Best approximations in normed vector spaces

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Abstract

We investigate which subsets of normed vector spaces are Chebyshev, that is, they admit a unique best approximation for every vector. We show that a subset of a strictly convex uniformly smooth finite-dimensional normed vector space is Chebyshev if, and only if, it is non-empty closed and convex. We also show that any non-empty closed convex subset of a strictly convex reflexive normed vector space is Chebyshev. We finally take a look at a few examples of applicable normed vector spaces and a few counter examples to some intuitions one might have.

Acknowledgments

Thanks to Fabian Ziltener for getting me interested in the topic and for being my supervisor.

Contents

1 Introduction

1.1 Motivation

The project of this thesis started out with the following theorem.

Theorem 1.1 (best approximation theorem for Hilbert spaces)**.** *Let H be a Hilbert space and $\emptyset \neq A \subset H$ a non-empty closed convex subset. Then for any vector $x \in H$ there is a unique vector $y_0 \in A$ such that $d(x, y_0) = d(x, A)$.*

A proof can be found in a book by Rynne and Youngsen [RY, Theorem 3.32] or in an article by Chang Hai Bin [Bin, Theorem 1.10].

At first glance, this theorem seems arbitrarily restricted. Would a Banach space not be good enough? Does the subset really need to be convex? What if we only require the vector y_0 to exist, and not to be unique? In this thesis we are going to answer these questions and more.

1.2 On functional analysis

This thesis is on the topic of functional analysis. In functional analysis we study spaces of functions which can be represented as vector spaces, often equipped with a norm or even an inner product. The study of this field turned out to be particularly useful in solving differential equations. For example, the Riesz representation theorem can be used to solve differential equations of the form

$$-\Delta f + f = q.$$

Here Δ is the Laplace operator

$$\Delta f := \sum_{i=1}^{n} \frac{\partial^2 f}{(\partial x_i)^2}.$$

Ultimately, the Riesz representation theorem uses the best approximation theorem for Hilbert spaces, Theorem 1.1. Therefore it might turn out to be useful to try and generalize this theorem.

1.3 Main results: Classifying Chebyshev sets in finite dimensions and in strictly convex reflexive normed vector spaces

To state the main results, we are going to need the following definitions.

Definition 1.2 (best approximation)**.** Let (V, d) be a metric space and $A \subset V$ a subset. For an element $x \in V$ a best approximation for x in A is an element $y_0 \in A$ such that $d(x, y_0) = d(x, A)$.

Definition 1.3 (Chebyshev set)**.** Let (V, d) be a metric space and $A \subset V$ a subset. We call A Chebyshev if for every $x \in V$ there is a unique best approximation for x in A.

In this thesis we are mainly concerned with the following question.

Question 1.4 (classification of Chebyshev sets in normed vector spaces)**.** *When is a subset of a normed vector space Chebyshev?*

We will first focus on finite-dimensional normed vector spaces. This will lead us to our first main result, which is the following theorem.

Theorem 1.5 (Chebyshev sets in finite dimensions)**.** *For a finite-dimensional normed vector space, the following are equivalent:*

(i) The vector space is strictly convex and uniformly smooth.

(ii) A subset is Chebyshev if, and only if, it is non-empty, closed and convex.

The special case of the implication (i)⇒(ii) where the normed vector space has the Euclidean norm has been proven before [Bin, Theorem 1.2]. In this thesis we generalize this known result from Euclidean spaces to strictly convex uniformly smooth finite-dimensional normed vector spaces. It looks like this result is new. We also show that, for this generalization, the reverse implication holds as well, which means that it is optimally generalized.

This result gives us a good reason to mainly focus on non-empty closed convex sets. This will lead us to our other main result, which is the following theorem.

Theorem 1.6 (Chebyshev sets in strictly convex reflexive spaces)**.** *For a normed vector space, the following are equivalent:*

(i) The vector space is strictly convex and reflexive.

(ii) All non-empty closed convex subsets are Chebyshev.

The special case of this theorem where we consider a Banach space has been proven before [FM14, Corollary 2.40 and Theorem 2.41]. In this thesis, we generalize this known result to also apply to incomplete normed vector spaces. Because incomplete normed vector spaces are irreflexive, meaning (i) does not hold, we will show that (ii) does not hold either. It also looks like this result is new.

Remark. For the definitions of strictly convex, uniformly smooth and reflexive, see pages 7, 9 and 27 respectively.

These results are great tools to classify Chebyshev sets. But there is definitely more to discover. The biggest open question now in approximation theory is the following.

Question 1.7 (Chebyshev set problem)**.** *Are all Chebyshev sets in Hilbert spaces convex?*

A counterexample has been found for an incomplete inner product space. The first such counterexample was due to Gordon G. Johnson [Joh87]. In this thesis, we will not continue the research on this question.

1.4 Organization of this thesis

In Section 2 we will prove Theorem 1.5, our first main result. It will require four subsections, each with its own main result related to Theorem 1.5. Some subsections will have main results which are more general than necessary for Theorem 1.5, because they are interesting on their own.

In Section 3 we will prove Theorem 1.6, our other main result. We will first prove this theorem for incomplete spaces, as we discussed underneath the statement of this theorem. We will then combine this with the known proof for Banach spaces to finish the proof of Theorem 1.6.

Finally, in Section 4 we will take a look at some examples of interesting normed vector spaces. We will first consider uniformly convex normed vector spaces and we will use our research to give an alternative proof of the Milman-Pettis theorem. We will then see that inner product spaces and the ℓ^p and L^p spaces (for $p \in (1, \infty)$) have basically every property related to this thesis. We will finally give some counterexamples to some intuitions one might have.

2 Chebyshev sets in finite dimensions: proof of Theorem 1.5

In this section we prove Theorem 1.5. In order to do so, we split the problem of characterizing Chebyshev sets into two distinct problems. We do this by defining proximal sets, for which best approximations always exist, and non-polyproximal sets, for which best approximations are always unique.

Theorem 1.5 references the notions of strict convexity and uniform smoothness. The intuitions behind these notions are that a normed vector space is strictly convex if the unit sphere does not contain any line segment and uniformly smooth if the unit sphere does not contain any corners. The definitions we will actually use, see p.7 and p.9 respectively, will be more algebraïc.

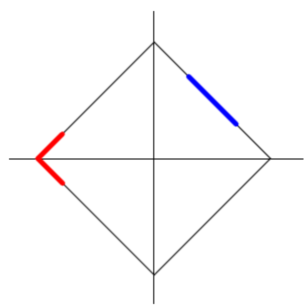

Figure 1: The norm $\|\cdot\|_1$ is neither strictly convex nor uniformly smooth in $\mathbb{R}^2$, because its unit sphere contains a line segment (blue) and a corner (red).

We will first show that a subset of a finite-dimensional normed vector space is proximal if, and only if, it is non-empty and closed. We will then show that exactly the strictly convex normed vector spaces have the property that any convex subset is non-polyproximal. We will then show that all Chebyshev subsets of uniformly smooth finite-dimensional normed vector spaces are convex. We will then show that any strictly convex finite-dimensional normed vector space that is not uniformly smooth does admit a non-convex Chebyshev subset. It looks like these last two results are new. Finally, we will combine all results to prove Theorem 1.5.

2.1 Proximal sets and the Heine-Borel property in finite dimensions

In this subsection we will first give the definition of a proximal set and deduce its basic properties. We will then give the definition of the Heine-Borel property, which states that every bounded sequence has a convergent subsequence. This notion which will turn out to be crucial for analyzing finite-dimensional normed vector spaces, so it will also be used in further subsections. The main result of this subsection will be that in finite dimensions we always have the Heine-Borel property and all non-empty closed subsets are proximal.

So to start off, we will look at proximal sets.

Definition 2.1 (proximal set)**.** Let (V, d) be a metric space and $A \subset V$ a subset. We call A proximal if, for every $x \in V$, there exists a best approximation for x in A.

Remark. This is different from Chebyshev sets, as we don't require the best approximation to be unique.

We will now deduce some basic properties of proximal sets.

Proposition 2.2 (non-empty closed proximal sets)**.** *The following statements hold.*

(i) All proximal subsets of non-empty metric spaces are non-empty.

(ii) All proximal sets are closed.

Proof of Proposition 2.2. **(i):** Let (V, d) be a non-empty metric space and $A \subset V$ a proximal subset. Because V is non-empty, there exists some $x \in V$. Because A is proximal, there exists a best approximation for x in A, which is an element of A. So A is non-empty.

This proves (i).

(ii): Let (V, d) be a metric space and $A \subset V$ a proximal subset. So for all $x \in A^c$ there exists $y_0 \in A$ such that $d(x, A) = d(x, y_0) > 0$. Because $B_{d(x,A)}(x) \subset A^c$, we can conclude that A^c is open, so A is closed.

This proves (ii) and completes the proof of Proposition 2.2. □

We can now conclude the following.

Corollary 2.3 (non-empty closed proximal sets in normed vector spaces)**.** *All proximal subsets of normed vector spaces are non-empty and closed.*

Proof of Corollary 2.3. Because all vector spaces contain a zero vector, they are non-empty. So according to Proposition 2.2, all proximal subsets of normed vector spaces are non-empty and closed. □

The following definition will turn out to be crucial for analyzing finite-dimensional normed vector spaces. It will therefore also be applied in further subsections.

Definition 2.4 (Heine-Borel property)**.** A metric space has the Heine-Borel property if every bounded sequence has a convergent subsequence.

We can now state the main result of this subsection.

Theorem 2.5 (proximal sets in finite dimensions)**.** *For a normed vector space, the following are equivalent.*

(i) The vector space is finite-dimensional.

(ii) The vector space has the Heine-Borel property.

(iii) All non-empty closed subsets are proximal.

Remark. Combining Theorem 2.5 and Corollary 2.3 we get that, in a normed vector space, the proximal subsets are exactly those that are non-empty and closed if, and only if, the space is finite-dimensional.

To prove this theorem, we are going to use the following proposition.

Proposition 2.6 (the Heine-Borel property and proximal sets)**.** *If a non-empty subset of a metric space has the Heine-Borel property, then it is proximal.*

Proof of Proposition 2.6. Let (V, d) be a metric space and $\emptyset \neq A \subset V$ a non-empty subset with the Heine-Borel property and let $x \in V$. Because A is non-empty, we have $d(x, A) \in [0, \infty)$. It follows that there exists a sequence $(y_i)_{i\in\mathbb{N}}$ in A, such that $\lim_{i\to\infty} d(x, y_i) = d(x, A)$. Note that this sequence is bounded, so because A has the Heine-Borel property, there exists a convergent subsequence $(z_i)_{i\in\mathbb{N}}$ with limit $y_0 \in A$. Consequently, we have $\lim_{i\to\infty} d(x, z_i) = d(x, A)$. Because metrics are always continuous with respect to themselves, we get $d(x, y_0) = d(x, A)$. So y_0 is a best approximation for x in A. Because this works for any $x \in V$, the set A is proximal. □

We are now ready to prove Theorem 2.5.

Proof of Theorem 2.5. **(i)⇒(ii):** Because finite-dimensional normed vector spaces with the same dimension are isomorphic, it follows from the Bolzano-Weierstrass theorem that all finite-dimensional normed vector spaces have the Heine-Borel property.

(ii)⇒(iii): Because all closed subsets of a metric space with the Heine-Borel property also have the Heine-Borel property, according to Proposition 2.6 all non-empty such sets are proximal.

(iii)⇒(i): Let $(X, \|\cdot\|)$ be an infinite-dimensional normed vector space. By using Riesz's lemma inductively, we can construct a sequence $(x_i)_{i\in\mathbb{N}}$ in $S_1^X(0)$ such that any two elements are at least at distance $\frac{3}{4}$ from each other. We can now define the sequence $(y_i)_{i\in\mathbb{N}}$ in X by $y_i := \left(1 + \frac{1}{i}\right) x_i$. We clearly get $\|y_i\| = 1 + \frac{1}{i}$ for all $i \in \mathbb{N}$, and for all $i, j \geq 4$ with $i \neq j$, we get

$$\|y_i - y_j\| \geq \|x_i - x_j\| - \left\|\frac{1}{i}x_i\right\| - \left\|\frac{1}{j}x_j\right\| \geq \frac{3}{4} - \frac{1}{4} - \frac{1}{4} = \frac{1}{4}.$$

We can conclude that $(y_i)_{i\in\mathbb{N}}$ does not have any converging subsequence, so $A := \{y_i | i \in \mathbb{N}\}$ is non-empty and closed. However, for all $i \in \mathbb{N}$ we get

$$\|0 - y_i\| = 1 + \frac{1}{i} > 1 + \frac{1}{i+1} = \|0 - y_{i+1}\| .$$

So there is no best approximation for 0 in A, so A is not proximal.

By contraposition, this proves (iii)⇒(i) and completes the proof of Theorem 2.5. □

2.2 Non-polyproximal sets and strict convexity

In this subsection we will first give the definition of a non-polyproximal set. We will then give the definition of a strictly convex normed vector space. The main result of this subsection will be that exactly the strictly convex normed vector spaces have the property that all convex subsets are non-polyproximal.

So to start off, we will look at non-polyproximal sets.

Definition 2.7 (non-polyproximal set)**.** Let (V, d) be a metric space and $A \subset V$ a subset. We call A non-polyproximal if, for every $x \in V$, there exists at most one best approximation for x in A.

Remark. (i) This is different from Chebyshev sets, as we allow such a best approximation to not exist.

(ii) A subset of a metric space is Chebyshev if, and only if, it is both proximal and non-polyproximal.

The following definition will turn out to be crucial for analyzing non-polyproximal sets in normed vector spaces. It will even be applied in Section 3.

Definition 2.8 (strictly convex)**.** We call a normed vector space $(X, \|\cdot\|)$ strictly convex if, for all $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$, we have $\|x + y\| < 2$.

Example 2.9 (strictly convex normed vector spaces)**.** Uniformly convex normed vector spaces. See Definition 4.1 and Example 4.2 for the definition and examples of uniformly convex normed vector spaces. The proof is on p.30.

We can now state the main result of this subsection.

Theorem 2.10 (non-polyproximal sets and strictly convex spaces)**.** *For a normed vector space, the following are equivalent.*

(i) The vector space is strictly convex.

(ii) All convex subsets are non-polyproximal.

(iii) All one-dimensional real linear subspaces are non-polyproximal.

(iv) The unit sphere contains no line segments.

To prove this theorem, we are going to use the following proposition.

Proposition 2.11 (norm on line segments and lines)**.** *Let $(X, \|\cdot\|)$ be a normed vector space with two elements $x, y \in X$ with $\|x\| = \|y\| = r$ and let v be on the line through x and y. Then $\|v\| \leq r$ holds if v is between x and y and $\|v\| \geq r$ holds otherwise.*

Proof of Proposition 2.11. Let $v = \lambda x + (1 - \lambda)y$ be on the line through x and y. If v is on the line segment between x and y, then $\lambda \in [0, 1]$, so $\|v\| \leq \lambda \|x\| + (1 - \lambda) \|y\| = r$. Otherwise, we can assume without loss of generality that $\lambda > 1$, giving $\|v\| \geq \lambda \|x\| - (\lambda - 1) \|y\| = r$. □

We are now ready to prove Theorem 2.10.

Proof of Theorem 2.10. **(i)⇒(ii):** Let $(X, \|\cdot\|)$ be a normed vector space with a convex subset $A \subset X$ that is not non-polyproximal. So there exists $x \in X$ with $y_0, z_0 \in A$ two distinct best approximations of x in A. We define

$$c := d(x, A), y' := \frac{1}{c}(x - y_0), z' := \frac{1}{c}(x - z_0).$$

This is well-defined, because $2c = d(x, y_0) + d(x, z_0) \geq d(y_0, z_0) > 0$, which gives $c > 0$. We get $y' \neq z'$ and $\|y'\| = \|z'\| = 1$. Because A is convex, we get $v := \frac{1}{2}(y_0 + z_0) \in A$, so

$$\|y' + z'\| = \frac{2}{c} d(x, v) \geq \frac{2}{c} d(x, A) = 2.$$

We conclude that X is not strictly convex.

By contraposition, this proves (i)⇒(ii).

(ii)⇒(iii): This follows from the fact that one-dimensional real linear subspaces of vector spaces are always convex.

(iii)⇒(iv): If the unit sphere of a normed vector space contains a line segment I, according to Proposition 2.11 all elements of I are best approximations of 0 in the line L extended from I, so L is not non-polyproximal. If we translate L such that it passes through the origin, this property is preserved, so we find a one-dimensional real linear subspace that is not non-polyproximal.

By contraposition, this proves (iii)⇒(iv).

(iv)⇒(i): Let $(X, \|\cdot\|)$ be a normed vector space that is not strictly convex. So there exist $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$ and $\|x + y\| = 2$. We define $z := \frac{1}{2}(x + y)$ such that z is strictly between x and y and $\|x\| = \|y\| = \|z\| = 1$.

Let v be a point between x and z. Notice that this also means that v is on the extended part of the line through z and y. According to Proposition 2.11, we get both $\|v\| \leq 1$ and $\|v\| \geq 1$, so $\|v\| = 1$. Because this holds for all points v between x and z, the line segment from x to z is contained in the unit sphere.

By contraposition, this proves (iv)⇒(i) and completes the proof of Theorem 2.10. □

2.3 Convexity of Chebyshev subsets of uniformly smooth normed vector spaces

In this subsection we will first give the definition of a uniformly smooth normed vector space. In this subsection and in the next one, we will use this notion to characterize finite-dimensional normed vector spaces where all Chebyshev subsets are convex. In this subsection we will give sufficient conditions and in the next subsection we will give necessary conditions. The main result of this subsection will be that any Chebyshev subset of a uniformly smooth finite-dimensional normed vector space is convex.

So to start off, we will look at uniformly smooth normed vector spaces.

Definition 2.12 (uniformly smooth)**.** We call a normed vector space $(X, \|\cdot\|)$ uniformly smooth, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for all $x, y \in X$ we have

$$\|x\| = 1, \|y\| \leq \delta \Rightarrow \|x + y\| + \|x - y\| \leq 2 + \varepsilon \|y\| .$$

Example 2.13 (uniformly smooth normed vector spaces)**.** (i) Inner product spaces. The proof is on p.33.

(ii) The ℓ^p and L^p spaces for $p \in (1, \infty)$. The proof is on p.33.

We can now state the main result of this subsection.

Theorem 2.14 (Chebyshev sets in uniformly smooth finite dimensional normed vector spaces)**.** *All Chebyshev subsets of uniformly smooth finite-dimensional normed vector spaces are convex.*

Before we prove this theorem, we need to introduce the following definition.

Definition 2.15 (projection map)**.** Let (V, d) be a metric space and $A \subset V$ a Chebyshev subset. Because this means that for all $x \in V$ there is a unique best approximation for x in A, we can define a function $\gamma_A : V \to A$ such that $\gamma_A(x)$ is the best approximation for x in A. We call the function γ_A the projection map of A.

Intuitively, one might think that the projection map is always continuous, or at least if the corresponding Chebyshev set is a linear subspace. However, this is not the case. See Example 4.4.

Luckily, many 'naturally occurring' projection maps are continuous. For example, we have the following proposition about the continuity of the projection map.

Proposition 2.16 (continuous projection maps)**.** *For any Chebyshev set with the Heine-Borel property, its projection map is continuous.*

To prove this proposition, we are going to use the following lemma.

Lemma 2.17 (convergence and subsequences of subsequences)**.** *Let $(X, \mathcal{T})$ be a topological space with an element $x \in X$ and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X, such that every subsequence $(x_{n_i})_{i\in\mathbb{N}}$ has itself a subsequence $(x_{n_{i_j}})_{j\in\mathbb{N}}$ with limit x. Then the sequence $(x_n)_{n\in\mathbb{N}}$ converges to x.*

Proof of Lemma 2.17. The proof is by contraposition. If the sequence $(x_n)_{n\in\mathbb{N}}$ does not converge to x, then by definition there exists a neighborhood of x such that infinitely many elements of the sequence $(x_n)_{n\in\mathbb{N}}$ are not inside this neighborhood. These elements form a subsequence which itself does not have a subsequence with limit x. □

Proof of Proposition 2.16. Let (V, d) be a metric space and let $A \subset V$ be a Chebyshev subset with the Heine-Borel property. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in V with limit $x \in V$. We will show that $(\gamma_A(x_n))_{n\in\mathbb{N}}$ has limit $\gamma_A(x)$ by using Lemma 2.17.

First note that, for every convergent subsequence $(\gamma_A(x_{n_i}))_{i\in\mathbb{N}}$ of $(\gamma_A(x_n))_{n\in\mathbb{N}}$ with limit $y \in A$, we get

$$\begin{aligned} d(x, y) &= \lim_{i\to\infty} d(x_{n_i}, \gamma_A(x_{n_i})) \\ &\leq \lim_{i\to\infty} d(x_{n_i}, \gamma_A(x)) && \text{by the definition of } \gamma_A \\ &= d(x, \gamma_A(x)). \end{aligned}$$

Because A is Chebyshev, this gives $y = \gamma_A(x)$.

Because the sequence $(x_n)_{n\in\mathbb{N}}$ is convergent, it is bounded. It follows that the sequence $(\gamma_A(x_n))_{n\in\mathbb{N}}$ is also bounded, so all of its subsequences are bounded as well. By the Heine-Borel property, this means that every subsequence has itself a convergent subsequence with limit in A, for which we already deduced the limit is $\gamma_A(x)$. By Lemma 2.17, this means that the sequence $(\gamma_A(x_n))_{n\in\mathbb{N}}$ itself converges with limit $\gamma_A(x)$. Therefore, γ_A is continuous.

This proves Proposition 2.16. □

We are now going to use this proposition to prove the following lemma, which we will use to prove Theorem 2.14.

Lemma 2.18 (projection map on a ray)**.** *Let $(X, \|\cdot\|)$ be a normed vector space and let $A \subset X$ be a Chebyshev subset and let $x \in X$ be a point. Furthermore, let $\lambda \in [0, \infty)$ and take $y := \gamma_A(x) + \lambda(x - \gamma_A(x))$. The following statements hold.*

(i) If $\lambda \in [0, 1]$ then $\gamma_A(y) = \gamma_A(x)$. So all points between $\gamma_A(x)$ and x have the same best approximation.

(ii) If X is finite-dimensional, then the same still holds if $\lambda \in (1, \infty)$. So all points on the ray from $\gamma_A(x)$ through x have the same best approximation.

Proof of Lemma 2.18. Let $(X, \|\cdot\|)$ be a normed vector space and let $A \subset X$ be a Chebyshev subset and let $x \in X$ be a point. Furthermore, let $\lambda \in [0, \infty)$ and take

$$y := \gamma_A(x) + \lambda(x - \gamma_A(x)).$$

(i): We assume $\lambda \in [0, 1]$. We get

$$\begin{aligned}
\|x - \gamma_A(y)\| &\leq \|x - y\| + \|y - \gamma_A(y)\| \\
&\leq \|x - y\| + \|y - \gamma_A(x)\| && \text{by definition of } \gamma_A \\
&= (1 - \lambda)\|x - \gamma_A(x)\| + \lambda \|x - \gamma_A(x)\| && \text{by definition of } y \\
&= \|x - \gamma_A(x)\|.
\end{aligned}$$

Because A is Chebyshev, this gives $\gamma_A(y) = \gamma_A(x)$.

This proves (i).

(ii): We assume that X is finite-dimensional and that $\lambda \in (1, \infty)$. For $r := d(x, y)$ we define $B := \overline{B}_r^X(x)$ and $S := S_r^X(x) \subset B$. We define a map $\rho : B \to S$ where $\rho(v)$ is the unique point in S, such that x is between $\gamma_A(v)$ and $\rho(v)$.

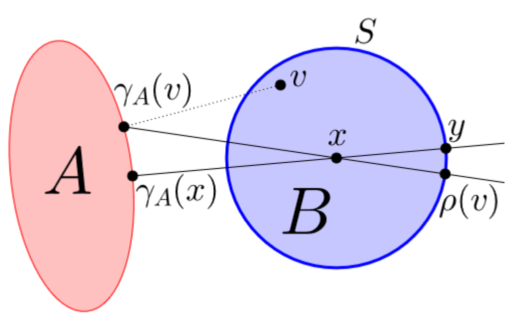

Figure 2: The function ρ.

Because X is finite-dimensional, it follows from Theorem 2.5 (i)$\Rightarrow$(ii) that X has the Heine-Borel property. Because A is Chebyshev, by Corollary 2.3 it is closed, so it also has the Heine-Borel property. Therefore it follows from Proposition 2.16 that γ_A is continuous, which makes ρ continuous as well.

By Brouwer's fixed point theorem, we find some $v \in S$ with $\rho(v) = v$. This means that x is between $\gamma_A(v)$ and v. From (i) it follows that $\gamma_A(v) = \gamma_A(x)$. We get that v is past x on the ray from $\gamma_A(x)$ through x. Because $v \in S$, we also get that $d(x, v) = d(x, y)$. We are forced to conclude that $v = y$ and hence $\gamma_A(y) = \gamma_A(x)$.

This proves (ii) and finishes the proof of Lemma 2.18. □

We can now finally prove Theorem 2.14.

Proof of Theorem 2.14. Let $(X, \|\cdot\|)$ be a uniformly smooth normed vector space and let $A \subset X$ be a Chebyshev subset that is not convex. Using Corollary 2.3 we get that A is closed. Because A is closed, but not convex, we can find $a_1, a_2 \in A$ such that $x := \frac{1}{2}(a_1 + a_2) \notin A$. Notice that $a_1 \neq a_2$ and

$$x - a_2 = -(x - a_1). \tag{2.1}$$

We define y_0 as the best approximation of x in A.

We now define

$$\varepsilon := 2\frac{\|x - y_0\|}{\|x - a_1\|} > 0.$$

Because X is uniformly smooth, there exists $\delta > 0$ such that for all $v, w \in X$ we have

$$\|v\| = 1, \|w\| \leq \delta \Rightarrow \|v + w\| + \|v - w\| \leq 2 + \varepsilon \|w\|. \tag{2.2}$$

We define

$$\begin{aligned}
\lambda &:= \frac{\|x - a_1\|}{\delta \|x - y_0\|} + 1 \in (1, \infty) \\
z &:= y_0 + \lambda(x - y_0) \\
z' &:= (\lambda - 1)(x - y_0) \\
v &:= \frac{1}{\|z'\|} z' \\
w &:= \frac{1}{\|z'\|}(x - a_1).
\end{aligned}$$

Notice that

$$z = z' + x. \tag{2.3}$$

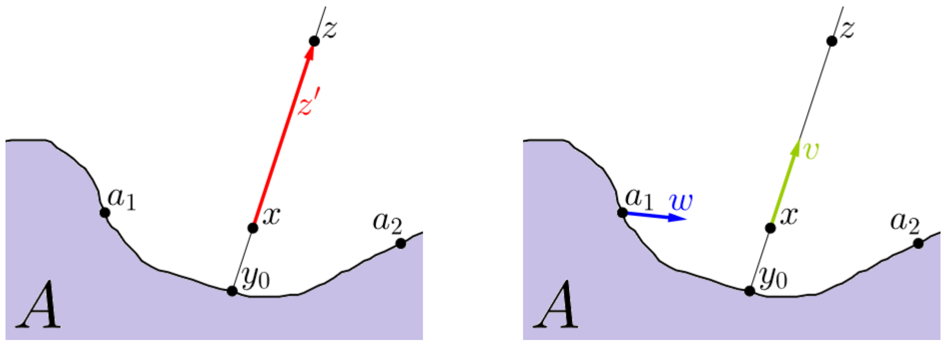

Figure 3: All the defined points and vectors.

We get $\|v\| = 1$ and from the definition of λ we get

$$\|w\| = \frac{\|x - a_1\|}{\|z'\|} = \frac{\|x - a_1\|}{(\lambda - 1)\|x - y_0\|} = \delta.$$

From (2.2) we get

$$\|v + w\| + \|v - w\| \leq 2 + \varepsilon \|w\|. \tag{2.4}$$

Combining everything, we get

$$\begin{aligned}
\|z - a_1\| + \|z - a_2\| &= \|z' + x - a_1\| + \|z' - (x - a_1)\| && \text{by (2.3) and (2.1)}\\
&= \|z'\| (\|v + w\| + \|v - w\|) && \text{definitions } v \text{ and } w\\
&\leq \|z'\| (2 + \varepsilon \|w\|) && \text{by (2.4)}\\
&= 2(\|z'\| + \|x - y_0\|) && \text{definitions } \varepsilon \text{ and } w\\
&= 2\|z - y_0\|. && \text{definitions } z' \text{ and } z
\end{aligned}$$

This inequality implies that either $\|z - a_1\| < \|z - y_0\|$ or $\|z - a_2\| < \|z - y_0\|$ or $\|z - a_1\| = \|z - a_2\| = \|z - y_0\|$ holds. But $a_1 \neq a_2$, so from Lemma 2.18 it follows that X can not be finite-dimensional, because otherwise we get that y_0 is the unique best approximation of z in A.

By contraposition, this proves Theorem 2.14. □

Remark. This proof, especially Proposition 2.16 and Lemma 2.18, was inspired by a proof of a weaker theorem by Chang Hai Bin [Bin, Theorem 1.2].

2.4 Non-convex Chebyshev subsets of normed vector spaces

In this subsection we will continue characterizing finite-dimensional normed vector spaces where all Chebyshev subsets are convex. In this subsection we will give necessary conditions. The main result of this subsection will be that any strictly convex finite-dimensional normed vector space that is not uniformly smooth admits a Chebyshev subset that is not convex.

We will start by stating the main result of this subsection. This is one of the main ingredients of the proof of Theorem 1.5.

Lemma 2.19 (non-convex Chebyshev sets)**.** *For any strictly convex finite-dimensional normed vector space that is not uniformly smooth, there exists a Chebyshev subset that is not convex.*

To prove this lemma, we are going to use the following lemma.

Lemma 2.20 (distinct equidistant linear subspaces)**.** *Let* $(X, \|\cdot\|)$ *be a real finite-dimensional normed vector space that is not uniformly smooth. Then there exist two distinct linear subspaces* $Y_1, Y_2 \subset X$ *and a point* $x \in \operatorname{span}(Y_1 \cup Y_2)$ *with* $d(x, Y_1) = d(x, Y_2) = \|x\| = 1$.

To prove this lemma, we are going to use the following proposition.

Proposition 2.21 (derivatives of concave functions)**.** *Let* I *be an open interval and let* $f : I \to \mathbb{R}$ *be a concave function. Then* f *is left and right differentiable.*

Remark. A function $f : I \to \mathbb{R}$ is called concave if the set $\{(x, y) \in I \times \mathbb{R} | y \leq f(x)\}$ is convex. Equivalently, the 'concavity inequality' $f(\lambda x + (1 - \lambda)y) \geq \lambda f(x) + (1 - \lambda)f(y)$ holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

Proof of Proposition 2.21. Let $x \in I$. Because I is open, there exists $r > 0$ such that $[x - r, x + r) \subset I$. We can therefore define the function $g : [-r, r) \setminus \{0\} \to \mathbb{R}$ by $g(h) := \frac{1}{h}(f(x + h) - f(x))$.

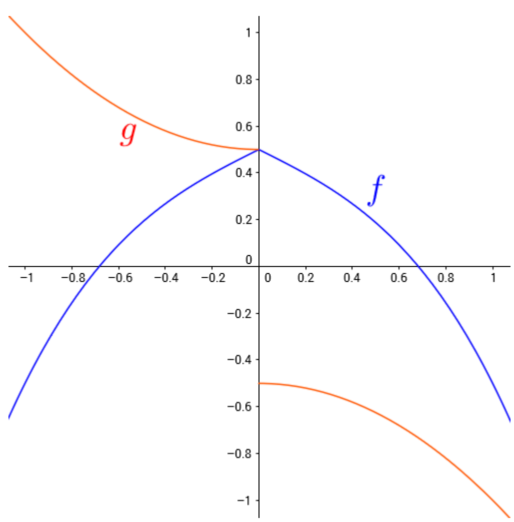

Figure 4: Example of g where $f : I := \mathbb{R} \to \mathbb{R}$, $f(x) := \frac{1}{2}(1 - |x|^3 - |x|)$ and $x := 0 \in I$.

Let $h, h' \in (0, r)$ with $h \leq h'$. For $\lambda := 1 - \frac{h}{h'} \in [0, 1]$ we get

$$\begin{aligned} g(h) &= \frac{f(x+h) - f(x)}{h} \\ &\geq \frac{\lambda f(x) + (1-\lambda) f(x+h') - f(x)}{h} && \text{concavity inequality} \\ &= \frac{f(x+h') - f(x)}{h'} \\ &= g(h'). \end{aligned}$$

So g is non-increasing on $(0, r)$. By a similar argument, $g(h)$ is bounded by $g(-r)$ for $h \in (0, r)$. To see this, take $\lambda := 1 + \frac{h}{r} > 1$ and use the concavity inequality backwards. We get that

$$\lim_{h \to 0^+} g(h) = \sup_{h \in (0,r)} g(h) \in \mathbb{R}. \tag{2.5}$$

By definition, this means f is right differentiable at x. Because this holds for all $x \in I$, f is right differentiable. By symmetry, we get that f is also left differentiable.

This proves Proposition 2.21. □

We are now ready to prove Lemma 2.20.

Proof of Lemma 2.20. Let $(X, \|\cdot\|)$ be a real finite-dimensional normed vector space that is not uniformly smooth.

Claim 1. There exist $\varepsilon > 0$ and $x, y \in X$ with $\|x\| = \|y\| = 1$ such that for all $\lambda > 0$ we have $\|x + \lambda y\| + \|x - \lambda y\| \geq 2 + \varepsilon\lambda$.

Because X is not uniformly smooth, there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist $x, y \in X$ such that $\|x\| = 1$, $\|y\| \leq \delta$ and $\|x + y\| + \|x - y\| > 2 + \varepsilon \|y\|$. Take any sequence $(\delta_i)_{i \in \mathbb{N}}$ in $(0, \infty)$ which converges to 0. For all $i \in \mathbb{N}$ we find $x_i, \tilde{y}_i \in X$ such that

$$\|x_i\| = 1, \|\tilde{y}_i\| \leq \delta_i, \|x_i + \tilde{y}_i\| + \|x_i - \tilde{y}_i\| > 2 + \varepsilon \|\tilde{y}_i\|. \tag{2.6}$$

Note that $\tilde{y}_i$ can not be 0, so $\|\tilde{y}_i\| > 0$, so we can define $y_i := \frac{1}{\|\tilde{y}_i\|}\tilde{y}_i$, which gives $\|y_i\| = 1$. Because X is finite-dimensional, according to Theorem 2.5 (i)⇒(ii) it has the Heine-Borel property. Because $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ are bounded, passing to a subsequence we can assume without loss of generality that these sequences have limits $x, y \in X$ respectively with $\|x\| = \|y\| = 1$.

Claim 2. For all $\lambda > 0$ we have $\|x + \lambda y\| + \|x - \lambda y\| \geq 2 + \varepsilon\lambda$.

Let $\lambda > 0$. We get

$$\|x + \lambda y\| + \|x - \lambda y\| = \lim_{i \to \infty} \left(\left\| x_i + \frac{\lambda}{\|\tilde{y}_i\|} \tilde{y}_i \right\| + \left\| x_i - \frac{\lambda}{\|\tilde{y}_i\|} \tilde{y}_i \right\| \right).$$

In evaluating this limit, we can assume i to be large enough such that $\|\tilde{y}_i\| \leq \lambda$. Because $\|\tilde{y}_i\| > 0$, we can define $\lambda_i := \frac{\lambda}{\|\tilde{y}_i\|} \geq 1$. Using inequality (2.6) and $\|x_i\| = 1$, we get

$$\begin{aligned} \|x + \lambda y\| + \|x - \lambda y\| &= \lim_{i\to\infty} \left(\|x_i + \lambda_i \tilde{y}_i\| + \|x_i - \lambda_i \tilde{y}_i\| \right) \\ &\geq \lim_{i\to\infty} \left(\lambda_i \|x_i + \tilde{y}_i\| - (\lambda_i - 1)\|x_i\| + \lambda_i \|x_i - \tilde{y}_i\| - (\lambda_i - 1)\|x_i\| \right) \\ &\geq \lim_{i\to\infty} \left(\lambda_i (2 + \varepsilon \|\tilde{y}_i\|) - 2\lambda_i + 2\right) \\ &= 2 + \varepsilon\lambda. \end{aligned}$$

This proves Claim 2 and finishes the proof of Claim 1.

We choose $\varepsilon > 0$ and $x, y \in X$ as in Claim 1. So we have $\|x\| = \|y\| = 1$ and

$$\forall \lambda > 0 : \|x + \lambda y\| + \|x - \lambda y\| \geq 2 + \varepsilon\lambda. \tag{2.7}$$

We define the function $f : (-1, 1) \to (0, \infty)$ by $f(a) := \sup \{b \in \mathbb{R} |\, \|ay + bx\| \leq 1\} > 0$.

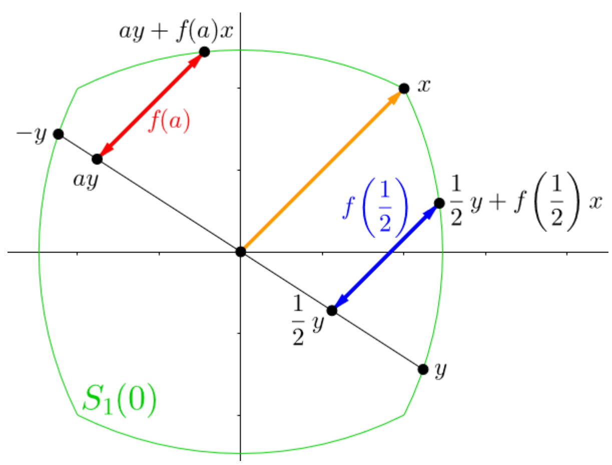

Figure 5: Here $S_1(0)$ represents the unit sphere, so $f(a)$ measures the distance from ay to the unit sphere in the direction x.

It follows from $\|y\| = 1$ and the properties of a norm that f is well-defined. From the continuity of norms with respect to themselves it follows that

$$\forall a \in (-1, 1) : \|ay + f(a)x\| = 1. \tag{2.8}$$

From the triangle inequality it follows that f is concave. Using Proposition 2.21 we get that f is left and right differentiable. We denote L^- as the left derivative at 0 and L^+ as the right derivative at 0.

Claim 3. We have $L^+ < L^-$.

We define $\varepsilon' := \frac{1}{4}\varepsilon$. Note that $f(0) = 1$, so by definition of left and right derivatives, there exists $\delta > 0$ with

$$\left|1 - f(-\delta) - \delta L^-\right|, \left|f(\delta) - 1 - \delta L^+\right| < \delta\varepsilon'. \tag{2.9}$$

Therefore we have

$$|f(\delta) - f(-\delta)| \leq |f(\delta) - 1| + |1 - f(-\delta)| < \delta(L^+ + L^- + 2\varepsilon'). \tag{2.10}$$

We define $\lambda := \frac{\delta}{f(\delta)} > 0$. Because $\|x\| = 1$, we get

$$\begin{aligned} 2 + |f(\delta) - f(-\delta)| &= \|f(\delta)x + \delta y\| + \|f(-\delta)x - \delta y\| + \|(f(\delta) - f(-\delta))x\| && \text{by (2.8)} \\ &\geq \|f(\delta)x + \delta y\| + \|f(\delta)x - \delta y\| \\ &= f(\delta)(\|x + \lambda y\| + \|x - \lambda y\|) \\ &\geq f(\delta)(2 + \varepsilon\lambda) && \text{by (2.7)} \\ &= 2(f(\delta) + \varepsilon'\delta) + 2\varepsilon'\delta \\ &> 2(f(\delta) + \left|f(\delta) - 1 - \delta L^+\right|) + 2\varepsilon'\delta && \text{by (2.9)} \\ &\geq 2(1 + \delta L^+) + 2\varepsilon'\delta. \end{aligned}$$

By combining this with inequality (2.10) we get that $L^+ < L^-$.

This proves Claim 3.

We define $Y_1 := \text{span}\{y + L^+x\}$ and $Y_2 := \text{span}\{y + L^-x\}$. Because $L^+ < L^-$, it follows that Y_1 and Y_2 are two distinct linear subspaces of X with $x \in \text{span}(Y_1 \cup Y_2)$.

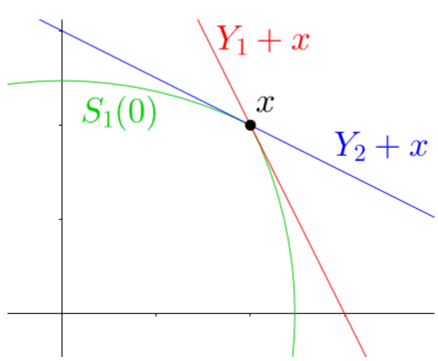

Figure 6: Because $a \mapsto y + f'(a)x$ is the derivative, with respect to a, of $a \mapsto ay + f(a)x$, which maps to the unit sphere, we find $\text{span}\{y + f'(a)x\} + ay + f(a)x$ as tangent to the unit sphere. Substituting $a = 0$ we find that the lines $Y_1 + x$ and $Y_2 + x$ are tangent to the unit sphere at x.

Claim 4. We have $d(x, Y_1) = d(x, Y_2) = 1$.

Let $\lambda > 0$. Let $\varepsilon > 0$. By definition of L^+ we find a $\delta > 0$ with

$$\left|f(\delta) - 1 - \delta L^+\right| < \delta\varepsilon.$$

We assume without loss of generality that $\delta < \lambda$. We get

$$\begin{aligned}\left\|\lambda y + (1 + \lambda L^+)x\right\| &\geq \frac{\lambda}{\delta}\left\|\delta y + (1 + \delta L^+)x\right\| - \left(\frac{\lambda}{\delta} - 1\right)\|x\| \\ &\geq \frac{\lambda}{\delta}\big(\left\|\delta y + f(\delta)x\right\| - \left|f(\delta) - 1 - \delta L^+\right|\big) - \frac{\lambda}{\delta} + 1 && \|x\| = 1 \\ &> 1 - \varepsilon\lambda. && \text{by (2.8)}\end{aligned}$$

Because this holds for all $\varepsilon > 0$, we get $\|\lambda y + (1 + \lambda L^+)x\| \geq 1$. Note that this holds for all $\lambda > 0$. It follows that for any $\lambda > 0$, if we take $\mu := \lambda(L^- - L^+) + 1 > 1$ and $\lambda' := \lambda/\mu > 0$, we get $\|\lambda y + (1 + \lambda L^-)x\| = \mu\,\|\lambda' y + (1 + \lambda' L^+)x\| > 1$ as well.

So we have shown that

$$\forall\lambda > 0 : \left\|\lambda y + (1 + \lambda L^+)x\right\|, \left\|\lambda y + (1 + \lambda L^-)x\right\| \geq 1.$$

Similarly, by interchanging the roles of L^+ and L^-, we can show that

$$\forall\lambda < 0 : \left\|\lambda y + (1 + \lambda L^+)x\right\|, \left\|\lambda y + (1 + \lambda L^-)x\right\| \geq 1.$$

For all $v = \lambda(y + L^+x) \in Y_1$ we get

$$d(x, v) = \left\|(-\lambda)y + (1 + (-\lambda)L^+)x\right\| \geq 1.$$

If $\lambda = 0$ we actually have equality, because $\|x\| = 1$. We therefore have $d(x, Y_1) = 1$. Similarly, we have $d(x, Y_2) = 1$.

This proves Claim 4 and finishes the proof of Lemma 2.20. □

The following proposition will also be used to prove Lemma 2.19.

Proposition 2.22 (distance preserving subspace extension)**.** *Let $(X, \|\cdot\|)$ be a real normed vector space with a point $x \in X$ and a closed linear subspace $Y \subset X$ with codimension at least 2, such that $d(x, Y) = 1$. Then there exists a linear subspace $Y \subset Z \subset X$ in which Y has codimension 1, such that $d(x, Z) = 1$.*

Remark. The codimension of a linear subspace of a vector space is the dimension of the quotient space induced by it. In finite dimensions, you can just subtract the dimension of the subspace from the dimension of the vector space to obtain the codimension.

To prove this proposition, we are going to use the following proposition.

Proposition 2.23 (disjoint sets with positive distance)**.** *Let* (V, d) *be a metric space with a closed subset* $A \subset V$ *and a compact subset* $B \subset V$*. If* A *and* B *are disjoint, then* $d(A, B) > 0$*.*

Proof of Proposition 2.23. The proof is by contraposition. If $d(A, B) = 0$, then there is a sequence $(x_i)_{i\in\mathbb{N}}$ in B such that $d(x_i, A)$ has limit 0. Because B is compact, it is sequentially compact, so we find a convergent subsequence with limit $x \in B$. It follows from the continuity of metrics with respect to themselves that $d(x, A) = 0$. Because A is closed, we get $x \in A$, so $x \in A \cap B$, so A and B are not disjoint. □

We are now ready to prove Proposition 2.22.

Proof of Proposition 2.22. Because Y has codimension at least 2, there exists $v \in X \setminus \operatorname{span}(Y \cup \{x\})$. We define $V := \operatorname{span}\{x, v\}$. We now define the function $f : S := S_1^V(0) \to [0, 1]$ by

$$f(z) := d(x, Y + [0, \infty) \cdot z).$$

To see why this is well-defined, note that $f(z) \le d(x, Y) = 1$.

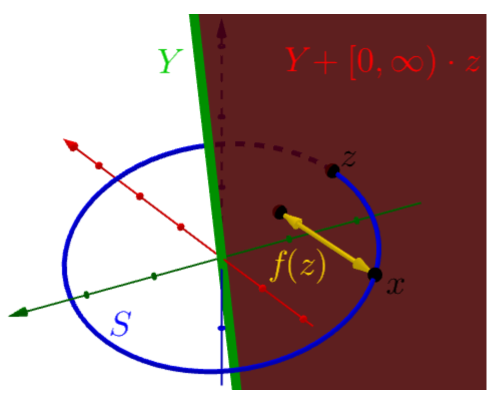

Figure 7: The set $Y + [0, \infty) \cdot z$ and the function f in the case $X = \mathbb{R}^3$ and Y one-dimensional. Here $Y + [0, \infty) \cdot z$ is half a plane, and f calculates the distance from x to that plane. The idea is that for $Z := \operatorname{span}(Y \cup \{z\})$ we have $d(x, Z) = \min\{f(z), f(-z)\}$, so if $f(z) = f(-z) = 1$, then $d(x, Z) = 1$.

Claim 1. The function f is continuous.

Because V is finite-dimensional, according to Theorem 2.5 (i)⇒(ii) it has the Heine-Borel property. Because S is a closed bounded subset of V, it follows that it is compact. Because Y is closed and disjoint from S, it follows from Proposition 2.23 that $r := d(S, Y) > 0$. For all $\lambda > s := \frac{1}{r}(\|x\| + 1)$ and $z \in S$ we get

$$d(x, Y + \lambda z) \ge \lambda \cdot d(z, Y) - \|x\| \ge \lambda r - \|x\| > 1.$$

Therefore, we have $f(z) = d(x, Y + [0, s] \cdot z)$ for all $z \in S$. It follows that we have $|f(z) - f(z')| \leq s \cdot d(z, z')$ for all $z, z' \in B$, which implies the continuity of f.

This proves Claim 1.

We can use the Borsuk-Ulam theorem for $n = 1$ to find $z \in S$ with $f(z) = f(-z)$.

Claim 2. We have $f(z) = 1$.

The proof is by contradiction. If we assume that $f(z) < 1$, then we can find $y_1, y_2 \in Y$ and $\lambda_1, \lambda_2 \geq 0$ with $d(x, y_1 + \lambda_1 z), d(x, y_2 - \lambda_2 z) < 1$. Notice that it follows from $d(x, Y) = 1$ that $\lambda_1, \lambda_2 > 0$. For $\mu_i := \lambda_i/(\lambda_1 + \lambda_2) > 0$ $(i \in \{1, 2\})$ we get

$$\|x - (\mu_2 y_1 + \mu_1 y_2)\| \leq \mu_2 \|x - (y_1 + \lambda_1 z)\| + \mu_1 \|x - (y_2 - \lambda_2 z)\| < \mu_2 + \mu_1 = 1$$

This contradicts $d(x, Y) = 1$, which proves Claim 2.

It follows that for $Z := \operatorname{span}(Y \cup \{z\})$ we get $d(x, Z) = 1$, since $Z = Y + \mathbb{R} \cdot z = (Y + [0, \infty) \cdot z) \cup (Y + [0, \infty) \cdot (-z))$.

This proves Proposition 2.22. □

We are now ready to prove Lemma 2.19.

Proof of Lemma 2.19. Let $(X, \|\cdot\|)$ be a strictly convex finite-dimensional normed vector space that is not uniformly smooth. We assume without loss of generality that X is a real vector space. With Lemma 2.20 we find two distinct linear subspaces $Y_1, Y_2 \subset X$ and a point $x \in \operatorname{span}(Y_1 \cup Y_2)$ with $d(x, Y_1) = d(x, Y_2) = \|x\| = 1$.

For all $i \in \{1, 2\}$ we use Proposition 2.22 inductively to find a hyperspace $Y_i \subset Z_i \subset X$ such that $d(x, Z_i) = 1$. Because X is strictly convex and finite-dimensional, and because Z_i is a non-empty closed convex subset, by Theorem 2.5 (i)⇒(iii) and Theorem 2.10 (i)⇒(ii) the set Z_i is Chebyshev. Because $d(x, Z_i) = \|x\| = 1$, we get that 0 is the best approximation for x in Z_i, and we get that $x \notin Z_i$. From this it follows that for each $v \in X$ there is a unique way to write it as $v = z_i + \lambda_i x$ where $z_i \in Z_i$ and $\lambda_i \in \mathbb{R}$.

We now define $Z_i' := Z_i + (-\infty, 0] \cdot x$ for $i \in \{1, 2\}$ and $A := Z_1' \cup Z_2'$. We get that $x \notin A$. But it is given that $x \in \operatorname{span}(Y_1 \cup Y_2)$, so because Y_1 and Y_2 are linear subspaces, we get that x is in the convex hull of $Y_1 \cup Y_2$. Since $Y_1 \cup Y_2 \subset A$ we also get that x is in the convex hull of A. It follows that A is not convex.

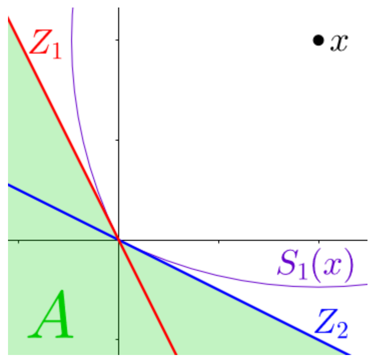

Figure 8: In $X = \mathbb{R}^2$ hyperspaces Z_1 and Z_2 are one-dimensional and A extends them.

Claim 1. The set A is Chebyshev.

Let $v = z_1 + \lambda_1 x = z_2 + \lambda_2 x \in X$. If $\lambda_1 \leq 0$ or $\lambda_2 \leq 0$, then $v \in A$, so v is the unique best approximation for v in A. Otherwise, for all $i \in \{1, 2\}$ it follows from the fact that 0 is the unique best approximation of x in Z_i that for all $v' = z_i' + \lambda_i' x \in Z_i'$ with $v' \neq z_i$ we get

$$\begin{aligned} d(v, v') &= \|(z_i - z_i') + (\lambda_i - \lambda_i')x\| \\ &= (\lambda_i - \lambda_i') \left\| \frac{1}{\lambda_i - \lambda_i'}(z_i - z_i') + x \right\| \\ &> \lambda_i \|0 - x\| \\ &= d(v, z_i). \end{aligned}$$

Here $\lambda_i - \lambda_i' > 0$ holds, because $\lambda_i > 0$ and $\lambda_i' \leq 0$, and the inequality is strict, because from $v' \neq z_i$ it follows that at least one of $\lambda_i' < 0$ and $z_i \neq z_i'$ holds. So z_i is the unique best approximation of v in Z_i'.

It follows that either z_1 or z_2 or both are best approximations of v in A. But if both are best approximations of v in A, then $d(v, z_1) = d(v, z_2)$, so it follows that $\lambda_1 = \lambda_2$, which implies that $z_1 = z_2$. So the best approximation of v in A is always unique, which means that A is Chebyshev.

This proves Claim 1 and finishes the proof of Lemma 2.19. □

2.5 Proof of Theorem 1.5

In this subsection we finally prove Theorem 1.5, the main result of this section.

Proof of Theorem 1.5. **(i)⇒(ii):** This is a direct combination of Theorem 2.5 (i)⇒(iii), Theorem 2.10 (i)⇒(ii), Corollary 2.3 and Theorem 2.14.

(ii)⇒(i): This is a direct combination of Theorem 2.10 (ii)⇒(i) and Lemma 2.19.

This completes the proof of Theorem 1.5. □

3 Chebyshev sets in strictly convex reflexive normed vector spaces: proof of Theorem 1.6

In this section we prove Theorem 1.6. As explained in the introduction of this thesis, we will first show that the theorem holds for incomplete spaces, which comes down to showing that (ii) does not hold. It looks like this result is new. We will then show the known proof that the theorem holds for Banach spaces. Finally, we will combine these results to prove Theorem 1.6.

3.1 Proximal sets and completeness

In this subsection we will show that any incomplete normed vector space admits a non-empty closed convex subset that is not proximal. This is also the main result of this subsection. In the next subsection we will introduce reflexive normed vector spaces and we will see that incomplete normed vector spaces can not be reflexive. Therefore the main result of this subsection implies that Theorem 1.6 holds for incomplete spaces.

The following lemma is the main result of this subsection. It is a lemma for Theorem 1.6, the main result of this section.

Lemma 3.1 (proximal sets in incomplete normed vector space)**.** *Any incomplete normed vector space admits a non-empty closed convex subset that is not proximal.*

To prove this lemma, we will introduce the following two definitions.

Definition 3.2 (cone)**.** Let $(X, \|\cdot\|)$ be a normed vector space and $x \in X \setminus \{0\}$ a point. The cone of x is defined as

$$\widehat{x} := \bigcup_{r \in [0,\infty)} \overline{B}_r^X \left(\left(1 + \frac{2r}{\|x\|} \right) x \right).$$

For an example, see Figure 9.

Definition 3.3 (liminf of sets)**.** Let $(A_i)_{i \in \mathbb{N}}$ be a sequence of sets. The liminf of the sequence is defined as

$$\liminf_{i \to \infty} A_i := \bigcup_{N \in \mathbb{N}} \bigcap_{i \geq N} A_i.$$

Before we use these definitions to prove Lemma 3.1, we will first prove some properties of these definitions.

Lemma 3.4 (convex cones)**.** *Any cone is convex.*

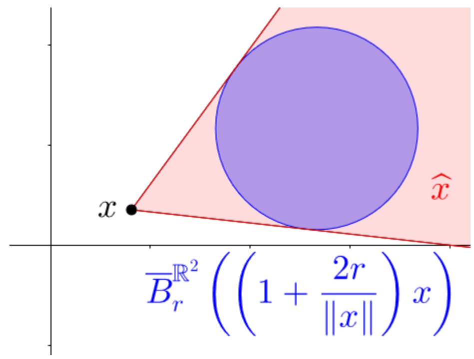

Figure 9: A cone in $\mathbb{R}^2$.

Proof of Lemma 3.4. Let $(X, \|\cdot\|)$ be a normed vector space and $x \in X \setminus \{0\}$ a vector. Let $v, w \in \widehat{x}$ and $\lambda \in [0,1]$. So we have $v \in \overline{B}_r^X((1+\frac{2r}{\|x\|})x)$ and $w \in \overline{B}_s^X((1+\frac{2s}{\|x\|})x)$ for some $r, s \in [0,\infty)$. We define $t := \lambda r + (1-\lambda)s \in [0,\infty)$. It follows from the triangle inequality that $\lambda v + (1-\lambda)w \in \overline{B}_t^X((1+\frac{2t}{\|x\|})x) \subset \widehat{x}$, so $\widehat{x}$ is convex. $\square$

Lemma 3.5 (convex liminf)**.** *The liminf of a sequence of convex sets is convex.*

Proof of Lemma 3.5. Let $(A_i)_{i\in\mathbb{N}}$ be a sequence of convex sets. Let $x, y \in \liminf_{i\to\infty} A_i$ and $\lambda \in [0,1]$. So there exists $N \in \mathbb{N}$ such that for all $i \geq N$ we have $x, y \in A_i$. Now for all $i \geq N$, because A_i is convex, we have $\lambda x + (1-\lambda)y \in A_i$, so $\lambda x + (1-\lambda)y \in \liminf_{i\to\infty} A_i$, so $\liminf_{i\to\infty} A_i$ is convex. $\square$

We are now ready to prove Lemma 3.1.

Proof of Lemma 3.1. Let $(X, \|\cdot\|)$ be a normed vector space where every non-empty closed convex subset is proximal.

Claim 1. The normed vector space X is complete.

Let $(x_i)_{i\in\mathbb{N}}$ be a Cauchy sequence in X. We define

$$c := \lim_{i\to\infty} \|x_i\|.$$

If $c = 0$ then $(x_i)_{i\in\mathbb{N}}$ converges to 0. So from now on, we assume $c > 0$, which means we can also assume without loss of generality that $c > 2$ and that $\|x_i\| > 0$ holds for all $i \in \mathbb{N}$. We define

$$A := \liminf_{i\to\infty} \widehat{x_i}.$$

Combining Lemma 3.4 and Lemma 3.5, we get that A is convex, so $\overline{A}$ is closed and convex.

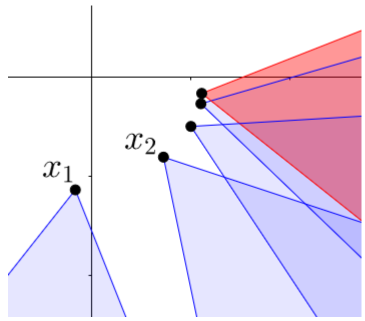

Figure 10: As points get closer, their cones will intersect more and more. Therefore A is some sort of limit cone. Because the tip of a cone represents the origin of that cone, we will see that the tip of A will be the limit of the given sequence. Since A will turn out to be proximal, the tip is the best approximation for 0 in A.

Claim 2. We have

$$\inf_{x \in \overline{A}} \|x\| \leq c.$$

Let $d > c$ and take

$$\varepsilon := \min\left\{1, \frac{d-c}{c+2}, \frac{c-2}{3}\right\} > 0.$$

Now there exists $N \in \mathbb{N}$ such that

$$\forall i \geq N : c - \varepsilon \leq \|x_i\| \leq c + \varepsilon, \tag{3.1}$$

$$\forall i, j \geq N : \|x_i - x_j\| \leq \varepsilon. \tag{3.2}$$

We define $x := (1+\varepsilon)x_N$, such that

$$\begin{aligned}
\|x\| &= (1+\varepsilon)\|x_N\| \\
&\leq (1+\varepsilon)(c+\varepsilon) && \text{by (3.1)} \\
&= c + \varepsilon(c+1+\varepsilon) \\
&\leq c + \varepsilon(c+2) && \varepsilon \leq 1 \\
&\leq c + d - c && \varepsilon \leq \frac{d-c}{c+2} \\
&= d.
\end{aligned}$$

Let $i \geq N$ and define $r_i := \frac{\|x_i\|}{2}\varepsilon > 0$. We get

$$\begin{aligned}
\left\| x - \left(1 + \frac{2r_i}{\|x_i\|}\right) x_i \right\| &= (1+\varepsilon)\,\|x_N - x_i\| \\
&\leq \frac{1}{2}(2 + 3\varepsilon - \varepsilon)\varepsilon && \text{by (3.2)} \\
&\leq \frac{1}{2}(2 + c - 2 - \varepsilon)\varepsilon && \varepsilon \leq \frac{c-2}{3} \\
&= \frac{c-\varepsilon}{2}\varepsilon \\
&\leq r_i. && \text{by (3.1)}
\end{aligned}$$

So $x \in \overline{B}^X_{r_i}((1 + \frac{2r_i}{\|x_i\|})x_i) \subset \widehat{x_i}$. Because this holds for all $i \geq N$, we have $x \in A$, so we have

$$\inf_{x \in A} \|x\| \leq d.$$

Because this holds for all $d > c$, and because $A \subset \overline{A}$, we have

$$\inf_{x \in \overline{A}} \|x\| \leq \inf_{x \in A} \|x\| \leq c.$$

This proves Claim 2.

Note that Claim 2 in particular implies that $\overline{A}$ is non-empty. By choice of X, there exists a best approximation for 0 in $\overline{A}$. That is, a $y \in \overline{A}$ such that

$$\|y\| = \inf_{x \in \overline{A}} \|x\| \leq c.$$

Claim 3. The sequence $(x_i)_{i \in \mathbb{N}}$ converges to y.

By definition of closure, there exists a sequence $(y_i)_{i \in \mathbb{N}}$ in A with limit y, giving

$$\lim_{i \to \infty} \|y_i\| = \|y\| \leq c.$$

Let $\varepsilon > 0$. Now there exists $N \in \mathbb{N}$ such that

$$\forall i \geq N : \|y_i - y\| \leq \varepsilon, \tag{3.3}$$

$$\forall i \geq N : \|y_i\| \leq c + \varepsilon, \tag{3.4}$$

$$\forall i \geq N : \|x_i\| \geq c - \varepsilon. \tag{3.5}$$

Because $y_N \in A$, there exists $N' \in \mathbb{N}$ such that for all $i \geq N'$ we get $y_N \in \widehat{x_i}$. We assume without loss of generality that $N' \geq N$. Now let $i \geq N'$. Because $y_N \in \widehat{x_i}$, for some $r_i \in [0, \infty)$ we get

$$y_N \in \overline{B}^X_{r_i}\left(\left(1 + \frac{2r_i}{\|x_i\|}\right) x_i\right). \tag{3.6}$$

This gives

$$\begin{aligned}
c - \varepsilon + r_i &\leq \|x_i\| + 2r_i - r_i && \text{by (3.5)} \\
&\leq \left(1 + \frac{2r_i}{\|x_i\|}\right)\|x_i\| - \left\|y_N - \left(\left(1 + \frac{2r_i}{\|x_i\|}\right)x_i\right)\right\| && \text{by (3.6)} \\
&\leq \|y_N\| \\
&\leq c + \varepsilon. && \text{by (3.4)}
\end{aligned}$$

We get $r_i \leq 2\varepsilon$, so

$$\begin{aligned}
\|x_i - y\| &\leq \left\|\frac{2r_i}{\|x_i\|}x_i\right\| + \left\|y_N - \left(\left(1 + \frac{2r_i}{\|x_i\|}\right)x_i\right)\right\| + \|y_N - y\| \\
&\leq 2r_i + r_i + \varepsilon && \text{by (3.6) and (3.3)} \\
&\leq 7\varepsilon.
\end{aligned}$$

So the sequence $(x_i)_{i\in\mathbb{N}}$ converges to y.

This proves Claim 3 and finishes the proof of Claim 1. By contraposition, this proves Lemma 3.1. □

3.2 Proximal sets and reflexivity

In this subsection we show the known proof that Theorem 1.6 holds for Banach spaces. In this subsection we will use some uncommon terms. We will give their definitions, but we will not go into detail about the properties that we will use. The main goal of this subsection is to give an idea of how you can prove Theorem 1.6 for Banach spaces.

We recall the definition of a reflexive normed vector space, which requires the definition of the dual space.

Definition 3.6 (dual space)**.** The dual space of a normed vector space $(X, \|\cdot\|)$ is defined as $X' := B(X, \mathbb{K})$, which is the set of all bounded linear functions $x' : X \to \mathbb{K}$.

Remark. Together with the operator norm $\|\cdot\|' : X' \to \mathbb{K}$, $\|x'\|' := \sup_{x\in\overline{B}_1^X(0)} |x'(x)|$, the dual space is a normed vector space. The bidual of X is $X'' := (X')'$.

Definition 3.7 (reflexive space)**.** We call a normed vector space $(X, \|\cdot\|)$ reflexive if the canonical map to its bidual $\iota_X : X \to X''$, $\iota_X(x)(x') := x'(x)$ is surjective.

Remark. Since ι_X is always a linear isometry, if X is reflexive, it becomes an isometric isomorphism. Because completeness is preserved through isomorphisms, and because any dual space is complete, it follows that any reflexive space is complete.

Example 3.8 (reflexive space)**.** (i) Finite-dimensional normed vector spaces. Since their bidual has the same dimension, any injection is also a surjection.

(ii) Uniformly convex Banach spaces. See Theorem 4.3.

Before we state the main result of this subsection, we need to introduce the weak topology.

Definition 3.9 (weak topology)**.** Let $(X, \|\cdot\|)$ be a normed vector space. By definition, all elements of X' are continuous. We define the weak topology on X as the initial topology of X', so the coarsest topology such that all elements of X' are continuous.

Remark. We speak about weakly closed, weak convergence, etc. when we talk about closedness, convergence, etc. with respect to the weak topology.

We can now state the main result of this subsection.

Theorem 3.10 (proximal sets in reflexive spaces)**.** *For a Banach space, the following are equivalent:*

(i) The vector space is reflexive.

(ii) All non-empty weakly closed subsets are proximal.

(iii) All non-empty closed convex subsets are proximal.

(iv) All closed linear subspaces are proximal.

(v) All closed hyperplanes are proximal.

The proof of this uses the following definition.

Definition 3.11 (lower semi-continuous)**.** Let $(X, \mathcal{T})$ be a topological space. A function $f : X \to \mathbb{R} \cup \{\infty\}$ is called lower semi-continuous, if for all $\lambda \in \mathbb{R}$ the set $f^{-1}((-\infty, \lambda])$ is closed.

Proof of Theorem 3.10. **(i)⇒(ii):** Let $(X, \|\cdot\|)$ be a reflexive normed vector space and $A \subset X$ a non-empty weakly closed subset. Let $x \in X$. Since $B := \overline{B}^X_{d(x,A)+1}(x) \cap A$ is weakly closed and bounded, it follows from Alaoglu's theorem that it is weakly compact. Since $y \mapsto \|x - y\|$ is weakly lower semi-continuous, it follows that it attains a minimum when restricted to B, which gives us a best approximation of x in A.

(ii)⇒(iii): This follows from the fact that closed convex sets are weakly closed.

(iii)⇒(iv): This follows from the fact that linear subspaces are always non-empty and convex.

(iv)⇒(v): This follows from the fact that hyperplanes are linear subspaces.

(v)⇒(i): Let $(X, \|\cdot\|)$ be an irreflexive Banach space. By James' theorem we find an element $x' \in X'$ which does not attain its norm. It follows that the kernal of x' is a non-proximal hyperplane, since otherwise you can find $x \in X$ with $d(x, \ker(x')) = \|x\| = 1$, from which you can obtain $\|x'\| = |x'(x)|$. □

Remark. (i) A more detailed proof is given by James Fletcher and Warren B. Moors [FM14, Theorem 2.41].

(ii) Together with Theorem 2.10 (i) $\iff$ (ii) this proves that Theorem 1.6 holds for Banach spaces.

3.3 Proof of Theorem 1.6

In this subsection we finally prove Theorem 1.6, the main result of this section.

Proof of Theorem 1.6. **(i)⇒(ii):** This follows directly from combining Theorem 2.10 (i)⇒(ii) and Theorem 3.10 (i)⇒(iii).

(ii)⇒(i): Because only Banach spaces can be reflexive, this follows from combining Lemma 3.1, Theorem 3.10 (iii)⇒(i) and Theorem 2.10 (ii)⇒(i). □

4 Examples and counterexamples

4.1 Uniformly convex normed vector spaces

We have seen that strictly convex normed vector spaces have interesting properties regarding Chebyshev sets. According to Example 2.9, which we are about to prove, all uniformly convex normed vector spaces are strictly convex, which makes uniformly convex normed vector spaces interesting as well. In this subsection we will prove Example 2.9 and we will give examples of uniformly convex normed vector spaces, which in turn gives us examples of strictly convex normed vector spaces. We will also give a proof of the Milman-Pettis theorem, which is based on our research on Chebyshev sets. This appears to be a new method. In combination with Theorem 1.6, the Milman-Pettis theorem is another great tool for classifying Chebyshev sets.

Definition 4.1 (uniformly convex)**.** We call a normed vector space $(X, \|\cdot\|)$ uniformly convex if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for all $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$, we have $\|x + y\| \leq 2 - \delta$.

Example 4.2 (uniformly convex normed vector spaces)**.**

(i) Finite-dimensional strictly convex normed vector spaces.

(ii) Inner product spaces. The proof is on p.32.

(iii) The ℓ^p and L^p spaces for $p \in (1, \infty)$. The proof is on p.33.

Proof of Example 4.2 (i). Let $(X, \|\cdot\|)$ be a finite-dimensional normed vector space that is not uniformly convex. So there exists $\varepsilon > 0$ such that for all $\delta > 0$ there exist $x, y \in X$ with $\|x\|, \|y\| \leq 1$, $\|x - y\| \geq \varepsilon$ and $\|x + y\| > 2 - \delta$.

Take any sequence $(\delta_i)_{i \in \mathbb{N}}$ in $(0, \infty)$ which converges to 0. For all $i \in \mathbb{N}$ we find x_i and y_i such that $\|x_i\|, \|y_i\| \leq 1$, $\|x_i - y_i\| \geq \varepsilon$ and $\|x_i + y_i\| > 2 - \delta_i$. Since X is finite-dimensional, by Theorem 2.5 (i)$\Rightarrow$(ii), it has the Heine-Borel property. Since the sequences $(x_i)_{i \in \mathbb{N}}$ and $(y_i)_{i \in \mathbb{N}}$ are bounded, passing to a subsequence we can assume without loss of generality that they have limits $x, y \in X$ respectively. By continuity of norms with respect to themselves, we get $\|x\|, \|y\| \leq 1$ and $\|x + y\| \geq 2$ and $\|x - y\| \geq \varepsilon > 0$, also giving $x \neq y$. So X is not strictly convex.

By contraposition, this proves Example 4.2 (i). □

We can now finally prove Example 2.9.

Proof of Example 2.9. Let $(X, \|\cdot\|)$ be a uniformly convex normed vector space. Now let $x, y \in X$ with $x \neq y$ and $\|x\| = \|y\| = 1$. Because X is uniformly convex, we get that for $\varepsilon := \|x - y\| > 0$ there exists $\delta > 0$ such that

$$\|x + y\| \leq 2 - \delta < 2.$$

So X is strictly convex. $\square$

With our research on best approximations, we find an alternative proof of the Milman-Pettis theorem.

Theorem 4.3 (Milman-Pettis theorem)**.** *Any uniformly convex Banach space is reflexive.*

Proof of Theorem 4.3. Let $(X, \|\cdot\|)$ be a uniformly convex Banach space.

Claim 1. All non-empty closed convex subsets of X are proximal.

Let $\emptyset \neq A \subset X$ be a non-empty closed convex subset of X and let $x \in X$. We define

$$c := \inf_{y \in A} \|x - y\|.$$

This gives a sequence $(y_i)_{i \in \mathbb{N}}$ in A such that

$$\lim_{i \to \infty} \|x - y_i\| = c.$$

Claim 2. The sequence $(y_i)_{i \in \mathbb{N}}$ is Cauchy.

Since the case $c = 0$ is trivial, we assume $c > 0$. Let $\varepsilon > 0$ and define

$$\varepsilon' := \frac{\varepsilon}{c + \varepsilon} > 0.$$

Because X is uniformly convex, there exists $\delta > 0$ such that for all $v, w \in X$ with $\|v\|, \|w\| \leq 1$ and $\|v - w\| \geq \varepsilon'$ we have $\|v + w\| \leq 2 - \delta$. We define

$$\varepsilon'' := \min \left\{ \frac{1}{2} c\delta, \varepsilon \right\} > 0.$$

This gives an $N \in \mathbb{N}$ such that for all $i > N$ we have $\|x - y_i\| \leq c + \varepsilon''$.

Assume for the contrary that there exist $i, j > N$ such that $\|y_i - y_j\| \geq \varepsilon$. We define

$$u := \frac{1}{c + \varepsilon''}(x - y_i), v := \frac{1}{c + \varepsilon''}(x - y_j).$$

We get

$$\|u\|, \|v\| \leq 1, \|u - v\| = \frac{\|y_j - y_i\|}{c + \varepsilon''} \geq \frac{\varepsilon}{c + \varepsilon} = \varepsilon'.$$

Therefore we have

$$\begin{aligned} \left\| x - \frac{1}{2}(y_i + y_j) \right\| &= \frac{c + \varepsilon''}{2} \|u + v\| \\ &\leq \frac{c + \varepsilon''}{2}(2 - \delta) \\ &= c + \varepsilon'' - \frac{1}{2}c\delta - \frac{1}{2}\varepsilon''\delta \\ &< c. \end{aligned}$$

Because A is convex, we have $\frac{1}{2}(y_i + y_j) \in A$, so this contradicts the definition of c. So for all $i, j > N$ we have $\|y_i - y_j\| < \varepsilon$.

This proves Claim 2.

Because X is complete, the sequence $(y_i)_{i \in \mathbb{N}}$ converges to some $y_0 \in X$. Because A is closed, we also have $y_0 \in A$. Because norms are always continuous with respect to themselves, we get

$$\|x - y_0\| = \lim_{i \to \infty} \|x - y_i\| = c = \inf_{y \in A} \|x - y\| .$$

So y_0 is a best approximation for x in A. Because this works for any $x \in X$, the set A is proximal.

This proves Claim 1.

Using Theorem 3.10 (iii)⇒(i), we find that X must be reflexive.

This proves Theorem 4.3. □

Remark. (i) This theorem can not be generalized to strictly convex spaces. See Example 4.5.

(ii) This theorem does not work the other way around. Not every reflexive space is uniformly convex. Not even every strictly convex reflexive space is uniformly convex. See Example 4.6.

4.2 Properties of inner product spaces

Throughout this thesis, we have made several claims about inner product spaces. We will prove all these claims in this subsection.

Proof of Example 4.2 (ii). Let $(X, \|\cdot\|)$ be an inner product space. Let $\varepsilon > 0$ and assume without loss of generality that $\varepsilon < 2$, such that we can define $\delta := 2 - \sqrt{4 - \varepsilon^2} > 0$. Now let $x, y \in X$ with $\|x\|, \|y\| \leq 1$ and $\|x - y\| \geq \varepsilon$. Using the parallelogram law, we get

$$\|x + y\|^2 = 2(\|x\|^2 + \|y\|^2) - \|x - y\|^2 \leq 4 - \varepsilon^2.$$

Using this, we get

$$\|x + y\| \leq \sqrt{4 - \varepsilon^2} = 2 - \delta.$$

So X is uniformly convex. □

Proof of Example 2.13 (i). Let $(X, \|\cdot\|)$ be an inner product space. Let $\varepsilon > 0$ and assume without loss of generality that $\varepsilon < 2$, such that we can define

$$\delta := \frac{4\varepsilon}{4 - \varepsilon^2} > 0.$$

Now let $x, y \in X$ with $\|x\| = 1$ and $\|y\| \leq \delta$. We get

$$\begin{aligned} \|y\| &\leq \frac{4\varepsilon}{4 - \varepsilon^2} \\ \left(1 - \frac{1}{4}\varepsilon^2\right) \|y\|^2 &\leq \varepsilon \|y\| \\ 1 + \|y\|^2 &\leq 1 + \varepsilon \|y\| + \frac{1}{4}\varepsilon^2 \|y\|^2 \\ &= \left(1 + \frac{1}{2}\varepsilon \|y\|\right)^2. \end{aligned}$$

Combining this with Young's inequality for $p = 2$ and the parallelogram law, we get

$$\begin{aligned} (\|x + y\| + \|x - y\|)^2 &\leq 2(\|x + y\|^2 + \|x - y\|^2) \\ &\leq 4(\|x\|^2 + \|y\|^2) \\ &= 4(1 + \|y\|^2) \\ &\leq 4\left(1 + \frac{1}{2}\varepsilon \|y\|\right)^2. \end{aligned}$$

So finally, we get

$$\|x + y\| + \|x - y\| \leq 2 + \varepsilon \|y\|.$$

So X is uniformly smooth.

This proves Example 2.13 (i). □

4.3 Properties of the ℓ^p and L^p spaces for $p \in (1, \infty)$

Throughout this thesis, we have made several claims about the ℓ^p and L^p spaces for $p \in (1, \infty)$. We will prove all these claims in this subsection.

Proof of Example 4.2 (iii). A proof of this is given by Clarkson [Cla36][Corollary to Theorem 2]. □

Proof of Example 2.13 (ii). For all $p \in (1, \infty)$ the dual of ℓ^p is $\ell^{p'}$ and the dual of L^p is $L^{p'}$ where $p' = \frac{p}{p-1} \in (1, \infty)$. According to Example 4.2 (iii), this means the duals of ℓ^p and L^p are uniformly convex. Chimude and Charles proved that this means ℓ^p and L^p are uniformly smooth [CC09][Theorem 2.10]. □

4.4 Non-trivial normed vector spaces with unusual properties

Throughout this thesis, we have made a few remarks that certain theorems cannot be generalized a certain way. In this subsection, we will give some counterexamples which will prove these remarks to be true.

Example 4.4 (Chebyshev linear subspace of a reflexive strictly convex Banach space with a discontinuous projection map)**.** An example is given by James Fletcher and Warren B. Moors [FM14][Example 7].

Example 4.5 (strictly convex irreflexive Banach space)**.** The normed vector space $(\ell^1, \|\cdot\|_1 + \|\cdot\|_2)$.

Proof of Example 4.5. Let us consider the normed vector space $(X, \|\cdot\|) := (\ell^1, \|\cdot\|_1 + \|\cdot\|_2)$. Because $(\ell^2, \|\cdot\|_2)$ is an inner product space, according to Example 4.2 (ii) and Example 2.9 it is strictly convex. Because ℓ^1 is a linear subspace of ℓ^2, it easily follows that $(\ell^1, \|\cdot\|_2)$ must also be strictly convex, so it follows that $(X, \|\cdot\|)$ is strictly convex.

Because we have $\|\cdot\|_2 \leq \|\cdot\|_1$, we get $\|\cdot\|_1 \leq \|\cdot\| \leq 2\,\|\cdot\|_1$, so the norms $\|\cdot\|_1$ and $\|\cdot\|$ are equivalent on ℓ^1. Because $(\ell^1, \|\cdot\|_1)$ is irreflexive and complete, and because reflexivity and completeness are preserved through equivalent norms, this means $(X, \|\cdot\|)$ is irreflexive and complete.

This proves Example 4.5. $\square$

Example 4.6 (separable strictly convex reflexive Banach space not isomorphic to any uniformly convex space)**.** An example is given by Mahlon M. Day [Day41][Theorem 1].

A Notation

- For a metric space (V, d) with subsets $A, B \subset V$ we denote

  $$d(A, B) := \inf_{x \in A, y \in B} d(x, y).$$

  If either A or B is a singleton, we only write down its element. Here we use the convention that the infimum of the empty set is ∞. This particular value will never be used, but we need some definition in order for a best approximation to be well-defined for empty sets.

- For sets X, Y, Z with a binary operation $\circ : X \times Y \to Z$ and subsets $A \subset X$ and $B \subset Y$ we denote

  $$A \circ B := \{a \circ b | a \in A, b \in B\}.$$

  If either A or B is a singleton, we only write down its element.

- For a metric space (V, d) with an element $x \in V$ and a number $r \in [0, \infty)$ we denote

  $$B_r^V(x) := \{y \in V | d(x, y) < r\},$$

  $$\overline{B}_r^V(x) := \{y \in V | d(x, y) \leq r\},$$

  $$S_r^V(x) := \{y \in V | d(x, y) = r\}.$$

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