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BLACK HOLES IN HIGHER DIMENSIONS

EXPLORING THE POSSIBILITIES OF FIVE DIMENSIONAL GENERAL RELATIVITY

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Abstract

We investigate the comparability of rotating four dimensional and five dimensional black holes in asymptotically flat vacuum spacetime. First a concise introduction to the theory of general relativity and a description of the four dimensional Schwarzschild and Kerr black holes are given. Then these black hole solutions are generalised to higher dimensions, focusing on the five dimensional Myers-Perry black hole and black ring. Lastly Kaluza-Klein theory on the reduction of dimensionality is shortly discussed.

Contents

1 INTRODUCTION

In 1916 Einstein published his theory of general relativity, a theory of space, time, and gravity [1]. He proposed space and time are unified in a spacetime and that this spacetime can be curved. The curvature expresses itself as what we experience as gravity. Gravity is thus something inherent to (curved) spacetime and not an additional field placed upon it. Einstein was led to this idea by the Principle of Equivalence. He formulated the Einstein Equivalence Principle, or EEP, as: *In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.* This statement implies that gravity is inescapable and acceleration due to gravity can not be clearly defined, since there is no object with zero gravity. We will define "unaccelerated" as "freely falling in the present gravitation field", which destroys the notion of gravity as a force, since a force leads to acceleration. The EEP thus suggest gravity could be attributed to something as universal as the curvature of spacetime, but locally should not be observable. The mathematical framework that lends itself perfectly to this theory is that of manifolds, which locally look like flat space but overall can be curved. In the center of general relativity stands Einstein's equation relating the curvature of spacetime to the presence of energy and momentum.

General relativity has successfully predicted and described several phenomena, which have also been experimentally confirmed; such as perihelion precession [2, §40.5], gravitational time dilation [2, §38.5], gravitational waves [3], gravitational lensing [4], black holes [5], and many more [6]. This motivates our view of general relativity as a particularly good foundation for astrophysics and cosmology. In this work I will focus on black holes. Several black hole metrics have been found as exact solutions of Einstein's equation. Black holes are not just a peculiarity of the theory but are real objects in the universe. They form as massive stars collapse at the end of their lifetime and are believed to be in the center of most galaxies [7]. The presence of black holes can be derived from the motion of planets orbiting around it, its interaction with matter and electromagnetic waves, and the creation of gravitational waves as two black holes merge [8]. In April 2019 the first direct image of a black hole was published [9].

In four dimensions the most prominent exact solutions of Einstein's equation are the static Schwarzschild, charged Reissner-Nordström and rotating Kerr black hole [7]. Although not initially intended, the dimensionality of general relativity can be freely chosen. This way also higher dimensional black hole solutions can be found. These are a great tool to explore what more dimensions can bring to physics as the number of degrees of freedom increase with dimensionality. To scratch the surface of the possibilities I want to compare five dimensional rotating black holes to four dimensional ones. The study of higher dimensional black holes can tell us which properties are independent of dimensionality and thus fundamental to the idea of a black hole. In present times there is also another application. A prominent current proposal is, namely, that matter fields and forces can be explained via higher dimensional spacetimes [10]. These extra dimensions, however, should not be directly observable to accurately describe our four dimensional world.

This proposal has its roots in the theory formulated by Kaluza in 1921 [11]. He found a way to unify gravity, described by Einstein's equations, with electromagnetism, described by Maxwell's equations, in a higher dimensional theory. In five dimensional general relativity the metric tensor, describing spacetime, has 15 independent components. To not observe the extra fifth dimension all derivatives of the metric components with respect to the extra coordinate are set to zero (the so-called cylinder condition). This results in 14 independent components set by 14 field equations. These turn out to be 10 Einstein equations specifying the four dimensional metric, and the 4 Maxwell equations. Klein later showed how to implement quantum effects by curving the extra dimension into a circle, creating Kaluza-Klein theory [12]. This has developed into several modern Kaluza-Klein theories (see [11] for a review) among which are induced matter theory and membrane theory in five dimensions [10].

Induced matter theory tells us how the fifth dimension of an empty flat spacetime gives rise to a four dimensional spacetime with matter sources. In membrane theory a five dimensional space is considered on which there exists a four dimensional singular hypersurface which we call spacetime. The field equations derived from these two theories have been shown to be equivalent. In dimensions greater than five, other theories connected to higher dimensional general relativity are for instance 10D and 11D supergravities (review of supergravity [13]; review of Kaluza-Klein supergravity [14]; review of 11D supergravity [15]), M-Theory and string theory [16].

I will compare four to five dimensional black holes on horizon topology, solution uniqueness and angular momentum range. The topic will be restricted to asymptotically flat vacuum spacetimes and spherically or axial symmetric black holes that are static or stationary rotating with no charge. Chapter 2 is dedicated to the introduction of the necessary mathematical concepts and basic notions and equations of general relativity. The four dimensional Schwarzschild and Kerr solutions are also discussed. In chapter 3 those well known solutions are generalised to higher dimensions, the first to n dimensions and the latter to five dimensions. In five dimensions we will discover solutions are no longer uniquely given by their mass and angular momentum. We will describe the known solutions and focus on the Myers-Perry black hole and black ring. In the last chapter Kaluza-Klein theory is discussed, which introduces the idea of dimensional reduction. This way I will try to pave a way for undergraduate level physics students into the realm of higher dimensional black holes, by building more on intuition than mathematical definitions.

2 INTRODUCTION TO GENERAL RELATIVITY

Einstein's general relativity is a geometric theory of gravity. Space and time are linked in spacetime and described by a metric. Mass, energy and momentum curve spacetime as told by Einstein's equation. In turn this curvature manifests itself as gravity and influences particles in their way through space. The paths free falling particles move on are described by the geodesic equation. Spacetime also contains singularities where the curvature becomes infinite and geodesics seize to exist or spontaneously arise. Some of these singularities are associated with black holes. In this chapter we will first discuss the necessary mathematical constructs to describe spacetime and its curvature and then make our way into general relativity and black holes. As basis and guideline for this chapter we use the book *Spacetime and Geometry: An Introduction to General Relativity* by Sean Carroll [7].

2.1 Mathematical Basis

2.1.1 Manifolds

In general relativity spacetime is modeled by a manifold. Why we choose to do so is motivated in the introduction of section 2.2. A manifold is a space that locally looks like flat Euclidean space $\mathbb{R}^n$, but globally can be curved and more complicated. By smoothly pasting together these local regions the entire manifold is built. If all those pieces locally look like n-dimensional Euclidean space, then we say the manifold is of dimension n. Some examples of manifolds are:

- Euclidean space $\mathbb{R}^n$ itself of course.
- The n-sphere S^n;
  you can view this manifold as the collection of points a fixed distance from an origin in $\mathbb{R}^{n+1}$.
- The n-torus T^n;
  this manifold is made by gluing together opposite sides of a n-dimensional cube. A familiar example is the two-torus T^2 that looks like a doughnut.
- The direct product of two manifolds is again a manifold. The resulting manifold has a dimension of the dimensions of the original manifolds added.

We will not cover the formal mathematical definition of a manifold (but can be found in [7]), since an intuitive idea of what it is suffices for the following discussion.

2.1.2 Vectors & Tensors

Generally for n-dimensional spacetimes we will denote a vector x by its components $x^\mu = (x_0, x_1, \ldots, x_{n-1})$, where μ runs from 0 to $n-1$. With x^1 the second vector component x_1 is meant, while x^μ represents the whole vector. In four dimensional spacetime vectors are called four-vectors. When considering curved spacetime we lose the familiar properties of vectors we know from flat spacetimes. Vectors no longer stretch from one point in the manifold to another and we can not move vectors arbitrarily across the manifold; the path you take will alter the resulting vector at the endpoint.

The tangent space T_p at a point p is the set of all possible vectors at that point and can be identified with the space of directional derivative operators along curves through p. Consider we have a coordinate chart with coordinates x^μ. The set of n partial derivatives $\partial_\mu = \frac{\partial}{\partial x^\mu}$ at p then forms a good basis for the vector space of directional derivatives and therefore also for the tangent space. The partial derivative with respect to x^μ is by definition the directional derivative along a curve defined by $x^\nu =$ constant for all $\nu \neq \mu$, parameterised by x^μ. The tangent vector of a curve parameterised by λ is then

$$\frac{d}{d\lambda} = \frac{dx^\mu}{d\lambda}\partial_\mu \,, \tag{2.1}$$

where there should be summed over μ according to Einstein's summation convention. A general vector V can be written as $V = V^\mu \partial_\mu$ and should be unchanged under a change of coordinates $x^\mu \to x^{\mu'}$. The vector components will change accordingly:

$$\begin{aligned} V^\mu \partial_\mu &= V^{\mu'} \partial_{\mu'} \\ &= V^{\mu'} \frac{\partial x^\mu}{\partial x^{\mu'}} \partial_\mu \\ \to \;\; V^{\mu'} &= \frac{\partial x^{\mu'}}{\partial x^\mu} V^\mu \,, \end{aligned} \tag{2.2}$$

where we have used the chain rule. This is called the vector transformation law. How an object transforms under a coordinate change can also be used as a definition of that object. So when an object transforms according to the vector transformation law, you know it is a vector. A scalar is defined to be invariant under a change of coordinates.

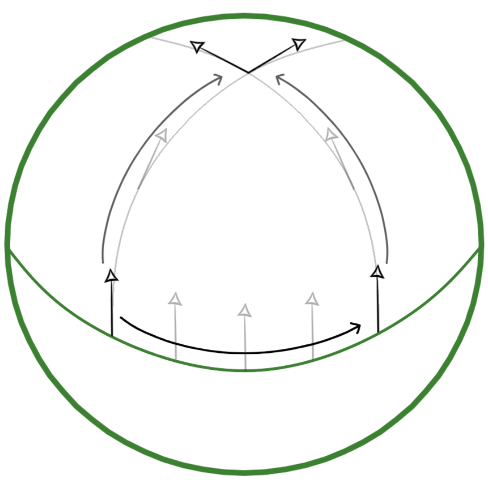

Figure 1: Parallel transporting two identical vectors on a two-sphere via different routes gives different results. Adapted from [7].

In flat spacetime you can arbitrarily move a vector while keeping it constant to another vector to compare the two. Keeping a vector constant along a path is called parallel transport (a more detailed definition in given in section 2.2.1). In curved spacetime the result of parallel transporting a vector depends on the path taken. This can best be envisioned by considering a two-sphere (described in spherical coordinates by $r =$ constant, $0 \leq \phi \leq 2\pi$, $0 \leq \theta \leq \pi$) with two identical vectors on the same point on the equator pointing along constant ϕ to $\theta = 0$. One vector is parallel transported directly up via constant ϕ to $\theta = 0$. The other vector is first parallel transported a bit along the equator and then up along constant ϕ to $\theta = 0$. The two vectors will now be at the same endpoint but are rotated relative to each other. This is illustrated in figure 1.

Another relevant vector space is the dual vector space. The dual space is the space of all linear maps from the original vector space to the real numbers. Dual vectors are labeled with lower indices in contrast to vectors with upper indices. The simplest example of a dual vector is the gradient of a scalar function $\mathrm{d}\phi$. The gradient is the set of partial derivatives with respect to the coordinates x^μ:

$$(\mathrm{d}\phi)_\mu = \frac{\partial\phi}{\partial x^\mu}\,. \tag{2.3}$$

The dual space to the tangent space T_p is called the cotangent space T_p^*. Elements of T_p are called contravariant vectors or simply vectors, and elements of T_p^* are called covariant or dual vectors. The action of a dual vector ω on a vector V is written as

$$\omega(V) = \omega_\mu\, V^\mu \in \mathbb{R}\,. \tag{2.4}$$

You can also think of vectors as linear maps of dual vectors; the dual space to the dual space is just the original space and

$$V(\omega) \equiv \omega(V) = \omega_\mu V^\mu\,. \tag{2.5}$$

The transformation law for dual vectors is quite straightforward:

$$\omega_{\mu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\,\omega_\mu\,. \tag{2.6}$$

Vectors and dual vectors can be generalised to tensors. A tensor T of type or rank (k,l) is a multilinear map from a collection of vectors and dual vectors to $\mathbb{R}$:

$$T : \underbrace{T_p^* \times \ldots \times T_p^*}_{(k\,\text{times})} \times \underbrace{T_p \times \ldots \times T_p}_{(l\,\text{times})} \to \mathbb{R}\,, \tag{2.7}$$

with $\times$ the Cartesian product. Multilinearity means the map is linear in all its arguments. Just as for vectors we denote a tensor T by its components $T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}$, where the order of indices is important. With this definition we can view a scalar as a type $(0,0)$ tensor, a vector as a type $(1,0)$ tensor and a dual vector as a type $(0,1)$ tensor.
Another definition of a tensor is by its transformation law:

$$T^{\mu'_1\cdots\mu'_k}{}_{\nu'_1\ldots\nu'_l} = \frac{\partial x^{\mu'_1}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu'_k}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial x^{\nu'_1}}\cdots\frac{\partial x^{\nu_l}}{\partial x^{\nu'_l}}\,T^{\mu_1\cdots\mu_k}{}_{\nu_1\ldots\nu_l}\,. \tag{2.8}$$

If we have a type (k,l) tensor T and a type (m,n) tensor S we can define a tensor product of the two. The tensor product is denoted by $\otimes$ and results in a $(k+m, l+n)$ tensor. Its definition is

$$\begin{aligned} &T \otimes S(\omega^{(1)},\ldots,\omega^{(k)},\ldots,\omega^{(k+m)},V^{(1)},\ldots,V^{(l)},\ldots,V^{(l+n)}) \\ &\quad = T(\omega^{(1)},\ldots,\omega^{(k)},V^{(1)},\ldots,V^{(l)}) \times S(\omega^{(k+1)},\ldots,\omega^{(k+m)},V^{(l+1)},\ldots,V^{(l+n)})\,, \end{aligned} \tag{2.9}$$

where $\omega^{(i)}$ and $V^{(i)}$ represent the whole (dual) vector and not just components thereof. Generally tensor products do not commute; $T \otimes S \neq S \otimes T$.

We have defined a tensor as a linear map to $\mathbb{R}$, but we need not always let the tensor act on a full set of arguments. We can let a $(1,1)$ tensor act on a vector and obtain a vector:

$$T^{\mu}{}_{\nu} : V^{\nu} \to T^{\mu}{}_{\nu}\, V^{\nu} \,. \tag{2.10}$$

$T^{\mu}{}_{\nu} V^{\nu}$ is again a vector because it transforms according to the vector transformation law. We can also let a $(2,1)$ tensor act on a part of a $(1,2)$ tensor to get a $(1,1)$ tensor (again according to how it transforms):

$$T^{\mu\rho}{}_{\sigma} : S^{\sigma}{}_{\rho\nu} \to T^{\mu\rho}{}_{\sigma}\, S^{\sigma}{}_{\rho\nu} = U^{\mu}{}_{\nu} \,. \tag{2.11}$$

Another thing we can do is called contraction and turns a (k,l) tensor into a $(k-1,l-1)$ tensor. It is done by summing over one upper and one lower index:

$$T^{\mu\nu\rho}{}_{\sigma\nu} = S^{\mu\rho}{}_{\sigma} \,. \tag{2.12}$$

Note that the order of the indices matters, so $T^{\mu\nu\rho}{}_{\sigma\nu} \neq T^{\mu\rho\nu}{}_{\sigma\nu}$.

Lastly we define a tensor to be symmetric in any of its indices if it is unchanged under exchange of those indices. For example $S_{\mu\nu\rho}$ is symmetric in its first two indices if

$$S_{\mu\nu\rho} = S_{\nu\mu\rho} \,. \tag{2.13}$$

A tensor is said to be antisymmetric if it changes sign under exchange of those indices. Thus $A_{\mu\nu\rho}$ is antisymmetric in μ and ρ if

$$A_{\mu\nu\rho} = -A_{\rho\nu\mu} \,. \tag{2.14}$$

One important example of a tensor is the energy-momentum tensor $T^{\mu\nu}$. It is a symmetric $(2,0)$ tensor that contains all information about the energy aspects of a system like pressure, energy density, stress, and so on. In vacuum space $T^{\mu\nu} = 0$, since there is no matter, energy, pressure, and so on.

The power of tensor calculus lies in the idea of general covariance. That is the belief that the law's of physics take the same form in all coordinate systems. Both the right and left hand side of a tensor equation transform in the same way under coordinate transformations, resulting in the same form as the original equation. This underlines that physical laws fundamentally describe nature.

2.1.3 The Metric

Another very important tensor in general relativity is the metric tensor $g_{\mu\nu}$ (or simply metric). It entails all the causal and geometric properties of spacetime and replaces the gravitational potential in classical mechanics. The metric is a symmetric $(0,2)$ tensor that is usually nondegenerate, i.e. its determinant $g = |g_{\mu\nu}|$ does not vanish. When it is nondegenerate, we can define the inverse metric $g^{\mu\nu}$, a symmetric $(2,0)$ tensor, by

$$g^{\mu\nu} g_{\nu\sigma} = g_{\lambda\sigma} g^{\lambda\mu} = \delta^{\mu}_{\sigma} \,. \tag{2.15}$$

The metric and its inverse can be used to raise and lower indices on tensors. We can for instance turn vectors and dual vectors into each other:

$$V_\mu = g_{\mu\nu} V^\nu \tag{2.16}$$
$$\omega^\mu = g^{\mu\nu} \omega_\nu \,, \tag{2.17}$$

and define new tensors from the known tensor $T^{\mu\nu}_{\rho\sigma}$:

$$T^{\mu\nu\rho}{}_{\sigma} = g^{\rho\alpha} \, T^{\mu\nu}{}_{\alpha\sigma} \tag{2.18}$$
$$T_{\mu\nu}{}^{\rho\sigma} = g_{\mu\alpha} \, g_{\nu\beta} \, g^{\rho\gamma} \, g^{\sigma\delta} \, T^{\alpha\beta}{}_{\gamma\delta} \,. \tag{2.19}$$

From the metric we can define a line element ds^2 in spacetime by

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,, \tag{2.20}$$

where dx^μ is a dual vector given by the gradient of a coordinate function (so should actually be written as $\mathrm{d}x^\mu$), but informally can be regarded as an infinitesimal displacement. From the previous section we know the right hand side of equation (2.20) is a scalar, which means the line element ds^2 is coordinate independent, as it should be.

The metric also defines the inner product between two vectors V^μ and W^ν:

$$\langle V, W \rangle = g_{\mu\nu} V^\mu W^\nu \,. \tag{2.21}$$

Two vectors are called orthogonal when their inner product vanishes. The norm of a vector is defined to be the inner product of the vector with itself, and is not positive definite in the spacetimes of interest in general relativity. We introduce the following nomenclature to the norm of a vector:

$$\text{if} \quad g_{\mu\nu} V^\mu V^\nu \quad \text{is} \begin{cases} < 0 \,, & V^\mu \text{ is timelike} \\ = 0 \,, & V^\mu \text{ is lightlike or null} \\ > 0 \,, & V^\mu \text{ is spacelike.} \end{cases} \tag{2.22}$$

The metric for four dimensional flat Minkowski spacetime $\eta_{\mu\nu}$ in Cartesian coordinates $\{t, x, y, z\}$ is

$$\eta_{\mu\nu} = \mathrm{diag}(-1, 1, 1, 1) \,. \tag{2.23}$$

According to equation (2.20), the line element of Minkowski space is then

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \,, \tag{2.24}$$

as we would expect. We always use the convention that the speed of light $c = 1$. The Minkowski metric in Cartesian coordinates is called its canonical form; all elements are either $1, -1$ or 0. Its signature, the number of positive and negative eigenvalues, is minus-plus-plus-plus. If the metric is continuous and nondegenerate (none of the eigenvalues are zero), its signature will be the same at every point. We will only consider such metrics. A metric is called Euclidean, Riemannian or positive definite if all signs are positive. If there is a single minus sign, the metric is called Lorentzian or pseudo-Riemannian. Other metrics with some positive and some negative signs are called indefinite. In general relativity we study Lorentzian metrics.

2.1.4 Covariant Derivatives

We have come across the partial derivative $\partial_\mu = \frac{\partial}{\partial x^\mu}$. This derivative depends on the chosen coordinate system and we would like a derivative independent of coordinates. So we want to define a covariant derivative ∇ - a derivative that transforms as a tensor on an arbitrary manifold - with the same function as the partial derivative. Concretely we want the covariant derivative to have the following properties:

- it is a map from (k, l) tensors to $(k, l+1)$ tensors,
- it is linear: $\nabla(T + S) = \nabla T + \nabla S$,
- it obeys the Leibniz rule: $\nabla(T \otimes S) = (\nabla T) \otimes S + T \otimes (\nabla S)$,
- it commutes with contractions: $\nabla_\mu (T^\lambda{}_{\lambda\rho}) = (\nabla T)_\mu{}^\lambda{}_{\lambda\rho}$,
- it reduces to the partial derivative on scalars: $\nabla_\mu \phi = \partial_\mu \phi$.

The covariant derivative has the form:

$$\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\lambda} V^\lambda \qquad \text{for vectors,} \tag{2.25}$$

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma^\lambda_{\mu\nu} \omega_\lambda \qquad \text{for dual vectors,} \tag{2.26}$$

$$\begin{aligned} \nabla_\sigma T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} = {} & \partial_\sigma T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} \\ & + \Gamma^{\mu_1}_{\sigma\lambda} T^{\lambda \mu_2 \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} + \Gamma^{\mu_2}_{\sigma\lambda} T^{\mu_1 \lambda \ldots \mu_k}{}_{\nu_1 \ldots \nu_l} + \ldots \\ & - \Gamma^\lambda_{\sigma\nu_1} T^{\mu_1 \ldots \mu_k}{}_{\lambda \nu_2 \ldots \nu_l} - \Gamma^\lambda_{\sigma\nu_2} T^{\mu_1 \ldots \mu_k}{}_{\nu_1 \lambda \ldots \nu_l} - \ldots \quad \text{for } (k,l) \text{ tensors.} \end{aligned} \tag{2.27}$$

The known partial derivative is corrected by n $(n \times n)$ matrices $\Gamma^\nu_{\mu\lambda}$ called connection coefficients. $\Gamma^\nu_{\mu\lambda}$ is not a tensor and transforms is a way such that ∇ does transform as a tensor:

$$\Gamma^{\nu'}_{\mu'\lambda'} = \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial x^{\nu'}}{\partial x^\nu} \Gamma^\nu_{\mu\lambda} - \frac{\partial x^\mu}{\partial x^{\mu'}} \frac{\partial x^\lambda}{\partial x^{\lambda'}} \frac{\partial^2 x^{\nu'}}{\partial x^\mu \partial x^\lambda} \,. \tag{2.28}$$

There are many connections you can define on a given manifold, which all give a good definition for a covariant derivative. In general relativity, however, we use the Christoffel connection given by

$$\Gamma^\lambda_{\mu\nu} = \frac{1}{2} g^{\lambda\rho} (\partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu}) \,. \tag{2.29}$$

The coefficients are called Christoffel symbols. This connection has the following nice properties:

- it is symmetric in its lower indices: $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ (this is also known as torsion-free),
- the corresponding covariant derivative is metric compatible: $\nabla_\rho g_{\mu\nu} = 0$.

2.1.5 Curvature

So far we have talked about flat and curved spacetimes intuitively. We would like a mathematical structure to locally describe the curvature at each point and a measure for curvature independent of coordinates. The Riemann (or curvature) tensor $R^{\rho}{}_{\sigma\mu\nu}$ is defined as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_{\mu}\,\Gamma^{\rho}_{\nu\sigma} - \partial_{\nu}\,\Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\,\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\,\Gamma^{\lambda}_{\mu\sigma}\,, \tag{2.30}$$

and contains the information we want. Its definition is true for any connection. We say a manifold is flat when $R^{\rho}{}_{\sigma\mu\nu} = 0$. Since the Christoffel connection is constructed from the metric, we can associate the curvature with the metric itself when we use this connection in the definition for the Riemann tensor. We can then say the metric is flat or curved for a vanishing or nonvanishing Riemann tensor, respectively. When using the Christoffel connection we can also make the following statements:

- If there is a coordinate system for which the metric components are constant, the Riemann tensor will vanish.
- If the Riemann tensor vanishes, there is always a coordinate system in which the metric components are constant.

From the Riemann tensor we can construct the Ricci tensor $R_{\mu\nu}$ by contraction:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}\,. \tag{2.31}$$

For the Christoffel connection this is the only independent contraction (all others vanish or are related to this one), and the Ricci tensor is symmetric. The trace of the Ricci tensor is the Ricci (or curvature) scalar:

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu}\,. \tag{2.32}$$

The Ricci scalar gives a quantitative measure of curvature independent of coordinates.

2.2 General Relativity

Einstein related gravity to the curvature of spacetime. He was led to this idea by the Principle of Equivalence. There are three forms of the equivalence principle. The first is the weak equivalence principle or WEP, which states that the inertial and gravitational mass of any object are equal. The inertial mass is the proportionality constant between the force exerted on an object and the acceleration it undergoes, as stated in Newton's second law $F = m_i\, a$. The gravitational mass is the proportionality constant in Newton's law of gravity, relating the gravitational force to the gradient of a scalar field Φ: $F_g = -m_g \nabla\Phi$. Long ago Galileo showed - as the story goes by dropping weights from the Leaning Tower of Pisa - that the acceleration due to gravity is the same for every object, independent of composition or mass. As a consequence the behaviour of freely falling particles is universal to all particles. We can apply the WEP to a thought-experiment. Imagine one physicist on earth in a box closed off from the outside world measuring the gravitational field by dropping test particles. Image another physicist in a box closed off from the outside world, but this time accelerating through space with the same acceleration as the gravitational acceleration on the earth's surface. She is doing the same experiments as the first physicist. The WEP then implies that there is no way for the physicists to know if they are on Earth or floating through space; there is no way to differentiate the effects of a gravitational field from those of being in a uniformly accelerating frame by observing freely falling particles. We should note that this is only true in small enough regions of spacetime, such that tidal forces due to inhomogeneities in the gravitational field are not detectable.

Einstein expanded the WEP to include the new theory of special relativity and to be true for every kind of experiment and not only for dropping test particles. This became the Einstein Equivalence Principle or EEP: *In small enough regions of spacetime, the laws of physics reduce to those of special relativity; it is impossible to detect the existence of a gravitational field by means of local experiments.* The EEP only applies to "gravitational laws of physics". The Strong Equivalence Principle includes all laws of physics, to allow gravitational experiments and self-gravitating objects.

The EEP states that gravity can not be distinguished from uniform acceleration and since there is no gravitationally neutral object that can act as reference point, gravity is inescapable. Acceleration due to gravity can then not be reliably defined. This leads us to define "unaccelerated" as "freely falling in the present gravitational field". This deprives gravity from its status of being a force, since a force is something that causes acceleration. As a consequence it is impossible to construct a global inertial coordinate system as in special relativity, because at one point freely falling particles will seem to be accelerating with respect to such a system. The best we can do is define locally inertial frames, that follow the motion of individual freely falling particles in small enough regions of spacetime. A downside is that we no longer can compare velocities of greatly separated objects. The mathematical framework that supports these ideas is that of describing spacetime as a manifold. A manifold locally looks like flat spacetime which matches the local reduction of physics to special relativity. On the whole, however, a manifold can be curved. The universality of gravity then motivates the attribution of gravity to the curvature of spacetime.

2.2.1 Spacetime

We have described why n-dimensional spacetime is modeled as a manifold and how it can be represented in a Lorentzian metric. There is one time dimension of negative signature and $n-1$ space dimensions of positive signature. Now we can get more into what this physically tells us.

We have seen that the metric defines the spacetime interval between two events as

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu . \tag{2.33}$$

When this interval is negative we call the events timelike separated, when it is positive we call them spacelike separated and when it is zero we call them lightlike or null separated. To see why they get these names we look at the spacetime diagram for Minkowski space (figure 2).

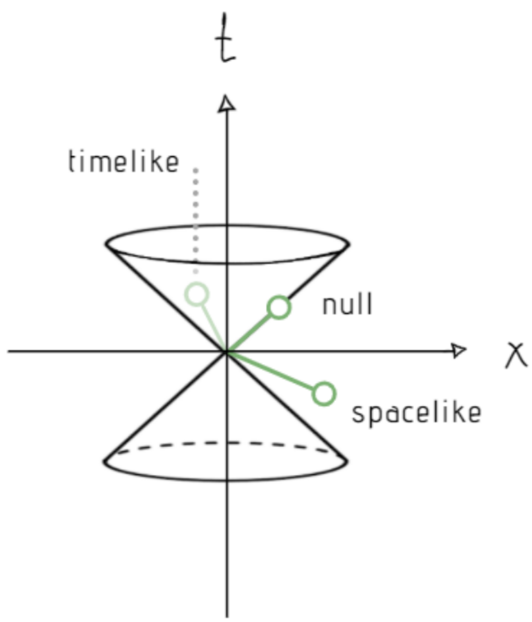

Figure 2: Spacetime diagram of Minkowski space including a light cone in the origin made out of all paths light could take through the origin, and the relation of timelike, null and spacelike paths to this light cone. Adapted from [7].

In the figure the x and t axes are depicted. We also see this diabolo shape at a 45 degree angle. This is called the light cone and consists of all points connected to one event by straight moving particles with the speed of light. So the future light cone is all spacetime points where light could go to departing from one event, and the past light cone is all the points light could have come from to reach that one event. If $dy = dz = 0$ and $c = 1$, light moves on the lines $x = \pm t$. That is why the cones are at a 45 degree angle and the light cone is defined by $ds^2 = 0$, as can be seen from equation (2.33). All points inside the future and past light cone, with $ds^2 < 0$ to the origin, are called timelike. All points outside the light cone, with $ds^2 > 0$ to the origin, are called spacelike. Nothing can move faster than light, so the light cone can also be seen as a boundary between events that are and events that are not causally linked; the event at the origin can only effect events inside and on the light cone.

For timelike paths we define the proper time τ to be

$$d\tau^2 = -ds^2 \qquad \text{or} \qquad \Delta\tau = \int \sqrt{-g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}}\, d\lambda \, . \tag{2.34}$$

The proper time gives the time between two events measured by an observer moving on a straight line.

From classical mechanics we know that unaccelerated particles take the path of shortest distance. In flat spacetime this is a straight line. In curved spacetime the shortest path is called a geodesic and is given by the geodesic equation. The easiest way to get to the geodesic equation is by reviewing and generalising a second definition of the straight line in flat space. A straight line can also be defined as a path along which its tangent vector stays constant.[1] For a tensor $T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l}$ to remain constant along the curve $x^\mu(\lambda)$ means that all its components have to remain constant:

$$\frac{d}{d\lambda}T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} = \frac{dx^\sigma}{d\lambda}\frac{\partial}{\partial x^\sigma}T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} = 0\,. \tag{2.35}$$

To generalise this to curved spacetime we need to make this equation coordinate-independent by replacing the partial derivative by the covariant derivative. We also define the directional covariant derivative to be

$$\frac{D}{d\lambda} = \frac{dx^\sigma}{d\lambda}\nabla_\sigma\,. \tag{2.36}$$

Equation 2.35 then becomes

$$\left(\frac{D}{d\lambda}T\right)^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} \equiv \frac{dx^\sigma}{d\lambda}\nabla_\sigma T^{\mu_1\mu_2\dots\mu_k}{}_{\nu_1\nu_2\dots\nu_l} = 0\,. \tag{2.37}$$

When this requirement is met, i.e. the covariant derivative of T along $x^\mu(\lambda)$ vanishes, we say there is parallel transport of T. For a vector V^μ this condition takes the form

$$\frac{d}{d\lambda}V^\mu + \Gamma^\mu_{\sigma\rho}\frac{dx^\sigma}{d\lambda}V^\rho = 0\,. \tag{2.38}$$

Now we can define a geodesic as a path $x^\mu(\lambda)$ along which the tangent vector $dx^\mu/d\lambda$ is parallel-transported and get the resulting geodesic equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma^\mu_{\rho\sigma}\frac{dx^\rho}{d\lambda}\frac{dx^\sigma}{d\lambda} = 0\,. \tag{2.39}$$

You can also derive the geodesic equation from the shortest distance definition by looking at small variations of the integral definition of the proper time and searching for critical points. Geodesics will turn out to correspond to paths of maximum proper time.

When a geodesic appears to end on a specific point of a manifold, this point is called a singularity (that is excluding singularities due to a specific choice of coordinates). The manifold is then said to be geodesically incomplete. The minimum requirement for spacetime singularities is timelike and null geodesic incompleteness. Spacelike geodesic incompleteness is of less importance because nothing moves on those curves. Spacetime singularities already arise under quite general assumptions, so even the quite simple Schwarzschild spacetime features real singularities (see section 2.2.3).[17][18]

[1] These two definitions are actually only the same when you use the Christoffel connection.

2.2.2 Einstein's Equations

Einstein's equation tells us how the curvature of spacetime is linked to the distribution of energy-momentum, represented in the energy-momentum tensor. Energy and momentum create curvature. Curvature acts as gravity and tells matter how to move via the geodesic equation. We will derive Einstein's equation with the principle of least action. From classical field theory you might be familiar with this method, but to use it in general relativity we first need to generalise it to curved spacetime.

The classical solutions for a field theory, where the dynamical variables are a set of fields $\Phi^i(x)$, will be the critical points of an action S. The action is the integral of a Lagrange density $\mathcal{L}$, a function of the fields and their covariant derivatives, over space:

$$S = \int d^n x \ \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i) \, . \tag{2.40}$$

We write the Lagrangian as

$$\mathcal{L} = \sqrt{-g} \hat{\mathcal{L}} \, , \tag{2.41}$$

with $\hat{\mathcal{L}}$ a scalar. To find the critical points of S we vary the fields and require the action to be unchanged. So we begin by varying the fields and making an appropriate Taylor expansion of the Lagrangian:

$$\begin{aligned} \Phi^i &\to \Phi^i + \delta\Phi^i \\ \nabla_\mu \Phi^i &\to \nabla_\mu \Phi^i + \nabla_\mu(\delta\Phi^i) \\ \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i) &\to \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i) + \frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi^i)}\nabla_\mu(\delta\Phi^i) \\ &= \mathcal{L} + \delta\mathcal{L} \, . \end{aligned} \tag{2.42}$$

The action then follows with

$$\begin{aligned} S &\to S + \delta S \\ &= \int d^n x \ \mathcal{L}(\Phi^i, \nabla_\mu \Phi^i) + \int d^n x \left[\frac{\partial \mathcal{L}}{\partial \Phi^i}\delta\Phi^i + \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi^i)}\nabla_\mu(\delta\Phi^i) \right] . \end{aligned} \tag{2.43}$$

We want δS to be zero so the actions remains unchanged under field variations. To reformulate this condition we factor out the term $\delta\Phi^i$ in δS. We do this by integrating the second term by parts:

$$\begin{aligned} \int d^n x \ \frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi^i)}\nabla_\mu(\delta\Phi^i) &= - \int d^n x \ \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi^i)} \right) \delta\Phi^i + \int d^n x \ \nabla_\mu \left(\frac{\partial \mathcal{L}}{\partial(\nabla_\mu \Phi^i)} \delta\Phi^i \right) \\ &= - \int d^n x \ \nabla_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\nabla_\mu \Phi^i)} \right) \sqrt{-g} \, \delta\Phi^i \\ &\qquad + \int d^n x \ \nabla_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\nabla_\mu \Phi^i)} \delta\Phi^i \right) \sqrt{-g} \\ &= - \int d^n x \ \nabla_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial(\nabla_\mu \Phi^i)} \right) \sqrt{-g} \, \delta\Phi^i \, , \end{aligned} \tag{2.44}$$

where we have used equation (2.41), a metric compatible covariant derivative, Stokes's theorem and our choice to set the variation of the field to zero at the boundary (infinity). Stokes's theorem in curved spacetime is

$$\int d^n x \; \nabla_\mu V^\mu \sqrt{|g|} = \int d^{n-1} x \; n_\mu V^\mu \sqrt{|\gamma|} \,, \tag{2.45}$$

with V^μ a vector field over a region Σ with boundary $\partial\Sigma$, n_μ normal to $\partial\Sigma$, and γ_{ij} the induced metric on $\partial\Sigma$. Now we have an expression for δS and we also know its form by definition:

$$\begin{aligned} \delta S &= \int d^n x \; \sqrt{-g} \left[\frac{\partial \hat{\mathcal{L}}}{\partial \Phi^i} - \nabla_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \right) \right] \delta \Phi^i && (2.46) \\ &= \int d^n x \; \frac{\delta S}{\delta \Phi^i} \delta \Phi^i \,, && (2.47) \end{aligned}$$

again using equation (2.41). For δS to be zero we need $\delta S / \delta \Phi^i$ to be zero, so

$$\frac{\partial \hat{\mathcal{L}}}{\partial \Phi^i} - \nabla_\mu \left(\frac{\partial \hat{\mathcal{L}}}{\partial (\nabla_\mu \Phi^i)} \right) = 0 \,. \tag{2.48}$$

These are called the Euler-Lagrange equations. Solutions to the field theory also satisfy these equations.

Now we want to apply this method to general relativity, a field theory with the metric as dynamic variable. The only independent scalar you can construct from the metric with no derivatives higher than the first one is the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$. So the Lagrangian and corresponding action, called the Hilbert action, are

$$\mathcal{L} = \sqrt{-g} R \qquad \text{and} \qquad S_H = \int \sqrt{-g} R \, d^n x \,. \tag{2.49}$$

The Hilbert action is not of the same form as equation (2.40) (it can not be written in terms of the metric and its covariant derivative), so we can not simply plug the Lagrangian in the Euler-Lagrange equation to obtain the field equations and instead have to explicitly variate the action with respect to the metric. It turns out to be easier to variate with respect to the inverse metric $g^{\mu\nu}$ and stationary points coming from variations in $g^{\mu\nu}$ are equivalent to ones from variations in $g_{\mu\nu}$. This can be seen from varying the expression

$$\begin{aligned} g^{\mu\lambda} g_{\lambda\nu} &= \delta^\mu_\nu \\ (g^{\mu\lambda} + \delta g^{\mu\lambda})(g_{\lambda\nu} + \delta g_{\lambda\nu}) &\stackrel{!}{=} \delta^\mu_\nu \\ &= \delta^\mu_\nu + g^{\mu\lambda} \delta g_{\lambda\nu} + g_{\lambda\nu} \delta g^{\mu\lambda} + \mathcal{O}(\delta^2) \\ \rightarrow \delta g_{\mu\nu} &= - g_{\mu\lambda} g_{\nu\rho} \delta g^{\lambda\rho} \,. \end{aligned} \tag{2.50}$$

We can start by expressing the variation of the Hilbert action as

$$\delta S_H = (\delta S)_1 + (\delta S)_2 + (\delta S)_3 \,, \tag{2.51}$$

with

$$
\begin{aligned}
(\delta S)_1 &= \int d^n x \; R\, \delta\sqrt{-g} \\
(\delta S)_2 &= \int d^n x \; \sqrt{-g}\, g^{\mu\nu}\, \delta R_{\mu\nu} \\
(\delta S)_3 &= \int d^n x \; \sqrt{-g} R_{\mu\nu}\, \delta g^{\mu\nu}\,.
\end{aligned}
\tag{2.52}
$$

We want each of these to have a separate $\delta g^{\mu\nu}$ term, just as in equation (2.46). $(\delta S)_3$ is already in the right form. We will first look at $(\delta S)_1$. We need to know what the variation of $g = \det\, g_{\mu\nu}$ is. For a general square matrix M with non vanishing determinant the following is true

$$
\ln(\det M) = \mathrm{Tr}(\ln M)\,. \tag{2.53}
$$

The variation of this is

$$
\frac{1}{\det M}\delta(\det M) = \mathrm{Tr}(M^{-1}\delta M)\,. \tag{2.54}
$$

So for the metric we have

$$
\delta g = g(g^{\mu\nu}\delta g_{\mu\nu}) = -g(g_{\mu\nu}\delta g^{\mu\nu})\,, \tag{2.55}
$$

using equation (2.50). Then we now know

$$
\begin{aligned}
\delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\,\delta g \\
&= \frac{1}{2}\frac{g}{\sqrt{-g}} g_{\mu\nu}\,\delta g^{\mu\nu} \\
&= -\frac{1}{2}\sqrt{-g}\, g_{\mu\nu}\,\delta g^{\mu\nu}\,,
\end{aligned}
\tag{2.56}
$$

and $(\delta S)_1$ becomes

$$
(\delta S)_1 = \int d^n x \; \sqrt{-g}\left[-\frac{1}{2}R g_{\mu\nu}\right]\delta g^{\mu\nu}\,. \tag{2.57}
$$

Now we will look at $(\delta S)_2$. The variation of the Riemann tensor $\delta R_{\mu\nu}$ can be found by varying its definition (equation (2.30)) with respect to the Christoffel symbols and then inserting the variation of the Christoffel symbols with respect to the inverse metric. The expression you get can be converted into a boundary integral by Stokes's theorem and can be set to zero by making the variation vanish at infinity. The remaining terms give the variation of the Hilbert action:

$$
\delta S_H = \int d^n x \; \sqrt{-g}\left[R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right]\delta g^{\mu\nu}\,. \tag{2.58}
$$

At the critical points of the action $\delta S_H/\delta g^{\mu\nu} = 0$ and comparing equation (2.58) to equation (2.47) we see that

$$
\frac{1}{\sqrt{-g}}\frac{\delta S_H}{\delta g^{\mu\nu}} = R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu} = 0\,. \tag{2.59}
$$

This is Einstein's equation in vacuum. To get the field equations of general relativity coupled to matter we have to include an extra term S_M, representing the matter field, in the action:

$$S = \frac{1}{16\pi G} S_H + S_M \,, \tag{2.60}$$

with G Newton's gravitational constant. We used a normalisation on the Hilbert action we know will yield the right equation. When we apply the same method as above we get

$$\frac{1}{\sqrt{-g}} \frac{\delta S}{\delta g^{\mu\nu}} = \frac{1}{16\pi G} \left(R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} \right) + \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} = 0 \,. \tag{2.61}$$

Now we need a new (and correct) definition of the energy-momentum tensor

$$T_{\mu\nu} = -2 \frac{1}{\sqrt{-g}} \frac{\delta S_M}{\delta g_{\mu\nu}} \,, \tag{2.62}$$

such that equation (2.61) becomes

$$R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu} = 8\pi G \, T_{\mu\nu} \qquad \text{or} \qquad G_{\mu\nu} = 8\pi G \, T_{\mu\nu} \,. \tag{2.63}$$

This is the complete Einstein's equation. $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R \, g_{\mu\nu}$ is fittingly called the Einstein tensor. In four dimensions these are 16 second-order differential equations for the metric. But because both sides of equation (2.63) are symmetric tensors there are only 10 independent equations. This corresponds to the 10 unknown metric components.
We can rewrite Einstein's equation, using $R = -8\pi \, G \, T$, in a form that is very convenient when considering vacuum spaces:

$$R_{\mu\nu} = 8\pi \, G \left(T_{\mu\nu} - \frac{1}{2} T \, g_{\mu\nu} \right) \,, \tag{2.64}$$

$$R_{\mu\nu} = 0 \qquad \text{in vacuum} \,. \tag{2.65}$$

2.2.3 Schwarzschild Black Hole

We will take a look at our first example of a physically relevant metric, the Schwarzschild metric. The metric describes spherically symmetric and static sources for gravity in vacuum space, and is a good approximation to describe the field created by the Earth or the Sun at distances far greater than the Schwarzschild radius (defined shortly). The Schwarzschild metric in spherical coordinates is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \,, \tag{2.66}$$

where M can be interpreted as the mass of the source and $d\Omega^2$ is the metric on a unit two-sphere:

$$d\Omega^2 = d\theta^2 + \sin^2\theta \, d\phi^2 \,. \tag{2.67}$$

The metric can be derived from the Minkowski metric $ds^2 = -dt^2 + dr^2 + r^2 d\Omega^2$ and some set conditions. To ensure you maintain spherical symmetry, the term $d\Omega^2$ needs to retain its form and can only have an overall coefficient, i.e. the $d\phi^2$ term should be $\sin^2\theta$ times the $d\theta^2$ term. A static source means it remains unchanged in time, so "it does nothing". To describe a static source the metric components should be independent of t and the metric should be time reversal invariant, so there should not be any space and time coordinate cross terms $(dtdx^i + dx^i dt)$. And finally Einstein's equation should be considered in vacuum, so $R_{\mu\nu} = 0$. A full derivation can be found in the book by Carroll [7].
When $M \to 0$ Minkowski space is recovered, but also as $r \to \infty$ the metric becomes more and more like Minkowski space. When a metric has this last property we say that it is asymptotically flat.

When looking at equation (2.66), we assume the points $r = 0$ and $r = 2GM$ will represent special places in Minkowski space as the metric coefficients become infinite there. And indeed they will respectively turn out to be a singularity and an event horizon of a black hole, as we will show and define now.

The metric coefficients are coordinate dependent, so we need to rule out that $r = 0$ is just a coordinate singularity where the chosen coordinate systems fails to describe the spacetime. We already pointed out that a singularity is a place where geodesics suddenly seize to exist. An easier and sufficient condition is that the curvature becomes infinite. To measure this independent of coordinates we look at scalars constructed from the Riemann tensor, like the Ricci scalar $R = g^{\mu\nu}R_{\mu\nu}$ or higher order scalars such as $R^{\mu\nu}R_{\mu\nu}$, $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$, $R_{\mu\nu\rho\sigma}R^{\rho\sigma\lambda\tau}R_{\lambda\tau}{}^{\mu\nu}$, and so on. We then say a point is a singularity, if any of these scalars goes to infinity as we approach that point. For the Schwarzschild metric we have

$$R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} = \frac{48G^2M^2}{r^6} \,. \tag{2.68}$$

We then see that $r = 0$ is indeed a singularity. At $r = 2GM$ all curvature scalars remain finite, so it is not a singularity.

To examine what does happen at $r = 2GM$, we explore the causal structure of the spacetime. To do so we change to different coordinates:

$$v = t + r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \,, \tag{2.69}$$

$$u = t - r + 2GM \ln\left(\frac{r}{2GM} - 1\right) \,. \tag{2.70}$$

Infalling radial null geodesics are represented by $v =$ constant, outgoing ones by $u =$ constant. In equation (2.66) we change the timelike coordinate t to the new coordinate v and keep the original r coordinate (these are known as the Eddington-Finkelstein coordinates), and the metric becomes

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dv^2 + (dvdr + drdv) + r^2 d\Omega^2 \,. \tag{2.71}$$

The causal structure can best be seen from the behaviour of light cones, outlined by radial null geodesics. Here, these are described by

$$\frac{dv}{dr} = \begin{cases} 0 & \text{for infalling radial null geodesics,} \\ 2\left(1 - \frac{2GM}{r}\right)^{-1} & \text{for outgoing radial null geodesics.} \end{cases} \tag{2.72}$$

The light cones are depicted in figure 3. We see that at $r = 2GM$ the light cones tilt over and all future-directed paths for $r < 2GM$ have to go in the direction of decreasing r. This means that once a particle crosses $r = 2GM$ it is trapped inside that boundary. $r = 2GM$ is called the Schwarzschild radius and is an event horizon. An event horizon is a surface past which particles can never escape to infinity. The region of spacetime separated from infinity by an event horizon is called a black hole; a suiting name because we can not see inside. Singularities can occur without an event horizon covering it, these are called naked singularities. According to the belief of the cosmic censorship conjecture, however, they can not form under normal circumstances[2], so whenever we encounter them we will not regard them as proper solutions.

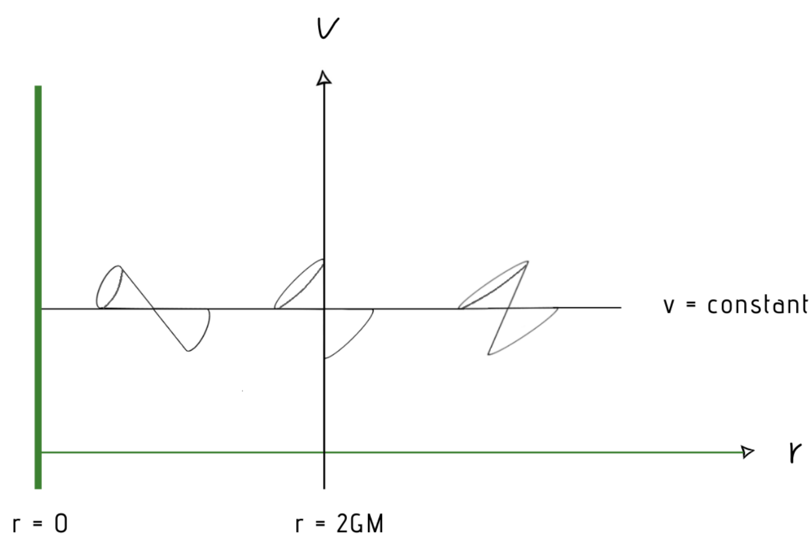

Figure 3: Light cones in Schwarzschild spacetime in Eddington-Finkelstein coordinates for future-directed paths, tilt over at $r = 2GM$, making this point an event horizon. Adapted from [7].

[2]Normal circumstances mean in gravitational collapse from generic, initially nonsingular states in an asymptotically flat spacetime obeying the dominant energy condition. The cosmic censorship conjecture has not been proven yet, although many have tried [19].

Another and more generally applicable way of finding event horizons is by its more formal definition. The causal past of a region is the set of all points in spacetime you can reach from that region by following past-directed timelike paths. An event horizon is then the boundary of the causal past of infinity [3]; nothing that has crossed this boundary can reach infinity anymore. Since causal influences can not move faster than the speed of light, the horizon has to be a null hypersurface. A hypersurface Σ is defined by some function $f(x) =$ constant. The gradient $\partial_\mu f$ is normal to Σ. The hypersurface is said to be null when the normal vector is null. The normal vector is then also tangent to Σ.[4]

To get to a concrete condition for an event horizon we use the advantage that we will only consider stationary and asymptotically flat metrics. For stationary metrics we can choose the metric components to be independent of time. We can also choose coordinates in a clever way such that $r =$ constant hypersurfaces will be timelike at infinity and stay timelike for decreasing r until some fixed $r = r_H$ where the hypersurface will be null everywhere. Timelike paths in the region $r < r_H$ will then never be able to cross this boundary to infinity and r_H is clearly the event horizon. To find the point r_H where $r =$ constant hypersurfaces become null, we need to examine where its normal vector $\partial_\mu r$ becomes a null vector:

$$g^{\mu\nu}(\partial_\mu r)(\partial_\nu r) = g^{rr}(r_H) = 0\,. \tag{2.73}$$

We can now apply this to the Schwarzschild metric and check we get the same result as before:

$$g^{rr}(r_H) = 1 - \frac{2GM}{r_H} = 0 \quad \rightarrow \quad r_H = 2GM\,. \tag{2.74}$$

The Schwarzschild metric can be maximally extended in Kruskal coordinates to describe every region there is in this spacetime (a so called white hole and parallel universe). If you are interested, I again recommend Carroll [7] for a more thorough analysis of this metric.

The Schwarzschild metric is the unique static solution for asymptotically flat vacuum space with spherical symmetry as stated by Birkhoff's theorem. This means the static solution to Einstein's equation is completely determined by the mass of the system.

2.2.4 Kerr Black Hole

Another black hole in asymptotically flat vacuum spacetime is the rotating Kerr black hole. This solution does not have spherical but axial symmetry around its axis of rotation $\theta = 0, \pi$. The Kerr metric is given by

$$\begin{aligned} ds^2 = &-\left(1 - \frac{2GM\,r}{\rho^2}\right)dt^2 - \frac{2GM\,a\,r\sin^2\theta}{\rho^2}(dtd\phi + d\phi dt) + \frac{\rho^2}{\Delta}dr^2 \\ &+ \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left((r^2+a^2)^2 - a^2\Delta\sin^2\theta\right)d\phi^2\,, \end{aligned} \tag{2.75}$$

[3] We should actually be more specific and note that asymptotically flat spacetimes look like Minkowski space at infinity, and have the same structure at conformal future and past null infinity $\mathscr{I}^\pm$ and spatial infinity i^0. An event horizon is the boundary of the causal past of future null infinity $\mathscr{I}^+$.

[4] This looks very counter-intuitive, but remember that tangent vectors are orthogonal to normal vectors and the inner product is defined with the Lorentzian metric. Null vectors are orthogonal to themselves.

with

$$\Delta(r) = r^2 - 2GM\,r + a^2 \tag{2.76}$$

and

$$\rho^2(r,\theta) = r^2 + a^2\cos^2\theta\,. \tag{2.77}$$

The used coordinates $\{t, r, \theta, \phi\}$ are called Boyer-Lindquist coordinates. M is again the mass of the source and a is the angular momentum per unit mass. When $M \to 0$ flat spacetime in ellipsoidal coordinates is recovered, and as $a \to 0$ the metric reduces to the Schwarzschild metric. The Kerr metric describes a stationary solution; the black hole rotates in exactly the same way for all time. The metric components are independent of t but the metric is not time-reversal invariant. When time is reversed the black hole will spin the other way around.

For this metric the curvature scalar $R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma}$ diverges at $\rho = 0$, so at that point there is a curvature singularity. ρ only vanishes when both $r = 0$ and $\theta = \pi/2$; this describes a ring on the edge of the disk $r = 0$ (as can be seen by plugging $t = r = 0$ and $\theta = \pi/2$ into the metric).

When searching for event horizons, we again look for those values of r for which g^{rr} vanishes (we have chosen the right coordinates to use this method). Here $g^{rr} = \Delta/\rho^2$, and since $\rho^2 \geq 0$ we want

$$\Delta(r) = r^2 - 2GM\,r + a^2 = 0\,. \tag{2.78}$$

There are three possibilities: $GM < a$, $GM = a$ and $GM > a$. The first one describes a naked singularity; a singularity without an event horizon around it. The second one is an unstable solution. We will focus on the case $GM > a$ because this one is of the most physical interest. It has been proven that for $GM > a$ the Kerr metric is the unique stationary solution to Einstein's equation in asymptotically flat vacuum spacetime [20]. The mass and angular momentum of the system uniquely determine the stationary solution. The two event horizons are in this case

$$r_{H\pm} = GM \pm \sqrt{G^2M^2 - a^2}\,. \tag{2.79}$$

A new feature of rotating black holes is the existence of an ergosphere; a space around the black hole where it is impossible to stand still. A necessary condition for an observer to stand still is that its world line, his path in space time, is timelike. Suppose we try to stay at one fixed point of the coordinate system, the world line is then

$$X^\mu(t) = (t, r_0, \theta_0, \phi_0)\,, \tag{2.80}$$

and the corresponding tangent vector is

$$T^\mu = \frac{dX^\mu}{dt} = (1,0,0,0)\,. \tag{2.81}$$

For the world line to be timelike we need

$$g_{\mu\nu}T^\mu T^\nu = g_{tt} < 0\,, \tag{2.82}$$

or specifically for the Kerr metric

$$g_{tt} = -\left(1 - \frac{2GM\,r}{\rho^2}\right) < 0\,. \tag{2.83}$$

The solutions to $g_{tt} = 0$ are

$$r_{E\pm} = GM \pm \sqrt{G^2M^2 - a^2\cos^2(\theta)}\,, \tag{2.84}$$

and are called stationary limit surfaces. The condition stated in equation (2.83) is met for $r > r_{E+}$, so for these r an observer can stand still. For the region $r_{E-} < r < r_{E+}$, on the contrary, the condition is not met and it is impossible to stand still. The region between r_{H+} and r_{E+} is called the ergosphere. Inside the ergosphere the observer has to move in the direction of the rotation of the black hole, but can move freely towards or away from the event horizon. r_{E+} is sometimes called an ergosurface.

When comparing equation (2.79) to equation (2.84), we see that $r_{E+} \geq r_{H+} \geq r_{H-} \geq r_{E-}$ and that the stationary limit surfaces coincide with the event horizons when $\theta = 0, \pi$. For $\theta = \pi/2$ the inner stationary limit surface touches the singularity. These characteristics of the Kerr metric are depicted in figure 4. [21]

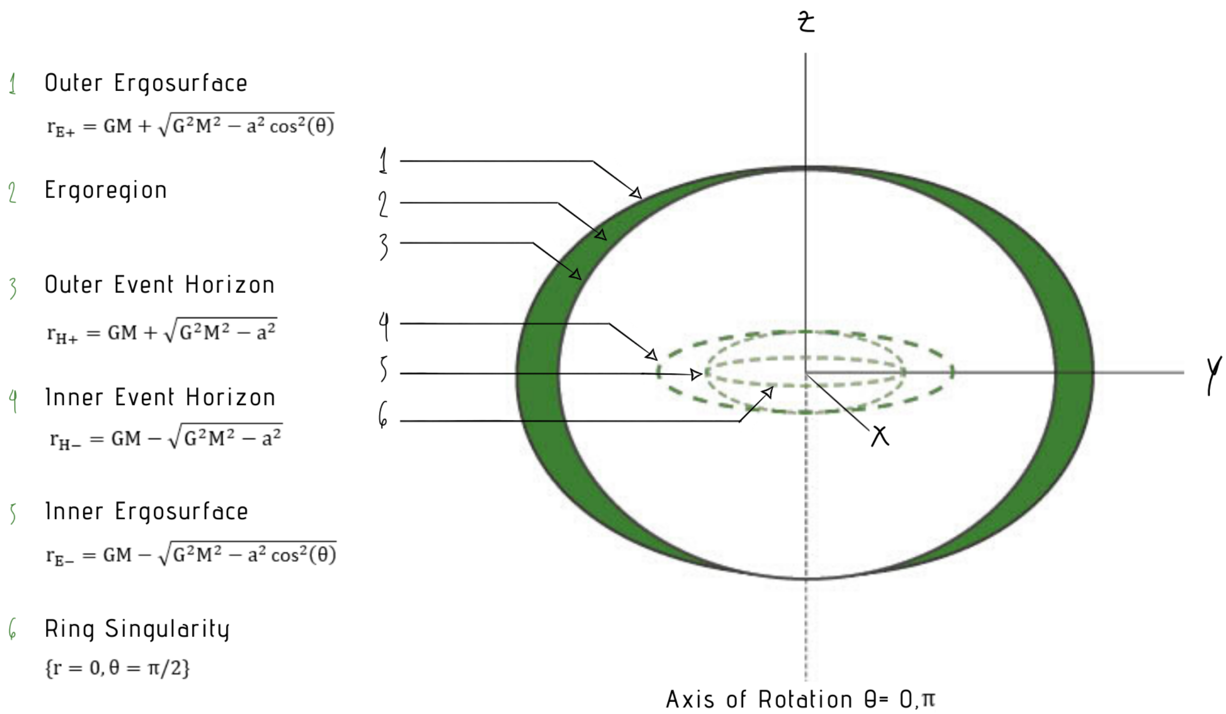

Figure 4: Location of the singularity, event horizons and ergosphere of a Kerr black hole. Adapted from [21].

3 HIGHER DIMENSIONAL BLACK HOLES

So far we have looked at the Schwarzschild and Kerr metric in four dimensions. The discussed results like vectors, tensors, metrics, the geodesic equation and Einstein's equation were described for a general number of dimensions n. It was not hard to describe the general theory in n dimensions, since tensor equations are not restricted by their dimensionality. In this chapter, however, we will discuss some explicit examples of higher dimensional spacetimes and experience the increasing difficulty of the extra degrees of freedom that come with more dimensions. In higher dimensions, solutions will also fail to be uniquely described by mass and angular momenta as in four dimensions. We will describe the generalisation of the Schwarzschild metric to n dimensions called the Schwarzschild-Tangherlini black hole. Then we will cover black holes in five dimensions, focusing on the Myers-Perry black hole and black rings.

3.1 Schwarzschild-Tangherlini Black Hole

Tangherlini generalised the Schwarzschild metric to n dimensions [22][23] and got the following expression for it:

$$ds^2 = -\left(1 - \frac{\mu}{r^{n-3}}\right) dt^2 + \left(1 + \frac{\mu}{r^{n-3}}\right)^{-1} dr^2 + r^2 d\Omega_{n-2}^2 \, , \tag{3.1}$$

where

$$\mu = \frac{16\pi G\, M}{(n-2)\Omega_{n-2}} \quad \text{is the reduced mass,} \tag{3.2}$$

$$\Omega_{n-2} = \frac{2\pi^{(n-1)/2}}{\Gamma(\frac{n-1}{2})} \quad \text{is the area of a unit } (n-2)\text{-sphere,} \tag{3.3}$$

and

$$d\Omega_{n-2}^2 = d\theta_1^2 + \sin^2(\theta_1) d\theta_2^2 + \ldots + (\sin^2(\theta_1) \ldots \sin^2(\theta_{n-3})) d\theta_{n-2}^2 \tag{3.4}$$

is the metric of a unit $(n-2)$-sphere.

We should note that here G is the n dimensional gravitational constant as given in the n dimensional Einstein equation (2.63).

It is truly a generalisation of the Schwarzschild metric since it describes static, spherically symmetric, asymptotically flat, vacuum solutions. The metric components are independent of t and invariant under time reversal. For all constant t and r the metric describes a $(n-2)$-sphere. As $\mu \to 0$, and as $r \to$ flat spacetime is recovered. And this metric is a solution to Einstein's equation in vacuum $R_{\mu\nu} = 0$. Moreover there is a singularity at $r^{n-3} = 0$ and an event horizon at $r^{n-3} = \mu$. The event horizon is again determined by the condition $g^{rr} = 0$, just like in four dimensions.

The Schwarzschild-Tangherlini black hole is the unique static solution in asymptotically flat spacetime [24]. So for a given mass there is no other static solution to Einstein's equation. In five dimensions, however, we will see in the next section that there is another black hole solution with zero angular momentum: a black saturn. This does not contradict the uniqueness of the Schwarzschild-Tangerlini black hole because a black saturn is stationary, but not static.

3.2 Five Dimensional Black Holes

We continue by considering stationary black holes in five dimensional spacetime. These black holes are restricted to have the following event horizon topologies: a three-sphere S^3, or a ring $S^1 \times S^2$ [25]. The first corresponds to the static Schwarzschild-Tangherlini black hole or the stationary Myers-Perry black hole, the second to a black ring.

In five dimensions there are also solutions constructed from two separate black holes called multiple-black-hole solutions:

- A black saturn;
  a rotating spherical black hole surrounded by a concentric rotating black ring kept in equilibrium by angular momentum. The black hole and ring can counter-rotate and achieve zero total angular momentum. [26]
- A black di-ring;
  two concentric black rings rotating in the same plane. There are an infinite amount of different di-rings for the same mass and angular momentum, so uniqueness of solutions in five dimensions is definitely off the table. [27]
- A bicycling black ring;
  two concentric spinning black rings in planes orthogonal to each other. [28]

We will now go over the specifics of the Myers-Perry black holes and black rings.

3.2.1 Myers-Perry Black Hole

Myers-Perry black holes are generalisations of the spinning Kerr black hole [29]. Before writing its metric, we will first take a look at the flat space metric for an odd number of dimensions $n = 2d + 1$, $d \geq 2$:

$$ds^2 = -dt^2 + \sum_{i=1}^{d}(dx_i^2 + dy_i^2) \tag{3.5}$$

$$= -dt^2 + dr^2 + r^2 \sum_{i=1}^{d}(d\mu_i^2 + \mu_i^2\, d\phi_i^2)\,. \tag{3.6}$$

In the first line we grouped together the spatial Cartesian coordinates (x_i, y_i) in d orthogonal planes. In the second line we converted to polar coordinates by the following transformations:

$$x_i = r\,\mu_i \cos\phi_i \qquad \text{and} \qquad y_i = r\,\mu_i \sin\phi_i\,, \tag{3.7}$$

such that $r^2 = \sum_{i=1}^{d}(x_i^2 + y_i^2)$. This puts a constraint on the direction cosines μ_i and makes them dependent on each other:

$$\sum_{i=1}^{d} \mu_i^2 = 1\,. \tag{3.8}$$

The Myers-Perry metric will be in the same form as (3.6) and approach it asymptotically.

The Myers-Perry metric for an odd number of dimensions is

$$\begin{aligned} ds^2 = -dt^2 + \frac{\mu r^2}{\Pi F}\left(dt + \sum_{i=1}^{d} a_i \mu_i^2 d\phi_i\right)^2 + \frac{\Pi F}{\Pi - \mu r^2} dr^2 \\ + \sum_{i=1}^{d}(r^2 + a_i^2)(d\mu_i^2 + \mu_i^2 d\phi_i^2)\,, \end{aligned} \tag{3.9}$$

where

$$F = 1 - \sum_{i=1}^{d} \frac{a_i^2 \mu_i^2}{r^2 + a_i^2} \tag{3.10}$$

$$\Pi = \prod_{i=1}^{d}(r^2 + a_i^2)\,. \tag{3.11}$$

The positive parameters μ and a_i specify the mass and angular momentum of the black hole:

$$M = \frac{(n-2)\Omega_{n-2}}{16\pi G}\,\mu\,, \tag{3.12}$$

$$J^{x_i y_i} = \frac{\Omega_{n-2}}{8\pi G}\,\mu\, a_i = \frac{2}{n-2}\, M\, a_i\,, \tag{3.13}$$

with Ω_{n-2} the area of a unit $(n-2)$-sphere, as in (3.3). When the spin parameters a_i are zero, this metric reduces to the n dimensional Schwarzschild-Tangherlini metric (3.1). When both $a_i = 0$ and $\mu = 0$ the flat spacetime metric (3.6) is recovered. The Myers-Perry solution for $n = 4$ is the same as the known Kerr solution.

We will now focus on the solution in five dimensions and choose $a_1 = a$, $a_2 = b$, $\mu_1 = \sin\theta$, $\mu_2 = \cos\theta$, $\phi_1 = \psi$ and $\phi_2 = \phi$ for later convenience. The Myers-Perry metric in five dimensions is then

$$\begin{aligned} ds^2 = -dt^2 + \frac{\mu}{\Sigma}(dt + a\sin^2\theta\, d\psi + b\cos^2\theta\, d\phi)^2 + \frac{r^2\,\Sigma}{\Pi - \mu r^2} dr^2 \\ + \Sigma\, d\theta^2 + (r^2 + a^2)\sin^2\theta\, d\psi^2 + (r^2 + b^2)\cos^2\theta\, d\phi^2\,, \end{aligned} \tag{3.14}$$

where

$$\Sigma = r^2 + a^2\cos^2\theta + b^2\sin^2\theta\,, \tag{3.15}$$

$$\Pi = (r^2 + a^2)(r^2 + b^2)\,. \tag{3.16}$$

We see that various metric components diverge for $\Sigma = 0$ and $\Pi - \mu r^2 = 0$, and suspect this metric contains singularities, event horizons and ergospheres. The situation will be different depending on whether a spin parameter vanishes or not. We will go over all possibilities now.

Singularities

To find a singularity we examine where the curvature scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges. It is explicitly given by

$$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} = \frac{24\mu^2}{\Sigma^6}(4r^2 - 3\Sigma)(4r^2 - \Sigma)\,. \tag{3.17}$$

- $b = 0$
  When one of the spin parameters vanishes, say $b = 0$, the curvature scalar becomes

  $$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}|_{b=0} = \frac{24\mu^2}{(r^2 + a^2\cos^2\theta)^6}(r^2 - 3a^2\cos^2\theta)(3r^2 - a^2\cos^2\theta)\,. \tag{3.18}$$

  At $r = 0$ we see the curvature scalar diverges for $\theta \to \pi/2$. If we chose $a = 0$, the singularity would be at $r = 0$, $\theta = 0$.

- $a, b \neq 0$
  When neither spin parameter vanishes, the curvature scalar remains finite. But we can introduce the coordinate change $\rho = r^2$ and explore negative values of r^2. When $a = b$ the curvature scalar becomes

  $$R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}|_{a=b} = \frac{24\mu^2}{(r^2 + a^2)^6}(r^2 - 3a^2)(3r^2 - a^2)\,, \tag{3.19}$$

  and we see the entire surface $\rho = -a^2$ is singular. If $a < b$ the curvature scalar diverges for $\Sigma = 0$, or

  $$\sin^2\theta = \frac{-\rho - a^2}{b^2 - a^2}\,, \tag{3.20}$$

  where ρ is required to stay in the domain $-b^2 \leq \rho \leq -a^2$.

All discussed singularities correspond with the divergence of the metric components containing a μ/Σ term.

Event horizons

The event horizon is determined by the condition $g^{rr} = 0$, or in this specific case

$$\Pi - \mu r^2 = (r^2 + a^2)(r^2 + b^2) - \mu r^2 = 0\,. \tag{3.21}$$

This is a quadratic equation in r^2 and its solutions are

$$r^2_{H\pm} = \frac{1}{2}\left(\mu - a^2 - b^2 \pm \sqrt{(\mu - a^2 - b^2)^2 - 4a^2b^2}\right)\,. \tag{3.22}$$

We want real solutions, so require that

$$\mu \geq a^2 + b^2 + 2ab\,. \tag{3.23}$$

- $b = 0$
  In this case the mass condition becomes $\mu \geq a^2$ and the corresponding event horizons are
  $$r_{H+}^2 = \mu - a^2 \geq 0 \qquad \text{and} \qquad r_{H-}^2 = 0\,. \tag{3.24}$$
  The event horizon area $\mathcal{A}_H$ is [22]
  $$\mathcal{A}_H = 4\pi\mu\sqrt{\mu - a^2}\,. \tag{3.25}$$
  The singularity at $r = 0$, $\theta = \pi/2$ is concealed if $\mu > a^2$, otherwise it is a naked singularity.

- $a, b \neq 0$
  We saw we can extend the coordinates to negative values of ρ in this case and that the singularities are in that region. To avoid naked singularities we need $\rho_H = r_{H\pm}^2 > -a^2$. The mass condition ensures that $\rho_H > 0$, thus there are no naked singularities.[5]

We can rewrite the mass condition (3.23) in terms of the (real) mass M and angular momenta $J^{x_i y_i}$:
$$M^3 \geq \frac{27\pi}{32G}(J_\psi^2 + J_\phi^2 + 2J_\psi J_\phi)\,, \tag{3.26}$$
where $J_\psi = J^{x_1 y_1}$ and $J_\phi = J^{x_2 y_2}$. Just like the Kerr black hole the angular momentum is restricted by the mass. In dimensions greater than five this restriction is not present and there are event horizons for arbitrarily large angular momentum. These are called ultra-spinning black holes.

Ergosphere

The ergosphere is the region between a stationary limit surface and the outer event horizon. A stationary limit surface is given by $g_{tt} = 0$:
$$\begin{aligned} g_{tt} &= 1 - \frac{\mu}{\Sigma} = 0 \\ &\to \Sigma - \mu = 0 \\ &\to r_E^2 = \mu - a^2\cos^2\theta - b^2\sin^2\theta\,. \end{aligned} \tag{3.27}$$

- $b = 0$
  When the spin parameter $b = 0$, the ergosurface is
  $$r_E^2 = \mu - a^2\cos^2\theta\,, \tag{3.28}$$
  and
  $$\mu - a^2 \leq r_E^2 \leq \mu\,. \tag{3.29}$$
  When comparing this to the location of the event horizons, we see the event horizon and ergosurface touch at $\theta = 0,\, \pi$.

- $a, b \neq 0$
  When both spin parameters are nonvanishing, the event horizon and ergosurface (the positive root of (3.27)) never touch.

[5] There are event horizons in the range $-a^2 < \rho < 0$ (with $a < b$) for negative μ. However, these negative mass solutions contain causality violating regions. In the range $-b^2 < \rho < -a^2$ there are also event horizons but these do not entirely conceal the singularities, so these are still naked singularities.

3.2.2 Black Ring

A black ring is a rotating black hole with an event horizon of topology $S^1 \times S^2$. The ring can have one angular momentum (rotating solely around S^1 or S^2), or have two angular momenta. We will begin by describing a black ring with one angular momentum rotating around S^1 [30][31]. Its metric is most conveniently represented in ring coordinates $\{t, x, y, \phi, \psi\}$. These coordinates are more elaborately discussed regarding flat space in appendix A. The metric of the black ring is

$$\begin{aligned} ds^2 = &-\frac{F(y)}{F(x)}\left(dt + CR\,\frac{1+y}{F(y)}d\psi\right)^2 \\ &+ \frac{R^2}{(x-y)^2}F(x)\left[-\frac{G(y)}{F(y)}d\psi^2 - \frac{1}{G(y)}dy^2 + \frac{1}{G(x)}dx^2 + \frac{G(x)}{F(x)}d\phi^2\right], \end{aligned} \tag{3.30}$$

where

$$F(\xi) = 1 + \lambda\xi\,, \qquad\qquad G(\xi) = (1-\xi^2)(1+\nu\xi)\,, \tag{3.31}$$

and

$$C = \sqrt{\lambda(\lambda-\nu)\frac{1+\lambda}{1-\lambda}}\,. \tag{3.32}$$

The coordinates and dimensionless parameters, λ and ν, have the following domains:

$$-1 \le x \le 1\,, \qquad\qquad -\infty \le y < -1\,, \tag{3.33}$$

$$0 < \nu \le \lambda < 1\,. \tag{3.34}$$

R sets the radius of the ring, λ is an indicator of the rotational velocity of the ring and ν of the shape of the ring. When $\lambda = \nu = 0$ flat spacetime in ring coordinates is recovered. The mass M_1 and spin $J_{1,\psi}$ are obtained by examining the metric near asymptotic infinity $x, y \to -1$. We find

$$M_1 = \frac{3\pi R^2}{4G}\frac{\lambda}{1-\nu}\,, \tag{3.35}$$

$$J_{1,\psi} = \frac{\pi R^3}{2G}\frac{\sqrt{\lambda(\lambda-\nu)(1+\lambda)}}{(1-\nu)^2}\,, \tag{3.36}$$

and we see the system is characterised by the parameters R, λ and ν. We would actually expect there to be one less parameter when the system is in equilibrium. For a given mass and angular momentum, the radius brings the tension and self-attraction of the ring in equilibrium with the centrifugal force. This leads to the requirement that the angular variables, ϕ and ψ, are periodic:

$$\Delta\phi = \Delta\psi = 2\pi\frac{\sqrt{1-\lambda}}{1-\nu}\,, \tag{3.37}$$

and to the following relationship between λ and ν:

$$\lambda = \frac{2\nu}{1+\nu^2}\,. \tag{3.38}$$

We will now turn to the singularity and horizons of this metric. The curvature scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ diverges at $y = -\infty$, which represents a ring of radius R around the axis of rotation of ψ. The event horizon is located at $y_H = -1/\nu$ and has topology $S^1 \times S^2$. This horizon is found by considering $g^{yy} = 0$ instead of $g^{rr} = 0$ as in the previous cases; in this metric y has some sort of radial function. The event horizon area $\mathcal{A}_H$ is

$$\mathcal{A}_H = 8\pi^2 R^3 \frac{\nu^{3/2}\sqrt{\lambda(1-\lambda^2)}}{(1-\nu)^2(1+\nu)} \,. \tag{3.39}$$

As $\nu \to 1$ the horizon disappears and a naked singularity is left. The ergosurface can be found by imposing $g_{tt} = 0$ and is located at $y_E = -1/\lambda$. It has the same topology as the event horizon.

To get more insight in the parameters of this solution we change to coordinates $\{t, r, \theta, \phi, \psi\}$ and redefine the parameters $(\nu, \lambda) \to (r_0, \sigma)$:

$$r = -\frac{R}{y}\,, \qquad \cos\theta = x\,, \tag{3.40}$$

$$\nu = \frac{r_0}{R}\,, \qquad \lambda = \frac{r_0\cosh^2\!\sigma}{R}\,. \tag{3.41}$$

The metric takes the form:

$$\begin{aligned} ds^2 = &-\frac{\hat{f}}{\hat{g}}\left(dt + r_0\sinh\sigma\cosh\sigma\sqrt{\frac{R + r_0\cosh^2\!\sigma}{R - r_0\cosh^2\!\sigma}\,\frac{\frac{r}{R}-1}{r\hat{f}}}\,R\,d\psi\right)^2 \\ &+ \frac{\hat{g}}{(1+\frac{r\cos\theta}{R})^2}\left[\frac{f}{\hat{f}}\left(1-\frac{r^2}{R^2}\right)R^2 d\psi^2 + \frac{1}{(1-\frac{r^2}{R^2})f}dr^2 + \frac{r^2}{g}d\theta^2 + \frac{g}{\hat{g}}\,r^2\sin^2\!\theta\, d\phi^2\right], \end{aligned} \tag{3.42}$$

where

$$f = 1 - \frac{r_0}{r}\,, \qquad \hat{f} = 1 - \frac{r_0\cosh^2\!\sigma}{r}\,, \tag{3.43}$$

and

$$g = 1 + \frac{r_0}{R}\cos\theta\,, \qquad \hat{g} = 1 + \frac{r_0\cosh^2\!\sigma}{R}\cos\theta\,. \tag{3.44}$$

In these coordinates the singularity is located at $r = 0$, the event horizon at $r_H = r_0$ and the ergosurface at $r_E = r_0\cosh^2\!\sigma$. Now we see that ν can be interpreted as the ratio between the radius of S^2 at the horizon and the radius R of the ring S^1. Smaller ν correspond to thinner rings. The ratio λ/ν measures the speed of the rotation of the ring.

Black rings could also rotate along S^2, but these can not support themselves against the centrifugal tension of the rotation. The only option left is for the black ring to have two independent angular momenta [22][32].

The metric for a black ring with two angular momenta is

$$\begin{aligned} ds^2 = &-\frac{H(y,x)}{H(x,y)}(dt+\Omega)^2 - \frac{F(x,y)}{H(y,x)}d\psi^2 - 2\frac{J(x,y)}{H(y,x)}d\psi d\phi + \frac{F(y,x)}{H(y,x)}d\phi^2 \\ &+ \frac{2k^2 H(x,y)}{(x-y)^2(1-\nu)^2}\left(\frac{1}{G(x)}dx^2 - \frac{1}{G(y)}dy^2\right) , \end{aligned} \tag{3.45}$$

with

$$\begin{aligned} \Omega = &\frac{2k\lambda\sqrt{(1+\nu)^2-\lambda^2}}{H(y,x)} [\, (1-x^2)y\sqrt{\nu}\, d\phi \\ &+ \frac{1+y}{1-\lambda+\nu}(1+\lambda-\nu+x^2y\nu(1-\lambda-\nu)+2\nu x(1-y))d\psi \,] , \end{aligned} \tag{3.46}$$

$$G(\xi) = (1-\xi^2)(1+\lambda\xi+\nu\xi^2) , \tag{3.47}$$

$$H(\xi,\eta) = 1+\lambda^2-\nu^2+2\eta\lambda\nu(1-\xi^2)+2\xi\lambda(1-\eta^2\nu^2)+\xi^2\eta^2\nu(1-\lambda^2-\nu^2) , \tag{3.48}$$

$$J(\xi,\eta) = \frac{2k^2(1-\xi^2)(1-\eta^2)\lambda\sqrt{\nu}}{(\xi-\eta)(1-\nu)^2}\left[1+\lambda^2-\nu^2+2(\xi+\eta)\lambda\nu-\xi\eta\nu(1-\lambda^2-\nu^2)\right] , \tag{3.49}$$

$$\begin{aligned} F(\xi,\eta) = &\frac{2k^2}{(\xi-\eta)^2(1-\nu)^2} [\, G(\xi)(1-\eta^2)\, [\, ((1-\nu)^2-\lambda^2)(1+\nu) \\ &+ \eta\lambda(1-\lambda^2+2\nu-3\nu^2)\,] + G(\eta)\, [\, 2\lambda^2+\xi\lambda((1-\nu)^2+\lambda) \\ &+ \xi^2((1-\nu)^2-\lambda)(1+\nu)+\xi^3\lambda(1-\lambda^2-3\nu^2+2\nu^3) \\ &- \xi^4(1-\nu)\nu(-1+\lambda^2+\nu^2)\,]\,] . \end{aligned} \tag{3.50}$$

These metric components only take the forms (3.46)-(3.50) when there is a balance of forces in the ring (otherwise they would have the much more complicated expressions as described in [33]). I chose for an overall positive sign in the expressions of Ω, such that the black ring rotates around ψ the same way as the black ring with just one angular momentum does. The same coordinates are used as for the black ring metric with one angular momentum, with the same domains as in (3.33). It should be noted that the definitions of the functions F and G are different from the previous case. Also, the angular coordinates ψ and ϕ have been rescaled to have periodicity 2π.

For this solution to have an event horizon and a positive mass, the parameters λ and ν are restricted to the following domains:

$$0 \le \nu < 1 \qquad \text{and} \qquad 2\sqrt{\nu} \le \lambda < 1+\nu . \tag{3.51}$$

The parameter k sets the scale of the solution. Flat spacetime is recovered as $\lambda = 0$, and the black ring metric with only one angular momentum (3.30) can be recovered from this metric by setting $\nu \to 0$, $R^2 = 2k^2(1+\lambda^2)$ and $\lambda \to \nu$.

The mass M_2 and two angular momenta $J_{2,\phi}$, $J_{2,\psi}$ of this system are

$$M_2 = \frac{3k^2\pi\lambda}{G(1-\lambda+\nu)}\,, \tag{3.52}$$

$$J_{2,\phi} = \frac{4k^3\pi\lambda\sqrt{\nu}\sqrt{(1+\nu)^2-\lambda^2}}{G(1-\nu)^2(1-\lambda+\nu)}\,, \qquad \text{and} \tag{3.53}$$

$$J_{2,\psi} = \frac{2k^3\pi\lambda\,(1+\lambda-6\nu+\lambda\nu+\nu^2)\sqrt{(1+\nu)^2-\lambda^2}}{G(1-\nu)^2(1-\lambda+\nu)^2}\,. \tag{3.54}$$

The condition $g^{yy}=0$ gives us again the location of the event horizon:

$$y_{H\pm} = \frac{-\lambda \pm \sqrt{\lambda^2-4\nu}}{2\nu}\,. \tag{3.55}$$

If λ would exceed its limit, $2\sqrt{\nu}\le\lambda$, there would be a naked singularity. As $\nu\to1$, $\lambda\to2$ and the extremal Myers-Perry black hole with $\mu=a_1^2+a_2^2+2a_1a_2$ is recovered.

3.2.3 Black Hole versus Black Ring

To adequately compare the Myers-Perry black hole and black ring solutions, we will first inspect if the two solutions can be recovered from each other and then study their spin parameter ranges.

The metric of the Myers-Perry black hole with one angular momentum ((3.14) with $b=0$) can be recovered from the metric of a black ring with one angular momentum (3.30) [31], and likewise for the case with two independent angular momenta [22]. To do so for the first case, one needs to set $\lambda,\nu\to1$ and $R\to0$, while keeping the new parameters μ and a constant:

$$\mu = \frac{2R^2}{1-\nu}\,, \qquad \text{and} \qquad a^2 = 2R^2\frac{\lambda-\nu}{(1-\nu)^2}\,. \tag{3.56}$$

The coordinates should be transformed as $(x,y)\to(r,\theta)$ with

$$x = -1+2\left(1-\frac{a^2}{\mu}\right)\frac{R^2\cos^2\theta}{r^2-(\mu-a^2)\cos^2\theta}\,, \tag{3.57}$$

$$y = -1-2\left(1-\frac{a^2}{\mu}\right)\frac{R^2\sin^2\theta}{r^2-(\mu-a^2)\cos^2\theta}\,, \tag{3.58}$$

and the angular coordinates ψ and ϕ need to be rescaled to have periodicity 2π:

$$(\psi,\phi)\to\sqrt{\frac{\mu-a^2}{2R^2}}(\psi,\phi)\,. \tag{3.59}$$

The resulting metric, which describes a Myers-Perry black hole rotating in the ψ direction, is

$$ds^2 = -\left(1-\frac{\mu}{\Sigma}\right)\left(dt - \frac{\mu\, a \sin^2\theta}{\Sigma-\mu}d\psi\right)^2 + \Sigma\left(\frac{1}{\Delta}dr^2 + d\theta^2\right) + \frac{\Delta \sin^2\theta}{1-\mu/\Sigma}d\psi^2 + r^2\cos^2 d\phi^2\,, \tag{3.60}$$

with

$$\Delta = r^2 + a^2 - \mu\,, \qquad \Sigma = r^2 + a^2\cos^2\theta\,. \tag{3.61}$$

To recover the Myers-Perry black hole with two angular momenta we unfortunately can not use the expressions of the metric components given in (3.46)-(3.50), but need to consider the case where the ring is not necessarily in equilibrium and use the general components as given in [33].

When comparing spin parameter ranges, it is easiest to make use of the reduced angular momentum j and event horizon area a_H. The mass M is fixed and the angular momentum J and horizon area $\mathcal{A}_H$ are divided by powers of M or GM to obtain dimensionless variables. With convenient normalisation the reduced quantities take the form

$$j^2 \equiv \frac{27\pi}{32G}\frac{J^2}{M^3}\,, \tag{3.62}$$

$$a_H \equiv \frac{3}{16}\sqrt{\frac{3}{\pi}}\frac{\mathcal{A}_H}{(GM)^{3/2}}\,. \tag{3.63}$$

We will first focus on the black holes with one angular momentum [30]. When the black ring is in equilibrium, there is only one independent parameter and the ring's reduced angular momentum j_{BR} and horizon area $a_{H,BR}$ are related. This can be parameterised by:

$$j_{BR}^2 = \frac{(1+\nu)^3}{8\nu}\,, \qquad a_{H,BR} = 2\sqrt{\nu(1-\nu)}\,, \qquad \text{for} \quad 0<\nu<1\,, \tag{3.64}$$

using the given expressions for the mass (3.35), spin (3.36) and horizon area (3.39). The reduced angular momentum takes on a minimum at $\nu = 1/2$ with value $j_{BR}^2 = 27/32$. This minimum corresponds to a maximum event horizon area. The black ring regime is divided into two types called thin black rings for $0 < \nu < 1/2$ and fat black rings for $1/2 \leq \nu < 1$. As $\nu \to 0$, the ring's angular momentum becomes infinite and its event horizon area zero. For fat black rings the reduced angular momentum is bound from above by $j_{BR}^2 = 1$ for $\nu \to 1$. The reduced horizon area of the Myers-Perry black hole $a_{H,MP}$ has the following parameterisation:

$$a_{H,MP} = 2\sqrt{2(1-j_{MP}^2)}\,, \qquad \text{for} \quad 0 \leq j_{MP} < 1\,, \tag{3.65}$$

with j_{MP} its reduced angular momentum, for given mass (3.12) and spin (3.13) (with $n = 5, i = 1$), and event horizon area (3.25). The curves (3.64) and (3.65) are plotted in figure 5.

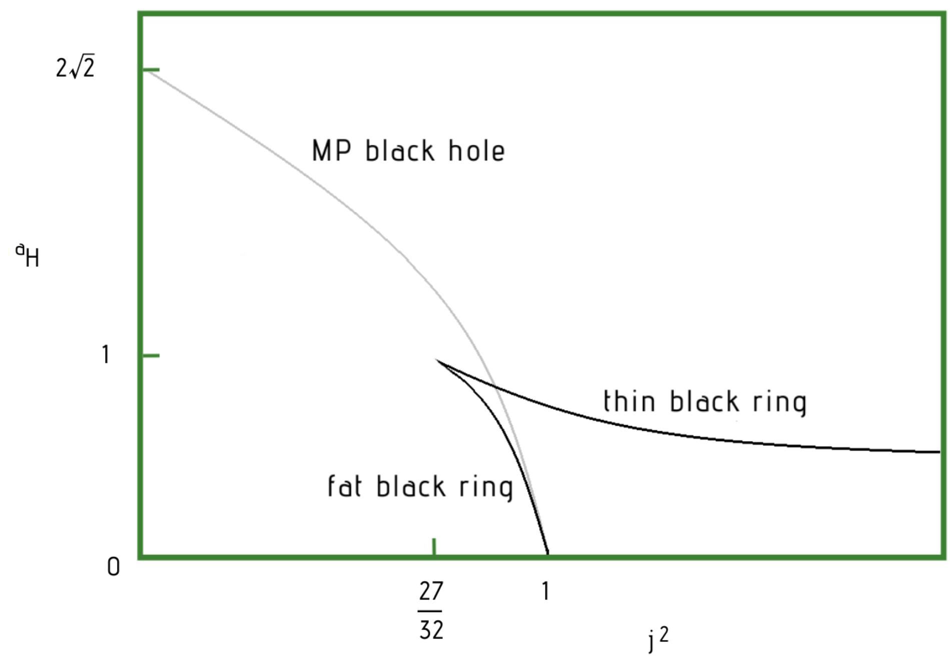

Figure 5: Reduced event horizon area a_H of a Myers-Perry black hole and thin and fat black ring with one angular momentum plotted against their reduced angular momentum squared j^2. The light gray line represents the Myers-Perry black hole, and the black line the thin and fat black rings. Adapted from [30].

We clearly see there exist three types of black holes in the regime $27/32 \leq j^2 < 1$ for the same mass; a Myers-Perry black hole and a thin and fat black ring. There is three-fold black hole non-uniqueness. Another unusual feature is the fact that the angular momentum of a black ring is bound from below, but not from above as is the case for the Myers-Perry black hole. Lastly, it can be seen that the naked singularity limit $\nu \to 1$ for the fat black ring coincides with the extremal limit $\mu \to a^2$ of the Myers-Perry black hole.

To compare the black holes with two angular momenta, we focus on the phase spaces of their reduced angular momenta (j_1, j_2) [22]. For Myers-Perry black holes this is easily derived from the mass condition stated in (3.26). This condition can be rewritten as

$$1 \geq j_{1,MP}^2 + j_{2,MP}^2 + 2j_{1,MP}\, j_{2,MP}\,, \tag{3.66}$$

where $j_{1,MP}$ corresponds with J_ψ and $j_{2,MP}$ with J_ϕ. This condition represents a square in the phase space $(j_{1,MP}, j_{2,MP})$ with the corners on the axes (see figure 7). The phase space for the black rings is a bit more difficult. The reduced angular momenta can be derived from (3.52)-(3.54). When examining their ranges we find [28]

$$j_{1,BR} \geq \frac{3}{4}\,, \qquad \text{and} \qquad j_{2,BR} \leq \frac{1}{4}\,, \tag{3.67}$$

where $j_{1,BR}$ corresponds with $J_{2,\psi}$ and $j_{2,BR}$ with $J_{2,\phi}$.

We also noted that as $\nu \to 1$, $\lambda \to 2$ and the limit coincides with the extremal Myers-Perry black hole. The solutions should touch in the phase diagram at points where the equality in (3.66) holds. The black rings phase space is restricted by three curves:

- Nonextremal minimally spinning black rings with minimal $j_{1,BR}$ for given $j_{2,BR}$:
  this curve extends between ($j_{1,BR} = 4/5$, $j_{2,BR} = 1/5$) and
  ($j_{1,BR} = \sqrt{27/32}$, $j_{2,BR} = 0$).
- Extremal black rings with maximal $j_{2,BR}$ for given $j_{1,BR}$:
  this curve extends between ($j_{1,BR} = 3/4$, $j_{2,BR} = 1/4$) and ($j_{1,BR} \to \infty$, $j_{2,BR} \to 0$).
- The line of extremal Myers-Perry black holes:
  $j_{1,MP}^2 + j_{2,MP}^2 + 2j_{1,MP}\, j_{2,MP} = 1\,.$

The black ring phase space for $j_{1,BR} > j_{2,BR} \geq 0$ is depicted in figure 6. As in the case for black holes with one angular momentum there is a range of momenta for which there exist a Myers-Perry black hole and a thin and fat black ring with the same mass. We again encounter black hole non-uniqueness in five dimensions.

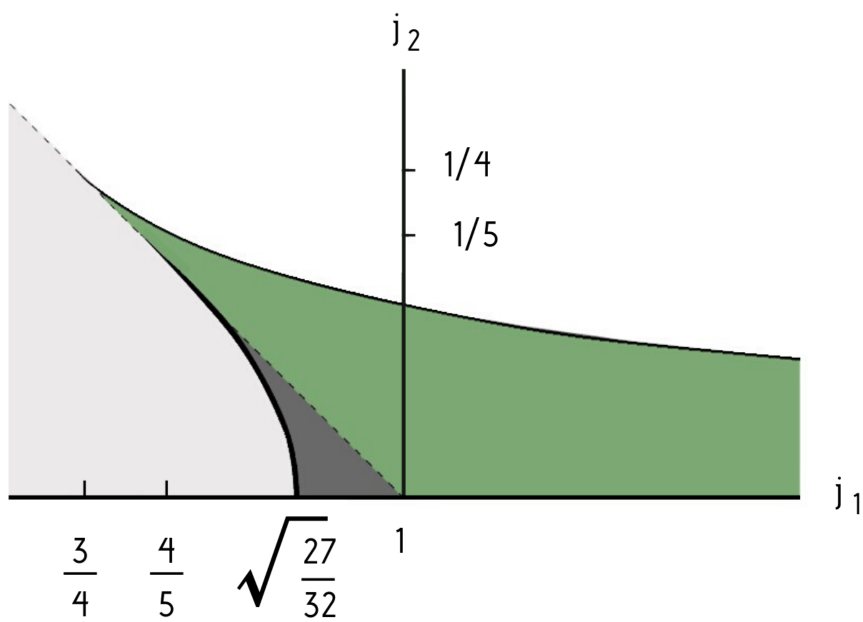

Figure 6: Phase space of black rings with two angular momenta for $j_1 \equiv j_{1,BR} > j_2 \equiv j_{2,BR} \geq 0$. The upper thin black line represents the boundary of extremal black rings, the dashed line represents extremal Myers-Perry black holes, and the lower thick black line represents nonextremal minimally spinning black rings. The values of j_1, j_2 for which these lines meet each other or the j_1-axis are indicated. In the green area there exist only thin black rings, in the dark gray area there exist thin and fat black rings and Myers-Perry black holes, in the light gray area there are only Myers-Perry black holes. Adapted from [22].

The complete phase space for all values of reduced angular momentum can be constructed by replicating figure 6 and taking $\pm j_1 \leftrightarrow \pm j_2$. This is shown in figure 7.

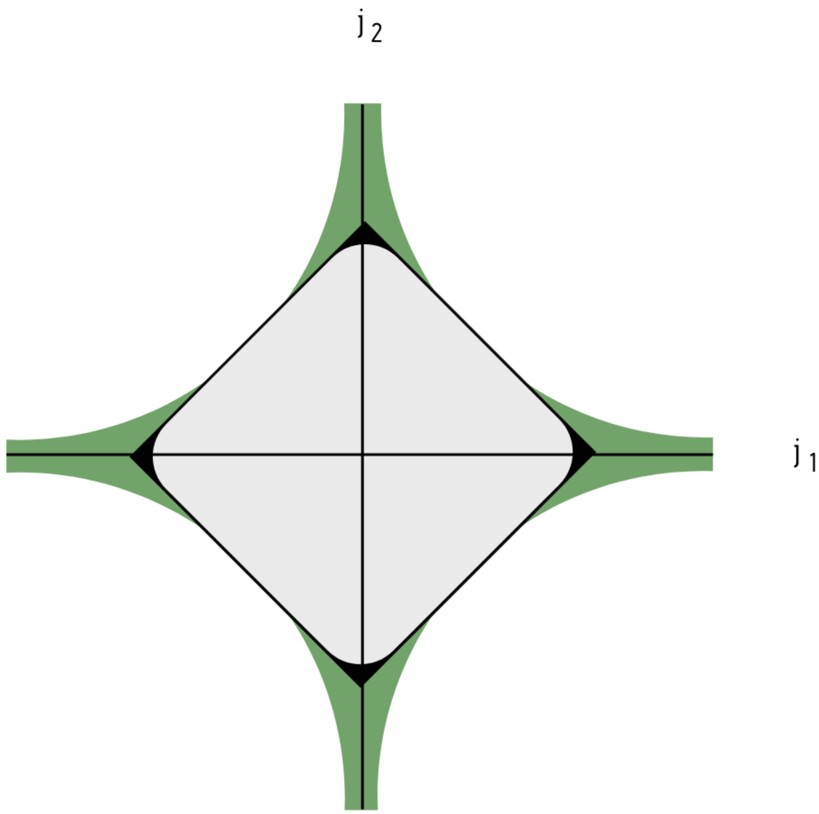

Figure 7: Complete phase space of Myers-Perry black holes and black rings with two angular momenta. Inside the square there exist Myers-Perry solutions, with the black boundary representing the extremal black holes. In the green areas there exist only thin black rings. In the light gray area there exist only Myers-Perry black holes. In the black rounded off corners there exist Myers-Perry black holes and thin and fat black rings. Adapted from [22].

4 DIMENSIONAL REDUCTION

Higher dimensional theories can be used as an explanation for properties of our four dimensional world. To do so, one needs to reduce the dimensionality of the theory. The first to propose this interesting idea was Kaluza in 1912 [11]. Kaluza unified gravity with electromagnetism using general relativity in five dimensions. Four dimensional matter appears to arise from the geometry of empty five dimensional spacetime. Nowadays this same way of thinking is used in 10D and 11D supergravities, M-Theory and string theory, and the original Kaluza theory has branched out into several modern varieties.

Kaluza's way of dealing with the extra fifth dimension was to set all derivatives with respect to the fifth coordinate to zero. This is called the cylinder condition. This forces the physics to depend only on the first four coordinates, and not the fifth one. There was, however, no reason given why this must be so. The derivation starts with Einstein's equation in five dimensional vacuum spacetime:

$$\hat{G}_{AB} = 0\,, \qquad \text{or} \qquad \hat{R}_{AB} = 0\,, \tag{4.1}$$

where $\hat{G}_{AB}$ and $\hat{R}_{AB}$ are the five dimensional Einstein tensor and Ricci tensor, respectively. Capital Latin indices run from 0 to 4 and five dimensional quantities are represented with hats. The Christoffel symbol and Ricci tensor are given by

$$\hat{\Gamma}^C_{AB} = \frac{1}{2}\,\hat{g}^{CD}\left(\partial_A\,\hat{g}_{DB} + \partial_B\,\hat{g}_{DA} - \partial_D\,\hat{g}_{AB}\right), \tag{4.2}$$

$$\hat{R}_{AB} = \partial_C\,\hat{\Gamma}^C_{AB} - \partial_B\,\hat{\Gamma}^C_{AC} + \hat{\Gamma}^C_{AB}\,\hat{\Gamma}^D_{CD} - \hat{\Gamma}^C_{AD}\,\hat{\Gamma}^D_{BC}\,. \tag{4.3}$$

These are the same expressions as in equations (2.29) and (2.31).

For a given five dimensional metric we can then calculate its Christoffel symbols and Ricci tensor and obtain the 15 equations contained in (4.1). A general and convenient way to parameterise the metric is by

$$\hat{g}_{AB} = \begin{pmatrix} g_{\alpha\beta} + \kappa^2\phi^2\,A_\alpha A_\beta & \kappa\phi^2 A_\alpha \\ \kappa\phi^2 A_\beta & \phi^2 \end{pmatrix}, \tag{4.4}$$

where $g_{\alpha\beta}$ is the four dimensional metric tensor, A_α the electromagnetic potential, ϕ a scalar field, and κ a scaling parameter. Small Greek indices run from 0 to 3 and we set $c = 1$. The four dimensional metric signature is $(+ - --)$.

We plug this parameterisation of the five dimensional metric into (4.2) and (4.3), while dropping all derivatives with respect to the fifth coordinate. The $\alpha\beta$-, $\alpha 4$- and 44- components of (4.1) then reduce, respectively, to the following equations in four dimensions:

$$G_{\alpha\beta} = \frac{\kappa^2\phi^2}{2}\,T^{EM}_{\alpha\beta} - \frac{1}{\phi}\left[\,\nabla_\alpha(\partial_\beta\phi) - g_{\alpha\beta}\,\Box\,\phi\,\right], \tag{4.5}$$

$$\nabla^\alpha\,F_{\alpha\beta} = -3\frac{\partial^\alpha\phi}{\phi}F_{\alpha\beta}\,, \tag{4.6}$$

$$\Box\,\phi = \frac{\kappa^2\phi^3}{4}\,F_{\alpha\beta}\,F^{\alpha\beta}\,, \tag{4.7}$$

where $G_{\alpha\beta}$ is the four dimensional Einstein tensor, $T^{EM}_{\alpha\beta} = \frac{1}{4} g_{\alpha\beta} F_{\gamma\delta} F^{\gamma\delta} - F^{\gamma}_{\alpha} F_{\beta\gamma}$ is the electromagnetic energy-momentum tensor, $F_{\alpha\beta} \equiv \partial_\alpha A_\beta - \partial_\beta A_\alpha$ is the Faraday tensor, and $\Box \equiv g^{\alpha\beta} \nabla_\alpha \nabla_\beta$ the d'Alembert operator. (4.7) is the massless Klein-Gordon equation (a relativistic wave equation) for the scalar field.

If the scalar field ϕ is constant through spacetime, the first two equations become the Einstein and Maxwell equations:

$$G_{\alpha\beta} = 8\pi G \phi^2 \, T^{EM}_{\alpha\beta} \,, \qquad\qquad \nabla^\alpha F_{\alpha\beta} = 0 \,, \tag{4.8}$$

where we chose the scaling parameter κ to be $\kappa = 4\sqrt{\pi G}$. Pure geometry in five dimensions leads thus to matter sources (or electromagnetic radiation at least) in four dimensions. Kaluza's method produces a nice result, but it is not substantiated why the metric should be independent of the fifth dimension.

Klein added to Kaluza's method a reason why the fifth dimension is not observable or of influence in the four dimensional theory, creating the Kaluza-Klein theory [12]. He proposed that the extra fifth dimension is compactified, i.e. it forms a circle with very small radius $\hat{R}$. We can then think of five dimensional spacetime as a manifold $M = M_4 \times S^1$, with M_4 four dimensional spacetime and S^1 a circle. We shall denote the coordinates x^A as $x^A = \{x^\mu, y\}$. Since the fifth coordinate y is now periodic, any field quantity $F(x, y)$ can be expanded in a Fourier expansion:

$$F(x,y) = \sum_{n=-\infty}^{\infty} F_n(x) \exp\left(\frac{iny}{\hat{R}}\right) \,, \qquad\qquad \text{for} \quad 0 \leq y \leq 2\pi\hat{R} \,. \tag{4.9}$$

The quantities $g_{\mu\nu}$, A_μ, and ϕ can be expanded in this way. When only the zero mode of the expansions is considered, they will depend only on x^μ, as desired. Why we are allowed to do so is substantiated shortly.

Instead of working with Einstein's equation, we could also use the five dimensional Hilbert action:

$$\hat{S}_H = -\frac{1}{16\pi\hat{G}} \int d^4x \, dy \, \sqrt{-\hat{g}} \hat{R} \,. \tag{4.10}$$

Plugging in the five dimensional metric (4.4) and its corresponding Ricci tensor $\hat{R}$ for the zero mode quantities and then integrating over y, results in the four dimensional action:

$$S_H = -\int d^4x \sqrt{-g} \, \phi \left[\frac{R}{16\pi G} + \frac{1}{4}\phi^2 F_{\alpha\beta} F^{\alpha\beta} + \frac{1}{24\pi G} \frac{\partial^\alpha \phi \, \partial_\alpha \phi}{\phi^2} \right] , \tag{4.11}$$

where we chose $\kappa = 4\sqrt{\pi G}$, and defined G as $G \equiv \hat{G}/2\pi\hat{R}$. If the scalar field ϕ is constant, the first two components are the Einstein-Maxwell action for gravity and electromagnetic radiation. The third component is the action for a massless Klein-Gordon scalar field. These are the same results as in the Kaluza method. Kaluza-Klein theory in more than five dimensions allows for the incorporation of strong and weak nuclear interactions.

To understand the neglection of the non-zero Fourier modes we examine the incorporation of a massless five dimensional scalar field $\hat{\psi}(x,y)$. The field can be expanded as[6]:

$$\hat{\psi}(x,y) = \sum_{n=-\infty}^{n=\infty} \psi_n(x) \exp\left(\frac{iny}{\hat{R}}\right) , \tag{4.12}$$

and plugged in the five dimensional wave equation:

$$(\partial_t^2 - \nabla_x^2 - \partial_y^2) \sum_{n=-\infty}^{n=\infty} \psi_n(x) \exp\left(\frac{iny}{\hat{R}}\right) = 0 \,. \tag{4.13}$$

This results in the following expression for the Fourier modes:

$$(\partial_t^2 - \nabla_x^2 - m_n^2)\, \psi_n(x) = 0 \,, \tag{4.14}$$

with

$$m_n^2 = \left(\frac{n}{\hat{R}}\right)^2 \,. \tag{4.15}$$

So the fields $\psi_n(x)$ are solutions to the four dimensional Klein-Gordon equation with mass m_n. The radius $\hat{R}$ is thought to be of the order of the Planck length, which motivates why the extra dimension is unobservable. When looking at the expression for the mass of the fields (4.15), we see that all modes $|n| \geq 1$ then correspond to particles of at least one Planck mass. However, all known and observed particles are much lighter than the Planck mass, which is why we are justified to only consider the massless $n = 0$ mode.

To get more insight in the Kaluza-Klein method we will apply it to the black ring metric with one angular momentum (3.30). We will compactify the dimension described by ψ, since this coordinate is already periodic and describes a circle. All metric components are independent of ψ so we do not have to use Fourier expansions. We want to identify the metric with the parameterisation (4.4) and can find the unknown $g_{\mu\nu}, A_\mu$, and ϕ in three steps. Since the metric only contains a cross term between t and ψ, A_μ will only have an A_t term and only g_{tt} will be altered.

1. The component $\hat{g}_{\psi\psi}$ corresponds to ϕ^2, so

$$\phi^2 = -C^2 R^2 \frac{(1+y)^2}{F(x)F(y)} - R^2 \frac{F(x)G(y)}{F(y)(x-y)^2} \,. \tag{4.16}$$

2. The cross term $\hat{g}_{t\psi}$ corresponds to $\kappa\,\phi^2 A_t$:

$$\begin{aligned} \kappa\,\phi^2 A_t &= -CR\,\frac{1+y}{F(x)} \\ \rightarrow \quad \kappa\, A_t &= -CR\,\frac{1+y}{F(x)}\frac{1}{\phi^2} \,. \end{aligned} \tag{4.17}$$

All other components of A_μ are zero.

[6]The Fourier modes can be written as four dimensional quantities, leaving (4.12) a general Fourier expansion, since the fifth dimension is compactified.

3. The term A_t alters the g_{tt} component as

$$\begin{aligned} \hat{g}_{tt} &= -\frac{F(y)}{F(x)} = g_{tt} + \phi^2\kappa^2 A_t^2 \\ \rightarrow \quad g_{tt} &= -\frac{F(y)}{F(x)} - C^2R^2\frac{(1+y)^2}{F(x)^2}\frac{1}{\phi^2} \\ &= -\frac{F(x)F(y)G(y)}{C^2(1+y)^2(x-y)^2 + F(x)^2G(y)} \\ &\equiv T(x,y)\,. \end{aligned} \tag{4.18}$$

   The other metric components remain the same as in the five dimensional black ring metric.

This results in the four dimensional metric

$$ds^2 = T(x,y)\,dt^2 + \frac{R^2}{(x-y)^2}\left[\frac{F(x)}{G(x)}dx^2 - \frac{F(x)}{G(y)}dy^2 + G(x)d\phi^2\right]. \tag{4.19}$$

The derived quantities $g_{\mu\nu}$, A_μ, and ϕ satisfy the equations (4.5)-(4.7) and we have successfully derived matter sources in four dimensions from an empty five dimensional spacetime.

5 RESULTS & OUTLOOK

Results

We have described the four dimensional Kerr black hole and five dimensional rotating black holes focusing on horizon topology, solution uniqueness and angular momentum range. The Kerr black hole has an event horizon (and ergosurface) with spherical topology, which is the only allowed horizon topology for black holes in four dimensions. At $\theta = 0, \pi$ the ergosurface and event horizon touch. The five dimensional Myers-Perry black hole also has a spherical horizon topology. When there is only one spin parameter the ergosurface and event horizon touch at $\theta = 0, \pi$. When both spin parameters are nonvanishing they never touch. In five dimensions there are also two types of black rings with an event horizon (and ergosurface) of ring topology. We see that a higher dimensionality allows for more horizon topologies.

In section 3.2.3 we identified a range of angular momenta for which there are three different five dimensional black hole solutions of the same mass: a Myers-Perry black hole and a thin and fat black ring, for black holes with one or two angular momenta alike. In five dimensions there is no longer uniqueness of solutions. The quantities mass and angular momenta are no longer sufficient to completely characterise the solution, so another quantity is needed. Horizon topology is not an option, since the thin and fat black ring have the same topology. The Kerr black hole, on the other hand, is uniquely described by its mass and angular momentum.

The uniqueness of the Kerr solution has only been proven in the case that $GM > a$. In the limit $GM \to a$ the solution becomes a naked singularity, and for $GM < a$ the solution is unstable. The angular momentum a is restricted for the solution to be of physical relevance. We did the same for the Myers-Perry black holes where the restriction on the angular momenta J_ψ and J_ϕ was

$$M^3 \geq \frac{27\pi}{32G}(J_\psi^2 + J_\phi^2 + 2J_\psi J_\phi)\,. \tag{5.1}$$

For the black ring with one angular momentum we found the following expression for its reduced angular momentum:

$$j_{BR}^2 = \frac{(1+\nu)^3}{8\nu}, \qquad \text{for} \quad 0 < \nu < 1\,. \tag{5.2}$$

Fat black rings have their angular momentum bound from below (with minimum $j^2 = 27/32$) and above (with maximum $j^2 = 1$). The angular momentum of thin black rings is only bound from below (with the same minimum as fat black rings) and ranges all the way to infinity. For the black ring with two angular momenta we found the following bounds on the reduced angular momenta:

$$j_{1,BR} \geq \frac{3}{4}\,, \qquad \text{and} \qquad j_{2,BR} \leq \frac{1}{4}\,. \tag{5.3}$$

Both reduced angular momenta of fat black rings are bound from above as well as below; $4/5 \leq j_{1,BR} < 1$ and $0 < j_{2,BR} \leq 1/5$ for $j_{1,BR}$, $j_{2,BR} > 0$. For thin black rings one angular momentum is only limited from below and the other bound from below and above; for instance $3/4 \leq j_{1,BR}$ and $0 < j_{2,BR} \leq 1/4$ for $j_{1,BR}$, $j_{2,BR} > 0$. In five dimensions the thin black ring has an ultra-spinning regime. In four dimensions, on the contrary, there are no ultra-spinning black hole solutions.

The Myers-Perry black hole with one angular momentum is the most similar to the Kerr black hole, being equal in horizon topology, touch points of ergosurface and event horizon, and having a similar bound on its angular momentum. The existence of the additional solutions in five dimensions, the Myers-Perry black hole with two angular momenta and the black rings, showcases the extra degrees of freedom in higher dimensions. In addition to the horizon topology, uniqueness and angular momentum range, we could also have looked into the stability against several types of perturbations to compare the different solutions.

In the last chapter Kaluza-Klein theory was discussed and applied to the black ring metric with one angular momentum. The theory describes a way to reduce five dimensional general relativity in vacuum to four dimensions, under the assumption that the extra dimension is a compactified circle. The results in four dimensions are Einstein's equations of gravity and Maxwell's equations of electromagnetism, together with the massless Klein-Gordon equation for a scalar field. Four dimensional matter sources appear from pure geometry in five dimensions.

Outlook

The classification of all five dimensional vacuum black holes with two rotational symmetries is almost complete. A good contender for the missing quantity to uniquely describe the black hole solutions is "rod structure", as has been shown by Hollands and Yazadjiev for solutions with two axial symmetries [34]. The rod structure entails information about the relative position and lengths of the various axes and the horizon. The last thing to be proven is that the rod structures describing regular black hole solutions are indeed the known Myers-Perry and black rings solutions. The next goal is to chart all stationary asymptotically flat black hole solutions to the vacuum Einstein equation in six and more dimensions. This is called the classification problem. We have seen how much one extra spatial dimension brings to the collection of possible solutions. So for even more dimensions, we can count on the problem to get more and more intricate.

In five dimensions the horizon topology is constricted to S^3 or $S^1 \times S^2$. In $n \geq 6$ there are less restrictions and the number of possible horizon topologies grows [25]. Likewise, the ways in which the black hole can rotate increase with dimensionality. The horizon topology together with the way of rotation create new dynamical situations (as we saw for the black ring in five dimensions), which can get very complicated.

It has been proven that a stationary rotating black hole must be axisymmetric in $n \geq 4$ [35], so there is at least one rotational symmetry. In five dimensions we saw the Myers-Perry and black ring solutions admit two rotational symmetries. This raises the question of how many symmetries a general stationary black hole must have. Is there a reason black holes must have maximal symmetry, or are we only able to find these solutions?

Even if we assume solutions to have the maximum number of rotational symmetries, the classification in $n \geq 6$ is not complete. Emparan et al. proposed a new kind of black hole, the "pinched" black holes, acting as a connection between Myers-Perry black holes and black rings, and between Myers-Perry black holes and black saturns [36]. The current proposal for the phase diagram of black holes with one angular momentum is given by Emparan and Figueras [37]. There are, however, still questions left. The specifics of the transitions between solutions are unclear, and a classification system is needed.

Kaluza-Klein theory was a great finding but has been shown to be flawed. As mentioned, the five dimensional theory is not fit to describe particles with mass in accordance to observations. This problem can be solved in several ways and has led to the modern Kaluza-Klein theories and other ways of dimensional reduction. A nice review is given by Overduin and Wesson [11]. These ideas are used in supergravity theories, M-Theory and string theory, giving relevance to the study of higher dimensional spacetimes.

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A RING COORDINATES

To describe black rings in section 3.2.2 we used ring coordinates. To get a bit more familiar with this set of coordinates we will describe four dimensional flat space with it [30].

We start by grouping together Cartesian coordinates in planes and introducing polar coordinates on those:

$$x_1 = r_1 \cos\phi\,, \qquad y_1 = r_1 \sin\phi\,, \tag{A.1}$$

$$x_2 = r_2 \cos\psi\,, \qquad y_2 = r_2 \sin\psi\,. \tag{A.2}$$

The flat space metric then takes the form:

$$ds^2 = dr_1^2 + r_1^2 d\phi^2 + dr_2^2 + r_2^2 d\psi^2\,. \tag{A.3}$$

Rotation in the (x_1, y_1) and (x_2, y_2) plane give rise to angular momentum J_ϕ and J_ψ respectively. We will describe a ring rotating along ψ in the (x_2, y_2) plane, located at $r_1 = 0$, $r_2 = R$ and $0 \le \psi < 2\pi$.

To implement the ring coordinates $\{x, y, \phi, \psi\}$, we need the following transformations:

$$x = \frac{R^2 - r_1^2 - r_2^2}{\Sigma}\,, \qquad y = -\frac{R^2 + r_1^2 + r_2^2}{\Sigma}\,, \tag{A.4}$$

where

$$\Sigma = \sqrt{(R^2 + r_1^2 + r_2^2)^2 - 4R^2\, r_2^2}\,. \tag{A.5}$$

Their inverse are

$$r_1 = R\frac{\sqrt{1-x^2}}{x-y}\,, \qquad r_2 = R\frac{\sqrt{y^2-1}}{x-y}\,. \tag{A.6}$$

The coordinate domains are

$$-1 \le x \le 1\,, \qquad -\infty \le y \le -1\,. \tag{A.7}$$

The location of the ring is at $y = -\infty$ and asymptotic infinity is reached as $x, y \to -1$. The axis of rotation around ψ is $y = -1$ and the axis of rotation around ϕ is divided into two pieces: for inside the ring $r_2 \le R$ it is $x = 1$, and outside the ring $r_2 \ge R$ it is $x = -1$.

In these coordinates the metric becomes:

$$ds^2 = \frac{R^2}{(x-y)^2}\left[(y^2-1)d\psi^2 + \frac{1}{y^2-1}dy^2 + \frac{1}{1-x^2}dx^2 + (1-x^2)d\phi^2\right]\,. \tag{A.8}$$

In figure 8 a section at constant ϕ and ψ (and $\phi + 2\pi$, $\psi + 2\pi$) of this metric is illustrated.

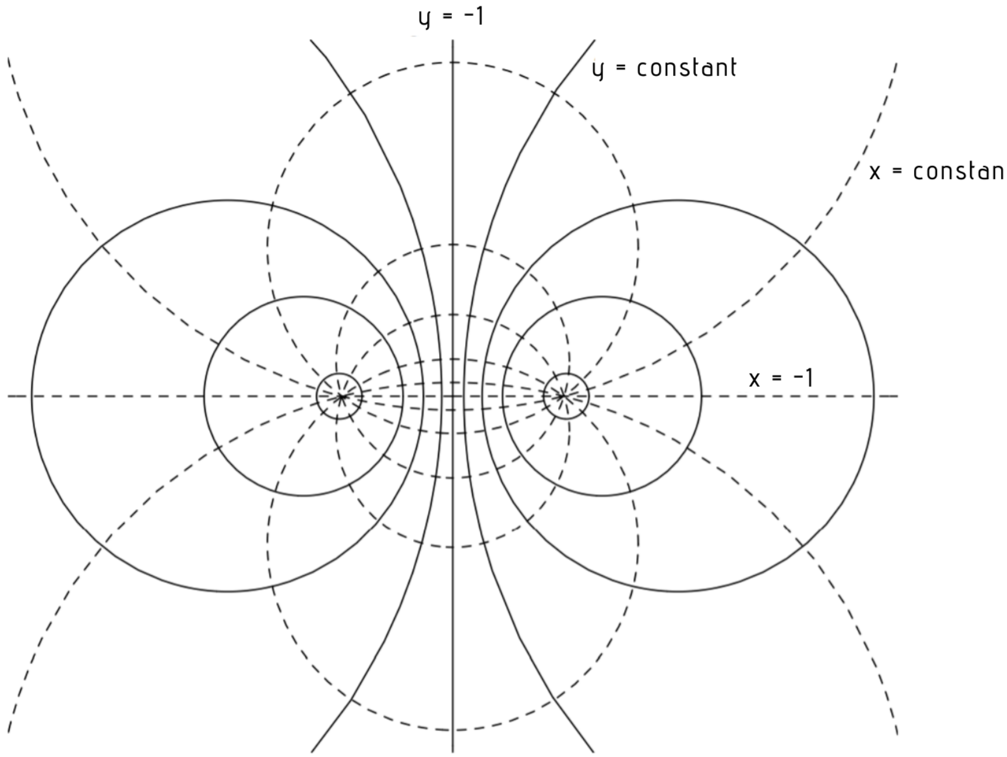

Figure 8: Four dimensional flat spacetime in ring coordinates at constant ϕ and ψ (and $\phi + 2\pi$, $\psi + 2\pi$). The dashed circles are spheres of constant $x \in [-1, 1]$, the solid circles of constant $y \in [-\infty, -1]$. At the location of the ring of radius R, $y = -\infty$, the spheres of constant y collapse to zero size. The disc bound by the ring is $x = 1$, its complement is $x = -1$. Adapted from [30].

We will do one last coordinate transformation to coordinates that are particularly nice in the near region around the ring. We will make the following changes:

$$r = -\frac{R}{y}\,, \qquad \cos\theta = x\,, \tag{A.9}$$

with

$$0 \le r \le R\,, \qquad 0 \le \theta \le \pi\,. \tag{A.10}$$

Metric (A.8) then takes the form:

$$ds^2 = \frac{1}{(1+\frac{r\cos\theta}{R})^2}\left[\left(1-\frac{r^2}{R^2}\right)R^2 d\psi^2 + \frac{1}{1-r^2/R^2}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{A.11}$$

Note that the point $r = R$ represents the axis of rotation around ψ although it looks like a (coordinate) singularity. In this metric form we can more easily see that surfaces of constant r have a ring shaped topology $S^1 \times S^2$, where the S^1 is parameterised by ψ, and the S^2 by θ and ϕ. The ring of radius R can be found at $r = 0$, with corresponding metric:

$$ds^2 = R^2 d\psi^2\,. \tag{A.12}$$