A Lie group integrator for the Landau-Lifshitz equation

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November 10, 2019

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1 Introduction

The Landau-Lifshitz equation describes the magnetization of a ferromagnetic material. The magnetization is described by a time dependent vector function on $\mathcal{D} \times \mathbb{R}$, where $\mathcal{D} = \mathbb{R}^d$ for d = 1, 2 or 3, depending on the number of spatial dimensions used for the problem at hand.

The Landau-Lifshitz equation is a Hamiltonian partial differential equation with a Lie-Poisson structure [8]. Our goal is to numerically solve the Landau-Lifshitz equation given some initial conditions. When choosing such a numerical method it is desireable that conserved quantities, like the total amount of energy of the system, remain conserved as much a possible.

When it comes to choosing a time discretization, there is a strong preference for symplectic methods [12]. For Poisson systems such methods, however, are based on splitting, and they in turn rely on finding an integrable splitting [8]. To avoid the need for finding such an integrable splitting, we will introduce an alternate formulation of the Landau-Lifshitz equation on the Lie group SO^3, the set of 3×3 orthogonal matrices. This structure gives rise to a canonical Hamiltonian structure.

In section 2, we will start with the formulation of a Lie-group description of the Landau-Lifshitz equation. We will start this section by introducing the Landau-Lifshitz equation in its traditional form and discuss some of the mathematical properties of that description.

In section 3, we will reformulate the Landau-Lifshitz equation in terms of a description in SO^3, and derive some of its properties within this context. However, before doing that we will first introduce some fundamental concepts.

In section 4, we will show how a numerical method may be implemented based on this Lie group description of the equation. We will describe and derive a method to discretize both space and time, and discuss how we can solve the equations at each time step.

In our fifth and final section we will apply the numerical method developed in section 4 on a so called 'soliton wave' and provide graphs that we have generated based on the constructed method.

2 The traditional Landau-Lifshitz equation

2.1 The Landau-Lifshitz equation

In this section we focus on the Landau-Lifshitz equation. Because we work in a one-dimensional domain we will define a function $m : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^3$, where $m(x,t)$ gives the magnetization at position x and time t.

As said in the introduction, we will represent the magnetization of a ferromagnetic material by a time-dependent vector function $m : \mathbb{R}^d \times \mathbb{R} \to \mathbb{R}^3$, with d representing the number of physical dimensions under consideration. This magnetization must obey the Landau-Lifshitz equation [9], that is given by:

$$\frac{\partial m(x,t)}{\partial t} = m(x,t) \times \Delta m(x,t) + m(x,t) \times D\, m(x,t),$$

where Δ is the Laplace operator over the spatial dimensions and $D = \mathrm{diag}(D_1, D_2, D_3)$ is a diagonal matrix. Note that when $D = I$, the second term on the right vanishes. Thus, without loss of generality, we assume $D_j > 0$. This equation is an example of a Hamiltonian partial differential equation with Poisson structure. The total energy of such a system is given in terms of its Hamiltonian; the Hamiltionian of this system is given by ([4], Eqn. (1.34), p. 287):

$$\mathcal{H}(m) = \frac{1}{2} \int_{\mathcal{D}} |\nabla m(x,t)|^2 - m(x,t) \cdot D m(x,t),$$

and the Poisson structure is implied by:

$$\frac{\partial m(x,t)}{\partial t} = -m(x,t) \times \frac{\delta \mathcal{H}}{\delta m},$$

where $\frac{\delta}{\delta m}$ refers to the variational derivative with respect to m. Note that the skew-symmetry of the cross product implies the conservation of energy:

$$\frac{d}{dt}\mathcal{H} = 0,$$

as one would expect. Another conserved quantity is the total magnetization, as it can be checked that:

$$\frac{d}{dt}\frac{1}{2}\int_D |m(x,t)|^2 dt = 0.$$

In fact, the strength of magnetization $|m(x,t)|$ is conserved locally for all x. Because we are mostly interested in the orientation of the magnetization, we will assume the strength of the magnetization to be constant. Throughout this thesis we will assume that $|m(x,t)| = 1$.

2.2 The spatially discrete Landau-Lifshitz equation

We begin by discretizing space. For our purposes we will work in one dimension and will assume a periodic domain of length L with N points, with $\Delta x = L/N$

as the distance between them. One could visualize this system as a grid of points where each point has a vector associated to it, that tells us about the magnetization at that point, and this vector also changes in time. As such we consider the discrete set of magnetization vectors $m_i(t) = m(x_i, t)$.

Before constructing a discrete version of the Landau-Lifshitz equation, we examine what the second derivative looks like within this discrete context. For any differentiable function in one variable the derivative is simply defined as:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

In similar fashion we approximate the derivative at m_i with respect to x to be the difference between m_{i+1} and m_i divided over the length of the interval between them. That gives us:

$$m_i' \approx \frac{m_{i+1} - m_i}{\Delta x},$$

and this gives us a set of first differences for i from 1 to N-1. For the second difference we apply a variation of this formula to obtain:

$$m_i'' \approx \frac{m_i' - m_{i-1}'}{\Delta x} = \frac{m_{i+1} - 2m_i + m_{i-1}}{\Delta x^2},$$

which assumes a uniform grid spacing. Substituting this into the Landau-Lifshitz equation then gives us the semi-discrete approximation equation:

$$\frac{dm_i}{dt} = m_i \times \frac{m_{i+1} - 2m_i + m_{i-1}}{\Delta x^2} - m_i \times Dm_i, \tag{1}$$

using this discretization, we can also reformulate the Hamiltonian and the total magnetization in a spatially discrete form. For the Hamiltonian this gives us:

$$\mathcal{H} = \sum_{i=1}^{N} \frac{1}{2} |\frac{m_{i+1} - m_i}{\Delta x}|^2 - m_i \cdot Dm_i.$$

and the total magnetization is:

$$\mathcal{C} = \frac{1}{2} \sum_{i=1}^{N} ||m_i||^2.$$

Again, one can easily check that both of these quantities are invariant under the dynamics (1). If we would also like to discretize time we prefer a sympletic method [12], but for Hamiltonian systems with Poisson structure (like this one), the only available sympletic methods are based on splitting and that requires us to find an integrable splitting. Such splittings are known [7]. However, is is unsatisfying that splitting is problem-specific and not an generic approach. Instead we will now introduce the alternate formulation of the Landau-Lifshitz where we write the system in canonical form on the cotangent bundle T^*SO^3 to make it accessible to standard symplectic Runga-Kutta methods.

3 The Landau-Lifshitz equation in SO^3

3.1 Matrix calculus, Lie algebras and Lie groups

In order to properly derive the Landau-Lifshitz equations, some preliminary results are required. This section is devoted to establishing these results. In order to do this we must first establish some terminology.

In the next subsection we will work with matrices, and specifically orthonormal matrices. An orthonormal matrix is a square matrix satisfying the condition $A^T A = I$. These matrices form a group under matrix multiplication. This group contains a subgroup SO^n called the special orthogonal group (see e.g. Olver [14], p. 16).

A Lie group is a group consisting of a set that is also a smooth manifold, such that the group operation and its inverse are smooth operations between manifolds ([14], p. 15). To interpret SO^3 as a manifold we interpret each 3×3 matrix as a 9 dimensional vector, and thus consider SO^3 as a subspace of $\mathbb{R}^9$ with the standard metric, topology and smooth structure.

Another important notion we will use is the notion of a Lie algebra. We will devote some space to introduce this concept, as it will be important for our treatment of the Landau-Lifshitz equation later on.

In the theory of smooth manifolds and Lie groups it is well established that every Lie group has a unique associated Lie algebra [10]. For the group SO^3, the corresponding Lie algebra is the algebra $\mathfrak{so}(3)$, the set of skew symmetric matrices, equipped with the commutator operation with $[A, B] = AB - BA$. One of the reasons our formulation in SO^3 will turn out to be so powerful, is that this Lie algebra is isomorphic to $\mathbb{R}^3$ equipped with the cross product.

To see that consider the isomorphism$\widehat{\ }$: $\mathbb{R}^3 \to so(3)$:

$$\widehat{r} = \begin{bmatrix} 0 & -z & y \\ z & 0 & -x \\ -y & x & 0 \end{bmatrix}$$

With $r = (x, y, z)^T$. From this we can see that:

$$[r, r'] = rr' - r'r = \begin{bmatrix} 0 & x'y - y'x & x'z - z'x \\ y'x - x'y & 0 & y'z - z'y \\ z'x - x'z & z'y - y'z & 0 \end{bmatrix} = \widehat{r \times r'}$$

In the next section we will use this expression to replace the cross product with matrix notation. But in order to make sense of the Landau-Lifshitz equation in terms of matrices, we should first establish how to interpret certain derivatives in the context of a matrix. In the simplest case the matrix is a function of one real variable and we interpret the derivative with respect to this variable as the matrix that results from evaluating the derivative at every entry. In that case we have:

$$M(t) : \mathbb{R} \to \mathbb{R}^{n \times m}$$

The derivative of M is obtained by differentiating all entries in M with respect to t. In that case $(\frac{dM}{dt})_{i,j} = \frac{dM_{i,j}}{dt}$. We will, however, encounter another kind of derivative [2]. As we will see in the next section, some of our expressions will involve the trace of a matrix, and it will be convenient to assign a derivative to these functions, this time with respect to a matrix.

What does it even mean to differentiate with respect to a matrix? Let's consider a function $F : \mathbb{R}^{n\times m} \to \mathbb{R}$. One can interpret this function as a function from an nm dimensional space to $\mathbb{R}$. As such one can consider the partial derivative $\frac{\partial F}{\partial M_{i,j}}$ of such a function. Taken over all i, j, this set of derivatives is used to define a new matrix in $\mathbb{R}^{n\times m}$.

Now that we can make sense of derivatives with respect to a particular matrix entry, we have a clear way of assigning a meaning to a derivative with respect to a matrix. In the case of a general function $F : \mathbb{R}^{n\times n} \to \mathbb{R}$ this interpretation gives:

$$\frac{\partial F}{\partial A} = \begin{bmatrix} \frac{\partial F}{\partial A_{1,1}} & \cdots\cdots\cdots & \frac{\partial F}{\partial A_{1,n}} \\ \vdots & \ddots & \vdots \\ \frac{\partial F}{\partial A_{n,1}} & \cdots\cdots & \frac{\partial F}{\partial A_{n,n}} \end{bmatrix}.$$

To illustrate this consider the function $F(A, B) = \mathrm{tr}(A^T B)$. Using the definition of the trace and matrix manipulation we obtain:

$$F(A, B) = \sum_{l=1}^{n} \sum_{k=1}^{n} A_{l,k} B_{l,k}. \tag{2}$$

Differentiating this with respect to $A_{i,j}$ gives:

$$\frac{\partial}{\partial A_{i,j}} \mathrm{tr}(A^T B) = B_{i,j}.$$

Using this definition we can clearly see that:

$$\frac{\partial}{\partial A} \mathrm{tr}(A^T B) = B, \quad \frac{\partial}{\partial B} \mathrm{tr}(A^T B) = A. \tag{3}$$

The expression $\mathrm{tr}(A^T B)$ is the natural inner product on matrices, giving rise to the Frobenius norm [3]. This expression will be used in our next section, in order to derive an alternate expression for the Landau-Lifshitz equation.

3.2 Obtaining the Landau-Lifshitz equation

In this paragraph, we introduce a different point of view. Instead of using a vector centred description of the magnetization we use a matrix formulation. In the vector centred notation the function $m(x,t)$ told us the magnetization at position x and time t. Alternatively, we can look for a matrix description.

In this section we will derive a matrix form for the Landau-Lifshitz equation. The exact relation between this matrix form and the magnetization vector as described before will be clarified as we develop the theory. Our first order of business is to make sense of the cross product in terms of matrixes. Let us recall the Landau-Lifshitz equation:

$$\frac{\partial m(x,t)}{\partial t} = m(x,t) \times \Delta m(x,t) + m(x,t) \times Dm(x,t) = m(x,t) \times (\Delta + D)m(x,t)$$

In the previous section we worked out an expression for the cross product in terms of skew-symmetric matrices, using this isomorphism we see that the Landau-Lifshitz equation can be written in the notation:

$$\frac{\partial \widehat{m}(x,t)}{\partial t} = [\widehat{m}(x,t), \widehat{v}(x,t)],$$

with $v(x,t) = (\Delta + D)m(x,t)$. Before discretizing this equation, we reformulate the Landau-Lifshitz equation in yet another form.

The Landau-Lifshitz equations share the same Lie-Poisson structure as the Euler equations for the rigid body. Motivated by the work of Moser and Veselov [13] and Marsden, Pekarsky and Shkoller [11], in an unpublished note Frank and McLachlan [5] developed a numerical discretization of the Landau-Lifshitz equation on the SO^3. They show that the equation can be derived from the action integral

$$\mathcal{L} = \frac{1}{2} \int \int_D \mathrm{tr}((q^{-1}\dot{q})\mathcal{A}(q^{-1}\dot{q})^T) dxdt,$$

where $q(x,t) \in SO^3$ for all $x \in D$ and all $t \in \mathbb{R}$. The operator $\mathcal{A}$ is defined via its inverse. For $\widehat{m}(x,t)$ a smooth skew-symmetric matrix function of x and t,

$$\mathcal{A}^{-1}(\widehat{m}) = \Delta\widehat{m} + \tilde{D}\widehat{m}\tilde{D} = (\Delta + D)m(x,t),$$

where $\tilde{D} = \mathrm{diag}(\tilde{D}_1, \tilde{D}_2, \tilde{D}_3)$ and $\tilde{D}_j = \sqrt{D_1 D_2 D_3}/D_j$. The invertibility of $\mathcal{A}$ can be shown using the Fourier transform, depending on boundary conditions, and noting that D can be shifted by a multiple of the identity without loss of generality. In the language of [11], the action integral above is left invariant with respect to the Lie group SO^3. The operator $\mathcal{A}$ is symmetric with respect to the trace inner product of matrices, symmetric with respect to the L^2 inner product on $\mathcal{D}$, and preserves the skew-symmetry of its argument.

In this thesis we take a different approach to [5]. Instead of working with the Lagrangian formulation on the tangent bundle TSO^3, we work with the Hamiltonian formulation on the cotangent bundle T^*SO^3. Let the Langrangian

L denote the integrand of $\mathcal{L}$. We now formally introduce the conjugate momentum p using the partial derivative rule (3):

$$p = \frac{\partial L}{\partial \dot{q}} = (\mathcal{A}(q^{-1}\dot{q})^T q^{-1})^T.$$

Using the expression for the derivative of the trace derived in the previous section and the chain rule, The above equation can be solved for $q^{-1}\dot{q} \in \mathfrak{so}(3)$:

$$q^{-1}\dot{q} = \mathcal{A}^{-1}(p^T q).$$

We now substitute the above to define the Hamiltonian via the Legendre transformation:

$$\mathcal{H}(q,p) = \frac{1}{2}\int \mathrm{tr}(p^T\dot{q}) - L(q,\dot{q}(q,p))dx = \frac{1}{2}\int \mathrm{tr}[p^T q\mathcal{A}^{-1}(q^T p)]dx.$$

Note that the Legendre transformation maps the tangent bundle TSO^3 to the cotangent bundle $T^*SO(3)$ [1]. Now the quantity $q^T p$ is an element of the dual to the Lie algebra, $\mathfrak{so}(3)^*$, which again may be identified with the skew-symmetric matrices[1].

Hamilton's equations then are obtained using the partial derivative rule (3):

$$\dot{q} = \frac{\partial H}{\partial p} = q\mathcal{A}^{-1}q^T p \tag{4}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -(\mathcal{A}^{-1}(q^T p)p^T)^T. \tag{5}$$

By manipulating the first equation and assuming $q^T q = I$ and $p^T q + q^T p = 0$ we obtain:

$$\frac{\partial}{\partial t}(q^T q) = \dot{q}^T q + q^T\dot{q} = \mathcal{A}^{-1}(p^T q)q^T q + q^T q\mathcal{A}^{-1}(q^T p) = \mathcal{A}^{-1}(p^T q) + \mathcal{A}^{-1}(q^T p) = 0,$$

which shows that if the initial condition satisfies $q(x,t) \in SO^3$ then it will remain so. Under the same assumption we check that

$$\dot{p}^T q + p^T\dot{q} + \dot{q}^T p + q^T\dot{p} = [p^T q, \mathcal{A}^{-1}(p^T q)^T] + [\mathcal{A}^{-1}(p^T q), q^T p] = 0,$$

which shows that, if $q^T p \in \mathfrak{so}(3)^*$, it will remain so.

From Hamilton's equations we obtain:

$$\frac{\partial}{\partial t}(p^T q) = p^T\dot{q} + \dot{p}^T q = p^T q\mathcal{A}^{-1}q^T p - \mathcal{A}^{-1}(p^T q)^T p^T q = -[p^T q, \mathcal{A}^{-1}q^T p],$$

and this is again equivalent to the Landau-Lifshitz equation if we take $\hat{m} = p^T q$ and $\mathcal{A}^{-1} = \Delta + D$. Now the relation between the magnetization vector as established before and our matrices q and p have been clarified. We have now established the theoretical framework necessary for discretizing the Landau-Lifshitz equation within this SO^3 context.

[1] The Lie algebra $\mathfrak{so}(3)$ and its dual $\mathfrak{so}(3)^*$ are linearly isomorphic via the trace inner product (2) and can be identified with the space of skew-symmetric matrices (and with R^3). The tangent spaces $T_q SO(3)$ and $T_q^* SO(3)$ can also be identified since inner product (2) is invariant with respect to left multiplication by elements of $SO(3)$.

4 A sympletic discretization

In this section we will discretize the Landau-Lifshitz equation as derived in the previous section. We will make a few symplifying assumptions. Throughout this section we will assume that $D = I$, so $\mathcal{A}^{-1}$ becomes identical to the Laplace operator. In our case where we study one dimension, $\mathcal{A}^{-1}$ becomes the second derivative operator.

In order to discretize this function we will be using the midpoint rule. Given a differential equation of the form

$$\frac{dy}{dt} = f(y),$$

this gives us a discretization:

$$\frac{y_{n+1} - y_n}{\Delta t} = f(\frac{y_{n+1} + y_n}{2}).$$

When we define the shorthand:

$$y_{n+1/2} = \frac{y_{n+1} + y_n}{2}$$

and apply the midpoint rule to (4)–(5) we obtain:

$$\begin{cases} \frac{q_{n+1} - q_n}{\Delta t} = q_{n+1/2}\mathcal{A}^{-1}(p_{n+1/2}^T q_{n+1/2})^T \\ \frac{p_{n+1} - p_n}{\Delta t} = -(\mathcal{A}(p_{n+1/2}^T q_{n+1/2})^T p_{n+1/2}^T)^T. \end{cases} \tag{6}$$

One of the favourable properties of this discretization is that it automatically preserves any quadratic first integral of the equation, and this is a property shared by all Gauss-Legendre collocation methods [6], a class of methods to which the midpoint rule is the lowest order member.

Now suppose $G(y) = \frac{1}{2}y^T Ay$ is a quadratic conserved quantity. This would imply that:

$$\frac{d}{dt}G(y) = y^T Af(y) = 0$$

Inserting $y_{n+1/2}$ in the second equality gives:

$$y_{n+1/2}Af(y_{n+1/2}) = y_{n+1/2}A\frac{y_{n+1} - y_n}{\Delta t} = 0$$

Expanding the right hand side gives:

$$(\frac{y_{n+1} + y_n}{2})^T A(\frac{y_{n+1} - y_n}{\Delta t}) = \frac{y_{n+1}^T Ay_{n+1}}{2\Delta t} - \frac{y_n^T Ay_n}{2\Delta t} =$$

$$\frac{G(y_{n+1}) - G(y_n)}{\Delta t} = 0,$$

which proves conservation of $G(y)$ by the midpoint rule.

As we observed in the previous section two quadratic weak invariants, $q^T q = I$ and $p^T q + q^T p = 0$, hold if the initial conditions satisfy these. From there we can easily see that this implies that if $q_n^T q_n = I$ then $q_{n+1}^T q_{n+1} = I$ as well. Whenever we start with $q_0 \in SO^3$ we will remain on this manifold. We have also seen that $q^T p$ is skew symmetric and the function $G(q,p) = q^T p + p^T q = 0$ is also a quadratic first integral of the equations (4)–(5). Because the midpoint rule is an example of a Gauss-Legendre collocation method, it preserves this quantity, and therefore

$$q_n^T p_n + p_n^T q_n = q_{n+1}^T p_{n+1} + p_{n+1}^T q_{n+1} = 0,$$

hence, $q_{n+1} p_{n+1}$ is skew symmetric whenever $q_n^T p_n$ is.

In our next section we will apply the numerical method discussed here to an interesting example.

5 Numerical illustration

As mentioned in the introduction, we now apply the constructed theory to study the behaviour of a so called soliton, also known as a solitary wave [15]. In order to apply the constructed theory we need to construct some initial conditions. We assume D = I, so the operator $\mathcal{A}^{-1}$ can be replaced by the Laplacian and this in turn is the second derivate operator $\frac{\partial^2}{\partial x^2}$ as we work in one dimension.

In our discretization of space we use a spatial domain of length 30, centred around $x = 0$. So we will be working with the interval $[-15, 15]$, this will be cut into 1000 discrete points, giving us a spatial discretization of length $\Delta x = \frac{30}{1000} = 3 * 10^{-2}$. For the time discretization we will take time steps equal to $\tau = 2 * 10^{-3}$. We will let the time vary between $t = 0$ and $t = 2$, giving us 1000 time steps.

In order to know how our system will evolve in time, it is necessary to give an initial condition at time $t = 0$. As our initial condition we will have some function of x, we will start by specifying the function $m(x, 0)$ and then look for a suitable way to translate this into an initial condition on the matrices p_0 and q_0. In order to make sure that $|m(x, 0)| = 1$ we will use a parametrization of the unit circle in SO^3 and write down seperate functions of x based on the 2 angle parameters, θ and ϕ.

A parametrization of the unit sphere is given by:

$$S(\theta, \phi) = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))^T,$$

where θ varies from 0 to π and ϕ from 0 to 2π. We will now start defining some functions that will be used to define the initial conditions.

We use the following set of functions to define our initial conditions [15]. Let V and b be constants and define $\omega = \frac{V^2}{1-b^2}$,

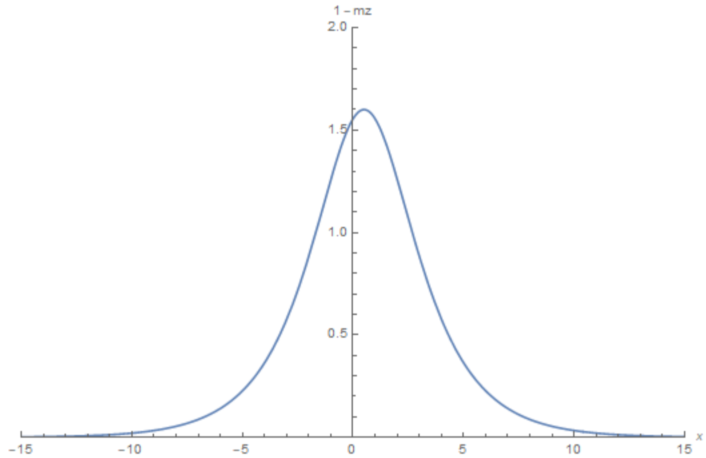
$$\begin{cases} \eta(x) = x - x_0 \\ \theta(x) = \arccos(1 - 2b^2 \operatorname{sech}(b\sqrt{\omega}\eta(x))) \\ \phi(x) = 0.5V(x - x_0) + \operatorname{sign}(V)\arctan(\sqrt{(}\omega)\eta(x)) \\ m(x, 0) = S(\theta(x), \phi(x)) \end{cases}$$

This now defines our initial condition. Now that we have formulated the initial magnetization it is time to translate this into the language of SO^3. In order to do so we need to define initial matrices $q_0(x)$ and $p_0(x)$ that may depend on x. As long as $p_0^T q_0 = \widehat{m(x, 0)}$, this will guarantee that q remains on SO^3 and $p^T q$ remains in $\mathfrak{so}^3$, as we saw in the previous section. At any point in time we may construct $p^T q = \widehat{m}$ to reconstruct the magnetization vector.

For our purposes we will choose $q_0 = I(3)$, the 3×3 identity matrix and $p_0 = \widehat{m(x, 0)}$. With these initial conditions we will use the equations derived from the midpoint rule to propagate the magnetization forward in time.

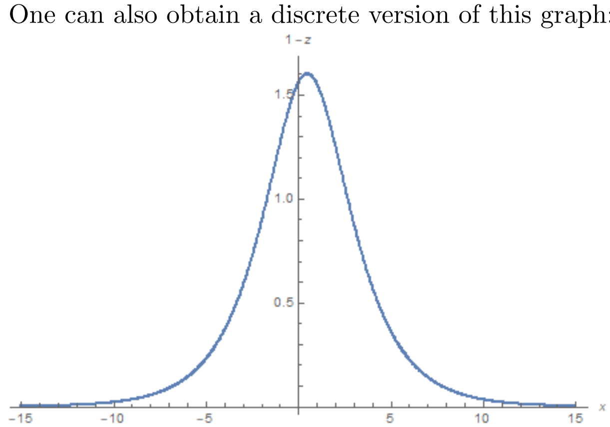
When evaluating the equation we choose $x_0 = 10, V = 0.5, b = 0.8$.

Using these parameters we construct the following graph:



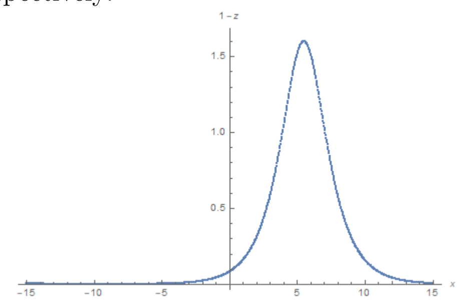
In this graph we have plotted $1 - m_z$ as a function of x, m_z being the z component of the magnetization vector. As we see in this picture this magnetization vector takes the form of a wave. This graph has been made with Mathematica using the initial conditions as described.

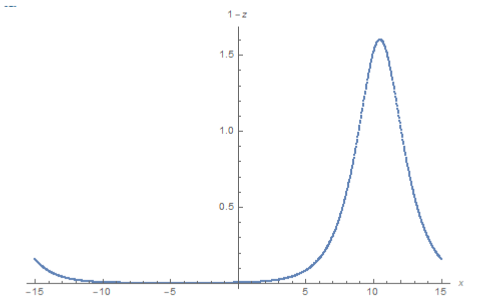
One can also obtain a discrete version of this graph:

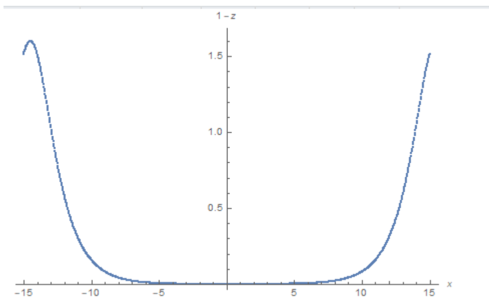


As can be seen it is almost identical to the continuous case, this is a listplot consisting of 1000 data points, each spread apart by $\Delta x = 3 * 10^{-2}$.

When numerically evaluating this function for other times, using the midpoint method, we obtain the following series for $t = 0.5, t = 1$ and $t = 1.5$ respectively:







As can be seen in these graphs, the evolution of the function acts like a traveling wave. Remarkably, the soliton is stable with respect to perturbations caused by numerical approximation. This example demonstrates the usefulness of the developed method.

6 Conclusion

In this thesis we have constructed a way of numerically solving the Landau-Lifshitz equation based on an SO^3 interpretation of the same equation. The advantage of basing the numerical method on the SO^3 interpretation is, that it allows us to preserve quadratic first integrals of the differential equation (which define the geometry of this problem).

Our thesis has shown, that when our constructed method is applied to a soliton wave, this results in graphs that show a wave that keeps the same shape over time and seems to move at a constant speed. Intuitively, one would not expect a soliton wave necessarily to be stable under numerical approximation.

This thesis has by no means provided an exhaustive analysis of the Landau-Lifshitz equations in SO^3. We could have applied the method developed in this thesis, to a system with initial conditions that are more complicated than the one we have used now. For example, we could have applied this method to a set of 2 soliton waves moving towards each other. We could also have investigated the influence of the anisotropy matrix D on the dynamics of systems. Considering dynamical systems in a 2 or 3 dimensional space of magnetization vectors instead of only one, could have been an option as well.

However, this thesis has provided a proof of concept. We have applied the constructed method to a relatively simple system and this has shown us that it produced the same results as the traditional method [8]. This proof of concept may encourage further research into possibilities of applications in more complicated systems.

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