Algebraically Compact Categories in the Effective Topos

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Chapter 1

Introduction

1.1 Fixpoints, -maps and -objects

In analysis, we know that a *fixpoint* for a function $f : \mathbb{R} \to \mathbb{R}$ is a real number x for which $fx = x$. We can generalize this to endofunctions of arbitrary sets, but also to arrows of different categories. In topology for example, one encounters spaces that have fixpoints for all of their continuous endofunctions. The interval $[0, 1]$ has this property for example[1]. There is a condition that guarantees the existence of fixpoints:

Lemma 1.1.1 (Lawvere) *Given a category $\mathcal{C}$ with sufficient structure and within that category a pair of objects X and Y and a (regular) epimorphism $p : X \twoheadrightarrow Y^X$. Y has the fixpoint property: the property that every endomorphism has a fixpoint.*

Proof. Informally we can define $\omega x := pxx$. We then have for any function $f : Y \to Y$ that $f \circ \omega : X \to Y$ and therefore that there is a $y : X$ such that $py = f \circ \omega$. By definition $\omega y = pyy = f(\omega y)$ and therefore ωy is a fixpoint. $\square$

I'll leave it at this, although a more formal proof can be given by using generalized elements (and by determining what structure exactly is sufficient). The proof is constructive, meaning that we don't need to use the principle of the excluded middle or the axiom of choice to show that a fixpoint exists. Many important results in mathematical logic can be seen as special cases of this lemma, for example:

Cantor's paradox Allowing a subset $x \subset y$ to be equivalent to a function $x : y \to \mathbf{2}$ (where $\mathbf{2} = \{0, 1\}$), we can see that if $y \simeq \mathbf{2}^y$ for any set y $\mathbf{2}$ has to satisfy the fixpoint property. But what is the fixed point of the element swapping function g defined by $g0 = 1$ and $g1 = 0$?

Russell's paradox In this case we consider the class of all sets V, and this time we consider to interpretation of predicates as functions $V \to \mathbf{2}$. For any set x we can make a predicate $Py = y \in x$, and this mapping is surjective if unrestricted comprehension holds, since that means we have a set $\{y|Py\}$ which is mapped to an equivalent predicate. Again this implies that $\mathbf{2}$ has the fixed point property, which is absurd.

Recursion Theorem Moving to another branch of mathematics: there is a partial recursive coding of all partial recursive functions[2], so within a category of

[1] Given $f : [0, 1] \to [0, 1]$, we know that it is impossible that $fx < x$ or $fx > x$ for all $x \in [0, 1]$ because they imply $f0 < 0$ and $f1 > 1$ respectively. Therefore either $fx = x$ for all x, or there are both x with $fx - x < 0$ and x' with $fx' - x' < 0$. In the latter case there must be some x'' between x and x' where $fx'' = x''$, because of the intermediate value theorem.

[2] I use '$\dashrightarrow$' for partial functions

> *partial* recursive maps, we have an arrow $\phi : \mathbb{N} \dashrightarrow \mathbb{N}^{\mathbb{N}}$, giving $\mathbb{N}$ the fixed point property for every partial recursive function $\mathbb{N} \dashrightarrow \mathbb{N}$. Now we can use surjectivity of ϕ on last time to obtain: for every $f \in \mathbb{N}$ there is an $n \in \mathbb{N}$ such that fnx converges to y iff nx converges to y for all x, $y \in \mathbb{N}$.

In this thesis I'm looking for categories with a fixpoint property. To be more specific:

Definition 1.1.2 Given a category $\mathcal{C}$ for any endofunctor $F : (\mathcal{C}^{op})^m \times \mathcal{C}^n \to \mathcal{C}$ a *fixmap* is an isomorphism $F(X, \ldots, X) \to X$ (or equivalently an isomorphism $X \to F(X, \ldots, X)$) for some object X. A *fixobject* X is an object for which such a fixmap $f : F(X, \ldots, X) \to X$ exists.

So now I'm searching for categories that have fixmaps for their endofunctors, but also for categories which have fixmaps for any functor of the form $F : (\mathcal{C}^{op})^m \times \mathcal{C}^n \to \mathcal{C}$.

1.2 Recursive Domain Equations

Categories with fixpoint properties have a use: they allow a mathematical interpretation of recursive types in programming languages. Recursive types are fixobjects of type constructors, and usually their structure – the programs defined on the type by its definition – is determined by the fixmap.

Example 1.2.1 Given any type X the type LX of lists over X is determined by a fixmap $f : LX \to 1 + X \times LX$. This fixmap determines that a list $l \in LX$ is either empty when $fx : 1$, or consists of an element of X and a sublist when $fx : X \times LX$.

Example 1.2.2 When we have a type constructor for the type of programs Y^X between to types, a fixmap $f : L \to L^L$ is a model for the λ-calculus. Note that this is also a case were we can apply the fixpoint lemma 1.1.1: here it affirms that all λ-term have fixpoints.

Equations of the form $LX \simeq 1 + X \times LX$ and $L \simeq L^L$ are called *recursive domain equations*.

We shall see that when classical set theory is used as foundation, categories with fixpoint properties are rather simple and not appropriate for modelling programming languages. But programming languages are inherently connected to computability, and that suggests that we may be able to find models with a recursive universe of sets. The effective topos is precisely such a recursive universe of sets. Based on the realizability interpretation of arithmetic, it models how truth can be *observed* algorithmically.

Since the effective topos contains the category of sets as a subcategory, it can also be seen as extending set theory by new types of object. The new objects have unusual properties contradicting classical theorems, and allow the construction of more exotic internal categories. Therefore my search will include internal categories of the effective topos.

1.3 Literature

Fortunately, other people lead the way to fixpoint categories. My thesis is mainly based on these three articles: 'Algebraically Compact Categories' by P. Freyd [1], 'Extensional PERs' by P. Freyd, P. Mulry, G. Rosolini and D. Scott [2] and 'Recursive types Reduced to Inductive Types' by P. Freyd [3]. Spread over these three articles, strong properties implying fixmaps are defined, an internal subcategory of the effective topos is identified, and its fixpoint property is proved.

Unfortunately, besides the inconvenient fact that none of these papers contains all of this information at once, the second *crucial* article 'Extensional PERs' contains a strong claim about internal functors in the effective topos that I haven't been able to verify (or refute, by the way). I found that the main result, namely that the effective topos has an algebraically compact internal category, can be proved from other properties of internal functors. I'll get into the details in section 4.3.1, and in the last chapter of my thesis.

Chapter 2

Algebraic Compactness

2.1 Complete Categories

In a complete poset, we can always find a least and a greatest fixpoint: for any endomorphism f the least fixpoint is $\inf\{x|fx \leq x\}$ and the greatest is $\sup\{x|x \leq fx\}$. Since posets can be considered to be a kind of categories, on can try to generalize these constructions to obtain categories with fixpoint properties.

First we generalize the subset $\{x|fx \leq x\}$: given a category $\mathcal{C}$ and an endofunctor F we can construct the category of F-algebras. The objects are pairs $(X, f : FX \to X)$ and an arrow $a : (X, f) \to (Y, g)$ is just an arrow $a : X \to Y$ that commutes with the algebra structures: $a \circ f = g \circ a$. From this category $F-$alg the mapping U that maps (X, f) to X and $a : (X, f) \to (Y, g)$ to $a : X \to Y$, is a functor called the *underlying object functor*.

Lemma 2.1.1 *For any endofunctor F of any category $\mathcal{C}$, the underlying object functor $U : F-\text{alg} \to \mathcal{C}$ creates limits.*

Proof. The mapping α satisfying $\alpha(X, f) = f$ is a natural transformation $FU \to U$: Algebra morphisms commute with algebra structures and that yields the naturality of this mapping. Given a functor $G : \mathcal{D} \to F$-alg and a limiting cone $\kappa : Z \to UG$, we can construct a new cone $\alpha \circ F\kappa : FZ \to UG$, which factorizes into $z \circ \kappa$ for a unique $z : FZ \to Z$. This is an algebra structure, and more importantly it is the *only* algebra structure that lets the morphisms $\kappa_D : Z \to UGD$ for $D \in \mathcal{D}$ be algebra morphisms. This means that G has a limiting cone if UG has one for any functor G whose domain is any category $\mathcal{D}$, and this is how U creates limits. □

Corollary 2.1.2 *For any endofunctor F of a complete category $\mathcal{C}$ the category of F-algebras is complete.*

In small complete categories there will be a limit for the identity functor of $F-$alg. This is the same thing as an initial object, or in this case an initial F-algebra.

Definition 2.1.3 A category $\mathcal{C}$ is *algebraically complete* if every endofunctor F has an initial algebra.

Theorem 2.1.4 *Every small and complete category is algebraically complete*

Proof. As we have seen, for any small and complete category $\mathcal{C}$ and any endofunctor F, the category of F-algebras is small and complete, and the initial F-algebra is just the limit of the identity functor. □

Because the initial algebra $(X, a : FX \to X)$ is the limit of the category of all algebras, we can get into some of the details of its construction. X is a subobject of the product $\prod_{(Y,g)\in F-\text{alg}} Y$, and therefore any (generalized) element $x : X$ can be seen as a function that maps objects to elements of other objects: $x_{(Y,g)} : (Y, g)$. Of course, for any algebra morphism $m : (Y, g) \to (Z, h)$ we need that $mx_{(Y,g)} = x_{(Z,h)}$, and therefore:

$$X \simeq \left\{ x \in \prod_{(Y,g)\in F-\text{alg}} Y \;\middle|\; \begin{array}{l} \forall\ A, B, C \in F\text{-alg}, \\ \forall\ m : A \to C, m' : B \to C. \\ m(x_A) = m'(x_B) \end{array} \right\} \tag{2.1}$$

Let $\pi_{(Y,g)}$ be the mapping $x \mapsto x_{(Y,g)} : X \to Y$. The algebra structure $f : FX \to X$ must satisfy $\pi_{(Y,g)} \circ f = g \circ F\pi_{(Y,g)}$, or

$$(fz)_{(Y,g)} = g(F\pi_{(Y,g)}z) \tag{2.2}$$

for all $z : FX$, and this equation determines it uniquely.

Lemma 2.1.5 (Lambek) *Initial algebras are fixmaps.*

Proof. If $a : FZ \to Z$ is an initial algebra, there must be a unique algebra morphism $b : (Z, a) \to (FZ, Fa)$, because (FZ, Fa) is another algebra. $b \circ a = 1_Z$, because 1_Z is the only algebra endomorphism of Z, and $b \circ a = Fa \circ Fb = 1_{FZ}$, because b is an algebra morphism. Therefore a is an isomorphism of Z and FZ.□

So our intuition was right: we can find categories with fixpoint properties, by generalizing complete posets. Of all these results above the duals hold for small cocomplete categories. This gives us terminal coalgebras which are fixmaps for endofunctors et cetera. What makes that especially interesting is:

Lemma 2.1.6 *Small complete categories are cocomplete.*

Proof. For any functor $F : \mathcal{D} \to \mathcal{C}$, we can construct the 'category of cocones', which objects are objects of $\mathcal{C}$ combined with cocones on F. In this case, there is an underlying object functor that creates limits too, and therefore the category of cocones is complete. But this means there must be an initial or colimiting cocone for F. The dual of this lemma holds to, and can be proved by showing there is a terminal object in the category of cones. □

Fixmaps all are algebras and coalgebras at the same time, and therefore an initial algebra is an initial object in the category of fixobjects. Similarly the terminal coalgebras are terminal objects in the category.

2.2 Functors

Lemma 2.2.1 *For any two algebraically complete categories $\mathcal{A}$ and $\mathcal{B}$ and any pair of functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$, if (Y, g) is an initial GF-algebra, then (FY, Fg) is an initial FG-algebra.*

Proof. Let (X, f) be an initial FG-algebra. (GX, Gf) is another GF-algebra, and therefore we have a unique morphism $m : (Y, g) \to (GX, Gf)$. Similarly there is a unique morphism $n : (X, f) \to (FY, Fg)$. Remember Lambek's lemma that (FGX, FGf) is isomorphic to (X, f) by f, and notice that $Fm \circ n$ is inverse to that isomorphism, so Fm is a kind of left inverse of n. On the other hand $Gn \circ m$ is also an isomorphism and so is $FGn \circ Fm$, so Fm is also a kind of right inverse of n. Conclusion: $(GX, Gf) \simeq (Y, g)$ and $(FY, Fg) \simeq (X, f)$. □

In the case that $\mathcal{B} = \mathcal{A}$ and $G = F$, we see that if (X, f) is an initial algebra of F^2, then $(FX, Ff) \simeq (X, f)$. The isomorphism $n : X \to FX$ derived above, is the inverse of an initial algebra of F, and $f = n^{-1} \circ Fn^{-1}$, showing how the two are connected. As a consequence the following is a theorem:

$$\begin{tikzcd} (X,f) \arrow[r, "n"] & (FY,Fg) \arrow[r, "Fm"] & (GFX,GFf) \arrow[ll, bend left, "f"] \end{tikzcd}$$

Figure 2.1: For initial algebras $f : FGX \to X$ and $g : GFY \to Y$, we have $(FY, Fg) \simeq (X, f)$

Theorem 2.2.2 (Freyd's Iterated Square [1]) *For any category $\mathcal{A}$ and any endofunctor F suppose that F^2 has an initial algebra. Then so does F.*

In 2.2.1 we can also consider $\mathcal{B} = \mathcal{A}^{op}$ and $G = F^{op}$, for $\mathcal{A}$ being both algebraically complete and cocomplete. However, in this case we may get a separate initial algebra (X, f) and terminal coalgebra (Y, g) for F^2. F just maps one into the other.

Definition 2.2.3 Given a category $\mathcal{C}$ that is algebraically complete and cocomplete, there is for any endofunctor F a canonical map from the initial algebra to the inverse of the terminal coalgebra. If this map is an isomorphism, for every endofunctor, the category is *algebraically compact.*

Lemma 2.2.4 (Freyd [1]) *Algebraically compact categories have fixmaps for contravariant functors.*

Proof. Any contravariant functor F swaps the initial and terminal algebras of F^2. But whenever (X, f) is an initial algebra for F^2, we have an isomorphism $(X, f) \to (FX, Ff^{-1})$, whose underlying arrow is a fixmap for F. □

If $f : FX \to X$ is an initial algebra in some algebraically compact category $\mathcal{C}$, we know that given a coalgebra $g : Y \to GY$, there is a unique coalgebra morphism $h : (Y, g) \to (X, f^{-1})$. It has to satisfy $f^{-1} \circ h = Fh \circ g$ or

$$h = f \circ Fh \circ g \tag{2.3}$$

This is a recursive equation that uniquely determines h in any algebraically compact category.

Given a natural transformation $\eta : F \Rightarrow G$ between two endofunctors of an algebraically compact category $\mathcal{C}$, we can use is to turn G-algebras into F-algebras: given any G-algebra $g : GY \to Y$, $\eta_* g := g \circ \eta_Y : FY \to Y$ is an F-algebra. Note that if (Y, g) is initial, and (X, f) is an initial algebra of F, then there is a unique arrow $\eta' : (X, f) \to (Y, \eta_* g)$. This suggests there is a 'fixmap functor' that sends endofunctors to initial algebras, and natural transformations to precisely these maps between the underlying objects. In order to do this, we have to verify the mapping is functorial.

By chasing the diagram below one can see that for any triple of natural transformations $\eta : F \Rightarrow G$, $\theta : G \Rightarrow H$, and $\zeta : F \Rightarrow H$, we have $\zeta' = \theta' \circ \eta'$ whenever

$\zeta = \theta \circ \eta$ (where θ' and ζ' are derived from θ and ζ like η' from η above).

$$\begin{tikzcd}
& & FZ \arrow[dr, "\eta_Z"] \arrow[ddrr, bend left, "\zeta_Z"] & & \\
& FY \arrow[ur, "F\theta'"] \arrow[dr, "\eta_Y"] & & GZ \arrow[dr, "\theta_Z"] & \\
FX \arrow[ur, "F\eta'"] \arrow[uurr, bend left, "F\zeta'"] \arrow[d, "f"'] & & GY \arrow[ur, "G\theta'"] \arrow[d, "g"'] & & HZ \arrow[d, "h"] \\
X \arrow[rr, "\eta'"] \arrow[rrrr, bend right, "\zeta'"'] & & Y \arrow[rr, "\theta'"] & & Z
\end{tikzcd}$$

Lemma 2.2.5 *Given an algebraically complete category $\mathcal{C}$ and a map y that chooses an initial algebra for every endofunctor, y can be extended to a functor $\mathcal{C}^{\mathcal{C}} \to \mathcal{C}$ in a unique way*

Proof. We just let $y\eta = \eta'$, for every natural transformation η. When $\eta = 1_F$ for some functor F, automatically $\eta' = 1_X$ (the initial algebra of F again being (X, f)), because it is the underlying arrow of the unique algebra endomorphism of the initial algebra. $\square$

Functors of this form are *fixmap functors*, and they are unique up to isomorphism. By the axiom of choice there is a map y that chooses an initial algebra for every endofunctor for every small and algebraically complete category $\mathcal{C}$, and therefore a fixmap functor $\mathcal{C}^{\mathcal{C}} \to \mathcal{C}$.

Lemma 2.2.6 (Freyd [1]) *The product of algebraically complete categories is algebraically complete.*

Proof. Let $H : \mathcal{A} \times \mathcal{B} \to \mathcal{A} \times \mathcal{B}$. H consists of two functors $H_1 : \mathcal{A} \times \mathcal{B} \to \mathcal{A}$ and $H_2 : \mathcal{A} \times \mathcal{B} \to \mathcal{B}$, and both have transposes $H_1^t : \mathcal{B} \to \mathcal{A}^{\mathcal{A}}$ and $H_2^t : \mathcal{A} \to \mathcal{B}^{\mathcal{B}}$. By composing with fixmap functors associated with $\mathcal{A}$ and $\mathcal{B}$, we obtain functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$, and the pair of initial algebras of their composites is an initial algebra of H. $\square$

Consider the case that $\mathcal{B} = \mathcal{A}^{op}$, and that $\mathcal{A}$ is algebraically compact. For any functor $T : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}$, we can make a functor $T' : \mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}^{op} \times \mathcal{A}$ by letting $T(X, Y) = (TYX, TXY)$. This will have an initial algebra $(f, g) : (TYX, TXY) \to (X, Y)$, but because of the symmetries of the product, and the algebraic compactness of $\mathcal{A}$, the algebra $(g^{-1}, f^{-1})(TXY, TYX) \to (Y, X)$ is initial too. Conclusion: $X \simeq Y$ and is a fixobject for T.

Theorem 2.2.7 (Freyd [1]) *Algebraically compact categories have fixmaps for co-, contra-, and bivariant endofunctors of any arity.*

Any functor $F : (\mathcal{A}^{op})^m \times \mathcal{A}^n \to \mathcal{A}$ has a fixmap, because F can be composed with diagonal functors to obtain a simple bifunctor $\mathcal{A}^{op} \times \mathcal{A} \to \mathcal{A}$, and those have fixmaps as just proved.

Corollary 2.2.8 (Freyd [1]) *Algebraically compact categories have 0-objects*

Proof. A *0-object* is an initial object that is also terminal. The identity functor has an initial and terminal algebra, and since every object is a fixobject for the identity, this object is initial and terminal in this category. $\square$

2.3 Triviality

We obtained categories with fixpoint properties by generalizing complete partial orders. Except for complete posets we don't have any examples of complete categories that are algebraically complete. The following lemma helps explain why we don't have such examples:

Lemma 2.3.1 *Let x, y, z be any triple of sets, let $x \subset y$ and let $p : x \twoheadrightarrow z^y$ be a surjective mapping. z is empty or has only one element.*

Proof. Either $x = \emptyset$ or $x \neq \emptyset$. If $x = \emptyset$, then $p : x \twoheadrightarrow z^y$ forces $z^y = \emptyset$, and therefore $z = \emptyset$. So let x be nonempty, so that there is some $a \in x$. Define $q : y \to x$ with $qb = b$ for $b \in x$ and $qc = a$ for $b \notin x$. Now we have $p \circ q : y \to z^y$, so z has the fixed point property.

$1_z : z \to z$ exists and has a fixed point, so now z cannot be empty. Suppose that $a, b \in z$ with $a \neq b$, and define $f : z \to z$ by $fa = b$ and $fx = a$ whenever $x \neq a$. f has a fixed point c, and $c = fc = a$ whenever $c \neq a$, or $fc = b \neq a$ whenever $c = a$: contradiction. So all elements of z are equal. □

This lemma shows how classical mathematics opposes the fixpoint property, mainly because of the principle of the excluded middle (PEM). It is needed to justify $x = \emptyset$ or $x \neq \emptyset$, to find $a \in x$ when x is non empty and to show that q and f are total functions.

Let $\mathcal{C}$ be a small and complete category. Such categories necessarily have initial and terminal objects, and won't be empty. Now let $B_{i \in I}$ be a constant family of objects indexed by I, so $B_x = B$ for all $x \in I$. This family has a limit B^I, and that means that $\hom(A, B^I) \simeq \hom(A, B)^I$ for all A. Now here's a problem: if there are two or more morphisms between A and B, then the cardinality of $\hom(A, B)^I$ is strictly greater than the cardinality of I. Let I be the set of all morphisms...

Lemma 2.3.2 (Freyd) *Small complete categories are preordered sets*

Proof. We have $\hom(A, B)^I \simeq \hom(A, B^I)$, and $\hom(A, B^I) \subset I$, when I is the set of all functions. From 2.3.1, we deduce that $\hom(A, B)$ has 0 or 1 elements, making the category a preordered set. □

Of course, being a preordered set does not prohibit a category from being algebraically compact. But remember (2.2.8) that algebraically compact categories have 0-objects, in the preordered set case an object 0 with $0 \leq x \leq 0$ for any x in the set. The only examples of small categories both complete and algebraically complete are therefore trivial one object one arrow categories.

Now we might still have some hope for large complete categories, which don't need to be preordered sets. But if the *contravariant* functor $TX = B^{\hom(A,X)}$ (for any A and B, where $B^{\hom(A,X)}$ is the product of $\# \hom(A, X)$ copies of B) has a fixed point, we have an object D for which $D \simeq B^{\hom(A,D)}$, therefore $\hom(A, D) \simeq \hom(A, B^{\hom(A,D)}) \simeq \hom(A, B)^{\hom(A,D)}$.

Theorem 2.3.3 (Freyd) *Every complete and algebraically compact category is trivial.*

Proof. This is just the worst case scenario: on one hand, $\hom(A, B)$ has zero or one elements because of 2.3.1, and $\hom(A, B)$ cannot be empty, because then there is no way that $\hom(A, D) \simeq \hom(A, B)^{\hom(A,D)}$ holds. We see that in any algebraically compact category there is exactly one arrow between any pair of objects. □

By the way: let $T'X = B^{\mathbf{P}(\hom(A,X))}$, where $\mathbf{P}(\hom(A, X))$ is the powerset of $(\hom(A, X))$, then we have a covariant functor, and for a map $D \to T'D$ we have $\hom(A, D) \simeq \hom(A, B)^{\mathbf{P}(\hom(A,D))}$. By the singleton mapping $x \mapsto \{x\}$, we have as subset $S \subset \mathbf{P}(\hom(A, X))$ such that $S \simeq \hom(A, B)^{\mathbf{P}(\hom(A,D))}$.

Theorem 2.3.4 (Freyd[1]) *If a complete category has fixmaps for all endofunctors, it is a preorder. This includes all algebraically complete categories.*

Proof. Again from 2.3.1 deduce that $\hom(A, B)$ has zero or one elements for any pair of objects A and B. $\square$

2.4 Summary

From completeness in small categories we derived a fixpoint property: algebraic completeness. Then we found that a simple extension of this fixpoint property – algebraic compactness – allowed us to find fixmaps for contra- and bivariant functors too. But completeness forces a category to be a preordered set, and algebraic compactness forces a category to be trivial.

We already knew that complete posets have fixpoint properties, and found that the same holds for small complete categories, but ironically small complete categories are just preordered sets. We also found that the homsets of algebraically compact categories have the fixpoint property – the part mimicking the whole – which would be useful if algebraically compact categories weren't trivial because of it.

Chapter 3

The Effective Topos

3.1 Constructive Logic

From the classical principle of the excluded middle I derived lemma 2.3.1, which gives a strong restriction on any kind of category with a fixpoint property. This lemma is equivalent to the principle of the excluded middle: let P be a proposition, x be a singleton and $y := \{z \in x | P\}$. Then we have the function $k : y \to y^x$ that maps elements of y to constant functions, and that is an isomorphism, because x is a singleton. Now it follows from the lemma that y is empty or a singleton, and therefore, x has only two subsets. If $y = \emptyset$, then $\neg P$ and if $y = x$ then P, and therefore $P \vee \neg P$. Therefore, if we drop the principle of the excluded middle, we also loose the lemma, and non trivial categories with fixpoint properties become possible.

Is it reasonable to drop the principle of the excluded middle? I suppose that it expresses a valid property of truth: it makes sense to me that all propositions that are objective, unambiguous and so on are either true or false. We may however never be able to determine the truth of such propositions as "there is intelligent life on planets other then Earth", because of physical limitations. Because of those limitations, the *absolute independent truth* that classical logic is all about, sometimes has little practical value to us.

The various kinds of constructive logic can be seen as capturing notions of observability: they do not only ask whether a proposition is true, but also whether we can 'observe' that in some way. For example: whether "there is intelligent life on planets other then Earth" or not, is currently not observable, and possibly never will be. Within constructive logic we assume for any pair of propositions P and Q that $P \vee Q$ is observably true iff we can observe which one of them is true. So for the statement about extraterrestrial intelligence, it doesn't make sense to say that we can observe this statement is true or false, because we have no means to observe which is the case.

Of course, constructive logic usually concerns mathematical statements, rather than extraterrestrials. Within mathematics observability usually is restricted only for infinite objects. Even if a set has 10^{100} elements, we don't worry about the effort it would take to check whether all of those elements share some property. Only when a set is infinite we start to worry whether statements like $\forall x.px$ are still observable (provided px is observable for all x). When we're dealing with sets of infinite sets, statements like $\exists x.px$ are also problematic, because how can we ever finish constructing a single example of an object x satisfying px, if the only sets satisfying it are infinite? Because they deal with this latter concern, these logics are called constructive.

The kind of observability or constructability treated in this chapter is *effective computability*. Informally we want to know when a computer of unrestricted capabilities in an indefinite amount of time can determine by running an algorithm that a true statement is true. Except for determining truth of falsity, the computer may go into a loop, and potentially run forever without producing any result. For every computer there must be a true statement that makes it go into such a loop as a consequence of the incompleteness theorems and the halting problem. This is how observability is restricted, and show why PEM doesn't hold for *effectively computable truth*.

3.2 Topos Theory in General

To interpret the theory of small categories, we will need to reason with sets of sets and so on. This means we need an interpretation of higher order intuitionistic logic. Toposes are a kind of category that can be used as alternative set theories, because in toposes an interpretation of higher order intuitionistic logic is possible.

3.2.1 Elementary Toposes

Definition 3.2.1 An *elementary topos*, or *topos* for short, is a Cartesian closed category with finite limits and a *subobject classifier* $\top : \mathbf{1} \to \Omega$. This means any monic arrow $f : A \rightarrowtail B$ is the pullback of $\top$ along a unique arrow $\kappa : B \to \Omega$. Here, κ is the *characteristic arrow* of the subobject of B that f represents, and Ω is the *truth values object*.

$$\begin{tikzcd} A \arrow[r, tail, "f"] \arrow[d] & B \arrow[d, "\kappa"] \\ \mathbf{1} \arrow[r, "\top"] & \Omega \end{tikzcd}$$

Figure 3.1: Every monic it the pullback of $\top : \mathbf{1} \to \Omega$

$\mathbf{P}A := \Omega^A$ is called the *power object* of A because it generalizes the notion of power sets in set theory: global elements $\mathbf{1} \to \Omega^A$ are in bijective correspondence to arrows $A \to \Omega$ which are in bijective correspondence to subobjects. If we consider arrows $A \to \Omega$ to be relations on A, then this gives us a way to define subsets by separation.

With just this structure we can interpret most of set theory and higher order logic, although that is not easy too see from just the assumptions above:

1. Given Cartesian products, we have diagonal subobjects $\Delta_X : X \rightarrowtail X \times X$, and therefore we have $=_X : X \times X \to \Omega$ classifying them.

2. Given two subobjects of X, their pullback is a new subobject of X. Because of the bijective correspondence between subobjects and maps $X \to \Omega$, we get an operation $\wedge : \mathbf{P}X \times \mathbf{P}X \to \mathbf{P}X$. The greatest subobject of X is X itself of course, and this gives us $\top : \Omega^X$, the unit of $\wedge$.

3. Because of Cartesian closure, we have objects like $\mathbf{P}^2X$, including the operations $\top$, $\wedge$ and $=$ defined above, allowing second and higher order reasoning. Now we can interpret all the other logical symbols too:

$$
\begin{array}{llcl}
\forall & \forall x \in X.\phi & := & (\lambda x : X.\phi) = \top \\
\bot & \bot & := & \forall \chi \in \Omega^A.\chi \\
\to & \phi \to \psi & := & \phi = \phi \wedge \psi \\
\neg & \neg\phi & := & \forall \chi \in \Omega^A.\phi \to \chi \\
\vee & \phi \vee \psi & := & \forall \chi \in \Omega^A.(\phi \to \chi) \wedge (\psi \to \chi) \to \chi \\
\exists & \exists x \in X.\phi & := & \forall \chi \in \Omega^A.(\forall x \in X.\phi \to \chi) \to \chi
\end{array}
$$

We can reason about the objects of toposes as if they were sets, and talk about their elements and subsets. This will not precisely be a set theory, because in general we have no element relation between the objects. The power objects help a bit, but to define what $Y \in \mathbf{P}X$ is supposed to mean, we need a mapping to choose for every arrow $X \to \Omega$ a monic that represents the subobject it classifies and that for every object X.

When reasoning within a topos, it may be helpful to think of objects as sets anyway, but with elements that aren't objects of the topos themselves. Based on that interpretation, we may reason from the following axioms:

Terminal Object There exists an object $\mathbf{1}$ such that for every pair x and $y \in \mathbf{1}$, $x = y$.

Products Given any pair of objects X and Y, their product $X \times Y$ exists, which elements are pairs (x, y), and which satisfies $(x, y) \in X \times Y$ iff $x \in X$ and $y \in Y$. The projections $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ satisfying $\pi_1(x, y) = x$ and $\pi_2(x, y) = y$, are arrows in the topos.

Homsets For any pair of objects X and Y, there is an object of arrows Y^X. This means on one hand that $f \in Y^X$ for every $f : X \to Y$ in the topos, and on the other that we have an evaluation maps $(f, x) \mapsto fx : Y^X \times X \to Y$ for all pairs of objects X and Y. When we have an arrow $f : X \times Y \to Z$, we can use λ-abstraction to create the 'arrow valued arrow' $g : X \to Z^Y$ that is the unique arrow satisfying $gx = \lambda y.f(x, y)$.

Equalizers Given any parallel pair of arrows f and $g : X \to Y$, there is an object $\{x|fx = gx\}$ satisfying $x \in \{x \in X|fx = gx\}$ iff $fx = gx$.

Subobject Classifier There is an object Ω, and an element $\top \in \Omega$ that classifies all subobjects. This means that for every monic $f : X \rightarrowtail Y$, there exists an arrow $\kappa : Y \to \Sigma$, such that for all $x \in X$ $\kappa(fx) = \top$ and such that there is a unique isomorphism $f^{-1} : \{y \in Y|\kappa y = \top\} \to X$ such that $f \circ f^{-1}$ equals the inclusion of $\{y \in Y|\kappa y = \top\}$ into Y.

Using higher order logic, we may then derive the following useful principles:

Empty Set There exists an empty object $\emptyset$

Power Set For every object X, an object $\mathbf{P}X$ representing the class of all subobjects exists.

Restricted Comprehension Whenever ϕ is a monic predicate in which all quantifiers are restricted to objects, the object $\{x|\phi x\}$ exists.

We need to reason constructively, because some toposes don't satisfy the principle of the excluded middle, or the axiom of choice. But constructive logic, together with the axioms above, allow us to pretend any topos is a class of sets.

3.2.2 Bounded Limits and Colimits

Because we can represent a family of subobjects of some object Y indexed over some other object X, as an arrow $f : X \to \mathbf{P}Y$, we can define the product of such a family: $\prod_{x \in X} fx := \{g \in Y^X | \forall x \in X.gx \in fx\}$. Even more general: given a directed graph represented by:

$$X_1 \overset{d_0}{\underset{d_1}{\rightrightarrows}} X_0$$

where X_1 is the object of edges, X_0 the object of vertices, where d_0 maps edges to source vertices, and d_1 to target vertices; given also a map $f_0 : X_0 \to \mathbf{P}Y$ and a map $f_1 : X_1 \to \mathbf{P}(Y^2)$ such that for every $x \in X_1$ f_1x is (a graph of) a function $f_0(d_0x) \to f_0(d_1x)$, then we can define the limit as the set:

$$\{g \in Y^{X_0} | \forall x \in X_1.f_1x(g(d_0x)) = g(d_1x)\}$$

Given such an $f : X \to \mathbf{P}Y$, we can also construct the product of the power objects, because they are subobjects of $\mathbf{P}Y$. We use this to construct coproducts as sets of equivalence classes. For a set $u \in \mathbf{P}(X_0 \times X)$ to be an equivalence class for the same graph (X_0, X_1, d_0, d_1) and mappings $f_0 : X_0 \to \mathbf{P}Y$ and $f_1 : X_1 \to \mathbf{P}(Y^2)$ as above, we need first that for all $x \in X_1$ $(y, x) \in u$ implies $y \in f_0x$, and second that given any $x' \in X_1$ $(y, d_0x') \in u$ iff $(f_1x'y, d_1x) \in u$. The colimit is then defined as the subobject of all equivalence classes in $\mathbf{P}(X_0 \times Y)$.

Although this construction of colimits works just fine in set theory, it is non trivial that it also works in any topos. We need Paré's theorem [7] to prove that the object of equivalence classes is indeed the colimit of the diagram defined by the graph above.

Definition 3.2.2 A Y*-bounded family indexed over* X is an arrow $X \to \mathbf{P}Y$. A Y*-bounded diagram* is a diagram (a map of a directed graph into a category) whose objects are embedded into Y. Y*-bounded limits and colimits* are limits and colimits over Y-bounded diagrams. *bounded* in general is Y-bounded to some object Y.

Because we can construct limits and colimits of any bounded diagrams as subobjects of $\mathbf{P}(X_0 \times Y)$ ($Y^{X_0} \subset \mathbf{P}(X_0 \times Y)$ by the function to graph mapping), we may state:

Lemma 3.2.3 *All toposes are bounded complete and cocomplete, where* bounded complete *means that all bounded limits exist and* bounded cocomplete *that all bounded colimits exist.*

3.2.3 Natural Number Objects

What's still missing for a full interpretation of mathematic is the object of natural numbers. It is basically defined as an initial algebra for the functor that maps X to $X + \mathbf{1}$. Such an algebra $f : X + \mathbf{1} \to X$ splits into a global element $0 : \mathbf{1} \to X$ and a successor function $s : X \to X$. Being initial makes the induction principle true: given an object Y, a global element $x_0 : \mathbf{1} \to Y$ and an arrow $g : Y \to Y$, there is a unique function $h : X \to Y$ satisfying $h \circ 0 = x_0$ and $h \circ s = g \circ h$. In a topos with natural number objects, we generally pick one and call it N. Symbol '$\mathbb{N}$' is preserved for the standard natural numbers.

Using products and equivalence relations, we can construct integer and rational number objects. From rational number objects we can define real number objects using Dedekind cuts. With real numbers every topos allows an interpretation of constructive analysis, and of more advanced theories in its wake.

$$\begin{tikzcd} \mathbf{1} \arrow[r, "0"] \arrow[dr, "x"'] & N \arrow[r, "s"] \arrow[d, dotted, "\exists!"] & N \arrow[d, dotted, "\exists!"] \\ & X \arrow[r, "f"'] & X \end{tikzcd}$$

Figure 3.2: universal property of natural number objects

Because a topos is Cartesian closed, we can give an internal proposition on 'recursive definition' which follows from the existence of a natural number object N:

Theorem 3.2.4 *There is an object N, with an element 0 and an endomorphism s, such that for every object X, $x \in X$ and $f : X \to X$, there is a unique map $g : N \to X$ satisfying $g0 = x$ and $g \circ s = f \circ g$.*

I will use this theorem of toposes with natural number objects, as another axiom of the internal logic.

3.3 Realizability

3.3.1 Partial Recursive Functions

The partial recursive functions are basically the partial functions $\mathbb{N}^n \dashrightarrow \mathbb{N}$ that are computable[1]. Important here, is that there is a recursive coding of pairs of natural numbers allowing for functions of an arbitrary arity to be expressed as functions of arity 1. I'll use $\langle n, m \rangle$ for the coding of the pairs (n, m) by an arbitrary recursive paring functions, and when x is number coding a pair, I'll use x_1 and x_2 to denote its elements.

Even more important is that there is a *universal* partial recursive function $\phi : \mathbb{N}^2 \dashrightarrow \mathbb{N}$, such that for every $f : \mathbb{N} \dashrightarrow \mathbb{N}$ there is an $e \in \mathbb{N}$ such that whenever fx is defined $\phi(e, x)$ is defined and equal to fx. This function ϕ is of what makes the recursion theorem (1.1) possible, but also shows that there must be severe restrictions on what is recursively definable.

There is a primitive recursive function S such that whenever $\phi(f, \langle x, y \rangle)$ is defined, $\phi(\phi(Sf, x), y)$ is defined and equal to it. Using this function, we can give an interpretation to the definitions of recursive functions by λ-abstraction.

Example 3.3.1 Take $\lambda xyz.xz(yz)$ for example:

1. First we consider this as the partial recursive function $\mathbb{N}^3 \to \mathbb{N}$ defined by:

$$(x, y, z) \mapsto \phi(\phi(x, z), \phi(y, z))$$

2. Then we use our coding of pairs to turn this into:

$$\langle \langle x, y \rangle, z \rangle \mapsto \phi(\phi(x, z), \phi(y, z))$$

3. Because this function is partial recursive, there is some e such that

$$\phi(e, \langle \langle x, y \rangle, z \rangle) \equiv \phi(\phi(x, z), \phi(y, z))$$

   as functions in x, y and z. This deserves some explanation: the idea here, is to use the (partial) recursive mappings $\pi_1 \circ \pi_1$, $\pi_2 \circ \pi_1$ and π_2 to obtain x, y and z from a code for $\langle \langle x, y \rangle, z \rangle$.

[1] Again, I'll use $\dashrightarrow$ to distinguish partial from total functions

4. With S we get $\phi(\phi(\phi(S(Se), x), y), z) = \phi(\phi(Se, \langle x, y\rangle), z) = \phi(e, \langle\langle x, y\rangle, z\rangle)$.

Thus we find that $S(Se)xyz \equiv xz(yz)$ and that therefore $S(Se)$ interprets the λ-term $\lambda xyz.xz(yz)$.

Rather then worrying about how partial recursive function are coded explicitly, we can describe partial recursive function informally, or formally with help of λ-terms, and then just assume a code exists. Of course, I will make use of this convention throughout the rest of this thesis.

Definition 3.3.2 I'll write $nx \to y$ for $\phi(n, x)$ is defined and equal to y. $nx \downarrow$ means $nx \to y$ for some y, in other words: nx is defined. $nx \uparrow$ means $nx \to y$ for no y, in other words: nx is undefined. And $n \equiv m$ whenever $nx \to y$ iff $mx \to y$ for all x and y. When convergence is obvious, I'll write $nx = y$.

It's nice to view the subsets of $\mathbb{N}$ as a category, using the partial recursive functions as arrows: $f : A \to B$ for subsets A and B of $\mathbb{N}$ iff fx is defined and in B for all $x \in A$. I extend this notation to numbers, by letting $n : A \to B$ iff $\phi(n, -) = f$ for some partial recursive function $f : A \to B$. But keep in mind that composition is defined on functions, and only works for numbers upto $\equiv$. The category is Cartesian closed, because we can use the coding of pairs to create a product: $A \times B := \{\langle x, y\rangle | x \in A, y \in B\}$, and the sets $B^A := \{n | n : A \to B\}$ can be used as internal homsets.

3.3.2 Realizability

Intuitionistic logic is connected to computability through the Curry Howard isomorphism. We can think of propositions as data types, and of proofs as computer programs that transform individuals of one type into the other. By modelling this notion with natural numbers and partial recursive functions, we arrive at the notion of realizability. Originally realizability was introduced by Kleene specifically for arithmetic, meaning that the data types are arithmetic propositions and the programs proofs of arithmetic theorems.

For a number n and a proposition A, n realizes A, symbolically written as $n \vdash A$, can be interpreted as n is a code for something of type A. Then given a code of a partial recursive function f that has the property that if $x \vdash A$ then $fx \downarrow$ and $fx \vdash B$ for all $x \in \mathbb{N}$, we consider it a program, transforming codes of elements of A into codes of elements of B, and therefore itself a code of something of type $A \to B$.

This interpretation is extended to all formulas by the rules:

$\wedge$	$n \vdash A \wedge B$	iff	$n_1 \vdash A$ and $n_2 \vdash B$
$\vee$	$n \vdash A \vee B$	iff	$n_1 = 1$ and $n_2 \vdash A$, or
			$n_1 = 2$ and $n_2 \vdash B$
$\to$	$n \vdash A \to B$	iff	for all $x \in \mathbb{N}$ if $x \vdash A$,
			then $nx \downarrow$ and $nx \vdash B$
$\neg$	$n \vdash \neg A$	iff	no $m \vdash A$
$\perp$	$n \vdash \perp$		never happens
$\forall_{\mathbb{N}}$	$n \vdash \forall x \in \mathbb{N}.\phi$	iff	$nm \vdash \phi[m/x]$ for all $m \in \mathbb{N}$
$\exists_{\mathbb{N}}$	$n \vdash \exists x \in \mathbb{N}.\phi$	iff	$n_1 \vdash \phi[n_2/x]$
$=$	$n \vdash x = y$	iff	$x = n$ and $y = n$

Here $\phi[m/x]$ stands for the formula ϕ with all free occurrences of x replaced by (the symbol for) m.

Now the logical symbols have become type constructors, similar to familiar constructions on sets in set theory: $\wedge$ is the product, $\vee$ the disjoint sum, $\to$ the exponential, and $\perp$ the empty set. The interpretation of quantifiers in realizability may look a bit peculiar, but we restrict quantification to $\mathbb{N}$, so $\forall x \in \mathbb{N}.\phi$ is equivalent to $\forall x.x \in \mathbb{N} \to \phi$ and $\exists x \in \mathbb{N}.\phi$ is equivalent to $\exists x.x \in \mathbb{N} \wedge \phi$. Now if we let $n \vdash n \in \mathbb{N}$, the interpretation of the quantifiers above follows naturally.

Realizers for the universal quantifier can be thought of as array: $n \vdash \forall x \in \mathbb{N}.\phi$ contains for every $i \in \mathbb{N}$, a realizer of $\phi[i/x]$. It is interpreted as a product indexed by $\mathbb{N}$, and dually the existential quantifier becomes an $\mathbb{N}$-indexed coproduct.

3.4 The Effective Topos

The effective topos is an extension of realizability to the higher order theory of natural numbers and sets of natural numbers. For that purpose, we construct a large category of objects with realizable properties. It will include a natural numbers object on which the realizable properties are exactly the realizable theorems. I'll use $\mathcal{E}\!f\!f$ to refer to the effective topos here on.

I think it is easiest to trace the interpretation of logical formulas from the simplest to the most complex. That means that I will start with propositional logic, and work may way up back to higher order logic again. To start, for propositional logic we can take a look at the table for realizability in arithmetic, because for propositional connectives the interpretations remain the same.

3.4.1 Equality

Going beyond propositional logic requires us to introduce sets of individuals to reason about, and quantify over. But before we look at quantification, we'll have to take a look at equality.

What we need on any set of individuals X is an interpretation of $x = y$. Sometimes we can get away with $[x = y] = \mathbb{N}$ if $x = y$ and $[x = y] = \emptyset$ if $x \neq y$. However, this doesn't capture the constructive notion of equality in every case. Take for example $\mathbb{N}$ above, which has $[x = y] = \{x\} \cap \{y\}$. Therefore we will request that our set X of individuals comes with a function $R : X^2 \to \mathcal{P}\mathbb{N}$, which can be used as an interpretation of $=$ ($[x = y] = R(x, y)$): this defines what an object of $\mathcal{E}\!f\!f$ is.

To be useful as an equivalence relation, we'll need the axioms of symmetry $x = y \to y = x$ and transitivity $x = y \wedge y = z \to x = z$ to be realized. That means we'll need an $n : R(x, y) \to R(y, x)$ and an $m : R(x, y) \times R(y, z) \to R(y, z)$ (for all x,y and $z \in X$). Such a relation is an effective partial equivalence relation.

Notice that we don't ask that there is some $n \in [x = x]$ for all x. Partly this is because we don't need to: symmetry and transitivity imply reflexivity, at least for all x whose equivalence to some y is realized. That, however, leaves space for 'ghost' elements z with $R(z, z) = \emptyset$. We do not require R to keep different element separate either: having $n \in R(x, y)$ does not mean that $x = y$ in the underlying set.

Now for $\mathcal{PN}$ we define $R(X, Y) = (X \to Y) \times (Y \to X)$, and let $\Omega := (\mathcal{PN}, R)$. Similarly, we let $N = (\mathbb{N}, S)$ with $S(n, m) = \{n\} \cap \{m\}$. These objects are our truth value object and natural number object respectively.

3.4.2 Predicate Logic

A relation on X is a function $f : X \to \mathcal{P}\mathbb{N}$. In order to make it work properly in combination with our equivalence relation, we demand a realizer for $f(x) \wedge x = y \to f(y)$, which makes substitution work properly. We'll also need a realizer $f(x) \to x = x$, to restrict the domain of f to (X, R).

To look further then unary relations on a single object (X, R) of individuals, we'll need products. But $(X, R) \times (Y, S)$ will simply be $(X \times Y, R \otimes S)$ where $(R \otimes S)((x, y)(x', y')) = R(x, x') \times S(y, y')$ (which in realizability is equivalent to saying that $(x, y) = (x', y')$ iff $x = x'$ and $y = y'$). Of course an n-ary relation is the same thing as unary relation on a product. Now we have products, we will also need a terminal object. We can take any singleton $\{x\}$, and let $R(x, x)$ be any nonempty set of numbers. With Cartesian product and terminal objects the category is Cartesian.

No we can quantify over relations, and quantification in realizability is quite straightforward. $[\exists x.\phi x] = \bigcup_{x \in X}[\phi x]$. Because $[\phi x \to x = x]$ is realized, we don't need to incorporate $[x = x]$ into the definition of $\exists$, even for N. For universal quantification, we do need to restrict the domain of quantification, therefore $[\forall x.\phi x] = \bigcap_{x \in X}[x = x \to \phi x]$.

3.4.3 Arrows and Higher Order Logic

Only now will I introduce arrows, which in $\mathcal{E}\!f\!f$ are binary relations of a special kind. F is an arrow $(X, R) \to (Y, S)$, if it is a relation '$fx = y$' on $(X, R) \times (Y, S)$ that is total on (X, R), meaning that $\forall x \in X \exists y \in Y. fx = y$ is realized, and that takes unique values in (Y, S), meaning that $fx = y \wedge fx = y' \to y = y'$ is realized.

The equality relations part of every object satisfy these conditions easily, and form the identity arrows of the objects. Composition is defined as in set theory: $(f \circ g)x = z$ iff $\exists y. fy = z \wedge gx = y$, or $F \circ G(x, z) = \bigcup_y F(y, z) \times G(x, y)$. This defines a composition that is associative and that behaves properly with respect to identity arrows, so that we indeed have a category.

In the set of relations over (X, R), call it $P(X, R)$, the equivalence of two relations f and g is given by point wise equivalence in X, so $(f \Leftrightarrow g) := \bigcup_{x \in X} fx^{gx} \times gx^{fx}$. Any relation f is now an element of an equivalence class in $\mathbf{P}(X, R) := (P(X, R), \Leftrightarrow)$. We can use this relation to define a subobject of (X, R) by letting $R_f(x, y) = fx \times R(x, y)$. Simultaneously R_f is a new equality on X, and an inclusion $R_f : (X, R_f) \to (X, R)$. In this way the separation scheme is realized: given a predicate ϕ interpreted as a relation on $[x] = (X, R)$, we get $[\{y \in x | \phi y\}] = (X, R_{[\phi]})$.

Note that there is a subobject of $\mathbf{P}((X, R) \times (Y, S))$ of functional relations on $(X, R) \times (Y, S)$. That subobject is the exponential Y^X, and because it exists for any pair of objects X and Y, the category is Cartesian closed.

Given a monic arrow f, its image is the subobject defined by the relation $gy := [\exists x. fx = y] = \bigcup_x .F(x, y)$, and using this relation we obtain an equivalent subobject of (X, R). Now equivalent relations will give us equivalent subobjects, so there actually is bijection between equivalence classes of subobjects on one side, and of relations on the other.

A bijection between $\hom((X, R), \Omega)$ and $\mathbf{P}(X, R)$ is given by $\forall x \in X, \sigma \in \Omega. F(x, \sigma) = fx \Leftrightarrow \sigma$ - this allows us to obtain a relation from an arrow by letting $fx = F(x, \mathbb{N})$. There is a bijection between equivalence classes of subobject of (X, R) and of arrows $(X, R) \to \Omega$, through the object $(P(X, R), \Leftrightarrow)$. So Ω has all the properties that a subobject classifier must have.

And with finite products, exponentiation, a subobject classifier and a natural numbers object, the category we defined is actually a topos. The standard interpretation of set theory in toposes mentioned in the previous section is equivalent to the realizability interpretation given here.

3.5 Axiomatics

Intuitionistic first order logic, lacking the principle of the excluded middle, is weaker than classical logic. When we start quantifying over predicates however, the weaker predicates allow for stronger higher order theorems. Now $\mathcal{E}\!f\!f$ is a topos that doesn't satisfy the principle of the excluded middle, or a general axiom of choice, but replaces these higher order propositions by others. I will consider some of these strong higher order theorems as axioms for set theory in $\mathcal{E}\!f\!f$:

Axiom 1 (Countable Axiom of Choice (CAC)) *Given $A \subset N$ with $\neg\neg(n \in A) \to n \in A$ for all $n \in N$, an arbitrary object X and a relation R on $N \times X$ such that for every $n \in A$ there is a $x \in X$ satisfying $R(n, a)$, there is a choice function $f : A \to N$ such that $R(n, fn)$ for all $n \in A$.*

Definition 3.5.1 It is nice in general to look at subsets Y of any object X that satisfy $\neg\neg(n \in Y) \to n \in Y$ for all $n \in N$. Such subsets are called $\neg\neg$-*closed* or simply *closed* subobjects. Closed subobjects can be considered to be classical in the sense that they satisfy the double negation elimination property by definition, although we also have *decidable* subsets, satisfying $x \in Y \vee x \notin Y$, which is a strictly stronger property in this topos. It nice to know that closed subsets of N are externally just ordinary subsets of $\mathbb{N}$.

To explain CAC: we can make a choice in any non empty set, but to make infinitely many choices to define functions from N to any other object X, we need an extra principle like CAC. Of course, this statement holds classically, as it is just a special case of the axiom of choice. That it hold in the effective topos is a consequence of primitive recursion in recursion theory, and the axiom of choice in set theory: we use the latter to show that the function exists, and the former to construct its realizer. Note that many possible choice functions are connected to a few realizers here.

Axiom 2 (Extended Church Thesis (ECT)) *Given a closed subset $A \subset N$ and a function $f : A \to N$, then for some $n \in N$ $nx \to fx$ for all $x \in A$*

All of recursion theory, including the recursive enumeration of partial recursive functions, can be done within $\mathcal{E}\!f\!f$. Here however, all partial functions $N \dashrightarrow N$ are recursive. This would be one of the strong propositions that hold in the effective topos and that contradict the axiom of choice and the principle of the excluded middle. Partial recursive functions $\mathbb{N} \dashrightarrow \mathbb{N}$ almost realize themselves, while non recursive functions have no realizers at all.

Axiom 3 (Markov's Principle (MP)) *In any non empty decidable subset $A \subset N$, there is a least element.*

In classical mathematics, all subsets of N are decidable, and this just states N is a well ordered set. But in the effective topos, some subsets of N are rather fuzzy, and determining any element, let alone the least one, may be impossible. This axioms is connected to μ-recursion in recursion theory: a decidable subset $S \subset N$ can be represented as a function $f : N \to \{0, 1\}$, with $fx = 0$ iff $x \in S$. By applying μ to f, we find the least element of S.

3.5.1 Uniformity Principle

For the next axiom I will have to introduce some terminology. The topos of sets is included in the effective topos as a reflective subcategory. Given a set X we can construct ∇X by creating a trivial equivalence relation:

$$R(x, y) := \begin{cases} \mathbb{N} & \text{if } x = y \\ \emptyset & \text{if } x \neq y \end{cases}$$

and letting $\nabla X := (X, R)$. In the definition of R above, we can replace $\mathbb{N}$ by any non empty $S \subset \mathbb{N}$, because the resulting functors are isomorphic.

The reflectivity of this category is due to the fact that the effective topos is locally small: $\hom(\mathbf{1}, X)$ is still a *set* of arrows $\mathbf{1} \to X$. These arrows with domain $\mathbf{1}$ are called the *global elements* of X. So the set $\Gamma X := \hom(\mathbf{1}, X)$ of global elements of X is a set, and every map $X \to \nabla Y$ (where Y is a set of course), it factors over $\nabla(\Gamma X)$.

Classical sets, and general quotients of classical sets, are called *uniform* objects. Spelled out: Y is uniform, if there is a *set* X and regular epimorphism $p : X \twoheadrightarrow Y$. For such objects, the following principle holds:

Axiom 4 (Uniformity Principle (UP)) *Given a uniform object Y and a relation R on $Y \times N$ such that for all $x \in Y$ there is an $n \in N$ satisfying $R(x, n)$, there is an $n \in N$ such that $R(x, n)$ for all $x \in Y$*

I think of this as another restricted form of the axiom of choice, because it allows for $\forall\exists$-propositions to be turned into a $\exists\forall$-propositions. However, in this case the restriction is quite severe: all choice functions are constant. When Y is uniform, we can apply UP to the relation $fx = y$, for any function $f : Y \to N$ to get:

Corollary 3.5.2 *Every $f : Y \to N$ is constant, when Y is uniform and non empty.*

The inclusion of a set X into $\mathcal{E}\!f\!f$ is done by letting the equality $R : X^2 \to \mathcal{P}\mathbb{N}$ be as simple as possible: $R(x, y) = \mathbb{N}$ if $x = y$ and $\emptyset$ if $x \neq y$. By looking at global sections $\Gamma Y = \hom(\mathbf{1}, Y)$, for arbitrary objects of $\mathcal{E}\!f\!f$, we obtain a set.

Example 3.5.3 Important examples of uniform objects form the power objects $\mathbf{P}X$. The underlying set of $\mathbf{P}X$ is the subset of relations $PX \subset \mathbf{P}\mathbb{N}^{|X|}$ (where for any X $|X|$ is the underlying set of X), and the equality $\Leftrightarrow$ on $\mathbf{P}X$ defines a regular epimorphism $PX \twoheadrightarrow \mathbf{P}X$: it just maps every relation to its equivalence class.

3.5.2 Countable Uniform Covers

Uniform objects can be thought of as 'connected' in the topological sense. We can make this more precise by saying $n \in \mathbb{N}$ is a connection between x and y iff $n \vdash x = x$ and $n \vdash y = y$. Note that this notion of connection doesn't take equality into account: $m \vdash x = y$, doesn't guarantee there is a connection between x and y. In uniform objects we have connections between any pair of elements. If $n \vdash x = x$ and $n \vdash y = y$ in N on the other hand, then $x = y$ because $x = n$ and $y = n$. For this reason N is called *discrete*, and this is what makes the uniformity principle hold.

Thinking topologically this way, one may consider looking at the connected subspaces – meaning the uniform subobjects – of arbitrary objects. If we use the notion of connection above, we're going to have to deal with the fact that equality doesn't give connectedness. The solution is to use a bundle: a regular epimorphism $\pi : E \twoheadrightarrow X$, such that the connections exists in E, and the equalities in X.

Let $|E| := \{(x, n) \in |X| \times N | n \vdash x = x\}$, and $m \vdash (x, n) =_E (y, p)$ iff $x = y$ and $m = n = p$. Now $q \vdash \pi(x, n) = y$ is just $q \vdash x = y$ is determined by the equality on X. So there we have our 'covering map' covering X – and this is really interesting – by an N-indexed coproduct of *sets*. This actually shows that objects of the effective topos are all colimits of countable diagrams of sets. But we are interested in uniform subobjects, and that is why we define X_n to be the image of π when restricted to the set $E_n := \{(x, m) \in E | m = n\}$.

Lemma 3.5.4 *X_n is uniform.*

Proof. E_n is a set and $\pi : E_n \twoheadrightarrow X_n$ is a regular epimorphism. By definition X_n is uniform. $\square$

Note that $\emptyset$ is a set and therefore uniform. So X_n may be empty for some $n \in N$. The object of indexes of non empty uniform subobjects $A := \{n|\exists x \in |X|.(x, n) \in E\}$ is therefore a subobject of N, but always a closed subobject: $n \in A$ iff $X_n \neq \emptyset$ which is a negative and therefore $\neg\neg$-closed property.

I want to summarize these properties into one last axiom for the internal logic of the effective topos:

Axiom 5 (Countable Uniform Cover (CUC)) *For every object X there is a subobject $B \subset X \times N$, such that the projection $\pi_1 : B \twoheadrightarrow X$ is a regular epimorphism, and the objects $X_n := \{x \in X|(x, n) \in B\}$ are uniform.*

By letting $b : E \to N$ be the map that satisfies $b(x, n) = m$ iff $n = m$, we construct B as the image of $(\pi, b) : E \to X \times N$. B is an N-indexed and therefore countable disjoint sum of uniform subobjects of X, and it covers X completely. Therefore every object of the topos has a 'countable uniform cover'.

3.6 Summary

The effective topos is a universe of mathematics in which everything is computable: any proposition is true iff there is a realizer for it. It is also an extension of set theory with new non classical objects in it, meaning that ordinary sets are included as a category of the effective topos, but there are also new objects that are no sets. Pivotal among these new objects is the new natural number object: all of the non classical properties of the effective topos have to do with this object.

Because $\mathcal{E}\!f\!f$ is a topos that has a natural numbers object, we may reason about its objects as if they were sets, and as if one of those sets were the set of natural numbers. We found five theorems about the internal logic of the effective topos, which can be used as axioms for such set theoretic reasoning. In the next chapters I'll be reasoning in the effective topos precisely that way. However, I want to use the words 'set' and 'function' in their classical meaning, however, since the topos of sets is a subcategory of $\mathcal{E}\!f\!f$. Therefore other objects will simply be 'objects', and other arrows 'arrows'.

Chapter 4

PERs and Modest Sets

4.1 Internal Category Theory

Now we've moved to a new universe of mathematics, we start looking for fixpoint categories again. First we'll have to define what precisely is a category in the $\mathcal{E}\!f\!f$. Basically, we take the normal definition from set theory and adapt it to the new environment:

Definition 4.1.1 An *internal category* $\mathcal{C}$ consists of two objects C_0 and C_1, three arrows $d_0, d_1 : C_1 \to C_0$ and $1 : C_0 \to C_1$, and a partial operator $\circ : C_1^2 \dashrightarrow C_1$. C_0 is the object of objects, C_1 the object of arrows, d_0 maps arrows to their domains, d_1 maps arrows to their codomains, 1 maps objects to their identity arrows and $\circ$ is the composition operator. Of course these have to satisfy the usual axioms of category theory, which state when $\circ$ is defined and what equations it satisfies when it is defined.
Given two objects a and b, for all $f \in C_1$ $f : a \to b$ means $d_0 f = a$ and $d_1 f = b$. $\hom(a, b) := \{f \in C_1 | f : a \to b\}$. Usually, $1x$ is written 1_x to avoid confusion.

Inside these categories, we have to usual kinds of objects and arrows, like 'isomorphisms', or 'initial objects' and so on. The definition above gives sufficient structure to interpret the properties associated with such objects. The *dual* $\mathcal{C}^{op}$ of a category $\mathcal{C}$ can be obtained by interchanging d_0 and d_1, and by letting $x \circ^{op} y = y \circ x$.

We want to know when categories have fixpoint for all of there endofunctors, so a definition of what an internal functor is will come in handy too:

Definition 4.1.2 An *internal functor* F from a category $\mathcal{C} = (C_0, C_1, d_0, d_1, 1, \circ)$ to a category $\mathcal{D} = (D_0, D_1, d_0', d_1', 1', \circ')$, consists of two arrows: $f_0 : C_0 \to D_0$ and $f_1 : C_1 \to D_1$. f_0 is the object map, and f_1 is the arrow map. Of course if $g : a \to b$, then $f_1 g : f_0 a \to f_0 b$, and f_1 has to preserve composite and identity arrows. Usually, when $x \in C_0$ Fa denotes $f_0 a$, while Fg denotes $f_1 g$ when $g \in C_1$.

For completeness we also add natural transformations:

Definition 4.1.3 Given two categories $\mathcal{C}$ and $\mathcal{D}$, and two functors F and $G : \mathcal{C} \to \mathcal{D}$ an *internal natural transformation* is a map $\eta : C_0 \to D_1$, satisfying $\eta_a : Fa \to Ga$, and given any $g : a \to b$, $\eta_b \circ Fg = Gg \circ \eta_a$. Here η_a is the conventional way of writing ηa.

We now can define higher properties of categories like 'initial algebra' or 'regular epimorphism' too. The category of all internal categories for any topos is closed under a lot of category constructions, which are somehow derived from objects constructions themselves. For example: because $\mathcal{E}\!f\!f$ is Cartesian closed, we can construct the category of internal functors between any pair of internal categories.

Example 4.1.4 In the spirit of bounded limits and colimits, we may think of full subcategories of the effective topos, whose objects are subobjects of Y. Its object of objects $E_{0,Y}$ is $\mathbf{P}Y$. Its objects of arrows is $\coprod_{A,B\in\mathbf{P}Y} B^A$ which can be constructed as a subobject $E_{1,Y}$ of $(\mathbf{P}Y)^2 \times \mathbf{P}(Y^2)$: $(A, B, F) \in E_{1,Y}$ iff F is the graph of a function $A \to B$. Obviously $d_0(A, B, F) = A$ and $d_1(A, B, F) = B$. $1_A := (A, A, \Delta_A)$, where Δ_A is the diagonal of A. Finally, since $(A, B, F) \in E_{1,Y}$ iff $F : A \to B$, we can define $(A, B, F) \circ (B', C, G) = (A, C, G \circ F)$ when $B = B'$, and leave it undefined when $B' \neq B$. This data defines the category $\mathcal{E}_Y$, which is supposed to be equivalent to the full subcategory of representatives of subobjects of Y within the effective topos.

According to lemma 3.2.3, given any functor F from any small category $\mathcal{D}$ to $\mathcal{E}_Y$, there exists a limit and a colimit within the effective topos, although not necessarily as a subobject of Y. Instead, It will be a subobject of $\mathbf{P}(D_0 \times Y)$, so at least we find a new upper bound for this limits.

Definition 4.1.5 A *Y-bounded functor* F from an internal category $\mathcal{C}$ to $\mathcal{E}\!f\!f$, is an internal functor $F : \mathcal{C} \to \mathcal{E}_Y$. A *bounded functor* is a Y-bounded functor for some object Y.

Note that the constructions in this section don't really depend on the structure of the effective topos. Therefore they work in all toposes.

4.1.1 Algebraic Completeness and Compactness for Internal Categories

We generalize the results of chapter 2 on small categories to internal categories, to see where we have to make adjustments. Fortunately most of the proofs are constructive, so:

Theorem 4.1.6 *Every complete internal category is algebraically complete.*

Proof. Given an endofunctor F of a category $\mathcal{C}$, we prove this by constructing the category of algebras and taking the limit of the underlying object functor. This has a unique algebra structure, and that makes it an initial algebra. □

Lemma 4.1.7 *Complete internal categories are cocomplete.*

Proof. Use the construction of the category of cocones again. Here to we take the limit of the underlying object functor. This just happens to be a colimiting cone. □

Lemma 4.1.8 *For any two algebraically complete categories $\mathcal{A}$ and $\mathcal{B}$ and any pair of functors $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{A}$, if (Y, g) is an initial GF-algebra, then (FY, Fg) is an initial FG-algebra.*

Proof. We can use the proof of 2.2.1, because it is constructive □

Both the Iterated Square and the fixmaps of contravariant functors of algebraically compact categories follow as a corollary of the theorem above. Although we know how that initial algebras exists and how they are structured, without the axiom of choice, it is hard to find a mapping that takes any endofunctor to one of its fixmaps. Therefore we can make a distinction between algebraically complete categories based on the existence of such a fixmap functor:

Definition 4.1.9 An algebraically complete category with a fixmap functor is *strongly* algebraically complete. Algebraic completeness without fixmap functor is called *weak* algebraic completeness.

Now we can also generalized to following theorems:

Theorem 4.1.10 (Freyd [1]) *The product of strongly algebraically complete categories is strongly algebraically complete.*

Proof. Take the proof of the original theorem: the new fixmap functor of the product is constructed as a combination of fixmap functors of the terms. □

Finally this really interesting result, which makes algebraically compact categories so useful, holds too:

Theorem 4.1.11 (Freyd [1]) *Algebraically compact categories have fixmaps for co-, contra-, and bivariant functors of any arity.*

Here I assume that algebraically compact categories are always strongly algebraically complete.

4.2 PERs

Within the effective topos, there is a closed $A \subset N$ with a regular epimorphism $p : A \twoheadrightarrow N^N$: as a consequence of ECT (axiom 2), every function $N \to N$ is represented by a number. A is the subobject of all numbers representing a total function, and this happens to be a closed subset. p is the arrow that satisfies $pex = \phi(e, x)$ (where ϕ is the universal partial recursive function).

We may expect that the class of objects X that like N^N have a regular epimorphism $p : A \twoheadrightarrow X$ from a closed subset A of N, form a subcategory of the effective topos that is Cartesian closed: if we have two of these objects X with $p : A \twoheadrightarrow X$ and X' with $p' : A' \twoheadrightarrow X'$, we can use the coding of pairs to combine the epis into an new epi for the product:

$$C \twoheadrightarrow A \times A' \overset{p\times p'}{\twoheadrightarrow} X \times X'$$

where $C := \{\langle x, x'\rangle | x \in A, x' \in A'\}$. For the exponential we may construct $q : D \twoheadrightarrow X^{X'}$, letting:

$$D := \{n : A' \to A | \forall x, y \in A'.p'x = p'y \to p(nx) = p(ny)\}$$

where we use arrows $A' \to A$ to represent arrows $X' \to X$.

Object of this form, which will be called *modest sets*, may be represented by subobjects of N^2: given $p : A \twoheadrightarrow X$, we just let $S \subset N^2$ be the set $\{(n, m)|pn = pm\}$. Because $\mathbf{P}(N^2)$ is an object of the effective topos, we can construct an internal category, that is in some sense equivalent to the large category of all modest sets. This small category is Cartesian closed too. In this case however, this means that there is an object inside the internal category representing the object of arrows between corresponding modest sets in the effective topos. This provides an interesting example of internal category.

4.2.1 The Category of PERs

Definition 4.2.1 A PER R is a closed subset of N^2 that as a binary relation on N is symmetric and transitive [4]. The name PER is an acronym for partial equivalence relation, and PERs are $\neg\neg$-closed partial equivalence relations on N. In other words $R \subset N^2$ is a PER iff:

1. if $\neg\neg((x,y) \in R)$, then $(x,y) \in R$
2. if $(x,y) \in R$ and $(y,z) \in R$, then $(y,z) \in R$
3. if $(x,y) \in R$, then $(y,x) \in R$

Given any set of PERs, its intersection is another PER, therefore PERs form a complete lattice of subsets of N^2. I'm a bit fond of writing $\overline{R}$ for any sort of closure of R, but for subsets of N^2, it will usually be *the least PER greater then R*.

Obviously PERs correspond bijectively to *separated* quotients of closed subsets of N: when we let $\operatorname{dom} R := \{n|(n,n) \in R\}$, then the closed subquotient we're looking for is $(\operatorname{dom} R)/R$. Separated means precisely that when $\neg\neg(x = y)$ for all x, y in $(\operatorname{dom} R)/R$, then $x = y$. Because the partial equivalence relation R is a total equivalence relation on $\operatorname{dom} R$, this is just the set theoretical construction of the quotient, as the set of non empty R-equivalence classes, which is:

$$\{U \in \mathbf{P}N | \exists n \in \operatorname{dom} R.n \in U \wedge \forall n \in U, m \in N.(n,m) \in R \leftrightarrow m \in U\} \tag{4.1}$$

Using this correspondence, we can define arrows to and from PERs, as arrows to and from the corresponding subquotients of N. So $f : X \to R$ means $f : X \to (\operatorname{dom} R)/R$ and $g : S \to Y$ means $g : (\operatorname{dom} S)/S \to Y$. As one might expect, this lets arrows $f : S \to R$ between PERs be equivalence classes of normal arrows $\operatorname{dom} S \to \operatorname{dom} R$, where $f \sim g$ iff $(fx, gy) \in R$ for every pair $(x,y) \in S$.

Lemma 4.2.2 *The object of arrows $S \to R$ is isomorphic to a PER.*

Proof. Note that because s is closed, so is $\operatorname{dom} S$. A function $f : S \to R$ corresponds to and equivalence class of functions $\operatorname{dom} S \to \operatorname{dom} R \subset N$, and according to ECT (axiom 2) functions $\operatorname{dom} R \to N$ are partial recursive functions represented by numbers. $\Box$

We use this fact to define the PER R^S:

Definition 4.2.3

$$S^R := \{(n,m)| \text{ for all } x, y \in N \text{ if } (x,y) \in R \text{ then } nx\downarrow, my\downarrow \text{ and } (nx, my) \in S\}$$

I'll write $n : S \to R$ when $(n,n) \in R^S$, using natural numbers to represent equivalence classes of partial recursive functions. We have an isomorphism $R^S \to \hom((\operatorname{dom} S)/S, (\operatorname{dom} R)/R)$, defined by sending numbers to the functions they represent.

Given any two PERs R and S, we can use the pairing of numbers to construct their product:

Definition 4.2.4

$$R \times S := \{(\langle r,s\rangle, \langle r',s'\rangle)|(r,r') \in R, (s,s') \in S\}$$

With this product the category of PERs is a Cartesian closed category internal to $\mathcal{E}\!f\!f$.

Definition 4.2.5 The *category of PERs*, from now on denoted $\mathcal{P}$, can be defined by the following data:

- Let P_0 be the object of all PERs.

- We define P_1 to be to coproduct:

$$\coprod_{R \in P_0, S \in P_0} S^R \subset \mathbf{P}(P_0^2 \times \mathbf{P}N)$$

as defined in 3.2.3: $(R, S, A) \in P_1$ iff A is an equivalence class of arrows $R \to S$.

- We let $d_0(R, S, A) = R$ and $d_1(R, S, A) = S$, so that the definitions of $f : R \to S$ agree. Identity arrows are given by $1_R := (R, R, [\lambda x.x])$, where $[\lambda x.x]$ is the equivalence class that contains a code for $\lambda x.x$.

- $(R, S, A) \circ (S, T, B) = (R, T, A \circ B)$, where

$$A \circ B := \{p \in N | \exists n \in A, m \in B.(p, n \circ m) \in T^R\}$$

For pairs $((R, S, A), (S', T, A))$ with $S' \neq S$ the operator $\circ$ is left undefined.

Definition 4.2.6 The mapping $R \mapsto (\operatorname{dom} R)/R$ extends to a bounded functor from $\mathcal{P}$. Let $Y := \mathbf{P}N$, and then define $F : \mathcal{P} \to \mathcal{E}_Y$ by:

$$f_0 R := \{u \in \mathbf{P}N | \forall n \in u, m \in N.m \in u \leftrightarrow (n, m) \in R\} \subset \mathbf{P}N$$

and

$$f_1(R, S, A) := (f_0 R, f_0 S, \{(u, v) \in f_0 R \times f_0 S | Au = v\})$$

where $Au := \{n \in N | \exists a \in A, m \in u.am \to n\}$

By this definition FR is the object of equivalence classes of R, and Ff tells us how f acts on these equivalence classes. For any PER R we have a quotient map $q : \operatorname{dom} R \twoheadrightarrow FR$, where $qn = A$ iff $n \in A$. dom, R is closed and q is a regular epimorphism. We generalize this property of FR to obtain a large category of object that are isomorphic to PERs:

Definition 4.2.7 A *modest set* is an object X for which a regular epi $p : A \twoheadrightarrow X$ exists, where A is any closed subset of N, and where the object $\{(m, n) \in N^2 | pm = pn\}$ is closed. An object is called *modest* if it is a modest set.

The modesty of these objects lies in there size relative to N. Note that modest sets aren't sets, because they don't belong to the subcategory of sets in the effective topos.

Lemma 4.2.8 *An object is a modest set iff it is isomorphic to a PER*

Proof. X is isomorphic to R iff X is isomorphic to FR. When X is modest, let $p : A \twoheadrightarrow X$ for $A \subset_{\neg\neg} N$, be one of the regular epimorphisms. Let $R := \{(n, m) | pn = pm\}$. R is a closed equivalence relation on A, and therefore a PER, and X is isomorphic to this PER. On the other hand, when there is an isomorphism $f : FR \to X$, we can compose it with the quotient map $q : \operatorname{dom} R \twoheadrightarrow FR$, which is a regular epimorphism from a closed subset of N. □

This lemma tells us that the categories of modest sets and of PERs are indeed equivalent. Note that internally we don't have functors mapping the categories into each other: we only have a bounded functor form $\mathcal{P}$, mapping PERs to modest sets.

4.2.2 Completeness and Reflectivity

Let X be any object of $\mathcal{E}\!f\!f$, and let $f : X \to N$. Now for a PER R we may say $f : X \to R$ if $fx \in \mathrm{dom}R$. Remember axiom 5 (CUC) that says there is a subobject $R_X \subset X \times N$, such that for all $n \in N$ $X_n := \{x|(x,n) \in R_X\}$ is uniform, and from 4 (UP) that therefore functions $f : X_n \to N$ are constant (if $X_n \neq \emptyset$ of course).

We can make the result above stronger, by noting that if $x \in X_n \cap X_m$, then for all y and $y' \in X$ if $(y,n) \in R$ and $(y',m) \in R$, then $fy = fy'$ because $fy = fx$ and $fy' = fx$. So given a countable uniform cover $R \subset X \times N$ of X, let S be the least PER with $(n,m) \in S$ when $X_n \cap X_m$ is non empty. Then N^X is actually isomorphic to N^S. N^S is isomorphic to the PER $(\Delta_N)^S$, where Δ_N is the diagonal: $\{(n,m) \in N^2|n = m\}$.

We can generalize even further: given any PER R, we let $R^X := FR^X$.

Lemma 4.2.9 *Given any PER R, R^X is modest.*

Proof. Given an arrow $f : X \to (\mathrm{dom}\, R)/R$, we look at its pullback along the quotient map $q : \mathrm{dom}\, R \twoheadrightarrow (\mathrm{dom}\, R)/R$:

$$\begin{tikzcd} Y \arrow[r, "q^*f"] \arrow[d, two heads, "p"'] & \mathrm{dom}\, R \arrow[d, two heads, "q"] \\ X \arrow[r, "f"'] & (\mathrm{dom}\, R)/R \end{tikzcd}$$

The map q^*f is an element of N^Y, and since Y has a countable uniform cover, there is some PER S such that $N^Y \simeq N^S$.

Of the arrows $g : Y \to N$, we only need the ones that satisfy: if $px = px'$ then $(gx, gx') \in R$. All functions of the form q^*f satisfy this property, and if a function g satisfies this property, we can define $f : X \to R$: $fx = y$ iff $\exists z \in Y.pz = x \wedge gx \in y$ (think of y as an equivalence class defined by R). This class of function forms a closed subobject $C \subset N^Y$, and by isomorphism a closed subobject $C' \simeq N^S$. Now C' is isomorphic to R^X, but also to the PER R^S, making R^X a modest set. $\square$

Corollary 4.2.10 *The category of PERs is complete.*

Proof. A family of PERs may be represented as an arrow $f : X \to \mathbf{P}(N^2)$, where fx is a PER for all $x \in X$. Its product is a closed subquotient of N^X: an arrow $g : X \to N$ represents an element of $\prod f$ iff $(gx, gx) \in fx$ for all $x \in X$, and two such representations g_1 and g_2 stand for the same element iff $(g_1x, g_2x) \in fx$ for all $x \in X$. Since there is a PER S such that $N^X \simeq N^S$, and since closed subquotients of PERs are (isomorphic to) PERs, there is a PER $\prod f'$ isomorphic to $\prod f$. So the category of PERs has all products.

Given two PERs R and S, and two arrows f and $g : R \to S$, we can define the equalizer of f and g by $\{(x,y) \in N^2|(x,y) \in R \wedge (fx, gy) \in R\}$. So the category of PERs has all internal products and all equalizers, and therefore is complete. $\square$

Now the following result follows from weak equivalence:

Corollary 4.2.11 *The category of modest sets is bounded complete.*

Proof. A family of modest sets is harder to represent then a family of PERs, because we lack the notion of 'object valued arrow' in the effective topos. Usually one takes a bundle: $\pi : E \twoheadrightarrow X$, where the fibers $E_x := \{y \in E|\pi y = x\}$ are modest sets. The latter condition means there is a regular epimorphism $p : A \twoheadrightarrow E$, where $A \subset X \times N$, $\pi(p(x,n)) = x$ for all $x \in X$ and $A_x := \{n|(x,n) \in A\}$ is $\neg\neg$-closed.

Given $\pi : E \twoheadrightarrow X$ and $p : A \twoheadrightarrow E$ we define $f : X \to \mathbf{P}(N \times N)$ by

$$fx := \{(n, m) | (x, n), (x, m) \in A, p(x, n) = p(x, m)\}$$

to turn it into a family of PERs. Now the product $\prod_{x \in X} E_x$ is isomorphic to $\prod f$, which is modest.

That modest sets are closed under equalizers is a consequence of the fact that regular epis are preserved in pullbacks:

$$\begin{tikzcd}
A' \arrow[r, tail] \arrow[d, "p'"'] & A \arrow[d, "p"] & \\
X \arrow[r, tail, "f", "g"'] & Y \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Z
\end{tikzcd}$$

Modesty of Z helps the subobject A' to be closed. With all internal products and equalizers the category of modest sets is complete too. □

Lemma 4.2.12 *The category of modest sets is a reflective subcategory of the effective topos.*

Proof. *Reflective* means there is a left adjoint to the inclusion. In this case we have for any object X an arrow $\eta_X : X \to N^{N^X}$ defined by $\eta_X x f = fx$. The latter object N^{N^X} modest, and therefore the closure of the image of η_X, the subobject $X' := \{\chi : N^{N^X} | \neg\neg(\exists x \in X. \forall f \in N^X. fx = \chi f)\}$, is modest too. This is a 'least modest set' X' with a morphism $\eta_X : X \to X'$.

When X is modest, $\eta_X : X \to X'$ is an isomorphism. In that case let $f : Y \to X$ for any object Y. We can factor f into $\eta_Y : Y \to N^{N^Y}$ and $f^t := \eta_X^{-1} \circ N^{N^f} : N^{N^Y} \to X$, and because η_Y maps into Y', f actually factors over Y': this is the sense in which Y' is the least modest set with a morphism $Y \to Y'$. It also shows that the mapping $X \mapsto X'$ extends to a functor that is left adjoint to the inclusion of modest sets in $\mathcal{E}\!f\!f$. □

The bounded completeness of the category of modest sets follows directly from its reflectivity in a bounded complete category – namely $\mathcal{E}\!f\!f$ itself. The category of PERs shares any property defined up to isomorphism, like bounded completeness, and is therefore also bounded complete. But because every PER is a subset of N^2, every internal diagram of PERs is bounded and has a limit.

Now that we know that $\mathcal{P}$ is complete, we may conclude that is satisfies all properties of complete internal categories, among which algebraic completeness. This gives us the first example of a fixpoint category that is not automatically a poset, thanks to the constructive internal logic of the effective topos. Up to now however, we do not have a fixmap functor, which allows us to generalize this property to products of $\mathcal{P}$.

Once one defines what is supposed to be an *internal* endofunctor of the equivalent category of modest sets, we see that such functors must also have initial algebras and coalgebras due to the (internal) equivalence with the category of PERs.

4.3 Functors and Natural Transformations

The object of PERs P_0, is uniform. The reason is that $\mathbf{P}N$ is uniform (see 3.5.3), and that there is a regular epi $\mathbf{P}N \to \mathcal{P}_0$ given by:

$$U \mapsto \overline{\{(n, m) | \langle n, m \rangle \in U\}}$$

for all $U \in \mathbf{P}N$ (the closure is still the least PERs greater then . . .).

Lemma 4.3.1 *For every functor $F : \mathcal{P} \to \mathcal{P}$, there is a number f, such that for every pair of PERs X and Y and every number a representing a function $X \to Y$, we have $[fa] = F[a]$ (where $[x]$ stands for the equivalence class containing x).*

Proof. First of all, the object of maps $\hom(X, Y) \to \hom(FX, FY)$ is isomorphic to a PER, and therefore we can consider the arrow mapping of F to be a number in this set anyway. Now we have for any pair of PERs X and Y an $f_{X,Y} : \hom(X, Y) \to \hom(FX, FY)$ such that for all $a \in N$ if $[a] : X \to Y$, then $[f_{X,Y}a] = F[a]$. But because of uniformity (see axiom 4) that means there must be a single $f \in N$, such that for every pair of PERs X, Y and for all $a : X \to Y$ fa is defined, and in $F[a]$. $\square$

For natural transformations between endofunctors, say $\eta : F \Rightarrow G$, we have some $n \in N$ such that $\eta_X \equiv n$ for every PER X, for similar reasons. We may as well say that $n : F \Rightarrow G$ iff $Gx \circ n \equiv n \circ Fx$ for all $x \in \mathbb{N}$. Note that by this definition we can define $\mathrm{nat}(F, G)$ to be another PER. In this way $\mathcal{P}^{\mathcal{P}}$ is enriched over $\mathcal{P}$.

Theorem 4.3.2 (Freyd [2]) *The category of PERs is strongly algebraically complete*

Proof. For arbitrary endofunctors, we construct the category of algebras and define its initial algebra. The structure of the initial algebra is described by equations (2.1) and (2.2) just below the definition of algebraically complete categories (definition 2.1.3). By taking advantage of the uniformity of the class of all PERs, we can use these equations as a guideline for the construction of a PER, and an algebra structure on it, which together form an initial algebra in this category.

Given any endofunctor $F : \mathcal{P} \to \mathcal{P}$, we construct a category equivalent to the category of algebras, by letting the object be pairs (R, a) of PERs and natural numbers, such that $a : FR \to R$: a represents an algebra on R. Let $f : (R, a) \to (S, b)$ if $f : R \to S$ and $(f \circ a, b \circ Ff) \in S^{FR}$. Of course this category has an underlying PER functor U, based on an arrow $F\text{-alg}_0 \to \mathbf{P}(N^2)$. Therefore it has a canonical limit, defined as a subset of $(N^2)^{F\text{-alg}_0}$. A map $f : F\text{-alg}_0 \to N^2$ is equivalent to a pair of partial recursive functions (f_1, f_2), first of all because $(N^2)^{F\text{-alg}_0} \simeq (N^{F\text{-alg}_0})^2$, and second because $f(R, a)$ is constant in its first variable: it only depends on a, because the object of PERs is uniform.

So we may define a PER $\lim U$ that is a limit of U:

$$\lim U := \left\{ (f, f') \in N^2 \;\middle|\; \begin{array}{l} \forall\ (R, a), (S, b), (T, c) \in F\text{-alg}_0, \\ \forall\ m : (R, a) \to (T, c), m' : (S, b) \to (T, c). \\ (m(fa), m'(f'b)) \in T \end{array} \right\} \tag{4.2}$$

To explain this limit: for every a we have a least PER R_a such that $a : FR_a \to R_a$: just take the intersection. Now if $(f_1, f_2) \in \lim U$, then $(f_1a, f_2a) \in R_a$. We get this from the defining predicate above by letting m and m' be the identity arrows of R_a. A further restriction is that these pairs of maps should commute with all non identity algebra morphisms too, and this is assured by requiring $(m(fa), m'(f'b)) \in T$ for any pair of algebra morphisms m and m' with common codomain.

Given an algebra (R, a), we get the projection map $\pi_a : \lim U \to R$, simply defined by evaluation: $\pi_a(f) = fa$. This is the limiting cone, which justifies calling $\lim U$ the limit of U. Obviously, any algebra structure c on $\lim U$ has to make the following diagram commute for any algebra (R, a):

$$\begin{tikzcd} F(\lim U) \arrow[r, "F\pi_a"] \arrow[d, "c"] & FR \arrow[d, "a"] \\ \lim U \arrow[r, "\pi_a"] & R \end{tikzcd}$$

That means that for all $(x, y) \in F(\lim U)$ $(cxa, a(F\pi_a y)) \in R$. Letting $c := \lambda xa.a(F\pi_a x)$ achieves this. This determines representatives of c up to equivalence, and depends only on the representation of the recursive map $F : \hom(R, S) \to \hom(FR, FS)$.

Now the mapping $F \mapsto \lambda xa.a(F\pi_a x)$ can be extended into a fixmap functor $y : \mathcal{P}^{\mathcal{P}} \to \mathcal{P}$, and therefore the category of PERs is strongly algebraically complete. $\square$

So we do have a fixmap functor on $\mathcal{P}$. Now the further results on strongly algebraically complete categories hold for $\mathcal{P}$ too.

4.3.1 Monotony

Given a code i for the identity function $\lambda x.x$ and two PERs R and S, $i : R \to S$ precisely when $R \subset S$. So if F is an endofunctor of PERs *and* $Fi \equiv i$, then F is monotone: Fi is another code for the identity, and since $Fi : FR \to FS$, we find $FR \subset FS$. In the article 'Extensional PERs' [2] endofunctors are defined to preserve a specific code for the identity (meaning $Fi = i$ for all F and some i), and thus endofunctors are monotone by definition.

The motivation given for this restrictive definition of endofunctor is that 'it was noticed as the "externalisation" of the condition that the functor exist in the realizability universe'. I assume that the realizability universe is the effective topos, but I fail to see how preservation of identity externally follows from the same condition internally. In fact it seem that even internally, $(Fi, i) \in \bigcap_R FR^{FR}$ is all what is needed to realize that F represents a functor. Nonetheless:

Theorem 4.3.3 *Every endofunctor of $\mathcal{P}$ is naturally isomorphic to a monotone endofunctor*

Here *monotone* means that $FR \subset FS$ whenever $R \subset S$.

Proof. We know because of Yoneda's lemma that $FX \simeq \mathrm{nat}(\hom(X, -), F)$ naturally in both F and X, and when F is and endofunctor of $\mathcal{P}$, then the mapping $X \mapsto \mathrm{nat}(\hom(X, -), F)$ can be turned into an endofunctor too, by using the PER of natural transformations and noting that for every arrow f $\mathrm{nat}(\hom(f, -), F)$ must be (a code for) $\lambda xy.x \circ y \circ f$. Now the latter functor $\mathrm{nat}(\hom(X, -), F)$ happens to be monotone.

When $X \subset Y$ and $n : Y \to Z$, then $n : X \to Z$ because $(nx, ny) \in Z$ whenever $(x, y) \in Y$ and $(x, y) \in Y$ whenever $(x, y) \in X$. Therefore $\hom(Y, -) \subset \hom(X, -)$ point wise. Furthermore: for $f : Z \to Z'$ we have $\hom(X, f) : \hom(X, Z) \to \hom(X, Z')$ given by $x \mapsto f \circ x$, which is independent from the PERs as it should be [because of the uniformity of the object of PERs], so $\hom(X, f) = \hom(Y, f)$. As a consequence $i : \hom(Y, -) \Rightarrow \hom(X, -)$ if i is any code for the identity function.

Let $i : F \Rightarrow G$ for two functors F and G, and let $n : G \Rightarrow H$, then $n : F \Rightarrow G$, because $n \circ i \equiv n$. Therefore $\mathrm{nat}(G, -) \subset \mathrm{nat}(F, -)$, and even $i : \mathrm{nat}(G, -) \Rightarrow \mathrm{nat}(F, -)$ point wise. As a consequence the functor $\lambda x.\mathrm{nat}(\hom(x, -), F)$ is a monotonic functor. $\square$

Of contravariant endofunctors we can prove similarly, that they all are isomorphic to *antitone* contravariant endofunctor: contravariant functors that also satisfy $FS \subset FR$ if $R \subset S$.

Since PERs form a complete lattice of subsets of N^2, monotonic endofunctors F have fixobjects given by $\bigcap\{R|FR \subset R\}$ - the least fixobject - and $\overline{\bigcup\{R|R \subset FR\}}$ - the greatest fixobject. For these fixobjects, i is a fixmap. So for any functor F, there is a least object X with $X = \mathrm{nat}(\hom(X, -), F) \simeq FX$, which confirms the strong algebraic completeness 4.3.2 of $\mathcal{P}$ in another way:

Theorem 4.3.4 (Freyd [2]) *The category of PERs is strongly algebraically complete.*

Proof. Map any endofunctor F to:

$$R_F := \bigcap\{R|\text{nat}(\text{hom}(R,-),F) \subset R\}$$

This mapping is a fixmap functor, where the fixmaps are given by the isomorphism $FR_F \xrightarrow{\sim} \text{nat}(\text{hom}(R_F,-),F) = R_F$. This isomorphism stems from the bijection $\eta_S f = Ffx$ for all $S \in \mathcal{P}$ and $f : R_0 \to S$ between natural transformations $\eta : \text{hom}(R_F,-) \to F$ and equivalence classes x of R. Now any code for $\lambda xf.Ffx$ represents a fixmap $FR_F \to R_F$ for all F. $\square$

Another way to use Yoneda is to obtain colimits in a different way: colim $F := 1_{\mathcal{P}}(\text{colim } F) \simeq \text{nat}(\text{hom}(\text{colim } F,-),1_{\mathcal{P}}) \simeq \text{nat}(\text{Cocone}(F,-),1_{\mathcal{P}})$, where 1 is the identity functor, and Cocone(F,X) the PER of cocones $F \to X$ for any PER X (remember that cocones are a kind of natural transformation). Whatever internal category the domain of F is, Cocone(F,X) and nat(Cocone$(F,-),1_{\mathcal{P}})$ are modest sets. This reaffirms cocompleteness of $\mathcal{P}$, in a way that doesn't depend on the category of cocones.

4.3.2 Algebraic Compactness

We now know that $\mathcal{P}$ is complete, therefore cocomplete, algebraically complete and algebraically cocomplete. However, the initial algebra of the identity functor $1_{\mathcal{P}}$ is the initial PER (which is empty), and any terminal PER is a terminal coalgebra, and not isomorphic to the initial object. So again:

Theorem 4.3.5 *The category of PERs is not algebraically compact.*

Take $\mathcal{P}_*$, the category of pointed PERs. Every object here is a PER R together with a number n such that $(n,n) \in R$. In this case, we want the arrows to preserve the point, so $f : (R,n) \to (S,m)$ iff $f : R \to S$ and $(fn,m) \in S$. Since every pointed PER has an element, any terminal object of $\mathcal{P}$ is a 0-object of $\mathcal{P}_*$, and therefore both initial algebra and terminal coalgebra of 1. However:

Theorem 4.3.6 *The category of pointed PERs is not algebraically compact either.*

Proof. An initial algebra $(X, a : FX \to X)$ is a terminal coalgebra $(X, a^{-1} : X \to FX)$, if there is a unique coalgebra morphism m to it from every coalgebra $g : Y \to YF$. This means that m has to be the unique arrow satisfying $m = a \circ Fm \circ g$.

Let's take the functor L with $L(X,x) = (X+1,0)$, $Lf0 = 0$ and $Lf(x+1) = fx+1$ if $x > 0$. Here $X+1 := \{(n+1,m+1)|(n,m) \in X\} \cup \{(0,0)\}$. For this functor, $(\Delta_N, i, 0)$, where i is a code of the identity function and the PER Δ_N is the diagonal of N, is an initial algebra.

Given any coalgebra (R,n,a), we need a morphism $h : R \to N$ satisfying $h = Lh \circ a$, or $hx = h(ax-1)+x$. Let R be any *decidably* pointed PER, so $(x,n) \in R \vee (x,n) \notin R$. We can define $a : (R,n) \to (LR,0)$ by $ax = 0$ if $(x,n) \in R$ and $ax = x+1$ if $(x,n) \notin R$. Furthermore, there are examples of decidably pointed PERs with elements different from the point: $(\Delta_N, 0)$ to name one. In those examples we need h to satisfy $hx = hx+1$ whenever $(x,n) \notin R$, which is impossible. $\square$

There is a subcategory of $\mathcal{P}$ that is algebraically compact, but we are going to have to look around little harder to find it.

Chapter 5

Extensional PERs

5.1 Decidable and Semidecidable Subobjects

5.1.1 Decidable Subobjects

The principle of the excluded middle doesn't hold in general. But for some propositions – we have already used some of them above – a decision procedure does exist. For such *decidable* propositions $A \vee \neg A$ holds. By generalizing this notion to predicates we get:

Definition 5.1.1 A subobject $X \subset Y$ is *decidable* when for all $x \in Y$, $x \in X \vee x \notin X$.

It is nice to know that decidable subobjects are classified by $1 \in \mathbf{2} := \{0, 1\} \subset N$: given an arrow $f : X \to \mathbf{2}$, the relation $fx = 1$ is decidable (and $fx \neq 1$ iff $fx = 0$). Therefore $\{x \in X | fx = 1\}$ is a decidable subset of X. On the other hand, when $Y \in X$ and $x \in Y$ is decidable, then we can define $fx = 1$ iff $x \in Y$, and thereby determine an arrow $f : X \to 2$. We conclude:

Lemma 5.1.2 *Let* $\mathbf{2} = \{0, 1\} \subset N$*, then* $\mathbf{2}^X$ *is isomorphic to the class of decidable subobjects of* X*.*

Finite intersections and unions of decidable objects are decidable, as are complements, because we can combine the decision procedures on these objects to obtain decision procedures for the composites. Another way to see is, is to see that these operations can be defined as operations on $\mathbf{2}$.

About the decidable subobjects of various kinds of objects:

Lemma 5.1.3 *The only decidable subobjects of a uniform object* X*, are* X *and* $\emptyset$*.*

Proof. Let X be uniform, then any function $f : X \to \mathbf{2}$ is constant, because $\mathbf{2} \subset N$. □

Lemma 5.1.4 (Cantor) $\mathbf{2}^N$ *is uncountable*

Proof. Let $f : N \to \mathbf{2}^N$, and define $g\langle x, y\rangle = 1$ iff $fxy = 0$. Obviously g determines another decidable subset of N. Now were $g = fx$ for some x, then $g\langle x, x\rangle = fxx$ - which is impossible. Therefore $g \neq fx$ for all x. □

Within the effective topos, various notions of countability take on different meanings. The existence of an *enumeration* – a function $f : N \twoheadrightarrow X$ – as supposed above, makes the object non empty and *recursively enumerable*. This is a consequence of axiom 2. Another consequence is:

Lemma 5.1.5 $\mathbf{2}^N$ *is modest*

Proof. Let $(n, m) \in R$ if for all $x \in N$ $nx \to 0$ or $nx \to 1$ and $m \equiv n$. $R \simeq \mathbf{2}^N$ because any arrow $f : N \to \mathbf{2} \subset N$ can be represented by a code n, and R is just the equivalence relations on codes of arrows $f : N \to \mathbf{2}$. An object is modest whenever it is isomorphic to a PER. $\square$

Note that being modest means that there is some closed subset A of N with a regular epi $p : A \twoheadrightarrow \mathbf{2}^N$. So being uncountable doesn't exclude the possibility of being modest - a subquotient of N.

Theorem 5.1.6 (Turing) *There is an undecidable subset of* N

Proof. Let A be the set of codes of arrows $N \to \mathbf{2}$. If A were decidable, there would be a function $f : N \to \mathbf{2}$ such that $fx = 0$ iff $x \in A$ and $xx = 1$. f has a code $q \in A$, and $fq = 0$ iff $qq = fq = 1$ which in not possible. $\square$

Turing own proof was based on the halting set: the set $K := \{n | nn \downarrow\}$, the decidability of which leads to a contradiction in a similar way. These undecidable sets show how mathematics in $\mathcal{E}\!f\!f$ is different from classical mathematics.

5.1.2 Semidecidable Subobjects

Definition 5.1.7 A *semidecidable subobject* is a subobject of the form $\{x \in X | \exists n \in N(x, n) \in R\}$, where R is a decidable subobject of $X \times N$.

Semidecidable subobjects are actually objects like $K \subset N$: countable, or rather enumerable unions of decidable subobjects. Intuitively, semidecidable subobjects are objects for which it is easier to determine that some element is in them, than it is to determine that some element is not in them. When $S \subset X$ is semidecidable, and therefore of the form $\{x \in X | \exists n \in N(x, n) \in R\}$ for some decidable $R \subset X \times N$, and if $x \in S$, we can determine that fact by trying $(x, n) \in R$ for one $n \in N$ at the time. Eventually we will find to least n (consequence of MP, axiom 3) for which $(x, n) \in R$, and thus we verify that $x \in S$. If $x \notin S$ however, the same procedure makes us look for the least $n \in N$ such that $(x, n) \in R$ forever.

Lemma 5.1.8 *Semidecidable subobjects are* $\neg\neg$*-closed*

Proof. A semidecidable subobject $A \subset X$ is of the form $\{x \in X | \exists n \in N(x, n) \in R\}$, where R is a decidable subobject of $X \times N$. Given a fixed $x \in R$ the objects $R_x := \{n | (x, n) \in R\}$ are decidable in particular, and now we can apply Markov's Principle (axiom 3) to these sets. MP States that R_x has a least element iff it is non empty. When $\neg\neg n \in R_x$ we cannot have $\forall n \in N.n \notin R_x$, because that leads to a contradiction. Therefore R_x is non empty, and has a least element $n \in R_x$. Now $\neg\neg n \in R_x$ is equivalent to $(x, n) \in R$, so when $\neg\neg(\exists n \in N.(x, n) \in R)$ then $\exists n \in N.(x, n) \in R$. $\square$

Because semidecidable subobjects are enumerable unions, enumerable unions of such subobjects are also semidecidable. Closure under finite intersection is retained from decidable subsets. Other nice properties:

Lemma 5.1.9 *Given an arrow* $f : X \to Y$, f^{-1} *preserves semidecidable subobjects.*

Proof. A semidecidable subobject S of Y is determined by some function $g : Y \times N \to \mathbf{2}$. Now $x \in f^{-1}(S)$ iff $fx \in S$ iff $\exists n \in N.g(fx, n) = 1$. But the map $(x, n) \mapsto g(fx, n)$ just determines that $f^{-1}(S)$ is a decidable subset of X $\square$

$$\begin{tikzcd}
\mathbf{2} \arrow[r, "1"] & \mathbf{2} \\
X \times N \arrow[u, "g\circ(f\times 1)"] \arrow[r, "f\times 1"] \arrow[d, "\pi_1"'] & Y \times N \arrow[u, "g"'] \arrow[d, "\pi_1"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

Definition 5.1.10 Σ is the class of semidecidable subobjects of $\mathbf{1}$.

This means $\Sigma \subset \Omega$, and $\sigma \in \Sigma$ when σ is equivalent to $\exists n \in N.n \in S$ for some decidable subset S of N (One might say that Σ^X is the class of subsets that satisfy Kripke's scheme). The subobject classifier $\top \in \Omega$ – we replace $\top : \mathbf{1} \to \Omega$ by it image – is also an element of Σ: here it classifies semidecidable subobjects. Therefore Σ^X is isomorphic to the class of semidecidable subobjects of X.

5.2 Extensional Objects and PERs

Lemma 5.2.1 Σ *is modest.*

Proof. Let $(n, m) \in \Sigma'$ if n is a code for non empty decidable subobjects of N – so $n : N \to \mathbf{2}$ and $nx = 1$ for some x – whenever m is. Nonempty elements of $\mathbf{2}^N$ form a semidecidable subobject: $\bigcup_{n\in N}\{m \in \mathbf{2}^N | mn = 1\}$, which is closed by 5.1.8. Therefore Σ' is closed, and the relation is obviously an equivalence, therefore a PER. Let $f : \Sigma' \to \Omega$ satisfy fn iff $\exists m \in N.nm \to 1$. f is a isomorphism between Σ and a PER, therefore Σ is modest. $\square$

Σ^X is modest because Σ is isomorphic to the PER Σ', and 4.2.9. There is a canonical mapping $\eta_X : X \to \Sigma^{\Sigma^X}$, and when the image of this mapping is $\neg\neg$-closed, we get a special kind of modest set:

Definition 5.2.2 An *extensional object* is an object X for which a closed monomorphism to Σ^Y exists for some object Y; An *extensional PER* is a PER that is also an extensional object.

Given a closed mono $f : X \rightarrowtail \Sigma^Y$. There is a regular epimorphism $p : A \to \Sigma^Y$ for some closed subset A of N, and by pulling it back along f, we get a closed subset of N, namely $f^*A := \{n \in N | \exists x \in X.pn = fx\}$ (closed because f is a closed mono), and a regular epimorphism $f^*p : f^*A \to X$. This proves that extensional objects are indeed modest.

By restricting the bounded functor from the category of PERs to the category of modest sets, we find that the category of extensional PERs is equivalent with the category of extensional objects.

Lemma 5.2.3 *For any object X Σ^X is extensional.*

Proof. The identity arrow of Σ^X is a closed mono. $\square$

Just like we derive all kinds of completeness properties of modest sets and PERs from the modesty of N^X for all X, we can derive completeness properties of extensional objects from the extensionality of Σ^X for all X.

Corollary 5.2.4 *The category of extensional objects is reflective in $\mathcal{E}\!f\!f$. The category of extensional PERs is reflective in $\mathcal{P}$.*

Proof. The proof of the first part is similar to the proof of theorem 4.2.12, but N^{N^X} is replaced by Σ^{Σ^X}. The same holds for the second part, but here we have to use Σ' in stead of Σ, and the PER Σ'^R for any PER R.

Given any object X let $X' := \{\chi \in \Sigma^{\Sigma^X} | \neg\neg(\exists x \in X. \forall f \in \Sigma^X. \chi f = fx)\}$. Then X' is extensional, and there is a mapping $X \to X'$ defined by $x \mapsto \lambda y.yx$. If X is extensional, then $X' \simeq X$, in which case any function $f : Y \to X$ factors over Y'. Therefore the mapping $X \mapsto X'$ extends into a left adjoint of the inclusion, making the category of extensional objects reflective in $\mathcal{E}\!f\!f$.

Given a PER R define R' similarly, as:

$$\{(\phi, \psi) \in N^2 | \exists n \in N \forall (f, g) \in (\Sigma')^R. (\phi f, \psi g) \in \Sigma' \wedge (\phi f, fn) \in \Sigma'\}$$

where we may leave out $\neg\neg$ because of Markov's principle. Just as above this defines a left adjoint to the inclusion of the category of extensional PERs into the category of PERs. $\square$

Corollary 5.2.5 *The categories of extensional objects and extensional PERs are (bounded) complete.*

Proof. Informally, given any extensional object valued bounded functor F plus an object X with a limiting cone κ for F the maps $\kappa_Y : X \to FY$ factors over $\eta_X : X \to X'$. Because κ is a limiting cone, and we have a factored cone $\kappa/\eta_X : X' \to F$, there is an inverse to η_X. So $X \simeq X'$ and X is extensional. And since limits of bounded functors always exists, the categories of extensional objects is bounded complete.

For extensional PERs the situation is simpler. We have a definition of what and internal functor is, and we know that limits for these functors exist within the category of PERs, whether the functor is extensional PER valued or not. For the rest of the prove we use the same reasoning: given a limiting cone $\kappa : X \to F$, it factors over the 'extensional closure' X' of X, and therefore these PERs must be isomorphic. Finally this says X' is in the category of extensional PERs itself. $\square$

We find that the category of extensional PERs is another complete internal category, and therefore satisfies the fixpoint properties of having initial and terminal fixmaps for all internal endofunctors. This category also has a fixmap functor, constructed in the same way as the functor wise limit of the categories of algebras. Therefore we have strong completeness, and that extends to all powers of this category too. Unfortunately,

Corollary 5.2.6 *Neither of these categories is algebraically compact.*

Proof. $\emptyset$ is extensional. Surely the unique map $\emptyset \to \Sigma^X$ is a closed mono, where $\emptyset$ stands for an initial object. Initial objects are initial algebras for the identity functor, while terminal objects are terminal coalgebras, and surely they are not isomorphic. Since the categories have no 0-object, they cannot be algebraically compact. $\square$

5.3 The Algebraically Compact Subcategory

Let F be an endofunctor of a complete category $\mathcal{C}$, and let $f : FX \to X$ be its initial algebra. $f^{-1} : X \to FX$ is the terminal coalgebra of the category whenever for any coalgebra $g : Y \to FY$, there is a unique $h : Y \to X$ such that $h = f \circ Fh \circ g$ (2.3). Note that this defines h as a fixpoint for a map $X^Y \to X^Y$.

We know that all complete partial ordered sets have the fixpoint property. Actually, we don't need all unions or intersections to find a fixpoint: given a least element $\perp$ and meets of ascending chains, for any *continuous* – that is: ascending meet preserving – endomorphism f a fixpoint is given by $f^\infty \perp = \bigvee_{n \in N} f^n \perp$. In such a category of posets, we may be able to define h above as the least morphism satisfying $h = f \circ Fh \circ g$.

Extensional objects happen to be (isomorphic to) closed subclasses of the class of semidecidable subobjects for some (other) object of the effective topos. Therefore, they are naturally partially ordered. Also, enumerable unions of semidecidable subobjects like $f^\infty \perp$, exists. So if endomorphisms of extensional objects are continuous, we only need to look at extensional objects containing $\emptyset$ and the unions of enumerable ascending chains, to find objects with a fixpoint property.

5.3.1 Monotony and Continuity of Maps between Extensional Objects

Let $m : X \rightarrowtail \Sigma^Y$ be a closed mono. X is partially ordered by $x \leq y$ iff $mx \subset my$. In general I will treat extensional object as if they were closed subobjects of the object of semidecidable subobjects, and write $x \subset y$ instead of $x \leq y$. Similarly $\emptyset \in X$ means: there is a $x \in X$ with $mx = \emptyset$.

Definition 5.3.1 In an extensional object X, an *ascending chain* is a map $f : N \to X$ with $fn \subset f(n+1)$ for all $n \in N$. An extensional object is *complete* if it has the unions of all ascending chains. A map between extensional object is *continuous*, if it preserves unions of ascending chains.

For any arrow $f : X \to Y$ between extensional objects X and Y the following diagram commutes:

$$\begin{tikzcd}
\Sigma^{\Sigma^X} \arrow[r, "\Sigma^{\Sigma^f}"] & \Sigma^{\Sigma^Y} \\
X \arrow[u, "\eta_X"] \arrow[r, "f"'] & Y \arrow[u, "\eta_Y"']
\end{tikzcd}$$

Σ^{Σ^f} is continuous: it is an inverse image mapping and preserves arbitrary unions of subobjects. It is also monotone. Because η_X and η_Y are (closed) monomorphisms, f has to preserve all unions that X contains. To summarize:

Lemma 5.3.2 *Any arrow $X \to Y$ between extensional objects X and Y between extensional PERs is continuous and monotone.*

5.3.2 Pointed Complete Extensional PERs

Definition 5.3.3 A *pointed complete extensional PER* – or *pointed CEPER* – is a complete extensional PER that contains $\emptyset$. A map of pointed CEPERs is any map, $\emptyset$ preserving or not, of extensional PERs

Given an arbitrary extensional PER R, we find a least pointed CEPER CR. We can on one hand, add $\emptyset$ and unions of ascending chains to R, and define CR as a sort of 'free algebra' over R, or on the other hand take the intersection of all pointed CEPERs greater than R. We can turn C into a functor: given $f : R \to S$, we only need to define $Cf\emptyset$, and there is a standard choice for this: $\emptyset$. For unions of ascending chains, we have continuity, so that $Cf(\bigcup g)$ must be $\bigcup Cfg$. Finally, if S is pointed complete, then $S = CS$ and therefore any arrow $f : R \to S$ extends to $Cf : CR \to S$. We see that:

Lemma 5.3.4 *The category of pointed CEPER is reflective in the category of extensional PERs*

Proof. If S is pointed complete, then $\hom(R, S) = \hom(CR, S)$. □

Thanks to reflectivity, we find a new example of a fixpoint category:

Corollary 5.3.5 *The category of pointed CEPERs is strongly algebraically complete.*

Proof. As a reflective subcategory of the category of extensional PERs, the category of pointed CEPERs inherits internal completeness. Because it is an internal category we can construct the initial algebra of any endofunctor F, as the limit of the underlying PER functor of the category of algebra's of F. This gives us a strongly algebraically complete category. □

5.3.3 Algebraic Compactness

Given a pointed CEPER R and any endomorphism f, then because $\emptyset \subset f\emptyset$, we have $f^n\emptyset \subset f^{n+1}\emptyset$. This means that we have an ascending chain $x_i := f^i\emptyset$ in R, which has a union $yf := \bigcup_{n\in N} f^n\emptyset$. Now f is continuous (see 5.3.2), so $f(yf) = f(\bigcup_{n\in N} f^n\emptyset) = \emptyset \cup \bigcup_{n\in N} f^{n+1}\emptyset = yf$. Therefore yf is a fixed point of f.

Let for some other x $fx = x$ too, then because $\emptyset \subset x$ we have $f^n\emptyset \subset x$ and therefore $yf \subset x$. Therefore yf is the least fixpoint of f. To summarize this internal application of the *Kleene fixpoint theorem*:

Theorem 5.3.6 *For any pointed CEPER X, there is a map $y_X : X^X \to X$ such that yf is the least fixpoint of f, whenever f is an endomorphism of X.*

For every pair of pointed CEPERs R and S $R^S \simeq \hom(S, R)$ is a pointed CEPER, and has the fixpoint property. As we have seen in the proof of 2.3.3, this is a property enjoyed by algebraically compact categories. It also allows us to determine functions $S \to R$ by recursive equations.

Lemma 5.3.7 (Freyd) *Given an endofunctor F on the category of CEPERs, and an initial algebra $f : FX \to X$.*

$$y(\lambda x.f \circ Fx \circ f^{-1}) = 1_X$$

Proof. Let $p = y(\lambda x.f \circ Fx \circ f^{-1})$. Per definition $p = f \circ Fp \circ f^{-1}$, or $p \circ f = f \circ Fp$. Therefore, p is an algebra morphism of the initial algebra. But the only algebra morphism of the initial algebra is the identity. □

Lemma 5.3.8 (Freyd) *Given an endofunctor F on the category of CEPERs, an initial algebra $f : FX \to X$ and an arbitrary coalgebra $g : Y \to FY$. There is a least morphism $h : Y \to X$ satisfying $h = f \circ Fh \circ g$.*

Proof. Notice that the map $\lambda x.f \circ Fx \circ g$ is an endomorphism of X^Y, and that $y(\lambda x.f \circ Fx \circ g)$ gives us a least fixed point. Let $h = y(\lambda x.f \circ Fx \circ g)$, then of course $h = f \circ Fh \circ g$. □

Now h is a morphism of coalgebras $(Y, g) \to (X, f^{-1})$. So from every coalgebra there is a least coalgebra morphism to the inverse of the initial algebra.

Theorem 5.3.9 (Freyd) *The category of pointed CEPERs is algebraically compact.*

Proof. We already know that there is a least morphism h from any coalgebra to the inverse of the initial algebra, so now we only need to show it is unique. Given any coalgebra morphism $h' : (Y, g) \to (X, f^{-1})$, we have $h' := 1_X \circ h' = y(\lambda x. f \circ Fx \circ f^{-1}) \circ h'$. Let $kx = f \circ Fx \circ f^{-1}$, so that $yk \circ h' = h'$. Because composition is continuous and $yk := \bigcup_{n \in N} k^n \emptyset$ – here $\emptyset$ stands for the constant function that maps everything to $\emptyset$ – $h' = yk \circ h' = \bigcup_{n \in N}((k^n \emptyset) \circ h')$.

Now $(k^0 \emptyset) \circ h' = \emptyset \circ h' = \emptyset$, and $(k^{n+1}\emptyset) \circ h = f \circ F(k^n \emptyset) \circ f^{-1} \circ h'$. h' is a coalgebra morphism $(Y, g) \to (X, f^{-1})$ so $f \circ F(k^n \emptyset) \circ f^{-1} \circ h' = f \circ F(k^n \emptyset) \circ F h' \circ g = f \circ F(k^n \emptyset \circ h') \circ g$. When we let $lx = f \circ Fx \circ g$, we get $h' = \bigcup_{n \in N}((k^n \emptyset) \circ h') = \bigcup_{n \in N} l^n \emptyset = y(\lambda x. f \circ Fx \circ g) = h$ – h as defined in the previous theorem.

So any coalgebra morphism $h' : (Y, g) \to (X, f^{-1})$ equals the least coalgebra morphism $h : (Y, g) \to (X, f^{-1})$. Therefore the inverse of the initial algebra is a terminal coalgebra, and the category is algebraically compact. □

Algebraic compactness in this case is strong, because the category is already strongly algebraically complete. So we have a fixmap functor pointing to the initial algebras, which just happen to be the inverses of terminal coalgebras.

Corollary 5.3.10 *Let $\mathcal{C}$ be the category of pointed CEPERs. For all $n \in N$ $\mathcal{C}^n$ is algebraically compact. Every functor $F : (\mathcal{C}^{op})^m \times \mathcal{C}^n \to \mathcal{C}$ has a fixmap.*

Finally, we've found the holy grail!

5.4 Summary

We've found a category that on one hand consists of some kind complete posets and therefore has homsets that satisfy the fixpoint property, but that on the other hand is an internal reflective subcategory of the effective topos. These properties combined make the category not only algebraically compact, but also equivalent to a full and complete subcategory of the effective topos. This in sharp contrast to the situation in the topos of sets, because all categories satisfying a fixpoint property are preordered sets, and the only full subcategories of the topos of sets, that are also preordered sets, must consist solely of $\emptyset$ and singletons.

The effective topos is a category with sufficient structure to interpret most of mathematics in it. Now we know it also has a subcategory in which domain equations can be solved, and that therefore allows interpretations of various programming languages.

Chapter 6

Dependencies

This last chapter is here to answer two questions:

1. Which of the axioms of the effective topos are needed to prove that the category of pointed CEPERs is algebraically compact?
2. Why is the category of pointed CEPERs algebraically complete, even if not all endofunctors are monotone?

The answer to the first question should allow us to generalize the theorems above to other toposes: if another topos $\mathcal{E}$ is a model for the necessary axioms, it must have a natural numbers object (because none of the axioms make sense without it), and therefore an internal category of pointed CEPERs that is algebraically compact.

The second question gives me a chance to outline what small part of the proof is my own contribution, what I have done to circumvent the monotony property of endofunctor I haven't been able to prove or disprove.

6.1 Dependency on the Axioms

Let's go by them one by one:

- Countable Axiom of Choice (CAC)

  This is used implicitly in the definition of arrows between PERs or modest sets. Given regular epis $p : A \twoheadrightarrow X$ and $q : B \twoheadrightarrow Y$, where A and B are closed subsets of N. Given an arrow $f : X \to Y$, this choice property on N is what allows us to show there is an arrow $n : A \to B$ such that $q \circ n = f \circ p$. This allows us to replace arrows between PERs by equivalence classes of functions $N \to N$.

- Extended Church Thesis (ECT)

  The fact that the class of arrows $R \to S$ between any two PERs R and S is isomorphic to a PER, follows from the fact that these arrows already correspond to equivalence classes of arrows $\mathrm{dom}R \to N$ according to CAC and that those arrows are represented by natural numbers because of ECT.

- Markov's Principle (MP)

  I need MP to prove that semidecidable subobjects are closed, and that therefore Σ is a PER in section 5.1.

- Uniformity Principle (UP)

- Countable Uniform Cover (CUC)

  These axioms imply that N^X is a modest set for any object X. This is turn proves the reflectivity of the category of modest sets, and various other subcategories. UP alone is also needed to prove that $\mathrm{nat}(F, G)$ is modest, when F and G are endofunctors of the category of PERs, and to prove that the internal categories are strongly algebraically complete.

We see that *all* of these properties of the effective topos are involved in proving that the category of PERs is algebraically compact.

6.2 Monotony

Were all endofunctors of $\mathcal{P}$ monotonic, their initial algebra would be the identity $\lambda x.x$ on their least fixpoint. This gives us a very simple construction for the fixmap functor required to constructively prove algebraic completeness and compactness results involving products and contravariant functors. This makes the assumption that all functors are monotone very attractive. Furthermore, all functor have to preserve identity arrows, and since of class of PERs is uniform, there must be a single number i representing identity arrows for all objects (the least code for $\lambda x.x$ will do). That makes is even likely that $Fi = i$, and that all functor are monotone therefore. However $\forall R \in P_0.F1_R = 1_{FR}$, means that $(Fi, i) \in \bigcap_{R \in P_0} FR^{FR}$, and nothing more. How can we deduce from that, that $Fi = i$?

Anyway, constructing the initial algebra as a limit, isn't very hard to do in this case, or in general. When a category is small and has all small limits, we know that is must exists. We also know what is must look like: an equationally defined subset of the product of all algebras. What is special about the category of PERs is that because the object of PERs is uniform, we can combine these two facts into the construction of an initial PER.

Also as we've seen in subsection 4.3.1, every functor is isomorphic to a monotonic functor. Using this isomorphism together with the least fixpoint construction gives us another way of proving strong algebraic completeness, taking advantage of monotony indirectly. Because of the isomorphism, we can restrict to monotonic functors – and maybe even to functors satisfying $Fi = i$ – without loss of generality.

Once we have strong algebraic completeness for the category of PERs, we have it for all its reflective subcategories, like the category of pointed CEPERs. So from this point on the proof given in this thesis is parallel to the proof given partly in [2] and partly in [3]. I just thought it would be more convenient to put everything in one place here.

Appendix A

Notation Habits

A.1 Notation of Functions and Formulas

I base my notation of the standard notation of λ-terms, because when lots of function applications are combined, it saves a lot of brackets.

- Application is simple juxtaposition: I write fx instead of $f(x)$.
- To denote conjunctions in formulas, both ',' and '$\wedge$' are used. The interesting possibility of '$\alpha, \beta \wedge \gamma$' however, is not.
- In quantification, I use '.' to separate the formula part from the bound variable part. So 'for all x ϕx' is written as '$\forall x.\phi$', mimicking the way '.' is used in λ-terms, e.g. $\lambda x.x$. When needed, brackets are used to delimit the scope of the quantifiers, like in '$(\forall y.\phi) \to \psi$'.
- Quantification may be restricted to some domain, in which case some typing information is added to the variable. The most common are of the forms '$\exists x \in X.\phi$' or '$\forall f : X \to Y.\psi$', but other possibilities are not excluded, e.g. '$\forall X \subset Y.\phi$'.
- Expressions like '$x, y \in z$' or 'x and $y \in z$' stand for $x \in z$ and $y \in z$, relation symbols other then '$\in$' may get a similar treatment when used often.
- Repeated quantifiers are avoided by using commas: $\exists x.\exists y.\phi$ is written as $\exists x, y.\phi$. Combining this with the last two conventions, one may find expressions like '$\forall x, y \in N.\psi$' meaning '$\forall x \in N.\forall y \in N.\psi$'
- Regarding combinations of arrows in categories, '(f, g)' stands for the unique factorization of a cone consisting of $f : X \to Y$ and $g : X \to Z$ over a product cone:

$$\begin{tikzcd} & X \arrow[dl, "f"'] \arrow[d, dotted, "(f,g)"] \arrow[dr, "g"] & \\ Y & Y \times Z \arrow[l] \arrow[r] & Z \end{tikzcd}$$

  The parallel combination is denoted '$f \times g$':

$$\begin{tikzcd} A \arrow[d, "f"'] & A \times B \arrow[l] \arrow[r] \arrow[d, dotted, "(f \times g)"] & B \arrow[d, "g"] \\ C & C \times D \arrow[l] \arrow[r] & D \end{tikzcd}$$

- As juxtaposition is used for application, I will sometimes use '$\cdot$' to distinguish functor composition, so $(F \cdot G)X = F(GX)$.

And the rest is as found in most mathematics literature.

A.2 Table of Symbols and Acronyms

$\mathcal{E}\mathit{ff}$	the effective topos
dom	the domain of a PER (see 4.2.1)
$\mathbf{P}$	power set or object
nat	the set of natural transformations
$\mathbb{N}$	the set of natural numbers
$\mathcal{P}$	the category of PERs
P_0	the object of PERs
$\mathcal{P}_*$	the category of pointed PERs
$\mathbf{1}$	terminal object (in any category that has one)
$\mathbf{2}$	$= \{0, 1\} \subset \mathbb{N}$ or N
$\dashrightarrow$	partial – as opposed to total – arrow
CAC	Countable Axiom of Choice
CEPER	Complete Extensional Partial Equivalence Relation
CUC	Countable Uniform Cover
ECT	Extended Church's Thesis
MP	Markov's Principle
PEM	Principle of the Excluded Middle
PER	Partial Equivalence Relation
UP	Uniformity Principle

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