Faculteit Bètawetenschappen

Phase Transitions in the Transverse-field Ising Ladder

Bachelor Thesis

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Abstract

In this thesis I have successfully recreated the results that describe the behavior of a transverse-field Ising chain. There is a phase transition when the parameter that controls the influence of the transverse field is one (that is: the factor in front of the transverse-field term and the interaction term are equally big). However, I could not determine exactly what the order parameter should be.
Then for the transverse-field Ising ladder I have used perturbation theory in the parameter that controls the interaction between the chains. I have found that the phase transition doesn't change position, but that the ground state is a different state compared to the transverse-field Ising chain, when the transverse-field term is smaller than the interaction.

Contents

1 Introduction

1.1 Background Theory

The transverse field Ising ladder is a double transverse field Ising chain lined up parallel in space. An Ising chain is a set of spins lined up in one dimension with a constant distance a between them. This distance is small enough that the spins have influence on their nearest neighbors, but influence from spins that are further away will be neglected in this thesis. The words "transverse field" specify that this Ising chain/ladder is set in an external magnetic field that is constant in space, which points in a direction perpendicular to the direction that the spins would attempt to align if the external field wasn't there, and perpendicular to the alignment of the spins. This gives us the following Hamiltonian for the Ising chain

$$H = -J\sum_{j=1}^{N}(\sigma_j^x\sigma_{j+1}^x + g\sigma_j^z) \tag{1}$$

where N is the number of sites, J is a coupling constant, Jg is equal to the strength of the external field, and periodic boundary conditions are assumed for the spin operators. That is

$$\sigma_{N+1}^i = \sigma_1^i \tag{2}$$

where i can be x, z or $+$ (for the raising operator) etc.
Notice the difference in the dimension that the spin operators apply to in expression (1). This causes that this Hamiltonian when diagonalized will not be diagonalized for both the z- and x-spin operators, because those do not commute with each other.

The Hamiltonian for the Ising ladder is as follows

$$H = -J\sum_{l=1}^{2}\sum_{j=1}^{N}(\sigma_{l,j}^x\sigma_{l,j+1}^x + g\sigma_{l,j}^z) - Jh\sum_{j=1}^{N}\sigma_{1,j}^z\sigma_{2,j}^z \tag{3}$$

where h is the parameter that controls how much the two chains influence each other, and the first index denotes the position in the x-direction (see sketch 1 below).

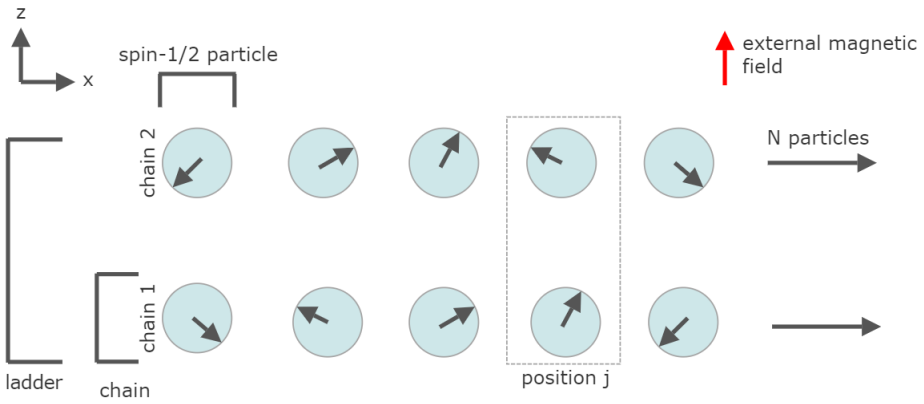

Figure 1: Sketch of the Ising ladder. The x-, y-, z-axes have been chosen to optimize clarity, and don't have to be oriented as in the sketch.

1.2 Outline

The goal of this thesis is to first describe the phase transition(s) for the Ising chain (at which point does it occur and what is the order parameter) and compare this to results found in literature. After that the same will be done for the Ising ladder using perturbation theory in coupling parameter h (see expression (3)), for which no comparison can be made with any literature.

2 Diagonalization of the Transverse-field Ising Chain

For the diagonalization of this Hamiltonian a couple of transformations should be done. For this I followed [**1**] and [**2**] to get an understanding of how these transformations should be done.

2.1 The Jordan-Wigner Transformation

As the reader may already know, raising and lowering operators can be defined as follows

$$\sigma_j^{\pm} = \frac{1}{2}(\sigma_j^x \pm i\sigma_j^y) \tag{4}$$

These operators' same site anti-commutator is equal to 1 and they commute on different sites, that is

$$\sigma_j^+\sigma_j^- = 1 - \sigma_j^-\sigma_j^+ \tag{5}$$

and

$$\sigma_j^+\sigma_k^- = \sigma_k^-\sigma_j^+ \tag{6}$$

for $j \neq k$.
Fermion annihilation and creation operators on the other hand anti-commute on different sites, while their anti-commutator also becomes 1 on the same site, that is

$$c_j^\dagger c_k = \delta_{jk} - c_k c_j^\dagger \tag{7}$$

where c_j is the annihilation operator for the j-th site, and $c_j^\dagger$ the creation operator for that site. Furthermore, we have

$$c_j c_k = -c_k c_j \tag{8}$$

$$c_j^\dagger c_k^\dagger = -c_k^\dagger c_j^\dagger \tag{9}$$

With a relatively simple transformation called the Jordan-Wigner transformation new operators can be introduced which obey the commutation relations of the fermion operators (hence making them indistinguishable from fermion operators). This is desirable, since the problem will be reduced to a problem with a chain of fermions, to which there is a solution. The transformation is as follows: the j-th raising or lowering operator gets a phase change of π (that is: a factor -1), if the number of fermions (or equivalently: the number of down spins) to the left of that site (if the first site is all the way to the left and the N-th spin to the right) is odd.
In mathematical language

$$\sigma_j^- = \prod_{\substack{k=1\\k<j}} (1 - 2c_k^\dagger c_k) c_j \tag{10}$$

and

$$\sigma_j^+ = \prod_{\substack{k=1\\k<j}} (1 - 2c_k^\dagger c_k) c_j^\dagger \tag{11}$$

Now everything is put in terms of fermion annihilation/creation operators, respectively c_j and $c_j^\dagger$. The $\prod_{\substack{j=0\\k<j}}(1-2c_k^\dagger c_k)$-factor basically gives a factor -1 for every fermion to the left of site j, so the transformation is as described above.

σ_j^z and σ_j^x transform likewise

$$\sigma_j^z = 1 - 2\sigma_j^-\sigma_j^+ = 1 - 2c_j^\dagger c_j \tag{12}$$

$$\sigma_j^x = \sigma_j^- + \sigma_j^+ = \prod_{\substack{k=1\\k<j}} (1 - 2c_k^\dagger c_k)(c_j^\dagger + c_j) \tag{13}$$

The terms in the Hamiltonian that will give the most difficulty would be the $\sigma_j^x\sigma_{j+1}^x$ terms, and the complete derivation is given in appendix A. Only in the last term (which looks like this: $\sigma_N^x\sigma_1^x$) the large product

remains, since σ_1^x doesn't have any sites to the left of it. This large product is either 1 or -1 depending on the total number of fermions on the chain, since it can be rewritten as

$$\prod_{\substack{k=1\\k<N}} (1-2c_k^\dagger c_k) = \prod_{\substack{k=1\\k\leq N}} (1-2c_k^\dagger c_k)(1-2c_N^\dagger c_N) \tag{14}$$

which would be in contradiction with the boundary conditions described in expression (2).
This problem can be solved by letting the boundary conditions of the creation and annihilation operators depend on the number of fermions on the chain.

If we choose the so-called anti-periodic boundary conditions in the case of an even number of fermions on the chain:

$$c_{N+1} = -c_1 \tag{15}$$

and the usual periodic boundary conditions in the case of an odd number

$$c_{N+1} = c_1 \tag{16}$$

then we get in both cases

$$(\sigma_j^x \sigma_{j+1}^x)|_{j=N} = (c_N^\dagger - c_N)(c_1^\dagger + c_1) \tag{17}$$

just as the rest of the terms. The Hamiltonian becomes

$$H = -J\sum_{j=1}^{N}(c_j^\dagger c_{j+1}^\dagger + c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j + c_{j+1}c_j + g - 2gc_j^\dagger c_j) \tag{18}$$

Thus we have reinterpreted the problem as a problem with a chain of fermions. In the following subsections I will explain how this system's Hamiltonian can be diagonalized.

2.2 The Fourier Transformation

The next step would be a Fourier transform to get rid of operators that act on the "next" $((j+1)$-th) site.

$$\begin{aligned} c_j &= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} c_{k_n} e^{-ik_n ja} \\ c_j^\dagger &= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1} c_{k_n}^\dagger e^{-ik_n ja} \end{aligned} \tag{19}$$

if the distance between the sites is a. The inverse transformation would look like

$$\begin{aligned} c_{k_n} &= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} c_j e^{ik_n ja} \\ c_{k_n}^\dagger &= \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} c_j^\dagger e^{ik_n ja} \end{aligned} \tag{20}$$

from which it is relatively easy deduced that the operators from $\{c_{k_n}\}$ obey the same commutation relations as the ones from $\{c_j\}$, that is

$$\begin{aligned} c_{k_n}^\dagger c_{k_m} &= \delta_{k_n k_m} - c_{k_m} c_{k_n}^\dagger \\ c_{k_n} c_{k_m} &= -c_{k_m} c_{k_n} \\ c_{k_n}^\dagger c_{k_m}^\dagger &= -c_{k_m}^\dagger c_{k_n}^\dagger \end{aligned} \tag{21}$$

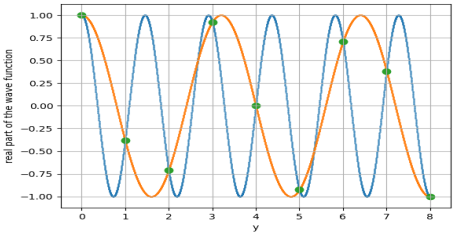

Figure 2: A wave with $n = 2$ and one with $n = 2 - N$ both result in the same manifestation on the equidistant points (here a wave number as (22) is chosen).

which means that these operators are creation and annihilation operators for fermions, which we'll call Fourier-fermions.
Here, however, the different boundary conditions we have chosen for the cases with an even and odd number of fermions have consequences for the quantization of the wave number.
For an even number it becomes

$$k_n = \frac{2\pi(n+\frac{1}{2})}{Na} \text{ for } n = 0, \ldots, N-1 \tag{22}$$

and for an odd number it becomes

$$k_n = \frac{2\pi n}{Na} \text{ for } n = 0, \ldots, N-1 \tag{23}$$

Furthermore, only the states $n = 0$ through $n = N-1$ are summed over, because the fact that the spins are on equidistant points makes that the N-th state is equivalent to the zeroth (see figure 2). This can easily be proved by adding N to n in the expression for any wave number, (22) or (23) (it is done explicitly in appendix A).

A relatively simple calculation will render the following result for the sum of the second and third terms in the Hamiltonian as expressed in (18).

$$\sum_{j=1}^{N}(c_j^\dagger c_{j+1} + c_{j+1}^\dagger c_j) = \sum_{n=0}^{N-1} 2c_{k_n}^\dagger c_{k_n} \cos(k_n a) \tag{24}$$

The first and fourth terms require a little more effort (the full derivation is in appendix A). The result is

$$\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger = \sum_{n=0}^{N-1} i c_{-k_n}^\dagger c_{k_n}^\dagger \sin(k_n a) \tag{25}$$

$$\sum_{j=1}^{N} c_{j+1} c_j = \sum_{n=0}^{N-1} i c_{-k_n} c_{k_n} \sin(k_n a) \tag{26}$$

where the first term (at $n = 0$) vanishes in the case of an odd number of fermions, so they could be written as

$$\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger = \sum_{n=1}^{N-1} i c_{-k_n}^\dagger c_{k_n}^\dagger \sin(k_n a) \tag{27}$$

$$\sum_{j=1}^{N} c_{j+1} c_j = \sum_{n=1}^{N-1} i c_{-k_n} c_{k_n} \sin(k_n a) \tag{28}$$

The resulting Hamiltonian for an even number of fermions is

$$H = J \sum_{n=0}^{N-1} \Big[2c_{k_n}^\dagger c_{-k_n} (g - \cos(k_n a)) - i \sin(k_n a)(c_{-k_n}^\dagger c_{k_n}^\dagger + c_{-k_n} c_{k_n}) - g \Big] \tag{29}$$

and for an odd number

$$H = J \left\{ \sum_{n=1}^{N-1} \Big[2c_{k_n}^\dagger c_{-k_n} (g - \cos(k_n a)) - i \sin(k_n a)(c_{-k_n}^\dagger c_{k_n}^\dagger + c_{-k_n} c_{k_n}) - g \Big] + 2c_0^\dagger c_0 (g-1) - g \right\} \tag{30}$$

We now have expressed our Hamiltonian in Fourier space. We notice a small difference between the Hamiltonian for the case of an even number of fermions, and one with an odd amount of fermions. This detail will reveal itself to be crucial in finding the phase transition. Notice that the Hamiltonian is still not diagonalized; we will need still one transformation.

2.3 The Bogoliubov Transformation

Now a new problem has arisen that prevents the Hamiltonian from being diagonal, namely the terms with a negative index. For this the following transformation will be used:

$$\begin{aligned} c_{k_n} &= u_{k_n}\gamma_{k_n} + iv_{k_n}\gamma^\dagger_{-k_n} \\ c^\dagger_{k_n} &= u_{k_n}\gamma^\dagger_{k_n} - iv_{k_n}\gamma_{-k_n} \end{aligned} \tag{31}$$

which is called the Bogoliubov transformation.
The inverse transformation would look like

$$\begin{aligned} \gamma_{k_n} &= u_{k_n}c_{k_n} - iv_{k_n}c^\dagger_{-k_n} \\ \gamma^\dagger_{k_n} &= u_{k_n}c^\dagger_{k_n} + iv_{k_n}c_{-k_n} \end{aligned} \tag{32}$$

from which it is easily deduced that the operators from $\{\gamma_{k_n}\}$ and $\{\gamma^\dagger_{k_n}\}$ follow the same commutation relations as the $\{c_{k_n}\}$ and $\{c^\dagger_{k_n}\}$, which in turn follow the same commutation relations as $\{c_j\}$ and $\{c^\dagger_j\}$, that is

$$\begin{aligned} \gamma^\dagger_{k_n}\gamma_{k_m} &= \delta_{k_nk_m} - \gamma_{k_m}\gamma^\dagger_{k_n} \\ \gamma_{k_n}\gamma_{k_m} &= -\gamma_{k_m}\gamma_{k_n} \\ \gamma^\dagger_{k_n}\gamma^\dagger_{k_m} &= -\gamma^\dagger_{k_m}\gamma^\dagger_{k_n} \end{aligned} \tag{33}$$

which means that these operators are creation and annihilation operators for fermions, which we'll call γ-fermions. It can be assumed that the momentum of any γ-fermion is the same as the Fourier-fermion it is transformed from (that is: $\gamma^\dagger_{k_n}$ creates a γ-fermion with momentum k_n).
Now these u_{k_n} and v_{k_n} must be chosen such that in the Hamiltonian the terms with a k_n and a $-k_n$ disappear. This turns out to be the case when we choose

$$\begin{aligned} u_{k_n} &= \cos(\theta_{k_n}/2) \\ v_{k_n} &= \sin(\theta_{k_n}/2) \end{aligned} \tag{34}$$

with

$$e^{i\theta_{k_n}} = \frac{g - e^{ik_na}}{\sqrt{1+g^2-2g\cos(k_na)}} \tag{35}$$

In appendix A this is made clear, and it is shown that eventually you will find the next expression for the case of an even number of fermions.

$$H = \sum_{n=0}^{N-1} \epsilon_{k_n}\Big(\gamma^\dagger_{k_n}\gamma_{k_n} - \frac{1}{2}\Big) \tag{36}$$

and in the case of an odd number of fermions

$$H = \sum_{n=1}^{N-1} \epsilon_{k_n}\Big(\gamma^\dagger_{k_n}\gamma_{k_n} - \frac{1}{2}\Big) + 2J(g-1)\Big(\gamma^\dagger_0\gamma_0 - \frac{1}{2}\Big) \tag{37}$$

with $\epsilon_{k_n} = 2J\sqrt{1+g^2-2g\cos(k_na)}$. This ϵ_{k_n} is the energy that is added to the total energy of a given state, if there is a γ-fermion with momentum k_n in that given state, and subtracted if there isn't.

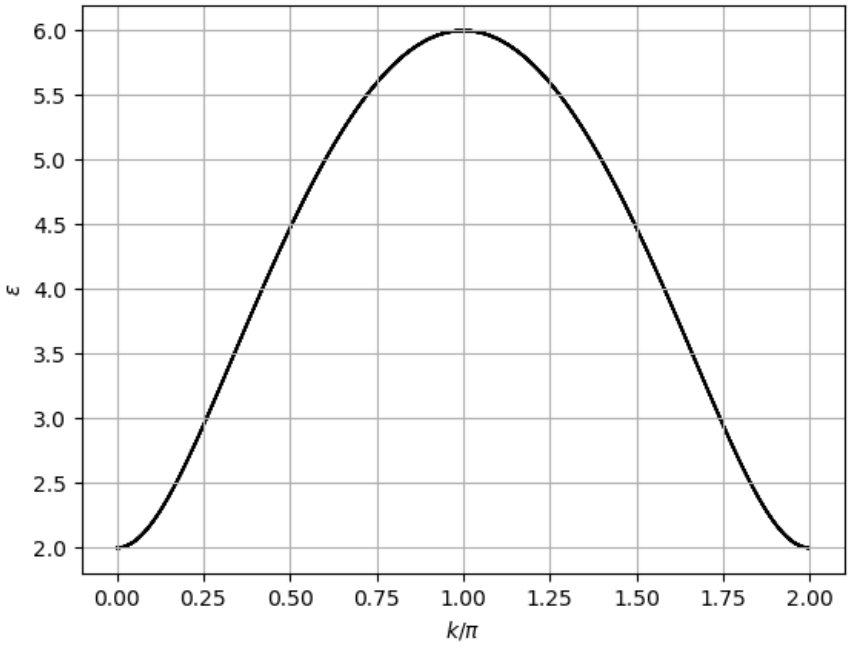

Figure 3: Expression (2.3) for $g = 2$, $a = 1$ and $J = 1$.

Now we have an expression for the Hamiltonian that is diagonalized. We can now start to find the phase transition, after we have tested our result.

3 The Phase Transition of a Transverse-field Ising Chain

As can be seen in the last section, the case of an odd number of fermions leaves us with a Hamiltonian that is almost identical to the one where the number of fermions is even; only the term with $n = 0$ is slightly different. As we will see this detail will have a drastic effect on the behavior of the transverse-field Ising chain. But first we should check the validity of our analytic result.

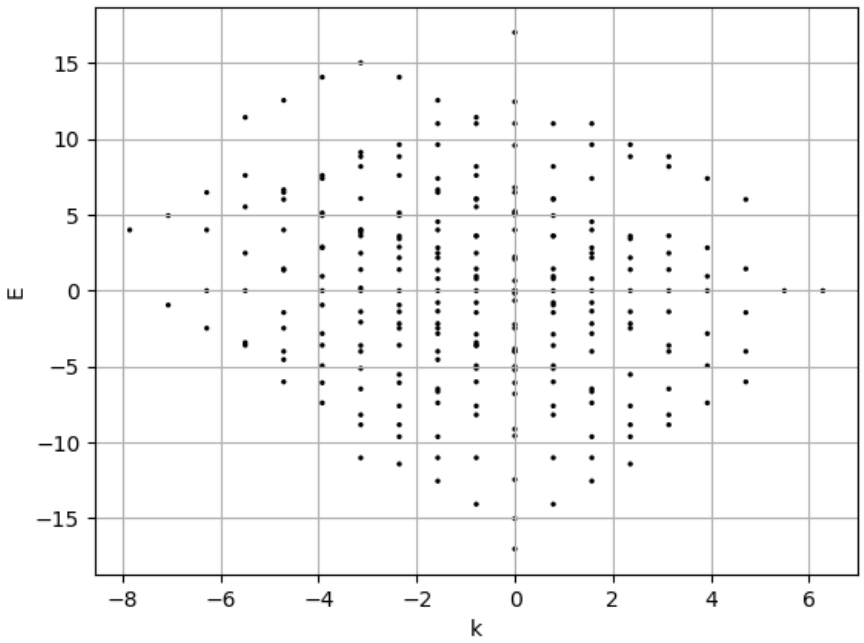

Figure 4: The analytically deduced energy spectrum for $N = 8$ and $g = 2$ for the states with $n = -N/2, \ldots, N/2-1$. The wave number that is laid out on the x-axis, is proportional to the momentum. Note that this plot is not symmetrical since the choice has to be made whether to include the state $n = -N/2$ or $n = N/2$ (since they're identical). One could also choose to plot the states $n = 0, \ldots, N-1$ and their respective momenta, and the plot would also be valid and very asymmetrical. This does however not mean that the system is asymmetrical.

3.1 Testing the Validity of the Analytical Result

In the table to the left I have listed the energy values (for $N = 4$, $g = 2$ and $J = 1$) that the analytical result predicts (analytic values) and the results that I have retrieved (numeric values) by letting Python diagonalize the original Hamiltonian (see appendix C). I have rounded the numbers to the number of decimals reported by Python for the numeric values, where necessary. It should be noted that one has to be careful to use the right analytic result (either (36) or (37)) depending on the number of fermions the concerning state has, or otherwise one will obtain a double amount of results compared to the numerically obtained results. One could take a state with one γ-fermion and plug it into expression (36) (so the 'wrong' one) and obtain one of the energies that doesn't appear from the direct, numeric diagonalization

numeric value	analytic value
8.54311682	8.54311682
-8.54311682	-8.54311682
-6.47213595	-6.47213595
2.47213595	2.47213595
2.64861379	2.64861379
-2.64861379	-2.64861379
6.47213595	6.47213595
4.00000000	4
-2.47213595	-2.47213595
-4.00000000	-4
-4.00000000	-4
4.00000000	4
0.0	0
0.0	0
0.0	0
0.0	0

of the Hamiltonian of the spin system. This energy would however be present in a system of a chain of fermions; the only reason why they are left out is because we are actually discussing a spin system where the boundary conditions should be as expression (2).

Another check would be to compare the results in the limits of $g = 0$ and $g \gg 1$, which render Hamiltonians that are diagonalized from the beginning (that is: it isn't necessary to employ the transformations used in section 1). Logically for $g = 0$ the ground state would be any state where all spins align in the x-direction if $J > 0$ (that is: all up or all down), and anti-align in the x-direction if $J < 0$. These states both would have an energy of $-JN$ (see expression (1)). For $g \gg 1$ we get a state where all spins align along the z-axis in the plus or minus direction dependent of the sign of Jg. These states both have an energy of $-JgN$ (see expression (1)).

First one could check the numeric result with these limiting cases by tweaking the parameters J, g and N in the code described in appendix C. You will find that it is consistent with the above results.

Now if we look at the final Hamiltonians ((36) and (37)), we see a couple of things:

[1] that for $J > 0$, one of the ground states should always be one where there are no γ-fermions, since this is the only state where every term in the sum is either negative or zero for all g; that for $J < 0$, one of the ground states should always be one where all the γ-fermions are present, since this is the only state where every term in the sum is either positive or zero for all g.

[2] that for $g = 0$ we find that ϵ_{k_n} becomes $2J$ for all k_n.

[3] that for $g \gg 1$ we find that ϵ_{k_n} becomes $2Jg$ for all k_n.

Putting the first two observations together and we find that the ground state energy for $g = 0$ becomes $-JN$ regardless of the sign of J. Using the first and the third observation we find that the ground state energy for $g \gg 1$ becomes $-JgN$ regardless of the sign of Jg.

Now we can conclude that the numeric and analytic results are consistent with the most interesting limiting cases of the original expression for the Hamiltonian (1). One could also compare the first (couple) correction(s) in the energy with perturbation theory for around $g = 0$ and $g \gg 1$.

3.2 Locating the Critical Point

When we take the thermodynamic limit ($N \gg 1$) we find a clear cut phase transition at $g = 1$. This will be made clear analytically in the following.

A clear indication of a phase transition is whenever a different state becomes the ground state or when the the degeneracy of the ground state changes. Let it be clear that the state with no γ-fermions is always a ground state when $J > 0$, and the state with all γ-fermions when $J < 0$ (by throwing a quick glance at (36) and (37)). In the following, I will discuss the case with $J > 0$, but the calculation for $J < 0$ is equivalent, and the result even identical, because of symmetry in the following transformation

$$\begin{aligned} \sigma_i^x &\to (-1)^i \sigma_i^x \\ \sigma_i^y &\to (-1)^i \sigma_i^y \end{aligned} \tag{38}$$

which would create a minus sign in expression (1), and which shouldn't alter the physics of the system since it is just a phase change.

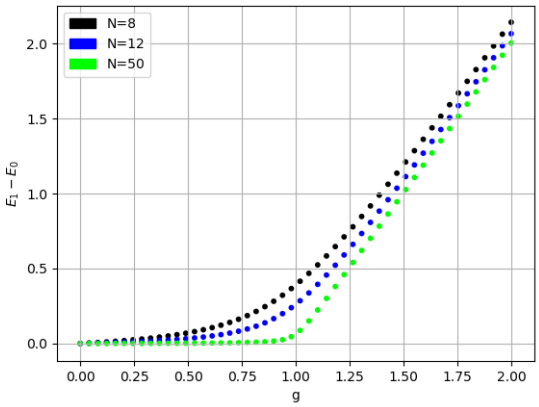

Figure 5: The energy gap between the ground state and the first excited state as a function of g for different values of N.

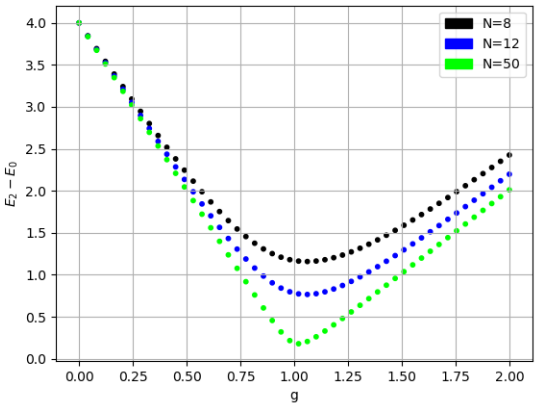

Figure 6: The energy gap between the ground state and the second excited state as a function of g for different values of N.

The state with no γ-fermions has an energy that can be written as

$$E_0 = -\frac{1}{2}\sum_{n=0}^{N-1}\epsilon_{k_n} = -J\sum_{n=0}^{N-1}\sqrt{1+g^2-2g\cos\left(\frac{2\pi(n+\frac{1}{2})}{N}\right)} \tag{39}$$

The smallest energy that could be added by adding a γ-fermion is the one with $k_n = 0$, since then the cosine in ϵ_{k_n} is maximal. So this state is always a first excited state or a ground state. Its energy can be written as

$$E_1 = -J\sum_{n=1}^{N-1}\sqrt{1+g^2-2g\cos\left(\frac{2\pi n}{N}\right)} + J(g-1) \tag{40}$$

Now if we take the thermodynamic limit, we can't simply assume $2\pi n/N$ to be zero, since at $n = N$ for instance $2\pi n/N$ should still be equal to 2π. We can however write $2\pi n/N = k$ where k is a continuous variable with $0 \leq k < 2\pi$ if $n = 0, \ldots, N-1$. We get in the thermodynamic limit for an odd number of fermions

$$k_n = \frac{2\pi n}{N} \to k \tag{41}$$

and for an even number

$$k_n \to k + \lim_{N\to\infty}\frac{\pi}{N} = k \tag{42}$$

where the arrow denotes taking the thermodynamic limit. This means we can express the (absolute) difference between E_0 and E_1 as

$$\Delta E_1 = J\left(g - 1 + \sqrt{1+g^2-2g\cos\left(\frac{\pi}{N}\right)}\right) \to J(g-1+|g-1|) \tag{43}$$

Or more insightfully

$$\Delta E_1 = \begin{cases} 0 & for\ g \leq 1 \\ 2J(g-1) & for\ g > 1 \end{cases} \tag{44}$$

This means that for $g < 1$ there are two ground states - one without any γ-fermions and one with only the γ-fermion with $n = 0$ - and for $g > 1$ there is just one - the one without any γ-fermions; thus the ground state degeneracy changes. This leaves us with a strong indication towards a phase transition at $g = 1$. The left figure on the next page (figure 5) shows a visual representation of how the difference between the ground state and the first excited state seem to converge to the above relation (43) as N grows. This was created

using the lowest energy levels of the data that was used for figure 4 (which are taken from the analytic result).

Similarly one could do the same for the difference between the ground state and the second excited state. The states that are always at most the second excited state are the states with one γ-fermion with either $n = 1$ or $n = N - 1$, since these have the same energy and the lowest apart from the state with $n = 0$ according to (36) and (37). So the energy of the second excited state in the thermodynamic limit (keeping (41) in mind) is

$$E_2 \to -J \sum_{n=2}^{N-1} \sqrt{1 + g^2 - 2g\cos(k)} - J(g-1) + J\sqrt{1 + g^2 - 2g\cos(dk)} \tag{45}$$

where dk is the smallest increment in k (which is infinitesimal since k is a continuous variable).
The difference between the ground state and the second excited state in the thermodynamic limit (keeping (42) in mind) would become

$$\Delta E_2 = -J(g-1) + J|g-1| + J\sqrt{1 + g^2 - 2g\cos(k)} + J\sqrt{1 + g^2 - 2g\cos(dk)} \tag{46}$$

dk can easily be set to zero, since there is no integration and it is only multiplied by something finitely big. Eventually we find

$$\Delta E_2 = -J(g-1) + 3J|g-1| \tag{47}$$

or more insightfully

$$\Delta E_2 = \begin{cases} 4J(1-g) & \textit{for } g \leq 1 \\ 2J(g-1) & \textit{for } g > 1 \end{cases} \tag{48}$$

The right figure on the previous page (figure 6) shows a visual representation of how the difference between the ground state and the second excited state seem to converge to the above relation (47) as N grows, which was created using the lowest energy levels of the data that was used for figure 4.

All in all, this of course suggests that there is a phase transition at $g = 1$, since the degeneracy of the ground state changes from 2 to 1 when g increases past $g = 1$.

3.3 Physical Interpretation

In this section I will attempt to make clear what that phase transition looks like that supposedly occurs at $g = 1$.

To visualize the phase transition it is useful to look at the expectation values of σ_j^x and σ_j^z in the ground state.

We start with σ_j^z. First of all we should transform it like we did with the Hamiltonian in section 2.

$$\sigma_j^z = 1 - 2c_j^\dagger c_j = 1 - \frac{2}{N} \sum_{n=0}^{N-1} c_{k_n}^\dagger e^{ik_n a} \sum_{m=0}^{N-1} c_{k_m} e^{-ik_m a} \tag{49}$$

applying the Jordan-Wigner and Fourier transformations.
Now for any configuration of k_n (either (22) or (23)) we get

$$\sigma_j^z = \frac{1}{N} \sum_{n=0}^{N-1} \sum_{m=0}^{N-1} \left[1 - 2c_{k_n}^\dagger c_{k_m} e^{2\pi i(n-m)/N} \right] \tag{50}$$

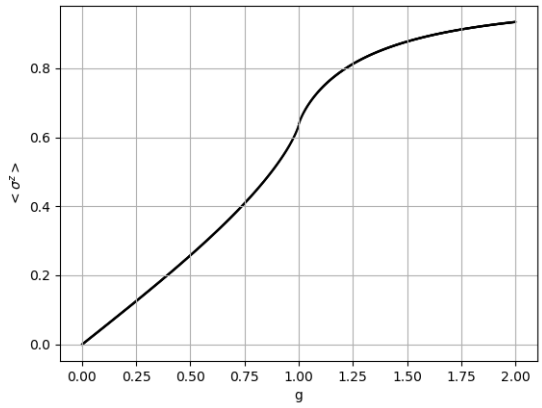

Figure 7: Expression (54) for $\langle 0|\,\sigma_j^z\,|0\rangle$ evaluated numerically.

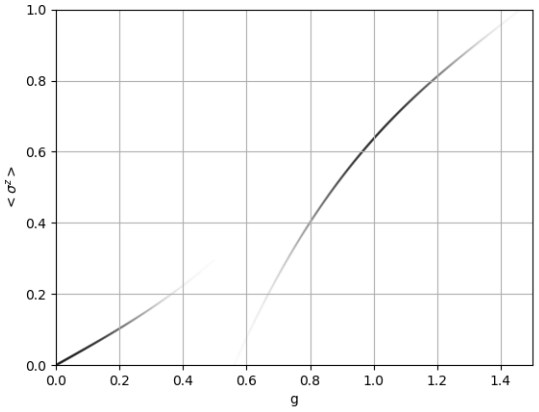

Figure 8: Expression (54) for $\langle 0|\,\sigma_j^z\,|0\rangle$ approximated with a third order Taylor expansion at $g = 0$ and $g = 1$ (for $g \gg 1$ it goes to 1 up to the third order in the Taylor series). Transparency denotes inaccuracy of the Taylor approximation at that point.

Since we want to know the expectation value of this operator at the ground state and we know that the γ-states form an orthogonal basis, we conclude that the only terms that remain after the Bogoliubov transformation are the ones with a $\gamma_{k_n}\gamma_{k_n}^\dagger$ (since we know that the state with zero γ-fermions is always a ground state). This knowledge we will use to prevent extremely large expressions.

$$\langle 0|\,\sigma_j^z\,|0\rangle = \frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}\Big[1 - 2\,\langle 0|\,\gamma_{k_{-n}}\gamma_{k_{-m}}^\dagger\,|0\rangle \sin(\theta_{k_{-n}}/2)\sin(\theta_{k_{-m}}/2)e^{2\pi i(n-m)/N}\Big] \tag{51}$$

This expression only contains terms with $m = n$. This results in

$$\langle 0|\,\sigma_j^z\,|0\rangle = \frac{1}{N}\sum_{n=0}^{N-1}\Big[1 - 2\,\langle 0|\,\gamma_{k_{-n}}\gamma_{k_{-n}}^\dagger\,|0\rangle \sin^2(\theta_{k_{-n}}/2)\Big] \tag{52}$$

Now obviously $\langle 0|\,\gamma_{k_{-n}}\gamma_{k_{-n}}^\dagger\,|0\rangle = 1$. This results into

$$\langle 0|\,\sigma_j^z\,|0\rangle = \frac{1}{N}\sum_{n=0}^{N-1}\cos(\theta_{k_{-n}}) = \frac{1}{N}\sum_{n=0}^{N-1}\frac{g - \cos(k_{-n}a)}{\sqrt{1 + g^2 - 2g\cos(k_{-n}a)}} \tag{53}$$

In the thermodynamic limit (see last subsection 3.2) this reduces to

$$\langle 0|\,\sigma_j^z\,|0\rangle = \frac{1}{2\pi}\int_0^{2\pi} dk \frac{g - \cos(k)}{\sqrt{1 + g^2 - 2g\cos(k)}} \tag{54}$$

which is in accordance with this article [**3**].
This integral we can't evaluate analytically for any g, but for certain values it reduces to a simple expression which can be evaluated.

$$\langle 0|\,\sigma_j^z\,|0\rangle = \begin{cases} 0 & \text{for } g = 0 \\ \frac{2}{\pi} & \text{for } g = 1 \\ 1 & \text{for } g \gg 1 \end{cases} \tag{55}$$

At these points a Taylor expansion can be done and the result is shown in 8. Furthermore I have used an Euler forward to integrate this expression numerically in Python for various values of g. The result can

be found in figure 7 and the code to recreate this result can be found in appendix C. There seems to be a discontinuity in either the first derivative or a higher derivative at $g = 1$, but other than that this result doesn't show much of a phase transition.

So we move on to the expectation value of σ_j^x in the ground state. Since for $g = 0$ the ground state is one with all the spins pointing in either the positive or negative x-direction, the expectation value of σ_j^x in the ground state is simply zero. This doesn't change for bigger g. So the full ground state will not give us useful information about the phase transition, but looking at how any of the ground states individually evolves as g changes could be interesting. We will consider the case with all the spins in the positive x-direction.

For this perturbation theory will be used, since (of course) we don't know what the ground state looks like for arbitrary g. We can ignore the degenerate perturbation correction, since we know that for small perturbations in g starting from $g = 0$ don't change which state(s) construct the ground state (while it does change what those states look like, since the Bogoliubov transformation is dependent on g, see expression (31)). In general the first order non-degenerate perturbation theory for any state (which could be found in almost any textbook on quantum mechanics) looks like

$$|\chi_n\rangle = \alpha\Big(|\chi_n^{(0)}\rangle + \lambda \sum_{m\neq n}^{M} \frac{\langle \chi_m^{(0)}| H_1 |\chi_n^{(0)}\rangle}{E_n^{(0)} - E_m^{(0)}} |\chi_m^{(0)}\rangle + O(\lambda^2)\Big) \tag{56}$$

with α the normalization constant; $|\chi_n\rangle$ the state after the perturbation, in our case the ground state (we'll call it $|\chi_0\rangle$); $|\chi_n^{(0)}\rangle$ the state before the perturbation, in our case this is the state with all the spins in the positive x-direction (we'll call it $|\chi_0^{(0)}\rangle$); λ is the parameter that realizes the perturbation, in our case that's g; M is the number of states, in our case 2^N; $|\chi_m^{(0)}\rangle$ for any $m = 0, \ldots, n-1, n+1, \ldots, M$ are the rest of the states; H_1 is the part of the Hamiltonian of order g, in our case that's $-J\sum_{j=1}^{N}\sigma_j^z$; E_l for any $l = 0, \ldots, M$ including $l = n$ is the energy of state $|\chi_l^{(0)}\rangle$ before the perturbation. Let's plug this in first.

$$|\chi_0\rangle = \alpha\Big(|\chi_0^{(0)}\rangle + g \sum_{n=1}^{2^N-2} \frac{\langle \chi_n^{(0)}| \left(-J\sum_{j=1}^{N}\sigma_j^z\right) |\chi_0^{(0)}\rangle}{E_0^{(0)} - E_n^{(0)}} |\chi_n^{(0)}\rangle + O(g^2)\Big) \tag{57}$$

Since $|\chi_0^{(0)}\rangle$ is the state with all the spins to the right, its energy for $g = 0$ is equal to $E_0^{(0)} = -NJ$ (see (1)). Note that I have left out the last state, since the state with all the spins in the negative x-direction is also a ground state. The state with the n-th spin in the positive x-direction can be written as the normalized sum of an up and a down spin. So when σ_n^z acts on this, it turns into a spin in the negative x-direction. To clarify

$$\sigma_n^z |{\rightarrow_n}\rangle = \sigma_n^z \frac{1}{\sqrt{2}}(|{\uparrow_n}\rangle + |{\downarrow_n}\rangle) = \frac{1}{\sqrt{2}}(|{\uparrow_n}\rangle - |{\downarrow_n}\rangle) = |{\leftarrow_n}\rangle \tag{58}$$

where the positive x-direction is assumed to be to the right. So the only terms in $\sum_{n=1}^{2^N-2}\langle\chi_n^{(0)}|\sum_{j=1}^{N}\sigma_j^z|\chi_0^{(0)}\rangle$ which don't vanish are the ones with $n = 1, \ldots, N$ if $j = n$. Here I assumed the state $|\chi_n^{(0)}\rangle$ is the state with one spin in the negative x-direction on the n-th site for $n = 1, \ldots, N$. The energy of any such state is equal to $E_n^{(0)} = (4 - N)J$ for $g = 0$ (again easily deduced by glancing at (1)). The result:

$$|\chi_0\rangle = \alpha\Big(|\chi_0^{(0)}\rangle + g \sum_{n=1}^{N} \frac{\langle \chi_n^{(0)}| \sigma_n^z |\chi_0^{(0)}\rangle}{4} |\chi_n^{(0)}\rangle + O(g^2)\Big) \tag{59}$$

Now letting σ_n^z act on the ket, and remembering that $\{|\chi_n^{(0)}\rangle\,|n = 0, \ldots, 2^N - 1\}$ is an orthonormal basis, we get

$$|\chi_0\rangle = \alpha\Big(|\chi_0^{(0)}\rangle + \frac{g}{4} \sum_{n=1}^{N} |\chi_n^{(0)}\rangle + O(g^2)\Big) \tag{60}$$

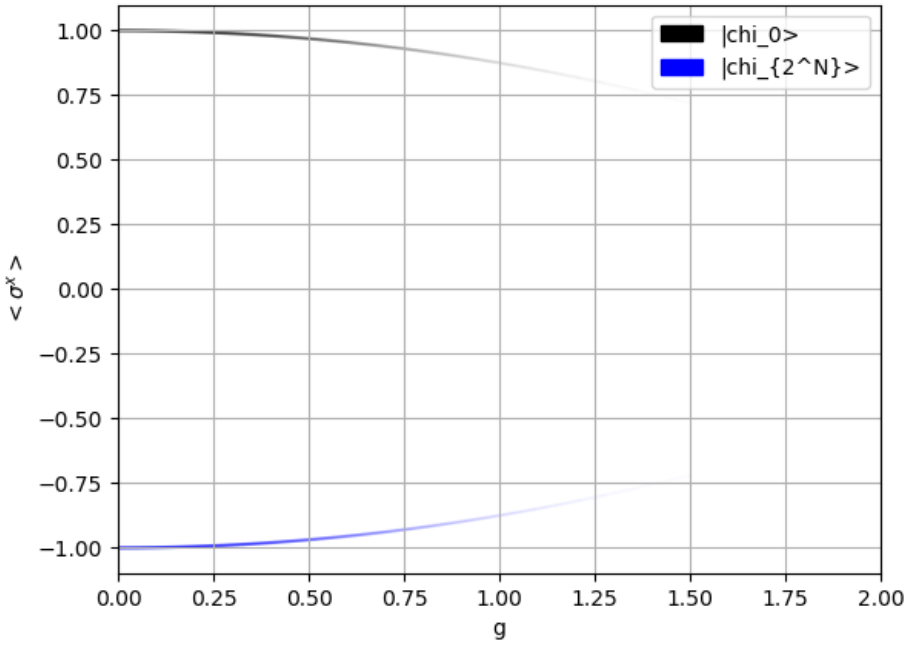

Figure 9: $\langle 0|\,\sigma_j^x\,|0\rangle$ approximated with first order perturbation theory. Transparency denotes inaccuracy of the Taylor approximation at that point.

Now the normalization constant α should be found. We know the inner product of $|\chi_0\rangle$ with itself should be zero:

$$\langle\chi_0|\chi_0\rangle = \alpha^2\Big(\,\langle\chi_0^{(0)}|\chi_0^{(0)}\rangle + \frac{g^2}{16}\sum_{n=1}^{N}\sum_{m=1}^{N}\langle\chi_n^{(0)}|\chi_m^{(0)}\rangle\,\Big) = \alpha^2\Big(1+\frac{Ng^2}{16}\Big) = 1 \tag{61}$$

where again orthonormality should be kept in mind. The result is that the normalization constant is as follows.

$$\alpha = \frac{1}{\sqrt{1+\frac{Ng^2}{16}}} \tag{62}$$

And thus the expectation value of σ_j^x can be calculated.

$$\langle\chi_0|\,\sigma_j^x\,|\chi_0\rangle = \frac{1}{1+\frac{Ng^2}{16}}\Big(\,\langle\chi_0^{(0)}|\,\sigma_j^x\,|\chi_0^{(0)}\rangle + \frac{g^2}{16}\sum_{n=1}^{N}\sum_{m=1}^{N}\langle\chi_n^{(0)}|\,\sigma_j^x\,|\chi_m^{(0)}\rangle + O(g^3)\Big) \tag{63}$$

Now knowing that the basis $\{|\chi_n^{(0)}\rangle\,|n=0,\ldots,2^N-1\}$ consists of eigenvectors of any σ_j^x, we get

$$\langle\chi_0|\,\sigma_j^x\,|\chi_0\rangle = \Big(1-\frac{Ng^2}{16}\Big)\Big(1+\frac{(N-2)g^2}{16}+O(g^3)\Big) = 1-\frac{g^2}{8}+O(g^3) \tag{64}$$

Where I have Taylor expanded the normalization factor to the second order and have neglected every term of order g^3 and smaller.
Similar for the state with all spins in the negative x-direction ($|\chi_{2^N}\rangle$) one would get

$$\langle\chi_{2^N}|\,\sigma_j^x\,|\chi_{2^N}\rangle = -1+\frac{g^2}{8}+O(g^3) \tag{65}$$

Now for $g \gg 1$ we have a single ground state with just up spins (this state we'll call $|\phi_0^{(0)}\rangle$). Now since the new basis $(\{|\phi_n^{(0)}\rangle\,|n=0,\ldots,2^N-1\}$ where $|\phi_j^{(0)}\rangle$ are states that are all perpendicular to $|\phi_0^{(0)}\rangle$ for

$j = 1, \ldots, 2^N - 1$) consists only of eigenvectors of the correction in the Hamiltonian (H_1 of (56)), we find that $\sum_{m \neq n}^{M} \langle \phi_m^{(0)} | \, H_1 \, | \phi_n^{(0)} \rangle$ is always zero. And thus the first correction of $|\phi_0\rangle$ is zero, and the expectation value of σ_j^x remains unchanged up to second order, that is: equal to zero.

From all this we can conclude that there is an indication that $\langle 0 | \, \sigma_j^x \, | 0 \rangle$ could be going to zero at $g = 1$, after which it stays there, which would make σ_j^x the order parameter of our system. But no hard proof is found with the techniques used in this subsection.

4 The Ising Ladder

In this section I will transform expression (3) for the Hamiltonian of the Ising ladder. Using perturbation theory I will show approximate values of the perturbed energy, the ground state and expectation values of σ_j^z and σ_j^x.

4.1 The Transformation

The first part of (3) is just twice the Hamiltonian of an Ising chain, so these can be transformed individually as explained in section 2.

$$\begin{aligned} &-J\sum_{l=1}^{2}\sum_{j=1}^{N}(\sigma_{l,j}^x\sigma_{l,j+1}^x+g\sigma_{l,j}^z) \\ &=\begin{cases} \sum_{l=1}^{2}\sum_{n=0}^{N-1}\epsilon_{l,k_n}\left(\gamma_{l,k_n}^\dagger\gamma_{l,k_n}-\frac{1}{2}\right) \text{ for an even number of fermions} \\ \sum_{l=1}^{2}\left[\sum_{n=1}^{N-1}\epsilon_{l,k_n}\left(\gamma_{l,k_n}^\dagger\gamma_{l,k_n}-\frac{1}{2}\right)+2J(g-1)\left(\gamma_{l,0}^\dagger\gamma_{l,0}-\frac{1}{2}\right)\right] \text{ for an odd number of fermions} \end{cases} \end{aligned} \tag{66}$$

The second part needs to be transformed from scratch. It is convenient to take the thermodynamic limit in this case as is made clear in appendix B, where the complete derivation of the following result is also given.

$$\begin{aligned} &-J\sum_{j=1}^{N}\sigma_{1,j}^z\sigma_{2,j}^z\to\frac{-JN}{2\pi}\int_0^\pi dk\sigma_{1,k}^z\sigma_{2,k}^z= \\ &\frac{-JN}{(2\pi)^2}\Bigg[4\int_0^{2\pi}dk_1\frac{(g-\cos(k_1))\gamma_{k_1}^\dagger\gamma_{k_1}+\frac{i}{2}(\gamma_{-k_1}\gamma_{k_1}+\gamma_{-k_1}^\dagger\gamma_{k_1}^\dagger)}{\sqrt{1+g^2-2g\cos(k_1)}} \\ &\int_0^{2\pi}dk_2\frac{(g-\cos(k_2))\gamma_{k_2}^\dagger\gamma_{k_2}+\frac{i}{2}(\gamma_{-k_2}\gamma_{k_2}+\gamma_{-k_2}^\dagger\gamma_{k_2}^\dagger)}{\sqrt{1+g^2-2g\cos(k_2)}} \\ &+\Big(\int_0^{2\pi}dk\frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}}\Big)^2 \\ &-4\int_0^{2\pi}dk\frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}}\Big(\int_0^{2\pi}dk_1\frac{(g-\cos(k_1))\gamma_{k_1}^\dagger\gamma_{k_1}+\frac{i}{2}(\gamma_{-k_1}\gamma_{k_1}+\gamma_{-k_1}^\dagger\gamma_{k_1}^\dagger)}{\sqrt{1+g^2-2g\cos(k_1)}} \\ &+\int_0^{2\pi}dk_2\frac{(g-\cos(k_2))\gamma_{k_2}^\dagger\gamma_{k_2}+\frac{i}{2}(\gamma_{-k_2}\gamma_{k_2}+\gamma_{-k_2}^\dagger\gamma_{k_2}^\dagger)}{\sqrt{1+g^2-2g\cos(k_2)}}\Big)\Bigg] \end{aligned} \tag{67}$$

where the arrow denotes that the thermodynamic limit is taken; in k_j the subscript denotes which chain is concerned (note that there is no difference between k_1 and k_2, but there certainly is a difference between γ_{k_1} and γ_{k_2}), and k denotes a wave number for which it is unspecified which chain it concerns; and $dk_1=dk_2=dk=\frac{2\pi}{N}$.

We can now calculate the first order correction of the ground state energy using

$$E_n^{(1)}=\langle n^{(0)}|\,V\,|n^{(0)}\rangle \tag{68}$$

In this case V is equal to the expression above (67), and the states we are going to look at are one where the ladder contains zero γ-fermions (denoted by $|(0,0)^{(0)}\rangle$), one where one chain contains a γ-fermion of momentum zero (denoted by $|(\gamma_0,0)^{(0)}\rangle$), and one where both chains contain one γ-fermion of momentum zero (denoted by $|(\gamma_0,\gamma_0)^{(0)}\rangle$).

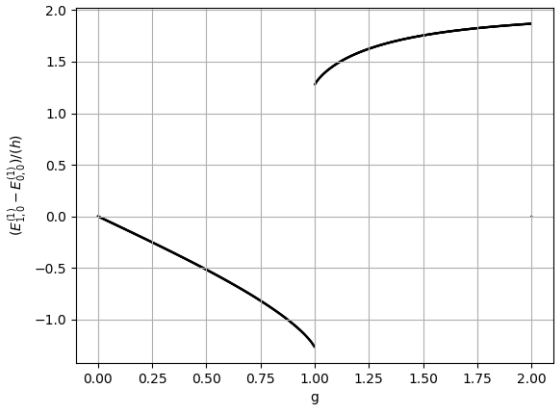

Figure 10: The energy gap between the first order corrections of the state with no γ-fermions and with a γ-fermion on one chain (for $J = 1$).

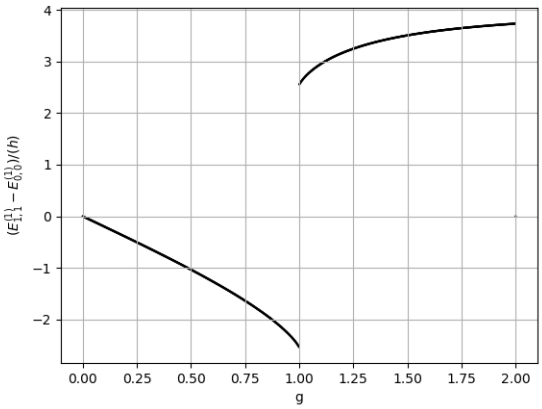

Figure 11: The energy gap between the first order corrections of the state with no γ-fermions and with a γ-fermion on both chains (for $J = 1$).

We find

$$\langle (0,0)^{(0)} | \, V \, | (0,0)^{(0)} \rangle = \frac{-JN}{(2\pi)^2} \Big(\int_0^{2\pi} dk \frac{g - \cos(k)}{\sqrt{1 + g^2 - 2g\cos(k)}} \Big)^2 \tag{69}$$

$$\begin{aligned} \langle (\gamma_0,0)^{(0)} | \, V \, | (\gamma_0,0)^{(0)} \rangle =& \frac{-JN}{(2\pi)^2} \Big(\int_0^{2\pi} dk \frac{g - \cos(k)}{\sqrt{1 + g^2 - 2g\cos(k)}} \Big)^2 \\ &+ \frac{2J}{2\pi} \frac{g-1}{|g-1|} \int_0^{2\pi} dk \frac{g - \cos(k)}{\sqrt{1 + g^2 - 2g\cos(k)}} \end{aligned} \tag{70}$$

$$\begin{aligned} \langle (\gamma_0,\gamma_0)^{(0)} | \, V \, | (\gamma_0,\gamma_0)^{(0)} \rangle =& \frac{-JN}{(2\pi)^2} \Big(\int_0^{2\pi} dk \frac{g - \cos(k)}{\sqrt{1 + g^2 - 2g\cos(k)}} \Big)^2 \\ &+ \frac{4J}{2\pi} \frac{g-1}{|g-1|} \int_0^{2\pi} dk \frac{g - \cos(k)}{\sqrt{1 + g^2 - 2g\cos(k)}} - \frac{4J}{N} \end{aligned} \tag{71}$$

or (the last two) more insightfully

$$\begin{aligned} &\langle (\gamma_0,0)^{(0)} | \, V \, | (\gamma_0,0)^{(0)} \rangle \\ &= \begin{cases} \frac{-JN}{(2\pi)^2} \Big(\int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} \Big)^2 - \frac{2J}{2\pi} \int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} & for\ 0 < g < 1 \\ \frac{-JN}{(2\pi)^2} \Big(\int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} \Big)^2 + \frac{2J}{2\pi} \int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} & for\ g > 1 \end{cases} \end{aligned} \tag{72}$$

$$\begin{aligned} &\langle (\gamma_0,\gamma_0)^{(0)} | \, V \, | (\gamma_0,\gamma_0)^{(0)} \rangle \\ &= \begin{cases} \frac{-JN}{(2\pi)^2} \Big(\int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} \Big)^2 - \frac{4J}{2\pi} \int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} - \frac{4J}{N} & for\ 0 < g < 1 \\ \frac{-JN}{(2\pi)^2} \Big(\int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} \Big)^2 + \frac{4J}{2\pi} \int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} - \frac{4J}{N} & for\ g > 1 \end{cases} \end{aligned} \tag{73}$$

See figures 10 and 11 (which have been made using the same Euler forward as in subsection 3.2, see appendix C).
Combining this with the result of subsection 3.2 we get for the total energy gap (so this is not what is displayed in the figures) between the state with zero γ-fermions on both chains (we call it $E_{0,0}$) and the state with one γ-fermion of momentum zero on one chain (we call it $E_{1,0}$)

$$E_{1,0} - E_{0,0} = \begin{cases} -h\frac{2J}{2\pi} \int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} + O(h^2) & for\ 0 < g < 1 \\ 2J(g-1) + h\frac{2J}{2\pi} \int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} + O(h^2) & for\ g > 1 \end{cases} \tag{74}$$

and for the energy gap between the state with zero γ-fermions on both chains and the state with one γ-fermion of momentum zero on both chains (we call it $E_{1,1}$)

$$E_{1,1} - E_{0,0} = \begin{cases} -h\left(\frac{4J}{2\pi}\int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} - \frac{4J}{N}\right) + O(h^2) & \text{for } 0 < g < 1 \\ 4J(g-1) + h\left(\frac{4J}{2\pi}\int_0^{2\pi} dk \frac{g-\cos(k)}{\sqrt{1+g^2-2g\cos(k)}} - \frac{4J}{N}\right) + O(h^2) & \text{for } g > 1 \end{cases} \tag{75}$$

We see that for $0 < g < 1$ the ground state is not anymore the one with zero γ-fermions on both chains, but the one with one γ-fermion of momentum zero on both chains. But for $g > 1$ the state with zero γ-fermions is again the ground state. Furthermore, this means that the critical point of the phase transition hasn't moved from its position on $g = 1$.

4.2 Discussion

To be more certain about whether the critical point really stays at $g = 1$, higher order corrections in the energy could be calculated for the states discussed in the last subsection. For this first the first correction for the states themselves should be calculated, for which degenerate perturbation should be used, since the ground state is degenerate for $h = 0$. So this will take a considerable amount of time and effort.
There also seems to be a discontinuity in the $E_{1,0}$ and $E_{1,1}$ at $g = 1$. This may disappear when taking the higher order corrections for these energies, and if not, it will be interesting to think about why this discontinuity appears when there is an interaction between the two chains, and what happens exactly at $g = 1$.

A Derivation of the Diagonalization of the Hamiltonian

As is seen in the second section of this paper we start with (1) as the Hamiltonian that should be diagonalized.

A.1 The Jordan-Wigner Transformation

We start by inserting identities (12) and (13) in this expression

$$H = -J\sum_{j=1}^{N}(\sigma_j^x\sigma_{j+1}^x + g\sigma_j^z) = -J\sum_{j=1}^{N}\left[\prod_{\substack{j=0\\k<j}}(1-2c_k^\dagger c_k)(c_j^\dagger + c_j)\prod_{\substack{j=0\\k<j+1}}(1-2c_k^\dagger c_k)(c_{j+1}^\dagger c_{j+1}) + g(1-2c_j^\dagger c_j)\right] \quad (76)$$

Since $\sigma_k^x\sigma_k^x = 1$ for all k, we have $\prod_{\substack{j=0\\k<j}}(1-2c_k^\dagger c_k)\prod_{\substack{j=0\\k<j}}(1-2c_k^\dagger c_k) = 1$, which leaves us with

$$H = -J\sum_{j=1}^{N}\Big[(c_j^\dagger + c_j)(1-2c_j^\dagger c_j)(c_{j+1}^\dagger + c_{j+1}) + g(1-2c_j^\dagger c_j)\Big] \quad (77)$$

where commutation relations (7), (8) and (9) allowed us to switch the positions of $\prod_{\substack{j=0\\k<j}}(1-2c_k^\dagger c_k)$ and $(c_j^\dagger + c_j)$, and the boundary conditions of (15) and (16) are already implemented. It should also be kept in mind that $c_jc_j = c_j^\dagger c_j^\dagger = 0$.

If we now rewrite this expression and use $c_jc_j^\dagger c_j = c_j$ knowing the physical interpretation of what these operators do (see subsection 2.1), we get expression (18) for the Hamiltonian.

A.2 The Fourier Transformation

To understand the expression for the Hamiltonian after the Fourier transform (18), one should first understand equations (24), (25) and (26). First we substitute the Fourier transform in the second term

$$\sum_{j=1}^{N}c_j^\dagger c_{j+1} = \sum_{j=1}^{N}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}c_{k_n}^\dagger e^{ik_nja}c_{k_m}e^{-ik_m(j+1)a} = \sum_{j=1}^{N}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}c_{k_n}^\dagger c_{k_m}e^{ik_njae^{-ik_mja}}e^{-ik_ma} \quad (78)$$

where k_n are expressions (22) and (23) for an even and odd number of fermions respectively. Now because

$$\sum_{j=1}^{N}e^{-ik_njae^{ik_mja}} = \delta_{nm} \quad (79)$$

we have

$$\sum_{j=1}^{N}c_j^\dagger c_{j+1} = \sum_{j=1}^{N}\frac{1}{N}\sum_{n=0}^{N-1}c_{k_n}^\dagger c_{k_n}e^{-ik_na} = \sum_{n=0}^{N-1}c_{k_n}^\dagger c_{k_n}e^{-ik_na} \quad (80)$$

Equivalently for the third term in the Hamiltonian as expressed in (18)

$$\sum_{j=1}^{N}c_{j+1}^\dagger c_j = \sum_{n=0}^{N-1}c_{k_n}^\dagger c_{k_n}e^{ik_na} \quad (81)$$

The sum of these two become equation (24).

Equation (25) goes slightly different, but the idea is the same. First we again substitute the Fourier transform in the first and fourth term respectively of the Hamiltonian as expressed in (18).

$$\sum_{j=1}^{N}c_j^\dagger c_{j+1}^\dagger = \sum_{j=1}^{N}\frac{1}{N}\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}c_{k_n}^\dagger e^{ik_nja}c_{k_m}^\dagger e^{ik_m(j+1)a} \quad (82)$$

$$\sum_{j=1}^{N} c_{j+1}c_j = \sum_{j=1}^{N} \frac{1}{N} \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} c_{k_n} e^{-ik_n(j+1)a} c_{k_m} e^{-ik_m ja} \tag{83}$$

Now because the sites are only on discrete equidistant points, the states with $n = -N, \ldots, -1$ are equivalent to the states with $n = 0, \ldots, N-1$ respectively, since

$$c_{k_{n\pm N}} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} c_j e^{2\pi i(n\pm N)j/N} = \frac{1}{\sqrt{N}} \sum_{j=1}^{N} c_j e^{2\pi inj/N} \cdot 1 = c_{k_n} \tag{84}$$

and similar for the case of an even number of fermions (this phenomenon is also demonstrated in figure 2). Using this fact and identity (79) it can be shown that for an even number of fermions

$$\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger = \sum_{n=0}^{N-1} c_{k_{-n-1}}^\dagger c_{k_n}^\dagger e^{2\pi i(n+\frac{1}{2})/N} = \frac{1}{2} \sum_{n=-N}^{N-1} c_{k_{-n-1}}^\dagger c_{k_n}^\dagger e^{2\pi i(n+\frac{1}{2})/N} \tag{85}$$

This can be rewritten as

$$\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger = \frac{1}{2}\left[\sum_{n=-N}^{-1} c_{k_{-n-1}}^\dagger c_{k_n}^\dagger e^{2\pi i(n+\frac{1}{2})/N} + \sum_{n=0}^{N-1} c_{k_{-n-1}}^\dagger c_{k_n}^\dagger e^{2\pi i(n+\frac{1}{2})/N} \right] \tag{86}$$

$$\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger = \frac{1}{2} \sum_{n=0}^{N-1} \left[c_{k_n}^\dagger c_{k_{-n-1}}^\dagger e^{-2\pi i(n+\frac{1}{2})/N} + c_{k_{-n-1}}^\dagger c_{k_n}^\dagger e^{2\pi i(n+\frac{1}{2})/N} \right] \tag{87}$$

and using (9)

$$\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger = \frac{1}{2} \sum_{n=0}^{N-1} \left[c_{k_{-n-1}}^\dagger c_{k_n}^\dagger (e^{2\pi i(n+\frac{1}{2})/N} - e^{-2\pi i(n+\frac{1}{2})/N}) \right] = \sum_{n=0}^{N-1} c_{-k_n}^\dagger c_{k_n}^\dagger i\sin(k_n a) \tag{88}$$

Equivalently for the fourth term

$$\sum_{j=1}^{N} c_{j+1}c_j = \sum_{n=0}^{N-1} c_{-k_n} c_{k_n} i\sin(k_n a) \tag{89}$$

The same approach is used for the case with an odd number of fermions.

$$\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger = \sum_{j=1}^{N} \frac{1}{N} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} c_{k_m}^\dagger c_{k_n}^\dagger e^{2\pi i(m+n)j/N} e^{-2\pi in/N} = \sum_{n=0}^{N-1} c_{k_{-n}}^\dagger c_{k_n}^\dagger e^{2\pi in/N} \tag{90}$$

Now we have a bit of a problem, because the first term consists of two of the same operators ($c_0 c_0$), so we can't turn it into a sine. However, this term vanishes anyway because of Pauli's exclusion principle for fermions. So in the end we can write this term for an odd number of fermions as

$$\sum_{j=1}^{N} c_j^\dagger c_{j+1}^\dagger = \sum_{n=1}^{N-1} c_{-k_n}^\dagger c_{k_n}^\dagger i\sin(k_n a) \tag{91}$$

Equivalently for the fourth term

$$\sum_{j=1}^{N} c_{j+1}c_j = \sum_{n=1}^{N-1} c_{-k_n} c_{k_n} i\sin(k_n a) \tag{92}$$

Now, knowing that the last term turns into

$$\sum_{j=1}^{N} c_j^\dagger c_j = \sum_{n=0}^{N-1} c_{k_n}^\dagger c_{k_n} \tag{93}$$

We find expressions (29) and (30) for the Hamiltonian.

A.3 The Bogoliubov Transformation

Now only the Bogoliubov transformation remains to be explained (31). After simply inserting the transformation into the Hamiltonian for an even number of fermions, we find

$$\begin{aligned} H = J\sum_{n=0}^{N-1}\Big[&2(g-\cos(k_n a))(u_{k_n}\gamma_{k_n}^\dagger - iv_{k_n}\gamma_{-k_n})(u_{k_n}\gamma_{k_n} + iv_{k_n}\gamma_{-k_n}^\dagger) \\ &- i\sin(k_n a)((u_{k_n}\gamma_{-k_n}^\dagger + iv_{k_n}\gamma_{k_n})(u_{k_n}\gamma_{k_n}^\dagger - iv_{k_n}\gamma_{-k_n}) + (u_{k_n}\gamma_{-k_n} - iv_{k_n}\gamma_{k_n}^\dagger)(u_{k_n}\gamma_{k_n} + iv_{k_n}\gamma_{-k_n}^\dagger)) - g\Big] \end{aligned} \tag{94}$$

For convenience I split the Hamiltonian in two parts ($H_1 + H_2 = H$), where the first part contains the diagonalized part.

$$\begin{aligned} H_1 = J\sum_{n=0}^{N-1}\Big[&2(g-\cos(k_n a))(u_{k_n}^2\gamma_{k_n}^\dagger\gamma_{k_n} + v_{k_n}^2\gamma_{-k_n}\gamma_{-k_n}^\dagger) \\ &- i\sin(k_n a)(iu_{k_n}v_{k_n}\gamma_{k_n}\gamma_{k_n}^\dagger - iu_{k_n}v_{k_n}\gamma_{-k_n}^\dagger\gamma_{-k_n} + iu_{k_n}v_{k_n}\gamma_{-k_n}\gamma_{-k_n}^\dagger - iu_{k_n}v_{k_n}\gamma_{k_n}^\dagger\gamma_{k_n}) - g\Big] \\ H_2 = J\sum_{n=0}^{N-1}\Big[&2(g-\cos(k_n a))(iu_{k_n}v_{k_n}\gamma_{k_n}^\dagger\gamma_{-k_n}^\dagger - iu_{k_n}v_{k_n}\gamma_{-k_n}\gamma_{k_n}) \\ &- i\sin(k_n a)(u_{k_n}^2\gamma_{-k_n}^\dagger\gamma_{k_n}^\dagger + v_{k_n}^2\gamma_{k_n}\gamma_{-k_n} + u_{k_n}^2\gamma_{-k_n}\gamma_{k_n} + v_{k_n}^2\gamma_{k_n}^\dagger\gamma_{-k_n}^\dagger)\Big] \end{aligned} \tag{95}$$

Using this and inserting (34) into H_2, and then applying some trigonometric identities, we find

$$\begin{aligned} H_2 = J\sum_{n=0}^{N-1}\Big[&(g-\cos(k_n a))(i\sin(\theta_{k_n})\gamma_{k_n}^\dagger\gamma_{-k_n}^\dagger - i\sin(\theta_{k_n})\gamma_{-k_n}\gamma_{k_n}) \\ &- i\sin(k_n a)(\cos(\theta_{k_n})\gamma_{-k_n}\gamma_{k_n} - \cos(\theta_{k_n})\gamma_{k_n}^\dagger\gamma_{-k_n}^\dagger)\Big] \end{aligned} \tag{96}$$

At last we use (35) to obtain

$$\begin{aligned} H_2 = J\sum_{n=0}^{N-1}\frac{1}{\sqrt{1+g^2-2g\cos(k_n a)}}\Big[&(g-\cos(k_n a))i\sin(k_n a)(\gamma_{-k_n}\gamma_{k_n} - \gamma_{k_n}^\dagger\gamma_{-k_n}^\dagger) \\ &- i\sin(k_n a)(g-\cos(k_n a))(\gamma_{-k_n}\gamma_{k_n} - \gamma_{k_n}^\dagger\gamma_{-k_n}^\dagger)\Big] \end{aligned} \tag{97}$$

which obviously equals zero.

Now H_1 should be rewritten to something more concise and useful. For this it is necessary to assess how terms like $\gamma_{k_n}\gamma_{k_n}$ and $\gamma_{-k_n}\gamma_{-k_n}$ are related to each other. We use (84) to prove that $\sum_{n=0}^{N-1} c_{k_n}c_{k_n} = \sum_{n=0}^{N-1} c_{-k_n}c_{-k_n}$, and from this it can be easily deduced that $\sum_{n=0}^{N-1}\gamma_{k_n}\gamma_{k_n} = \sum_{n=0}^{N-1}\gamma_{-k_n}\gamma_{-k_n}$. Applying this to H_1 (which we now know equals H) in the case of an even number of fermions

$$\begin{aligned} H = J\sum_{n=0}^{N-1}\Big[&2(g-\cos(k_n a))(u_{k_n}^2\gamma_{k_n}^\dagger\gamma_{k_n} + v_{k_n}^2\gamma_{k_n}\gamma_{k_n}^\dagger) \\ &- i\sin(k_n a)(iu_{k_n}v_{k_n}\gamma_{k_n}\gamma_{k_n}^\dagger - iu_{k_n}v_{k_n}\gamma_{k_n}^\dagger\gamma_{k_n} + iu_{k_n}v_{k_n}\gamma_{k_n}\gamma_{k_n}^\dagger - iu_{k_n}v_{k_n}\gamma_{k_n}^\dagger\gamma_{k_n}) - g\Big] \end{aligned} \tag{98}$$

And then inserting (34), and after applying (7) and some commutation relations, we find

$$\begin{aligned} H = 2J\sum_{n=0}^{N-1}\Big\{&\gamma_{k_n}^\dagger\gamma_{k_n}[\cos(\theta_{k_n})(g-\cos(k_n a)) - \sin(\theta_{k_n})\sin(k_n a)] \\ &+ \sin^2(\theta_{k_n}/2)(g-\cos(k_n a)) + \frac{1}{2}\sin(\theta_{k_n})\sin(k_n a) - g/2\Big\} \end{aligned} \tag{99}$$

Now, choosing N to be an even number, we see that - without any loss of generality - $\sum_{n=0}^{N-1}\cos(k_n a)=0$, so such a term can be added at preference anywhere up to a factor (in this case J). This means we get

$$\begin{aligned} H = 2J\sum_{n=0}^{N-1}\Big\{&\gamma_{k_n}^\dagger\gamma_{k_n}[\cos(\theta_{k_n})(g-\cos(k_n a))-\sin(\theta_{k_n})\sin(k_n a)] \\ &-\frac{1}{2}[\cos(\theta_{k_n})(g-\cos(k_n a))-\sin(\theta_{k_n})\sin(k_n a)+g]\Big\} \end{aligned} \tag{100}$$

Now if we insert (35), the expressions in the square brackets both reduce to ϵ_{k_n} as in (2.3), and we are left with (36).

For an odd number of spins, the terms with $n=1,\ldots,N-1$ follow an identical derivation. However, because it misses the term with $n=0$, we have $\sum_{n=1}^{N-1}\cos(k_n a)=-1$, so there needs to be added up a term J in order to leave the Hamiltonian unchanged. The term with $n=0$ becomes

$$2J\Big[c_0^\dagger c_0(g-1)-g/2+\frac{1}{2}\Big]=2J\Big[\gamma_0^\dagger\gamma_0\frac{(g-\cos(k_n a))^2}{\mid g-1\mid}(g-1)-\frac{1}{2}(g-1)\Big]=2J(g-1)\Big[\gamma_0^\dagger\gamma_0-\frac{1}{2}\Big] \tag{101}$$

Add this up to the rest of the Hamiltonian and we find (37).

B Transformation of the Ising Ladder

First I write down the expression after the Jordan-Wigner, Fourier and Bogoliubov transformation respectively. If this is confusing, take a look at appendix A. So first the Jordan-Wigner transformation.

$$-J\sum_{j=1}^{N}\sigma_{1,j}^{z}\sigma_{2,j}^{z}=-J\sum_{j=1}^{N}(1-2c_{1,j}^{\dagger}c_{1,j})(1-2c_{2,j}^{\dagger}c_{2,j}) \tag{102}$$

Then the Fourier transform.

$$\begin{aligned}-J\sum_{j=1}^{N}\sigma_{1,j}^{z}\sigma_{2,j}^{z}=&-J\sum_{j=1}^{N}(1-\frac{2}{N}\sum_{n=0}^{N-1}\sum_{p=0}^{N-1}c_{k_{1,n}}^{\dagger}c_{k_{1,p}}e^{ik_{1,n}ja}e^{-ik_{1,p}ja})(1-\frac{2}{N}\sum_{m=0}^{N-1}\sum_{q=0}^{N-1}c_{k_{2,m}}^{\dagger}c_{k_{2,q}}e^{ik_{2,m}ja}e^{-ik_{2,q}ja})\\ =&-J\Big[N-2\sum_{n=0}^{N-1}(c_{k_{1,n}}^{\dagger}c_{k_{1,n}}+c_{k_{2,n}}^{\dagger}c_{k_{2,n}})\\ &+\frac{4}{N^{2}}\sum_{j=1}^{N}\sum_{n=0}^{N-1}\sum_{p=0}^{N-1}\sum_{m=0}^{N-1}\sum_{q=0}^{N-1}c_{k_{1,n}}^{\dagger}c_{k_{1,p}}c_{k_{2,m}}^{\dagger}c_{k_{2,q}}e^{ik_{1,n}ja}e^{-ik_{1,p}ja}e^{ik_{2,m}ja}e^{-ik_{2,q}ja}\Big]\end{aligned} \tag{103}$$

Now the fourth term of this expression with the quadruple summation (plus the summation over j) is only non-zero when $n=p$ and $m=q$, or $n=q$ and $m=p$, or $n=-m$ and $p=-q$, if $\{k_{1,n}|n=0,\ldots,N-1\}=\{k_{k_{2,n}}|n=1,\ldots,N-1\}$ (that is: if both chains have an equal number of fermions, and thus the same boundary conditions (see subsections 2.1 and 2.2)), and otherwise only when $n=p$ and $m=q$. However, when we take the thermodynamic limit, the two wave numbers become identical (see expressions (41) and (42)), and we only have to take into account the first case. So implementing this and taking the thermodynamic limit, we get for the last term in the last expression

$$\begin{aligned}&\frac{4}{N^{2}}\sum_{j=1}^{N}\sum_{n=0}^{N-1}\sum_{p=0}^{N-1}\sum_{m=0}^{N-1}\sum_{q=0}^{N-1}c_{k_{1,n}}^{\dagger}c_{k_{1,p}}c_{k_{2,m}}^{\dagger}c_{k_{2,q}}e^{ik_{1,n}ja}e^{-ik_{1,p}ja}e^{ik_{2,m}ja}e^{-ik_{2,q}ja}\\ =&\frac{4}{N}\Big[\sum_{n=0}^{N-1}\sum_{m=0}^{N-1}c_{k_{1,n}}^{\dagger}c_{k_{1,n}}c_{k_{2,m}}^{\dagger}c_{k_{2,m}}+\sum_{n=0}^{N-1}\sum_{\substack{m=0\\m\neq n}}^{N-1}(c_{k_{1,n}}^{\dagger}c_{k_{1,m}}c_{k_{2,m}}^{\dagger}c_{k_{2,n}}+c_{k_{1,n}}^{\dagger}c_{k_{1,m}}c_{k_{2,-n}}^{\dagger}c_{k_{2,-m}})\Big]\\ \to&\frac{4N}{(2\pi)^{2}}\Big[\int_{0}^{2\pi}\int_{0}^{2\pi}dk_{1}dk_{2}c_{k_{1}}^{\dagger}c_{k_{1}}c_{k_{2}}^{\dagger}c_{k_{2}}+\ldots\Big]\end{aligned} \tag{104}$$

where the arrow denotes that the thermodynamic limit is taken, and the dots denote the second and third term which turn to zero when taken the expectation value of it in any state that only consists of γ-fermions (see definition (31)).

Taking the thermodynamic limit for the second and third term in (103)

$$2\sum_{n=0}^{N-1}c_{k_{j,n}}^{\dagger}c_{k_{j,n}}\to\frac{2N}{2\pi}\int_{0}^{2\pi}dk_{j}c_{k_{j}}^{\dagger}c_{k_{j}} \tag{105}$$

with $j=1,2$.
And then after the Bogoliubov transformation

$$\begin{aligned}\frac{2N}{2\pi}\int_{0}^{2\pi}dk_{j}c_{k_{j}}^{\dagger}c_{k_{j}}=&N+\frac{2N}{2\pi}\int_{0}^{2\pi}dk\Big[(g-\cos(k_{j}))\gamma_{k_{j}}^{\dagger}\gamma_{k_{j}}+\frac{i}{2}(\gamma_{-k_{j}}\gamma_{k_{j}}+\gamma_{-k_{j}}^{\dagger}\gamma_{k_{j}}^{\dagger})\Big]\frac{1}{\sqrt{1+g^{2}-2g\cos(k_{j})}}\\ &-\frac{N}{2\pi}\int_{0}^{2\pi}dk\frac{g-\cos(k_{j})}{\sqrt{1+g^{2}-2g\cos(k_{j})}}\end{aligned} \tag{106}$$

where no new techniques have been used compared with appendix A. This expression can be used repeatedly to retrieve expression (67).

C Code

This is the code that I used for several numerical results shown in my thesis. This code can be copied and pasted in an environment in which Python works, preferably in Spyder, and the numeric results from my thesis should be the output. The packages that are necessary for this code are laid out in the first seven lines.

```
from IPython import get_ipython
import numpy as nn
import matplotlib.pyplot as pp
from matplotlib.patches import Rectangle
import itertools
get_ipython().magic('matplotlib qt5')

def flat(l): #flattens list
    return list(itertools.chain.from_iterable(l))
def block(x,d=1,off=0): #block function
    x=nn.ceil((x+1 off)/d)
    if x%2>0:
        return 1
    else:
        return  1
def dot(m,n): #dot product
    if len(m[0])==len(n):
        o=nn.zeros((len(m),len(n[0])))
        for j in range(len(m)):
            for k in range(len(n[0])):
                for l in range(len(n)):
                    o[j,k]=o[j,k]+m[j,l]*n[l,k]
        return o
    else:
        return 'no'
def plot(d,name=1,y='',x='',mark='o',legends=1,alph=1,alphx=False,lims=1): #plots data
    if len(nn.shape(d))==1:
        for k in range(len(d)):
            for j in range(len(d[k])):
                pp.scatter(d[k][j][0],d[k][j][1],s=500/len(d[k][j][0]),
                           marker=mark,alpha=alph)
    else:
        if len(nn.shape(d[0]))==1:
            rgba=nn.zeros((len(d[0]),4))
            if alphx==False:
                rgba[:,3]=alph
                pp.scatter(d[0],d[1],s=500/len(d[0]),marker=mark,color=rgba)
            else:
                rgba[:,3]=nn.e**(nn.linspace(1,1,len(d[0]))**2*5)/nn.e
                pp.scatter(d[0],d[1],s=500/len(d[0]),marker=mark,color=rgba)
        else:
            rgba=nn.zeros((len(d)+1,len(d[0][0]),4))
            for j in range(len(d)):
                rgba[j+1,:,2 j%3]=int((j%3+1)/(j+1))
                if alphx==False:
                    rgba[j,:,3]=alph
                    pp.scatter(d[j][0],d[j][1],s=500/len(d[0][0]),
                               marker=mark,color=rgba[j])
                else:
```

```
                    rgba[j,:,3]=nn.e**( nn.linspace( 1,1,len(d[j][0]))**2*5)/nn.e
                    pp.scatter(d[j][0],d[j][1],s=500/len(d[0][0]),
                               marker=mark,color=rgba[j])
    pp.xlabel(x)
    pp.ylabel(y)
    pp.grid(True)
    if lims==1:
        lims=lims
    else:
        pp.xlim(lims[0],lims[1])
        pp.ylim(lims[2],lims[3])
    if legends==1:
        x=x
    else:
        handles=[Rectangle((0,0),1,1,color=rgba[j,0,:3]) for j in range(len(d))]
        pp.legend(handles,legends)
    if name==1:
        return
    else:
        pp.savefig(name)
        return
#%% creating the numerical values in the table of my thesis
J=1 #the coupling constant
N=4 #number of sites
g=2 #as the 'g' in the theory
sz=nn.zeros((N,2**N,2**N))
for j in range(N):
    for k in range(2**N):
        sz[j,k,k]=block(k,2**N/2**(j+1))
        #sz[n] is the n th z spin operator in the usual basis
sx=nn.zeros((N,2**N,2**N))
for j in range(N):
    for k in range(2**N int(2**N/2**(j+1))):
        sx[j,int(k+2**N/2**(j+1)),k]=(block(k,2**N/2**(j+1))+1)/2
        sx[j,k,int(k+2**N/2**(j+1))]=(block(k,2**N/2**(j+1))+1)/2
        #sx[n] is the n th x spin operator in the usual basis
H=nn.zeros((2**N,2**N))
for j in range(N):
    H=H J*(dot(sz[j],sz[(j+1)%N]) g*sx[j]) #the Hamiltonian in the usual basis
#%%
for j in range(2**N):
    print(nn.real(nn.linalg.eig(H)[0][j])*(abs(nn.linalg.eig(H)[0][j])>10**(-15)))
#%% to test the accuracy of the Euler forward for g=1
N=1000000 #number of steps in Euler forward
n=nn.linspace(0,2*nn.pi,N)
I=0
for k in n:
    I=I+(1 nn.cos(k))**.5/(2**.5*N) #the integration
print(2/nn.pi,I) #should be equal to one another
#%% Euler forward applied to several values of g
N=1000
G=1000
n=nn.linspace(0,2*nn.pi,N)
g=nn.linspace(0,2,G)
```

```
I=nn.zeros(G)
for gg in range(G):
    for k in n:
        I[gg]=I[gg]+(g[gg] nn.cos(k))/((1+g[gg]**2 2*g[gg]*nn.cos(k))**.5*N)
#%%
plot([g,I],x='g',y='$<\sigma^z>$')
#%%
N=1000
G=1000
n=nn.linspace(0,2*nn.pi,N)
g=nn.linspace(0,2,G)
I=nn.zeros(G)
for gg in range(G):
    for k in n:
        I[gg]=I[gg]+(g[gg] nn.cos(k))/(N*(1+g[gg]**2 2*g[gg]*nn.cos(k))**.5)
#%%
dE10=nn.zeros(G)
dE10[:int(G/2)]= 2*I[:int(G/2)] #E_{1,0}^{(1)} E_{0,0}^{(1)} for 0<g<1
dE10[int(G/2):1]=2*I[int(G/2):1] #E_{1,0}^{(1)} E_{0,0}^{(1)} for g>1
#%%
plot([g,dE10],x='g',y='$(E_{1,0}^{(1)} E_{0,0}^{(1)})/(h)$')
#%%
dE11=nn.zeros(G)
dE11[:int(G/2)]= 4*I[:int(G/2)] 4/N #E_{1,1}^{(1)} E_{0,0}^{(1)} for 0<g<1
dE11[int(G/2):1]=4*I[int(G/2):1] 4/N #E_{1,1}^{(1)} E_{0,0}^{(1)} for g>1
#%%
plot([g,dE11],x='g',y='$(E_{1,1}^{(1)} E_{0,0}^{(1)})/(h)$')
```

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