Morse-Smale Homology

for compact manifolds

and

Invariant Sets

University of Utrecht

Author:
Oihana Garayalde

Supervisors:
Prof. Dr. R.C.A.M. Vandervorst
Dr.A.G. Henriques

August 13, 2011

Contents

1.INTRODUCTION

Morse Theory gives another way to determine the topology of a given manifold M by considering *Morse functions* $f : M \to \mathbb{R}$. For example, as J. Milnor proves in [17], if a compact manifold admits a smooth function with only one maximum and one minimum, then we know that it must be a sphere. We say that f is a Morse function if all its critical points are *non-degenerate*, that is, the Hessian matrix is non-singular in these critical points.

One of the consequences of these functions is that given two regular values $a < b \in \mathbb{R}$ with no critical values in between, then $f^{-1}[a, b]$ is compact in M. Moreover, $M^a = f^{-1}(-\infty, a]$ is isomorphic to $M^b = f^{-1}(-\infty, b]$ and M^a is a deformation retract of M^b. From these results, in fact, we can analyse the decomposition of the cell structure of the manifold M. However, in order to construct Morse-Smale homology groups, S.Smale added one more condition to these Morse functions, that is, the *Morse-Smale transversality condition* for the stable and unstable manifolds (we will give the definitions in Section 2.1).

In this paper, we first consider a finite dimensional smooth and compact Riemannian manifold together with a Morse-Smale function in order to define the Morse-Smale complexes and the corresponding homology groups. As an application of such a homology groups we get the so called *Morse inequalities.* However, the main objective of this paper is to give a detailed construction of such chain complexes and homology groups over $\mathbb{Z}_2$ (in order to simplify the computations).

Note that the chain complexes are constructed by considering the critical points of a given Morse-Smale function, that is, by the points on M in which the differential is equal to zero. Equivalently, we can consider a gradient-flow generated by a Morse-Smale function and analyse its solutions. Recall that a gradient flow is defined by the following equation:

$$\begin{cases} x' = -\nabla f(x) \\ x(0) = x_0 \end{cases}$$

We denote by $\phi : M \times \mathbb{R} \to M$ the dynamical system obtained by $\dot{\phi}(x, t) = x(t; x)$ where $x(t; x)$ is a solution curve and which satisfies the following properties:

(a) ϕ is continuous

(b) $\phi(0, x) = x$ and $\phi(t + s, x) = \phi(t, \phi(s, x))$ for all $t, s \in \mathbb{R}$ and $x \in M$

(c) $\phi(\cdot, t) \in Diff(M)$

Furthermore, Smale proved in [23] that if we consider a Morse Smale function, then this will also define a Morse-Smale gradient system. We consider gradient-flows so that the given Morse-Smale function decreases along the flow and thus, the flow lines will connect distinct critical points without having closed orbits.

Even though all the concepts and results in this paper have already been proved in [23], [20] and [2], we want to approach them from a different point of view, by fixing a metric and by perturbing the smooth functions or without making use of *Conley index theory.*

Nevertheless, if a manifold is not compact, then the Morse inequalities do not hold in general. Whenever M is not compact, we consider *isolating invariant sets* $S \subset M$, under the given Morse-Smale gradient-flow $\phi(\cdot, t)$. As Conley claims in [5], these isolated invariant sets can be continued nearby due to their stability. Furthermore, this continuation is determined by its *isolating neighborhood* N which always exists and it has the property that S is the maximal invariant set that is contained in the interior of N. Equivalently,

$$S = \{x \in N : \phi(x,t) \in N, \forall t \in \mathbb{R}\} \subset int(N)$$

Note that in this case we are considering a compact set with boundary together with a Morse-Smale function and thus, we need to be sure that the flow lines do not touch the boundary of N, denoted by ∂N. In particular, we need to make sure that whenever we consider homotopies (or more generally, homotopies of homotopies), the connecting flow lines do not intersect the boundary. After avoiding this situation, we can just construct the chain complexes and the corresponding homology groups as in Section 2.

We would also like to point out that the construction we developed in Section 2 is just a particular case for the isolated invariant sets, because when M is compact itself, it is enough to consider $S = M = N$.

1.1 Overview

In the second section, we state the basic definitions for Morse-Smale functions and for chain complexes as transversality condition, the space of connecting flow lines and we also recall some important results as the *Gluing theorem* in [8]. Actually, these concepts are important for the compactification of the manifold of flow lines. Since all these groups will be defined under the assumption that the given function is Morse-Smale, we also show the existence of such a functions by the *Kupka-Smale theorem* or *the Morse-lemma.* Moreover, we also prove that under small perturbations of the function we do not loose the transversality condition. This will be a very important result in order to prove invariance properties of the homology groups in the next chapter. In the end of this section, we prove that such a chain generated by the critical points with the boundary operator is a chain complex and thus, we can define the corresponding homology groups. We will also give some examples.

In the third section, we show the invariance of the homology groups. That is, we first prove that the homology groups do not depend on the particular Morse-Smale function we choose. In order to show this invariance, we consider homotopies or homotopy of homotopies between two given Morse-Smale functions over compact manifolds (e.g. $M \times S^1$). Then, we consider the gradient systems generated by these functions and thus, construct chain complexes exactly as in Section 2. Finally, we conclude from these chain complexes that the homotopies defined before, induce isomorphism on homology.

Moreover, we define the homology groups for a generic smooth function and finally, we state the well known *Morse-Homology theorem* which says that the Morse homology group is isomorphic to the singular homology.

In the fourth section, we consider the isolated invariant sets so that we can construct the chain complexes and the corresponding Morse-Smale homology groups by using the results from Section 2. However, we first give some basic definitions together with some useful results and then start constructing the chain complexes. As for the compact manifolds, we do expect to have the same invariance as before and we just recall that we can conclude the same results as before by observing that we also have a compact set together with a Morse-Smale gradient system. The unique difference is that in this case N is compact with boundary and thus, we need to show something more. That is, we need to prove that whenever we either perturb some smooth function to a Morse-Smale function or we define homotopies, then non of the complete solutions of the differential equations touches the boundary of N, denoted by ∂N.

2. Morse-Smale function and Homology

2.1 Morse-Smale function

In this section we consider (M, g) a m-dimensional smooth, oriented and compact Riemannian manifold and we define Morse-Smale function on it. We would like to give basic definitions and results so that we can analyse the topology of the connecting orbits. In fact, it is important to conclude that the connecting trajectory space is a smooth manifold which can always be compactified by the *broken flow lines*. Furthermore, the boundary map will be defined depending on these flow lines.

Since we want to define a Morse-Smale Homology we need to define a Morse-Smale chain complex as well as its boundary operator which for simplicity will be done over $\mathbb{Z}_2$.

2.1.1. Basic definitions and Transversality

Definition 1. *Let $f\colon M \to \mathbb{R}$ be a smooth function. Then f is a Morse function if all its critical points (i.e. $p \in M$ such that $df(p) = 0$) are non-degenerate, that is, if the Hessian at such a points is non-degenerate.*

One of the consequences of considering these kind of functions is that the critical points are *isolated*. Consider the Hessian of a critical point p which by definition is non-degenerate. Then, by the *implicit function theorem*, the gradient of such a function is a local diffeomorphism in a neighborhood of p and thus, this point will be the unique critical point.

Furthermore we associate an *index* λ_p to each point which by definition is the number of negative eigenvalues of the hessian, $Hess_p(f)$.

Consider the gradient flow $\dot{\phi}(x, t) = -(\nabla f \circ \phi(x, t))$ where $\phi(\cdot, t) \in Diff(M)$ is a 1-parameter family such that $\phi(x, 0) = x$ and $\phi(x, t + s) = \phi((\phi(x, s), t)$, $\forall t, s \in \mathbb{R}$ and $\forall x \in M$. Define the *stable* and *unstable manifolds* of the critical point $p \in Cr(f)$ as follows:

$W^s(p) = \{x \in M : \lim_{t\to\infty} \phi(x, t) = p\}$

$W^u(p) = \{x \in M : \lim_{t\to-\infty} \phi(x, t) = p\}$

Definition 2. *Let $f\colon M \to R$ be a Morse function. Then, f is said to be a Morse-Smale function if it satisfies the Morse-Smale transversality condition, that is, for every p, q critical points of f, $W^u(q)$ and $W^s(p)$ intersect transversally. By definition, transversal intersection means that for every $r \in W^s(p) \cap W^u(q)$, then $T_rM = T_rW^s(q) + T_rW^u(p)$ and we denote it by $W(q, p) = W^s(p) \pitchfork W^u(q)$.*

Even more, for such Morse-smale functions, $W^s(p)$ and $W^u(q)$ are $m - \lambda_p$ and λ_q dimensional embedded submanifolds of M, respectively, as it is proven in [3] (check appendix B to see the definition of immersion and submersion). Furthermore, under the assumption of transversal intersection of these manifolds, we can also claim that their intersection will also be an smooth manifold. In order to show this, we first recall the following statement which can again be found in [3] (section 5.2):

Theorem 1 (Inverse Image Theorem)**.** *Let $Z \subset N$ be an immersed submanifold and $f : M \to N$ a smooth map. If f and Z intersect transversally, the $f^{-1}(Z)$ is a submanifold of M whose codimension*[1] *in M is the same as the codimension of Z in N, i.e.*

$$\dim M - \dim f^{-1}(Z) = \dim N - \dim Z.$$

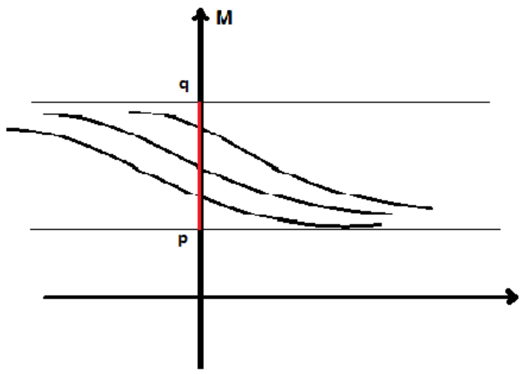
Now, what we would like to show by making use of this theorem is that in fact, $W(q,p) = W^s(p) \pitchfork W^u(q)$ is also a smooth submanifold of M. So, consider $W^s(p) \subset M$ an immersed submanifold of M and consider $i : W^u(q) \to M$ the inclusion map (smooth). Note that we have $i \pitchfork W^u(p)$. Then, by the *Inverse Image Theorem*, $i^{-1}(W^u(p))$ is a smooth submanifold of M. In addition, by the *local immersion theorem* the inclusion map can be given by: $(x_1, \cdots, x_s) \to (x_1, \cdots, x_s, 0, \cdots, 0)$ and hence,

$$i^{-1}(W^u(p)) = \{(x_1, \cdots, x_s) : (x_1, \cdots, x_s, 0, \cdots, 0) \in W^u(p)\} = W^s(p) \cap W^u(q)$$

is a smooth submanifold of M.

This definition of the manifold shows that there is an order relation between the critical points since if $W(q,p) \neq \emptyset$ then there is at least one flow line from q to p and in this case, it is said that q is *successed* by p; $q \succeq p$. Moreover, for every pair (q,p) of critical points with indices λ_q and λ_p respectively, $W(q,p)$ is a $\lambda_q - \lambda_p$ dimensional manifold (from the Theorem 1 as well). Note that this implies that $W(q,p)$ is invariant under the flow and that there is a free action of $\mathbb{R}$ on it.

We observe that such an action identifies all the flow lines that can be translated by the following transformation $t \to t + c$. In order to illustrate the effect of this action we consider the picture in the right side. Note that all the flow lines are identified by one flow line which is just the vertical line (in red) from q to p.



Define $M(q,p) = W(q,p)/\mathbb{R}$ as the *space of connecting flow lines from q to p*. Furthermore, this space is a $\lambda_q - \lambda_p - 1$ dimensional manifold. There is also an equivalent definition of this space in terms of level hypersurfaces [13]:

$M(q,p) = W(q,p) \cap f^{-1}(c)$ where $c \in (f(p), f(q))$ is a regular value.

[2]If $N \subset M$ is a submanifold, the the *codimension* of N is given by: codim(N)= dim(M)-dim(N). Note that the codimension can only be defined for subspaces.

2.1.2. Compactness

In order to define the Morse-Smale chain complex consider the case $\lambda_q - \lambda_p = 1$. Note that $W(q,p)$ is a 1-dimensional manifold and thus, $M(q,p)$ is a zero dimensional manifold. Furthermore, $W(q,p)$ has finitely many components because on one hand, we know that there is no any other critical point between q and p and therefore, this manifold contains closed trajectories. That is, every flow line in $W(q,p)$ starts in q and ends up in p, so $W(q,p)$ is closed. On the other hand, since M is compact $W(q,p) \subset M$ is also compact. Then, the flow lines from q to p cover the manifold and by compactness there is a finite covering for $W(q,p)$, i.e, it has finitely many components. So, from these lines we also get that $M(q,p)$ contains finitely many points and hence, it is a compact submanifold as we will see later and it is proven in [13]. In this case, we define *n(q,p)* as the number of elements corresponding to the flow lines from q to p in $M(q,p)$, i.e., $n(q,p) = \sharp M(q,p)$ in order to define a boundary map.

However, observe that if $\lambda_q - \lambda_p > 1$, $M(q,p)$ may not be compact and thus, $n(q,p)$ will not be well-defined. Nevertheless, if we regard $M(q,p)$ as an intersection of $W(q,p)$ and $f^{-1}(c)$ for some $c \in (f(p), f(q))$ regular value, the topology on the space of gradient flow lines together with the *Gluing Theorem* show the existence of the compactification of $M(q,p)$, denoted by $\overline{M}(q,p)$. As we can imagine such a compactification is not induced from the topology of M and thus, the compactification is done by adding the so-called *broken trajectories*, that is, flow lines which go through critical points. Moreover, such a flow lines will be the boundary or ends of $\overline{M}(q,p)$.

These results can be found in [8] (lemma2.5 and lemma2.6) with the corresponding proofs. Nonetheless, we are only interested on the case that $\lambda_q - \lambda_p$ is 1 or 2 and therefore, we will state these lemmas for such a cases:

Lemma 1. *Let $p, q \in Cr(f)$ with $\lambda_q - \lambda_p = 2$ and let $\{\gamma_i\}$ be a sequence of gradient flow lines in $M(q,p)$, then there exists a subsequence $\{\gamma_{i_j}\}$, a critical point r and real numbers c_1, c_2 with $c_1 \in (f(q), f(r))$ and $c_2 \in (f(r), f(p))$ such that:*

i) $\lambda_q > \lambda_r > \lambda_p$.

ii) the points $b_{1j} = \gamma_{i_j}(s)$ and $b_{2j} = \gamma_{i_j}(s)$ with $f(b_{1j}) = c_{1j}$ and $f(b_{2j}) = c_{2j}$ converge to a regular point in $M(q,r)$ for c_{1j} and to another regular point in $M(r,p)$ for c_{2j}.

Remark 1. Note that this lemma gives the definition of the compactification of the space of connecting orbits denoted by $\overline{M}(q,p)$. Any flow line in $M(q,p)$ either converges to a trajectory in $M(q,p)$ or to a broken flow line (boundary of $M(q,p)$). In fact, whenever the manifold M is not compact such a compactification is determined by the so called *Palais-Smale condition* : given any sequence (x_n) in M with $f(x_n)$ bounded and $df(x_n) \to 0$, then there is a subsequence of (x_n) converging to a critical point.

Theorem 2 (Gluing theorem)**.** *Suppose that p, q, r are critical points of $f : M \to \mathbb{R}$ with $\lambda_q = \lambda_r + 1 = \lambda_q + 2$. Then, for sufficiently small ϵ, there is a diffeomorphism called* gluing map :

$$G : M(q,p) \times M(r,p) \times (0,\epsilon) \longrightarrow M(q,p)$$

mapping onto an open set in $M(q,p)$.

There always exists the compactification of $M(q,p)$ by the broken flow lines so that $n(q,p) = \sharp\overline{M}(q,p)$ is always finite. Moreover, these results will be used to prove that the boundary operator defined by $n(q,p)$ gives a Morse-Smale chain complex. In fact, we can define $n(q,p) = \sharp\overline{M}(q,p)$ where $n(q,p)$ counts the number of the broken trajectories over $\mathbb{Z}_2$.

Before going to chain complex section, we should realise that all these definitions and statements are done under the assumption that $f : M \to \mathbb{R}$ is a Morse-Smale function. It is a natural question to ask how we can find such functions or whether this theory could be applied to any given smooth function. We will see that in fact, given any f smooth and a fixed metric g on M, we can get a Morse-Smale function either by perturbing f or g.

2.2 Existence of Morse-Smale functions

Suppose that M is a smooth finite dimensional compact Riemannian manifold with a metric g. As we said before, we want to know if given any $f : M \to \mathbb{R}$ smooth function we can get a *Morse-Smale* function by either perturbing f or g.

First, suppose that $f : M \to \mathbb{R}$ is a smooth function and perturb it so that f is a *Morse* function. In fact, Y.Matsumoto proved in [22] that if M is a closed and finite dimensional manifold, f can be perturbed to a Morse function for a fixed metric g. In order to show this, Y.Matsumoto first proved it for a function defined over $\mathbb{R}^m$ using *Sard's theorem* and then, he generalized it for a function defined over M as above.

Given a smooth function f as before, it can be perturbed to a Morse function so that its critical points are isolated and non-degenerate. Furthermore, we will prove that if we perturb f to $\tilde{f} = f + h$ for a small perturbation h, their critical points are in one to one correspondence.In order to show this, we will need the following result which is proven in [4] (pages 52-54).

Lemma 2 (Morse-Lemma)**.** *Let $f : M \to \mathbb{R}$ be a smooth function and let $p \in M$ be a non-degenerate critical points of index k. Then, there exists a smooth chart $\psi : U \to \mathbb{R}^m$ where U is an open neighborhood of p and $\psi(p) = 0$ such that:*

$$(f \circ \psi)(x_1, \ldots, x_m) = f(p) - x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2$$

Now, we would like to get a Morse-Smale function from a given f Morse. On one hand, M.Schwarz proved in [20] that any Morse function can be transformed to a Morse-Smale function by perturbing a metric g on M.

However, we are interested in keeping the metric unchanged so far, and thus, we state the following theorem which can be found in [4], [24].

Theorem 3 (Kupka-Smale theorem)**.** *If (M, g) is a finite dimensional compact smooth Riemannian manifold, then the set of Morse-Smale gradient vector field of class C^r is a generic subset of the set of all gradient vector fields on M of class C^r for every $r \geq 1$.*

Definition 3. *Suppose that X is a set and A is a subset of X. Then, A is a generic subset of X if it contains a residual set B which is a countable intersection of open dense subsets.*

We can also find the following result in [8] in Appendix B which is a variant of the *Kupka-Smale theorem*:

Proposition 1. *Let M be a finite dimensional, smooth and compact Riemmanian manifold with a fixed metric g. Then, there is a Baire set $C^{\infty}_{MS}(M,\mathbb{R}) \subset C^{\infty}(M,\mathbb{R})$ so that $f \in C^{\infty}_{MS}(M,\mathbb{R})$ is a Morse-Smale function.*

Its proof can also be found in the Appendix B of [8] in which they show that the universal parametrization of the stable and unstable manifolds are transversal to the diagonal by perturbing the gradient flow.

We conclude that given $f : M \to \mathbb{R}$ smooth function, it can be perturbed to a Morse-Smale function for a fixed metric g on M.

2.3 Stability of the Morse-Smale transversality condition

In this section we want to prove some nice properties of the *Morse-Smale transversality* condition, that is, we show the invariance of this property under small perturbations. In order to show the following results, we will recall the Proposition 1 written in [8].

Lemma 3. *Let M be a finite dimensional compact manifold and suppose that $f : M \to \mathbb{R}$ is a smooth function. Then, there is a Morse-Smale function $\tilde{f} : M \to \mathbb{R}$ defined by a small perturbation with the property that the critical points and their indices of f and $\tilde{f}$ are in one-to-one correspondence.*

Proof. Let $f : M \to \mathbb{R}$ be any smooth function defined on M. Then, from the previous proposition, we know that there exists a $C^{\infty}-$close Morse-Smale function $\tilde{f}$, to f. Therefore, we may write $\tilde{f} = f + h$ where $\|h\|_{C^r} < \epsilon$ for some $\epsilon > 0$ small and $r \geq 1$. Actually, since we will consider a critical point of $\tilde{f}$ and then, analyse f in its neighborhood, it is enough to consider a perturbation h that is zero outside of the neighborhoods of the critical points. Then, we know that whenever we are outside of such a neighborhood $|\nabla \tilde{f}(x)| \geq \delta > 0$ because f is Morse and thus, the critical points are isolated. Moreover, we will also have that

$$|\nabla f(x)| = |\nabla(\tilde{f} - h)(x)| \geq \frac{\delta}{2} > 0 \tag{*}$$

In this way, we make sure that we do not create new critical points outside of the neighborhoods.

Observe that this lower bound *(*)* is also valid on the boundary of the neighborhood so that the critical point of f will also be isolated. What we want to show is that given any $\tilde{p}$ critical point of $\tilde{f}$ and after choosing a neighborhood of $\tilde{p}$, say $U_{\tilde{p}}$, then there is a unique critical point p of f in $U_{\tilde{p}}$ such that $\lambda_p = \lambda_{\tilde{p}}$.

Given $\tilde{p}$ a critical point of index k for $\tilde{f}$, choose $U_{\tilde{p}}$ a small enough open neighborhood, so that by the Morse-lemma, there exists coordinates ψ in $U_{\tilde{p}}$ such that:

$$\begin{gathered}\overline{f}(\mathbf{x}) = (f \circ \psi)(x_1, \ldots, x_m) = \\ = \tilde{f}(p) - a_1 x_1^2 - \cdots - a_k x_k^2 + a_{k+1} x_{k+1}^2 + \cdots a_m x_m^2 - (h \circ \psi)(x_1, \ldots, x_m) = \\ = \tilde{f}(p) - \sum_{i=1}^{k} a_i x_i^2 + \sum_{i=k+1}^{m} a_i x_i^2 - \overline{h}(\mathbf{x}).\end{gathered}$$

Observe that in this expression all the coefficients a_i are positive values.

On one hand, note that given a critical point of f, we have to compute its *Hessian matrix* in order to compute its index and we would like to have the same number of negative eigenvalues for both Hessian matrices which are given as follows:

$$H_p(\tilde{f}) = \begin{pmatrix} -2a_1 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 2a_m \end{pmatrix}$$

and

$$H_p(f) = \begin{pmatrix} -2a_1 - \partial^2_{x_1x_1}\overline{h} & \cdots & -\partial^2_{x_1x_m}\overline{h} \\ \vdots & & \vdots \\ -\partial^2_{x_mx_1}\overline{h} & \cdots & 2a_m - \partial^2_{x_mx_m}\overline{h} \end{pmatrix}$$

Note that by hypothesis $\|h\|_{C^\infty} < \epsilon$, so h and its derivatives are very small values. In particular, if we compare $-\partial_{x_ix_i}\overline{h}$ with $2a_i$ for $i = 0, \ldots, m$ then we may assume that this derivative will not change the sign of such a term $2a_i$. That is, whenever $2a_i$ is positive (by assumption), then $2a_i - \partial^2_{x_ix_i}\overline{h}$ is also positive. Further, the contribution of the mixed second derivatives will also be very small. Then, we claim that both matrices have the same number of negative eigenvalues and thus, both critical points will have the same index k. Not only this, but for any point in this neighborhood the number of negative eigenvalues remain unchanged.

On the other hand, in order to show that f has a unique critical point in $U_{\tilde{p}}$, we will make use of the *degree theory* (for basic definitions and results, see appendix A). We first consider the vector fields generated by the gradient of these two functions and show that their degree coincide. This will imply that since $\tilde{f}$ has a ciritcal point and hence, the vector field has one zero, then f has at least one critical point in this neighborhood. The next step will be to show the uniqueness of such a critical point.

First, consider the following vector fields:

$$X = \nabla\overline{f} : \mathbb{R}^m \to \mathbb{R}^m$$
$$(x_1, \ldots, x_m) \to (\partial_{x_1}\overline{f}, \ldots, \partial_{x_m}\overline{f})$$

and

$$\tilde{X} = \nabla\overline{\tilde{f}} : \mathbb{R}^m \to \mathbb{R}^m$$
$$(x_1, \ldots, x_m) \to (\partial_{x_1}\overline{\tilde{f}}, \ldots, \partial_{x_m}\overline{\tilde{f}})$$

Now we want show that they have the same degree so that we prove that X has at least one zero. First, note that the zero of $\nabla\overline{\tilde{f}}$ is isolated and thus, $0 \notin \tilde{X}(\partial U_{\tilde{p}})$. We want $0 \notin X(\partial U_p)$ so that we can apply the invariance property of the degree through the homotopies.

Choose a homotopy between these two vector fields X and $\tilde{X}$, $H_t : U_{\tilde{p}} \times [0,1] \to \mathbb{R}$ given by $H_t(x) = (1-t)X + t\tilde{X}$ with $0 \notin H_t(\partial U_{\tilde{p}})$. Then, we have that:

$$deg(X, U_{\tilde{p}}, 0) = deg(\tilde{X}, U_{\tilde{p}}, 0)$$

which implies that X has at least one zero in $U_{\tilde{p}}$, i.e., f has at least one critical point in this neighborhood.

The next step would be to check if this critical point is in fact unique. So, suppose that $\tilde{X}$ has more than one, say s. Then, by definition, the degree of this vector field is given by the determinant of the Jacobian matrix which in this case, is the same as the Hessian matrix of $\tilde{f}$:

$$H_p(\overline{f}) = \begin{pmatrix} -2a_1 - \partial^2_{x_1x_1}\overline{h} & \cdots & -\partial^2_{x_1x_m}\overline{h} \\ \vdots & & \vdots \\ -\partial^2_{x_mx_1}\overline{h} & \cdots & 2a_m - \partial^2_{x_mx_m}\overline{h} \end{pmatrix}$$

Then, the *Jacobian* is just the determinant of the matrix, that is, $J_X(x) = det(H_p(f))$.

Recall that for every point in the neighborhood this matrix has the same number of negative eigenvalues; that is, the sign of the determinant of this matrix remains unchanged. Furthermore, the local degree is always either 1 or -1. So, this implies that if we compute the absolute value of the degree by using the definition, then the sum will give just the number of zeros that the vector field has in the neighborhood, that is,

$$|deg(X, U_{\tilde{p}}, 0)| = |\sum_{x \in X^{-1}(0)} sign(J_X(0))| = s > 1$$

This will contradict that X and $\tilde{X}$ have the same degree. Therefore, X only has one zero in this neighborhood or equivalently, f has a unique critical point in $U_{\tilde{p}}$.

Hence, we have shown that given any critical point of index k of $\tilde{f}$ there is a unique critical point with the same index in a small neighborhood of it for f. If we consider this result and apply in a small neighborhood of every critical point of $\tilde{f}$ by knowing that we do not create more critical points outside these $U_{\tilde{p}}$, then we get the desired result. $\square$

Corollary 1. Given any $F : M \times S^1 \to \mathbb{R}$ Morse function with the property that $F(0, .)$ and $F(1, .)$ are Morse-Smale functions and M is a finite dimensional compact manifold, then there is a C^1-close function $\tilde{F} = F + h$ with the property that $\tilde{F}$ defines a Morse-Smale gradient-system in $M \times S^1$.

2.4 Chain complex and Homology

Let $C_k(f)$ be a free abelian group generated by the critical points of index k for f: $M \to \mathbb{R}$ Morse-Smale function and we define the *Morse-Smale boundary operator* as follows:

$$\partial_k : C_k \to C_{k-1}$$

given by $\partial_k(q) = \sum_{p \in C_{k-1}} n(q,p)p$ where as we said before $n(q,p) = \sharp\overline{M}(q,p)$ counts the number of (broken-) flow lines from q to p over $\mathbb{Z}_2$ and thus, these values are just 0 or 1 (even or odd) modulo 2.

We need to show that the above relation gives a chain complex, that is, ∂_k decreases the degree by one and $\partial_{k+1} \circ \partial_k = 0$. The first condition is clear from the way we have defined $n(q,p)$. For the second condition, suppose that $q \in C_k(f)$ and compute:

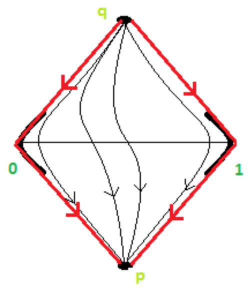
$$(\partial_{k-1} \circ \partial_k)(q) = \partial_{k-1}(\sum_{r \in C_{k-1}} n(q,r)r) = \sum_{r \in C_{k-1}} n(q,r)\partial_{k-1}(r) =$$

$$= \sum_{r \in C_{k-1}} n(q,r)(\sum_{p \in C_{k-2}} n(r,p)p) =$$

$$= (\sum_{r \in C_{k-1}} n(q,r). \sum_{p \in C_{k-2}} n(r,p))p =$$

$$= \sum_{r \in C_{k-1}} \sum_{p \in C_{k-2}} n(q,r)n(r,p)p$$

Note that $n(q,r)n(r,p)$ represents the number of elements of $\overline{M}(q,p)$ where $M(q,p) = W(q,p)/\mathbb{R}$ has dimension 1 since $W(q,p) = W^u(q) \cap W^s(p)$ is a $\lambda_q - \lambda_p = 2$-dimensional manifold as we have seen in Section 2.1.1. Then, $M(q,p)$ can only contain closed orbits (circles) and open intervals (say, (0,1) $\simeq \mathbb{R}$) and the compactification of such manifolds can be done by adding one or two points; or equivalently, by adding one or two broken flow lines. Note that if we consider a circle without one point and assume that it can be compactified by adding just one point, then we can approximate to such a point in two different directions of the circle. But, this conclusion would contradict the *Gluing theorem* from [8] which implies the uniqueness of such a compatification. Therefore, there must be an even number of such a broken lines for the circles. Similarly, if we consider the open interval (0,1) we will conclude that it must also be compactified by two points. As an illustration, we consider the following picture:



Indeed, as it can be seen in the picture, if we look at the ends of the interval (0,1) when compactified [0,1], we need to add two flow lines. If not, supposed that it can be done with just one broken trajectory. Then, if we connect q to 0 with a broken line, then this type of trajectories cannot compactify the other side. So, there must be two flow lines and in general, there will be even number of them.

In fact, this implies the following result:

$$\sum_{r \in C_{k-1}} \sum_{p \in C_{k-2}} n(q,r)n(r,p) = 0$$

over $\mathbb{Z}_2$. Hence, $\partial_{k-1} \circ \partial_k = 0$ for every $k \geq 1$.

We have seen that $(C_*(f), \partial_*(f))$ is a chain complex which will be called the *Morse-Smale chain complex of f.*

Lemma 4. *The Morse-Smale complex $(C_*(f), \partial_*(f))$ defined as above is a chain complex, that is, $(\partial_*(f))^2 = 0$.*

It is well-known that whenever there is a chain complex, then there exist the corresponding homology groups and we get the following result:

Theorem 4. *Given f a Morse-Smale function on a compact, smooth and m-dimensional Riemmanian manifold M, the Morse-Smale homology is given by:*

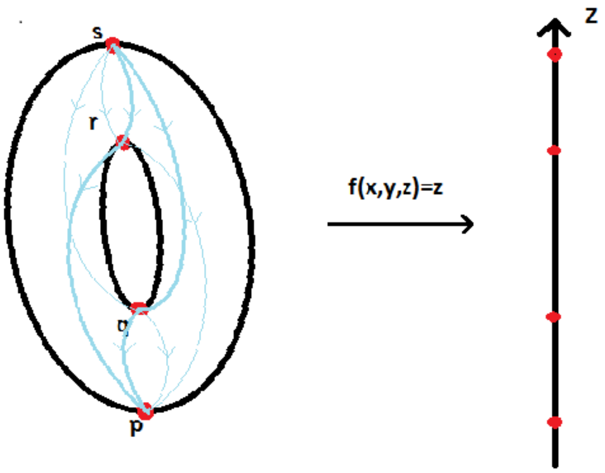
$$HM_k(C_*, \partial_*) = \frac{ker\partial_k}{Im\partial_{k+1}}$$

for every $k \geq 0$.

2.5 Example

Example 1 (1.The Tilted Torus)**.** Let $\mathbb{T}^2$ be a torus embedded in $\mathbb{R}^3$ and consider f: $\mathbb{T}^2$ $\to R$ a height function $f(x, y, z) = z$. We first realise that if we just consider a torus with this function, then it is not a Morse-Smale function because the flow would go from the point r to q which have the same index (see the picture). However, we can perturb (in this case, tilt) it a bit so that we get a Morse-Smale function, called f again. This can be assumed by Kupka-Smale *Theorem* 3 which says that the space of all C^r Morse-Smale function is dense in $C^r(M, (R))$. Furthermore, this perturbed function f will also have the same number and indices of critical points as the height function by Lemma 3.

So, we start analysing the critical points: there are 4 critical points, namely, p, q, r and s with indices $\lambda_p = 0$, $\lambda_q = \lambda_r = 1$ and $\lambda_s = 2$.



Then, we compute the following free abelian groups:

1. $C_0 = \mathbb{Z}_2\langle p\rangle \simeq \mathbb{Z}_2$,
2. $C_1 = \mathbb{Z}_2\langle q\rangle \oplus \mathbb{Z}_2\langle r\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
3. $C_2 = \mathbb{Z}_2\langle s\rangle \simeq \mathbb{Z}_2$

from which we get the next Morse-Smale chain complex:

$$0 \longrightarrow \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \oplus \mathbb{Z}_2 \longrightarrow \mathbb{Z}_2 \longrightarrow 0$$

Now, we consider the critical points of relative index one and compute $n(q,p)$ for every p,q critical points, that is, the number of flow lines from q to p. Note that since we are working over $\mathbb{Z}_2$ we do not need to consider the orientation of the flow, but the number of the flow lines (even or odd):

1. Clearly, $\partial_0 = 0$
2. $n(s,r) = 1+1 = 0 = n(s,q) \to \partial_2 = 0$
3. $n(r,p) = n(s,p) = 1+1 = 0 \to \partial_1 = 0$

Then, since all the boundary maps are zero, we have:

1. $H_0(T^2;\mathbb{Z}_2) = \mathbb{Z}_2$,
2. $H_1(T^2;\mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2$
3. $H_2(T^2;\mathbb{Z}_2) = \mathbb{Z}_2$

3.Invariance of Morse-Smale Homology

3.1 Independence of a Morse-Smale function

The main objective of this section is to prove the following theorem:

Theorem 5. *Let $f, \tilde{f}$ be two Morse-Smale functions on a smooth, compact, oriented m-dimensional Riemannian manifold M. Then, the Morse-Smale homology groups are isomorphic, that is,*

$$HM_*(f) \cong HM_*(\tilde{f})$$

In order to show that this result holds, first consider a homotopy between f and $\tilde{f}$ over $M \times S^1$ and use this function to define a Morse-Smale function called $F : M \times S^1 \to \mathbb{R}$. Since this function is defined on a compact, smooth and finite dimensional manifold $M \times S^1$ and it is a Morse-Smale function then, from Section 2, if we consider the gradient-system defined by this function, there is a chain complex defined by the critical points of F. More generally, we could consider a Morse-Smale function defined over $M \times S^1 \times \cdots \times S^1$ to get the same results.

After observing this fact, our idea is to construct a homotopy that will induce a homomorphism between the homology groups and then, consider the homotopy of homotopies to conclude that such a map is indeed an isomorphism.

(i) Homotopy:

Recall that a linear homotopy from $\tilde{f}$ to f can be given as follows:

$h_\lambda : M \times [0,1] \to \mathbb{R}$ where $h_\lambda(x) = \lambda f(x) + (1-\lambda)\tilde{f}(x)$, with $\lambda \in [0,1]$.

Nevertheless, we want to define this homotopy over $S^1 \simeq \mathbb{R}/2\mathbb{Z}$ and we define a function $\omega : S^1 \to [0,1]$ which will be used to define the linear homotopy. Furthermore, this function will have the following properties:

1. $\omega(\mu) = \omega(2-\mu)$ for $\mu \in S^1$.
2. $\omega(\mu) = 0$ for $\mu \in [0, \delta]$ and $\omega(\mu) = 1$ for $\mu \in [1-\delta, 1+\delta]$ and some $\delta \geq 0$.

Observe that the first condition shows the symmetry that the circle has when we identify as a quotient of $\mathbb{R}/2\mathbb{Z}$. Note that the homotopy is constant for some values of μ by the second condition as it is shown in Figure 1.

The reason for which we choose a linear homotopy is that in this way we make sure that the connecting flow lines going from $\tilde{f}$ to f do not intersect their self and thus, there is not any non-transversal intersection in between.

We define the function $F : M \times S^1 \to \mathbb{R}$ as follows:

$$F(x, \mu) = h_{\omega(\mu)}(x) + \kappa\Lambda(\mu)$$

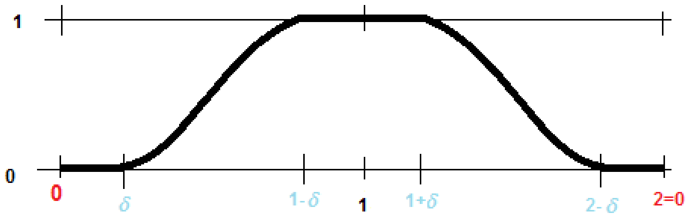


Figure 1: Function $\omega(\mu)$

where $\mu \in [0,2]$, $\Lambda(\mu) = 1 + cos(\pi\mu)$ and κ will be a "sufficiently large" constant. As we said before, we have defined this function F on a compact manifold and thus, now we want to define a gradient-flow defined by it:

$$\begin{cases} x' = -\frac{\partial}{\partial x} h_{\omega(\mu)}(x) \\ \mu' = -\kappa\Lambda'(\mu) - \frac{\partial}{\partial \mu} h_{\omega(\mu)}(x) \end{cases}$$

where $\Lambda'(\mu) = \pi \sin(\pi\mu)$ and $\mu \in [0,2]$.

Note that the system has been perturbed by adding the term $\frac{\partial}{\partial \mu} h_{\omega(\mu)}(x)$. But, we know that depending on the value of δ, near to 0 and 1 we again get the original dynamical system. Although this perturbation will change the values of the system, we would like to define a system with the same behavior as the original one. Therefore, we need this new term to be small compared to $\kappa\Lambda'(\mu)$. Our idea is to get an upper bound for $\frac{\partial}{\partial \mu} h_{\omega(\mu)}(x)$ so that we can choose some larger value κ and make sure that the behavior of such a system remains unchanged.

First write:

$$\frac{\partial}{\partial \mu} h_{\omega(\mu)}(x) = \omega'(\mu)f(x) - \omega'(\mu)\tilde{f}(x) = \omega'(\mu)(f(x) - \tilde{f}(x))$$

On one hand, we define

$$K := max\{|f(x) - \tilde{f}(x)| : x \in M\}$$

so that $|\frac{f(x)-\tilde{f}(x)}{K}| \leq 1$ for all $x \in M$.

On the other hand, $\omega'(\mu)$ can also be bounded by 2 if we choose $\delta \in [0, \frac{1}{4}]$ because the slope is determined as follows for $\mu \in [0,1]$:

$$m = \frac{1-0}{(1-\delta)-\delta} = \frac{1}{1-2\delta} \leq 2 \leftrightarrow 1 \leq 2(1-2\delta) \leftrightarrow \delta \leq \frac{1}{4}.$$

Hence, we get the following upper bound:

$$|\frac{\partial}{\partial \mu} h_{\omega(\mu)}(x)| = |\omega'(\mu)(f(x) - \tilde{f}(x))| = |\omega'(\mu)||(f(x) - \tilde{f}(x))| \leq 2K.$$

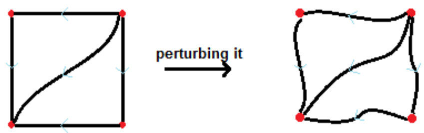
Finally, choose $\kappa \geq 2K$ so that the perturbation does not change the behavior of the dynamics neither in $(\delta, 1-\delta)$ nor in $(1+\delta, 2-\delta)$.

(ii) Morse-Smale transversality condition:

In the previous step we have constructed a system by defining a homotopy between f and $\tilde{f}$ which satisfies the Morse-Smale transversality condition near to $\mu = 0, 1 mod 2$. However, we would like to conclude that the system itself can be perturbed to a Morse-Smale system. First of all, we know that its critical points are precisely the critical points of f and $\tilde{f}$ which by definition are non-degenerate. So, this implies that the original system itself is Morse.

Furthermore, by Lemma 3 in Section 2.3, F can be perturbed to a Morse-Smale system, say $\tilde{F}$, so that the critical points of F and $\tilde{F}$ are in one to one correspondence. Moreover, since we have defined F by a linear homotopy, we know that the transversality conditions will not be lost under such a perturbation and that there will not be more transversal intersection in the connecting orbits. In fact, we conclude that $\tilde{F}$ is Morse-Smale.

So, even though F is a Morse system, we know that it can be perturbed to a Morse-Smale system which will have the same amount of critical points with the corresponding indices. Note that this is all we need to define the free abelian groups and therefore, the homology groups of the system. Hence, denote the perturbed function by F again so that we can construct its homology groups as we have done in the previous section.



Remark 2. Note that in the images above we have drawn straight lines even though we have defined the homotopy over S^1. Recall that S^1 has been regarded as $\mathbb{Z}_2$ and thus, we have a periodic homotopy from f to f through $\tilde{f}$ for the intervals of length two, say [0,2]. Similarly, the periodic homotopy can be considered from $\tilde{f}$ to $\tilde{f}$ through f. However, the circle is symmetric and the homotopy can be regarded as a unique "double" line from $\tilde{f}$ to f of length 1. So, for simplicity we will consider straight lines to represents the homotopies as we have done above.

(iii) Chain maps:

We have already seen that the critical points of $F(x,\mu)$ are given by the critical points of f and $\tilde{f}$. Furthermore, we also know that such a function is defined on a finite dimensional, smooth, compact and oriented manifold $M \times S^1$ by hypothesis and therefore, by Lemma 4 from Section 2.3 we know that there exists the corresponding chain complex denoted by $(C_*(F), \partial_*(F))$.

So, we define $C_k(F) := C_k(f) \oplus C_{k-1}(\tilde{f})$ to be the free abelian group generated by the critical points of $F(x,\mu)$. In order to define the boundary map, we need to consider the maps that connect critical points of relative index one. That is, the boundary maps ∂_* and $\tilde{\partial}_*$ corresponding to the chain complexes of f and $\tilde{f}$ respectively, and the homomorphism induced by the homotopy which connects the critical points of f and $\tilde{f}$ of relative index one. Note that in order to get the critical points of index k of F we need to consider the critical points of index $k-1$ of $\tilde{f}$ and due to this construction we need to define the boundary map as follows:

$$\partial_k(F) := \begin{pmatrix} \partial_k & \overline{h}_{k-1} \\ 0 & \tilde{\partial}_{k-1} \end{pmatrix}$$

where ∂_* and $\tilde{\partial}_*$ are the boundary maps associated to f and $\tilde{f}$ respectively and $\overline{h}_* : C_*(\tilde{f}) \to C_*(f)$. We have defined a Morse-Smale chain complex $(C_*(F), \partial_*(F))$ and as we said before this will give the following relation:

$$\partial_{k-1}(F) \circ \partial_k(F) := \begin{pmatrix} \partial_{k-1} \circ \partial_k & \partial_{k-1} \circ \overline{h}_{k-1} + \overline{h}_{k-2} \circ \tilde{\partial}_{k-1} \\ 0 & \tilde{\partial}_{k-2} \circ \tilde{\partial}_{k-1} \end{pmatrix}$$

From $\partial_{k-1} \circ \partial_k = 0$ we get that $\partial_{k-1} \circ \overline{h}_{k-1} + \overline{h}_{k-2} \circ \tilde{\partial}_{k-1} = 0$, that is, $\partial_{k-1} \circ \overline{h}_{k-1} = \overline{h}_{k-2} \circ \tilde{\partial}_{k-1}$ over $\mathbb{Z}_2$. This implies that $\overline{h}_*$ also induces a homomorphism on homology. In order to show this, we have to check if this map preserves the kernel and the image:

i) if $x \in Ker\tilde{\partial}_k$, then $\tilde{\partial}_k(x) = 0 \in C_{k-1}(\tilde{f})$. Since $\overline{h}_{k-1}$ is a homomorphism $0 = \overline{h}_{k-1}\tilde{\partial}_k(x) = \partial_k \overline{h}_k(x) \Leftrightarrow \overline{h}_k(x) \in Ker\partial_k$. So, the kernels are in correspondence.

ii) if $x \in Im\tilde{\partial}_{k+1}$ with $\tilde{\partial}_{k+1}(x') = x$, then $\overline{h}_k(\partial_{k+1}(x')) \neq 0$. Now, from the above equality: $\partial_k(\overline{h}_k(x')) \neq 0 \Rightarrow \overline{h}_k(x') \in Im\partial_k$. Hence, the images are also in correspondence.

Therefore, we have seen that $\overline{h}_*$ induces a homomorphism on homology denoted by $\overline{h}_* : HM_*(\tilde{f}) \to HM_*(f)$.

(iv) Homotopy of homotopies:

In order to show that the previous homomorphism is an isomorphism, suppose that we are given four Morse-Smale functions on M, namely, f^1, f^2, f^3 and f^4; that is, $f^i : M \to \mathbb{R}$ for $i = 1,2,3,4$. Then, the linear homotopy can be defined as follows:

$$h_{\lambda_1,\lambda_2} : M \times S^1 \times S^1 \to \mathbb{R}$$

given by

$$h_{\lambda_1,\lambda_2}(x) = \lambda_2(\lambda_1 f^1(x) + (1-\lambda_1) f^2(x)) + (1-\lambda_2)(\lambda_1 f^3(x) + (1-\lambda_1) f^4(x))$$

for $\lambda_1, \lambda_2 \in [0,1]$.

Nonetheless, we define a linear homotopy over $M \times S^1 \times S^1$ and we again identify $S^1 \simeq \mathbb{R}/2\mathbb{Z}$. Hence, we need to follow the same steps as before.

First, let $\omega(\mu_1), \omega(\mu_2)$ to be defined as in Figure 1 and denote by $h_{\omega(\mu_1),\omega(\mu_2)}(x)$ the linear homotopy. Now, define:

$$F(x, \mu_1, \mu_2) = h_{\omega(\mu_1),\omega(\mu_2)}(x) + \kappa_1 \Lambda(\mu_1) + \kappa_2 \Lambda(\mu_2)$$

over $M \times S^1 \times S^1$, where $\mu_1, \mu_2 \in [0,2]$, $\Lambda(\mu_1) = 1 + cos(\mu_1 \pi)$, $\Lambda(\mu_2) = 1 + cos(\mu_2 \pi)$ and κ_1, κ_2 are sufficiently large constants.

Consider the gradient-system generated by F, that is:

$$\begin{cases} x' = -\nabla h_{\omega(\mu_1),\omega(\mu_2)}(x) \\ \mu_1' = -\kappa_1 \Lambda'(\mu_1) - \frac{\partial}{\partial \mu_1} h_{\omega(\mu_1),\omega(\mu_2)}(x) \\ \mu_2' = -\kappa_2 \Lambda'(\mu_2) - \frac{\partial}{\partial \mu_2} h_{\omega(\mu_1),\omega(\mu_2)}(x) \end{cases}$$

Clearly, we have the same situation as with the system defined by the previous homotopy, that is, we have perturbed the original system by adding the terms $\frac{\partial}{\partial \mu_1} h_{\omega(\mu_1),\omega(\mu_2)}(x)$ and $\frac{\partial}{\partial \mu_2} h_{\omega(\mu_1),\omega(\mu_2)}$. In order to make sure that the behavior of the original system does not change we can always find an upper bound for these values because we have considered $\delta \in [0, \frac{1}{4}]$ and then, it is enough κ_1 and κ_2 to be large compared to such an upper bounds.

Furthermore, since under the perturbation the non-degeneracy of the critical points will still remain unchanged and in every "corner" the system will still be Morse-Smale, we can assume that the whole system will also be a Morse-Smale system by applying the results from Section 2.2 as before.

Now, we will consider the Morse-Smale chain complex of this Morse-Smale gradient system. So, denote by $C_*(F)$ the free abelian group generated by the critical points of $F(x, \mu_1, \mu_2)$. Note that, this group is also determined by the critical points of the given Morse-Smale functions, namely, f^1, f^2, f^3 and f^4. More precisely, we have the following relation:

$$C_k(M \times S^1 \times S^1, F) \simeq C_k(M, f^1) \oplus C_{k-1}(M, f^2) \oplus C_{k-1}(M, f^3) \oplus C_{k-2}(M, f^4)$$

We are only interested in the case in which $f^1 = f^3 = f$ and $f^2 = f^4 = \tilde{f}$ and we will develop the theory for this case because both the notation and the computations will be easier than for the general case (in fact, the general case can be found in [13]). So, for this case, the groups generated by critical points are given as follows for $k \geq 0$:

$$C_k(M \times S^1 \times S^1, F) \simeq C_k(M, f^1) \oplus C_{k-1}(M, f^2) \oplus C_{k-1}(M, f^1) \oplus C_{k-2}(M, f^2)$$

In order to define the boundary map associated to $C_k(M \times S^1 \times S^1, F)$, we need to consider all the critical points of relative index one of f^1 and f^2. Clearly, $\partial_k^i : C_k(f^i) \to C_{k-1}(f^i)$ for $i = 1, 2$ are the boundary maps of f^1 and f^2 and let $h_k^{12} : C_k(f^2) \to C_k(f^1)$

for $\mu_2 = 0$ and $\tilde{h}^{12}_k : C_k(f^2) \to C_k(f^1)$ for $\mu_2 = 1$ be homomorphisms induced by the homotopy which also relate the critical points of relative index one.

Note that for us, the induced homomorphisms h^{31}_* and h^{42}_* are just the identity maps and finally, we define an extra map, namely, $G_k : C_k(f^2) \to C_{k+1}(f^1)$.

Now we can define a boundary map associated to $C_k(M \times S^1 \times S^1, F)$ given by:

$$\partial_k(V) := \begin{pmatrix} \partial^1_k & h^{12}_{k-1} & \mathrm{I} & G_{k-2} \\ 0 & \partial^2_{k-1} & 0 & \mathrm{I} \\ 0 & 0 & \partial^1_{k-1} & \tilde{h}^{12}_{k-2} \\ 0 & 0 & 0 & \partial^2_{k-2} \end{pmatrix}$$

Therefore, we have defined a Morse-Smale chain complex $(C_*(F), \partial_*(F))$ and by Lemma 2 in Section 2.2 we know that $\partial^2(F) = 0$. Note that this composition is given by the following matrix:

$$\begin{pmatrix} \partial^1_{k-1}\partial^1_k & \partial^1_{k-1}h^{12}_{k-1} + h^{12}_{k-2}\partial^2_{k-1} & \partial^1_{k-1} + \partial^1_{k-2} & \partial^1_{k-1}G_{k-2} + h^{12}_{k-2} + \tilde{h}^{12}_{k-2} + G_{k-3}\partial^2_{k-2} \\ 0 & \partial^2_{k-2}\partial^2_{k-1} & 0 & \partial^2_{k-2} + \partial^2_{k-2} \\ 0 & 0 & \partial^1_{k-2}\partial^1_{k-1} & \partial^1_{k-2}\tilde{h}^{12}_{k-2} + \tilde{h}^{12}_{k-3}\partial^2_{k-2} \\ 0 & 0 & 0 & \partial^2_{k-3} \circ \partial^2_{k-2} \end{pmatrix}$$

Since this matrix is zero, we get the following results:

(I) $\partial^i_{k-1} \circ \partial^i_k = 0$ for $i = 1, 2$ (clearly, because these maps are boundary maps)

(II) $\partial^1_{k-1}h^{12}_{k-1} + h^{12}_{k-2}\partial^2_{k-2} = 0$ and $\partial^1_{k-2}\tilde{h}^{12}_{k-2} + \tilde{h}^{12}_{k-3}\partial^2_{k-2} = 0$ This implies that h^{12}_* and $\tilde{h}^{12}_*$ induce a homomorphism on homology as we showed in the previous section.In particular, if $h^{ij} = id$.

(III) $\partial^1_{k-1}G_{k-2} + h^{12}_{k-2} + \tilde{h}^{12}_{k-2} + G_{k-3}\partial^2_{k-2} = 0 \Longleftrightarrow$
$\partial^1_{k-1}G_{k-2} + G_{k-3}\partial^2_{k-2} = h^{12}_{k-2} + \tilde{h}^{12}_{k-2}$
This equality implies that G_* induces homotopic maps h^{12}_* and $\tilde{h}^{12}_*$, that is, these maps induce the same homomorphism on homology. In order to show this, note that since the homology groups are just quotient groups by the kernel and the image (or boundary) of the chain maps, the two elements (in our case, two functions) are the same, if their difference equals to an element on the image. Therefore, suppose that $x \in Ker\partial^2_{k-2}$. Then, $G_{k-3}\partial^2_{k-2}(x) = 0$ because G_* is a homomorphism. Then, note that the remaining term in the left side of the equation $\partial^1_{k-1}G_{k-2}(x)$ is just the boundary of some element $y = G_{k-2}(x)$. Hence, h^{12}_{k-2} and $\tilde{h}^{12}_{k-2}$ differ by an element in the boundary and thus, they induce the same homomorphism on homology.

Remark 3. Note that in the last case, we have said that h^{12}_* and $\tilde{h}^{12}_*$ differ by an element on the boundary even if in the right side we have a sum of these values. Recall that this is still true because we are working over $\mathbb{Z}_2$.

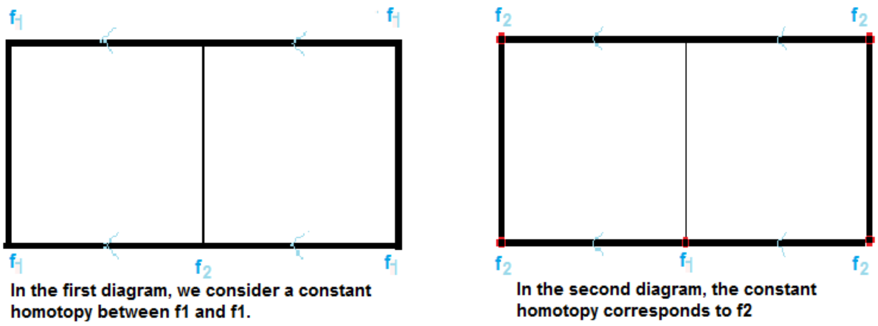
Now, what we want to prove is that these homomorphisms on homology are isomorphisms, that is, for the homomorphism $\overline{h}^{ij}_* : HM_*(f^i) \to HM_*(f^j)$ we want to see that the following holds:

a) $\overline{h}^{ii} = 1$

b) $\overline{h}^{ij} = (\overline{h}^{ji})^{-1}$ for $i = 1, 2$.

Note that the first result is clear because the identity maps induce identity maps on homology, and therefore, this is done. In order to show the second result we will consider the diagrams drawn below:

Figure 2: Diagram of homotopies between f and $\tilde{f}$



Consider the circle S^1 as the following diagrams in Figure 2 where the horizontal lines "represent" the interval $[0, 2]$. Note that if we move on the circle from 0 to 2, this can be understood as a constant homotopy from f^1 to f^1 or as a composition of two maps, namely, $h^{12} \circ h^{21}$. Furthermore, we know that the constant homotopies induce an identity map on homology and thus, we have:

$$\overline{h^{12} \circ h^{21}} = \overline{h}^{12} \circ \overline{h}^{21} = \overline{h}^{11} = I$$

Equivalently, we can also move in the circle from -1 to 1 and this can be understood as a constant homotopy from f^2 to f^2 or as a composition of two maps, namely, $h^{21} \circ h^{12}$. As in the previous case, the constant homotopies induce identity maps on homology and therefore,

$$\overline{h}^{21} \circ \overline{h}^{12} = I$$

Hence, we conclude that $\overline{h}_*^{12} = (\overline{h}_*^{21})^{-1}$, that is, these maps are isomorphism and in particular, $HM_*(f) \cong HM_*(\tilde{f})$.

3.2 Morse-Smale Homology for smooth functions

In the previous section we have seen that given two any Morse-Smale functions over a smooth, oriented, compact and m-dimensional manifold, then their Morse-Homology groups are well-defined and moreover, they are isomorphic. Now, the next question would be what happens if we just have a smooth function. We have seen in Section 2.2 that given any smooth function f, we can find a Morse-Smale function $\tilde{f}$ by *Kupka-Smale Theorem*. Even more, we have also shown in Lemma 3 that for such a smooth function we can find a Morse-Smale function with the same number of critical points and with the same indices. So, the Morse-Smale homology group for any function is defined in the following statement:

Definition 4. *Given any smooth function on M as above, then denote by $\tilde{f}$ any approximation to a Morse-Smale function. Then,*

$$HM_*(f) := HM_*(\tilde{f})$$

is its Morse-Smale homology group.

Observe that this homology group is an invariant since it does not depend on the perturbation of f. That is, if a given smooth function f can be perturbed to two different Morse-Smale functions, say f^1, f^2, then by Theorem 3 of Section 2, we know that the corresponding Morse-Smale homology groups are well-defined. Moreover, by the previous theorem we know that they are isomorphic and thus,

$$HM_*(f^1) \cong HM_*(f^2) \cong HM_*(f)$$

3.3 Computation of Morse-Smale Homology

3.3.1 Main Result and Examples:

From the previous sections we know that for any Morse-Smale function its Morse-Smale homology group is well-defined. Even more, we have seen that given any smooth function, its Morse-Smale homology groups is also well-defined. This implies that these groups do not depend on the function we consider. However, now we want to state a more general result:

Theorem 6 (The Morse Homology Theorem)**.** *Let $(C_*(f), \partial_*(f))$ be a Morse-Smale chain complex. Then, its homology is isomorphic to the singular homology $H_*(M; \mathbb{Z}_2)$ of a manifold.*

Instead of showing this result, we give an example as in the previous section.

Example 2 (Homology group of a manifold M of genus g)**.** Let M be a smooth, finite dimensional, compact riemannian manifold of genus g embedded in $\mathbb{R}^3$. Consider $f : M \to \mathbb{R}$ a height function. Note that as in the example of the torus (g=1), this function connects the consecutive critical points of index one, namely all the saddle points. Therefore, we need to tilt it so that we "break" such connections.

Let f be a height function defined over a manifold M of genus g. Then, this function has one critical point of index 2 namely, the maximum, denoted by p_{2g+2}; there are $2g$ critical points of index 1, that is, the saddle points $p_2, \cdots p_{2g+1}$ and there is a unique

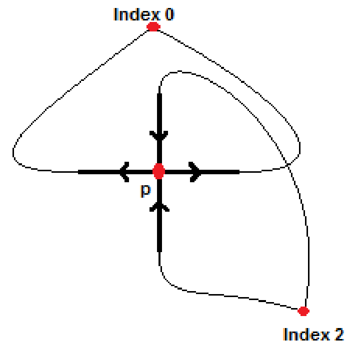


Figure 3: Saddle point

minimum p_1 of index 0. Furthermore, from Section 3, we know that this function can be perturbed to a Morse-Smale function $\tilde{f}$ so that both the critical points and their indices are in one-to-one correspondence. As a result, $\tilde{f}$ does not connect the critical points of the same index, that is, there is no any gradient- flow line between the saddle points and this holds as follows:

Consider any saddle point, say p_k. Note that the unstable and stable manifolds, $W^s(p_k)$, $W^u(p_k)$ have dimension 1 as it can be seen in *Figure* 3. Every point in the unstable manifold of p_k converges to a critical point of index $< \lambda_{p_k} = 1$ since we are considering a gradient flow generated by a Morse-Smale function $\tilde{f}$. Observe that the unique critical point of index 0 is p_1 and thus, both flow lines in the *Figure* 3 converge to it. Similarly, every point in the stable manifold of p_k converges to a critical point of index $> \lambda_{p_k} = 1$ in negative time. Note that the unique critical point of index 2 is in fact, the maximum p_{2g}. Therefore, both flow lines converge to the maximum.

So, even though there are infinitely many flow lines going out from the maximum p_{2g}, there are exactly two of them going to the saddle points and from each saddle point exactly two flow lines converge to the minimum p_1.

In order to compute the homology groups, we first determine the free abelian groups generated for these critical points together with the boundary maps:

1. $C_0 = \mathbb{Z}_2\langle p_1\rangle \simeq \mathbb{Z}_2$,
2. $C_1 = \mathbb{Z}_2\langle p_2\rangle \oplus \mathbb{Z}_2\langle p_3\rangle \oplus \cdots \oplus \mathbb{Z}_2\langle p_{2g+1}\rangle \simeq \mathbb{Z}_2^{2g}$,
3. $C_2 = \mathbb{Z}_2\langle p_{2g+2}\rangle \simeq \mathbb{Z}_2$

We have the following chain complex:

$$0 \xrightarrow{\partial_3} \mathbb{Z}_2 \xrightarrow{\partial_2} \mathbb{Z}_2^{2g} \xrightarrow{\partial_1} \mathbb{Z}_2 \xrightarrow{\partial_0} 0$$

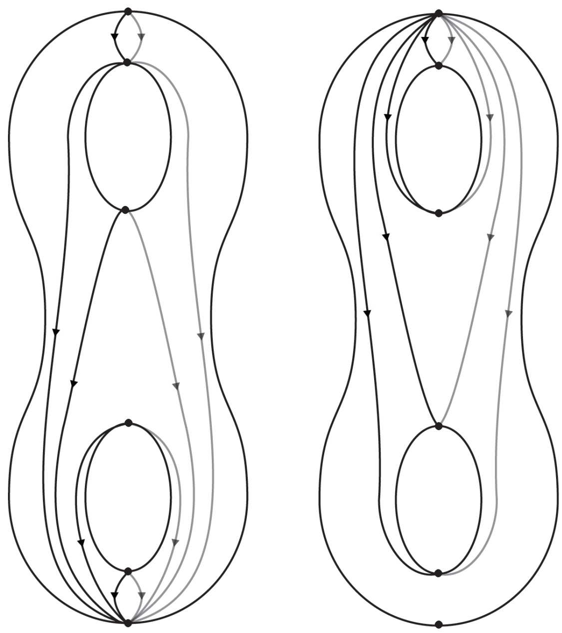


Figure 4: 2-torus

Finally, we compute the boundary maps in order to construct the corresponding homology groups:

I Clearly, $\partial_0 = 0$, and thus, $Ker\partial_0 \simeq \mathbb{Z}_2$.

II Note that for every for every p_i with $i = 2, 3, \cdots, 2g+1$, $\partial_1(p_i) = 2p_1 = 0$ because from every unstable manifold there are two flow lines converging to p_1. And thus, $< p_2, \cdots, p_{2g+1} > \in Ker(\partial_1) \subseteq \mathbb{Z}_2^{2g}$. Then, since $2g$ different points span a rank $2g$ group in $\mathbb{Z}_2$, $Ker(\partial_1) \simeq \mathbb{Z}_2^{2g}$.

III In the last case, we only have to compute one sum for p_{2g+2}, that is, $\partial_2(p_{2g+2}) = 2p_2 + 2p_3 + \cdots + 2p_{2g+1} = 0$ over $\mathbb{Z}_2$ because again, every stable manifold of a saddle point contains two flow lines. Thus, $\partial_2 = 0$ so that $Ker\partial_2 \simeq \mathbb{Z}_2$.

Hence, the boundary maps $\partial_i : C_i(M) \to C_{i-1}(M)$ for $i = 0, 1, 2$ are given by the following matrices respectively:

$$\begin{pmatrix} 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 \cdots 0 \end{pmatrix}$$

and

$$\begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

and therefore, the homology groups are given as follows:

$$HM_k(M) = \begin{cases} \mathbb{Z}_2 & \textit{if } k = 0, 2. \\ \mathbb{Z}_2^{2g} & \textit{if } k = 1. \end{cases}$$

In fact, note that these homology groups coincide with the *singular homology groups* $H_*(M)$ as we expected.

3.3.2 The relation between manifolds and their homology groups:

By the previous example we have seen that the homology groups of a finite dimensional, smooth and compact Riemannian manifold do not depend on the function defined on M. Therefore, it is a natural question to ask whether there is a relation between the homology groups and the manifold itself. That is, we would like to know if there is any equivalence relation between isomorphic homology groups and isomorphic manifolds. Intuitively, we claim that if the given two manifolds are isomorphic (or diffeomorphic) then so are the corresponding homology groups. However, the other direction does not seem to be true, because the isomorphism or diffeomorphism relation is stronger than the homotopy relation. Thus, it would be enough to give an contraexample.

On one hand, let N, M be diffeomorphic finite dimensional compact Riemannian manifolds with $f_1 : M \to \mathbb{R}$ and $f_2 : N \to \mathbb{R}$ Morse-Smale functions and $\psi : M \to N$ a diffeomorphism .

We want to show that $HM_*(M, f_1) \simeq HM_*(N, f_2)$.

First note that given $\psi : M \to N$ a diffeomorphism and $f_2 : N \to \mathbb{R}$ a Morse-Smale function, then $\tilde{f_2} = f_2 \circ \psi : M \to \mathbb{R}$ is also a Morse-Smale function. Under this relation, there exists an induced map ψ^* between the free abelian groups generated by the critical points of $\tilde{f_2}$ and f_1. The idea is to consider the gradient-flows generated by f_1 and $\tilde{f_2}$ and construct the chain complexes connected by ψ^*. Furthermore, by the following remark we conclude that $HM_*(M, \tilde{f_2}) \simeq HM_*(M, f_1)$.

Remark 4. Given $f_1 : M \to \mathbb{R}$ and $\tilde{f_2} = f_2 \circ \psi : M \to \mathbb{R}$, consider the following gradient-system:

$$\begin{cases} x' = -\nabla_{g_1} f_1(x) \\ x' = -\nabla_{\tilde{g}_1} \tilde{f_2}(x) \end{cases}$$

where g_1 is a metric on M and $\tilde{g}_1 = \psi^*(g_2)$ is also a metric on M defined by the pull-back of the metric g_2 on N. Even though we have not proved the following result, we claim that these two gradient-systems are independent of the metric we choose and thus, equivalent.

Hence, by this remark and by the invariance of the Morse-Smale homology groups, we conclude that $HM_*(M, f_1) \simeq HM_*(M, \tilde{f_2})$.

As we said before, after considering the chain complexes corresponding to f_2 and $\tilde{f_2}$, we get the following diagram:

$$\cdots \to C_k(M,\tilde{f}_2) \to C_{k-1}(M,\tilde{f}_2) \to \cdots \to C_0(M,\tilde{f}_2) \to 0$$
$$\downarrow \psi^*_{k-1}$$
$$\cdots \to C_k(N,f_2) \to C_{k-1}(N,f_2) \to \cdots \to C_0(N,f_2) \to 0$$

We need to show that the kernels and boundaries are in one-to-one correspondence to get that the Morse-Smale homology groups are isomorphic.

In order to show this, consider $\tilde{f}_1 = f_1 \circ \psi^{-1} : N \to \mathbb{R}$ a Morse-Smale function and denote by $\varphi = \psi^{-1}$. By the previous remark, we know that $HM_*(N,f_2) \simeq HM(N,\tilde{f}_1)$. Denote by $\varphi^* = (\psi^{-1})^*$ to get the following diagram:

$$\cdots \to C_k(M,\tilde{f}_1) \to C_{k-1}(M,\tilde{f}_1) \to \cdots \to C_0(M,\tilde{f}_1) \to 0$$
$$\downarrow \varphi^*_{k-1}$$
$$\cdots \to C_k(N,\tilde{f}_2) \to C_{k-1}(N,\tilde{f}_2) \to \cdots \to C_0(N,\tilde{f}_2) \to 0$$

From these two diagrams above we conclude that $\phi^* \circ \varphi^* = id$ and $\varphi^* \circ \psi = id$, that is, ψ^* and φ^* are isomorphisms. Then, the kernels and boundaries are in one-to-one correspondence and therefore, we have that the homology groups of both chain complexes are isomorphic, i.e., $HM_*(M,\tilde{f}_1) \simeq HM_*(N,\tilde{f}_2)$.

Furthermore, note that those homology groups do not depend on the function we choose and we conclude that given two diffeomorphic finite dimensional compact manifolds as above, their homology groups are isomorphic $HM_*(M) \simeq HM_*(N)$.

On the other hand, as we said before, the fact that the homology groups are isomorphic does not imply that the manifolds are diffeomorphic in general. Nevertheless, people made use of the fundamental groups in order to classify the 3-dimensional manifolds. Thus, there are some specific classifications as for Seifter manifolds for which the homotopy equivalence implies the diffeomorphism of the manifolds. Waldhausen [11] and Heil [26] also proved that if $M, \tilde{M}$ are Haken 3 closed manifolds with isomorphic fundamental groups (homotopy equivalent), then $M \simeq \tilde{M}$. But, there are examples of non-homeomorphic and homotopic Haken manifolds with boundary as the circle of a thrice-punctured sphere and the product with the circle of a once-punctured torus given in [14].

So, even for 3-dimensional manifolds there is no a generalized relation between homotopic and diffeomorphic manifolds and it gets worse if the dimension of the manifold is increased.

3.3.3 Morse Inequalities and Poincaré Polynomial:

In the introduction we have said that one of the applications of the homology groups are the so called *Morse inequalities* in Morse Theory. As an illustration, consider $f : M \to \mathbb{R}$ a Morse-Smale function defined on a finite dimensional smooth manifold M and consider the gradient-system $x' = -\nabla f(x)$ so that we define a chain complex $(C_*(f), \partial_*(f))$ as in Section 2.

Now, suppose that we consider for the same function the following gradient-system:

$$x' = -\nabla(-f(x)) = \nabla f(x)$$

and define $C^k(f)$ to be the free abelian group generated by the critical points of f or equivalently, by the zeroes of $\nabla f(x)$ of index k.

Observe that the zeroes of both gradient-systems coincide and thus, $C_*(f) = C^*(f)$ where $C_*(f) = \oplus_{k\geq 0} C_k(f)$ and $C^*(f) = \oplus_{k\geq} C^k(f)$.

Furthermore, if $p \in Cr(f)$ has index k, then the Hessian of f at p has k negative eigenvalues. If we compute the Hessian of $-f$ at p, this will have $m-k$ negative eigenvalues. Hence, we get the following relation between the free abelian groups:

$$C_k(f) \simeq C^{m-k}(f)$$

for every $k \geq 0$.

Moreover, for the latter gradient-system, the flow lines go from critical points of index k to index $k+1$ since the flows are in correspondence by reversing the time $t \to -t$. This also implies that the boundary map δ_k corresponding to $C^*(f)$ is defined as follows:

$$\delta_k : C^k(f) \to C^{k+1}$$

with $\delta_k(q) = \sum_{p\in C^{k+1}} n(q,p)p$

Observe that δ_k and ∂_k are equivalent since $C^k(f) \simeq C_{m-k}$ and $C^{k+1} \simeq C_{m-k-1}(f)$.

It is well-known that if we consider the chain complex generated by this last gradient system, $(C^*(f), \delta_*(f))$, then the corresponding homology groups are called *the cohomology groups of f.* Hence, we get the following result:

Theorem 7. *If M is a m-dimensional smooth, oriented and compact manifold with no boundary, then the k^{th} cohomology group is isomorphic to the $(m-k)^{th}$ homology group. That is:*

$$HM^k(M) \simeq HM_{m-k}(M)$$

for every $k \geq 0$.

Now, we can define b_k to be the *kth Betti number*, that is, the dimension of $HM_k(M, \mathbb{Z}_2)$ and c_k to be the number of critical points of index k so that we can get the *Morse inequalities* :

$$\sum_{k=0}^{m} (-1)^k c_k \geq \sum_{k=0}^{m} (-1)^k b_k.$$

Note that this inequality says that the *kth Betti number* gives the low bound of the number of the critical point of index k.

In addition, the *Poincaré Polynomial* is very related to these concepts since it is defined as follows:

$$P_t(M) = \sum_{i=0}^{m} b_i t^i$$

Note that if we consider the Poincaré Polynomial at $t = -1$, then we get the well-known *Euler characteristic*, that is:

$$P_{-1}(M) = \sum_{i=0}^{m} b_i(-1)^i = \chi(M)$$

For example, if we consider as an example a *torus*, then we have that the Poincaré Polynomial is given by $P_t(T) = 1 + 2t + t^2$ which is called *perfect* because the Betti numbers coincide with the number of the critical points. Finally, note that in this case $\chi(T) = 0$ as we already know.

4. Morse Homology for isolated invariant sets

4.1 Basic definitions and results

If M is an infinite dimensional manifold we are not able to define the Morse-Homology groups as in Section 2. Therefore, in this section we consider isolated invariant sets in any manifold M so that we can construct the chain complexes with the corresponding homology groups as before.

Nevertheless, Conley showed in [5] that these isolated invariant sets might be too complicated to be analysed since for example, they are not stable under small perturbation and thus, their structure can be dramatically changed. Due to this instability, Conley proposed to approach these invariant sets by analysing the so called *isolating neighborhoods* which are stable under small perturbations as it is proven in [6] (and as we will show).

Suppose that M is any manifold and $f : M \to \mathbb{R}$ is a Morse-Smale function. As in Section 2, suppose that $\phi : M \times \mathbb{R} \to M$ is the corresponding gradient-flow. Now, we give some definitions:

Definition 5. *A set $S \subset M$ is called* invariant *under the (gradient-) flow $\phi(\cdot, t)$ if $\phi(S, t) = S$ for every $t \in \mathbb{R}$.*

Definition 6. *An invariant set S is called* isolated *if there exists a* compact *neighborhood N of S in which S is the maximal invariant set. That is,*

$$S = Inv(N) = \{x \in N | \phi(x, t) \subset N, \forall t\} \subset int(N)$$

Remark 5. This isolating neighborhood is not unique and furthermore, F.Wesley Wilson, JR and J.A. Yorke proved in [10] that such an isolating neighborhood always admits an *isolating block* contained in N which has some nice properties. Moreover, he also proved that it is equivalent to work with such blocks or isolating neighborhoods.

Lemma 5. *Let $f : M \to \mathbb{R}$ be any smooth function on M defining a gradient dynamical system $\phi(\cdot, t)$ and let N be an isolated neighborhood for $S \subset M$. Now, consider $\tilde{\phi}(\cdot, t)$ to be the gradient dynamical system defined by $\tilde{f} = f + h$, where $\|h\|_{C^r} < \epsilon$ for all $r \geq 0$ and for a sufficiently small $\epsilon > 0$. Then, N is also an isolating neighborhood for $\tilde{\phi}(\cdot, t)$.*

Proof. We would like to show that $\tilde{S}_\epsilon = Inv(N, \tilde{\phi}(\cdot, t)) \subset int(N)$ is isolated by N for a sufficiently small $\epsilon > 0$ where $\tilde{\phi}(\cdot, t)$ is the gradient-flow generated by the following equation:

$$x' = -\nabla \tilde{f}(x) = -\nabla f(x) - \nabla h(x)$$

Assume the contrary, that is, suppose $\tilde{S}_\epsilon$ is not an isolating invariant set by N for $\tilde{\phi}(\cdot, t)$ and for any sufficiently small ϵ. This implies that there exists a sequence $\{\epsilon_n\}$ such that $\epsilon_n \xrightarrow{n \to \infty} 0$ with $f_\epsilon = f + h_\epsilon$, $\|h_\epsilon\| \leq \epsilon_n$ and S_ϵ is not isolated by N.

Equivalently, this means that there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n \xrightarrow{n \to \infty} 0$ and that there is a sequence of orbits $\{\gamma_{x_n}\} \subset S_\epsilon$ such that $\gamma_{x_n}(0) = x_n \in \partial N$ (after rescaling the time, if necessary).

Note that $\{\gamma_{x_n}\}$ are solutions of the differential equation defining the flow with $x_n \in \partial N$. Furthermore, ∂N is compact and this implies that there exists a subsequence $\{x_{n_k}\}$ converging to a point in the boundary, say $x_{n_k} \to x \in \partial N$.

Even more, each of these orbits $\gamma_{x_{n_k}} = x_{n_k}(t) \subset N$ is a complete trajectory that satisfies the following differential equation:

$$\gamma'_{x_{n_k}} = -\nabla f(\gamma_{x_{n_k}}) - \nabla h(\gamma_{x_{n_k}})$$

We know that the right hand side is a bounded function (it is a continuous function defined in a compact set N) and thus, there exists a finite number $C > 0$ such that $\|\gamma'_{x_{n_k}}\| < C$.

So, since we are considering the trajectories $\{\gamma_{n_k}\}$ in a compact set, these complete orbits are also bounded. Hence, we can apply *Arzela-Ascoli* theorem to conclude that this sequence converges uniformly to a flow line in N for any compact interval $I = [-T, T] \subset \mathbb{R}$ for some $T > 0$.

Remark 6. *Arzela-Ascoli theorem:* If a sequence of real-valued functions $\{f_n\}$ is bounded and equicontinuous,[2]then it has a subsequence converging uniformly in any interval $I \subset \mathbb{R}$.

Moreover, note that I is any compact interval and we can make it as large as we want. This trajectory can be extended to a complete solution of the differential equation, since the sequence is uniformly convergent. The latter implies that the convergence does not depend on t at all.

More precisely, fix any $t_0 > T$ and consider a sequence $\{\gamma_n(t)\}$ in the interval $[-t_0, t_0]$ with $\gamma_n(0) = x_n \in \partial N$. We know that since ∂N is compact there is a subsequence $\gamma_{n_k}(0) = x_{n_k} \in \partial N$ which converges uniformly to some $x \in \partial N$.

On the other hand, by Arzela-Ascoli theorem, these orbits converge uniformly in any interval, in particular in $[-t_0, t_0]$ and this limit is the same as the previous one in the interval $[-T, T]$. Therefore, we can extend the interval $I = [-T, T]$ to $[-t_0, t_0]$ for an arbitrary $t_0 > T$ and thus, I can be extended to $\mathbb{R}$.

In conclusion, we have shown that the subsequence of trajectories $\{\gamma_{n_k}(t)\}$ has a limit function $x(t)$ for every $t \in \mathbb{R}$ such that $x(t) \subset N$ and $x(0) = x \in \partial N$.

Now,we would like to conclude that this complete orbits satisfies the original differential equation $x' = -\nabla f(x)$ with x(0) $\in \partial N$. Note that this would contradict our hypothesis, that is, N would not be an isolating neighborhood of S by $\phi(\cdot, t)$.

In order to show this, we want to prove that $\gamma'_{n_k}(t) \xrightarrow{n_k \to \infty} x'(t)$ and $-\nabla \tilde{f}(\gamma_{n_k}) \xrightarrow{n_k \to \infty} -\nabla f(x(t))$.

The latter convergence is clear because when $n_k \to \infty$, then $\epsilon \to 0$ and $x_{n_k} \to x(t)$. Furthermore, by hypothesis $\|h\|_{C^\infty} < \epsilon$ and thus,

$$-\nabla f(\gamma_{n_k}(t)) - \nabla h(\gamma_{n_k}(t)) \xrightarrow{n_k \to \infty} -\nabla f(x(t))$$

[2]Since the sequence of the derivatives is bounded, then the sequence is equicontinuous, i.e. if $d(x, y) < \delta \to \|f(x) - f(y)\| < \epsilon$

So, we need to prove that $\gamma'_{n_k}(t) \to x'(t)$. However, we have seen that every γ'_{n_k} is bounded on a compact set and thus, $\{\gamma'_{n_k}\}$ converges in N. Moreover, this convergence is uniform in N and thus, we know that $x'_{n_k} \xrightarrow{n_k \to \infty} x'(t)$.

Therefore, we conclude that for $n_k \to \infty$, $x'(t) = -\nabla f(x(t))$ and hence, $x(t)$ satisfies the original differential equation with the property that $x(0) \in \partial N$. So, this contradicts our hypothesis of N being an isolating neighborhood for S by $\phi(\cdot, t)$.

In conclusion, we have shown that $\tilde{S}_\epsilon = Inv(N, \tilde{\phi}(\cdot, t))$ for a sufficiently small ϵ and where $\tilde{\phi}(\cdot, t)$ is the gradient-flow generated by the perturbed function $\tilde{f}$.

□

4.2 Chain complex and Homology

In this section we are not going to give a detailed construction of a chain complex for S because we have already done it in Section 2 for compact manifolds. So, let $f : M \to \mathbb{R}$ be a Morse-Smale function with $W^s(q) \pitchfork W^u(p)$ for every $p, q \in Cr(f)$. Since we want to define the Morse-Smale homology groups for S, we first restrict the stable and unstable manifolds to S as follows:

$W^s_S(p) = \{x \in S : \lim_{t \to \infty} \phi(x, t) = p\}$

$W^u_S(p) = \{x \in S : \lim_{t \to -\infty} \phi(x, t) = p\}$

Note that S is invariant under the flow and therefore, $W^s_S(q), W^u_S(p) \subset S$ for every $p, q \in Cr(f)$. In particular, $W_S(q,p) = W^s_S(q) \pitchfork W^u_S(p)$ is a $\lambda_q - \lambda_p$- dimensional manifold contained in $S \subset int(N)$.

Denote by $M_S(q,p) = W(q,p)/\mathbb{R}$ the space of connecting flow lines of dimension $\lambda_q - \lambda_p - 1$. Note that all such flow lines connecting the critical points of f in S are contained in $S \subset int(N)$. Then, $M_S(q,p)$ satisfies the same properties as in Section 2 because it is contained in the interior of N with N compact. Finally, define $n(q,p)$ to be the number of broken trajectories modulo $\mathbb{Z}_2$, that is, $n(q,p) = \sharp \overline{M}_S(q,p)$ as in Section 2 where $\overline{M}_S(q,p)$ is the compactification of $M_S(q,p)$.

In order to define the chain complex, denote by $C_k(N,f)$ the free abelian group generated by the critical points of f in N (in fact, by the critical points of f in S) and define the boundary map $\partial_k : C_k(N,f) \to C_{k-1}(N,f)$ by $\partial_k(q) = \sum_{p \in C_{k-1}(N,f)} n(q,p)p$

Corollary 2. *$(C_k(N,f), \partial_k)$ given as above is a* chain complex, *that is, $\partial_k \circ \partial_{k+1} = 0$ for every $k \geq 0$.*

We do not need to prove this corollary because it is just a consequence of the Lemma 4 in Section 2 and we can define the corresponding homology groups restricted to N :

$$HM_k(N,f) \simeq \frac{Ker \partial_k}{Im \partial_{k+1}}$$

for every $k \geq 0$.

Finally, in this point, we are able to define the homology groups corresponding to the isolated invariant set S.

Definition 7. *Let $f : M \to R$ be a Morse-Smale function that defines a flow ϕ_t on M. Suppose that N is an isolating neighborhood for ϕ_t with $S = inv(N)$ an isolated invariant set. Then, the Morse-Smale homology group of S can be defined as follows:*

$$HC_*(S) := HM_*(N, f)$$

So, in order to show that this homology group is well-defined, we need to verify that $HM_*(S)$ does not depend neither on f nor in N. But, note that since the critical points are in S and not in $N \setminus S$, then if we take any other isolating neighborhood N', we will still have the same chain groups generated by the critical points and thus, the homology groups will also be isomorphic, that is:

$$HC_*(S) \simeq HM_*(N, f) \simeq HM_*(N', f)$$

4.3 Independence of the function:

In this section we would like to show that the homology group of an isolated invariant set S is independent of the function f we choose. In order to prove this, we follow the same steps as in Section 3.1. That is, first define a homotopy between two given Morse-Smale functions $f, \tilde{f}$ with the same isolating neighborhood N so that we show the existence of an induced homomorphism between the corresponding homology groups, $HM_*(N, f)$ and $HM_*(N, \tilde{f})$. Finally, we consider the homotopy of homotopies to conclude, in fact, that this homomorphism is an isomorphism.

However, on one hand note that by a homotopy we define, N will always be the same isolating neighborhood, but the invariant set S might change a bit for $\mu \in (0, 1)$. Nevertheless, this will not be a problem for us since we will consider the critical points of f and $\tilde{f}$ in order to construct the new chain complex and clearly, both have the same isolated invariant set S. On the other hand, we also need to make sure that non of the connecting orbits touches the boundary of N, say ∂N, for all $\mu \in [0, 1]$.

Hence, we first state the results that we would like to prove thorough this section:

Theorem 8. *Let $f, \tilde{f} : M \to \mathbb{R}$ be Morse-Smale functions with the same isolated invariant set S by N. Suppose that there is a family of functions f_μ for $\mu \in [0, 1]$ connecting f with $\tilde{f}$ such that N is an isolating neighborhood of S_μ for all $\mu \in [0, 1]$. Then:*

$$H_*(S) \simeq H_*(N, f_\mu)$$

for all $\mu \in [0, 1]$.

As we said before, we are not going to construct all the homotopies with the corresponding Morse-Smale gradient systems, because we have already done this work in Section 2. So, the main objective of this section is to show that we can still define a homotopy between two Morse-Smale functions, say $f, \tilde{f}$, so that the perturbed homotopy F defined in Section 3 has the property that non of the connecting flow lines touches the boundary of $N \times S^1$, that is, $\partial N \times S^1$, for every $\mu \in [0, 1]$.

First, we assume that N is an isolating neighborhood for every $\mu \in [0, 1]$, that is, $S_\mu = inv(N, \phi^\mu(\cdot, t) \subset int(N)$ for every $\mu \in [0, 1]$.

In order to show that it is not possible to exist a connecting flow line from $\tilde{f}$ to f touching the boundary, we will use the same construction as in Lemma 5. That is, we would like to consider a sequence of connecting flow lines touching the boundary for some point of time and some $\mu \in (0,1)$ so that after considering their limit function we get a contradiction by showing that such a complete solution satisfies the original differential equation.

First, consider the gradient system generated by $F(x,\mu) = h_{\omega(\mu)}(x) + \kappa\Lambda(\mu)$ and we consider the metric corresponding to the system, denoted by $g \oplus g_{S^1}$. The idea is to perturb it so that the gradient-system depends on the sequence of small values $\{\epsilon_n\}$.

Recall that any differential equation $x' = -\nabla_g f(x)$ can also be defined by the metric as follows:

$$df(x).\xi = g(\nabla f(x), \xi)$$

Now, suppose we perturb the metric g to $\tilde{g}$ so that

$$df(x).\xi = g(\nabla f(x), \xi) = \tilde{g}(\tilde{\nabla} f(x), \xi)$$

In our case, we choose a small $\epsilon > 0$ so that $dF.\xi = \epsilon^{-1}\tilde{g}(-\tilde{\nabla}F, \xi)$ where $-\tilde{\nabla}F = \epsilon\nabla F$. Note that in this way, we perturb the original system to the following one:

$$\begin{cases} x' = -\nabla h_{\omega(\mu)}(x) \\ \mu' = -\epsilon\kappa\Lambda'(\mu) - \epsilon\frac{\partial}{\partial\mu} h_{\omega(\mu)}(x) \text{ for } \mu \in [0,1] \end{cases}$$

The idea is to consider a sequence of $\{\epsilon_n\}$ converging to zero so that the metric is perturbed (but note that these ϵ's are never zero) and to consider a sequence of complete solutions $z_n(t) = (x_n(t), \mu_n(t))$ with $z_n(0) = z_n \in \partial N \times S^1$.

Note that $N \times S^1$ and in particular, $\partial(N \times S^1) = \partial N \times S^1$ are also compact sets and therefore, we can apply the same results as before. That is, by *Arzela-Ascoli Theorem*, we conclude that there exists a sequence $\{z_n(t)\} = \{(x_n(t), \mu_n(t))\}$ with a uniformly convergent subsequence $\{z_{n_k}(t)\} \subset N \times S^1$ for any interval $I = [-T, T]$ as in the proof of Lemma 5. Its limit will be denoted by $z(t) = (x(t), \mu(t))$.

In addition, this solution can be extended to $\mathbb{R}$ because it is a uniform convergence.

Finally, we would like to see that this solutions satisfies the original system we considered above. So, take the sequence of solutions, substitute in the previous system and take the limit to get the following:

$$-\nabla h_{\omega(\mu_{n_k})}(x_{n_k}) \xrightarrow{n_k \to \infty} -\nabla h_{\omega(\mu(t))}(x(t))$$

and

$$\mu' = -\epsilon\kappa\Lambda'(\mu_{n_k}) - \epsilon\frac{\partial}{\partial\mu} h_{\omega(\mu_{n_k})}(x_{n_k}) \xrightarrow{n_k \to \infty} 0.$$

On the other hand, as in the Lemma 5, we have that the derivatives of the sequence converge to the derivative of the solution, that is:

$$z'_{n_k} \xrightarrow{n_k \to \infty} z'(t) = (x'(t), \mu'(t)).$$

Hence, we conclude that $\mu'(t) = 0$, that is, $\mu'(t) = \mu_0$ for some $\mu_0 \in [0, 1]$ and every $t \in \mathbb{R}$ and thus, the solution $x(t)$ satisfies the following equality: $x'(t) = -\nabla h_{\mu_0}(x(t))$. Therefore, this means that we have found a complete solution $z(t) = (x(t), \mu(t))$ in $N \times \{\mu_0\}$ for some $\mu_0 \in [0, 1]$ with the property that $z(0) \in \partial N \times \mu_0$. But, note that this contradicts our assumption of N being an isolating neighborhood for every $\mu \in [0, 1]$ (note that here μ_0 is just a constant value representing the function $\mu(t)$.)

We have defined a homotopy F between $f, \tilde{f}$ defining a Morse-Smale gradient system with $S_\mu = Inv(N, \phi^\mu(\cdot, t) \subset int(N) \times \{\mu\}$ for every $\mu \in [0, 1]$ and $z(t) \subset int(N \times S^1)$ $\forall t \in \mathbb{R}$. Hence, from Section 3.3 we conclude that this homotopy shows that the existence of the homomorphism between the corresponding homology groups, that is, $F_* : HM_*(N, f) \to HM_*(N, \tilde{f})$ is a homomorphism.

Recall that the next step is to consider the homotopy of homotopies to conclude that the homomorphism F obtained above is an isomorphism. Denote by $F(x, \mu_1, \mu_2)$ the perturbed homotopy defined in *section* (iv). Actually, we also want to make sure that there is no any complete orbit in $N \times S^1 \times S^1$ passing through its boundary $\partial N \times S^1 \times S^1$. So, first assume that N is an isolating neighborhood for the gradient-flows defined through the homotopy of homotopies, i.e.,

$$S_{\mu_1,\mu_2} = Inv(N, \phi^{\mu_1,\mu_2}(\cdot, t)) \subset int(N \times T^1).$$

We can again perturb the metric $g \oplus g_{S^1} \oplus g_{S^1} = g \oplus g_{T^1}$ to $g \oplus \epsilon^{-1} g_{T^1}$ so that we define the following Morse-Smale gradient system:

$$\begin{cases} x' = -\nabla h_{\omega(\mu_1),\omega(\mu_2)}(x) \\ \mu_1' = -\epsilon\kappa_1 \Lambda'(\mu_1) - \epsilon \frac{\partial}{\partial \mu_1} h_{\omega(\mu_1),\omega(\mu_2)}(x) \\ \mu_2' = -\epsilon\kappa_2 \Lambda'(\mu_2) - \epsilon \frac{\partial}{\partial \mu_2} h_{\omega(\mu_1),\omega(\mu_2)}(x) \end{cases}$$

for $\mu_1, \mu_2 \in [0, 1]$.

Observe that in this case $N \times T^1$ is also compact and thus, if we consider the sequence of complete solutions touching the boundary, we will get the same results as for the homotopy by *Arzela-Ascoli*, that is, this sequence will converge to a complete solution $z(t) = (x(t), \mu_1(t), \mu_2(t)) \subset N \times T^1$ satisfying the equation $x' = -\nabla h_{\omega(\mu_1),\omega(\mu_2)}(x)$ for some $\mu_1, \mu_2 \in [0, 1]$ and with $z(0) \in N \times T^1$ for some $\mu_1, \mu_2 \in [0, 1]$. Hence, this contradicts our assumption and we get the desired result.

4.3.1. Morse-Smale homology for smooth functions

Consider f any smooth function from which we define the gradient flow $\phi(\cdot, t)$. Let N be an isolating neighborhood for S by $\phi(\cdot, t)$ and now, from Lemma 3, we know that this function can be perturbed to a $Morse - Smale$ function without changing the number and the indices of the critical points. Moreover, we choose such a perturbation so that N is still an isolating neighborhood for the perturbed flow, say $\tilde{\phi}(\cdot, t)$. Clearly, this can be done by Lemma 5. Therefore, we can assume that given any smooth function with an isolating neighborhood N of S and for $\phi(\cdot, t)$, this flow can be perturbed to a *Morse-Smale gradient-flow* with N being an isolating neighborhood for S' close to S.

Definition 8. *Let $f : N \to \mathbb{R}$ be any restricted smooth function defined on any smooth manifold M. Suppose that N is an isolating neighborhood for S by $\phi(\cdot, t)$ where $\phi(\cdot, t)$ is a gradient flow generated by f. Consider $\tilde{f} = f + h$ the perturbed Morse-Smale function with*

$\|h\|_{C^\infty} < \epsilon$ and for a sufficiently small ϵ. Then, the Morse-Smale homology group defined for an isolating neighborhood N is given by:

$$HC_*(S) \simeq HM_*(N, f) \simeq HM_*(N, \tilde{f})$$

Note that the definition of the Morse-Smale homology groups is well-defined and that it does not depend on which perturbation of f we choose. That is, if we consider another one, say $\tilde{f}$ with N being also an isolating neighborhood for S, then its Morse-Smale homology groups are isomorphic to the previous ones.

Finally, we would like to point out (although we will not re-write them now) that all the results we obtained in Section 3 will hold for isolated invariant sets as the *Morse-inequalities* as well as the definition of the *Poincaré Polynomial*.

APPENDIX

Appendix A: Degree Theory

Degree Theory is a topological tool that can be used to get the number of solutions of some function (or vector field in our case). There are different kind of degrees: for smooth maps, the Brouwer degree, Leray-Schauder, etc. However, we are interested in recalling some results for smooth function over $\mathbb{R}^m$, because we have used this theory for a manifold M in coordinates.

Suppose that $f : U \subset \mathbb{R}^m \to \mathbb{R}^m$ is a smooth function and $p \notin f(\partial U)$ is a regular value, then by the *inverse function theorem*, $f^{-1}(p)$ is a finite set and thus, the *degree of f at p* can be defined as follows:

$$deg(f, U, p) := \sum_{x \in f^{-1}(p)} sign(J_f(x))$$

where

$$J_f(x) = det \begin{pmatrix} \partial_{x_1} f_1 & \cdots & \partial_{x_m} f_1 \\ \vdots & & \vdots \\ \partial_{x_1} f_m & \cdots & \partial_{x_m} f_m \end{pmatrix}$$

Now we recall some basic properties that hold from the above definition:

(i) deg(id, U, p)=1, because there is always a unique point in the pre-image of any regular value and its Jacobian is always constant equal to 1.

(ii) deg(f,U,p)=deg(f-p,U,0) because the Jacobian of f at p and the Jacobian of $f - p$ at 0 coincide, i.e., $J_f(p) = J_{f-p}(0)$.

(iii) If $U = D_1 \cup D_2$, then:

$$deg(f, U, p) = deg(f, D_1, p) + deg(f, D_2, p)$$

However, in the beginning of this section we have said that the degree is a topological tool and therefore, it is a topological invariant. On one hand, we can consider a homotopy f_t between two given smooth functions f and $\tilde{f}$ and consider a regular value p such that $p \notin f_t(\partial U)$. Then, we have that:

$$deg(f, U, p) = deg(\tilde{f}, U, p) = deg(f_1, U, p)$$

On the other hand, we can also consider regular values p_0 and p_1 connected by some curve, $t \to p_t$ and we have the following result:

$$deg(f, U, p_0) = deg(f, U, p_1) = deg(f, U, p_t).$$

Note that this last result implies that the degree can also be defined for critical points, because we can always connect them to a sufficiently closed regular value by some curve. Furthermore, the existence of such a regular values hold from *Sard's theorem*.

Appendix B: Immersion and Submersion

In this section we would like to recall some results (without the proofs) that can be found in [3] :

Definition 9. *Given M, N two smooth manifolds of dimension m and n, respectively; $f : M \to N$ is said to be an* immmersion *if for every point $x \in M$ with $y = f(x)$, the differential map $df_x : T_xM \to T_yN$ is injective.*

Definition 10. *Given M, N two smooth manifolds of dimension m and n, respectively; $f : M \to N$ is said to be an* submersion *if for every point $x \in M$ with $y = f(x)$, the differential map $df_x : T_xM \to T_yN$ is surjective.*

Corollary 3. *Given a smooth map $f : M \to N$ between smooth manifolds, then:*

(i) If f is an immersion at $x \in M$, then f is an immersion on some neighborhood of x.

(ii) If f is a submersion at $x \in M$, then f is a submersion on some neighborhood of x.

Even though we have already defined what the transversal intersection means, whenever the given manifolds are embeddings, there is an equivalent definition given as follows:

Definition 11. *Let $f : M \to N$ and $g : Z \to N$ be smooth maps between smooth manifolds. Then, f is said to be transversed to g, $f \pitchfork g$, if and only if $df_x(T_xM) + dg_x(T_xZ) = T_yN$ with $y = f(x) = g(x)$.*

Furthermore, if $Z \subset N$ and $g : Z \to N$ is the inclusion map with f and g are transverse, we denote it by: $f \pitchfork Z$.

Finally, we also state the following theorems, because we have made use of one of them to show that $W^u(q) \pitchfork W^s(q)$ is a manifold:

Theorem 9 (Local Immersion Theorem)**.** *Let $f : M \to N$ be an immersion between two manifolds and let $x \in M$ with $y = f(x) \in N$. Then, f is locally an inclusion, that is, there are coordinates around these two points such that f can be given as follows:*

$$f(x_1, \cdots, x_m) = (x_1, \cdots, x_m, 0, \cdots, 0)$$

Theorem 10 (Local Submersion Theorem)**.** *Let $f : M \to N$ be a submersion between two manifolds and let $x \in M$ with $y = f(x) \in N$. Then, f is locally a projection, that is, there are coordinates around these two points such that f can be given as follows:*

$$f(x_1, \cdots, x_m) = (x_1, \cdots, x_m)$$

Appendix C: Isolating Blocks and Lyapunov function

Suppose that $\phi : M \times \mathbb{R} \to M$ is a (gradient-)flow defined by an ordinary differential equation (in our case, we have a gradient-flow defined by a Morse-Smale function). We have already defined in section 4 the isolated invariant sets and isolating neighborhoods, that is,

$$S = inv(N) = \{x \in N : \phi_t(x) \in N, \forall t \in \mathbb{R}\} \subset N$$

is an isolated invariant set with N an isolating neighborhood.

As we said before, the invariant sets are not stable under small perturbation and their structure might be very complicated. In order to avoid their direct study, C.Conley introduced the concept of the isolating neighborhoods in **??**. Furthermore, he also showed that these isolating neighborhoods determine the Morse index of the isolating neighborhood (even though we do not know what S exactly is).

Note however that these isolating neighborhoods are not unique since if we consider any $S \subset N' \subset N$ open subset of M, then N' is also an isolating neighborhood for S.

Moreover, C.Conley and R.W. Easton defined the so-called *isolating blocks* in [7]. Now, we define these submanifolds:

Definition 12. *$B \subset M$ is an* isolating block *for an invariant set $S \subset M$ if B is a m-dimensional compact submanifold and its boundary is decomposed as $\partial B = B^+ \cup B^- \cup \tau$ where*

(a) $B^+ = \{x \in \partial B : \phi_{(-\epsilon,0)}(x) \cap B = \emptyset\}$ *for some* $\epsilon > 0$.

(b) $B^- = \{x \in \partial B : \phi_{(0,\epsilon)}(x) \cap B = \emptyset\}$ *for some* $\epsilon > 0$.

(c) $\tau = \{x \in \partial B :$ *the flow is tangent to the boundary*$\}$

(d) $\tau = B^+ \cap B^-$

Therefore, any point in the boundary of B leaves in one of the directions defined above immediately.

It has also been proved that it is equivalent to work with isolating neighborhoods or isolating blocks and thus, we could also define an isolated invariant set as follows:
$S \subset M$ is an isolated invariant set if it is invariant, i.e. $\phi_t(S) = S$ for every $t \in \mathbb{R}$ and there exists an isolating block such that $S = inv(B) \subset int(B)$.

Moreover, the interior of B, $int(B)$, is an isolating neighborhood for S. Even more, the *Conley index* can also be defined by them and in both cases for isolating neighborhoods and isolating block we get the same invariance (for more details see [7]).

Another important result that F. Wesley Wilson and James A. Yorke proved in [10] is that the existence of such isolating blocks is equivalent to the existence of the *Lyapunov functions.* We give the following definition of Lyapunov function although in [10] they distinguish between *generalized, monotone* and *hyperbolic* Lyapunov functions.

Definition 13. *Given $U \subset M$ an open neighborhood of an isolated invariant set S, a continuous function $V : U \to \mathbb{R}$ is a* Lyapunov function *if:*

(i) $V(x) = 0 \leftrightarrow x \in S$

(ii) $\frac{d}{dt}V(\phi_t(x)) < 0$ *for all $x \in U$ S, i.e. V decreases along the flow lines.*

As an illustration of such a Lyapunov function we could consider a gradient-flow over a compact manifold defined by a Morse function so that the critical points are isolated. Then, every critical point is an isolated invariant set where a unit ball (for example) is an isolating neighborhood for it. Note that in the critical point the flow is constant (in fact, it is zero) and outside of the critical point the function decreases along the flow lines.

CONCLUSION AND FUTURE QUESTIONS

In this paper we have considered gradient-flows generated by Morse-Smale functions, because on one hand the gradient-flow decreases along the regular values and on the other hand Morse-Smale functions do not allow to connect critical points of the same index. Furthermore, the compactness of the manifold has been an important property to assume the compactness of the manifold of connecting flow lines.

Therefore, these facts allow us to define the boundary map by counting finitely many flow lines connecting the critical points of index k with index $k-1$ for every $k \geq 1$.

We concluded that the Morse-Smale homology defined under these conditions in isomorphic to the singular homology and in particular, we have proved its invariance of the function.

However, note that we can also consider any manifold M and if M is not compact, the above construction fails. So, we consider isolated invariant sets in the manifold and define the corresponding Morse-Smale homology groups exactly in the same way as before. The unique difference is that M was a closed manifold with no boundary and the isolated invariant set, denoted by N, is a compact manifold with boundary and that is why we had to do a bit more work. That is, we have shown that the connecting flow lines considered in the homotopies do not intersect the boundary of the Morse-Smale gradient system.

Note that we could consider M to be an infinite dimensional manifold and try to construct the homology groups as Floer did in [2]. Moreover, another question we could think of is if we can generalize this construction for non-gradient or gradient-like flows. That is, we could generalize these results by analysing them for any flow generated by a general vector field $X : M \to \mathbb{R}$.

ACKNOWLEDGMENTS

Thank you very much to Rob Vandervorst for his help during this months since after knowing that he is a great teacher, he has also shown me that he is even a better supervisor. Specially, because in every meeting he is able to transmit his enthusiasm and thus, it has been a pleasure to work with him.

On the other hand, I would also like to show my gratitude to all my friends in Utrecht. Some of them gave me mathematical advices and helped improving my knowledge. Some other friends distracted me from my studies by enjoying ourself with unforgettable moments. Thank you very much.

Bibliography

[1] A. Hatcher, *Algebraic Topology.*

[2] A.Floer, *Witten's Complex and infinite dimensional Morse Theory.* J.Differential Geom. Volume 30, Number 1 (1989), 207-221.

[3] A. Banyaga and D.E. Hurtubise, *Morse-Bott Homology.* Trans. Amer. Math. Soc. 362(2010), no. 8, 3997-4043.

[4] A. Banyaga and D.E. Hurtubise, *Lectures on Morse Homology.* ISBN 1-4020-2695 (HB), ISBN 1-4020-2696-X (e-book). Published by Kluwer Academic Publishers, Dordrecht, The Netherlands.

[5] C. Conley, A qualitative singular perturbation theorem, *Global Theory of Dynamical Systems* (eds. Z.Niteki and C.Robinson) Lectures Note on math. 819 Springer 1980 65-89.

[6] C.Conley, *Isolated invariant sets and Morse index*, CBMS Lecture notes 38 A.M.S. Providence, R.I. 1978.

[7] C. C. Conley and R. W. Easton, *Isolated invariant sets and isolating blocks*, Trans. AMS, 158, 1 (1971) 35-60. MR 43:5551

[8] D.M. Austin and P.J. Braam, *Morse-Bott theory and equivariant cohomology.* Floer Memorial volume, 1995.

[9] D.Mc Duff, *Floer Theory and low dimensional topology* (2005). Bulletin of the American Mathematical Society. Volume 43, number 1, p.25-43, S 0273-0979(05)01080-3.

[10] F.Wesley Wilson, JR and J.A. Yorke, *Lyapunov functions and isolating blocks.* Journal of diff. equations, volume 13, 106-123 (1973).

[11] F.Waldhausen, *On irreducible 3-manifolds which are sufficiently large.* Ann. Math. 87 (1968), 56-88.

[12] James F. Reinek, *continuation to gradient flows.* IMA preprint Series ♯ 666, July 1990.

[13] Joa Webber, *The Morse-Witten complex via dynamical systems.* arXiv: math 0411465v2 [math.GT] 26 Oct, 2005.

[14] Joel Hass and Peter Scott, *Homotopy equivalent and homeomorphism of 3-manifolds.* Topology Vol 31. No 3. pp 493-512, 1992.

[15] Joel W. Robbin and Dietman A. Salamon, *Lyapunov maps, simplicial complexes and the stone functors* (1992). Ergodic Theory and Dynamical Systems, 12, pp. 153-183. doi: 10.1017/ S 0143385700006647.

[16] John M. Franks, *Homology and Dynamical Systems* (1982). CBMS Regional Conf. Ser. in Math., volume 48, Amer. Math. Soc. Providence, R.I.

[17] John W. Milnor, *Topology from the differentiable viewpoint.* ISBN: 0813901812, University of Virginia Press.

[18] John W. Milnor, *Morse Theory* (1969). Princeton University Press, ISBN: 0-691-08008-9.

[19] Jurgen Jost, *Riemannian GEometry and Geometric Analysis.* 5th edition, Springer, ISBN: 978-3-540-77340-5.

[20] M. Schwarz, *Morse Homology.* Springer London, ISBN: 9783764329044; ISBN: 3764329041.

[21] R.Van der Vorst, *Introduction to Differentiable manifolds* (2009).

[22] Y.Matsumoto, *An introduction to Morse Theory.*

[23] S. Smale, *Morse inequalities for a dynamical system.* Bull. American Mathematical Society 66 (1960) 43-49.

[24] S. Smale, *Stable manifolds for differential equations and diffeomorphism.* Ann. Scuola Norm. Sup. Pisa(3) 17 (1963), 97 (116. MR 29 $\sharp$ 2818b).

[25] T. Kluck, *Simplectic Field Theory and Integrable hierarchies* (2010). Korteweg-de Vries Institute for Mathematics, Universiteit van Amsterdam.

[26] W. Heil, *On* $\mathbb{P}^2$*- irreducible 3-manifolds.* Bull. Amer. Math. Soc. 75 (1973) 772-775.