An examination of the solvability of a sliding puzzle on a hexagonal grid

Joep Hamersma

9 May 2018

Utrecht University

Abstract

Puzzles in which pieces have to be moved to a specific place or configuration have existed for decades. From the 15-puzzle to the Rubik's cube and more recently Rush Hour and similar mobile games, these puzzles provide great experiences, but also interesting research topics. Questions about solvability and optimal solutions have come up and some of those questions have been solved for those famous puzzles. This paper takes a look at some of these puzzles and the previous research on them, before describing a new sliding puzzle on a hexagonal grid where pieces connect two hexagons of the grid. This puzzle explores possibilities of a puzzle on a hexagonal grid, which is rare in puzzles. From a design perspective, whether the puzzle is solvable and how fast are logical questions. A proof is given to show that any configuration can be transformed into any other configuration on all but one hole-free 2-connected boards. Furthermore the proof shows that it is possible in polynomial number of moves. Other results are shown for other boards as well as different pieces.

Contents

1 Introduction

1.1 Sliding puzzles

When talking about sliding puzzles, it is nearly impossible not to mention the 15-puzzle. In the most common variant, a 4×4 grid is filled with 15 tiles, with numbers on them ranging from 1 to 15. In order to solve the puzzle, the tiles have to be rearranged into order. A possible move consists of moving a tile onto the empty space. Different board sizes are possible to create variations, both rectangular grids and square grids. Besides the 4×4 grid, the 3×3 grid (9-puzzle) is the most common. The main mechanics of the puzzle stay the same regardless of boardsize.

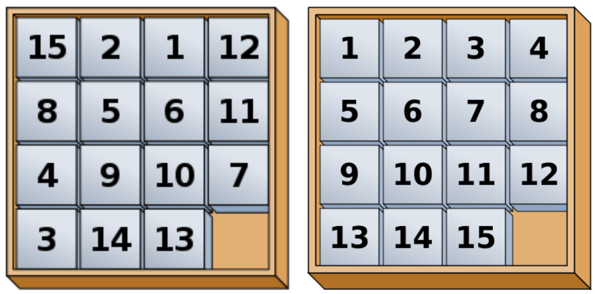

Figure 1: A general configuration and a solved configuration of the 15-puzzle

Multiple other sliding puzzles have been released, one of the more popular ones being Rush Hour. In Rush Hour, a grid (usually 6×6) is partially filled with rectangular blocks of size 1×2 (cars) and 1×3 (trucks). The goal here is to move a particular car to a specific location (usually the edge of the grid). The blocks have to stay on the line they were initially placed on, but can be moved back and forth along this line, provided the space they are moving to is empty.

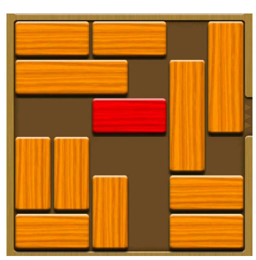

Figure 2: The mobile phone game Unblock me, based on Rush Hour

Another famous puzzle sharing similarities with the 15-puzzle is the Rubik's cube. The normal Rubik's cube is a $3 \times 3 \times 3$ cube with each of the six faces in a different color. Then, by rotating a face, the colors on the four adjacent faces can be mixed up. Continuous rotation of faces scrambles the cube and then the goal is to return the cube to a state where each face only consists of one color.

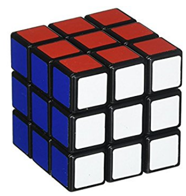

Figure 3: The Rubik's cube

The rise of mobile phones and mobile phone gaming caused a big increase in puzzle games. In puzzle games, reaction time is usually not so relevant and controls can be adjusted to work on a smaller screen, making puzzle games and mobile devices a great combination.

1.2 A new type of puzzle

In this paper a new type of sliding puzzle will be presented as well as analyzed. Instead of a traditional rectangular board, this puzzle is played on a hexagonal grid. Aligning sides of regular hexagons of equal size with each other leads to a space filling pattern. Any collection of hexagons can be considered a **board** to play the puzzle on. These hexagons that are part of the board will be called **tiles**.

One can place a **piece** on any two adjacent tiles. A piece consists of two halves. Each half of the piece can be colored. Both halves can be the same color or they can be colored differently. Given a piece with two colors, there are two **orientations** for a piece.

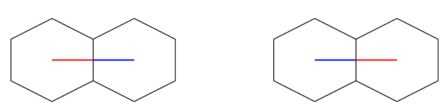

Figure 4: Two orientations of a red-blue piece

Given the pieces and the board, we want to be able to change the configuration. However, there are some rules for moving pieces around. In particular, to move a piece, there needs to be an empty tile. A **move** then consists of moving a piece to cover an empty tile. This can be done by either sliding the piece over or rotating the piece onto the empty tile.

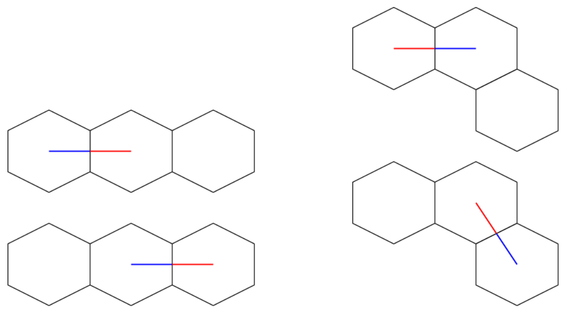

Figure 5: Two sliding move examples

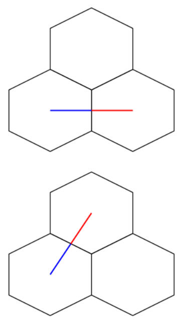

Figure 6: Rotating move

Given a set of pieces and a board of hexagonal tiles, one can create a **conguration** by placing the pieces on the board, with every tile of the board containing at most one piece. This can generally be done in multiple ways. One question that arises when viewing multiple configurations of the same pieces on one board is whether there exists a sequence of moves that starts with one configuration and ends with the other configuration. Such a sequence of moves will be called a **transformation**. In particular, one can define a final configuration and a (random) starting configuration and finding a transformation between these two configuration then is a puzzle. From a puzzle design perspective it is interesting to know when such a transformation exists between configurations and when it does not.

By choosing the board, the pieces and the (final) configuration carefully, aesthetically pleasing images or certain patterns can be obtained as in figure 7.

This paper will provide a proof to show that the puzzle is always solvable if the board has a few specific properties. In fact, an exact algorithm to transform a configuration to another configuration is given. This also brings some bounds on the number of moves necessary. Furthermore, some insights on variation in boards that do not have the required properties as well as puzzles with longer pieces are shared.

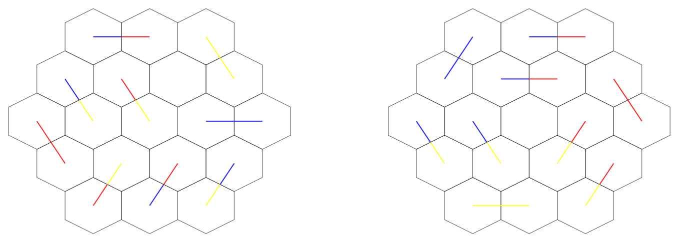

Figure 7: Starting configuration and final configuration of a possible puzzle

2 Related work

Especially the 15-puzzle and the Rubik's cube have been studied extensively. This chapter gives an overview of some of the results from previous research most relevant to this paper.

2.1 15-puzzle

For the 15-puzzle, out of all starting configurations of tiles on a 4   4 grid, exactly half of them are solvable. Story [1] showed this result by using the parity of the permutation of the 16 squares plus the parity of the taxicab distance of the empty space to the bottom-right corner. Brüngger, Marzetta, Fukuda and Nievergelt [2] showed that the most single-tile moves any configuration of the 15-puzzle requires to be solved is 80. Norskog [3] further showed that if sliding multiple tiles at once in a direction is considered as one move, the maximum number of moves needed is 43.

Story [1] also showed that the extended n n version of the 15-puzzle has a polynomial solution. Ratner and Warmuth [4] proved that finding the shortest solution to the extended version of the 15-puzzle is NP-hard. Goldreich [5] showed that finding a shortest solution to the graph-generalized 15-puzzle is NP-hard.

2.2 Rush Hour

For Rush Hour, on the 6 6 grids, a lot of solvable configurations have come up in the board game or digital implementations of the game, but there is no easy way of consistently telling the solvability of a general n n board.

Collette, Raskin and Servais [6] show that any configuration on the regular 6 6 grid can be solved in 93 moves or less. Furthermore, there is exactly one configuration that needs 93 moves, shown in figure 8.

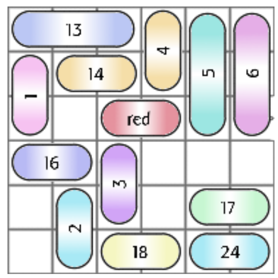

Figure 8: The hardest configuration of Rush Hour

2.3 Rubik's cube

While there are 43 quintillion possible configurations of the Rubik's cube, Rokicki, Kociemba, Davidson and Dethridge [7] show every single one of them can be solved in 20 moves considering the Half-Turn-Metric and in 26 moves considering the Quarter-Turn-Metric. One of the configurations requiring 20 moves is the so-called superflip, where every corner is correct, but every edge is flipped, shown in figure 9.

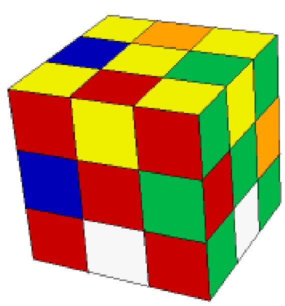

Figure 9: The superflip

Similar to the 15-puzzle, the Rubik's cube can be arranged in an unsolvable state. Doing so requires removal of a piece and inserting it in a different orientation. When scrambled from a solved state, the Rubik's cube is always solvable by doing all the moves to scramble in reverse.

3 Connectivity and Hamiltonian cycles

To proof every configuration can be transformed into every other configuration on some boards, some definitions will be introduced. Considering the middle of every tile as a node and any two neighbouring tiles as an edge, the board can be transferred to a graph. This gives rise to the notion of **p-connected** boards. A board is **p-connected** if its corresponding graph is p-connected. Furthermore, a board is **hole-free** if all tiles that are not part of the board are connected.

Similarly, a Hamiltonian cycle on a board is nothing else than a Hamiltonian cycle on its corresponding graph. Define a part of a Hamiltonian cycle connecting two tiles as a **link**. Therefore a Hamiltonian cycle on a board of 10 tiles consists of 10 links. A Hamiltonian cycle divides space in a part inside the cycle and a part outside the cycle. Any two adjacent links of the Hamiltonian cycle make up a 60, 120, 180, 240 or 300 angle on the inside of the cycle. This will be noted as an **inside corner**. This division of a Hamiltonian cycle into an inside and an outside, leads to the notion of **hole-free Hamiltonian cycles**. These are Hamiltonian cycles of which every tile that is part of the inside of the cycle is also part of the board. For hole-free boards, all Hamiltonian cycles will be hole-free, but boards with holes can have hole-free Hamiltonian cycles as in figure 10.

Figure 10: A hole-free cycle on a board with a hole

Furthermore, the collection of all 2-connected hole-free boards with an odd number of tiles except the 13-tiled star-shaped board in figure 11 will be called **standard boards**.

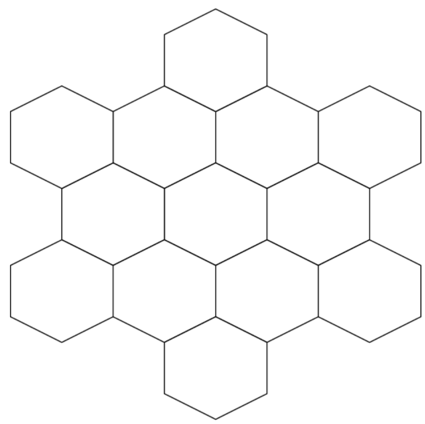

Figure 11: Star-shaped board

Throughout this paper, it will be shown that any configuration with $b(n=2)c$ pieces, where n is the odd number of tiles, of the puzzle on a standard board can be transformed into any other. First, it is shown that uniform identical pieces can be put into any possible configuration. Then by showing that one specific configuration can be achieved with any piece in any place in any orientation, the conclusion follows.

4 From and to Hamiltonian configurations

There is at least one Hamiltonian cycle on all standard boards. When considering the centers of the hexagons as nodes and two adjacent hexagons in the board as links, the board transforms into a graph. On these graphs, Polishchuk, Arkin and Mitchell [8] show that every graph corresponding to a standard board is Hamiltonian. Therefore, on every standard board a Hamiltonian cycle is possible.

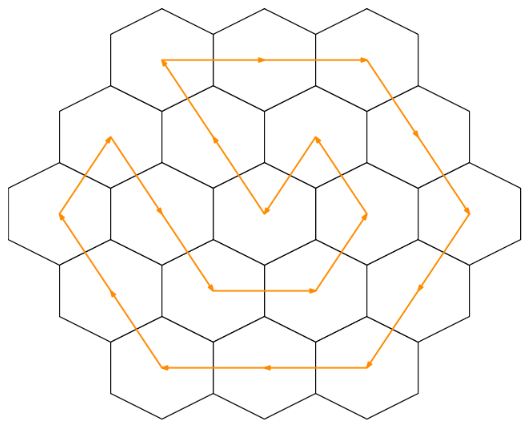

Figure 12: A Hamiltonian cycle with orientation

Hamiltonian cycles give rise to special configurations of the pieces. A **Hamiltonian configuration** is a configuration on the board where all pieces align with a chosen Hamiltonian cycle on the board. This means every piece is on two tiles of the grid that are connected in the chosen Hamiltonian cycle by a link.

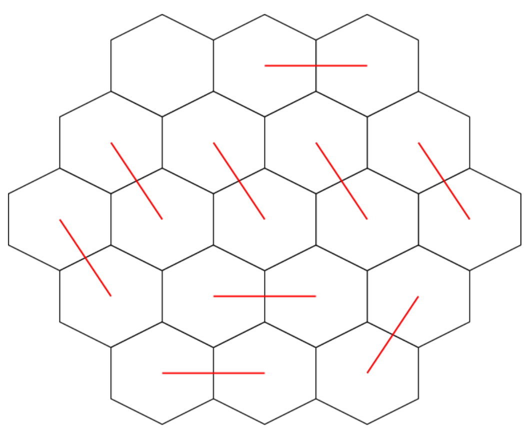

Figure 13: A Hamiltonian configuration corresponding to the cycle in figure 12

Theorem 1. *Any possible configuration on a standard board with $2n+1$ tiles can be transformed into any Hamiltonian configuration in $O(n^2)$ moves.*

Proof. Given an initial configuration and any Hamiltonian cycle on a board with n pieces, choose an orientation of the Hamiltonian cycle, that is, for every link of the cycle, it points to the next tile in the cycle (two ways possible). If all pieces align with the Hamiltonian cycle, the board is in Hamiltonian configuration. If not, suppose there are $n-k$ pieces not aligned yet and consider the piece that is on the tile to which the orientation is pointing from the empty tile. Pull this piece onto the empty tile. If the piece was not aligned, this move can be done such that the piece will be aligned. This increases the number of aligned pieces by one. Otherwise a piece is pulled that was already aligned. The empty tile then moves two places along the Hamiltonian cycle and points to a new piece. This can happen for all k currently aligned pieces, but eventually a piece that is not aligned will be pulled. Again this increases the number of aligned pieces by one. This means that if the board is not in Hamiltonian configuration, after at most k pulls, there is one more piece aligned. If a new piece is aligned, the argument can be applied repeatedly to the new situation until all n pieces are aligned.

The above algorithm shows that after at most $k+(k+1)+\dots+n$ pulls, all pieces are aligned with the Hamiltonian cycle and thus in Hamiltonian configuration. This also means that the worst-case scenario using this strategy is $1+2+3+\dots+n=\frac{n\cdot(n+1)}{2}$ moves, which happens for the configuration in figure 14. □

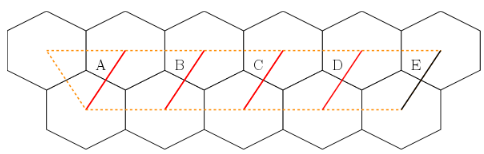

Figure 14: A worst case scenario board for the strategy with eleven tiles

In figure 14 with a clockwise orientation, piece A will be pulled to lay horizontally, moving the empty tile to the bottom-left. Then, piece A will be pulled again to move the empty tile to the second tile of the top row. Now piece B is pulled to be placed horizontally, after which again piece A has to be moved, followed by piece B. Continuing this way, the full sequence of pieces moved is:

(A,A,B,A,B,C,A,B,C,D,A,B,C,D,E)

for a total of 15 moves to reach a Hamiltonian configuration. Note that moving A to be horizontal, then moving B to be horizontal under A, moving C to be horizontal, moving D to be horizontal under C and moving E to be horizontal also reaches a Hamiltonian configuration in only 5 moves. It is possible that such a linear (or at least better than $O(n^2)$) strategy exists for all boards, however that was not found during the research in this paper.

Corollary 1. *Any possible configuration with n uniform identical pieces on a standard board with $2n+1$ uniform identical tiles can be transformed into any other possible configuration.*

Proof. Because the moves in the puzzle are reversible, two configurations can be transformed to each other by choosing a Hamiltonian cycle. Both configurations transform into a Hamiltonian configuration on the given cycle. The difference between the two configurations can be solved by moving the empty spot along the cycle by pulling pieces. Therefore, to transform any possible configuration into any other, one can first transform into a Hamiltonian configuration and then to the desired configuration. □

From Hamiltonian cycles on standard boards with $2n+1$ tiles we can note:

> The empty spot can be moved to any tile of the board: Consider a Hamiltonian configuration. Label the tiles starting from the empty spot into a chosen direction 0 to $2n$. Pulling a piece in the chosen direction moves the empty spot from tile m to tile $(m+2) \bmod (2n+1)$. Therefore, pulling in the chosen direction visits tiles $1, 3, 5, \ldots, 2n+1$ in order. The next pull moves the empty piece to tile 2, while following pulls move the empty tile to $4, 6, 8, \ldots, 2n$. Therefore, the empty tile can be moved anywhere.

> Furthermore, after $2n+1$ pulls, the board is back into its initial state, except that the pieces shifted one place in their order. With uniform identical pieces, this does not matter, but for distinct pieces, it is important. So not only can the empty spot be on any tile of the board, but one of the pieces that is next to it can also be chosen. Choosing that specific piece and the location of the empty tile does restrict the other pieces to specific places without breaking the Hamiltonian configuration.

5 Splitting Hamiltonian cycles

The goal is to use these Hamiltonian cycles to arrange pieces in order and orientate them in any way. However, just the basic cycle is not sufficient to do so. Therefore, we show that these cycles, with a fex exceptions, can always be decomposed into two smaller cycles.

The four cycles in figure 15 are the only cycles, where every link is part of at least one inside 60 corner, that can not be transformed into a different cycle.

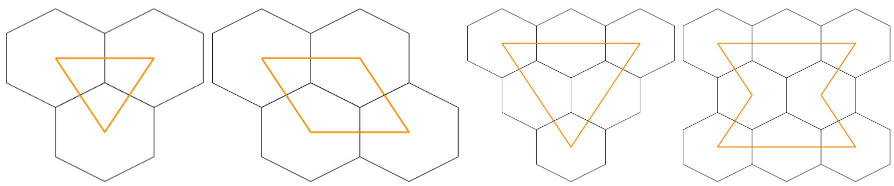

Figure 15: Four special boards

One way to show this result is looking at several cases based on the inside corners that are allowed:

1. Only 60 inside corners. Allowing only 60 corners leads to the smallest cycle, the 3-tiled triangle, the first in figure 15.

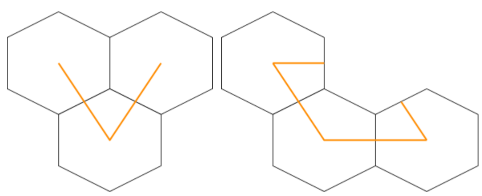

Figure 16: The starting shape for 60 and 120 corners

2. 60 and 120 corners. As every link has to be part of at least one 60 corner, the 120 corner has to be connected on both sides to a 60 corner as in figure 16. This leaves one option for the last corner, resulting in the 4-tiled diamond, the second in figure 15.

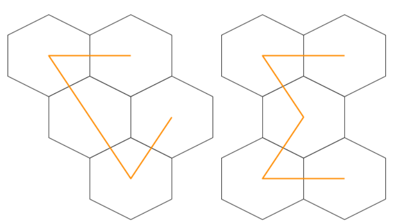

Figure 17: The starting shape for 180 and 240 corners

3. 60 , 120 and 180 corners. The combination of a 60 , 180 and another 60 corner results in the left shape in figure 17 that can be closed in two

ways with the corners we have. One is by adding two more links in a 60 corner, leading to the 6-tiled triangle, third in figure 15. The other way consists of adding one link, however this link will then be part of two 120 corners and thus not part of a 60 corner.

4. 60 , 120 , 180 and 240 corners. Starting with two 60 and a 240 corner, we get an m-shape as on the right in figure 17. Connecting the same shape rotated 180 gives the 8-tiled board, last in figure 15. A completely alternating shape of 60 and 240 corners leads to the star shaped cycle as in figure 18, but the corresponding board is not hole-free.

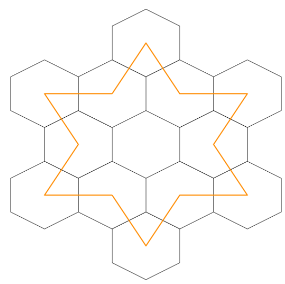

Figure 18: The star shape

5. 300 corners. Allowing 300 corners seems to open up the number of possibilities as two consequent corners are now not convex anymore. However, a w-shape formed by the 60 corners and a 300 corner, always allows for a different cycle, consisting of a 120 , a 60 and a 300 corner, preceded by something else than a 60 corner, as seen in figure 19. Therefore, any Hamiltonian cycle with this pattern can be rearranged into one with a link that is not part of a 60 corner.

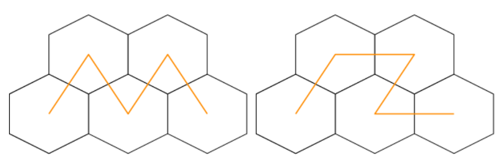

Figure 19: The original m-shape and the alternative

Theorem 2. *Any hole-free Hamiltonian cycle, with the exception of those in gure 15, can be transformed into two hole-free Hamiltonian cycles on two separate boards sharing exactly one tile.*

Proof. If every link in the cycle is part of a 60 inside corner and the cycle is not one of those in figure 15, transform the cycle into a new one, using the alternative of the m-shape in figure 19. Changing the cycle this way adds no holes, so the resulting cycle is also hole-free. Any link that is not part of a 60 inside corner in the Hamiltonian cycle gives rise to a possible splitting point. Every link touches the inside of the cycle as well as the outside. This means that the tile closest to the link on the inside of the cycle has to be part of the board as well, as there are no holes inside the cycle. This also means the Hamiltonian cycle travels through this tile. By removing the candidate link and attaching the two loose ends to the tile on the inside, two new cycles are created. Every tile is still visited by either of the cycles, so the new cycles are Hamiltonian on their respective boards. As every Hamiltonian cycle that is not one of the four depicted has at least one candidate link or a adjusted cycle with one candidate link, every hole-free Hamiltonian cycle can be split into two Hamiltonian cycles sharing only one tile.

As the Hamiltonian cycle connects every tile with two neighbours, the individual boards containing these cycles are 2-connected themselves. As the split occurs along the inside of the original Hamiltonian cycle, the replacement of the old link with two new links adds a small triangle to the outside region. This triangle is always connected to the outside and can therefore not create any holes. So the new cycles are hole-free. Also, as standard boards contain no holes, the cycles on them are also hole-free, so the proof applies to cycles on standard boards.

□

In fact, if it is possible on a board with odd tiles to split a hole-free Hamiltonian cycle up into two pieces, it is possible to split the Hamiltonian cycle on the board into two hole-free Hamiltonian cycles of odd length.

Consider a candidate splitting link (the link connecting the light blue tiles), then without loss of generality there are four cases:

Case I:

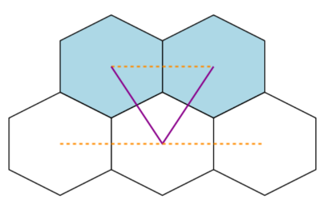

Figure 20: Case I

Consider this situation. If the new Hamiltonian cycles are not of odd length, a different splitting link can be chosen, shortening one cycle by one link and lengthening the other by one as seen in figure 21. Either choice works. This is always possible, because if either piece was part of a 60 corner, the original splitting point would create two cycles of odd length (namely one of length three).

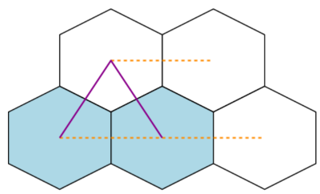

Figure 21: Case I alternatives

Case II and III:

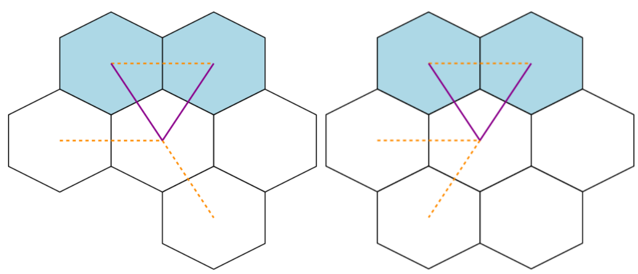

Figure 22: Case II and III

Case II and III are very similar to Case I, the only difference is that the new splitting link has one clear option (on the left). Again, if this was not possible, because there is a 60 corner, the original splitting link would suffice.

Case IV:

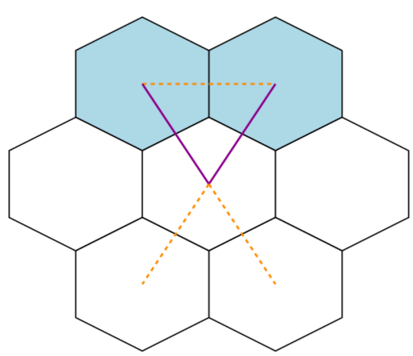

Figure 23: Case IV

The most interesting case is IV. If the original splitting link leads to two cycles of even length, there is no obvious new candidate. However, consider the two links from the bottom layer to the middle tile. As the original split is on the inside of the original cycle, the left-middle and right-middle tiles are part of the board. If they were not, either the original board had a hole (not standard) or the split is on the outside, neither of which is possible. This means that both bottom links are a new candidate. There are now two possibilities: Either at least one of the links is part of a 60 corner or both are not. If at least one of the links is part of a 60 corner, for example the left one, the original chosen Hamiltonian cycle can be transformed into a different cycle as in figure 24.

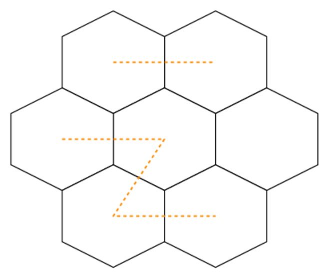

Figure 24: Case IV alternative cycle

This cycle has two candidates similar to case II and case III corners, of which one will result in two cycles of odd length.

If both links are not part of a 60 corner, they are new candidate splitting links. These new splitting links can be any of the four cases. If either is a case I, II, III or IV with at least one 60 corner, it follows exactly the same logic as those cases. However, it is possible that both links are case IV without 60 corners again. Then part of the cycle is given as in figure 25.

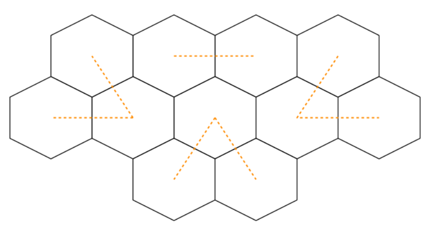

Figure 25: Case IV extended

However, in this case, the cycle can also be altered to the following cycle:

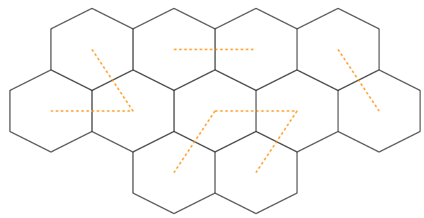

Figure 26: Case IV alternate cycle

Given this cycle, the situation is similar to that of the other cases with the two parallel links. Splitting one of them will lead to two cycles of even length and one of them will lead to two cycles of odd length. Thus it is always possible to split a Hamiltonian cycle of odd length in two cycles of odd length sharing exactly one tile.

6 Transforming any con guration

With the Hamiltonian cycles and the decomposition of a cycle into two smaller ones, all the necessary ideas are in place to show the final result. Combining the results from above, we can now prove the following:

Theorem 3. *Given unique pieces, a possible initial con guration and a possible nal con guration on a standard board with* $2n+1$ *tiles, there exists a sequence of moves transforming the initial con guration into the nal con guration in* $O(n^3)$ *moves.*

Proof. The 3-tiled triangle board is trivial, so we assume our board has at least 5 tiles. Given a desired configuration there is a sequence of moves to turn it into any configuration and vice versa, as shown in Corollary 1 for identical pieces. Therefore if we can rearrange and rotate the pieces in one configuration in any possible way, any configuration with unique pieces is possible. One useful thing shown after Corollary 1 is that the empty tile can be moved along a Hamiltonian cycle to any spot. Furthermore, by moving all pieces twice along the Hamiltonian cycle, every piece shifts one place in the cycle but the empty tile comes back to the original position. After bringing the puzzle to a Hamiltonian configuration and given a split of the cycle in to two smaller cycles of odd length, we can move the empty tile to the shared tile of the smaller cycles. This results in the pieces aligning precisely with the two smaller cycles as well. The splits from the previous chapter result in one of the four cases in figure 27.

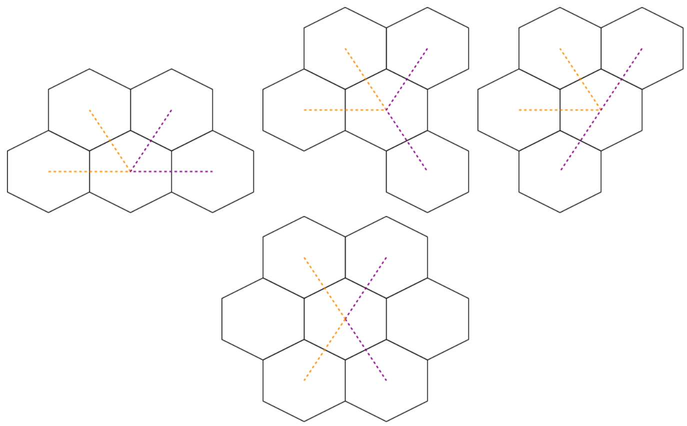

Figure 27: Possible situations after splitting a cycle

From here there are two steps to align the pieces in any variation of the current configuration:

First order all but one of the pieces on the orange cycle

Order the magenta cycle and the final piece of the orange cycle

First off: ordering most of the orange cycle. Given a position in the orange cycle, we can decide which piece we place there in what orientation. First, bring the desired piece to the magenta cycle if it's not already there. By moving the pieces along the orange cycle, we can bring the desired piece onto the shared tile as in figure 28.

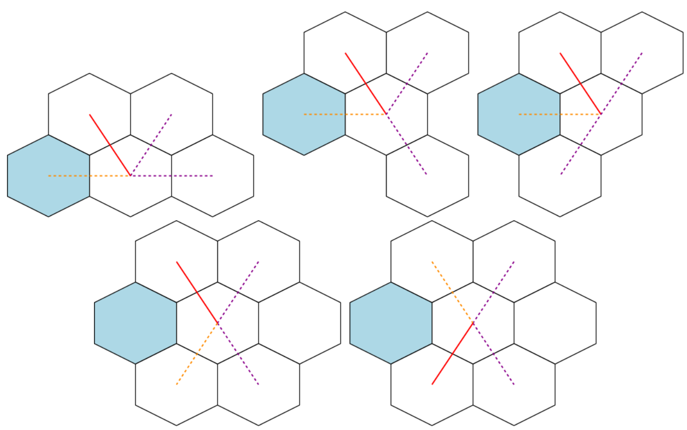

Figure 28: The set-up for the exchange, step 1

Then the piece can be moved to align it with the magenta cycle, moving the empty tile back to the shared position as in figure 29.

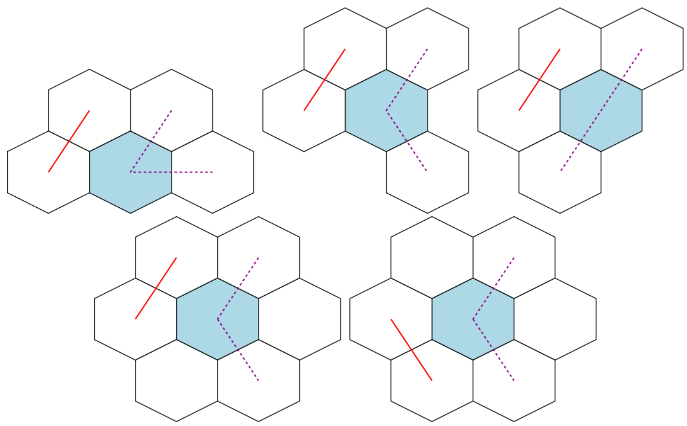

Figure 29: The set-up for the exchange, step 2

Now the magenta cycle can include this piece in its cycle by following the path in figure 30. By moving all the pieces along this extended magenta cycle, we can shift all pieces in the extended magenta cycle by one position.

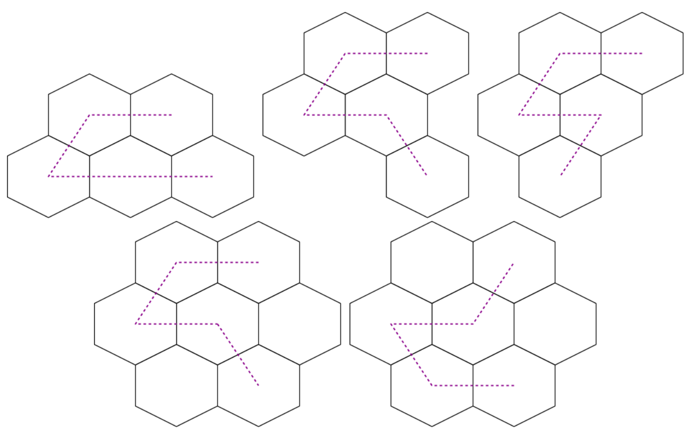

Figure 30: The extended cycles

By now undoing the alignment with the magenta cycle, we can bring a new piece back in the orange cycle. Then the orange cycle can be rotated such that the piece that is now in the desired position of the desired piece is aligned as in figure 28. By following the exact moves as before, this piece can be added to the magenta cycle. By shifting all the pieces in the reverse order of the last time, we can now rotate the desired piece in the shared position. If the piece started in the magenta cycle, we might have to rotate the cycle more to bring it to the correct position. Then we can once more undo the moves of alignment to fully include it in the orange cycle and orient it. Then moving the empty tile along the orange cycle to the shared tile brings us back to the initial position.

After this is done for all but one piece, we align the last piece with the magenta cycle, so that we can rotate the extended magenta cycle as well as the normal magenta cycle. Then we can order and orient all the remaining pieces any way we want. First of all, the piece that is closest to the orange cycle can be oriented either way by moving the normal magenta cycle around until the empty tile is next to this piece. By rotating the extended magenta cycle, we can put all pieces in this position and orient them. Then, by rotating the smaller cycle, we can change the position of the piece closest to the orange cycle in the extended magenta cycle. By simply positioning each piece by first moving it to the spot of the extended cycle and then rotating the rest of the pieces along the normal magenta cycle, we can order all the pieces in the extended magenta cycle in any way we want.

Therefore we can put any piece in any orientation in any position of this configuration. As we can do this from (and therefore to) any configuration, we can transform any configuration into any other configuration.

Moving and rotating every piece individually around the cycles can be done in $O(n^2)$ moves where n is the number of pieces in the cycle for each piece. If every piece needs to move multiple places along the cycles, this will lead to $O(n^3)$ worst case moves. Therefore, in the worst case scenario, it will take $O(n^3)$ moves.

□

7 Other boards

7.1 1-connected boards

Of course boards are not limited to the 2-connected variants described above. As long as the number of tiles is odd, one can move the pieces around. Therefore, 1-connected boards might be worth looking at. However, 1-connectedness severely limits the movement options for pieces. In particular, it allows for the construction in figure 31, in which one of the two pieces is immobile.

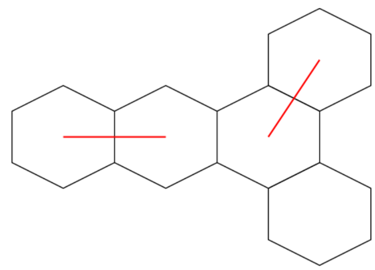

Figure 31: Immobile piece on a 1-connected board

In fact, configurations on 1-connected boards with at least 2 unique pieces are not transformable into each other. The 1-connectedness either immobilizes pieces or restricts pieces to specific parts of the board. As in figure 32, the pieces can never swap places.

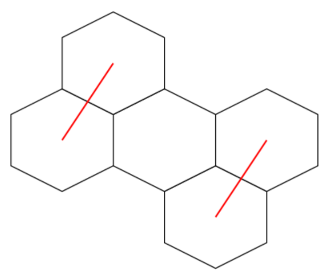

Figure 32: A 1-connected board where pieces can not be swapped

7.2 Boards with holes

In the previous sections, boards were considered that had no holes. In other words, all tiles not part of the board were connected. However, the puzzle can be played fine on boards with holes. In particular, two types of boards with holes are always solvable. The first of these two types are boards with an odd number of tiles where there exists at least one hole-free Hamiltonian cycle as in figure 33. As the strategy relies on the Hamiltonian cycle not to contain holes, it still applies.

A different set of solvable boards with holes are boards with an odd number of tiles containing the specific subshape in figure 34 and allow a hamiltonian cycle that passes through the 5 tiles of this subshape in order.

Figure 33: A board with a hole but a hole-free cycle

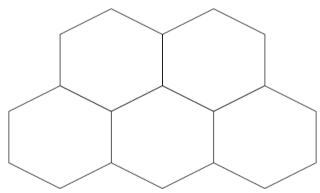

Figure 34: A swapping station

On these boards, pieces can be brought into Hamiltonian configuration as before. Then any two consequent pieces can be swapped and oriented at the subshape, until all pieces are in order and oriented correctly.

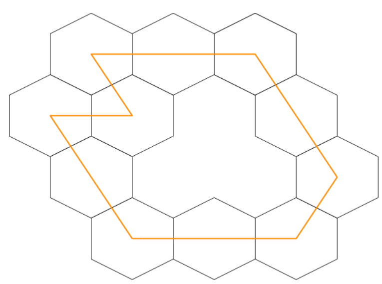

Figure 35: A solvable board with a hole

However, matters can become complicated when either the Hamiltonian cycle does not pass through the five tiles of the subshape in order or when there is no Hamiltonian cycle possible as in the boards in figure 36.

A conclusive answer when these boards allow complete transformations to all configurations was not found.

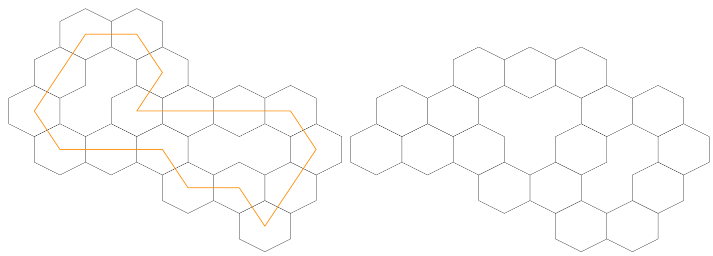

Figure 36: Two difficult boards

7.3 Length 3 pieces

A logical variation of the puzzle described above is a 2-connected compact board with flexible pieces of length 3.

However, with these pieces, immovable boards can be constructed quite easily, as shown in figure 37.

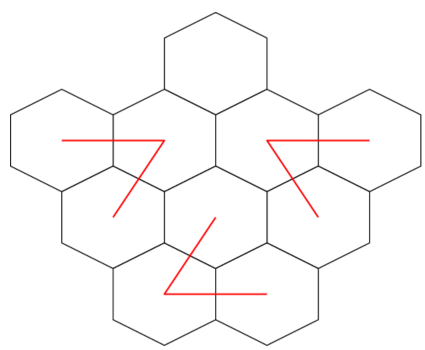

Figure 37: An immovable board with one empty tile

Furthermore, adding empty tiles does not solve the issue, as similar situations can be created elsewhere to create an immovable board with as many empty tiles as wanted. An example of an immovable board with two empty tiles is shown in figure 38.

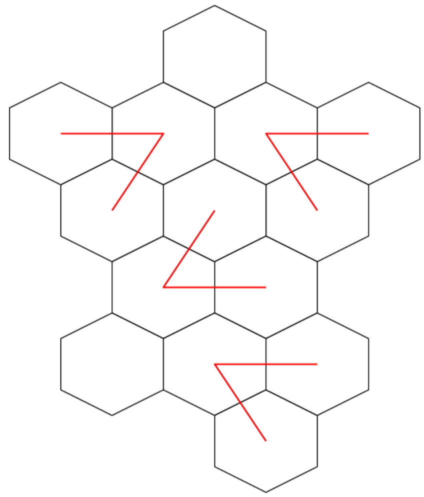

Figure 38: An immovable board with two empty tiles

8 Conclusions and future work

In this paper it is shown that a hexagonal tile based puzzle allows transformations from any configuration to any other on all standard boards. Furthermore, 1-connected boards do not allow pieces to move freely. Boards with holes come in both solvable variations and unsolvable variations, but no conclusive property has been found yet.

For future research, more detailed study of boards with holes is recommended. Hamiltonian cycles or subcycles on the board might be relevant. Also, pieces of length 3 are not completely studied either. Allowing them to rotate in place might be sufficient to prevent locking pieces in place and allow for interesting puzzles.

A 3-dimensional variation of the puzzle might be interesting to look at, but this requires a natural extension of the grid. In addition, the 2-dimensional variant lends itself to practical applications.

The algorithms described in this paper are most likely not optimal. Further investigation of bounds on the algorithms and improvements to the algorithm are also valid options for further research. In particular finding the solution that uses minimum moves to transform one specific configuration into another specific one could be NP-hard, similar to the case of the extended 15-puzzle [4].

9 Acknowledgements

I like to thank prof. dr. M. J. van Kreveld for introducing me to the topic of the paper, as well as the vivid conversations about puzzles in general and meetings. He also provided feedback on numerous occassions and suggestions for improvement. Furthermore i want to thank MSc T.C. van der Zanden for sharing his insights as well as making time for meetings and providing feedback.

References

[1] Story, W.E.: Note on the "15" puzzle. in American Journal of Mathematics. **2**, 399-404 (1879)

[2] Brüngger, A. Marzetta, A., Fukuda, K., Nievergelt, J.: The parallel search bench ZRAM and its applications. in Annals of Operations Research. **90**, 45-63 (1999)

[3] Norskog, B.: The Fifteen Puzzle can be solved in 43 "moves". available at `http://cubezzz.duckdns.org/drupal/?q=node/view/223` (2010)

[4] Ratner, D., Warmuth, M.: Finding a shortest solution for the NxN extension of the 15-puzzle is intractable. in Proceedings of AAAI. **86**, 168-172 (1986)

[5] Goldreich, O.: Finding the Shortest Move-Sequence in the Graph-Generalized 15-Puzzle Is NP-Hard. in Studies in Complexity and Cryptography: Miscellanea on the interplay between Randomness and Computation. Springer, Berlin, Heidelberg (2011)

[6] Collette, S.,Raskin, J., Servais, F.: On the symbolic computation of the hardest configurations of the Rush Hour game. in Computers and Games. **2006**. Springer, Berlin, Heidelberg (2006)

[7] Rokicki, T., Kociemba, H., Davidson, M., Dethridge, J.: God's Number is 20. available at `http://www.cube20.org/`

[8] Polishchuk, V., Arkin, E.M., Mitchell, J.S.M: Hamiltonian cycles in triangular grids. in Proceedings of CCCG. **2006**, 63-66 (2006)