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Ordinal arithmetic and natural arithmetic

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Contents

Introduction

The concept of an ordinal (number) expands upon the notion of a natural number to include quantities that are not necessarily finite. The arithmetic on the natural numbers can likewise be generalised to include the transfinite ordinals. In this document we will see two different ways in which we can generalise the usual arithmetic operations on natural numbers to all ordinals.

In Chapters 1 and 2 we set the stage. We introduce ordinals and the concepts needed to work with them, and state some of the elementary properties of ordinals that underpin the rest of the contents of this document.

Chapter 3 deals with *ordinal arithmetic*, the most well-known manner in which the arithmetical operations on the natural numbers can be extended to the transfinite ordinals. We define ordinal addition, multiplication, and exponentiation and review several algebraic and order-theoretic properties of these operations.

These properties are put to good use in Chapter 4, where we prove the basis representation theorem for ordinals. In that same chapter we use basis representation to introduce the Cantor normal form, and illustrate how it interacts with the operations of ordinal arithmetic.

In Chapter 5 we use the Cantor normal form to define the operations of natural addition and natural multiplication for ordinals, which were first considered by Gerhard Hessenberg in 1906 [2]. We review their algebraic and order-theoretic properties, and briefly compare these to those of the operations in Chapter 3.

In the two appendices we fill in some details that, while noteworthy, are not part of the main subject matter of this document. Appendix A deals with some set-theoretical matters while Appendix B concerns classes and class functions.

Chapter 1

Preliminaries

This chapter very briefly discusses some preliminary concepts that we will need to grasp before we can delve into the world of ordinal numbers. The contents of this chapter are taken from Moerdijk & Van Oosten [4], specifically pages 17, 25, and 107.

1.1 Transitive and well-ordered sets

The first concept we introduce is that of a *transitive set*.

Definition 1.1.1 A set x is called *transitive* if and only if for every $y \in x$ and $z \in y$ we have $z \in x$.

Note that the above definition is equivalent to requiring that $x \subseteq \mathcal{P}(x)$, i.e. stipulating that every element of x also be a subset of x.

The second concept we will need is that of a *partially ordered set*.

Definition 1.1.2 A (strictly) *partially ordered set* or *poset* is a tuple $(x, <)$ consisting of a set x and a relation $<$ on x satisfying the following properties:

1. The relation $<$ is irreflexive on x: for all $a \in x$ we have $\neg(a < a)$.
2. The relation $<$ is transitive on x: for all $a, b, c \in x$, if $a < b$ and $b < c$ then $a < c$.

If transitivity is assumed, then irreflixivity is equivalent to asymmetry:

3. The relation $<$ is asymmetric on x: for all $a, b \in x$, if $a < b$ then $\neg(b < a)$.

In the context of posets we also introduce the following notation: $a \leq b$ will be defined as $(a < b) \vee (a = b)$.

Some posets are *totally ordered*, a property which is defined below.

Definition 1.1.3 A poset $(x, <)$ is said to be *totally ordered* or to be a *total order* if and only if for every two elements $a, b \in x$ we have $(a \leq b) \vee (b \leq a)$.

We are especially interested in posets that are not just total orders, but *well-orders*. See the definition below.

Definition 1.1.4 A poset $(x, <)$ is said to be *well-ordered* or to be a *well-order* if and only if every nonempty subset $y \subseteq x$ contains a $<$-least element, i.e. an element $a \in y$ such that for all $b \in y$ we have $a \leq b$. We denote this least element a as $\min y$.

Note that a well-ordered set is automatically totally ordered: given elements a, b in a well-order x, the nonempty subset $\{a, b\} \subseteq x$ must have a $<$-least element, hence either $a \leq b$ or $b \leq a$.

Chapter 2

Ordinal numbers

This chapter introduces the ordinal numbers, the central concept of this document. We state and prove several key properties of ordinal numbers and the relations between them.

2.1 Definition and basic properties

The definitions and results in this section are adapted from Section 4.2 on pages 107-108 of Moerdijk & Van Oosten [4], with the exception of Fact 2.1.3. The proofs presented here are also derived from that same location, except those of Theorem 2.1.4 and Theorem 2.1.7.

The definition of an ordinal number is given below.

Definition 2.1.1 (Ordinal number) An *ordinal number* or simply *ordinal* is a transitive set that is well-ordered by the relation $\in$.

In this document we will typically denote ordinals by letters from the greek alphabet.

Example 2.1.2 The empty set $\varnothing$ is vacuously an ordinal. Other examples of ordinals include $\{\varnothing\}$ and $\{\varnothing, \{\varnothing\}\}$. The set $\{\{\varnothing\}\}$ is not an ordinal, because it is not transitive.

We will now prove some basic properties of ordinals. However, to do so we will need the following fact.

Fact 2.1.3 There does not exist a nonempty finite sequence $(x_i)_{i=0}^{n}$ of sets such that $x_i \in x_{i+1}$ for all $0 \leq i < n$ and $x_n \in x_0$. In particular, no set is an element of itself.

This fact is provable from the axioms of Zermelo-Fraenkel set theory: this is done in Appendix A.2. Fact 2.1.3 serves to exclude certain membership chains with undesirable properties. If we for example had sets x and y such that $x \in y$ and $y \in x$, then we could write

$$x = \{y, \cdots\} = \{\{x, \cdots\}, \cdots\} = \{\{\{y, \cdots\}, \cdots\}, \cdots\} = \{\{\{\{x, \cdots\}, \cdots\}, \cdots\}, \cdots\}$$

leading to the set x having an infinite "depth" in terms of membership.

The first property of ordinals that we prove concerns their internal structure. It is proven in Theorem 2.1.4 below. The proof also illustrates the interplay between transitivity and $\in$-well-orderedness quite well.

Theorem 2.1.4 *Any element of an ordinal is itself an ordinal.*

Proof. Let α a nonempty ordinal and $x \in \alpha$. Suppose $y \in x$ and $z \in y$. Since $y \in x \in \alpha$ and α is transitive, $y \in \alpha$. But then $z \in y \in \alpha$, so $z \in \alpha$. Since α is well-ordered by $\in$ it is in particular totally ordered. Hence for $x, z \in \alpha$ exactly one of $x \in z$, $x = z$, and $z \in x$ holds.

If $x \in z$ then since $z \in y$ and $y \in x$ we obtain a contradiction with Fact 2.1.3.

If $x = z$ then $y \in x = z$ and $x = z \in y$ hence we again obtain a contradiction with Fact 2.1.3.

We conclude that $z \in x$. Thus x is a transitive set.

Now suppose that $\varnothing \neq A \subseteq x$ is a nonempty subset of x. Then since α is transitive and $x \in \alpha$ we have $A \subseteq x \subseteq \alpha$, hence A is a nonempty subset of α. Since α is well-ordered by $\in$, A must have a $\in$-least element. Thus x is well-ordered by $\in$. $\square$

The next property of ordinals that we prove concerns their external structure, by which we mean the relations between different ordinals. In a transitive set, any member is also a (necessarily strict) subset. As Theorem 2.1.5 below proves, when all sets involved are ordinals the converse also holds: a strict subset of an ordinal is also a member of that ordinal, provided it is itself an ordinal.

Theorem 2.1.5 *For ordinals α and β we have that $\alpha \subsetneq \beta$ if and only if $\alpha \in \beta$.*

Proof. Suppose $\alpha \in \beta$. Then because β is transitive, $\alpha \subseteq \beta$. If $\alpha = \beta$ then $\alpha \in \beta = \alpha$, contradicting Fact 2.1.3. Thus $\alpha \subsetneq \beta$.

Now suppose $\alpha \subsetneq \beta$. Then $\beta \smallsetminus \alpha$ is a nonempty subset of the well-order β, hence has a $\in$-least element γ. Note that $\gamma \subseteq \alpha$, because for any element $x \in \gamma \smallsetminus \alpha$ we have $x \in \beta \smallsetminus \alpha$ since β is transitive, contradicting the minimality of γ. Now let $y \in \alpha$. Then because $\alpha \subsetneq \beta$ we have $y \in \beta$, hence because β is a total order we have $y = \gamma \vee \gamma \in y \vee y \in \gamma$. If $x = \gamma$ then $\alpha \ni y = \gamma \in \beta \smallsetminus \alpha$, a contradiction. If $\gamma \in y$ then because α is transitive we have $\alpha \supseteq y \ni \gamma \in \beta \smallsetminus \alpha$, again a contradiction. Hence $y \in \gamma$ and so $\alpha \subseteq \gamma$. We conclude that $\alpha = \gamma \in \beta$. □

Corollary 2.1.6 *For ordinals α and β we have that $\alpha \subseteq \beta$ if and only if $\alpha \in \beta$ or $\alpha = \beta$.*

From Theorem 2.1.5 (or rather Corollary 2.1.6) we can easily prove Theorem 2.1.7 below. This result is interpreted further in the next section.

Theorem 2.1.7 *Given ordinals α and β, exactly one of the following conditions holds:*

1. $\alpha = \beta$
2. $\alpha \in \beta$
3. $\beta \in \alpha$.

Proof. First notice that Fact 2.1.3 guarantees that at most one of the conditions holds. Suppose $\alpha \neq \beta$ and $\alpha \notin \beta$, i.e. $\neg(\alpha = \beta \vee \alpha \in \beta)$. Then by Corollary 2.1.6 we have $\alpha \not\subseteq \beta$. Hence $\alpha \smallsetminus \beta$ is a nonempty subset of the well-order α, with $\in$-least element γ. Note that we must have $\gamma \subseteq \beta$, because if there exists some $x \in \gamma \smallsetminus \beta$ then since α is transitive we have $x \in \alpha \smallsetminus \beta$, contradicting the minimality of γ. Observe that $\gamma \in \beta$ contradicts the construction of γ, hence by Corollary 2.1.6 we have $\beta = \gamma \in \alpha$. □

2.2 The class of ordinals

A *class* is an informal collection of sets sharing some property. Some classes are themselves sets: any set x can be thought of as the class of all sets y satisfying the property $y \in x$. A *proper class* is a class that is not a set. An example of a proper class is Ord, the class of all ordinals. For more on classes, see Appendix B.

Theorem 2.1.7 can be concisely reformulated as saying that the relation of set membership is trichotomous on Ord. Set membership is also transitive on Ord (since ordinals are transitive sets) and hence we see that set membership defines a strict total order on Ord. As such, for ordinals α and β we introduce the notation $\alpha < \beta$ which we stipulate to be equivalent to $\alpha \in \beta$. Theorem 2.1.5 then tells us that we have the equivalences

$$\alpha < \beta \iff \alpha \in \beta \iff \alpha \subsetneq \beta \quad \text{and} \quad \alpha \leq \beta \iff (\alpha \in \beta \vee \alpha = \beta) \iff \alpha \subseteq \beta.$$

The total order $<$ on Ord is actually is some sense a well-order: this is made precise in Theorem 2.2.1 below. Its statement and proof are those of Theorem 4.2.2(a) in Moerdijk & Van Oosten [4].

Theorem 2.2.1 *Let ϕ be a property such that $\phi(\alpha)$ holds for at least one ordinal α. Then there exists a least ordinal β such that $\phi(\beta)$ holds.*

Proof. Let α be an ordinal such that $\phi(\alpha)$ holds. Consider the set $A := \{\gamma \in \alpha \mid \phi(\gamma)\}$.

If $A = \varnothing$ then α is the least ordinal satisfying ϕ. Indeed, if $\beta < \alpha$ is an ordinal then $\beta \in \alpha$ and therefore $\phi(\beta)$ does not hold, since if it did then we would have $\beta \in A = \varnothing$, a contradiction.

If $A \neq \varnothing$ then since α is well-ordered $\min A$ exists, and is the least ordinal satisfying ϕ. Indeed, if $\beta < \min A$ then $\beta \in \min A \in \alpha$ implies $\beta \in \alpha$ (since α is transitive) hence $\phi(\beta)$ cannot hold, since if it did we would have $A \ni \beta < \min A$. □

Theorem 2.2.1 above establishes that any nonempty subclass of Ord has a least element. If we now reinterpret Theorem 2.1.4 as saying that Ord is a transitive class, then it becomes apparent that Ord, apart from not being a set but a proper class, satisfies exactly those properties that we demand ordinals have. In this sense we can informally think of Ord as the "greatest ordinal", although we must be careful not to confuse it with actual ordinals. (Which are sets!) Being well-ordered in this manner also makes Ord susceptible to induction arguments, as proven in Theorem 2.2.2 below. As a sidenote on notation, hereafter we may occasionally write "$\alpha \in \mathrm{Ord}$" for "α is an ordinal."

Theorem 2.2.1 also tells us that there must exist a least ordinal, which by inspection must be the empty set $\varnothing$ (which is vacuously an ordinal). In the context of ordinals (especially ordinal arithmetic, see Chapter 3) we adopt the notation $0 := \varnothing$.

Theorem 2.2.2 below is adapted from Theorem 7.17 in Takeuti & Zaring [6]: the proof is our own.

Theorem 2.2.2 (Transfinite Induction on Ord) *Let ϕ be a property. Suppose the following condition is satisfied:*

$$(\forall \alpha \in \mathit{Ord})\,[(\forall \beta \in \mathit{Ord})(\beta < \alpha \to \phi(\beta)) \to \phi(\alpha)]$$

i.e. if ϕ holds for all ordinals less than α, then ϕ also holds for α. Then ϕ holds for all ordinals.

Proof. Suppose the condition is satisfied but there exists an ordinal for which ϕ does not hold. Then by Theorem 2.2.1 there exists a least such ordinal: denote it α. Then $\phi(\beta)$ holds for all $\beta < \alpha$, lest the minimality of α be violated. Hence the condition implies that $\phi(\alpha)$ also holds, producing a contradiction. Hence no such ordinal α can exists, consequently ϕ holds for all ordinals. □

The technique of transfinite induction will be our main method of proving that some property holds for all ordinals. It is employed numerous times in this document.

2.3 Successor and limit ordinals

Our next goal will be to sort the ordinals into two subclasses, the *successor ordinals* and the *limit ordinals*.

Definition 2.3.1 Let α an ordinal. Then the *successor of α* is defined to be the least ordinal greater than α, and is denoted $S\alpha$. An ordinal that is of the form $S\alpha$ for some ordinal α is called a *successor ordinal*.

The existence of successor ordinals is proven in Theorem 2.3.2 below. The statement is adapted from Theorem 4.2.2(c) in Moerdijk & Van Oosten [4], although the proof is our own.

Theorem 2.3.2 *If α is an ordinal, then so is $\alpha \cup \{\alpha\}$, and it is the successor of α.*

Proof. Let $x \in y \in \alpha \cup \{\alpha\}$. Then either $x \in y = \alpha \subseteq \alpha \cup \{\alpha\}$; or $x \in y \in \alpha$ hence $x \in \alpha \subseteq \alpha \cup \{\alpha\}$ since α is transitive. In either case we have $x \in \alpha \cup \{\alpha\}$ and so $\alpha \cup \{\alpha\}$ is transitive.

Now let $x \subseteq \alpha \cup \{\alpha\}$ be a nonempty subset. Then either $x \cap \alpha$ is nonempty, in which case it has a least element that is automatically also the least element of x (since $x \setminus (x \cap \alpha) \subseteq \{\alpha\}$); or $x \cap \alpha = \varnothing$ in which case the nonemptiness of x implies $x = \{\alpha\}$ hence $\min x = \alpha$. Thus $\alpha \cup \{\alpha\}$ is well-ordered.

Now let β be any ordinal satisfying $\alpha < \beta$, i.e. $\alpha \in \beta$. Then also $\alpha \subseteq \beta$ since β is transitive. Hence we see that $\alpha \cup \{\alpha\} \subseteq \beta$, which implies $\alpha \cup \{\alpha\} \leq \beta$ by Corollary 2.1.6. □

Examples of successor ordinals include $1 := S0 = 0 \cup \{0\} = \{\varnothing\}$ as well as $2 := S1 = 1 \cup \{1\} = \{0, 1\} = \{\varnothing, \{\varnothing\}\}$ and $3 = S2$ et cetera. The ordinal 0 together with those ordinals obtained from 0 by finitely many operations of succession are referred to as the *natural numbers* in this document.

It is worth noting that a successor ordinal has a unique predecessor: this is proven in Theorem 2.3.3 below.

Theorem 2.3.3 *Suppose that α and β are ordinals such that $S\alpha = S\beta$. Then $\alpha = \beta$.*

Proof. We argue by contraposition. Suppose that $\alpha \neq \beta$ and assume without loss of generality that $\alpha < \beta$. Then $S\alpha \leq \beta$ by the definition of the succcessor, consequently $S\alpha \leq \beta < S\beta$ and so $S\alpha \neq S\beta$. □

Having defined an order on Ord, it makes sense to consider the *supremum* (least upper bound) of a set of ordinals. See Theorem 2.3.4 below. The statement is adapted from Theorem 4.2.2(b) in Moerdijk & Van Oosten [4], although the proof is our own.

Theorem 2.3.4 *Let x be a set of ordinals. Then $\bigcup x$ is an ordinal, and it is the supremum of x (with respect to the order on Ord).*

Proof. First we prove that $\bigcup x$ is an ordinal.

If $z \in y \in \bigcup x$ then there exists an ordinal $\alpha \in x$ such that $z \in y \in \alpha$. Since α is transitive it follows that $z \in \alpha \subseteq \bigcup x$.

If $a \subseteq \bigcup x$ is nonempty then there exists an ordinal $\alpha \in x$ such that $a \cap \alpha$ is nonempty: since furthermore α is well-ordered $\gamma := \min(a \cap \alpha)$ exists, and we claim it is equal to $\min a$. Indeed, let $y \in a$ and suppose $y < \gamma$. Then $y \notin \alpha$, lest the minimality of γ be violated. Hence by Theorem 2.1.7 we must have $\alpha \leq y$. But then $y < \gamma < \alpha \leq y$, which is a contradiction. Thus no such y exists, and we conclude that $\gamma = \min a$.

Next we prove that $\bigcup x = \sup x$.

Note that if $\alpha \in x$, then $\alpha \subseteq \bigcup x$ and so $\alpha \leq \bigcup x$. Hence $\bigcup x$ is an upper bound for x.

Now let β be an upper bound for x. Then for all $\alpha \in x$ we have $\alpha \leq \beta$ hence $\alpha \subseteq \beta$. Consequently $\bigcup x \subseteq \beta$ i.e. $\bigcup x \leq \beta$. Thus $\bigcup x$ is the supremum of x. $\square$

Since any ordinal α is a transitive set, it follows that $\sup \alpha = \bigcup \alpha \leq \alpha$. The particular case $\sup \alpha = \alpha$ is worthy of special nomenclature: see Definition 2.3.5 below.

Definition 2.3.5 An ordinal α is called a *limit ordinal* if and only if it satisfies $\alpha = \sup \alpha$.

Examples of limit ordinals include 0 and ω, where ω is the least infinite ordinal. With the definition of the natural numbers above, we think of ω as the set of natural numbers:

$$\omega = \{0, 1, 2, \cdots\}.$$

Using the axioms of Zermelo-Fraenkel set theory, we construct ω in Appendix A.3.

Having established the concepts of successor and limit ordinals, we are ready to prove the classifying Theorem 2.3.6 below.

Theorem 2.3.6 *Every ordinal is either a successor ordinal or a limit ordinal.*

Proof. Let α an ordinal. Clearly $\sup \alpha \leq \alpha$. Hence exactly one of $\sup \alpha = \alpha$ or $\sup \alpha < \alpha$ holds.

If $\sup \alpha = \alpha$ then α is a limit ordinal.

If $\sup \alpha < \alpha$ then consider $\gamma = S(\sup \alpha)$. Clearly $\gamma \leq \alpha$. If $\gamma < \alpha$ then $\sup \alpha < \gamma \in \alpha$, a contradiction. Hence $\gamma = \alpha$ and so $\alpha = S(\sup \alpha)$ is a successor ordinal.

Further, no ordinal can simultaneously be a successor and a limit. Indeed if $\alpha = S\beta$ is a successor ordinal, then $\sup \alpha = \beta < \alpha$. $\square$

With Theorem 2.3.6 in hand, we will typically structure induction arguments in the following manner:

1. Prove $\phi(0)$.
2. Prove $\phi(\alpha) \to \phi(S\alpha)$ for all ordinals α.
3. Prove that if $\phi(\beta)$ for all ordinals $\beta < \alpha$ with α any nonzero limit ordinal, then $\phi(\alpha)$.

Theorem 2.3.7 below establishes a very useful relation between suprema of sets of ordinals. Its statement is that of Theorem 8.6 in Takeuti & Zaring [6], although the proof is our own.

Theorem 2.3.7 *Let x and y be sets of ordinals such that for every $\alpha \in x$ there exists a $\beta \in y$ such that $\alpha \leq \beta$. Then $\sup x \leq \sup y$.*

Proof. Suppose to the contrary that $\sup y < \sup x$. Then by the definition of $\sup x$ there must exist $\alpha \in x$ such that $\sup y < \alpha$. But by assumption there exists $\beta \in y$ such that $\alpha \leq \beta$, hence $\sup y < \beta$ and so $\sup y$ is not an upper bound for y, contradicting the definition of $\sup y$. We conclude that $\sup x \leq \sup y$. $\square$

The sets $\{2k \mid k < \omega\}$ and $\{2k + 1 \mid k < \omega\}$ illustrate that we cannot sharpen the preceding lemma to preserve strict inequalities when passing to the suprema. Note that here we use the operations of addition and multiplication of natural numbers, which we assume to be intuitively understood. To formalise these operations and extend them to the transfinite ordinals, we turn to the next chapter.

Chapter 3

Ordinal arithmetic

In this chapter we define the operations of ordinal addition, ordinal multiplication, and ordinal exponentiation, and we prove several properties that these operations possess. What we do not prove explicitly is that these operations reduce to conventional arithmetic when restricted to the natural numbers, but this is easily seen simply from the way we have defined the natural numbers in the previous chapter.

The operations we consider in this chapter are defined using the technique of *transfinite recursion*: for details, see Theorem B.3.1 in Appendix B.

The content of this chapter can be found in Chapter 8 of Takeuti & Zaring [6]. The proofs presented here are our own.

3.1 Ordinal addition

Intuitively, the operation of ordinal addition is obtained by transfinitely iterating the successor operator. We give a formal definition below.

Definition 3.1.1 We define the *ordinal sum* of two ordinals α and β by transfinite recursion on β:

$$\alpha + \beta := \begin{cases} \alpha & \text{if } \beta = 0 \\ S(\alpha + \gamma) & \text{if } \beta = S\gamma \\ \sup\limits_{\gamma < \beta}(\alpha + \gamma) & \text{if } \beta = \sup\limits_{\gamma < \beta} \gamma \neq 0. \end{cases}$$

In the expression $\alpha + \beta$, we will call the left argument α the *augend* (from the Latin *augendum*, "that which is to be increased") and the right argument β the *addend* (from the Latin *addendum*, "that which is to be added") [5, p. 80]. We elect to use this terminology over the more common practice of referring to both α and β as *terms*, because (as we will shortly see) ordinal addition is not commutative. Thus the roles of the two arguments are different, and we deem them worthy of separate nomenclature.

On the natural numbers, ordinal addition reduces to the conventional addition of natural numbers, with nice algebraic properties such as associativity, commutativity and cancellativity. When transfinite ordinals are involved however, some of these properties may fail. For example, the ordinal sum $\omega + 1$ evaluates to

$$\omega + 1 = S(\omega + 0) = S\omega$$

while the ordinal sum $1 + \omega$ evaluates to

$$1 + \omega = 1 + \sup\{n \mid n < \omega\} = \sup\{1 + n \mid n < \omega\} = \sup\{n \mid n < \omega\} = \omega.$$

The second-to-last equality in the above calculation follows from applying Theorem 2.3.7 twice. We see that ordinal addition is certainly not commutative, as $\omega \neq S\omega$. Consequently the question arises as to which algebraic properties ordinal addition *does* satisfy, and it is this question which will occupy our attention for the rest of this section.

The definition of ordinal addition tells us that $\alpha + 0 = \alpha$ for all ordinals α. Theorem 3.1.2 below establishes that $0 + \alpha$ also evaluates to α, for any α. Thus we may call 0 a neutral element with respect to ordinal addition.

Theorem 3.1.2 *The empty ordinal* 0 *is a two-sided neutral element with respect to ordinal addition.*

Proof. By the definition of ordinal addition we have that $\alpha + 0 = \alpha$ for all ordinals α. To prove that we also have $0 + \alpha = \alpha$ for all ordinals α we employ transfinite induction (Theorem 2.2.2) over α.

Initial case. If $\alpha = 0$ then $0 + \alpha = 0 + 0 = 0 = \alpha$.

Successor case. If $\alpha = S\beta$ and $0 + \beta = \beta$, then

$$0 + \alpha = 0 + (S\beta) = S(0 + \beta) = S\beta = \alpha.$$

Limit case. If $\alpha = \sup \alpha$ and $0 + \beta$ for all $\beta < \alpha$, then

$$0 + \alpha = 0 + \sup_{\beta<\alpha} \beta = \sup_{\beta<\alpha}(0 + \beta) = \sup_{\beta<\alpha} \beta = \alpha. \qquad \square$$

Ordinal addition, importantly, respects the order-theoretic structure present on the class of ordinals, at least with respect to the addend. See Theorem 3.1.3 below.

Theorem 3.1.3 *Ordinal addition is strictly increasing in the addend. I.e. if α, β, γ are ordinals such that $\beta < \gamma$, then $\alpha + \beta < \alpha + \gamma$.*

Proof. We argue by transfinite induction on γ.

Initial case. If $\gamma = 0$ then no $\beta < \gamma$ exists and the statement holds vacuously.

Successor case. Suppose that $\gamma = S\eta$ and $\beta < \gamma$. Then either $\beta < \eta$ or $\beta = \eta$.

If $\beta < \eta$ then since $\eta < \gamma$ we have $\alpha + \beta < \alpha + \eta$ by the inductive hypothesis. It follows that

$$\alpha + \beta < \alpha + \eta < S(\alpha + \eta) = \alpha + (S\eta) = \alpha + \gamma.$$

If $\beta = \eta$ then $\alpha + \beta = \alpha + \eta < S(\alpha + \eta) = \alpha + (S\eta) = \alpha + \gamma$.

Limit case. Suppose that $\gamma = \sup \gamma$ and $\beta < \gamma$. Then since $\beta < \sup \gamma$ there exists $\eta \in \gamma$ such that $\beta < \eta$. Since $\eta < \gamma$ we have $\alpha + \beta < \alpha + \eta$ by the inductive hypothesis. It follows that

$$\alpha + \beta < \alpha + \eta \leq \sup_{\xi<\gamma}(\alpha + \xi) = \alpha + \sup_{\xi<\gamma} \xi = \alpha + \gamma. \qquad \square$$

Theorem 3.1.3 above has a very useful consequence, namely that ordinal addition is left cancellative. This is worked out in Corollary 3.1.4 below.

Corollary 3.1.4 *Let α, β, γ be ordinals such that $\alpha + \beta = \alpha + \gamma$. Then $\beta = \gamma$.*

Proof. We argue by contraposition. Suppose $\beta < \gamma$. Then by Theorem 3.1.3 we have $\alpha+\beta < \alpha+\gamma$. Analogously $\gamma < \beta$ implies $\alpha + \gamma < \alpha + \beta$. Hence if $\beta \neq \gamma$ then $\alpha + \beta \neq \alpha + \gamma$. $\square$

Our example $1 + \omega = \omega$ at the beginning of this chapter was used to show that ordinal addition is not commutative. However, it is also emblematic of a general trend in ordinal arithmetic: the operations behave less nicely with respect to the left argument than to the right argument. For example, even though $0 < 1$ we have $0 + \omega = \omega \not< \omega = 1 + \omega$, so ordinal addition is not strictly increasing in the augend. Nondecreasement however still holds, as Theorem 3.1.5 below proves.

Theorem 3.1.5 *Ordinal addition is nondecreasing in the augend. I.e. if α, β, γ are ordinals such that $\alpha \leq \beta$, then $\alpha + \gamma \leq \beta + \gamma$.*

Proof. Suppose that $\alpha \leq \beta$. We argue by transfinite induction on γ.

Initial case. If $\gamma = 0$ then $\alpha + \gamma = \alpha \leq \beta = \beta + \gamma$.

Successor case. Suppose that $\gamma = S\eta$. Then $\eta < \gamma$ hence by the inductive hypothesis we have $\alpha + \eta \leq \beta + \eta$. By the definition of the successor it follows that

$$\alpha + \gamma = \alpha + (S\eta) = S(\alpha + \eta) \leq S(\beta + \eta) = \beta + (S\eta) = \beta + \gamma.$$

Limit case. Suppose that $\gamma = \sup \gamma$. By the inductive hypothesis we have $\alpha + \eta \leq \beta + \eta$ for all $\eta < \gamma$. Consequently we may apply Theorem 2.3.7 to conclude that

$$\alpha + \gamma = \alpha + \sup_{\eta<\gamma} \eta = \sup_{\eta<\gamma}(\alpha + \eta) \leq \sup_{\eta<\gamma}(\beta + \eta) = \beta + \sup_{\eta<\gamma} \eta = \beta + \gamma. \qquad \square$$

The counterexample $1+\omega = 2+\omega$ illustrates that ordinal addition does not admit right cancellation.

Even though there are no "negative ordinals," ordinal addition does admit some form of substraction on the left, or rather additive decomposition. This is proven in Theorem 3.1.6 below.

Theorem 3.1.6 *Let α and β ordinals such that $\beta \leq \alpha$. Then there exists a unique ordinal γ such that $\alpha = \beta+\gamma$.*

Proof. Since 0 is neutral with respect to ordinal addition (Theorem 3.1.2) we have $\alpha = 0+\alpha$. Since ordinal addition is nondecreasing in the augend (Theorem 3.1.5) and $0 \leq \beta$ by the definition of 0, we have $\alpha = 0+\alpha \leq \beta+\alpha$. This demonstrates that there exist ordinals ξ for which $\alpha \leq \beta+\xi$. By Theorem 2.2.1 let γ be the least such ordinal. Then exactly one of $\alpha < \beta+\gamma$ and $\alpha = \beta+\gamma$ holds.

Suppose $\alpha < \beta+\gamma$. We derive a contradiction.

Suppose γ is a successor ordinal, say $\gamma = S\eta$. Then since $\alpha < \beta+(S\eta) = S(\beta+\eta)$ we have $\alpha \leq \beta+\eta$, contradicting the minimality of γ.

Suppose γ is a limit ordinal. Then $\alpha < \beta+\gamma = \sup\{\beta+\eta \mid \eta < \gamma\}$. Hence there exists $\eta < \gamma$ such that $\alpha < \beta+\eta$. But that contradicts the minimality of γ.

We conclude that $\alpha = \beta+\gamma$. Unicity of γ now follows from left-cancellativity of ordinal addition (Corollary 3.1.4). □

The counterexample $1+\omega = 2+\omega$ also shows that even if ordinal addition admitted some form of right substraction, that the result would not be unique. In fact ordinal addition does not admit right substraction for arbitrary ordinals: there is no ordinal α such that $\alpha+\omega = S\omega$. This can be seen as follows: if $\alpha < \omega$ then $\alpha+\omega = \omega < S\omega$, and if $\alpha \geq \omega$ then $\alpha+\omega \geq \omega+\omega > S\omega$.

One property that ordinal addition satisfies no less well than conventional addition of natural numbers is associativity. See Theorem 3.1.7 below.

Theorem 3.1.7 *Ordinal addition is associative. I.e. if α, β, and γ are ordinals, then*

$$(\alpha+\beta)+\gamma = \alpha+(\beta+\gamma).$$

Proof. We argue by transfinite induction on γ.

Initial case. If $\gamma = 0$ then

$$(\alpha+\beta)+\gamma = (\alpha+\beta)+0 = \alpha+\beta = \alpha+(\beta+0) = \alpha+(\beta+\gamma).$$

Successor case. If $\gamma = S\eta$ and $(\alpha+\beta)+\eta = \alpha+(\beta+\eta)$ then

$$\begin{aligned}(\alpha+\beta)+\gamma &= (\alpha+\beta)+(S\eta) = S((\alpha+\beta)+\eta) = S(\alpha+(\beta+\eta)) \\ &= \alpha+S(\beta+\eta) = \alpha+(\beta+(S\eta)) = \alpha+(\beta+\gamma).\end{aligned}$$

Limit case. If $\gamma = \sup\gamma$ and $(\alpha+\beta)+\eta = \alpha+(\beta+\eta)$ for all $\eta < \gamma$ then

$$\begin{aligned}(\alpha+\beta)+\gamma &= (\alpha+\beta)+\sup_{\eta<\gamma}\eta = \sup_{\eta<\gamma}((\alpha+\beta)+\eta) = \sup_{\eta<\gamma}(\alpha+(\beta+\eta)) \\ &= \alpha+\sup_{\eta<\gamma}(\beta+\eta) = \alpha+(\beta+\sup_{\eta<\gamma}\eta) = \alpha+(\beta+\gamma).\end{aligned}$$ □

Having established the associativity of ordinal addition, we can unambiguously write down an expression like $\alpha+\beta+\gamma$, secure in the knowledge that the two interpretations $(\alpha+\beta)+\gamma$ and $\alpha+(\beta+\gamma)$ coincide. In fact, for any finite ordered sequence $(\alpha_i)_{i<n}$ of ordinals (here $n < \omega$) we can unambiguously talk about the sum of this sequence. See Definition 3.1.8 below.

Definition 3.1.8 Given a natural number $n < \omega$ and a finite sequence $(\alpha_i)_{i<n}$ of ordinals, we recursively define the *ordinal sum* of $(\alpha_i)_{i<n}$ as

$$\sum_{i<n}\alpha_i := \begin{cases} 0 & \text{if } n = 0 \\ \left(\sum_{i<m}\alpha_i\right)+\alpha_m & \text{if } n = Sm. \end{cases}$$

For a nonempty sequence $(\beta_i)_{i<n+1}$ we will also occasionally write

$$\sum_{i\leq n}\beta_i \qquad \text{or} \qquad \beta_0+\cdots+\beta_n$$

instead of $\sum_{i<n+1}\beta_i$.

3.2 Ordinal multiplication

Ordinal multiplication is obtained by transfinitely iterating ordinal addition. See Definition 3.2.1 below.

Definition 3.2.1 We define the *ordinal product* of two ordinals α and β by transfinite recursion on β:

$$\alpha\beta := \begin{cases} 0 & \text{if } \beta = 0 \\ (\alpha\gamma) + \alpha & \text{if } \beta = S\gamma \\ \sup\limits_{\gamma<\beta}(\alpha\gamma) & \text{if } \beta = \sup\limits_{\gamma<\beta} \gamma \neq 0. \end{cases}$$

We wil occasionally denote $\alpha\beta$ as $\alpha \cdot \beta$, in situations where confusion might otherwise arise.

In the expression $\alpha\beta$, we will call the left argument α the *multiplicand* (From the Latin *multiplicandum*, “thing to be multiplied”) and the right argument β the *multiplier* [5, p. 449]. As with ordinal addition, we choose to revivify this somewhat obscure terminology because ordinal multiplication is in general not commutative. Thus we prefer to have separate names for both arguments rather than refer to both of them as *factors*.

Like ordinal addition, ordinal multiplication admits a neutral element. Theorem 3.2.2 below proves this claim, and asserts that in keeping with what we might expect this neutral element is 1.

Theorem 3.2.2 *The ordinal* 1*, the successor of the empty ordinal, is a two-sided neutral element with respect to ordinal multiplication.*

Proof. For any ordinal α we have

$$\alpha \cdot 1 = (\alpha \cdot 0) + \alpha = 0 + \alpha = \alpha.$$

To prove that $1 \cdot \alpha = \alpha$ as well, we employ transfinite induction (Theorem 2.2.2) over α.

Initial case. If $\alpha = 0$, then

$$1 \cdot \alpha = 1 \cdot 0 = 0 = \alpha.$$

Successor case. If $\alpha = S\beta$ and $1 \cdot \beta = \beta$ then

$$1 \cdot \alpha = 1 \cdot (S\beta) = (1 \cdot \beta) + 1 = \beta + 1 = S\beta = \alpha.$$

Limit case. If $\alpha = \sup \alpha \neq 0$ and $1 \cdot \beta = \beta$ for all $\beta < \alpha$ then

$$1 \cdot \alpha = 1 \cdot \sup_{\beta<\alpha} \beta = \sup_{\beta<\alpha}(1 \cdot \beta) = \sup_{\beta<\alpha} \beta = \alpha. \qquad \square$$

Again mirroring the situation we had with ordinal addition, ordinal multiplication behaves more nicely with respect to the second argument than to the first. For example, $\omega 1 = \omega < \omega + \omega = \omega 2$ whereas $1\omega = \omega = 2\omega$. This first example illustrates the general fact that ordinal multiplication is strictly increasing in the multiplier (for nonzero multiplicands), as proven in Theorem 3.2.3 below.

Theorem 3.2.3 *Ordinal multiplication is strictly increasing in the multiplier provided the multiplicand is nonzero. I.e. if α, β, and γ are ordinals such that $0 < \alpha$ and $\beta < \gamma$, then $\alpha\beta < \alpha\gamma$.*

Proof. We argue by transfinite induction on γ.

Initial case. If $\gamma = 0$ then no $\beta < 0$ exists and the statement follows vacuously.

Successor case. Suppose that $\gamma = S\eta$ and $\beta < \gamma$. Then either $\beta < \eta$ or $\beta = \eta$. If $\beta < \eta$ then by the inductive hypothesis we have $\alpha\beta < \alpha\eta$. If $\beta = \eta$ then clearly $\alpha\beta = \alpha\eta$. Hence in either case we have $\alpha\beta \leq \alpha\eta$. Now since $0 < \alpha$ and ordinal addition in strictly increasing in the addend (Theorem 3.1.3) we have

$$\alpha\beta \leq \alpha\eta < \alpha\eta + \alpha = \alpha(S\eta) = \alpha\gamma.$$

Limit case. Suppose that $\gamma = \sup \gamma$ and $\beta < \gamma$. Then there exists $\eta < \gamma$ such that $\beta < \eta$. By the inductive hypothesis we have $\alpha\beta < \alpha\eta$, consequently

$$\alpha\beta < \alpha\eta \leq \sup_{\xi<\gamma}(\alpha\xi) = \alpha \sup_{\xi<\gamma} \xi = \alpha\gamma. \qquad \square$$

We already saw that $1\omega = \omega = 2\omega$, proving that ordinal multiplication is not strictly increasing in the multiplicand. However as with ordinal addition we can salvage nondecreasement, see Theorem 3.2.4 below.

Theorem 3.2.4 *Ordinal multiplication is nondecreasing in the multiplicand. I.e. if α, β, and γ are ordinals such that $\alpha \le \beta$, then $\alpha\gamma \le \beta\gamma$.*

Proof. Suppose $\alpha \le \beta$. We argue by transfinite induction on γ.

Initial case. If $\gamma = 0$ then the statement reduces to the tautology $0 \le 0$.

Successor case. Suppose that $\gamma = S\eta$ and that $\alpha\eta \le \beta\eta$. Then since ordinal addition is nondecreasing in the augend (Theorem 3.1.5) we have $(\alpha\eta) + \alpha \le (\beta\eta) + \alpha$. Since $\alpha \le \beta$ and ordinal addition is strictly increasing in the addend (Theorem 3.1.3) we also have $(\beta\eta) + \alpha \le (\beta\eta) + \beta$. All in all we conclude that

$$\alpha\gamma = \alpha(S\eta) = (\alpha\eta) + \alpha \le (\beta\eta) + \alpha \le (\beta\eta) + \beta = \beta(S\eta) = \beta\gamma.$$

Limit case. Suppose that $\gamma = \sup \gamma$ and that $\alpha\eta \le \beta\eta$ for all $\eta < \gamma$. Then by Theorem 2.3.7 we have

$$\alpha\gamma = \alpha \sup_{\eta<\gamma} \eta = \sup_{\eta<\gamma}(\alpha\eta) \le \sup_{\eta<\gamma}(\beta\eta) = \beta \sup_{\eta<\gamma} \eta = \beta\gamma. \qquad \square$$

Ordinal multiplication also satisfies a distributive property over ordinal addition, albeit only from the left. See Theorem 3.2.5 below. The fact that the analogous right distributive property does not hold can be illustrated with the counterexample $(1+1)\omega = 2\omega = \omega \ne \omega + \omega = (1\omega) + (1\omega)$.

Theorem 3.2.5 *Ordinal multiplication satisfies a left-distributive property with respect to ordinal addition. I.e. if α, β, and γ are ordinals, then*

$$\alpha(\beta + \gamma) = (\alpha\beta) + (\alpha\gamma).$$

Proof. We argue by transfinite induction on γ.

Initial case. If $\gamma = 0$ then

$$\alpha(\beta + \gamma) = \alpha(\beta + 0) = \alpha\beta = (\alpha\beta) + 0 = (\alpha\beta) + (\alpha 0) = (\alpha\beta) + (\alpha\gamma).$$

Successor case. If $\gamma = S\eta$ and $\alpha(\beta + \eta) = (\alpha\beta) + (\alpha\eta)$ then

$$\begin{aligned} \alpha(\beta + \gamma) &= \alpha(\beta + (S\eta)) = \alpha(S(\beta + \eta)) = (\alpha(\beta + \eta)) + \alpha = ((\alpha\beta) + (\alpha\eta)) + \alpha \\ &= (\alpha\beta) + ((\alpha\eta) + \alpha) = (\alpha\beta) + (\alpha(S\eta)) = (\alpha\beta) + (\alpha\gamma). \end{aligned}$$

Here we have made use of the associativity of ordinal addition (Theorem 3.1.7).

Limit case. If $\gamma = \sup \gamma$ and $\alpha(\beta + \eta) = (\alpha\beta) + (\alpha\eta)$ for all $\eta < \gamma$ then

$$\begin{aligned} \alpha(\beta + \gamma) &= \alpha(\beta + \sup_{\eta<\gamma} \eta) = \alpha(\sup_{\eta<\gamma}(\beta + \eta)) = \sup_{\eta<\gamma}(\alpha(\beta + \eta)) = \sup_{\eta<\gamma}((\alpha\beta) + (\alpha\eta)) \\ &= (\alpha\beta) + (\sup_{\eta<\gamma}(\alpha\eta)) = (\alpha\beta) + (\alpha(\sup_{\eta<\gamma} \eta)) = (\alpha\beta) + (\alpha\gamma). \qquad \square \end{aligned}$$

In expressions of the form $(\alpha\beta) + \gamma$ or similar we will typically omit the parentheses for brevity. Thus we will write $\alpha\beta + \gamma := (\alpha\beta) + \gamma$, not to be confused with $\alpha(\beta + \gamma)$, wherein we will not omit the parentheses.

Like ordinal addition, ordinal multiplication is associative. See Theorem 3.2.6 below.

Theorem 3.2.6 *Ordinal multiplication is associative. I.e. if α, β, and γ are ordinals then*

$$(\alpha\beta)\gamma = \alpha(\beta\gamma).$$

Proof. We argue by transfinite induction on γ.

Initial case. If $\gamma = 0$ then

$$(\alpha\beta)\gamma = (\alpha\beta)0 = 0 = \alpha 0 = \alpha(\beta 0) = \alpha(\beta\gamma).$$

Successor case. If $\gamma = S\eta$ and $(\alpha\beta)\eta = \alpha(\beta\eta)$ then

$$(\alpha\beta)\gamma = (\alpha\beta)(S\eta) = ((\alpha\beta)\eta) + (\alpha\beta) = (\alpha(\beta\eta)) + (\alpha\beta) = \alpha((\beta\eta) + \beta) = \alpha(\beta(S\eta)) = \alpha(\beta\gamma).$$

Here we have made use of the distributivity of ordinal multiplication over ordinal addition (Theorem 3.2.5).

Limit case. If $\gamma = \sup \gamma$ and $(\alpha\beta)\eta = \alpha(\beta\eta)$ for all $\eta < \gamma$ then

$$\begin{aligned}(\alpha\beta)\gamma &= (\alpha\beta)(\sup_{\eta<\gamma} \eta) = \sup_{\eta<\gamma}((\alpha\beta)\eta) = \sup_{\eta<\gamma}(\alpha(\beta\eta)) \\ &= \alpha(\sup_{\eta<\gamma}(\beta\eta)) = \alpha(\beta(\sup_{\eta<\gamma} \eta)) = \alpha(\beta\gamma).\end{aligned}$$ □

The Euclidean division algorithm (restricted to the natural numbers) states that for natural numbers a and $b \neq 0$ there exist unique natural numbers q and r such that $a = bq + r$. This statement can be generalised to include the transfinite ordinals, as is done in Theorem 3.2.7 below.

Theorem 3.2.7 (Euclidean Division for Ordinals) *Let α and $0 < \beta$ ordinals. Then there exist unique ordinals θ and ρ such that $\alpha = \beta\theta + \rho$ and $\rho < \beta$.*

Proof. First we prove existence.

Since $1 \leq \beta$ we have $\alpha \leq \beta\alpha$ because ordinal multiplication is nondecreasing in the multiplicand (Theorem 3.2.4). Since ordinal multiplication is strictly increasing in the multiplier (Theorem 3.2.3) we also have $\beta\alpha < \beta(S\alpha)$. Hence $\alpha < \beta(S\alpha)$. This demonstrates that there exist ordinals ξ for which $\alpha < \beta\xi$. By Theorem 2.2.1 let γ be the least such ordinal.

If γ is a limit ordinal, then since $\alpha < \beta\gamma = \sup_{\zeta<\gamma}(\beta\zeta)$ there must exist a $\zeta < \gamma$ such that $\alpha < \beta\zeta$, contradicting the minimality of γ.

Thus γ is a successor ordinal, say $\gamma = S\theta$. Since $\theta < \gamma$ we must have $\beta\theta \leq \alpha$ by the minimality of γ, hence we may apply left substraction (Theorem 3.1.6) to conclude that there exists a unique ordinal ρ such that $\alpha = \beta\theta + \rho$.

We claim that $\rho < \beta$. Indeed, suppose that $\beta \leq \rho$. Then because ordinal addition is strictly increasing in the addend (Theorem 3.1.3) we would have

$$\beta\gamma = \beta(S\theta) = \beta\theta + \beta \leq \beta\theta + \rho = \alpha$$

contradicting the construction of γ. Thus $\rho < \beta$.

Secondly we prove unicity.

Suppose $\alpha = \beta\theta + \rho = \beta\theta' + \rho'$ where $\rho, \rho' < \beta$. Then because ordinal addition is strictly increasing in the addend we have

$$\beta\theta \leq \beta\theta + \rho = \alpha = \beta\theta' + \rho' < \beta\theta' + \beta = \beta(S\theta').$$

Hence since ordinal multiplication is strictly increasing in the multiplier (Theorem 3.2.3) we have $\theta < S\theta'$, thus $\theta \leq \theta'$ by the definition of the successor. By symmetry we also have $\theta' \leq \theta$ and we conclude that $\theta = \theta'$. The equality $\rho = \rho'$ now follows from left-cancellativity of ordinal addtion (Corollary 3.1.4). □

We will use Euclidean division for ordinals when proving the existence of a basis representation for all ordinals in Chapter 4.

3.3 Ordinal exponentiation

Ordinal exponentiation is obtained by transfinitely iterating ordinal multiplication. See Definition 3.3.1 below.

Definition 3.3.1 We define *ordinal exponentiation* with base α by transfinite recursion on the exponent β:

$$\alpha^\beta := \begin{cases} 1 & \text{if } \beta = 0 \\ (\alpha^\gamma)\alpha & \text{if } \beta = S\gamma \\ \sup_{\gamma<\beta}(\alpha^\gamma) & \text{if } \beta = \sup_{\gamma<\beta} \gamma \neq 0. \end{cases}$$

Ordinal exponentiation is strictly increasing in the exponent, provided the base exceeeds 1. See Theorem 3.3.2 below.

Theorem 3.3.2 *Ordinal exponentiation is strictly increasing in the exponent for bases exceeding* 1. *I.e. if α, β, and γ are ordinals such that $1 < \alpha$ and $\beta < \gamma$, then $\alpha^\beta < \alpha^\gamma$.*

Proof. We argue by transfinite induction (Theorem 2.2.2) on γ.

Initial case. If $\gamma = 0$ then no $\beta < \gamma$ exists and the statement follows vacuously.

Successor case. If $\gamma = S\eta$ and $\beta < \gamma$, then either $\beta < \eta$ or $\beta = \eta$. If $\beta < \eta$ then by the inductive hypothesis we have $\alpha^\beta < \alpha^\eta$. If $\beta = \eta$ then clearly $\alpha^\beta = \alpha^\eta$. In either case we have $\alpha^\beta \leq \alpha^\eta$. Now since $1 < \alpha$ and multiplication is strictly increasing in the multiplier (Theorem 3.1.3) we have

$$\alpha^\beta \leq \alpha^\eta < \alpha^\eta \alpha = \alpha^{S\eta} = \alpha^\gamma.$$

Limit case. If $\gamma = \sup \gamma$ and $\beta < \gamma$ then there must exist some $\eta < \gamma$ such that $\beta < \eta$. Hence by the inductive hypothesis we have $\alpha^\beta < \alpha^\eta$, therefore

$$\alpha^\beta < \alpha^\eta \leq \sup_{\xi<\gamma} \alpha^\xi = \alpha^{\sup_{\xi<\gamma}(\xi)} = \alpha^\gamma. \qquad \square$$

As was the case with ordinal addition and ordinal multiplication, ordinal exponentiation is less well-behaved in the first argument than in the second. More specifically ordinal exponentiation is not strictly increasing in the base, as the counterexample $2^\omega = 3^\omega$ shows. However, as in the previous two cases, we can at least recover nondecreasement. See Theorem 3.3.3 below.

Theorem 3.3.3 *Ordinal exponentiation is nondecreasing in the base. I.e. if α, β, and γ are ordinals such that $\alpha \leq \beta$, then $\alpha^\gamma \leq \beta^\gamma$.*

Proof. Suppose $\alpha \leq \beta$. We argue by transfinite induction on γ.

Initial case. If $\gamma = 0$ then the statement reduces to the tautology $1 \leq 1$.

Successor case. Suppose that $\gamma = S\eta$. By the inductive hypothesis we have $\alpha^\eta \leq \beta^\eta$. Hence since ordinal multiplication is nondecreasing in the multiplicand (Theorem 3.2.4) we have $\alpha^\eta\alpha \leq \beta^\eta\alpha$. Further, since $\alpha \leq \beta$ and ordinal multiplication is strictly increasing in the multiplier (Theorem 3.2.3) we have $\beta^\eta\alpha \leq \beta^\eta\beta$. It follows that

$$\alpha^\gamma = \alpha^{S\eta} = \alpha^\eta\alpha \leq \beta^\eta\alpha \leq \beta^\eta\beta = \beta^{S\eta} = \beta^\gamma.$$

Limit case. Suppose that $\gamma = \sup \gamma$. By the inductive hypothesis we have $\alpha^\eta \leq \beta^\eta$ for every $\eta < \gamma$. Hence since the supremum preserves nonstrict inequalities (Theorem 2.3.7) it follows that

$$\alpha^\gamma = \alpha^{\sup_{\eta<\gamma}(\eta)} = \sup_{\eta<\gamma} \alpha^\eta \leq \sup_{\eta<\gamma} \beta^\eta = \beta^{\sup_{\eta<\gamma}(\eta)} = \beta^\gamma. \qquad \square$$

In this chapter we have established several key properties of ordinal addition, multiplication, and exponentiation. In the next chapter, we will combine all three of these operations to prove what is arguably the most difficult result in this document.

Chapter 4

Basis representation for ordinals

When we write down a number like "97" we make use of decimal notation, wherein we express a natural number as a sum of powers of 10, with coefficients less than 10:

$$97 = 10^1 \cdot 9 + 10^0 \cdot 7.$$

This also referred to as writing the number 97 in *base 10 expansion.* This chapter is devoted to developing a similar method of expansion for ordinals.

The main result of this chapter is the theorem stated below, which is Theorem 8.44 in Takeuti & Zaring [6].

Theorem 4.0.1 (Basis Representation for Ordinals) *Let $\beta > 1$ be an ordinal. Then for every ordinal $\alpha > 0$, there exist a unique natural number r and unique ordinals $0 < \alpha_0, \cdots, \alpha_r < \beta$ and $\gamma_0 > \cdots > \gamma_r$ such that*

$$\alpha = \sum_{i \leq r} \beta^{\gamma_i} \alpha_i = \beta^{\gamma_0} \alpha_0 + \cdots + \beta^{\gamma_r} \alpha_r.$$

The case $\alpha = 0$ is omitted in the above theorem because any expansion of 0 in any basis would obviously have all coefficients equal to zero, and we need to demand that the coefficients α_i be nonzero to guarantee their unicity.

We prove this theorem in two parts: the existence of a basis representation is proven in Section 4.1, and its unicity is proven in Section 4.2. The results presented in Sections 4.1 and 4.2 are from Takeuti & Zaring [6] with the exception of Lemma 4.2.2. The proofs are our own.

At the end of this chapter we will apply Theorem 4.0.1 to define the *Cantor Normal Form* of any ordinal, which leads into the definition of natural arithmetic on ordinals in the next chapter.

4.1 Existence of the basis representation

To prove the existence of a basis representation for every ordinal, we will need two preparatory lemmas. These establish some inequalities involving ordinal exponentiation that are interesting in their own right.

Lemma 4.1.1 *Let α and β be ordinals such that $1 < \beta$. Then $\alpha \leq \beta^\alpha$.*

Proof. We argue by transfinite induction (Theorem 2.2.2) on α.

Initial case. If $\alpha = 0$ then the statement reduces to the tautology $0 \leq 1$.

Successor case. Suppose $\alpha = S\eta$. By the inductive hypothesis we have $\eta \leq \beta^\eta$. Since $1 < \beta$ and ordinal exponentiation is strictly increasing in the exponent for bases exceeding 1 (Theorem 3.3.2) we have $\beta^\eta < \beta^{S\eta} = \beta^\alpha$. Hence $\eta < \beta^\alpha$, consequently by the definition of a successor ordinal we have $\alpha = S\eta \leq \beta^\alpha$.

Limit case. Suppose $\alpha = \sup \alpha$. By the inductive hypothesis we have $\eta \leq \beta^\eta$ for all $\eta < \alpha$. Since the supremum preserves nonstrict inequalities (Theorem 2.3.7) it follows that

$$\alpha = \sup_{\eta<\alpha} \eta \leq \sup_{\eta<\alpha} \beta^\eta = \beta^\alpha. \qquad \square$$

The way we will use Lemma 4.1.1 above is just to show that for nontrivial bases, ordinal exponentiation can yield arbitrarily large results.

Lemma 4.1.2 below asserts that for a given nontrivial base, we can "trap" any ordinal between two successive powers of that base.

Lemma 4.1.2 *Let α and β be ordinals such that $0 < \alpha$ and $1 < \beta$. Then there exists a unique ordinal γ such that $\beta^\gamma \leq \alpha < \beta^{S\gamma}$.*

Proof. By Lemma 4.1.1 which we just proved we have $S\alpha \leq \beta^{S\alpha}$, therefore $\alpha < S\alpha \leq \beta^{S\alpha}$. This demonstrates that there exist ordinals ξ for which $\alpha < \beta^\xi$. By Theorem 2.2.1 let η be the least such ordinal.

If η is a limit ordinal, then we have $\alpha < \beta^\eta = \sup_{\zeta<\eta}(\beta^\zeta)$ which implies the existence of a $\zeta < \eta$ for which $\alpha < \beta^\zeta$. This however contradicts the minimality of η. Thus η is a successor ordinal, say $\eta = S\gamma$. Then by the minimality of η we have $\beta^\gamma \leq \alpha < \beta^{S\gamma}$, as desired. The uniqueness of γ follows from the minimality of η. □

With these two results established it is now relatively straightforward to prove the existence of a basis representation of any nonzero ordinal, and this is done in Theorem 4.1.3 below.

Theorem 4.1.3 (Existence of the Basis Representation) *Let $\beta > 1$ be an ordinal. Then for every ordinal $\alpha > 0$, there exist a natural number r and ordinals $0 < \alpha_0, \cdots, \alpha_r < \beta$ and $\gamma_0 > \cdots > \gamma_r$ such that*

$$\alpha = \sum_{i \leq r} \beta^{\gamma_i}\alpha_i = \beta^{\gamma_0}\alpha_0 + \cdots + \beta^{\gamma_r}\alpha_r.$$

This proves the existence clause of Theorem 4.0.1.

Proof. We argue by transfinite induction on α.

Since $1 < \beta$ and $0 < \alpha$, by Lemma 4.1.2 there exists a unique ordinal γ_0 such that $\beta^{\gamma_0} \leq \alpha < \beta^{S\gamma_0}$.

Since $1 < \beta$ and exponentiation is strictly increasing in the exponent for bases exceeding 1 (Theorem 3.3.2) we have $\beta^{\gamma_0} \geq \beta^0 = 1 > 0$, hence by Theorem 3.2.7 there exist unique ordinals α_0 and $\rho < \beta^{\gamma_0}$ such that $\alpha = \beta^{\gamma_0}\alpha_0 + \rho$.

If $\rho = 0$, then $\alpha = \beta^{\gamma_0}\alpha_0$ is a basis representation for α. If $\rho > 0$, then since $\rho < \beta^{\gamma_0} \leq \alpha$ we may apply the inductive hypothesis to conclude that there exist a natural number r and ordinals $0 < \alpha_1, \cdots, \alpha_r < \beta$ and $\gamma_1 > \cdots > \gamma_r$ such that $\rho = \beta^{\gamma_1}\alpha_1 + \cdots + \beta^{\gamma_r}\alpha_r$. In this case $\alpha = \beta^{\gamma_0}\alpha_0 + \cdots + \beta^{\gamma_r}\alpha_r$ is a basis representation of α. □

4.2 Unicity of the basis representation

Unicity of the basis representation is slightly harder to prove. Our proof hinges upon the fact that two basis representations can be compared exponent by exponent and coefficient by coefficient. To establish this fact we will need the following two preparatory lemmas.

Lemma 4.2.1 below expresses that if one basis representation features a greater exponent than another, then the one with the greater exponent represents a greater ordinal, regardless of the coefficients involved (as long as they are nonzero of course). This is analogous to such an inequality as

$$999 = 10^2 9 + 10^1 9 + 10^0 9 < 10^3 1 = 1000.$$

Lemma 4.2.1 *Let $1 < \beta$ and γ be ordinals. Then for every $r < \omega$ and for all ordinals $\gamma_0, \cdots, \gamma_r$ satisfying*

$$\gamma_r < \cdots < \gamma_0 < \gamma$$

and all ordinals $\alpha_0, \cdots, \alpha_r$ satisfying

$$0 < \alpha_0, \cdots \alpha_r < \beta$$

we have

$$\beta^{\gamma_0}\alpha_0 + \cdots + \beta^{\gamma_r}\alpha_r < \beta^\gamma.$$

Proof. We argue by transfinite induction (Theorem 2.2.2) on γ.

Initial case. If $\gamma = 0$ then no $\gamma_0 < \gamma$ exists and the statement is vacuously true.

Successor case. Suppose $\gamma = S\eta$. Since $\gamma_0 < \gamma$ we may apply the inductive hypothesis to conclude that

$$\beta^{\gamma_1}\alpha_1 + \cdots + \beta^{\gamma_r}\alpha_r < \beta^{\gamma_0}.$$

Consequently since ordinal addition is strictly increasing in the addend (Theorem 3.1.3) we have

$$\beta^{\gamma_0}\alpha_0 + \beta^{\gamma_1}\alpha_1 + \cdots + \beta^{\gamma_r}\alpha_r < \beta^{\gamma_0}\alpha_0 + \beta^{\gamma_0} = \beta^{\gamma_0}(S\alpha_0).$$

Now $\alpha_0 < \beta$ by assumption, so $S\alpha_0 \leq \beta$. Likewise $\gamma_0 < \gamma$ so $S\gamma_0 \leq \gamma$. Hence by applying the strict increasement of ordinal multiplication in the multiplier (Theorem 3.2.3) and the strict increasement of ordinal exponentiation in the exponent (Theorem 3.3.2) we find

$$\beta^{\gamma_0}(S\alpha_0) \leq \beta^{\gamma_0}\beta = \beta^{S\gamma_0} \leq \beta^{\gamma}.$$

Thus we are able to conclude that

$$\beta^{\gamma_0}\alpha_0 + \cdots + \beta^{\gamma_r}\alpha_r < \beta^{\gamma}.$$

Limit case. Suppose $\gamma = \sup\gamma \neq 0$. Since $\gamma_0 < \gamma$ there exists $\eta < \gamma$ such that $\gamma_0 < \eta$. Hence by the inductive hypothesis and the definition of exponentiation with a limit exponent we have

$$\beta^{\gamma_0}\alpha_0 + \cdots + \beta^{\gamma_r}\alpha_r < \beta^{\eta} \leq \sup_{\xi<\gamma} \beta^{\xi} = \beta^{\gamma}. \qquad \square$$

Lemma 4.2.2 below states that when two basis representations have coinciding greatest exponents but one of them has a greater coefficient in the term with that exponent, then the representation with the greater coefficient must represent a greater ordinal, regardless of the coefficients at lesser exponents. This is analogous to such an inequality as

$$799 = 10^2 7 + 10^1 9 + 10^0 9 < 10^2 8 = 800.$$

Lemma 4.2.2 *Let $1 < \beta$ and $0 < \alpha_0 < \alpha < \beta$ be ordinals. Then for all ordinals $\gamma_0, \cdots, \gamma_r$ satisfying*

$$\gamma_r < \cdots < \gamma_0$$

and all ordinals $\alpha_1, \cdots, \alpha_r$ satisfying

$$0 < \alpha_1, \cdots \alpha_r < \beta$$

we have

$$\beta^{\gamma_0}\alpha_0 + \cdots + \beta^{\gamma_r}\alpha_r < \beta^{\gamma_0}\alpha.$$

Proof. By Lemma 4.2.1 which we just proved we have

$$\beta^{\gamma_1}\alpha_1 + \cdots + \beta^{\gamma_r}\alpha_r < \beta^{\gamma_0}.$$

By assumption $\alpha_0 < \alpha$, hence $S\alpha_0 \leq \alpha$. Now by using strict increasement of ordinal addition in the addend (Theorem 3.1.3) and of ordinal multiplication in the multiplier (Theorem 3.2.3) we conclude that

$$\beta^{\gamma_0}\alpha_0 + \beta^{\gamma_1}\alpha_1 + \cdots + \beta^{\gamma_r}\alpha_r < \beta^{\gamma_0}\alpha_0 + \beta^{\gamma_0} = \beta^{\gamma_0}(S\alpha_0) \leq \beta^{\gamma_0}\alpha. \qquad \square$$

The above two lemmas together serve to prove Theorem 4.2.3 below, which guarantees unicity of the basis representation.

Theorem 4.2.3 (Unicity of the Basis Representation) *Let $1 < \beta$ and $r, s < \omega$ and $\gamma_r < \cdots < \gamma_0$ and $\gamma'_s < \cdots < \gamma'_0$ and $0 < \alpha_0, \cdots, \alpha_r < \beta$ and $0 < \alpha'_0, \cdots, \alpha'_s < \beta$ be ordinals. Consider the two ordinals*

$$\eta = \sum_{i \leq r} \beta^{\gamma_i}\alpha_i \qquad \text{and} \qquad \eta' = \sum_{i \leq s} \beta^{\gamma'_i}\alpha'_i.$$

If $\eta = \eta'$, then $r = s$ and $\gamma_i = \gamma'_i$ and $\alpha_i = \alpha'_i$ for all $i \leq r$. This proves the unicity clause of Theorem 4.0.1.

Proof. We argue by contraposition. Assume the conclusion is false. We consider two cases:

1. There exists $j \leq \min\{r, s\}$ such that $\gamma_j \neq \gamma_j$ or $\alpha_j \neq \alpha'_j$.
2. $\gamma_i = \gamma'_i$ and $\alpha_i = \alpha'_i$ for all $i \leq \min\{r, s\}$, but $r \neq s$.

Case 1. Suppose that there exists $j \leq \min\{r, s\}$ for which $\gamma_j \neq \gamma'_j$ or $\alpha_j \neq \alpha'_j$. Without loss of generality we may assume that j is minimal with this property. If $\gamma_j \neq \gamma'_j$ then without loss of generality we may assume that $\gamma_j < \gamma'_j$. Hence by Lemma 4.2.1 we have

$$\sum_{j\leq i\leq r} \beta^{\gamma_i}\alpha_i < \beta^{\gamma'_j} \leq \sum_{j\leq i\leq r} \beta^{\gamma'_i}\alpha'_i.$$

Since j is minimal we have $\gamma_i = \gamma'_i$ and $\alpha_i = \alpha'_i$ for all $i < j$. Consequently

$$\begin{aligned}\eta = \sum_{i\leq r} \beta^{\gamma_i}\alpha_i &= \sum_{i<j} \beta^{\gamma_i}\alpha_i + \sum_{j\leq i\leq r} \beta^{\gamma_i}\alpha_i = \sum_{i<j} \beta^{\gamma'_i}\alpha'_i + \sum_{j\leq i\leq r} \beta^{\gamma_i}\alpha_i \\ &< \sum_{i<j} \beta^{\gamma'_i}\alpha'_i + \sum_{j\leq i\leq r} \beta^{\gamma'_i}\alpha'_i = \sum_{i\leq r} \beta^{\gamma'_i}\alpha'_i = \eta'\end{aligned}$$

and so $\eta \neq \eta'$. If $\gamma_j = \gamma'_j$ but $\alpha_j \neq \alpha'_j$ then without loss of generality we may assume that $\alpha_j < \alpha'_j$. Hence by Lemma 4.2.2 we have

$$\sum_{j\leq i\leq r} \beta^{\gamma_i}\alpha_i < \beta^{\gamma_j}\alpha'_j = \beta^{\gamma'_j}\alpha'_j \leq \sum_{j\leq i\leq r} \beta^{\gamma'_i}\alpha'_i.$$

Since j is minimal we have $\gamma_i = \gamma'_i$ and $\alpha_i = \alpha'_i$ for all $i < j$. Consequently

$$\begin{aligned}\eta = \sum_{i\leq r} \beta^{\gamma_i}\alpha_i &= \sum_{i<j} \beta^{\gamma_i}\alpha_i + \sum_{j\leq i\leq r} \beta^{\gamma_i}\alpha_i = \sum_{i<j} \beta^{\gamma'_i}\alpha'_i + \sum_{j\leq i\leq r} \beta^{\gamma_i}\alpha_i \\ &< \sum_{i<j} \beta^{\gamma'_i}\alpha'_i + \sum_{j\leq i\leq r} \beta^{\gamma'_i}\alpha'_i = \sum_{i\leq r} \beta^{\gamma'_i}\alpha'_i = \eta'\end{aligned}$$

and so $\eta \neq \eta'$.

Case 2. Suppose that $\gamma_i = \gamma'_i$ and $\alpha_i = \alpha'_i$ for all $i \leq \min\{r, s\}$ but $r \neq s$. Without loss of generality we may assume that $r < s$. Then we have

$$\eta = \sum_{i\leq r} \beta^{\gamma_i}\alpha_i = \sum_{i\leq r} \beta^{\gamma'_i}\alpha'_i < \sum_{i\leq r} \beta^{\gamma'_i}\alpha'_i + \sum_{r<i\leq s} \beta^{\gamma'_i}\alpha'_i = \sum_{i\leq s} \beta^{\gamma'_i}\alpha'_i = \eta'$$

since $0 < \alpha'_i$ for all $i \leq s$. Hence $\eta \neq \eta'$. □

From the proof of Theorem 4.2.3 above we can extract the following corollary.

Corollary 4.2.4 *The proof of Theorem 4.2.3 above yields an algorithm to compare two ordinals that have been represented with respect to a basis* $\beta > 1$ *as*

$$\eta = \sum_{i\leq r} \beta^{\gamma_i}\alpha_i \quad \textit{and} \quad \eta' = \sum_{i\leq s} \beta^{\gamma'_i}\alpha'_i.$$

First compare γ_0 *and* γ'_0*, then* α_0 *and* α'_0*, then* γ_1 *and* γ'_1*, et cetera. At the first difference, whichever ordinal has the larger component is the larger ordinal. If the components do not differ until one representation terminates before the other, then the ordinal whose representation terminates last is the larger ordinal. Finally, if both representations coincide then of course so do the ordinals themselves.*

With Theorems 4.1.3 and 4.2.3 established, Theorem 4.0.1 is now proven. We proceed to put it to use in the next section.

4.3 The Cantor Normal Form

A special case of the Basis Representation Theorem 4.0.1 is the so called *Cantor Normal Form* (see Section 2 in Altman [1]), which is simply basis representation with respect to the basis $\beta = \omega$.

Definition 4.3.1 (Cantor Normal Form) Let $\alpha > 0$ an ordinal. The *Cantor Normal Form* of α is the unique expression of the form

$$\alpha = \sum_{i \leq r} \omega^{\gamma_i} a_i = \omega^{\gamma_0} a_0 + \cdots + \omega^{\gamma_r} a_r.$$

where $r < \omega$ and $0 < a_0, \cdots, a_r < \omega$ are natural number and $\gamma_0 > \cdots > \gamma_r$ are ordinals. Theorem 4.0.1 guarantees the existence and unicity of the Cantor normal form of any nonzero ordinal. In addition to this, the ordinal 0 is also considered to be in Cantor normal form.

As mentioned when proving unicity of the basis representation, we need to treat 0 separately from all other ordinals because its Cantor normal form cannot have nonzero coefficients, while for all other ordinals we need to demand that the coefficients be nonzero to guarantee their unicity.

In the remainder of this section we state some useful facts concerning the behaviour of the Cantor normal form when performing ordinal arithmetic.

When performing ordinal addition on expressions in Cantor normal form, the main fact one should keep in mind is the fact that greater powers of ω in an ordinal sum annihilate lesser powers of ω to their left. For details, see Theorem 4.3.2 below. It is adapted from the proof of Proposition 2.2(5) in Lipparini [3].

Theorem 4.3.2 *If $\alpha < \beta$ then $\omega^\alpha + \omega^\beta = \omega^\beta$.*

Proof. We argue by transfinite induction (Theorem 2.2.2) on β.

Initial case. Suppose that $\beta = 0$. Then no $\alpha < \beta$ exists and the statement holds vacuously.

Successor case. Suppose that $\beta = S\eta$ and $\alpha < \beta$. Then $\alpha \leq \eta$, consequently we have

$$\omega^\beta \leq \omega^\alpha + \omega^\beta \leq \omega^\eta + \omega^\beta.$$

However, a straightforward computation yields the fact that

$$\omega^\eta + \omega^\beta = \omega^\eta + \omega^{S\eta} = \omega^\eta + \omega^\eta \omega = \omega^\eta (1 + \omega) = \omega^\eta \omega = \omega^{S\eta} = \omega^\beta$$

and therefore the above inequality implies that $\omega^\alpha + \omega^\beta = \omega^\beta$.

Limit case. Suppose that $\beta = \sup \beta$ and $\alpha < \beta$. Then there exists $\eta < \beta$ such that $\alpha < \eta$. Hence by the inductive hypothesis we have that $\omega^\alpha + \omega^\xi = \omega^\xi$ for all $\eta < \xi < \beta$. It follows that

$$\omega^\alpha + \omega^\beta = \omega^\alpha + (\sup_{\xi < \beta} \omega^\xi) = \omega^\alpha + (\sup_{\eta < \xi < \beta} \omega^\xi) = \sup_{\eta < \xi < \beta} (\omega^\alpha + \omega^\xi) = \sup_{\eta < \xi < \beta} \omega^\xi = \omega^\beta. \qquad \square$$

The following definition is from Section 2 in Lipparini [3].

Definition 4.3.3 Given an ordinal

$$\alpha = \omega^{\alpha_0} a_0 + \cdots + \omega^{\alpha_r} a_r$$

in Cantor normal form, we define the ordinal $\alpha^{\restriction \eta}$, pronounced α *truncated at the η^{th} exponent of* ω, as

$$\alpha^{\restriction \eta} := \omega^{\alpha_0} a_0 + \cdots + \omega^{\alpha_s} a_s$$

where s is the greatest index such that $\alpha_s \geq \eta$. In the case that $\alpha < \omega^\eta$ we simply define $\alpha^{\restriction \eta} := 0$.

With this definition established, Theorem 4.3.2 has as immediate consequence Corollary 4.3.4 below, which is Proposition 2.2(5) in Lipparini [3].

Corollary 4.3.4 *For ordinals α and β we have $\alpha + \beta = \alpha^{\restriction \eta} + \beta$, where η is the leading exponent in the Cantor normal form of β.*

When performing ordinal multiplication on expressions in Cantor normal form, Theorem 4.3.5 below aids in simplifying computations. It is adapted from Theorem 8.46 in Takeuti & Zaring [6], although the proof is our own.

Theorem 4.3.5 *Given an ordinal*

$$\alpha = \omega^{\alpha_0} a_0 + \cdots + \omega^{\alpha_r} a_r > 0$$

in Cantor normal form, for every $0 < n < \omega$ and $0 < \gamma$ we have

$$\alpha n = \omega^{\alpha_0} a_0 n + \omega^{\alpha_1} a_1 + \cdots + \omega^{\alpha_r} a_r \qquad \text{and} \qquad \alpha \omega^\gamma = \omega^{\alpha_0 + \gamma}.$$

Proof. We prove the first claim via induction on n.

Initial case. Suppose that $n = 1$. Then the statement reduces to the tautology $\alpha = \alpha$.

Successor case. Suppose that $n = Sm$ and that the statement holds for m. Then by Corollary 4.3.4 above we have

$$\begin{aligned}\alpha n = \alpha(Sm) = \alpha m + \alpha &= (\omega^{\alpha_0}a_0 m + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r) + (\omega^{\alpha_0}a_0 + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r)\\ &= \omega^{\alpha_0}a_0 m + \omega^{\alpha_0}a_0 + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r = \omega^{\alpha_0}a_0(Sm) + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r\\ &= \omega^{\alpha_0}a_0 n + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r.\end{aligned}$$

Thus the first claim holds for all $0 < n < \omega$.

We now prove the second claim via transfinite induction on γ.

Initial case. Suppose that $\gamma = 1$. Note that

$$\omega^{\alpha_0}a_0 n \leq \omega^{\alpha_0}a_0 n + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r < \omega^{\alpha_0}a_0(Sn)$$

by Lemma 4.2.2. Consequently we have

$$\sup_{n<\omega}(\omega^{\alpha_0}a_0 n) \leq \sup_{n<\omega}(\omega^{\alpha_0}a_0 n + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r) \leq \sup_{n<\omega}(\omega^{\alpha_0}a_0(Sn)).$$

But $\sup_{n<\omega}(\omega^{\alpha_0}a_0(Sn)) = \sup_{n<\omega}(\omega^{\alpha_0}a_0 n)$, therefore the above inequality implies that

$$\sup_{n<\omega}(\omega^{\alpha_0}a_0 n + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r) = \sup_{n<\omega}(\omega^{\alpha_0}a_0 n).$$

Thus we have

$$\begin{aligned}\alpha\omega^{\gamma} = \alpha\omega = \sup_{n<\omega}(\alpha n) &= \sup_{n<\omega}(\omega^{\alpha_0}a_0 n + \omega^{\alpha_1}a_1 + \cdots + \omega^{\alpha_r}a_r) = \sup_{n<\omega}(\omega^{\alpha_0}a_0 n)\\ &= \omega^{\alpha_0}(\sup_{n<\omega} a_0 n) = \omega^{\alpha_0}\omega = \omega^{S\alpha_0} = \omega^{\alpha_0+1} = \omega^{\alpha_0+\gamma}.\end{aligned}$$

Successor case. Suppose that $\gamma = S\eta$ and that $\alpha\omega^{\eta} = \omega^{\alpha_0+\eta}$. Then we have

$$\alpha\omega^{\gamma} = \alpha\omega^{S\eta} = \alpha\omega^{\eta}\omega = \omega^{\alpha_0+\eta}\omega = \omega^{S(\alpha_0+\eta)} = \omega^{\alpha_0+(S\eta)} = \omega^{\alpha_0+\gamma}.$$

Limit case. Suppose that $\gamma = \sup\gamma \neq 0$ and that $\alpha\omega^{\eta} = \omega^{\alpha_0+\eta}$ for all $\eta < \gamma$. Then we have

$$\alpha\omega^{\gamma} = \alpha\omega^{\sup_{\eta<\gamma}(\eta)} = \alpha(\sup_{\eta<\gamma}\omega^{\eta}) = \sup_{\eta<\gamma}(\alpha\omega^{\eta}) = \sup_{\eta<\gamma}(\omega^{\alpha_0+\eta}) = \omega^{\sup_{\eta<\gamma}(\alpha_0+\eta)} = \omega^{\alpha_0+\sup_{\eta<\gamma}(\eta)} = \omega^{\alpha_0+\gamma}. \quad \square$$

Example 4.3.6 With these results in hand, we can quickly perform some elementary ordinal arithmetic, both addition

$$(\omega^4 2 + \omega^2 1 + \omega^0 3) + (\omega^3 5 + \omega^1 7) = \omega^4 2 + \omega^3 5 + \omega^1 7$$

and multiplication

$$\begin{aligned}(\omega^4 2 + \omega^2 1 + \omega^0 3)(\omega^3 5 + \omega^1 7) &= (\omega^4 2 + \omega^2 1 + \omega^0 3)\omega^3 5 + (\omega^4 2 + \omega^2 1 + \omega^0 3)\omega^1 7\\ &= \omega^{4+3}5 + \omega^{4+1}7 = \omega^7 5 + \omega^5 7.\end{aligned}$$

Chapter 5

Natural arithmetic

The Cantor normal form allows us to express any ordinal as a "polynomial in ω." This leads us to define the operation of *natural addition* for ordinals, which was first considered by Gerhard Hessenberg in 1906 [2] and as such is also referred to as *Hessenberg addition*. Natural addition of ordinals expressed in Cantor normal form mimics addition of polynomials, with ω playing the role of the variable. Likewise we will define *natural multiplication* or *Hessenberg multiplication*, which behaves on ordinals in Cantor normal form in a manner analogous to multiplication of polynomials. We state and prove several properties of these natural operations.

5.1 Natural addition

The following definition is Definition 2.1 in Altman [1].

Definition 5.1.1 (Natural addition) Given ordinals α and β, we define their *natural sum* as follows. Let r and $a_0, \cdots, a_r, b_0, \cdots, b_r$ be natural numbers and let $\gamma_0 > \cdots > \gamma_r$ be ordinals such that

$$\alpha = \sum_{i \leq r} \omega^{\gamma_i} a_i = \omega^{\gamma_0} a_0 + \cdots + \omega^{\gamma_r} a_r \quad \text{and} \quad \beta = \sum_{i \leq r} \omega^{\gamma_i} b_i = \omega^{\gamma_0} b_0 + \cdots + \omega^{\gamma_r} b_r.$$

Then the natural sum of α and β is defined as

$$\alpha \oplus \beta := \sum_{i \leq r} \omega^{\gamma_i}(a_i + b_i) = \omega^{\gamma_0}(a_0 + b_0) + \cdots + \omega^{\gamma_r}(a_r + b_r).$$

The existence and unicity of the Cantor normal form for all ordinals guarantees that natural addition is well-defined. It may be necessary to pad the Cantor normal form with terms with coefficient zero to make the indices match. See the example below.

Example 5.1.2 When computing the natural sum

$$(\omega 2 + 3) \oplus (\omega^2 4 + \omega 5 + 6)$$

we pad the left argument with the null term $\omega^2 0$, and sum linearly:

$$\begin{aligned}(\omega 2 + 3) \oplus (\omega^2 4 + \omega 5 + 6) &= (\omega^2 0 + \omega^1 2 + \omega^0 3) \oplus (\omega^2 4 + \omega^1 5 + \omega^0 6) \\ &= \omega^2(0 + 4) + \omega^1(2 + 5) + \omega^0(3 + 6) \\ &= \omega^2 4 + \omega 7 + 9.\end{aligned}$$

When computing the natural sum, all arithmetic is reduced to arithmetic on natural numbers (the coefficients in the Cantor normal form). As such, it is easy to see that when we restrict natural addition to the natural numbers, that we again recover conventional addition. Further, natural addition satisfies some very nice algebraic properties.

The following theorem is part (1) of Proposition 2.2 in Lipparini [3].

Theorem 5.1.3 *Natural addition is commutative, associative, both left and right cancellative, and strictly increasing in both arguments. I.e. for all ordinals α, β, and γ we have the following:*

1. $\alpha \oplus \beta = \beta \oplus \alpha$

2. $(\alpha \oplus \beta) \oplus \gamma = \alpha \oplus (\beta \oplus \gamma)$
3. $\alpha \oplus \beta = \alpha \oplus \gamma \to \beta = \gamma$
4. $\beta < \gamma \to \alpha \oplus \beta < \alpha \oplus \gamma$.

Note that if commutativity is assumed, left cancellativity implies right cancellativity and vice versa. Similarly, being strictly increasing in either argument implies being strictly increasing in the other argument.

Proof. Commutativity, associativity, and cancellativity follow from expressing all ordinals involved in Cantor normal form and applying commutativity, associativity and cancellativity of addition of natural numbers. To prove that natural addition is strictly increasing, apply Corollary 4.2.4 to reduce the comparison of ordinals to the comparison of the coefficients of their Cantor normal forms. □

Commutativity of natural addition means that from a purely algebraic point of view it appears to be more nicely behaved than ordinal addition. However, the fact that for example

$$1 \oplus \omega = 1 \oplus \sup_{n<\omega} n \neq \sup_{n<\omega}(1 \oplus n) = \omega$$

means that natural addition is not *continuous* (in either argument): it does not commute with suprema, which correspond to the taking of limits in the order topology of Ord. Ordinal addition is at least continuous in the addend, if not in the augend. Hence from a topological point of view ordinal addition is the "nicer" operation.

Since natural addition is associative, we can unambiguously define the natural sum of any finite sequence of ordinals.

Definition 5.1.4 Given a natural number $n < \omega$ and a finite sequence $(\alpha_i)_{i<n}$ of ordinals, we recursively define the *natural sum* of $(\alpha_i)_{i<n}$ as

$$\bigoplus_{i<n} \alpha_i := \begin{cases} 0 & \text{if } n = 0 \\ \left(\bigoplus_{i<m} \alpha_i \right) \oplus \alpha_n & \text{if } n = Sm. \end{cases}$$

For a nonempty sequence $(\beta_i)_{i<n+1}$ we will also occasionally write

$$\bigoplus_{i \leq n} \beta_i \qquad \text{or} \qquad \beta_0 \oplus \cdots \oplus \beta_n$$

instead of $\bigoplus_{i<n+1} \beta_i$.

Note that because natural addition is commutative, any permutation of a finite sequence has the same natural sum as the original sequence. Consequently we can unambiguously interpret the natural sum of any finite set of ordinals. We will use this finite natural summation to define natural multiplication in the next section.

5.2 Natural multiplication

The following definition is Definition 2.2 in Altman [1].

Definition 5.2.1 (Natural multiplication) Given ordinals α and β, we define their *natural product* as follows. Let $r, s < \omega$ and $0 < a_0, \cdots, a_r, b_0, \cdots, b_s < \omega$ be natural numbers and let $\alpha_0 > \cdots > \alpha_r$ and $\beta_0 > \cdots > \beta_s$ be ordinals such that

$$\alpha = \sum_{i \leq r} \omega^{\alpha_i} a_i = \omega^{\alpha_0} a_0 + \cdots + \omega^{\alpha_r} a_r \quad \text{and} \quad \beta = \sum_{j \leq s} \omega^{\beta_j} b_j = \omega^{\beta_0} b_0 + \cdots + \omega^{\beta_s} b_s.$$

Then the natural product of α and β is defined as

$$\alpha \otimes \beta := \bigoplus_{\substack{i \leq r \\ j \leq s}} \omega^{\alpha_i \oplus \beta_j} a_i b_j.$$

As with natural addition, the existence of a Cantor normal form for all ordinals guarantees that natural multiplication is well-defined.

Note that while in the definition of natural multiplication we demand that the coefficients a_i and b_j be nonzero, we can just as easily drop this requirement: this just introduces some additional zero terms in the final summation.

Example 5.2.2 We calculate some simple natural products.

$$2 \otimes (\omega + 1) = (\omega^0 2) \otimes (\omega^1 1 + \omega^0 1) = \omega^{0 \oplus 1}(2 \cdot 1) \oplus \omega^{0 \oplus 0}(2 \cdot 1) = \omega^1 2 \oplus \omega^0 2 = \omega 2 + 2$$

$$\omega^3 \otimes (\omega^\omega + \omega 2) = (\omega^3 1) \otimes (\omega^\omega 1 + \omega^1 2) = \omega^{3 \oplus \omega}(1 \cdot 1) \oplus \omega^{3 \oplus 1}(1 \cdot 2) = \omega^{\omega + 3} + \omega^4 2$$

The following theorem is Lemma 2.5 in Altman [1]. The proof is our own.

Theorem 5.2.3 *Natural multiplication is commutative and associative and distributes over natural addition. I.e. for all ordinals α, β, γ we have*

1. $\alpha \otimes \beta = \beta \otimes \alpha$
2. $\alpha \otimes (\beta \otimes \gamma) = (\alpha \otimes \beta) \otimes \gamma$
3. $\alpha \otimes (\beta \oplus \gamma) = (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma)$

Note that if commutativity is assumed, then left distributivity implies right distributivity and vice versa.

Proof. Commutativity and associativity of natural multiplication follow readily from commutativity and associativity of both natural addition of ordinals and addition of natural numbers.

For distributivity, let $r < \omega$ and $a_0, \cdots a_r, b_0, \cdots, b_r, c_0, \cdots, c_r < \omega$ be natural numbers and let $\xi_0 > \cdots > \xi_r$ be ordinals such that

$$\alpha = \sum_{i \le r} \omega^{\xi_i} a_i \quad \text{and} \quad \beta = \sum_{i \le r} \omega^{\xi_i} b_i \quad \text{and} \quad \gamma = \sum_{i \le r} \omega^{\xi_i} c_i.$$

Then we have

$$\begin{aligned} \alpha \otimes (\beta \oplus \gamma) &= \left(\sum_{i \le r} \omega^{\xi_i} a_i \right) \otimes \left(\left(\sum_{i \le r} \omega^{\xi_i} b_i \right) \oplus \left(\sum_{i \le r} \omega^{\xi_i} c_i \right) \right) = \left(\sum_{i \le r} \omega^{\xi_i} a_i \right) \otimes \left(\sum_{i \le r} \omega^{\xi_i} (b_i + c_i) \right) \\ &= \bigoplus_{i,j \le r} \omega^{\xi_i \oplus \xi_j} a_i (b_j + c_j) \end{aligned}$$

by the definitions of the natural arithmetical operations. Now using the distributivity of ordinal multiplication over ordinal addition (Theorem 3.2.5) we find

$$\omega^{\xi_i \oplus \xi_j} a_i (b_j + c_j) = \omega^{\xi_i \oplus \xi_j} (a_i b_j + a_i c_j) = \omega^{\xi_i \oplus \xi_j} a_i b_j \oplus \omega^{\xi_i \oplus \xi_j} a_i c_j.$$

Using this equality and the commutativity of natural addition we conclude that

$$\begin{aligned} \bigoplus_{i,j \le r} \omega^{\xi_i \oplus \xi_j} a_i (b_j + c_j) &= \bigoplus_{i,j \le r} \left(\omega^{\xi_i \oplus \xi_j} a_i b_j \oplus \omega^{\xi_i \oplus \xi_j} a_i c_j \right) \\ &= \left(\bigoplus_{i,j \le r} \omega^{\xi_i \oplus \xi_j} a_i b_j \right) \oplus \left(\bigoplus_{i,j \le r} \omega^{\xi_i \oplus \xi_j} a_i c_j \right) \\ &= \left(\left(\sum_{i \le r} \omega^{\xi_i} a_i \right) \otimes \left(\sum_{i \le r} \omega^{\xi_i} b_i \right) \right) \oplus \left(\left(\sum_{i \le r} \omega^{\xi_i} a_i \right) \otimes \left(\sum_{i \le r} \omega^{\xi_i} c_i \right) \right) \\ &= (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma) \end{aligned}$$

and therefore

$$\alpha \otimes (\beta \oplus \gamma) = (\alpha \otimes \beta) \oplus (\alpha \otimes \gamma).$$

□

Similar to the situation we had with natural addition, we may remark that the commutativity of natural multiplication has come at the price of right-continuity when we compare it to ordinal multiplication. The algebraist in us revels in the fact that

$$2 \otimes \omega = \omega \otimes 2 = \omega 2$$

while our inner topologist is dismayed to find that

$$2 \otimes \omega = 2 \otimes \sup_{n < \omega} n \neq \sup_{n < \omega} (2 \otimes n) = \omega.$$

Appendix A

Set theory

The theory of ordinals is an elementary part of the theory of sets. As such, we sometimes need to appeal fairly directly to the axioms of Zermelo-Fraenkel set theory when proving certain claims about ordinals. In this appendix we in particular prove Fact 2.1.3 and provide a construction of ω, the least infinite ordinal.

A.1 The axioms of Zermelo-Fraenkel set theory

Below we present a list of the eight axioms of Zermelo-Fraenkel set theory, slightly adapted from how they appear on pages 105 and 106 of Moerdijk & Van Oosten [4]:

1. *Axiom of extensionality*
   $\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$
   Two sets are equal whenever they have the same members.

2. *Axiom of pairing*
   $\forall x \forall y \exists z \forall w (w \in z \leftrightarrow (w = x \vee w = y))$
   For every set x and set y, there is a set $\{x, y\}$ whose members are exactly x and y.

3. *Axiom of union*
   $\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w))$
   For every set x, there is a set $\bigcup x$ whose members are exactly the sets that are members of members of x.

4. *Axiom of power set*
   $\forall x \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in x))$
   For every set x, there exists a set $\mathcal{P}(x)$ whose members are exactly the subsets of x.

5. *Axiom scheme of separation*
   For every formula ϕ not containing the variable y and with at most the free variable z, we have an axiom
   $\forall x \exists y \forall z (z \in y \leftrightarrow (z \in x \wedge \phi))$
   For every set x and property ϕ, there exists a set $\{z \in x \mid \phi(z)\}$ whose members are exactly the members of x satisfying ϕ.

6. *Axiom scheme of replacement*
   For every formula ϕ not containing the variable y and with two free variables, we have an axiom $\forall a \exists b \forall c (\phi(a, c) \leftrightarrow c = b) \quad \rightarrow \quad \forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge \phi(w, z)))$
   The condition above expresses that ϕ defines an operation on sets (see Section B.2). The conclusion states that for every set x, the image of x under the operation defined by ϕ is a set.

7. *Axiom of infinity*
   Assuming the existence of any set (otherwise set theory would be quite dull), by applying the axiom of separation with a formula no set can satisfy (e.g. $\neg(z = z)$) we find that there exists a set with no elements. This set is unique by the axiom of extensionality, and we denote it $\varnothing$. Further, for any set y, by the axiom of pairing we have the set $\{y\} := \{y, y\}$. Applying the axiom of pairing to y and $\{y\}$ yields the set $\{y, \{y\}\}$, and applying the axiom of union to this third set we obtain $y \cup \{y\} := \bigcup \{y, \{y\}\}$. All these sets are unique by the axiom of extensionality. With this notation, the axiom of infinity becomes
   $\exists x (\varnothing \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))$
   There exists a set with infinitely many elements.

8. *Axiom of regularity*
   $\forall x(\neg(x = \varnothing) \to \exists y(y \in x \wedge \forall z \neg(z \in x \wedge z \in y)))$
   Every nonempty set admits a member from which it is disjoint.

A.2 Proof of Fact 2.1.3

We now deliver a proof of Fact 2.1.3 using the axiom of regularity.

Theorem A.2.1 (Fact 2.1.3) *There does not exist a nonempty finite sequence $(x_i)_{i=0}^n$ of sets such that $x_i \in x_{i+1}$ for all $0 \leq i < n$ and $x_n \in x_0$. In particular, no set is an element of itself.*

Proof. Suppose we had such a sequence $(x_i)_{i=0}^n$. Then by repeated application of the axioms of paring and union we find that

$$X := \{x_i\}_{i=0}^n$$

is a nonempty set. Hence by the axiom of regularity there must exist $x_j \in X$ such that $x_j \cap X = \varnothing$. But by assumption we have that $x_i \in x_{i+1} \cap X$ for $i < n$, and $x_n \in x_0 \cap X$. Thus no such x_j can exists, producing a contradiction. □

A.3 The axiomatic construction of ω

The simplest examples of ordinals are sets such as 0, 1, and 2 which contain finitely many elements and which are easily verified to be ordinals. However, the study of ordinals would be nowhere near as interesting as it is if there did not also exist infinite ordinals. In this subsection we will construct the least infinite ordinal, which is typically denoted ω.

We begin with a definition.

Definition A.3.1 We say that a set x is an *inductive set* if and only if it satisfies the property

$$\Phi(x) := \varnothing \in x \wedge \forall y \in x\,((y \cup \{y\}) \in x).$$

With this terminology, the axiom of infinity (see Section A.1) can be succinctly reformulated in words as follows:

There exists an inductive set.

A fundamental property of inductive sets is that the intersection of a collection of inductive sets is again an inductive set. See Theorem A.3.2 below.

Theorem A.3.2 *Let x be a set whose members are inductive sets. Then the set*

$$\bigcap x := \{z \in x \mid \forall y \in x(z \in y)\}$$

which exists by the axiom scheme of separation, is an inductive set.

Proof. Let $y \in x$. Then y is an inductive set, therefore $\varnothing \in y$. Now since $\varnothing$ is a member of every member of x it follows that $\varnothing \in \bigcap x$.

Let $z \in \bigcap x$. Let $y \in x$. Then by the definition of $\bigcap x$ we have $z \in y$. Hence since y is inductive it follows that $z \cup \{z\} \in y$. Since $z \cup \{z\}$ is a member of every member of x we conclude that $z \cup \{z\} \in \bigcap x$. Consequently $\bigcap x$ is an inductive set. □

This leads us to the following definition of ω, slightly adapted from how it appears on page 109 in Moerdijk & Van Oosten [4].

Definition A.3.3 By the axiom of infinity, let x be any inductive set. We define the set ω_x using the axiom scheme of separation as follows:

$$\omega_x := \{z \in x \mid \forall a \subseteq x(\Phi(a) \to z \in a)\}.$$

In words: ω_x is the intersection of all inductive subsets of x. Note that by Theorem A.3.2 the set ω_x is itself an inductive set.

The subscripted x in ω_x suggests that the above construction depends on the initial choice of inductive set x. This is not actually the case, as Theorem A.3.4 below proves.

Theorem A.3.4 *Let x and y be inductive sets. Then $\omega_x = \omega_y$.*

Proof. Let $z \in \omega_x$. Let $b \subseteq y$ be an inductive subset. Then $x \cap b$ is the intersection of two inductive sets and is therefore itself an inductive set by Theorem A.3.2. Now since $x \cap b \subseteq x$ is an inductive subset of x we have $z \in x \cap b$ by the definition of ω_x. Thus $z \in b$. Now since z is a member of every inductive subset of y it follows that $z \in \omega_y$ by the definition of ω_y. Consequently every member of ω_x is also a member of ω_y and we have $\omega_x \subseteq \omega_y$. By symmetry, we conclude that $\omega_y \subseteq \omega_x$ as well and so the statement follows. □

We are thus justified in ommitting the subscripted x when we define $\omega := \omega_x$. We call ω the *minimal inductive set*, because it is a subset of any inductive set: if x is any inductive set, then $\omega = \omega_x \subseteq x$. It follows that ω is her own sole inductive subset. See Theorem A.3.5.

Theorem A.3.5 *Let $a \subseteq \omega$ be an inductive subset. Then $a = \omega$.*

Proof. Since a is an inductive set, $\omega = \omega_a \subseteq a$. Since $a \subseteq \omega$ we conclude that $a = \omega$. □

Theorem A.3.5 allows us to easily prove some core properties of ω, such as Theorem A.3.6 below.

Theorem A.3.6 *The set ω is transitive.*

Proof. Consider the set $a := \{x \in \omega \mid x \subseteq \omega\}$. We prove that is it an inductive set, and therefore $a = \omega$ by Theorem A.3.5. This is turn implies that ω is a transitive set.

Since ω is an inductive set, $\varnothing \in \omega$. The statement $\varnothing \subseteq \omega$ is vacuously true. Hence $\varnothing \in a$.

Suppose that $x \in a$. Then $x \in \omega$ and $x \subseteq \omega$, therefore $x \cup \{x\} \subseteq \omega$. Since $x \in \omega$ and ω is inductive we also have $x \cup \{x\} \in \omega$. Thus $x \cup \{x\} \in a$ and so a is an inductive set. □

Theorem A.3.6 above together with Theorem A.3.7 below will serve to prove that ω is an ordinal, in accordance with our desires.

Theorem A.3.7 *Every member of ω is either the empty ordinal or a successor ordinal.*

Proof. Let $\mathrm{ord}(x)$ be the formula expressing that x is an ordinal: see Section B.1. Consider the set

$$a := \{x \in \omega \mid x = \varnothing \vee \exists y(\mathrm{ord}(y) \wedge x = y \cup \{y\})\}.$$

We show that a is an inductive set, which by Theorem A.3.5 implies $a = \omega$.

The empty set $\varnothing$ lies in ω since ω is inductive. It follows that $\varnothing \in a$.

Let $x \in a$. Then $x \in \omega$, hence $x \cup \{x\} \in \omega$ since ω is inductive. Since $x \in a$ either $x = \varnothing$ or $x = y \cup \{y\}$ for some ordinal y. The empty set is vacuously an ordinal, and $y \cup \{y\} = Sy$ is an ordinal by Theorem 2.3.2. Hence x is an ordinal in either case. Therefore, again by Theorem 2.3.2, we know that $x \cup \{x\}$ is the successor of x, in particular it is a successor ordinal. Therefore $x \cup \{x\} \in a$. □

We are now ready to prove the foremost result of this section, Theorem A.3.8 below.

Theorem A.3.8 *The set ω is the least nonzero limit ordinal.*

Proof. We first prove that ω is an ordinal. Theorem A.3.6 established that ω is a transitive set, and by Theorem A.3.7 we know that ω is a set of ordinals. As a consequence of Theorem 2.2.1 every set of ordinals is well-ordered by $\in$. Consequently ω is an ordinal.

If ω is not a limit ordinal, then it is a successor ordinal by Theorem 2.3.6. Let $\omega = S\alpha$. Then $\alpha \in \omega$, hence $\alpha \cup \{\alpha\} \in \omega$ since ω is an inductive set. But $\alpha \cup \{\alpha\} = S\alpha$ by Theorem 2.3.2, and so we have

$$\omega = S\alpha = \alpha \cup \{\alpha\} \in \omega$$

which contradicts Theorem A.2.1. Thus we conclude that ω is a limit ordinal.

Now if $\alpha < \omega$ is a limit ordinal, then by Theorem A.3.7 we have $\alpha = \varnothing$. Thus ω is the least nonzero limit ordinal. □

In this section, we have constructed ω and proven that it is the least nonzero limit ordinal. From this we would like to deduce that ω is the least infinite ordinal. However, we are unable to do so without specifying what exactly we *mean* (in the formal language of set theory) when we say “infinite.” Unfortunately, a proper discussion about the definition of (in)finity falls outside the scope of this document. In this document we will consider a set *finite* if it is in bijective correspondence with a member of ω. This is not very satisfying, as it makes the statement “ω is the least infinite ordinal” a tautology. However, as the focus of this document is on ordinal arithmetic this definition will suffice for our purposes.

Appendix B

Classes and class functions

In this chapter we discuss classes and class functions, two concepts that, while not central to this document, are certainly more than tangential to its subject matter.

Our definition of a class as it appears in this chapter is taken from page 106 in Moerdijk & Van Oosten [4]. The concept of a class function is adapted from the concept of a formula that defines an operation on sets as is appear on page 109 in Moerdijk & Van Oosten [4]. The statement of Theorem B.3.1 is that of Theorem 4.2.3 in Moerdijk & Van Oosten [4]; for the idea of its proof we thank Dr. Van Oosten specifically.

B.1 Classes

In this section we will frequently use the word *property*, which the definition below now gives a precise meaning.

Definition B.1.1 A *property* is a formula with a single free variable.

For a proper introduction to formulas, we refer to Moerdijk & Van Oosten [4].

A *class* is an informal collection of sets. Specifically, a class is the collection of all sets satisfying a given property. Every set is a class: the set x is the class of all sets y satisfying the property $y \in x$. Conversely, there are classes which are not sets: these are called *proper classes*. The prime example of a proper class is the class of all sets, which is characterised by (among others) the property $x = x$. It is not a set, because if it were then it would be a member of itself, contradicting the axiom of regularity. This is the famous *Russel's Paradox* (see also the introduction to Chapter 4 in Moerdijk & Van Oosten [4]).

The primary proper class we concern ourselves with in this document is Ord, the class of all ordinals. It is characterised by the property that expresses that x is an ordinal, i.e. that x is both a transitive set and a well-order under $\in$. The property $\phi(x)$ that expresses that x is transitive is this one:

$$\forall y \forall z((y \in x \land z \in y) \to z \in x).$$

The property $\psi(x)$ that x is a well-order under $\in$ can be formulated as

$$\forall y((\forall u(u \in y \to u \in x) \land \exists v(v \in y))) \to \exists z(z \in y \land \forall w(w \in y \to (z = w \lor z \in w))))$$

or more concisely (i.e. using notational abbreviation) as

$$\forall y \subseteq x(y \neq \varnothing \to \exists z \in y \forall w \in y(z = w \lor z \in w)).$$

We can then define the property $\mathrm{ord}(x)$ as the coniunction $\phi(x) \land \psi(x)$, such that the class characterised by ord is precisely Ord. For any class C characterised by a property χ we may write (with a slight abuse of notation) $x \in C$ for $\chi(x)$; in particular when we write $x \in \mathrm{Ord}$ we mean $\mathrm{ord}(x)$.

B.2 Class functions

Like classes, class functions are informal objects characterised by a formula of a specific type. For classes this formula was a property; for class functions it is a formula that *defines an operation*.

Definition B.2.1 A formula $\phi(x, y)$ with two free variables is said to *define an operation on sets* if and only if

$$\forall x \exists y \forall z(\phi(x, z) \leftrightarrow z = y)$$

holds. Expanding on this, a formula $\phi(x, y)$ is said to *define an operation on the class* C if and only if

$$\forall x(x \in C \to \exists y \forall z(\phi(x, z) \leftrightarrow z = y))$$

holds. Finally, a formula $\phi(x, y)$ is said to *define an operation from the class* C *to the class* D if and only if

$$\forall x(x \in C \to \exists y(y \in D \wedge \forall z(\phi(x, z) \leftrightarrow z = y)))$$

holds.

A class function F from the class C to the class D is characterised by a formula ϕ that defines an operation from C to D; we think of it as the (informal) operation that assigns to a set $x \in C$ the unique set $F(x) \in D$ such that $\phi(x, F(x))$ holds.

B.3 Transfinite recursion

In this section we prove the theorem of transfinite recursion, which was used to define the arithmetic operations of addition, multiplication, and exponentiation for ordinals. While transfinite induction proves that a property holds for all ordinals, transfinite recursion constructs a sequence of objects indexed by the ordinals. Such a sequence is called a *transfinite sequence*. While an "ordinary" sequence indexed by the natural numbers is characterised by a function from the natural numbers into whichever set the sequence lies in, a transfinite sequence is characterised by a class function from the class of all ordinals into the class of all sets (or some smaller class).

Theorem B.3.1 (Transfinite Recursion) *Let G be a class function from the class of all sets to itself. There there exists a unique class function F from Ord to the class of all sets satisfying*

$$F(\alpha) = G(\{F(\beta) \mid \beta < \alpha\}) \tag{B.1}$$

for all $\alpha \in$ Ord.

Proof. We first prove a more general unicity statement relating to property (B.1). Suppose H and H' are two class functions, both of whose domains contain all ordinals no greater than some ordinal α and both of which satisfy property (B.1) for all ordinals no greater than α. Suppose they do not agree on all ordinals no greater than α. Then since Ord is a well-ordered class (Theorem 2.2.1) there exists a least ordinal $\beta \leq \alpha$ such that $H(\beta) \neq H'(\beta)$. Since β is minimal we have $H(\gamma) = H'(\gamma)$ for all $\gamma < \beta$, therefore

$$H(\beta) = G(\{H(\gamma) \mid \gamma < \beta\}) = G(\{H'(\gamma) \mid \gamma < \beta\}) = H'(\beta)$$

which is a contradiction. Thus H and H' agree on all ordinals no greater than α. In particular if H and H' are defined on all ordinals, then they agree on all ordinals.

We now employ transfinite induction (Theorem 2.2.2) to prove that for every ordinal α, there exists a unique class function F_α defined on exactly all ordinals no greater than α mapping into the class of all sets satisfying (B.1) on its domain.

Firstly we define F_0 by $F_0(0) := G(0)$. Then we have

$$F_0(0) = G(0) = G(\varnothing) = G(\{F_0(\gamma) \mid \gamma < 0\})$$

and so F_0 satisfies (B.1). It is clearly unique with this property, since its domain is a singleton set.

Now suppose for all $\beta < \alpha$ that F_β has been defined and satisfies the stated property. Then we define F_α by

$$F_\alpha(\beta) = \begin{cases} F_\beta(\beta) & \text{if } \beta < \alpha \\ G(\{F_\gamma(\gamma) \mid \gamma < \alpha\}) & \text{if } \beta = \alpha. \end{cases}$$

We claim that F_α satisfies the stated property.

To see this, first note that if $\gamma < \beta < \alpha$ then F_β and F_γ are defined and satisfy property (B.1) on all ordinals no greater than γ, hence they agree on all ordinals no greater than γ.

Now for all $\beta < \alpha$ we have

$$F_\alpha(\beta) = F_\beta(\beta) = G(\{F_\beta(\gamma) \mid \gamma < \beta\}) = G(\{F_\gamma(\gamma) \mid \gamma < \beta\}) = G(\{F_\alpha(\gamma) \mid \gamma < \beta\}),$$

furthermore

$$F_\alpha(\alpha) = G(\{F_\gamma(\gamma) \mid \gamma < \alpha\}) = G(\{F_\alpha(\gamma) \mid \gamma < \alpha\})$$

and so F_α is defined on exactly the ordinals no greater than α and satisfies property (B.1) on its domain. It is therefore unique, by the unicity statement above.

By transfinite induction, F_α exists, satisfies property (B.1), and is unique for all ordinals α.

We now define a class function F on all ordinals by

$$F(\alpha) := F_\alpha(\alpha).$$

Note that for any ordinal α we have

$$F(\alpha) = F_\alpha(\alpha) = G(\{F_\alpha(\beta) \mid \beta < \alpha\}) = G(\{F_\beta(\beta) \mid \beta < \alpha\}) = G(\{F(\beta) \mid \beta < \alpha\})$$

and so F satisfies property (B.1) for all ordinals, and is therefore unique by the unicity statement above. $\square$

As an example application of Theorem B.3.1 above, we use it to define ordinal addition. We will apply transfinite resursion on the addend, that is to say, we define a class function taking the ordinal β to the ordinal $\alpha + \beta$ for a fixed ordinal α. The class function G we use is

$$G(x) := \alpha \cup \bigcup \{y \cup \{y\} \mid y \in x\}.$$

From this definition of addition it can be proved that $\alpha + \beta$ is an ordinal, that $\alpha + \gamma < \alpha + \beta$ whenever $\gamma < \beta$, and that $\alpha + \beta$ takes on exactly the values specified in Definition 3.1.1. For example,

$$\alpha + 0 := F(0) = G(\{F(\gamma) \mid \gamma < 0\}) = G(\varnothing) = \alpha \cup \bigcup \{y \cup \{y\} \mid y \in \varnothing\} = \alpha \cup \varnothing = \alpha.$$

Closing word

If time and other resources had permitted it, this document would have been twice as long. We regret that we could not delve deeper into the world of ordinal- and natural arithmetic, to highlight further differences between the two. In particular we would have liked to have been able to discuss the infinitary variants of both arithmetics, as is done in Altman [1] and Lipparini [3]. Transfinite sums and products illustrate very well exactly what problems a lack of contuinity causes.

Likewise, in Appendix A we very quickly table the discussion of what exactly we call a “finite set” in favour of focusing on the newly constructed ordinal ω and how it interacts with ordinal arithmetic. We would have been delighted to study different definitions of finity and the relations between them, as well as the possible presence of a nonstandard element in ω.

Nevertheless, we have enjoyed the construction of this document and hope that you, the reader, have derived some pleasure from its consumption. We would also like to take this opportunity to thank Dr. Van Oosten for supervising this project.

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