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Induced transformations generalized to non-commutative ergodic theory

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Acknowledgments

In dedication

- ... to my former teachers, who allowed me to stand on their shoulders.
- ... to my supervisor Karma Dajani, whose invaluable guidance taught me patience and without which I most certainly would have failed.
- ... to new friends Ambi, Wilbert and Sebastian, without whom this would have been a very lonely and unfulfilling experience.
- ... to old friends Christopher, Dawie, Alexandra and Gerhard, who are as much a part of me as I am of them, and whom I miss more than they know.
- ... to my brothers and my parents, for whom my love is only bounded by their support.

> *A scientist worthy of the name, above all a mathematician, experiences in his work the same impression as an artist; his pleasure is as great and of the same nature.*
>
> — Henri Poincaré

> *Without music, life would be a mistake.*
>
> — Friedrich Nietzsche

> *We can forgive a man for making a useful thing as long as he does not admire it. The only excuse for making a useless thing is that one admires it intensely. All art is quite useless.*
>
> — Oscar Wilde

> *When you like music more than life, something's wrong.*
>
> — K's Choice

Contents

Chapter 1

Introduction

The premise of this thesis is to (attempt to) generalize the construction of induced transformations of measure preserving dynamical systems (which we will call classical systems) to general (not necessarily commutative) C*-dynamical systems.

The fact that classical systems can be written in the C*-formulation of dynamical systems without much difficulty (where the obtained C*-algebra is always commutative), begs the question whether induced transformations of classical systems can also be translated into the C*-formulation.

A deeper question that also arises, is whether in general, we can altogether forget about commutativity in the C*-setting and still construct induced transformations which remain consistent with the classical construction of induced transformations (when the C*-algebra is indeed commutative).

Classical induced transformations also, to some extent, inherit 'mixing properties' from the systems that they are induced from. We raise the question as to whether this can be translated to the (not necessarily commutative) C*-formulation in a consistent manner.

The structure of this document is divided into two layers. Part I is a layer which introduces and concerns itself purely with classical systems. Part II presents the 'denser' theory of the non-commutative. Appropriately, being denser, Part II has sunk to the bottom.

In Chapter 2 we provide a quick survey of measure preserving dynamical systems and well known results which we will need subsequently. The author assumes the reader has a reasonable knowledge of Measure Theory, as terms like 'σ-algebra' and 'measure' are not defined in the text.

Chapter 3 introduces classical recurrence by means of the Poincaré Recurrence Theorem, which is crucial to the construction of classical induced transformations. We present the construction of classical induced transformations in detail and provide proof of how 'mixing properties' are inherited by induced systems.

The first taste of the non-commutative is introduced in Chapter 4. Some essential preliminaries are provided in highly concentrated form, which may be difficult to swallow whole for a reader encountering the concepts for the first time. The basic definitions and results C*-dynamical are presented before we prove the equivalence of classical systems to their C*-formulations.

Chapter 5 is the main chapter of this thesis. Our goal in this chapter is to generalize induced transformations to the non-commutative case. This is easier said than done. As will be seen, a big drawback to working with general C*-dynamical systems is the absence of point sets and a σ-algebra, which can be 'got at' when working with classical measure preserving dynamical systems (or commutative C*-dynamical systems). However, if the C*-algebra in a C*-dynamical system is not commutative we are at a loss, since we cannot obtain (anything remotely like) a classical probability space to work with. This is quite a significant problem because the Poincaré Recurrence Theorem was crucial in the classical construction of induced transformations, and in no way can it be applied, as it now stands, in this setting. Therefore, in achieving our goal of generalizing induced transformations to the C*-setting, we are forced to find some analogue or generalization to the Poincaré Recurrence Theorem in the C*-setting. This we do, and then in much detail, construct generalized induced transformations in close analogy to what was presented in Chapter 3.

In Chapter 6 we present illustrative examples and some slightly unrelated (though not uninteresting) theory which were of immense help to the author in understanding and developing the theory of the directly previous chapter. For reasons that will become clear, the examples presented here float somewhere on the interface of the classical and the denser non-commutative theory. Dense enough to sink through the classical, penetrate the non-commutative, yet not quite weighty enough to sink any further.

Much work is being done in the fields of non-commutative geometry, dynamical systems and stochastics. The interested reader might do well to read some of the work of L. Accardi [2] and the Fields medalist A. Connes [4], both of whom have published prolifically in the field with a view toward applications in quantum mechanics.

Part I

Classical Dynamical Systems

Chapter 2

Introducing Measure Preserving Dynamical Systems

2.1 Definitions and Basic Results

This section is based on [13].

Definition 2.1.1. Let X be a set, Σ a σ-algebra over X, and μ a probability measure defined on Σ. We call the triple (X, Σ, μ) a *probability space.*

Definition 2.1.2. Let (X, Σ, μ) be a probability space. A measurable transformation $T : X \to X$ is called *measure preserving (with respect to μ)* if $\mu(T^{-1}A) = \mu(A)$ for all $A \in \Sigma$.

Definition 2.1.3. The tuple (X, Σ, μ, T), where (X, Σ, μ) forms a probability space and the map $T : X \to X$ is measure preserving, is called a *measure preserving dynamical system*.

Definition 2.1.4. Two measure preserving dynamical systems $(X_i, \Sigma_i, \mu_i, T_i)$, $i = 1, 2$ are said to be *isomorphic* if there exists sets $N_i \in \Sigma_i$ such that $\mu_i(X_i \backslash N_i) = 0$ and $T_i N_i \subset N_i$ and a measurable bijection $\psi : N_1 \to N_2$ such that $\mu_1(\psi^{-1}(C)) = \mu_2(C)$ for all $C \in \Sigma_2$ and $T_2 \circ \psi = \psi \circ T_1$.

Definition 2.1.5. A measure preserving dynamical system (X, Σ, μ, T) is called *ergodic* when for any $A \in \Sigma$, $T^{-1}A = A$ implies that either $\mu(A) = 1$ or $\mu(A) = 0$.

We will denote the vector space of all μ-measurable functions on a measure space (X, Σ, μ) by $L^0(X, \Sigma, \mu)$. For any $p > 0$, the vector space of all (equivalence classes of) p-integrable, μ-measurable functions (that are equal μ-almost everywhere or μ-a.e.) will be denoted by $L^p(X,\Sigma,\mu)$. It is well known that with the norm defined for every $f \in L^p(X,\Sigma,\mu)$ by

$$\|f\|_p := \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}},$$

that $L^p(X,\Sigma,\mu)$ is a Banach space, and $L^2(X,\Sigma,\mu)$ is a separable Hilbert space. We will denote the vector space of (equivalence classes of) essentially bounded μ-measurable functions (that are equal μ-almost everywhere or μ-a.e.) by $L^\infty(X,\Sigma,\mu)$ which is a Banach space when endowed with the norm defined for every $f \in L^\infty(X,\Sigma,\mu)$ by

$$\|f\|_\infty := \text{ess sup}_{x\in X}|f(x)|.$$

By definition, it is clear that $L^p(X,\Sigma,\mu) \subseteq L^0(X,\Sigma,\mu)$ for every $0 < p \leq \infty$. More detail can be found in [3].

Definition 2.1.6. Let (X,Σ,μ,T) be a measure preserving dynamical system. We define the operator $U_T : L^0(X,\Sigma,\mu) \to L^0(X,\Sigma,\mu)$ by $U_T f := f \circ T$. The operator U_T is sometimes called the *Koopman operator*.

The properties of the Koopman operator U_T can be summarized in the following result

Theorem 2.1.7. *[13, p. 25] The operator U_T has the following properties:*

1. *U_T is linear*
2. *For all $f,g \in L^0(X,\Sigma,\mu)$,$U_T(fg) = U_T(f)U_T(g)$*
3. *If $c \in L^0(X,\Sigma,\mu)$ is constant μ-a.e. then $U_T c = c$.*
4. *If $B \in \Sigma$ then $U_T\chi_B = \chi_{T^{-1}B}$.*
5. *For every $f \in L^0(X,\Sigma,\mu)$, $\int_X U_T f d\mu = \int_X f d\mu$, where if either side of the equation does not exist or is infinite, the other side has the same property.*
6. *For $p \geq 1$, $U_T L^p(X,\Sigma,\mu) \subset L^p(X,\Sigma,\mu)$ and $\|U_T f\|_p = \|f\|_p$ for every $f \in L^p(X,\Sigma,\mu)$.*

Theorem 2.1.8. *[13, p. 28] Let (X,Σ,μ,T) be a measure preserving dynamical system. The following are equivalent:*

1. *(X,Σ,μ,T) is ergodic.*
2. *If $f \in L^0(X,\Sigma,\mu,T)$ and $(U_T f)(x) = f \circ T(x) = f(x)$ for every $x \in X$, then f is constant μ-a.e.*
3. *If $f \in L^0(X,\Sigma,\mu,T)$ and $(U_T f)(x) = f \circ T(x) = f(x)$ for μ- almost every $x \in X$, then f is constant μ-a.e.*

4. *If $f \in L^2(X, \Sigma, \mu, T)$ and $(U_T f)(x) = f \circ T(x) = f(x)$ for every $x \in X$, then f is constant μ-a.e.*

5. *If $f \in L^2(X, \Sigma, \mu, T)$ and $(U_T f)(x) = f \circ T(x) = f(x)$ for μ- almost every $x \in X$, then f is constant μ-a.e.*

The following famous result is known as 'Birkhoff's Ergodic Theorem'. (Note that we do not state the result in its full generality.)

Theorem 2.1.9. *[13, p. 34] (Birkhoff's Ergodic Theorem) Let (X, Σ, μ, T) be a measure preserving dynamical system and let $f \in L^1(X, \Sigma, \mu)$ be arbitrary. Then there exists a function $f^* \in L^1(X, \Sigma, \mu)$ such that*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = f^*(x)$$

for μ-almost every $x \in X$. Moreover $f^* \circ T = f^*$ μ-almost everywhere, and $\int_X f^* d\mu = \int_X f d\mu$.

Combining the two previous results yields the interesting corollary:

Corollary 2.1.10. *[13, p. 34] If (X, Σ, μ, T) is ergodic then for every $f \in L^1(X, \Sigma, \mu)$ and μ-almost every $x \in X$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ T^k(x) = \int_X f d\mu.$$

Corollary 2.1.11. *[13, p. 37] Let (X, Σ, μ, T) be a measure preserving dynamical system. (X, Σ, μ, T) is ergodic if and only if for for every $A, B \in \Sigma$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T^{-k} B) = \mu(A)\mu(B)$$

In light of the previous corollary, stronger 'mixing' notions have been defined.

Definition 2.1.12. [13, p. 40] Let (X, Σ, μ, T) be a measure preserving dynamical system.

1. (X, Σ, μ, T) is called *weakly mixing* if for every $A, B \in \Sigma$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\mu(A \cap T^{-k} B) - \mu(A)\mu(B)\right| = 0.$$

2. (X, Σ, μ, T) is called *strongly mixing* if for every $A, B \in \Sigma$

$$\lim_{n\to\infty} \mu(A \cap T^{-n} B) = \mu(A)\mu(B).$$

It is possible to weaken the hypotheses of the previous corollary and definition to obtain the slightly stronger, though not unexpected result.

Theorem 2.1.13. *[13, p. 41] Let (X, Σ, μ, T) be a measure preserving dynamical system and $\mathcal{S}$ a generating semi-algebra for Σ.*

1. *(X, Σ, μ, T) is ergodic if and only if for for every $A, B \in S$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T^{-k}B) = \mu(A)\mu(B)$$

2. *(X, Σ, μ, T) is weakly mixing if and only if for every $A, B \in \mathcal{S}$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \left|\mu(A \cap T^{-k}B) - \mu(A)\mu(B)\right| = 0.$$

3. *(X, Σ, μ, T) is strongly mixing if and only if for every $A, B \in \mathcal{S}$*

$$\lim_{n\to\infty} \mu(A \cap T^{-n}B) = \mu(A)\mu(B).$$

2.2 Example

Let $S^1 := \{e^{i\theta\pi} : \theta \in [0, 2)\}$ denote the unit circle in $\mathbb{C}$, with $\mathcal{B}$ the Borel σ-algebra on S^1 and λ normalized Lebesgue measure. Let $\alpha \in \mathbb{R}$ be fixed. The map $T_\alpha : S^1 \to S^1$ is defined by $T_\alpha e^{i\theta\pi} := e^{i(\theta+\alpha)\pi}$. The map T_α is measurable, and since Lebesgue measure is translation invariant, T_α is measure preserving.

Therefore $(S^1, \mathcal{B}, \lambda, T_\alpha)$ is a measure preserving dynamical system, which we will henceforth call a *rotation*, and when α is irrational, we will call the system an *irrational rotation.*

Proposition 2.2.1. *Rotations are ergodic if and only if they are irrational.*

Proof. Let $(S^1, \mathcal{B}, \lambda, T_\alpha)$ be a rational rotation. We prove that it is not ergodic.

If $\alpha = \frac{p}{q} \neq 0 \bmod 2$ is rational ($p, q \in \mathbb{Z}$ with $\frac{p}{q} \leq 2$, $\gcd(p, q) = 1$). We define the set $A_\varepsilon := \{e^{i\theta\pi} : \theta \in [0, \varepsilon]\}$, then $T_\alpha^{2q} A_\varepsilon = A_\varepsilon$. We can choose $\varepsilon > 0$ small enough such that $0 < \lambda \left(\bigcup_{k=1}^{2q} A_\varepsilon\right) < 1$, while $T_\alpha^{-1} \left(\bigcup_{k=1}^{2q} A_\varepsilon\right) = \bigcup_{k=1}^{2q} A_\varepsilon$. Hence the system is not ergodic.

Conversely, let $(S^1, \mathcal{B}, \lambda, T_\alpha)$ be an irrational rotation.

The collection $\{e_n\}_{n\in\mathbb{Z}} \in L^2(S^1, \mathcal{B}, \lambda)$ defined by $e_n(e^{ix}) := e^{inx}$, where $x \in [0, 2\pi]$, forms an orthonormal basis for $L^2(S^1, \mathcal{B}, \lambda)$.

In light of Theorem 2.1.8, if we can show that if $f \circ T_\alpha = f$ implies that f is constant (λ-a.e.) for any $f \in L^2(S^1, \mathcal{B}, \lambda)$, we are done. Let $f \in L^2(S^1, \mathcal{B}, \lambda)$ be such that $f \circ T_\alpha = f$.

We express f as the Fourier series $f = \sum_{n\in\mathbb{Z}} a_n e_n$ (λ-a.e.). Now

$$\sum_{n\in\mathbb{Z}} a_n e_n = f = f \circ T_\alpha = \sum_{n\in\mathbb{Z}} a_n e_n \circ T_\alpha,$$

so that for every $x \in [0, 2\pi]$

$$\begin{aligned} 0 &= \sum_{n\in\mathbb{Z}} a_n(e_n(e^{ix}) - e_n \circ T_\alpha(e^{ix})) \\ &= \sum_{n\in\mathbb{Z}} a_n(e^{inx} - e^{in(x+\alpha)}) \\ &= \sum_{n\in\mathbb{Z}} a_n(1 - e^{in\alpha})e^{inx} \\ &= \sum_{n\in\mathbb{Z}} a_n(1 - e^{in\alpha})e_n(e^{ix}). \end{aligned}$$

By the uniqueness of the Fourier expansion we must have that $a_n(1-e^{in\alpha}) = 0$ for all $n \in \mathbb{Z}$. But $1 - e^{in\alpha} \neq 0$ for all $n \in \mathbb{Z}\backslash\{0\}$, because α is irrational, and hence for all $n \in \mathbb{Z}\backslash\{0\}$, $a_n = 0$. Therefore $f = \sum_{n\in\mathbb{Z}} a_n e_n = a_0 e_0 = a_0$, i.e. constant ($\lambda$-a.e.), the result now follows from Theorem 2.1.8. $\square$

Chapter 3

Classical Recurrence and Induced Transformations

3.1 Poincaré Recurrence

The following famous, (though not difficult, and seemingly unassuming), little theorem by Poincaré lays the foundation toward the definition of induced transformations.

Theorem 3.1.1. *[13, p. 26] (Poincaré's Recurrence Theorem) Let (X, Σ, μ, T) be a measure preserving dynamical system and $A \in \Sigma$ be such that $\mu(A) > 0$. Then the set*

$$F := \{x \in A | T^k x \notin A \text{ for all } k \in \mathbb{N}\}$$

has measure zero.

Proof. We note that $T^{-k}F \cap F = \emptyset$ for all $k \in \mathbb{N}$, since if $x \in T^{-k}F \cap A$ then $x \notin F$ because $T^k x \in F \subseteq A$ and if $x \in T^{-k}F \cap (X \backslash A)$ then $x \notin A$ and hence $x \notin F$ since $F \subseteq A$. Then for $m, n \in \mathbb{N}$, $m \neq n$ (we may assume $n > m$ without loss)

$$\emptyset = T^{-m}\emptyset = T^{-m}(T^{-(n-m)}F \cap F) = T^{-n}F \cap T^{-m}F.$$

Therefore the sets in the collection $\{T^{-k}F\}_{k \in \mathbb{N}}$ are pairwise disjoint and hence

$$1 = \mu(X) \geq \mu(\bigcup_{k \in \mathbb{N}} T^{-k}F) = \sum_{k \in \mathbb{N}} \mu(T^{-k}F) = \sum_{k \in \mathbb{N}} \mu(F).$$

Now, if $\mu(F) > 0$, the right of the above will diverge, while being bounded above by 1, which is absurd. We conclude that $\mu(F) = 0$. $\square$

Corollary 3.1.2. *Let $A \in \Sigma$ be such that $\mu(A) > 0$. Then there exists an $n \in \mathbb{N}$ such that $\mu(T^{-n}A \cap A) > 0$.*

3.2 Induced Transformations

The following two sections are based on work outlined in [1].

Throughout this section (X, Σ, μ, T) will be an arbitrary measure preserving dynamical system and $A \in \Sigma$ an arbitrary measurable set with $\mu(A) > 0$.

The Poincaré Recurrence Theorem, proved in the previous section, allows us to define the following function.

Definition 3.2.1. For any $A \in \Sigma$ with $\mu(A) > 0$. The function $n_A : X \to \mathbb{N}$ defined by

$$n_A(x) := \inf\{n \in \mathbb{N} : T^n x \in A\}$$

is called the *first return time function to A (under T).*

Remark 3.2.2. By invoking the Poincaré Recurrence Theorem from the previous section, we may assume that $n_A(x) < \infty$ for all $x \in A$, by removing the set of measure zero of all $x \in A$ where $n_A(x) = \infty$. Measure theoretically, we have not changed A if we view measurable sets that differ by a set of measure zero as equivalent. Henceforth we assume that $n_A(x) < \infty$ for all $x \in A$.

We define the subsets $\{A_j\}_{j=1}^{\infty}$ of A and $\{B_j\}_{j=1}^{\infty}$ of $X \backslash A$ recursively as follows, for all $j \in \mathbb{N}$,

$$\begin{aligned} A_1 &:= T^{-1}A \cap A \\ B_1 &:= T^{-1}A \cap (X \backslash A), \end{aligned}$$

and

$$\begin{aligned} A_j &:= (T^{-j}A \cap A) \backslash \bigcup_{k=1}^{j-1} A_k \\ B_j &:= (T^{-j}A \cap (X \backslash A)) \backslash \bigcup_{k=1}^{j-1} B_k. \end{aligned}$$

It takes no great stretch of the imagination to see from their definition that both the collections $\{A_j\}_{j=1}^{\infty}$ and $\{B_j\}_{j=1}^{\infty}$ are pairwise disjoint collections, and that $A_j = \{x \in A : n_A(x) = j\}$ and $B_j = \{x \in X \backslash A : n_A(x) = j\}$ for all $j \in \mathbb{N}$.

Proposition 3.2.3. *Let $A \in \Sigma$ with $\mu(A) > 0$. The map n_A, when restricted to A, is measurable with respect to the induced σ-algebra $\Sigma \cap A$.*

Proof. Notice that for any $a \in \mathbb{R}$, with $a > 0$,

$$n_A^{-1}(a, \infty) = \bigcup_{j=\lceil a \rceil}^{\infty} A_j,$$

where the right lies in the σ-algebra $\Sigma \cap A$, since each $A_j \in \Sigma \cap A$ and $\Sigma \cap A$ is closed under countable unions. □

By the Poincaré Recurrence Theorem, and the fact that we assumed that $n_A(x) < \infty$ for all $x \in A$, we see that $A = \bigcup_{j=1}^{\infty} A_j$. The collection $\{A_j\}_{j=1}^{\infty}$ being pairwise disjoint, forms a partition of A.

Definition 3.2.4. We will call the partition $\{A_j\}_{j=1}^{\infty}$ of A, the *return time partition of A (under T).*

We are now in a position to define induced transformations.

Definition 3.2.5. The map $T_A : A \to A$ defined by

$$T_A x := T^{n_A(x)} x$$

is called *the induced transformation of T on A* or if no confusion arises just *the induced transformation.*

The induced transformation T_A has somewhat surprising properties. First of all:

Proposition 3.2.6. *The map T_A, is a measurable function with respect to the induced σ-algebra $\Sigma \cap A$.*

Proof. Let $B \in \Sigma \cap A$ be arbitrary, then

$$\begin{aligned} T_A^{-1}B &= \{x \in A : T^{n_A(x)}x \in B\} \\ &= A \cap \{x \in A : T^{n_A(x)}x \in B\} \\ &= \bigcup_{j=1}^{\infty} A_j \bigcap \{x \in A : T^{n_A(x)}x \in B\} \\ &= \bigcup_{j=1}^{\infty} A_j \cap \{x \in A : T^j x \in B\} \\ &= \bigcup_{j=1}^{\infty} A_j \cap T^{-j}B. \end{aligned}$$

The map T, being measurable, ensures that $T^{-j}B \in \Sigma$, so that $A_j \cap T^{-j}B \in \Sigma \cap A$ for every j. Being a σ-algebra, $\Sigma \cap A$ is closed under countable unions, therefore $T_A^{-1}B = \bigcup_{j=1}^{\infty} A_j \cap T^{-j}B \in \Sigma \cap A$. $\square$

Lemma 3.2.7. *For every $j \in \mathbb{N} \cup \{0\}$, $T^{-1}B_j = B_{j+1} \cup A_{j+1}$ when $B_0 := A$*

Proof. We first note that

$$\begin{aligned} T^{-1}A &= T^{-1}A \cap A \bigcup T^{-1}A \cap (X \backslash A) \\ &= A_1 \cup B_1. \end{aligned}$$

For any $j \in \mathbb{N}$,

$$\begin{aligned} T^{-1}B_j &= \{x \in X : Tx \in B_j\} \\ &= \{x \in X : n_A(x) = j+1\} \\ &= \{x \in A : n_A(x) = j+1\} \cup \{x \in X \backslash A : n_A(x) = j+1\} \\ &= A_{j+1} \cup B_{j+1}. \end{aligned}$$

□

More surprising is

Proposition 3.2.8. *The map T_A, is measure preserving with respect to the measure μ.*

Proof. Let $C \in \Sigma \cap A$ be arbitrary. As was shown in Proposition 3.2.6, $T_A^{-1}C = \bigcup_{j=1}^{\infty} A_j \cap T^{-j}C$. Therefore, by the fact the the sets in $\{A_j\}_{j=1}^{\infty}$ are pairwise disjoint and the additivity of μ,

$$\mu(T_A^{-1}C) = \mu\left(\bigcup_{j=1}^{\infty} A_j \cap T^{-j}C\right) = \sum_{j=1}^{\infty} \mu(A_j \cap T^{-j}C).$$

The collection $\{B_j\}_{j=1}^{\infty}$ is also pairwise disjoint and therefore additivity of μ also implies

$$1 \geq \mu\left(\bigcup_{j=1}^{\infty} T^{-j}C \cap B_j\right) = \sum_{j=1}^{\infty} \mu(T^{-j}C \cap B_j),$$

while each $\mu(T^{-j}C \cap B_j) \geq 0$. We may conclude $\mu(T^{-j}C \cap B_j) \to 0$ as $j \to \infty$.

Also, since T is measure preserving with respect to μ, by the previous lemma for every $n \in \mathbb{N}$,

$$\begin{aligned} & \mu(C) \\ =\ & \mu(T^{-1}C) \\ =\ & \mu(T^{-1}(C \cap A)) \\ =\ & \mu(T^{-1}C \cap T^{-1}A) \\ =\ & \mu(T^{-1}C \cap (A_1 \cup B_1)) \\ =\ & \mu(T^{-1}C \cap A_1) + \mu(T^{-1}C \cap B_1) \\ =\ & \mu(T^{-1}C \cap A_1) + \mu(T^{-1}(T^{-1}C \cap B_1)) \\ =\ & \mu(T^{-1}C \cap A_1) + \mu(T^{-2}C \cap (A_2 \cup B_2)) \\ \vdots\ & \\ =\ & \sum_{j=1}^{n} \mu(T^{-j}C \cap A_j) + \mu(T^{-n}C \cap B_n). \end{aligned}$$

Taking the limit as $n \to \infty$ on both sides above we see, by addivity of μ,

$$\begin{aligned} & \mu(C) \\ = \; & \lim_{n\to\infty} \sum_{j=1}^{n} \mu(T^{-j}C \cap A_j) + \lim_{n\to\infty} \mu(T^{-n}C \cap B_n) \\ = \; & \sum_{j=1}^{\infty} \mu(T^{-j}C \cap A_j) \\ = \; & \mu\left(\bigcup_{j=1}^{\infty} T^{-j}C \cap A_j\right) \\ = \; & \mu(T_A^{-1}C). \end{aligned}$$

□

Remark 3.2.9. We define μ_A as the normalized probability measure on $\Sigma \cap A$, i.e. for every $B \in \Sigma \cap A$ we define $\mu_A(B) := \mu(B)/\mu(A)$. Since T_A is measure preserving with respect to μ, it is clear that it is also measure preserving with respect μ_A. Hence $(A, \Sigma \cap A, \mu_A, T_A)$ is a measure preserving dynamical system!

Definition 3.2.10. We will call the measure preserving dynamical system $(A, \Sigma \cap A, \mu_A, T_A)$ *the system induced from* (X, Σ, μ, T) *onto* A, or just the *induced system*. We call the act of constructing $(A, \Sigma \cap A, \mu_A, T_A)$, *inducing onto* A.

3.3 Inherited Ergodicity of Induced Transformations

We continue with the same definitions and notations as in the previous section. What is somewhat surprising, is that induced transformations inherit ergodicity from the original system.

Theorem 3.3.1. *[1, p. 42] If* (X, Σ, μ, T) *is ergodic then so is* $(A, \Sigma \cap A, \mu_A, T_A)$.

Proof. Let $C \in \Sigma \cap A$ be such that $T_A^{-1}C = C$. We show that either $\mu_A(C) = 1$ or $\mu_A(C) = 0$.

Since $A = \bigcup_{j=1}^{\infty} A_j$,

$$C = T_A^{-1}C = \bigcup_{j=1}^{\infty} A_j \cap T_A^{-1}C = \bigcup_{j=1}^{\infty} A_j \cap T^{-j}C.$$

We define $E := \bigcup_{j=1}^{\infty} B_j \cap T^{-j}C$. Note that E and C are disjoint since $\bigcup_{j=1}^{\infty} A_j \subset A$ and $\bigcup_{j=1}^{\infty} B_j \subset X \backslash A$. We define $F := E \cup C$. By Lemma

3.2.7,

$$\begin{aligned}
& T^{-1}F \\
= \; & T^{-1}C \cup T^{-1}E \\
= \; & T^{-1}(C \cap A) \bigcup T^{-1}\left(\bigcup_{j=1}^{\infty} B_j \cap T^{-j}C\right) \\
= \; & \left(T^{-1}C \cap (A_1 \cup B_1)\right) \bigcup \left(\bigcup_{j=1}^{\infty} T^{-1}B_j \cap T^{-j-1}C\right) \\
= \; & \left(T^{-1}C \cap A_1\right) \bigcup \left(T^{-1}C \cap B_1\right) \bigcup \left(\bigcup_{j=2}^{\infty} B_j \cap T^{-j}C\right) \cup \left(\bigcup_{j=2}^{\infty} A_j \cap T^{-j}C\right) \\
= \; & \left(\bigcup_{j=1}^{\infty} B_j \cap T^{-j}C\right) \cup \left(\bigcup_{j=1}^{\infty} A_j \cap T^{-j}C\right) \\
= \; & E \cup C \\
= \; & F.
\end{aligned}$$

But (X, Σ, μ, T) is ergodic, hence $\mu(F) = 1$ or $\mu(F) = 0$.

If $\mu(F) = 0$ then $\mu(C) = 0$, and hence $\mu_A(C) = 0$, since $C \subset F$.

If $\mu(F) = 1$, then $\mu(X \backslash F) = 0$, and $X \backslash F = (A \backslash C) \cup (X \backslash A) \backslash E \supseteq A \backslash C$ implies

$$\mu(A \backslash C) \leq \mu((A \backslash C) \cup (X \backslash A) \backslash E) = \mu(X \backslash F) = 0,$$

and since $C \subset A$, $\mu(C) = \mu(A)$, so that $\mu_A(C) = 1$.

We conclude that $(A, \Sigma \cap A, \mu_A, T_A)$ is ergodic. $\square$

Even more surprising is that ergodicity of the original system can (under certain circumstances) be derived from ergodicity of an induced system.

Proposition 3.3.2. *[1, p. 42] If $(A, \Sigma \cap A, \mu_A, T_A)$ is ergodic and $\mu\left(\bigcup_{k=0}^{\infty} T^{-k}A\right) = 1$ then (X, Σ, μ, T) is ergodic.*

Proof. Let $C \in \Sigma$, such that $T^{-1}C = C$. We will show that either $\mu(C) = 1$ or $\mu(C) = 0$.

We show that $T_A^{-1}(A \cap C) = A \cap C$. If $x \in T_A^{-1}(A \cap C)$, then $x \in T_A^{-1}A = A$ and $T^{n(x)}x = T_A x \in C$ implies $x \in T^{-n(x)}C = C$. Therefore $T_A^{-1}(A \cap C) \subseteq A \cap C$. Conversely, if $x \in A \cap C$, then $T_A x \in A$ and hence $x \in T_A^{-1}A$, and $T_A x = T^{n(x)}x \in C$ since $T^{-n(x)}C = C$. Therefore $A \cap C \subseteq T_A^{-1}(A \cap C)$, and hence $T_A^{-1}(A \cap C) = A \cap C$.

Since $(A, \Sigma \cap A, \mu_A, T_A)$ is ergodic, either $\mu_A(A \cap C) = 0$ or $\mu_A(A \cap C) = 1$.

If $\mu_A(A \cap C) = 0$, then $\mu(A \cap C) = 0$ and since T is measure preserving with respect to μ and $T^{-1}C = C$ we see for all $k \in \mathbb{N}$

$$0 = \mu(A \cap C) = \mu(T^{-1}(A \cap C)) = \mu(T^{-1}A \cap C) = \mu(T^{-k}A \cap C).$$

Since $\mu\left(\bigcup_{k=0}^{\infty} T^{-k}A\right) = 1$, by sub-additivity

$$\mu(C) = \mu\left(\bigcup_{k=0}^{\infty} T^{-k}A \cap C\right) \le \sum_{k=0}^{\infty} \mu\left(T^{-k}A \cap C\right) = 0.$$

Hence $\mu(C) = 0$.

On the other hand when $\mu_A(A \cap C) = 1$ then $\mu(A \cap C) = \mu(A)$ and we see

$$\mu(A \backslash C) = \mu(A) - \mu(A \cap C) = 0$$

so that since since T is measure preserving with respect to μ and $T^{-1}C = C$

$$0 = \mu(A \backslash C) = \mu(T^{-k}A \backslash C).$$

Therefore, by sub-additivity

$$\mu(X \backslash C) = \mu\left(\bigcup_{k=0}^{\infty} T^{-k}A \backslash C\right) \le \sum_{k=0}^{\infty} \mu\left(T^{-k}A \backslash C\right) = 0.$$

We conclude $\mu(C) = 1$ and hence that (X, Σ, μ, T) is ergodic. □

3.4 Example

Let $(S^1, \mathcal{B}, \lambda, T_\alpha)$ be an irrational rotation. We will point out some interesting specific induced transformations of this system. We can of course induce onto any $A \in \mathcal{B}$, and by the results in the previous section the induced system $(A, \mathcal{B} \cap A, \lambda_A, T_A)$ will be ergodic since $(S^1, \mathcal{B}, \lambda, T_\alpha)$ is ergodic.

For certain sets $A \in \mathcal{B}$, the induced transformation is quite interesting.

Choosing $A := \{e^{i\theta\pi} : \theta \in [0, \alpha]\}$, we see that the induced system $(A, \mathcal{B} \cap A, \lambda_A, T_A)$ is ergodic and isomorphic to $([0, \alpha], \mathcal{C}, \lambda', S)$, where $\mathcal{C}$ is the Borel σ-algebra, and λ' is normalized Lebesgue measure and S is given by $Sx = x + \gamma' \bmod \alpha$, with $\gamma' = 2 - \left\lfloor \frac{2}{\alpha} \right\rfloor \alpha$. We may recognize this system as (being isomorphic to) a rotation $(S^1, \mathcal{B}, \lambda, T_{\gamma(\alpha)})$, where $\gamma(\alpha) = 2\gamma'/\alpha$, by scaling appropriately. The system $(S^1, \mathcal{B}, \lambda, T_{\gamma(\alpha)})$ is ergodic, being isomorphic to the ergodic induced transformation, hence we conclude that $\gamma(\alpha)$ is irrational from Proposition 2.2.1.

Choosing $A := \{e^{i\theta\pi} : \theta \in [0, \beta]\}$, with $\beta \in (\alpha, 2)$ we see that the system $(A, \mathcal{B} \cap A, \lambda_A, T_A)$ is ergodic and isomorphic to $([0, \beta], \mathcal{C}, \lambda', S)$, where $\mathcal{C}$ is the Borel σ-algebra, and λ' is normalized Lebesgue measure. With $\gamma := (2 - \beta) - \left\lfloor \frac{2-\beta}{\alpha} \right\rfloor \alpha$, S is given by

$$Sx = \begin{cases} x + \alpha & x \in [0, \beta - \alpha] \\ x - (\beta - \alpha) + \alpha - \gamma & x \in (\beta - \alpha, \beta - \alpha + \gamma] \\ x - (\beta - \alpha) - \gamma & x \in (\beta - \alpha + \gamma, \beta]. \end{cases}$$

Paying careful attention to S, we see that it merely exchanges the three intervals according to

$$\begin{array}{rcl} [0, \beta - \alpha] & \to & [\alpha, \beta] \\ (\beta - \alpha, \beta - \alpha + \gamma] & \to & (\alpha - \gamma, \alpha] \\ (\beta - \alpha + \gamma, \beta] & \to & (0, \alpha - \gamma]. \end{array}$$

For the obvious reason, such a system is called a *three interval exchange.*

Part II

Non-commutative Dynamical Systems

Chapter 4

Introducing C*-Dynamical Systems

4.1 Preliminaries

In making the transition from classical measure preserving dynamical systems to C*-dynamical systems, we will need to introduce some fundamental concepts in highly concentrated form.

Definition 4.1.1. [5, p. 187] An *algebra* over a field $\mathbb{F}$ is a vector space $\mathcal{A}$ endowed with a multiplication that makes it into a ring, such that for all $\alpha \in \mathbb{F}$ and $a, b \in \mathcal{A}$, $\alpha(ab) = (\alpha a)b = a(\alpha b)$.

Definition 4.1.2. [5, p. 232] An *involution* on an algebra $\mathcal{A}$ over $\mathbb{C}$ is a map such that $a \in \mathcal{A} \mapsto a^* \in \mathcal{A}$ satisfying

1. $(a^*)^* = a$
2. $(ab)^* = b^*a^*$
3. $(\alpha a + b)^* = \overline{\alpha}a^* + b^*$

for all $a, b \in \mathcal{A}$ and $\alpha \in \mathbb{C}$. An algebra endowed with an involution will be called a **-algebra*. We will assume that every *-algebra $\mathcal{A}$ is endowed with an identity element, denoted by $1_{\mathcal{A}}$. Such *-algebras are called *unital*.

Definition 4.1.3. Let $\mathcal{A}, \mathcal{B}$ be *-algebras. A linear map $h : \mathcal{A} \to \mathcal{B}$ is called a **-homomorphism* if $h(ab) = h(a)h(b)$, and $h(a^*) = h(a)^*$ for all $a, b \in \mathcal{A}$.

Definition 4.1.4. Let $\mathcal{A}$ be a *-algebra.

An element $a \in \mathcal{A}$ will be called *positive* if there exists some $r \in \mathcal{A}$ such that $a = r^*r$ and we will write $a \geq 0$ or $a > 0$ when we know $a \neq 0$. For any elements $a, b \in \mathcal{A}$, we define $a \geq b$ to mean $a - b \geq 0$.

We call an element $p \in \mathcal{A}$ a *projection* if $pp = p$ and $p^* = p$.

We call a projection $q \in \mathcal{A}$ a *sub-projection* of p if $pq = qp = q$, and we say p and q are *orthogonal* if $pq = qp = 0$.

Proposition 4.1.5. *Let $\mathcal{A}$ be a *-algebra with $p, q \in \mathcal{A}$ projections, such that q is a sub-projection of p. Then $(p - q)$ is a sub-projection of p and $p \geq q$.*

Proof. That $(p-q)$ is a projection follows from

$$(p-q)(p-q) = pp - pq - qp + qq = p - q - q + q = (p-q)$$

and

$$(p-q)^* = p^* - q^* = (p-q).$$

Now

$$p(p-q) = pp - pq = p - q = pp - qp = (p-q)p$$

implies that $(p-q)$ is a sub-projection of p.

Projections are positive since, $p = pp = p^*p$. Therefore $p - q \geq 0$ implies $p \geq q$. □

Definition 4.1.6. [5, p. 187] A *Banach algebra* $\mathcal{A}$ is an algebra over a field $\mathbb{F}$, endowed with a norm $\|\cdot\|$ making it a Banach space, while the norm satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$.

Definition 4.1.7. [5, p. 232] A *C*-algebra* $\mathcal{A}$ is a Banach algebra that is also a *-algebra, satisfying $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$.

Proposition 4.1.8. *[5, p. 247, p.234] If $\mathcal{A}_i$, $i = 1, 2$ are C*-algebras and $\tau : \mathcal{A}_1 \to \mathcal{A}_2$ a *-homomorphism, then $\|\tau(a)\| \leq \|a\|$ for all $a \in \mathcal{A}_1$, the image $\tau(\mathcal{A}_1)$ is closed in $\mathcal{A}_2$, hence a C*-sub-algebra of $\mathcal{A}_2$ and $\tau(1_{\mathcal{A}_1}) = 1_{\tau(\mathcal{A}_1)}$ (note that it is not necessarily true that $1_{\tau(\mathcal{A}_1)} = 1_{\mathcal{A}_2}$).*

If $\mathcal{H}$ is a Hilbert space, we denote the C*-algebra of all bounded linear operators from $\mathcal{H}$ into $\mathcal{H}$ by $\mathcal{B}(\mathcal{H})$. We note that the projections in $\mathcal{B}(\mathcal{H})$, as defined above, are exactly the orthogonal projections on $\mathcal{H}$. Moreover if $p, q \in \mathcal{B}(\mathcal{H})$ are projections, q is a sub-projection of p exactly when range$(q) \subseteq$ range(p), and p and q are orthogonal exactly when range$(q) \perp$range(p).

Definition 4.1.9. [5, p. 248] Let $\mathcal{A}$ be a C*-algebra. A *representation* of $\mathcal{A}$ is a pair $(\pi, \mathcal{H})$ where $\mathcal{H}$ is a Hilbert space, and $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *-homomorphism such that $\pi(1_{\mathcal{A}}) = 1_{\mathcal{B}(\mathcal{H})}$. A representation is called *cyclic* if there exists a vector $e \in \mathcal{H}$ such that $\mathcal{H} = \overline{\pi(\mathcal{A})e}$ (norm-closure).

Definition 4.1.10. [5, p. 250] Let $\mathcal{A}$ be a C*-algebra. A bounded linear functional $f : \mathcal{A} \to \mathbb{C}$ is called positive if $f(a) \geq 0$, when $a \geq 0$ and f is called *hermitian* if $f(a^*) = \overline{f(a)}$ for every $a \in \mathcal{A}$.

The following is known as the Gelfand-Naimark-Segal Construction, or the GNS-construction.

Theorem 4.1.11. *[5, p. 250] (GNS-construction) Let $\mathcal{A}$ be a C*-algebra and $f : \mathcal{A} \to \mathbb{C}$ a positive bounded linear functional. Then there exists a cyclic representation $(\pi_f, \mathcal{H}_f)$ of $\mathcal{A}$ with a cyclic vector $e \in \mathcal{H}_f$ such that $f(a) = \langle \pi_f(a)e, e\rangle$ for all $a \in \mathcal{A}$.*

Remark 4.1.12. We give an outline for the proof of the above theorem. We define the closed, left ideal

$$\mathcal{L} := \{x \in \mathcal{A} : f(x^*x) = 0\}.$$

Viewing the quotient $\mathcal{A}/\mathcal{L}$ as a vector space. We denote the bounded linear map $\iota : a \mapsto a + \mathcal{L}$, and define an inner product on $\mathcal{A}/\mathcal{L}$ by $\langle \iota(x), \iota(y)\rangle := f(y^*x)$ for all $x, y \in \mathcal{A}$, making $\mathcal{A}/\mathcal{L}$ an inner product space, which we can complete to obtain the Hilbert space $\mathcal{H}_f$.

The representation $\pi_f : \mathcal{A} \to \mathcal{B}(\mathcal{H}_f)$ is then be defined as follows. For any $x \in \mathcal{A}$ we define $\pi_f(x)$ on $\mathcal{A}/\mathcal{L}$ by $\pi_f(x)(y+\mathcal{L}) = xy+\mathcal{L}$, which can be uniquely extended to $\mathcal{H}_f$, because $\mathcal{A}/\mathcal{L}$ is dense in $\mathcal{H}_f$. The vector $e := 1_{\mathcal{A}} + \mathcal{L}$ is then a cyclic vector for π_f.

Definition 4.1.13. A net of operators $\{T_\lambda\} \subset \mathcal{B}(\mathcal{H})$ *is* said to *converge strongly* to $T \in \mathcal{B}(\mathcal{H})$ if $T_\lambda u \to Tu$, for all $u \in \mathcal{H}$.

Theorem 4.1.14. *[10, p. 114] Let $\{p_\lambda\} \subset \mathcal{B}(\mathcal{H})$ be an increasing net (in the sense of Definition 4.1.4) of projections. The net $\{p_\lambda\}$ converges strongly to the projection $p \in \mathcal{B}(\mathcal{H})$ which projects onto the closed vector subspace $\overline{(\cup_\lambda p_\lambda(\mathcal{H}))}$ (norm-closure).*

Definition 4.1.15. [10, p. 116] A *von Neumann algebra* $\mathcal{A}$ acting on a Hilbert space $\mathcal{H}$, is a strongly closed C*-sub-algebra of $\mathcal{B}(\mathcal{H})$.

Definition 4.1.16. Let $\mathcal{A}$ be any *-sub-algebra of $\mathcal{B}(\mathcal{H})$. We define

$$\mathcal{A}' := \{T \in \mathcal{B}(\mathcal{H}) : TA = AT \text{ for all } A \in \mathcal{A}\}.$$

We call $\mathcal{A}'$ *the commutant* of $\mathcal{A}$. A von Neumann algebra $\mathcal{A}$ satisfying $\mathcal{A}\cap\mathcal{A}' = \mathbb{C}1_{\mathcal{B}(\mathcal{H})}$ is called a *factor*.

The following is often useful.

Theorem 4.1.17. *[10, p. 119] A von Neumann algebra $\mathcal{A}$ equals the norm-closure of the linear span of its projections.*

4.2 Definitions and Basic Results

This section is based on definitions and results in [8, 6].

Definition 4.2.1. Let $\mathcal{A}$ be a C*-algebra. A positive, hermitian bounded linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ is called a *state* (on $\mathcal{A}$) if $\varphi(1_{\mathcal{A}}) = 1$.

Definition 4.2.2. Let $\mathcal{A}$ be a C*-algebra, φ a state on $\mathcal{A}$. A bounded linear transformation $\tau : \mathcal{A} \to \mathcal{A}$ is called a *Markov operator* if $\tau(1_{\mathcal{A}}) = 1_{\mathcal{A}}$ and is called *state preserving (with respect to φ)* if $\varphi \circ \tau(a) = \varphi(a)$ for all $a \in \mathcal{A}$.

Definition 4.2.3. Let $\mathcal{A}$ be a C*-algebra, φ a state on $\mathcal{A}$ and $\tau : \mathcal{A} \to \mathcal{A}$ state preserving, Markov operator. We call the tuple $(\mathcal{A}, \varphi, \tau)$ a *state preserving C*-dynamical system* or just a *C*-dynamical system*.

Remark 4.2.4. With $(\mathcal{A}, \varphi, \tau)$ a state preserving C*-dynamical system, we may often require that $\mathcal{A}$, φ and/or τ possess further properties, for example $\mathcal{A}$ being a von Neumann algebra, φ possessing some additivity property and τ being a *-homomorphism in addition to being just linear. These extra assumptions will be explicitly mentioned as they arise.

Definition 4.2.5. Let $(\mathcal{A}, \varphi, \tau)$ be a state preserving C*-dynamical system.

- We call $(\mathcal{A}, \varphi, \tau)$ *ergodic* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \varphi(a\tau^k(b)) = \varphi(a)\varphi(b)$$

  for all $a, b \in \mathcal{A}$.

- We call $(\mathcal{A}, \varphi, \tau)$ *weakly mixing* if

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \left|\varphi(a\tau^k(b)) - \varphi(a)\varphi(b)\right| = 0$$

  for all $a, b \in \mathcal{A}$.

- We call $(\mathcal{A}, \varphi, \tau)$ *strongly mixing* if

$$\lim_{n\to\infty} \varphi(a\tau^n(b)) = \varphi(a)\varphi(b)$$

  for all $a, b \in \mathcal{A}$.

The following will often be useful to us.

Let $(\mathcal{A}, \varphi, \tau)$ be a C*-dynamical system, and $(\pi, \mathcal{H})$ the cyclic representation representation, with cyclic vector $e \in \mathcal{H}$, obtained from applying the GNS-construction on $(\mathcal{A}, \varphi)$.

Recalling the definitions in Remark 4.1.12, we define the bounded linear operator $U : \mathcal{A}/\mathcal{L} \to \mathcal{A}/\mathcal{L}$ by $U\iota(x) := \iota(\tau(x))$ for all $x \in \mathcal{A}$. Since $\mathcal{A}/\mathcal{L}$ is dense in $\mathcal{H}$, U can be uniquely extended as a bounded linear operator to the whole of $\mathcal{H}$, [7, p. 100] (also denoted by U). The fixed point space of $U : \mathcal{H} \to \mathcal{H}$ is defined to be $\{x \in \mathcal{H} : Ux = x\}$.

Theorem 4.2.6. *[6, p. 47] A C*-dynamical system $(\mathcal{A}, \varphi, \tau)$ is ergodic if and only if the fixed point space of $U : \mathcal{H} \to \mathcal{H}$ (as defined above) is one-dimensional.*

Remark 4.2.7. Duvenhage uses a different definition as we do for ergodicity when proving the previous theorem, he does however later prove [6, p. 53] that his definition is equivalent to ours above.

Remark 4.2.8. We implore the reader to consider the similarity of the operator U and the previous theorem to the Koopman operator U_T and Theorem 2.1.8 in Section 2.1.

4.3 Classical dynamical systems in the C*-formulation

In this section we show that our *-algebraic formulation of dynamical systems generalizes classical measure preserving dynamical systems.

We will show that for any given measure preserving dynamical system, we can construct an analogous C*-dynamical system which has exactly the same 'mixing properties' as the measure preserving system. Conversely, we will show that it is possible for each of a small class of C*-dynamical systems we can construct a measure preserving dynamical system that has exactly the mixing properties of the C*-dynamical system.

Proposition 4.3.1. *Let (X, Σ, μ, T) be any measure preserving dynamical system. Then defining $\mathcal{A} := L^\infty(X, \Sigma, \mu)$, $\varphi : \mathcal{A} \to \mathbb{C}$ by $\varphi(f) := \int_X f d\mu$ and $\tau : L^\infty(X, \Sigma, \mu) \to L^\infty(X, \Sigma, \mu)$ by $\tau(f) := f \circ T$, the tuple $(\mathcal{A}, \varphi, \tau)$ forms a C*-dynamical system.*

Proof. The algebra $\mathcal{A}$ is in fact a von Neumann algebra (acting on $L^2(X, \Sigma, \mu)$) hence a C*-algebra. That $\varphi(1_{\mathcal{A}}) = 1$ follows from μ being a probability measure. That φ is positive and hermitian is clear. That τ is linear and state preserving follows readily from Theorem 2.1.8. □

Proposition 4.3.2. *Let (X, Σ, μ, T) and $(\mathcal{A}, \varphi, \tau)$ be as in the previous result. The system (X, Σ, μ, T) is ergodic, (weakly mixing, strongly mixing) if and only if $(\mathcal{A}, \varphi, \tau)$ is ergodic, (weakly mixing, strongly mixing).*

Proof. Let $(\mathcal{A}, \varphi, \tau)$ be ergodic, then

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \varphi(a\tau^k(b)) = \varphi(a)\varphi(b)$$

for all $a, b \in \mathcal{A}$. Let $A, B \in \Sigma$ be arbitrary, so their characteristic functions

χ_A, χ_B are elements of $\mathcal{A}$ and hence

$$\begin{aligned}
& \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \mu(A \cap T^{-k}B) \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \int \chi_A \chi_{T^{-k}B} d\mu \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \int \chi_A (\chi_B \circ T^k) d\mu \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \int \chi_A \tau^k(\chi_B) d\mu \\
= & \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \varphi\left(\chi_A \tau^k(\chi_B)\right) \\
= & \varphi(\chi_A)\varphi(\chi_B) \\
= & \left(\int \chi_A d\mu\right)\left(\int \chi_B d\mu\right) \\
= & \mu(A)\mu(B)
\end{aligned}$$

establishes the ergodicity of (X, Σ, μ, T) by Theorem 2.1.13.

Conversely, let (X, Σ, μ, T) be ergodic. To show that $(\mathcal{A}, \varphi, \tau)$ is ergodic we are required to show that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n-1} \varphi(a\tau^k(b)) = \varphi(a)\varphi(b)$$

for all $a, b \in \mathcal{A}$. Now the argument is more complicated, yet follows from standard measure theoretic arguments.

Let $r, s \in \mathcal{A} = L^\infty(X, \Sigma, \mu)$ be arbitrary simple functions, hence there exists a finite partition $\{X_0, \ldots, X_M\}$ of X into measurable sets of positive measure so that we may write

$$r = \sum_{m=0}^{M} r_m \chi_{X_m} \qquad s = \sum_{m'=0}^{M} s_{m'} \chi_{X_{m'}}.$$

Now by Theorem 2.1.13, taking the limit as $n \to \infty$ in the fourth step,

$$\begin{aligned}
& \frac{1}{n}\sum_{k=0}^{n-1} \varphi(r\tau^k(s)) \\
= \quad & \frac{1}{n}\sum_{k=0}^{n-1} \int \left(\sum_{m=0}^{M} r_m \chi_{X_m}\right)\left(\sum_{m'=0}^{M} s_{m'}(\chi_{X_{m'}} \circ T^k)\right) d\mu \\
= \quad & \sum_{m,m'=0}^{M} r_m s_{m'} \frac{1}{n}\sum_{k=0}^{n-1} \int \chi_{X_m}\chi_{T^{-k}X_{m'}} d\mu \\
= \quad & \sum_{m,m'=0}^{M} r_m s_{m'} \frac{1}{n}\sum_{k=0}^{n-1} \mu(X_m \cap T^{-k}X_{m'}) \\
\to \quad & \sum_{m,m'=0}^{M} r_m s_{m'} \mu(X_m)\mu(X_{m'}) \\
= \quad & \left(\sum_{m=0}^{M} r_m \mu(X_m)\right)\left(\sum_{m'=0}^{M} s_{m'}\mu(X_{m'})\right) \\
= \quad & \left(\int r d\mu\right)\left(\int s d\mu\right) \\
= \quad & \varphi(r)\varphi(s).
\end{aligned}$$

Therefore the required result holds for all simple functions in $\mathcal{A} = L^\infty(X, \Sigma, \mu)$.

Let $a, b \in \mathcal{A} = L^\infty(X, \Sigma, \mu)$ be arbitrary non-negative functions, and let $\{r_l\}$ and $\{s_l\}$ be increasing sequences of non-negative simple functions converging pointwise to a and b respectively.

Then for every l and $n \in \mathbb{N}$,

$$\begin{aligned}
& \left|\frac{1}{n}\sum_{k=0}^{n-1} \varphi(a\tau^k(b)) - \varphi(a)\varphi(b)\right| \\
\leq \quad & \frac{1}{n}\sum_{k=0}^{n-1} \left|\varphi(a\tau^k(b)) - \varphi(r_l\tau^k(b))\right| \\
& + \frac{1}{n}\sum_{k=0}^{n-1} \left|\varphi(r_l\tau^k(b)) - \varphi(r_l\tau^k(s_l))\right| \\
& \quad + \frac{1}{n}\sum_{k=0}^{n-1} \left|\varphi(r_l\tau^k(s_l)) - \varphi(r_l)\varphi(s_l)\right| \\
& \quad\quad + |\varphi(r_l)\varphi(s_l) - \varphi(a)\varphi(s_l)| \\
& \quad\quad\quad + |\varphi(a)\varphi(s_l) - \varphi(a)\varphi(b)|.
\end{aligned}$$

We now treat each term on the right separately.

For the first term, notice that for every n

$$
\begin{aligned}
& \frac{1}{n}\sum_{k=0}^{n-1}\left|\varphi(a\tau^k(b)) - \varphi(r_l\tau^k(b))\right| \\
= \;& \frac{1}{n}\sum_{k=0}^{n-1}\left|\varphi((a-r_l)\tau^k(b))\right| \\
\leq \;& \frac{1}{n}\sum_{k=0}^{n-1}\int |a-r_l||b\circ T^k|d\mu
\end{aligned}
$$

where for each k, the sequence $\{|a-r_l||b\circ T^k|\}_l$ is a monotone decreasing sequence converging pointwise to zero, hence by the Lebesgue Monotone Convergence Theorem taking the limit at $l\to\infty$ on both sides of the above inequality yields for every n that

$$
\begin{aligned}
& \lim_{l\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\left|\varphi(a\tau^k(b)) - \varphi(s_l\tau^k(b))\right| \\
\leq \;& \frac{1}{n}\sum_{k=0}^{n-1}\lim_{l\to\infty}\int |a-s_l||b\circ T^k|d\mu \\
= \;& \frac{1}{n}\sum_{k=0}^{n-1}\int \lim_{l\to\infty}|a-s_l||b\circ T^k|d\mu \\
= \;& 0.
\end{aligned}
$$

For the second term, for every n, since $s_l\leq b$ and $|r_l|\leq \|a\|_\infty$ for all l and $n\in\mathbb{N}$, we see

$$
\begin{aligned}
& \frac{1}{n}\sum_{k=0}^{n-1}\left|\varphi(r_l\tau^k(b)) - \varphi(r_l\tau^k(s_l))\right| \\
\leq \;& \frac{1}{n}\sum_{k=0}^{n-1}\left|\int r_l\tau^k(b-s_l)d\mu\right| \\
\leq \;& \frac{1}{n}\sum_{k=0}^{n-1}\int |r_l||\tau^k(b-s_l)|d\mu \\
\leq \;& \|a\|_\infty\frac{1}{n}\sum_{k=0}^{n-1}\int |\tau^k(b-s_l)|d\mu \\
= \;& \|a\|_\infty\frac{1}{n}\sum_{k=0}^{n-1}\int |(b-s_l)\circ T^k|d\mu \\
= \;& \|a\|_\infty\frac{1}{n}\sum_{k=0}^{n-1}\int |b-s_l|d\mu
\end{aligned}
$$

by Theorem 2.1.7. As before $\{|b - s_l|\}$ is a monotone decreasing sequence converging pointwise to zero, hence by the Lebesgue Monotone Convergence Theorem applying the limit as $l \to \infty$ on both sides yields

$$\begin{aligned}
& \lim_{l\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \left|\varphi(r_l\tau^k(b)) - \varphi(r_l\tau^k(s_l))\right| \\
\leq \;& \|a\|_\infty \frac{1}{n}\sum_{k=0}^{n-1} \lim_{l\to\infty} \int |b - s_l| d\mu \\
= \;& \|a\|_\infty \frac{1}{n}\sum_{k=0}^{n-1} \int \lim_{l\to\infty} |b - s_l| d\mu \\
= \;& 0.
\end{aligned}$$

A similar argument as for the first and second terms establish for the fourth and fifth terms that

$$\begin{aligned}
\lim_{l\to\infty} |\varphi(r_l)\varphi(s_l) - \varphi(a)\varphi(s_l)| &= 0 \\
\lim_{l\to\infty} |\varphi(a)\varphi(s_l) - \varphi(a)\varphi(b)| &= 0.
\end{aligned}$$

Therefore, given an arbitrary $\varepsilon > 0$, it is possible to choose a fixed l_0 such that for every n

$$\begin{aligned}
& \left|\frac{1}{n}\sum_{k=0}^{n-1} \varphi(a\tau^k(b)) - \varphi(a)\varphi(b)\right| \\
< \;& \left|\frac{1}{n}\sum_{k=0}^{n-1} \varphi(r_{l_0}\tau^k(s_{l_0})) - \varphi(r_{l_0})\varphi(s_{l_0})\right| + \frac{\varepsilon}{2}
\end{aligned}$$

and now, since r_{l_0} and s_{l_0} are simple, choosing n large enough, our result for simple functions establishes

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} \varphi(a\tau^k(b)) - \varphi(a)\varphi(b)\right| < \varepsilon.$$

We conclude that

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} \varphi(a\tau^k(b)) = \varphi(a)\varphi(b)$$

for all non-negative $a, b \in \mathcal{A}$.

From here, the result in its full generality follows easily by decomposing $a, b \in \mathcal{A}$ into their four constituent, respectively positive and negative, real and imaginary parts and applying the previous to finally obtain

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} \varphi(a\tau^k(b)) = \varphi(a)\varphi(b)$$

for all $a, b \in \mathcal{A}$.

(Proof for weakly mixing and strongly mixing follows by the same argumentation) □

Remark 4.3.3. In rewriting classical measure preserving dynamical systems in the C*-formulation, by construction, the obtained C*-algebra is always commutative, and is in fact a von Neumann algebra as well.

We will now present a class of C*-dynamical systems that allow us to regain an underlying classical measure preserving dynamical system.

We will need the following quite famous results from [10].

Theorem 4.3.4. *[10, p. 135] Let $\mathcal{A}$ be a commutative von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$, which contains the identity operator on $\mathcal{H}$ and has a cyclic vector. Then there exists a second countable compact Hausdorff space X, a positive, regular Borel measure μ on (the Borel σ-algebra of) X, and a unitary operator $u : \mathcal{H} \to L^2(X, \mu)$, such that $u\mathcal{A}u^*$ is the von Neumann algebra of all multiplication operators on $L^2(X, \mu)$.*

Theorem 4.3.5. *[10, p. 136] Let $\mathcal{A}$ be a commutative von Neumann algebra acting on a separable Hilbert space $\mathcal{H}$. Then there exists a second countable compact Hausdorff space X, and a positive regular Borel measure μ such that $\mathcal{A}$ is *-isomorphic to the C*-algebra $L^\infty(X, \mu)$.*

Let $(\mathcal{A}, \varphi, \tau)$ be a C*-dynamical system, such that $\mathcal{A}$ is commutative and τ a surjective *-homomorphism. Applying the GNS-construction to $(\mathcal{A}, \varphi)$ we obtain a cyclic representation $(\pi, \mathcal{H})$, with cyclic vector $e \in \mathcal{H}$ (see Remark 4.1.12). We assume that $\mathcal{H}$ is separable and $\pi(\mathcal{A}) \subset \mathcal{B}(\mathcal{H})$ is a von Neumann algebra acting on $\mathcal{H}$.

The two theorems above applied to $\pi(\mathcal{A})$ yield a second countable compact Hausdorff space X and positive regular Borel measure μ on (the Borel σ-algebra, Σ, of) X such that $\pi(\mathcal{A})$ is *-isomorphic to $L^\infty(X, \Sigma, \mu)$ and there exists a unitary operator $u : \mathcal{H} \to L^2(X, \Sigma, \mu)$, such that $u\pi(\mathcal{A})u^*$ is the von Neumann algebra of all multiplication operators on $L^2(X, \Sigma, \mu)$, i.e. $u\pi(\mathcal{A})u^* = L^\infty(X, \Sigma, \mu)$.

We will assume that $ue = \chi_X \in L^2(X, \Sigma, \mu)$. This is not unreasonable, because $\chi_X \in L^2(X, \Sigma, \mu)$ is a cyclic vector for $L^\infty(X, \Sigma, \mu)$ as a von Neumann algebra acting on $L^2(X, \Sigma, \mu)$ through multiplication.

Now, for any $A \in \Sigma$, there exists an $a \in \mathcal{A}$ such that $u\pi(a)u^* = \chi_A$ (as a multiplication operator, or M_{χ_A} if the reader is pedantic). Moreover $\pi(a)$ is idempotent and self-adjoint as can be seen from

$$u\pi(a)^*u^* = (u\pi(a)u^*)^* = \chi_A^* = \chi_A = u\pi(a)u^*$$

and

$$u\pi(a)\pi(a)u^* = u\pi(a)u^*u\pi(a)u^* = \chi_A\chi_A = \chi_A = u\pi(a)u^*$$

implying $\pi(a)^* = \pi(a)$ and $\pi(a)\pi(a) = \pi(a)$ by canceling out the u and u^*'s.

Since $u\pi(\cdot)u^*$ is a *-homomorphism, it should be immediately clear that $u\pi(1_\mathcal{A})u^* = \chi_X$, by Proposition 4.1.8.

Therefore we may conclude

$$\begin{aligned} & \mu(A) \\ = & \int \chi_A d\mu \\ = & \int (\chi_A \chi_X)^* (\chi_A \chi_X) d\mu \\ = & \langle \chi_A \chi_X, \chi_A \chi_X \rangle_2 \\ = & \langle u^* \chi_A u u^* \chi_X, u^* \chi_A u u^* \chi_X \rangle_2 \\ = & \langle \pi(a)e, \pi(a)e \rangle_{\mathcal{H}} \\ = & \langle \pi(a)^* \pi(a)e, e \rangle_{\mathcal{H}} \\ = & \langle \pi(a)e, e \rangle_{\mathcal{H}} \\ = & \varphi(a), \end{aligned}$$

and in particular that $\mu(X) = \varphi(1_{\mathcal{A}}) = 1$, establishing that (X, Σ, μ) is a probability space.

We denote the collection of all sets of measure zero in Σ by Σ_0. By Σ/Σ_0 we mean the σ-algebra of equivalence classes of sets in Σ whose symmetric difference lies in Σ_0.

Using the map $c : \chi_A \in L^\infty(X, \Sigma, \mu) \mapsto A \in \Sigma/\Sigma_0$ defined on measurable characteristic functions, we define the transformation $T^{-1} : \Sigma \to \Sigma/\Sigma_0$ by

$$T^{-1}A := c\left(u(\pi \circ \tau(a))u^*\right)$$

where $a \in \mathcal{A}$ is any such element that $u\pi(a)u^* = \chi_A$. The map c makes sense in this context, because when a is self adjoint and idempotent, so is $u(\pi \circ \tau(a))u^* \in L^\infty(X, \Sigma, \mu)$ which implies that it equals a characteristic function (μ-a.e) and we can apply c to it with a clear conscience.

Remark 4.3.6. It is extremely important for the reader to note that the resemblance of the map T^{-1} as the 'inverse image map' of some point map $T : X \to X$, is (as of yet) *purely notational*. In the sequel we will construct just such a point set map, such that T^{-1} is (analogous) to its inverse images.

The map, $T^{-1} : \Sigma \to \Sigma/\Sigma_0$ as defined above, is well defined, for if $a_1, a_2 \in \mathcal{A}$ are both such that $u\pi(a_i)u^* = \chi_A$ for $i = 1, 2$ we have that

$$u(\pi(a_1 - a_2))u^* = u\pi(a_1)u^* - u\pi(a_2)u^* = \chi_A - \chi_A = 0$$

implies that $0 = \pi(a_1 - a_2) \in \mathcal{B}(\mathcal{H})$. Therefore for all $x \in \mathcal{A}$ we have $\pi(a_1 - a_2)(x + \mathcal{L}) = 0$ hence $(a_1 - a_2)x \in \mathcal{L}$ (see Remark 4.1.12) and, since τ is state preserving,

$$\begin{aligned} 0 &= \varphi\left(((a_1 - a_2)x)^*((a_1 - a_2)x)\right) \\ &= \varphi \circ \tau\left(((a_1 - a_2)x)^*((a_1 - a_2)x)\right) \\ &= \varphi \circ \left((\tau(a_1 - a_2)\tau(x))^*(\tau(a_1 - a_2)\tau(x))\right). \end{aligned}$$

Now since τ was assumed to be surjective we obtain

$$0 = \varphi \circ ((\tau(a_1 - a_2)x)^*(\tau(a_1 - a_2)x))$$

for all $x \in \mathcal{A}$, which implies that $0 = \pi \circ \tau(a_1 - a_2) \in \mathcal{B}(\mathcal{H})$, hence $u(\pi \circ \tau(a_1 - a_2))u^* = 0$, and therefore

$$u(\pi \circ \tau(a_1))u^* = u(\pi \circ \tau(a_2))u^*$$

as characteristic functions in $L^\infty(X, \Sigma, \mu)$. Therefore the symmetric difference of $c\,(u(\pi \circ \tau(a_1))u^*)$ and $c\,(u(\pi \circ \tau(a_2)u^*)$ must have measure zero. Hence T^{-1} is indeed well defined.

By what we have established previously and by definition of T^{-1} we see $\chi_{T^{-1}A} = u(\pi \circ \tau(a))u^*$ when $a \in \mathcal{A}$ is such that $u\pi(a)u^* = \chi_A$, so that

$$\mu(T^{-1}A) = \varphi \circ \tau(a) = \varphi(a) = \mu(A).$$

We will now show that we can regain a point set map $S : X\backslash X_0 \to X$ where $\mu(X_0) = 0$ such that the inverse images of this map coincide with the forward images of T^{-1}.

We will need the following from [12].

Definition 4.3.7. [12, p. 318] If Σ_1 and Σ_2 are σ-algebras on X_1 and X_2 a map $\Phi : \Sigma_1 \to \Sigma_2$ is called a *σ-homomorphism* when $\Phi\left(\bigcup_{k=1}^\infty A_k\right) = \bigcup_{k=1}^\infty \Phi(A_k)$ for any $\{A_k\}_{k=1}^\infty \subset \Sigma_1$.

Theorem 4.3.8. *[12, p. 329] Let (X, Σ, μ) be a measurable space, Y a complete separable metric space, and Φ a σ-homomorphism from the Borel sets of Y into Σ/Σ_0, with $\Phi(Y) = X$. Then there exists a set $X_0 \in \Sigma_0$ and a point map $\phi : X\backslash X_0 \to Y$ such that for every Borel set B of Y, $\phi^{-1}(B) = \Phi(B)$ (modulo a set of measure zero).*

In light of the hypothesis of the previous, the following famous theorems allow a very welcome transmogrification of X.

Theorem 4.3.9. *[9, p. 215] (Urysohn's Metrization Theorem) Every regular second countable topological space is metrizable.*

Theorem 4.3.10. *[9, p. 276] (Heine Borel Theorem) A metric space is compact if and only if it is complete and totally bounded.*

Theorem 4.3.11. *[9, p. 191] If a topological space is second countable, it has a countable dense subset, i.e. it is separable.*

We recall that X is a compact (hence regular, since it is Haussdorff), second countable Hausdorff space and therefore is metrizable by Urysohn's metrization theorem, complete by the Heine-Borel Theorem and separable by Theorem 4.3.11.

We would like to have that T^{-1} is a σ-homomorphism. In aid of showing this, we first prove:

Lemma 4.3.12. *Let* $\{a_\lambda\} \subset \mathcal{A}$ *be any net such that* $u\pi(a_\lambda)u^* \to u\pi(a)u^*$ *strongly, for some* $a \in \mathcal{A}$*, as multiplication operators in* $L^\infty(X, \Sigma, \mu)$ *acting on* $L^2(X, \Sigma, \mu)$*. Then* $u\pi \circ \tau(a_\lambda)u^* \to u\pi \circ \tau(a)u^*$ *strongly.*

Proof. Let $b \in \mathcal{A}$ be arbitrary, so that $b + \mathcal{L} \in \mathcal{H}$ (see Remark 4.1.12) is an arbitrary element of a dense subset of $\mathcal{H}$. Now, since u is unitary, hence an isometry,

$$\begin{aligned} & \|u\pi \circ \tau(a_\lambda)u^*u(b + \mathcal{L}) - u\pi \circ \tau(a)u^*u(b + \mathcal{L})\|_2^2 \\ = \; & \|\pi \circ \tau(a_\lambda)(b + \mathcal{L}) - \pi \circ \tau(a)(b + \mathcal{L})\|^2 \\ = \; & \|\pi \circ \tau(a_\lambda - a)(b + \mathcal{L})\|^2 \\ = \; & \varphi((\tau(a_\lambda - a)b)^* \tau(a_\lambda - a)b). \end{aligned}$$

Since τ is assumed to be surjective, there exists a $b' \in \mathcal{A}$ such that $\tau(b') = b$ so that, τ being state preserving and a *-homomorphism then implies

$$\begin{aligned} & \|u\pi \circ \tau(a_\lambda)u^*u(b + \mathcal{L}) - u\pi \circ \tau(a)u^*u(b + \mathcal{L})\|_2^2 \\ = \; & \varphi((\tau(a_\lambda - a)b)^* \tau(a_\lambda - a)b) \\ = \; & \varphi((\tau(a_\lambda - a)\tau(b'))^* \tau(a_\lambda - a)\tau(b')) \\ = \; & \varphi(((a_\lambda - a)b')^*(a_\lambda - a)b') \\ = \; & \|\pi(a_\lambda - a)(b' + \mathcal{L})\|^2 \\ = \; & \|u\pi(a_\lambda)u^*u(b' + \mathcal{L}) - u\pi(a)u^*u(b' + \mathcal{L})\|^2 \\ \to \; & 0 \end{aligned}$$

with λ, by hypothesis. So since $(b + \mathcal{L})$ was arbitrary in a dense subset of $\mathcal{H}$, we have that $u\pi \circ \tau(a_\lambda)u^* \to u\pi \circ \tau(a)u^*$strongly in $L^\infty(X, \Sigma, \mu)$. □

We can now show

Proposition 4.3.13. *The map* $T^{-1} : \Sigma \to \Sigma/\Sigma_0$ *is a* σ*-homomorphism satisfying* $T^{-1}X = X$ *(modulo a set of measure zero).*

Proof. First let $A, B \in \Sigma$ be arbitrary and $a, b \in \mathcal{A}$ such that $u\pi(a)u^* = \chi_A$ and $u\pi(b)u^* = \chi_B$. Then

$$u\pi(ab)u^* = u\pi(a)u^*u\pi(b)u^* = \chi_A\chi_B = \chi_{A\cap B},$$

so that by definition $u\pi \circ \tau(ab)u^* = \chi_{T^{-1}(A\cap B)}$ and hence

$$\begin{aligned} & \chi_{T^{-1}A\cap T^{-1}B} \\ = \; & \chi_{T^{-1}A}\chi_{T^{-1}B} \\ = \; & u\pi \circ \tau(a)u^*u\pi \circ \tau(b)u^* \\ = \; & u\pi \circ \tau(ab)u^* \\ = \; & \chi_{T^{-1}(A\cap B)} \end{aligned}$$

as elements of $L^\infty(X,\Sigma,\mu)$, hence the symmetric difference of $T^{-1}(A\cap B)$ and $T^{-1}A\cap T^{-1}B$ has measure zero.

Now also notice that

$$\begin{aligned}\chi_{A\cup B} &= \chi_A+\chi_B-\chi_{A\cap B}\\ &= u\pi(a)u^*+u\pi(b)u^*-u\pi(ab)u^*\\ &= u\pi(a+b-ab)u^*,\end{aligned}$$

so that by definition, $\chi_{T^{-1}(A\cup B)} = u\pi\circ\tau(a+b-ab)u^*$ implies

$$\begin{aligned}&\chi_{T^{-1}(A\cup B)}\\ =\ & u\pi\circ\tau(a+b-ab)u^*\\ =\ & u\pi\circ\tau(a)u^*+u\pi\circ\tau(b)u^*-u\pi\circ\tau(ab)u^*\\ =\ & \chi_{T^{-1}A}+\chi_{T^{-1}B}-\chi_{T^{-1}A\cap T^{-1}B}\\ =\ & \chi_{T^{-1}A\cup T^{-1}B}\end{aligned}$$

as elements of $L^\infty(X,\Sigma,\mu)$. So the symmetric difference of $T^{-1}A\cup T^{-1}B$ and $T^{-1}(A\cup B)$ has measure zero.

Now, let $\{A_l\}_{l=1}^\infty\subset\Sigma$ and $\{a_k\}\in\mathcal{A}$ such that $u\pi(a_k)u^*=\chi_{\bigcup_{l=1}^k A_l}$. It should be clear that $u\pi(a_k)u^*\to\chi_{\bigcup_{l=1}^\infty A_l}$ strongly in $L^\infty(X,\Sigma,\mu)$, as $k\to\infty$ by Theorem 4.1.14. Let $a\in\mathcal{A}$ be such that $\chi_{\bigcup_{l=1}^\infty A_l}=u\pi(a)u^*$. Now by the previous paragraphs and the previous lemma, we have that

$$\begin{aligned}&\chi_{\bigcup_{l=1}^k T^{-1}A_l}\\ =\ & \chi_{T^{-1}\left(\bigcup_{l=1}^k A_l\right)}\\ =\ & u\pi\circ\tau(a_k)u^*\\ \to\ & u\pi\circ\tau(a)u^*\\ =\ & \chi_{T^{-1}\left(\bigcup_{l=1}^\infty A_l\right)}\end{aligned}$$

strongly in $L^\infty(X,\Sigma,\mu)$ as $k\to\infty$. But it should be clear that the strong limit of the sequence $\{\chi_{\bigcup_{l=1}^k T^{-1}A_l}\}$ is $\chi_{\bigcup_{l=1}^\infty T^{-1}A_l}$ by Theorem 4.1.14, therefore we conclude that the symmetric difference of $T^{-1}\left(\bigcup_{l=1}^\infty A_l\right)$ and $\bigcup_{l=1}^\infty T^{-1}A_l$ have measure zero.

Since $\tau(1_\mathcal{A})=1_\mathcal{A}$,

$$\chi_{T^{-1}X}=u\pi\circ\tau(1_\mathcal{A})u^*=u\pi(1_\mathcal{A})u^*=\chi_X,$$

implies that the symmetric difference of $T^{-1}X$ and X has measure zero.

Therefore we may conclude that $T^{-1}:\Sigma\to\Sigma/\Sigma_0$ is a σ-homomorphism such that $T^{-1}X=X$ (modulo a set of measure zero). □

Now T^{-1} satisfies the hypothesis of Theorem 4.3.8, hence there exists a set $X_0\in\Sigma$ of measure zero and a map $S:X\backslash X_0\to X$ such that for every Borel set B of X, $S^{-1}B=T^{-1}B$ (modulo a set of measure zero).

We define the set $X' := X \setminus \bigcup_{k=1}^{\infty} S^{-1}X_0$. Since $\mu(\bigcup_{k=1}^{\infty} S^{-1}X_0) = 0$ we still have $\mu(X') = 1$. We define the map $T' : X' \to X'$ as the restriction of S to X'. Then $(X', \Sigma \cap X', \mu, T')$ is a measure preserving dynamical system.

Moreover, $(X', \Sigma \cap X', \mu, T')$ preserves the 'mixing properties' of $(\mathcal{A}, \varphi, \tau)$. To see this, suppose $(\mathcal{A}, \varphi, \tau)$ is ergodic, let $A, B \in \Sigma \cap X'$, and $a, b \in \mathcal{A}$ be any such elements that $u\pi(a)u^* = \chi_A$ and $u\pi(b)u^* = \chi_B$. We convince ourselves that $u(\pi \circ \tau^k(b))u^* = \chi_{T^{-k}B} = \chi_B \circ T'^k$ because $T^{-1}B := c\,(u(\pi \circ \tau(b))u^*)$ and hence $u(\pi \circ \tau(b))u^* = \chi_{T^{-1}B} = \chi_B \circ T'$. We then investigate

$$\begin{aligned} & \chi_{A \cap T^{-k}B} \\ = \;& \chi_A \chi_{T^{-k}B} \\ = \;& u\pi(a)u^* u(\pi \circ \tau^k(b))u^* \\ = \;& u(\pi(a)\pi \circ \tau^k(b))u^* \\ = \;& u\pi(a\tau^k(b))u^*, \end{aligned}$$

which, with what we had established before implies

$$\mu(A \cap T^{-k}B) = \varphi(a\tau^k(b))$$

and therefore

$$\begin{aligned} & \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \mu(A \cap T'^{-k}B) \\ = \;& \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(a\tau^k(b)) \\ = \;& \varphi(a)\varphi(b) \\ = \;& \mu(A)\mu(B) \end{aligned}$$

establishes the ergodicity of $(X', \Sigma \cap X', \mu, T')$ by Theorem 2.1.13. (Similar argumentation establishes the weakly mixing, or strongly mixing of $(X', \Sigma \cap X', \mu, T')$ when $(\mathcal{A}, \varphi, \tau)$ possesses the same respective property).

4.4 Examples

As outlined in the previous section, we can obviously construct a C*-dynamical system from any given measure preserving dynamical system. For more interesting examples (where the C*-algebra is actually non-commutative) we refer the reader to Chapter 6.

Chapter 5

Generalized Recurrence and Induced Transformations

5.1 Classical induced transformations in the C*-setting

Let (X, Σ, T, μ) be a measure preserving dynamical system and $A \in \Sigma$ with $\mu(A) > 0$. Let $(A, \Sigma \cap A, \mu_A, T_A)$ be the system induced onto A from (X, Σ, T, μ). Let $(\mathcal{A}, \varphi, \tau)$ and $(\mathcal{A}_A, \varphi_A, \tau_A)$ be the C*-dynamical systems derived from (X, Σ, T, μ) and $(A, \Sigma \cap A, \mu_A, T_A)$ respectively as in Proposition 4.3.1. Note that the C*-algebras $\mathcal{A} = L^\infty(X, \Sigma, \mu)$ and $\mathcal{A}_A = L^\infty(A, \Sigma \cap A, \mu_A)$ are von Neumann algebras acting on the Hilbert spaces $L^2(X, \Sigma, \mu)$ and $L^2(A, \Sigma \cap A, \mu_A)$ respectively through multiplication, therefore we may refer to von Neumann algebra theory.

If our goal of defining induced transformations for general C*-dynamical systems is indeed achievable, it should certainly be possible to obtain the system $(\mathcal{A}_A, \varphi_A, \tau_A)$ directly from $(\mathcal{A}, \varphi, \tau)$ without relying on the underlying classical systems (X, Σ, T, μ) and $(A, \Sigma \cap A, \mu_A, T_A)$ (too much).

We notice that $\chi_A \in \mathcal{A}$ is a projection and $\mathcal{A}_A^{(1)} := \chi_A \mathcal{A} \chi_A = \chi_A \mathcal{A}$ is a von Neumann algebra acting on $L^2(X, \Sigma, \mu)$ [10, p. 116], while we actually want it to act on $L^2(A, \Sigma \cap A, \mu_A)$. It should however be clear that $\mathcal{A}_A^{(1)} = \chi_A \mathcal{A}$ is *-isomorphic to $\mathcal{A}_A$ through the map $\chi_A f \mapsto f|_A$. Moreover, it is clear that this *-isomorphism is weakly continuous, so that we may view $\mathcal{A}_A^{(1)} = \chi_A \mathcal{A}$ as a von Neumann algebra acting on $L^2(A, \Sigma \cap A, \mu_A)$ [10, p. 132]. Hence, not more than a moment's reflection convinces us that $\mathcal{A}_A^{(1)}$ and $\mathcal{A}_A$ are actually 'the same' von Neumann algebra, as they act on the same Hilbert space and their elements are in bijective correspondence respectively through restriction to A and extension to X from A by zero.

We can define a state $\varphi_A^{(1)} : \mathcal{A}_A^{(1)} \to \mathbb{C}$ by $\varphi_A^{(1)}(f') := \varphi(f')/\varphi(\chi_A)$ for all

$f' \in \mathcal{A}_A^{(1)}$. On comparison with the state φ_A we see for any $f \in \mathcal{A}_A$, by denoting f's extension to X by zero by $f' \in \mathcal{A}_A^{(1)}$,

$$\begin{aligned}
\varphi_A(f) &= \int_A f d\mu_A \\
&= \frac{1}{\mu(A)} \int_A f d\mu \\
&= \frac{1}{\varphi(\chi_A)} \int_X f' d\mu \\
&= \frac{1}{\varphi(\chi_A)} \varphi(f') \\
&= \varphi'_A(f')
\end{aligned}$$

that they are indeed equal.

If we partition $A \in \Sigma$ by return time, say $\{A_j\}_{j=1}^{\infty}$ according to Definition 3.2.4, we can define a transformation $\tau_A^{(1)} : \mathcal{A}_A^{(1)} \to \mathcal{A}_A^{(1)}$ as follows. For any $f' \in \mathcal{A}_A^{(1)}$

$$\tau'_A(f') := \sum_{j=1}^{\infty} \chi_{A_j} \tau^j(f') = \sum_{j=1}^{\infty} \chi_{A_j} f' \circ T^j.$$

When we restrict both sides to A (as they are L^{∞} functions on X, which equal zero (μ-almost everywhere) outside A), we see that for (μ-almost) every $x \in A$, there exists a unique j_0 such that $x \in A_{j_0}$ and hence $n_A(x) = j_0$, since $\{A_j\}_{j=1}^{\infty}$ is a partition of A, so that when we denote the restriction of $f' \in \mathcal{A}_A^{(1)}$ to A by $f \in \mathcal{A}_A$

$$\begin{aligned}
\left(\tau_A^{(1)}(f')\right)(x) &= \left(\sum_{j=1}^{\infty} \chi_{A_j} f' \circ T^j\right)(x) \\
&= \sum_{j=1}^{\infty} \chi_{A_j}(x) f' \circ T^j(x) \\
&= f' \circ T^{j_0}(x) \\
&= f' \circ T^{n(x)}(x) \\
&= f \circ T_A(x) \\
&= (\tau_A(f))(x).
\end{aligned}$$

It should also be easily seen that the fact that $\tau_A^{(1)}$ is state preserving with respect to $\varphi_A^{(1)}$ is implied by τ_A being state preserving with respect to φ_A. Moreover $\tau_A^{(1)}$ is a *-homomorphism, since τ_A is.

Hence $(\mathcal{A}_A^{(1)}, \varphi_A^{(1)}, \tau_A^{(1)})$ is not only a C*-dynamical system, but is 'the same' as $(\mathcal{A}_A, \varphi_A, \tau_A)$, and obviously therefore possesses the same 'mixing properties' as $(\mathcal{A}_A, \varphi_A, \tau_A)$ which it had inherited from $(A, \Sigma \cap A, \mu_A, T_A)$, which $(A, \Sigma \cap A, \mu_A, T_A)$ in turn, had inherited them from (X, Σ, μ, T).

The above discussion gives us a good clue as to how to go about defining induced transformations for general C*-dynamical systems.

We might make the educated guess that constructing an induced transformation on some general $(\mathcal{A}, \varphi, \tau)$ C*-dynamical system will revolve around a (suitable) projection $p \in \mathcal{A}$, in analogy to the projection χ_A (as a multiplication operator) as used above, so that we may work with the *-algebra $\mathcal{A}_p := p\mathcal{A}p$. Also, because the return time map n_A looses all meaning and use in this general setting, we will likely require that the projection p admits a 'partition' $\{p_j\}_{j=1}^{\infty}$ of (hopefully) mutually orthogonal projections, such that $p = \sum_{j=1}^{\infty} p_j$ (at least strongly, when $\mathcal{A}$ happens to be a von Neumann algebra). This would permit us to define an analogous induced transformation $\tau_p : \mathcal{A}_p \to \mathcal{A}_p$ by

$$\tau_p(a) := \sum_{j=1}^{\infty} p_j \tau^j(a)$$

for any $a \in \mathcal{A}_p$, in analogy to what was done above, which is then expected to be state preserving with respect to the state $\varphi_p : \mathcal{A}_p \to \mathbb{C}$ defined by $\varphi_p(a) := \varphi(a)/\varphi(p)$ for all $a \in \mathcal{A}$.

If this is managed, $(\mathcal{A}_p, \varphi_p, \tau_p)$ would be a C*-dynamical system, which we might guess inherit 'mixing properties' from $(\mathcal{A}, \varphi, \tau)$ in analogy to the way that $(A, \Sigma \cap A, \mu_A, T_A)$ inherits 'mixing properties' from (X, Σ, T, μ).

5.2 Generalized Recurrence

This section is based on [6]. The reader will do well to remind himself/herself of the definitions in Section 4.1.

Definition 5.2.1. [6, p. 57] Let $\mathcal{A}$ be a *-algebra and $\mathcal{B}$ a unital *-algebra. Let $\varphi : \mathcal{A} \to \mathcal{B}$ be a positive mapping (i.e. $\varphi(a^*a) \geq 0$ for all $a \in \mathcal{A}$).

We call φ *additive* if

$$\sum_{k=1}^{n} \varphi(p_k) \leq 1_{\mathcal{B}}$$

for any projections $p_1, \dots, p_n \in \mathcal{A}$ for which $\varphi(p_k p_l p_k) = 0$ if $k < l$.

We call φ *faithful* if it is linear, $\mathcal{A}$ is unital, $\varphi(1_{\mathcal{A}}) = 1_{\mathcal{B}}$ and $\varphi(a^*a) > 0$ when $a \in \mathcal{A}$ is non-zero. (This obviously requires that $a^*a \neq 0$ when $a \neq 0$, which at least holds for all C*-algebras).

We will say φ is *tracial* if it has the property $\varphi(ab) = \varphi(ba)$ for all $a, b \in \mathcal{A}$.

The following can be viewed as a generalization of the Poincaré Recurrence Theorem

Theorem 5.2.2. *[6, p. 57] Let $\mathcal{A}$ be a *-algebra, $\mathcal{B}$ a unital C*-algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ an additive map. Let $\tau : \mathcal{A} \to \mathcal{A}$ be a *-homomorphism such that $\varphi(\tau(pqp)) = \varphi(pqp)$ for all projections $p, q \in \mathcal{A}$. Then for any projection $p \in \mathcal{A}$ with $\varphi(p) > 0$, there exists a positive integer n such that $\varphi(p\tau^n(p)p) > 0$.*

Proof. Note that since τ is a *-homomorphism, it preserves projections (i.e. if $p \in \mathcal{A}$ is a projection so is $\tau(p)$). Therefore

$$p\tau^n(p)p = p\tau^n(p)\tau^n(p)p = (\tau^n(p)^*p^*)^*\tau^n(p)p = (\tau^n(p)p)^*\tau^n(p)p$$

implies

$$\varphi(p\tau^n(p)p) = \varphi((\tau^n(p)p)^*\tau^n(p)p) \geq 0.$$

Suppose, contrary to what we want to prove, that $\varphi(p\tau^n(p)p) = 0$ for all $n \in \mathbb{N}$. Then for all $k, n \in \mathbb{N}$ we see, by hypothesis, and our supposition, that

$$\varphi(\tau^k(p)\tau^{k+n}(p)\tau^k(p)) = \varphi(\tau^k(p\tau^n(p)p)) = \varphi(p\tau^n(p)p) = 0.$$

Therefore $\{\tau^k(p)\}_{k\in\mathbb{N}}$ forms a collection of projections such that $\varphi(\tau^k(p)\tau^n(p)\tau^k(p)) = 0$ when $k < n$.

Now, since φ is additive, for every $n \in \mathbb{N}$

$$\sum_{k=1}^{n} \varphi(\tau^k(p)) \leq 1_{\mathcal{B}}.$$

Moreover, since p is a projection, $p = ppp$, so that

$$\sum_{k=1}^{n} \varphi(\tau^k(p)) = \sum_{k=1}^{n} \varphi(\tau^k(ppp)) = \sum_{k=1}^{n} \varphi(p) = n\varphi(p).$$

Therefore $n\varphi(p) \leq 1_{\mathcal{B}}$, and $n\|\varphi(p)\| \leq \|1_{\mathcal{B}}\| = 1$ by [10, p. 46], which implies that $\|\varphi(p)\| = 0$ and therefore $\varphi(p) = 0$, contrary to our hypothesis that $\varphi(p) > 0$.

In conclusion, we have established the existence of an $n \in \mathbb{N}$ such that $\varphi(p\tau^n(p)p) > 0$.

□

We can adapt the proof of the previous theorem to obtain the related result

Theorem 5.2.3. *Let $\mathcal{A}$ be a *-algebra, $\mathcal{B}$ a unital C^*-algebra and $\varphi : \mathcal{A} \to \mathcal{B}$ an additive map. Let $\tau : \mathcal{A} \to \mathcal{A}$ be a *-homomorphism such that $\varphi(\tau(pqp)) = \varphi(pqp)$ for all projections $p, q \in \mathcal{A}$. Then for any projection $p \in \mathcal{A}$ with $\varphi(p) > 0$, there exists infinitely many positive integers n such that $\varphi(p\tau^n(p)p) > 0$.*

Proof. Suppose to the contrary, that there exist only a finite number of integers n such that $\varphi(p\tau^n(p)p) > 0$. Then there exists an integer n_0 such that $n > n_0$ implies $\varphi(p\tau^n(p)p) = 0$. Then for any $k, n \in \mathbb{N}$ with $n > n_0$

$$\varphi(\tau^k(p)\tau^{k+n}(p)\tau^k(p)) = \varphi(\tau^k(p\tau^n(p)p)) = \varphi(p\tau^n(p)p) = 0,$$

so that the collection of projections $\{\tau^{k+n_0}(p)\}_{k\in\mathbb{N}}$ is such that

$$\varphi(\tau^{k+n_0}(p)\tau^{n+n_0}(p)\tau^{k+n_0}(p)) = 0$$

when $k < n$.

By additivity of φ we then obtain

$$\sum_{k=1}^{n} \varphi(\tau^{k+n_0}(p)) \leq 1_{\mathcal{B}}.$$

Since $p = ppp$, also

$$\sum_{k=1}^{n} \varphi(\tau^{k+n_0}(p)) = \sum_{k=1}^{n} \varphi(\tau^{k+n_0}(ppp)) = \sum_{k=1}^{n} \varphi(p) = n\varphi(p).$$

Therefore $n\varphi(p) \leq 1_{\mathcal{B}}$, and again $n\|\varphi(p)\| \leq \|1_{\mathcal{B}}\| = 1$ by [10, p. 46], which implies that $\|\varphi(p)\| = 0$ and therefore $\varphi(p) = 0$, contrary to our hypothesis that $\varphi(p) > 0$.

We conclude that there must exist infinitely many positive integers n such that $\varphi(p\tau^n(p)p) > 0$. □

Corollary 5.2.4. *If under the same hypothesis as the previous theorem, we assume that φ is tracial, then for any projection $p \in \mathcal{A}$ and any sub-projection $q \in \mathcal{A}$ of p such that $\varphi(q) > 0$, there exist infinitely many positive integers n such that $\varphi(p\tau^n(q)p) > 0$. Also, there exist infinitely many positive integers n such that $\varphi(q\tau^n(p)q) > 0$.*

Proof. Because q is a sub-projection of p, $(p-q)$ is a projection as can be seen from

$$(p-q)^* = (p^* - q^*) = (p-q)$$

and

$$(p-q)(p-q) = pp - qp - pq + qq = p - q - q + q = (p-q).$$

Since φ is positive and tracial by hypothesis and τ, being a *-homomorphism, preserves projections, we see that for all $n \in \mathbb{N}$

$$\begin{aligned}
& \varphi(p\tau^n(q)p) - \varphi(q\tau^n(q)q) \\
= \; & \varphi(pp\tau^n(q)) - \varphi(qq\tau^n(q)) \\
= \; & \varphi(p\tau^n(q)) - \varphi(q\tau^n(q)) \\
= \; & \varphi((p-q)\tau^n(q)) \\
= \; & \varphi((p-q)(p-q)\tau^n(q)) \\
= \; & \varphi((p-q)\tau^n(q)\tau^n(q)(p-q)) \\
= \; & \varphi((\tau^n(q)(p-q))^*\tau^n(q)(p-q)) \\
\geq \; & 0.
\end{aligned}$$

Hence $\varphi(p\tau^n(q)p) \geq \varphi(q\tau^n(q)q)$. But by Theorem 5.2.3 the existence of infinitely many $n \in \mathbb{N}$ such that $\varphi(q\tau^n(q)q) > 0$ implies

$$\varphi(p\tau^n(q)p) \geq \varphi(q\tau^n(q)q) > 0,$$

for such all such n, establishing the first result.

For the second result we investigate,

$$\begin{aligned} & \varphi(q\tau^n(p)q) - \varphi(q\tau^n(q)q) \\ = \; & \varphi(q\tau^n(p)) - \varphi(q\tau^n(q)) \\ = \; & \varphi(q\tau^n(p-q)) \\ = \; & \varphi(q\tau^n(p-q)\tau^n(p-q)q) \\ = \; & \varphi((\tau^n(p-q)q)^*\tau^n(p-q)q) \\ \geq \; & 0, \end{aligned}$$

which holds since φ is tracial and positive, $(p-q)$ is a projection and τ is a *-homomorphism. Therefore

$$\varphi(q\tau^n(p)q) \geq \varphi(q\tau^n(q)q) > 0$$

for infinitely many positive integers n by Theorem 5.2.3, establishing the second result. □

The same proof as for the previous result allows us to also establish:

Corollary 5.2.5. *If under the same hypothesis as Theorem 5.2.3, $p \in \mathcal{A}$ any projection and $q \in \mathcal{A}$ any sub-projection of p such that $\varphi(q) > 0$, $p\tau^n(q) = \tau^n(q)p$ and $q\tau^n(q) = \tau^n(q)q$ for all $n \in \mathbb{N}$, there exist infinitely many positive integers n such that $\varphi(p\tau^n(q)p) > 0$.*

And also:

Corollary 5.2.6. *If under the same hypothesis as Theorem 5.2.3, $p \in \mathcal{A}$ any projection and $q \in \mathcal{A}$ any sub-projection of p such that $\varphi(q) > 0$, $q\tau^n(p) = \tau^n(p)q$ and $q\tau^n(q) = \tau^n(q)q$ for all $n \in \mathbb{N}$, there exist infinitely many positive integers n such that $\varphi(q\tau^n(p)q) > 0$.*

We will use these results in the coming sections to define generalized induced transformations.

5.3 Generalized Return Time Partitions

In this section we will investigate the possibility of defining a consistent return time partition as was done for classical measure preserving dynamical systems in Section 3.2.

Let $(\mathcal{A}, \varphi, \tau)$ be a C*-dynamical system, with $\mathcal{A}$ a von Neumann algebra (that is not a factor) acting on a Hilbert space denoted by $\mathcal{H}$. We assume that $\mathcal{H}$ is the same Hilbert space as is obtained from the GNS-construction on $(\mathcal{A}, \varphi)$ (see Remark 4.1.12), and that the representation π obtained is just an inclusion map. We therefore suppress mention of π and justify writing $\iota(ab) = a\iota(b)$ for all $a, b \in \mathcal{A}$. Let τ be a *-homomorphism and let p be any projection in $\mathcal{A} \cap \mathcal{A}'$, such that $\varphi(p) > 0$ and $\tau^j(p) \in \mathcal{A} \cap \mathcal{A}'$ for all $j \in \mathbb{N}$.

We recursively define for all $j \in \mathbb{N}$

$$\begin{aligned} p_1 &:= \tau(p)p \\ p_j &:= \tau^j(p)(p - \sum_{k=1}^{j-1} p_k) \end{aligned}$$

and

$$\begin{aligned} q_1 &:= \tau(p)(1_{\mathcal{A}} - p) \\ q_j &:= \tau^j(p)(1_{\mathcal{A}} - p - \sum_{k=1}^{j-1} q_k). \end{aligned}$$

Because $\mathcal{A} \cap \mathcal{A}'$ is a von Neumann algebra [10, p. 117], we can conclude that $p_j, q_j \in \mathcal{A} \cap \mathcal{A}'$ for all $j \in \mathbb{N}$, since their constituents lie in $\mathcal{A} \cap \mathcal{A}'$. Moreover, since both $p, (1_{\mathcal{A}} - p) \in \mathcal{A} \cap \mathcal{A}'$, p_j and q_j are sub-projections of the projections p and $(1_{\mathcal{A}} - p)$ respectively for all $j \in \mathbb{N}$.

Remark 5.3.1. We will remark that when $(\mathcal{A}, \varphi, \tau)$ is constructed from a measure preserving dynamical system (X, Σ, μ, T) as in Section 4.3, then for any $A \in \Sigma$ with $\mu(A) > 0$ we have that $\chi_A \in \mathcal{A} = L^\infty(X, \Sigma, \mu)$ also lies in $\mathcal{A}'$, by the commutativity of $\mathcal{A}$. In this case, it takes no great stretch of the imagination to see that $p_j = \chi_{A_j}$ and $q_j = \chi_{B_j}$ for each $j \in \mathbb{N}$ where $\{A_j\}_{j=1}^\infty$ and $\{B_j\}_{j=1}^\infty$ are as they are were defined in Section 3.2. Therefore $\{p_j\}_{j=1}^\infty$ is indeed a partition of p (in some sense), since $\{A_j\}_{j=1}^\infty$ is a partition of A. This hints that we are indeed on the right track in our search for a consistent definition of a generalized return time partition.

We can prove the following:

Proposition 5.3.2. *The collections $\{p_j\}_{j=1}^\infty \subset \mathcal{A} \cap \mathcal{A}'$ and $\{q_j\}_{j=1}^\infty \subset \mathcal{A} \cap \mathcal{A}'$ respectively consist of pairwise orthogonal projections.*

Proof. We can easily see that $p_1 \in \mathcal{A}$ is a projection, since $p \in \mathcal{A} \cap \mathcal{A}'$ and τ is a *-homomorphism,

$$p_1 p_1 = p\tau(p)p\tau(p) = p^2\tau(p^2) = \tau(p)p = p_1$$

and

$$p_1^* = (p\tau(p))^* = \tau(p)^* p^* = \tau(p^*)p = \tau(p)p = p_1.$$

We can show that $p_1 p_2 = p_2 p_1 = 0$, since $p_j \in \mathcal{A} \cap \mathcal{A}'$ for all $j \in \mathbb{Z}_{\geq 0}$, $p_1 p_2 = p_2 p_1$ is clear. By definition,

$$\begin{aligned} & p_1 p_2 \\ =\ & \tau(p)p\tau^2(p)(p - p_1) \\ =\ & \tau(p)p\tau^2(p)(p - \tau(p)p) \\ =\ & p\tau(p)\tau^2(p) - p\tau(p)\tau^2(p) \\ =\ & 0. \end{aligned}$$

By strong induction we can now show that the p_j are mutually orthogonal projections. Suppose for some $k_0 \in \mathbb{N}$, for each $l \leq k_0$, p_l is a projection and the collection $\{p_l\}_{l=1}^{k_0}$ of projections are pairwise orthogonal.

We can show that p_{k_0+1} is a projection, for the pairwise orthogonality of $\{p_l\}_{l=1}^{k_0}$ and the fact that all operators involved lie in $\mathcal{A}'$, implies

$$\begin{aligned}
p_{k_0+1}p_{k_0+1} &= \tau^{k_0+1}(p)\left(p-\sum_{k=1}^{k_0}p_k\right)\tau^{k_0+1}(p)\left(p-\sum_{k=1}^{k_0}p_k\right) \\
&= p\tau^{k_0+1}(p)\left(1_{\mathcal{A}}-\sum_{k=1}^{k_0}p_k\right)\left(1_{\mathcal{A}}-\sum_{k=1}^{k_0}p_k\right) \\
&= p\tau^{k_0+1}(p)\left(1_{\mathcal{A}}-\sum_{k=1}^{k_0}p_k-\sum_{k=1}^{k_0}p_k+\left(\sum_{k=1}^{k_0}p_k\right)\left(\sum_{r=1}^{k_0}p_r\right)\right) \\
&= p\tau^{k_0+1}(p)\left(1_{\mathcal{A}}-\sum_{k=1}^{k_0}p_k-\sum_{k=1}^{k_0}p_k+\sum_{k,r=1}^{k_0}p_kp_r\right) \\
&= p\tau^{k_0+1}(p)\left(1_{\mathcal{A}}-\sum_{k=1}^{k_0}p_k-\sum_{k=1}^{k_0}p_k+\sum_{k=1}^{k_0}p_k\right) \\
&= \tau^{k_0+1}(p)\left(p-\sum_{k=1}^{k_0}p_k\right) \\
&= p_{k_0+1},
\end{aligned}$$

and

$$\begin{aligned}
p_{k_0+1}^* &= \left(\tau^{k_0+1}(p)\left(p-\sum_{k=1}^{k_0}p_k\right)\right)^* \\
&= \tau^{k_0+1}(p)^*\left(p^*-\sum_{k=1}^{k_0}p_k^*\right) \\
&= \tau^{k_0+1}(p)\left(p-\sum_{k=1}^{k_0}p_k\right) \\
&= p_{k_0+1},
\end{aligned}$$

establishing that p_{k_0+1} is indeed a projection.

We can now show that the projections in $\{p_l\}_{l=1}^{k_0+1}$ are pairwise orthogonal. By assumption, the projections in $\{p_l\}_{l=1}^{k_0}$ are pairwise orthogonal, thus it only remains to show that p_{k_0+1} is orthogonal to every element in $\{p_l\}_{l=1}^{k_0}$. It is clear that $p_{k_0+1}p_l = p_lp_{k_0+1}$ for every $l \leq k_0$ since $p_j \in \mathcal{A} \cap \mathcal{A}'$ for all $j \in \mathbb{N}$. Since the projections in$\{p_l\}_{l=1}^{k_0}$ are pairwise orthogonal and each is a sub-projection of p, we see that

$$\begin{aligned}
& p_{k_0+1}p_l \\
= \;& \tau^{k_0+1}(p)\left(p - \sum_{k=1}^{k_0} p_k\right) p_l \\
= \;& \tau^{k_0+1}(p)\left(p_l - \sum_{k=1}^{k_0} p_k p_l\right) \\
= \;& \tau^{k_0+1}(p)\,(p_l - p_l) \\
= \;& 0,
\end{aligned}$$

implies that the projections in $\{p_l\}_{l=1}^{k_0+1}$ are pairwise orthogonal.

But the induction hypothesis holds when $k_0 = 1$, therefore each element of the collection $\{p_j\}_{j=1}^{\infty}$ is a projection, and the projections in the collection are pairwise orthogonal.

Analogous argumentation establishes the result for $\{q_j\}_{j=1}^{\infty}$. □

Definition 5.3.3. If $r \in \mathcal{A}$ is a projection, we will call a countable collection of pairwise orthogonal projections in $\mathcal{A}$ a *potential partition* of r if each projection in the collection is a sub-projection of r.

We call $\mathcal{P}_0 := \{p_j\}_{j=1}^{\infty}$ the *basic return time potential partition* of the projection p.

Let $\mathcal{P}' := \{p'_j\}_{j=1}^{\infty} \subset \mathcal{A}$ and $\mathcal{P}'' := \{p''_j\}_{j=1}^{\infty} \subset \mathcal{A}$ both be potential partitions of some projection $r \in \mathcal{A}$. We will say $\mathcal{P}''$ is a *compatible expansion* of $\mathcal{P}'$ and write $\mathcal{P}' \leq \mathcal{P}''$ if for all $j, j' \in \mathbb{N}$

$$p'_j p''_{j'} = p''_{j'} p'_j = \delta_{jj'} p'_{j'}$$

Proposition 5.3.4. *The relation '$\leq$' is a partial order on the collection of all compatible expansions of $\mathcal{P}_0$.*

Proof. Clearly, if $\mathcal{P}' := \{p'_j\}_{j=1}^{\infty}$ is a compatible expansion of $\mathcal{P}_0$, then $\mathcal{P}' \leq \mathcal{P}'$ because the pairwise orthogonality the projections in $\mathcal{P}'$ establishes $p'_j p'_{j'} = p'_{j'} p'_j = \delta_{jj'} p'_{j'}$ for all $j, j' \in \mathbb{N}$. Hence the relation is reflexive.

If $\mathcal{P}' := \{p'_j\}_{j=1}^{\infty}$ and $\mathcal{P}'' := \{p''_j\}_{j=1}^{\infty}$ are compatible expansions of $\mathcal{P}_0$ such that $\mathcal{P}' \leq \mathcal{P}''$ and $\mathcal{P}'' \leq \mathcal{P}'$, then

$$p'_j p''_{j'} = p''_{j'} p'_j = \delta_{jj'} p'_{j'}$$

and

$$p'_j p''_{j'} = p''_{j'} p'_j = \delta_{jj'} p''_{j'}$$

implies $\delta_{jj'} p'_{j'} = p'_j p''_{j'} = \delta_{jj'} p''_{j'}$, and therefore $p''_j = p'_j$ for all $j \in \mathbb{N}$. Hence $\mathcal{P}' = \mathcal{P}''$. Therefore the relation is anti-symmetric.

If $\mathcal{P}^{(i)} := \{p_j^{(i)}\}_{j=1}^{\infty}$ for $i = 1, 2, 3$ are compatible expansions of $\mathcal{P}_0$ such that $\mathcal{P}^{(1)} \leq \mathcal{P}^{(2)}$ and $\mathcal{P}^{(2)} \leq \mathcal{P}^{(3)}$, then

$$p_j^{(1)} p_{j'}^{(2)} = p_{j'}^{(2)} p_j^{(1)} = \delta_{jj'} p_{j'}^{(1)}$$

and

$$p_j^{(2)} p_{j'}^{(3)} = p_{j'}^{(3)} p_j^{(2)} = \delta_{jj'} p_{j'}^{(2)}.$$

But then

$$p_j^{(1)} p_{j'}^{(3)} = p_j^{(1)} p_j^{(2)} p_{j'}^{(3)} = \delta_{jj'} p_j^{(1)} p_{j'}^{(2)} = \delta_{jj'} p_{j'}^{(1)},$$

and similarly $p_{j'}^{(3)} p_j^{(1)} = \delta_{jj'} p_{j'}^{(1)}$, which establishes $\mathcal{P}^{(1)} \leq \mathcal{P}^{(3)}$ and therefore transitivity of the relation.

We conclude that the relation '$\leq$' on compatible expansions of $\mathcal{P}_0$ is indeed a partial order. □

Proposition 5.3.5. *Every chain of compatible expansions of $\mathcal{P}_0$ ordered by '$\leq$' has an upper bound.*

Proof. Let

$$\left\{ \mathcal{P}^{(n)} := \{p_j^{(n)}\}_{j=1}^{\infty} \right\}_{n \geq 0}$$

be an arbitrary chain of compatible expansions of $\mathcal{P}_0$. We may as well assume that $\mathcal{P}^{(0)} = \mathcal{P}_0$ because $\mathcal{P}_0$ is a lower bound for the collection of all compatible expansions of $\mathcal{P}_0$.

We define $\overline{\mathcal{P}} := \{\overline{p_j}\}_{j=1}^{\infty}$ as follows. For each $j \in \mathbb{N}$, $\{p_j^{(n)}\}_{n \geq 0}$ is an increasing net of projections in $\mathcal{A}$, therefore converges strongly to a projection in $\mathcal{A}$, by Theorem 4.1.14, which we denote by $\overline{p_j}$.

We will now show that $\overline{\mathcal{P}}$ is an upper bound for the chain.

Let $n \in \mathbb{N}$ be arbitrary but fixed, and $m \in \mathbb{N}$ arbitrary such that $m > n$. Then for all $j, j' \in \mathbb{N}$, and $u \in \mathcal{H}$

$$\begin{aligned}
& \|\overline{p_j} p_{j'}^{(n)} u - \delta_{jj'} p_{j'}^{(n)} u\| \\
= \; & \|\overline{p_j} p_{j'}^{(n)} u - p_j^{(m)} p_{j'}^{(n)} u + p_j^{(m)} p_{j'}^{(n)} u - \delta_{jj'} p_{j'}^{(n)} u\| \\
\leq \; & \|(\overline{p_j} - p_j^{(m)}) p_{j'}^{(n)} u\| + \|\delta_{jj'} p_{j'}^{(n)} u - \delta_{jj'} p_{j'}^{(n)} u\| \\
= \; & \|(\overline{p_j} - p_j^{(m)}) p_{j'}^{(n)} u\| \\
\to \; & 0
\end{aligned}$$

as $m \to \infty$, while the left hand side is independent of m. Therefore for all $n \geq 0$,

$$\overline{p_j} p_{j'}^{(n)} = \delta_{jj'} p_{j'}^{(n)}.$$

Similarly, because for any projection p in a C*-algebra we have $\|p\| = \|p^* p\| = \|p\|^2$, implying that $\|p\|$ equals either 0 or 1. Hence

$$\begin{aligned}
& \|p_{j'}^{(n)} \overline{p_j} u - \delta_{jj'} p_{j'}^{(n)} u\| \\
= \; & \|p_{j'}^{(n)} \overline{p_j} u - p_{j'}^{(n)} p_j^{(m)} u + p_{j'}^{(n)} p_j^{(m)} u - \delta_{jj'} p_{j'}^{(n)} u\| \\
\leq \; & \|p_{j'}^{(n)} \overline{p_j} u - p_{j'}^{(n)} p_j^{(m)} u\| + \|p_{j'}^{(n)} p_j^{(m)} u - \delta_{jj'} p_{j'}^{(n)} u\| \\
\leq \; & \|p_{j'}^{(n)}\|_{\mathcal{H}} \|\overline{p_j} u - p_j^{(m)} u\| + \|\delta_{jj'} p_{j'}^{(n)} u - \delta_{jj'} p_{j'}^{(n)} u\| \\
\leq \; & \|\overline{p_j} u - p_j^{(m)} u\| \\
\to \; & 0
\end{aligned}$$

as $m \to \infty$ while the left is again independent of m. Therefore $p_{j'}^{(n)}\overline{p_j} = \delta_{jj'}p_{j'}^{(n)}$. So we may conclude that for all $j, j' \in \mathbb{N}$ and all $n \geq 0$,

$$p_{j'}^{(n)}\overline{p_j} = \overline{p_j}p_{j'}^{(n)} = \delta_{jj'}p_{j'}^{(n)}.$$

We now prove that the projections in $\overline{\mathcal{P}}$ are pairwise orthogonal. Let $j, j' \in \mathbb{N}$, $u \in \mathcal{H}$ and $m \geq 0$ be arbitrary. Then, by invoking the just established fact $p_{j'}^{(m)}\overline{p_j} = \overline{p_j}p_{j'}^{(m)} = \delta_{jj'}p_{j'}^{(m)}$,

$$\begin{array}{rl}
& \|\overline{p_j p_{j'}}u - \delta_{jj'}\overline{p_j}u\| \\
= & \|\overline{p_j p_{j'}}u - p_{j'}^{(m)}\overline{p_j}u + p_{j'}^{(m)}\overline{p_j}u - \delta_{jj'}\overline{p_j}u\| \\
\leq & \|\overline{p_j p_{j'}}u - \overline{p_j}p_{j'}^{(m)}u\| + \|p_{j'}^{(m)}\overline{p_j}u - \delta_{jj'}\overline{p_j}u\| \\
\leq & \|\overline{p_j}\|_{\mathcal{H}}\|\overline{p_{j'}}u - p_{j'}^{(m)}u\| + \delta_{jj'}\|p_{j'}^{(m)}u - \overline{p_j}u\| \\
\leq & \|\overline{p_{j'}}u - p_{j'}^{(m)}u\| + \delta_{jj'}\|p_{j'}^{(m)}u - \overline{p_j}uH\| \\
\to & 0
\end{array}$$

as $m \to \infty$ while the left is again independent of m. Hence $\overline{p_j p_{j'}} = \delta_{jj'}\overline{p_j}$, establishing the pairwise orthogonality of the projections in $\overline{\mathcal{P}}$.

We now show that every projection in the collection $\overline{\mathcal{P}}$ is a sub-projection of p. Let $m \geq 0$, $j \in \mathbb{N}$ and $u \in \mathcal{H}$ be arbitrary

$$\begin{array}{rl}
& \|p\overline{p_j}u - \overline{p_j}u\| \\
= & \|p\overline{p_j}u - pp_j^{(m)}u + pp_j^{(m)}u - \overline{p_j}u\| \\
\leq & \|p\overline{p_j}u - pp_j^{(m)}u\| + \|pp_j^{(m)}u - \overline{p_j}u\| \\
= & \|p\|\|\overline{p_j}u - p_j^{(m)}u\| + \|p_j^{(m)}u - \overline{p_j}u\| \\
\to & 0
\end{array}$$

as $m \to \infty$, while the left is independent of m, hence $p\overline{p_j} = \overline{p_j}$.

Similarly

$$\begin{array}{rl}
& \|\overline{p_j}pu - \overline{p_j}u\| \\
= & \|\overline{p_j}pu - p_j^{(m)}pu + p_j^{(m)}pu - \overline{p_j}u\| \\
\leq & \|\overline{p_j}pu - p_j^{(m)}pu\| + \|p_j^{(m)}pu - \overline{p_j}u\| \\
= & \|(\overline{p_j} - p_j^{(m)})pu\| + \|p_j^{(m)}u - \overline{p_j}u\| \\
\to & 0
\end{array}$$

as $m \to \infty$, while the left is independent of m, hence $\overline{p_j}p = \overline{p_j}$.

We conclude that $\overline{\mathcal{P}}$ is a compatible expansion of $\mathcal{P}_0$ and $\mathcal{P}^{(n)} \leq \overline{\mathcal{P}}$ for all $n \geq 0$, hence is an upper bound for the chain. $\square$

Corollary 5.3.6. *There exists a maximal compatible expansion of $\mathcal{P}_0$.*

Proof. The result follows immediately by invoking Zorn's Lemma on the previous results. □

As an aside we might raise the following, in analogy with so many other statements of this form:

Conjecture 5.3.7. *The existence of an upper bound for every chain of compatible expansion of $\mathcal{P}_0$ is equivalent to Zorn's Lemma.*

Proposition 5.3.8. *Let $\mathcal{P}^{(\infty)} := \{p_j^{(\infty)}\}_{j=1}^{\infty}$ any maximal compatible expansion of $\mathcal{P}_0$. The strong limit $\sum_{j=1}^{\infty} p_j^{(\infty)}$ exists and is a sub-projection of p.*

Proof. Since the projections $\{p_j^{(\infty)}\}_{j=1}^{\infty}$ are pairwise orthogonal, $\{\sum_{j=1}^{k} p_j^{(\infty)}\}_{k=1}^{\infty}$ is an increasing sequence of projections which, by Theorem 4.1.14, to a projection in $\mathcal{A}$, which we denote by $p' := \sum_{j=1}^{\infty} p_j^{(\infty)}$. We show that p' is a sub-projection of p. The assumption $p \in \mathcal{A} \cap \mathcal{A}'$ establishes $pp' = p'p$. For any $u \in \mathcal{H}$, and any $k \in \mathbb{N}$, since each p_j^{∞} is a sub-projection of p

$$
\begin{aligned}
& \|pp'u - p'u\| \\
= \;& \|pp'u - \left(\sum_{j=1}^{k} p_j^{(\infty)}\right) u + \left(\sum_{j=1}^{k} p_j^{(\infty)}\right) u - p'u\| \\
\leq \;& \|pp'u - p\left(\sum_{j=1}^{k} p_j^{(\infty)}\right) u\| + \|\left(\sum_{j=1}^{k} p_j^{(\infty)}\right) u - p'u\| \\
\leq \;& \|p\|_{\mathcal{H}} \|p'u - \left(\sum_{j=1}^{k} p_j^{(\infty)}\right) u\| + \|\left(\sum_{j=1}^{k} p_j^{(\infty)}\right) u - p'u\| \\
\to \;& 0
\end{aligned}
$$

as $k \to \infty$ while the left is independent of k. Hence $pp' = p'p = p'$. □

Proposition 5.3.9. *Let $\mathcal{P}^{(\infty)} := \{p_j^{(\infty)}\}_{j=1}^{\infty}$ any maximal compatible expansion of $\mathcal{P}_0$. Then*

$$\varphi\left(\sum_{j=1}^{\infty} p_j^{(\infty)}\right) = \varphi(p),$$

(where the limit of the series is the strong limit).

Proof. By the previous proposition the strong limit $p' := \sum_{j=1}^{\infty} p_j^{(\infty)}$ exists and is a sub-projection of p. Therefore $p_0 := (p - p')$ is also a sub-projection of p and

$$\varphi(p_0) = \varphi(p - p') = \varphi((p - p')^*(p - p')) \geq 0$$

so that $\varphi(p) \geq \varphi(p')$.

Suppose that $\varphi(p_0) = \varphi(p - p') > 0$. By Corollary 5.2.6, there exists an $n \in \mathbb{N}$ such that $\varphi(p_0 \tau^n(p) p_0) > 0$. Let n_0 be the least such n. The assumption that $\tau^j(p) \in \mathcal{A} \cap \mathcal{A}'$ and the fact that p' is a sub-projection of p then implies

$$\begin{array}{rcl} 0 & < & \varphi(p_0 \tau^{n_0}(p) p_0) \\ & = & \varphi(\tau^{n_0}(p)(p - p')) \\ & = & \varphi(p \tau^{n_0}(p)(1_{\mathcal{A}} - p')) \\ & = & \varphi(p'_{n_0}) \end{array}$$

where we define $p'_{n_0} := p \tau^{n_0}(p)(1_{\mathcal{A}} - p') \in \mathcal{A}$. It is clear that p'_{n_0} is indeed a projection. Now for any $p_j^{(\infty)} \in \{p_j^{(\infty)}\}_{j=1}^{\infty}$, the fact that $p_j^{(\infty)}$ is a sub-projection of p' establishes

$$\begin{array}{rcl} p'_{n_0} p_j^{(\infty)} & = & p \tau^{n_0}(p)(1_{\mathcal{A}} - p') p_j^{(\infty)} \\ & = & p \tau^{n_0}(p)(p_j^{(\infty)} - p' p_j^{(\infty)}) \\ & = & p \tau^{n_0}(p)(p_j^{(\infty)} - p_j^{(\infty)}) \\ & = & 0, \end{array}$$

and since both $p, \tau^{n_0}(p) \in \mathcal{A} \cap \mathcal{A}'$

$$\begin{array}{rcl} p_j^{(\infty)} p'_{n_0} & = & p_j^{(\infty)} p \tau^{n_0}(p)(1_{\mathcal{A}} - p') \\ & = & p \tau^{n_0}(p)(p_j^{(\infty)} - p_j^{(\infty)} p') \\ & = & p \tau^{n_0}(p)(p_j^{(\infty)} - p_j^{(\infty)}) \\ & = & 0, \end{array}$$

so that p'_{n_0} is orthogonal to every $p_j^{(\infty)}$.

We define the collection of projections $\mathcal{P}$ as $\mathcal{P}^{(\infty)}$ where $p_{n_0}^{(\infty)}$ is replaced with with $p_{n_0}^{(\infty)} + p'_{n_0}$. Because $p'_{n_0} p_j^{(\infty)} = 0$ for all $j \in \mathbb{N}$, $\mathcal{P}$ is a collection of mutually orthogonal projections, and is indeed a compatible expansion of $\mathcal{P}^{(\infty)}$.

We now show that $\mathcal{P}$ is a compatible expansion of $\mathcal{P}_0$. For any projection $p_j \in \mathcal{P}_0 \subset \mathcal{A} \cap \mathcal{A}'$, $p'_{n_0} p_j = p_j p'_{n_0}$ and

$$\begin{array}{rcl} p'_{n_0} p_j & = & p \tau^{n_0}(p)(1_{\mathcal{A}} - p') p_j \\ & = & p \tau^{n_0}(p)(p_j - p' p_j) \\ & = & p \tau^{n_0}(p)(p_j - p_j) \\ & = & 0, \end{array}$$

which then implies that

$$(p_{n_0}^{(\infty)} + p'_{n_0}) p_j = p_{n_0}^{(\infty)} p_j = p_j p_{n_0}^{(\infty)} = \delta_{j n_0} p_j$$

and for any $j' \in \mathbb{N}$ with $j' \neq n_0$, by definition of $\mathcal{P}^{(\infty)}$,

$$p_{j'}^{(\infty)} p_j = p_j p_{j'}^{(\infty)} = \delta_{j j'} p_j.$$

Therefore $\mathcal{P}$ is a compatible expansion of both $\mathcal{P}_0$ and $\mathcal{P}^{(\infty)}$.

But since $\varphi(p'_{n_0}) > 0$ we cannot have that $\mathcal{P} = \mathcal{P}^{(\infty)}$, because

$$\varphi(p_{n_0}^{(\infty)}) \neq \varphi(p_{n_0}^{(\infty)} + p'_{n_0}).$$

Hence $\mathcal{P}$ is strictly greater than $\mathcal{P}^{(\infty)}$ as a compatible expansion of $\mathcal{P}_0$. But this is absurd, because $\mathcal{P}^{(\infty)}$ was assumed to be a maximal compatible expansion of $\mathcal{P}_0$.

We therefore conclude that $\varphi(p_0) = \varphi(p - p') = 0$, and hence

$$\varphi\left(\sum_{j=1}^{\infty} p_j^{(\infty)}\right) = \varphi(p') = \varphi(p).$$

□

Remark 5.3.10. The result above is not unexpected. For a C*-dynamical system, $(\mathcal{A}, \varphi, \tau)$, constructed from a classical measure preserving dynamical system, (X, Σ, μ, T), the Poincaré Recurrence Theorem immediately implies that $\mathcal{P}_0$, as constructed above, is already a maximal compatible expansion of itself.

5.4 Generalized Induced Transformations

In this section our aim is to construct generalized induced transformations, which are consistent with classical induced transformations. We continue under the assumptions made in the previous section.

We will assume that $\mathcal{P}_0 := \{p_j\}_{j=1}^{\infty}$ is a maximal compatible expansion of itself. This assumption is not unreasonable, as this is the case classically, and for the example in Section 6.2. We define $p_0 := p - \sum_{j=1}^{\infty} p_j$, where the limit of the series is the strong limit. As mentioned in the previous section p_0 is a sub-projection of p, and then $\sum_{j=0}^{\infty} p_j = p$ (strongly).

We can prove the following result which is suspiciously similar to the classical result, Lemma 3.2.7.

Lemma 5.4.1. *With $\{p_j\}_{j=1}^{\infty}$ and $\{q_j\}_{j=1}^{\infty}$ as defined in Section 5.3,*

$$\tau(q_j) = p_{j+1} + q_{j+1}$$

for all $j \in \mathbb{N} \cup \{0\}$ when we take $q_0 := p$.

Proof. The first equality follows easily:

$$\tau(p) = \tau(p)(1_{\mathcal{A}} - p) + \tau(p)p = q_1 + p_1.$$

We will prove the second by strong induction. Suppose for some $k_0 \in \mathbb{N}$,

$\tau(q_k) = q_{k+1} + p_{k+1}$ for all $k \leq k_0$. Then, since τ is a *-homomorphism,

$$\begin{aligned}
\tau(q_{k_0+1}) &= \tau\left(\tau^{k_0+1}(p)(1_{\mathcal{A}} - p - \sum_{k=1}^{k_0} q_k)\right) \\
&= \tau^{k_0+2}(p)(\tau(1_{\mathcal{A}}) - \tau(p) - \sum_{k=1}^{k_0} q_{k+1} - \sum_{k=1}^{k_0} p_{k+1}) \\
&= \tau^{k_0+2}(p)(1_{\mathcal{A}} - \tau(p)p - \tau(p)(1_{\mathcal{A}} - p) - \sum_{k=2}^{k_0+1} q_k - \sum_{k=2}^{k_0+1} p_k) \\
&= \tau^{k_0+2}(p)(1_{\mathcal{A}} - p_1 - q_1 - \sum_{k=2}^{k_0+1} q_k - \sum_{k=2}^{k_0+1} p_k) \\
&= \tau^{k_0+2}(p)(1_{\mathcal{A}} - p - \sum_{k=1}^{k_0+1} q_k + p - \sum_{k=1}^{k_0+1} p_k) \\
&= \tau^{k_0+2}(p)(1_{\mathcal{A}} - p - \sum_{k=1}^{k_0+1} q_k) + \tau^{k_0+2}(p)(p - \sum_{k=1}^{k_0+1} p_k) \\
&= q_{k_0+2} + p_{k_0+2}.
\end{aligned}$$

Hence the relation holds for $k_0 + 1$ if it holds for all $k \leq k_0$, The required result follows. □

Remark 5.4.2. The previous result is remarkable. For every $a \in p\mathcal{A}p$, since τ is state preserving and φ is linear, it allows us to write

$$\begin{aligned}
& \varphi(a) \\
&= \varphi(ap) \\
&= \varphi(\tau(ap)) \\
&= \varphi(\tau(a)\tau(p)) \\
&= \varphi(\tau(a)(p_1 + q_1)) \\
&= \varphi(\tau(a)p_1) + \varphi(\tau(a)q_1) \\
&= \varphi(\tau(a)p_1) + \varphi(\tau^2(a)\tau(q_1)) \\
&\vdots \\
&= \varphi\left(\sum_{j=1}^{k} p_j \tau^j(a)\right) + \varphi(\tau^k(a)q_k)
\end{aligned}$$

for every $k \in \mathbb{N}$. Therefore if we can show that for every $a \in p\mathcal{A}p$ the sequence $\{\sum_{j=1}^{k} p_j \tau^j(a)\}_{k=1}^{\infty}$ converges in some sense in $p\mathcal{A}p$, while $\{\varphi(\tau^k(a)q_k)\}_{k=1}^{\infty}$ converges to zero, we are close to establishing that the map $a \mapsto \sum_{j=1}^{\infty} p_j \tau^j(a)$ preserves φ, as it happens classically. Comparing the proof of Proposition 3.2.8 with this remark, might give the observant reader an eerie sense of déjà vu.

Proposition 5.4.3. *For any (fixed) $a \in p\mathcal{A}p$ the sequence $\{\sum_{j=1}^{k} p_j \tau^j(a)\}_{k=1}^{\infty}$ converges strongly in $p\mathcal{A}p$.*

Proof. Since $\{\sum_{j=0}^{k} p_j\}_{k=1}^{\infty}$ converges strongly to p, we have that the tail $\{\sum_{j=k}^{\infty} p_j\}_{k=1}^{\infty}$ converges strongly to zero.

Let $u \in \mathcal{H}$ and $a \in p\mathcal{A}p$ be fixed but arbitrary. For any $m, n \in \mathbb{N}$ with $m > n$, since τ is a *-homomorphism $\|\tau^j(a)\| \leq \|a\|$ for all $j \in \mathbb{N}$ by Proposition 4.1.8. We remember that $\{p_j\}_{j=1}^{\infty} \subset \mathcal{A}' \cap \mathcal{A}$ and is pairwise orthogonal, so by the Pythagorean theorem,

$$\begin{aligned}
&\left\|\sum_{j=0}^{m} p_j \tau^j(a)u - \sum_{j=0}^{n} p_j \tau^j(a)u\right\|^2 \\
=& \left\|\sum_{j=n+1}^{m} \tau^j(a)p_j u\right\|^2 \\
=& \sum_{j=n+1}^{m} \left\|\tau^j(a)p_j u\right\|^2 \\
\leq& \sum_{j=n+1}^{m} \|\tau^j(a)\|^2 \|p_j u\|^2 \\
\leq& \|a\|^2 \sum_{j=n+1}^{m} \|p_j u\|^2 \\
\leq& \|a\|^2 \left\|\sum_{j=n+1}^{\infty} p_j u\right\|^2
\end{aligned}$$

hence choosing n 'large enough' $\left\|\sum_{j=n+1}^{\infty} p_j u\right\|^2$ can be made arbitrarily small. We conclude that $\{\sum_{j=0}^{k} p_j \tau^j(a)u\}_{k=1}^{\infty} \subset \mathcal{H}$ is a Cauchy sequence and converges since $\mathcal{H}$ is complete. Therefore, for each $a \in p\mathcal{A}p$, the sequence

$$\{\sum_{j=0}^{k} p_j \tau^j(a)\}_{k=1}^{\infty} \subset p\mathcal{A}p$$

converges strongly. Since $p\mathcal{A}p$ is a von Neumann algebra, it is strongly closed, hence the strong limit of the mentioned sequence lies in $p\mathcal{A}p$. $\square$

We define the map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$ by

$$\tau_p(a) := \sum_{j=0}^{\infty} p_j \tau^j(a),$$

where the limit is the strong limit, which exists by the previous result.

Proposition 5.4.4. *The map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$, as defined above, is linear.*

Proof. The a finite linear combination of strongly convergent nets converges strongly to the corresponding finite linear combination of their strong limits. $\square$

Proposition 5.4.5. *For every $a \in p\mathcal{A}p$ the sequence $\left\{\left\|\sum_{j=0}^{k} p_j \tau^j(a)\right\|\right\}_{k=1}^{\infty}$ is bounded (by $\|a\|$).*

Proof. Let $u \in \mathcal{H}$ and $k \in \mathbb{N}$ be arbitrary. By the Pythagorean theorem, since $\{p_j\}_{j=0}^{k} \subset \mathcal{A}' \cap \mathcal{A}$ is pairwise orthogonal and $\sum_{j=0}^{\infty} p_j = p$ (strongly)

$$\begin{aligned}
&\left\|\sum_{j=0}^{k} p_j \tau^j(a) u\right\|^2 \\
= &\sum_{j=0}^{k} \left\|\tau^j(a) p_j u\right\|^2 \\
\leq &\sum_{j=0}^{k} \|\tau^j(a)\|^2 \, \|p_j u\|^2 \\
\leq &\sum_{j=0}^{k} \|a\|^2 \, \|p_j u\|^2 \\
= &\|a\|^2 \sum_{j=0}^{k} \|p_j u\|^2 \\
\leq &\|a\|^2 \sum_{j=0}^{\infty} \|p_j u\|^2 \\
= &\|a\|^2 \left\|\sum_{j=0}^{\infty} p_j u\right\|^2 \\
= &\|a\|^2 \, \|pu\|^2 \\
\leq &\|a\|^2 \|p\|^2 \|u\|^2 \\
\leq &\|a\|^2 \|u\|^2.
\end{aligned}$$

Taking the supremum over all such u that $\|u\| = 1$ on both sides above, establishes the result. $\square$

Corollary 5.4.6. *The map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$, as defined above, is bounded as a linear operator on $p\mathcal{A}p$, and $\|\tau_p\| \leq 1$.*

Proof. Implied by the two previous results. $\square$

Proposition 5.4.7. *The map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$, as defined above, preserves products in $p\mathcal{A}p$.*

Proof. Let $a, b \in p\mathcal{A}p$ and $u \in \mathcal{H}$ be arbitrary. Then for any $k \in \mathbb{N}$, since the $\{p_j\}_{j=0}^{\infty} \subset \mathcal{A}' \cap \mathcal{A}$ are pairwise orthogonal and τ is a *-homomorphism

$$\begin{aligned}
& \|\tau_p(ab)u - \tau_p(a)\tau_p(b)u\| \\
\leq & \left\|\tau_p(ab)u - \sum_{j=0}^{k} p_j \tau^j(ab)u\right\| \\
& + \left\|\sum_{j=0}^{k} p_j \tau^j(ab)u - \sum_{j=0}^{k} p_j \tau^j(a)\tau_p(b)u\right\| \\
& + \left\|\sum_{j=0}^{k} p_j \tau^j(a)\tau_p(b)u - \tau_p(a)\tau_p(b)u\right\| \\
= & \left\|\tau_p(ab)u - \sum_{j=0}^{k} p_j \tau^j(ab)u\right\| \\
& + \left\|\left(\sum_{j=0}^{k} p_j \tau^j(a)\right)\left(\sum_{j'=0}^{k} p_{j'} \tau^{j'}(b)\right)u - \sum_{j=0}^{k} p_j \tau^j(a)\tau_p(b)u\right\| \\
& + \left\|\sum_{j=0}^{k} p_j \tau^j(a)\tau_p(b)u - \tau_p(a)\tau_p(b)u\right\| \\
\leq & \left\|\tau_p(ab)u - \sum_{j=0}^{k} p_j \tau^j(ab)u\right\| \\
& + \left\|\sum_{j=0}^{k} p_j \tau^j(a)\right\| \left\|\sum_{j'=0}^{k} p_{j'} \tau^{j'}(b)u - \tau_p(b)u\right\| \\
& + \left\|\left(\sum_{j=0}^{k} p_j \tau^j(a) - \tau_p(a)\right)\tau_p(b)u\right\| \\
\to & \ 0
\end{aligned}$$

with increasing k, because $\{\sum_{j=0}^{k} p_j \tau^j(a)\}_{k=1}^{\infty}$ converges strongly in $p\mathcal{A}p$ and the sequence $\{\left\|\sum_{j=0}^{k} p_j \tau^j(a)\right\|\}_{k=1}^{\infty}$ is bounded by Proposition 5.4.5. But the left is independent of k, hence we conclude that $\tau_p(ab) = \tau_p(a)\tau_p(b)$. $\square$

We cannot conclude by a similar argument as in the previous result that τ_p also preserves involutions, since involutions are not necessarily strongly continuous, as is illustrated by the example in [10, p. 113]. We can however obtain this result through other means.

Lemma 5.4.8. *The map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$, as defined above, preserves projections in $p\mathcal{A}p$.*

Proof. Because r is a projection, $\{p_j\}_{j=0}^{\infty} \subset \mathcal{A}' \cap \mathcal{A}$ is a collection of pairwise orthogonal projections and τ is a *-homomorphism, we see for each j that

$$(p_j\tau^j(r))(p_j\tau^j(r)) = p_j^2\tau^j(r^2) = p_j\tau^j(r)$$

and

$$(p_j\tau^j(r))^* = \tau^j(r^*)p_j^* = p_j\tau^j(r),$$

so that each $p_j\tau^j(r)$ is a projection. By the pairwise orthogonality of the collection $\{p_j\}_{j=0}^{\infty}$ and the fact that each p_j lies in the commutant of $\mathcal{A}$ it is easy to conclude that the sequence

$$\{\sum_{j=0}^{k} p_j\tau^j(r)\}_{k=1}^{\infty}$$

is an increasing sequence of projections, which converges strongly to a projection by Theorem 4.1.14. By definition, $\tau_p(r)$ is equal to this projection. $\square$

Proposition 5.4.9. *The map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$, as defined above, preserves involutions in $p\mathcal{A}p$.*

Proof. We first show that the result holds for finite linear combinations of projections. Let $\{d_j\}_{j=1}^{n} \subset \mathbb{C}$ be arbitrary and let $\{r_j\}_{j=1}^{n} \subset p\mathcal{A}p$ be arbitrary projections. Then by linearity of τ_p, conjugate linearity of the involution and the previous lemma,

$$\begin{aligned}
\tau_p\left(\left(\sum_{j=1}^{n} d_j r_j\right)^*\right) &= \sum_{j=1}^{n} \overline{d_j}\tau_p\left(r_j^*\right) \\
&= \sum_{j=1}^{n} \overline{d_j}\tau_p\left(r_j\right) \\
&= \sum_{j=1}^{n} \overline{d_j}\tau_p\left(r_j\right)^* \\
&= \left(\sum_{j=1}^{n} d_j\tau_p\left(r_j\right)\right)^* \\
&= \left(\tau_p\left(\sum_{j=1}^{n} d_j r_j\right)\right)^*.
\end{aligned}$$

The linear span of projections in $p\mathcal{A}p$ is norm-dense in $p\mathcal{A}p$ by Theorem 4.1.17. Therefore for any $a \in p\mathcal{A}p$ there exists a net $\{s_\lambda\} \subset p\mathcal{A}p$ of finite linear combinations in $p\mathcal{A}p$ converging (in norm) to a. The linear map τ_p is bounded by Corollary 5.4.6, and the involution is an isometry, so for any λ

$$\begin{array}{ll} & \|\tau_p(a^*) - \tau_p(a)^*\| \\ \leq & \|\tau_p(a^*) - \tau_p(s_\lambda^*)\| + \|\tau_p(s_\lambda^*) - \tau_p(a)^*\| \\ = & \|\tau_p(a^* - s_\lambda^*)\| + \|\tau_p(s_\lambda)^* - \tau_p(a)^*\| \\ \leq & \|a^* - s_\lambda^*\| + \|\tau_p(s_\lambda) - \tau_p(a)\| \\ \leq & 2\|a - s_\lambda\| \\ \to & 0 \end{array}$$

with increasing λ, while the left is independent of λ. Therefore $\tau_p(a^*) = \tau_p(a)^*$. □

Corollary 5.4.10. *The map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$, as defined above, is a *-homomorphism.*

Proof. Follows from the linearity, and preservation of products and involutions, all shown above. □

Remark 5.4.11. Comparing Corollary 5.4.6 together with the previous result, is reassuring as to the correctness of our argument, when considered in light of Proposition 4.1.8.

We will now set out to prove that τ_p is preserves φ.

We we first prove the following:

Lemma 5.4.12. *Let $\{p_\lambda\} \subset \mathcal{A}$ be an increasing net of projections (in $\mathcal{B}(\mathcal{H})$) converging strongly to $p \in \mathcal{A}$, then $\varphi(p_\lambda) \to \varphi(p)$.*

Proof. Since $\{p_\lambda\}$ is increasing, $p - p_\lambda$ is a projection. Therefore $\varphi(p - p_\lambda) \geq 0$ and by Remark 4.1.12,

$$\begin{array}{rcl} \varphi(p - p_\lambda) & = & \varphi((p - p_\lambda)^*(p - p_\lambda)) \\ & = & \|\iota(p - p_\lambda)\|^2 \\ & = & \|\iota(p1_\mathcal{A}) - \iota(p_\lambda 1_\mathcal{A})\|^2 \\ & = & \|p\iota(1_\mathcal{A}) - p_\lambda\iota(1_\mathcal{A})\|^2 \\ & \to & 0 \end{array}$$

with increasing λ, since $\{p_\lambda\}$ converges strongly to p. Therefore $\varphi(p_\lambda) \to \varphi(p)$. □

Corollary 5.4.13. *Let $\{p_j\} \subset \mathcal{A}$ be a collection of pairwise orthogonal projections then $\sum_{j=1}^\infty \varphi(p_j) = \varphi\left(\sum_{j=1}^\infty p_j\right)$ where $\sum_{j=1}^\infty p_j$ denotes the strong limit.*

Proof. The sequence $\{\sum_{j=1}^k p_j\}$ is an increasing sequence of projections, therefore has a strong limit, denoted $\sum_{j=1}^\infty p_j$, by Theorem 4.1.14. Hence, by the previous proposition, $\sum_{j=1}^k \varphi(p_j) = \varphi\left(\sum_{j=1}^k p_j\right) \to \varphi\left(\sum_{j=1}^\infty p_j\right)$ as $k \to \infty$. Hence $\sum_{j=1}^\infty \varphi(p_j) = \varphi\left(\sum_{j=1}^\infty p_j\right)$. □

Lemma 5.4.14. *The map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$, as defined above, preserves the φ-states of projections in $p\mathcal{A}p$.*

Proof. Let $r \in p\mathcal{A}p$ be any projection. Since the collection of projections $\{q_j\}_{j=1}^{\infty}$ are pairwise orthogonal and lie in the commutant of $\mathcal{A}$, the sequence $\{q_j\tau^j(r)\}_{k=1}^{\infty}$ is a sequence of pairwise orthogonal projections. By the previous corollary,

$$\sum_{j=1}^{\infty}\varphi(q_j\tau^j(r)) = \varphi\left(\sum_{j=1}^{\infty} q_j\tau^j(r)\right) < \infty,$$

Because, $\varphi(q_j\tau^j(r)) \geq 0$ since projections are positive, we must have that $\varphi(q_j\tau^j(r)) \to 0$ with increasing j.

Now, by the statements made in Remark 5.4.2, for every $k \in \mathbb{N}$

$$\varphi(r) = \varphi\left(\sum_{j=1}^{k} p_j\tau^j(r)\right) + \varphi(q_k\tau^k(r)).$$

Taking the limit as $k \to \infty$ on both sides above we see, by the previous corollary, that

$$\begin{aligned}
\varphi(r) &= \lim_{k\to\infty}\varphi\left(\sum_{j=1}^{k} p_j\tau^j(r)\right) + \lim_{k\to\infty}\varphi(q_k\tau^k(r)) \\
&= \lim_{k\to\infty}\sum_{j=1}^{k}\varphi(p_j\tau^j(r)) + 0 \\
&= \sum_{j=1}^{\infty}\varphi(p_j\tau^j(r)) \\
&= \varphi\left(\sum_{j=1}^{\infty} p_j\tau^j(r)\right) \\
&= \varphi\circ\tau_p(r).
\end{aligned}$$

□

Proposition 5.4.15. *The map $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$, as defined above, preserves φ.*

Proof. We first prove the result for arbitrary finite linear combinations of projections in $p\mathcal{A}p$. Let $\{r_j\}_{j=1}^{n} \subset p\mathcal{A}p$ be arbitrary projections, and $\{d_j\}_{j=1}^{n} \subset \mathbb{C}$ arbitrary. Then by linearity of τ_p and φ and the previous lemma

$$\varphi\circ\tau_p\left(\sum_{j=1}^{n} d_jr_j\right) = \sum_{j=1}^{n} d_j\varphi\circ\tau_p(r_j) = \sum_{j=1}^{n} d_j\varphi(r_j) = \varphi\left(\sum_{j=1}^{n} d_jr_j\right).$$

The linear combinations of projections in $p\mathcal{A}p$ is norm dense in $p\mathcal{A}p$, by Theorem 4.1.17. Let $a \in p\mathcal{A}p$ be arbitrary and $\{s_\lambda\}$ a net of finite linear combinations of projections in $p\mathcal{A}p$ such that $s_\lambda \to a$. By what was established in the previous paragraph, we see that for every λ, $\varphi \circ \tau_p(s_\lambda) = \varphi(s_\lambda)$. But both φ and τ_p are bounded, hence norm-continuous, therefore

$$\varphi \circ \tau_p(a) = \varphi \circ \tau_p(\lim_\lambda s_\lambda) = \lim_\lambda \varphi \circ \tau_p(s_\lambda) = \lim_\lambda \varphi(s_\lambda) = \varphi(\lim_\lambda s_\lambda) = \varphi(a).$$

□

Remark 5.4.16. In light of what was proved in this section, we define $\mathcal{A}_p := p\mathcal{A}p$ and $\varphi_p : \mathcal{A}_p \to \mathcal{A}_p$ by $\varphi_p(a) := \varphi(a)/\varphi(p)$, and can then conclude that $(\mathcal{A}_p, \varphi_p, \tau_p)$ is a C*-dynamical system!

Comparing the results of this chapter thus far, with those in Section 3.2, it doesn't take much imagination, to see that what we have achieved is a consistent generalization of induced transformations to this setting.

Definition 5.4.17. We will call the C*-dynamical system $(\mathcal{A}_p, \varphi_p, \tau_p)$ *the system induced from* $(\mathcal{A}, \varphi, \tau)$ *onto* p, or *just* the *induced system*. We will call the act of constructing $(\mathcal{A}_p, \varphi_p, \tau_p)$, *inducing onto* p.

5.5 Inherited Ergodicity of Generalized Induced Transformations

In this section we show that as in the classical case, ergodicity of a C*-dynamical system implies the ergodicity of an induced C*-dynamical system.

We continue under the assumptions imposed on $(\mathcal{A}, \varphi, \tau)$ in the previous two sections and will make the additional assumption that τ is strongly continuous, i.e. that when $a_\lambda \to a$ converges strongly in the von Neumann algebra $\mathcal{A}$, implies $\tau(a_\lambda) \to \tau(a)$ converges strongly.

Remark 5.5.1. The assumption of τ's strong continuity is not unreasonable. For a few classical systems (X, Σ, μ, T) this is indeed true. We define the Perron-Frobenius operator $\mathcal{L}$ associated with T by

$$(\mathcal{L}g)(x) := \sum_{y \in T^{-1}\{x\}} \frac{1}{|T'(y)|} g(y)$$

for any map $g : X \to \mathbb{C}$, where $|T'(y)|$ denotes the Jacobian determinant of T evaluated at y. If $\mathcal{L}$ is such that $\mathcal{L}(L^2(X, \Sigma, \mu)) \subseteq L^2(X, \Sigma, \mu)$, then $\tau : L^\infty(X, \Sigma, \mu) \to L^\infty(X, \Sigma, \mu)$ defined by $\tau(f) := f \circ T$ is strongly continuous. This can easily be seen by considering any net $f_\lambda \to f$ converging strongly as multiplication operators in $L^\infty(X, \Sigma, \mu)$ then for any $u \in L^2(X, \Sigma, \mu)$, by administering the change of variables $z = Tx$,

$$\begin{aligned}
& \|\tau(f_\lambda)u - \tau(f)u\|_2^2 \\
= & \int |f_\lambda \circ T(x)u(x) - f \circ T(x)u(x)|^2 d\mu(x) \\
= & \int |f_\lambda(z)(\mathcal{L}u)(z) - f(z)(\mathcal{L}u)(z)|^2 d\mu(z) \\
= & \|f_\lambda(\mathcal{L}u) - f(\mathcal{L}u)\|_2^2 \\
\to & \ 0.
\end{aligned}$$

It is clear that $\mathcal{L}(L^2(X,\Sigma,\mu)) \subseteq L^2(X,\Sigma,\mu)$ indeed happens for (the) classical measure preserving dynamical systems (like) irrational rotations, the Baker's map and β-transformations.

Similar to the definition of $\tau_p : p\mathcal{A}p \to p\mathcal{A}p$ in the previous section we can define $\rho_p : p\mathcal{A}p \to (1_\mathcal{A} - p)\mathcal{A}(1_\mathcal{A} - p)$ by

$$\rho_p(a) := \sum_{j=1}^{\infty} q_j \tau^j(a)$$

where the limit is again taken to be the strong limit. The existence of this limit follows by the same argument that established the values of τ_p as such strong limits. The linearity and boundedness, hence continuity of ρ_p, also follow exactly as it did for τ_p.

We establish the following

Lemma 5.5.2. *The following equality holds for all $a \in p\mathcal{A}p$*

$$\tau \circ \rho_p(a) = \rho_p(a) + \tau_p(a) - (p_1 + q_1)\tau(a).$$

Proof. Let $a \in p\mathcal{A}p$, $u \in \mathcal{H}$ and $k \in \mathbb{N}$ be arbitrary, then by Lemma 5.4.1 and the assumed strong continuity of τ,

$$\begin{aligned}
& \|\tau \circ \rho_p(a)u - \rho_p(a)u - \tau_p(a)u + (p_1 + q_1)\tau(a)u\| \\
= & \|\tau \circ \rho_p(a)u - \tau\left(\sum_{j=1}^{k} q_j \tau^j(a)\right) u\| \\
& + \|\tau\left(\sum_{j=1}^{k} q_j \tau^j(a)\right) u - \rho_p(a)u - \tau_p(a)u + (p_1 + q_1)\tau(a)u\| \\
= & \|\tau \circ \rho_p(a)u - \tau\left(\sum_{j=1}^{k} q_j \tau^j(a)\right) u\| \\
& + \|\sum_{j=1}^{k} q_{j+1}\tau^{j+1}(a)u + \sum_{j=1}^{k} p_{j+1}\tau^{j+1}(a)u - \rho_p(a)u - \tau_p(a)u + (p_1 + q_1)\tau(a)u\| \\
\leq & \|\tau \circ \rho_p(a)u - \tau\left(\sum_{j=1}^{k} q_j \tau^j(a)\right) u\| \\
& + \|\sum_{j=1}^{k+1} q_j \tau^j(a)u - \rho_p(a)u\| \\
& + \|\sum_{j=1}^{k+1} p_j \tau^j(a)u - \tau_p(a)u\| \\
\to & \ 0
\end{aligned}$$

with increasing k, while the left is independent of k. We conclude $\tau \circ \rho_p(a) = \rho_p(a) + \tau_p(a) - (p_1 + q_1)\tau(a)$. □

The inclusion $\iota : \mathcal{A} \to \mathcal{H}$ defined in the GNS-construction (see Remark 4.1.12), is a bounded linear map.

We define the following operators $U : \iota(\mathcal{A}) \to \mathcal{H}$, $U_p : \iota(p\mathcal{A}p) \to p\mathcal{H}$ and $R_p : \iota(p\mathcal{A}p) \to (1_{\mathcal{A}} - p)\mathcal{H}$ by

$$\begin{aligned}
U\iota(a) &:= \iota(\tau(a)) \\
U_p\iota(a) &:= \iota(\tau_p(a)) \\
R_p\iota(a) &:= \iota(\rho_p(a)).
\end{aligned}$$

Since the sets $\iota(\mathcal{A})$, $\iota(p\mathcal{A}p)$ are norm-dense in $\mathcal{H}$ and $p\mathcal{H}$ respectively, and the operators are bounded and densely defined, they can be uniquely extended to bounded linear operators (denoted by the same symbols) $U : \mathcal{H} \to \mathcal{H}$, $U_p : p\mathcal{H} \to p\mathcal{H}$ and $R_p : p\mathcal{H} \to (1_{\mathcal{A}} - p)\mathcal{H}$ [7, p. 100].

We can prove the following

Corollary 5.5.3. *For any $u \in p\mathcal{H}$, $UR_pu = R_pu + U_pu - (p_1 + q_1)Uu$*

Proof. The result follows from the previous lemma, and the definitions above. □

Lemma 5.5.4. *For any $u \in p\mathcal{H}$, $Uu = (p_1 + q_1)Uu$*

Proof. Let $u \in p\mathcal{H}$ be arbitrary, then since $\iota(p\mathcal{A}p)$ is norm dense in $p\mathcal{H}$, there exists a net $\{a_\lambda\} \subset p\mathcal{A}p$ such that $\iota(a_\lambda) \to u$ in norm. Since $\{a_\lambda\} \subset p\mathcal{A}p$, $pa_\lambda = a_\lambda$, by Lemma 5.4.1, $\tau(p) = p_1 + q_1$.

Then for any λ, because τ is a *-homomorphism to boot,

$$\begin{aligned}
& \|Uu - (p_1 + q_1)Uu\| \\
= \; & \|Uu - U\iota(a_\lambda) + U\iota(a_\lambda) - (p_1 + q_1)Uu\| \\
\leq \; & \|Uu - U\iota(a_\lambda)\| + \|\iota(\tau(a_\lambda)) - (p_1 + q_1)Uu\| \\
\leq \; & \|U\|\|u - \iota(a_\lambda)\| + \|\iota(\tau(pa_\lambda)) - (p_1 + q_1)Uu\| \\
= \; & \|U\|\|u - \iota(a_\lambda)\| + \|\iota(\tau(p)\tau(a_\lambda)) - (p_1 + q_1)Uu\| \\
= \; & \|U\|\|u - \iota(a_\lambda)\| + \|\iota((p_1 + q_1)\tau(a_\lambda)) - (p_1 + q_1)Uu\| \\
= \; & \|U\|\|u - \iota(a_\lambda)\| + \|(p_1 + q_1)\iota(\tau(a_\lambda)) - (p_1 + q_1)Uu\| \\
= \; & \|U\|\|u - \iota(a_\lambda)\| + \|(p_1 + q_1)U\iota(a_\lambda) - (p_1 + q_1)Uu\| \\
\leq \; & \|U\|\|u - \iota(a_\lambda)\| + \|p_1 + q_1\|\|U\|\|\iota(a_\lambda) - u\| \\
\to \; & 0
\end{aligned}$$

with increasing λ, while the left is independent of λ. We conclude $Uu = (p_1 + q_1)Uu$. □

We can now prove the main result of this thesis.

Theorem 5.5.5. *If $(\mathcal{A}, \varphi, \tau)$ is ergodic, then the induced system $(\mathcal{A}_p, \varphi_p, \tau_p)$ is ergodic.*

Proof. By Theorem 4.2.6, it is enough to prove that the fixed point space of the operator $U_p : p\mathcal{H} \to p\mathcal{H}$ is one dimensional, (equal to $\mathbb{C}\iota(p)$). To this end, let $c \in p\mathcal{H}$ be an arbitrary fixed point of U_p.

Then, by the previous two results

$$\begin{aligned}
U(c + R_pc) &= Uc + UR_pc \\
&= (p_1 + q_1)Uc + R_pc + U_pc - (p_1 + q_1)Uc \\
&= U_pc + R_pc \\
&= c + R_pc,
\end{aligned}$$

hence $c + R_pc$ is a fixed point of U.

By Theorem 4.2.6, Since $(\mathcal{A}, \varphi, \tau)$ is ergodic, the operator $U : \mathcal{H} \to \mathcal{H}$ has fixed point space $\mathbb{C}\iota(1_\mathcal{A})$. Hence there exists some $d \in \mathbb{C}$ such that $c + R_pc = d\iota(1_\mathcal{A})$. But $R_pc \in (1_\mathcal{A} - p)\mathcal{H}$, hence $pR_pc = p(1_\mathcal{A} - p)R_pc = (p - p)R_pc = 0$. Also $c \in p\mathcal{H}$ so that $pc = c$. It is then seen that

$$c = pc = p(c + R_pc) = dp\iota(1_\mathcal{A}) = d\iota(p1_\mathcal{A}) = d\iota(p).$$

We conclude that the fixed point space of U_p is one-dimensional and, by Theorem 4.2.6, $(\mathcal{A}_p, \varphi_p, \tau_p)$ is ergodic. $\square$

Generalizing Theorem 3.3.2 requires some more work.

It can easily be seen that the proofs of results 5.4.3 up to and including 5.4.10, go through when $a \in \mathcal{A}$ just as well as when $a \in p\mathcal{A}p$. This allows us to define the *-homomorphism $\overline{\tau_p} : \mathcal{A} \to p\mathcal{A}p$, just as before

$$\overline{\tau_p}(a) := \sum_{j=0}^{\infty} p_j \tau^j(a)$$

where the limit is the strong limit. As before, we define the operator $\overline{U}_p : \mathcal{H} \to p\mathcal{H}$ as the unique bounded linear extension of $\overline{U}_p : \iota(\mathcal{A}) \to \iota(p\mathcal{A}p)$ defined by $\overline{U}_p\iota(a) := \iota(\overline{\tau_p}(a))$.

Lemma 5.5.6. *The equality $\tau_p(pa) = \overline{\tau_p}(a)$ holds for all $a \in \mathcal{A}$ and $U_p p = \overline{U_p}$ holds.*

Proof. Let $a \in \mathcal{A}$ and $u \in \mathcal{H}$ be arbitrary. Since each p_j is a sub-projection of $\tau^j(p)$, by their definition in Section 5.3, for any $k \in \mathbb{N}$, by the definition of the images of $\overline{\tau_p}$ and τ_p as strong limits,

$$\begin{aligned}
& \|\tau_p(pa)u - \overline{\tau_p}(a)u\| \\
\leq \quad & \|\tau_p(pa)u - \sum_{j=0}^{k} p_j\tau^j(pa)u\| + \|\sum_{j=0}^{k} p_j\tau^j(pa)u - \overline{\tau_p}(a)u\| \\
= \quad & \|\tau_p(pa)u - \sum_{j=0}^{k} p_j\tau^j(pa)u\| + \|\sum_{j=0}^{k} p_j\tau^j(p)\tau^j(a)u - \overline{\tau_p}(a)u\| \\
= \quad & \|\tau_p(pa)u - \sum_{j=0}^{k} p_j\tau^j(pa)u\| + \|\sum_{j=0}^{k} p_j\tau^j(a)u - \overline{\tau_p}(a)u\| \\
\to \quad & 0,
\end{aligned}$$

with increasing k, while the left is independent of k. We conclude that $\tau_p(pa) = \overline{\tau_p}(a)$.

Now, for any $u \in \mathcal{H}$, by the density of $\iota(\mathcal{A})$ in $\mathcal{H}$, there exist a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that $\iota(a_n) \to u$ in norm. By the just established equality we see

$$\begin{aligned}
& \|U_p pu - \overline{U_p}u\| \\
\leq \quad & \|U_p pu - U_p p\iota(a_n)\| - \|U_p p\iota(a_n) - \overline{U_p}u\| \\
= \quad & \|U_p pu - U_p p\iota(a_n)\| - \|\iota(\tau_p(pa_n)) - \overline{U_p}u\| \\
= \quad & \|U_p pu - U_p p\iota(a_n)\| - \|\iota(\overline{\tau_p}(a_n)) - \overline{U_p}u\| \\
= \quad & \|U_p pu - U_p p\iota(a_n)\| - \|\overline{U_p}\iota(a_n) - \overline{U_p}u\| \\
\leq \quad & \|U_p p\|\|u - \iota(a_n)\| - \|\overline{U_p}\|\|\iota(a_n) - u\| \\
\to \quad & 0
\end{aligned}$$

with increasing n, while the left is independent of n. We conclude $U_p p = \overline{U_p}$

$\square$

Lemma 5.5.7. *The following equalities* $p_j\overline{\tau_p}(a) = p_j\tau^j(a)$, *for all* $a \in \mathcal{A}$, *and* $p_j\overline{U_p} = p_jU^j$ *hold for all* $j \in \mathbb{N} \cup \{0\}$.

Proof. Let $a \in \mathcal{A}$, $u \in \mathcal{H}$, and $k, j \in \mathbb{N}$ be arbitrary. Then since $\left\{\sum_{j=0}^{k} p_j\tau^j(a)\right\}_{k=1}^{\infty}$ converges strongly to $\overline{\tau_p}(a)$ and the projections in $\{p_j\}_{j=0}^{\infty}$ are pairwise orthogonal, we see for $k > j$

$$
\begin{aligned}
& \|p_j\overline{\tau}_p(a)u - p_j\tau^j(a)u\| \\
= \quad & \|p_j\overline{\tau}_p(a)u - p_j\sum_{j'=0}^{k} p_{j'}\tau^{j'}(a)u\| + \|p_j\sum_{j'=0}^{k} p_{j'}\tau^{j'}(a)u - p_j\tau^j(a)u\| \\
\leq \quad & \|p_j\|\|\overline{\tau}_p(a)u - \sum_{j'=0}^{k} p_{j'}\tau^{j'}(a)u\| + \|p_j\tau^j(a)u - p_j\tau^j(a)u\| \\
\leq \quad & \|\overline{\tau}_p(a)u - \sum_{j'=0}^{k} p_{j'}\tau^{j'}(a)u\| \\
\to \quad & 0
\end{aligned}
$$

with increasing $k > j$, while the left is independent of k. We conclude that $p_j\overline{\tau}_p(a) = p_j\tau^j(a)$, establishing the first equality.

For the second equality, let $a \in \mathcal{A}$ be arbitrary. We first note that by the first equality proved above,

$$p_j\overline{U_p}\iota(a) = p_j\iota(\overline{\tau_p}(a)) = \iota(p_j\overline{\tau_p}(a)) = \iota(p_j\tau^j(a)) = p_j\iota(\tau^j(a)) = p_jU^j\iota(a).$$

Now, for any $u \in \mathcal{H}$, by the density of $\iota(\mathcal{A})$ in $\mathcal{H}$, there exist a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that $\iota(a_n) \to u$ in norm. By the above equality we see that

$$
\begin{aligned}
& \|p_j\overline{U_p}u - p_jU^ju\| \\
\leq \quad & \|p_j\overline{U_p}u - p_j\overline{U_p}\iota(a_n)\| + \|p_j\overline{U_p}\iota(a_n) - p_jU^ju\| \\
= \quad & \|p_j\overline{U_p}u - p_j\overline{U_p}\iota(a_n)\| + \|p_jU^j\iota(a_n) - p_jU^ju\| \\
\leq \quad & \|p_j\|\|\overline{U_p}\|\|u - \iota(a_n)\| + \|p_j\|\|U^j\|\|\iota(a_n) - u\| \\
\to \quad & 0,
\end{aligned}
$$

with increasing n, while the left is independent of n. Hence $p_j\overline{U_p} = p_jU^j$. $\square$

Lemma 5.5.8. *The sequence of operators* $\left\{\sum_{j=0}^{k} p_jU^j\right\}_{k=1}^{\infty}$ *converges strongly to* $\overline{U_p}$.

Proof. Let $u \in \mathcal{H}$, and $k \in \mathbb{N}$ be arbitrary. By the previous lemma, the fact that $p\overline{U_p} = \overline{U_p}$ since $\overline{U_p}$ maps into $p\mathcal{H}$, and since $\left\{\sum_{j=0}^{k} p_j\right\}_{k=1}^{\infty}$ converges strongly to p, we see that

$$\begin{aligned}
& \|\sum_{j=0}^{k} p_j U^j u - \overline{U_p} u\| \\
= & \|\sum_{j=0}^{k} p_j \overline{U_p} u - p\overline{U_p} u\| \\
= & \|\left(\sum_{j=0}^{k} p_j - p\right) \overline{U_p} u\| \\
\to & 0,
\end{aligned}$$

with increasing k, establishes the result. □

Lemma 5.5.9. *The equality $\tau^j(p)U^j = U^j p$ holds for every $j \in \mathbb{N} \cup \{0\}$*

Proof. Let $a \in \mathcal{A}$ and $j \in \mathbb{N}\cup\{0\}$ be arbitrary, then since τ is a *-homomorphism

$$\tau^j(p)U^j\iota(a) = \tau^j(p)\iota(\tau^j(a)) = \iota(\tau^j(p)\tau^j(a)) = \iota(\tau^j(pa)) = U^j p\iota(a).$$

Now, for any $u \in \mathcal{H}$, by the density of $\iota(\mathcal{A})$ in $\mathcal{H}$, there exist a sequence $\{a_n\}_{n=1}^{\infty} \subset \mathcal{A}$ such that $\iota(a_n) \to u$ in norm. By the above equality we see that

$$\begin{aligned}
& \|\tau^j(p)U^j u - U^j pu\| \\
\leq & \|\tau^j(p)U^j u - \tau^j(p)U^j\iota(a_n)\| + \|\tau^j(p)U^j\iota(a_n) - U^j pu\| \\
= & \|\tau^j(p)U^j u - \tau^j(p)U^j\iota(a_n)\| + \|U^j p\iota(a_n) - U^j pu\| \\
\leq & \|\tau^j(p)U^j\|\|u - \iota(a_n)\| + \|U^j\|\|p\|\|\iota(a_n) - u\| \\
\to & 0,
\end{aligned}$$

with increasing n, while the left is independent of n. We conclude $\tau^j(p)U^j = U^j p$. □

Theorem 5.5.10. *If the induced system $(\mathcal{A}_p, \varphi_p, \tau_p)$ is ergodic and $p+\sum_{j=1}^{\infty} q_j = 1_{\mathcal{A}}$ (the limit is the strong limit), then $(\mathcal{A}, \varphi, \tau)$ is ergodic.*

Proof. By Theorem 4.2.6 it is sufficient to prove that the operator U has a one dimensional fixed point space. To this end let $c \in \mathcal{H}$ be an arbitrary fixed point of U, i.e. $Uc = c$. Now by Lemmas 5.5.6 and 5.5.8, and the fact that

$\left\{\sum_{j=0}^{k} p_j\right\}_{k=1}^{\infty}$ converges strongly to p,

$$\begin{aligned}
& \|U_p pc - pc\| \\
\leq\ & \|\overline{U_p}c - \sum_{j=0}^{k} p_j U^j c\| + \|\sum_{j=0}^{k} p_j U^j c - pc\| \\
=\ & \|\left(\overline{U_p} - \sum_{j=0}^{k} p_j U^j\right) c\| + \|\left(\sum_{j=0}^{k} p_j - p\right) c\| \\
\to\ & 0,
\end{aligned}$$

with increasing k, while the left is independent of k. We conclude that $U_p pc = pc$. But $(\mathcal{A}_p, \varphi_p, \tau_p)$ is ergodic by hypothesis and hence by Theorem 4.2.6, U_p has a one dimensional fixed point space equal to $\mathbb{C}\iota(p)$. Hence there exists some $d \in \mathbb{C}$ such that $pc = d\iota(p)$.

For every $j \in \mathbb{N}$ we see that, by definition of q_j, and Lemma 5.5.9,

$$\begin{aligned}
q_j c &= q_j U^j c \\
&= \tau^j(p)(1_{\mathcal{A}} - p - \sum_{k=1}^{j-1} q_k) U^j c \\
&= (1_{\mathcal{A}} - p - \sum_{k=1}^{j-1} q_k)\tau^j(p) U^j c \\
&= (1_{\mathcal{A}} - p - \sum_{k=1}^{j-1} q_k) U^j pc \\
&= (1_{\mathcal{A}} - p - \sum_{k=1}^{j-1} q_k) U^j d\iota(p) \\
&= d(1_{\mathcal{A}} - p - \sum_{k=1}^{j-1} q_k)\iota(\tau^j(p)) \\
&= d\iota(\tau^j(p)(1_{\mathcal{A}} - p - \sum_{k=1}^{j-1} q_k)) \\
&= d\iota(q_j) \\
&= dq_j\iota(1_{\mathcal{A}}).
\end{aligned}$$

We define $q := \sum_{j=1}^{\infty} q_j$ (strong limit), which exists by Theorem 4.1.14. Then

$$\begin{aligned}
& \|qc - dq\iota(1_{\mathcal{A}})\| \\
\leq\ & \|qc - \sum_{j=1}^{k} q_j c\| + \|\sum_{j=1}^{k} q_j c - dq\iota(1_{\mathcal{A}})\| \\
=\ & \|qc - \sum_{j=1}^{k} q_j c\| + \|\sum_{j=1}^{k} dq_j\iota(1_{\mathcal{A}}) - dq\iota(1_{\mathcal{A}})\| \\
=\ & \|\left(q - \sum_{j=1}^{k} q_j\right) c\| + |d| \|\left(\sum_{j=1}^{k} q_j - q\right) \iota(1_{\mathcal{A}})\| \\
\to\ & 0,
\end{aligned}$$

with increasing k, while the left is independent of k. We conclude that $qc = dq\iota(1_{\mathcal{A}}) = d\iota(q)$.

By hypothesis $p + q = p + \sum_{j=1}^{\infty} q_j = 1_{\mathcal{A}}$ (strong limit), so that

$$c = 1_{\mathcal{A}} c = (p+q)c = pc + qc = d\iota(p) + d\iota(q) = d\iota(p+q) = d\iota(1_{\mathcal{A}})$$

and we conclude that the fixed point space of U is one dimensional. And hence that $(\mathcal{A}, \varphi, \tau)$ is ergodic, by Theorem 4.2.6. $\square$

Chapter 6

Illustrative Examples

In this chapter we (attempt to) present concrete examples of non-commutative C*-dynamical systems to which the theory outlined in the previous sections is applicable.

We may note to the reader, that the examples given here illustrate well the relationship between classical return times and the generalized recurrence results presented in Section 5.2 where return times are meaningless. The example presented in Section 6.2 was indeed a great aid to the author in understanding and developing the theory presented in the previous chapter.

To the author's great frustration, he must confess that he was unable to construct an example of an *ergodic* non-commutative C*-dynamical system to which the previous theory can be applied. This remains a great and lamentable shortcoming of this thesis. The author however still remains convinced of the following:

Conjecture 6.0.11. *There exists an ergodic C*-dynamical system $(\mathcal{A}, \varphi, \tau)$, with $\mathcal{A}$ a non-commutative von Neumann algebra (that is not a factor), acting on the Hilbert space obtained from the GNS-construction on $(\mathcal{A}, \varphi)$ and τ a strongly continuous *-homomorphism, to which the theory of the previous sections may be applied.*

The following will present interesting musings encountered along the road to (attempted) proof of the previous conjecture. These musings lie on a strange interface of classical measure preserving dynamical systems and non-commutative C*-dynamical systems – dense enough to sink through the classical and to penetrate the 'denser' theory of the non-commutative, yet still floats (for lack of weight that proof of the previous conjecture would provide).

The most promising example that might provide proof for the above conjecture is given in Section 6.3.

6.1 (Not entirely uninteresting) Preliminaries

Let (X, Σ, μ) be a probability space

Define the algebra

$$\mathcal{A}_1 := \{A : X \to M_2(\mathbb{C}) : A^{(ij)} : X \to \mathbb{C} \text{ measurable; ess sup}_{x\in X}\|A_x\|_2 < \infty\}$$

where for $A, B \in \mathcal{A}$ multiplication is defined by $(AB)_x := A_x B_x$. We will define the quotient algebra $\mathcal{A} := \mathcal{A}_1/\{A \in \mathcal{A}_1 : \text{ess sup}_{x\in X}\|A_x\|_2 = 0\}$.

We note that for every $M \in M_2(\mathbb{C})$, $\operatorname{tr}(M^*M) = \sum_{i,j=1}^{2} |M^{(ij)}|^2$, that $\|M\|_2 = \sqrt{\lambda_{\max}(M^*M)}$. Also that $\operatorname{tr}(M^*M) = \lambda_{\max}(M^*M) + \lambda_{\min}(M^*M)$ (where $\lambda_{\max}(\cdot)$ and $\lambda_{\max}(\cdot)$ respectively denote the maximum and minimum eigenvalues of a positive matrix in $M_2(\mathbb{C})$).

Proposition 6.1.1. *$A \in \mathcal{A}$ if and only if its entries $A^{(ij)} : X \to \mathbb{C}$ lie in $L^\infty(X, \Sigma, \mu)$.*

Proof. Let $A \in \mathcal{A}$, we see that

$$\begin{aligned}
\infty &> \text{ess sup}_{x\in X}\|A_x\|_2^2 \\
&= \text{ess sup}_{x\in X}\lambda_{\max}(A_x^*A_x) \\
&\geq \text{ess sup}_{x\in X}\frac{1}{2}\left(\lambda_{\max}(A_x^*A_x) + \lambda_{\min}(A_x^*A_x)\right) \\
&= \frac{1}{2}\text{ess sup}_{x\in X}\operatorname{tr}(A_x^*A_x) \\
&= \frac{1}{2}\text{ess sup}_{x\in X}\left(\sum_{i,j=1}^{2}|A_x^{(ij)}|^2\right) \\
&\geq \frac{1}{2}\text{ess sup}_{x\in X}|A_x^{(ij)}|^2
\end{aligned}$$

for any $i, j = 1, 2$. Therefore for $i, j = 1, 2$ ess $\sup_{x\in X}|A_x^{(ij)}| < \infty$, and we conclude that the components of A as functions of $x \in X$, lie in $L^\infty(X, \Sigma, \mu)$.

Conversely, let $A^{(ij)} : X \to \mathbb{C}$ lie in $L^\infty(X, \Sigma, \mu)$, for $i, j = 1, 2$, and denote the components of $A : X \to M_2(\mathbb{C})$

$$\begin{aligned}
&\text{ess sup}_{x\in X}\|A_x\|_2^2 \\
&= \text{ess sup}_{x\in X}\lambda_{\max}(A_x^*A_x) \\
&\leq \text{ess sup}_{x\in X}\left(\lambda_{\max}(A_x^*A_x) + \lambda_{\min}(A_x^*A_x)\right) \\
&= \text{ess sup}_{x\in X}\operatorname{tr}(A_x^*A_x) \\
&= \text{ess sup}_{x\in X}\sum_{i,j=1}^{2}|A_x^{(ij)}|^2 \\
&\leq \sum_{i,j=1}^{2}\text{ess sup}_{x\in X}|A_x^{(ij)}|^2 \\
&< \infty.
\end{aligned}$$

Therefore $A \in \mathcal{A}$. $\square$

Proposition 6.1.2. *With the norm $\|A\|_\infty :=$ ess sup$_{x\in X}\|A_x\|_2$, $\mathcal{A}$ is a Banach algebra.*

Proof. We prove that $\mathcal{A}$ is complete. Let $\{A_{(\lambda)}\} \subset \mathcal{A}$ be an arbitrary Cauchy net. Then, given any $\varepsilon > 0$ there exists a λ_0 such that $\lambda_1, \lambda_2 \geq \lambda_0$ implies

$$\varepsilon > \|A_{(\lambda_1)} - A_{(\lambda_2)}\| = \text{ess sup}_{x\in X}\|A_{(\lambda_1),x} - A_{(\lambda_2),x}\|_2.$$

Therefore for a.e. $x \in X$, $\{A_{(\lambda),x}\} \subset M_2(\mathbb{C})$ is a Cauchy net, hence converges, because $M_2(\mathbb{C})$ is complete. We denote this limit by A_x, and define the map $A : X \to M_2(\mathbb{C})$ accordingly, (setting $A_x := 0$ when for the almost no $x \in X$ where $\{A_{(\lambda),x}\}$ is not Cauchy). Hence $\|A_{(\lambda_1),x} - A_x\|_2 < \varepsilon$ for a.e. $x \in X$ when $\lambda_1 \geq \lambda_0$, and therefore

$$\|A_{(\lambda_1)} - A\| = \text{ess sup}_{x\in X}\|A_{(\lambda_1),x} - A_x\|_2 < \varepsilon,$$

and hence $A_{(\lambda)} \to A$. We prove that $A \in \mathcal{A}$. Since $A_{(\lambda)} \in \mathcal{A}$, for every λ, there exists a number $k_\lambda < \infty$ such that ess sup$_{x\in X}\|A_{(\lambda),x}\| \leq k_\lambda$. Hence for a fixed λ, satisfying $\lambda \geq \lambda_0$, for a.e. $x \in X$

$$\|A_x\|_2 \leq \|A_x - A_{(\lambda),x}\|_2 + \|A_{(\lambda),x}\|_2 < \varepsilon + k_\lambda,$$

implies that ess sup$_x\|A_x\|_2 <$ ess sup$_x(\varepsilon + k_\lambda) = \varepsilon + k_\lambda < \infty$.

Clearly for $A, B \in \mathcal{A}$,

$$\begin{aligned}
\|AB\|_\infty &= \text{ess sup}_{x\in X}\|A_xB_x\|_2 \\
&\leq \text{ess sup}_{x\in X}\|A_x\|_2\|B_x\|_2 \\
&\leq \text{ess sup}_{x\in X}\|A_x\|_2 \text{ess sup}_{x\in X}\|B_x\|_2 \\
&= \|A\|_\infty\|B\|_\infty.
\end{aligned}$$

We conclude that $\mathcal{A}$ is a Banach algebra. $\square$

Proposition 6.1.3. *With involution defined by $(A^*)_x := A_x^*$, $\mathcal{A}$ is a C*-algebra.*

Proof. We only need to verify that C*-identity. Let $A \in \mathcal{A}$ be arbitrary, then since $M_2(\mathbb{C})$ with norm $\|\cdot\|_2$ is a C*-algebra

$$\begin{aligned}
\|A^*A\|_\infty &= \text{ess sup}_{x\in X}\|A_x^*A_x\|_2 \\
&= \text{ess sup}_{x\in X}\|A_x\|_2^2 \\
&= (\text{ess sup}_{x\in X}\|A_x\|_2)^2 \\
&= \|A\|_\infty^2.
\end{aligned}$$

$\square$

We define the state $\varphi : \mathcal{A} \to \mathbb{C}$ by $\varphi(A) := \int_X \frac{1}{2}\text{tr}(A_x)d\mu(x)$.

Proposition 6.1.4. *For all $A \in \mathcal{A}$, $\varphi(A^*A) = 0$ if and only if $\|A\|_\infty = 0$.*

Proof. Let $\varphi(A^*A) = 0$, so that

$$0 = \varphi(A^*A) = \int_X \frac{1}{2}\mathrm{tr}(A_x^*A_x)d\mu(x) = \frac{1}{2}\sum_{i,j=1}^{2}\int_X |A_x^{(ij)}|^2 d\mu(x).$$

Therefore $\int_X |A_x^{(ij)}|^2 d\mu(x) = 0$, so that that $A_x^{(ij)} = 0$ implying that $\|A_x\|_2 = 0$ for a.e. $x \in X$. Hence $\|A\|_\infty = \mathrm{ess\,sup}_{x\in X}\|A\|_2 = 0$.

Conversely, let $\|A\|_\infty = 0$. Because $0 = \|A\|_\infty = \mathrm{ess\,sup}_{x\in X}\|A\|_2$, we have that $0 = \|A_x\|_2^2 = \lambda(A_x^*A_x)$ for a.e. $x \in X$. Since $A_x^*A_x$ is positive, its spectrum lies in $[0,\infty)$, and therefore $0 \leq \lambda_{\min}(A_x^*A_x) \leq \lambda_{\max}(A_x^*A_x) = 0$, so that $\mathrm{tr}(A_x^*A_x) = 0$, for a.e. $x \in X$ and therefore

$$\varphi(A^*A) = \int_X \frac{1}{2}\mathrm{tr}(A_x^*A_x)d\mu(x) = 0.$$

□

Corollary 6.1.5. *The ideal $\mathcal{I} := \{A \in \mathcal{A} : \varphi(A^*A) = 0\}$ is a singleton and in particular only contains the zero element of $\mathcal{A}$. Hence the quotient $\mathcal{A}/\mathcal{I}$ which arises in the GNS-construction is (isometrically isomorphic to) $\mathcal{A}$.*

We define the set

$$\mathcal{H}_1 := \{A : X \to M_2(\mathbb{C}) : A^{(ij)} : X \to \mathbb{C} \text{ measurable}; \int_X \frac{1}{2}\mathrm{tr}(A_x^*A_x)d\mu(x) < \infty\}.$$

We define $\varphi : \mathcal{H}_1 \to \mathbb{C}$ by $\varphi(A) := \int_X \frac{1}{2}\mathrm{tr}(A_x)d\mu(x)$. Note that we use the same symbol φ, for the map on $\mathcal{A}$, and $\mathcal{H}_1$. We define the inner product space $\mathcal{H} := \mathcal{H}_1/\{A \in \mathcal{H}_1 : \varphi(A^*A) = 0\}$with inner product $\langle A, B\rangle := \varphi(B^*A)$.

Proposition 6.1.6. *$A \in \mathcal{H}$ if and only if its entries $A^{(ij)} : X \to \mathbb{C}$ lie in $L^2(X, \Sigma, \mu)$.*

Proof. Let $A \in \mathcal{H}$, then

$$\infty > \int_X \frac{1}{2}\mathrm{tr}(A_x^*A_x)d\mu(x) = \frac{1}{2}\sum_{i,j=1}^{2}\int_X |A_x^{(ij)}|^2 d\mu(x),$$

hence $\int_X |A_x^{(ij)}|^2 d\mu(x) < \infty$ for $i, j = 1, 2$. We conclude that the entries of A as functions on X are elements of $L^2(X, \Sigma, \mu)$.

Conversely let $\int_X |A_x^{(ij)}|^2 d\mu(x) < \infty$ for $i, j = 1, 2$, then

$$\int_X \frac{1}{2}\mathrm{tr}(A_x^*A_x)d\mu(x) = \frac{1}{2}\sum_{i,j=1}^{2}\int_X |A_x^{(ij)}|^2 d\mu(x) < \infty$$

and we conclude that $A : X \to M_2(\mathbb{C})$ with entries $A^{(ij)} : X \to \mathbb{C}$ lies in $\mathcal{H}$. □

Proposition 6.1.7. *$\mathcal{H}$ is a Hilbert space.*

Proof. That $\langle\cdot,\cdot\rangle$ is an inner product is clear. We denote the norm induced by this inner product by $\|\cdot\|_\varphi$.

We show that $\mathcal{H}$ is complete. Let $\{A_{(\lambda)}\} \subset \mathcal{H}$ be an arbitrary Cauchy net. For any given $\varepsilon > 0$, there exists a λ_0 such that when $\lambda_1, \lambda_2 \geq \lambda_0$

$$\begin{aligned}
\varepsilon^2 &> \|A_{(\lambda_1)} - A_{(\lambda_2)}\|_\varphi^2 \\
&= \int_X \frac{1}{2}\mathrm{tr}((A_{(\lambda_1),x} - A_{(\lambda_2),x})^*(A_{(\lambda_1),x} - A_{(\lambda_2),x}))d\mu(x) \\
&= \int_X \frac{1}{2}\mathrm{tr}((A_{(\lambda_1),x} - A_{(\lambda_2),x})^*(A_{(\lambda_1),x} - A_{(\lambda_2),x}))d\mu(x) \\
&= \frac{1}{2}\sum_{i,j=1}^{2}\int_X |A^{(ij)}_{(\lambda_1),x} - A^{(ij)}_{(\lambda_2),x}|^2 d\mu(x)
\end{aligned}$$

from which we see $\int_X |A^{(ij)}_{(\lambda_1),x} - A^{(ij)}_{(\lambda_2),x}|^2 d\mu(x) < 2\varepsilon^2$ for $i,j = 1,2$ when $\lambda_1, \lambda_2 \geq \lambda_0$ and, by the previous proposition, conclude that $\{A^{(ij)}_{(\lambda)}\} \subset L^2(X,\Sigma,\mu)$ is a Cauchy net for $i,j = 1,2$. Since $L^2(X,\Sigma,\mu)$ is complete, this net converges to a limit in $L^2(X,\Sigma,\mu)$ which we denote by $A^{(ij)} \in L^2(X,\Sigma,\mu)$ and define the map $A : X \to M_2(\mathbb{C})$ accordingly to have entries $A^{(ij)}$, for $i,j = 1,2$. Since $A^{(ij)} \in L^2(X,\Sigma,\mu)$ it is clear that $A \in \mathcal{H}$, because

$$\int_X \frac{1}{2}\mathrm{tr}(A_x^* A_x)d\mu(x) = \frac{1}{2}\sum_{i,j=1}^{2}\int_X |A_x^{(ij)}|^2 d\mu(x) < \infty.$$

Also choosing λ_3 'large enough' that $\int_X |A^{(ij)}_{(\lambda),x} - A^{(ij)}_x|^2 d\mu(x) < \frac{\varepsilon^2}{2}$ when $\lambda \geq \lambda_3$, we then see

$$\begin{aligned}
\|A_{(\lambda)} - A\|_\varphi^2 &= \int_X \frac{1}{2}\mathrm{tr}((A_{(\lambda),x} - A_x)^*(A_{(\lambda),x} - A_x))d\mu(x) \\
&= \frac{1}{2}\sum_{i,j=1}^{2}\int_X |A^{(ij)}_{(\lambda),x} - A^{(ij)}_x|^2 d\mu(x) \\
&< \varepsilon^2
\end{aligned}$$

showing that $A_{(\lambda)} \to A$. We conclude that $\mathcal{H}$ is a Hilbert space. □

Corollary 6.1.8. *$\mathcal{A}$ is dense in $\mathcal{H}$.*

Proof. Since $L^\infty(X,\Sigma,\mu)$ is dense in $L^2(X,\Sigma,\mu)$, an element in $\mathcal{H}$ can be approximated by elements in $\mathcal{A}$ entrywise. □

Corollary 6.1.9. *The Hilbert space arising from the GNS-construction on $(\mathcal{A},\varphi)$ is $\mathcal{H}$.*

We remind ourselves that $L^\infty(X,\Sigma,\mu)$ is a von Neumann algebra acting on the Hilbert space $L^2(X,\Sigma,\mu)$ when viewed as a subset of the bounded linear operators on $L^2(X,\Sigma,\mu)$ defined by multiplication.

Proposition 6.1.10. *The C^*-algebra $\mathcal{A}$ is a von-Neumann algebra acting on the Hilbert space $\mathcal{H}$.*

Proof. It is sufficient to prove that $\mathcal{A}$ is a strongly closed subset of $\mathcal{B}(\mathcal{H})$. It is clear from the GNS construction that $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ when an element $A \in \mathcal{A}$ is viewed as defining a bounded linear operator on $\mathcal{H}$ through pointwise matrix multiplication, i.e. for every $h \in \mathcal{H}$, $Ah \in \mathcal{H}$ is defined by $(Ah)_x := A_x h_x$ for every $x \in X$.

It remains to prove that $\mathcal{A}$ is strongly closed. To this end, let $\{A_{(\lambda)}\} \subseteq \mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$ be an arbitrary net converging strongly to some $A \in \mathcal{B}(\mathcal{H})$, i.e. for every $h \in \mathcal{H}$ the net $\{A_{(\lambda)}h\} \subset \mathcal{H}$ converges to $Ah \in \mathcal{H}$. We aim to show that $A \in \mathcal{A}$.

We define the following operators $A^{(ij)} \in \mathcal{B}(L^2(X,\mu))$ by

$$A^{(ij)}f := \left(A\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}\right)^{(ij)}$$

for $i,j = 1,2$ and every $f \in L^2(X,\Sigma,\mu)$. Linearity of these four operators is clear, and their boundedness follows from

$$\begin{aligned}
&\|A^{(ij)}f\|_2^2 \\
=\ & \left\|\left(A\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}\right)^{(ij)}\right\|_2^2 \\
\leq\ & \sum_{i,j=1}^{2}\left\|\left(A\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}\right)^{(ij)}\right\|_2^2 \\
=\ & \left\|A\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}\right\|_\varphi^2 \\
\leq\ & \|A\|_\varphi^2\left\|\begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix}\right\|_\varphi^2 \\
=\ & \|A\|_\varphi^2\,\|f\|_2^2,
\end{aligned}$$

for every $i,j = 1,2$ and $f \in L^2(X,\Sigma,\mu)$.

Now for any given $\varepsilon > 0$ and $f \in L^2(X,\Sigma,\mu)$, the strong convergence of

$\{A_{(\lambda)}\}$ to A implies that there exists a λ_0 such that $\lambda \geq \lambda_0$ implies

$$\begin{aligned}
\varepsilon^2 &> \left\| A_{(\lambda)} \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} - A \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \right\|_{\varphi}^2 \\
&= \frac{1}{2} \sum_{i,j=1}^{2} \left\| \left(A_{(\lambda)} \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} - A \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \right)^{(ij)} \right\|_2^2 \\
&= \frac{1}{2} \sum_{i,j=1}^{2} \left\| \left(A_{(\lambda)} \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \right)^{(ij)} - \left(A \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \right)^{(ij)} \right\|_2^2 \\
&= \frac{1}{2} \sum_{i,j=1}^{2} \left\| \left(A_{(\lambda)} \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \right)^{(ij)} - A^{(ij)} f \right\|_2^2 \\
&\geq \frac{1}{2} \left\| \begin{bmatrix} A_{(\lambda)}^{(11)} f & A_{(\lambda)}^{(12)} f \\ A_{(\lambda)}^{(21)} f & A_{(\lambda)}^{(22)} f \end{bmatrix}^{(ij)} - A^{(ij)} f \right\|_2^2 \\
&= \frac{1}{2} \left\| A_{(\lambda)}^{(ij)} f - A^{(ij)} f \right\|_2^2 .
\end{aligned}$$

Implying that $A_{(\lambda)}^{(ij)} \to A^{(ij)}$ strongly, as operators defined on $L^2(X, \Sigma, \mu)$ for each $i, j = 1, 2$. But since $\{A_{(\lambda)}\} \subset \mathcal{A}$, we have that $A_{(\lambda)}^{(ij)} \in L^\infty(X, \Sigma, \mu)$ for every λ and $i, j = 1, 2$ by Proposition 6.1.1 and since $L^\infty(X, \Sigma, \mu)$ is a strongly closed subspace of $\mathcal{B}(L^2(X, \Sigma, \mu))$ we have that the operators $A^{(ij)}$ lie in $L^\infty(X, \Sigma, \mu)$. Therefore $A' := \begin{bmatrix} A^{(11)} & A^{(12)} \\ A^{(21)} & A^{(22)} \end{bmatrix} \in \mathcal{A}$, again by Proposition 6.1.1.

We show now show that $A_{(\lambda)} \to A'$ strongly. For any $h := \begin{bmatrix} h^{(11)} & h^{(12)} \\ h^{(21)} & h^{(22)} \end{bmatrix} \in$

$\mathcal{H}$, where each $h^{(ij)}$ lies in $L^2(X,\Sigma,\mu)$ by Proposition 6.1.6,

$$\begin{aligned}
&\|A_{(\lambda)}h - A'h\|_\varphi^2 \\
= \;& \frac{1}{2}\sum_{i,j=1}^{2}\left\|\left(A_{(\lambda)}h - A'h\right)^{(ij)}\right\|_2^2 \\
= \;& \frac{1}{2}\left\|(A_{(\lambda)}^{(11)} - A^{(11)})h^{(11)} + (A_{(\lambda)}^{(12)} - A^{(12)})h^{(21)}\right\|_2^2 \\
& + \frac{1}{2}\left\|(A_{(\lambda)}^{(11)} - A^{(11)})h^{(12)} + (A_{(\lambda)}^{(12)} - A^{(12)})h^{(22)}\right\|_2^2 \\
& + \frac{1}{2}\left\|(A_{(\lambda)}^{(21)} - A^{(21)})h^{(11)} + (A_{(\lambda)}^{(22)} - A^{(22)})h^{(21)}\right\|_2^2 \\
& + \frac{1}{2}\left\|(A_{(\lambda)}^{(21)} - A^{(21)})h^{(12)} + (A_{(\lambda)}^{(22)} - A^{(22)})h^{(22)}\right\|_2^2 \\
\leq \;& \frac{1}{2}\left\|(A_{(\lambda)}^{(11)} - A^{(11)})h^{(11)}\right\|_2^2 + \frac{1}{2}\left\|(A_{(\lambda)}^{(12)} - A^{(12)})h^{(21)}\right\|_2^2 \\
& + \frac{1}{2}\left\|(A_{(\lambda)}^{(11)} - A^{(11)})h^{(12)}\right\|_2^2 + \frac{1}{2}\left\|(A_{(\lambda)}^{(12)} - A^{(12)})h^{(22)}\right\|_2^2 \\
& + \frac{1}{2}\left\|(A_{(\lambda)}^{(21)} - A^{(21)})h^{(11)}\right\|_2^2 + \frac{1}{2}\left\|(A_{(\lambda)}^{(22)} - A^{(22)})h^{(21)}\right\|_2^2 \\
& + \frac{1}{2}\left\|(A_{(\lambda)}^{(21)} - A^{(21)})h^{(12)}\right\|_2^2 + \frac{1}{2}\left\|(A_{(\lambda)}^{(22)} - A^{(22)})h^{(22)}\right\|_2^2 \\
\to \;& 0
\end{aligned}$$

with λ, since each $h^{(ij)} \in L^2(X,\mu)$ and each net $\{A_{(\lambda)}^{(ij)}\}$ converges strongly to $A^{(ij)}$ when viewed as bounded linear operators on $L^2(X,\Sigma,\mu)$. By the uniqueness of limits we can now conclude that $A = A' \in \mathcal{A}$, and finally that $A \in \mathcal{A}$, establishing that $\mathcal{A}$ is strongly closed, and hence $\mathcal{A}$ is a von Neumann algebra acting on $\mathcal{H}$. □

6.2 Irrational rotations, with a twist

We denote the four Pauli spin matrices as follows

$$\sigma_0 := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}; \quad \sigma_1 := \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}; \quad \sigma_2 := \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}; \quad \sigma_3 := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix};$$

The set $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ forms an orthonormal basis for the vector space $M_2(\mathbb{C})$ with inner product defined by $\langle A, B\rangle := \frac{1}{2}\mathrm{tr}(B^*A)$ for all $A, B \in M_2(\mathbb{C})$ [11, p. 10].

Let $(S^1, \mathcal{B}, \lambda, T_\alpha)$ be an irrational rotation. On the system $(S^1, \mathcal{B}, \lambda, T_\alpha)$ we construct the von Neumann algebra $\mathcal{A}$ with state $\varphi(A) := \frac{1}{2}\int_{S^1}\mathrm{tr}(A_x)d\lambda(x)$ as in Section 6.1.

We define the unitary element $R \in \mathcal{A}$ by rotation matrices

$$R_{e^{i\theta\pi}} := \begin{bmatrix} \cos\theta\pi & \sin\theta\pi \\ -\sin\theta\pi & \cos\theta\pi \end{bmatrix}.$$

We define the map $\tau : \mathcal{A} \to \mathcal{A}$ by $\tau(A)_x := R^*_{T_\alpha x} A_{T_\alpha x} R_{T_\alpha x}$ for all $x \in S^1$. Defined as such, τ is a state preserving *-homomorphism, hence $(\mathcal{A}, \varphi, \tau)$ is a C*-dynamical system.

The multiplication by the 'rotation element' $R \in \mathcal{A}$ is introduced in an attempt to 'mix' the non-commutative part of this system to such an extent as to hopefully make it ergodic. The extension of τ from $\mathcal{A}$ to $\mathcal{H}$, denoted $U : \mathcal{H} \to \mathcal{H}$ takes the 'same form' as τ, i.e. that $(UA)_x = R^*_{T_\alpha x} A_{T_\alpha x} R_{T_\alpha x}$ for all $A \in \mathcal{H}$.

Alas, the two elements in $\mathcal{H}$ which equal σ_0 and σ_2 almost everywhere on S^1 are linearly independent and are both fixed points of U. Therefore by Theorem 4.2.6 $(\mathcal{A}, \varphi, \tau)$ is not ergodic.

We may however still consider what happens when inducing onto an appropriate projection.

Projections in $\mathcal{A}' \cap \mathcal{A}$, are exactly the projections which are equal to σ_0 on a set of positive measure in $\mathcal{B}$, and zero elsewhere. It is easy to see that for any such projection $P \in \mathcal{A}' \cap \mathcal{A}$, that $\tau^j(P) \in \mathcal{A}' \cap \mathcal{A}$ for all $j \in \mathbb{N}$ and is again such a projection as described. Therefore, inducing on such a projection $P \in \mathcal{A}' \cap \mathcal{A}$, is indeed possible by the previous theory to yield the induced C*-dynamical system $(\mathcal{A}_P, \varphi_P, \tau_P)$.

The 'underlying' classical measure preserving dynamical system, makes it easier to make the connection between return times and the classical Poincaré Recurrence Theorem and the recurrence results in Section 5.2

Why do we require, in the theory presented in the previous chapter, that a projection that we induce on lies in $\mathcal{A}' \cap \mathcal{A}$? We may ask what happens if we were to attempt to induce onto a projection that does not lie in $\mathcal{A}' \cap \mathcal{A}$. For example onto $P \in \mathcal{A}$ with $P_x = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ for every $x \in S^1$. The element $\tau(P)$ is again a projection, but (its range, when (incorrectly) visualized as a copy of $\mathbb{R}$ at every point of S^1, not unlike an infinite mobius band) is 'twisted' around S^1 through the multiplication by $R \in \mathcal{A}$, and λ-almost nowhere coincides with (the range of) P. Even though it may happen that $\varphi(P\tau(P)P) \neq 0$ or even > 0, the disparity of (the ranges) of P and $\tau(P)$ makes P a lousy candidate to induce onto and clouds our minds (or certainly the author's) as to how to go about defining an induced system onto this projection.

6.3 Irrational rotations, with a striped twist

Let $([0, 1), \mathcal{B}, \lambda, T_\alpha)$ be (isomorphic) to an irrational rotation, such that $\mathcal{B}$ is the Borel σ-algebra, λ is the Lebesgue measure, and $T_\alpha x = x + \alpha \bmod 1$ for all $x \in [0, 1)$ (remember that α is irrational).

Let φ be the state and $\mathcal{A}$ be the von Neumann algebra (acting on $\mathcal{H}$) be as constructed from $([0, 1), \mathcal{B}, \lambda)$ in the preliminaries of this chapter.

We define the unitary matrices

$$W := \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad R := \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$$

where $\beta \in (\mathbb{R}\backslash\mathbb{Q})\pi$ is fixed, and define $S \in \mathcal{A}$ by

$$S_x : \quad = \quad \begin{cases} R^n & x \in [\frac{2^{n-1}-1}{2^{n-1}}, \frac{2^n-1}{2^n}) \\ W & \text{else} \end{cases}$$

for any $x \in [0, 1)$, and thereby define the transformation $\tau : \mathcal{A} \to \mathcal{A}$ by

$$(\tau(A))_x := S_x^* A_{T_\alpha x} S_x.$$

Then $(\mathcal{A}, \varphi, \tau)$ is a C*-dynamical system, with τ a *-homomorphism. The form of $S \in \mathcal{A}$ suggests the name 'striped twist' for the *-homomorphism τ.

We will assume that $\mathcal{A} \subset \mathcal{H}$ (as sets) and hence suppress mention of ι from the GNS construction. In attempting to prove that this system is ergodic, by Theorem 4.2.6 it is enough to show that the bounded operator $U : \mathcal{H} \to \mathcal{H}$ defined (densely) by $UA = \tau(A)$ for all $A \in \mathcal{A} \subset \mathcal{H}$, has a one dimensional fixed point space.

We define $\Omega \in \mathcal{A} \subset \mathcal{H}$ by

$$\Omega_x := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

for all $x \in [0, 1)$.

Suppose $A \in \mathcal{H}$ is such that A_x is constant in $x \in [0, 1)$ almost everywhere (this implies $A \in \mathcal{A}$), and that it is a fixed point of U, i.e.

$$A_x = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

for almost every $x \in [0, 1)$, and $UA = A$. But this implies that

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = R \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} R^* = R^2 \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} R^{*2} = W \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} W^*$$

which reduces (with some work, best done by some computer algebra software) to $a_{11} = a_{22}$ and $a_{12} = a_{21} = 0$. Therefore the only element $A \in \mathcal{H}$ such that A_x is constant in $x \in [0, 1)$ and $UA = A$, must equal a multiple of Ω.

This already a step up from the example in the previous section, for which U had (at least) two fixed points which were constant in $x \in [0, 1)$.

Suppose $A \in \mathcal{H}$ is such that the entries of A_x are simple functions in $x \in [0, 1)$ almost everywhere (this implies $A \in \mathcal{A}$, and we will call such elements of $\mathcal{A}$ and $\mathcal{H}$ *simple* elements), and that it is a fixed point of U. Then there exists a *finite* partition of $[0, 1)$ into intervals, such that A_x is constant almost everywhere on every interval of the partition. There then exists an interval $(1-\varepsilon, 1)$ with $\varepsilon > 0$ such that A_x is constant in $x \in [0, 1)$ almost everywhere on both the intervals $T_\alpha^{-1}(1-\varepsilon, 1)$ and $(1-\varepsilon, 1)$. We will assume

$$A_x = \begin{cases} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} & x \in T_\alpha^{-1}(1-\varepsilon, 1) \\ \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & x \in (1-\varepsilon, 1). \end{cases}$$

Let n_0 be large enough that $(2^{n_0-1}-1)/2^{n_0-1} > (1-\varepsilon)$. Then $UA = A$ establishes,

$$\begin{array}{ccccc} R^{n_0}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} R^{*n_0} & = & \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} & = & R^{n_0+1}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} R^{*n_0+1} \\ & & \| & & \\ & & W\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} W^* & & \end{array}$$

which implies (with some work, best done by some computer algebra software) that $a_{11} = a_{22}$ and $a_{12} = a_{21} = 0$, and hence that A_x equals a multiple of the identity almost everywhere on both intervals $T_\alpha^{-1}(1-\varepsilon, 1)$ and $(1-\varepsilon, 1)$. But since $([0,1), \mathcal{B}, \lambda, T_\alpha)$ is ergodic, for almost every $y \in [0,1)$ there exists an m such that $y \in T_\alpha^{-m}(1-\varepsilon, 1)$ and so that for some $t \in \mathbb{C}$,

$$\begin{aligned} A_y &= (\tau^m(A))_y \\ &= S_y \ldots S_{T_\alpha^{m-1}y} A_{T_\alpha^m y} S^*_{T_\alpha^{m-1}y} \ldots S^*_y \\ &= S_y \ldots S_{T_\alpha^{m-1}y} \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix} S^*_{T_\alpha^{m-1}y} \ldots S^*_y \\ &= \begin{bmatrix} t & 0 \\ 0 & t \end{bmatrix}. \end{aligned}$$

We conclude that all the simple elements $A \in \mathcal{H}$ that are fixed points of U, must equal a multiple of Ω.

Here is where the water gets muddied. The above arguments do not yet imply that all the fixed points of U lie in a one dimensional subspace of $\mathcal{H}$. (As of yet) the author has been unable to prove the offending statement or find a counterexample (of two linearly independent fixed points of U), yet remains quite convinced of:

Conjecture 6.3.1. *The system $(\mathcal{A}, \varphi, \tau)$, as described above, is ergodic.*

Still we can make some interesting observations about fixed points of U and consider some programs devised for proving that the fixed point space of U is indeed one dimensional.

Program 6.3.2. Since the simple functions are dense in $L^\infty([0,1), \mathcal{B}, \lambda)$, given any fixed point $A \in \mathcal{H}$ of U and any $\varepsilon > 0$, we can find a simple element $S \in \mathcal{A} \subset \mathcal{H}$ such that $\|A - S\| < \varepsilon$. If it is possible to prove that the assumption implies $\|S - \varphi(A)\Omega\| < \varepsilon$, we would be done, since every net of simple elements converging to A also converges to $\varphi(A)\Omega$, and by the uniqueness of limits, we would have that $A = \varphi(A)\Omega$ – a multiple of Ω.

Program 6.3.3. For every $A \in \mathcal{H}$, we write

$$A_x = \begin{bmatrix} a_{11}(x) & a_{12}(x) \\ a_{21}(x) & a_{22}(x) \end{bmatrix},$$

and can define a 'eigenvalue functions' by

$$\gamma^{\pm}(x) := \frac{(a_{22}(x) + a_{11}(x)) \pm \sqrt{(a_{22}(x) + a_{11}(x))^2 - 4a_{12}(x)a_{21}(x)}}{2}.$$

If $A \in \mathcal{H}$ is a fixed point of U then $A_x = (UA)_x = S_x^* A_{T_\alpha x} S_x$ for almost every $x \in [0, 1)$. Since for each $x \in [0, 1)$, S_x is unitary, the matrices A_x and $A_{T_\alpha x}$ are similar for almost every $x \in [0, 1)$. But similar matrices have the same eigenvalues, therefore $\gamma^{\pm}(x) = \gamma^{\pm}(T_\alpha x)$, and since $([0, 1), \mathcal{B}, \lambda, T_\alpha)$ is ergodic, by Theorem 2.1.8, both γ^+ and γ^- must be constant almost everywhere on $[0, 1)$. Suppose that A is linearly independent of Ω, then we might as well assume that $A \perp \Omega$, and hence $0 = \langle A, \Omega \rangle = \varphi(A) = \gamma^+ + \gamma^-$, implying $\gamma^+ = -\gamma^-$. By scaling appropriately, we may assume that $\gamma^+ = 1$, and hence that A_x is self-adjoint and unitary for almost every $x \in [0, 1)$. If we can now somehow show that $\gamma^+ = \gamma^-$, it would imply that $A = 0$, and we would be done.

Program 6.3.4. We may (perhaps) define (an analogy to) a weak derivative. We call an element $\Phi \in \mathcal{A}$ a ***test element*** if its entries, $\phi_{ij} : [0, 1) \to \mathbb{C}$, $i, j = 1, 2$, are continuously differentiable and $\phi_{ij}(x) \to \phi_{ij}(0)$ and $\phi'_{ij}(x) \to \phi'_{ij}(0^+)$ when $x \to 1^-$, and say $B : [0, 1) \to M_2(\mathbb{C})$ is the weak derivative of $A : [0, 1) \to M_2(\mathbb{C})$ if

$$\langle A, \Phi' \rangle = \langle B, \Phi \rangle$$

for all test elements Φ, where $\Phi' : [0, 1) \to M_2(\mathbb{C})$ is defined to be the to have as 'entry functions' the derivatives ϕ'_{ij}. We might guess, when denoting the 'entry functions' of A by $a_{ij} : [0, 1) \to \mathbb{C}$, and and their weak derivative by a'_{ij}, and assuming $((\Phi')^*)_x = ((\Phi^*)')_x$, the the following

$$\begin{aligned}
\langle A, \Phi' \rangle &= \frac{1}{2} \int \mathrm{tr}(\Phi'^*_x A_x) d\lambda(x) \\
&= \frac{1}{2} \int \overline{\phi'_{11}} a_{11} d\lambda + \frac{1}{2} \int \overline{\phi'_{21}} a_{21} d\lambda + \frac{1}{2} \int \overline{\phi'_{12}} a_{12} d\lambda + \frac{1}{2} \int \overline{\phi'_{22}} a_{22} d\lambda \\
&= \frac{1}{2} \int \overline{\phi_{11}} a'_{11} d\lambda + \frac{1}{2} \int \overline{\phi_{21}} a'_{21} d\lambda + \frac{1}{2} \int \overline{\phi_{12}} a'_{12} d\lambda + \frac{1}{2} \int \overline{\phi_{22}} a'_{22} d\lambda \\
&= \langle A', \Phi \rangle
\end{aligned}$$

may well hold for all test elements Φ, where A' has a'_{ij} as entry functions.

Let the author be the first to criticize the above for its vagueness. However, if the theory described in the previous can be developed, it might be used to prove that any fixed point of U must have a weak derivative equal to zero, implying that it is constant almost everywhere, and hence a multiple of Ω. Developing this theory (if even possible) will take us too far afield.

To the author's great frustration, all three of the proposed programs above have been fruitless (thus far) in proving $(\mathcal{A}, \varphi, \tau)$ ergodic.

6.4 Irrational rotations, with secondary irrational rotations

It is very important to note that the transformations in this and the next two sections, are not *-homomorphisms. This prevents us from applying the theory developed in the previous chapter.

Let $(S^1, \mathcal{B}, \lambda, T_\alpha)$ be an irrational rotation. (see Section 2.2).

On the system $(S^1, \mathcal{B}, \lambda, T_\alpha)$ we construct the von Neumann $\mathcal{A}$ with state $\varphi(A) := \frac{1}{2}\int_{S^1} \mathrm{tr}(A_x) d\lambda(x)$ as in Section 6.1.

It is clear that we may decompose every element $A \in \mathcal{A}$ as $A = \sum_{j=0}^{3} f_j \sigma_j$ where $f_j \in L^\infty(S^1, \mathcal{B}, \lambda)$ for each $j = 0, 1, 2, 3$.

We define a linear transformation $\tau_\alpha : \mathcal{A} \to \mathcal{A}$ as follows. Let $\beta_0, \beta_1, \beta_2, \beta_3 \in \mathbb{R}$, be such that $\beta_0 = 0$ and $\beta_j \in (0,2) \backslash \{n\alpha \bmod 2 : n \in \mathbb{Z}\}$. For all $f_j \in L^\infty(S^1, \mathcal{B}, \lambda)$ we define $\tau_\alpha(\sum_{j=0}^{3} f_j \sigma_j) := \sum_{i=0}^{3} e^{i\beta_j \pi}(f_j \circ T_\alpha)\sigma_j$.

Multiplication by the $e^{i\beta_j\pi}$ could be viewed as a secondary rotations to the primary rotation T_α (which can be viewed as multiplication by $e^{i\alpha\pi}$, see Section 2.2).

Proposition 6.4.1. *The linear transformation τ_α is state preserving with respect to φ.*

Proof. Let $A \in \mathcal{A}$ be arbitrary and consider, by Theorem 2.1.8,

$$\begin{aligned}
\varphi \circ \tau_\alpha(A) &:= \frac{1}{2}\int_{S^1} \mathrm{tr}(\tau(A)_x) d\lambda(x) \\
&= \frac{1}{2}\int_{S^1} \mathrm{tr}\tau\left(\sum_{i=0}^{3} f_j(\cdot)\sigma_j\right)_x d\lambda(x) \\
&= \frac{1}{2}\int_{S^1} \mathrm{tr}\left(\sum_{i=0}^{3} e^{i\beta_j\pi} f_j \circ T_\alpha(x)\sigma_j\right) d\lambda(x) \\
&= \frac{1}{2}\int_{S^1} \sum_{i=0}^{3} e^{i\beta_j\pi} f_j \circ T_\alpha(x)\mathrm{tr}(\sigma_j) d\lambda(x) \\
&= \int_{S^1} f_0 \circ T_\alpha(x) d\lambda(x) \\
&= \int_{S^1} f_0(x) d\lambda(x) \\
&= \frac{1}{2}\int_{S^1} \sum_{i=0}^{3} f_j(x)\mathrm{tr}(\sigma_j) d\lambda(x) \\
&= \frac{1}{2}\int_{S^1} \mathrm{tr}\left(\sum_{i=0}^{3} f_j(x)\sigma_j\right) d\lambda(x) \\
&= \frac{1}{2}\int_{S^1} \mathrm{tr}\left(A_x\right) d\lambda(x) \\
&= \varphi(A)
\end{aligned}$$

which establishes the result. □

Therefore $(\mathcal{A}, \varphi, \tau_\alpha)$ is a C*-dynamical system.

Let $\mathcal{H}$ be the Hilbert space arising from the GNS construction on $(\mathcal{A}, \varphi)$, see Corollary 6.1.9. As shown in the Section 6.1, $\mathcal{A}$ is dense in $\mathcal{H}$ (with respect to the norm induced by the inner product on $\mathcal{H}$). It is again clear that we may express any element $A \in \mathcal{H}$ as $A = \sum_{j=0}^{3} f_j\sigma_j$ with $f_j \in L^2(S^1, \mathcal{B}, \lambda)$ for each $j = 0, 1, 2, 3$. Let $U : \mathcal{H} \to \mathcal{H}$ be the unique bounded linear extension of τ_α to the whole of $\mathcal{H}$.

Proposition 6.4.2. *The operator U takes on the same form as τ, i.e.*

$$U\left(\sum_{j=0}^{3} f_j\sigma_j\right) = \sum_{i=0}^{3} e^{i\beta_j\pi}(f_j \circ T_\alpha)\sigma_j$$

for any $f_j \in L^2(S^1, \mathcal{B}, \lambda)$ and any $j = 0, 1, 2, 3$.

Proof. It is clear that the map $\sum_{j=0}^{3} f_j\sigma_j \mapsto \sum_{i=0}^{3} e^{i\beta_j\pi}(f_j \circ T_\alpha)\sigma_j$ equals τ when restricted to $\mathcal{A}$, therefore if we show that said map is bounded (with respect to the norm $\|\cdot\|_\varphi$ on $\mathcal{H}$) it must equal U, since U is the unique bounded linear extension of τ. To that end, let $A \in \mathcal{H}$ be arbitrary which we express as $A = \sum_{j=0}^{3} f_j\sigma_j$ with $f_j \in L^2(S^1, \mathcal{B}, \lambda)$ for each $j = 0, 1, 2, 3$. Now consider

$$\begin{aligned}
& \left\| \sum_{j=0}^{3} e^{i\beta_j\pi}(f_j \circ T_\alpha)\sigma_j \right\|_\varphi^2 \\
= \;& \varphi\left(\left(\sum_{j=0}^{3} e^{i\beta_j\pi}(f_j \circ T_\alpha)\sigma_j\right)^* \left(\sum_{k=0}^{3} e^{i\beta_k\pi}(f_k \circ T_\alpha)\sigma_k\right)\right) \\
= \;& \varphi\left(\sum_{j,k=0}^{3} e^{-i\beta_j\pi}e^{i\beta_k\pi}(\overline{f_j} \circ T_\alpha)(f_k \circ T_\alpha)\sigma_j\sigma_k\right) \\
= \;& \frac{1}{2}\int_{S^1} \sum_{j,k=0}^{3} e^{-i\beta_j\pi}e^{i\beta_k\pi}(\overline{f_j} \circ T_\alpha)(f_k \circ T_\alpha)\mathrm{tr}(\sigma_j\sigma_k)d\lambda(x) \\
= \;& \int_{S^1} \sum_{j,k=0}^{3} e^{-i\beta_j\pi}e^{i\beta_k\pi}(\overline{f_j} \circ T_\alpha)(f_k \circ T_\alpha)\delta_{jk}d\lambda(x) \\
= \;& \sum_{j=0}^{3} \int_{S^1} |f_j \circ T_\alpha|^2 d\lambda(x) \\
= \;& \sum_{j=0}^{3} \int_{S^1} |f_j|^2 d\lambda(x) \\
= \;& \frac{1}{2}\int_{S^1} \sum_{j,k=0}^{3} \overline{f_j}(x)f_k(x)\mathrm{tr}(\sigma_j\sigma_k)d\lambda(x) \\
= \;& \frac{1}{2}\int_{S^1} \mathrm{tr}\left(\left(\sum_{j=0}^{3} f_j(x)\sigma_j\right)^* \left(\sum_{k=0}^{3} f_k(x)\sigma_k\right)\right) d\lambda(x) \\
= \;& \left\| \sum_{j=0}^{3} f_j\sigma_j \right\|_\varphi^2 \\
= \;& \|A\|_\varphi^2
\end{aligned}$$

which establishes the boundedness of the map $\sum_{j=0}^{3} f_j\sigma_j \mapsto \sum_{i=0}^{3} e^{i\beta_j}(f_j \circ T_\alpha)\sigma_j$ on $\mathcal{H}$, hence U equals this map.

□

Proposition 6.4.3. *The system $(\mathcal{A}, \varphi, \tau_\alpha)$ is ergodic.*

Proof. By Theorem 4.2.6 it is enough to show that the map $U : \mathcal{H} \to \mathcal{H}$ has a one-dimensional fixed point space.

Let $F \in \mathcal{H}$ be an arbitrary fixed point of U, i.e. $UF = F$. We will show that $F = c\sigma_0$ where $c \in L^2(S^1, \mathcal{B}, \lambda)$ is constant λ-a.e. on S^1. Since the (λ-a.e.) constant $L^2(S^1, \mathcal{B}, \lambda)$ functions is a one dimensional subspace of $L^2(S^1, \mathcal{B}, \lambda)$, this would establish the result.

We express $F = \sum_{j=0}^{3} f_j \sigma_j$, with each $f_j \in L^2(S^1, \mathcal{B}, \lambda)$, and consider

$$\sum_{j=0}^{3} f_j \sigma_j = F = UF = U\left(\sum_{j=0}^{3} f_j \sigma_j\right) = \sum_{i=0}^{3} e^{i\beta_j \pi}(f_j \circ T_\alpha)\sigma_j$$

which together with the linear Independence of $\{\sigma_0, \sigma_1, \sigma_2, \sigma_3\}$ in $M_2(\mathbb{C})$ implies that $f_j - e^{i\beta_j \pi}(f_j \circ T_\alpha) = 0$ for each $j = 0, 1, 2, 3$.

Since $\beta_0 = 0$, the ergodicity of T_α together with $f_0 = f_0 \circ T_\alpha$ implies that f_0 is constant (λ-a.e.).

Now, for each $j \neq 0$, we consider the Fourier expansion of f_j. We write $f_j(e^{ix}) = \sum_{n \in \mathbb{Z}} c_{j,n} e^{inx}$ for all $x \in [0, 2\pi]$ and substitute into $f_j - e^{i\beta_j}(f_j \circ T_\alpha) = 0$ to obtain

$$\begin{aligned} 0 &= \sum_{n \in \mathbb{Z}} c_{j,n} e^{inx} - e^{i\beta_j \pi} c_{j,n} e^{in(x+\alpha\pi)} \\ &= \sum_{n \in \mathbb{Z}} c_{j,n}(1 - e^{i(\beta_j + n\alpha)\pi}) e^{inx}. \end{aligned}$$

The uniqueness of the Fourier expansion implies that $c_{j,n}(1 - e^{i(\beta_j + n\alpha)\pi}) = 0$ for each $n \in \mathbb{Z}$. If $1 - e^{i(\beta_j + n\alpha)\pi} = 0$, then $(\beta_j + n\alpha)\pi = 2k\pi$ for some $k \in \mathbb{Z}$, which implies $\beta_j = -n\alpha \bmod 2$ which is impossible by the assumption on β_j, for $j = 1, 2, 3$, and hence $c_{j,n}$ must equal zero for all $n \in \mathbb{Z}$ and $j = 1, 2, 3$. This implies that $f_j = 0$ for $j = 1, 2, 3$.

We conclude that F assumes the form that was required to establish the result.

□

Remark 6.4.4. It would be desirable to have the transformation τ_α to be a *-homomorphism and still remain ergodic, because this would place this example firmly on the footing of the previous chapter. Unfortunately, this can only happen when an even number of the numbers $\beta_0, \beta_1, \beta_2, \beta_3$ equal zero and the others equal 1. But because we already assume $\beta_0 = 0$, this implies that at least one of the $\beta_1, \beta_2, \beta_3$ must equal 0, say $\beta_{j_0} = 0$ for some $j_0 = 1, 2, 3$, and then as a consequence for the (λ-a.e.) constant 1 function $1 \in L^2(S^1, \mathcal{B}, \lambda)$, it is easily seen that $1\sigma_0$ and $1\sigma_{j_0}$ are linearly independent fixed points of the operator U, thereby breaking the ergodicity by Theorem 4.2.6.

6.5 Induced transformations (1)

We continue under the definitions of the previous section. The fact that τ_α is not a *-homomorphism prevents us from directly using the results in the previous chapter.

For simplicity we only consider a small class of possible induced transformations.

As in Section 3.4, let $(S^1, \mathcal{B}, \lambda, T_{\gamma(\alpha)})$ be the ergodic (system isomorphic to the) system induced onto $C := \{e^{i\theta\pi} : \theta \in [0, \alpha]\}$ from $(S^1, \mathcal{B}, \lambda, T_\alpha)$. As shown previously, $\gamma(\alpha) = \frac{2\gamma'}{\alpha}$ where $\gamma' = 2 - \left\lfloor \frac{2}{\alpha} \right\rfloor \alpha$.

We may now ask as in Theorems 3.3.1 and 5.5.5, whether the induced system $(\mathcal{A}, \varphi, \tau_{\gamma(\alpha)})$, with the $\beta_0, \beta_1, \beta_2, \beta_3$ kept as before, is ergodic. For this to happen it is sufficient for $j = 1, 2, 3$ that $\beta_j \in (0, 2) \backslash \{n\gamma(\alpha) \bmod 2 : n \in \mathbb{Z}\}$. Suppose to the contrary that for some $j = 1, 2, 3$ and some $n \in \mathbb{Z}$

$$\begin{aligned}
\beta_j \bmod 2 &= n\gamma(\alpha) \\
\beta_j \bmod 2 &= n\frac{2\gamma'}{\alpha} \\
\beta_j \bmod 2 &= n\frac{2(2 - \left\lfloor \frac{2}{\alpha} \right\rfloor \alpha)}{\alpha} \\
\beta_j \alpha \bmod 2 &= 4n - 2\left\lfloor \frac{2}{\alpha} \right\rfloor \alpha \\
\beta_j \alpha \bmod 2 &= -2\left\lfloor \frac{2}{\alpha} \right\rfloor \alpha \\
\beta_j \bmod 2 &= -2\left\lfloor \frac{2}{\alpha} \right\rfloor \\
\beta_j \bmod 2 &= 0
\end{aligned}$$

which is false by hypothesis. Therefore $(\mathcal{A}, \varphi, \tau_{\gamma(\alpha)})$ is indeed ergodic, by Proposition 6.4.3.

Remark 6.5.1. The previous argument confirms our intuition on a very small class of possible induced transformations - specifically inducing onto an interval of S^1 of length α.

We may investigate the situation of inducing onto an arbitrary interval of S^1. In this case the classical induced transformation becomes a so-called three interval exchange map as was seen in Section 3.2. This complicates matters, for our 'Fourier expansion argument' used in Theorem 6.4.3, is significantly complicated in that the Fourier expansion of $f \circ T$ facilitates establishing the result quite well when T is a rotation map, but this becomes less obvious when T is a different type of map to a rotation.

We may also ask whether the result goes through for an arbitrary subset of S^1 of positive measure as it does in Theorems 3.3.1 and 5.5.5. For the same reason as the in the previous paragraph, answering this question is complicated.

We are still prompted to raise the following:

Conjecture 6.5.2. *Consider $(\mathcal{A}, \varphi, \tau_\alpha)$ (as constructed above from $(S^1, \mathcal{B}, \lambda, T_\alpha)$). For any $C \in \mathcal{B}$ of positive measure, let the system $(\mathcal{A}_C, \varphi_C, \tau_C)$ with $\mathcal{A}_C$ and φ_C constructed from the classical induced system $(C, \mathcal{B} \cap C, \lambda', T_C)$ as described* *in Section 6.1 and τ_C defined by*

$$\tau_C(\sum_{j=0}^{3} f_j\sigma_j) := \sum_{i=0}^{3} e^{i\beta_j\pi}(f_j \circ T_C)\sigma_j.$$

where each $f_j \in L^\infty(C, \mathcal{B}\cap C, \lambda')$.

The ergodicity of $(\mathcal{A}, \varphi, \tau_\alpha)$ implies the ergodicity of $(\mathcal{A}_C, \varphi_C, \tau_C)$.

6.6 Induced transformations (2)

We again continue under the definitions of Section 6.4.

Another candidate for an induced transformation can be constructed as follows. It is indeed closer to our definition in the previous Chapter than what was investigated in the previous section, as the induced transformation will apply τ_α iteratively according to the classical first return time function defined in Section 3.2.

Let $C \in \mathcal{B}$ be a set of positive measure, and let $(C, \mathcal{B}\cap C, \lambda', T_C)$ be induced from $(S^1, \mathcal{B}, \lambda, T_\alpha)$. The system $(C, \mathcal{B}\cap C, \lambda', T_C)$ is ergodic, by Theorem 3.3.1, since $(S^1, \mathcal{B}, \lambda, T_\alpha)$ is ergodic. Let $\mathcal{A}_C$ and φ_C be constructed on $(C, \mathcal{B}\cap C, \lambda', T_C)$ as in the Section 6.1. We define $\tau'_C : \mathcal{A}_C \to \mathcal{A}_C$ as follows for any $x \in C$ and $f_j \in L^\infty(C, \mathcal{B}\cap C, \lambda')$

$$\left(\tau'_C(\sum_{j=0}^{3} f_j\sigma_j)\right)_x := \sum_{j=0}^{3} e^{i\beta_j n_C(x)\pi} f_j \circ T_C(x)\sigma_j$$

where $n_C(x)$ denotes the first return time of $x \in C$ under T_α to C.

Partitioning C into sets according to return times, say $\{C_k\}_{k=1}^{\infty}$ and defining $P_k := \chi_{C_k}\sigma_0 \in \mathcal{A}_C$, it is easily seen that for any $A \in \mathcal{A}_C$

$$\tau'_C(A) = \sum_{k=1}^{\infty} P_k \tau_\alpha^k(A)$$

which is more reminiscent of the induced transformations defined in Sections 5.1 and 5.4 than the transformation τ_C from the previous section.

However, our efforts to prove that ergodicity of $(\mathcal{A}, \varphi, \tau_\alpha)$ implies ergodicity of $(\mathcal{A}_C, \varphi_C, \tau'_C)$ is again met with difficulties in even the simplest of examples.

Consider again $C := \{e^{i\theta\pi} : \theta \in [0, \alpha]\}$. The first return time map n_C then only takes on two values, $l_0 := \lfloor \frac{2}{\alpha} \rfloor + 1$ or $l_0 + 1$, on C. Hence τ'_C takes on the form

$$\tau'_C(\sum_{j=0}^{3} f_j\sigma_j) := \sum_{j=0}^{3} h_j(f_j \circ T_C)\sigma_j$$

where $h_j : C \to \mathbb{C}$ is given by

$$h_j(x) = \begin{cases} e^{i\beta_j l_0\pi} & n_C(x) = l_0 \\ e^{i\beta_j(l_0+1)\pi} & n_C(x) = l_0 + 1 \end{cases}.$$

Since T_C is again a 'rotation map', we might express this system as we did in the previous section, in the hope to apply the 'Fourier expansion technique' used in Proposition 6.4.3 to prove ergodicity of $(\mathcal{A}_C, \varphi_C, \tau'_C)$, but the fact that the h_j, for $j = 1, 2, 3$ are no longer constant, significantly complicates things in taking that approach.

We might also try an approach using the original definition of ergodicity as in Definition 4.2.5, but this path is also complicated by the structure that iterates of τ'_C take on, because the h_j are not constant. We see that for any $k \in \mathbb{N}$

$$\begin{aligned}
(\tau'_C)^k(\sum_{j=0}^{3} f_j\sigma_j) &= (\tau'_C)^{k-1} \circ \tau'_C(\sum_{j=0}^{3} f_j\sigma_j) \\
&= (\tau'_C)^{k-1}(\sum_{j=0}^{3} h_j(f_j \circ T_C)\sigma_j) \\
&= (\tau'_C)^{k-2}(\sum_{j=0}^{3} h_j(h_j \circ T_C)(f_j \circ T_C^2)\sigma_j) \\
&= \vdots \\
&= \sum_{j=0}^{3} \left(\prod_{r=0}^{k-1} h_j \circ T_C^r\right)(f_j \circ T_C^k)\sigma_j,
\end{aligned}$$

and the product $\prod_{r=0}^{k-1} h_j \circ T_C^r$ significantly complicates things when considering averages of terms involving this iterate.

We still raise the following:

Conjecture 6.6.1. *Consider $(\mathcal{A}, \varphi, \tau_\alpha)$ (as constructed above from $(S^1, \mathcal{B}, \lambda, T_\alpha)$). For an arbitrary $C \in \mathcal{B}$ of positive measure, let the system $(\mathcal{A}_C, \varphi_C, \tau'_C)$ with $\mathcal{A}_C$ and φ_C constructed from the classical induced system $(C, \mathcal{B} \cap C, \lambda', T_C)$ as in Section 6.1 and τ'_C defined by*

$$\left(\tau'_C(\sum_{j=0}^{3} f_j\sigma_j)\right)_x := \sum_{i=0}^{3} e^{i\beta_j n_C(x)\pi}(f_j \circ T_C)(x)\sigma_j.$$

where each $f_j \in L^\infty(C, \mathcal{B} \cap C, \lambda')$.

The ergodicity of $(\mathcal{A}, \varphi, \tau_\alpha)$ implies the ergodicity of $(\mathcal{A}_C, \varphi_C, \tau'_C)$.

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