))(v) &= \lim_{i\to\infty} (\varsigma(\varphi u))(\chi\psi_i) = \lim_{i\to\infty} (\varphi u)(\chi\psi_i) = \lim_{i\to\infty} u(\varphi\chi\psi_i) \\ &= \lim_{i\to\infty} u(\varphi'\chi\psi_i) = \lim_{i\to\infty} (\varphi' u)(\chi\psi_i) = \lim_{i\to\infty} (\varsigma(\varphi' u))(\chi\psi_i) \\ &= (\varsigma(\varphi' u))(v),\end{aligned}$$

where we have used that $\varphi\chi\psi_i = \varphi'\chi\psi_i$ for every $i \in I$, which is true because $\mathrm{supp}(\chi\psi_i) \subseteq \mathrm{supp}(\chi) \subseteq U_\varphi \cap U_{\varphi'}$ while both φ and φ' equal 1 on $U_\varphi \cap U_{\varphi'}$. Hence, the definition of $\hat{u}(v)$ is indeed independent of the choice of φ.

Next, we want to show that $\hat{u}$ is a continuous linear map. As usual, the linearity is easily verified, so we focus on the continuity. On the strength of Proposition A.3.2, it suffices to prove that for every $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$, $\hat{u}$ is continuous as map from $\mathscr{F}_K$ into $\mathbb{K}$. So fix $K \in \mathscr{P}_{\mathrm{c}}(\Omega)$ and let $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of K. Clearly $\hat{u}$ coincides with $\varsigma(\varphi u)$ on $\mathscr{F}_K$ and since $\varsigma(\varphi u)$ is continuous on $\mathscr{F}_K$, $\hat{u}$ is continuous on $\mathscr{F}_K$ as well.

Finally, we must verify that $\hat{u}$ is indeed an extension of u, but this is easy. Indeed, let $\psi \in \mathscr{D}(\Omega)$ and take $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of $\operatorname{supp}(\psi)$. Then $\hat{u}(\psi) = (\varsigma(\varphi u))(\psi) = (\varphi u)(\psi) = u(\varphi\psi) = u(\psi)$. Altogether, we conclude that $u \in (\mathscr{F}_{\text{comp}})'$ and hence that $(\mathscr{F}')_{\text{loc}} \subseteq (\mathscr{F}_{\text{comp}})'$.

It remains to be shown that the inclusion $(\mathscr{F}')_{\text{loc}} \subseteq (\mathscr{F}_{\text{comp}})'$ is continuous. To this end, let B be a bounded subset of $\mathscr{F}_{\text{comp}}$ and let p_B be the associated seminorm of $(\mathscr{F}_{\text{comp}})' \simeq (\mathscr{F}_{\text{comp}})^*$. On behalf of Lemma 2.5.15, we find an $K \in \mathscr{P}_{\text{c}}(\Omega)$ such that B is a bounded subset of $\mathscr{F}_K$. So B is in particular a bounded subset of $\mathscr{F}$ and p_B is also a seminorm of the standard inducing collection of seminorms for $\mathscr{F}' \simeq \mathscr{F}^*$. It should not come as a surprise that we again take an $\varphi \in \mathscr{D}(\Omega)$ such that φ equals 1 on an open neighborhood of K. By definition of $(\mathscr{F}')_{\text{loc}}$, $q_{p_B,\varphi}\colon u \mapsto p_B(\varsigma(\varphi u))$ is an element of the standard inducing collection of seminorms for $(\mathscr{F}')_{\text{loc}}$, so the continuity of the inclusion follows from the observation that for every $u \in (\mathscr{F}')_{\text{loc}}$,

$$p_B(u) = \sup_{v\in B} |\hat{u}(v)| = \sup_{v\in B} |(\varsigma(\varphi u))(v)| = q_{p_B,\varphi}(u).$$

□

Proposition 2.10.10. *For every normal functional space $\mathscr{F}$ on Ω, we have*

$$(\mathscr{F}_{\text{comp}})' = (\mathscr{F}')_{\text{loc}}.$$

Proof: Combine the two previous lemmas. □

Since we have already seen that $\mathscr{E}(\Omega)$ is local (see Example 2.6.7) and that $(\mathscr{E}(\Omega))' = \mathscr{E}'(\Omega)$, the next example shows that the class of (normal) local functional spaces is not closed under $\mathscr{F} \mapsto \mathscr{F}'$.

Example 2.10.11. Using the previous proposition, we find

$$(\mathscr{E}'(\Omega))_{\text{loc}} = ((\mathscr{E}(\Omega))')_{\text{loc}} = ((\mathscr{E}(\Omega))_{\text{comp}})' = (\mathscr{D}(\Omega))' = \mathscr{D}'(\Omega),$$

so $\mathscr{E}'(\Omega)$ is *not* local. ⊘

As promised, the class of (normal) semi-local functional spaces shows better behaviour.

Lemma 2.10.12. *For every normal functional space $\mathscr{F}$ on Ω, we have*

$$\mathscr{F}' \subseteq_{\text{c}} ((\mathscr{F}_{\text{semi}})')_{\text{semi}}.$$

Proof: Because of Corollary 2.8.4, it suffices to prove that for every $\varphi \in \mathscr{E}(\Omega)$, m_φ restricts to a continuous linear map from $\mathscr{F}'$ into $(\mathscr{F}_{\text{semi}})'$ (indeed, if this is the case, application of Corollary 2.8.4 to the inclusion $\mathscr{F}' \hookrightarrow \mathscr{D}'(\Omega)$ gives the desired continuous inclusion). So let $\varphi \in \mathscr{E}(\Omega)$. Then m_φ can be viewed as continuous linear map from $\mathscr{F}_{\text{semi}}$ into $\mathscr{F}$ and as a consequence we get a continuous linear map m'_φ from $\mathscr{F}'$ into $(\mathscr{F}_{\text{semi}})'$. As in the proof of Lemma 2.10.8, we readily deduce that m'_φ coincides with the restriction of $m_\varphi\colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ to $\mathscr{F}'$ and hence that m_φ indeed restricts to a continuous linear map from $\mathscr{F}'$ into $(\mathscr{F}_{\text{semi}})'$. □

Proposition 2.10.13. *Let $\mathscr{F}$ be a normal functional space on Ω. If $\mathscr{F}$ is semi-local, then so is $\mathscr{F}'$.*

Proof: The statement that $\mathscr{F}'$ is semi-local is equivalent to $(\mathscr{F}')_{\mathrm{semi}} = \mathscr{F}'$ and since $(\mathscr{F}')_{\mathrm{semi}} \subseteq_{\mathrm{c}} \mathscr{F}'$ is automatic, it suffices to prove $\mathscr{F}' \subseteq_{\mathrm{c}} (\mathscr{F}')_{\mathrm{semi}}$. For this, use the previous lemma and the fact that $\mathscr{F}_{\mathrm{semi}} = \mathscr{F}$. □

Since $\mathscr{F}'$ is linearly topologically isomorphic to $\mathscr{F}^*$, the next result is nothing more than a restatement of some well-known facts from functional analysis.

Proposition 2.10.14. *Let $\mathscr{F}$ be a normal functional space on Ω and let P be short for: normable, Banach or Hilbert. Then $\mathscr{F}$ is P implies $\mathscr{F}'$ is P.*

Remark 2.10.15. It is also known that metrizablity and being Fréchet are *not* preserved when taking strong duals. ⊘

2.11 Normalizing

In the previous section we have seen the importance of normality: for normal functional spaces we can naturally view the ordinary dual (in the sense of locally convex vector spaces) as a functional space. However, not all important functional spaces are normal (for example, $H^1(\Omega)$ is in general not normal) and therefore it would be handy to have a procedure that 'normalizes' functional spaces.

Definition 2.11.1. Let $\mathscr{F}$ be a functional space on Ω. We define $\mathscr{F}_0$ to be the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}$ endowed with the subspace topology. ⊘

Proposition 2.11.2. *$\mathscr{F}_0$ is a normal functional space on Ω and $\mathscr{F}_0 \subseteq_{\mathrm{c}} \mathscr{F}$.*

Proof: Most of the things that we should check are trivial. The only thing that might need a little explanation is why for every $\varphi \in \mathscr{D}(\Omega)$ the continuous linear map $m_\varphi \colon \mathscr{F} \to \mathscr{F}$ restricts to a continuous linear map from $\mathscr{F}_0$ into $\mathscr{F}_0$, but this is a consequence of the next, independently proven, lemma. □

Lemma 2.11.3. *If $\mathscr{F}$ and $\mathscr{G}$ are functional spaces on Ω and $T \colon \mathscr{F} \to \mathscr{G}$ is a continuous linear map that maps $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$, then T restricts to a continuous linear map from $\mathscr{F}_0$ into $\mathscr{G}_0$.*

Proof: Since $\mathscr{F}_0$ and $\mathscr{G}_0$ carry the subspace topology, it suffices to prove that T maps $\mathscr{F}_0$ into $\mathscr{G}_0$. So let $u \in \mathscr{F}_0$. Because $\mathscr{F}_0$ is the closure of $\mathscr{D}(\Omega)$ in $\mathscr{F}$, we find a net $\{\varphi_i\}_{i\in I}$ in $\mathscr{D}(\Omega)$ such that $\varphi_i \to u$ in $\mathscr{F}$. The continuity of T then implies that $T\varphi_i \to Tu$ in $\mathscr{G}$, while $\{T\varphi_i\}_{i\in I}$ is a net in $\mathscr{D}(\Omega)$ because T maps $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$. Thus Tu is the limit in $\mathscr{G}$ of a net in $\mathscr{D}(\Omega)$ and therefore an element of $\mathscr{G}_0$. □

Another consequence of the previous lemma is that the assignment $\mathscr{F} \mapsto \mathscr{F}_0$ is a construction functor. Just as $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{loc}}$ and $\mathscr{F} \mapsto \mathscr{F}_{\mathrm{semi}}$, localize, respectively semi-localize, (semi-)functional spaces, $\mathscr{F} \mapsto \mathscr{F}_0$ normalizes functional spaces.

Proposition 2.11.4. *The assignment $\mathscr{F} \mapsto \mathscr{F}_0$ is a functor from the category of functional spaces on Ω to the category of functional spaces on Ω.*

Proof: If $\mathscr{F}$ and $\mathscr{G}$ are functional spaces on Ω with $\mathscr{F} \subseteq_c \mathscr{G}$, then the inclusion map $\mathscr{F} \hookrightarrow \mathscr{G}$ is a continuous linear map that maps $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$. Applying the previous lemma then shows that this inclusion map restricts to a continuous linear map from $\mathscr{F}_0$ into $\mathscr{G}_0$, which precisely means that $\mathscr{F}_0 \subseteq_c \mathscr{G}_0$. □

It is clear from the definition of $\mathscr{F}_0$ that a functional space $\mathscr{F}$ on Ω is normal if and only if $\mathscr{F} = \mathscr{F}_0$, which places normality in a similar setting as locality and semi-locality (being properties that are directly related to a construction functor). Furthermore, it is also evident from the definition of $\mathscr{F}_0$ that $\mathscr{F}_0$ is a *closed* subspace of $\mathscr{F}$ and we know that this implies that the 'topological vector space properties' are preserved.

Proposition 2.11.5. *Let $\mathscr{F}$ be a functional space on Ω and let P be short for: metrizable, normable, complete, Fréchet, Banach or Hilbert. Then $\mathscr{F}$ is P implies $\mathscr{F}_0$ is P.*

Also the preservation of semi-locality is easily derived.

Proposition 2.11.6. *Let $\mathscr{F}$ be a functional space on Ω. If $\mathscr{F}$ is semi-local, then so is $\mathscr{F}_0$.*

Proof: Let $\varphi \in \mathscr{E}(\Omega)$. Because $\mathscr{F}$ is semi-local, m_φ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$ and because m_φ maps $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$, Lemma 2.11.3 then tells us that m_φ restricts to a continuous linear map from $\mathscr{F}_0$ into $\mathscr{F}_0$. □

2.12 Negative powers

Our last construction functor, that we will introduce right away, extends the concept $\mathscr{F} \mapsto \mathscr{F}^k$ of 'taking powers' to negative integers.

Definition 2.12.1. Let $\mathscr{F}$ be a normal reflexive functional space on Ω and let $k \in \mathbb{N}_\infty$. We define $\mathscr{F}^{-k}$ by

$$\mathscr{F}^{-k} := (((\mathscr{F}')^k)_0)'. \qquad \oslash$$

Remark 2.12.2. It is imperative to 'normalize' $(\mathscr{F}')^k$ before we can dualize again: $L^2(\Omega)$ is normal and reflexive, but it is in general not true that

$$((L^2(\Omega))')^1 = (L^2(\Omega))^1 = H^1(\Omega)$$

is normal. ⊘

Because the assignment $\mathscr{F} \mapsto \mathscr{F}^{-k}$ is a composition of two covariant and two contravariant functors from (a subcategory of) the category of functional spaces on Ω to the category of functional spaces on Ω, it follows that the assignment $\mathscr{F} \mapsto \mathscr{F}^{-k}$ is itself a covariant functor from the category of normal reflexive functional spaces on Ω to the category of functional spaces on Ω. Hence, in particular, $\mathscr{F}^{-k}$ is always a functional space on Ω.

Example 2.12.3. Let $1 < p < \infty$ and $k \in \mathbb{N}$ and let $1 < q < \infty$ be the Hölder conjugate of p. Then

$$(L^p(\Omega))^{-k} = ((((L^p(\Omega))')^k)_0)' = (((L^q(\Omega))^k)_0)' = ((W^{q,k}(\Omega))_0)'$$

and combining this with [13, Definition 31.3 and Proposition 31.3] leads to the conclusion that $(L^p(\Omega))^{-k}$ equals the Sobolev space $W^{p,-k}(\Omega)$ (which consists of those distributions on Ω that are equal to a finite sum of derivatives up to order k of elements of $L^p(\Omega)$). So for $1 < p < \infty$, the equality $W^{p,k}(\Omega) = (L^p(\Omega))^k$, which we have discussed for $k \in \mathbb{N}$, in fact holds for every $k \in \mathbb{Z}$. ⊘

It is clear why, in the definition of $\mathscr{F}^{-k}$, we require $\mathscr{F}$ to be normal. After all, otherwise $\mathscr{F}'$ would not be defined. But why do we require $\mathscr{F}$ to be reflexive? Taking $k = 0$ and looking at Lemma 2.10.7 makes this clear:

$$\mathscr{F}^{-0} = (((\mathscr{F}')^0)_0)' = ((\mathscr{F}')_0)' = (\mathscr{F}')' = \mathscr{F}.$$

That is, we need to require that $\mathscr{F}$ is reflexive to make sure that $\mathscr{F}^{-0} = \mathscr{F}$, which is obviously rather desirable.

Proposition 2.12.4. *For every normal reflexive functional space $\mathscr{F}$ on Ω and all $k \in \mathbb{N}_\infty$, $\mathscr{F}^{-k} \subseteq_{\mathrm{c}} \mathscr{F}^{-(k+1)}$.*

Proof: We already know that $(\mathscr{F}')^{k+1} \subseteq_{\mathrm{c}} (\mathscr{F}')^k$ for every $k \in \mathbb{N}_\infty$. First applying the covariant functor $\mathscr{F} \mapsto \mathscr{F}_0$ to this inclusion and then the contravariant functor $\mathscr{F} \mapsto \mathscr{F}'$ gives the desired result. □

So for every normal reflexive functional space $\mathscr{F}$ on Ω, we have the following chain of continuous inclusions:

$$\begin{gathered}\mathscr{D}(\Omega) \subseteq_{\mathrm{c}} \mathscr{F}^\infty \subseteq_{\mathrm{c}} \ldots \subseteq_{\mathrm{c}} \mathscr{F}^{k+1} \subseteq_{\mathrm{c}} \mathscr{F}^k \subseteq_{\mathrm{c}} \ldots \subseteq_{\mathrm{c}} \mathscr{F}^1 \subseteq_{\mathrm{c}} \mathscr{F} \\ \subseteq_{\mathrm{c}} \mathscr{F}^{-1} \subseteq_{\mathrm{c}} \ldots \subseteq_{\mathrm{c}} \mathscr{F}^{-k} \subseteq_{\mathrm{c}} \mathscr{F}^{-(k+1)} \subseteq_{\mathrm{c}} \ldots \subseteq_{\mathrm{c}} \mathscr{F}^{-\infty} \subseteq_{\mathrm{c}} \mathscr{D}'(\Omega)\end{gathered}$$

and it turns out that even with these negative exponents we can use partial derivatives to 'walk through' this chain.

Lemma 2.12.5. *Let $\mathscr{F}$ be a normal reflexive functional space on Ω and let $k \in \mathbb{N}_\infty$. For all multi-indices α and all $|\alpha| \le \ell$, $\partial^\alpha \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}^{-k}$ into $\mathscr{F}^{-k-\ell}$.*

Proof: By Proposition 2.9.17, $\partial^\alpha \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $(\mathscr{F}')^{k+\ell}$ into $(\mathscr{F}')^{(k+\ell)-\ell} = (\mathscr{F}')^k$. Because ∂^α maps $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$, Lemma 2.11.3 subsequently tells us that ∂^α restricts to a continuous linear map from $((\mathscr{F}')^{(k+\ell)})_0$ into $((\mathscr{F}')^k)_0$ and then clearly also $(-1)^{|\alpha|}\partial^\alpha$ can be viewed as a continuous linear map from $((\mathscr{F}')^{(k+\ell)})_0$ into $((\mathscr{F}')^k)_0$. As a consequence, we obtain a continuous linear map $((-1)^{|\alpha|}\partial^\alpha)'$ from $\mathscr{F}^{-k-\ell} = \mathscr{F}^{-(k+\ell)} = (((\mathscr{F}')^{(k+\ell)})_0)'$ into $\mathscr{F}^{-k} = (((\mathscr{F}')^k)_0)'$ and we easily check that $((-1)^{|\alpha|}\partial^\alpha)'$ in fact coincides with the restriction of ∂^α to $\mathscr{F}^{-k-\ell}$. □

Proposition 2.12.6. *Let $\mathscr{F}$ be a normal reflexive functional space on Ω and $k \in \mathbb{Z}_\infty$. For all multi-indices α and all $|\alpha| \le \ell$, $\partial^\alpha \colon \mathscr{D}'(\Omega) \to \mathscr{D}'(\Omega)$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-\ell}$.*

Proof: Note that $\mathscr{F}^{k-|\alpha|} \subseteq_c \mathscr{F}^{k-\ell}$, so it suffices to prove that ∂^α restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\alpha|}$. If $k \leq 0$, this is a direct consequence of the previous lemma and if $k \geq |\alpha|$, it is a direct consequence of Proposition 2.9.17. Hence we may assume that $0 < k < |\alpha|$. Thanks to this assumption, we find multi-indices β and γ with $|\gamma| = k$ and $|\beta| = |\alpha| - k$ such that $\partial^\alpha = \partial^\beta \circ \partial^\gamma$. Then by Proposition 2.9.17, ∂^γ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$, while by the previous lemma ∂^β restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}^{0-(|\alpha|-k)} = \mathscr{F}^{k-|\alpha|}$, so $\partial^\alpha = \partial^\beta \circ \partial^\gamma$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-|\alpha|}$. □

Similar to the discussion following Proposition 2.9.17, we now see that every differential operator on Ω of order $\ell \in \mathbb{N}$ with constant or compactly supported coefficients restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-\ell}$ for every $k \in \mathbb{Z}_\infty$ and if $\mathscr{F}$ is semi-local we can even remove the restrictions on the coefficients. That also the spaces $\mathscr{F}^{-k}$, with $k \in \mathbb{N}$, are often 'good enough' to work with follows by combining the preservation results for $\mathscr{F} \mapsto \mathscr{F}^k$, $\mathscr{F} \mapsto \mathscr{F}'$ and $\mathscr{F} \mapsto \mathscr{F}_0$ with the fact that $\mathscr{F} \mapsto \mathscr{F}^{-k}$ is a composition of those functors.

Proposition 2.12.7. *Let $\mathscr{F}$ be a normal reflexive functional space on Ω, let $k \in \mathbb{N}$ and let P be short for: normable, Banach or Hilbert. Then $\mathscr{F}$ is P implies $\mathscr{F}^{-k}$ is P.*

Proposition 2.12.8. *Let $\mathscr{F}$ be a normal reflexive functional space on Ω and let $k \in \mathbb{N}_\infty$. If $\mathscr{F}$ is semi-local, then so is $\mathscr{F}^{-k}$.*

Part II

3

Distributions on vector bundles

The theory of functional spaces that we have introduced in the previous chapter is already quite nice, but there is one obvious limitation: we only talk about functions and distributions on open subsets of Euclidean space. Although this Euclidean space is very convenient to work with, the natural setting for many geometric problems is the setting of differential geometry. In this setting, the 'solution spaces' for linear partial differential equations are typically spaces of sections of a vector bundle, so it would be nice if we could generalize our theory of functional spaces to the context of vector bundles. To be able to do this, we have to leave the realm of functional spaces for a moment to return to pure distribution theory.

In this chapter, we present a nice way to introduce the concept of a distribution in the context of vector bundles and we develop the corresponding theory. However, the title of this chapter might be a bit misleading (but short and catchy, so a fine title anyway): if $E \to M$ is a vector bundle, the space of distributions $\mathscr{D}'(M, E)$ that we are going to define will generalize (sufficiently well-behaved) sections of $E \to M$, so intuitively speaking the distributions are defined 'on' M and have 'values' in a vector bundle. In the literature, these 'distributions on vector bundles' are also known as generalized or distributional sections.

Throughout this chapter, M denotes an n-dimensional (second-countable smooth) manifold and $E \to M$ denotes a rank r vector bundle over M.

3.1 Test functions

We recall the following definition from differential geometry:

Definition 3.1.1. A *total trivialization triple* (U, κ, ρ) of the vector bundle $E \to M$ consists of an open subset $U \subseteq M$, a diffeomorphism κ from U onto an open subset of $\mathbb{R}^{\mathrm{n}}$ and a smooth map $\rho = (\rho^1, \dots, \rho^{\mathrm{r}})$ from $E|_U = \pi^{-1}(U)$ onto $\mathbb{K}^{\mathrm{r}}$, such that:

1. for every $x \in U$, $\rho|_{E_x} : E_x \to \mathbb{K}^{\mathrm{r}}$ is an isomorphism of vector spaces and
2. $(\pi|_{E_U}, \rho)\colon E_U \to U \times \mathbb{K}^{\mathrm{r}}$ is a diffeomorphism. ⊘

Remark 3.1.2. Of course, it is always possible to cover our manifold M with such total trivialization triples and because M is second-countable, it is even possible to find countable covers of trivialization triples. ⊘

Let us consider the vector space $\Gamma^\infty(M,E)$ of smooth sections of E. To define an appropriate locally convex topology on this vector space, choose some collection $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ of total trivialization triples such that $\{U_i\}_{i\in I}$ is an open cover of M. Then for every $i\in I$, $1\le \ell\le r$ and $\varphi\in\Gamma^\infty(M,E)$, $(\rho_i)^\ell\circ\varphi\circ\kappa_i^{-1}$ is an element of $\mathscr{E}(\kappa(U_i))$. Hence, for every 4-tuple (i,ℓ,K,k) with $i\in I$, $1\le\ell\le r$, $K\in\mathscr{P}_{\mathrm{c}}(U_i)$ and $k\in\mathbb{N}$,

$$\|\cdot\|_{i,\ell,K,k}\colon \Gamma^\infty(M,E)\to\mathbb{R}\colon \varphi\mapsto \|(\rho_i)^\ell\circ\varphi\circ\kappa_i^{-1}\|_{\kappa_i(K),k} \tag{3.1}$$

is a seminorm on $\Gamma^\infty(M,E)$ and we define $\mathscr{E}(M,E)$ to be $\Gamma^\infty(M,E)$ endowed with the locally convex topology induced by those seminorms.

To see that the topology on $\Gamma^\infty(M,E)$ that we have just described is independent of the choice of total trivialization cover, let us also present an alternative way for introducing this topology. To this end, choose some (vector bundle) metric g on E (i.e., a smoothly varying family of inner products on the fibers). We then define, for every $K\in\mathscr{P}_{\mathrm{c}}(M)$ and every linear partial differential operator $P\colon\Gamma^\infty(M,E)\to\Gamma^\infty(M,E)$, a seminorm $\|\cdot\|^g_{K,P}\colon\Gamma^\infty(M,E)\to\mathbb{R}$ by

$$\|\varphi\|^g_{K,P} := \sup_{x\in K}\sqrt{g_x((P\varphi)(x),(P\varphi)(x))} = \sup_{x\in K}|(P\varphi)(x)|^g,$$

where we abbreviate $\sqrt{g_x(e_x,e_x)}$ by $|e_x|^g$.

Claim. The collection of seminorms

$$\{\|\varphi\|^g_{K,P}\mid K\in\mathscr{P}_{\mathrm{c}}(M)\text{ and }P\in\mathrm{Diff}(E,E)\},$$

where $\mathrm{Diff}(E,E)$ denotes the space of linear partial differential operators from E to E, induces the same topology on $\Gamma^\infty(M,E)$ as the seminorms associated to some total trivialization cover.

Proof: Let $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ be a collection of total trivialization triples such that $\{U_i\}_{i\in I}$ is an open cover of M. Thanks to Corollary A.1.4, the problem basically reduces to giving two estimates.

Let $i\in I$, $1\le\ell\le r$, $K\in\mathscr{P}_{\mathrm{c}}(U_i)$ and $k\in\mathbb{N}$. Moreover, let $(e_1,\dots,e_r)$ be the frame over U_i that is mapped (pointwise) onto the standard basis of $\mathbb{K}^r$ by ρ_i. For every $|\alpha|\le k$, the map

$$P_i^\alpha\colon\Gamma^\infty(U_i,E_{U_i})\to\Gamma^\infty(U_i,E_{U_i})$$

defined by

$$P_i^\alpha(\varphi) = P_i^\alpha(\varphi_1e_1+\cdots+\varphi_re_r) := \left(\partial^\alpha_{\kappa_i}\varphi_\ell\right)e_\ell$$

is clearly a differential operator from E_{U_i} to E_{U_i}. To 'extend' this differential operator to M, let ψ be a compactly supported smooth function on M such that ψ equals 1 on an open neighborhood of K and $\mathrm{supp}(\psi)\subseteq U_i$ (note that it is an easy consequence of the existence of smooth partitions of unity on M subordinate to any cover that such functions exist). Then $P_i^\alpha\circ m_\psi$ can be viewed as a differential operator P^α from E to E and because for every $\varphi\in\Gamma^\infty(M,E)$, $\psi\varphi$ and φ coincide on an open neighborhood of K, we have that $(P^\alpha\varphi)(x) = (P_i^\alpha(\psi\varphi))(x) = (P_i^\alpha\varphi)(x)$ for all $x\in K$. Using this, we deduce that for every $\varphi \in \Gamma^\infty(M,E)$,

$$\begin{aligned}\|\varphi\|_{i,\ell,K,k} = \|\varphi_\ell \circ \kappa_i^{-1}\|_{\kappa_i(K),k} &= \sum_{|\alpha|\le k} \sup_{x\in\kappa_i(K)} |\partial^\alpha(\varphi_\ell \circ \kappa_i^{-1})(x)| \\ &= \sum_{|\alpha|\le k} \sup_{x\in K} |(P_i^\alpha \varphi)_\ell(x)| = \sum_{|\alpha|\le k} \sup_{x\in K} \frac{|(P^\alpha\varphi)(x)|^g}{|e_\ell(x)|^g} \\ &\le \left(\sup_{x\in K} \frac{1}{|e_\ell(x)|^g}\right) \sum_{|\alpha|\le k} \|\varphi\|^g_{K,P^\alpha},\end{aligned}$$

where we have used that for every $\hat{\varphi} \in \Gamma^\infty(M,E)$, $|\hat{\varphi}_\ell e_\ell|^g = |\hat{\varphi}_\ell||e_\ell|^g$ on U_i and that $(P_i^\alpha\varphi)_\ell e_\ell = P_i^\alpha \varphi$ on U_i. Since $x \mapsto |e_\ell(x)|^g = \sqrt{g_x(e_\ell(x), e_\ell(x))}$ is a continuous function on U_i and therefore attains a minimum on K, this completes the derivation of the first estimate.

Next, let $K \in \mathscr{P}_c(M)$ and $P \in \mathrm{Diff}(E,E)$. Moreover, let $k \in \mathbb{N}$ such that P is a differential operator of order at most k. It is easy to see that we can find $K_0, \ldots, K_n \in \mathscr{P}_c(M)$ such that for every $0 \le j \le n$ there exists an $i \in I$ with $K_j \subseteq U_i$ and such that $K \subseteq \cup_{j=0}^n K_j$ (just cover M with compact discs with small enough radii and use that K is compact). We have that

$$\|\varphi\|^g_{K,P} = \sup_{x\in K} |(P\varphi)(x)|^g \le \sum_{j=0}^n \sup_{x\in K_j} |(P\varphi)(x)|^g$$

for every $\varphi \in \Gamma^\infty(M,E)$. Therefore, to find an estimate of the desired type for $\|\varphi\|^g_{K,P}$, it suffices to find such estimates for $\sup_{x\in K_j} |(P\varphi)(x)|^g$ with $0 \le j \le n$. So fix $0 \le j \le n$. Let $i \in I$ such that $K_j \subseteq U_i$ and let $(e_1, \ldots, e_r)$ be the frame over U_i that is mapped (pointwise) onto the standard basis of $\mathbb{K}^r$ by ρ_i. As above, we denote the ℓ^{th} 'component' of a section $\varphi \in \Gamma^\infty(M,E)$ on U_i by φ_ℓ (formally, $\varphi_\ell = \rho_i^\ell \circ \varphi$). Because P is a differential operator of order at most k, we find for every $|\alpha| \le k$ a smooth matrix C^α on U_i such that for all $1 \le \ell \le r$ and $\varphi \in \Gamma^\infty(M,E)$

$$(P\varphi)_\ell = \sum_{\ell'=1}^r \sum_{|\alpha|\le k} C^\alpha_{\ell\ell'} \partial^\alpha_{\kappa_i} \varphi_{\ell'}.$$

We now deduce that for every $\varphi \in \Gamma^\infty(M,E)$,

$$\begin{aligned}\sup_{x\in K_j} |(P\varphi)(x)|^g &= \sup_{x\in K_j} |(\sum_{\ell=1}^r (P\varphi)_\ell e_\ell)(x)|^g \le \sum_{\ell=1}^r \sup_{x\in K_j} |(P\varphi)_\ell(x) e_\ell(x)|^g \\ = \sum_{\ell=1}^r \sup_{x\in K_j} |(P\varphi)_\ell(x)||e_\ell(x)|^g &\le \sum_{\ell,\ell'=1}^r \sum_{|\alpha|\le k} \sup_{x\in K_j} |C^\alpha_{\ell\ell'}(x)\partial^\alpha_{\kappa_i}\varphi_{\ell'}(x)||e_\ell(x)|^g \\ &\le C \sum_{\ell'=1}^r \sum_{|\alpha|\le k} \sup_{x\in\kappa_i(K_j)} |\partial^\alpha(\varphi_{\ell'} \circ \kappa_i^{-1})(x)| = C \sum_{\ell'=1}^r \|\varphi\|_{i,\ell',K_j,k},\end{aligned}$$

where

$$C := r \sum_{\ell,\ell'=1}^r \sum_{|\alpha|\le k} \sup_{x\in K_j} |C^\alpha_{\ell\ell'}(x)||e_\ell(x)|^g.$$

Since this is an estimate of the desired form, we are done. □

Remark 3.1.3. The previous claim simultaneously shows that the topology of $\mathscr{E}(M,E)$ does not depend on the choice of trivialization cover and that the topology induced by $\{\|\varphi\|^g_{K,P} \mid K \in \mathscr{P}_c(M)$ and $P \in \operatorname{Diff}(E,E)\}$ does not depend on the choice of the metric g. ⊘

Now that we have properly defined $\mathscr{E}(M,E)$, we can define $\mathscr{E}_K(M,E)$, with $K \in \mathscr{P}_c(M)$, and $\mathscr{D}(M,E)$ in a completely similar way as in the first chapter (so we will not repeat it here). Nevertheless, there is also a difference with the first chapter: as we will soon see, not the spaces $\mathscr{E}(M,E)$ and $\mathscr{D}(M,E)$ but two closely related spaces are the spaces of test functions in this setting (but the title of this section *will* be accurate).

Remark 3.1.4. Let (U,κ,ρ) be a total trivialization triple of E. Being an open subset of a manifold, U is a manifold itself, while the restricted bundle E_U is a vector bundle over U. Hence, we can consider $\mathscr{E}(U,E_U)$. It readily follows from the definition of $\mathscr{E}(M,E)$ and the observation that (U,κ,ρ) by itself forms a total trivialization cover of $E_U \to U$ that

$$\mathscr{E}(U,E_U) \to (\mathscr{E}(\kappa(U)))^{\times r} : \varphi \mapsto (\rho^1 \circ \varphi \circ \kappa^{-1}, \dots, \rho^r \circ \varphi \circ \kappa^{-1})$$

is a linear topological isomorphism which maps, for all $K \in \mathscr{P}_c(U)$, $\mathscr{E}_K(U,E_U)$ onto $(\mathscr{E}_{\kappa(K)}(\kappa(U)))^{\times r}$. Using the nice interaction between inductive limits and products (see Lemma A.5.3), we even find that this linear topological isomorphism restricts to a linear topological isomorphism from $\mathscr{D}(U,E_U)$ onto $(\mathscr{D}(\kappa(U)))^{\times r}$. (We use the notation ${}^{\times r}$ for the products to distinguish them from the 'power construction' $\mathscr{F} \mapsto \mathscr{F}^r$.) ⊘

If M equals an open subset Ω of $\mathbb{R}^n$ and E equals the trivial line bundle $M \times \mathbb{K}$ over M, the previous remark tells us that $\mathscr{E}(M,E) \simeq \mathscr{E}(\Omega)$ and that $\mathscr{D}(M,E) \simeq \mathscr{D}(\Omega)$. Under the usual identification of smooth sections of the trivial line bundle with smooth functions (clearly, a smooth section of the line bundle is always of the form $x \mapsto (x,\varphi(x))$, with φ a smooth function) these isomorphisms even become identities: $\mathscr{E}(\Omega, M \times \mathbb{K}) = \mathscr{E}(\Omega)$ and $\mathscr{D}(\Omega, M \times \mathbb{K}) = \mathscr{D}(\Omega)$. As a consequence, for general manifolds M, it is safe to abbreviate $\mathscr{E}(M, M \times \mathbb{K})$ and $\mathscr{D}(M, M \times \mathbb{K})$ by $\mathscr{E}(M)$ and $\mathscr{D}(M)$, respectively (after all, if $M = \Omega \subseteq \mathbb{R}^n$, the old and new definitions coincide, so no confusion can arise).

Relevant results

Many of the 'relevant results' from Section 1.1 are still valid (and relevant) in the setting of vector bundles. We specifically mention:

Proposition 3.1.5. $\mathscr{D}(M,E) \subseteq_c \mathscr{E}(M,E)$.

Corresponding result: Proposition 1.1.1.

Lemma 3.1.6. *$\mathscr{D}(M,E)$ and $\mathscr{E}(M,E)$ are both Hausdorff.*

Corresponding result: Remark 1.1.2.

Lemma 3.1.7. *For every $K \in \mathscr{P}_c(M)$, $\mathscr{E}_K(M,E)$ is closed in $\mathscr{E}(M,E)$.*

Corresponding result: Lemma 1.1.3.

Lemma 3.1.8. *$\mathscr{D}(M,E)$ can be realized as a strict inductive limit of the form $\cup_{i\in\mathbb{N}}\mathscr{E}_{K_i}(M,E)$ for a collection $\{K_i\}_{i\in\mathbb{N}}$ of compact subsets of M.*

Corresponding result: Remark 1.1.13.

Lemma 3.1.9. *A subset B of $\mathscr{D}(M,E)$ is bounded if and only if there exists an $K \in \mathscr{P}_{\mathrm{c}}(M)$ such that B is a bounded subset of $\mathscr{E}_K(M,E)$.*

Corresponding result: Lemma 1.1.14.

Proposition 3.1.10. *$\mathscr{D}(M,E)$ is sequentially dense in $\mathscr{E}(M,E)$.*

Corresponding result: Proposition 1.1.15.

Lemma 3.1.11. *$\mathscr{D}(M,E)$ and $\mathscr{E}(M,E)$ are both reflexive.*

Corresponding result: Lemma 1.1.16.

The proofs of most of these results are obtained by making very simple adaptations to the proofs of their corresponding results (a lot of readers will not even need a piece of scrap paper for this). For the reflexivity of $\mathscr{D}(M,E)$ and $\mathscr{E}(M,E)$ we again use that these spaces are Montel spaces; a fact that is well-known, but for which it is hard to find a good reference. Furthermore, we should spend a few words on Lemma 3.1.8.

The 'problem' with Lemma 3.1.8, when compared to Remark 1.1.13, is that on our manifold M an exhaustion by compacts $\{K_i\}_{i\in\mathbb{N}}$ does not always have the property that $\mathrm{int}(K_{i+1}) \setminus K_i$ is nonempty; something which is true for exhaustions of an open subset Ω of $\mathbb{R}^{\mathrm{n}}$ because open subsets of $\mathbb{R}^{\mathrm{n}}$ cannot contain compact clopens (a clopen is a subset which is both closed and open) and which we silently used in Remark 1.1.13. In fact, an exhaustion by compacts $\{K_i\}_{i\in\mathbb{N}}$ of M does not necessarily lead to a *strict* system of spaces $\{\mathscr{E}_{K_i}(M,E)\}_{i\in\mathbb{N}}$. For example, if M is compact, K_i might equal M for all $i \in \mathbb{N}$ (but also if M itself is noncompact but has compact connected components things might go wrong). Nevertheless, after a moment's thought we realize that it is always possible to *choose* an exhaustion by compacts which does have the property that $\mathrm{int}(K_{i+1}) \setminus K_i$ is nonempty for every $i \in \mathbb{N}$ and it is easy to see that for such an exhaustion the inclusions $\mathscr{E}_{K_i}(M,E) \subseteq_{\mathrm{c}} \mathscr{E}_{K_{i+1}}(M,E)$ are strict. Indeed, just pick an element x of the nonempty open subset $\mathrm{int}(K_{i+1}) \setminus K_i$ and use a trivializing open neighborhood U of x that is contained in $\mathrm{int}(K_{i+1}) \setminus K_i$ to construct a nontrivial smooth section φ of E with compact support inside U (such a section obviously belongs to $\mathscr{E}_{K_{i+1}}(M,E)$ but not to $\mathscr{E}_{K_i}(M,E)$).

3.2 Distributions

We already have $\mathscr{D}(M,E)$ around, so following the analogy with the first chapter, we could define the space $\mathscr{D}'(M,E)$ of distributions on M with 'values' in E as the dual of $\mathscr{D}(M,E)$. However, if there is no canonical choice of integration available on M (like Lebesgue integration on $\mathbb{R}^{\mathrm{n}}$), there is no canonical way to identify $\mathscr{D}(M,E)$ and $\mathscr{E}(M,E)$ with subspaces of $(\mathscr{D}(M,E))^*$ and since the *key* feature of ordinary distributions is that they generalize (locally integrable) functions, the existence of such identifications is essential. Therefore,

$\mathscr{D}'(M,E) := (\mathscr{D}(M,E))^*$ is not a suitable definition if we want our theory to be applicable to manifolds without having to choose a way of integration.

So how could we solve this? The trick is to include the choice of integration in the test functions. To make this precise, we define the *functional dual* of a vector bundle E by

$$E^\vee := \mathrm{Hom}(E, D) \ (= E^* \otimes D),$$

with D the density bundle of M (see Section B.1). Basically, a section of $E^\vee$ (which is again a vector bundle over M of rank r) also includes a choice of density, hence a choice of integration. More formally, we have a bilinear map

$$\langle \cdot\,, \cdot \rangle \colon \mathscr{E}(M, E^\vee) \times \mathscr{E}(M,E) \to \mathscr{E}(M,D),$$

given by $\langle \varphi, \psi \rangle(x) = (\varphi(x))(\psi(x))$, which becomes $\langle \varphi, \psi \rangle(x) = \langle \varphi(x), \psi(x) \rangle$ if we use the conventional bracket notation for '$\varphi(x)$ applied to $\psi(x)$' (which we will actually do a lot from now on to improve the readability). Since clearly $\mathrm{supp}(\langle \varphi, \psi \rangle) \subseteq \mathrm{supp}(\varphi) \cap \mathrm{supp}(\psi)$ and compactly supported continuous sections of D are canonically integrable, we subsequently get 'pairings'

$$\begin{aligned} [\cdot\,, \cdot] \colon \mathscr{D}(M, E^\vee) \times \mathscr{E}(M,E) \to \mathbb{K} \colon \int_M \langle \varphi, \psi \rangle \quad \text{and} \\ [\cdot\,, \cdot] \colon \mathscr{E}(M, E^\vee) \times \mathscr{D}(M,E) \to \mathbb{K} \colon \int_M \langle \varphi, \psi \rangle. \end{aligned}$$

Using these pairings, we can view elements of $\mathscr{E}(M,E)$ as linear forms on $\mathscr{D}(M,E^\vee)$ and elements of $\mathscr{D}(M,E)$ as linear forms on $\mathscr{E}(M,E^\vee)$, so in view of the above discussion about extending ordinary sections, we see that it makes sense to define

$$\mathscr{D}'(M,E) := (\mathscr{D}(M,E^\vee))^* \quad \text{and} \quad \mathscr{E}'(M,E) := (\mathscr{E}(M,E^\vee))^*.$$

In analogy with the first chapter, we then denote the canonical identification $\mathscr{E}(M,E) \to \mathscr{D}'(M,E) \colon \psi \mapsto u_\psi$, with $u_\psi \colon \mathscr{D}(M,E^\vee) \to \mathbb{K} \colon \varphi \mapsto [\varphi, \psi]$, by $\jmath$ and the canonical identification $\mathscr{D}(M,E) \to \mathscr{E}'(M,E) \colon \psi \mapsto \hat{u}_\psi$, with $\hat{u}_\psi \colon \mathscr{E}(M,E^\vee) \to \mathbb{K} \colon \varphi \mapsto [\varphi, \psi]$, by $\hat{\jmath}$.

Of course, we should prove that $\jmath$ and $\hat{\jmath}$ are well-defined (that is, we should prove that u_ψ and $\hat{u}_\psi$ are continuous) and we would also like to have that $\jmath$ and $\hat{\jmath}$ are injective continuous linear maps. Although this is indeed the case, we prefer to prove this after we have discussed some 'relevant results'. Assuming that we do not make false promises, together with the continuous inclusion $\imath \colon \mathscr{D}(M,E) \hookrightarrow \mathscr{E}(M,E)$ and the adjoint $(\imath^\vee)^*$ of the continuous inclusion $\imath^\vee \colon \mathscr{D}(M,E^\vee) \hookrightarrow \mathscr{E}(M,E^\vee)$, we again get a nice square of injective continuous linear identification maps:

$$\begin{array}{ccc} \mathscr{D}(M,E) & \xrightarrow{\ \imath\ } & \mathscr{E}(M,E) \\ {\scriptstyle \hat{\jmath}}\big\downarrow & & {\scriptstyle \jmath}\big\downarrow \\ \mathscr{E}'(M,E) & \xrightarrow{(\imath^\vee)^*} & \mathscr{D}'(M,E) \end{array}$$

(note that $(\imath^\vee)^*$ is injective since $\mathscr{D}(M,E^\vee)$ is dense in $\mathscr{E}(M,E^\vee)$; after all, the statements from the previous section are equally valid if we replace E by $E^\vee$).

Remark 3.2.1. Just as in the Euclidean case, $\mathscr{E}'(M,E)$ and $\mathscr{D}'(M,E)$ are strong duals of locally convex vector spaces and therefore Hausdorff. $\oslash$

Remark 3.2.2. As described in Section B.1, every chart (U,κ) of M induces a trivialization $\rho_\kappa\colon D_U \to \mathbb{K}$ of $D \to M$ over U via the nowhere vanishing smooth section $|d\kappa_1 \wedge \cdots \wedge d\kappa_n|$ of the rank 1 bundle $D_U \to U$. In other words, for any chart (U,κ) of M there is a total trivialization triple (U,κ,ρ_κ) of $D \to M$ that is associated in a natural way to (U,κ). Using this, we see that every total trivialization triple (U,κ,ρ) of $E \to M$ gives rise to a total trivialization triple $(U,\kappa,\rho_\kappa^\vee)$ of $E^\vee \to M$: if ρ_κ denotes the trivialization of $D \to M$ over U associated to (U,κ), then $\rho_\kappa^\vee\colon (E^\vee)_U = (E_U)^\vee \to \mathbb{K}^r$ is defined by stipulating that for $x \in U$, $T_x \in \mathrm{Hom}(E_x, D_x)$ is mapped to $\rho_\kappa \circ T_x \circ (\rho|_{E_x})^{-1} \in (\mathbb{K}^r)^* \simeq \mathbb{K}^r$.

Now let $\varphi \in \mathscr{E}(M,E^\vee)$ and $\psi \in \mathscr{E}(M,E)$ and let $\tilde{\varphi} := \rho_\kappa^\vee \circ \varphi \circ \kappa^{-1}$ and $\tilde{\psi} := \rho \circ \psi \circ \kappa^{-1}$ be the vector-valued functions on $\kappa(U)$ which correspond to φ, respectively ψ, under these trivializations. As a direct consequence of the given definitions, we then have

$$\rho_\kappa \circ \langle \varphi, \psi \rangle \circ \kappa^{-1} = \sum_{j=1}^{r} \tilde{\psi}_j \tilde{\varphi}_j. \tag{3.2}$$

That is, the bilinear map $\langle \cdot\,,\cdot \rangle\colon \mathscr{E}(M,E^\vee) \times \mathscr{E}(M,E) \to \mathscr{E}(M,D)$ that we have defined above, basically corresponds to the standard 'bilinear product' on $\mathbb{K}^r$ if we use the 'coherent' trivializations that we have just discussed (note that if $\mathbb{K} = \mathbb{C}$, the standard 'bilinear product' is not the same as the standard inner product). $\oslash$

Remark 3.2.3. By combining the previous remark with Remark 3.1.4, we see that for every total trivialization triple (U,κ,ρ) of the bundle $E \to M$, we not only have $\mathscr{E}(U,E_U) \simeq (\mathscr{E}(\kappa(U)))^{\times r}$ and $\mathscr{D}(U,E_U) \simeq (\mathscr{D}(\kappa(U)))^{\times r}$, but also

$$\begin{aligned} &\mathscr{E}(U,(E_U)^\vee) = \mathscr{E}(U,(E^\vee)_U) \simeq (\mathscr{E}(\kappa(U)))^{\times r} \quad \text{and} \\ &\mathscr{D}(U,(E_U)^\vee) = \mathscr{D}(U,(E^\vee)_U) \simeq (\mathscr{D}(\kappa(U)))^{\times r}. \end{aligned}$$

Using the neat interaction between products and duals, which is described in Lemma A.5.2, we subsequently obtain linear topological isomorphisms

$$\begin{aligned} &\mathscr{E}'(U,E_U) = (\mathscr{E}(U,(E_U)^\vee))^* \simeq (\mathscr{E}'(\kappa(U)))^{\times r} \quad \text{and} \\ &\mathscr{D}'(U,E_U) = (\mathscr{D}(U,(E_U)^\vee))^* \simeq (\mathscr{D}'(\kappa(U)))^{\times r}. \end{aligned}$$

These isomorphisms are very natural and respect the structure: for example, we readily verify that the 'inclusion' $\jmath\colon \mathscr{E}(U,E_U) \subseteq_{\mathrm{c}} \mathscr{D}'(U,E_U)$ that we have defined above (with M replaced by U and E replaced by E_U) corresponds to the product

$$\overbrace{\jmath \times \cdots \times \jmath}^{r}$$

of the 'original' $\jmath\colon \mathscr{E}(\kappa(U)) \to \mathscr{D}'(\kappa(U))$ under these isomorphisms. $\oslash$

If M is equal to an open subset Ω of $\mathbb{R}^{\mathrm{n}}$ and E is equal to the trivial line bundle $M \times \mathbb{K}$, the previous remark shows that $\mathscr{D}'(M,E) \simeq \mathscr{D}'(\Omega)$ and $\mathscr{E}'(M,E) \simeq \mathscr{E}'(\Omega)$. So, for general manifolds M, we can safely abbreviate $\mathscr{D}'(M, M \times \mathbb{K})$ and $\mathscr{E}'(M, M \times \mathbb{K})$ by, respectively, $\mathscr{D}'(M)$ and $\mathscr{E}'(M)$.

Relevant results

Lemma 3.2.4. *The canonical integration map*

$$\int_M : \Gamma_c^0(M,D) \to \mathbb{K}$$

restricts to a continuous linear map from $\mathscr{D}(M,D)$ into $\mathbb{K}$.

Proof: According to Proposition A.3.2, it suffices to prove that $\int_M$ restricts to a continuous linear map from $\mathscr{E}_K(M,D)$ into $\mathbb{K}$ for every $K \in \mathscr{P}_c(M)$. So fix $K \in \mathscr{P}_c(M)$. Moreover, let $\{(U_i,\kappa_i)\}_{i\in I}$ be a collection of charts of M such that $\{U_i\}_{i\in I}$ is an open cover of M by precompact subsets, let $\{\eta_i\}_{i\in I}$ be a (smooth) partition of unity subordinate to $\{U_i\}_{i\in I}$ and let K_i denote the support of η_i (which is compact because $\mathrm{supp}(\eta_i)$ is a closed subset of the compact subset $\mathrm{cl}(U_i)$). Since $\{\mathrm{supp}(\eta_i)\}_{i\in I}$ is locally finite, we find a finite subset I_K of I with the property that $\mathrm{supp}(\eta_i) \cap K \neq \emptyset$ if and only if $i \in I_K$. Using the total trivialization triples $(U_i, \kappa_i, \rho_{\kappa_i})$ of $D \to M$ that are naturally associated to the charts (U_i,κ_i) and the definition of $\int_M$ as given in Section B.1, we see that

$$\int_M \omega = \sum_{i\in I_K} \int_M \eta_i\omega = \sum_{i\in I_K} \int_{\kappa_i(U_i)} \rho_{\kappa_i} \circ \eta_i\omega \circ \kappa_i^{-1}\, d\lambda$$

for every $\omega \in \mathscr{E}_K(M,D)$. Now let $\|\cdot\|_{i,\ell,K,k}$ (with $i \in I$, $1 \le \ell \le \mathrm{rank}(D)$, $K \in \mathscr{P}_c(U_i)$ and $k \in \mathbb{N}$) be the seminorms of $\mathscr{E}(M,D)$ associated to the total trivialization cover $\{(U_i,\kappa_i,\rho_{\kappa_i})\}_{i\in I}$ as introduced in the previous section (see equation (3.1)). We deduce that for every $\omega \in \mathscr{E}_K(M,D)$,

$$\begin{aligned}
\left|\int_M \omega\right| &\le \sum_{i\in I_K} \int_{\kappa_i(U_i)} |\rho_{\kappa_i} \circ \eta_i\omega \circ \kappa_i^{-1}|\, d\lambda \\
&\le \sum_{i\in I_K} \lambda(\kappa_i(K_i)) \sup_{x\in\kappa_i(K_i)} |(\rho_{\kappa_i} \circ \eta_i\omega \circ \kappa_i^{-1})(x)| \\
&\le \sum_{i\in I_K} \lambda(\kappa_i(K_i)) \sup_{x\in\kappa_i(K_i)} |(\eta_i \circ \kappa_i^{-1})(x)(\rho_{\kappa_i} \circ \omega \circ \kappa_i^{-1})(x)| \\
&\le \sum_{i\in I_K} \lambda(\kappa_i(K_i)) \sup_{x\in K_i} |\eta_i(x)| \|\omega\|_{i,1,K_i,0} \\
&\le \left(\sum_{i\in I_K} \lambda(\kappa_i(K_i))\right) \sum_{i\in I_K} \|\omega\|_{i,1,K_i,0}
\end{aligned} \tag{3.3}$$

and since the seminorms $\|\cdot\|_{i,\ell,K,k}$ by definition induce the topology of $\mathscr{E}(M,D)$ and $\mathscr{E}_K(M,D)$ carries the restricted topology, this estimate shows that $\int_M$ indeed restricts to a continuous linear map from $\mathscr{E}_K(M,D)$ into $\mathbb{K}$. □

Lemma 3.2.5. *The bilinear map*

$$\begin{aligned}
\langle\cdot,\cdot\rangle &: \mathscr{E}(M,E^\vee) \times \mathscr{E}(M,E) \to \mathscr{E}(M,D) \\
&: (\varphi,\psi) \mapsto (x \mapsto \langle\varphi(x),\psi(x)\rangle)
\end{aligned}$$

is continuous.

Proof: Let $\{(U_i, \kappa_i, \rho_i)\}_{i\in I}$ be some collection of total trivialization triples of $E \to M$ such that $\{U_i\}_{i\in I}$ is an open cover of M. As a consequence of Remark 3.2.2, this total trivialization cover of $E \to M$ naturally induces total trivialization covers of $D \to M$ and $E^\vee \to M$ and we denote the seminorms from the inducing collections associated to these total trivialization covers by $\|\cdot\|^{E}_{i,\ell,K,k}$, $\|\cdot\|^{E^\vee}_{i,\ell,K,k}$ and $\|\cdot\|^{D}_{i,\ell,K,k}$ (see equation (3.1) for their definition).

Now fix $i \in I$, $K \in \mathscr{P}_{\mathrm{c}}(U_i)$ and $k \in \mathbb{N}$. As in Remark 3.2.2, we denote the vector-valued functions on $\kappa_i(U_i)$ which correspond to $\varphi \in \mathscr{E}(M, E^\vee)$ and $\psi \in \mathscr{E}(M, E)$ under the trivializations by $\tilde{\varphi}$ and $\tilde{\psi}$. Using equation (1.3) and equation (3.2), we find that

$$\begin{aligned} \|\langle\varphi,\psi\rangle\|^{D}_{i,1,K,k} &= \|\rho_{\kappa_i}\circ\langle\varphi,\psi\rangle\circ\kappa_i^{-1}\|_{\kappa_i(K),k} \\ &= \|\sum_{j=1}^{r}\tilde{\varphi}_j\tilde{\psi}_j\|_{\kappa_i(K),k} \le \sum_{j=1}^{r}\|\tilde{\varphi}_j\tilde{\psi}_j\|_{\kappa_i(K),k} \\ &\le \sum_{j=1}^{r}\sum_{|\alpha|\le k}\sum_{\beta\le\alpha}\|\tilde{\varphi}_j\|_{\kappa_i(K),k}\|\tilde{\psi}_j\|_{\kappa_i(K),k} \\ &= \left(\sum_{|\alpha|\le k}\sum_{\beta\le\alpha}1\right)\sum_{j=1}^{r}\|\varphi\|^{E^\vee}_{i,j,K,k}\|\psi\|^{E}_{i,j,K,k} \end{aligned} \tag{3.4}$$

for every $\varphi \in \mathscr{E}(M, E^\vee)$ and $\psi \in \mathscr{E}(M, E)$ and since, according to Lemma A.5.4, the continuity of $\langle\cdot\,,\cdot\rangle$ is a consequence of the existence of such estimates, this finishes the proof. □

Corollary 3.2.6. *For every $\psi \in \mathscr{E}(M, E)$,*

$$v_\psi\colon \mathscr{D}(M, E^\vee) \to \mathscr{D}(M, D)\colon \varphi \mapsto \langle\varphi,\psi\rangle$$

is a continuous linear map.

Proof: Let $K \in \mathscr{P}_{\mathrm{c}}(M)$. From the continuity and bilinearity of $\langle\cdot\,,\cdot\rangle$ as map from $\mathscr{E}(M, E^\vee) \times \mathscr{E}(M, E)$ into $\mathscr{E}(M, D)$ and the fact that

$$\operatorname{supp}(\langle\varphi,\psi\rangle) \subseteq \operatorname{supp}(\varphi) \cap \operatorname{supp}(\psi)$$

it follows that v_ψ is a continuous linear map from $\mathscr{E}_K(M, E^\vee)$ into $\mathscr{E}_K(M, D)$ (note that continuity on the product implies seperate continuity). The result follows by using Proposition A.3.2 and the fact that $\mathscr{E}_K(M, D) \subseteq_{\mathrm{c}} \mathscr{D}(M, D)$. □

Corollary 3.2.7. *For every $\psi \in \mathscr{D}(M, E)$,*

$$\hat{v}_\psi\colon \mathscr{E}(M, E^\vee) \to \mathscr{D}(M, D)\colon \varphi \mapsto \langle\varphi,\psi\rangle$$

is a continuous linear map.

Proof: From the continuity of $\langle\cdot\,,\cdot\rangle$ as map from $\mathscr{E}(M, E^\vee) \times \mathscr{E}(M, E)$ into $\mathscr{E}(M, D)$ and the fact that $\operatorname{supp}(\langle\varphi,\psi\rangle) \subseteq \operatorname{supp}(\psi) \cap \operatorname{supp}(\psi)$ it follows that $\hat{v}_\psi$ is a continuous linear map from $\mathscr{E}(M, E^\vee)$ into $\mathscr{E}_{\operatorname{supp}(\psi)}(M, D)$. Combining this with $\mathscr{E}_{\operatorname{supp}(\psi)}(M, D) \subseteq_{\mathrm{c}} \mathscr{D}(M, D)$ gives the desired result. □

Lemma 3.2.8. *Let* $\psi \in \mathscr{E}(M, E)$*. Then* $\psi = 0$ *if and only if* $[\varphi, \psi] = 0$ *for every* $\varphi \in \mathscr{D}(M, E^\vee)$*.*

Proof: The direct implication is trivial, so assume that $[\varphi, \psi] = 0$ for every $\varphi \in \mathscr{D}(M, E^\vee)$. Fix an arbitrary $x_0 \in M$. We will show that $\psi(x_0) = 0$. Choose some total trivialization triple (U, κ, ρ) of $E \to M$ such that $x_0 \in U$ and let $\rho_\kappa^\vee$ be the induced trivialization of $E^\vee \to M$ over U. If we denote the the vector-valued function on $\kappa(U)$ that corresponds to ψ under the trivialization by $\tilde{\psi}$, then

$$\varphi' \colon U \to E^\vee \colon x \mapsto \left(\rho_\kappa^\vee|_{(E^\vee)_x}\right)^{-1} \left(\overline{\tilde{\psi}(\kappa(x))}\right),$$

where the horizontal bar stands for 'complex conjugate of' (so if $\mathbb{K} = \mathbb{R}$ it does nothing), is a smooth section of $(E^\vee)_U \to U$. Now pick some $\chi \in \mathscr{D}(M)$ such that $\operatorname{im}(\chi) \subseteq [0, 1]$, $\operatorname{supp}(\chi) \subseteq U$ and $\chi(x_0) = 1$. Then $\varphi := \chi\varphi'$, where $\chi\varphi'$ is interpreted as a function on M in the obvious way, is an element of $\mathscr{D}(M, E^\vee)$ which corresponds to $(\chi \circ \kappa^{-1})\overline{\tilde{\psi}}$ under the trivialization triple $(U, \kappa, \rho_\kappa^\vee)$. As a consequence,

$$\begin{aligned} [\varphi, \psi] &= \int_M \langle \varphi, \psi \rangle = \int_{\kappa(U)} \rho_\kappa \circ \langle \varphi, \psi \rangle \circ \kappa^{-1} \, d\lambda \\ &= \int_{\kappa(U)} \sum_{j=1}^r (\chi \circ \kappa^{-1}) \overline{\tilde{\psi}_j} \tilde{\psi}_j \, d\lambda = \int_{\kappa(U)} (\chi \circ \kappa^{-1}) \|\tilde{\psi}\| \, d\lambda, \end{aligned}$$

where $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{K}^r$ and where we have used the fact that $\operatorname{supp}(\varphi) \subseteq U$ to obtain the second equality and equation (3.2) to obtain the third equality. But by assumption $[\varphi, \psi] = 0$, so it follows that

$$\int_{\kappa(U)} (\chi \circ \kappa^{-1}) \|\tilde{\psi}\| \, d\lambda = 0$$

and since $(\chi \circ \kappa^{-1})\|\tilde{\psi}\|$ is a real-valued nonnegative smooth function, this implies that $(\chi \circ \kappa^{-1})\|\tilde{\psi}\| = 0$. Therefore, in particular

$$\|\tilde{\psi}(\kappa(x_0))\| = (\chi \circ \kappa^{-1})(\kappa(x_0))\|\tilde{\psi}(\kappa(x_0))\| = 0$$

from which we conclude that $\tilde{\psi}(\kappa(x_0)) = 0$ and hence that $\psi(x_0) = 0$. □

Lemma 3.2.9. *Let* $\varphi \in \mathscr{E}(M, E^\vee)$*. Then* $\varphi = 0$ *if and only if* $[\varphi, \psi] = 0$ *for every* $\psi \in \mathscr{D}(M, E)$*.*

Proof: The proof is completely analogous to the proof of the previous lemma. □

Now that we have discussed the 'relevant results', the time has come to fulfill a promise.

Claim. $\jmath$ and $\hat{\jmath}$ are well-defined injective continuous linear maps.

Proof: Let us start with $\jmath \colon \mathscr{E}(M, E) \to \mathscr{D}'(M, E)$. If $\psi \in \mathscr{E}(M, E)$, then $u_\psi \colon \mathscr{D}(M, E^\vee) \to \mathbb{K} \colon \varphi \mapsto [\varphi, \psi]$ is equal to the composition $\int_M \circ v_\psi$ and since $\int_M$ and v_ψ are both continuous linear maps (see Lemma 3.2.4 and Corollary 3.2.6), u_ψ is a continuous linear map as well. That is, $u_\psi \in \mathscr{D}'(M, E)$, thus $\jmath$ is well-defined. That $\jmath$ is linear is of course clear and that $\jmath$ is injective follows straight from Lemma 3.2.8, so it only remains to be proven that $\jmath\colon \mathscr{E}(M,E) \to \mathscr{D}'(M,E)$ is continuous. We do this via Lemma A.1.2, i.e., by finding suitable estimates.

Let B be a bounded subset of $\mathscr{D}(M,E^\vee)$ and let q_B be the associated seminorm of $\mathscr{D}'(M,E) = (\mathscr{D}(M,E^\vee))^*$. In addition, let $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ be a total trivialization cover of $E \to M$ with precompact domains U_i and denote the corresponding seminorms for $\mathscr{E}(M,E)$ and $\mathscr{E}(M,E^\vee)$ (via the induced total trivialization cover of $E^\vee \to M$) by $\|\cdot\|^E_{i,\ell,K,k}$, respectively $\|\cdot\|^{E^\vee}_{i,\ell,K,k}$ (where $i \in I$, $1 \le \ell \le r$, $K \in \mathscr{P}_\mathrm{c}(U_i)$ and $k \in \mathbb{N}$). Thanks to Lemma 3.1.9, we find an $K \in \mathscr{P}_\mathrm{c}(M)$ such that B is a bounded subset of $\mathscr{E}_K(M,E^\vee)$ and if we look at the proofs of Lemma 3.2.4 and Lemma 3.2.5 and combine equation (3.3) and equation (3.4), we see that there exist a constant $C \ge 0$, a finite subset I_K of I and compact subsets $K_i \in \mathscr{P}_\mathrm{c}(M)$ for every $i \in I_K$, such that for all $\varphi \in B$ and $\psi \in \mathscr{E}(M,E)$

$$\left|\int_M \langle\varphi,\psi\rangle\right| \le C \sum_{i\in I_K}\sum_{j=1}^r \|\varphi\|^{E^\vee}_{i,j,K_i,0}\|\psi\|^E_{i,j,K_i,0}.$$

Now since B is bounded in $\mathscr{E}(M,E^\vee)$, we can find constants D_{ij} for $i \in I_K$ and $1 \le j \le r$ such that $\|\varphi\|^{E^\vee}_{i,j,K_i,0} \le D_{ij}$ for all $\varphi \in B$ and if we define $D := \max\{D_{ij} \mid i \in I_K \text{ and } 1 \le j \le r\}$, we get that

$$q_B(\jmath(\psi)) = \sup_{\varphi\in B}\left|\int_M \langle\varphi,\psi\rangle\right| \le CD \sum_{i\in I_K}\sum_{j=1}^r \|\psi\|^E_{i,j,K_i,0}$$

for all $\psi \in \mathscr{E}(M,D)$. Because this is an estimate of the desired form, we conclude that $\jmath$ is indeed continuous.

The discussion for $\hat\jmath$ is very similar. If $\psi \in \mathscr{D}(M,E)$, $\hat u_\psi = \int_M \circ\hat v_\psi$, hence $\hat u_\psi \in \mathscr{E}'(M,E)$ by Lemma 3.2.4 and Corollary 3.2.7 and thus $\hat\jmath$ is well-defined. The linearity of $\hat\jmath$ is clear and the injectivity again follows from Lemma 3.2.8. According to Proposition A.3.2, to prove that $\hat\jmath\colon \mathscr{D}(M,E) \to \mathscr{E}'(M,E)$ is continuous, it suffices to prove that $\hat\jmath$ is continuous on $\mathscr{E}_K(M,E)$ for every $K \in \mathscr{P}_\mathrm{c}(M)$. So fix $K \in \mathscr{P}_\mathrm{c}(M)$ and let B be a bounded subset of $\mathscr{E}(M,E^\vee)$. As above, we choose a total trivialization cover $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ of $E \to M$ with precompact domains U_i and we find a constant $C \ge 0$, a finite subset I_K of I and compact subsets $K_i \in \mathscr{P}_\mathrm{c}(M)$ for every $i \in I_K$, such that for all $\varphi \in B$ and $\psi \in \mathscr{E}_K(M,E)$

$$\left|\int_M \langle\varphi,\psi\rangle\right| \le C \sum_{i\in I_K}\sum_{j=1}^r \|\varphi\|^{E^\vee}_{i,j,K_i,0}\|\psi\|^E_{i,j,K_i,0}.$$

Exploiting the fact that B is bounded in $\mathscr{E}(M,E^\vee)$ in the same way as above, we arrive at our desired estimate. □

3.3 Support of a distribution

Despite of the fact that the definition of a distribution in the setting of vector bundles is a bit more technical, the arguments and definitions from the first chapter in general only need small modifications if we want to use them in the current setting. Regarding the definition of the support of a distribution there is, compared to Section 1.4, in fact only one noteworthy step (which is still really obvious): restriction of distributions with 'values' in E is the adjoint of extension of sections of $E^\vee$. To be complete and a bit more precise, we give a quick summary.

Let U and V be open subsets of M with $V \subseteq U$. Then U and V are also manifolds and we have restricted bundles E_U and E_V over U and V, respectively. Clearly, 'extension by zero' gives a continuous linear map

$$\mathrm{ext}^E_{V,U} : \mathscr{D}(V, E_V) \to \mathscr{D}(U, E_U)$$

and because $E^\vee$ is a vector bundle over M as well, we also have

$$\mathrm{ext}^{E^\vee}_{V,U} : \mathscr{D}(V, (E_V)^\vee) = \mathscr{D}(V, (E^\vee)_V) \to \mathscr{D}(U, (E^\vee)_U) = \mathscr{D}(U, (E_U)^\vee).$$

Taking the adjoint of $\mathrm{ext}^{E^\vee}_{V,U}$ then gives a continuous linear 'restriction' map

$$\mathrm{res}^E_{U,V} := (\mathrm{ext}^{E^\vee}_{V,U})^* : \mathscr{D}'(U, E_U) \to \mathscr{D}'(V, E_V).$$

Similar arguments as before show that together with these restriction maps the assignment $U \mapsto \mathscr{D}'(U, E_U)$ is a sheaf over M. As a consequence, we have a well-defined notion of support for elements of $\mathscr{D}'(M, E)$ and it is easy to check that this extended notion of support coincides with the ordinary one on $\mathscr{E}(M, E)$. Furthermore, locally finite families of distributions and local linear maps are defined in the same way as in the Euclidean case.

Relevant results

All of the 'relevant results' from Section 1.4 are still valid (and relevant) in the setting of vector bundles. For completeness and convenience we reformulate them here. The proofs of these results are trivial adaptations of the proofs of their corresponding results from Section 1.4.

Lemma 3.3.1. *For every open subset U of M and every $u \in \mathscr{D}'(M, E)$*

$$\mathrm{supp}(u|_U) \subseteq \mathrm{supp}(u)$$

and if $\mathrm{supp}(u) \subseteq U$, *then even*

$$\mathrm{supp}(u|_U) = \mathrm{supp}(u).$$

Corresponding result: Lemma 1.4.9.

Lemma 3.3.2. *Let U be an open subset of M and $u \in \mathscr{D}'(M, E)$. Then u vanishes on U if and only if $u(\varphi) = 0$ for every $\varphi \in \mathscr{D}(M, E^\vee)$ with the property that* $\mathrm{supp}(\varphi) \subseteq U$.

Corresponding result: Lemma 1.4.10.

Lemma 3.3.3. *For every closed subset A of M and every $u \in \mathscr{D}'(M, E)$, we have* $\mathrm{supp}(u) \subseteq A$ *if and only if $u(\varphi) = 0$ for every $\varphi \in \mathscr{D}(M, E^\vee)$ with* $\mathrm{supp}(\varphi) \subseteq M \setminus A$.

Corresponding result: Lemma 1.4.11.

Lemma 3.3.4. *Let $u, v \in \mathscr{D}'(M,E)$ and $\mu \in \mathbb{K}$. Then*

1. $\operatorname{supp}(\mu u) = \emptyset$ *if* $\mu = 0$,
2. $\operatorname{supp}(\mu u) = \operatorname{supp}(u)$ *if* $\mu \neq 0$ *and*
3. $\operatorname{supp}(u+v) \subseteq \operatorname{supp}(u) \cup \operatorname{supp}(v)$.

Corresponding result: Lemma 1.4.12.

Lemma 3.3.5. *If $u \in \mathscr{D}'(M,E)$ and $\varphi \in \mathscr{D}(M,E^\vee)$ with the property that* $\operatorname{supp}(\varphi) \cap \operatorname{supp}(u) = \emptyset$, *then* $u(\varphi) = 0$.

Corresponding result: Lemma 1.4.13.

Lemma 3.3.6. *If $u \in \mathscr{D}'(M,E)$ and $\varphi \in \mathscr{D}(M,E^\vee)$ vanishes on an open neighborhood of* $\operatorname{supp}(u)$, *then* $u(\varphi) = 0$.

Corresponding result: Lemma 1.4.14.

Lemma 3.3.7. *If $u \in \mathscr{D}'(M,E)$ and $\varphi, \psi \in \mathscr{D}(M,E^\vee)$ such that φ and ψ coincide on an open neighbordhood of* $\operatorname{supp}(u)$, *then* $u(\varphi) = u(\psi)$.

Corresponding result: Lemma 1.4.15.

Lemma 3.3.8. *Let $u \in \mathscr{D}'(M,E)$. If there exists an open subset U of M such that* $\operatorname{supp}(u) \subseteq U$ *and* $u|_U = 0$, *then* $u = 0$.

Corresponding result: Lemma 1.4.16.

Lemma 3.3.9. *Let u, $v \in \mathscr{D}'(M,E)$. If there exists an open subset U of M such that* $\operatorname{supp}(u) \subseteq U$, $\operatorname{supp}(v) \subseteq U$ *and* $u|_U = v|_U$, *then* $u = v$.

Corresponding result: Lemma 1.4.17.

Lemma 3.3.10. *Let U and V be open subsets of M and let $u \in \mathscr{D}'(U,E_U)$ and $v \in \mathscr{D}'(V,E_V)$ such that $u|_{U\cap V} = v|_{U\cap V}$. Then $u(\varphi|_U) = v(\varphi|_V)$ for all $\varphi \in \mathscr{D}(M,E)$ with* $\operatorname{supp}(\varphi) \subseteq U \cap V$.

Corresponding result: Lemma 1.4.18.

Lemma 3.3.11. *Suppose that $\{u_i\}_{i\in I}$ is a locally finite family of distributions on (M,E). Then*

$$\sum_{i\in I} u_i \colon \mathscr{D}(M,E^\vee) \to \mathbb{K} \colon \varphi \mapsto \sum_{i\in I} u_i(\varphi)$$

is a well-defined distribution on (M,E).

Corresponding result: Lemma 1.4.19.

Lemma 3.3.12. *Let $\{u_i\}_{i\in I}$ be a net in $\mathscr{D}'(M,E)$, $u \in \mathscr{D}'(M,E)$ and K a compact subset of M such that $u_i \to u$ in $\mathscr{D}'(M,E)$ and* $\operatorname{supp}(u_i) \subseteq K$ *for every $i \in I$. Then also* $\operatorname{supp}(u) \subseteq K$.

Corresponding result: Lemma 1.4.20.

3.4 Pushforward of a distribution

Although most of the material in this chapter generalizes concepts from the first chapter, this section covers a principle intrinsic to the setting of vector bundles: pushforwards under vector bundle homomorphisms. In addition to the vector bundle E over M of rank r that is around in the entire chapter, F will denote a vector bundle over M of rank s in this section.

Let $T\colon E \to F$ be a vector bundle homomorphism (recall that all vector bundle homomorphisms are assumed to be the identity on the base space, see 'Notation and conventions'). Then we have a well-known pushforward $T_*\colon \mathscr{E}(M,E) \to \mathscr{E}(M,F)$ defined by $(T_*\varphi)(x) := T\varphi(x)$ and it is not difficult to check that:

Claim. T_* is a continuous linear map.

Proof: Let $\{(U_i,\kappa_i,\rho_i^E,\rho_i^F)\}_{i\in I}$ be a collection of 4-tuples such that for every $i \in I$, (U_i,κ_i,ρ_i^E) is a total trivialization triple of $E \to M$ and (U_i,κ_i,ρ_i^F) is a total trivialization triple of $F \to M$ and such that $\{U_i\}_{i\in I}$ is an open cover of M. To prove that T_* is continuous, we will give an estimate in terms of the seminorms associated to the trivialization covers $\{(U_i,\kappa_i,\rho_i^E)\}_{i\in I}$ and $\{(U_i,\kappa_i,\rho_i^F)\}_{i\in I}$ (note that it is evident that T_* is linear). Fix $i \in I$, $1 \le \ell \le s$, $K \in \mathscr{P}_{\mathrm{c}}(U_i)$ and $k \in \mathbb{N}$. To avoid cumbersome notation, we denote for every $\varphi \in \mathscr{E}(M,E)$, $\rho_i^E \circ \varphi \circ \kappa_i^{-1}$ by $\tilde{\varphi}$ and we define $\tilde{T}\colon \kappa(U_i) \times \mathbb{K}^r \to \mathbb{K}^s$ by

$$\tilde{T}(x)v := \left(\rho_i^F \circ T \circ \left(\rho_i^E\big|_{E_{\kappa^{-1}(x)}}\right)^{-1}\right) v.$$

Of course, $\tilde{T}$ is nothing more than a smooth $s \times r$ matrix on $\kappa_i(U_i)$ and we have

$$\rho_i^F \circ T_*\varphi \circ \kappa_i^{-1} = \tilde{T}\tilde{\varphi}.$$

Therefore, using equation (1.3), we find that for every $\varphi \in \mathscr{E}(M,E)$

$$\begin{aligned}
\|T_*\varphi\|^F_{i,\ell,K,k} &= \Big\| \sum_{m=1}^r \tilde{T}_{\ell m}\tilde{\varphi}_m \Big\|_{\kappa_i(K),k} \le \sum_{m=1}^r \|\tilde{T}_{\ell m}\tilde{\varphi}_m\|_{\kappa_i(K),k} \\
&\le \sum_{m=1}^r \sum_{|\alpha|\le k} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \|\tilde{T}_{\ell m}\|_{\kappa_i(K),k} \|\tilde{\varphi}_m\|_{\kappa_i(K),k} \\
&\le \left(\sum_{n=1}^r \sum_{|\alpha|\le k} \sum_{\beta \le \alpha} \binom{\alpha}{\beta} \|\tilde{T}_{\ell n}\|_{\kappa_i(K),k} \right) \sum_{m=1}^r \|\varphi\|^E_{i,m,K,k},
\end{aligned}$$

which proves that T_* is continuous on behalf of Lemma A.1.2. □

Since clearly $\operatorname{supp}(T_*\varphi) \subseteq \operatorname{supp}(\varphi)$ for every $\varphi \in \mathscr{E}(M,E)$, it subsequently follows that T_* restricts to a continuous linear map from $\mathscr{D}(M,E)$ into $\mathscr{D}(M,F)$. Now observe that if $T\colon E \to F$ is a vector bundle homomorphism, then also its 'functional adjoint'

$$T^\vee\colon F^\vee = \operatorname{Hom}(F,D) \to \operatorname{Hom}(E,D) = E^\vee,$$

which sends $L_x \in \operatorname{Hom}(F_x, D_x)$ to $L_x \circ T \in \operatorname{Hom}(E_x, D_x)$, is a vector bundle homomorphism. Taking the pushforward of $T^\vee$ then gives a continuous linear map $(T^\vee)_*\colon \mathscr{D}(M, F^\vee) \to \mathscr{D}(M, E^\vee)$ and by subsequently taking the adjoint of this map, we get a continuous linear map $((T^\vee)_*)^*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$.

Claim. $((T^\vee)_*)^*$ extends T_*.

Proof: Note that for all $\psi \in \mathscr{E}(M, E)$, $\varphi \in \mathscr{D}(M, F^\vee)$ and $x \in M$,

$$\begin{aligned} \langle (T^\vee)_*\varphi, \psi\rangle(x) &= \langle ((T^\vee)_*\varphi)(x), \psi(x)\rangle = \langle T^\vee\varphi(x), \psi(x)\rangle \\ &= \langle \varphi(x), T\psi(x)\rangle = \langle \varphi(x), (T_*\psi)(x)\rangle = \langle \varphi, T_*\psi\rangle(x), \end{aligned} \tag{3.5}$$

hence

$$\begin{aligned} (((T^\vee)_*)^*\hat{u}_\psi)(\varphi) &= \hat{u}_\psi((T^\vee)_*\varphi) = [(T^\vee)_*\varphi, \psi] \\ &= \int_M \langle (T^\vee)_*\varphi, \psi\rangle = \int_M \langle \varphi, T_*\psi\rangle = [\varphi, T_*\psi] = \hat{u}_{T_*\psi}(\varphi). \quad \square \end{aligned}$$

On behalf of this claim, we can safely denote $((T^\vee)_*)^*$ again by T_* and we conclude that every vector bundle homomorphism $T\colon E \to F$ gives rise to a nice continuous linear pushforward map $T_*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$ on the level of distributions that extends the ordinary continuous linear pushforward $T_*\colon \mathscr{E}(M, E) \to \mathscr{E}(M, F)$ on the level of smooth sections.

Remark 3.4.1. It is possible to combine the concept of pushforwards under vector bundle homomorphisms with the concept of pushforwards under diffeomorphisms that we have discussed in Section 1.7. For this, we would start with isomorphic vector bundles $E \to M$ and $F \to N$ over possibly different base manifolds (which would, however, be diffeomorphic since a vector bundle isomorphism induces a diffeomorphism of the base manifolds) and given a vector bundle isomorphism $T\colon E \to F$, we would combine the approach of this section and Section 1.7 to obtain a linear topological isomorphism $T_*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(N, F)$. When $N = M$ and $F = E$, this T_* then resembles a ‘change of coordinates’ for the vector bundle $E \to M$. ⊘

Relevant results

Lemma 3.4.2. *For every vector bundle homomorphism $T\colon E \to F$,*

$$T_*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$$

is $\mathscr{E}(M)$-linear.

Proof: Let $u \in \mathscr{D}'(M, E)$, $\psi \in \mathscr{E}(M)$ and $\varphi \in \mathscr{D}(M, F^\vee)$. Using the trivial observation that ordinary pushforwards of vector bundle homomorphisms on the level of smooth sections are $\mathscr{E}(M)$-linear, we deduce

$$\begin{aligned} (T_*m_\psi u)(\varphi) &= (m_\psi u)((T^\vee)_*\varphi) = u(m_\psi (T^\vee)_*\varphi) \\ &= u((T^\vee)_* m_\psi\varphi) = (T_*u)(m_\psi\varphi) = (m_\psi T_*u)(\varphi). \quad \square \end{aligned}$$

Lemma 3.4.3. *For every vector bundle homomorphism $T\colon E \to F$ and all $u \in \mathscr{D}'(M, E)$,*

$$\operatorname{supp}(T_* u) \subseteq \operatorname{supp}(u).$$

(In other words, $T_*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$ is local.)

Proof: Let $\varphi \in \mathscr{D}'(M, F^\vee)$ with the property that $\operatorname{supp}(\varphi) \subseteq M \setminus \operatorname{supp}(u)$. On behalf of Lemma 3.3.3, it suffices to prove that $(T_* u)(\varphi) = 0$. As observed before, it is clearly true that ordinary pushforwards on the level of smooth sections are local, so $\operatorname{supp}((T^\vee)_*\varphi) \subseteq \operatorname{supp}(\varphi) \subseteq M \setminus \operatorname{supp}(u)$ and in combination with Lemma 3.3.3 this implies

$$(T_* u)(\varphi) = u((T^\vee)_*\varphi) = 0. \qquad \square$$

Lemma 3.4.4. *Let E, F and H be vector bundles over M and $T\colon E \to F$ and $L\colon F \to H$ vector bundle homomorphisms. Then $(L \circ T)_* = L_* \circ T_*$.*

Proof: If we view the pushforwards as maps on the spaces of (compactly supported) smooth sections, this is obviously true, so the statement is really about the newly defined generalized pushforwards on the spaces of distributions. That is, we should prove that $(((L \circ T)^\vee)_*)^* = ((L^\vee)_*)^* \circ ((T^\vee)_*)^*$. But it follows straight from the definition of the functional adjoint of a vector bundle homomorphism that $(L \circ T)^\vee = T^\vee \circ L^\vee$ and $((L \circ T)^\vee)_*$ is meant as an ordinary pushforward of sections, so we indeed obtain

$$(((L \circ T)^\vee)_*)^* = ((T^\vee \circ L^\vee)_*)^* = ((T^\vee)_* \circ (L^\vee)_*)^* = ((L^\vee)_*)^* \circ ((T^\vee)_*)^*.$$

$\square$

Lemma 3.4.5. *Let E, F be vector bundles over M and let $T\colon E \to F$ and $L\colon E \to F$ be vector bundle homomorphisms. Then $(L + T)_* = L_* + T_*$.*

Proof: Again, $(L + T)_* = L_* + T_*$ clearly holds for the ordinary pushforwards on smooth sections, so we should prove that

$$(((T + L)^\vee)_*)^* = ((T^\vee)_*)^* + ((L^\vee)_*)^*.$$

Since the adjoint $\cdot^*$ preserves addition and, as we have just mentioned, the same is true for the ordinary pushforward, this boils down to proving that $(T + L)^\vee = T^\vee + L^\vee$, which is a trivial consequence of the definition of the functional adjoint. $\square$

Lemma 3.4.6. *For every vector bundle homomorphism $T\colon E \to F$, the pushforward $T_*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$ restricts to a continuous linear map from $\mathscr{E}'(M, E)$ into $\mathscr{E}'(M, F)$.*

Proof: As we already know, the ordinary pushforward of $T^\vee\colon F^\vee \to E^\vee$ is both a continuous linear map from $\mathscr{D}(M, F^\vee)$ into $\mathscr{D}(M, E^\vee)$ and from $\mathscr{E}(M, F^\vee)$ into $\mathscr{E}(M, E^\vee)$. Considering $(T^\vee)_*$ as map from $\mathscr{E}(M, F^\vee)$ into $\mathscr{E}(M, E^\vee)$ and taking its adjoint gives a continuous linear map from $\mathscr{E}'(M, E)$ into $\mathscr{E}'(M, F)$ that clearly equals the restriction of $T_*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$ to $\mathscr{E}'(M, E)$. $\square$

To establish an analogue of Corollary 1.3.2 (which we will need in the next chapter to deal with duals of functional spaces in the context of vector bundles), we consider the canonical map

$$\nu\colon E \to (E^\vee)^\vee = \mathrm{Hom}(\mathrm{Hom}(E, D), D),$$

which sends $e_x \in E$ to 'evaluation in e_x'. To be more precise, the image $\nu(e_x)\colon \mathrm{Hom}(E_x, D_x) \to D_x$ of e_x under ν is defined by $\nu(e_x)T_x := T_x e_x$.

Claim. ν is a vector bundle isomorphism.

Proof: Using some well-known vector bundle isomorphisms, we deduce

$$\begin{aligned}(E^\vee)^\vee &= \mathrm{Hom}(\mathrm{Hom}(E, D), D) \simeq \mathrm{Hom}(E^* \otimes D, D)\\ &\simeq (E^* \otimes D)^* \otimes D \simeq ((E^*)^* \otimes D^*) \otimes D\\ &\simeq (E^*)^* \otimes (D^* \otimes D) \simeq (E^*)^* \otimes (M \times \mathbb{K})\\ &\simeq (E^*)^* \simeq E,\end{aligned}$$

where the steps are taken so small that it should be clear what the explicit isomorphisms are. That the 'backwards' composition of these explicit isomorphisms equals ν is readily verified on a piece of scrap paper. □

As a consequence of this claim, ν_* is a topological isomorphism from $\mathscr{D}(M, E)$ onto $\mathscr{D}(M, (E^\vee)^\vee)$. Moreover, we easily check that for every $\varphi \in \mathscr{D}(M, E^\vee)$ and $\tilde{\psi} \in \mathscr{D}(M, (E^\vee)^\vee)$,

$$[\tilde{\psi}, \varphi] = [\varphi, (\nu_*)^{-1}\tilde{\psi}],$$

where the brackets on the left hand side are the brackets of the pairing

$$[\cdot\,, \cdot]\colon \mathscr{D}(M, (E^\vee)^\vee) \times \mathscr{E}(M, E^\vee) \to \mathbb{K}$$

and the brackets on the right hand side are the brackets of the pairing

$$[\cdot\,, \cdot]\colon \mathscr{D}(M, E^\vee) \times \mathscr{E}(M, E) \to \mathbb{K}.$$

We are now ready to prove the desired analogon of Corollary 1.3.2. Let $\hat{\imath}$ be the canonical topological isomorphism from $\mathscr{D}(M, E^\vee)$ onto $((\mathscr{D}(M, E^\vee))^*)^* = (\mathscr{D}'(M, E))^*$ (recall that $\mathscr{D}(M, E^\vee)$ is reflexive), let $\jmath$, $\imath$ and $\imath^\vee$ be as in the previous section and let $\jmath^\vee\colon \mathscr{E}(M, E^\vee) \to \mathscr{D}'(M, E^\vee)$ be the 'version' of $\jmath$ for the vector bundle $E^\vee$. Then we have:

Lemma 3.4.7. $((\nu_*)^{-1})^* \circ (\jmath \circ \imath)^* \circ \hat{\imath} = \jmath^\vee \circ \imath^\vee$.

Proof: For all $\varphi \in \mathscr{D}(M, E^\vee)$ and $\tilde{\psi} \in \mathscr{D}(M, (E^\vee)^\vee)$,

$$\begin{aligned}(((\nu_*)^{-1})^*(\jmath \circ \imath)^*\hat{\imath}\varphi)(\tilde{\psi}) &= ((\jmath \circ \imath)^*\hat{\imath}_\varphi)((\nu_*)^{-1}\tilde{\psi}) = (\hat{\imath}_\varphi)((\jmath \circ \imath)(\nu_*)^{-1}\tilde{\psi})\\ &= (\hat{\imath}_\varphi)(u_{(\nu_*)^{-1}\tilde{\psi}}) = u_{(\nu_*)^{-1}\tilde{\psi}}(\varphi) = [\varphi, (\nu_*)^{-1}\tilde{\psi}]\\ &= [\tilde{\psi}, \varphi] = ((\jmath^\vee \circ \imath^\vee)\varphi)(\tilde{\psi}).\end{aligned}$$

□

3.5 Multiplication by smooth functions

For multiplication by smooth functions things are pretty obvious. Every element φ of $\mathscr{E}(M)$ gives rise to a continuous linear map

$$m_\varphi\colon \mathscr{E}(M,E) \to \mathscr{E}(M,E)\colon \psi \mapsto (x \mapsto \varphi(x)\psi(x))$$

which restricts to a continuous linear map from $\mathscr{D}(M,E)$ into $\mathscr{D}(M,E)$. Since $E^\vee$ is a vector bundle as well, we in particular have a continuous linear map $m_\varphi\colon \mathscr{D}(M,E^\vee) \to \mathscr{D}(M,E^\vee)$ and the adjoint of this map gives a continuous linear extension of $m_\varphi\colon \mathscr{E}(M,E) \to \mathscr{E}(M,E)$ to $\mathscr{D}'(M,E)$.

Instead of checking all these statements, we observe that every $\varphi \in \mathscr{E}(M)$ gives rise to a vector bundle homomorphism $T_\varphi\colon E \to E$ by 'pointwise multiplication', i.e., by sending $e_x \in E_x$ to $\varphi(x)e_x$. Clearly, the ordinary pushforward of T_φ on the level of smooth sections equals the map m_φ given above, which is therefore indeed a continuous linear map from $\mathscr{E}(M,E)$ into $\mathscr{E}(M,E)$ that restricts to a continuous linear map from $\mathscr{D}(M,E)$ into $\mathscr{D}(M,E)$, and we readily check that the 'distributional' pushforward $(T_\varphi)_*$ equals the adjoint of $m_\varphi\colon \mathscr{D}(M,E^\vee) \to \mathscr{D}(M,E^\vee)$. Indeed, for every $\psi \in \mathscr{D}(M,E^\vee)$ and $x \in M$,

$$(((T_\varphi)^\vee)_*\psi)(x) = (T_\varphi)^\vee(\psi(x)) = \varphi(x)\psi(x) = (m_\varphi\psi)(x),$$

so

$$(T_\varphi)_* = (((T_\varphi)^\vee)_*)^* = (m_\varphi)^*.$$

Thus, the statement that the adjoint of $m_\varphi\colon \mathscr{D}(M,E^\vee) \to \mathscr{D}(M,E^\vee)$ gives a continuous linear extension of $m_\varphi\colon \mathscr{E}(M,E) \to \mathscr{E}(M,E)$ to $\mathscr{D}'(M,E)$ is just a special case of the extension of pushforwards to distributions that we have treated in the previous section.

Relevant results

All of the 'relevant results' from Section 1.5 are still valid (and relevant) in the setting of vector bundles. For completeness and convenience we reformulate them here. The proofs of these results are trivial adaptations of the proofs of their corresponding results from Section 1.5.

Lemma 3.5.1. *For every $u \in \mathscr{D}'(M,E)$ and $\psi \in \mathscr{E}(M)$*

$$\operatorname{supp}(\psi u) \subseteq \operatorname{supp}(\psi) \cap \operatorname{supp}(u).$$

Corresponding result: Lemma 1.5.3.

Lemma 3.5.2. *Let $u \in \mathscr{D}'(M,E)$, $\psi \in \mathscr{E}(M)$ and U an open subset of M. Then*

$$(\psi u)|_U = \psi|_U\, u|_U\,.$$

Corresponding result: Lemma 1.5.4.

Lemma 3.5.3. *Let $u \in \mathscr{D}'(M,E)$ and $\psi \in \mathscr{E}(M)$. If $x \in \operatorname{supp}(u)$ and ψ equals 1 on an open neighborhood $U \subseteq M$ of x, then $x \in \operatorname{supp}(\psi u)$.*

Corresponding result: Lemma 1.5.5.

Lemma 3.5.4. *If $u \in \mathscr{D}'(M,E)$ and $\psi \in \mathscr{E}(M)$ equals 1 on an open neighborhood of* $\operatorname{supp}(u)$*, then $\psi u = u$.*

Corresponding result: Lemma 1.5.6.

Lemma 3.5.5. *For all $u \in \mathscr{D}'(M,E)$ and $\psi \in \mathscr{E}(M)$ with the property that* $\operatorname{supp}(\psi) \cap \operatorname{supp}(u) = \emptyset$*, $\psi u = 0$.*

Corresponding result: Lemma 1.5.7.

Lemma 3.5.6. *If $\psi \in \mathscr{E}(M,E)$, U is an open subset of M and $u \in \mathscr{E}'(U, E_U)$, then*

$$\psi(\operatorname{ext}_{U,M} u) = \operatorname{ext}_{U,M}(\psi|_U\, u).$$

Corresponding result: Lemma 1.5.8.

Lemma 3.5.7. *If $u \in \mathscr{E}'(M,E)$ and U is an open subset of Ω with the property that* $\operatorname{supp}(u) \subseteq U$*, then $u|_U \in \mathscr{E}'(U, E_U)$ and*

$$\operatorname{ext}_{U,\Omega}(u|_U) = u.$$

Corresponding result: Lemma 1.5.9.

Lemma 3.5.8. *For every open subset U of M and every $u \in \mathscr{E}'(U, E_U)$*

$$\operatorname{supp}(\operatorname{ext}_{U,M} u) = \operatorname{supp}(u).$$

Corresponding result: Lemma 1.5.10.

Lemma 3.5.9. *Let $\{u_i\}_{i\in I}$ be a net in $\mathscr{D}'(M,E)$ and $u \in \mathscr{D}'(M,E)$. Then $u_i \to u$ in $\mathscr{D}'(M,E)$ if and only if $\varphi u_i \to \varphi u$ in $\mathscr{D}'(M,E)$ for every $\varphi \in \mathscr{D}(M)$.*

Corresponding result: Lemma 1.5.11.

We can now prove the following generalization of Lemma 1.3.3.

Lemma 3.5.10. *$\mathscr{D}(M,E)$ is sequentially dense in $\mathscr{E}'(M,E)$ and $\mathscr{D}'(M,E)$.*

Proof: We will start with the claim that $\mathscr{D}(M,E)$ is sequentially dense in $\mathscr{D}'(M,E)$. So let $u \in \mathscr{D}'(M,E)$. To find a sequence in $\mathscr{D}(M,E)$ that converges to u in $\mathscr{D}'(M,E)$, we first choose some countable collection $\{(U_i, \kappa_i, \rho_i)\}_{i\in\mathbb{N}}$ of total trivialization triples of $E \to M$ such that $\{U_i\}_{i\in\mathbb{N}}$ is a locally finite open cover of M by precompact subsets (this is possible because M is assumed to be second-countable). Then for every $i \in \mathbb{N}$,

$$\mathscr{D}(U_i, E_{U_i}) \simeq (\mathscr{D}(\kappa_i(U_i)))^{\times r} \quad \text{and} \quad \mathscr{D}'(U_i, E_{U_i}) \simeq (\mathscr{D}'(\kappa_i(U_i)))^{\times r},$$

while the inclusions $\mathscr{D}(U_i, E_{U_i}) \subseteq_{\mathrm{c}} \mathscr{D}'(U_i, E_{U_i})$ correspond to the (product of the) inclusions $\mathscr{D}(\kappa_i(U_i)) \subseteq_{\mathrm{c}} \mathscr{D}'(\kappa_i(U_i))$ (see Remark 3.2.3). Therefore, on behalf of Lemma 1.3.3 and the trivial fact that the product of sequentially dense subsets is sequentially dense, $\mathscr{D}(U_i, E_{U_i})$ is sequentially dense in $\mathscr{D}'(U_i, E_{U_i})$ for every $i \in \mathbb{N}$. Using this, we find for every $i \in \mathbb{N}$ a sequence $\{\varphi^i_j\}_{j\in\mathbb{N}}$ in $\mathscr{D}(U_i, E_{U_i})$ such that $\varphi^i_j \to u|_{U_i}$ in $\mathscr{D}'(U_i, E_{U_i})$ when $j \to \infty$.

Next, choose some smooth partition of unity $\{\eta_i\}_{i\in\mathbb{N}}$ subordinate to $\{U_i\}_{i\in\mathbb{N}}$. Since the U_i are precompact, we have that $\eta_i \in \mathscr{D}(M)$ for every $i \in \mathbb{N}$ (observe that $\mathrm{supp}(\eta_i) \subseteq \mathrm{cl}(U_i)$, so $\mathrm{supp}(\eta_i)$ is a closed subset of a compact subset) and because $\mathrm{supp}(\eta_i) \subseteq U_i$, we also have $\eta_i|_{U_i} \in \mathscr{D}(U_i)$. Just as in the Euclidean case, multiplication by a compactly supported smooth function is a continuous linear map from the space of distributions into the space of compactly supported distributions, so we find that for every $i \in \mathbb{N}$, $\eta_i|_{U_i}\varphi^i_j \to \eta_i|_{U_i} u|_{U_i} = (\eta_i u)|_{U_i}$ in $\mathscr{E}'(U_i, E_{U_i})$ when $j \to \infty$. Applying the continuous linear 'extension by zero' map $\mathrm{ext}_{U,M}\colon \mathscr{E}'(U_i, E_{U_i}) \to \mathscr{E}'(M,E) \subseteq_{\mathrm{c}} \mathscr{D}'(M,E)$, subsequently shows that for every $i \in \mathbb{N}$, $\eta_i\varphi^i_j \to \eta_i u$ in $\mathscr{D}'(M,E)$ when $j \to \infty$, where $\eta_i\varphi^i_j$ is interpreted as an element of $\mathscr{D}(M,E)$ in the obvious way (so it equals zero outside $\mathrm{supp}(\eta_i) \subseteq U_i$).

We now claim that the sequence $\{\psi_j\}_{j\in\mathbb{N}}$, defined by

$$\psi_j := \sum_{i=0}^{j} \eta_i\varphi^i_j \in \mathscr{D}(M,E),$$

converges to u in $\mathscr{D}'(M,E)$. To prove this, let B be a bounded subset of $\mathscr{D}(M,E^\vee)$ and let q_B be the associated seminorm of $\mathscr{D}'(M,E) = (\mathscr{D}(M,E^\vee))^*$. On the strength of Lemma 3.1.9, we then find an $K \in \mathscr{P}_{\mathrm{c}}(M)$ such that B is a bounded subset of $\mathscr{E}_K(M,E^\vee)$ and because $\{U_i\}_{i\in\mathbb{N}}$ is locally finite, we subsequently find an $N \in \mathbb{N}$ such that $U_i \cap K = \emptyset$ for all $i > N$. If we combine this with Lemma 3.3.5, we see that for all $\chi \in B$ and $j \geq N$,

$$\psi_j(\chi) = \sum_{i=0}^{j}(\eta_i\varphi^i_j)(\chi) = \sum_{i=0}^{N}(\eta_i\varphi^i_j)(\chi) \quad \text{and}$$
$$u(\chi) = \sum_{i=0}^{\infty}(\eta_i u)(\chi) = \sum_{i=0}^{N}(\eta_i u)(\chi).$$

So, for all $j \geq N$,

$$\begin{aligned} q_B(u-\psi_j) &= \sup_{\chi\in B}|u(\chi)-\psi_j(\chi)| = \sup_{\chi\in B}\Big|\sum_{i=0}^{N}(\eta_i u - \eta_i\varphi^i_j)(\chi)\Big| \\ &\leq \sum_{i=0}^{N} q_B(\eta_i u - \eta_i\varphi^i_j) \end{aligned}$$

and since for every $i \in \mathbb{N}$, $\eta_i\varphi^i_j \to \eta_i u$ in $\mathscr{D}'(M,E)$ when $j \to \infty$, this shows that $q_B(u-\psi_j) \to 0$ when $j \to \infty$. Hence, $\{\psi_j\}_{j\in\mathbb{N}}$ indeed converges to u in $\mathscr{D}'(M,E)$ and we conclude that $\mathscr{D}(M,E)$ is sequentially dense in $\mathscr{D}'(M,E)$.

That $\mathscr{D}(M,E)$ is also sequentially dense in $\mathscr{E}'(M,E)$ is an easy consequence of the just proven fact. Take $u \in \mathscr{E}'(M,E)$. Then in particular $u \in \mathscr{D}'(M,E)$, so we find a sequence $\{\psi_j\}_{j\in\mathbb{N}}$ such that $\psi_j \to u$ in $\mathscr{D}'(M,E)$. Now take $\varphi \in \mathscr{D}(M)$ such that φ equals 1 on an open neighborhood of $\mathrm{supp}(u)$. Then m_φ is a continuous linear map from $\mathscr{D}'(M,E)$ into $\mathscr{E}'(M,E)$ and $m_\varphi u = u$, so $\{\varphi\psi_j\}_{j\in\mathbb{N}}$ is a sequence in $\mathscr{D}(M,E)$ such that $\varphi\psi_j \to u$ in $\mathscr{E}'(M,E)$. □

3.6 Differentiation

The situation regarding differentiation is a bit more complex. On a manifold we in general do not have globally defined partial derivatives, but we do have linear partial differential operators between vector bundles which can locally be expressed in terms of partial derivatives. To discuss these differential operators, we agree that F again denotes a vector bundle over M of rank s in this section and that

$$\begin{aligned} \langle\cdot\,,\cdot\rangle_E &: \mathscr{E}(M, E^\vee) \times \mathscr{E}(M, E) \to \mathscr{E}(M, D) \quad \text{and} \\ \langle\cdot\,,\cdot\rangle_F &: \mathscr{E}(M, F^\vee) \times \mathscr{E}(M, F) \to \mathscr{E}(M, D) \end{aligned}$$

are the 'pairings' that we have discussed earlier.

As we briefly recall in Section B.4, a (smooth linear partial) differential operator P from E to F, notation $P \in \mathrm{Diff}(E, F)$, is a special type of linear map between the spaces of smooth sections of E and F. Since we have endowed these spaces of smooth sections with locally convex topologies, it is natural to ask whether partial differential operators are continuous.

Proposition 3.6.1. *Every $P \in \mathrm{Diff}(E, F)$ is a continuous linear map from $\mathscr{E}(M, E)$ into $\mathscr{E}(M, F)$.*

Proof: According to Theorem B.4.3, there exist vector bundle homomorphisms $T_0, \ldots, T_m \in \mathrm{Hom}(E, F)$ and differential operators $P_0, \ldots, P_m \in \mathrm{Diff}(E, E)$ such that $P = \sum_{j=0}^{m} (T_j)_* \circ P_j$ and because we already know that the pushforwards $(T_j)_*$ are continuous linear maps from $\mathscr{E}(M, E)$ into $\mathscr{E}(M, F)$, we see that it suffices to show that every differential operator from E to E is a continuous linear map from $\mathscr{E}(M, E)$ into $\mathscr{E}(M, E)$. Thanks to the work that we have done earlier in this chapter, this is actually really easy. Fix $P_0 \in \mathrm{Diff}(E, E)$, choose some vector bundle metric g on E and recall from the beginning of this chapter that the topology of $\mathscr{E}(M, E)$ is induced by the seminorms

$$\|\varphi\|_{K,P}^g := \sup_{x \in K} |(P\varphi)(x)|^g$$

with $K \in \mathscr{P}_\mathrm{c}(M)$ and $P \in \mathrm{Diff}(E, E)$. Since a composition of differential operators is again a differential operator, we simply have

$$\|P_0\varphi\|_{K,P}^g = \sup_{x \in K} |(PP_0\varphi)(x)|^g = \|\varphi\|_{K,PP_0}^g$$

for all $\varphi \in \mathscr{E}(M, E)$, $K \in \mathscr{P}_\mathrm{c}(M)$ and $P \in \mathrm{Diff}(E, E)$ and on the strength of Lemma A.1.2, this proves that $P_0 \colon \mathscr{E}(M, E) \to \mathscr{E}(M, E)$ is continuous. □

Corollary 3.6.2. *Every $P \in \mathrm{Diff}(E, F)$ restricts to a continuous linear map from $\mathscr{D}(M, E)$ into $\mathscr{D}(M, F)$.*

Proof: Because differential operators are local, it follows from the previous proposition that P is a continuous linear map from $\mathscr{E}_K(M, E)$ into $\mathscr{E}_K(M, F)$ for every $K \in \mathscr{P}_\mathrm{c}(M)$, so the result follows by using that $\mathscr{E}_K(M, F) \subseteq_\mathrm{c} \mathscr{D}(M, F)$ and applying Proposition A.3.2. □

Of course, since this chapter is about distributions, we are interested in extending differential operators $P\colon \mathscr{E}(M,E)\to\mathscr{E}(M,F)$ to maps between the spaces of distributions on (M,E) and (M,F). To realize this, we introduce the concept of *formal adjoints*.

Definition 3.6.3. A *formal adjoint* of a differential operator $P\in\operatorname{Diff}(E,F)$ is a differential operator $Q\in\operatorname{Diff}(F^\vee,E^\vee)$ such that

$$\int_M \langle\varphi,P\psi\rangle_F=\int_M\langle Q\varphi,\psi\rangle_E$$

for all $\varphi\in\mathscr{E}(M,F^\vee)$ and $\psi\in\mathscr{E}(M,E)$ with the property that $\operatorname{supp}(\varphi)\cap\operatorname{supp}(\psi)$ is compact. ⊘

Looking back, we see that we have actually already encountered a formal adjoint. Indeed, if $T\colon E\to F$ is a vector bundle homomorphism, then T_* is a partial differential operator (of order 0) and equation (3.5) shows that $(T^\vee)_*$ is a formal adjoint of T_*. Moreover, in this specific case we have seen that the adjoint (as a linear map) of the formal adjoint extended the original map, something which is true in general:

Lemma 3.6.4. *If a differential operator $P\colon\mathscr{E}(M,E)\to\mathscr{E}(M,F)$ has a formal adjoint $Q\colon\mathscr{E}(M,F^\vee)\to\mathscr{E}(M,E^\vee)$, then the (linear topological) adjoint $Q^*\colon\mathscr{D}'(M,E)\to\mathscr{D}'(M,F)$ of the restriction $Q\colon\mathscr{D}(M,F^\vee)\to\mathscr{D}(M,E^\vee)$ extends P.*

Proof: Let $\psi\in\mathscr{E}(M,E)$ and $\varphi\in\mathscr{D}(M,F^\vee)$. Then

$$(Q^*u_\psi)(\varphi)=u_\psi(Q\varphi)=\int_M\langle Q\varphi,\psi\rangle_E=\int_M\langle\varphi,P\psi\rangle_F=u_{P\psi}(\varphi).\qquad\square$$

Because the formal adjoint Q in the lemma above is by definition again a differential operator and therefore a *continuous* linear map from $\mathscr{D}(M,F^\vee)$ into $\mathscr{D}(M,E^\vee)$, the extension $Q^*\colon\mathscr{D}'(M,E)\to\mathscr{D}'(M,F)$ of P is also a continuous linear map. The following result therefore shows that every differential operator $P\in\operatorname{Diff}(E,F)$ has a continuous linear extension to a map between the spaces of distributions on (M,E) and (M,F).

Theorem 3.6.5. *Every $P\in\operatorname{Diff}(E,F)$ has a unique formal adjoint.*

Sketch of the proof: The uniqueness is easy. Suppose that both Q and Q' are formal adjoints of P. Then we have for all $\varphi\in\mathscr{E}(M,F^\vee)$ and $\psi\in\mathscr{D}(M,E)$,

$$\begin{aligned}[(Q'-Q)\varphi,\psi]_E&=\int_M\langle(Q'-Q)\varphi,\psi\rangle_E\\&=\int_M\langle Q'\varphi,\psi\rangle_E-\int_M\langle Q\varphi,\psi\rangle_E\\&=\int_M\langle\varphi,P\psi\rangle_F-\int_M\langle\varphi,P\psi\rangle_F=0.\end{aligned}$$

Hence, on behalf of Lemma 3.2.9, $(Q'-Q)\varphi=0$ for all $\varphi\in\mathscr{E}(M,F^\vee)$, which proves that $Q'=Q$.

For the existence part of the statement, we only give a sketch of the proof. Take $k \in \mathbb{N}$ such that P is of order at most k. Because the assignment $U \mapsto \mathrm{Diff}_k(E_U, F_U)$ is a sheaf over M (we briefly mention how restriction of a differential operator works in Section B.4), we may reduce the problem of finding a formal adjoint for P to finding a formal adjoint for $P|_U$ for every simultaneous total trivialization 'triple' $(U, \kappa, \rho^E, \rho^F)$ of $E \to M$ and $F \to M$. By definition, $P|_U$ can be written as $\sum_{|\alpha| \leq k} (C_\alpha)_* \circ \partial_\kappa^\alpha$ for certain vector bundle homomorphisms $C_\alpha \colon E_U \to F_U$ and by using the trivializations to switch to ordinary Lebesgue integrals and using partial integration, we see that

$$Q_U := \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \, \partial_\kappa^\alpha \circ (C_\alpha^\vee)_*$$

is a formal adjoint for $P|_U$. □

There are two things that are relevant to note here. First, we see from the sketch of the proof given above that if P is of order at most k, also its formal adjoint Q is of order at most k. And, second, one should be aware of the fact that the uniqueness of the formal adjoint of P does not necessarily imply that the extension of P to a continuous linear map from $\mathscr{D}'(M, E)$ into $\mathscr{D}'(M, F)$ is unique. However, the latter is true anyway, because $\mathscr{E}(M, E)$ is dense in $\mathscr{D}'(M, E)$ and $\mathscr{D}'(M, F)$ is Hausdorff.

Remark 3.6.6. In the case of open subsets of $\mathbb{R}^{\mathrm{n}}$ and trivial line bundles, we easily verify that the formal adjoint of a partial derivative ∂_i, with $1 \leq i \leq \mathrm{n}$, equals $-\partial_i$. So although on $\mathbb{R}^{\mathrm{n}}$ it often seemed that we were taking ordinary adjoints, we have actually been working with formal adjoints from the start. ⊘

4

Functional spaces on vector bundles

As discussed in the introduction of the previous chapter, we would like to generalize our theory of functional spaces from the Euclidean context to the context of vector bundles. This might sound like a lot of work, but for a large part of the theory this generalization is actually rather painless. The reason for this is that most of the technicalities that differential geometry brings along are already incorporated in our theory of distributions on vector bundles. Compared to the Euclidean context, our 'new' spaces of distributions basically have the same properties and our 'toolbox' is filled with similar results. There are only two topics that need some extra attention, namely duals and powers of functional spaces.

So, if this chapter would only be about generalizing the material from the second chapter, we would be finished relatively quick. However, generalizing the formal theory of functional spaces is only part of the work and not even the most important part. When working with partial differential equations on vector bundles, much more than a framework to deal with 'solution spaces', we would namely like to have a framework to *create* 'solution spaces'. In other words, we would like to be able to 'bring' the familiar and well-behaved solution spaces from the Euclidean context 'with us' when going from Euclidean space to vector bundles, so that we can use the 'same' solution spaces for problems that require more geometry. In this chapter, we will show that this wish can basically be granted. Of course, due to the different context, spaces of (distributional) sections of a nontrivial vector bundle cannot be literally the same as a functional space (i.e., a 'solution space') on $\mathbb{R}^{\mathrm{n}}$, but it *is* possible to create functional spaces on vector bundles that 'look like' sufficiently well-behaved functional spaces on $\mathbb{R}^{\mathrm{n}}$. We can even do this in such a way that many important properties are preserved.

As in the previous chapter, M denotes an n-dimensional (second-countable smooth) manifold and $E \to M$ denotes a rank r vector bundle over M.

4.1 Generalization to vector bundles

As indicated in the introduction above, most of the definitions and results from the second chapter generalize in a trivial and straightforward manner. For example, the definition of a functional space becomes:

Definition 4.1.1. A *functional space* on (M, E) is a linear subspace $\mathscr{F}$ of $\mathscr{D}'(M, E)$ that contains $\mathscr{D}(M, E)$ and carries a locally convex topology such that:

1. $\mathscr{D}(M,E) \subseteq_{\mathrm{c}} \mathscr{F} \subseteq_{\mathrm{c}} \mathscr{D}'(M,E)$ and
2. for every $\varphi \in \mathscr{D}(M)$, $m_\varphi \colon \mathscr{D}'(M,E) \to \mathscr{D}'(M,E)$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$. $\oslash$

Compared to our 'original' definition of a functional space, Definition 2.1.1, the only difference is that all occurences of Ω are replaced by (M,E) or M. And, as surprising as it may sound, such replacements are in fact sufficient to make the step from $\mathbb{R}^n$ to vector bundles for the majority of the definitions and results (to be precise: Ω should be replaced by M when we pick elements for multiplication and by (M,E) otherwise). There are only a few exceptions:

- The generalization of invariance needs additional explanation, but because we will not really need the generalized version, we exclude the definition and results concerning invariance from the generalization procedure.
- Lemma 2.6.22 and Lemma 2.6.23 are only true under the extra assumption that M is noncompact. Clearly, if M is compact, $\mathscr{F}_{\mathrm{comp}} = \mathscr{F} = \mathscr{F}_{\mathrm{loc}}$ for every semi-functional space $\mathscr{F}$ on (M,E) (for the latter equality, observe that the constant 1 function on M is compactly supported in this case).
- The definitions and results from Section 2.9, Section 2.10 and Section 2.12 need extra attention and we shall give this extra attention in the next two sections.

That most of the material from the second chapter generalizes so easily confirms that we have really developed a nice, independent theory with intrinsic arguments. The definitions, results and proofs stay almost literally the same; we just have to substitute some symbols and replace the supporting results from the first chapter by their generalized version from the third chapter.

4.2 Special attention: Duals

The basic identities

$$(\mathscr{D}'(M,E))^* = ((\mathscr{D}(M,E^\vee))^*)^* \simeq \mathscr{D}(M,E^\vee)$$

and

$$(\mathscr{E}'(M,E))^* = ((\mathscr{E}(M,E^\vee))^*)^* \simeq \mathscr{E}(M,E^\vee)$$

already indicate what the 'problem' with duals is in the context of vector bundles: the strong dual of a normal functional space $\mathscr{F}$ on (M,E) is in general *not* (canonically) a functional space on (M,E) but a functional space on $(M,E^\vee)$. Of course, this is not really a problem, it is just something that should be observed and dealt with and that requires a little more attention. In order to give a proper definition, let

$$\jmath \circ \imath \colon \mathscr{D}(M,E) \to \mathscr{D}'(M,E), \quad \nu \colon E \to (E^\vee)^\vee \quad \text{and}$$
$$\hat{\imath} \colon \mathscr{D}(M,E^\vee) \to ((\mathscr{D}(M,E^\vee))^*)^* = (\mathscr{D}'(M,E))^*$$

be the maps that we have discussed in Section 3.2 and Section 3.4.

Definition 4.2.1. Let $\mathscr{F}$ be a normal functional space on (M, E) and let

$$\mu\colon (\jmath \circ \imath)(\mathscr{D}(M,E)) \to \mathscr{F} \quad \text{and} \quad \mu'\colon \mathscr{F} \to \mathscr{D}'(M,E)$$

be the inclusion mappings. Because $(\jmath \circ \imath)(\mathscr{D}(M,E))$ is dense in $\mathscr{F}$ and $\mathscr{F}$ is dense in $\mathscr{D}'(\Omega)$, the adjoints of the continuous linear maps

$$\mu \circ \jmath \circ \imath\colon \mathscr{D}(M,E) \to \mathscr{F} \quad \text{and} \quad \mu'\colon \mathscr{F} \to \mathscr{D}'(M,E),$$

denoted by

$$(\mu \circ \jmath \circ \imath)^*\colon \mathscr{F}^* \to (\mathscr{D}(M,E))^* \quad \text{and} \quad (\mu')^*\colon (\mathscr{D}'(M,E))^* \to \mathscr{F}^*,$$

are injective continuous linear maps and we already know that

$$\hat{\imath}\colon \mathscr{D}(M,E^\vee) \to ((\mathscr{D}(M,E^\vee))^*)^* = (\mathscr{D}'(M,E))^* \quad \text{and}$$
$$((\nu_*)^{-1})^*\colon (\mathscr{D}(M,E))^* \to (\mathscr{D}(M,(E^\vee)^\vee))^* = \mathscr{D}'(M,E^\vee),$$

where the pushforward ν_* is the ordinary pushforward on compactly supported smooth sections, are linear topological isomorphisms. We define $\mathscr{F}'$ to be the vector subspace $(((\nu_*)^{-1})^* \circ (\mu \circ \jmath \circ \imath)^*)(\mathscr{F}^*)$ of $\mathscr{D}'(M,E^\vee)$ endowed with the topology that turns $((\nu_*)^{-1})^* \circ (\mu \circ \jmath \circ \imath)^*$ into a linear topological isomorphism from $\mathscr{F}^*$ onto $\mathscr{F}'$. ⊘

With a proof very similar to the proof of Proposition 2.10.2, we can show that $\mathscr{F}'$ is a functional space on $(M,E^\vee)$. Of course, this time the adaption of the proof requires a little more imagination than substituting some symbols, but thanks to the fact that we have already proven the correct analogue of Corollary 1.3.2 in Section 3.4, it should still be possible to do this without a piece of paper (but be aware of the difference in notation).

Remark 4.2.2. In this situation, it is very convenient to use $(\nu_*)^*$ to identify distributions on $(M,E^\vee)$ with continuous linear forms on $\mathscr{D}(M,E)$. Because ν_* is $\mathscr{E}(M)$-linear, this identification also handles multiplication correctly (multiplication by $\varphi \in \mathscr{E}(M)$ on $\mathscr{D}'(M,E^\vee)$ corresponds to the obvious multiplication, given by the adjoint of $m_\varphi\colon \mathscr{D}(M,E) \to \mathscr{D}(M,E)$, on $(\mathscr{D}(M,E))^*$), so this identification really allows us to forget about $\mathscr{D}'(M,E^\vee)$ and to work with $(\mathscr{D}(M,E))^*$ instead. Under this indentification, the elements of $\mathscr{F}'$ are precisely those continuous linear forms on $\mathscr{D}(M,E)$ that allow a continuous extension to $\mathscr{F}$; a description which should sound familiar and is easy to work with. ⊘

Because the adaptations of proofs and results regarding $\mathscr{F}'$ require a bit more than blindly substituting some symbols and we do not want to spend too much time on juggling with notation and identifications, we only mention the following results:

Proposition 4.2.3. *The assignment $\mathscr{F} \mapsto \mathscr{F}'$ is a contravariant functor from the category of normal functional spaces on (M,E) to the category of functional spaces on $(M,E^\vee)$.*

Lemma 4.2.4. *Let $\mathscr{F}$ be a normal functional space on (M,E). If $\mathscr{F}$ is reflexive (as locally convex vector space), then $\mathscr{F}'$ is again normal.*

Lemma 4.2.5. *For every normal reflexive functional space $\mathscr{F}$ on (M,E)*

$$\nu_*\colon \mathscr{D}'(M,E) \to \mathscr{D}'(M,(E^\vee)^\vee)$$

restricts to a linear topological isomorphism from $\mathscr{F}$ onto $(\mathscr{F}')'$.

4.3 Special attention: Powers

Also the construction $\mathscr{F} \mapsto \mathscr{F}^k$ needs special attention when making the step from $\mathbb{R}^n$ to vector bundles. The reason is simple: we used partial derivatives to introduce $\mathscr{F}^k$ for $k \in \mathbb{N}_\infty$ and on a manifold the concept of 'partial derivatives' usually only makes sense locally (e.g., on a coordinate chart). So, to start with, in this more general setting we have to come up with a different definition of $\mathscr{F}^k$ for $k \in \mathbb{N}_\infty$. The obvious solution is to use differential operators instead of partial derivatives:

Definition 4.3.1. Let $\mathscr{F}$ be a semi-functional space on (M, E), $\mathscr{P}$ an inducing collection of seminorms for $\mathscr{F}$ and $k \in \mathbb{N}_\infty$. As a set, we define

$$\mathscr{F}^k := \{u \in \mathscr{D}'(M, E) \mid Pu \in \mathscr{F} \text{ for all } P \in \mathrm{Diff}_k(E, E)\},$$

where $\mathrm{Diff}_k(E, E)$ is the space of (linear partial) differential operators from E to E of order at most k. Subsequently, we define for each $p \in \mathscr{P}$ and $P \in \mathrm{Diff}_k(E, E)$

$$p_P : \mathscr{F}^k \to \mathbb{R} : u \mapsto p(Pu).$$

We easily see that $\mathscr{F}^k$ is a vector subspace of $\mathscr{D}'(M, E)$ and that the p_P, with $p \in \mathscr{P}$ and $P \in \mathrm{Diff}_k(E, E)$, are seminorms on $\mathscr{F}^k$. We endow $\mathscr{F}^k$ with the topology induced by these seminorms. ⊘

Remark 4.3.2. $\mathrm{Diff}_\infty(E, E)$ is just the space $\mathrm{Diff}(E, E)$ of all linear differential operators from E to E. ⊘

However, there is a catch: in order to make sure that $\mathscr{F}^0 = \mathscr{F}$ we need to assume that $\mathscr{F}$ has the following property.

Definition 4.3.3. We say that a semi-functional space $\mathscr{F}$ on (M, E) is *top invariant* if for every vector bundle homomorphism $T : E \to E$ the pushforward

$$T_* : \mathscr{D}'(M, E) \to \mathscr{D}'(M, E)$$

restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$. ⊘

In fact, $\mathscr{F}^0 = \mathscr{F}$ if and only if $\mathscr{F}$ is top invariant (note that differential operators of order at most 0 are pushforwards of vector bundle homomorphisms and vice versa) and to avoid confusion we agree to only take 'powers' of semi-functional spaces that have this property.

Proposition 4.3.4. *Every top invariant semi-functional space $\mathscr{F}$ on (M, E) is semi-local.*

Proof: Let $\varphi \in \mathscr{E}(M)$. As discussed in Section 3.5, $m_\varphi : \mathscr{D}'(M, E) \to \mathscr{D}'(M, E)$ is equal to $(T_\varphi)_*$, where $T_\varphi : E \to E$ is the vector bundle homomorphism given by 'pointwise multiplication'. As a consequence, the top invariance of $\mathscr{F}$ implies that $m_\varphi = (T_\varphi)_*$ restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$. □

In view of the previous proposition and the discussion following Proposition 2.9.17, we see that the agreement to restrict our attention to top invariant semi-functional spaces not only makes sure that $\mathscr{F}^0 = \mathscr{F}$ but that it also makes sure that there is no conflict between the generalized and the 'original' definition of $\mathscr{F}^k$ (as given in the second chapter) when M is an open subset of $\mathbb{R}^n$ and E is the trivial line bundle over this open subset.

Remark 4.3.5. The same discussion also mentions differential operators with constant coefficients and compactly supported differential operators, so why did we choose to work with top invariant spaces and arbitrary differential operators? Well, on manifolds we can in general not speak about differential operators with constant coefficients and if we would define $\mathscr{F}^k$ in terms of compactly supported differential operators, the generalized definition would only coincide with the original one on local spaces. That is, by using compactly supported differential operators for the characterization, we would have to limit ourselves to local spaces on (M, E) when considering $\mathscr{F} \mapsto \mathscr{F}^k$. Moreover, after a moment's thought we realize that on local spaces the characterization in terms of compactly supported differential operators in fact coincides with the characterization in terms of arbitrary differential operators, so we really would not gain anything by restricting to local spaces and using compactly supported differential operators. ⊘

Of course, we should also generalize the definition of $\mathscr{F}^{-k}$ for $k \in \mathbb{N}_\infty$, but before we do this we first investigate $\mathscr{F}^k$ a bit more. With arguments that should be familiar by now, we deduce that for every $P \in \mathrm{Diff}_k(E, E)$, $P\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, E)$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$ and that the topology of $\mathscr{F}^k$ is the smallest locally convex topology with this property. Moreover, we have two familiar results:

Lemma 4.3.6. *Let $\mathscr{F}$ be a top invariant semi-functional space on (M, E), k an element of $\mathbb{N}_\infty$, $\mathscr{X}$ a locally convex vector space and $T\colon \mathscr{X} \to \mathscr{F}^k$ a linear map. Then T is continuous if and only if for every $P \in \mathrm{Diff}_k(E, E)$*

$$P \circ T\colon \mathscr{X} \to \mathscr{F}$$

is continuous.

Corollary 4.3.7. *Let $\mathscr{F}$ be a top invariant semi-functional space on (M, E), $k \in \mathbb{N}_\infty$, $\mathscr{X}$ a locally convex vector space and $T\colon \mathscr{X} \to \mathscr{D}'(M, E)$ a linear map. If for every $P \in \mathrm{Diff}_k(E, E)$, $P \circ T$ is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}$, then T is a continuous linear map from $\mathscr{X}$ into $\mathscr{F}^k$.*

We certainly do not have the ambition to generalize and discuss every single result from Section 2.9 here, but the least we can do is show that $\mathscr{F}^k$ is actually a semi-functional space.

Proposition 4.3.8. *Let $\mathscr{F}$ be a top invariant semi-functional space on (M, E). Then for every $k \in \mathbb{N}_\infty$, $\mathscr{F}^k$ is a semi-functional space on (M, E). Furthermore, $\mathscr{F}^{k+1} \subseteq_{\mathrm{c}} \mathscr{F}^k$, $\mathscr{F}^0 = \mathscr{F}$ and if $\mathscr{F}$ is a functional space on (M, E), then $\mathscr{F}^k$ is a functional space on (M, E) as well.*

Proof: Let $\ell, \ell' \in \mathbb{N}_\infty$ with $\ell \leq \ell'$. Then every differential operator of order at most ℓ is also a differential operator of order at most ℓ', so applying Corollary 4.3.7 to the inclusion map $\mathscr{F}^{\ell'} \hookrightarrow \mathscr{D}'(M, E)$ immediately shows that $\mathscr{F}^{\ell'} \subseteq_{\mathrm{c}} \mathscr{F}^{\ell}$. As a consequence, we in particular have that $\mathscr{F}^{k+1} \subseteq_{\mathrm{c}} \mathscr{F}^k$ and that $\mathscr{F}^k \subseteq_{\mathrm{c}} \mathscr{F}^0$.

Next, let us explain in more detail why the top invariance of $\mathscr{F}$ implies $\mathscr{F}^0 = \mathscr{F}$. Because the identity map $\imath\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, E)$ is a differential operator of order at most 0, we directly get from the characterizing property of

$\mathscr{F}^0$ that $\imath$ restricts to a continuous linear map from $\mathscr{F}^0$ into $\mathscr{F}$, hence $\mathscr{F}^0 \subseteq_{\mathrm{c}} \mathscr{F}$. For the converse inclusion, recall that every differential operator P from E to E of order at most 0 can be written as a pushforward T_* of a vector bundle homomorphism $T\colon E \to E$. As a result, the top invariance of $\mathscr{F}$ tells us that every differential operator P from E to E of order at most 0 restricts to a continuous linear map from $\mathscr{F}$ into $\mathscr{F}$, hence by applying Corollary 4.3.7 to the inclusion map $\mathscr{F} \hookrightarrow \mathscr{D}'(M, E)$ we obtain that $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{F}^0$.

Combining $\mathscr{F}^k \subseteq_{\mathrm{c}} \mathscr{F}^0$ and $\mathscr{F}^0 = \mathscr{F}$, we get $\mathscr{F}^k \subseteq_{\mathrm{c}} \mathscr{F} \subseteq_{\mathrm{c}} \mathscr{D}'(M, E)$. Moreover, if $\varphi \in \mathscr{D}(M)$, then m_φ is a differential operator from E to E of order at most 0, which by the 'composition property' of differential operators implies that $P \circ m_\varphi$ is a differential operator of order at most k for all $P \in \mathrm{Diff}_k(E, E)$. Applying Corollary 4.3.7 to $m_\varphi\colon \mathscr{F}^k \to \mathscr{D}'(M, E)$ then shows that m_φ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^k$, so we conclude that $\mathscr{F}^k$ is a semi-functional space on (M, E).

If $\mathscr{F}$ is a functional space on (M, E), then $\mathscr{D}(M, E) \subseteq_{\mathrm{c}} \mathscr{F}$. So for all differential operators $P \in \mathrm{Diff}_k(E, E)$, $P\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, E)$, which a priori restricts to a continuous linear map from $\mathscr{D}(M, E)$ into $\mathscr{D}(M, E)$, also restricts to a continuous linear map from $\mathscr{D}(M, E)$ into $\mathscr{F}$. Applying Corollary 4.3.7 to the inclusion map $\mathscr{D}(M, E) \hookrightarrow \mathscr{D}'(M, E)$ then shows that $\mathscr{D}(M, E) \subseteq_{\mathrm{c}} \mathscr{F}^k$ and we conclude that $\mathscr{F}^k$ is a functional space as well. □

Many of the results from Section 2.9 are still valid in the vector bundle setting for the more general definition of $\mathscr{F}^k$, but some of the proofs need significant changes or additional supporting results about differential operators and we simply cannot discuss everything. To stress that it *is* possible to develop the theory in the same fashion as in Section 2.9 and to show that looking for generalizations of proofs might even be enlightening, we present a selection of three generalized results here.

Proposition 4.3.9. *For every $k \in \mathbb{N}_\infty$, the assignment $\mathscr{F} \mapsto \mathscr{F}^k$ is a functor from the category of top invariant (semi-)functional spaces on (M, E) to the category of (semi-)functional spaces on (M, E).*

Proof: Suppose that $\mathscr{F}$ and $\mathscr{G}$ are top invariant (semi-)functional spaces on (M, E) such that $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$. By the characterizing property of $\mathscr{F}^k$, every $P \in \mathrm{Diff}_k(E, E)$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$. If we combine this with the assumption that $\mathscr{F} \subseteq_{\mathrm{c}} \mathscr{G}$, we obtain that every $P \in \mathrm{Diff}_k(E, E)$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{G}$, so we can apply Corollary 4.3.7 to the inclusion map $\mathscr{F}^k \hookrightarrow \mathscr{D}'(M, E)$ to get $\mathscr{F}^k \subseteq_{\mathrm{c}} \mathscr{G}^k$. □

Lemma 4.3.10. *For every top invariant semi-functional space $\mathscr{F}$ on (M, E) and all $k \in \mathbb{N}_\infty$, we have*

$$(\mathscr{F}^k)_{\mathrm{loc}} = (\mathscr{F}_{\mathrm{loc}})^k.$$

Proof: Similar to the proof of Lemma 2.9.29, it suffices to prove that for every $P \in \mathrm{Diff}_k(E, E)$ and $\varphi \in \mathscr{D}(M)$

1. $P \circ m_\varphi$ restricts to a continuous linear map from $(\mathscr{F}_{\mathrm{loc}})^k$ into $\mathscr{F}$ and

2. $m_\varphi \circ P$ restricts to a continuous linear map from $(\mathscr{F}^k)_{\mathrm{loc}}$ into $\mathscr{F}$.

However, due to the lack of a nice Leibniz rule for differential operators, the approach that we have used in the proof of Lemma 2.9.29 to show that (the analogues of) these statements are valid seems useless here. Forced to think outside the box, one might discover that there is a much easier proof that does generalize directly. Indeed, take $\psi \in \mathscr{D}(M)$ such that ψ equals 1 on an open neighborhood of the compact subset $\operatorname{supp}(\varphi) \subseteq M$. Then one readily verifies that

$$m_\psi \circ P \circ m_\varphi = P \circ m_\varphi \quad \text{and} \quad m_\varphi \circ P \circ m_\psi = m_\varphi \circ P.$$

(The first equality is a direct consequence of the fact that P is local. For the second equality recall that $\mathscr{E}(M, E)$ is dense in $\mathscr{D}'(M, E)$ and observe that for $\chi \in \mathscr{E}(M, E)$ and $x \in \operatorname{supp}(\varphi)$, $(P(\psi\chi))(x) = (P\chi)(x)$ since $\psi\chi$ and χ coincide on an open neighborhood of x, while for $x \notin \operatorname{supp}(\varphi)$, $\varphi(x)(P(\psi\chi))(x) = 0 = \varphi(x)(P\chi)(x)$.) Using these equalities, the two statements above follow straight from the characterizing properties and the 'multiplication property' of the involved semi-functional spaces. □

Proposition 4.3.11. *If $\mathscr{F}$ is a top invariant semi-functional space on (M, E), $k \in \mathbb{N}_\infty$ and $0 \leq \ell < k+1$, then every $P \in \operatorname{Diff}_\ell(E, E)$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}^{k-\ell}$.*

Proof: On behalf of Corollary 4.3.7, it suffices to prove that $Q \circ P$ restricts to a continuous linear map from $\mathscr{F}^k$ into $\mathscr{F}$ for every $Q \in \operatorname{Diff}_{k-\ell}(E, E)$. But for every $Q \in \operatorname{Diff}_{k-\ell}(E, E)$, $Q \circ P \in \operatorname{Diff}_k(E, E)$, so this follows from the characterizing property of $\mathscr{F}^k$. □

Now that we have investigated $\mathscr{F}^k$ for $k \in \mathbb{N}_\infty$, it is time to give the generalized definition of $\mathscr{F}^{-k}$. In the second chapter, we defined $\mathscr{F}^{-k}$ as $(((\mathscr{F}')^k)_0)'$. Also in the current context the expression $(((\mathscr{F}')^k)_0)'$ makes perfect sense. However, if $\mathscr{F}$ is a normal reflexive top invariant functional space on (M, E), $(((\mathscr{F}')^k)_0)'$ is a functional space on $(M, (E^\vee)^\vee)$ rather than on (M, E) (note that we should check that $\mathscr{F} \mapsto \mathscr{F}'$ preserves top invariance, but this is not difficult). As is already suggested by Lemma 4.2.5, we can resolve this by using the pushforward

$$\nu_* \colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, (E^\vee)^\vee)$$

of the vector bundle isomorphism $\nu \colon E \to (E^\vee)^\vee$.

Definition 4.3.12. Let $\mathscr{F}$ be a normal reflexive top invariant functional space on (M, E) and let $k \in \mathbb{N}_\infty$. We define $\mathscr{F}^{-k}$ to be the vector subspace

$$(\nu_*)^{-1}((((\mathscr{F}')^k)_0)')$$

of $\mathscr{D}'(M, E)$ endowed with the topology that turns ν_* into a linear topological isomorphism from $\mathscr{F}^{-k}$ onto $(((\mathscr{F}')^k)_0)'$. ⊘

Because the linear topological isomorphism ν_* is $\mathscr{E}(M)$-linear and restricts to a linear topological isomorphism from $\mathscr{D}(M, E)$ into $\mathscr{D}(M, (E^\vee)^\vee)$, we directly obtain that $\mathscr{F}^{-k}$ is functional space on (M, E).

Remark 4.3.13. Of course, this 'method' for translating the functional space $(((\mathscr{F}')^k)_0)'$ on $(M, (E^\vee)^\vee)$ into a functional space $\mathscr{F}^{-k}$ on (M, E) is more widely applicable: the linear topological isomorphism ν_* in fact gives a one-to-one correspondence between (semi-)functional spaces on (M, E) and (semi-)functional spaces on $(M, (E^\vee)^\vee)$ (where it is understood that also the topologies are translated via ν_*). $\oslash$

After one has learned to 'look through' the thick layer of identifications that is hidden in the definition of $\mathscr{F}^{-k}$, it should also be possible to generalize most of the results about $\mathscr{F}^{-k}$ that we have seen in the second chapter, but we will not go into this here.

4.4 Families of functional spaces

Apart from some details that we did not discuss, we have now finished the generalization of the theory of functional spaces from the Euclidean to the vector bundle setting. By doing this, we have provided a framework to work with 'solution spaces' on vector bundles; for example, we can express the 'quality' of a 'solution space' by listing the properties that it has as a functional space, we can use our 'toolbox', which is filled with results concerning functional spaces, to investigate it and we can 'improve' it by applying construction functors.

However, in the introduction of this chapter we have promised to deliver *two* frameworks: one to work with 'solution spaces' on vector bundles and one to create them. Before we really start working on the second framework, we discuss a concept that is in some sense 'intermediate', namely the concept of 'families of functional spaces'. These 'families' of functional spaces are not just arbitrary indexed collections of similar objects (for which the word 'family' is also loosely used in this text), but collections of really coherent (semi-)functional spaces, one for each vector bundle over M, with strong 'family bonds'.

Definition 4.4.1. We define $\mathrm{VB}(M)$ to be the category with vector bundles over M as objects and vector bundle homomorphisms (that are the identity on M) as arrows. Moreover, we let LCVS be the category with locally convex vector spaces as objects and continuous linear maps as arrows. $\oslash$

Definition 4.4.2. A *functorial family* of (semi-)functional spaces on M is a functor $\mathscr{F}_M \colon \mathrm{VB}(M) \to \mathrm{LCVS}$ such that:

1. $\mathscr{F}_M(E)$ is a (semi-)functional space on (M, E) for every $E \in \mathrm{VB}(M)$ and
2. for all E, $F \in \mathrm{VB}(M)$ and every vector bundle homomorphism $T \colon E \to F$, $\mathscr{F}_M(T)$ equals the restriction of $T_* \colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$ to $\mathscr{F}_M(E)$. $\oslash$

In other words, a functorial family of (semi-)functional spaces on M is a family $\{\mathscr{F}_{M,E}\}_{E \in \mathrm{VB}(M)}$, with $\mathscr{F}_{M,E}$ being a (semi-)functional space on (M, E), such that for every vector bundle homomorphism $T \colon E \to F$ between vector bundles E, $F \in \mathrm{VB}(M)$, T_* restricts to a continuous linear map from $\mathscr{F}_{M,E}$ into $\mathscr{F}_{M,F}$. We call such families functorial to emphasize that they have a rich, formally defined, structure of 'family bonds' that can be captured using the concept of a functor. The concept of functorial families is 'intermediate' between the two frameworks, because it is in principle an addition to our formal theory of functional spaces on vector bundles (i.e., the first framework), but this addition is strongly motivated by our method for 'transferring' functional spaces from $\mathbb{R}^n$ to vector bundles (i.e., the second framework). Indeed, as we will soon see, the typical way to get a functorial family of functional spaces, is by using a sufficiently well-behaved functional space on $\mathbb{R}^n$ as a 'model'.

Remark 4.4.3. Clearly, semi-functional spaces that are part of a functorial family are always top invariant and therefore in particular always semi-local. ⊘

Example 4.4.4. Of course, $\{\mathscr{D}'(M,E)\}_{E\in \mathrm{VB}(M)}$ is a functorial family of functional spaces on M. Moreover, given vector bundles E and F over M and a vector bundle homomorphism $T\colon E \to F$, we have seen that T_* restricts to a continuous linear map from $\mathscr{E}'(M,E)$ into $\mathscr{E}'(M,F)$, from $\mathscr{E}(M,E)$ into $\mathscr{E}(M,F)$ and from $\mathscr{D}(M,E)$ into $\mathscr{D}(M,F)$. Hence, also $\{\mathscr{D}(M,E)\}_{E\in \mathrm{VB}(M)}$, $\{\mathscr{E}(M,E)\}_{E\in \mathrm{VB}(M)}$ and $\{\mathscr{E}'(M,E)\}_{E\in \mathrm{VB}(M)}$ are functorial families of functional spaces on M. ⊘

Example 4.4.5. From now on it is always assumed that the spaces $\Gamma^k(M,E)$, with $k \in \mathbb{N}$ and $E \in \mathrm{VB}(M)$, are endowed with a topology similar to the topology of $\mathscr{E}(M,E)$. That is, if $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ is a cover of M by total trivialization triples of $E \to M$, then $\Gamma^k(M,E)$ is assumed to be endowed with the locally convex topology that is induced by the seminorms

$$\Gamma^k(M,E) \to \mathbb{R}\colon \varphi \mapsto \|(\rho_i)^\ell \circ \varphi \circ \kappa_i^{-1}\|_{\kappa_i(K),k},$$

with $K \in \mathscr{P}_{\mathrm{c}}(U_i)$ and $1 \le \ell \le \mathrm{rank}(E)$. Just as for $\mathscr{E}(M,E)$, this topology does not depend on the choice of total trivialization cover and we can give an alternative description using a vector bundle metric on E. When endowed with this topology, $\Gamma^k(M,E)$ is a functional space on (M,E) and $\{\Gamma^k(M,E)\}_{E\in \mathrm{VB}(M)}$ is a functorial family of functional spaces on M. ⊘

Thanks to the power of category theory, we obtain the following result almost without effort.

Lemma 4.4.6. *Let $\mathscr{F}_M$ be a functorial family of semi-functional spaces on M and let E and F be vector bundles over M. Then*

$$\mathscr{F}_M(E \oplus F) \simeq \mathscr{F}_M(E) \times \mathscr{F}_M(F).$$

Proof: According to [9, Proposition VIII.2.4], it suffices to prove that the functor $\mathscr{F}_M$ is additive. That is, we should prove that for any two vector bundles E and F over M and any two vector bundle homomorphisms $T\colon E \to F$ and $L\colon E \to F$, we have

$$\mathscr{F}_M(T + L) = \mathscr{F}_M(T) + \mathscr{F}_M(L).$$

But by definition of a functorial family, $\mathscr{F}_M(T{+}L) = (T{+}L)_*$, $\mathscr{F}_M(T) = T_*$ and $\mathscr{F}_M(L) = L^*$, so this simply follows by restricting the identity $(T{+}L)_* = T_*{+}L_*$ (see Lemma 3.4.5) from $\mathscr{D}'(M,E)$ to $\mathscr{F}_M(E)$. □

4.5 Constructions on families

A question that naturally arises now that we have added functorial families to our theory, is whether our construction functors, which were initially 'designed' to work on 'single' functional spaces, respect the 'family bonds' of a functorial family. To clarify what we mean by this, let $K \in \mathscr{P}_c(M)$. Then for any $E \in \mathrm{VB}(M)$, the assignment $\mathscr{F} \mapsto \mathscr{F}_K$ is a functor from the category of (semi-)functional spaces on (M, E) to the category of semi-functional spaces on (M, E). So given a functorial family $\mathscr{F}_M$ of (semi-)functional spaces on M, we can create a new 'family' $(\mathscr{F}_M)_K$ of semi-functional spaces by stipulating that $(\mathscr{F}_M)_K(E) := (\mathscr{F}_M(E))_K$ and we would like to know whether $(\mathscr{F}_M)_K$ is in fact a functorial family. For this, we need to check whether for all E, $F \in \mathrm{VB}(M)$ and every vector bundle homomorphism $T\colon E \to F$, T_* restricts to a continuous linear map from $(\mathscr{F}_M(E))_K$ into $(\mathscr{F}_M(F))_K$.

Claim. $(\mathscr{F}_M)_K$ is indeed a functorial family of semi-functional spaces on M.

Proof: Let E, $F \in \mathrm{VB}(M)$ and let $T\colon E \to F$ be a vector bundle homomorphism. Because $\mathscr{F}_M$ is a functorial family, $T_*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$ restricts to a continuous linear map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$ and because T_* is local (see Lemma 3.4.3) and $(\mathscr{F}_M(E))_K$ and $(\mathscr{F}_M(F))_K$ carry the restricted topology, we see that T_* also restricts to a continuous linear map from $(\mathscr{F}_M(E))_K$ into $(\mathscr{F}_M(F))_K$. □

Let us capture this in a proposition.

Proposition 4.5.1. *If $\mathscr{F}_M$ is a functorial family of semi-functional spaces on M and $K \in \mathscr{P}_c(M)$, then also*

$$(\mathscr{F}_M)_K\colon \mathrm{VB}(M) \to \mathrm{LCVS}\colon E \mapsto (\mathscr{F}_M(E))_K$$

is a functorial family of semi-functional spaces on M.

Of course, we have similar results for other construction functors:

Proposition 4.5.2. *If $\mathscr{F}_M$ is a functorial family of (semi-)functional spaces on M, then also*

$$(\mathscr{F}_M)_{\mathrm{comp}}\colon \mathrm{VB}(M) \to \mathrm{LCVS}\colon E \mapsto (\mathscr{F}_M(E))_{\mathrm{comp}}$$

is a functorial family of (semi-)functional spaces on M.

Proof: Let E and F be vector bundles over M and let $T\colon E \to F$ be a vector bundle homomorphism. Thanks to the previous proposition we already know that $T_*\colon \mathscr{D}'(M, E) \to \mathscr{D}'(M, F)$ restricts to a continuous linear map from $(\mathscr{F}_M(E))_K$ into $(\mathscr{F}_M(F))_K$ for each $K \in \mathscr{P}_c(M)$ and this implies (via Proposition A.3.2 and the fact that $(\mathscr{F}_M(F))_K \subseteq_c (\mathscr{F}_M(F))_{\mathrm{comp}}$) that T_* restricts to a continuous linear map from $(\mathscr{F}_M(E))_{\mathrm{comp}}$ into $(\mathscr{F}_M(F))_{\mathrm{comp}}$. □

Proposition 4.5.3. *If $\mathscr{F}_M$ is a functorial family of (semi-)functional spaces on M, then also*

$$(\mathscr{F}_M)_{\mathrm{loc}}\colon \mathrm{VB}(M) \to \mathrm{LCVS}\colon E \mapsto (\mathscr{F}_M(E))_{\mathrm{loc}}$$

is a functorial family of (semi-)functional spaces on M.

Proof: Let $E, F \in \mathrm{VB}(M)$ and let $T\colon E \to F$ be a vector bundle homomorphism. Because $\mathscr{F}_M$ is a functorial family, $T_*\colon \mathscr{D}'(M,E) \to \mathscr{D}'(M,F)$ restricts to a continuous linear map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$. On behalf of the generalization of Lemma 2.6.2 to the setting of vector bundles, to prove that T_* also restricts to a continuous linear map from $(\mathscr{F}_M(E))_{\mathrm{loc}}$ into $(\mathscr{F}_M(F))_{\mathrm{loc}}$, it suffices to prove that $m_\varphi \circ T_*$ restricts to a continuous linear map from $(\mathscr{F}_M(E))_{\mathrm{loc}}$ into $\mathscr{F}_M(F)$ for every $\varphi \in \mathscr{D}(M)$. So fix $\varphi \in \mathscr{D}(M)$. By $\mathscr{E}(M)$-linearity of T_* (see Lemma 3.4.2), $m_\varphi \circ T_* = T_* \circ m_\varphi$ and we know that m_φ restricts to a continuous linear map from $(\mathscr{F}_M(E))_{\mathrm{loc}}$ into $\mathscr{F}_M(E)$ and that T_* restricts to a continuous linear map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$, hence $m_\varphi \circ T_* = T_* \circ m_\varphi$ indeed restricts to a continuous linear map from $(\mathscr{F}_M(E))_{\mathrm{loc}}$ into $\mathscr{F}_M(F)$. □

Proposition 4.5.4. *If $\mathscr{F}_M$ is a functorial family of (semi-)functional spaces on M, then also*

$$(\mathscr{F}_M)_{\mathrm{semi}}\colon \mathrm{VB}(M) \to \mathrm{LCVS}\colon E \mapsto (\mathscr{F}_M(E))_{\mathrm{semi}}$$

is a functorial family of (semi-)functional spaces on M.

Proof: The proof is competely analogous to the proof of the previous proposition. Just use the generalization of Lemma 2.8.3 instead of the generalization of Lemma 2.6.2. □

Also the construction functor $\mathscr{F} \mapsto \mathscr{F}^k$ interacts nicely with functorial families of (semi-)functional spaces; that is, we have a similar statement as for the other construction functors. But in the case of $\mathscr{F} \mapsto \mathscr{F}^k$ there is more: we have a nice 'family version' of Proposition 4.3.11.

Proposition 4.5.5. *Let $\mathscr{F}_M$ be a functorial family of top invariant semi-functional spaces on M, $k \in \mathbb{N}_\infty$ and $0 \le \ell < k+1$. Then every $P \in \mathrm{Diff}_\ell(E,F)$ restricts to a continuous linear map from $(\mathscr{F}_M(E))^k$ into $(\mathscr{F}_M(F))^{k-\ell}$.*

Proof: On behalf of Corollary 4.3.7, it suffices to prove that $Q \circ P$ restricts to a continuous linear map from $(\mathscr{F}_M(E))^k$ into $\mathscr{F}_M(F)$ for every $Q \in \mathrm{Diff}_{k-\ell}(F,F)$. So fix $Q \in \mathrm{Diff}_{k-\ell}(F,F)$. Then $Q \circ P \in \mathrm{Diff}_k(E,F)$ and on the strength of Theorem B.4.3, we find vector bundle homomorphisms $T_0, \ldots, T_m \in \mathrm{Hom}(E,F)$ and differential operators $P_0, \ldots, P_m \in \mathrm{Diff}_k(E,E)$ such that

$$Q \circ P = \sum_{j=0}^{m} (T_j)_* \circ P_j. \tag{4.1}$$

Now because of the characterizing property of $(\mathscr{F}_M(E))^k$, P_j restricts to a continuous linear map from $(\mathscr{F}_M(E))^k$ into $\mathscr{F}_M(E)$ for every $0 \le j \le m$ and because of the functorial family axioms, $(T_j)_*$ restricts to a continuous linear map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$ for every $0 \le j \le m$. Together with equation (4.1) these observations imply the desired statement: namely that $Q \circ P$ restricts to a continuous linear map from $(\mathscr{F}_M(E))^k$ into $\mathscr{F}_M(F)$. □

Corollary 4.5.6. *If $\mathscr{F}_M$ is a functorial family of (semi-)functional spaces on M and $k \in \mathbb{N}_\infty$, then also*

$$(\mathscr{F}_M)^k\colon \mathrm{VB}(M) \to \mathrm{LCVS}\colon E \mapsto (\mathscr{F}_M(E))^k$$

is a functorial family of (semi-)functional spaces on M.

Proof: Let E, $F \in \mathrm{VB}(M)$ and let $T\colon E \to F$ be a vector bundle homomorphism. Then $T_* \in \mathrm{Diff}_0(E,F)$, so if we take $\ell = 0$ and $P = T_*$ in the previous proposition, we get that T_* restricts to a continuous linear map from $(\mathscr{F}_M(E))^k$ into $(\mathscr{F}_M(F))^k$. □

4.6 Creating families: from $\mathbb{R}^{\mathrm{n}}$ to vector bundles

In this section we finally develop the promised 'second framework'; i.e., the method to create functional spaces on vector bundles that 'look like' a sufficiently well-behaved functional space $\mathscr{F}$ on $\mathbb{R}^{\mathrm{n}}$ which serves as a 'model'. The idea is actually quite simple: we cover a vector bundle $E \to M$ with 'patches' on which the distributional sections of E reduce to tuples of ordinary 'Euclidean distributions', which allows us to decide which distributional sections of E are locally of 'type $\mathscr{F}$', and the 'subspace' of all distributional sections of E that are locally of 'type $\mathscr{F}$' is the proposed functional space on (M,E). The formal realization of this idea looks as follows:

Definition 4.6.1. Let $\mathscr{F}$ be a local invariant functional space on $\mathbb{R}^{\mathrm{n}}$, E a vector bundle over M of rank r and $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ a collection of total trivialization triples of $E \to M$ such that $\{U_i\}_{i\in I}$ is an open cover of M. Moreover, denote for $i \in I$ the linear topological isomorphism from $\mathscr{D}'(U_i, E_{U_i})$ onto $(\mathscr{D}'(\kappa_i(U_i)))^{\times r}$ that we have described in Remark 3.2.3 by h_i. As a set, we define

$$\mathscr{F}(M,E) := \{u \in \mathscr{D}'(M,E) \mid h_i(u|_{U_i}) \in (\mathscr{F}(\kappa_i(U_i)))^{\times r} \text{ for every } i \in I\},$$

where $\mathscr{F}(\kappa_i(U_i))$ is the functional space on $\kappa_i(U_i) \subseteq \mathbb{R}^{\mathrm{n}}$ as introduced in Section 2.7. Clearly, $\mathscr{F}(M,E)$ is a linear subspace of $\mathscr{D}'(M,E)$ and we have an injective linear map

$$H\colon \mathscr{F}(M,E) \to \prod_{i\in I}(\mathscr{F}(\kappa_i(U_i)))^{\times r} \colon u \mapsto \{h_i(u|_{U_i})\}_{i\in I}.$$

We endow $\mathscr{F}(M,E)$ with the unique topology that turns H into a linear topological embedding. ⊘

Remark 4.6.2. The dimension of the manifold M and $\mathbb{R}^{\mathrm{n}}$ is assumed to be the same, namely our 'differently typesetted' n. ⊘

It is clear why we need to assume that $\mathscr{F}$ is local: the domains of the total trivialization triples are diffeomorphic to open subsets of $\mathbb{R}^{\mathrm{n}}$, so we need to be able to 'restrict' $\mathscr{F}$ to open subsets of $\mathbb{R}^{\mathrm{n}}$ in a consistent fashion and we have discussed in Section 2.7 that requiring this 'sheaf like behaviour' is equivalent to requiring locality. It is also not very hard to imagine why we need to assume that $\mathscr{F}$ is invariant. As the word 'invariant' already suggests, the invariance of $\mathscr{F}$ makes sure that:

Claim. The definition of $\mathscr{F}(M,E)$ is independent of the choice of total trivialization cover.

Proof: Let $\{(\tilde{U}_j,\tilde{\kappa}_j,\tilde{\rho}_j)\}_{j\in J}$ be another total trivialization cover of $E \to M$ and let $\tilde{\mathscr{F}}(M,E)$ be the 'version' of $\mathscr{F}(M,E)$ that we get when we use this 'alternative' cover. Both $\mathscr{F}(M,E)$ and $\tilde{\mathscr{F}}(M,E)$ are by definition linear subspaces of $\mathscr{D}'(M,E)$ and by symmetry it suffices to prove that $\mathscr{F}(M,E) \subseteq_c \tilde{\mathscr{F}}(M,E)$ to prove that they are equal.

Fix $i \in I$ and $j \in J$ for a moment and assume that $U_i \cap \tilde{U}_j \neq \emptyset$. By combining the trivializations ρ_i and $\tilde{\rho}_j$, we obtain a smooth invertible $r \times r$ matrix T^{ij} on $\kappa_i(U_i \cap \tilde{U}_j)$, which is formally defined by

$$T^{ij}(x)v := \left(\rho_i \circ \left(\tilde{\rho}_j|_{E_{\kappa_i^{-1}(x)}}\right)^{-1}\right) v.$$

Because $\mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j))$ is local (see Lemma 2.7.10) and therefore also semi-local (see Proposition 2.8.10), acting with T^{ij} on $(\mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j)))^{\times r}$ defines a continuous linear map from $(\mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j)))^{\times r}$ into itself which is in fact a linear topological isomorphism due to the invertibility of T^{ij}. Furthermore, because $\mathscr{F}$ is invariant and $\tilde{\kappa}_j \circ \kappa_i^{-1}$ is a diffeomorphism from $\kappa_i(U_i \cap \tilde{U}_j)$ onto $\tilde{\kappa}_j(U_i \cap \tilde{U}_j)$, the pushforward $(\tilde{\kappa}_j \circ \kappa_i^{-1})_*$ is a linear topological isomorphism from $\mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j))$ onto $\mathscr{F}(\tilde{\kappa}_j(U_i \cap \tilde{U}_j))$ (see Lemma 2.7.12). We denote the composition of the r^{th} 'power' of $(\tilde{\kappa}_j \circ \kappa_i^{-1})_*$ with T^{ij}, which is a linear topological isomorphism from $(\mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j)))^{\times r}$ onto $(\mathscr{F}(\tilde{\kappa}_j(U_i \cap \tilde{U}_j)))^{\times r}$, by R^{ij}. When $U_i \cap \tilde{U}_j = \emptyset$, the situation is much simpler: $(\mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j)))^{\times r}$ and $(\mathscr{F}(\tilde{\kappa}_j(U_i \cap \tilde{U}_j)))^{\times r}$ both have just one element and we define R^{ij} to be the unique map between those one point spaces.

So what do these linear topological isomorphisms R^{ij} have to do with proving that $\mathscr{F}(M,E) \subseteq_c \tilde{\mathscr{F}}(M,E)$? Well, on behalf of Lemma 2.7.18 and the observation that $U_i = \cup_{j \in J}(U_i \cap \tilde{U}_j)$ for every $i \in I$, we see that $u \mapsto \{u|_{\kappa_i(U_i \cap \tilde{U}_j)}\}_{j \in J}$ defines a linear topological embedding of $\mathscr{F}(\kappa_i(U_i))$ into $\prod_{j \in J} \mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j))$ for every $i \in I$ and as a consequence we get a natural linear topological embedding of $\prod_{i \in I}(\mathscr{F}(\kappa_i(U_i)))^{\times r}$ into $\prod_{i \in I}\prod_{j \in J}(\mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j)))^{\times r}$. Using the obvious fact that we also have a 'tilde analogue' of this embedding, we get a diagram:

$$\begin{array}{ccccc}
\mathscr{F}(M,E) & \xrightarrow{H} & \prod_{i \in I}(\mathscr{F}(\kappa_i(U_i)))^{\times r} & \xrightarrow{I} & \prod_{i \in I}\prod_{j \in J}(\mathscr{F}(\kappa_i(U_i \cap \tilde{U}_j)))^{\times r} \\
 & & & & \Big\downarrow {\scriptstyle \prod_{i \in I}\prod_{j \in J} R^{ij}} \\
\tilde{\mathscr{F}}(M,E) & \xrightarrow{\tilde{H}} & \prod_{j \in J}(\mathscr{F}(\tilde{\kappa}_j(\tilde{U}_j)))^{\times r} & \xrightarrow{\tilde{I}} & \prod_{i \in I}\prod_{j \in J}(\mathscr{F}(\tilde{\kappa}_j(U_i \cap \tilde{U}_j)))^{\times r}
\end{array}$$

All arrows in this diagram still make perfect sense if we replace $\mathscr{F}$ by $\mathscr{D}'$, so we can pick an $u \in \mathscr{F}(M,E)$ and check whether $(\prod_{i \in I}\prod_{j \in J} R^{ij} \circ I \circ H)u = (\tilde{I} \circ \tilde{H})u$. This is indeed the case and after a moment's thought we realize that this means that $\mathscr{F}(M,E) \subseteq \tilde{\mathscr{F}}(M,E)$ and that 'walking through' the diagram from $\mathscr{F}(M,E)$ to $\tilde{\mathscr{F}}(M,E)$ coincides with applying the inclusion map. Because all arrows in the diagram are linear topological isomorphisms or linear topological embeddings, this implies that the inclusion map is continuous, hence $\mathscr{F}(M,E) \subseteq_c \tilde{\mathscr{F}}(M,E)$. □

Remark 4.6.3. It is a direct consequence of the previous claim that a distribution $u \in \mathscr{D}'(M,E)$ belongs to $\mathscr{F}(M,E)$ if and only if $h(u|_U) \in (\mathscr{F}(\kappa(U)))^{\times r}$ for every total trivialization triple (U,κ,ρ) of E (where $h\colon \mathscr{D}'(U,E_U) \to (\mathscr{D}'(\kappa(U)))^{\times r}$ is the linear topological isomorphism associated to the trivialization triple). ⊘

Before we check that $\mathscr{F}(M,E)$ is a functional space on (M,E), let us verify that $\mathscr{F}(M,E)$ 'looks as expected' for two very simple choices of $\mathscr{F}$.

Example 4.6.4. In the second chapter we have seen that $\mathscr{F} := \mathscr{E}(\mathbb{R}^{\mathrm{n}})$ is a local invariant functional space on $\mathbb{R}^{\mathrm{n}}$ and of course we should have that $\mathscr{F}(M,E)$ as defined above coincides with $\mathscr{E}(M,E)$ as defined in Section 3.1 (this is in particular desirable because $\mathscr{E}(\mathbb{R}^{\mathrm{n}})$ is often abbreviated as $\mathscr{E}$, so otherwise $\mathscr{E}(M,E)$ would have two different meanings). So why is this the case?

Let $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ be a total trivialization cover of $E \to M$. Then

$$H\colon \mathscr{F}(M,E) \to \prod_{i\in I} (\mathscr{F}(\kappa_i(U_i)))^{\times r} \colon u \mapsto \{h_i(u|_{U_i})\}_{i\in I}$$

is a linear topological embedding and since

$$\mathscr{F}(\kappa_i(U_i)) = (\mathscr{E}(\mathbb{R}^{\mathrm{n}}))(\kappa_i(U_i)) = \mathscr{E}(\kappa_i(U_i))$$

for every $i \in I$ (see Example 2.7.15) and h_i restricts to a linear topological isomorphism between $\mathscr{E}(U_i, E_{U_i})$ and $(\mathscr{E}(\kappa_i(U_i)))^{\times r}$, we see that

$$\hat{H}\colon \mathscr{F}(M,E) \to \prod_{i\in I} \mathscr{E}(U_i, E_{U_i}) \colon u \mapsto \{u|_{U_i}\}_{i\in I}$$

is also a linear topological embedding. Furthermore, using the definition of $\mathscr{F}(M,E)$, we see that a distribution $u \in \mathscr{D}'(M,E)$ is an element of $\mathscr{F}(M,E)$ if and only if $u|_{U_i} \in \mathscr{E}(U_i, E_{U_i})$ for all $i \in I$. But by the generalization of Example 2.6.7 and Lemma 2.7.18 to the context of vector bundles, $\mathscr{E}(M,E)$ has the very same properties. That is, $u \mapsto \{u|_{U_i}\}_{i\in I}$ defines a linear topological embedding of $\mathscr{E}(M,E)$ into $\prod_{i\in I} \mathscr{E}(U_i, E_{U_i})$ and a distribution $u \in \mathscr{D}'(M,E)$ is an element of $\mathscr{E}(M,E)$ if and only if $u|_{U_i} \in \mathscr{E}(U_i, E_{U_i})$ for all $i \in I$. It now clearly follows that $\mathscr{F}(M,E) = \mathscr{E}(M,E)$. ⊘

Example 4.6.5. By arguments that are completely analogous to the arguments in the previous example, if $\mathscr{F} := \mathscr{D}'(\mathbb{R}^{\mathrm{n}})$, we have that $\mathscr{F}(M,E)$ as defined above coincides with $\mathscr{D}'(M,E)$. Just use Example 2.7.16 instead of Example 2.7.15 and Example 2.6.5 instead of Example 2.6.7. ⊘

We now easily verify that:

Proposition 4.6.6. *$\mathscr{F}(M,E)$ is a functional space on (M,E).*

Proof: Let $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ be a total trivialization cover of $E \to M$ and denote for every $i \in I$ the linear topological isomorphism from $\mathscr{D}'(U_i, E_{U_i})$ onto $(\mathscr{D}'(\kappa_i(U_i)))^{\times r}$ associated to the total trivialization triple (U_i,κ_i,ρ_i) by h_i. The previous two examples and the definition of $\mathscr{F}(M,E)$ together tell us that

$$H\colon \mathscr{D}'(M,E) \to \prod_{i\in I} (\mathscr{D}'(\kappa_i(U_i)))^{\times r} \colon u \mapsto \{h_i(u|_{U_i})\}_{i\in I}$$

is a linear topological embedding which restricts to a linear topological embedding of $\mathscr{F}(M,E)$ into $\prod_{i\in I}(\mathscr{F}(\kappa_i(U_i)))^{\times r}$ and to a linear topological embedding of $\mathscr{E}(M,E)$ into $\prod_{i\in I}(\mathscr{E}(\kappa_i(U_i)))^{\times r}$. Now, because for every $i \in I$, $\mathscr{F}(\kappa_i(U_i))$ is a local functional space on U_i, we have

$$\prod_{i\in I} (\mathscr{E}(\kappa_i(U_i)))^{\times r} \subseteq_{\mathrm{c}} \prod_{i\in I} (\mathscr{F}(\kappa_i(U_i)))^{\times r} \subseteq_{\mathrm{c}} \prod_{i\in I} (\mathscr{D}'(\kappa_i(U_i)))^{\times r}$$

and a quick mental verification suffices to see that this implies that

$$\mathscr{D}(M,E) \subseteq_{\mathrm{c}} \mathscr{E}(M,E) \subseteq_{\mathrm{c}} \mathscr{F}(M,E) \subseteq_{\mathrm{c}} \mathscr{D}'(M,E).$$

Next, let $\varphi \in \mathscr{E}(M)$. We readily verify that H translates multiplication by φ on $\mathscr{D}'(M,E)$ into componentwise multiplication by $\varphi \circ \kappa_i^{-1}$ on $\prod_{i\in I}(\mathscr{D}'(\kappa_i(U_i)))^{\times r}$ (by which we mean that the ℓ^{th} component of the i^{th} component gets multiplied by $\varphi \circ \kappa_i^{-1}$) and combining this with the fact that the $\mathscr{F}(\kappa_i(U_i))$ are local and hence semi-local shows that $m_\varphi \colon \mathscr{D}'(M,E) \to \mathscr{D}'(M,E)$ restricts to a continuous linear map from $\mathscr{F}(M,E)$ into $\mathscr{F}(M,E)$. □

So, like we promised, for every sufficiently well-behaved (i.e., local and invariant) functional space $\mathscr{F}$ on $\mathbb{R}^{\mathrm{n}}$ and every vector bundle $E \to M$ over an n-dimensional manifold, we have a functional space $\mathscr{F}(M,E)$ on (M,E) that 'looks like' $\mathscr{F}$. Of course, the phrase 'looks like $\mathscr{F}$' should be considered as very informal and is only meant to give a rough, intuitive impression of what is going on. If we want to make this description slightly more accurate, we could say that $\mathscr{F}(M,E)$ *locally* 'looks like' $\mathscr{F}$ or that $\mathscr{F}(M,E)$ is 'modeled after' $\mathscr{F}$.

It trivially follows from the definition of $\mathscr{F}(M,E)$ that the procedure that turns a local invariant functional space on $\mathbb{R}^{\mathrm{n}}$ into a functional space on (M,E) can be regarded as a construction functor.

Proposition 4.6.7. *For every vector bundle E over an* n*-dimensional manifold M, the assignment $\mathscr{F} \mapsto \mathscr{F}(M,E)$ is a functor from the category of local invariant functional spaces on $\mathbb{R}^{\mathrm{n}}$ to the category of functional spaces on (M,E).*

It should not come as a surprise that for a local invariant functional space $\mathscr{F}$ on $\mathbb{R}^{\mathrm{n}}$ and vector bundles E and F over M, the functional spaces $\mathscr{F}(M,E)$ and $\mathscr{F}(M,F)$ are 'related'. After all, intuitively speaking, $\mathscr{F}(M,E)$ and $\mathscr{F}(M,F)$ are both modeled after $\mathscr{F}$ and are therefore of 'the same type'.

Proposition 4.6.8. *Let $\mathscr{F}$ be a local invariant functional space on $\mathbb{R}^{\mathrm{n}}$. The family $\{\mathscr{F}(M,E)\}_{E\in \mathrm{VB}(M)}$ is a functorial family of functional spaces on M.*

Proof: Let E and F be vector bundles over M of respectively rank r and rank s and let $T\colon E \to F$ be a vector bundle homomorphism. Moreover, let $\{(U_i, \kappa_i, \rho_i^E, \rho_i^F)\}_{i\in I}$ be a simultaneous total trivialization cover of $E \to M$ and $F \to M$ and let H^E and H^F be the associated linear topological embeddings of $\mathscr{D}'(M,E)$ into $\prod_{i\in I}(\mathscr{D}'(\kappa_i(U_i)))^{\times r}$ and of $\mathscr{D}'(M,F)$ into $\prod_{i\in I}(\mathscr{D}'(\kappa_i(U_i)))^{\times s}$. It is not difficult to see that $T_*\colon \mathscr{D}'(M,E) \to \mathscr{D}'(M,F)$ is translated by H^E and H^F into multiplication by smooth $s \times r$ matrices $\tilde{T}^i$ on $\kappa_i(U_i)$. It is then a consequence of the fact that H^E restricts to a linear topological embedding of $\mathscr{F}(M,E)$ into $\prod_{i\in I}(\mathscr{F}(\kappa_i(U_i)))^{\times r}$, the fact that H^F restricts to a linear topological embedding of $\mathscr{F}(M,F)$ into $\prod_{i\in I}(\mathscr{F}(\kappa_i(U_i)))^{\times s}$ and the fact that the spaces $\mathscr{F}(\kappa_i(U_i))$ are semi-local that T_* restricts to a continuous linear map from $\mathscr{F}(M,E)$ into $\mathscr{F}(M,F)$. □

Remark 4.6.9. In a similar fashion, but with a bit of extra effort to account for the change of base manifold, one can show that for isomorphic vector bundles $E \to M$ and $F \to N$, $\mathscr{F}(M,E)$ and $\mathscr{F}(N,F)$ are linearly topologically isomorphic. ⊘

Remark 4.6.10. It not a coincidence that for a local invariant functional space $\mathscr{F}$ on $\mathbb{R}^{\mathrm{n}}$, $\{\mathscr{F}(M,E)\}_{E\in\mathrm{VB}(M)}$ is a functorial family. As proclaimed when we introduced the notion of a functorial family of (semi-)functional spaces, the families of the form $\{\mathscr{F}(M,E)\}_{E\in\mathrm{VB}(M)}$ are actually a strong motivation and the key example for functorial families. However, this does not mean that *all* functorial families are of this form. Indeed, $\{\mathscr{D}(M,E)\}_{E\in\mathrm{VB}(M)}$ and $\{\mathscr{E}'(M,E)\}_{E\in\mathrm{VB}(M)}$ do not come from a local invariant functional space on $\mathbb{R}^{\mathrm{n}}$. ⊘

That we can create functional spaces on vector bundles that are modeled after local invariant functional spaces on $\mathbb{R}^{\mathrm{n}}$ would be of limited value if the theory ends here. Indeed, the underlying motivation for developing this part of the theory was to get nice 'solution spaces' on vector bundles, so we should make sure that there exist good 'preservation results' for the construction functor $\mathscr{F}\mapsto\mathscr{F}(M,E)$. After all, if the desirable properties of a local invariant functional space $\mathscr{F}$ on $\mathbb{R}^{\mathrm{n}}$ are not inherited by $\mathscr{F}(M,E)$, there is no point in considering $\mathscr{F}(M,E)$. To begin with, $\mathscr{F}(M,E)$ is again local.

Proposition 4.6.11. *For every local invariant functional space $\mathscr{F}$ on $\mathbb{R}^{\mathrm{n}}$ and every vector bundle E over M, $\mathscr{F}(M,E)$ is local.*

Proof: Let r be the rank of E, let $\{(U_i,\kappa_i,\rho_i)\}_{i\in I}$ be a total trivialization cover of $E\to M$ such that $\{U_i\}_{i\in I}$ is an open cover of M by precompact subsets and denote for every $i\in I$ the associated linear topological isomorphism from $\mathscr{D}'(U_i,E_{U_i})$ onto $(\mathscr{D}'(\kappa_i(U_i)))^{\times r}$ by h_i. Fix $i\in I$ for a moment. Because U_i is precompact, $\mathrm{cl}(U_i)$ is compact, so we find an $\varphi\in\mathscr{D}(M)$ such that φ equals 1 on an open neighborhood of $\mathrm{cl}(U_i)$. Now, on behalf of the characterizing property of $(\mathscr{F}(M,E))_{\mathrm{loc}}$, m_φ is a continuous linear map from $(\mathscr{F}(M,E))_{\mathrm{loc}}$ into $\mathscr{F}(M,E)$, while it follows from the definition of $\mathscr{F}(M,E)$ that $h_i\circ\mathrm{res}_{M,U_i}$ is a continuous linear map from $\mathscr{F}(M,E)$ into $(\mathscr{F}(\kappa_i(U_i)))^{\times r}$. As a result,

$$u\mapsto h_i(\,(\varphi u)|_{U_i})=h_i(\varphi|_{U_i}\,u|_{U_i})=h_i(u|_{U_i})$$

is a continuous linear map from $(\mathscr{F}(M,E))_{\mathrm{loc}}$ into $(\mathscr{F}(\kappa_i(U_i)))^{\times r}$. But this holds actually for every $i\in I$ and it then follows in a totally straightforward manner from the definition of $\mathscr{F}(M,E)$ that $(\mathscr{F}(M,E))_{\mathrm{loc}}\subseteq_{\mathrm{c}}\mathscr{F}(M,E)$. □

The following lemma almost directly leads to the next preservation result. To keep things readable, we stipulate that in the remainder of this section, unless explicitly specified otherwise, $\mathscr{F}$ denotes a local invariant functional space on $\mathbb{R}^{\mathrm{n}}$ and E denotes a vector bundle over M of rank r.

Lemma 4.6.12. *If $K\in\mathscr{P}_{\mathrm{c}}(M)$ such that K is contained in the domain of a total trivialization triple (U,κ,ρ) of $E\to M$, $(\mathscr{F}(M,E))_K$ is linearly topologically isomorphic to $(\mathscr{F}_{\kappa(K)})^{\times r}$.*

Proof: For notational convenience, we denote (U,κ,ρ) by (U_0,κ_0,ρ_0) and we exploit the assumption that M is second-countable to find a countable collection $\{(U_i,\kappa_i,\rho_i)\}_{i\geq 1}$ of total trivialization triples of $E\to M$ with the property that $\{U_0\}\cup\{U_i\}_{i\geq 1}=\{U_i\}_{i\in\mathbb{N}}$ is an open cover of M and $U_i\cap K=\emptyset$ for all $i\geq 1$ (it is easy to see that such a collection exists). As usual, we denote for every

$i \in \mathbb{N}$ the linear topological isomorphism from $\mathscr{D}'(U_i, E_{U_i})$ onto $(\mathscr{D}'(\kappa_i(U_i)))^{\times r}$ associated to the total trivialization triple (U_i, κ_i, ρ_i) by h_i. Then

$$H \colon \mathscr{F}(M,E) \to \prod_{i\in\mathbb{N}} (\mathscr{F}(\kappa_i(U_i)))^{\times r} \colon u \mapsto \{h_i(u|_{U_i})\}_{i\in\mathbb{N}}$$

is a linear topological embedding and because $(\mathscr{F}(M,E))_K$ carries the restricted topology from $\mathscr{F}(M,E)$, H restricts to a linear topological embedding of $(\mathscr{F}(M,E))_K$ into $\prod_{i\in\mathbb{N}}(\mathscr{F}(\kappa_i(U_i)))^{\times r}$. Now note that because $U_i \cap K = \emptyset$ for all $i \geq 1$, only the 0^{th} component of this restriction is nonzero and as a consequence, $u \mapsto h_0(u|_{U_0})$ is a linear topological embedding of $(\mathscr{F}(M,E))_K$ into

$$(\mathscr{F}(\kappa_0(U_0)))^{\times r} = (\mathscr{F}(\kappa(U)))^{\times r}.$$

We readily verify that the image of this embedding equals $((\mathscr{F}(\kappa(U)))_{\kappa(K)})^{\times r}$ and combining this with Lemma 2.7.8 shows that

$$(\mathscr{F}(M,E))_K \simeq (\mathscr{F}_{\kappa(K)})^{\times r}. \qquad \square$$

Theorem 4.6.13. *Let P be short for: metrizable, normable, complete, Fréchet, Banach or Hilbert. Then $\mathscr{F}$ is locally P implies $\mathscr{F}(M,E)$ is locally P.*

Proof: According to the generalization of Proposition 2.4.2 to the context of vector bundles, it suffices to show that every $x \in M$ admits a compact neighborhood K such that $(\mathscr{F}(M,E))_K$ is P. So fix $x \in M$. Now let (U,κ,ρ) be a total trivialization triple of $E \to M$ such that $x \in U$ and let K be a compact neighborhood of x such that $K \subseteq U$. On the strength of the previous lemma, we then have $(\mathscr{F}(M,E))_K \simeq (\mathscr{F}_{\kappa(K)})^{\times r}$ and since $\mathscr{F}$ is locally P and all properties that P can resemble are preserved under finite products, we conclude that $(\mathscr{F}(M,E))_K \simeq (\mathscr{F}_{\kappa(K)})^{\times r}$ is P. $\square$

The attentive reader might have noticed that we have not said anything about normality. Since normality is in fact preserved under $\mathscr{F} \mapsto \mathscr{F}(M,E)$ and the proof is not difficult, the reason for this is not of mathematical nature, but a matter of making choices; although we would like to, we cannot discuss everything and we decided to give priority to a different topic. Nevertheless, for completeness, we include normality in the following ‘summarizing’ result.

Theorem 4.6.14. *Let $\mathscr{F}$ be a local invariant functional space on $\mathbb{R}^{\mathrm{n}}$ and let M be an n-dimensional manifold. Then $\{\mathscr{F}(M,E)\}_{E\in\mathrm{VB}(M)}$ is a functorial family of local functional spaces on M and if $\mathscr{F}$ is locally Banach, locally Hilbert, Fréchet or normal, then so is $\mathscr{F}(M,E)$ for every $E \in \mathrm{VB}(M)$.*

Remark 4.6.15. Note that for local functional spaces, being locally Fréchet is equivalent to being Fréchet. ⊘

Example 4.6.16. Since we have seen in the second chapter that $\mathscr{E}(\mathbb{R}^{\mathrm{n}})$ is Fréchet (see Example 2.9.26) and we have seen above that $\mathscr{E}(M,E) = (\mathscr{E}(\mathbb{R}^{\mathrm{n}}))(M,E)$, it is a direct consequence of the previous theorem that $\mathscr{E}(M,E)$ is Fréchet. ⊘

Altogether, our framework to ‘create’ functional spaces on vector bundles from sufficiently well-behaved functional spaces on $\mathbb{R}^{\mathrm{n}}$ does a pretty good job:

the construction is functorial, we get functorial families as result and all important properties are preserved. But what if we encounter an interesting 'solution space' on $\mathbb{R}^{\mathrm{n}}$ that is not 'sufficiently well-behaved'? For example, the Sobolev spaces $W^{k,p}(\mathbb{R}^{\mathrm{n}})$, which are certainly interesting 'solution spaces', are in general neither local nor invariant. Luckily enough, if a functional space is not 'sufficiently well-behaved' (that is, local and invariant), we have an extensive 'toolbox', filled with the results from the second chapter, that can be used to investigate whether we can *make* such a functional space 'sufficiently well-behaved' and at what 'cost' (i.e., which properties do not survive this procedure). For example, since the spaces $W^{k,p}(\mathbb{R}^{\mathrm{n}})$ are locally invariant (see Example 2.9.38), the 'localized' Sobolev spaces $(W^{k,p}(\mathbb{R}^{\mathrm{n}}))_{\mathrm{loc}}$ are local and invariant (see Proposition 2.6.29) and therefore suitable as a 'model'. Moreover, our 'toolbox' tells us that the spaces $(W^{k,p}(\mathbb{R}^{\mathrm{n}}))_{\mathrm{loc}}$ are still locally Banach, so if we use the procedure of this section to create a functorial family of 'localized Sobolev spaces' on a *compact* manifold, the members of this functorial family will be Banach.

Part III

5

Functional spaces on fiber bundles

In this final chapter, we discuss how we can 'produce' functional spaces on fiber bundles. The big difference between vector bundles and fiber bundles is of course that on a fiber bundle the linear structure, which we have used extensively in the second part, is missing. As a consequence, we do not have a vector space structure on the (compactly supported) smooth sections of a fiber bundle, so those 'spaces' are no longer locally convex vector spaces and we cannot take duals. Therefore, we should in particular forget everything about distributions and revert back to spaces of 'ordinary' continuous sections. Rather than having the structure of infinite dimensional vector spaces, these spaces will have the structure of 'infinite dimensional manifolds'.

The approach that we follow to produce these infinite dimensional manifolds is, although adapted to our setting and generalized a bit, basically the same approach as can be found in [11]: we start with a sufficiently well-behaved functorial family of functional spaces on vector bundles over a *compact* manifold and extend it to fiber bundles. Of course, there is one particular type of functorial family to which we would like to apply this, namely functorial families that are created from functional spaces on $\mathbb{R}^{\mathrm{n}}$ by the procedure that we have described in Section 4.6.

Throughout this chapter, M will denote an n-dimensional (second-countable smooth) manifold and all vector spaces and vector bundles are over $\mathbb{R}$.

5.1 Fréchet manifolds

Baldly stated, like finite dimensional manifolds are topological spaces that are locally homeomorphic to $\mathbb{R}^{\mathrm{n}}$, infinite dimensional manifold are topological spaces that are locally homeomorphic to some infinite dimensional locally convex vector space. However, to get the theory going we cannot look at spaces 'modeled' on any locally convex vector space, additional properties are needed. In [7] a very general theory of infinite dimensional manifolds is developed based on so-called 'convenient' locally convex vector spaces, but here we stick to the even more convenient setting of Fréchet manifolds; infinite dimensional manifolds that are 'modeled' on Fréchet spaces. In this section we quickly mention a few basic definitions of this subject. For a more extensive introduction, see [5].

We start with the notion of differentiability for maps between Fréchet spaces.

Definition 5.1.1. Let $\mathscr{X}$ and $\mathscr{Y}$ be Fréchet spaces, U an open subset of $\mathscr{X}$ and $P\colon U \to \mathscr{Y}$ a continuous map. The *derivative* of P at the point $x \in U$ in the direction $x' \in \mathscr{X}$ is defined by

$$DP(x)x' := \lim_{t \to 0} \frac{P(x + tx') - P(x)}{t}.$$

If the limit exists, we say that P is *differentiable at x in the direction x'*. We say that P is *continuously differentiable* (often abbreviated as C^1), if the limit exists for all $x \in U$ and $x' \in \mathscr{X}$ and $DP\colon U \times \mathscr{X} \to \mathscr{Y}$ is continuous. Moreover, we define the *tangent* $TP\colon U \times \mathscr{X} \to \mathscr{Y} \times \mathscr{Y}$ of P by

$$TP(x)x' := (P(x), DP(x)x'). \qquad \oslash$$

Clearly, the tangent TP of P is defined and continuous if and only if the derivative DP is defined and continuous.

Definition 5.1.2. Let $\mathscr{X}$ and $\mathscr{Y}$ be Fréchet spaces, U an open subset of $\mathscr{X}$ and $P\colon U \to \mathscr{Y}$ a map. For $k \in \mathbb{N}$, we say that P is C^k if T^kP is defined and continuous (where, of course, $T^0P := P$ and $T^{n+1}P := T(T^nP)$ for $n \in \mathbb{N}$). We say that P is C^∞ or smooth if P is C^k for every $k \in \mathbb{N}$. $\oslash$

The following characterization of being continuously differentiable will be very useful.

Lemma 5.1.3. *Let $\mathscr{X}$ and $\mathscr{Y}$ be Fréchet spaces, U a convex open subset of $\mathscr{X}$ and $P\colon U \to \mathscr{Y}$ a continuous map. Then P is continuously differentiable if and only if there exists a continuous map $L\colon U \times U \times \mathscr{X} \to \mathscr{Y}$, linear in the last variable, such that for all x_0, $x_1 \in U$,*

$$P(x_1) - P(x_0) = L(x_0, x_1)(x_1 - x_0).$$

If this is the case, we have for $x \in U$ and $x' \in \mathscr{X}$,

$$DP(x)x' = L(x, x)x'.$$

Proof: See [5, Lemma I.3.3.1]. □

We can now tell what a Fréchet manifold is.

Definition 5.1.4. A *Fréchet manifold* is a Hausdorff topological space with an atlas of coordinate charts taking their value in Fréchet spaces, such that the coordinate transition functions are all smooth maps between Fréchet spaces. $\oslash$

This definition, which is literally taken from [5], might be a bit blunt, but the 'missing' details are completely analogous to the finite dimensional case. (So a coordinate chart of a Fréchet manifold $\mathscr{M}$ is a homeomorphism from an open subset of $\mathscr{M}$ onto an open subset of a Fréchet space F, we always assume the atlas to be maximal, etc.)

Remark 5.1.5. We do not require that the charts of a Fréchet manifold $\mathscr{M}$ all take values in the same Fréchet space, so informally speaking $\mathscr{M}$ might be 'modeled over' multiple Fréchet spaces. However, if $\mathscr{U}$ and $\mathscr{V}$ are overlapping charts domains of $\mathscr{M}$, the derivative of the coordinate transition function in any point of its domain is a linear topological isomorphism from the Fréchet space associated to $\mathscr{U}$ onto the Fréchet space associated to $\mathscr{V}$. As a consequence, every connected component of $\mathscr{M}$ is modeled up to isomorphism over a single Fréchet space (note that $\mathscr{M}$ is locally path-connected, so its connected components must be path-connected). $\oslash$

In the same style, we define what we mean by a smooth map between Fréchet manifolds.

Definition 5.1.6. Let $\mathscr{M}$ and $\mathscr{N}$ be Fréchet manifolds and $P\colon \mathscr{M} \to \mathscr{N}$ a map. We say that P is *smooth* if we can find charts around any point x in $\mathscr{M}$ and its image $P(x)$ in $\mathscr{N}$ such that the local representative of P in these charts is a smooth map of Fréchet spaces. ⊘

5.2 Simple functorial families

In the previous chapter, we have introduced functorial families of functional spaces on M. In this chapter, we discuss how such functorial families, which are a priori defined on the category of vector bundles over M, can be extended to (the category of) fiber bundles over M if we assume that M is compact. However, just like a functional space on $\mathbb{R}^{\mathrm{n}}$ can only be 'transferred' to vector bundles if it is local and invariant, a functorial family also needs to satisfy some additional conditions to be 'extendable'.

Definition 5.2.1. A *functorial family* $\mathscr{F}_M$ of (semi-)functional spaces on M is called *simple* if:

1. for every vector bundle E over M, $\mathscr{F}_M(E)$ is Fréchet and satisfies
$$\mathscr{F}_M(E) \subseteq_{\mathrm{c}} \Gamma^0(M, E)$$
(where $\Gamma^0(M, E)$ carries the topology as discussed in Example 4.4.5) and
2. for all vector bundles E and F over M and every fiber bundle homomorphism $f\colon E \to F$, f_* restricts to a continuous map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$. ⊘

Remark 5.2.2. Of course the word 'simple' has already a lot of meanings in mathematics, but it should not lead to confusions in this context and is actually quite appropriate; simple functorial families are the 'simple' ones, because we are dealing with ordinary continuous sections rather than with distributions and with Fréchet spaces instead of more general locally convex spaces. ⊘

Remark 5.2.3. We are mainly interested in functorial families that are created from a local invariant functional space $\mathscr{F}$ on $\mathbb{R}^{\mathrm{n}}$ by the procedure that we have discussed in Section 4.6, so whenever we talk about a functorial family $\mathscr{F}_M$, one should keep this 'key example' in mind. Note however, that it is certainly not true that every functorial family of this type is automatically simple: whether or not this is the case depends on the 'model' $\mathscr{F}$. In Section 5.6 we will say a bit more about this. ⊘

Let $\mathrm{FVB}(M)$ be the category of vector bundles over M with *fiber bundle* homomorphism as arrows and let $\mathrm{Frechet}^0$ be the category of Fréchet spaces with continuous maps as arrows. Then a simple functorial family $\mathscr{F}_M$ of semi-functional spaces on M is in particular a functor from $\mathrm{FVB}(M)$ into $\mathrm{Frechet}^0$ that maps an arrow $f\colon E \to F$ to the 'canonical arrow' f_*. Surprisingly enough, this canonical arrow $f_*\colon \mathscr{F}_M(E) \to \mathscr{F}_M(F)$, which is a priori only assumed to be continuous, turns out to be automatically smooth.

Proposition 5.2.4. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M, E and F vector bundles over M and $f\colon E \to F$ a fiber bundle homomorphism. Then $f_*\colon \mathscr{F}_M(E) \to \mathscr{F}_M(F)$ is smooth.*

The rest of this section will be devoted to the proof of this statement. Instead of directly proving that f_* is C^∞, we first prove that it is C^1.

Remark 5.2.5. If E and F are vector bundles over M and $\mathscr{F}_M$ is a simple functorial family of semi-functional spaces on M, then the linear topological isomorphism between $\mathscr{F}_M(E \oplus F) \subseteq \Gamma^0(M, E \oplus F)$ and $\mathscr{F}_M(E) \times \mathscr{F}_M(F) \subseteq \Gamma^0(M, E) \times \Gamma^0(M, F)$ from Lemma 4.4.6 just sends a function from M into $E \times F$ to a 2-tuple containing its components. That is, the linear topological isomorphism is just the identity and we have

$$\mathscr{F}_M(E \oplus F) = \mathscr{F}_M(E) \times \mathscr{F}_M(F). \qquad \oslash$$

Lemma 5.2.6. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M, E and F vector bundles over M and $f\colon E \to F$ a fiber bundle homomorphism. Then $f_*\colon \mathscr{F}_M(E) \to \mathscr{F}_M(F)$ is C^1.*

Proof: Let $\{(U_i, \kappa_i)\}_{i\in I}$ be a collection of charts of M such that $\{U_i\}_{i\in I}$ is an open cover of M with the property that for every $i \in I$ both E and F trivialize over U_i (it is clear that such covers exist). Furthermore, let r denote the rank of E and let k denote the rank of F. Then for every $i \in I$, the restriction $f\colon E_{U_i} \to F_{U_i}$ corresponds to a smooth function $\tilde{f}_i\colon \kappa(U_i) \times \mathbb{R}^r \to \mathbb{R}^k$ (to avoid cumbersome notation, we will not explicitly use trivialization maps ρ_i^E and ρ_i^F). For every $i \in I$, we now define $\tilde{\ell}_i\colon \kappa(U_i) \times \mathbb{R}^r \times \mathbb{R}^r \times \mathbb{R}^r \to \mathbb{R}^k$ by

$$\tilde{\ell}_i(x, y_0, y_1)z := \left(\int_0^1 D_2\tilde{f}_i(x, (1-t)y_0 + ty_1)\,dt\right) z,$$

where $D_2\tilde{f}_i(x, y)$ is the $k \times r$ matrix that is obtained by fixing x and then taking the matrix of partial derivatives of $y \mapsto \tilde{f}_i(x, y)$ and the integration over the matrix $D_2\tilde{f}_i(x, (1-t)y_0 + ty_1)$ is taken componentwise (resulting in a $k \times r$ matrix which acts on z). Then the $\tilde{\ell}_i$ are smooth maps which are, as already suggested by the notation, linear in their last component and it is routine to verify that

$$\tilde{\ell}_i(x, y_0, y_1)(y_1 - y_0) = \tilde{f}_i(x, y_1) - \tilde{f}_i(x, y_0)$$

for all $x \in \kappa(U_i)$ and y_0, $y_1 \in \mathbb{R}^r$. (The idea: define $g\colon [0,1] \to \mathbb{R}^k$ by $g(t) := \tilde{f}_i(x, (1-t)y_0 + ty_1)$ and compute $\tilde{f}_i(x, y_1) - \tilde{f}_i(x, y_0) = g(1) - g(0) = \int_0^1 g'(t)\,dt$.) Bringing the $\tilde{\ell}_i$ back to the abstract setting gives rise to fiber bundle homomorphisms $\ell_i\colon E_{U_i} \oplus E_{U_i} \oplus E_{U_i} \to F_{U_i}$ which are linear in the last component and satisfy $f_i(e_1) - f_i(e_0) = \ell_i(e_0, e_1)(e_1 - e_0)$ for all e_0, $e_1 \in E_x$ with $x \in U_i$. Now let $\{\eta_i\}_{i\in I}$ be a partition of unity subordinate to $\{U_i\}_{i\in I}$. Then

$$\ell\colon E \oplus E \oplus E \to F\colon (e_0, e_1, e_2) \mapsto \sum_{i\in I} \eta_i(\pi_E(e_0))\ell_i(e_0, e_1)e_2$$

is a (smooth) fiber bundle homomorphism, linear in the last component, such that $f(e_1) - f(e_0) = \ell(e_0, e_1)(e_1 - e_0)$. Combining the properties of ℓ with the conditions of a simple functorial family, we finally deduce that ℓ_* is a continuous map from $\mathscr{F}_M(E) \times \mathscr{F}_M(E) \times \mathscr{F}_M(E) = \mathscr{F}_M(E \oplus E \oplus E)$ into $\mathscr{F}_M(F)$ which is linear in the last component and satisfies

$$f_*(\varphi_1) - f_*(\varphi_0) = \ell_*(\varphi_0, \varphi_1)(\varphi_1 - \varphi_0)$$

for all φ_0, $\varphi_1 \in \mathscr{F}_M(E)$, which allows us to use Lemma 5.1.3 to conclude that $f_* \colon \mathscr{F}_M(E) \to \mathscr{F}_M(F)$ is C^1. □

Although it is nice to know that $f_* \colon \mathscr{F}_M(E) \to \mathscr{F}_M(F)$ is C^1, it would be even nicer to know what its derivative is. To give a nice formal expression of this derivative, we use (the 'translation' of) the vertical differential

$$\hat{\delta} f \colon E \oplus E \to F \oplus F$$

(see Section B.2 for more information).

Lemma 5.2.7. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M, E and F vector bundles over M and $f \colon E \to F$ a fiber bundle homomorphism. Consider f_* as a map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$. Then $Tf_* = (\hat{\delta} f)_*$.*

Proof: By the previous lemma, $f_* \colon \mathscr{F}_M(E) \to \mathscr{F}_M(F)$ is C^1, so we know that Tf_* is defined and continuous and we should prove that $Tf_* = (\hat{\delta} f)_*$. So let φ, $\psi \in \mathscr{F}_M(E)$ and $x \in M$. Looking at the relevant definitions, we immediately see that 'the first components' agree and that we should proof that $Df_*(\varphi)\psi$, which is an element of $\mathscr{F}_M(F) \subseteq_{\mathrm{c}} \Gamma^0(M, F)$, satisfies

$$(Df_*(\varphi)\psi)(x) = \left.\frac{d}{dt}\right|_{t=0} f(\varphi(x) + t\psi(x)).$$

By definition $Df_*(\varphi)\psi$ is the limit for $t \to 0$ in $\mathscr{F}_M(F)$ of

$$\frac{f_*(\varphi + t\psi) - f_*(\varphi)}{t}$$

and because $\mathscr{F}_M(F) \subseteq_{\mathrm{c}} \Gamma^0(M, F)$, we also have that

$$\frac{f_*(\varphi + t\psi) - f_*(\varphi)}{t} \to Df_*(\varphi)\psi$$

when $t \to 0$ in $\Gamma^0(M, F)$. Due to the topology of $\Gamma^0(M, F)$, we therefore in particular have pointwise convergence, hence

$$\begin{aligned}
(Df_*(\varphi)\psi)(x) &= \lim_{t \to 0} \frac{(f_*(\varphi + t\psi))(x) - (f_*(\varphi))(x)}{t} \\
&= \lim_{t \to 0} \frac{f(\varphi(x) + t\psi(x)) - f(\varphi(x))}{t} \\
&= \left.\frac{d}{dt}\right|_{t=0} f(\varphi(x) + t\psi(x)).
\end{aligned}$$ □

Proposition 5.2.4 now follows by repeatedly applying the two previous lemmas. For example, to prove that f_* is C^2, we should prove that Tf_* is C^1, but $Tf_* = (\hat{\delta} f)_*$ by Lemma 5.2.7 and $(\hat{\delta} f)_*$ is C^1 by Lemma 5.2.6. Clearly, we can repeat this argument to deduce that f_* is C^k for every $k \in \mathbb{N}$, thus f_* is smooth. Therefore, if $\mathrm{Frechet}^\infty$ denotes the category of Fréchet spaces with smooth maps as arrows, a simple functorial family $\mathscr{F}_M$ is in particular a functor from $\mathrm{FVB}(M)$ into $\mathrm{Frechet}^\infty$.

Example 5.2.8. It is not difficult to show that $\{\Gamma^k(M,E)\}_{E\in \mathrm{VB}(M)}$ is a simple functorial family of functional spaces on M for every $k \in \mathbb{N}_\infty$. (So in particular $\{\mathscr{E}(M,E)\}_{E\in \mathrm{VB}(M)} = \{\Gamma^\infty(M,E)\}_{E\in \mathrm{VB}(M)}$ is a simple functorial family.) ⊘

5.3 Vector bundle neighborhoods

Given a local invariant functional space $\mathscr{F}$ on $\mathbb{R}^n$, we basically created a functional space $\mathscr{F}(M,E)$ of distributional sections of a vector bundle $E \in \mathrm{VB}(M)$ of 'type $\mathscr{F}$' by covering M with 'patches' on which 'being of type $\mathscr{F}$' made sense and then we selected those distributional sections whose restrictions to these patches were of 'type $\mathscr{F}$'. The transition from vector bundles to fiber bundles is based on a similar principle: we cover a fiber bundle P by patches that have the structure of a vector bundle and then, given a simple functorial family of (semi-)functional spaces $\mathscr{F}_M$, we let $\mathscr{F}_M(P)$ be the space of those continuous sections of P that are of 'type $\mathscr{F}$' on these patches. The purpose of this section is to introduce those 'vector bundle'-type patches and some relevant results. In what follows, $P \to M$ always denotes a fiber bundle over M and we will often refer to it by just mentioning its total space P.

Definition 5.3.1. Let $\varphi \in \Gamma^0(M,P)$. A *vector bundle neighborhood* of φ in P is a vector bundle E over M which is an open subbundle of P and has the property that $\mathrm{im}(\varphi) \subseteq E$. ⊘

Remark 5.3.2. Note that $\mathrm{im}(\varphi) \subseteq E$ implies that $\varphi \in \Gamma^0(M,E)$. ⊘

Although we do not really need it, using the material from Section B.2, we easily see that vector bundle neighborhoods of continuous sections are essentially unique.

Lemma 5.3.3. *If $\varphi \in \Gamma^\infty(M,P)$ and E is a vector bundle neighborhood of φ in P, then $E \simeq T^v_\varphi(P)$.*

Proof: We already know that $T^v_\varphi(E) \simeq E$ and because E is an open subbundle of P, we clearly have $T^v_\varphi(P) = T^v_\varphi(E)$. □

Proposition 5.3.4. *If $\varphi \in \Gamma^0(M,P)$ and both E and F are a vector bundle neighborhood of φ in P, then $E \simeq F$.*

Proof: Since $E \cap F$ is an open subset of P that contains $\mathrm{im}(\varphi)$, Theorem B.3.1 tells us that there exists an $\psi \in \Gamma^\infty(M,P)$ such that $\mathrm{im}(\psi) \subseteq E \cap F$. But then both E and F are a vector bundle neighborhood of ψ in P, so by the previous lemma

$$E \simeq T^v_\varphi(P) \simeq F.$$ □

Far more important is the following existence theorem.

Theorem 5.3.5. *Let $\varphi \in \Gamma^0(M,P)$. Given an open neighborhood U of* $\mathrm{im}(\varphi)$ *in P, there exists a vector bundle neighborhood E of φ in P with $E \subseteq U$. Moreover, if $\varphi \in \Gamma^\infty(M,P)$, we can choose E such that φ is the zero section of E.*

Proof: See [11, Theorem 12.10]. □

Remark 5.3.6. The previous theorem has an interesting consequence for simple functorial families of (semi-)functional spaces. To explain this, let $\mathscr{F}_M$ be such a simple functorial family of semi-functional spaces on M. Furthermore, let E be a vector bundle over M and $\varphi \in \mathscr{E}(M,E)$. Because E is in particular a fiber bundle over M and φ is smooth, Theorem 5.3.5 tells us that there exists a vector bundle neighborhood F of φ in E such that φ is the zero section of F. Now we make two observations. First, since $\mathscr{F}_M(F)$ is a linear subspace of $\Gamma^0(M,F)$, the zero section, hence φ, is an element of $\mathscr{F}_M(F)$. And, second, because F is an open subbundle of E, the inclusion map $\imath\colon F \hookrightarrow E$ is a fiber bundle homomorphism. So, using the conditions of a simple functorial family, we deduce that $\imath_*\colon \Gamma^0(M,F) \to \Gamma^0(M,E)$, which is just the identity, restricts to a continuous map from $\mathscr{F}_M(F)$ into $\mathscr{F}_M(E)$. Hence, $\mathscr{F}_M(F) \subseteq_{\mathrm{c}} \mathscr{F}_M(E)$ and $\varphi \in \mathscr{F}_M(E)$ (after all, $\varphi \in \mathscr{F}_M(F)$). Since E and φ were arbitrarily chosen, we conclude that $\mathscr{E}(M,E) \subseteq \mathscr{F}_M(E)$ for every $E \in \mathrm{VB}(M)$. ⊘

Example 5.3.7. As a consequence of the previous remark, we see that

$$\{\mathscr{D}(M,E)\}_{E\in\mathrm{VB}(M)} \quad \text{and} \quad \{\mathscr{E}_K(M,E)\}_{E\in\mathrm{VB}(M)},$$

with $K \in \mathscr{P}_{\mathrm{c}}(M)$, are in general *no* simple functorial families of semi-functional spaces. ⊘

The intuitive explanation for the just observed fact that $\{\mathscr{D}(M,E)\}_{E\in\mathrm{VB}(M)}$ and $\{\mathscr{E}_K(M,E)\}_{E\in\mathrm{VB}(M)}$ are in general no simple functorial families is the following: simple functorial families are an intermediate step in removing the dependence on the linear structure that is present on vector bundles (which is needed to make the step to fiber bundles) and $\mathscr{D}(M,E)$ and $\mathscr{E}_K(M,E)$ really depend on the linear structure of the vector bundle E because the notion of support uses this linear structure (it depends on having a canonical zero section). However, for a neat extension to fiber bundles, we need to assume that M is compact anyway, so the functorial family $\{\mathscr{D}(M,E)\}_{E\in\mathrm{VB}(M)}$ will coincide with $\{\mathscr{E}(M,E)\}_{E\in\mathrm{VB}(M)}$ and therefore be a simple functorial family.

From now on, we assume that M is compact.

The following result is really 'clear from the picture', but we also give an indication of a formal argument.

Lemma 5.3.8. *Let E be a vector bundle over M, g a (vector bundle) metric on E, $\varphi \in \Gamma^0(M,E)$ and V an open neighborhood of* $\mathrm{im}(\varphi)$ *in E. Then there exists an $\varepsilon > 0$ such that for all $x \in M$ and $e_x \in E_x$, $|e_x - \varphi(x)|^g < \varepsilon$ implies $e_x \in V$.*

Proof: We can cover M by a finite number of compact subsets $K_0, \ldots, K_n$ such that each of these compact subsets lies in the domain of a special total trivialization triple (U,κ,ρ) with the property that $\rho\colon E_U \to \mathbb{R}^{\mathrm{rank}(E)}$ is an isometry on the fibers (it is well-known that M can be covered by such total trivialization triples and we can fit small compact discs inside the domains of these triples and use the compactness of M to get the desired cover). Now for the existence of an $\varepsilon_i > 0$, with $0 \le i \le n$, such that for all $x \in K_i$ and $e_x \in E_x$, $|e_x - \varphi(x)|^g < \varepsilon_i$ implies $e_x \in V$, we may assume that K_i lies in an open subset of $\mathbb{R}^{\mathrm{n}}$ and that E is the trivial bundle with the Euclidean metric. In this setting it is clear that such an ε_i indeed exists (this is basically a version of the tube lemma, but can also be derived directly by covering $\varphi(K_i)$ with a finite number of open 'squares' and using the equivalence of the Chebyshev and Euclidean metric) and taking $\varepsilon := \min_{0\le i\le n} \varepsilon_i$ then finishes the proof. $\square$

We already know that any two vector bundle neighborhoods of a continuous section φ of $P \to M$ are isomorphic. However, we do not have any 'control' over this vector bundle isomorphism and in general the pushforward of this isomorphism will not have φ as a fixed point; something which will be very desirable in the next section. Therefore, the next result is a very welcome addition to our 'toolbox'.

Lemma 5.3.9. *Let $\varphi \in \Gamma^0(M, P)$. If both E and F are a vector bundle neighborhood of φ in P, then there exists an injective fiber bundle homorphism $f\colon E \to F$ which equals the identity on an open neighborhood of* $\operatorname{im}(\varphi)$ *in E.*

Proof: Choose a (vector bundle) metric g on E. Since $E \cap F$ is an open neighborhood of $\operatorname{im}(\varphi)$ in E, the previous lemma gives us an $\varepsilon > 0$ such that for all $x \in M$ and $e_x \in E_x$, $|e_x - \varphi(x)|^g < \varepsilon$ implies $e_x \in F_x$ (well, the lemma only gives $e_x \in E \cap F \subseteq F$, but because E and F both are subbundles of P, we must have $e_x \in F_x$). Moreover, since $e \mapsto |e - \varphi(\pi_E(e))|^g$ is a continuous function on E, $U := \{e \in E \mid |e - \varphi(\pi_E(e))|^g < \frac{1}{4}\varepsilon\}$ is an open neighborhood of $\operatorname{im}(\varphi)$ in E, so according to Theorem B.3.1, we can find an $\psi \in \Gamma^\infty(M, E)$ such that $|\psi(x) - \varphi(x)|^g < \frac{1}{4}\varepsilon$ for all $x \in M$. Now let $\zeta\colon [0, \infty) \to [0, \frac{3}{4}\varepsilon)$ be a smooth injective map such that $\zeta(t) = t$ for all $0 \le t \le \frac{1}{2}\varepsilon$ and define $f\colon E \to F$ by

$$f(e_x) := \psi(x) + \frac{\zeta(|e_x - \psi(x)|^g)}{|e_x - \psi(x)|^g}(e_x - \psi(x))$$

(note that since $\zeta(|e_x - \psi(x)|^g)$ equals $|e_x - \psi(x)|^g$ if e_x 'comes close' to $\psi(x)$, there are no problems with division by zero). Then

$$\begin{aligned}|f(e_x) - \varphi(x)|^g &\le |f(e_x) - \psi(x)|^g + |\psi(x) - \varphi(x)|^g \\ &= \zeta(|e_x - \psi(x)|^g) + |\psi(x) - \varphi(x)|^g < \frac{3}{4}\varepsilon + \frac{1}{4}\varepsilon = \varepsilon,\end{aligned}$$

so $f(e_x)$ as defined above indeed belongs to F_x (from the defining expression it was a priori only clear that $f(e_x) \in E_x$) and we see that f is a (smooth) fiber bundle homomorphism. We readily verify that f is injective and it is also easy to check that f equals the identity on U. Indeed, if $e_x \in U$, then

$$|e_x - \psi(x)|^g \le |e_x - \varphi(x)|^g + |\varphi(x) - \psi(x)|^g < \frac{1}{4}\varepsilon + \frac{1}{4}\varepsilon = \frac{1}{2}\varepsilon,$$

so $\zeta(|e_x - \psi(x)|^g) = |e_x - \psi(x)|^g$ and hence $f(e_x) = e_x$. $\square$

5.4 From vector bundles to fiber bundles

Now everything is in place to discuss the extension of simple functorial families to fiber bundles. Also in this section, P denotes a fiber bundle over the (compact!) manifold M. Moreover, we introduce the following notation: $\mathrm{VSB}(P)$ denotes the set of all vector bundles over M that are open subbundles of P (note that because of dimensional reasons, all vector bundles in $\mathrm{VSB}(P)$ must have rank $\dim(P) - \dim(M)$).

Remark 5.4.1. Since every continuous section of an open subbundle of P is clearly also a continuous section of P, we have for every simple functorial family $\mathscr{F}_M$ of semi-functional spaces on M that $\mathscr{F}_M(E) \subseteq \Gamma^0(M, E) \subseteq \Gamma^0(M, P)$ for every $E \in \mathrm{VSB}(P)$. ⊘

Definition 5.4.2. Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M. As a set, we define

$$\mathscr{F}_M(P) := \bigcup_{E \in \mathrm{VSB}(P)} \mathscr{F}_M(E),$$

which should be viewed as a union of subsets of $\Gamma^0(M, P)$, and we endow $\mathscr{F}_M(P)$ with the largest topology such that $\mathscr{F}_M(E) \subseteq_{\mathrm{c}} \mathscr{F}_M(P)$ for all $E \in \mathrm{VSB}(P)$. ⊘

Remark 5.4.3. The topology on $\mathscr{F}_M(P)$ is the well-known *final topology* with respect to the inclusion maps $\{\mathscr{F}_M(E) \hookrightarrow \mathscr{F}_M(P)\}_{E \in \mathrm{VSB}(P)}$. So a subset $\mathscr{U}$ of $\mathscr{F}_M(P)$ is open in $\mathscr{F}_M(P)$ if and only if $\mathscr{U} \cap \mathscr{F}_M(E)$ is open in $\mathscr{F}_M(E)$ for all $E \in \mathrm{VSB}(P)$ and a map f from $\mathscr{F}_M(P)$ into a topological space X is continuous if and only if f restricts to a continuous map from $\mathscr{F}_M(E)$ into X for every $E \in \mathrm{VSB}(P)$. ⊘

Note that there is something to check here. If the fiber bundle P is actually some vector bundle E over M, then we should have that $\mathscr{F}_M(P)$ as defined above coincides with the original $\mathscr{F}_M(E)$. It is easy to see that this is indeed the case: because for every $F \in \mathrm{VSB}(E)$, the pushforward of the inclusion map $F \hookrightarrow E$ (which is a fiber bundle homomorphism) restricts to a continuous map from $\mathscr{F}_M(F)$ into $\mathscr{F}_M(E)$, we have that $\mathscr{F}_M(F) \subseteq_{\mathrm{c}} \mathscr{F}_M(E)$ for every $F \in \mathrm{VSB}(E)$ and together with the observation that $E \in \mathrm{VSB}(E)$, this shows that $\cup_{F \in \mathrm{VSB}(E)} \mathscr{F}_M(F)$ and $\mathscr{F}_M(E)$ are equal as topological spaces when the first is endowed with the final topology.

It is clear from the definition above that an element $\varphi \in \Gamma^0(M, P)$ belongs to $\mathscr{F}_M(P)$ precisely if there exists a vector bundle neighborhood E of φ in P such that $\varphi \in \mathscr{F}_M(E)$; i.e., if and only if φ is of 'type $\mathscr{F}$' on a vector bundle patch. The following result tells us that it does not matter which vector bundle patch around φ we use to determine whether φ is of 'type $\mathscr{F}$' (which is something that one expects and desires).

Lemma 5.4.4. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M. An element $\varphi \in \Gamma^0(M, P)$ belongs to $\mathscr{F}_M(P)$ if and only if $\varphi \in \mathscr{F}_M(E)$ for all vector bundle neighborhoods E of φ in P.*

Proof: Suppose that $\varphi \in \mathscr{F}_M(E)$ for all vector bundle neighborhoods E of φ in P. Because there always exists such a vector bundle neighborhood E of φ in P, we have $\mathscr{F}_M(E)$ for some $E \in \mathrm{VSB}(P)$, hence $\varphi \in \mathscr{F}_M(P)$.

Next, suppose that $\varphi \in \mathscr{F}_M(P)$. Then there exists a vector bundle neighborhood E of φ in P such that $\varphi \in \mathscr{F}_M(E)$. If F is a vector bundle neighborhood of φ in P as well, then Lemma 5.3.9 tells us that there exists a fiber bundle homomorphism $f\colon E \to F$ which equals the identity on an open neighborhood of $\mathrm{im}(\varphi)$. Clearly, this fiber bundle homomorphism f then satisfies $f_*(\varphi) = \varphi$, while f restricts to a continuous map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$ because $\mathscr{F}_M$ is a simple functorial family. Hence, the assumption that $\varphi \in \mathscr{F}_M(E)$ implies $\varphi = f_*(\varphi) \in \mathscr{F}_M(F)$. □

Now let $k \in \mathbb{N}_\infty$ and let $\Gamma^k_M\colon \mathrm{FVB}(M) \to \mathrm{Frechet}^\infty$ denote the simple functorial family $E \mapsto \Gamma^k(M,E)$. Another thing that one expects and desires is that $\Gamma^k_M(P)$ is as a set equal to $\Gamma^k(M,P)$, which is indeed the case because a trivial application of Theorem 5.3.5 shows that

$$\Gamma^k(M,P) = \bigcup_{E\in\mathrm{VSB}(P)} \Gamma^k(M,E).$$

So by applying Definition 5.4.2 to $\Gamma^k_M(P)$ we actually get a topology on $\Gamma^k(M,P)$ and from now on we always assume that $\Gamma^k(M,P)$ carries this topology (i.e., we stipulate that $\Gamma^k(M,P)$ and $\Gamma^k_M(P)$ are equal as topological spaces).

Lemma 5.4.5. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M. Then*

$$\mathscr{F}_M(P) \subseteq_{\mathrm{c}} \Gamma^0(M,P)$$

and if $\mathscr{F}_M$ is actually a family of functional spaces, then also

$$\Gamma^\infty(M,P) \subseteq_{\mathrm{c}} \mathscr{F}_M(P).$$

Proof: This is a direct consequence of the properties of final topologies and the fact that $\Gamma^\infty(M,E) = \mathscr{E}(M,E) = \mathscr{D}(M,E)$ for every vector bundle E over our compact manifold M. ☐

As promised, $\mathscr{F}_M(P)$ will be a Fréchet manifold and it is already pretty obvious what the charts of $\mathscr{F}_M(P)$ should look like: the charts are just the Fréchet spaces $\mathscr{F}_M(E)$ for $E \in \mathrm{VSB}(P)$. However, in order to show that these charts turn the topological space $\mathscr{F}_M(P)$ into a Fréchet manifold, we first have to discuss a pair of results that tells us a bit more about the topology of the Fréchet spaces $\mathscr{F}_M(E)$. (Be aware of the fact that the compactness of M plays an important role in the result below.)

Lemma 5.4.6. *Let E be a vector bundle over M and let U be an open subset of E. Then*

$$\mathscr{U} := \{\varphi \in \Gamma^0(M,E) \mid \mathrm{im}(\varphi) \subseteq U\}$$

is an open subset of $\Gamma^0(M,E)$.

Proof: If $\mathscr{U}$ is empty there is nothing to prove, so suppose that $\varphi \in \mathscr{U}$. Since φ is arbitrarily chosen, it suffices to find an open neighborhood $\mathscr{V}$ of φ in $\Gamma^0(M,E)$ that is contained in $\mathscr{U}$ to prove that $\mathscr{U}$ is open. To this end, choose some vector bundle metric g on E. By construction, U is an open neighborhood of $\mathrm{im}(\varphi)$ in E, so according to Lemma 5.3.8, there exists an $\varepsilon > 0$ such that for all $x \in M$ and $e_x \in E_x$, $|e_x - \varphi(x)|^g < \varepsilon$ implies $e_x \in U$. Now, since $\psi \mapsto \sup_{x\in M} |\psi(x)|^g$ is a seminorm from the inducing collection of seminorms for $\Gamma^0(M,E)$ associated to g (after all, $M \in \mathscr{P}_{\mathrm{c}}(M)$ because M is compact),

$$\mathscr{V} := \{\psi \in \Gamma^0(M,E) \mid \sup_{x\in M} |\psi(x) - \varphi(x)|^g < \varepsilon\}$$

is an open subset of $\Gamma^0(M,E)$ and it is clear that $\varphi \in \mathscr{V}$ and that $\mathscr{V} \subseteq \mathscr{U}$. ☐

Remark 5.4.7. The seminorm $\psi \mapsto \sup_{x\in M} |\psi(x)|^g$ from above is even a norm and the topology of $\Gamma^0(M,E)$ is also induced by this norm on its own. ⊘

Corollary 5.4.8. *Let $\mathscr{F}_M$ be some simple functorial family of semi-functional spaces on M, E a vector bundle over M and U an open subset of E. Then*

$$\{\varphi \in \mathscr{F}_M(E) \mid \operatorname{im}(\varphi) \subseteq U\}.$$

is an open subset of $\mathscr{F}_M(E)$.

Proof: Because $\mathscr{F}_M$ is a simple functorial family, the inclusion map

$$\imath \colon \mathscr{F}_M(E) \hookrightarrow \Gamma^0(M, E)$$

is continuous and $\{\varphi \in \mathscr{F}_M(E) \mid \operatorname{im}(\varphi) \subseteq U\}$ is the inverse image under $\imath$ of the open subset $\mathscr{U}$ of $\Gamma^0(M, E)$ that we have discussed in the previous lemma. □

Formally speaking, a chart of a Fréchet manifold $\mathscr{M}$ is a triple $(\mathscr{U}, \kappa, \mathscr{X})$ consisting of an open subset $\mathscr{U}$ of $\mathscr{M}$, a Fréchet space $\mathscr{X}$ and a homeomorphism κ between $\mathscr{U}$ and an open subset $\mathscr{X}$. So when we say that the $\mathscr{F}_M(E)$ with $E \in \mathrm{VSB}(P)$ will be the charts of $\mathscr{F}_M(P)$, we are not completely precise. What we actually mean is that the triples $(\mathscr{F}_M(E), \mathrm{id}_{\mathscr{F}_M(E)}, \mathscr{F}_M(E))$ will be the charts of $\mathscr{F}_M(P)$. That the charts can be of such a simple form is because the $\mathscr{F}_M(E)$ are simultaneously subsets of $\mathscr{F}_M(P)$ and Fréchet spaces. However, one should not be fooled by the apparent simplicity of the charts: in the first slot of $(\mathscr{F}_M(E), \mathrm{id}_{\mathscr{F}_M(E)}, \mathscr{F}_M(E))$, $\mathscr{F}_M(E)$ is seen as a subset of $\mathscr{F}_M(P)$ and therefore carries the restricted topology from $\mathscr{F}_M(P)$, while in the third slot $\mathscr{F}_M(E)$ is seen as Fréchet space and therefore carries its own locally convex topology. In other words, it is not a priori clear why $\mathrm{id}_{\mathscr{F}_M(E)}$ would be a homeomorphism. The following result takes care of this and simultaneously shows that $\mathscr{F}_M(E)$ is an open subset of $\mathscr{F}_M(P)$ (another requirement that needs to be fulfilled).

Proposition 5.4.9. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M and $E \in \mathrm{VSB}(P)$. The inclusion map $\mathscr{F}_M(E) \hookrightarrow \mathscr{F}_M(P)$ is an embedding onto an open subset of $\mathscr{F}_M(P)$.*

Proof: Considering the fact that we already know that the inclusion map is continuous and that it is clearly injective, we readily verify that the statement of the proposition follows if we can prove that every open subset $\mathscr{U}$ of $\mathscr{F}_M(E)$ is also open in $\mathscr{F}_M(P)$. So fix an open subset $\mathscr{U}$ of $\mathscr{F}_M(E)$ and let $F \in \mathrm{VSB}(P)$. To show that $\mathscr{U}$ is open in $\mathscr{F}_M(P)$ we have to show that $\mathscr{U} \cap \mathscr{F}_M(F)$ is open in $\mathscr{F}_M(F)$ (see Remark 5.4.3).

First of all, if $\mathscr{U} \cap \mathscr{F}_M(F)$ is empty there is nothing to prove, so suppose that $\varphi \in \mathscr{U} \cap \mathscr{F}_M(F) \subseteq \mathscr{F}_M(E) \cap \mathscr{F}_M(F)$. Then both E and F are a vector bundle neighborhood of φ in P, thus by Lemma 5.3.9 we find an injective fiber bundle homomorphism $f \colon F \to E$ such that f equals the identity on an open neighborhood V of $\operatorname{im}(\varphi)$ in F. Since E and F are open in P, $V \cap E$ is an open subset of both E and F, hence according to Corollary 5.4.8

$$\{\psi \in \mathscr{F}_M(E) \mid \operatorname{im}(\psi) \subseteq V \cap E\} \quad \text{and} \quad \{\psi \in \mathscr{F}_M(F) \mid \operatorname{im}(\psi) \subseteq V \cap E\}$$

are open subsets of respectively $\mathscr{F}_M(E)$ and $\mathscr{F}_M(F)$. But if $\psi \in \mathscr{F}_M(E)$ such that $\operatorname{im}(\psi) \subseteq V \cap E \subseteq F$, then F is a vector bundle neighborhood of ψ in P and $\psi \in \mathscr{F}_M(P)$, so by Lemma 5.4.4, $\psi \in \mathscr{F}_M(F)$. Using a similar argument for $\psi \in \mathscr{F}_M(F)$ with $\operatorname{im}(\psi) \subseteq V \cap E \subseteq E$, we deduce that:

$$\{\psi \in \mathscr{F}_M(E) \mid \operatorname{im}(\psi) \subseteq V \cap E\} = \{\psi \in \mathscr{F}_M(F) \mid \operatorname{im}(\psi) \subseteq V \cap E\}$$

and we denote this set, which is an open subset of both $\mathscr{F}_M(E)$ and $\mathscr{F}_M(F)$, by $\mathscr{V}$. We claim that $\mathscr{U} \cap \mathscr{V}$ is an open neighborhood of φ in $\mathscr{F}_M(F)$.

Clearly, $\varphi \in \mathscr{V}$ and since $\mathscr{U}$ is by assumption an open subset of $\mathscr{F}_M(E)$ that contains φ, we see that $\mathscr{U} \cap \mathscr{V}$ is an open neighborhood of φ in $\mathscr{F}_M(E)$. Now due to the fact that $\mathscr{F}_M$ is a simple functorial family, f_* restricts to a continuous map from $\mathscr{F}_M(F)$ into $\mathscr{F}_M(E)$, so $(f_*)^{-1}(\mathscr{U} \cap \mathscr{V})$ is an open subset of $\mathscr{F}_M(F)$. Moreover, because f is injective, f_* is injective and because f equals the identity on V, f_* equals the identity on $\mathscr{V}$. Using these properties of f_*, we obtain that $\mathscr{U} \cap \mathscr{V} = (f_*)^{-1}(\mathscr{U} \cap \mathscr{V})$ and hence that $\mathscr{U} \cap \mathscr{V}$ is open in $\mathscr{F}_M(F)$. Thus $\mathscr{U} \cap \mathscr{V}$ is indeed an open neighborhood of φ in $\mathscr{F}_M(F)$ and since $\mathscr{U} \cap \mathscr{V} \subseteq \mathscr{U} \cap \mathscr{F}_M(F)$ and φ was chosen arbitrarily, this proves that $\mathscr{U} \cap \mathscr{F}_M(F)$ is open in $\mathscr{F}_M(F)$. ☐

By using the just obtained information about the topological structure of $\mathscr{F}_M(P)$, we can neatly generalize Corollary 5.4.8 to fiber bundles.

Corollary 5.4.10. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M, P a fiber bundle over M and U an open subset of P. Then*

$$\mathscr{U} := \{\varphi \in \mathscr{F}_M(P) \mid \operatorname{im}(\varphi) \subseteq U\}.$$

is an open subset of $\mathscr{F}_M(P)$.

Proof: If $\mathscr{U}$ is empty there is nothing to prove, so suppose that $\psi \in \mathscr{U}$ and let E be a vector bundle neighborhood of ψ in P. Then $U \cap E$ is an open neigborhood of $\operatorname{im}(\psi)$ in E, hence

$$\mathscr{V} := \{\varphi \in \mathscr{F}_M(E) \mid \operatorname{im}(\varphi) \subseteq U \cap E\}$$

is an open neighborhood of ψ in $\mathscr{F}_M(E)$ (see Corollary 5.4.8). According to the previous proposition, $\mathscr{V}$ is then also an open neighborhood of ψ in $\mathscr{F}_M(P)$ and clearly $\mathscr{V} \subseteq \mathscr{U}$ (note that $\varphi \in \mathscr{F}_M(E)$ by definition implies $\varphi \in \mathscr{F}_M(P)$). So we have found an open neighborhood $\mathscr{V}$ of ψ in $\mathscr{F}_M(P)$ that is contained in $\mathscr{U}$ and because $\psi \in \mathscr{U}$ was chosen arbitrary, this proves that $\mathscr{U}$ is open. ☐

We are now ready to show that $\mathscr{F}_M(P)$ is a Fréchet manifold. It is understood that when we are talking about charts, $\mathscr{F}_M(E)$ actually represents the triple $(\mathscr{F}_M(E), \mathrm{id}_{\mathscr{F}_M(E)}, \mathscr{F}_M(E))$.

Theorem 5.4.11. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces over a compact manifold M. Then for every fiber bundle P over M, $\mathscr{F}_M(P)$ is a Fréchet manifold with $\{\mathscr{F}_M(E)\}_{E \in \mathrm{VSB}(P)}$ as atlas.*

Proof: Proposition 5.4.9 already shows that the $\mathscr{F}_M(E)$ with $E \in \mathrm{VSB}(P)$ are charts for $\mathscr{F}_M(P)$, so there are just two things that remain to be proven: that the collection $\{\mathscr{F}_M(E)\}_{E \in \mathrm{VSB}(P)}$ is an atlas and that $\mathscr{F}_M(P)$ is Hausdorff.

By definition $\{\mathscr{F}_M(E)\}_{E \in \mathrm{VSB}(P)}$ is a cover of $\mathscr{F}_M(P)$, thus to prove that this collection is an atlas, we only have to show that the transition functions are smooth. So let E, $F \in \mathrm{VSB}(P)$ and assume that $\mathscr{F}_M(E) \cap \mathscr{F}_M(F)$ is nonempty. The transition function is just the identity

$$\imath\colon \mathscr{F}_M(E) \cap \mathscr{F}_M(F) \to \mathscr{F}_M(E) \cap \mathscr{F}_M(F),$$

where in the domain $\mathscr{F}_M(E) \cap \mathscr{F}_M(F)$ is viewed as an open subset of the Fréchet space $\mathscr{F}_M(E)$, while in the codomain $\mathscr{F}_M(E) \cap \mathscr{F}_M(F)$ is viewed as an open subset of the Fréchet space $\mathscr{F}_M(F)$. Let $\varphi \in \mathscr{F}_M(E) \cap \mathscr{F}_M(F)$. Because smoothness is a local property, it suffices to find an open neighborhood $\mathscr{U}$ of φ in $\mathscr{F}_M(E) \cap \mathscr{F}_M(F) \subseteq \mathscr{F}_M(E)$ such that $\imath$ is a smooth map from $\mathscr{U}$ into $\mathscr{F}_M(F)$. To this end, let $f\colon E \to F$ be a fiber bundle homomorphism such that f equals the identity on an open neighborhood U of $\mathrm{im}(\varphi)$ in E (see Lemma 5.3.9). Then

$$\mathscr{U} := \{\psi \in \mathscr{F}_M(P) \mid \mathrm{im}(\psi) \subseteq U \cap F\}$$

is an open neighborhood of φ in $\mathscr{F}_M(P)$ (see the previous corollary) and using Lemma 5.4.4 we readily check that $\mathscr{U}$ is a subset of $\mathscr{F}_M(E) \cap \mathscr{F}_M(F)$ (indeed, if $\psi \in \mathscr{U}$, then E and F are vector bundle neighborhoods of ψ in P, hence $\psi \in \mathscr{F}_M(E)$ and $\psi \in \mathscr{F}_M(F)$). Moreover, since $\mathscr{F}_M(E) \subseteq_{\mathrm{c}} \mathscr{F}_M(P)$, $\mathscr{U}$ is also open in $\mathscr{F}_M(E)$. Finally, on behalf of Proposition 5.2.4, f_* restricts to a smooth map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$ and since f_* clearly coincides with $\imath$ on $\mathscr{U}$, we conclude that $\mathscr{U}$ has the desired properties.

To prove that $\mathscr{F}_M(P)$ is Hausdorff, take φ, $\psi \in \mathscr{F}_M(P)$ such that $\varphi \neq \psi$. Thanks to Corollary 5.4.10, it suffices to find open neighborhoods U and V of $\mathrm{im}(\varphi)$, respectively $\mathrm{im}(\psi)$, in P such that $\pi_P(U \cap V) \neq M$. Indeed, if we have such open neighborhoods, then

$$\begin{aligned} \mathscr{U} &:= \{\varphi' \in \Gamma^0(M,P) \mid \mathrm{im}(\varphi') \subseteq U\} \quad \text{and} \\ \mathscr{V} &:= \{\psi' \in \Gamma^0(M,P) \mid \mathrm{im}(\psi') \subseteq V\} \end{aligned}$$

are disjoint open neighborhoods of φ, respectively ψ, in $\Gamma^0(M,P)$. To find such open neighborhoods, let $x \in M$ such that $\varphi(x) \neq \psi(x)$. Since P is Hausdorff, we find disjoint open neighborhoods U_x and V_x of $\varphi(x)$, respectively $\psi(x)$, in P. Now because $W := M \setminus \{x\}$ is an open subset of M (after all, M is Hausdorff and $\{x\}$ is compact, so $\{x\}$ is closed), $P_W = \pi_P^{-1}(W)$ is an open subset of P and we claim that $U := U_x \cup P_W$ and $V := V_x \cup P_W$ have the desired properties. First of all, as unions of open sets, U and V are open. Second, we readily verify that $\mathrm{im}(\varphi) \subseteq U$ and $\mathrm{im}(\psi) \subseteq V$ (if $y \in M$ equals x, then $\varphi(y) = \varphi(x) \in U_x$ and $\psi(y) = \psi(x) \in V_x$ and if $y \neq x$, then $y \in W$ and thus $\varphi(y)$, $\psi(y) \in P_W$). Finally, since $U_x \cap V_x = \emptyset$,

$$\begin{aligned} \pi_P(U \cap V) &= \pi_P((U_x \cap V_x) \cup (U_x \cap P_W) \cup (V_x \cap P_W) \cup (P_W \cap P_W)) \\ &= \pi_P((U_x \cup V_x \cup P_W) \cap P_W) \subseteq \pi_P(P_W) = W, \end{aligned}$$

so $\pi_P(U \cap V) \neq M$. □

With the realization of $\mathscr{F}_M(P)$ as a Fréchet manifold, we have still not completely finished our 'quest'. After all, we promised that we were going to extend the simple functorial family $\mathscr{F}_M$ from the category of vector bundles over M to the category $\mathrm{FB}(M)$ of fiber bundles over M and so far we have only discussed the extension of the functor $\mathscr{F}_M$ to the objects of $\mathrm{FB}(M)$. So what about the arrows? Of course, we do not have much of a choice; in view of consistency with the vector bundle setting, a fiber bundle homomorphism $f\colon P \to Q$ (that is, an arrow in $\mathrm{FB}(M)$) between fiber bundles P and Q should be sent to the 'canonical arrow' f_*. The real question is whether f_* can be viewed as a suitable map between $\mathscr{F}_M(P)$ and $\mathscr{F}_M(Q)$.

Proposition 5.4.12. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M and let P and Q be fiber bundles over M. If $f\colon P \to Q$ is a fiber bundle homomorphism, then $f_*\colon \Gamma^0(M,P) \to \Gamma^0(M,Q)$ restricts to a smooth map from $\mathscr{F}_M(P)$ into $\mathscr{F}_M(Q)$.*

Proof: To prove that f_* restricts to a smooth map from $\mathscr{F}_M(P)$ into $\mathscr{F}_M(Q)$ it suffices to find for every $\varphi \in \mathscr{F}_M(P)$ an $E \in \mathrm{VSB}(P)$ and an $F \in \mathrm{VSB}(Q)$ such that $\varphi \in \mathscr{F}_M(E)$ and f_* restricts to a smooth map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$. So fix $\varphi \in \mathscr{F}_M(P)$. Using Theorem 5.3.5, we find a vector bundle neighborhood F of $f_*(\varphi) \in \Gamma^0(M,Q)$ in Q and a vector bundle neighborhood E of φ in P such that $E \subseteq f^{-1}(F)$ (note that $f^{-1}(F)$ is an open neighborhood of $\mathrm{im}(\varphi)$ in P). Then $E \in \mathrm{VSB}(P)$, $F \in \mathrm{VSB}(Q)$ and f restricts to a fiber bundle homomorphism from E into F. But then f_* restricts to a smooth map from $\mathscr{F}_M(E)$ into $\mathscr{F}_M(F)$ (see Proposition 5.2.4) and since by construction $\varphi \in \mathscr{F}_M(E)$, we are done. □

With the previous proposition our quest is really fulfilled: Definition 5.4.2, Theorem 5.4.11 and Proposition 5.4.12 together show how we can extend a simple functorial family $\mathscr{F}_M$ of semi-functional spaces on M, which is a functor from $\mathrm{VB}(M)$ into $\mathrm{Frechet}^0$, to a functor from $\mathrm{FB}(M)$ into the category $\mathrm{Frechet}_{\mathscr{M}}$ of Fréchet manifolds with smooth maps. Since for every $k \in \mathbb{N}_\infty$, $\{\Gamma^k(M,E)\}_{E\in\mathrm{VB}(M)}$ is a simple functorial family of functional spaces, we in particular see that the spaces of k-times continuously differentiable sections of P have the structure of Fréchet manifolds. If N is a finite dimensional manifold and we take P to be the trivial fiber bundle $M \times N$ over M, this gives the spaces $\mathscr{C}^k(M,N)$ of k-times continuously differentiable functions from M into N the structure of a Fréchet manifold.

Remark 5.4.13. What we call a Fréchet manifold is really a smooth Fréchet manifold (after all, we require the transition functions to be smooth). Nevertheless, even the space $\Gamma^0(M,P)$ of continuous sections of P has the structure of a smooth Fréchet manifold, so the differentiable structure of a space of sections is really something different than the differentiability of the sections! ⊘

Now that we know that $\Gamma^\infty(M,P)$ and $\Gamma^0(M,P)$ are Fréchet manifolds, Lemma 5.4.5 suddenly seems less satisfactory; we would not only like the inclusion maps to be continuous, we would like them to be smooth.

Proposition 5.4.14. *If $\mathscr{F}_M$ and $\mathscr{G}_M$ are simple functorial families of semi-functional spaces on M and $\mathscr{F}_M(E) \subseteq_{\mathrm{c}} \mathscr{G}_M(E)$ for every $E \in \mathrm{VB}(M)$, then for every $P \in \mathrm{FB}(M)$, $\mathscr{F}_M(P) \subseteq \mathscr{G}_M(P)$ and the inclusion map $\mathscr{F}_M(P) \hookrightarrow \mathscr{G}_M(P)$ is smooth.*

Proof: It is immediately clear from the definition of $\mathscr{F}_M(P)$ and $\mathscr{G}_M(P)$ as topological spaces and the properties of their (final) topologies that we have $\mathscr{F}_M(P) \subseteq_{\mathrm{c}} \mathscr{G}_M(P)$. Denote the inclusion map $\mathscr{F}_M(P) \hookrightarrow \mathscr{G}_M(P)$ by $\imath$. To prove that $\imath$ is smooth, it suffices to find for every $\varphi \in \mathscr{F}_M(P)$ a chart of $\mathscr{F}_M(P)$ that contains φ and a chart of $\mathscr{G}_M(P)$ that contains $\imath(\varphi)$ such that $\imath$ is smooth with respect to these charts. But this is trivial: if $\varphi \in \mathscr{F}_M(P)$, there exists an $E \in \mathrm{VSB}(P)$ such that $\varphi \in \mathscr{F}_M(E)$ and then $\mathscr{F}_M(E)$ is a chart of $\mathscr{F}_M(P)$ that contains φ and $\mathscr{G}_M(E)$ is a chart of $\mathscr{G}_M(P)$ that contains $\imath(\varphi) = \varphi$ (see Lemma 5.4.4), while $\imath$ is a continuous *linear* map with respect to these charts, hence smooth. □

Corollary 5.4.15. *Let $\mathscr{F}_M$ be a simple functorial family of semi-functional spaces on M. Then*

$$\mathscr{F}_M(P) \subseteq \Gamma^0(M, P)$$

and the inclusion map is smooth. If $\mathscr{F}_M$ is actually a family of functional spaces, then also

$$\Gamma^\infty(M, P) \subseteq \mathscr{F}_M(P)$$

with a smooth inclusion map.

Proof: This is a direct consequence of the previous proposition and the fact that $\Gamma^\infty(M, E) = \mathscr{E}(M, E) = \mathscr{D}(M, E)$ for every vector bundle E over our compact manifold M. □

Remark 5.4.16. For convenience, we have chosen to focus purely on functorial families of Fréchet semi-functional spaces and hence on Fréchet manifolds. However, because the notion of a smooth map is the same for Fréchet, Banach and Hilbert spaces, we can use precisely the same arguments to extend 'simple Banach' functorial families and 'simple Hilbert' functorial families to fiber bundles with Banach, respectively, Hilbert manifolds as result. ⊘

5.5 Noncompact manifolds

One might wonder if it is really necessary to restrict our attention to compact manifolds for the 'extension procedure' for simple functorial families. Our arguments clearly rely on the assumed compactness of the manifold M, but this in itself is not enough to abandon noncompact manifolds; after all, we might be able to find alternative proofs or lemmas to deal with the noncompact case. However, the problem is not just in the supporting lemmas and their proofs, it is already present in the intuitive idea behind the 'extension procedure'.

Recall that this intuitive idea can be summarized as follows: the procedure to extend a simple functorial family $\mathscr{F}_M$ to the category of fiber bundles over M, is to cover a fiber bundle P with 'patches' E that look like vector bundles and to take $\mathscr{F}_M(P)$ to be the collection of those elements of $\Gamma^0(M, P)$ that are of 'type $\mathscr{F}$' on these patches. This is a very natural approach and on the intuitive level it is very similar to our 'extension procedure' from $\mathbb{R}^n$ to vector bundles. When it comes to the formal definition one might debate about the question whether this should be

$$\mathscr{F}_M(P) := \{\varphi \in \Gamma^0(M, P) \mid \exists_{E \in \mathrm{VSB}(P)}\ \varphi \in \mathscr{F}_M(E)\}$$

or

$$\mathscr{F}_M(P) := \{\varphi \in \Gamma^0(M, P) \mid \forall_{E \in \mathrm{VSB}(P)}\ \mathrm{im}(\varphi) \subseteq E \Rightarrow \varphi \in \mathscr{F}_M(E)\}$$

(we have chosen the first option), but in view of Theorem 5.3.5 one expects these to be equivalent. Our current proof of the fact that this is indeed the case uses the compactness of M via Lemma 5.3.9, but it does not seem to be a problem to prove this lemma without assuming M to be compact: we first make a 'noncompact version' of Lemma 5.3.8, in which ε will be a smoothly varying positive function $\varepsilon(x)$ on M instead of a fixed $\varepsilon > 0$, and then we simply use a 'scaling function' ζ in the proof of Lemma 5.3.9 that also depends smoothly on x (so that ζ_x 'rescales' $[0,\infty)$ to something inside $[0,\frac{3}{4}\varepsilon(x))$). Moreover, even without adjusting any of our proofs, we directly see from Theorem 5.3.5 that when $\mathscr{F}_M$ equals Γ^0_M the two proposed definitions *are* equivalent, so for Γ^0_M it certainly does not matter which of two definitions we choose.

The argumentation above makes it very plausible that the formal definition of $\mathscr{F}_M(P)$ that we have chosen follows unambigously from the intuitive idea that we use for the extension. Well, the formal definition of $\mathscr{F}_M(P)$ as a set that is. What about the topology? We *chose* to endow $\mathscr{F}_M(P)$ with the final topology with respect to the inclusion maps $\mathscr{F}_M(E) \hookrightarrow \mathscr{F}_M(P)$ with $E \in \mathrm{VSB}(P)$. Well, apart from the fact that this seems to be the most natural topology in the first place, if we want to have any chance of $\mathscr{F}_M(P)$ being a Fréchet manifold with $\{\mathscr{F}_M(E)\}_{E\in\mathrm{VSB}(P)}$ as atlas (which is also a central idea in our extension procedure), we *have* to choose this final topology. Indeed, if $\{\mathscr{F}_M(E)\}_{E\in\mathrm{VSB}(P)}$ has to be an atlas for $\mathscr{F}_M(P)$, we should in particular have that for every $E \in \mathrm{VSB}(P)$ the inclusion map $\mathscr{F}_M(E) \hookrightarrow \mathscr{F}_M(P)$ is an embedding onto an open subset of $\mathscr{F}_M(P)$ (after all, the chart domains should be open and the identity maps

$$\mathscr{F}_M(P) \supseteq \mathscr{F}_M(E) \to \mathscr{F}_M(E)$$

should be homeomorphisms). As a consequence, $\mathscr{U} \subseteq \mathscr{F}_M(P)$ should be open in $\mathscr{F}_M(P)$ if and only if for every $E \in \mathrm{VSB}(P)$, $\mathscr{U} \cap \mathscr{F}_M(E)$ is open in $\mathscr{F}_M(E)$, which is precisely the characterization of the final topology with respect to the inclusion maps $\mathscr{F}_M(E) \hookrightarrow \mathscr{F}_M(P)$.

So it is quite safe to say that the formal definition of $\mathscr{F}_M(P)$ as a topological space that we gave in Definition 5.4.2 follows very naturally from our intuitive extension approach. However, when we want to apply this to noncompact manifolds, the theory very quickly ends after giving Definition 5.4.2. Even when $\mathscr{F}_M$ equals Γ^0_M it is in general not true that the intersections $\mathscr{F}_M(E)\cap\mathscr{F}_M(F)$, with $E\,F \in \mathrm{VSB}(P)$, are open; something which is essential if we want $\mathscr{F}_M(P)$ to be a Fréchet manifold with $\{\mathscr{F}_M(E)\}_{E\in\mathrm{VSB}(P)}$ as atlas. Indeed, if $\varphi \in \Gamma^0(M,E)\cap\Gamma^0(M,F)$, then $\Gamma^0(M,E)\cap\Gamma^0(M,F)$ open in $\Gamma^0(M,E)$ implies that there exist $K \in \mathscr{P}_{\mathrm{c}}(M)$ and $\varepsilon > 0$ such that all $\psi \in \Gamma^0(M,E)$ with $\sup_{x\in K} |\psi(x)|^g < \epsilon$ (with g some vector bundle metric on E) have their image in F, while it is clear that forcing $\psi \in \Gamma^0(M,E)$ to be 'close' to φ on a compact piece of the base manifold is not enough to ensure that $\psi(x)$ lies in F for all $x \in M$. For an explicit counterexample, one can just take $M = \mathbb{R}$, $P = \mathbb{R}\times\mathbb{R}$, $E = \mathbb{R}\times(-2,2)$ and $F = \mathbb{R}\times(-1,1)$ (where E and F are given the structure of a vector bundle by using that $(-2,2) \simeq \mathbb{R}$ and $(-1,1) \simeq \mathbb{R}$). Then $\mathbf{0}\colon x \mapsto (x,0)$ is in the intersection $\Gamma^0(M,E)\cap\Gamma^0(M,F)$, while for every compact $K \in \mathscr{P}_{\mathrm{c}}(M)$ and $\varepsilon > 0$ we have continuous sections of E that are ε-close to 0 on K, but that run out of F when leaving K.

Putting everything together, we see that we should really find a different approach for extending functorial families to fiber bundles if we want to include noncompact manifolds or that we should just be satisfied with ordinary topological spaces as extended objects instead of fancy infinite dimensional manifolds.

5.6 From $\mathbb{R}^n$ to fiber bundles

Although we have formulated the 'extension procedure' of Section 5.4 for arbitrary simple functorial families, in practice almost all interesting examples are functorial families that are 'modeled after' a local invariant functional space $\mathscr{F}$ on $\mathbb{R}^n$. In view of the principle of using familiar and well-behaved 'solution spaces' on $\mathbb{R}^n$ to obtain nice 'solution spaces' in settings that are geometrically more challenging, it is natural to ask whether we can combine the procedure discussed in Section 4.6 with the extension procedure of Section 5.4 to bring suitable solution spaces on $\mathbb{R}^n$ to the setting of global nonlinear analysis.

Of course, we could just start with a local invariant functional space $\mathscr{F}$ on $\mathbb{R}^n$, turn it into a functorial family $\mathscr{F}_M$ and then check whether $\mathscr{F}_M$ is simple. However, this is not very convenient. Instead, we would like to have a condition that determines directly whether a local invariant functional space $\mathscr{F}$ is suitable to serve as a model for infinite dimensional 'solution manifolds'.

Definition 5.6.1. Let $\mathscr{F}$ be a semi-functional space on Ω. We say that $\mathscr{F}$ is *simple* if:

1. $\mathscr{F}$ is Fréchet and $\mathscr{F} \subseteq_c \mathscr{C}(\Omega)$ and
2. for every $m \in \mathbb{N}$ and every smooth function $f\colon \mathbb{K}^m \to \mathbb{K}$,

$$(\mathscr{C}(\Omega))^m \to \mathscr{C}(\Omega)\colon (\varphi_1, \ldots, \varphi_m) \mapsto f(\varphi_1, \ldots, \varphi_m)$$

restricts to a continuous function from $\mathscr{F}^m$ into $\mathscr{F}$. ⊘

It is not difficult to see that this notion of simplicity is precisely the condition that we were looking for. Following the same approach as in the proof of Proposition 4.6.8, we see that if $\mathscr{F}$ is a simple local invariant functional space on $\mathbb{R}^n$, $\{\mathscr{F}(M,E)\}_{E\in \mathrm{VB}(M)}$ is a simple functorial family of functional spaces on M (recall that the property of being Fréchet is preserved by the construction functor $\mathscr{F} \mapsto \mathscr{F}(M,E)$). Combining the material of Section 4.6 and Section 5.4 then leads to the following theorem.

Theorem 5.6.2. *Let $\mathscr{F}$ be a simple local invariant functional space on $\mathbb{R}^n$ and let M be an n-dimensional compact manifold. For every fiber bundle P over M, there exists a Fréchet manifold $\mathscr{F}(M,P)$ of continuous sections of P of 'type $\mathscr{F}$'. Moreover, if Q is also a fiber bundle over M and $f\colon P \to Q$ is a fiber bundle homomorphism, then f_* restricts to a smooth map from $\mathscr{F}(M,P)$ into $\mathscr{F}(M,Q)$.*

Remark 5.6.3. Because the construction functor $\mathscr{F} \mapsto \mathscr{F}(M,E)$ also preserves being locally Banach and being locally Hilbert, starting with a simple local invariant functional space $\mathscr{F}$ on $\mathbb{R}^n$ that is locally Banach or locally Hilbert, would result in a family $\{\mathscr{F}(M,P)\}_{P\in \mathrm{FB}(M)}$ of Banach, respectively, Hilbert manifolds of continuous sections of 'type $\mathscr{F}$'. ⊘

Appendices

Appendix A

Locally convex vector spaces

In the main text we need quite a few things from the theory of locally convex vector spaces (a topic in functional analysis) and in this appendix we have collected a significant part of the required material. Nevertheless, it is still assumed that the reader already has some basic knowledge of functional analysis and locally convex vector spaces. For example, the two major equivalent definitions of a locally convex vector space and the concept of a (Cauchy) net are assumed to be known. Actually, this appendix is above all a weird mixture of very elementary results that do not appear in the desired form in one of our 'standard' references and precise statements of more complicated results together with a reference for the proof. For a decent introduction to functional analysis and locally convex vector spaces, we refer to [2] or [13].

A.1 Continuity

As we assume to be known, the topology of a locally convex vector space $\mathscr{X}$ is induced by a collection of seminorms on $\mathscr{X}$. Usually there are many different collections of seminorms that induce the same topology and a collection of seminorms on $\mathscr{X}$ that induces its topology will simply be called an *inducing collection of seminorms for* $\mathscr{X}$.

Lemma A.1.1. *Let $\mathscr{X}$ be a locally convex vector space and let $\mathscr{P}$ be an inducing collection of seminorms for $\mathscr{X}$. A seminorm $p\colon \mathscr{X} \to \mathbb{R}$ is continuous if and only if there exist $C \geq 0$ and $p_0, \ldots, p_n \in \mathscr{P}$ such that*

$$p(x) \leq C \sum_{i=0}^{n} p_i(x)$$

for every $x \in \mathscr{X}$.

Proof: Suppose that p is continuous. Then $p^{-1}((-1,1))$ is an open neighborhood of 0 in $\mathscr{X}$, so there are $p_0, \ldots, p_n \in \mathscr{P}$ and $\varepsilon_0, \ldots, \varepsilon_n > 0$ such that $\cap_{i=0}^{n} B_{\varepsilon_i}^{p_i}(0) \subseteq p^{-1}((-1,1))$, where $B_{\varepsilon_i}^{p_i}(0) := \{x \in \mathscr{X} \mid p_i(x) < \varepsilon_i\}$. Fix $x \in \mathscr{X}$. Clearly, for every $\delta > 0$

$$y := \frac{(\min_{0 \leq i \leq n} \varepsilon_i) x}{\delta + \sum_{i=0}^{n} p_i(x)}$$

satisfies $p_i(y) < \varepsilon_i$ for every $0 \leq i \leq n$, hence $y \in \cap_{i=0}^{n} B_{\varepsilon_i}^{p_i}(0) \subseteq p^{-1}((-1,1))$. Using this, we find that for every $\delta > 0$

$$p(x) < \frac{\delta + \sum_{i=0}^{n} p_i(x)}{\min_{0 \leq i \leq n} \varepsilon_i}$$

and taking the limit $\delta \to 0$ then shows

$$p(x) \leq \frac{1}{\min_{0 \leq i \leq n} \varepsilon_i} \sum_{i=0}^{n} p_i(x).$$

Since x was chosen arbitrarily, this proves that we have the desired estimate.

Next, suppose that there exist $C \geq 0$ and $p_0, \ldots, p_n \in \mathscr{P}$ such that

$$p(x) \leq C \sum_{i=0}^{n} p_i(x)$$

for every $x \in \mathscr{X}$. To prove that p is continuous, let $\{x_j\}_{j \in J}$ be a net in $\mathscr{X}$ and $x \in \mathscr{X}$ such that $x_j \to x$ in $\mathscr{X}$. Then $p_i(x - x_j) \to 0$ (if $j \to \infty$) for every $0 \leq i \leq n$ and using the above estimate, this implies that $p(x - x_j) \to 0$. Hence, by the reverse triangle inequality, $|p(x) - p(x_j)| \to 0$, which precisely means that $p(x_j) \to p(x)$ in $\mathbb{R}$. □

One of the key features of locally convex vector spaces is that the continuity of linear maps can be expressed in terms of seminorms.

Lemma A.1.2. *Let $\mathscr{X}$ and $\mathscr{Y}$ be locally convex vector spaces, let $\mathscr{P}$ and $\mathscr{Q}$ be inducing collections of seminorms for $\mathscr{X}$, respectively $\mathscr{Y}$, and let $T\colon \mathscr{X} \to \mathscr{Y}$ be a linear map. Then T is continuous if and only if for every $q \in \mathscr{Q}$ there exist $C \geq 0$ and $p_0, \ldots, p_n \in \mathscr{P}$ such that*

$$q(T(x)) \leq C \sum_{i=0}^{n} p_i(x)$$

for every $x \in \mathscr{X}$.

Proof: The direct implication is a straight consequence of the previous lemma and the observation that if T is continuous, $q \circ T$ is a continuous seminorm on $\mathscr{X}$ for every $q \in \mathscr{Q}$ (note that seminorms from an inducing collection are always continuous). For the converse implication, let $\{x_j\}_{j \in J}$ be a net in $\mathscr{X}$ and $x \in \mathscr{X}$ such that $x_j \to x$ in $\mathscr{X}$. To prove that T is continuous, we should prove that $T(x_j) \to T(x)$ in $\mathscr{Y}$, which is equivalent to the statement that $q(T(x) - T(x_j)) = q(T(x - x_j)) \to 0$ in $\mathbb{R}$ for every $q \in \mathscr{Q}$, which in turn trivially follows from the assumed existence of estimates and the fact that $p(x - x_j) \to 0$ in $\mathbb{R}$ for every $p \in \mathscr{P}$. □

Corollary A.1.3. *Let $\mathscr{X}$ and $\mathscr{Y}$ be locally convex vector spaces, let $\mathscr{Q}$ be an inducing collection of seminorms for $\mathscr{Y}$ and let $T\colon \mathscr{X} \to \mathscr{Y}$ be a linear map. Then T is continuous if and only if for every $q \in \mathscr{Q}$ there exists a continuous seminorm p on $\mathscr{X}$ such that*

$$q(T(x)) \leq p(x)$$

for every $x \in \mathscr{X}$.

Proof: The direct implication follows from the previous lemma and the fact that for every $C \geq 0$ and $p_0, \ldots, p_n \in \mathscr{P}$, $p := C \sum_{i=0}^{n} p_i$ is a continuous seminorm on $\mathscr{X}$. The converse implication is a simple combination of the previous two lemmas: first use Lemma A.1.1 to get an estimate for p, then combine this estimate with $q \circ T \leq p$ and apply Lemma A.1.2. □

Using the characterization of continuity given in Lemma A.1.2, we can give a useful criterion to determine whether or not two collections of seminorms induce the same topology.

Corollary A.1.4. *Let $\mathscr{X}$ be a vector space and let $\mathscr{P}$ and $\mathscr{P}'$ be collections of seminorms on $\mathscr{X}$. Then $\mathscr{P}$ and $\mathscr{P}'$ induce the same locally convex topology on $\mathscr{X}$ if and only if*

1. *for every $p' \in \mathscr{P}'$ there exist seminorms $p_0, \ldots, p_n \in \mathscr{P}$ and a constant $C \geq 0$ such that $p' \leq C \sum_{i=0}^{n} p_i$ and*
2. *for every $p \in \mathscr{P}$ there exist seminorms $p'_0, \ldots, p'_{n'} \in \mathscr{P}'$ and a constant $C' \geq 0$ such that $p \leq C' \sum_{i=0}^{n'} p'_i$.*

Proof: Let $\mathscr{T}_{\mathscr{P}}$ be the topology induced by $\mathscr{P}$ and let $\mathscr{T}_{\mathscr{P}'}$ be the topology induced by $\mathscr{P}'$. Then $\mathscr{T}_P = \mathscr{T}_{\mathscr{P}'}$ if and only if both $\mathrm{id}_{\mathscr{X}} : (\mathscr{X}, \mathscr{T}_P) \to (\mathscr{X}, \mathscr{T}_{\mathscr{P}'})$ and $\mathrm{id}_{\mathscr{X}} : (\mathscr{X}, \mathscr{T}_{P'}) \to (\mathscr{X}, \mathscr{T}_{\mathscr{P}})$ are continuous, which translates into the desired result via Lemma A.1.2. □

Remark A.1.5. With a *locally convex topology* on a vector space $\mathscr{X}$, we always mean a topology on $\mathscr{X}$ that turns $\mathscr{X}$ into a locally convex vector space. So, despite of the fact that the term 'locally convex topology' is not explicit about this, it is always assumed that addition and scalar multiplication are turned into continuous maps by locally convex topologies. ⊘

Corollary A.1.6. *Let $\mathscr{X}$ be a locally convex vector space and let $\mathscr{P}'$ be an inducing collection of seminorms for $\mathscr{X}$. If $\mathscr{P} \subseteq \mathscr{P}'$ such that for every $p' \in \mathscr{P}'$ there exists an $p \in \mathscr{P}$ with $p' \leq p$, then $\mathscr{P}$ also induces the topology of $\mathscr{X}$.*

Proof: $\mathscr{P}$ and $\mathscr{P}'$ trivially satisfy the criterion given by the previous result. □

Another useful consequence of Lemma A.1.2 is the following:

Lemma A.1.7. *A continuous linear map between locally convex vector spaces sends Cauchy nets to Cauchy nets.*

Proof: Let $\mathscr{X}$ and $\mathscr{Y}$ be locally convex vector spaces, let $\mathscr{P}$ and $\mathscr{Q}$ be inducing collections of seminorms for $\mathscr{X}$, respectively $\mathscr{Y}$, let $T : \mathscr{X} \to \mathscr{Y}$ be a linear map and let $\{x_j\}_{j \in J}$ be a Cauchy net in $\mathscr{X}$. We need to prove that $\{Tx_j\}_{j \in J}$ is a Cauchy net in $\mathscr{Y}$, hence that for every $q \in \mathscr{Q}$ and $\varepsilon > 0$ there exists an $j_\infty \in J$ such that $q(T(x_{j'}) - T(x_j)) < \varepsilon$ for all $j, j' \geq j_\infty$.

Fix $q \in \mathscr{Q}$ and $\varepsilon > 0$. By Lemma A.1.2, we find $C \geq 0$ and $p_0, \ldots, p_n \in \mathscr{P}$ such that $q \circ T \leq C \sum_{i=0}^{n} p_i$ and because $\{x_j\}_{j \in J}$ is a Cauchy net in $\mathscr{X}$, we find for every $0 \leq i \leq n$ an $j_i \in J$ such that $p_i(x_{j'} - x_j) < \frac{\varepsilon}{(n+1)(C+1)}$ for all $j, j' \geq j_i$. Now let j_∞ be any element of J such that $j_i \leq j_\infty$ for all $0 \leq i \leq n$ (such elements exist because J is a directed set). Then for all $j, j' \geq j_\infty$

$$\begin{aligned} q(T(x_{j'}) - T(x_j)) = q(T(x_{j'} - x_j)) &\leq C \sum_{i=0}^{n} p_i(x_{j'} - x_j) \\ &< C \sum_{i=0}^{n} \frac{\varepsilon}{(n+1)(C+1)} < \varepsilon. \end{aligned}$$ □

The next result is in fact true for arbitrary topological vector spaces, but since we want to emphasize that we only work with locally convex ones, we formulate it in terms of locally convex vector spaces anyway.

Lemma A.1.8. *If $\mathscr{X}$ and $\mathscr{Y}$ are locally convex vector spaces and $I\colon \mathscr{X} \to \mathscr{Y}$ and $P\colon \mathscr{Y} \to \mathscr{X}$ are continuous linear maps such that $P \circ I = \mathrm{id}_{\mathscr{X}}$, then I is a linear topological embedding with closed image.*

Proof: It is clear from $P \circ I = \mathrm{id}_{\mathscr{X}}$ that I is injective and by assumption I is continuous and linear, thus in order to prove that I is a linear topological embedding it only remains to be shown that I is an open map from $\mathscr{X}$ onto $I(\mathscr{X})$. So let U be an open subset of $\mathscr{X}$. Then, due to the continuity of P, $P^{-1}(U)$ is an open subset of $\mathscr{Y}$ and consequently $I(\mathscr{X}) \cap P^{-1}(U)$ is an open subset of $I(\mathscr{X})$. Since we readily check that $I(U) = I(\mathscr{X}) \cap P^{-1}(U)$, this implies that I is indeed an open map from $\mathscr{X}$ onto $I(\mathscr{X})$, hence a linear topological embedding. To see that $I(\mathscr{X})$ is closed in $\mathscr{Y}$, we simply observe that $I(\mathscr{X}) = \ker(\mathrm{id}_{\mathscr{Y}} - I \circ P)$. □

A.2 Density of subspaces

Lemma A.2.1. *A subspace A of a locally convex vector space $\mathscr{X}$ is dense in $\mathscr{X}$ if and only if every continuous linear functional $T\colon \mathscr{X} \to \mathbb{K}$ that vanishes on A is identically zero.*

Proof: See [2, Corollary IV.3.14]. □

Just to be clear: the remaining results of this section are all valid for arbitrary topological vector spaces.

Lemma A.2.2. *Let $\mathscr{Y}$ be a locally convex vector space, let $\mathscr{X}$ be a linear subspace of $\mathscr{Y}$ endowed with a locally convex topology such that $\mathscr{X} \subseteq_{\mathrm{c}} \mathscr{Y}$ and let A be a subset of $\mathscr{X}$. If A is dense in $\mathscr{X}$ and $\mathscr{X}$ is dense in $\mathscr{Y}$, then A is also dense in $\mathscr{Y}$.*

Proof: Let U be a nonempty open subset of $\mathscr{Y}$. By the continuity of the inclusion $\mathscr{X} \subseteq \mathscr{Y}$, $\mathscr{X} \cap U$ is open in $\mathscr{X}$ and since $\mathscr{X}$ is dense in $\mathscr{Y}$, $\mathscr{X} \cap U$ is in fact a nonempty open subset of $\mathscr{X}$. So, due to the fact that A is dense in $\mathscr{X}$, there must be some $a \in A$ such that $a \in \mathscr{X} \cap U \subseteq U$ and we conclude that every nonempty open subset of $\mathscr{Y}$ has a nonempty intersection with A. □

Lemma A.2.3. *Let $\mathscr{X}$ be a locally convex vector space and let U be an open neighborhood of the origin in $\mathscr{X}$. Then there exists for every $x \in \mathscr{X}$, an $m \in \mathbb{N}$ such that $x \in mU$.*

Proof: Fix $x \in \mathscr{X}$. As a consequence of the fact that scalar multiplication is a continuous map from $\mathbb{K} \times \mathscr{X}$ into $\mathscr{X}$, the sequence $\{\frac{1}{n}x\}_{n\in\mathbb{N}}$ converges to 0 in $\mathscr{X}$. Hence, we find an $m \in \mathbb{N}$ such that $\frac{1}{n}x \in U$ for every $n \geq m$, which in particular implies that $x \in mU$. □

Proposition A.2.4. *A proper subspace of a locally convex vector space has empty interior.*

Proof: Let $\mathscr{X}$ be a locally convex vector space and let A be a subspace of $\mathscr{X}$. Suppose that A has nonempty interior. Then we find an $a \in A$ and an open subset U of $\mathscr{X}$ such that $U \subseteq A$ and $a \in U$. By the continuity of subtraction and the fact that A is a subspace, we see that $V := U - \{a\}$ is an open neighborhood of the origin in $\mathscr{X}$ that is entirely contained in A. By the previous lemma, $\cup_{n\in\mathbb{N}} nV$ must then be equal to $\mathscr{X}$ and since $\cup_{n\in\mathbb{N}} nV$ must also be contained in A, we conclude that $A = \mathscr{X}$ (i.e., A is not a proper subspace). □

Corollary A.2.5. *A subspace A of a locally convex vector space $\mathscr{X}$ is either dense or nowhere dense (where the latter means that the closure of A has empty interior).*

Proof: The closure $\mathrm{cl}(A)$ of A is again a subspace of $\mathscr{X}$ and if A is not dense, $\mathrm{cl}(A)$ is a proper subspace of $\mathscr{X}$ and therefore has empty interior. □

A.3 Inductive limits

Definition A.3.1. Let $\mathscr{X}$ be a vector space and let $\{\mathscr{X}_i\}_{i\in I}$ be a family of vector subspaces of $\mathscr{X}$. Suppose furthermore that each $\mathscr{X}_i$ is equipped with some locally convex topology and that $\mathscr{X} = \cup_{i\in I}\mathscr{X}_i$. Then we define the *inductive limit topology* on $\mathscr{X}$ (relative to the family $\{\mathscr{X}_i\}_{i\in I}$) to be the largest locally convex topology such that $\mathscr{X}_i \subseteq_{\mathrm{c}} \mathscr{X}$ for every $i \in I$. When $\mathscr{X}$ is equipped with this topology, we say that $\mathscr{X}$ is the *inductive limit* of $\{\mathscr{X}_i\}_{i\in I}$ and we sometimes write $\mathscr{X} = \lim_i \mathscr{X}_i$. ⊘

Of course, we should explain why such a topology always exists, but this is quite easy. First of all, there always exists a locally convex topology on $\mathscr{X}$ such that $\mathscr{X}_i \subseteq_{\mathrm{c}} \mathscr{X}$ for every $i \in I$ (for example, the trivial topology). Now let $\{\mathscr{T}_j\}_{j\in J}$ be the collection of all locally convex topologies on $\mathscr{X}$ with this property and let $\mathscr{T}$ be the topology on $\mathscr{X}$ generated by $\cup_{j\in J}\mathscr{T}_j$. Then every element of $\mathscr{T}$ is a union of finite intersections of elements of $\cup_{j\in J}\mathscr{T}_j$. Because taking inverse images 'commutes' with unions and intersections, we directly see that $\mathscr{X}$ equipped with $\mathscr{T}$ is a topological vector space that still satisfies $\mathscr{X}_i \subseteq_{\mathrm{c}} \mathscr{X}$ for every $i \in I$. Moreover, because an intersection of convex sets is again convex, we see that $\mathscr{T}$ is in fact a locally convex topology on $\mathscr{X}$. That $\mathscr{T}$ is the largest locally convex topology on $\mathscr{X}$ with the desired property is an obvious consequence of its definition.

Proposition A.3.2. *Let $\mathscr{X}$ be a vector space and let $\{\mathscr{X}_i\}_{i\in I}$ be a family of vector subspaces of $\mathscr{X}$, each equipped with some locally convex topology, such that $\mathscr{X} = \cup_{i\in I}\mathscr{X}_i$. Furthermore, let $\mathscr{Y}$ be a locally convex vector space and let $T\colon \mathscr{X} \to \mathscr{Y}$ be a linear map. If $\mathscr{X}$ is equipped with the inductive limit topology (relative to the family $\{\mathscr{X}_i\}_{i\in I}$), then T is continuous if and only if $T|_{\mathscr{X}_i} : \mathscr{X}_i \to \mathscr{Y}$ is continuous for every $i \in I$.*

Proof: Since $\mathscr{X}_i \subseteq_{\mathrm{c}} \mathscr{X}$ for every $i \in I$, the direct implication is clear. For the converse implication it suffices to show that T is continuous at zero, which means that it suffices to show that for every neighborhood V of the origin in $\mathscr{Y}$, $T^{-1}(V)$ is a neighborhood of the origin in $\mathscr{X}$ and because $\mathscr{Y}$ is locally convex we may even assume that V is convex. But if V is convex, also $T^{-1}(V)$ is convex and it is an easy consequence of the definition of the inductive limit topology that a convex subset W of $\mathscr{X}$ is a neighborhood of the origin in $\mathscr{X}$ if and only if for every $i \in I$, $\mathscr{X}_i \cap W$ is a neighborhood of the origin in $\mathscr{X}_i$. Therefore, the observation that for every $i \in I$, $\mathscr{X}_i \cap T^{-1}(V) = \left(T|_{\mathscr{X}_i}\right)^{-1}(V)$, does the job.□

Definition A.3.3. Let $\mathscr{X}$ be a vector space and let $\{\mathscr{X}_n\}_{n\in\mathbb{N}}$ be a chain

$$\mathscr{X}_0 \subsetneq \mathscr{X}_1 \subsetneq \cdots \subsetneq \mathscr{X}_n \subsetneq \mathscr{X}_{n+1} \subsetneq \ldots$$

of vector subspaces of $\mathscr{X}$ that carry some locally convex topology such that $\mathscr{X} = \cup_{n\in\mathbb{N}} \mathscr{X}_n$ and such that for every $n \in \mathbb{N}$, the inclusion map $\mathscr{X}_n \hookrightarrow \mathscr{X}_{n+1}$ is a linear topological embedding with closed image. Then the inductive limit topology on $\mathscr{X}$ relative to the family $\{\mathscr{X}_n\}_{n\in\mathbb{N}}$ is called a *strict inductive limit topology* and when $\mathscr{X}$ is equipped with this topology, we say that $\mathscr{X}$ is the *strict inductive limit* of $\{\mathscr{X}_n\}_{n\in\mathbb{N}}$. ⊘

Proposition A.3.4. *Let $\mathscr{X}$ and $\{\mathscr{X}_n\}_{n\in\mathbb{N}}$ be as in the definition above. If $\mathscr{X}$ carries the strict inductive limit topology, a subset B of $\mathscr{X}$ is bounded if and only if there exists an $n \in \mathbb{N}$ such that B is a bounded subset of $\mathscr{X}_n$.*

Proof: See [2, Proposition IV.5.16]. □

A.4 Duality

Definition A.4.1. Let $\mathscr{X}$ be a locally convex vector space. The *continuous dual* $\mathscr{X}^*$ of $\mathscr{X}$ is by definition the set of all continuous linear maps from $\mathscr{X}$ into $\mathbb{K}$. ⊘

There are multiple ways to topologize $\mathscr{X}^*$, but in this text $\mathscr{X}^*$ will always carry the so-called *strong topology*, which we will now introduce. To this end, let $\mathscr{P}$ be an inducing collection of seminorms for $\mathscr{X}$ and let B be a bounded subset of $\mathscr{X}$. Then there exists for every $p \in \mathscr{P}$ an $r > 0$ such that $B \subseteq B^p_r(0)$. Hence, on behalf of Lemma A.1.2, $\{|u(x)| \mid x \in B\}$ is a bounded subset of $\mathbb{R}$ for every $u \in \mathscr{X}^*$ and thus

$$q_B\colon \mathscr{X}^* \to \mathbb{R}\colon u \mapsto \sup_{x\in B} |u(x)|$$

is a well-defined map. We easily check that q_B is in fact a seminorm on $\mathscr{X}^*$ and the strong topology on $\mathscr{X}^*$ is by definition the topology induced by the collection of seminorms $\{q_B \mid B \text{ a bounded subset of } \mathscr{X}\}$. When $\mathscr{X}^*$ carries this topology, it is called the *strong dual* of $\mathscr{X}$ and, as said before, in this text $\mathscr{X}^*$ will actually always denote the strong dual of $\mathscr{X}$.

Lemma A.4.2. *The strong dual of a locally convex vector space is Hausdorff.*

Proof: It is a trivial exercise to check that a locally convex vector space $\mathscr{X}$ with an inducing collection of seminorms $\mathscr{P}$ is Hausdorff if and only if

$$\bigcap_{p\in\mathscr{P}} \{x \in \mathscr{X} \mid p(x) = 0\} = \{0\}.$$

So in order to check that the strong dual $\mathscr{X}^*$ of a locally convex vector space $\mathscr{X}$ is Hausdorff, we should check that for $u \in \mathscr{X}^*$, $q_B(u) = 0$ for every bounded subset B of $\mathscr{X}$ implies $u = 0$. For this, just note that for every $x \in \mathscr{X}$, $\{x\}$ is a bounded subset of $\mathscr{X}$. □

Definition A.4.3. Let $\mathscr{X}$, $\mathscr{Y}$ be locally convex vector spaces and $T\colon \mathscr{X} \to \mathscr{Y}$ a continuous linear map. Then

$$T^*\colon \mathscr{Y}^* \to \mathscr{X}^*\colon v \mapsto v \circ T$$

is a well-defined linear map and we call T^* the *adjoint* of T. ⊘

It follows in fact automatically that the adjoint T^* is continuous: first observe that if B is a bounded subset of $\mathscr{X}$, $T(B)$ is a bounded subset of $\mathscr{Y}$ (this is a direct consequence of Lemma A.1.2, but is also very easily proven in the more general context of topological vector spaces) and then combine Lemma A.1.2 with the observation that

$$q_B(T^*v) = \sup_{x\in B} |(T^*v)(x)| = \sup_{x\in B} |v(Tx)| = \sup_{y\in T(B)} |v(y)| = q_{T(B)}(v).$$

Lemma A.4.4. *Let $\mathscr{X}$, $\mathscr{Y}$ be locally convex vector spaces and $T\colon \mathscr{X} \to \mathscr{Y}$ a continuous linear map with dense image. Then the adjoint $T^*\colon \mathscr{Y}^* \to \mathscr{X}^*$ of T is injective.*

Proof: Let $v \in \mathscr{Y}^*$. If $T^*v = v \circ T$ is identically zero on $\mathscr{X}$, then v vanishes on the dense subspace $T(\mathscr{X})$ of $\mathscr{Y}$ and since v is continuous, this implies that v is identically zero on $\mathscr{Y}$. In other words, $T^*v = 0$ implies $v = 0$ and because T^* is linear, this proves that T^* is injective. □

Definition A.4.5. Let $\mathscr{X}$ be a locally convex vector space. For every $x \in \mathscr{X}$,

$$\hat{\imath}_x\colon \mathscr{X}^* \to \mathbb{K}\colon u \mapsto u(x)$$

is a continuous linear map (use that $\{x\}$ is a bounded subset of $\mathscr{X}$), which is usually called 'evaluation in x', and as a consequence

$$\hat{\imath}\colon \mathscr{X} \to (\mathscr{X}^*)^*\colon x \mapsto \hat{\imath}_x$$

is a well-defined linear map. We say that $\mathscr{X}$ is *semi-reflexive* if $\hat{\imath}$ is bijective and we say that $\mathscr{X}$ is *reflexive* if $\hat{\imath}$ is a linear topological isomorphism. ⊘

Lemma A.4.6. *Let $\mathscr{X}$, $\mathscr{Y}$ be locally convex vector spaces and $T\colon \mathscr{X} \to \mathscr{Y}$ an injective continuous linear map. If $\mathscr{X}$ is semi-reflexive and $\mathscr{Y}$ is Hausdorff, the adjoint $T^*\colon \mathscr{Y}^* \to \mathscr{X}^*$ of T has dense image.*

Proof: We use Lemma A.2.1. So we suppose that $w\colon \mathscr{X}^* \to \mathbb{K}$ is a continuous linear map that vanishes on $T^*(\mathscr{Y}^*)$ and we want to prove that w is identically zero. First, observe that $w \in (\mathscr{X}^*)^*$. Because $\mathscr{X}$ is semi-reflexive, this implies that we find an $x \in \mathscr{X}$ such that $w = \hat{\imath}_x$. Next, observe that for every $v \in \mathscr{Y}^*$

$$v(Tx) = (T^*v)(x) = \hat{\imath}_x(T^*v) = w(T^*v) = 0.$$

Since the dual of a Hausdorff locally convex vector space seperates points (see [1, Corollary 5.82]), this implies that $Tx = 0$ and because T is injective, we obtain $x = 0$. Finally, $x = 0$ clearly implies $w = \hat{\imath}_x = 0$, so we are done. □

A.5 Products

If $\mathscr{X}$ and $\mathscr{Y}$ are locally convex vector spaces, then also the Cartesian product $\mathscr{X} \times \mathscr{Y}$ is a locally convex vector space when equipped with the usual product topology (i.e., the smallest topology such that the projections $\pi_{\mathscr{X}}\colon \mathscr{X} \times \mathscr{Y} \to \mathscr{X}$ and $\pi_{\mathscr{Y}}\colon \mathscr{X} \times \mathscr{Y} \to \mathscr{Y}$ are continuous). In fact, one easily checks that if $\mathscr{P}_{\mathscr{X}}$ and $\mathscr{P}_{\mathscr{Y}}$ are inducing collections of seminorms for respectively $\mathscr{X}$ and $\mathscr{Y}$, then $\{p_{\mathscr{X}} \circ \pi_{\mathscr{X}} + p_{\mathscr{Y}} \circ \pi_{\mathscr{Y}} \mid p_X \in \mathscr{P}_{\mathscr{X}} \text{ and } p_Y \in \mathscr{P}_{\mathscr{Y}}\}$ is an inducing collection of seminorms for $\mathscr{X} \times \mathscr{Y}$.

The interaction between products and duals turns out to be very elegant.

Lemma A.5.1. *Let $\mathscr{X}$ and $\mathscr{Y}$ be locally convex vector spaces. Then*

$$T\colon \mathscr{X}^* \times \mathscr{Y}^* \to (\mathscr{X} \times \mathscr{Y})^*\colon (u, v) \mapsto u \circ \pi_{\mathscr{X}} + v \circ \pi_{\mathscr{Y}}$$

is a linear topological isomorphism.

Proof: For all $u \in \mathscr{X}^*$ and $v \in \mathscr{Y}^*$, $u \circ \pi_{\mathscr{X}} + v \circ \pi_{\mathscr{Y}}$ is indeed a continuous linear map from $\mathscr{X} \times \mathscr{Y}$ into $\mathbb{K}$, so T is well-defined. Furthermore, it is clear that T is linear. To prove that T is continuous, we use Lemma A.1.2. Let B be a bounded subset of $\mathscr{X} \times \mathscr{Y}$ and let q_B be the associated seminorm of $(\mathscr{X} \times \mathscr{Y})^*$ from the standard inducing collection of seminorms. Because continuous functions send bounded sets to bounded sets, then also $B_{\mathscr{X}} := \pi_{\mathscr{X}}(B)$ and $B_{\mathscr{Y}} := \pi_{\mathscr{Y}}(B)$ are bounded. Let $q_{B_{\mathscr{X}}}$ and $q_{B_{\mathscr{Y}}}$ be the associated seminorms from the standard inducing collections for $\mathscr{X}^*$ respectively $\mathscr{Y}^*$. Then $(u, v) \mapsto q_{B_{\mathscr{X}}}(u) + q_{B_{\mathscr{Y}}}(v)$ is a seminorm from the standard inducing collection of seminorms for $\mathscr{X}^* \times \mathscr{Y}^*$, so on behalf of Lemma A.1.2 the following estimate proves that T is continuous:

$$\begin{aligned} q_B(T(u, v)) &= q_B(u \circ \pi_{\mathscr{X}} + v \circ \pi_{\mathscr{Y}}) \le q_B(u \circ \pi_{\mathscr{X}}) + q_B(v \circ \pi_{\mathscr{Y}}) \\ &= \sup_{(x,y) \in B} |(u \circ \pi_{\mathscr{X}})(x, y)| + \sup_{(x,y) \in B} |(v \circ \pi_{\mathscr{Y}})(x, y)| \\ &= \sup_{x \in \pi_{\mathscr{X}}(B)} |u(x)| + \sup_{y \in \pi_{\mathscr{Y}}(B)} |v(y)| = q_{B_{\mathscr{X}}}(u) + q_{B_{\mathscr{Y}}}(v). \end{aligned}$$

Now let $\imath_{\mathscr{X}}\colon \mathscr{X} \to \mathscr{X} \times \mathscr{Y}\colon x \mapsto (x, 0)$ and $\imath_{\mathscr{Y}}\colon \mathscr{Y} \to \mathscr{X} \times \mathscr{Y}\colon y \mapsto (0, y)$ be the continuous linear injections and consider

$$L\colon (\mathscr{X} \times \mathscr{Y})^* \to \mathscr{X}^* \times \mathscr{Y}^*\colon w \mapsto (w \circ \imath_{\mathscr{X}}, w \circ \imath_{\mathscr{Y}}).$$

Clearly, L is well-defined and linear and we readily check that T and L are inverse to each other. Hence, to prove that T is a linear topological isomorphism, it only remains to be shown that L is continuous. For this, we again use Lemma A.1.2. So let $(u, v) \mapsto q_{B_{\mathscr{X}}}(u) + q_{B_{\mathscr{Y}}}(v)$, with $B_{\mathscr{X}}$ a bounded subset of $\mathscr{X}$ and $B_{\mathscr{Y}}$ a bounded subset of $\mathscr{Y}$, be an element of the standard inducing collection of seminorms for $\mathscr{X}^* \times \mathscr{Y}^*$. Then $B := (B_{\mathscr{X}} \cup \{0\}) \times (B_{\mathscr{Y}} \cup \{0\})$ is a bounded subset of $\mathscr{X} \times \mathscr{Y}$. Indeed, if a seminorm $p_{\mathscr{X}}$ on $\mathscr{X}$ is bounded on $B_{\mathscr{X}}$ by $r_{\mathscr{X}} \in \mathbb{R}$ and a seminorm $p_{\mathscr{Y}}$ on $\mathscr{Y}$ is bounded on $B_{\mathscr{Y}}$ by $r_{\mathscr{Y}} \in \mathbb{R}$, then $p_{\mathscr{X}} \circ \pi_{\mathscr{X}} + p_{\mathscr{Y}} \circ \pi_{\mathscr{Y}}$ is bounded on $(B_{\mathscr{X}} \cup \{0\}) \times (B_{\mathscr{Y}} \cup \{0\})$ by $r_{\mathscr{X}} + r_{\mathscr{Y}}$. Let q_B be the associated seminorm of $(\mathscr{X} \times \mathscr{Y})^*$. Then the estimate

$$\begin{aligned} q_{B_{\mathscr{X}}}(w \circ \imath_{\mathscr{X}}) + q_{B_{\mathscr{Y}}}(w \circ \imath_{\mathscr{Y}}) &= \sup_{x \in B_{\mathscr{X}}} |w(x, 0)| + \sup_{y \in B_{\mathscr{Y}}} |w(0, y)| \\ &\le 2 \sup_{(x,y) \in B} |w(x, y)| = 2\, q_B(w) \end{aligned}$$

shows that L is continuous. □

Using a similar argument (or induction), we obtain the following:

Lemma A.5.2. *If $\mathscr{X}$ is a locally convex vector space and $n \in \mathbb{N}$, then*

$$(\mathscr{X}^*)^n \to (\mathscr{X}^n)^* \colon (u_1, \ldots, u_n) \mapsto \sum_{i=1}^{n} u_i \circ \pi_i,$$

where π_i denotes the projection from $\mathscr{X}^n$ onto the i^{th} component, is a linear topological isomorphism.

Also the interaction between products and inductive limits is perfect.

Lemma A.5.3. *Let $\mathscr{X}$ be a vector space and let $\{\mathscr{X}_i\}_{i\in I}$ be a family of vector subspaces of $\mathscr{X}$, each equipped with some locally convex topology, such that $\{\mathscr{X}_i \mid i \in I\}$ is a directed set under inclusion of sets and $\mathscr{X} = \cup_{i\in I}\mathscr{X}_i$. Then for every $n \in \mathbb{N}$,*

$$(\lim_i \mathscr{X}_i)^n = \lim_i \mathscr{X}_i^n.$$

Proof: Thanks to the assumption that $\{\mathscr{X}_i \mid i \in I\}$ is a directed set under inclusion, $(\cup_{i\in I}\mathscr{X}_i)^n = \mathscr{X}^n = \cup_{i\in I}\mathscr{X}_i^n$, so both $(\lim_i \mathscr{X}_i)^n$ and $\lim_i \mathscr{X}_i^n$ are equal to $\mathscr{X}^n$ as vector space. To establish the continuity of the inclusion $\lim_i \mathscr{X}_i^n \subseteq (\lim_i \mathscr{X}_i)^n$ it suffices, on behalf of Proposition A.3.2, to prove that $\mathscr{X}_i^n \subseteq_{\mathrm{c}} (\lim_i \mathscr{X}_i)^n$ for every $i \in I$. So fix $i \in I$. Because a map into the product $(\lim_i \mathscr{X}_i)^n$ is continuous if and only if its component maps are continuous, we take $1 \le m \le n$ and look at the m^{th} component of the inclusion map $\mathscr{X}_i^n \hookrightarrow (\lim_i \mathscr{X}_i)^n$. Since this component is nothing more than the composition of the projection π_m from $\mathscr{X}_i^n$ onto the m^{th} component (which is continuous by definition) with the inclusion $\mathscr{X}_i \hookrightarrow \lim_i \mathscr{X}_i$ (which is also continuous by definition), this component map is continuous and we conclude that the inclusion map $\mathscr{X}_i^n \hookrightarrow (\lim_i \mathscr{X}_i)^n$ is continuous.

To prove that also the inclusion $(\lim_i \mathscr{X}_i)^n \subseteq \lim_i \mathscr{X}_i^n$ is continuous, we consider the continuous linear injections

$$\imath_m^i \colon \mathscr{X}_i \to \mathscr{X}_i^n \colon x \mapsto (\overbrace{0, \ldots, 0}^{m-1}, x, \overbrace{0, \ldots, 0}^{n-m})$$

for $i \in I$ and $1 \le m \le n$. We first want to show that the injections

$$\imath_m \colon \lim_i \mathscr{X}_i \to \lim_i \mathscr{X}_i^n \colon x \mapsto (\overbrace{0, \ldots, 0}^{m-1}, x, \overbrace{0, \ldots, 0}^{n-m}),$$

for $1 \le m \le n$, are continuous as well (note that we already know that $\lim_i \mathscr{X}_i^n$ and $(\lim_i \mathscr{X}_i)^n$ are equal as sets, so we indeed have such injections). According to Proposition A.3.2, for this it suffices to show that for every $i \in I$, $\imath_m|_{\mathscr{X}_i} : \mathscr{X}_i \to \lim_i \mathscr{X}_i^n$ is continuous, which is the case because $\imath_m|_{\mathscr{X}_i}$ is simply the composition of $\imath_m^i$ with the continuous inclusion $\mathscr{X}_i^n \subseteq_{\mathrm{c}} \lim_i \mathscr{X}_i^n$. Thus, $\imath_m$ is indeed continuous for every $1 \le m \le n$. Now, since the inclusion map $(\lim_i \mathscr{X}_i)^n \hookrightarrow \lim_i \mathscr{X}_i^n$ evidently equals

$$\sum_{m=1}^{n} \imath_m \circ \pi_m$$

and $\pi_m : (\lim_i \mathscr{X}_i)^n \to \lim_i \mathscr{X}_i$ is by definition continuous for every $1 \le m \le n$, we deduce that $(\lim_i \mathscr{X}_i)^n \subseteq_{\mathrm{c}} \lim_i \mathscr{X}_i^n$. $\square$

As we have already mentioned, one of the key features of locally convex vector spaces is that the continuity of linear maps can be expressed in terms of seminorms. The following result shows that the same is true for bilinear maps.

Lemma A.5.4. *Let $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ be locally convex vector spaces, let $\mathscr{P}$, $\mathscr{Q}$ and $\mathscr{R}$ be inducing collections of seminorms for respectively $\mathscr{X}$, $\mathscr{Y}$ and $\mathscr{Z}$ and let $T : \mathscr{X} \times \mathscr{Y} \to \mathscr{Z}$ be a bilinear map. Then T is continuous if and only if for every $r \in \mathscr{R}$ there exist $C \ge 0$, $p_0, \ldots, p_n \in \mathscr{P}$ and $q_0, \ldots, q_m \in \mathscr{Q}$ such that*

$$r(T(x,y)) \le C \sum_{i=0}^{n} \sum_{j=0}^{m} p_i(x) q_j(y)$$

for all $(x,y) \in \mathscr{X} \times \mathscr{Y}$.

Proof: We first prove the direct implication. So suppose that T is continuous and let $r \in \mathscr{R}$. Then $T^{-1}(B_1^r(0))$ is an open neighborhood of 0 in $\mathscr{X} \times \mathscr{Y}$, so there are $p_0, \ldots, p_n \in \mathscr{P}$, $q_0 \ldots, q_m \in \mathscr{Q}$, $\varepsilon_0, \ldots, \varepsilon_n > 0$ and $\delta_0, \ldots, \delta_m > 0$ such that

$$\cap_{i=0}^n B_{\varepsilon_i}^{p_i}(0) \times \cap_{j=0}^m B_{\delta_j}^{q_j}(0) \subseteq T^{-1}(B_1^r(0)).$$

Fix $(x,y) \in \mathscr{X} \times \mathscr{Y}$. Clearly, for every $\gamma > 0$

$$x' := \frac{(\min_{0 \le i \le n} \varepsilon_i) x}{\gamma + \sum_{i=0}^n p_i(x)}$$

satisfies $p_i(x') < \varepsilon_i$ for every $0 \le i \le n$, hence $x' \in \cap_{i=0}^n B_{\varepsilon_i}^{p_i}(0)$. Similarly

$$y' := \frac{(\min_{0 \le j \le m} \delta_j) y}{\gamma + \sum_{j=0}^n q_j(y)}$$

satisfies $y' \in \cap_{j=0}^m B_{\delta_j}^{q_j}(0)$, hence

$$(x', y') \in \cap_{i=0}^n B_{\varepsilon_i}^{p_i}(0) \times \cap_{j=0}^m B_{\delta_j}^{q_j}(0) \subseteq T^{-1}(B_1^r(0)).$$

As a consequence, $r(T(x', y')) < 1$ and using the bilinearity of T, we find that for every $\gamma > 0$

$$r(T(x,y)) < \frac{(\gamma + \sum_{i=0}^n p_i(x))(\gamma + \sum_{j=0}^m q_j(y))}{(\min_{0 \le i \le n} \varepsilon_i)(\min_{0 \le j \le m} \delta_m)}.$$

Taking the limit $\gamma \to 0$ then shows

$$r(T(x,y)) \le \frac{1}{(\min_{0 \le i \le n} \varepsilon_i)(\min_{0 \le j \le m} \delta_j)} \sum_{i=0}^{n} \sum_{j=0}^{m} p_i(x) q_j(y)$$

and since (x,y) was chosen arbitrarily, this proves that we have the desired estimate.

For the converse implication, let $\{(x_\ell, y_\ell)\}_{\ell \in L}$ be a converging net in $\mathscr{X} \times \mathscr{Y}$ and $(x,y) \in \mathscr{X} \times \mathscr{Y}$ such that $(x_\ell, y_\ell) \to (x,y)$ in $\mathscr{X} \times \mathscr{Y}$. To prove that

T is continuous, we should prove that $T(x_\ell, y_\ell) \to T(x, y)$ in $\mathscr{Z}$, hence that $r(T(x, y) - T(x_\ell, y_\ell)) \to 0$ in $\mathbb{R}$ for every $r \in \mathscr{R}$. Fix $r \in \mathscr{R}$. By assumption there exist $C \geq 0$, $p_0, \ldots, p_n \in \mathscr{P}$ and $q_0, \ldots, q_m \in \mathscr{Q}$ such that

$$r(T(x', y')) \leq C \sum_{i=0}^{n} \sum_{j=0}^{m} p_i(x') q_j(y')$$

for all $(x', y') \in \mathscr{X} \times \mathscr{Y}$. Moreover, $(x_\ell, y_\ell) \to (x, y)$ in $\mathscr{X} \times \mathscr{Y}$ implies $x_\ell \to x$ in $\mathscr{X}$ and $y_\ell \to y$ in $\mathscr{Y}$, so $p_i(x - x_\ell) \to 0$ for every $0 \leq i \leq n$, $q_j(y - y_\ell) \to 0$ for every $0 \leq j \leq m$ and $q_j(y_\ell)$ is bounded by some constant $D_j \geq 0$ for every $0 \leq j \leq m$. Using the triangle inequality for r and the bilinearity of T, we now deduce

$$\begin{aligned} r(T(x, y) - T(x_\ell, y_\ell)) &= r(T(x, y) - T(x, y_\ell) + T(x, y_\ell) - T(x_\ell, y_\ell)) \\ &= r(T(x, y - y_\ell) + T(x - x_\ell, y_\ell)) \\ &\leq r(T(x, y - y_\ell)) + r(T(x - x_\ell, y_\ell)) \\ &\leq C \sum_{i=0}^{n} \sum_{j=0}^{m} p_i(x) q_j(y - y_\ell) + C \sum_{i=0}^{n} \sum_{j=0}^{m} D_j p_i(x - x_\ell) \end{aligned}$$

and since clearly

$$C \sum_{i=0}^{n} \sum_{j=0}^{m} p_i(x) q_j(y - y_\ell) + C \sum_{i=0}^{n} \sum_{j=0}^{m} D_j p_i(x - x_\ell) \to 0,$$

we conclude that $r(T(x, y) - T(x_\ell, y_\ell)) \to 0$. □

Appendix B

Differential geometry

In this appendix, we treat some concepts and results from differential geometry that are a bit more advanced and therefore might not be known by all readers. As mentioned in 'Notation and conventions', we are always working in the smooth setting. That is, manifolds, vector bundles, fiber bundles and maps between them are assumed to be smooth unless explicitly indicated otherwise. Moreover, manifolds are always assumed to be second-countable, vector bundles are $\mathbb{K}$-vector bundles of constant rank and fiber bundle homomorphisms between fiber bundles over the same manifold are assumed to be the identity on the base manifold (i.e., a fiber bundle homomorphism $f\colon P \to Q$ between fiber bundles P and Q over the same base manifold sends P_x into Q_x for all points x of the base manifold).

B.1 The density bundle

Definition B.1.1. Let V be an n-dimensional real vector space. A *density* on V is a map $\omega\colon V^n \to \mathbb{K}$ such that for every $T \in \operatorname{End}(V)$

$$T^*\omega := \omega \circ T^n = |\det T|\,\omega,$$

where $T^n\colon V^n \to V^n$ maps $(v_1, \ldots, v_n)$ to $(Tv_1, \ldots, Tv_n)$. ⊘

Under the obvious addition and scalar multiplication, the set of all densities on V is a linear space over $\mathbb{K}$. We will denote this space by $D(V)$. It is not hard to see that $D(V)$ is one dimensional. Indeed, if $(e_1, \ldots, e_n)$ is a basis for V, then $\omega \mapsto \omega(e_1, \ldots, e_n)$ is an isomorphism between $D(V)$ and $\mathbb{K}$.

Now let M be an n-dimensional manifold. Then it is possible to form a line bundle (that is, a rank 1 vector bundle) $D = D^M$ over M whose fiber over $x \in M$ equals $D(T_xM)$ and whose differentiable structure is such that for every chart (U, κ) of M, $|d\kappa_1 \wedge \cdots \wedge d\kappa_n|$ is a smooth nowhere vanishing section of $D_U \to U$. This line bundle D is called the *density bundle* of M. Because it has rank 1, the smooth nowhere vanishing section $|d\kappa_1 \wedge \cdots \wedge d\kappa_n|$ of $D_U \to U$ is in fact a frame over U and therefore it induces a trivialization $\rho_\kappa\colon D_U \to \mathbb{K}$ (for $x \in U$, an element $\omega_x \in D_x$ can be written as $\omega_x = \mu|d\kappa_1(x) \wedge \cdots \wedge d\kappa_n(x)|$ and ρ_κ sends ω_x to μ).

The crucial point about the density bundle is that we can *integrate* its compactly supported continuous sections. That is, there exists a natural linear integration map $\int_M\colon \Gamma_c^0(M, D) \to \mathbb{K}$. To define this map, we first consider an element ω of $\Gamma_c^0(M, D)$ such that $\operatorname{supp}(\omega)$ is contained in the domain of some chart (U, κ). Then ω is a compactly supported continuous section of $D_U \to U$ and the idea is to define

$$\int_M \omega := \int_{\kappa(U)} \rho_\kappa \circ \omega \circ \kappa^{-1} \, d\lambda$$

(note that $\rho_\kappa \circ \omega \circ \kappa^{-1} \in \mathscr{C}_c(\kappa(U))$, so this makes sense). However, we need to make sure that this does not depend on the chart (U, κ). So let (V, ν) be another chart of M such that $\operatorname{supp}(\omega) \subseteq V$. Then, for all $x \in U \cap V$,

$$|d\nu_1(x) \wedge \cdots \wedge d\nu_n(x)| = |\det(D\nu\kappa^{-1})(\kappa(x))| \, |d\kappa_1(x) \wedge \cdots \wedge d\kappa_n(x)|$$

(this is well-known, but can also easily be derived by using

$$d\nu_j = \sum_{i=1}^n \frac{\partial \nu_j}{\partial \kappa_i} d\kappa_i = \sum_{i=1}^n (\partial_i(\nu_j \circ \kappa^{-1}) \circ \kappa) d\kappa_i$$

and the permutation expression for the determinant). As a consequence, we have $\rho_\kappa|_{D_x} = |\det(D\nu\kappa^{-1})(\kappa(x))| \, \rho_\nu|_{D_x}$ for every $x \in U \cap V$ and using the change of variables theorem, we subsequently find

$$\begin{aligned} \int_{\nu(V)} \rho_\nu \circ \omega \circ \nu^{-1} \, d\lambda &= \int_{\nu(U \cap V)} \rho_\nu \circ \omega \circ \nu^{-1} \, d\lambda \\ &= \int_{\kappa(U \cap V)} (\rho_\nu \circ \omega \circ \kappa^{-1}) \, |\det D\nu\kappa^{-1}| \, d\lambda \\ &= \int_{\kappa(U \cap V)} \rho_\kappa \circ \omega \circ \kappa^{-1} \, d\lambda = \int_{\kappa(U)} \rho_\kappa \circ \omega \circ \kappa^{-1} \, d\lambda \end{aligned}$$

(by assumption $\operatorname{supp}(\omega) \subseteq U \cap V$, so we have $\operatorname{supp}(\rho_\nu \circ \omega \circ \nu^{-1}) \subseteq \nu(U \cap V)$ and $\operatorname{supp}(\rho_\kappa \circ \omega \circ \kappa^{-1}) \subseteq \kappa(U \cap V)$). Hence, the definition

$$\int_M \omega := \int_{\kappa(U)} \rho_\kappa \circ \omega \circ \kappa^{-1} \, d\lambda$$

is indeed independent of the chosen chart.

Now let $\{(U_i, \kappa_i)\}_{i \in I}$ be a collection of charts such that $\{U_i\}_{i \in I}$ is an open cover of M and let $\{\eta_i\}_{i \in I}$ be a (smooth) partition of unity subordinate to $\{U_i\}_{i \in I}$. For any $\omega \in \Gamma_c^0(M, D)$, there is a finite subset I_ω of I such that $\operatorname{supp}(\eta_i) \cap \operatorname{supp}(\omega) \neq \emptyset$ if and only if $i \in I_\omega$, so we can define

$$\int_M \omega := \sum_{i \in I} \int_M \eta_i \omega = \sum_{i \in I_\omega} \int_M \eta_i \omega$$

(note that for every $i \in I$, $\eta_i \omega$ is an element of $\Gamma_c^0(M, E)$ with support contained in some chart, so $\int_M \eta_i \omega$ is already defined). This is independent of the choice of cover and partition of unity, because if $\{(V_j, \nu_j)\}_{j \in J}$ is another collection of charts such that their domains cover M and $\{\chi_j\}_{j \in J}$ is a partition of unity subordinate to $\{V_j\}_{j \in J}$, then we readily check, using the linearity of the ordinary Lebesgue integral, that

$$\begin{aligned} \sum_{i \in I} \int_M \eta_i \omega &= \sum_{i \in I} \int_M \sum_{j \in J} \chi_j \eta_i \omega = \sum_{i \in I} \sum_{j \in J} \int_M \chi_j \eta_i \omega \\ &= \sum_{j \in J} \int_M \sum_{i \in I} \eta_i \chi_j \omega = \sum_{j \in J} \int_M \chi_j \omega. \end{aligned}$$

Moreover, the map

$$\int_M : \Gamma_c^0(M, D) \to \mathbb{K} : \omega \mapsto \sum_{i \in I} \int_M \eta_i \omega$$

is clearly linear, so we have found the desired integration map.

B.2 The vertical bundle and derivative

Let M be a manifold and $P \xrightarrow{\pi_P} M$ a fiber bundle over M. The *vertical bundle* of P, notation $T^v P$, is the vector subbundle of the tangent bundle TP whose fiber over $p \in P$ equals $T_p P_{\pi_P(p)}$ (i.e., informally speaking, $T^v P$ consists of those vectors in TP that are tangent to the fibers, which would in the traditional drawing of a fiber bundle be the vertical ones). Put differently, if we denote the derivative of $\pi_P \colon P \to M$ by $D\pi_P \colon TP \to TM$, then $T^v P = \ker(D\pi_P)$. For $\varphi \in \Gamma^\infty(M, P)$, we denote the pullback $\varphi^*(T^v P)$ of $T^v P$ under φ (which is a vector bundle over M) by $T^v_\varphi P$.

Hand-in-hand with the notion of vertical (tangent) bundle, comes the notion of vertical derivative. For this, let $Q \to M$ also be a fiber bundle over M and let $f \colon P \to Q$ be a fiber bundle homomorphism. Because f sends fibers to fibers, it follows that the derivative $Df \colon TP \to TQ$ maps $T^v P$ into $T^v Q$ and we call the restriction of Df to $T^v P$ the *vertical derivative* or *derivative along the fibers* of f, notation $\delta f \colon T^v P \to T^v Q$.

If $E \to M$ is a vector bundle over M, then $T^v E$ is canonically diffeomorphic to (the total space of the vector bundle) $E \oplus E$. Indeed, for any $x \in M$ and $e \in E_x$, $(T^v E)_e = T_e E_x \simeq E_x$, so for any $e \in E_x$, we have a copy of E_x. For $\varphi \in \Gamma^\infty(M, E)$, $T^v_\varphi E$ and E are even canonically isomorphic as vector bundles. If $F \to M$ is also a vector bundle over M and $f \colon E \to F$ is a *fiber bundle* homomorphism, then the 'translation' of δf to a map from $E \oplus E$ into $F \oplus F$ (which are both vector bundles over M) is a fiber bundle homomorphism as well. We will denote this fiber bundle homomorphism by $\hat{\delta} f$ and we easily see that $\hat{\delta} f \colon E \oplus E \to F \oplus F$ is characterized by

$$\hat{\delta} f(e, e') = (f(e), \left.\frac{d}{dt}\right|_{t=0} f(e + te'))$$

(note that e and e' belong to the same fiber of E).

B.3 Approximation of continuous sections

In the main text we need the following, very general, approximation theorem:

Theorem B.3.1. *Let M be a manifold and $P \to M$ a fiber bundle over M. If $\varphi \in \Gamma^0(M, P)$ and U is an open neighborhood of $\mathrm{im}(\varphi)$ in P, then there exists an $\psi \in \Gamma^\infty(M, P)$ such that $\mathrm{im}(\psi) \subseteq U$.*

Proof: See [15] (in the case at hand, it is not necessary to assume that M is connected; we can just approximate φ on all connected componentents) and [12, Section 6.7] for the original, more basic result (which evidently implies our theorem if M is compact). □

B.4 Differential operators

The concept of a differential operator is of course very well-known, but a quick (and incomplete) summary of a formal defintion is the following:

Definition B.4.1. Let M be a manifold and let E and F be vector bundles over M. A linear *partial differential operator* from E to F of *order* at most $k \in \mathbb{N}$ is a linear map $P\colon \Gamma^\infty(M,E) \to \Gamma^\infty(M,F)$ such that P is *local* (i.e., $\operatorname{supp}(P\varphi) \subseteq \operatorname{supp}(\varphi)$ for all $\varphi \in \Gamma^\infty(M,E)$) and such that for every total trivialization triple (U,κ,ρ) (see Definition 3.1.1) of $E \to M$ the restriction $P|_U$ can be written as $\sum_{|\alpha|\le k}(C_\alpha)_* \circ \partial_\kappa^\alpha$ for certain vector bundle homomorphisms $C_\alpha\colon E_U \to F_U$. ⊘

Here, the *partial derivatives* $\partial_\kappa^\alpha\colon \Gamma^\infty(U,E_U) \to \Gamma^\infty(U,E_U)$ act 'componentwise' on the sections (the trivialization $\rho\colon E_U \to \mathbb{K}^r$, with $r := \operatorname{rank}(E)$, determines a frame $(e_1,\ldots,e_r)$ of $E_U \to U$ and ∂_κ^α sends $\varphi = \varphi_1 e_1 + \cdots + \varphi_r e_r$ to $(\partial_\kappa^\alpha\varphi_1)e_1 + \cdots + (\partial_\kappa^\alpha\varphi_r)e_r$) and for a section $\varphi \in \Gamma^\infty(U,E_U)$ and a point $x \in U$, $(P|_U\,\varphi)(x)$ is computed by picking an $\varphi_x \in \Gamma^\infty(M,E)$ that coincides with φ on an open neighborhood of x in U and then calculating $(P\varphi_x)(x)$.

Although '(smooth) linear partial differential operator' is probably the most precise and correct name for a map P that satisfies the conditions of the above definition, such maps are often more conveniently just called 'differential operators'. In line with this, we will denote the space of all 'differential operators' from E to F of order at most k by $\operatorname{Diff}_k(E,F)$ and the space of all differential operators from E to F of *finite* order by $\operatorname{Diff}(E,F)$ (on noncompact manifolds, it is possible to define differential operators of 'infinite' order, but we will not consider such differential operators here).

The reason for spending some paper on differential operators in this appendix is that we would like to use a result about differential operators in the main text that does not seem to be very well-known. Before we present this result, we need one preliminary proposition.

Proposition B.4.2. *For every vector bundle $E \to M$ there exists a finite collection of total trivialization triples $\{(U_i,\kappa_i,\rho_i)\}_{i=0}^n$ of $E \to M$ such that $\{U_i\}_{i=0}^n$ is an open cover of M.*

Proof: An obvious adjustment of the proof of [14, Proposition III.4.1] works (observe that a countable collection of trivialization triples with mutually disjoint domains can be turned into one trivialization triple). □

Theorem B.4.3. *Let M be a manifold, let E and F be vector bundles over M and let $k \in \mathbb{N}_\infty$. For every differential operator $P \in \operatorname{Diff}_k(E,F)$ there exist vector bundle homomorphisms $T_0, \ldots, T_m \in \operatorname{Hom}(E,F)$ and differential operators $P_0, \ldots, P_m \in \operatorname{Diff}_k(E,E)$ such that $P = \sum_{j=0}^m (T_j)_* \circ P_j$.*

Proof: Let $\{(U_i,\kappa_i,\rho_i)\}_{i=0}^n$ be a finite collection of total trivilization triples of $E \to M$ such that $\{U_i\}_{i=0}^n$ is an open cover of M, let $\{\eta_i\}_{i=0}^n$ be a partition of unity subordinate to $\{U_i\}_{i=0}^n$ with the property that for every $0 \le i \le n$, $\eta_i = \mu_i^2$ for some $\mu_i \in \mathscr{C}^\infty(M)$ and fix some $k \in \mathbb{N}$ such that $P \in \operatorname{Diff}_k(E,F)$. Then we find for all $0 \le i \le n$ and $|\alpha| \le k$ a vector bundle homomorphism $C_\alpha^i \in \operatorname{Hom}(E_{U_i},F_{U_i})$ such that $P|_{U_i} = \sum_{|\alpha|\le k}(C_\alpha^i)_* \circ \partial_{\kappa_i}^\alpha$ for every $0 \le i \le n$.

Now for every $0 \leq i \leq n$ and $|\alpha| \leq k$, $\mu_i C^i_\alpha$ is a vector bundle homomorphism from E to F in the obvious way, while $\mu_i \partial^\alpha_{\kappa_i}$ becomes a differential operator from E to E of order at most k if we agree that $(\mu_i \partial^\alpha_{\kappa_i})(\varphi) = \mu_i(\partial^\alpha_{\kappa_i}\, \varphi|_{U_i})$ for $\varphi \in \Gamma^\infty(M, E)$ (note that $\operatorname{supp}(\mu_i) = \operatorname{supp}(\eta_i) \subseteq U_i$). We claim that

$$P = \sum_{i=0}^{n} \sum_{|\alpha| \leq k} (\mu_i C^i_\alpha)_* \circ (\mu_i \partial^\alpha_{\kappa_i}). \tag{B.1}$$

Indeed, if $\varphi \in \Gamma^\infty(M, E)$ and $x \in M$, then

$$\begin{aligned}
&((\sum_{i=0}^{n} \sum_{|\alpha| \leq k} (\mu_i C^i_\alpha)_* \circ (\mu_i \partial^\alpha_{\kappa_i}))(\varphi))(x) = \\
&\sum_{i=0}^{n} \sum_{|\alpha| \leq k} \mu_i(x) C^i_\alpha((\mu_i \partial^\alpha_{\kappa_i}\, \varphi|_{U_i})(x)) = \\
&\sum_{i=0}^{n} (\mu_i(x))^2 \sum_{|\alpha| \leq k} C^i_\alpha((\partial^\alpha_{\kappa_i}\, \varphi|_{U_i})(x)) = \\
&\sum_{i=0}^{n} \eta_i(x)(P|_{U_i}\, \varphi|_{U_i})(x) = \\
&\sum_{i=0}^{n} \eta_i(x)(P\varphi)(x) = (P\varphi)(x),
\end{aligned}$$

where the penultimate equality holds because for those i for which $x \in U_i$, φ coincides with $\varphi|_{U_i}$ on an open neighborhood of x, thus $(P|_{U_i}\, \varphi|_{U_i})(x) = (P\varphi)(x)$, while for those i for which $x \notin U_i$, $\eta_i(x)(P|_{U_i}\, \varphi|_{U_i})(x) = 0 = \eta_i(x)(P\varphi)(x)$.

So equation (B.1) indeed holds and since the right hand side of this equation is of the desired form, we are done. □

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