Preface

This thesis is the first (but hopefully by far not the last) chapter in my efforts to gain a better understanding of quantum field theories. The main objective does not look very impressive, as it is simply to understand a certain article on algebraic quantum field theory by Hans Halvorson and Michael Müger [19] [30]. More accurately, the aim was to understand the mathematics used in this paper, as this thesis is a math thesis. For the most part this thesis is concerned with Tannaka-Krein duality and Deligne's embedding theorem. These constructions are central players in the DHR analysis. A complete discussion of the DHR analysis and what it can tell us about superselection theory in AQFT is beyond the scope of this thesis. It would not be surprising if the DHR analysis has more secrets yet to be uncovered.

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The aim of science is to seek the simplest explanation of complex facts. We are apt to fall into the error of thinking that the facts are simple because simplicity is the goal of our quest. The guiding motto in the life of every natural philosopher should be, "Seek simplicity and distrust it".

-Alfred North Whitehead, *The Concept of Nature*

The Category Theory Behind the Doplicher Roberts Reconstruction Theorem

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Contents

Chapter 1

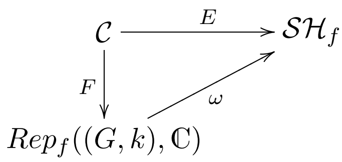
DHR Analysis

Category theory is becoming increasingly important as a tool for studying the foundations of quantum physics. This thesis deals with one particular application of category theory to the study of quantum field theories, namely the DHR analysis. The story told here was told before in the work of Halvorson and Müger [19] [30]. Originally the Doplicher-Roberts reconstruction theorem upon which the DHR analysis rests was proven by Doplicher and Roberts in [11] and [12]. In the work of Halvorson and Müger a different proof of the theorem is given, based on Deligne's embedding theorem [9] [4] and an unpublished manuscript of Roberts [33].

There are valuable lessons on the foundations of quantum field theory that can be learned from the DHR analysis. For a certain class of quantum field theories, made more precise later using the DHR selection criterion, the analysis shows how to construct a gauge group and a field algebra acting concretely on a Hilbert space from a net of local observable algebras and a state on this net. This may help to establish a connection between the algebraic formulation of quantum field theory and the canonical Hilbert space formulation. It also shows that algebraic quantum theory may be able to account for facts like the spin-statistics theorem as it is not as empty of field operators as it looks at first glance. The construction may also help shed light on the nature of the gauge group. Using Tannaka-Krein duality and Deligne's embedding theorem we can study compact (super)groups by studying symmetric TC*-categories like the DHR category that we will encounter later in this chapter.

A different reason why the DHR analysis is important is the following. An important issue in algebraic quantum theory is the role of inequivalent representations of the observables (the observables take the shape of the quasilocal algebra). The DHR analysis may help with this issue by means of superselection theory. One of the main engines of the DR reconstruction theorem is the following theorem which is a combination of Tannaka-Krein duality, Deligne's embedding theorem and some 'super'-terminology that is explained in Section 4.1.

Theorem 1.0.1. *Let $\mathcal{C}$ be a C^*-tensor category that has direct sums (biproducts), an irreducible tensor unit, subobjects, conjugates and a unitary symmetry and $\mathcal{SH}_f$ be the category of finite-dimensional super Hilbert spaces over $\mathbb{C}$. Then there exists a faithful tensor $*$-functor $E : \mathcal{C} \to \mathcal{SH}_f$. Furthermore there is a compact supergroup (G, k), unique up to isomorphism of supergroups such that $\mathcal{C}$ is equivalent as a tensor $*$-category to $Rep_f((G, k), \mathbb{C})$, the category of finite dimensional representations of the supergroup (G, k) over $\mathbb{C}$. The equivalence is given by a functor $F : \mathcal{C} \to Rep_f((G, k), \mathbb{C})$ such that the following diagram, where ω denotes the forgetful functor, commutes.*



A C*-tensor category as in the theorem will be called a STC* category. If the category does not have a symmetry but only a unitary braiding, we will call it a BTC* category. Chapters 2, 3 and 4 are devoted to proving the theorem assuming minimal prerequisites. The needed category theory for these chapters is developed during the text, so no familiarity with category theory is necessary. In the current chapter we assume the above theorem and use it in the DHR analysis. In Section 1.1 we discuss some of the basics of algebraic quantum field theory. Section 1.2 introduces the DHR category that has representations of the quasilocal algebra satisfying the DHR selection criterion as its objects. It is shown that for theories where the DHR selection criterion makes sense and spacetime is flat with dimension 3 or greater, the DHR category is a STC* category. If the spacetime dimension is 1 or 2, then it is a BTC* category. Section 1.3 introduces the notion of a field system with gauge symmetry. A field system with gauge symmetry contains a Hilbert space representation (H, π) of the quasilocal algebra $\mathfrak{U}$ and local fields acting on this space as a net of von Neumann algebras $\mathcal{O} \to \mathfrak{F}(\mathcal{O})$. A field system also contains a compact gauge group that has a unitary action on H. Dealing with only local fields, it turns out that every subrepresentation of the representation (H, π) of $\mathfrak{U}$ is a DHR representation. After this we start out with a net of observable algebras and a vacuum state on this net. Using Theorem 1.0.1 we construct a gauge group and a field algebra. We will find that there is a complete normal field system with gauge symmetry associated to the algebraic data, which is unique up to equivalence of field systems.

Next to nothing from this chapter will be used in the other chapters. The reader that finds this chapter more confusing than enlightening can safely skip it. Alternatively it can be read after the other chapters. The discussion of the DHR analysis presented in this chapter is brief and a lot of important details are omitted[1]. Even though a thorough discussion of the DHR analysis is beyond the scope of this text, the analysis is too important to skip entirely.

1.1 Algebraic Quantum Field Theory

The DHR analysis is part of the algebraic formulation of quantum field theories. Algebraic quantum field theory, or AQFT for short, is not that well-known so we will use this section to give a minimal account of AQFT. The reader that wants to learn more may find the references [18] [19] [34] helpful.

Definition 1.1.1. *Define a diamond $\mathcal{O}$ in Minkowski spacetime*[2] *as the intersection of the causal future of a spacetime point x with the causal past of a spacetime point y which is in the future of x. Let $\mathcal{K}$ denote the set of all diamonds in Minkowski spacetime.*

Next suppose that we have a mapping $\mathcal{O} \to \mathfrak{U}(\mathcal{O})$, which assigns to each $\mathcal{O} \in \mathcal{K}$ a unital C*-algebra $\mathfrak{U}(\mathcal{O})$.

Definition 1.1.2. *A Banach algebra is an algebra $\mathfrak{U}$ over $\mathbb{C}$ (or over $\mathbb{R}$, but we restrict our attention to the complex numbers) that has a norm $\|\cdot\|$ relative to which $\mathfrak{U}$ is a Banach space and that is submultiplicative. The latter means that for every $a, b \in \mathfrak{U}$ we have $\|ab\| \leq \|a\| \cdot \|b\|$. If the algebra is unital, i.e. has a multiplicative identity e, then we demand that $\|e\| = 1$. A C*-algebra is a Banach algebra $\mathfrak{U}$ with an involution $*$ such that for every $u \in \mathfrak{U}$ we have $\|u^*u\| = \|u\|^2$.*

Any C*-algebra that appears in this chapter is assumed to be unital unless stated otherwise. The multiplicative unit will most often be denoted by 1. We will consider sets of C*-algebras $\{\mathfrak{U}(\mathcal{O})|\mathcal{O} \in \mathcal{K}\}$ where the elements of $\mathfrak{U}(\mathcal{O})$ represent observables[3], localized in $\mathcal{O}$. We refer to the mapping $\mathcal{O} \to \mathfrak{U}(\mathcal{O})$ as the net of observable algebras. Thinking of elements of $\{\mathfrak{U}(\mathcal{O})|\mathcal{O} \in \mathcal{K}\}$ as local observables it is natural to make the following assumption.

Assumption 1.1.3. *Let $\{\mathfrak{U}(\mathcal{O})|\mathcal{O} \in \mathcal{K}\}$ be a net of observable algebras over Minkowski spacetime. Then the net is assumed to be an inductive system in*

[1] In particular Poincare covariance of the net of observables and of the DHR representations. The reader can find a discussion in [19]

[2] It is also possible to discuss AQFT in curved spacetime but we will not consider this any further. See [17] for the curved spacetime case.

[3] It is not trivial to say that local observables make up an algebra. Take for example the normal Hilbert space formalism of quantum mechanics where observables are represented by self-adjoint operators. If $\hat{Q}$ and $\hat{P}$ are two noncommuting self-adjoint operators then the composition $\hat{P}\hat{Q}$ is not self-ajoint.

the following sense. If $\mathcal{O}_1 \subset \mathcal{O}_2$ holds for the two diamonds, then there is an embedding (isometric $*$-homomorphism) $\mathfrak{U}(\mathcal{O}_1) \hookrightarrow \mathfrak{U}(\mathcal{O}_2)$.

Loosely translated this means that an observable localized in a diamond $\mathcal{O}_1$ is also an observable in each bigger spacetime region. The assumption of inductivity implies that we can talk about the inductive limit of the net. The C*-completion of this inducitve limit is called the quasilocal algebra $\mathfrak{U}$. The quasilocal algebra is a C*-algebra containing observables that can be uniformly approximated by local observables. Later on we will make some more assumptions on the net of observables.

Readers that are familiar with the canonical formulation of quantum field theory may be wondering if the discussion thusfar has anything to do with quantum field theory in the slightest. In the canonical formulation of quantum field theories one uses a Hilbert space and there are self-adjoint operators on this space corresponding to physical quantities. There are also field operators that need not be self-adjoint but are no less important tools. For a given Hilbert space H, the bounded linear operators on this space $\mathcal{B}(H)$ form a unital C*-algebra as the reader can check. But on the other hand, an abstract C*-algebra need not be equivalent to the bounded linear operators on some Hilbert space. In order to get any connection between the canonical formalism and the algebraic formulation we need to start thinking about representations.

Definition 1.1.4. *Let $\mathfrak{U}$ be a C*-algebra. A representation of $\mathfrak{U}$ is a pair (H, π), where H is a Hilbert space and π is a $*$-homomorphism $\pi : \mathfrak{U} \to \mathcal{B}(H)$. The representation is called faithful if the morphism π is an isomorphism. Two representations (H, π) and (H', ϕ) of $\mathfrak{U}$ are called unitarily equivalent if there exists an unitary map $U : H \to H'$ such that for each $a \in \mathfrak{U}$ we have $U\pi(a) = \phi(a)U$.*

Instead of just 1 Hilbert space, we can get a whole lot of different Hilbert spaces for the quasi-local algebra of a net. If all representations were unitarily equivalent there would be no problem, but in general there will be inequivalent reprsentations. What role will these inequivalent representations play in AQFT? We consider two somewhat opposite stances one can take regarding this issue. The first position is that of the socalled 'Algebraic Imperialist'. The algebraic imperialist claims that the physical content of the theory is in the net $\mathcal{O} \to \mathfrak{U}(\mathcal{O})$ and the states on the quasilocal algebra (explained in a moment) $\mathfrak{U}$[4]. Representations (H, π) are useful tools but have no ontological significance whatsoever. The second position is that of the 'Hilbert Space Conservatist', which looks like it is closer to the views of theoretical physicists working with the canonical QFT formulation. In this

[4]Actually one should also include a subgroup of $Aut(\mathfrak{U})$ corresponding to the symmetries, but we did not even discuss what a symmetry is in AQFT

position the theory is described by the net $\mathcal{O} \to \mathfrak{A}(\mathcal{O})$ and one representation (H, π) of $\mathfrak{A}$. The DHR analysis will shed some light on inequivalent representations, and we will come back to the two different stances at the end of this chapter. For now, let us shift out attention to the states in AQFT.

Definition 1.1.5. *Let $\mathfrak{A}$ be a C^*-algebra. Then a state ω on $\mathfrak{A}$ is a positive normalized linear functional on $\mathfrak{A}$. Positive means that for every $a \in \mathfrak{A}$ we have $\omega(a^*a) \geq 0$ and normalized means that for the multiplicative unit $e \in \mathfrak{A}$ we have $\omega(e) = 1$.*

Is there any connection between the previous definition of a state and the definition of a state as a ray in a Hilbert space? If we look at the Gelfand-Naimark-Sagal theorem then the answer is yes.

Theorem 1.1.6. *(Gelfand-Naimark-Sagal) Let $\mathfrak{A}$ be a C^*-algebra and ω be a state on $\mathfrak{A}$. Then there exists a representation (H, π), unique up to unitarily equivalence, such that the following holds. There is a unit vector $\Omega \in H$ such that*

- *$\omega(a) = \langle \Omega, \pi(a)\Omega \rangle$, for each $a \in \mathfrak{A}$.*
- *Ω is cyclic for $\pi(\mathfrak{A})$, i.e. $\pi(\mathfrak{A})\Omega$ is dense in H*

Proof. We will give a sketch of the proof leaving some of the details to the reader. First define the set $\mathcal{I} = \{x \in \mathfrak{A} | \omega(x^*x) = 0\}$. This set is closed in $\mathfrak{A}$ (with respect to the norm topology). It is also a left ideal of $\mathfrak{A}$ as we will show. Suppose that $a \in \mathfrak{A}$ and $x \in \mathcal{I}$, then

$$\omega((ax)^*(ax))^2 = \omega(x^*(a^*ax))^2 \leq \omega(x^*x)\omega(x^*a^*aa^*ax) = 0.$$

Next consider the vector space $\mathfrak{A}/\mathcal{I}$. For $x, y \in \mathfrak{A}$ define

$$\langle x + \mathcal{I}, y + \mathcal{I} \rangle = \omega(y^*x).$$

This map provides a well-defined inner product on $\mathfrak{A}/\mathcal{I}$. Define the Hilbert space H to be the completion of $\mathfrak{A}/\mathcal{I}$ with respect to the norm induced by this inner product. Consider, for an $a \in \mathfrak{A}$ the map $\pi(a) : \mathfrak{A}/\mathcal{I} \to \mathfrak{A}/\mathcal{I}$ defined by $(x + \mathcal{I}) \mapsto ax + \mathcal{I}$. As $\mathcal{I}$ is a left ideal this map is well-defined and linear. As we want to show that $\pi(a) \in \mathcal{B}(H)$, we need to show that this operator is bounded (note that we used $\pi(a)$ to denote the map on $\mathfrak{A}/\mathcal{I}$ as well as it's extension on H, which is sloppy but harmless). For $b \in \mathfrak{A}$ we write $b \geq 0$ if b is self-adjoint and the spectrum of b is a subset of $[0, \infty)$. The boundedness of $\pi(a)$ can be shown using that for every $x \in \mathfrak{A}$ we have $x^*a^*ax \leq \|a\|^2 x^*x$. Readers that want to prove this identity may find the following hints helpful. First of all it can be shown that $\|a^*a\| - a^*a$ is positive in $\mathfrak{U}$. This implies that for each $x \in \mathfrak{U}$ we have $0 \leq x^*(\|a^*a\| - a^*a)x$. The boundedness now follows from

$$\|ax + \mathcal{I}\|^2 = \omega(x^*a^*ax) \leq \|a\|^2 \omega(x^*x) = \|a\|^2 \|x + \mathcal{I}\|^2$$

which in turn implies that $\|\pi(a)\| \leq \|a\|$. We have a well-defined map $\pi : \mathfrak{U} \to \mathcal{B}(H)$. It is straightforward to check that this map defines a representation of $\mathfrak{U}$. The unit vector Ω is defined as $\Omega = e + \mathcal{I}$, where e is the multiplicative unit of $\mathfrak{U}$. It immediately follows that $\omega(a) = \langle e + \mathcal{I}, ae + \mathcal{I}\rangle = \langle \Omega, \pi(a)\Omega\rangle$. The cyclic property of Ω follows from $\pi(\mathfrak{U})(e + \mathcal{I}) = \mathfrak{U}/\mathcal{I}$ and the definition of H.

This proves the existence of the representation and we move onto uniqueness. Suppose that we have another such representation (K, ϕ) and (cyclic) unit vector Ω'. Subsequently $\langle \Omega, \pi(a)\Omega\rangle_H = \langle \Omega', \phi(a)\Omega'\rangle_K$ holds for each $a \in \mathfrak{U}$. Define U on a dense subspace of H as $U(\pi(a)\Omega) = \phi(a)\Omega'$. We find that U is well-defined and an isometry because

$$\begin{aligned}\|\pi(a)\Omega\|_H^2 &= \langle \pi(a)\Omega, \pi(a)\Omega\rangle_H = \langle \Omega, \pi(a^*a)\Omega\rangle_H \\ &= \langle \Omega', \phi(a^*a)\Omega'\rangle_K = \|\phi(a)\Omega'\|_K^2 .\end{aligned}$$

The isometry U extends to an isomorphism $H \to K$. Finally note that for every $x \in \mathfrak{U}$

$$U\pi(a)\pi(x)\Omega = U\pi(ax)\Omega = \phi(a)\phi(x)\Omega' = \phi(a)U\pi(x)\Omega$$

proving that the representations are unitarily equivalent. □

Every state gives a normalized vector in some Hilbert space. This raises questions like; how do we describe transition probabilities or measurement probabilities in the algebraic setting? These notions do have counterparts in the algebraic formalism [35], but we will not go into that here. Another problem in the algebraic formulation is that by focusing only on the observables we have closed our eyes to the field operators. How can we hope to account for facts like the spin-statistics theorem without using field operators? Again the DHR analysis will shed some light on this issue. In order to apply the DHR analysis we need additional assumptions on the net of observable algebras. The next definition will help with the formulation of the assumptions.

Definition 1.1.7. *Let $\mathcal{B}(H)$ denote the bounded linear operators on a Hilbert space H. A von Neumann algebra*[5] *$\mathfrak{R}$, is a $*$-subalgebra of $\mathcal{B}(H)$ satisfying*

- *The identity $e \in \mathfrak{R}$,*

[5] Von Neumann algebras are key players in AQFT but this thesis is not the place to elaborate on this.

- *If we denote* $\mathfrak{R}' = \{b \in \mathcal{B}(H) | [b, a] = 0, \forall a \in \mathfrak{R}\}$*, then* $(\mathfrak{R}')' = \mathfrak{R}$.

Instead of this algebraic definition of a von Neumann algebra which uses the commutant $\mathfrak{R}'$ of $\mathfrak{R}$, we could have given the equivalent topological definition. In the topological definition the second demand is replaced by the demand that $\mathfrak{R}$ is closed in $\mathcal{B}(H)$ with respect to the weak operator topology. We need the following data for the DHR analysis: a net of observables $\mathcal{O} \to \mathfrak{U}(\mathcal{O})$ and a state ω_0 on the quasilocal algebra $\mathfrak{U}$, which we will call the vacuum state. For the remainder of the thesis, we will assume that each of the following conditions are satisfied by this data.

Assumption 1.1.8. *(Microcausality) Let* $\mathcal{O}_1, \mathcal{O}_2$ *be diamonds in Minkowski spacetime that are spacelike separated. Then* $[\mathfrak{U}(\mathcal{O}_1), \mathfrak{U}(\mathcal{O}_2)] = \{0\}$.

The assumption of microcausality can be seen as a restriction coming from the theory of relativity[6].

Assumption 1.1.9. *(Property B in the vacuum sector) Let* (H_0, π_0) *be the GNS (Gelfand-Naimark-Sagal) representation of* $\mathfrak{U}$ *with respect to* ω_0*. The net of von Neumann algebras* $\mathcal{O} \to \mathfrak{R}_0(\mathcal{O}) = \pi_0(\mathfrak{U}(\mathcal{O}))''$ *satisfies the following property. Let* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *be 2 diamonds such that* $\overline{\mathcal{O}}_1 \subset \mathcal{O}_2$*. Then each nonzero projection* $E \in \mathfrak{R}(\mathcal{O}_1)$ *there exists an isometry* $V \in \mathfrak{R}(\mathcal{O}_2)$ *such that* $VV^* = E$.

This assumption looks a lot more technical. It can be seen as a consequence of other assumptions, the spectrum condition, additivity and microcausality that have a more clear interpretation. If the von Neumann algebras involved are type III factors, then property B is satisfied. We can think of property B as the assumption that the von Neumann algebras $\pi_0(\mathfrak{U}(\mathcal{O}))''$ are enough like type III factors. Property B is discussed in Section 2 and Section 7 of [19].

Assumption 1.1.10. *(Haag duality in the vacuum sector) For each diamond* $\mathcal{O} \in \mathcal{K}$ *The pair* $(\mathfrak{U}, \omega_0)$ *satisfies* $\pi_0(\mathfrak{U}(\mathcal{O}'))' = \pi_0(\mathfrak{U}(\mathcal{O}))''$*. Here* $\mathcal{O}'$ *stands for all spacetime points that are spacelike separated from* $\mathcal{O}$.

Assuming microcausality we have the following notion of locality. Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated. Then the elements of $\pi_0(\mathfrak{U}(\mathcal{O}_1))$ and $\pi_0(\mathfrak{U}(\mathcal{O}_2))$ commute pairwise. As a consequence we find for each diamond $\mathcal{O}$ that

$$\pi_0(\mathfrak{U}(\mathcal{O}))'' \subset \pi_0(\mathfrak{U}(\mathcal{O}'))'$$

holds. Haag duality is the stronger claim that the subset symbol can be replaced by an equality. It says that $\pi_0(\mathfrak{U}(\mathcal{O}))''$ cannot be enlarged without

[6]Here we implicitly assume that we will be dealing exclusively with relativistic field theories. An arbitrary quantum field theory need not relativistic, it is just a quantum theory with an infinite number of degrees of freedom.

violating locality. It is possible to perform the DHR analysis assuming essential duality instead of Haag duality. Essential duality is stronger than locality but weaker than Haag duality. For essential duality a bigger net is constructed that is assumed to satisfy Haag duality. Essential duality is discussed in Section 10.7 of [19] and in [34] [17]. Haag duality in the vacuum sector is connected to the absence of spontaneously broken gauge symmetries.

Assumption 1.1.11. *(Separability) The vacuum Hilbert space H_0 is separable.*

Assumption 1.1.12. *(Nontriviality) For each $\mathcal{O} \in \mathcal{K}$, $\pi_0(\mathfrak{U}(\mathcal{O}))$ contains at least one operator that is not a multiple of the identity.*

The last two assumptions play only a small role in the DHR analysis.

1.2 The DHR Category

This section is based on Section 8 of Halvorson [19], where full proofs of all the propositions and theorems cited here can be found. We start out with a net of observable algebras $\mathcal{O} \to \mathfrak{U}(\mathcal{O})$, the quasilocal algebra of this net $\mathfrak{U}$, and a vacuum state on the quasilocal algebra ω_0. Throughout this section we will denote the GNS representation on $\mathfrak{U}$ with respect to ω_0 by (H_0, π_0). We will restrict our attention to massive quantum field theories, i.e. field theories that have no long-range forces. Although work has been done in the more general case [5] [6], the massive case is involved enough for this chapter. In massive theories, where the fields can only have local excitations, the DHR selection criterion helps to pick out the physical representations of the quasilocal algebra.

DHR Selection Criterion: A representation (H, π) of $\mathfrak{U}$ is of interest, only if for each diamond $\mathcal{O}$

$$\pi|_{\mathfrak{U}(\mathcal{O}')} \cong \pi_0|_{\mathfrak{U}(\mathcal{O}')}$$

holds. This means that on $\mathcal{O}'$, the spacelike complement of $\mathcal{O}$, the representation is unitarily equivalent to the vacuum representation.

One of the main goals of this chapter is to get insight in the role that different (inequivalent) representations of the net of observables play in AQFT. It may seem weird to dismiss representations a priori when we do not know what representations mean in AQFT. On the other hand, the selection criterion for massive theories seems plausible when we think of the vacuum-like appearance that all states have with respect to measurements at great distances[7].

[7]For further motivation of the DHR selection criterion, see Remark 8.58 in [19]

Definition 1.2.1. *Define a DHR representation to be a representation of* $\mathfrak{A}$ *satisfying the DHR selection criterion. Then the DHR category* $DHR(\mathfrak{A}, \omega_0)$ *is defined to be the category that has DHR representations as objects and bounded intertwining operators as arrows.*

By definition $DHR(\mathfrak{A}, \omega_0)$ is a C*-category. To help give more structue to $DHR(\mathfrak{A}, \omega_0)$ we look at a different but related category.

Definition 1.2.2. *Let* ρ *be a* $*$*-endomorphism of* $\mathfrak{A}$ *and* $\mathcal{O}$ *be a diamond in spacetime. Then* ρ *is localized in* $\mathcal{O}$ *if*

$$\rho(a) = a, \quad \forall a \in \mathfrak{A}(\mathcal{O}')$$

holds. The morphism ρ *is called localized if there exists a diamond* $\mathcal{O}$ *such that* ρ *is localized in* $\mathcal{O}$*.*

Definition 1.2.3. *Let* ρ *be a* $*$*-endomorphism of* $\mathfrak{A}$*, localized in* $\mathcal{O}$*. Then* ρ *is called transportable if it satisfies the following property. Let* $\mathcal{O}_1$ *be any diamond. Then there exists a* $*$*-endomorphism* ρ_1 *localized in* $\mathcal{O}_1$ *and a unitary* $u \in \mathfrak{A}$ *such that*

$$u\rho(a) = \rho_1(a)u, \quad \forall a \in \mathfrak{A}.$$

Definition 1.2.4. *Define the category* Δ *as follows. The objects are the localized, transportable morphisms of* $\mathfrak{A}$*. The arrows are defined by*

$$Hom_\Delta(\rho, \rho') = \{t \in \mathfrak{A} | t\rho(a) = \rho'(a)t, \ \forall a \in \mathfrak{A}\}.$$

Composition of arrows is inherited from $\mathfrak{A}$*. If* $t \in Hom_\Delta(\rho, \rho')$ *and* $s \in Hom_\Delta(\rho', \sigma)$*, then* $s \circ t = st$*.*

The set of localized transportable morphisms that are localized in the diamond $\mathcal{O}$ *is denoted by* $\Delta(\mathcal{O})$*.*

It is clear that $s \circ t$ as defined above is an element of $Hom_\Delta(\rho, \sigma)$ as

$$st\rho(a) = s(t\rho(a)) = s(\rho'(a)t) = (s\rho'(a))t = (\sigma(a)s)t = \sigma(a)st.$$

Note that composition is associative and that for every localized transportable morphism ρ the unit 1 of $\mathfrak{A}$ acts as the identity arrow. Before connecting Δ to $DHR(\mathfrak{A}, \omega_0)$ we will first show that Δ is a C*-category. This amounts to showing that Δ is a $\mathbb{C}$-linear category with a positive $*$-operation and a norm $\|\cdot\|_{\rho,\rho'}$ on each $Hom_\Delta(\rho, \rho')$ that makes it into a Banach space. Furthermore the norms satisfy

$$\|st\|_{\rho,\sigma} \leq \|s\|_{\rho',\sigma} \|t\|_{\rho,\rho'}, \quad \|t^*t\|_{\rho,\rho} = \|t\|^2_{\rho,\rho}$$

for all $\rho, \rho', \sigma,\ s \in Hom_\Delta(\rho', \sigma)$ and $t \in Hom_\Delta(\rho, \rho')$. The C*-structure on Δ comes from $\mathfrak{A}$. Let $s \in Hom_\Delta(\rho, \rho')$ and $* : \mathfrak{A} \to \mathfrak{A}$ be the positive

$*$-operation on $\mathfrak{A}$. A straightforward check reveals that $s^* \in Hom_\Delta(\rho', \rho)$. The other structure on the Hom-sets, norms included can also be taken straight from $\mathfrak{A}$ as the reader can verify. In short Δ is a C*-category. The next proposition connects the category Δ to the category $DHR(\mathfrak{A}, \omega_0)$.

Proposition 1.2.5. *The mappings $F : \Delta \to DHR(\mathfrak{A}, \omega_0)$, given by $F(\rho) = \pi_0 \circ \rho$ on the objects of Δ, and $F(s) = \pi_0(s)$ on the morphisms of Δ define a functor of C*-categories. This functor defines an equivalence of categories.*

The proof of this proposition, given in [19], shows that mappings give a well-defined functor and subsequently shows that this functor is faithful, full and essentially surjective. In proving that the functor is full and essentially surjective, Haag duality of the vacuum sector is invoked. The functor F can be used to transfer structure of Δ to the category $DHR(\mathfrak{A}, \omega_0)$, structure that we will now explore. We start by showing that Δ has direct sums (biproducts) and subobjects. It is here that property B of the net of observables becomes important. Suppose that $\rho_1 \in \Delta(\mathcal{O}_1)$ and $\rho_2 \in \Delta(\mathcal{O}_2)$. Choose a diamond $\mathcal{O}$ such that $(\mathcal{O}_1 \cup \mathcal{O}_2)^- \subset \mathcal{O}$. Suppose that e is a non-zero projection in $\mathfrak{A}(\mathcal{O}_1)$. Using property B there are isometries $v_1, v_2 \in \mathfrak{A}(\mathcal{O})$ such that $v_1 v_1^* + v_2 v_2^* = 1$. Define

$$\rho(a) = v_1 \rho_1(a) v_1^* + v_2 \rho_2(a) v_2^*, \quad \forall a \in \mathfrak{A}.$$

Then ρ is localized in $\mathcal{O}$ and can be shown to be transportable. The object ρ defines a biproduct $\rho_1 \oplus \rho_2$ in Δ. For subobjects we make use of property B once again. Let $\rho \in \Delta$ be localized in $\mathcal{O}$ and $e \in Hom_\Delta(\rho, \rho)$ be a projection. By definition we have

$$ea = e\rho(a) = \rho(a)e = ae, \qquad \forall a \in \mathfrak{A}(\mathcal{O}').$$

Using Haag duality we can conclude that $e \in \mathfrak{A}(\mathcal{O})$. Choose a diamond $\mathcal{O}_1$ such that $\overline{\mathcal{O}} \subset \mathcal{O}_1$. By property B there is an isometry $v \in \mathfrak{A}(\mathcal{O}_1)$ such that $e = vv^*$. Define

$$\rho'(a) = v^* \rho(a) v, \qquad \forall a \in \mathfrak{A},$$

then ρ' can be shown to be a localized transportable morphism, thus an object of Δ. The identity

$$\rho'(a)v^* = v^*\rho(a)vv^* = v^*\rho(a)e = v^*e\rho(a) = v^*\rho(a), \quad \forall a \in \mathfrak{A}$$

shows that $v \in Hom_\Delta(\rho', \rho)$ defines an isometry. This shows that Δ has subobjects. Thusfar we have shown that Δ is a C*-category that has direct sums and subobjects.

The next step is to turn Δ into a C*-tensor category. The identity automorphism $\iota : \mathfrak{A} \to \mathfrak{A}$ will play the role of tensor unit. As an object of Δ it is irreducible in the sense that $Hom_\Delta(\iota, \iota) = \mathbb{C}id_\iota$. In order to define the tensor bifunctor we first note that if ρ and σ are objects of Δ, then so is $\rho\sigma$. If ρ is localized in $\mathcal{O}_1$ and σ is localized in $\mathcal{O}_2$, then $\rho\sigma$ is localized in $\mathcal{O}_1 \cup \mathcal{O}_2$. Now take $\mathcal{O}_3$ to be any diamond. As ρ and σ are transportable, there exist ρ' and σ' localized in $\mathcal{O}_3$ and unitaries $u \in Hom_\Delta(\rho, \rho')$, $v \in Hom_\Delta(\sigma, \sigma')$. Then $\rho'\sigma'$ is localized in $\mathcal{O}_3$ and $u\rho(v) \in Hom_\Delta(\rho\sigma, \rho'\sigma')$ is unitary and satisfies

$$u\rho(v)\rho\sigma(a) = u\rho(v\sigma(a)) = u\rho(\sigma'(a)v) = u\rho(\sigma'(a))\rho(v) = \rho'\sigma'(a)u\rho(v),$$

showing that $\rho\sigma$ is transportable. Define

$$\rho \otimes \sigma = \rho\sigma, \quad \rho, \sigma \in Obj(\Delta),$$

$$s \otimes t = s\rho(t), \quad s \in Hom_\Delta(\rho, \rho'), t \in Hom_\Delta(\sigma, \sigma'),$$

then $(\Delta, \otimes, \iota)$ defines a C*-tensor category. There are a lot of details that need to be checked, some are given in [19], the others are straightforward calculations.

In order to apply Tannaka-Krein duality and Deligne's embedding theorem to Δ we need conjugates and a unitary symmetry. We will define a unitary braiding on Δ in several steps. It will turn out that only for a dimension of spacetime of at least 3, this braiding is a symmetry. We start by defining arrows

$$c_{\rho_1,\rho_2}(u_1, u_2) : \rho_1 \otimes \rho_2 \to \rho_2 \otimes \rho_1,$$

of Δ that depend on socalled spectator morphisms. Suppose that $\rho_1 \in \Delta(\mathcal{O}_1)$ and that $\rho_2 \in \Delta(\mathcal{O}_2)$. Pick $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ to be spacelike separated diamonds. Because ρ_1 and ρ_2 are transportable there exist morphisms $\tilde{\rho}_1 \in \Delta(\tilde{\mathcal{O}}_1)$, $\tilde{\rho}_2 \in \Delta(\tilde{\mathcal{O}}_2)$ and unitaries $u_1 \in Hom_\Delta(\rho_1, \tilde{\rho}_1)$, $u_2 \in Hom_\Delta(\rho_2, \tilde{\rho}_2)$. The morphisms $\tilde{\rho}_i$ are the spectator morphisms. The following lemma helps us by showing that $\tilde{\rho}_1 \otimes \tilde{\rho}_2 = \tilde{\rho}_2 \otimes \tilde{\rho}_1$.

Lemma 1.2.6. *Let $\rho \in \Delta(\mathcal{O}_1)$ and $\sigma \in \Delta(\mathcal{O}_2)$, where $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated. Then $\sigma\rho = \rho\sigma$.*

Using $\tilde{\rho}_1 \otimes \tilde{\rho}_2 = \tilde{\rho}_2 \otimes \tilde{\rho}_1$ we define

$$c_{\rho_1,\rho_2}(u_1, u_2) = u_2^* \otimes u_1^* \circ u_1 \otimes u_2 = \rho_2(u_1^*)u_2^* u_1 \rho_1(u_2).$$

It is clear that the arrows $c_{\rho_1,\rho_2}(u_1, u_2)$ are unitary. To what extent do these arrows depend on the spectator morphisms? The next proposition is pivotal in answering this question.

Proposition 1.2.7. *The arrow $c_{\rho_1,\rho_2}(u_1, u_2)$ can be defined in terms of $\rho_1, \rho_2, \tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ and therefore does not depend on the choice of $\tilde{\rho}_1, \tilde{\rho}_2, u_1$ and u_2. The arrow $c_{\rho_1,\rho_2}(u_1, u_2)$ does not change if the diamonds $\tilde{\mathcal{O}}_i$, $i \in \{1, 2\}$ are replaced by diamonds $\hat{\mathcal{O}}_i$ that are spacelike separated and have the property that $\tilde{\mathcal{O}}_i \subset \hat{\mathcal{O}}_i$.*

As a consequence of the proposition $c_{\rho_1,\rho_2}(u_1, u_2)$ does not change if $(\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2)$ is translated or if the diamonds $\tilde{\mathcal{O}}_i$ are replaced by spacelike separated diamonds $\hat{\mathcal{O}}_i$ that contain $\tilde{\mathcal{O}}_i$ or are contained in it. This shows that $c_{\rho_1,\rho_2}(u_1, u_2)$ depends very little on the choice of the $\tilde{\mathcal{O}}_i$. In fact, if the spacetime has a dimension of at least 3, and as long as the diamonds $\tilde{\mathcal{O}}_1$ and $\tilde{\mathcal{O}}_2$ are spacelike separated, $c_{\rho_1,\rho_2}(u_1, u_2)$ is the same for every choice of the $\tilde{\mathcal{O}}_i$. If the dimension of spacetime is at most 2, 1-dimensional spacetime just being 1-dimensional space, then $c_{\rho_1,\rho_2}(u_1, u_2)$ depends only on the relative orientation of the $\tilde{\mathcal{O}}_i$. We write $\tilde{\mathcal{O}}_1 < \tilde{\mathcal{O}}_2$ if $\tilde{\mathcal{O}}_1$ is to the left of $\tilde{\mathcal{O}}_2$. For a fixed orientation and $\tilde{\mathcal{O}}_1, \tilde{\mathcal{O}}_2$ spacelike separated $c_{\rho_1,\rho_2}(u_1, u_2)$ is independent of the further choice of $\tilde{\mathcal{O}}_i$. In order to demonstrate the dependence on spatial orientation consider the following case. First take $\tilde{\mathcal{O}}_1 = \mathcal{O}_1$, $\tilde{\rho}_1 = \rho_1$, $u_1 = id_{\rho_1} = 1$ and $\tilde{\mathcal{O}}_2 < \tilde{\mathcal{O}}_1$. We find

$$c_{\rho_1,\rho_2}(1, u_2) = u_2^* \rho_1(u_2), \quad \tilde{\mathcal{O}}_2 < \tilde{\mathcal{O}}_1.$$

Reverse the orientation and take $\hat{\mathcal{O}}_1 = \tilde{\mathcal{O}}_2$, $\hat{\rho}_1 = \tilde{\rho}_2$, $u_1' = u_2$, $\hat{\mathcal{O}}_2 = \mathcal{O}_1$,$\hat{\rho}_2 = \rho_1$ and $u_2' = 1$. We find

$$c_{\rho_2,\rho_1}(u_2, 1) = \rho_1(u_2^*)u_2 = c_{\rho_1,\rho_2}(1, u_2)^*, \quad \hat{\mathcal{O}}_1 < \hat{\mathcal{O}}_2.$$

The arrows $c_{\rho_1,\rho_2}(u_1, u_2)$ depend only on the orientation, therefore we have shown that if two different spatial orientations are used, then

$$c_{\rho_1,\rho_2}(u_1, u_2) = c_{\rho_2,\rho_1}(u_1', u_2')^*.$$

We can now define a braiding on Δ.

Definition 1.2.8. *Let ρ_1 and ρ_2 be objects of Δ. Suppose that spacetime has a dimension of 2 or less. Define $c_{\rho_1,\rho_2} = c_{\rho_1,\rho_2}(u_1, u_2)$ where $\tilde{\mathcal{O}}_2 < \tilde{\mathcal{O}}_1$ with the $\tilde{\mathcal{O}}_i$ spacelike separated. Now assume that spacetime has a dimension of at least 3. Define $c_{\rho_1,\rho_2} = c_{\rho_1,\rho_2}(u_1, u_2)$ where the $\tilde{\mathcal{O}}_i$ are spacelike separated.*

Theorem 1.2.9. *The arrows c_{ρ_1,ρ_2} define a unitary braiding on the tensor category $(\Delta, \otimes, \iota)$. For spacetimes that have a dimension of at most 2, this is the unique braiding on Δ such that $c_{\rho_1,\rho_2} = 1$ when $\rho_i \in \Delta(\mathcal{O}_i)$ with the $\mathcal{O}_i$ spacelike separated and $\mathcal{O}_2 < \mathcal{O}_1$. For spacetimes that have a dimension of at least 3, this is the unique braiding on Δ such that $c_{\rho_1,\rho_2} = 1$ when $\rho_i \in \Delta(\mathcal{O}_i)$ with the $\mathcal{O}_i$ spacelike separated. For spacetimes that have a dimension of at least 3, the braiding is a symmetry.*

The only thing standing between the category Δ and the application of Tannaka-Krein duality with Deligne's embedding theorem is the absence of conjugates. An introduction to conjugates and the dimension of objects in terms of conjugates is given in Section 3.3. Instead of considering the whole category Δ we pick the full subcategory Δ_f that has conjugates.

Definition 1.2.10. *Let Δ_f be the full subcategory of Δ such that the objects of this category are those that admit a conjugate relative to which their dimension is finite. More precisely, ρ is an object of Δ_f, if there exists a solution $(\overline{\rho}, r, \overline{r})$ of the conjugate equations in Δ such that $d(\rho) = r^* \circ r$ is a finite number times id_ι.*

The cateogry Δ_f is closed under direct sums, subobjects and tensor products. Putting everything together, we found the following in this section. We started out with the C*-category $DHR(\mathfrak{U}, \omega_0)$ of DHR representations. This category is equivalent, as a C*-category, to the category Δ of localized transportable morphisms of the quasilocal algebra. The full tensor subcategory Δ_f of this category is a BTC*-category. If spacetime has a dimension of at least 3 this category is a STC*-category.

1.3 Field Systems with Gauge Symmetry

This section is based on Section 9 and Section 10 of [19]. First we define a field system with gauge symmetry. This is as close as we are going to get to the canonical formalism of QFT in this chapter. The field system is a representation (H, π) of the quasilocal algebra $\mathfrak{U}$ with local fields and an internal symmetry group acting on it. It turns out that every subrepresentation of $\mathfrak{U}$ contained in the field system is a DHR representation. Proofs and details are ommited. The proofs can be found in [19] but require a background on operator algebras and representation theory of compact groups[8]. After looking at field systems we start out with the algebraic data $(\mathfrak{U}, \omega_0)$. As discussed in the previous section, we can construct from this data an STC* category Δ_f that is equivalent to a full subcategory of $DHR(\mathfrak{U}, \omega_0)$. We apply Tannaka-Krein duality and Deligne's embedding theorem to this category to recognize it as the category of finite dimensional representations of a compact supergroup. Subsequently we take several steps to construct a field system with gauge symmetry from Δ_f, where the Tannaka supergroup acts as the gauge group.

Throughout this section, let $\mathfrak{U}$ be the quasilocal algebra of a net $\mathcal{O} \to \mathfrak{U}(\mathcal{O})$ of observable algebras. Let ω_0 be a vacuum state on $\mathfrak{U}$ and (H_0, π_0) the corresponding GNS representation.

Definition 1.3.1. *A field system with gauge symmetry for the data $(\mathfrak{U}, \omega_0)$ is a quadruple $(H, \pi, \mathfrak{F}, (G, k))$. Where*

- *(H, π) is a representation of $\mathfrak{U}$.*
- *$\mathfrak{F}$ comes from a net of von Neumann algebras $\mathcal{O} \to \mathfrak{F}(\mathcal{O})$. The net has an irreducible action on H.*

[8]The books by Kadison and Ringrose [22] [23] may aid in providing knowledge on operator algebras.

- *(G, k) is a compact*[9] *supergroup of acting faithfully on H through unitary operators $U(g)$. This means that G is a compact group and that $k \in G$ is central and of order 2.*

The following holds for the quadruple:

1. *The vacuum representation (H_0, π_0) is a subrepresentation of (H, π). This means that there is an isometry $V : H_0 \to H$ such that $V\pi(a) = \pi_0(a)V$ holds for every $a \in \mathfrak{A}$.*

2. *The isometry V maps H_0 into the subspace of H consisting of vectors that are invariant under the action of G.*

3. *Suppose that $\mathcal{O}_1$ and $\mathcal{O}_2$ are spacelike separated. Then the elements of $\mathfrak{F}(\mathcal{O}_1)$ commute pairwise with the elements of $\pi(\mathfrak{A}(\mathcal{O}_2))$.*

4. *The group G induces and action on each $\mathfrak{F}(\mathcal{O})$. Each $g \in G$ leaves $\mathfrak{F}(\mathcal{O})$ globally fixed for every $\mathcal{O}$. The fixed points of $\mathfrak{F}(\mathcal{O})$ under the action of $g \in G$ are the elements of $\pi(\mathfrak{A}(\mathcal{O}))'' \subset \mathfrak{F}(\mathcal{O})$.*

5. *For every diamond $\mathcal{O}$, $V(H_0)$ is cyclic for $\mathfrak{F}(\mathcal{O})$. This means that the closed linear span of $\mathfrak{F}(\mathcal{O})V(H_0)$ is H.*

Notice that the field algebra $\mathfrak{F}$ is made out of only local fields $\mathfrak{F}(\mathcal{O})$. This is because we restrict to field theories where the fields can only have local excitations. Regarding the use of supergroups (G, k) instead of just groups G, it will become a little more clear in Section 4.1. Despite the less natural appearance of a compact supergroup compared with a compact group, the $k \in G$ will play an important role regarding the statistics of the fields. We will only scratch the surface of the important subject that is statistics in the DHR analysis.

We will identitify H_0 with its image under de isometry V from demand (1). Demand (2) tells us that H_0 is a G-invariant subspace of H. It turns out that H_0 is precisely the G-invariant subspace of H.

By microcausality we know that observables which localized at spacelike separated regions commute. For field operators we do not expect the same to hold. In conventional quantum field theory bosonic field operators at spacelike separated regions commute while fermionic field operators at spacelike separated regions anticommute. Demand (3) can be seen as a weaker version of the normal commutation rules for field operators.

The action of G on the local fields from demand (4) is given by

$$\alpha_g(F) = U(g)FU(g)^*, \quad F \in \mathfrak{F}(\mathcal{O}), g \in G.$$

Demand (4) tells us that G is an internal symmetry group of the fields. The action of G on the local fields does not change the spacetime localization of

[9] compact with respect to the strong operator topology

the field operators. It also demonstrates the gauge-invariance of the local observables.

Demand (5) deserves a little more attention. It tells us that using fields only from $\mathfrak{F}(\mathcal{O})$ we can reach every subspace of H from the vacuum. This fact is connected to the Reeh-Schlieder theorem. As this theorem is one of the cornerstones of AQFT, we will briefly discuss it. For the moment we will consider two additional assumptions that can be made on the net of local observables. The first assumption is the spectrum condition. It is a more abstract version of the claim that the spectrum of the momentum operator is in the forward lightcone. The latter claim is equivalent to saying that the energy is positive in every Lorentz frame. In order to formulate the spectrum condition we need the following fact.

Lemma 1.3.2. *Let T be a group that has a strongly continuous action α on $\mathfrak{U}$ and suppose that ω_0 is T-invariant in the sense that $\omega_0(\alpha_x a) = \omega_0(a)$ holds for all $a \in \mathfrak{U}$ and $x \in T$. Then the GNS representation (H_0, π_0) of $\mathfrak{U}$ induced by ω_0 carries a strongly continuous unitary representation U of T such that*

1. $\pi(\alpha_x a) = U(x)\pi(a)U(x)^*, \quad \forall a \in \mathfrak{U}, x \in T,$
2. $U(x)\Omega_0 = \Omega_0, \quad \forall x \in T.$

The vector $\Omega_0 \in H_0$ is the cyclic vector from the GNS construction Theorem 1.1.6.

Assumption 1.3.3. *(Spectrum Condition) Let T be the translation group and assume that it carries a strongly continuous action on $\mathfrak{U}$. Further suppose that the vacuum state ω_0 is T-invariant. Then $(\mathfrak{U}, \omega_0)$ satisfies the spectrum condition if there is a subset $T_+ \subset T$ such that $T_+ \cap (-T_+) = \{0\}$ holds, with $-T_+ = \{-x | x \in T_+\}$, and the following condition is satisfied. Let U be the induced unitary representation of T on H_0. Then the spectrum of the representation is contained in T_+.*

The reader may wonder how the spectrum can be a subset of T as the spectrum of an operator is a subset of $\mathbb{C}$. But note that we are taking the spectrum of the representation, which is a family of operators. Giving a proper definition of the spectrum would take the discussion too far adrift. In Section II.5 of [18] some more background is provided. The second assumption that we need is additivity and claims that there is no smallest length scale in the theory.

Assumption 1.3.4. *(Additivity) Let T be the translation group and assume that it carries a strongly continuous action α on $\mathfrak{U}$. Using the notation $\mathfrak{R}(\mathcal{O}) = \pi(\mathfrak{U}(\mathcal{O}))''$, the net $\mathcal{O} \to \mathfrak{R}(\mathcal{O})$ satisfies additivity if for each diamond $\mathcal{O}$ the set $\{\mathfrak{R}(\mathcal{O} + x) | x \in T\}$ generates $\mathfrak{R}$ as a C^*-algebra. Here we used the notation $\mathfrak{R}(\mathcal{O} + x) = \pi(\alpha_x(\mathfrak{U}(\mathcal{O})))''$.*

Convincing reasons to adopt additivity are lacking, as well as convincing reasons to refute it. It may well be that in the future a succesful theory of quantum gravity is found where spacetime is quantized. This would be bad news for additivity. If we assume additivity as well as the spectrum condition we can get the Reeh-Schlieder theorem.

Theorem 1.3.5. *(Reeh-Schlieder) Let the spectrum condition and additivity be satisfied for the data* $(\mathfrak{A}, \omega_0)$. *Then for each diamond* $\mathcal{O}$, *the GNS vector* Ω_0 *is cyclic for* $\pi(\mathfrak{A}(\mathcal{O}))''$.

If the net $\mathcal{O} \to \pi(\mathfrak{A}(\mathcal{O}))''$ satisfies microcausality, even more can be said [19]. The theorem is of importance because of the nonlocal correlations of the vacuum state it entails. It implies for example that there can be no local number operators. Note the strong resemblance between the Reeh-Schlieder theorem and the demand (5) in the definition of a field system with gauge symmetry.

After this brief excursion we drop the spectrum condition and additivity and return to the field systems. Thusfar we did not comment on the role of $k \in G$ in the supergroup (G, k) of the field system. This central element of order 2 helps us to pick out the fermionic and bosonic field operators in the field algebra.

Definition 1.3.6. *Let* $(H, \pi, \mathfrak{F}, (G, k))$ *be a field system with gauge symmetry for* $(\mathfrak{A}, \omega_0)$, *and* $F \in \mathfrak{F}(\mathcal{O})$. *Then* F *is called a Bose field operator if* $\alpha_k(F) = F$. *The operator* F *is called a Fermi field operator if* $\alpha_k(F) = -F$.

Definition 1.3.7. *Let* $(H, \pi, \mathfrak{F}, (G, k))$ *be a field system with gauge symmetry for* $(\mathfrak{A}, \omega_0)$. *Then the fields are said to satisfy normal commutation relations if the following property holds for each pair of spacelike separated diamonds* $\mathcal{O}_1$ *and* $\mathcal{O}_2$. *Take field operators* $F_i \in \mathfrak{F}(\mathcal{O}_i)$ *satisfying* $\alpha_k(F_i) = \epsilon_i F_i$ *where* $\epsilon \in \{-1, 1\}$. *Then these local fields satisfy*

$$F_1 F_2 = (-1)^{(1-\epsilon_1)(1-\epsilon_2)/4} F_2 F_1.$$

Field systems are intimately connected to DHR representations. The following proposition shows that DHR representations are the building blocks of field systems. Later we will see the how DHR representations can be used to construct a field system. This proposition is the only result that makes use of Assumption 1.1.11, separability of H_0.

Proposition 1.3.8. *Let* $(H, \pi, \mathfrak{F}, (G, k))$ *be a field system with gauge symmetry for* $(\mathfrak{A}, \omega_0)$. *Let* $Rep_{\mathfrak{F}}(\mathfrak{A})$ *denote the category of subrepresentations of the representation* (H, π) *of* $\mathfrak{A}$, *viewed as a full subcategory of the category of representations of* $\mathfrak{A}$. *Then there exists a faithful functor*

$$F : Rep_{\mathfrak{F}}(\mathfrak{A}) \to DHR(\mathfrak{A}, \omega_0).$$

Using the equivalence $DHR(\mathfrak{A}, \omega_0) \cong \Delta$ *of the previous section this functor yields a faithful functor* $G : Rep_{\mathfrak{F}}(\mathfrak{A}) \to \Delta_f$ *into the BTC* category* Δ_f.

Field systems are thus made out of DHR representations (with conjugates), but can all DHR representations arise in this way? The DR reconstruction theorem answers this question affirmative, but before we can state it we need two more definitions.

Definition 1.3.9. *A field system with gauge symmetry $(H, \pi, \mathfrak{F}, (G, k))$ for $(\mathfrak{U}, \omega_0)$ is said to be complete if each DHR representation that corresponds to an object of Δ_f occurs as a subrepresentation of (H, π).*

Definition 1.3.10. *Let $(H_1, \pi_1, \mathfrak{F}_1, (G_1, k_1))$ and $(H_2, \pi_2, \mathfrak{F}_2, (G_2, k_2))$ be two field systems with gauge symmetry for $(\mathfrak{U}, \omega_0)$. Then the field systems are called equivalent if there is a unitary operator $W : H_1 \to H_2$ such that the following holds.*

1. $W\pi_1(a) = \pi_2(a)W, \quad \forall a \in \mathfrak{U},$
2. $WU_1(G_1) = U_2(G_2)W,$
3. $W\mathfrak{F}_1(\mathcal{O}) = \mathfrak{F}_2(\mathcal{O})W$ *for each diamond* $\mathcal{O}$.

We can now state the main theorem of this chapter.

Theorem 1.3.11. *(Doplicher-Roberts reconstruction theorem) For the data $(\mathfrak{U}, \omega_0)$ there exists a field system with gauge symmetry $(H, \pi, \mathfrak{F}, (G, k))$ that is complete and has normal commutation relations. Any field system with gauge symmetry for this data, which is complete and has normal commutation relations, is equivalent to $(H, \pi, \mathfrak{F}, (G, k))$.*

This theorem was first proven by Doplicher and Roberts [11] [12]. We will discuss the different version given in Halvorson and Müger [19] [30]. This proof applies Tannaka-Krein duality combined with Deligne's embedding theorem to the category Δ_f to obtain the gauge group. The field algebra is constructed using the functor $E : \Delta_f \to \mathcal{SH}_f$ provided by Deligne's embedding theorem and the work of Roberts [33]. We start with the construction of a gauge group for $(\mathfrak{U}, \omega_0)$. The first result that we need is Tannaka-Krein duality, which is proven in Chapter 3.

Theorem 1.3.12. *(Tannaka-Krein Duality) Let $\mathcal{C}$ be a STC* and $E : \mathcal{C} \to \mathcal{H}_f$ a symmetric $*$-preserving fiber functor into the category of finite dimensional Hilbert spaces. Let G_E be the compact group of monoidal natural transformations of E to itself. Then there exists a symmetric faithful tensor $*$-functor $F : \mathcal{C} \to Rep_f(G_E, \mathbb{C})$ where $Rep_f(G_E, \mathbb{C})$ is the category of finite dimensional representations of G_E. If $\omega : Rep_f(G_E, \mathbb{C}) \to \mathcal{H}_f$ is the forgetful functor, then we have $\omega \circ F = E$. The functor F is an equivalence of symmetric tensor $*$-categories.*

In order to apply this to Δ_f we need a functor E as above. Deligne's embedding theorem, which is proven in Chapter 4, comes to the rescue.

Theorem 1.3.13. *(Deligne's embedding theorem) Let $\mathcal{C}$ be an even STC*. Then there exists a symmetric $*$-preserving fiber functor $E : \mathcal{C} \to \mathcal{H}_f$.*

A STC* category is even if the canonical twist, defined in Section 4.1, is trivial. Only even STC* categories are enough like the category of finite dimensional representations of a compact group to admit an equivalence F. But if we shift our attention to the category of finite dimensional representations of compact supergroups, as explained in Section 4.1, and use the category of finite dimensional super Hilbert spaces, then we find

Theorem 1.3.14. *Let $\mathcal{C}$ be any STC* category. Then there exists a compact supergroup (G, k) which is unique up to an isomorphism of supergroups, and an equivalence $F : \mathcal{C} \to Rep_f((G, k), \mathbb{C})$ of symmetric tensor $*$-categories.*

In particular, if $\omega : Rep_f((G, k), \mathbb{C}) \to \mathcal{SH}_f$ is the forgetful functor, then the composition $E = \omega \circ F : \mathcal{C} \to \mathcal{SH}_f$ is a faithful symmetric $*$-preserving tensor functor into the STC category of finite dimensional super Hilbert spaces.*

We apply this to the data $(\mathfrak{U}, \omega_0)$. We have to restrict to Minkowski spacetimes that have a dimension of at least 3, otherwise Δ_f is not a STC* category but just a BTC* category. Applying Theorem 1.3.14 then gives us a compact supergroup (G, k) and an embedding $E : \Delta_f \to \mathcal{SH}_f$. Both are important in the construction of the field algebra which we now consider. Define $\mathfrak{F}_0$ as the set of triples

$$(a, \rho, \psi), \quad a \in \mathfrak{U}, \rho \in Obj(\Delta_f), \psi \in E(\rho),$$

subject to the following equivalence relation. If $s \in Hom_{\Delta_f}(\rho, \rho')$ and consequently $E(s) \in Hom_{\mathcal{SH}_f}(E(\rho), E(\rho'))$, then

$$(as, \rho, \psi) = (a, \rho', E(s)\psi).$$

The reader is invited to check that this defines an equivalence relation. In the next few steps we will add structure to $\mathfrak{F}_0$. The details of the constructions can be found in Section 10.2 and Section 10.3 of [19]. First we turn the set $\mathfrak{F}_0$ into a vector space. Scalar multiplication is defined by

$$\lambda(a, \rho, \psi) = (\lambda a, \rho, \psi) = (a, \rho, \lambda\psi).$$

The first equality is the definition while the second equality follows from the equivalence relation by using $E(\lambda id_\rho) = \lambda id_{E(\rho)}$. Addition is defined by

$$(a_1, \rho_1, \psi_1) + (a_2, \rho_2, \psi_2) = (a_1 w_1^* + a_2 w_2^*, \rho, E(w_1)\psi_1 + E(w_2)\psi_2),$$

where the $w_i \in Hom_{\Delta_f}(\rho_i, \rho)$ are isometries satisfying

$$w_1 w_1^* + w_2 w_2^* = id_\rho, \quad w_i^* w_j = \delta_{ij} id_{\rho_i}.$$

Such isometries always exist as Δ_f has direct sums. Using the equivalence relation in the definition of $\mathfrak{F}_0$ one can show that this definition does not depend on the choice of the isometries. If we identify $\mathfrak{U}$ with the subset $\{(a, \iota, e(1))|a \in \mathfrak{U}\} \subset \mathfrak{F}_0$, where ι is the tensor unit of Δ_f and $e : \mathbb{C} \to E(\iota)$ is the arrow belonging to the tensor functor E, then the following relation tells us that it is a linear subspace of $\mathfrak{F}_0$

$$(a_1, \rho, \psi) + (a_2, \rho, \psi) = (a_1 + a_2, \rho, \psi).$$

For each localized transportable morphism ρ in Δ_f the vector space $E(\rho)$ is also a linear subspace of $\mathfrak{F}_0$ if we identify it with $\{(1, \rho, \psi)|\psi \in E(\rho)\} \subset \mathfrak{F}_0$. This can be seen from

$$(a, \rho, \psi_1) + (a, \rho, \psi_2) = (a, \rho, \psi_1 + \psi_2).$$

We used direct sums to define addition in $\mathfrak{F}_0$, now we use tensor products to define multiplication, turning $\mathfrak{F}_0$ into an associative algebra. Define

$$(a_1, \rho_1, \psi_1)(a_2, \rho_2, \psi_2) = (a_1\rho_1(a_2), \rho_1 \otimes \rho_2, d_{\rho_1,\rho_2}(\psi_1 \otimes \psi_2)),$$

where $d_{\rho_1,\rho_2} : E(\rho_1) \otimes E(\rho_2) \to E(\rho_1 \otimes \rho_2)$ comes from the tensor functor E. The element $(1, \iota, e(1))$ acts as a multiplicative unit. The subspace $\mathfrak{U}$ is a subalgebra of $\mathfrak{F}_0$. The next step is to turn $\mathfrak{F}_0$ into a $*$-algebra, for which we need the following definition.

Definition 1.3.15. *Let H and H' be 2 Hilbert spaces. Define an anti-linear mapping*

$$\mathcal{J} : Hom_{\mathcal{H}}(H \otimes H', \mathbb{C}) \to Hom_{\mathcal{H}}(H, H'),$$
$$\langle \mathcal{J}(s)(x), x' \rangle_{H'} = s(x \otimes x'), \quad \forall x \in H, x' \in H',$$

where $s \in Hom_{\mathcal{H}}(H \otimes H', \mathbb{C})$.

The reader that has doubts if $\mathcal{J}(s)$ is well-defined may want to look up the Riesz representation theorem. Pick a $\rho \in Obj(\Delta_f)$ and a conjugate $(\overline{\rho}, r, \overline{r})$ for ρ, where $r : \iota \to \overline{\rho} \otimes \rho$ and $\overline{r} : \iota \to \rho \otimes \overline{\rho}$ satisfy the conjugate equations. Applying the previous definition to

$$e^{-1} \circ E(\overline{r}^*) \circ d_{\rho,\overline{\rho}} : E(\rho) \otimes E(\overline{\rho}) \to \mathbb{C},$$

we find a morphism $\mathcal{J}[e^{-1} \circ E(\overline{r}^*) \circ d_{\rho,\overline{\rho}}] : E(\rho) \to E(\overline{\rho})$.

We turn $\mathfrak{F}_0$ into a $*$-algebra by defining:

$$(a, \rho, \psi)^* = (r^*\overline{\rho}(a^*), \overline{\rho}, \mathcal{J}[e^{-1} \circ E(\overline{r}^*) \circ d_{\rho,\overline{\rho}}]\psi).$$

It follows that $\mathfrak{U}$ is a $*$-subalgebra of $\mathfrak{F}_0$.

Proposition 1.3.16. *Let $\mathcal{O}$ be a diamond. Define $\mathfrak{F}_0(\mathcal{O})$ to be the set whose elements are those $F = (a, \rho, \psi)$ in $\mathfrak{F}_0$ such that $a \in \mathfrak{U}(\mathcal{O})$ and $\rho \in \Delta(\mathcal{O})$. Then $\mathfrak{F}_0(\mathcal{O})$ is a $*$-subalgebra of $\mathfrak{F}_0$.*

The compact group G of the Tannaka supergroup (G,k) consists of monoidal natural transformations of E to itself. So if $g \in G$ then for each $\rho \in \Delta_f$ we have an arrow $g_\rho \in Hom_{\mathcal{SH}_f}(E(\rho), E(\rho))$. This can be used to define an action of G on $\mathfrak{F}_0$ by

$$\alpha_g(a,\rho,\psi) = (a,\rho,g_\rho\psi), \quad a \in \mathfrak{U},\ \psi \in E(\rho),\ g \in G.$$

Proposition 1.3.17. *The action given above defines a group isomorphism $g \mapsto \alpha_g$ of the Tannaka group G into the group $Aut_{\mathfrak{U}}(\mathfrak{F}_0)$ of $*$-automorphisms of $\mathfrak{F}_0$ that leave $\mathfrak{U}$ pointwise fixed.*

It is not hard to check that this action of G leaves every $\mathfrak{F}_0(\mathcal{O})$ globally fixed. Furthermore, the symmetry of $\mathcal{SH}_f$, the fact that E is a symmetric functor and the observation that for the symmetry of Δ_f, $c_{\rho_1,\rho_2} = id_{\rho_1\otimes\rho_2}$ holds whenever ρ_1 and ρ_2 are localized in spacelike separated regions, gives normal commutation relations. To be more precise.

Proposition 1.3.18. *Let $\mathcal{O}_1$ and $\mathcal{O}_2$ be spacelike separated diamonds. Let $F_i \in \mathfrak{F}_0(\mathcal{O})$ satisfy $\alpha_k(F_i) = \epsilon_i F_i$, where $\epsilon_i \in \{-1,1\}$. Then*

$$F_1F_2 = (-1)^{(1-\epsilon_1)(1-\epsilon_2)/4} F_2F_1.$$

We want to make a representation (H,π) of the $*$-algebra $\mathfrak{F}_0$. This is done by applying the GNS construction to the composition $\omega_0 \circ m : \mathfrak{F}_0 \to \mathbb{C}$, where $\omega_0 : \mathfrak{U} \to \mathbb{C}$ is the vacuum state and $m : \mathfrak{F}_0 \to \mathfrak{U}$ is a positive linear map which we will now define.

First note that because Δ_f is a STC* category, it is semisimple by Proposition 3.2.14. If $\rho \in Obj(\Delta_f)$, then $\rho = \rho_1 \oplus ... \oplus \rho_n$, where each ρ_i is irreducible. Define $p_\iota^\rho : \rho \to \rho$ to be the projection on the direct sum of those ρ_i that are isomorphic to ι. This projection is interesting because $E(p_\iota^\rho) = P_0^\rho$, where $P_0^\rho : E(\rho) \to E(\rho)$ is the projection in on the subspace of vectors that are invariant under the action $\pi_{E(\rho)}(g) = g_\rho$ of G.

Proposition 1.3.19. *Define the map $m : \mathfrak{F}_0 \to \mathfrak{U}$ by*

$$m(a,\rho,\psi) = (ap_\iota^\rho, \rho, \psi).$$

Then m is a well-defined linear map. It is positive in the sense that for each $F \in \mathfrak{F}_0$ we have $m(F^*F) \geq 0$. It is faithful in the sense that if $m(F^*F) = 0$, then $F = 0$. Let $A = (a,\iota,e(1)) \in \mathfrak{F}_0$ and $F \in \mathfrak{F}_0$, then $m(A) = A$ and $m(AF) = Am(F)$.

Thus $\omega = \omega_0 \circ m$ is a faithful state on the $*$-algebra $\mathfrak{F}_0$. Take the GNS representation[10] (H,π) of $\mathfrak{F}_0$ with respect to ω. Define $\mathfrak{F}$ to be the norm closure of $\pi(\mathfrak{F}_0)$ and $\mathfrak{F}(\mathcal{O})$ to be the weak closure of $\pi(\mathfrak{F}_0(\mathcal{O})$. As a

[10] Note that $\mathfrak{F}_0$ is not a C*-algebra, but only a $*$-algebra

consequence $\mathfrak{F}$ is the C*-inductive limit of the net of von Neumann algebras $\mathcal{O} \to \mathfrak{F}(\mathcal{O})$. We shall call $\mathfrak{F}$ the Roberts field net for the fiber functor E carrying the representation (H, π).

It was pionted out before that $E(p_\iota^\rho) = P_0^\rho$, where $P_0^\rho : E(\rho) \to E(\rho)$ is the projection on a pointwise G-invariant subspace. This entailts that for each $F \in \mathfrak{F}_0$ and $g \in G$ we have $m(\alpha_g F) = m(F)$. The state ω is G-invariant and by Lemma 1.3.2 the representation (H, π) carries a a unitary representation $g \mapsto U(g)$ of G.

Theorem 1.3.20. *Let (G, k) be the Tannaka supergroup belonging to the STC* category Δ_f. Take $\mathfrak{F}_E^R$ to be the Roberts field for the fiber functor $E : \Delta_f \to \mathcal{SH}_f$ carrying the representation (H_E^R, π_E^R). Then $(H_E^R, \pi_E^R, \mathfrak{F}_E^R, (G, k))$ is a field system with gauge symmetry for $(\mathfrak{U}, \omega_0)$. This field system is complete and has normal commutation relations.*

Now suppose that we use two different fiber functors $E_i : \Delta_f \to \mathcal{SH}_f$ to obtain two different Roberts fields $\mathfrak{F}_{E_i}^R$. By (the super version of) Theorem 3.6.2 there is a unitary monoidal natural transformation $E_1 \to E_2$. Using this natural transformation it can be shown that the field nets $\mathfrak{F}_{E_i}^R$ provide equivalent field systems ([19] Section 10.5). The construction of the field system does therefore not depend on the choice of the fiber functor.

But what if we take an arbitrary complete normal field system with gauge symmetry $(H, \pi, \mathfrak{F}, (G, k))$ for $(\mathfrak{U}, \omega_0)$? Why should this be equivalent to $(H_E^R, \pi_E^R, \mathfrak{F}_E^R, (G, k))$? The field algebra $\mathfrak{F}$ can be used to construct a fiber functor $E_{\mathfrak{F}} : \Delta_f \to \mathcal{SH}_f$. For this fiber functor we can construct a Roberts field algebra $\mathfrak{F}_{E_{\mathfrak{F}}}^R$. It can be shown that the resulting field system is equivalent to $(H, \pi, \mathfrak{F}, (G, k))$. Consequently it is equivalent to $(H_E^R, \pi_E^R, \mathfrak{F}_E^R, (G, k))$. This proves uniqueness up to equivalence of field systems. Details can be found in [19] Section 10.5.

1.4 Inequivalent Representations

As promised in Section 1.1, we come back to the algebraic imperialist and the Hilbert space conservatist. The algebraic imperialist claims that all the physical content of a quantum field theory is in the net $\mathcal{O} \to \mathfrak{U}(\mathcal{O})$, the states on the quasilocal algebra $\mathfrak{U}$, and the symmetries of $\mathfrak{U}$. The Hilbert space conservatist claims that we need the net, the symmetries and a representation (H, π). After walking through the DHR analysis, it may seem that the algebraic imperialist has a point. At least for theories that adhere to the DHR selection criterion and Minkowski spacetimes that have a dimension of at least 3, we were able to construct a global gauge group and a field algebra. So it looks like the gauge group and fields, despite being handy tools, are not fundamental. These can be obtained from the net of observable algebras and the states on the quasilocal algebra. However, we needed a little more than just the data of the algebraic imperialist. The DHR selection criterion helped to pick the physically relevant representations of $\mathfrak{U}$ from the set of all representations of $\mathfrak{U}$. This was a very important step in the DHR analysis as it was the structure of $DHR(\mathfrak{U}, \omega_0)$ rather than the structure of the set or category of all representations that allowed us to proceed with the DHR analysis. Actually, there was another step where we went from $DHR(\mathfrak{U}, \omega_0)$ to a full subcategory of it that is equivalent to Δ_f but we will come to that important point later.

The Hilbert space conservatist also seems to have a point. There is a preferred representation (H, π) of $\mathfrak{U}$. This is the representation in the complete normal field system corresponding to $(\mathfrak{U}, \omega_0)$. In this representation, the superselection sectors are given by the (unitary) equivalence classes of the DHR representations, and the field operators act as intertwiners between these sectors. But we do not need to add (H, π) to the data. We know enough to derive this representation if we have the vacuum state ω_0 and the category $DHR(\mathfrak{U}, \omega_0)$ at our disposal.

The stance that all the physical content of a quantum field theory is given by the net $\mathcal{O} \to \mathfrak{U}(\mathcal{O})$, the states on $\mathfrak{U}$, the symmetries of $\mathfrak{U}$ and the category $DHR(\mathfrak{U}, \omega_0)$ is called representation realism in [19]. The representation realist claims that inequivalent representations are of fundamental importance. We need the physically relevant inequivalent representations and especially the relations between them if we want to follow the DHR analysis. But how are the inequivalent representations related in AQFT? They are not related by dynamics. The Hamiltonian that generates the time evolution is an observable. The corresponding operator in the field system can only map vectors in a superselection sector to vectors in that same sector. Inequivalent representations give different sectors.

That inequivalent representations are not dynamically related does not mean that there are no important relations between them. An interesting relation is proposed in the work of Baker and Halvorson [3] on antimatter. Before we get into this matter (no pun intended), we first go back to the issue of conjugates. Before we could use Tannaka-Krein duality we first had to pass from the category Δ to the category Δ_f. This raises questions. How many DHR representations did we throw out? Why do only the DHR representations that have a conjugate representation matter? Is there a physical motivation for using conjugates? Notice that we do not fuss on the finiteness demand on the dimension. This is due to Corollary 4.6.6 that tells us that the dimension is automatically finite. The answer to the first of the three questions above is none if and only if we assume that each DHR representation satisfies finite statistics. The demand of finite statistics can be seen as a weakening of the Bose and Fermi statistics that also allows for parastatistics. A discussion of finite statistics and arguments for it to hold can be found in [13] [34] [15]. Note that in the previous section we claimed that starting out with a field system, the DHR representations that come from the subrepresentations (sectors that can be reached from the vacuum by the action of local fields) always have conjugates. So if we go from field systems to the DHR category, every representation obtained has conjugates, and assuming finite statistics, every DHR representation has conjugates.

Even though under mild conditions we lose nothing in passing from Δ to Δ_f, we can still ask if conjugation can be physically motivated. If this is the case, then we have also found a relation between inequivalent representations as a DHR-representation is often inequivalent to its conjugate. In [3] conjugation is strongly linked to antimatter as follows. Suppose that ω is a matter state on $\mathfrak{U}$ corresponding through the GNS construction to an object ρ of Δ_f. Then we can take the conjugate object $\overline{\rho}$. States $\overline{\omega}$ in the folium of $\overline{\rho}$ are then the corresponding antimatter states. Notice that different antimatter states give the same representations up to unitary equivalence. It is interesting to see antimatter in AQFT. In the foundations of QFT, particles seem to have no fundamental significance [14] [38] [20]. Antimatter is conventionally seen as 'matter' made out of antiparticles, yet AQFT makes no use of particles as these are not fundamental. Still there is a notion of antimatter.

The irreducible tensor unit ι of Δ_f corresponds to the vacuum representation π_0 in $DHR(\mathfrak{U}, \omega_0)$. Let ρ in Δ_f have a conjugate $(\overline{\rho}, r, \overline{r})$. Then $r : \iota \to \overline{\rho} \otimes \rho$ is an isometry up to a scalar. We see that the vacuum representation is a subobject of the DHR representations corresponding to $\overline{\rho} \otimes \rho$ and $\rho \otimes \overline{\rho}$. This fits well with the idea that matter and antimatter can annihilate.

For now we leave the subject of antimatter as well as the DHR analysis. Starting the next section we will set out to prove the main mathematical machinery behind the DHR analysis, namely Tannaka-Krein duality and Deligne's embedding theorem.

Chapter 2

Tannaka's Theorem

In this chapter we will prove Tannaka's theorem for compact Lie groups. Given a compact Lie group G this theorem shows how we can reconstruct this group from the structure of the representations of this group. This theorem is part of Tannaka-Krein duality for compact groups. The other part of this duality and the general theory will be the focus of the next chapter. We will prove Tannaka's theorem in two different ways that are initimately connected. The motivation for discussing two proofs comes in part from Chapter 3 where Tannaka-Krein duality is discussed with a heavy emphasis on category theory. A lot of the (additional) material presented here is intended to help clarify the more abstract definitions and constructions in the second chapter. The two different methods show different aspects of Tannaka-Krein duality.

The first two sections give some background material. In the first section we give an overview of the basics of representation theory of compact Lie groups. The reader familiar with representations of Lie groups can safely skip this section. the second section is devoted to proving the theorem of Peter and Weyl for compact Lie groups and some consequences of this theorem. In particular we will show that for any compact Lie group we can always find a faithful representation. The theory is put to work in the third and fourth sections. In the third section we will prove Tannaka's theorem for the first time. By making use of Hopf algebras we construct a topological group $G_{\mathbb{R}}$ from algebra homomorphisms of (real-valued) representative functions. This group is shown to be isomorphic to the original group G. The fourth section discusses the algebraic group $G_{\mathbb{C}}$ as the complexification of $G_{\mathbb{R}}$. The material presented in the fourth section is not necessary for understanding the rest of this thesis. This does, however not mean that it is not important or interesting. If the reader wants to go beyond the material presented in Chapter 3, and study for example Tannaka-Krein duality for quantum groups, Section 4 gives some of the basics that are used in the relevant literature [21]. But even without looking beyond the material of this thesis the material in Section 4 is interesting in its own right.

In the fifth section we more or less start over at the point where the second section ended. We start with a minimal introduction to category theory. We define the forgetful functor $\omega : Rep_f(G, \mathbb{C}) \to Vect_{\mathbb{C}}$ from the category of finite dimensional representations of G over $\mathbb{C}$ to the category of finite dimensional vector spaces over $\mathbb{C}$. We then construct a group $Aut^{\otimes}\omega$ using certain natural transformations of the forgetful functor to itself. This group plays the same role as $G_{\mathbb{R}}$ in Section 3. In Section 6 we will review the basics of Fourier theory for compact groups. The Fourier theory and some basic category theory are subsequently used to prove Tannaka's theorem for a second time.

2.1 Representation Theory

In this section G denotes a Lie group that is not assumed compact unless stated otherwise. Before we start with representations we first state the following fact for compact Lie groups. This fact, which we will occasionally need, is the existence of the invariant Haar integral defined by the invariant Haar measure. A proof can be found in Bröcker and tom Dieck [7].

Theorem 2.1.1. *Let G be a compact Lie group and $C^0(G, \mathbb{C})$ the $\mathbb{C}$-vector space of continuous complex-valued functions on G. Then there exists a unique invariant integral $C^0(G, \mathbb{C}) \to \mathbb{C}$, $f \mapsto \int_G f(x)dx$ that has the following properties:*

1. *It is linear, monotone and normalized ($\int 1 = 1$).*
2. *Left invariance $\int_G f(yx)dx = \int_G f(x)dx, \quad \forall y \in G$.*

The invariant Haar integral is also right invariant.

Definition 2.1.2. *Let G be any Lie group. A representation of G (also called a G-module) on a finite dimensional vector space V over the field $\mathbb{C}$ is a continuous action*

$$\rho : G \times V \to V$$

of G on V such that for each group element $x \in G$ the translation $\pi_V(x) : v \mapsto \rho(x, v)$ is a linear map. The space V is called the representation space and the dimension of V as a complex vector space is called the dimension $dim(V)$ of the representation.

From the definition of a group action we know the following facts. If $e \in G$ is the identity then $\pi_V(e) = id_V$. Let $x, y \in G$, then $\pi_V(x) \circ \pi_V(y) = \pi_V(xy)$. Combined these two relations imply that for every group element the translation $\pi_V(x)$ is a linear automorphism of V with inverse $\pi_V(x^{-1})$. The map $x \mapsto \pi_V(x)$ defines a homomorphism $\pi_V : G \to Aut(V)$.

Conversely, any such homomorphism π_V defines an action of G on V by $(x, v) \mapsto \pi_V(x)(v)$. By choosing a basis we can identitfy $Aut(V) \cong GL(n, \mathbb{C})$. This motivates the following definition.

Definition 2.1.3. *A matrix representation of G is a continuous homomorphism $\pi_V : G \to GL(n, \mathbb{C})$ of groups. A representation is called faithful if the associated homomorphism $G \to Aut(V)$ is injective.*

Definition 2.1.4. *A morphism $f : (V, \rho) \to (V', \rho')$ between representations is a linear map which is equivariant, i.e. which satisfies $f \circ \rho(x, v) = \rho'(x, f(v))$ for all $x \in G$ and $v \in V$. In shorthand notation this becomes $f(x.v) = x.f(v)$. These morphisms are also called intertwining operators. It is straightforward to check that this defines a category of representations.*

Example 2.1.5. *(Direct sum representation) Using direct sums, tensor products or dual spaces it is possible to construct new representations from old ones. Let (V, ρ) and (W, ρ') be two representations of G. We may construct the direct sum representation as follows. Take the vector space $V \oplus W$ and define the action of G by $x.(v, w) = (x.v, x.w)$. For matrix representations this amounts to the following. Given two matrix representations $G \to GL(m, \mathbb{C})$, $x \mapsto A(x)$ and $G \to GL(n, \mathbb{C})$, $x \mapsto B(x)$, we obtain the direct sum representation $G \to GL(m + n, \mathbb{C})$ by forming the block matrix.*

$$x \mapsto \begin{pmatrix} A(x) & 0 \\ 0 & B(x) \end{pmatrix}.$$

Example 2.1.6. *(Subrepresentations) Let (V, ρ) be a representation and $U \subset V$ be a subspace that is invariant under the action of G. By this we mean that for every $x \in G$ and $v \in U$ we have that $x.v \in U$. Then $(U, \rho|_{G \times U})$ defines a subrepresentation or submodule. The action of the subrepresentation is given by the restriction of the action to the subspace. A representation (V, ρ) is called irreducible if the only subrepresentations are V and 0.*

Example 2.1.7. *(Tensor product representation) Let (V, ρ) and (W, ρ') be representations of G. The tensor product representation is the representation that has as a vector space $V \otimes W$ and the following action $x.(v \otimes w) = x.v \otimes x.w$. Let $v_1, ..., v_n$ be a basis of V and $w_1, ..., w_m$ a basis of W. The nm elements of the form $v_i \otimes w_j$ form a basis of $V \otimes W$. If x acts on V and W via the matrices (r_{ij}) and (s_{ij}), then x acts on $V \otimes W$ via the matrix $(r_{ij}s_{kl})$ whose entry in the (i, k)th row and (j, l)th column is $r_{ij}s_{kl}$. More explicitly*

$$x.(v_j \otimes w_l) = \sum_i r_{ij}(x) v_i \otimes \sum_k s_{kl}(x) w_k = \sum_{i,k} r_{ij}(x) s_{kl}(x) v_i \otimes w_k.$$

Definition 2.1.8. *Define the trivial representation as the 1 dimensional vector space $\mathbb{C}$ with the homomorphism $G \to End\mathbb{C}$, $g \mapsto 1$. Every element acts as the identity.*

Example 2.1.9. *(Dual representation) The set $Hom(V, W)$ of intertwining operators has the structure of a vector space. We define a action of G on this space in the following way. Let $f \in Hom(V, W)$ and $x \in G$, then $(x.f)(v) = x.f(x^{-1}.v)$. If we take W to be the trivial representation $\mathbb{C}$, then this representation is called the dual representation of V and is denoted by $Hom(V, \mathbb{C}) = V^\vee$. The action on $V^\vee$ is given by $(x.f)(v) = f(x^{-1}.v)$. Let $v_1, ..., v_n$ be a basis for V and $v_1^\vee, ..., v_n^\vee$ the dual basis of $V^\vee$. Suppose that $x.v_j = \sum_i r_{ij}(x)v_i$ and that $x.v_j^\vee = \sum_i s_{ij}(x)v_i^\vee$, then the two matrices are related as follows*

$$s_{ij}(x) = (x.v_j^\vee)(v_i) = v_j^\vee(x^{-1}.v_i) = v_j^\vee\left(\sum_k r_{ki}(x^{-1})v_k\right) = r_{ji}(x^{-1}).$$

So x acts on the dual as the transpose of the inverse.

We can use the left-invariant normalized integral to define an invariant inner product on the representation space V. Take $\langle .,. \rangle_0$ to be any inner product on V. Then it is straightforward to check that

$$\langle u, v \rangle = \int_G \langle \pi_V(x)u, \pi_V(x)v \rangle_0 dx$$

defines an inner product. Because of the left-invariance of the integral this new inner product has the property that $\langle \pi_V(x)u, \pi_V(x)v \rangle = \langle u, v \rangle$ for all $u, v \in V$. Any orthogonal basis with respect to this inner product yields a unitary matrix representation.

Example 2.1.10. *(Conjugate representation) For a representation V we can also define the conjugate representation $\overline{V}$. The vector space of this representation has the same additive structure as V but scalar multiplication is defined by*

$$\mathbb{C} \times V \to V \qquad (z, v) \mapsto \overline{z}v$$

where at the righthand side the usual scalar multiplication is implied. The action on V also gives an action on $\overline{V}$ by $\pi_{\overline{V}}(x) = \overline{\pi_V(x)}$. Let $\langle .,. \rangle$ be an invariant inner product. Using this inner product we get an isomorphism of representations $\overline{V} \to V^\vee$ given by $\iota : \overline{v} \mapsto \langle -, v \rangle$. Lets check this explicitly

$$\iota(\overline{\pi_V(x)\overline{v}}) = (w \mapsto \langle w, \pi_V(x)v \rangle).$$

$$\pi_{V^\vee}(x)(\iota(\overline{v})) = \pi_{V^\vee}(x)(w \mapsto \langle w, v \rangle) = (w \mapsto \langle \pi_V(x^{-1})w, v \rangle).$$

Due to the invariance of the inner product these two identities are the same proving that this linear map is indeed an intertwining operator. It is an isomorphism because it maps a basis to a basis and therefore is an isomorphism of vector spaces.

Proposition 2.1.11. *Let G be a compact Lie group. If V is a submodule of the G-module U, then there is a complementary submodule W such that $U = V \oplus W$. Each module is the direct sum of irreducible submodules.*

Proof. Take an G-invariant inner product on U and let W be the orthogonal complement of V in U. It is straightforward to check that W is a submodule. The second claim follows from the first by induction on dimension of the representation space. □

So we can take direct sums and tensor products of representations. We can construct a dual and a conjugate representation for any representation. For now we are finished with constructions on representations and we will move onto the subject of representative functions. Let $G \to GL(n, \mathbb{C})$, $x \mapsto (r_{ij}(x))$ be a matrix representation. Then we have continuous functions $r_{ij} : G \to \mathbb{C}$. These functions are examples of representative functions. In fact, it will turn out that all representative functions are linear combinations of functions that are entries of some matrix representation.

Let $C^0(G, \mathbb{C})$ be the ring of continuous functions $G \to \mathbb{C}$. The translations in G induce actions of G on $C^0(G, \mathbb{C})$ in the following way

$$L : G \times C^0(G, \mathbb{C}) \to C^0(G, \mathbb{C}), \quad L(x, f)(y) = f(x^{-1}y).$$

$$R : G \times C^0(G, \mathbb{C}) \to C^0(G, \mathbb{C}), \quad R(x, f)(y) = f(yx).$$

Definition 2.1.12. *Let G act on $C^0(G, \mathbb{C})$ through R. A function $f \in C^0(G, \mathbb{C})$ is called a complex-valued representative function for G if f generates a finite-dimensional G-subspace of $C^0(G, \mathbb{C})$. By this we mean that the smallest G-invariant subspace containing f is finite dimensional.*

Let us take a look at the functions that come from the matrix representation $G \to GL(n, \mathbb{C})$, $x \mapsto (r_{ij}(x))$. For any matrix representation we have the following rule $r_{ij}(xy) = \sum_i r_{ik}(x) r_{kj}(y)$. So if we translate r_{ij} over y we have that the translated function $x \mapsto r_{ij}(xy)$ is a $\mathbb{C}$-linear combination of the functions r_{ik} with $k \in \{1, ..., n\}$. This shows that the functions coming from representations are indeed representative functions. In order see the converse, we take a slightly more abstract point of view. Let V be a representation of G and $V^\vee$ the corresponding dual representation. Given $v \in V$ and $f \in V^\vee$ we define $d_{f,v} \in C^0(G, \mathbb{C})$ by $d_{f,v}(x) = f(x.v)$. We obtain a linear map

$$s_V : V^\vee \otimes_{\mathbb{C}} V \to C^0(G, \mathbb{C}), \quad f \otimes v \mapsto d_{f,v}.$$

Let $S(V)$ denote the image of s_V, then $S(V)$ is a finite dimensional G-subspace of $(C^0(G, \mathbb{C}), R)$ and $(C^0(G, \mathbb{C}), L)$. To see that invariance under the given actions holds note that $L(g, d_{f,v}) = d_{gf,v}$ and $R(g, d_{f,v}) = d_{f,gv}$. Thus $S(V)$ consists of representative functions. Let $e_1, ..., e_n$ be a basis of V and $e_1^\vee, ..., e_n^\vee$ the corresponding dual basis of $V^\vee$. If we have $x.e_j =$

$\sum_i r_{ij}(x)e_i$ then we see that $d_{e_i^\vee, e_j} = r_{ij}$. So we recognize $S(V)$ as the space generated by the functions r_{ij} that come from the representation V.

In the proof of the next proposition we will see that representative functions always come from representations. The proposition also shows that the set of representations of G has additional structure.

Proposition 2.1.13. *If f is a representative function, then f generates a finite-dimensional G-subspace of $(C^0(G,\mathbb{C}), L)$. The representative functions form a $\mathbb{C}$-subalgebra $\mathcal{T}(G,\mathbb{C})$ closed under complex conjugation.*

Proof. Let f be a representative function. Then f generates a finite dimensional G-subspace V of $(C^0(G,\mathbb{C}), R)$. Let $e_1, ..., e_n$ be a basis of V and $e_1^\vee, ..., e_n^\vee$ the corresponding dual basis of $V^\vee$. Suppose that $R(x, f) = \sum_j a_j(x)e_j$, then we have

$$s_V(e_j^\vee, f)(x) = e_j^\vee\left(\sum_i a_i(x)e_i\right) = a_j(x).$$

$$f(x) = R(x, f)(1) = \sum_j a_j(x)e_j(1) = \sum_j s_V(e_j^\vee, f)(x)e_j(1).$$

Consequently f is an element of $S(V)$. The first assertion now follows from the fact that $S(V)$ is finite dimensional and invariant under the action of L. The space $\mathcal{T}(G,\mathbb{C})$ is generated as a $\mathbb{C}$-vector space by representative functions coming from matrix representations. So for the second claim we only look at representative functions that come from matrix representations. Suppose that $x \mapsto (r_{ij}(x))$ and $x \mapsto (t_{kl}(x))$ are two matrix representations. By looking at their direct sum and tensor product representations, we can see that $r_{ij} + t_{kl}$ and $r_{ij}t_{kl}$ are also representative functions. Looking at the conjugate representation shows that the $\overline{r_{ij}}$ are also representative functions. $\square$

2.2 The Theorem of Peter and Weyl

In this section we use some basic analysis on compact (Lie) groups to prove results that lead to the theorem of Peter and Weyl for compact Lie groups. We show that the representative functions lie dense in $C^0(G,\mathbb{C})$ with respect to the supremum norm topology. Thereafter we show that the $\mathbb{C}$-algebra $\mathcal{T}(G,\mathbb{C})$ of representative functions is the orthogonal direct sum of submodules of representative functions belonging to isomorphism classes of irreducible representations. We also prove some useful consequences of the theorem of Peter and Weyl such as the fact that every compact Lie group admits a faithful representation.

Take $C^0(G,\mathbb{C})$ to be the space of $\mathbb{C}$-valued continuous functions on G. Equipped with the supremum norm this space becomes a Banach space. We recall the Arzela-Ascoli theorem from functional analysis in this setting.

Theorem 2.2.1. *(Arzela-Ascoli) A subset L of the space $C^0(G, \mathbb{C})$ equipped with the supremum norm is compact if and only if L is closed, bounded and equicontinuous.*

We say that L is equicontinuous at a point $x \in G$ if for each $\epsilon > 0$ there is a neighborhood U of x such that $|f(y) - f(x)| < \epsilon$ for all $y \in U$ and $f \in L$. The set L is called equicontinuous if it is equicontinuous at each point of G. In this subsection we will encounter linear maps (linear operators) of normed spaces that take the following shape. Let H denote the vector space $C^0(G, \mathbb{C})$. This can be equipped with the supremum norm, but alternatively we can take the following inner product defined by using the normalized invariant integral

$$\langle u, v \rangle = \int_G u\bar{v}, \qquad \|u\|_2 = \langle u, u \rangle^{1/2}.$$

Completion of H with respect to this inner product gives a Hilbert space $\hat{H} = L^2(G)$. Take $k : G \times G \to \mathbb{C}$ to be a continuous function. Then the following map $K : (\hat{H}, \|.\|_2) \to (H, \|.\|_{sup})$, $f \mapsto Kf$ defined by

$$Kf(x) = \int_G k(x, y)f(y)dy$$

is a continuous linear operator. We know that for normed spaces a linear operator is continuous if and only if it is bounded. The operator is bounded because

$$|Kf(x)| \leq \int_G |k(x, y)| \, |f(y)| \, dy \leq (sup\{|k(x, y)| \, | x, y \in G\}) \, \|f\|_2 \, .$$

We can use the Arzela-Ascoli theorem to prove that this operator $K : (\hat{H}, \|.\|_2) \to (H, \|.\|_{sup})$ is compact. Composition with the identity map $(H, \|.\|_{sup}) \to (\hat{H}, \|.\|_2)$ then gives, with abuse of notation, a compact operator $K : (\hat{H}, \|.\|_2) \to (\hat{H}, \|.\|_2)$. Recall that a operator between normed spaces is called compact if it maps every bounded set into a precompact set. A precompact set is a set L such that every sequence in L has a subsequence that converges in L. Take $B \subset \hat{H}$ to be bounded relative to $\|.\|_2$ by a constant $C > 0$ and fix an $\epsilon > 0$. Choose a neighborhood V of the identity $e \in G$ such that for any $z \in G$ we have that $|k(x, z) - k(y, z)| < \epsilon C^{-1}$ when $xy^{-1} \in V$. Then

$$|Kf(x) - Kf(y)| = \left| \int_G (k(x, z) - k(y, z))f(z)dz \right| \leq \epsilon C^{-1} \|f\|_2 \leq \epsilon.$$

This shows that $K(B)$ is equicontinuous. As we have already seen that $K(B)$ is bounded, the Arzela-Ascoli theorem gives that $K(B)$ is precompact, and K is compact. Now suppose that the function k is symmetric in the sense that $k(x,y) = \overline{k(y,x)}$ for all $x, y \in G$. Then it follows from Fubini's theorem that the operator K is symmetric in the sense that $\langle Ku, v\rangle = \langle u, Kv\rangle$ holds for all $u, v \in L^2(G)$. Recall that we can use the norm on $L^2(G)$ to construct a norm on the operators by $\|K\| = sup\{\|Kf\| \mid \|f\| = 1\}$. Notice that we dropped the subscript 2 in the norm. The norm will always be assumed to be $\|.\|_2$ in this subsection unless stated otherwise. If K is a symmetric operator, then this definition is equivalent to $\|K\| = \{|\langle Kf, f\rangle| \mid \|f\| = 1\}$. A more complete discussion of the previous claims can be found in [37] or [8]. We are now ready to prove the following proposition.

Proposition 2.2.2. *Let K be a symmetric compact operator. Then $\|K\|$ or $-\|K\|$ is an eigenvalue of K.*

Proof. If $\|K\| = 0$ then $K = 0$ and the proposition holds trivially, so let's assume that $\|K\| \neq 0$. Because we have $\|K\| = \{|\langle Kf, f\rangle| \mid \|f\| = 1\}$, we can find a sequence $(f_n)_n$ such that $lim_{n\to\infty} |\langle Kf_n, f_n\rangle| = \|K\|$ and that has the property that $\|f_n\| = 1$. Therefore there exists a subsequence such that, with some abuse of notation, $\langle Kf_n, f_n\rangle$ converges to either $\|K\|$ or $-\|K\|$. Denote this limit by α. We have

$$\begin{aligned} 0 \le \|Kf_n - \alpha f_n\|^2 &= \|Kf_n\|^2 - 2\alpha \langle Kf_n, f_n\rangle + \alpha^2 \|f_n\|^2 \\ &\le 2\alpha^2 - 2\alpha \langle Kf_n, f_n\rangle . \end{aligned}.$$

The right-hand side converges to zero. We know that the operator K is compact so there is a subsequence (Kf_n) that converges to an element f. By the above calculation we know that (αf_n) also converges to f. Since $\alpha \neq 0$, we can define $h = \alpha^{-1} f \neq 0$. The (sub)sequence (f_n) converges to h. Now we know that $Kh = \alpha h$. □

After these preparations we can prove the spectral theorem for compact symmetric operators. First note that the eigenvalues of a symmetric operator must be real numbers. Also note that the eigenspaces for different eigenvalues are pairwise orthogonal.

Theorem 2.2.3. *(Spectral theorem) Let $K : (\hat{H}, \|.\|) \to (\hat{H}, \|.\|)$ be a compact symmetric operator and $H_\lambda = \{x | Kx = \lambda x\}$ be the eigenspace of the eigenvalue λ. Then for each $\epsilon > 0$ the subspace $\bigoplus_{|\lambda|\geq\epsilon} H_\lambda$ is finite dimensional and $\bigoplus_\lambda H_\lambda$ is dense in $\hat{H}$.*

Proof. Suppose that $\bigoplus_{|\lambda|\geq\epsilon} H_\lambda$ is not finite-dimensional. We can find a sequence of orthonormal vectors (f_n) such that $Kf_n = \lambda_n f_n$ and $\lambda_n \geq \epsilon$. But then we have that $\|Kf_n - Kf_m\|^2 \geq 2\epsilon^2$. This contradicts the compactness of the operator K. This proves the first claim. For the second claim take E to be the closure of $\bigoplus_\lambda H_\lambda$ in $\hat{H}$. Take F to be the orthogonal complement of E in $\hat{H}$. Let $e \in E$ and $f \in F$, then $0 = \langle Ke, f\rangle = \langle e, Kf\rangle$, so Kf is also in F. The operator K induces a linear map $F \to F$ that is symmetric and compact. By the previous proposition we know that K has an eigenvalue. This contradicts the construction of E unless $F = \{0\}$. $\square$

Now we can state the first main result of this section. It is one of the two ingredients of the theorem of Peter and Weyl.

Theorem 2.2.4. *The representative functions are dense in both $C^0(G, \mathbb{C})$ and $L^2(G, \mathbb{C})$.*

Proof. Let $f : G \to \mathbb{C}$ be continuous and $\epsilon > 0$. Because G is compact, it follows from uniform continuity that we can choose a neighborhood U of the identity $e \in G$ such that $U = U^{-1}$ and that $|f(x) - f(y)| < \epsilon$ whenever $x^{-1}y \in U$. Let $\delta : G \to [0, \infty)$ be a continuous function with support contained in U such that $\delta(x) = \delta(x^{-1})$ and $\int_G \delta = 1$. Now consider the linear operator

$$K : L^2(G) \to C^0(G), \quad f \mapsto Kf$$

defined by

$$Kf(x) = \int_G \delta(y)f(xy)dy = \int_G \delta(y^{-1}x)f(y)dy = (f * \delta)(x).$$

Here we used both the invariance of the integral and the properties of δ. We have the following

$$\|Kf - f\|_{sup} = \sup_{y\in G} \left| \int_G \delta(x)(f(yx) - f(y))dx \right| \leq \int_G \epsilon\delta(x)dx = \epsilon$$

where we used that if $\delta(x) \neq 0$, then $x \in U$ so $y^{-1}yx \in U$ and $|f(y) - f(xy)| < \epsilon$. We are finished with the proof if every function Kf can be approximated by a representative function. We have seen operators of this kind before and recognize them as symmetric and compact. Using invariance of the integral we can see that the operator is equivariant

$$K(y.f)(x) = \int_G f(y^{-1}z)\delta(z^{-1}x)dz = \int_G f(z)\delta(z^{-1}y^{-1}x)dz = (y.(Kf))(x).$$

Consequently the eigenspaces of K are G-invariant. Let $\lambda_0 = 0$ and $\lambda_n \neq 0$ be the eigenvalues of K. By the spectral theorem for compact symmetric operators $\bigoplus_{n\geq 0} H_{\lambda_n}$ is dense in $\hat{H}$. So $\bigoplus_{n\geq 0} KH_{\lambda_n}$ is dense in $K\hat{H}$. Furthermore $KH_0 = 0$ and $KH_{\lambda_n} = H_{\lambda_n}$ so $\bigoplus_{n\geq 0} KH_{\lambda_n} = \bigoplus_{n\geq 1} H_{\lambda_n}$. The spaces H_{λ_n} are finite dimensional and G-invariant, so $H_{\lambda_n} \subset \mathcal{T}(G, \mathbb{C})$. As $\bigoplus_{n\geq 1} H_{\lambda_n}$ is finite dimensional and consists of representative functions, the representative functions are dense in $K\hat{H}$. The spectral theorem has shown us that the representative functions are dense in $K\hat{H}$ and previously it was shown that $\|Kf - f\|_{sup} < \epsilon$ implying that the representative functions are dense in $C^0(G, \mathbb{C})$. $\square$

We will use this theorem to prove that any compact Lie group admits a faithful representation. In order to do so we need the descending chain property of compact Lie groups. This is the property that a descending chain of closed subgroups $K_1 \supseteq K_2 \supseteq ...$ of G is eventually constant. If $K_{i+1} \neq K_i$, then K_{i+1} either has a lower dimension than K_i or fewer connected components. As G is compact, it is clear that the chain must remain constant at some point.

Theorem 2.2.5. *Every compact Lie group admits a faithful representation.*

Proof. Assume that $G \neq \{e\}$. Then pick any $x \in G$ with $x \neq e$. There exists a continuous function $f : G \to \mathbb{C}$ such that $f(x) \neq f(e)$. By the previous theorem there is a representative function u such that $u(x) \neq u(e)$. The kernel K_1 of the corresponding representation is therefore a proper closed subgroup of G. If $K_1 \neq \{e\}$ take a new $x \in K_1 - \{e\}$. Use the previous argument to find a representation with a kernel K_2 such that $K_1 \cap K_2$ is properly contained in K_1. If π_1 is the representation that has K_1 as a kernel and π_2 has K_2 as a kernel, then the direct sum representation $\pi_1 \oplus \pi_2$ has $K_1 \cap K_2$ as a kernel. Repeating the arguments we find a descending chain of closed subgroups and by the descending chain property, this chain will become constant equal to $\{1\}$ in a finite number of steps. Take the representations belonging to $K_1 \cap ... \cap K_n = \{1\}$, that is take the $\pi_1, ...\pi_n$ and form the direct sum representation $\pi_1 \oplus ... \oplus \pi_n$. This representation is faithful. □

Using an invariant inner product, the above theorem gives the existence of a faithful unitary representation. Now we proceed to obtain the second ingredient for the theorem of Peter and Weyl. We start with Schur's lemma which we will ocassionaly need.

Proposition 2.2.6. *(Schur's Lemma) Let G be any group and let V and W be irreducible G-modules. An intertwining operator $V \to W$ is either an isomorphism or the zero map. Any intertwiner $f : V \to V$ has the form $f(v) = \lambda v$ for some $\lambda \in \mathbb{C}$.*

Proof. Let $f : V \to W$ be an intertwining operator. Then the kernel of f is an submodule of the irreducible module V. If the kernel is V then f is the zero map. If the kernel is not V and therefore $\{0\}$ then f is injective and $f(V)$ is a nonzero submodule of the irreducible module W. Consequently $f(V) = W$ and f is an isomorphism. For the second claim suppose that f is an isomorphism and let λ be an eigenvalue of f with eigenspace W. Then $W = \{v \in V, f(v) = \lambda v\}$ is a G-submodule of V. The claim follows by irreducibility of V. □

Theorem 2.2.7. *(Orthogonality Relations) Let V and W be nonisomorphic irreducible representations. Take $\langle ., . \rangle$ to be a G-invariant inner product on*

V or W. Let $v_1, v_2 \in V$ and $w_1, w_2 \in W$, then

$$\int_G \overline{\langle x.v_1, v_2\rangle}\,\langle x.w_1, w_2\rangle dx = 0.$$

Proof. Fix $v_1 = \alpha$ and $w_1 = \beta$ to get a bilinear map $b : V \times \overline{W} \to \mathbb{C}$ given by $(v, \overline{w}) \mapsto \int_G \langle v, x.\alpha\rangle \langle x.\beta, w\rangle dx$. This bilinear form is G-invariant and therefore defines a intertwining operator $b' : V \to Hom(\overline{W}, \mathbb{C}) = \overline{W}^{\vee} \cong W$. Applying Schur's lemma and keeping in mind that V and W are nonisomorphic, gives that this map is the zero map. □

Definition 2.2.8. *For a compact Lie group G, define $\hat{G}$ to be the set of isomorphism classes of all irreducible representations of G.*

Equipped with the orthogonality relations we can now prove the second ingredient for the theorem of Peter and Weyl. This proposition shows that $\mathcal{T}(G, \mathbb{C})$ is semisimple in the sense that it is the orthogonal direct sum of G-modules that come from functions of irreducible representations.

Proposition 2.2.9. *Let B be a G-submodule of $(\mathcal{T}(G, \mathbb{C}), R)$. Then the following holds:*

1. *B is the orthogonal direct sum of the sub-modules $B \cap S(U)$, where $S(U)$ ranges over the set $\hat{G}$.*
2. *B is closed in $\mathcal{T}(G, \mathbb{C})$ with respect to both the inner product and the supremum norm topologies.*

Proof. Take U an irreducible representation of G. We saw at the end of the first section that $S(U)$ was the space spanned by the representative functions coming from U. We will now use the orthogonality relations to show that the $S(U)$-spaces are pairwise orthogonal. Let U and V be two nonisomorphic irreducible representations. Take bases $u_1, ..., u_n$ for U and $v_1, ..., v_m$ for V. We get matrix representations $r_{ij}(x) = \langle x.u_i, u_j\rangle$ and $s_{kl}(x) = \langle x.v_k, v_l\rangle$. Applying the orthogonality relations gives $\int_G r_{ij}\overline{s_{kl}}dx = 0$. From the discussion in the first section and the proof of 2.1.13 we can see that the $S(U)$'s generate $\mathcal{T}(G, \mathbb{C})$. So we can conclude that $\mathcal{T}(G, \mathbb{C})$ is the direct sum of the $S(U)$'s where U ranges over all irreducible representations. To sum up, we have shown that $\mathcal{T}(G, \mathbb{C})$ is semisimple and that the summands corresponding to nonisomorphic irreducible representations are mutually orthogonal. Note also that Schur's lemma implies that if V and W are isomorphic irreducible representations, then these are canonically isomorphic. So it suffices to use isomorphism classes of irreducible representations instead of all irreducible representations. This proves the first claim.

To prove (2), let f be an element of $\mathcal{T}(G, \mathbb{C})$ lying in the closure of B with respect to the inner product topology. Now consider the orthogonal projection on $S(U)$ given by the map p_U. Since

$$\langle p_U(f - b), p_U(f - b)\rangle \leq \langle f - b, f - b\rangle$$

for every $b \in B$ we know that $p_U(f) \in \overline{p_U(B)}$, where the closure is in $S(U)$. But the space $S(U)$ is finite dimensional, hence $p_U(f) \in p_U(B)$. The function f is contained in the $p_U(B)$ where U ranges over the isomorphism classes of irreducible representations. By (1), f is in B. Finally we have $\|f\|_2 \leq \|f\|_{sup}$ so B is also closed with respect to the supremum norm topology. □

We can formulate the theorem of Peter and Weyl for compact Lie groups. We know that the representative functions are dense in $C^0(G, \mathbb{C})$ and therefore are dense in $L^2(G)$. We also know that $\mathcal{T}(G, \mathbb{C})$ is equal to the orthogonal direct sum of G-submodules $S(U)$. These modules $S(U)$ are constructed using isomorphism classes of irreducible representations U. Combining these observations gives the following theorem.

Theorem 2.2.10. *(Peter and Weyl) Let $\hat{G}$ denote the set of isomorphism classes of irreducible representations of the compact Lie group G. Then the space $L^2(G)$ decomposes as a Hilbert sum*

$$L^2(G) = \widehat{\bigoplus}_{\delta \in \hat{G}} S(U)_\delta.$$

Each term in the summand is an irreducible subspace that is invariant under both actions R and L of G.

The following result will also be quite helpful in proving Tannaka's theorem. It is the complex Stone-Weierstrass theorem. A proof of this theorem can for example be found in Conway [8] Paragraph V.8. Recall that a set of functions $B \subset C^0(X, \mathbb{C})$ is said to separate points if for each $x, y \in X$ with $x \neq y$ there is a $f \in B$ such that $f(x) \neq f(y)$.

Theorem 2.2.11. *(Stone-Weierstrass) Let G be a compact space. Take $B \subset C^0(G, \mathbb{C})$ to be a subalgebra of the algebra of continuous functions with the supremum norm. If B contains all complex constants, separates points and is closed under complex conjugation, then B is dense in $C^0(G, \mathbb{C})$ with respect to the supremum norm topology.*

Proposition 2.2.12. *Let $G \to GL(n, \mathbb{C})$, $x \mapsto (r_{ij}(x))$ be a faithful representation. Then the functions r_{ij} and $\overline{r_{ij}}$ generate $\mathcal{T}(G, \mathbb{C})$ as a $\mathbb{C}$-algebra.*

Proof. Let B be the $\mathbb{C}$-algebra generated by r_{ij} and $\overline{r_{ij}}$, then B is closed under complex conjugation and contains the complex constants. Because the representation is faithful, this algebra also separates points. By the Stone-Weierstrass theorem B is dense in $C^0(G, \mathbb{C})$. So it is also dense in $\mathcal{T}(G, \mathbb{C})$. The algebra B is also a G-module so it is closed in $\mathcal{T}(G, \mathbb{C})$ by Proposition 2.2.9(2). The algebra B equals $\mathcal{T}(G, \mathbb{C})$. □

2.3 Hopf Algebras and Tannaka's Theorem

In the previous two paragraphs we could just as easily have used the real numbers $\mathbb{R}$ instead of the field of complex numbers $\mathbb{C}$. If we replace $\mathbb{C}$ by $\mathbb{R}$ everywhere, all arguments would still hold. There would only be superficial changes. To name some examples: the conjugate representation $\overline{V}$ would be the same as V and the condition in the Stone-Weierstrass theorem that the set $B \subset C^0(G, \mathbb{R})$ is closed under conjugation would be redundant. For the largest part of this section we will work with a field K that can be either the real or complex numbers. The approach in this and the following section is based on the discussion in Bröcker and tom Dieck [7] sections III.7 and III.8.

Definition 2.3.1. *Take the set G_K to consist of all K-algebra homomorphisms $\mathcal{T}(G, K) \to K$, where K is either $\mathbb{R}$ or $\mathbb{C}$. The set $\tilde{G}_{\mathbb{R}}$ consists of all $\mathbb{C}$-algebra homomorphisms $\mathcal{T}(G, \mathbb{C}) \to \mathbb{C}$ with the property that for all $f \in \mathcal{T}(G, \mathbb{C})$ we have $s(\overline{f}) = \overline{s(f)}$. So $\tilde{G}_{\mathbb{R}}$ is a subset of $G_{\mathbb{C}}$.*

After adding enough structure to the sets $G_{\mathbb{K}}$, it will turn out that $G_{\mathbb{R}}$ is isomorphic to the Lie group G. The set $\tilde{G}_{\mathbb{R}}$, after adding enough structure, is also ismorphic to G as we will explain later. We start with adding more structure to the algebra $\mathcal{T}(G, K)$. This structure will help to make G_K into a group.

Lemma 2.3.2. *Let G and H be compact Lie groups. The K-algebra homomorphism*

$$t : \mathcal{T}(G, K) \otimes_K \mathcal{T}(H, K) \to \mathcal{T}(G \times H, K)$$

sending $u \otimes v$ to $(x, y) \to u(x)v(y)$ is an isomorphism.

Proof. Proving that the map defines a K-algebra homomorphism is straightforward so we start with injectivity. Suppose that for all $x \in G$ and $y \in H$ we have that $u(x)v(y) = 0$. Then it is easy to check that either u or v must be the zero map. The kernel of t is therefore equal to $\{0\}$. To show surjectivity, let $f \in \mathcal{T}(G \times H, K)$ be given and let $S \subset \mathcal{T}(H, K)$ be the space generated by the functions $y \mapsto f(x, y)$. Because $f \in \mathcal{T}(G \times H, K)$, this space is finite-dimensional. We can choose a basis $e_1, ..., e_n$ such that there are elements $y_1, ..., y_n \in H$ with $e_i(y_j) = \delta_{ij}$. We can now write $f(x, y) = \sum u_i(x)e_i(y)$. Surjectivity follows when $x \mapsto u_i(x)$ are in $\mathcal{T}(G, K)$. By the choice of the basis we have $u_i(x) = f(x, y_i)$, so these functions are indeed elements of $\mathcal{T}(G, K)$. □

Group multiplication $G \times G \to G$, $(x, y) \mapsto xy$ induces a homomorphism $\mathcal{T}(G, K) \to \mathcal{T}(G \times G, K)$ by $(x, y) \mapsto f(xy)$. Note that the arrow of the induced transformation is reversed. We should also check at this point that $(x, y) \mapsto f(xy)$ indeed defines a representative function for the group $G \times G$.

This can be done in a straightforward fashion. Using this induced multiplication map and the isomorphism of the above lemma gives the following K-algebra homomorphism

$$d : \mathcal{T}(G, K) \to \mathcal{T}(G \times G, K) \cong \mathcal{T}(G, K) \otimes_K \mathcal{T}(G, K)$$

that is called comultiplication. In a similar way the inverse map $x \mapsto x^{-1}$ induces a K-algebra homomorhism called the coinverse

$$c : \mathcal{T}(G, K) \to \mathcal{T}(G, K), \quad c(f)(x) = f(x^{-1}).$$

Evaluation at the unit element $e \in G$ gives the algebra homomorphism

$$\epsilon : \mathcal{T}(G, K) \to K$$

called the counit. These algebra homomorphisms satisfy the following properties. The properties are simply the translations of the axioms for the group G. Associativity of group multiplication yields the coassociativity of d

$$(d \otimes id) \circ d = (id \otimes d) \circ d.$$

The counit satisfies

$$(\epsilon \otimes id) \circ d = id = (id \otimes \epsilon) \circ d.$$

Let m and η be the multiplication and the unit in $\mathcal{T}$ (shorthand for $\mathcal{T}(G, K)$)

$$m : \mathcal{T} \otimes_K \mathcal{T} \to \mathcal{T}, \quad f \otimes g \mapsto f \cdot g$$

$$\eta : K \to \mathcal{T}, \quad 1 \mapsto 1$$

then we have for the coinverse

$$m \circ (c \otimes id) \circ d = \eta \circ \epsilon.$$

The algebra $\mathcal{T}(G, K)$ together with the above structure is called a Hopf algebra. The prefix co- that was used throughout is because the direction of all the arrows is reversed when compared with the arrows in the corresponding group operations.

Using comultiplication, we can multiply two homomorphisms $s, t \in G_K$ as follows

$$s \cdot t : \mathcal{T} \xrightarrow{d} \mathcal{T} \otimes_K \mathcal{T} \xrightarrow{s \otimes t} K \otimes K \cong K.$$

This map is again a K-algebra homomorphism. Suppose that $s, t \in G_{\mathbb{C}}$ have the property that $s(\overline{f}) = \overline{s(f)}$ and $t(\overline{f}) = \overline{t(f)}$ hold for all $f \in \mathcal{T}$. Then we again have $(s \cdot t)(\overline{f}) = \overline{(s \cdot t)(f)}$. So the set $\tilde{G}_{\mathbb{R}}$ is closed under this operation.

Proposition 2.3.3. *The above composition law makes G_K into a group.*

Proof. First check associativity using the above coassosiativity

$$(r \cdot s) \cdot t = m(id \otimes t)((r \cdot s) \otimes id)d = m(id \otimes t)(m \otimes id)(r \otimes s \otimes id)(d \otimes id)d.$$

$$r \cdot (s \cdot t) = m(r \otimes id)(id \otimes m)(id \otimes s \otimes t)(id \otimes d)d.$$

It is straightforward to show that

$$m(r \otimes id)(id \otimes m)(id \otimes s \otimes t) = m(id \otimes t)(m \otimes id)(r \otimes s \otimes id)$$

so applying coassociativity proves associativity. It is straightforward to check that $\epsilon \in G_K$ and that it functions as an identity element. The inverse of $s \in G_K$ is given by $s \circ c \in G_K$. This follows from

$$sc \cdot s = m(sc \otimes s)d = m(s \otimes s)(c \otimes id)d = sm(c \otimes id)d = s\eta\epsilon = \epsilon.$$

Thus $\langle G_K, \cdot \rangle$ satisfies all group axioms. □

Proposition 2.3.4. *Each $x \in G$ defines an evaluation homomorphism $i(x) \in G_K$ by $i(x) : \mathcal{T}(G,K) \to K$, $f \mapsto f(x)$. Thus we have a map $i : G \to G_K$, $x \mapsto ev_x$. This map is an injective homomorphism of groups.*

Proof. We will first show that the map is a homomorphism. Take $f \in \mathcal{T}(G,K)$ and write $d(f) = \sum_i u_i \otimes v_i$. We know that $(s \cdot t)(f) = \sum_i s(u_i)t(v_i)$ so $(i(x) \cdot i(y))(f) = \sum_i u_i(x)v_i(y)$. But the righthand side is just $f(xy) = i(xy)(f)$. Therefore $i(x) \cdot i(y) = i(xy)$ showing that the evaluation map is a homomorphism. Let $x \in G$ be in the kernel of i. This means that $i(x)(f) = \epsilon(f)$ for all $f \in \mathcal{T}(G,K)$. By the theorem of Peter and Weyl the representative functions separate points of the compact space G. The only way that $f(x) = f(1)$ holds for every $f \in \mathcal{T}(G,K)$ is if $x = e$ proving injectivity. □

The next step is to define a topology on G_K. Take the coarsest topology for which all the evaluation maps

$$\lambda_f : G_K \to K \quad s \mapsto s(f), \quad f \in \mathcal{T}(G,K).$$

are continuous. A map of topological spaces $\phi : X \to G_K$ is continuous with respect to this topology if and only if for each $f \in \mathcal{T}(G,K)$ we have that $\lambda_f \circ \phi$ is continuous. The reader may notice that this topology on G_K coincides with the product topology that is defined using

$$G_K = Hom_K(\mathcal{T}(G,K),K) \subset \prod_{f \in \mathcal{T}(G,,K)} K.$$

Proposition 2.3.5. *Equipped with the above topology G_K is a topological group. The map $i : G \mapsto G_K$ is a continuous map.*

Proof. The first claim amounts to showing that the multiplication and inverse maps are continuous. This is easily done when we look at the composition with λ_f for an arbitrary f. If we write $d(f) = \sum_i u_i \otimes v_i$, then the composition of multiplication $G_K \times G_K \to G_K$ and λ_f gives $(s,t) \mapsto \sum_i s(u_i)t(v_i)$, which is continuous. The approach for the inverse map is the same. The second claim that we need to prove is that the map i is continuous. The map $\lambda_f i$ is equal to $x \mapsto f(x)$ and is clearly continuous. □

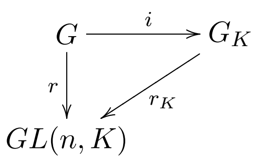
The next proposition will help to give $G_{\mathbb{R}}$ the structure of a compact Lie group. The topological group $G_{\mathbb{R}}$ will be identified with some subgroup of the orthogonal group. First we give a fact from the theory of Lie groups. This fact tells us that a subspace of a Lie group which is closed under the group operation is a Lie subgroup precisely when it is topologically closed in the Lie group. The proof can be found in Bröcker and tom Dieck [7] I.3.

Theorem 2.3.6. *An abstract subgroup H of a Lie group G is a submanifold of G if and only if H is closed in G.*

Let $r : G \to GL(n,K)$, $x \mapsto (r_{ij}(x))$ be a matrix representation. Then this induces a map

$$r_K : G_K \to GL(n,K), \quad s \mapsto (s(r_{ij})).$$

Proposition 2.3.7. 1. *The map r_K is a continuous homomorphism such that the following diagram commutes.*



2. *If the r_{ij} and $\overline{r_{ij}}$ generate $\mathcal{T}(G,K)$ as a K-algebra, then r_K is injective.*

3. *If r maps G into $O(n)$, then $r_{\mathbb{R}}G_{\mathbb{R}} \subset O(n)$ as a closed subgroup.*

Proof. Take $s,t \in G_K$ and write $r_{ij}(xy) = \sum_k r_{ik}(x)r_{kj}(y)$. We get

$$r_K(s \cdot t) = ((s \cdot t)(r_{ij})) = \left(\sum_k s(r_{ik})t(r_{kj}) \right) = r_K(s)r_K(t).$$

This proves that r_K is a homomorphism of groups. This map is continuous if all the maps $s \mapsto s(r_{ij})$ are continuous. These maps are continuous by the definition of the topology on G_K. Checking that the diagram commutes is straightforward, so we move on to (2). Under the assumption that the r_{ij} and $\overline{r_{ij}}$ generate $\mathcal{T}(G,K)$ any homomorphism $s : \mathcal{T}(G,K) \to K$ that satisfies is completely determined by its values on the r_{ij} and $\overline{r_{ij}}$. This proves (2). Let $r(G) \subset O(n)$. Because $(r_{ij})^t(r_{ij}) = E$, where E denotes the identity matrix, we have $(s(r_{ij}))^t(s(r_{ij})) = E$. It follows that the image of $r_{\mathbb{R}}$ is contained in $O(n)$. It remains to show that $r_{\mathbb{R}}G_{\mathbb{R}}$ is closed. To prove this, it is sufficient to show that $G_{\mathbb{R}}$ is a compact space. Consider any faithful representation $r : G \to O(n)$. By Proposition 2.2.12 the r_{ij} generate $\mathcal{T}(G, \mathbb{R})$. An element $f \in \mathcal{T}(G, \mathbb{R})$ is therefore a polynomial $f = P(r_{ij})$ in n^2 variables where the variables correspond to the entries of a matrix in $O(n)$. The Lie group $O(n)$ is compact, so the image of $O(n)$ in $\mathbb{R}$ under P is contained in some compact subset $X_f \subset \mathbb{R}$. Using $r_{\mathbb{R}}$ and P we obtain the following continuous map

$$\Lambda = (\lambda_f) : G_{\mathbb{R}} \to \prod_f X_f, \quad f \in \mathcal{T}(G, \mathbb{R}).$$

The space $\prod_f X_f$ is equipped with the product topology so it is compact by Tychonoff's theorem. The image of $G_{\mathbb{R}}$ is comprised of precisely those $(t_f)_f$ that satisfy $t_1 = 1$, $t_{f \cdot g} = t_f \cdot t_g$, $t_{f+g} = t_f + t_g$, $t_{af} = at_f$ and $t_{\overline{f}} = \overline{t_f}$ for $1, f, g \in \mathcal{T}(G, \mathbb{R})$ and $a \in \mathbb{R}$. This set is closed in $\prod_f X_f$. The image of $G_{\mathbb{R}}$ is closed in a compact space, hence compact. The map Λ is a topological embedding (an homeomorphism onto its image). Injectivity is straightforward and continuity of the inverse can again be checked by looking at the composition with arbitrary λ_f. Because the map is an embedding and the image is compact, the space $G_{\mathbb{R}}$ is compact. □

We can give a result for $\tilde{G}_{\mathbb{R}}$ analogues to that of $G_{\mathbb{R}}$ in part (3) of the above proposition. If r maps G into the unitary compact group $U(n)$, then $r_{\mathbb{C}}\tilde{G}_{\mathbb{R}} \subset U(n)$ is a closed subgroup. In this setting we have that $(r_{ij})^\dagger(r_{ij}) = E$ implies that $(s(r_{ij}))^\dagger(s(r_{ij})) = E$. Note that we need the additional demand $s(\overline{f}) = \overline{s(f)}$ for this to work. The rest of the proof works in the same way as above.

Given any faithful unitary representation $r : G \to O(n)$ we get a continuous injective homomorphism $r_{\mathbb{R}} : G_{\mathbb{R}} \to O(n)$. The image is a closed abstract group, so by Theorem 2.3.6 the image is a compact Lie group. An atlas $(\kappa_\alpha, U_\alpha)_\alpha$ of this compact Lie group can be pulled back to define an atlas of $G_{\mathbb{R}}$ by $(\kappa_\alpha \circ r_{\mathbb{R}}, r_{\mathbb{R}}^{-1}(U_\alpha))_\alpha$. We must check that this manifold structure is compatible with the group structure. Using that $r_{\mathbb{R}}$ is a group homomorphism it is straightforward to show that the group multiplication map and the inverse map are indeed smooth. The next proposition shows that this differentiable structure on the topological group $G_{\mathbb{R}}$ is unique. The proposition is also helpful in proving that the map $i : G \to G_{\mathbb{R}}$ is an isomorphism of Lie groups. The proof of the proposition can be found in Bröcker and tom Dieck [7] I.3.

Proposition 2.3.8. *Let $f : G \to H$ be a group homomorphism between Lie groups which is continuous as a map between manifolds. Then f is* *differentiable, hence a Lie group homomorphism. In particular, a topological group has at most one Lie group structure.*

Theorem 2.3.9. *(Tannaka's Theorem) The evaluation map $i : G \to G_{\mathbb{R}}$ is an isomorphism of Lie groups.*

Proof. We already established that the map is a continuous injective group homomorphism between Lie groups. Looking at the previous proposition it suffices to show that the map is surjective. We will do this by constructing an isomorphism of $\mathbb{R}$-algebras $\mathcal{T}(G_{\mathbb{R}}, \mathbb{R}) \to \mathcal{T}(G, \mathbb{R})$. Start with an arbitrary representative function $f \in \mathcal{T}(G, \mathbb{R})$. To each function $f \in \mathcal{T}(G, \mathbb{R})$ there corresponds a function $\lambda_f : G_{\mathbb{R}} \to \mathbb{R}$. The following calculation shows that this is a representative function for $G_{\mathbb{R}}$.

$$t \cdot \lambda_f(s) = \lambda_f(s \cdot t) = (s \cdot t)(f) = \sum_i s(u_i)t(v_i) = \sum_i \lambda_{u_i}(s)\lambda_{v_i}(t).$$

If f correspond to a matrix coefficient of a representation r, then λ_f is the corresponding coefficient of $r_{\mathbb{R}}$. We thus get a map

$$\lambda : \mathcal{T}(G, \mathbb{R}) \to \mathcal{T}(G_{\mathbb{R}}, \mathbb{R}), \quad f \mapsto \lambda_f.$$

This map is a homomorphism of algebras. The above calculation also shows that the image of λ is $G_{\mathbb{C}}$-invariant. It is straightforward to check that the λ_f separate points, contain the complex constants and that the image of λ is closed under complex conjugation. By the complex Stone-Weierstrass theorem the image of λ is dense in the space $\mathcal{T}(G_{\mathbb{R}}, \mathbb{R})$ with the supremum norm. Proposition 2.2.9(2) with the above observation of $G_{\mathbb{R}}$-invariance gives that the image of λ is equal to $\mathcal{T}(G_{\mathbb{R}}, \mathbb{R})$. Combining the surjective algebra homomorphism λ with the following algebra homomorphism gives the identity map.

$$i^* : \mathcal{T}(G_{\mathbb{R}}, \mathbb{R}) \to \mathcal{T}(G, \mathbb{R}), \quad s \mapsto si.$$

This implies that both λ and i^* are isomorphisms. Taking the completion of the above algebras with respect to the supremum norms yields the isomorphism $i^* : C^0(G_{\mathbb{R}}, \mathbb{R}) \to C^0(G, \mathbb{R})$. This in turn implies surjectivity of i, completing the proof. □

So we have proven that $G_{\mathbb{R}}$ is isomorphic to G. Using the discussion that followed the proof of Proposition 2.3.7, we could also prove that $\tilde{G}_{\mathbb{R}}$ is isomorphic to G. The proof goes in the same way as that of Tannaka's theorem above. In the next section we will see how $G_{\mathbb{C}}$ is related to G.

2.4 Complexifications

In the previous section it was shown that $G_{\mathbb{R}}$ is isomorphic to G as a Lie group. In this section we will see how $G_{\mathbb{C}}$ is related to G. The discussion that we present here follows that of Bröcker and tom Dieck [7] III.8 rather closely. Before we explore the structure of $G_{\mathbb{C}}$ we first need some terminology from algebraic geometry. Let k be a field that is algebraically closed and let $\mathbf{A}^n$ denote ordered n-tuples of elements of k. Take $A = k[x_1, ..., x_n]$ to be the ring of polynomials in n variables over k. We call a subset $Y \subset \mathbf{A}^n$ an algebraic set if there exists a subset $T \subset A$ such that $Y = Z(T)$, where we defined $Z(T) = \{p \in \mathbf{A}^n | f(p) = 0, \forall f \in T\}$. In some literature such a set $Z(T)$ is called an affine algebraic variety. We will only call an algebraic set an affine algebraic variety if it is irreducible in the sense that it cannot be written as the union of two proper subsets that are both algebraic.

We will see that $G_{\mathbb{C}}$ is an algebraic group. An algebraic group is a group that is also an algebraic set. Let $r : G \to GL(n, \mathbb{C})$ be a real faithful representation. Such a representation is always realizable by the construction in the proof of Theorem 2.2.5. By the theorem of Stone-Weierstrass the functions r_{ij} generate $\mathcal{T}(G, \mathbb{C})$ as a $\mathbb{C}$-algebra. We can identify $\mathcal{T}(G, \mathbb{C})$ as follows

$$\mathcal{T}(G, \mathbb{C}) \cong \mathbb{C}[X_{ij}]/I$$

where the ideal I is the kernel of the map $\mathbb{C}[X_{ij}] \to \mathcal{T}(G, \mathbb{C})$, $X_{ij} \mapsto r_{ij}$. Let $Z(I)$ be the algebraic set corresponding to the ideal I, $Z(I) = \{z \in \mathbb{C}^{n \cdot n} | f(z) = 0, f \in I\}$. Each $z \in Z(I)$ defines a $\mathbb{C}$-algebra homomorphism $\mathbb{C}[X_{ij}]/I \to \mathbb{C}$ by evaluation in z. Conversely each $\mathbb{C}$-algebra homomorphism $\mathbb{C}[X_{ij}]/I \to \mathbb{C}$ can be viewed as evaluation in a point $z \in Z(I)$. The coordinates z_{ij} of this point are given by the images of the $X_{ij} + I$ under the morphism. Under the identification $\mathcal{T}(G, \mathbb{C}) \cong \mathbb{C}[X_{ij}]/I$ the points of $Z(I)$ correspond one to one with $\mathbb{C}$-algebra homomorphisms $\mathcal{T}(G, \mathbb{C}) \to \mathbb{C}$. Thus we have a bijection

$$\sigma : V(I) \to G_{\mathbb{C}} \quad z \mapsto (f + I \mapsto f(z)).$$

Composition with $r_{\mathbb{C}}$ gives the identity on $Z(I)$, $r_{\mathbb{C}} \circ \sigma = id_{Z(I)}$.

$$z \mapsto \sigma_z \mapsto (\sigma_z(r_{ij})) = (\sigma_z(X_{ij} + I)) = (X_{ij}(z)) = (z_{ij}) = z.$$

We have thus proven the following proposition.

Proposition 2.4.1. *Let $r : G \to GL(n, \mathbb{C})$ be a representation such that the maps r_{ij} generate $\mathcal{T}(G, \mathbb{C})$ as a $\mathbb{C}$-algebra. Then the map $r_{\mathbb{C}}$ maps $G_{\mathbb{C}}$ bijectively onto $Z(I) \subset GL(n, \mathbb{C})$ with inverse σ.*

The set $Z(I)$ is a closed subgroup of $GL(n, \mathbb{C})$ so it is a Lie subgroup. The subspace topology on $G_{\mathbb{C}}$ defined by $r_{\mathbb{C}}$ coincides with the topology previously defined on $G_{\mathbb{C}}$ (coarsest topology rendering all projections λ_f continuous). Because of the algebraic nature of $Z(I)$ we can say more about it. It is a fact that for an algebraic set $Z(I)$ there is a open set $U \subset \mathbb{C}^{n \cdot n}$ such that $Z(I) \cap U$ is a analytic submanifold. A proof of this claim can be found in Mumford [31]. But $Z(I)$ is also a group so we can conclude that $Z(I)$ is an analytic submanifold. Group multiplication and inversion on $GL(n, \mathbb{C})$ are globally defined rational maps. Their restrictions to $Z(I)$ are therefore holomorphic.

Next we define an involution on $G_{\mathbb{C}}$, $* : G_{\mathbb{C}} \to G_{\mathbb{C}}$, $s \mapsto s^*$ by

$$s^* : \mathcal{T}(G, \mathbb{C}) \to \mathbb{C}, \quad f \mapsto \overline{s(\overline{f})}.$$

This map is a well defined automorphism that is involutive in the sense that $(s^*)^* = s$. The fixed points of this involution are exactly the elements of $\tilde{G}_{\mathbb{R}}$. We know that $\tilde{G}_{\mathbb{R}}$ is isomorphic to $G_{\mathbb{R}}$ because both are isomorphic to G by the discussion in the previous section. It is also easy to show directly that $\tilde{G}_{\mathbb{R}} \cong G_{\mathbb{R}}$. First note that each $f \in \mathcal{T}(G, \mathbb{C})$ can be written as $f_1 + i f_2$ where $f_1, f_2 \in \mathcal{T}(G, \mathbb{R})$. Define $\rho : \tilde{G}_{\mathbb{R}} \to G_{\mathbb{R}}$ as the restriction of $s \in \tilde{G}_{\mathbb{R}}$ to $\mathcal{T}(G, \mathbb{R})$. It is straightforward to check that this defines an isomorphism. The set $\tilde{G}_{\mathbb{R}}$ can be viewed as the real points of $G_{\mathbb{C}}$.

$$r_{\mathbb{C}} \tilde{G}_{\mathbb{R}} = r_{\mathbb{C}} G_{\mathbb{C}} \cap \mathbb{R}^{n \cdot n}.$$

Recall that r is a faithful real representation, otherwise the above identity would not hold. For the remainder of this section we demand that r is a unitary representation such that the r_{ij} generate $\mathcal{T}(G, \mathbb{C})$. We define an involution $* : GL(n, \mathbb{C}) \to GL(n, \mathbb{C})$ by $A^* = \overline{A}^{\vee} = (\overline{A}^t)^{-1}$. This involution is related through the previous one on $G_{\mathbb{C}}$ by the embedding $r_{\mathbb{C}}$. We have that $r_{\mathbb{C}}(s^*) = r_{\mathbb{C}}(s)^*$. A matrix is unitary exactly when $A^* = A$, so the unitary elements of $r_{\mathbb{C}} G_{\mathbb{C}} \subset GL(n, \mathbb{C})$ correspond to the elements of $r_{\mathbb{C}} \tilde{G}_{\mathbb{R}}$. We know that $\tilde{G}_{\mathbb{R}}$ are just the real points of $G_{\mathbb{C}}$. Therefore we can conclude that $r_{\mathbb{R}} G_{\mathbb{R}} = U(n) \cap r_{\mathbb{C}} G_{\mathbb{C}}$.

The next proposition will help us understand the topological structure of $G_{\mathbb{C}}$. Before we can state and prove it we need a bit more preparation. Any complex number z can be written in polar form $re^{i\theta}$ with $r \geq 0$. We can also give a polar decomposition of the elements of $GL(n, \mathbb{C})$. Let $P(n)$ denote the set of Hermitian positive definite matrices and $U(n)$ is again the set of unitary matrices. Every matrix $A \in GL(n, \mathbb{C})$ can be written as the product $A = HP$ where $H \in U(n)$ and $P \in P(n)$. This decomposition is unique. If we drop the uniqueness demand and allow P to be positive semidefinite then the decomposition holds for any complex square matrix, not just invertible ones. Take $A \in GL(n, \mathbb{C})$ arbitrary, then $M = A^\dagger A$ is a Hermitian positive definite matrix. This means that M can be written as $M = V \Lambda V^{-1}$ with V unitary and Λ a diagonal matrix with positive entries on the diagonal. Take $\Lambda^{1/2}$ to be the diagonal matrix with the entries on the diagonal equal to $(\Lambda^{1/2})_{ii} = \sqrt{\Lambda_{ii}}$. Define $M^{1/2} = V\Lambda^{1/2}V^{-1}$ then this is also an element of $P(n)$. Define $H = AM^{-1/2}$ where $M^{-1/2} = (M^{1/2})^{-1} = (M^{-1})^{1/2}$. It is easy to check that H is an unitary matrix and that $HM^{1/2} = A$. In short we have a bijection

$$U(n) \times P(n) \to GL(n, \mathbb{C}), \quad (H, P) \mapsto HP.$$

This map is in fact a homeomorphism. In the proof of the following proposition we will look at the Lie algebra of $U(n) \cap r_{\mathbb{C}}G_{\mathbb{C}}$. We do not need much from the theory of Lie algebras. In fact, we only need to know what a Lie algebra is and how the exponential map is defined. Because Lie algebras play a very small role in this thesis, the reader unfamiliar with Lie algebras is referred to sections I.2 and part of I.3 of Bröcker and tom Dieck [7]. There is more than enough background material in these sections.

Proposition 2.4.2. *1. Let $\check{G} = r_{\mathbb{C}}G_{\mathbb{C}}$. Suppose that $A \in \check{G}$ has a polar decomposition $A = HP$ with $H \in U(n)$ and $P \in P(n)$. Then both H and U are in $\check{G}$. Furthermore, the map*

$$(\check{G} \cap U(n)) \times (\check{G} \cap P(n)) \to \check{G}, \quad (H, P) \mapsto HP$$

is a homeomorphism.

2. *$\check{G} \cap P(n)$ is homeomorphic to a Euclidean space of dimension $dim(G) = dim(\check{G} \cap U(n))$.*

3. *$\check{G} \cap U(n)$ is a maximal compact subgroup of $\check{G}$*

Proof. We start with (1). We know that $A \in \check{G}$ has a polar decomposition in $H \in U(n)$ and $P \in P(n)$. There exists a unitary matrix V such that $VPV^{-1} = \Lambda$, with Λ a diagonal matrix with positive real numbers on the diagonal. Then there exists a diagonal matrix Y such that $D = exp(Y)$. Now we will make use of the fact that $G_{\mathbb{C}}$ is an algebraic set. This gives that $\check{G} \subset GL(n, \mathbb{C})$ is an algebraic set. The map $X_{ij} \mapsto (VXV^{-1})_{ij}$ defines an algebraic isomorphism $\mathbb{C}^{n \cdot n} \to \mathbb{C}^{n \cdot n}$. Consequently $V\check{G}V^{-1}$ is also an algebraic set. By definition it must be equal to the zero locus of some ideal $J \subset \mathbb{C}[X_{ij}]$, so we have $V\check{G}V^{-1} = Z(J)$. If $A = HP$ is in $\check{G}$, then $\bar{A}^t = (A^*)^{-1}$ is also in $\check{G}$ and so is $\bar{A}^t A = P^2$. This means that $\Lambda^2 \in Z(J)$. Let $Q(X_{ij})$ be a polynomial in J. Substituting $X_{ij} \mapsto 0$ if $i \neq j$ and $X_{ii} \mapsto X_i$ yields a polynomial $q(X_i, ..., X_n)$ in n variables. For any $k \in \mathbb{Z}$ we know that $\Lambda^{2k} \in Z(J)$. This implies that $q(exp(2ka_1), ..., exp(2ka_n)) = 0$ where $a_i = Y_{ii} > 0$. If this holds for all $k \in \mathbb{Z}$ one can prove that $q(exp(ta_1), ..., exp(ta_n)) = 0$ for all $t \in \mathbb{R}$ and all $Q \in J$. Setting $t = 1$ gives that $\Lambda \in Z(J) = V\check{G}V^{-1}$. This in turn shows that $P = V\Lambda V^{-1}$ is in $\check{G}$. The unitary matrix H is equal to AP^{-1} so it is also in $\check{C}$. We are left with the claim that the given map is a homeomorphism. This follows from the fact that the map is a continuous bijection and the restriction of the homeomorphism $U(n) \times P(n) \to GL(n, \mathbb{C})$.

We will prove (2) by showing that the Lie algebra L of the Lie groep $\check{G} \cap U(n)$ gives a homeomorphism $L \to \check{G} \cap P(n)$ through the exponential map $X \mapsto exp(iX)$. By the very definition of L it is an Euclidean space of dimension $dim(G) = dim(\check{G} \cap U(n))$. Suppose that $X \in L$, then we have that for all $t \in \mathbb{R}$, $exp(tX) \in \check{G} \cap U(n)$. The set $\check{G}$ is an algebraic set and can be written as $Z(I)$ for some ideal I. Take $p(X_{ij}) \in I$ and substitute the coordinates by the entries $exp(tX)$. This yields an entire analytic function that vanishes on all points $t \in \mathbb{R}$. Consequently it vanishes on all $t \in \mathbb{C}$. For every $t \in \mathbb{R}$ we have that $exp(itX)$ is in $\check{G}$. The Lie algebra L is contained in the Lie algebra $LU(n)$. This Lie algebra consists of skew-Hermitian matrices from which it follows that itX is Hermitian. The image under the exponential map $exp(itX)$ therefore lies in $\check{G} \cap P(n)$. Thusfar we have proven that the exponential map, maps L into $\check{G} \cap P(n)$. We must show that this map is an homeomorphism. We can use the proof of part (1) to show that the map is surjective. In that part of the proof we learned that every $P \in \check{G} \cap P(n)$ can be written as an exponent $P = exp(Y)$ where Y is Hermitian. We also saw that $exp(tY) \in \check{G}$ for all $t \in \mathbb{R}$. This proves surjectivity. The fact that the map $L \to \check{G} \cap P(n)$, $X \mapsto exp(iX)$ is an homeomorphism now follows from the fact that it is the restriction of a homeomorphism. This is the homeomorphism $SH(n) \to P(n)$, $X \mapsto exp(iX)$. Here $SH(n)$ denotes the set of skew-Hermitian $(n \times n)$ matrices.

We are left with proving (3). The set $\check{G} \cap U(n)$ is closed in $U(n)$ so it is compact. We only need to check that it is maximal. Suppose that there is a larger compact subgroup of $\check{G}$. Then this subgroup would contain an element of $\check{G} \cap P(n)$. This cannot be the case because of (2). □

Recall that earlier in this section we made the identification of $r_{\mathbb{R}} G_{\mathbb{R}}$ with $\check{G} \cap U(n)$. From this and the discussion in the previous section we can conclude that G is isomorphic to $\check{G} \cap U(n)$. Recall that L was used to denote the Lie algebra of $\check{G} \cap U(n)$. We can conclude that L is isomorphic to LG. In the proof of the previous theorem we learned that $L\check{G}$ was the direct sum of L and iL. If we put this together we have the following linear isomorphism

$$\mathbb{C} \otimes_{\mathbb{R}} LG \cong \mathbb{C} \otimes_{\mathbb{R}} L \cong L \oplus iL = L\check{G} \cong LG_{\mathbb{C}}.$$

All the isomorphisms that are involved preserve Lie brackets. We have thus shown the following.

Proposition 2.4.3. *We have $\mathbb{C} \otimes_{\mathbb{R}} LG \cong LG_{\mathbb{C}}$ as Lie algebras.*

Definition 2.4.4. *The homomorphism $i : G \to G_{\mathbb{C}}$ as well as the Lie group $G_{\mathbb{C}}$ are called the complexification of G.*

The complexification of G has the following universal property.

Proposition 2.4.5. *Given a representation $r : G \to GL(n, \mathbb{C})$ (that need not be faithful, unitary or real), there is a unique holomorphic representation $r_{\mathbb{C}} : G_{\mathbb{C}} \to GL(n, \mathbb{C})$ such that $r_{\mathbb{C}} \circ i = r$.*

Proof. We start with existence. We construct $r_{\mathbb{C}}$ as usual by $r_{\mathbb{C}}(s) = (s(r_{ij}))$. We can identify $\mathcal{T}(G, \mathbb{C})$ with $\mathbb{C}[X_1, ..., X_d]/I$ for some ideal I as before. Under this identification $G_{\mathbb{C}}$ is identified with $Z(I)$. Each function r_{ij} becomes an algebraic function $Z(I) \to \mathbb{C}$. The map $r_{\mathbb{C}}$ becomes an algebraic map $\mathbb{C}^d \to \mathbb{C}^{m \cdot m}$ and is therefore holomorphic. This settles existence. Now we prove uniqueness. We identify $G_{\mathbb{C}} \cong \check{G} \cong (\check{G} \cap U(n)) \times (\check{G} \cap P(n))$. Suppose that we have the values of $\check{G} \cap U(n)$ for a given holomorphic representation. If $P \in \check{G} \cap P(n)$ is of the form $exp(iX)$ we can determine the values of the representation on $exp(tX)$ for all $t \in \mathbb{R}$. Because the exponential map is holomorphic, this determines $exp(tX)$ for each complex number t. □

2.5 The Forgetful Functor and the Tannaka Group

In this section we will reformulate Tannaka's theorem in terms of category theory. The following chapters will only make use of Tannaka's theorem as it is stated in this section. This section is also used to show that this formulation is equivalent to that of the preceding sections. After completing this section the hurried reader can skip to the next chapter where full Tannaka duality is discussed. In the last section of this chapter we will prove Tannaka's theorem as it is stated here without using (most) of the material of Section 3. This does not amount to repeating the calculations of Section 3 with a different terminology. We will develop some new tools for this job, namely abstract Fourier theory of compact groups. This Fourier theory will not be used in later chapters but will provide useful if the reader wants to explore Tannaka duality beyond the material presented in this thesis. As the goals are set for the upcoming sections, we are ready to sail out once again. Just as in Section 3 we will start with a compact group G and use the representations of that group to construct a group that is isomorphic to it. In order to define this group, which is called the Tannaka group, we need some basic definitions from category theory. So that will be our starting point.

Definition 2.5.1. *We call $\mathcal{C}$ a category of it consists of*

1. *A class $Obj(\mathcal{C})$ of objects X, Y, Z...*
2. *To any two objects X, Y in $\mathcal{C}$ we can associate a set $Hom_{\mathcal{C}}(X, Y)$ of morphisms (also called arrows). Sometimes we write $f : X \to Y$ for $f \in Hom_{\mathcal{C}}(X, Y)$.*

3. *For any three objects X, Y and Z we have a function $\circ : Hom_{\mathcal{C}}(Y, Z) \times Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{C}}(X, Z)$, $(f, g) \mapsto f \circ g$. This function is associative in the sense that $(f \circ g) \circ h = f \circ (g \circ h)$ holds whenever it is defined.*

4. *For each object X there exists an arrow $id_X \in End_{\mathcal{C}}(X) = Hom_{\mathcal{C}}(X, X)$ called the identity of X. If $f : X \to Y$ is an arrow the identity arrows satisfy $id_Y \circ f = f$ and $f \circ id_X = f$. An arrow $f : X \to Y$ is called an isomorphism if it has a two-sided inverse $g : Y \to X$ in the sense that $f \circ g = id_Y$ and $g \circ f = id_X$.*

Let's look at some categories that play an important role here. Let $Vect_{\mathbb{C}}$ be the category of finite dimensional vector spaces over $\mathbb{C}$. This category has finite dimensional vector spaces over $\mathbb{C}$ as objects and linear maps as morphisms. The identity arrow is the identity map and the composition operation is just the usual composition of linear maps. Let $\mathcal{H}_f$ denote the category that has finite dimensional Hilbert spaces over $\mathbb{C}$ as objects and linear maps between them as arrows. This is a lame example as this is just $Vect_{\mathbb{C}}$ with a different name. We would only see a difference if we drop the finiteness condition on the dimension of the underlying vector spaces. Previously we also encountered $Rep_f(G, \mathbb{C})$, the category of continuous finite dimensional representations of a compact group G. The objects are the representations and the arrows are the intertwiners.

Definition 2.5.2. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ is a map that maps objects of $\mathcal{C}$ to objects of $\mathcal{D}$ and arrows of $\mathcal{C}$ to arrows of $\mathcal{D}$. Each arrow $f : X \to Y$ of $\mathcal{C}$ is mapped to an arrow $F(f) : F(X) \to F(Y)$. The functor respects the composition $\circ$ in the sense that $F(f \circ g) = F(f) \circ F(g)$ and $F(id_X) = id_{F(X)}$.*

Let $Rep_f(G, \mathbb{C})$ be the category of continuous finite dimensional representations of a compact group G and $Vect_{\mathbb{C}}$ be the category of finite dimensional vector spaces. We define the forgetful functor $\omega : Rep_f(G, \mathbb{C}) \to Vect_{\mathbb{C}}$ as follows. Each representation is mapped to the underlying vector space $\omega((V, \pi_V)) = V$ and intertwiners are mapped to the corresponding linear maps.

Definition 2.5.3. *Suppose that $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{C} \to \mathcal{D}$ are functors. A natural transformation u from F to G is defined as follows. For each object $X \in \mathcal{C}$ we have an arrow $u_X : F(X) \to G(X)$ of $\mathcal{D}$. For all objects X and Y of $\mathcal{C}$ and arrows $f \in Hom_{\mathcal{C}}(X, Y)$ the following square should commute.*

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle u_X}\downarrow & & \downarrow{\scriptstyle u_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array}$$

A natural isomorphism is a natural transformation u such that all arrows u_X are isomorphisms.

Let $End(\omega)$ denote all natural transformations of the forgetful functor to itself. Naturality of $u \in End(\omega)$ means that the following square commutes for each linear map $h : V \to W$ that comes from an intertwining operator $h : (V, \pi_V) \to (W, \pi_W)$.

$$\begin{CD} V @>{u_V}>> V \\ @V{h}VV @VV{h}V \\ W @>>{u_W}> W \end{CD}$$

In the above square we used the sloppy notation u_V for $u_{(V,\pi_V)}$ as the maps of u are indexed by the objects of $Rep_f(G, \mathbb{C})$ and not by the objects of $Vect_{\mathbb{C}}$. We will continue denoting representations by the underlying vector spaces whenever we think that no confusion would arise. Each element x of the group G defines a natural transformation $\pi(x) \in End(\omega)$. This transformation is given by $\pi(x)_{(V,\pi_V)} = \pi_V(x) \in End(V)$. From the above square we read that this transformation is natural if for each intertwiner $h : V \to W$ we have that $\pi_W(x) \circ h = h \circ \pi_V(x)$. This holds by definition of the intertwiners.

We want to consider the natural transformations that behave like the $\pi(x)$. We therefore consider only natural transformations u that are tensor preserving. By this we mean that $u_{V \otimes W} = u_V \otimes u_W$ holds for alle representations V and W. Furthermore, if $\mathbf{1}$ denotes the trivial representation (which, in the language of the following chapter, is a tensor unit) then we demand that $u_{\mathbf{1}} = id_{\mathbf{1}}$. We can also define a conjugation operation on $End(\omega)$ by using the conjugation operation on representations. This map $End(\omega) \to End(\omega)$, $u \mapsto \overline{u}$ is defined as follows. $\overline{u}_V(x) = \overline{u_{\overline{V}}(\overline{x})}$. We say that $u \in End(\omega)$ is selfconjugate if $\overline{u} = u$. Note that the transformations $\pi(x)$ are selfconjugate.

We define the Tannaka group $G_\omega = Aut^{\otimes}\omega$, which at this point is just a set, as the selfconjugate tensor preserving elements of $End(\omega)$. For each $x \in G$ the natural transformation $\pi(x)$ is an element of G_ω. It is straightforward to check that this set is closed under composition of natural transformations. Next we fix a topology for the Tannaka group. The set $End(\omega)$ has a natural topology. This is the coarsest topology that renders all projections $End(\omega) \to End(V)$, $u \mapsto u_V$ continuous. Note that the Tannaka group G_ω is a closed subset of $End(\omega)$ with respect to this topology.

Proposition 2.5.4. *The Tannaka group G_ω is a topological group.*

Proof. The group operation is given by composition. The identity element is the natural transformation $u \in G_\omega$ for which $u_V = id_V$ holds for each representation. This is clearly natural, tensor preserving and selfconjugate.

Finding an inverse is a bit more work. Take an arbitrary $u \in G_\omega$ and consider the projections $u_V \in End(V)$. We have for each representation V the dual representation $V^\vee$. We also have the intertwiner $\epsilon : V^\vee \otimes V \to \mathbb{C}$ given by $s \otimes v \mapsto s(v)$. Naturality of u implies that $\epsilon \circ u_{V^\vee \otimes V} = u_{\mathbf{1}} \circ \epsilon$ which amounts to $\epsilon = \epsilon \circ (u_{V^\vee} \otimes u_V)$. From this fact we can derive that $u_{V^\vee} \circ u_V^t = id_{V^\vee}$. If we define u' as $u'_V = (u_{V^\vee})^t$, then we would have $u' \circ u = u \circ u' = id$ where id is the identity transformation. It is straightforward to check that u' is an element of the Tannaka group. By looking at the projections on the representations one can verify that the inverse maps and multiplication are continuous. □

The map $\pi : G \to G_\omega$ given by $x \mapsto \pi(x)$ defines an continuous homomorphism of groups. Tannaka's theorem amounts to the claim that this map is an isomorphism of Lie groups. Starting the following section, the remainder of this chapter is concerned with deriving this result independent of the results of Section 3. In this section we will show that the Tannaka group is just the group $\tilde{G}_{\mathbb{R}}$ in disguise. In order to show this, it helps to consider isomorphism classes of irreducible representations instead of all representations.

Proposition 2.5.5. *Let $\hat{G}$ denote the set of isomorphism classes of irreducible representations of G. Then the map*

$$q : End(\omega) \to \prod_{\lambda \in \hat{G}} End(V_\lambda), \quad u \mapsto (u_\lambda | \lambda \in \hat{G})$$

is an isomorphism of topological algebras.

Proof. Because every element of the Tannaka group is a natural transformation, the following diagram commutes for any V and W.

$$\begin{array}{ccccc} V & \longrightarrow & V \oplus W & \longleftarrow & W \\ {\scriptstyle u_V}\downarrow & & {\scriptstyle u_{V\oplus W}}\downarrow & & \downarrow{\scriptstyle u_W} \\ W & \longrightarrow & V \oplus W & \longleftarrow & W \end{array}$$

This proves that $u_{V \oplus W} = u_V \oplus u_W$. We know from Section 2 that every representation is semisimple, so each $u \in G_\omega$ is completely determined by $q(u)$. This proves injectivity and we move on to surjectivity. Let $t = (t_\lambda | \lambda \in \hat{G})$ be and element of $\prod_{\lambda \in \hat{G}} End(V_\lambda)$. We seek a natural transformation such that for each $\lambda \in \hat{G}$ we have that $u_\lambda = t_\lambda$. Given a representation V and a irreducible representation V_λ, then V is called V_λ-isotypical if it is the direct sum of irreducible representations each of which is isomorphic to V_λ. We can decompose any representation S into isotypical parts as $S = \Sigma^{\oplus}_{\lambda \in \hat{G}} S_\lambda$. For each isomorphism class we have the following intertwining isomorphism $\Psi_\lambda : V_\lambda \otimes Hom(V_\lambda, S_\lambda) \to S_\lambda$ given by $\Psi_\lambda(v, f) = f(v)$. Define

$u_{S_\lambda} = \Psi_\lambda \circ (t_\lambda \otimes id) \circ \Psi_\lambda^{-1}$ and $u_S = \Sigma^{\oplus}_{\lambda \in \hat{G}} u_{S_\lambda}$. This gives us a natural transformation u such that $u_\lambda = t_\lambda$ holds for every $\lambda \in \hat{G}$. The rest of the proof is easy. □

The previous proposition will help us identify the Tannaka group as a compact Lie group.

Proposition 2.5.6. *The Tannaka group G_ω is a compact Lie group if G is a compact Lie group.*

Proof. The proof that G_ω is compact is performed in almost the same as proving that $G_\mathbb{R}$ is compact in Theorem 2.3.7(3). Since we presuppose that G is a compact Lie group, we now have a normalized invariant inner product at our disposal. This gives us the bilinear pairing

$$h : \overline{V} \otimes V \to \mathbb{C}, \quad h(\bar{x}, y) = \langle x, y \rangle .$$

As this is an intertwiner we know for every $u \in End(\omega)$ the identity $h \circ u_{\overline{V} \otimes V} = u_{\mathbf{1}} \circ h$ hold. In the Tannaka group this becomes $h(\overline{u_V(x)}, u_V(y)) = h(\bar{x}, y)$, or $\langle u_V(x), u_V(y) \rangle = \langle x, y \rangle$. The linear map u_V is an element of the unitary group $U(V)$. We can apply the same kind of reasoning as in the proof of 2.3.7(3) and conclude that the group is compact. The disussion following this proof shows how G_ω can be given the structure of a Lie group. □

As promised before we will show that in a natural way the compact Lie groups G_ω and $\tilde{G}_\mathbb{R}$ are isomorphic. Suppose that we are given an $\mathbb{C}$-algebra homomorphism $s \in \tilde{G}_\mathbb{R}$. Pick an irreducible representation V and choose a basis to get a matrix representation (r_{ij}). Define $u_V = (s(r_{ij})) \in End(V)$, then the map

$$k : \tilde{G}_\mathbb{R} \to G_\omega, \quad s \mapsto u = (u_{V_\lambda} = r_\mathbb{C}(s) | \lambda \in \hat{G})$$

defines an isomorphism of Lie groups. We need to check that the map is a homeomorphism and a group homomorphism, but first we need to check if the image is indeed formed by selfconjugate tensor preserving natural transformations of the forgetful functor to itself. As we only look at the isomorphism classes of irreducible representations, Schur's lemma gives that naturality should pose no problem. Using that s is a homomorphism of $\mathbb{C}$-algebras and the fact that $s(\overline{f}) = \overline{s(f)}$, it is straightforward to check that $k(s)$ is selfconjugate and tensor preserving. Therefore the map is well defined. We also know that the map is a homomorphism of groups because of Proposition 2.3.7, which gave us $r_\mathbb{C}(s \cdot t) = r_\mathbb{C}(s) r_\mathbb{C}(t)$. Continuity and injectivity of k are easy so we turn to surjectivity. Let $u \in G_\omega$ and consider $u_V \in End(V)$. Pick a basis to get matrices $u_V = (a_{ij})$ and $\pi_V = (r_{ij})$. Define $s(r_{ij}) = a_{ij}$. Using that u is selfconjugate, tensor preserving and that $u_{V \oplus W} = u_V \oplus u_W$ it follows that the defined map s is an element of

$\tilde{G}_{\mathbb{R}}$. In this way we define a continuous two sided inverse of k completing the proof that k is an isomorphism of Lie groups. Let $i : G \to \tilde{G}_{\mathbb{R}}$ be as in Section 2.3, up to identification of $\tilde{G}_{\mathbb{R}}$ with $G_{\mathbb{R}}$. The map k maps $i(x)$ to $\pi(x)$ and we have that $k = \pi \circ i^{-1}$. This proves that π is an isomorphism of Lie groups.

In the remainder of the chapter we will prove that $\pi(x) : G \to G_\omega$ is an isomorphism of Lie groups without using the theory of Section 2.3. We close this section with the definition of another involution on $End(\omega)$. We define

$$* : End(\omega) \to End(\omega), \quad u \mapsto u^* = (\bar{u})^\vee = \overline{(u^\vee)}$$

where we have that $(u^\vee)_V = (u_{V^\vee})^\vee$. There is a good reason to denote this involution by $*$. The map q of Proposition 2.5.5 transports this involution to the canonical C^*-algebra structure on each $End(V_\lambda)$. A natural transformation $u \in End(\omega)$ is called unitary if $u^*u = uu^* = id$. These elements form a subgroup that is isomorphic $\prod_{\lambda \in \hat{G}} U(dim(V_\lambda))$. The Tannaka group is a closed subgroup of this product.

2.6 Tannaka's Theorem Revisited

In this section we follow the discussion in Section 2 of Joyal and Street [21].

Definition 2.6.1. *Let $f \in C^0(G, \mathbb{C})$ be a continuous function. The Fourier transform of f, $\mathcal{F}f \in End(\omega)$ is defined by the integral expression*

$$(\mathcal{F}f)_V = \int_G f(x)\pi_V(x)dx.$$

Define the convolution of two functions $f, g \in C^0(G, \mathbb{C})$ as $(f * g) = \int_G f(xy^{-1})g(y)dy$. Also consider the $*$-involution on $C^0(G, \mathbb{C})$ given by $f^*(x) = \overline{f(x^{-1})}$. The Fourier transform satisfies the following properties with regard to these operations

$$\mathcal{F}(f * g) = \mathcal{F}f \circ \mathcal{F}g, \quad \mathcal{F}(f^*) = (\mathcal{F}f)^*.$$

We know from Proposition 2.5.5 that any element of $End(\omega)$ is completely determined by the projections on the isomorphism classes of the irreducible representations. The Fourier transform $\mathcal{F}f$ of f is determined by the components $(\mathcal{F}f)(\lambda) = (\mathcal{F}f)_{V_\lambda}$, where $\lambda \in \hat{G}$.

As an example we will look at the case that G is an abelian compact Lie group. This allows us to see a connection of the Fourier theory of compact groups with the usual Fourier theory. It also gives an opportunity to introduce the characters of a compact Lie group. Characters are important tools when studying abelian topological groups but we will sometimes use them in nonabelian cases. It can be shown (Bröcker and tom Dieck [7] I.3.7) that a compact abelian Lie group is isomorphic to

$$G \cong S^1 \times ... \times S^1 \times \mathbb{Z}/m_1 \times ... \times \mathbb{Z}/m_k$$

where the m_i are natural numbers. Given a topological group G and a representation V, the character of that representation is the function

$$\chi_V : G \to \mathbb{C}, \quad x \mapsto Tr(\pi_V(x)).$$

The character of an irreducible representation is called an irreducible character. Characters have numerous properties because of their definition as a trace. For instance we have that $\chi_{V\otimes W} = \chi_V \cdot \chi_W$ and $\chi_{V\oplus W} = \chi_V + \chi_W$. The characters of two isomorphic representations are the same. Let $f : V \to W$ be an isomorphism, then we have

$$\chi_V(x) = Tr(\pi_V(x)) = Tr(f^{-1}\pi_W(x)f) = Tr(\pi_W(x)) = \chi_W(x)$$

where we used the cyclic property of the trace. A consequence of this is that $\overline{\chi_V} = \chi_{\overline{V}} = \chi_{V^\vee}$. Define V^G to be the fixed point set $V^G = \{v \in V | \pi_V(x)v = v, \forall x \in G\}$. Define the following map that acts as a projector of V onto the fixed point set $p : V \to V^G$, $v \mapsto \int x.vdx$. Using this projection we get the following result.

$$dim(V^G) = Tr(p) = Tr(\int \pi_V(x)dx) = \int Tr(\pi_V(x))dx = \int \chi_V(x)dx.$$

We now apply the identity $\int \chi_V(x)dx = dim(V^G)$ to the case of $Hom(V,W)$ and note that $Hom(V,W)^G = Hom_{Rep_f(G)}(V,W)$. Using the relations $\chi_{Hom(V,W)} = \chi_{V^\vee \otimes W} = \chi_{V^\vee}\chi_W = \overline{\chi_V}\chi_W$ we get

$$\langle \chi_W, \chi_V \rangle = \int \bar{\chi}_V(x)\chi_W(x)dx = dim(Hom_{Rep_f(G)}(V,W)).$$

If V_λ and W_μ are both irreducible representations, Schur's lemma gives

$$\int \bar{\chi}_\lambda(x)\chi_\mu(x)dx = \delta_{\lambda\mu}.$$

Let V be any representation of G. We can write $V = \oplus_\lambda n_\lambda V_\lambda$. From this we can see that $\langle \chi_V, \chi_V \rangle = \sum_\lambda n_\lambda^2$. A representation is irreducible if and only if $\langle \chi_V, \chi_V \rangle = 1$. Suppose again that V and W are irreducible representations of G. The following computation shows that the representation $V \otimes W$ of $G \times G$ is also irreducible.

$$\begin{aligned}\int_{G\times G} \bar{\chi}_{V\otimes W}\chi_{V\otimes W} &= \int_{G\times G} \bar{\chi}_V(x)\bar{\chi}_W(y)\chi_V(x)\chi_W(y)dxdy \\ &= \int_G \bar{\chi}_V(x)\chi_V(x)dx \cdot \int_G \bar{\chi}_W(y)\chi_W(y)dy = 1\end{aligned}$$

Conversely, every irreducible representation of $G \times G$ can be written as a tensor product $V \otimes W$ where V and W are irreducible representations of G. We can prove this as follows. Suppose that U is a $G \times G$ module. The following map is an isomorphism of representations.

$$\phi : \bigoplus_{\lambda \in \hat{G}} Hom_G(W_\lambda, U) \otimes W_\lambda \to U, \quad f \otimes w \mapsto f(w).$$

This isomorphism allows us to decompose $U \cong \bigoplus_{\lambda,\mu} n_{\lambda\mu} V_\lambda \otimes W_\mu$. Let $Irr(G, \mathbb{C})$ denote the set of all irreducible representations of G, then the previous observations provide us the following bijection.

$$Irr(G, \mathbb{C}) \times Irr(G, \mathbb{C}) \to Irr(G \times G, \mathbb{C}), \quad (V, W) \mapsto V \otimes W.$$

The discussion of characters thusfar was for any topological group G. Now we restrict out attention once again to abelian topological groups. The above bijection allows us to derive the irreducible representations of any abelian compact Lie group from the irreducible representations of S^1 and $\mathbb{Z}/m$. Let V be an irreducible representation of an abelian G. Each $x \in G$ defines an intertwiner $\pi_V(x) : V \to V$. By Schur's lemma, this map must be a scalar multiple of the identity. This implies that every subspace of V is a G-invariant space and that V must be one dimensional. So every complex irreducible representation of an abelian G has complex dimension one. The representation is given by a single complex valued function that is equal to the irreducible character of the representation $\chi_V : G \to \mathbb{C}$. Because G is compact, we can make the representation unitary. The irreducible characters of S^1 are the continuous group homomorphisms $S^1 \to S^1$. These are just the functions $z \mapsto z^n$ with $n \in \mathbb{Z}$. In the case that $G = S^1$ the Fourier transform becomes

$$(\mathcal{F}f)_n = \int_G f(x) e^{2\pi i n x} dx.$$

The Fourier transform for the circle is just the familiar Fourier transform for functions on the circle. The irreducible characters for $\mathbb{Z}/m$ are given by $x \bmod m \mapsto exp(2\pi i x k/m)$ for $k \in \{0, 1, ..., m-1\}$. Finally we can give the Fourier coefficients for G of the form $S^1 \times ... \times S^1 \times \mathbb{Z}/m_1 \times ... \times \mathbb{Z}/m_k$.

$$(\mathcal{F}f)_{n_1,...,n_j,p_1,...,p_k} = \int f(x_1, ..., p_k) e^{2\pi i \Sigma_i n_i x_i} e^{2\pi i \Sigma_i p_i y_i / m} dx_1 ... dx_j dy_1 ... dy_k$$

where $n_i \in \mathbb{Z}$ and $p_i \in \{0, 1, ..., m_i - 1\}$. It can be shown that the characters of G form a group that is, in a natural way, isomorphic to G. This statement is the abelian analogue of Tannaka's theorem.

After this excursion we turn back to the case that G is any compact Lie group. We would like to define an inverse of the Fourier transform. The

Fourier transform gives us a map

$$\mathcal{F} : C^0(G, \mathbb{C}) \to \prod_{\lambda \in \hat{G}} End(V_\lambda).$$

This map cannot be surjective as each point of the image is bounded as $\|\mathcal{F}f(\lambda)\| \le \|f\|_\infty$. The norm on the left hand side is the norm of the C^*-algebra $End(V_\lambda)$. The image is contained in the set

$$\prod_{\lambda \in \hat{G}}^{bded} End(V_\lambda) \subset \prod_{\lambda \in \hat{G}} End(V_\lambda)$$

which is a C^*-algebra. We now define a partial inverse of $\mathcal{F}$.

$$\mathcal{F}^{-1} : \Sigma^{\oplus}_{\lambda \in \hat{G}} End(V_\lambda) \to C^0(G, \mathbb{C})$$

$$(\mathcal{F}^{-1}u)(x) = \Sigma_{\lambda \in \hat{G}} Tr(u(\lambda)\pi_\lambda(x)^{-1}) d_\lambda.$$

In the above identity d_λ is the dimension of the representation V_λ. The reader can check that for the circle S^1 the partial inverse gives the desired formula of the inverse Fourier transformation. In order to discuss the properties of the Fourier transform, we need to derive the orthogonality relations. Let $\lambda, \mu \in \hat{G}$ and $A \in Hom(V_\mu, V_\lambda)$ be an intertwiner. Then

$$\int_G \pi_\lambda(x) A \pi_\mu(x)^{-1} dx = \delta_{\lambda\mu} d_\lambda^{-1} Tr(A) id_{V_\lambda}.$$

If $\lambda \ne \mu$ it follows from Schur's lemma that the left hand side of the above equation is zero. If $\lambda = \mu$ then Schur's lemma gives us that the map is proportional to the identity. The constant of proportionality is found by taking the trace. Suppose that A is of the special form $A(y) = \phi(y)v$ where $\phi : V_\lambda \to \mathbb{C}$ is a linear form. For such an A the above identity becomes

$$\int_G \pi_\lambda(x)(v)\phi(\pi_\mu(x)^{-1}(w)) dx = \delta_{\lambda\mu} d_\lambda^{-1} \phi(v) w.$$

As this holds for any linear form ϕ we have proven the following lemma.

Lemma 2.6.2. *Let λ, $\mu \in \hat{G}$, then*

$$\int_G \pi_\lambda(x) v \otimes \pi_\mu(x)^{-1} w dx = \delta_{\lambda\mu} d_\lambda^{-1} w \otimes v.$$

Proposition 2.6.3. *(Orthogonality relations) Let $\lambda, \mu \in \hat{G}$, $A \in End(V_\lambda)$ and $B \in End(V_\mu)$, then*

$$\int_G Tr(A\pi_\lambda(x)) Tr(B\pi_\mu(x)^{-1}) dx = \delta_{\lambda\mu} d_\lambda^{-1} Tr(AB).$$

$$\int_G Tr(A\pi_\lambda(x)) \overline{Tr(B\pi_\mu(x))} dx = \delta_{\lambda\mu} d_\lambda^{-1} Tr(AB^\dagger).$$

Proof. We only prove the first relation as the second follows directly from it. The relation is linear in both A and B so it suffices to check the relation when A and B are of rank one. Take $Ax = \phi(x)v$ and $By = \psi(y)w$. The orthogonality relation can be obtained from the identity of the previous lemma by applying $\phi \otimes \psi$ to both sides and subsequently taking the trace. □

The trace defines an inner product by $\langle A, B\rangle = Tr(A^\dagger B)$. Keeping this in mind, the following relation can be proven from the orthogonality relations by applying $Tr(B, -)$ on both sides.

Corollary 2.6.4. *Let $\lambda, \mu \in \hat{G}$ and $A \in End(V_\lambda)$, then the following holds*

$$d_\lambda \int_G Tr(A\pi_\lambda(x))\pi_\mu(x)^{-1}dx = \delta_{\lambda\mu}A.$$

We will use the following inner products. On $C^0(G, \mathbb{C})$ we take $\langle f, g\rangle = \int_G \bar{f}(x)g(x)dx$ and on $\Sigma_\lambda^\oplus EndV_\lambda$ we take $\langle u, v\rangle = \Sigma_\lambda Tr(u^*(\lambda)v(\lambda))d_\lambda$. The inner product on $C^0(G, \mathbb{C})$ can be written as $\langle f, g\rangle = \epsilon(f^* * g)$, where $\epsilon : C^0(G, \mathbb{C}) \to \mathbb{C}$ is the evaluation map in the identity $\epsilon(f) = f(e)$. Now we will continue exploring the Fourier transform and its (partial) inverse.

Proposition 2.6.5. *The following identities hold*

1. $\mathcal{F}\mathcal{F}^{-1}u = u$,
2. $\mathcal{F}^{-1}(uv) = (\mathcal{F}^{-1}u) * (\mathcal{F}^{-1}v)$,
3. $\mathcal{F}^{-1}(u^*) = (\mathcal{F}^{-1}u)^*$,
4. $\langle \mathcal{F}^{-1}u, \mathcal{F}^{-1}v\rangle = \langle u, v\rangle$.

Proof. The first three parts are easy. Writing out the Fourier tranforms and the convolution that appear in identities (1) and (2), the statements follow straight from the orthogonality relations. We move onto the last part. First note the identity $\epsilon(\mathcal{F}^{-1}u) = \sum_\lambda Tr(u(\lambda))d_\lambda$.

$$\begin{aligned}\langle \mathcal{F}^{-1}u, \mathcal{F}^{-1}v\rangle &= \epsilon((\mathcal{F}^{-1}u)^* * (\mathcal{F}^{-1}v)) = \epsilon\mathcal{F}^{-1}(u^*v) \\ &= \sum_\lambda Tr(u^*(\lambda)v(\lambda))d_\lambda = \langle u, v\rangle\end{aligned}$$

□

We can complete the domain of $\mathcal{F}^{-1}$ with respect to the norm coming from the previously defined inner product. The completion is a Hilbert space isomorphic to the Hilbert sum $\widehat{\bigoplus}_\lambda End(V_\lambda)$ where each $End(V_\lambda)$ has the norm $\|A\|^2 = d_\lambda Tr(A^\dagger A)$. By virtue of property (4) of Proposition 2.6.5 the continuous extension of $\mathcal{F}^{-1}$ defines an isometric embedding

$$\mathcal{F}^{-1} : \widehat{\bigoplus}_\lambda End(V_\lambda) \to L^2(G).$$

Theorem 2.6.6. *(Plancherel theorem for compact groups) The Fourier transform $\mathcal{F}$ can be extended continuously to an isometry*

$$\mathcal{F} : L^2(G) \xrightarrow{\sim} \widehat{\bigoplus_\lambda} End(V_\lambda).$$

Proof. We know that $\mathcal{F}^{-1}$ is an isometric embedding. The subspace $Im(\mathcal{F}^{-1})$ of $L^2(G)$ is closed because it is complete. Let $f : G \to \mathbb{C}$ be a representative function. This function is a linear combination of functions coming from a representation so it can be written as $f(x) = Tr(A\pi_V(x))$. The subspace $Im(\mathcal{F}^{-1})$ contains all representative functions. These functions lie dense in $L^2(G)$ so $Im(\mathcal{F}^{-1}) = L^2(G)$. □

Recall that we have two left actions L and R of G on $C^0(G, \mathbb{C})$

$$L(x, f)(y) = f(x^{-1}y), \quad R(x, f)(y) = f(yx).$$

We also have two left actions on the Hilbert spaces $End(V_\lambda)$ given by

$$L(x, A) = \pi_\lambda(x)A, \quad R(x, A) = A\pi_\lambda(x^{-1}).$$

It is a straightforward check that the Fourier transform respects these actions

$$\mathcal{F}(L(x, f)) = L(x, \mathcal{F}f), \quad \mathcal{F}(R(x, f)) = R(x, \mathcal{F}f).$$

A class function is defined as a $f \in C^0(G, \mathbb{C})$ that is invariant under the combined action of L and R. By this we mean that

$$f(x^{-1}yx) = f(y) \ \ \forall y \in G.$$

It is an easy exercise to check that a function is a class function if and only if it is in the centre with respect to the convolution product. We are going to restrict the Fourier transform to the central part. The following proposition tells us what the centre of $End(\omega)$ looks like.

Proposition 2.6.7. *An element $u \in End(\omega)$ is central if and only if, for every representation V of G the linear map $u_V : V \to V$ is an intertwining operator.*

Proof. Suppose that u is central. Then for each $x \in G$ we have that $\pi(x)u = u\pi(x)$. In particular $\pi_V u_V = u_V \pi_V$ holds for each representation V. Conversely, let each $u_V : V \to V$ be an intertwiner. If $w \in End(\omega)$ then we have $u_V w_V = w_V u_V$. Therefore $uw = wu$ and u is central. □

Suppose that $f \in C^0(G, \mathbb{C})$ is a class function. Because the Fourier transforms respects the actions of G, $L(x, f) = R(x^{-1}, f)$ implies that $\pi(x)(\mathcal{F}f) = (\mathcal{F}f)\pi(x)$. This shows that $\mathcal{F}f$ is central and that for each

$\lambda \in \hat{G}$, the map $(\mathcal{F}f)(\lambda) : V_\lambda \to V_\lambda$ is an intertwiner. By Schur's lemma it is proportional to the identity. Call the constant of proportionality $(Tf)(\lambda)$.

$$\begin{aligned}(Tf)(\lambda) = \frac{1}{d_\lambda} Tr((\mathcal{F}f)(\lambda)) &= \frac{1}{d_\lambda}\int_G f(x) Tr(\pi_\lambda(x))dx \\ &= \frac{1}{d_\lambda}\int_G f(x)\chi_\lambda(x)dx.\end{aligned}$$

Here $\chi_\lambda(x)$ is the irreducible character of V_λ. The relation $(\mathcal{F}f)(\lambda) = (Tf)(\lambda) id_\lambda$ gives us the relations $T(f * g) = (Tf)(Tg)$ and $T(f^*) = \overline{Tf}$. The centre of $End(V_\lambda)$ is equal to $\mathbb{C} id_\lambda$. The centre of $End(\omega)$ is therefore isomorphic to $\mathbb{C}^{\hat{G}}$. Take any function $g : \hat{G} \to \mathbb{C}$ and define the inverse Fourier transform as the function $(T^{-1}g) : G \to \mathbb{C}$ given by

$$(T^{-1}g)(x) = \sum_\lambda g(\lambda)\overline{\chi_\lambda(x)} d_\lambda.$$

Using Proposition 2.6.6 and the definitions of T and T^{-1} we derive

1. $TT^{-1}g = g$,
2. $T^{-1}(gh) = (T^{-1}g) * (T^{-1}h)$,
3. $T^{-1}(g^*) = (T^{-1}g)^*$,
4. $\langle T^{-1}g, T^{-1}h\rangle = \langle g, h\rangle$.

In the last identity the inner product on the right hand side is defined by

$$\langle g, h\rangle = \sum_\lambda \bar{g}(\lambda)h(\lambda)d_\lambda^2.$$

Let $L^2(\hat{G})$ be the space of square summable functions with respect to the spectral measure. The spectral measure on $\hat{G}$ is the measure that assigns weights d_λ^2 to the singletons $\{\lambda\}$. Let $C(G)$ denote the set of conjugacy classes of G. There is a canonical measure on $C(G)$ by taking the image of the Haar measure along the projection $G \to C(G)$. The space $L^2(C(G))$ is isomorphic to the subspace of $L^2(G)$ of square summable class functions. The following theorem is nothing more than the restriction of the Fourier transform to the central parts.

Theorem 2.6.8. *The Fourier transforms T and T^{-1} have continuous extensions to mutually inverse isometries*

$$L^2(C(G)) \overset{\sim}{\leftrightarrows} L^2(\hat{G}).$$

A collection $\mathcal{X}$ of representations of G is called closed when it contains

1. π_V if it is isomorphic to some $\pi_W \in \mathcal{X}$,

2. π_V if it is a subrepresentation of some $\pi_W \in \mathcal{X}$,

3. $\pi_V \oplus \pi_W$ if π_V, $\pi_W \in \mathcal{X}$,

4. $\pi_V \otimes \pi_W$ if π_V, $\pi_W \in \mathcal{X}$,

5. $\overline{\pi}_V$ if $\pi_V \in \mathcal{X}$,

6. The trivial representation **1**.

The set of representative functions of the members of $\mathcal{X}$ make up a subalgebra $\mathcal{T}(\mathcal{X}, \mathbb{C})$ of the algebra $\mathcal{T}(G, \mathbb{C})$ of representative functions.

Lemma 2.6.9. *Let $\mathcal{X}$ be a closed collection of representations of G. Suppose that for every $x \in G$, $x \neq e$ there exists a representation $\pi_V \in \mathcal{X}$ such that $\pi_V(x) \neq e$. Then $\mathcal{X} = Rep(G, \mathbb{C})$.*

Proof. Suppose that $\mathcal{X} \neq Rep(G, \mathbb{C})$. This means that there is a $\lambda \in \hat{G}$ such that $\pi_\lambda \notin \mathcal{X}$. By the orthogonality relations (or the Peter Weyl theorem 2.2.10) there is a function that is orthogonal to the whole of $\mathcal{T}(\mathcal{X}, \mathbb{C})$. But the hypothesis implies that $\mathcal{T}(\mathcal{X}, \mathbb{C})$ separates points. By the Stone-Weierstrass theorem $\mathcal{T}(\mathcal{X}, \mathbb{C})$ is dense in $Rep(G, \mathbb{C})$. This is a contradiction. □

In Section 2.3, in the proof of Tannaka's theorem we showed that $i : G \to G_{\mathbb{R}}$ is an isomorphism by showing that there is an isomorphism of algebras $i^* : \mathcal{T}(G_{\mathbb{R}}, \mathbb{R}) \to \mathcal{T}(G, \mathbb{R})$. Here we do the same. We show that $\pi : G \to G_\omega$ induces, via restriction an isomorphism of algebras $\pi^* : \mathcal{T}(G_\omega, \mathbb{C}) \to \mathcal{T}(G, \mathcal{C})$. Before we can do this, we need some more category theory.

Definition 2.6.10. *Let $\mathcal{C}$ be a category. A subcategory $\mathcal{D} \subset \mathcal{C}$ is defined by a subsclass $Obj(\mathcal{D}) \subset Obj(\mathcal{C})$ and, for every X, $Y \in Obj(\mathcal{D})$, a subset $Hom_{\mathcal{D}}(X, Y) \subset Hom_{\mathcal{C}}(X, Y)$. The arrow id_X should be in $Hom_{\mathcal{D}}(X, X)$ for each object X in $\mathcal{D}$. The arrows in $\mathcal{D}$ should be closed under the composition $\circ$ of $\mathcal{C}$. A subcategory $\mathcal{D} \subset \mathcal{C}$ is called full if $Hom_{\mathcal{D}}(X, Y) = Hom_{\mathcal{C}}(X, Y)$ for all X and Y in $\mathcal{D}$.*

Definition 2.6.11. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called faithful, respectively full, if the map*

$$F_{X,Y} : Hom_{\mathcal{C}}(X, Y) \to Hom_{\mathcal{D}}(F(X), F(Y)),$$

is injective, respectively surjective, for all objects X and Y of $\mathcal{C}$. The functor is called essentially surjective if for every object Y of $\mathcal{D}$ there is an object X of $\mathcal{C}$ such that $F(X) \cong Y$.

Definition 2.6.12. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if there exists a functor $G : \mathcal{D} \to \mathcal{C}$ and natural isomorphisms $\eta : FG \to id_{\mathcal{D}}$ and $\epsilon : id_{\mathcal{C}} \to GF$. Two categories are called equivalent if there is an equivalence of categories.*

The proof of the following proposition can be found in MacLane [27] Section IV.4. As the proof makes use of adjunctions the reader may want to consult Section 3.3 of this thesis where a small introduction is given.

Proposition 2.6.13. *A functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories if and only if F is faithful, full and essentially surjective*

We have enough category theory to continue with the proof of Tannaka's theorem.

Lemma 2.6.14. *The restriction functor $\pi^* : Rep_f(G_\omega, \mathbb{C}) \to Rep_f(G, \mathbb{C})$ is an equivalence of categories.*

Proof. We define an extension functor and show that it defines an equivalence of categories. Define

$$E : Rep_f(G, \mathbb{C}) \to Rep_f(G_\omega, \mathbb{C})$$

as follows. For any representation V of G, the map $u \mapsto u_V$ defines a representation of G_ω on V. This defines $E(V, \pi_V) = (V, p_V)$. We have the following commutative triangle

$$\begin{array}{ccc} G & \xrightarrow{\pi_V} & GL(V) \\ {\scriptstyle \pi}\big\downarrow & \nearrow_{p_V} & \\ G_\omega & & \end{array}$$

From this diagram we also note that $\pi^*(p_V) = \pi_V$. Take any intertwiner $h : (V, \pi_V) \to (W, \pi_W)$ in $Rep_f(G, \mathbb{C})$ and let $e \in G_\omega$. By definition of the Tannaka group we know that $h \circ u_V = u_W \circ h$ and so $h : (V, \pi_V) \to (W, \pi_W)$ is an intertwining operator for the representations of G_ω. It is straightforward to prove that E preserves tensor products, direct sums and conjugation. In the terminology of the following chapter we will show that E is an $*$-preserving equivalence of tensor categories. By the previous proposition it satisfies to show that E is faithful, full and essentially surjective in order to prove that defines an equivalence. The extension functor preserves all the structure of $Rep_f(G, \mathbb{C})$ and it is clearly full and faithful. It is an equivalence of $Rep_f(G, \mathbb{C})$ with some full subcategory of $Rep_f(G_\omega, \mathbb{C})$. Note also that if V is irreducible, then $E(V)$ is also irreducible. We can conclude that the objects in the image of E form a closed collection $\mathcal{X}$ of representations of G_ω. Let $u \in G_\omega$ such that $u \neq id$. Then there is a representation V of G such that $u_V \neq id_V$. Consequently $p_V(u) \neq id_V$. This means that p_V separates u from e. The collection $\mathcal{X}$ separates points of the Tannaka group and by 2.6.9 the collection is equal to $Rep_f(G_\omega, \mathbb{C})$. Thus we have proven that E defines an equivalence of categories. By symmetry of the definition and the fact that $\pi^* \circ E = id$ the restriction functor π^* defines an equivalence of categories. We will encounter the constructions of this proof again in the next chapter. □

Lemma 2.6.15. *The restriction map* $\pi^* : \mathcal{T}(G_\omega, \mathbb{C}) \to \mathcal{T}(G, \mathbb{C})$ *is an isomorphism of algebras.*

Proof. We define an inverse $e : \mathcal{T}(G, \mathbb{C}) \to \mathcal{T}(G_\omega, \mathbb{C})$. Let $f \in \mathcal{T}(G, \mathbb{C})$ be a representative function. Then f can be written as

$$f(x) = \sum_{\lambda \in \hat{G}} Tr(g(\lambda)\pi_V(x))d_\lambda$$

where $g \in \sum_\lambda End(V_\lambda)$ is the Fourier transform of f, $g = \mathcal{F}f$. Define

$$e(f)(x) = \sum_{\lambda \in \hat{G}} Tr(g(\lambda)p_V(u))d_\lambda.$$

Lemma 2.6.14 proves that $\hat{G} = (\hat{G_\omega})$. This proves that e is a bijection. It is also clear that e preserves the algebra structure. The lemma follows from $\pi^* \circ e = id$ $\square$

Theorem 2.6.16. *(Tannaka's theorem) For any compact group* G*, the map* $\pi : G \to G_\omega$ *is an isomorphism of Lie groups*

Proof. Injectivity is a consequence of the Peter Weyl theorem. It suffices to show that the map is surjective. Suppose that the equality

$$\int_{G_\omega} f(u)du = \int_G f(\pi(x))dx$$

holds for all $f \in C^0(G_\omega, \mathbb{C})$. Suppose further that $Im(\pi) \neq G_\omega$. Let f be a positive function with support in the closed set that is the complement of $Im(\pi)$. Then we have

$$\int_{G_\omega} f(u)du > 0, \quad \int_G f(\pi(x))dx = 0.$$

This is a contradiction. It therefore suffices to prove the above equality. We will do this using Lemma 2.6.14 and the orthogonality relations. It suffices to check this for $f \in \mathcal{T}(G_\omega, \mathbb{C})$ as these lie dense in $C^0(G_\omega, \mathbb{C})$. Let f be such a representative function. Using Lemma 2.6.14 we can write it as

$$f(u) = \sum_{\lambda \in \hat{G}} Tr(g(\lambda)p_\lambda(u))d_\lambda,$$

so that

$$f(\pi(x)) = \sum_{\lambda \in \hat{G}} Tr(g(\lambda)\pi_\lambda(x))d_\lambda.$$

Applying the orthogonality relations gives the following relations that prove the desired equality.

$$\int_G Tr(g(\lambda)\pi_\lambda(x))dx = \begin{cases} g(I) & \text{if } \lambda = I \\ 0 & \text{if } \lambda \neq I \end{cases}$$

$$\int_{G_\omega} Tr(g(\lambda)p_\lambda(u))du = \begin{cases} g(I) & \text{if } \lambda = I \\ 0 & \text{if } \lambda \neq I \end{cases} \qquad \square$$

This proves Tannaka's theorem for the second time. The proof given in this section holds for any compact topological group, not just compact Lie groups. Nowhere in this section did we need results that were specific for Lie groups. The situation is different is Sections 2.3 and 2.4. In Section 2.3 the descending chain property was used and Section 2.4 used the theory of Lie groups at several points. In short, the approach presented in Sections 2.5 and 2.6 is more general as it extends Tanaka's theorem to all compact groups. The approach using Hopf algebras has the advantage that it could be used to reveal some of the algebraic structure of compact Lie groups.

Chapter 3

Tannaka-Krein Duality for Compact Groups

The starting point of the previous chapter was a compact group G. From this group we got the category $Rep_f(G, \mathbb{C})$ and the forgetful functor $\omega : Rep_f(G, \mathbb{C}) \to \mathcal{H}_f$. Here we used the category $\mathcal{H}_f$ instead of $Vect_{\mathbb{C}}$. The category $\mathcal{H}_f$ has finite dimensional Hilbert spaces over $\mathbb{C}$ as its objects and linear maps as arrows[1]. The main reason that we want Hilbert spaces rather than vector spaces is because we want to use the involution that is given by taking a Hermitian adjoint relative to the inner product. The only difference with $Vect_{\mathbb{C}}$ are the inner products. Recall that for every representations in $Rep_f(G, \mathbb{C})$ there is an invariant inner product, relative to which the representation is unitary. The functor ω maps a unitary represention $(\pi_V, V, \langle\cdot,\cdot\rangle)$ to the underlying Hilbert space $(V, \langle\cdot,\cdot\rangle)$. The content of Tannaka's theorem is that the group of unitary monoidal natural transformations of the forgetful functor to itself gives us our group G back up to an isomorphism. In this chapter we do something similair, but more abstract. Starting with a suitable abstract category $\mathcal{C}$, and a suitable functor $E : \mathcal{C} \to \mathcal{H}_f$, we prove that the set G_E of unitary monoidal natural transformations is a compact group. It is subsequently shown that $\mathcal{C}$ is equivalent as a tensor $*$-category to $Rep(G_E, \mathbb{C})$. We will concentrate on categories $\mathcal{C}$ of the kind STC*. This means that we look at tensor $*$-categories with finite dimensional hom-sets, biproducts, subobjects, conjugates, an irreducible tensor unit and a unitary symmetry. The functors under consideration are symmetric faithful functors of tensor $*$-categories, called $*$-preserving symmetric fiber functors. All terminology will be explained.

The first three sections give the background in category theory necessary to formulate and prove Tannaka Krein duality. Readers with experience in

[1]At first sight it seems more natural to take unitary maps or at least isometries as arrows instead of all linear maps, but this is not desirable. Thinking about applying the theory to quantum physics the use of only unitary maps is too restrictive.

category theory may want to skip these sections. In the fourth section Tannaka-Krein duality is stated and proven up to an (important) technical proposition. Sections 3.5 through 3.7 lead to a proof of this proposition. These sections also give insight in the diversity of $*$-preserving symmetric fiber functors that can occur for a given STC*. It turns out that any two such functors are isomorphic through a unitary monoidal natural isomorphism. Two different $*$-preserving symmetric fiber functors for the same STC* $\mathcal{C}$ automatically give the same group G_E up to an isomorphism. At the end of the chapter the reader can find some other incarnations of Tannaka-Krein duality. This very brief discussion is only meant to indicate the diversity of the subject and to direct the interested reader to relevant papers. The treatment of Tannaka duality as presented in Sections 3.1 through 3.7 is based on Müger [30].

3.1 Tensor Categories

We give a brief introduction in the language of monoidal categories. More material can be found in Mac Lane [27] or Deligne and Milne [10].

Definition 3.1.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. The product category $\mathcal{C} \times \mathcal{D}$ is the category that has as its objects $Obj(\mathcal{C} \times \mathcal{D}) = Obj(\mathcal{C}) \times Obj(\mathcal{D})$. The arrows are given by $Hom_{\mathcal{C}\times\mathcal{D}}(X \times Y, Z \times W) = Hom_{\mathcal{C}}(X, Z) \times Hom_{\mathcal{D}}(Y, W)$. For each object $X \times Y$ the identity arrow is given by $id_{X\times Y} = id_X \times id_Y$. Composition is defined as $(f \times g) \circ (h \times k) = (f \circ h) \times (g \circ k)$. A functor $F : \mathcal{B} \times \mathcal{C} \to \mathcal{D}$ from a product category is called a bifunctor.*

If one argument of a bifunctor is fixed, this will give an ordinary functor of the remaining argument. Suppose now we have a category $\mathcal{C}$ equipped with a bifunctor

$$\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}, \quad (X, Y) \mapsto X \otimes Y.$$

Suppose further that for any three objects X,Y and Z of $\mathcal{C}$ we have an isomorphism

$$\alpha_{X,Y,Z} : X \otimes (Y \otimes Z) \to (X \otimes Y) \otimes Z$$

such that the following diagram is commutative.

$$\begin{array}{ccccc}
X \otimes (Y \otimes (Z \otimes T)) & \xrightarrow{\alpha} & (X \otimes Y) \otimes (Z \otimes T) & \xrightarrow{\alpha} & ((X \otimes Y) \otimes Z) \otimes T \\
\big\downarrow{\scriptstyle id\otimes\alpha} & & & & \big\uparrow{\scriptstyle \alpha\otimes id} \\
X \otimes ((Y \otimes Z) \otimes T) & & \xrightarrow{\qquad\qquad\alpha\qquad\qquad} & & (X \otimes (Y \otimes Z)) \otimes T
\end{array}$$

We demand that the map α is a natural transformation and consequently a natural isomorphism. The above diagram is called the pentagon axiom and α is called an associativity constraint. In the diagram the subscripts for the maps were left out. We will continue to do this whenever this makes the diagrams more transparant. Let, for any two objects X and Y of $\mathcal{C}$ there be an isomorphism

$$c_{X,Y} : X \otimes Y \to Y \otimes X$$

coming from a natural transformation c. This natural map is called the commutativity constraint. The associativity constraint α and the commutativity constraint c are called compatible is for any three objects X, Y and Z of $\mathcal{C}$ the diagrams

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{\alpha} & (X \otimes Y) \otimes Z & \xrightarrow{c} & Z \otimes (X \otimes Y) \\ \downarrow{\scriptstyle id\otimes c} & & & & \downarrow{\scriptstyle \alpha} \\ X \otimes (Z \otimes Y) & \xrightarrow{\alpha} & (X \otimes Z) \otimes Y & \xrightarrow{c\otimes id} & (Z \otimes X) \otimes Y \end{array}$$

$$\begin{array}{ccccc} (X \otimes Y) \otimes Z & \xrightarrow{\alpha} & X \otimes (Y \otimes Z) & \xrightarrow{c} & (Y \otimes Z) \otimes X \\ \downarrow{\scriptstyle c\otimes id} & & & & \downarrow{\scriptstyle \alpha} \\ (Y \otimes X) \otimes Z & \xrightarrow{\alpha} & Y \otimes (X \otimes Z) & \xrightarrow{id\otimes c} & Y \otimes (Z \otimes X) \end{array}$$

are commutative. Together these diagrams are called the hexagon axiom. We will call the commutativity constraint a braiding if it is compatible with the associativity constraint. Finally consider the triple $(\mathbb{1}, \lambda, \rho)$ where $\mathbb{1}$ is an object of $\mathcal{C}$ called the identity object or tensor unit. The maps λ and ρ are natural isomorphisms $\lambda_X : \mathbb{1} \otimes X \to X$ and $\rho_X : X \otimes \mathbb{1} \to X$ such that following diagram is commutative for all objects X and Y in $\mathcal{C}$.

$$\begin{array}{ccc} X \otimes (\mathbb{1} \otimes Y) & \xrightarrow{\qquad\alpha\qquad} & (X \otimes \mathbb{1}) \otimes Y \\ & {\scriptstyle id\otimes\lambda}\searrow \quad \swarrow{\scriptstyle \rho\otimes id} & \\ & X \otimes Y & \end{array}$$

The natural isomorphisms λ and ρ must be compatible with the braiding in the sense that the following diagram is commutative.

$$\begin{array}{ccc} X \otimes \mathbb{1} & \xrightarrow{\quad c \quad} & \mathbb{1} \otimes X \\ & {\scriptstyle \rho}\searrow \quad \swarrow{\scriptstyle \lambda} & \\ & X & \end{array}$$

We demand that $\lambda_{\mathbb{1}} = \rho_{\mathbb{1}} : \mathbb{1} \otimes \mathbb{1} \to \mathbb{1}$. It can be shown that these properties imply that the following diagrams are commutative

$$\begin{array}{ccc} \mathbb{1} \otimes (X \otimes Y) & \xrightarrow{\alpha} & (\mathbb{1} \otimes X) \otimes Y \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle \lambda\otimes id} \\ & & X \otimes Y \end{array} \qquad \begin{array}{ccc} X \otimes (Y \otimes \mathbb{1}) & \xrightarrow{\alpha} & (X \otimes Y) \otimes \mathbb{1} \\ & {\scriptstyle id\otimes\rho}\searrow & \downarrow{\scriptstyle \rho} \\ & & X \otimes Y \end{array}$$

Definition 3.1.2. *A tensor category, or monoidal category, is a sixtuple* $(\mathcal{C}, \otimes, \mathbb{1}, \lambda, \rho, \alpha)$ *consisting of a category* $\mathcal{C}$*, a bifunctor* $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$*, an identity element* $\mathbb{1}$ *with natural maps* λ *and* ρ *as above and an associativity constraint* α*. A tensor category is usually denoted by the underlying category* $\mathcal{C}$*. A braided tensor category is a tensor category with a commutativity constraint* c *that is compatible with the associativity constraint. A braided tensor category such that* $c_{Y,X} \circ c_{X,Y} = id_{X \otimes Y}$ *holds for all objects* X *and* Y *of* $\mathcal{C}$*, is called a symmetric tensor category. If all the arrows in a tensor category that come from* α*,* λ *and* ρ *are the identity arrow, the category is called a strict tensor category.*

Using the familiar tensor products of vector spaces and representations, the categories $Vect_{\mathbb{C}}$ and $Rep_f(G, \mathbb{C})$ are tensor categories. The flip symmetry

$$\Sigma_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto w \otimes v$$

makes these categories into symmetric tensor categories. Note that this map is an intertwining operator for every pair X and Y. Also note that these categories are not strict tensor categories.

Definition 3.1.3. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *denote tensor categories. A tensor functor, or monoidal functor, is a triple* (F, d, e) *where* $F : \mathcal{C} \to \mathcal{D}$ *is a functor of categories. The map* d *is a natural isomorphism* $d_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$*. If* $\mathbb{1}$ *and* $\mathbb{1}'$ *are the tensor units of* $\mathcal{C}$ *and* $\mathcal{D}$*, respectively, then* e *is an arrow in* $\mathcal{D}$*,* $e : \mathbb{1}' \to F(\mathbb{1})$*. The following three diagrams must be commutative.*

$$\begin{array}{ccccc}
FX \otimes (FY \otimes FZ) & \xrightarrow{id \otimes d} & FX \otimes F(Y \otimes Z) & \xrightarrow{d} & F(X \otimes (Y \otimes Z)) \\
\downarrow{\scriptstyle \alpha'} & & & & \downarrow{\scriptstyle F\alpha} \\
(FX \otimes FY) \otimes FZ & \xrightarrow{d \otimes id} & F(X \otimes Y) \otimes FZ & \xrightarrow{d} & F((X \otimes Y) \otimes Z)
\end{array}$$

$$\begin{array}{ccc}
FX \otimes \mathbb{1}' & \xrightarrow{\rho'} & FX \\
{\scriptstyle id \otimes e}\downarrow & & \uparrow{\scriptstyle F\rho} \\
FX \otimes F\mathbb{1} & \xrightarrow[d]{} & F(X \otimes \mathbb{1})
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{1}' \otimes FX & \xrightarrow{\lambda'} & FX \\
{\scriptstyle e \otimes id}\downarrow & & \uparrow{\scriptstyle F\lambda} \\
F\mathbb{1} \otimes FX & \xrightarrow[d]{} & F(\mathbb{1} \otimes X)
\end{array}$$

In Mac Lane [27] functors of this kind are called strong tensor functors. In the definition of a tensor functor in Mac Lane it is not required that the natural transformation d is a natural isomorphism. It is easy to check that the composition of two tensor functors is again a tensor functor. A tensor functor is called strict if the arrows $d_{X,Y}$ and e are identities.

Definition 3.1.4. *Let* $\mathcal{C}$ *and* $\mathcal{D}$ *be braided (symmetric) tensor categories. A tensor functor* $F : \mathcal{C} \to \mathcal{D}$ *is braided (symmetric) if the following diagram* *is commutative for all objects X and Y of $\mathcal{C}$.*

$$\begin{array}{ccc} F(X\otimes Y) & \xleftarrow{d} & F(X)\otimes F(Y) \\ {\scriptstyle F(c)}\downarrow & & \downarrow{\scriptstyle c} \\ F(Y\otimes X) & \xleftarrow{d} & F(Y)\otimes F(X) \end{array}$$

Definition 3.1.5. *Let F and G be tensor functors $\mathcal{C} \to \mathcal{D}$. A natural transformation of tensor functors, or monoidal natural transformation, $\theta : (F, d^F, e^F) \to (G, d^G, e^G)$ is a natural transformation between the underlying functors $\theta : F \to G$ such that the following two diagrams are commutative for all objects X and Y in $\mathcal{C}$.*

$$\begin{array}{ccc} FX\otimes FY & \xrightarrow{d^F} & F(X\otimes Y) \\ {\scriptstyle \theta_X\otimes\theta_Y}\downarrow & & \downarrow{\scriptstyle \theta_{X\otimes Y}} \\ GX\otimes GY & \xrightarrow{d^G} & G(X\otimes Y) \end{array} \qquad \begin{array}{ccc} \mathbb{1}' & \xrightarrow{e^F} & F\mathbb{1} \\ & {\scriptstyle e^G}\searrow & \downarrow{\scriptstyle \theta_{\mathbb{1}}} \\ & & G\mathbb{1} \end{array}$$

Definition 3.1.6. *A tensor functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of tensor categories if there exists a tensor functor $G : \mathcal{D} \to \mathcal{C}$ and monoidal natural isomorphisms $GF \to id_{\mathcal{C}}$ and $FG \to id_{\mathcal{D}}$.*

Similar to Proposition 2.6.13 we have that a tensor functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of tensor categories if and only if it is faithful, full and essentially surjective [27]. From this point onwards we will only consider strict tensor categories. The diagrams and equations become more transparant without writing out the arrows coming from α, λ and ρ. Even though the categories of interest, like $Rep_f(G, \mathbb{C})$, are not strict we have no loss of generality. This is a consequence of the coherence theorem. A discussion of the coherence theorem as well as a proof of the following proposition which is a consequence thereof, can be found in Mac Lane [27].

Proposition 3.1.7. *Any monoidal category $\mathcal{C}$ is equivalent as a tensor category, via a (strong) monoidal functor $F : \mathcal{C} \to \mathcal{D}$ and a (strong) monoidal functor $F : \mathcal{D} \to \mathcal{C}$, to a strict monoidal category $\mathcal{D}$.*

3.2 Categories of type TC*

Definition 3.2.1. *A category $\mathcal{C}$ is called pre-additive or an Ab-category if the following two conditions hold. The set $Hom_{\mathcal{C}}(X, Y)$ is an abelian group for all objects X and Y of $\mathcal{C}$ and the composition $\circ$ is bi-additive with respect to this group operation.*

Definition 3.2.2. *Let $\mathcal{C}$ be an Ab-category and X,Y and Z be objects of $\mathcal{C}$. A biproduct diagram for the objects X and Y is a diagram*

$$X \underset{p_1}{\overset{i_1}{\rightleftarrows}} Z \underset{p_2}{\overset{i_2}{\leftrightarrows}} Y$$

with arrows p_1, p_2, i_1 and i_2, that satisfy the identities

$$p_1 i_1 = id_X, \quad p_2 i_2 = id_Y, \quad i_1 p_1 + i_2 p_2 = id_Z.$$

An Ab-category is said to have biproducts if it has biproducts for every pair of objects X and Y. We will often call biproducts direct sums. This name is motivated by the next example. If the Ab-category is a $*$-category (to be defined later in this section) we further require that $p_1 = i_1^*$ and $p_2 = i_2^*$. This amounts to demanding that i_1 and i_2 are isometries.

Later we will repeatedly use the following identities. Suppose that $Z \cong X \oplus Y$ with maps i_j and p_j. Then

$$p_1 i_2 = p_1(i_1 p_1 + i_2 p_2) i_2 = id_X \circ p_1 i_2 + p_1 i_2 \circ id_Y = p_1 i_2 + p_1 i_2$$

implying $p_1 i_2 = 0$. On similar grounds $p_2 i_1 = 0$. The argument can be extended to any finite direct sum by using induction.

A biproduct is best defined as a product that is also a coproduct. As we aim to develop no more category theory than we need for Tannaka-Krein duality, we will not go into this. Let $Vect_{\mathbb{C}}$ again be the category of vector spaces over $\mathbb{C}$. For vector spaces X and Y the set $Hom(X, Y)$ has the structure of a $\mathbb{C}$-vector space, so in particular the Hom sets are abelian groups with respect to addition. Composition of linear maps clearly respects addition so we have an Ab-category. It is easy to check that the direct sum $Z = X \oplus Y$ provides a biproduct for X and Y if we take the obvious maps for p and i. The category $Rep_f(G, \mathbb{C})$ is an Ab-category with direct sums on similar grounds. Note that the maps i and p are intertwining operators.

Definition 3.2.3. *Let $\mathcal{C}$ be a category. An object $\mathbf{0}$ is called a zero object if for every object X of $\mathcal{C}$ the sets $Hom(X, \mathbf{0})$ and $Hom(\mathbf{0}, X)$ are singletons.*

Note that two zero objects are automatically isomorphic. In the example of $Vect_{\mathbb{C}}$, the zero dimensional vector space $\{0\}$ acts as a zero object.

Definition 3.2.4. *A category $\mathcal{C}$ is called an additive category if it is an Ab-category that has biproducts (direct sums) and a zero object.*

Definition 3.2.5. *Let $\mathcal{C}$ be a category and k a field. Then $\mathcal{C}$ is called a k-linear category if $Hom_{\mathcal{C}}(X, Y)$ is a k-linear vector space for all objects X and Y and the composition map $\circ : (f, g) \mapsto f \circ g$ is bilinear. If $\mathcal{C}$ is a tensor category we also require that $\otimes : (f, g) \mapsto f \otimes g$ is bilinear.*

Thusfar we defined pre-additive, additive and k-linear categories. Next we define another kind of category by adding some more structure to a $\mathbb{C}$-linear category.

Definition 3.2.6. *A positive $*$-operation on a $\mathbb{C}$-linear category $\mathcal{C}$ is a family of maps that associates to every arrow $s \in Hom_{\mathcal{C}}(X,Y)$ an arrow $s^* \in Hom_{\mathcal{C}}(Y,X)$ such that the following properties hold. The map is involutive in the sense that $s^{**} = (s^*)^* = s$ and contravariant in the sense that $(s \circ t)^* = t^* \circ s^*$ holds whenever it is defined. The map is antilinear and is positive in the sense that $s^* \circ s = 0$ implies that $s = 0$. If $\mathcal{C}$ is a tensor category, we also demand that $(s \otimes t)^* = s^* \otimes t^*$. A $\mathbb{C}$-linear (tensor) category equipped with a positive $*$-operation is called a (tensor) $*$-category.*

Let $\mathcal{H}_f$ denote the category of finite dimensional Hilbert spaces over $\mathbb{C}$ with linear maps as arrows. Hermitian conjugation defines a positive $*$-operation on $\mathcal{H}_f$. Hermitian conjugation also works for $Rep_f(G,\mathbb{C})$. Take representations (X,π_X), (Y,π_Y) and an intertwining operator $s : X \to Y$. Use invariant inner products to make both representations unitary. The equality $\pi_Y(x)s = s\pi_X(x)$ implies $\bar{s}^t\pi_Y(x^{-1}) = \pi_X(x^{-1})\bar{s}^t$ proving that the linear map $\bar{s}^t = s^* : Y \to X$ is an intertwining operator.

Definition 3.2.7. *Let $\mathcal{C}$ be a $*$-category. An arrow $v : X \to Y$ in $\mathcal{C}$ is called an isometry if $v^* \circ v = id_X$. The arrow is called unitary if it satisfies $v^* \circ v = id_X$ and $v \circ v^* = id_Y$. A morphism $p \in End(X)$ is called a projector if $p = p \circ p = p^*$. We say that the category $\mathcal{C}$ has subobjects if for every projector $p \in End(X)$ there exists an isometry $v : X \to Y$ such that $v \circ v^* = p$.*

Definition 3.2.8. *A functor F between $*$-categories is called $*$-preserving if $F(s^*) = F(s)^*$ for every arrow s. The isomorphisms $d_{X,Y}$ and e coming with a functor between tensor $*$-categories are required to be unitary arrows.*

The category $Vect_{\mathbb{C}}$ has subobjects. In this category the notions of isometries, unitarity maps and projectors reduce to the familiar ones. Subobjects are just subspaces. For $Rep_f(G,\mathbb{C})$ the concepts also take on a familiar form; subobjects are G-invariant subspaces and projections are the usual projections on G-invariant subspaces. Next we define a conjugation on tensor $*$-categories. In the next section we will further explore this conjugation.

Definition 3.2.9. *(provisional definition) Let $\mathcal{C}$ be a tensor $*$-category and X an object of $\mathcal{C}$. An object $\overline{X}$ of $\mathcal{C}$ is called a conjugate object of X if there exist non-zero morphisms $r : \mathbf{1} \to \overline{X} \otimes X$ and $\overline{r} : \mathbf{1} \to X \otimes \overline{X}$ that satisfy the conjugate equations*

$$id_X \otimes r^* \circ \overline{r} \otimes id_X = id_X$$

$$id_{\overline{X}} \otimes \overline{r}^* \circ r \otimes id_{\overline{X}} = id_{\overline{X}}.$$

We say that $(\overline{X}, r, \overline{r})$ *is a conjugate of* X*. If every non-zero object of* $\mathcal{C}$ *has a conjugate we say that* $\mathcal{C}$ *has conjugates.*

The category $\mathcal{H}_f$ has conjugates. Take any vector space V not equal to $\{0\}$. Take for $\overline{X}$ the vector space dual to X. Pick a basis $e_1, e_2, ...$ of V and let $e_i^\vee$ denote the basis elements dual to that basis. Define the linear maps $r : \mathbb{C} \to X^\vee \otimes X$ and $\overline{r} : \mathbb{C} \to X \otimes X^\vee$ by $r(1) = \sum_i e_i^\vee \otimes e_i$ and $\overline{r}(1) = \sum_i e_i \otimes e_i^\vee$. It is straightforward to check that the conjugation equations are satisfied. These maps also work for $Rep_f(G, \mathbb{C})$. In order to prove that the trace map r is an intertwining operator, recall that $\pi_{V^\vee}(x) = \pi_V(x^{-1})^t$.

Definition 3.2.10. *An object in a* $\mathbb{C}$*-linear category is called irreducible if* $End(X) = \mathbb{C}id_X$*.*

In Chapter 2 irreducibility was defined for $Rep_f(G, \mathbb{C})$ as the property that a representation V has no proper subrepresentations, or equivalently the impossibility of writing V as a nontrivial direct sum $V_1 \oplus V_2$ of two subsepresentations. The above definition of irreducibility of Z in $\mathcal{C}$ coincides with the definition that Z is not isomorphic to a proper direct sum. By proper we mean that we exclude direct sums like $Z \oplus \mathbf{0}$. First suppose that $End(Z) = \mathbb{C}id_Z$ and $Z \cong X \oplus Y$. By irreducibility $i_1 \circ p_1 = \lambda_1 id_Z$ and $i_2 \circ p_2 = \lambda_2 id_Z$ for some constants $\lambda_1, \lambda_2 \in \mathbb{C}$. If we denote the zero element of $Hom_{\mathcal{C}}(X, Y)$ by $0_{X,Y}$ then we find

$$\lambda_1 p_2 = p_2 \circ \lambda_1 id_Z = p_2 i_1 p_1 = 0_{X,Y} \circ p_1 = 0_{Z,Y},$$

Implying that $\lambda_1 = 0$. Similarly we find $\lambda_2 = 0$, which contradicts $i_1 p_1 + i_2 p_2 = id_Z$. We can conclude that if $End(Z) = \mathbb{C}id_Z$, then Z is not isomorphic to a nontrivial biproduct. In the above calculation we used the identity $0_{X,Y} \circ p_1 = 0_{Z,Y}$. This follows from

$$\begin{aligned} 0_{Z,Y} &= 0_{Z,Y} \circ (i_1 p_1 + i_2 p_2) = 0_{Z,Y} i_1 p_1 + 0_{Z,Y} i_2 p_2 \\ &= (0_{Z,Y} i_1 + 0_{X,Y}) p_1 + 0_{Z,Y} i_2 p_2 = 0_{Z,Y}(i_1 p_1 + i_2 p_2) + 0_{X,Y} p_1 \\ &= 0_{Z,Y} + 0_{X,Y} \circ p_1 = 0_{X,Y} \circ p_1. \end{aligned}$$

From now on we will not fuss about the zero morphisms and denote each zero arrow by 0. Next assume that Z is not isomorphic to a nontrivial biproduct. Then the only projection in $End(X)$ is the identity arrow. If p_1 is another projection, then so is $id_Z - p_1$. By definition $\mathcal{C}$ has subobjects, providing isometries $v_1 : X \to Z$ and $v_2 : Y \to Z$ such that $Z \cong X \oplus Y$. Using the observation that the identity is the only projection, the proof of Proposition 3.2.14 shows that $End(Z) \cong \mathbb{C}$. This completes the proof that the definitions of irreducibility are equivalent.

Using terminology from this section we can define the type of categories that we will investigate in the following sections.

Definition 3.2.11. *A TC* category is a tensor ∗-category with finite dimensional hom-sets, conjugates, direct sums, subobjects and an irreducible tensor unit* **1**. *A BTC* category is a TC* category with a unitary braiding. An STC* category is a TC* category with a unitary symmetry.*

Thusfar we have seen that both $Rep_f(G, \mathbb{C})$ and $\mathcal{H}_f$ are examples of TC* categories. Both categories are also equipped with a natural symmetry, the flip symmetry $\Sigma_{X,Y} : X \otimes Y \to Y \otimes X$, given by $\Sigma_{X,Y}(x \otimes y) = y \otimes x$. As this map is unitary (in both categories) both categories are STC* categories. In the next section we take a closer look at the conjugates as defined in Definition 3.2.9. We close this section with the proof that TC* categories are semisimple. In order to do so we need the fact that for an object X in a TC* category $\mathcal{C}$ the finite dimensional $\mathbb{C}$-algebra $End(X)$ is semisimple. The following lemma helps to prove this. A proof of this lemma can be found in Lang [25].

Lemma 3.2.12. *(Nakayama's Lemma) Let R be a ring and M be a finitely generated left R-module. Let J be the radical of R, defined as the intersection of all maximal left ideals of R. If $JM = M$, then $M = \{0\}$.*

Lemma 3.2.13. *Let R be a finite dimensional algebra over $\mathbb{C}$. Suppose that the (unique) maximal nilpotent left-ideal of R is equal to $\{0\}$. Then R is semisimple in the sense that it is the direct sum of simple algebras. A simple algebra is one that has no non-trivial ideals.*

Proof. Take the radical J of R. This provides a decreasing sequence $J, J^2, ...$ of left-ideals of R. In particular it is a decreasing sequence of finite dimensional $\mathbb{C}$-vector spaces, hence it stabilizes. There is a $n \in \mathbb{N}$ such that $J^n = J^{n+1} = J \cdot J^n$. If we apply Nakayama's Lemma with $M = J^n$, then $J^n = \{0\}$. As J is nilpotent, by assumption we have $J = \{0\}$. As R is finitely generated, there can only be a finite number of maximal ideals. Suppose that $J = \cap_{i=1}^m M_i$, where the M_i are maximal ideals. Next consider the natural map

$$\gamma : R \to R/M_1 \times ... \times R/M_m.$$

The kernel of this ring homomorphism is equal to $J = \{0\}$. Thus we have obtained an embedding of R into a ring that is clearly semisimple. Consequently R is semisimple. □

Proposition 3.2.14. *A TC* category is semisimple in the sense that every object is the finite direct sum of irreducible objects.*

Proof. Let X be an object of $\mathcal{C}$. The space $A = End(X)$ is a finite dimensional $\mathbb{C}$-algebra with a positive involution given by $*$. By the previous lemma we know that A is semisimple if it has no non-zero nilpotent ideals. Suppose J is a nilpotent left ideal of A. Let $s \in J$, then $s^* \circ s \in J$. This means that there is a natural number k such that $(s^* \circ s)^k = 0$. Using positivity of the $*$-operation and the fact that $(s^* \circ s)^* = s^* \circ s$ this implies that $s^* \circ s = 0$. Invoking positivity once more gives $s = 0$. The radical of $End(X)$ is trivial, hence it is semisimple. The map $id_X \in End(X)$ is therefore the sum of projections p_i such that $p_i \circ End(X) \circ p_i \cong \mathbb{C}$. By assumption the category $\mathcal{C}$ has subobjects, therefore to each projector p_i there is an associated object X_i of $\mathcal{C}$. It is straightforward to check that $X \cong \oplus_i X_i$. □

3.3 Adjunctions and Conjugates

The mathematical concept underlying conjugates is that of an adjoint functor. One goal of this section gives an introduction into adjoints and to show how the conjugates as in Definition 3.2.9 can be viewed as such. For the remaining sections we will barely use adjoint functors explicitly. Even so, it is frequently the case that adjoint functors are hidden just below the surface of the material presented here. We start with some examples. Let Set denote the category of all (small) sets, where the arrows are functions. Take $Vct_{\mathbb{C}}$ to be the category of all (small) vector spaces (not necessarily finite dimensional) over $\mathbb{C}$, where the arrows are the linear maps. Consider the following two functors

$$Set \underset{U}{\overset{V}{\rightleftarrows}} Vct_{\mathbb{C}}.$$

The functor U sends each vector space to the underlying set and each linear map to the underlying function. The functor U is just the forgetful functor. A set X is mapped to the vector space $V(X)$ that has X as a basis. The vectors of $V(X)$ are formal linear combinations of $\sum_i c_i x_i$ with $c_i \in \mathbb{C}$ and $x_i \in X$. Take any function $g : X \to U(W)$. This function extends to a linear map $f : V(X) \to W$ given by $f(\sum c_i x_i) = \sum c_i g(x_i)$. The resulting correspondence $\psi : Hom_{Set}(X, U(W)) \to Hom_{Vct_{\mathbb{C}}}(V(X), W)$ has an inverse $\phi(f) = f|_X$. For each set X and vector space we get a bijection

$$\phi_{X,W} : Hom_{Vct_{\mathbb{C}}}(V(X), W) \cong Hom_{Set}(X, U(W)).$$

These maps are components of a natural transformation ϕ that is natural in both arguments. In order to see this we need some more terminology.

Definition 3.3.1. *Let $\mathcal{C}$ be any category. Define the opposite category $\mathcal{C}^0$ of $\mathcal{C}$ as follows. The objects of $\mathcal{C}^0$ are the same as for $\mathcal{C}$. For each arrow $f : X \to Y$ of $\mathcal{C}$ there is an arrow $f^0 : Y \to X$ of $\mathcal{C}^0$. Composition of arrows is given by $f^0 \circ g^0 = (g \circ f)^0$ whenever it is defined. A contravariant functor $F : \mathcal{C} \to \mathcal{D}$ is defined as a (covariant) functor $F : \mathcal{C}^0 \to \mathcal{D}$.*

Definition 3.3.2. *Suppose that $\mathcal{C}$ has small hom-sets and let X be any object of $\mathcal{C}$. We define the covariant hom-functor by*

$$k_{\mathcal{C}}(X) : \mathcal{C} \to Set, \quad Y \mapsto Hom_{\mathcal{C}}(X, Y), \quad f \mapsto f_*$$

where for $f : Y' \to Y$ and $h : X \to Y'$ we define $f_* h = f \circ h$. It is easy to see that this defines a functor. Likewise we define the contravariant hom-functor by

$$h_{\mathcal{C}}(X) : \mathcal{C}^0 \to Set, \quad Y \mapsto Hom_{\mathcal{C}}(Y, X), \quad f \mapsto f^*$$

where for $f : Y \to Y'$ and $h : Y' \to X$ we define $f^* h = h \circ f : Y \to X$.

With the hom-functors we recongnize the sets $Hom_{Vct_{\mathbb{C}}}(V(X), W)$ and $Hom_{Set}(X, U(W))$ as image objects of functors. Checking that $\phi_{X,W}$ is a natural transformation in both arguments amounts to showing that the following two diagrams are commutative for all vector spaces W and W', sets X and X', and linear maps $f : W \to W'$ and functions $h : X \to X'$.

$$\begin{array}{ccc} Vct_{\mathbb{C}}(VX, W) & \xrightarrow{\phi_{X,W}} & Set(X, UW) \\ \downarrow{\scriptstyle (Vh)^*} & & \downarrow{\scriptstyle h^*} \\ Vct_{\mathbb{C}}(VX', W) & \xrightarrow[\phi_{X',W}]{} & Set(X', UW) \end{array} \qquad \begin{array}{ccc} Vect_{\mathbb{C}}(VX, W) & \xrightarrow{\phi_{X,W}} & Set(X, UW) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle (Uf)^*} \\ Vct_{\mathbb{C}}(VX, W') & \xrightarrow[\phi_{X,W'}]{} & Set(X, UW') \end{array}$$

In the diagrams the shorthand notation $\mathcal{C}(X, Y) = Hom_{\mathcal{C}}(X, Y)$ was used. We will use this notation whenever it simplifies the diagrams or equations. In the next example we see something similair. Consider the following two functors

$$F^W : Rep(G, \mathbb{C}) \to Rep(G, \mathbb{C}), \quad V \mapsto V \otimes W, \quad f \mapsto f \otimes id_W$$

$$G^W : Rep(G, \mathbb{C}) \to Rep(G, \mathbb{C}), \quad V \mapsto Hom(W, V), \quad f \mapsto f_*$$

where $Hom(W, V)$ denotes a representation as in Example 2.1.9, not just a set. For each triple of representations U, V and W define map

$$\phi^W_{U,V} : Rep(G, \mathbb{C})(U \otimes W, V) \to Rep(G, \mathbb{C})(U, Hom(W, V)).$$

This map sends $f : U \otimes W \to V$ to $g = \phi f : U \to Hom(W, V)$ such that $g(u)$ maps w to $f(w \otimes u)$. In short $g(u, w) = f(w \otimes u)$. We need to check that intertwining operators are mapped to intertwining operators.

$$\begin{aligned} \pi_{Hom(W,V)}(x) g(u, w) &= \pi_V(x) g(u, \pi_W(x)^{-1} w) \\ &= \pi_V(x) f(\pi_W(x)^{-1} w \otimes \pi_U(x)^{-1} \pi_U(x) u) \\ &= \pi_V(x) \pi_V(x)^{-1} f(w \otimes \pi_U(x) u) = g(\pi_U(x) u, w) \end{aligned}$$

where in the third equality we used that f is an intertwining operator. This shows that the map is well-defined. Showing that it is a bijection is easy to check. Like the previous example, the maps $\phi^W_{U,V}$ form the components of a natural transformation. Checking that ϕ is natural in each of the three components is left to the reader. The examples motivate the following definition.

Definition 3.3.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories. An adjunction from $\mathcal{C}$ to $\mathcal{D}$ is a triple $(F, G, \phi) : \mathcal{C} \to \mathcal{D}$ where F and G are functors*

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}.$$

and ϕ is a function that assigns objects C in $\mathcal{C}$ and D in $\mathcal{D}$ a bijection $\phi_{C,D} : \mathcal{D}(FC, D) \cong \mathcal{C}(C, GD)$ which is natural in both C and D. F is called the left-adjoint of G and G the right-adjoint of F.

There is also an algebraic version of Tannaka-Krein duality, the Tannaka duality for affine group schemes. In this algebraic version, which shows strong similarities with Tannaka-Krein duality as it is discussed in this thesis, rigid monoidal categories play an important role. The existence of conjugates for TC* type categories is related to the demand of rigidity on categories in the algebraic case. The main motivation for showing that conjugates come from adjunctions is to make it easier to see a connection because rigidity is formulated using adjunctions. We will not discuss Tannaka duality for affine group schemes in this thesis. The interested reader can find further information in [10], [36]. As we get back to the task at hand, suppose that $\mathcal{C}$ is a TC*. For any nonzero object X there is a triple $(\overline{X}, r, \overline{r})$ such that these satisfy the conjugate equations given in Definition 3.2.9.

Lemma 3.3.4. *Take X, $\overline{X}$, r and $\overline{r}$ as above. If Y and Z are objects of $\mathcal{C}$ then the following maps are inverses of each other.*

$$s \mapsto id_{\overline{X}} \otimes s \circ r \otimes id_Y, \quad \mathcal{C}(X \otimes Y, Z) \to \mathcal{C}(Y, \overline{X} \otimes Z)$$

$$t \mapsto \overline{r}^* \otimes id_Z \circ id_X \otimes t, \quad \mathcal{C}(Y, \overline{X} \otimes Z) \to \mathcal{C}(X \otimes Y, Z).$$

Proof.

$$id_{\overline{X}} \otimes \overline{r}^* \otimes id_Z \circ id_{\overline{X} \otimes X} \otimes t \circ r \otimes id_Y = id_{\overline{X}} \otimes \overline{r}^* \otimes id_Z \circ r \otimes id_{\overline{X} \otimes Z} \circ t = t.$$

In the last step the conjugate equations were used. The other half of the proof can be given in the same way. □

Similarly there is a bijection $\mathcal{C}(Y \otimes X, Z) \to \mathcal{C}(Y, Z \otimes \overline{X})$ given by $t \mapsto t \otimes id_{\overline{X}} \circ id_Y \otimes \overline{r}$ with inverse $s \mapsto id_Z \otimes r^* \circ s \otimes id_X$. The lemma implies in particular that every arrow $r_Y : \mathbb{1} \to Y \otimes X$ is of the form $r_Y = t \otimes id_X \circ r$ for a unique arrow $t : \overline{X} \to Y$. The map r_Y defines a conjugate if and only if t is invertible. This can be seen by defining $\overline{r}_Y = id_X \otimes t^{*-1} \circ \overline{r}$ and checking that the conjugate equations are satisfied. The same reasoning holds for arrows $r_Z : \mathbb{1} \to \overline{X} \otimes Z$. Any such arrow can be written as $r_Z : id_{\overline{X}} \otimes s \circ r$ for a unique arrow $s : X \to Z$ and defines a conjugate for Z if and only if s is invertible.

The lemma provides bijections $\phi^X_{Y,Z} : \mathcal{C}(X \otimes Y, Z) \to \mathcal{C}(Y, \overline{X} \otimes Z)$ for any triple of objects. The map ϕ^X is natural in both arguments Y and Z. Proving this amounts to checking that the following two diagrams are commutative for every arrow $f : Y \to Y'$ and $g : Z \to Z'$.

$$\begin{array}{ccc} \mathcal{C}(X \otimes Y, Z) & \xrightarrow{\phi^X_{Y,Z}} & \mathcal{C}(Y, \overline{X} \otimes Z) \\ {\scriptstyle (id_X \otimes f)^*}\downarrow & & \downarrow{\scriptstyle f^*} \\ \mathcal{C}(X \otimes Y', Z) & \xrightarrow[\phi^X_{Y',Z}]{} & \mathcal{C}(Y', \overline{X} \otimes Z) \end{array} \qquad \begin{array}{ccc} \mathcal{C}(X \otimes Y, Z) & \xrightarrow{\phi^X_{Y,Z}} & \mathcal{C}(Y, \overline{X} \otimes Z) \\ {\scriptstyle g_*}\downarrow & & \downarrow{\scriptstyle (id_{\overline{X}} \otimes g)_*} \\ \mathcal{C}(X \otimes Y, Z') & \xrightarrow[\phi^X_{Y,Z'}]{} & \mathcal{C}(Y, \overline{X} \otimes Z') \end{array}$$

This proves that the functor $\langle X \otimes - \rangle : \mathcal{C} \to \mathcal{C}$ is a right adjoint for the functor $\langle \overline{X} \otimes - \rangle : \mathcal{C} \to \mathcal{C}$.

There is a different yet equivalent way to look at adjunctions. Suppose that (F, G, ϕ) is an adjunction from $\mathcal{C}$ to $\mathcal{D}$. From this data we can construct natural transformations $\eta : id_{\mathcal{C}} \to GF$ and $\epsilon : FG \to id_{\mathcal{D}}$ such that the composites

$$G \overset{\eta G}{\to} GFG \overset{G\epsilon}{\to} G, \quad F \overset{F\eta}{\to} FGF \overset{\epsilon F}{\to} F,$$

are the idenities. The transformations η and ϵ are called the unit and counit respectively. It is proven in Section IV.1 of MacLane [27] that the functors F and G together with such η and ϵ completely determine an adjunction. The bijection ϕ can thus be found from this data. Let us turn to the construction of η and ϵ from (F, G, ϕ). We have a bijection

$$\phi_{Y,Z} : \mathcal{D}(FY, Z) \to \mathcal{D}(Y, GZ).$$

If we pick $Z = FY$ then η_Y is the image of id_{FY} under $\phi_{Y,FY}$. In short $\eta_Y : Y \to GFY$ is given by $\eta_Y = \phi_{Y,FY}(id_{FY})$. If we set $Y = GZ$ then we take ϵ_Z as the image of id_{GZ} under $\phi^{-1}_{GZ,Z}$. The proof that these arrows make up suitable natural transformations is given in MacLane [27]. If we apply this construction to the functors $\langle X \otimes - \rangle : \mathcal{C} \to \mathcal{C}$ and $\langle \overline{X} \otimes - \rangle : \mathcal{C} \to \mathcal{C}$ and ϕ given by Lemma 3.3.4, we find $\eta_Y = r \otimes id_Y$ and $\epsilon_Y = \overline{r}^* \otimes id_Y$. The adjunction can completely be described by the functors $\langle X \otimes - \rangle$ and $\langle \overline{X} \otimes - \rangle$ and the arrows $r : \mathbb{1} \to \overline{X} \otimes X$ and $\overline{r} : \mathbb{1} \to X \otimes \overline{X}$.

Let X, Y be objects of $\mathcal{C}$ with conjugates $(\overline{X}, r_X, \overline{r}_X)$ and $(\overline{Y}, r_Y, \overline{r}_Y)$ and suppose that $s : X \to Y$ is a morphism of $\mathcal{C}$. Then in a natural way we can construct a morphism $\overline{s} : \overline{X} \to \overline{Y}$ by defining

$$\overline{s} = id_{\overline{Y}} \otimes \overline{r}^*_X \circ id_{\overline{Y}} \otimes t^* \otimes id_{\overline{X}} \circ r_Y \otimes id_{\overline{X}}.$$

As every TC* is semisimple, for any object X there exist isometries $w_i : X_i \to X$ such that $w_i^* \circ w_i = id_{X_i}$, $\sum_i w_i \circ w_i^* = id_X$ and every X_i is an irreducible object. It would be nice if the corresponding maps $\overline{w}_i : \overline{X}_i \to \overline{X}$ would give a decomposition of $\overline{X}$ into irreducible components. It turns out that we need to include another axiom in the definition of conjugates in order to guarantee this.

Definition 3.3.5. *(final version) Let $\mathcal{C}$ be a tensor $*$-category and X an object of $\mathcal{C}$. An object $\overline{X}$ of $\mathcal{C}$ is called a conjugate object of X if there exist morphisms $r : \mathbf{1} \to \overline{X} \otimes X$ and $\overline{r} : \mathbf{1} \to X \otimes \overline{X}$ that satisfy the conjugate equations. Let $s : X \to Y$ be a morphism in $\mathcal{C}$ and $\overline{s} : \overline{X} \to \overline{Y}$ be as above. Then we demand that $(\overline{s})^* = \overline{s^*}$. We say that $\mathcal{C}$ has conjugates if there is a conjugate $(\overline{X}, r, \overline{r})$ for each object and $(\overline{s})^* = \overline{s^*}$ holds for each morphism.*

Lemma 3.3.6. *Let $\mathcal{C}$ be a TC* category and X an object of $\mathcal{C}$. Suppose that $X \cong \oplus_i X_i$ effected by isometries $w_i : X_i \to X$, where the X_i are irreducible objects. Suppose that $(\overline{X}, r_X, \overline{r}_X)$ is a conjugate for X and $(\overline{X}_i, r_{X_i}, \overline{r}_{X_i})$ are conjugates for the X_i. Let $\overline{w_i} : \overline{X_i} \to \overline{X}$ be as above. Then the $\overline{w_i}$ are isometries effecting the decomposition $\overline{X} \cong \overline{X_i}$ in irreducible objects $\overline{X_i}$.*

Showing that $\overline{w_i}^* \circ \overline{w_i} = id_{\overline{X_i}}$ is a straightforward computation. One needs the conjugate equations and the relation $\overline{w_i}^* = \overline{w_i^*}$. As the involved identities are long and not very illuminating, the reader that wants to prove this lemma may want to consult Section 3.5 where a diagrammatic method is discussed that greatly simplifies the calculations.

Conjugates can be used to define a dimension of objects and a trace of endomorphisms. The next definition defines conjugates of particular interest. If we restrict to conjugates of this kind the notions of dimension and trace will be independent of the particular choice of r and $\overline{r}$.

Definition 3.3.7. *Let $\mathcal{C}$ be a TC* and X an object of $\mathcal{C}$ that has a conjugate $(\overline{X}, r, \overline{r})$. Let $X \cong \bigoplus_i X_i$ be the decomposition of X in irreducible components and $w_i : X_i \to X$ be the isometries corresponding to the projections p_i. By the previous lemma we have a similar decomposition $\overline{X} \cong \bigoplus_i \overline{X}_i$ with isometries $\overline{w}_i : \overline{X}_i \to \overline{X}$. A solution to the conjugate equations $r : \mathbb{1} \to \overline{X} \otimes X$, $\overline{r} : \mathbb{1} \to X \otimes \overline{X}$ is called a standard conjugate if it is of the form*

$$r = \sum_i (\overline{w}_i \otimes w_i) \circ r_i, \quad \overline{r} = \sum_i (w_i \otimes \overline{w}_i) \circ \overline{r}_i$$

where $(\overline{X}_i, r_i, \overline{r}_i)$ are conjugates for X_i which are normalized in the sense that $r_i^* \circ r_i = \overline{r}_i^* \circ \overline{r}_i$.

A conjugate $(\overline{X}, r, \overline{r})$ is standard if and only if we have normalized conjugates $(\overline{X}_i, (\overline{w}_i^* \otimes w_i^*) \circ r, (w_i^* \otimes \overline{w}_i^*) \circ \overline{r})$ for the X_i. This definition of standardness looks different from that given in Müger [30]. In a paper by Longo and Roberts [26] it is shown that both definitions are equivalent. However, this paper is about to C^*-tensor categories. As we will later see the Hom-sets of a TC* can be given unique norms rendering it a C^*-tensor category. Conversely, any C^*-tensor category with directs sums, subobjects, conjugates and an irreducible tensor unit is a TC* category. The equivalence of the different standardness definitions is presented in the following lemma.

Lemma 3.3.8. *Let $\mathcal{C}$ be a TC* and $(\overline{X}, r, \overline{r})$ be a conjugate for X. Then $(\overline{X}, r, \overline{r})$ is a standard conjugate if and only for every arrow $s : X \to X$ we have*

$$r^* \circ (id_{\overline{X}} \otimes s) \circ r = \overline{r}^* \circ (s \otimes id_{\overline{X}}) \circ \overline{r}.$$

We prove that standardness implies the stated identity. In order to do so, we use the relations $w_i^* \circ w_j = 0$ and $\overline{w}_i^* \circ \overline{w}_j = 0$ for $i \neq j$ that follow from the definition of biproducts. Assume that $(\overline{X}, r, \overline{r})$ is a standard conjugate.

$$\begin{aligned} r^* \circ (id_{\overline{X}} \otimes s) \circ r &= \sum_{i,j} r_i^* \circ ((\overline{w}_i^* \circ \overline{w}_j) \otimes (w_i^* \circ s \circ w_j)) \circ r \\ &= \sum_i r_i^* \circ (id_{\overline{X}_i} \otimes (w_i^* s w_i)) \circ r. \end{aligned}$$

By irreducibility $w_i^* \circ s \circ w_i : X_i \to X_i$ is of the form, λid_{X_i} for some $\lambda \in \mathbb{C}$. Writing out the right hand side of $r^* \circ (id_{\overline{X}} \otimes s) \circ r = \overline{r}^* \circ (s \otimes id_{\overline{X}}) \circ \overline{r}$ gives that the relation is satisfied if for each i we have that $r_i^* \circ r_i = \overline{r}_i^* \circ \overline{r}_i$. This holds by definition of a standard conjugate. A proof of the converse, which is the fact that the property of Lemma 3.3.8 implies standardness, can be found in Longo and Roberts [26] in the shape of Lemma 3.9.

For the remainder of this section we will not work out every detail of the presented material. This would take too much time and space. The interested reader can find full proofs of all claims (and more) in Sections 2 and 3 of Longo and Roberts [26]. The following lemma shows that restricting our attention to standard conjugates does not restrict the number of objects under consideration.

Lemma 3.3.9. *Let $\mathcal{C}$ be a TC*. Every non-zero object X has a standard conjugate.*

Proof. First consider the case that X is an irreducible object. A solution r, $\overline{r}$ to the conjugate equations is standard if and only if $r^* \circ r = \overline{r}^* \circ \overline{r}$. By rescaling both r and $\overline{r}$ this can be satisfied. Note that in order for the conjugate equations to hold we can only rescale as $r \to \lambda r$, $\overline{r} \to \overline{\lambda}^{-1} \overline{r}$, $\lambda \in \mathbb{C}$. By positivity of the $*$-operation, $r^* \circ r > 0$ so this restriction leaves us with enough freedom. Now take X to be any object, then it is a finite direct sum $X \cong \bigoplus_i X_i$ of irreducible objects X_i. For each X_i we can find a standard conjugate $(\overline{X}_i, r_i, \overline{r}_i)$. Put $\overline{X} = \bigoplus_i \overline{X}_i$. Let $w_i : X_i \to X$, $\overline{w}_i : \overline{X}_i \to \overline{X}$ be the isometries corresponding to the direct sum decompositions of X and $\overline{X}$. Define $r = \sum_i w_i \otimes \overline{w}_i \circ r_i$ and $\overline{r} = \sum_i \overline{w}_i \otimes w_i \circ \overline{r}_i$. This defines a standard conjugate for X. □

The following lemma shows how we can get a standard conjugate for a subobject of X if we know a standard conjugate of X. The subsequent lemma shows how we can get a standard conjugate for $X \otimes Y$ from standard conjugates of X and Y.

Lemma 3.3.10. *Let* $(\overline{X}, r, \overline{r})$ *be a standard conjugate of* X *and* $p : X \to X$ *a projection. Define*

$$\overline{p} = (r^* \otimes id_{\overline{X}}) \circ (id_{\overline{X}} \otimes p \otimes id_{\overline{X}}) \circ (id_{\overline{X}} \otimes \overline{r}) : \overline{X} \to \overline{X}.$$

Then $\overline{p}$ *is a projection. There are isometries such that* $v \circ v^* = p$ *and* $w \circ w^* = \overline{p}$*. The triple* $(\overline{Y}, w^* \otimes v^* \circ r, v^* \otimes w^* \circ \overline{r})$ *defines a standard conjugate for* Y*.*

Lemma 3.3.11. *Let* $(\overline{X}, r, \overline{r})$ *and* $(\overline{Y}, r', \overline{r}')$ *be standard conjugates of* X *and* Y *respectively. Defining* $r'' = (id_{\overline{Y}} \otimes r \otimes id_Y) \circ r'$ *and* $\overline{r}'' = (id_X \otimes \overline{r}' \otimes id_{\overline{X}}) \circ \overline{r}$ *the triple* $(\overline{Y} \otimes \overline{X}, r'', \overline{r}'')$ *defines a standard conjugate for* $X \otimes Y$*.*

Proving that the constructions satisfy the conjugate equations and are standard is left to the reader. In the following definition/proposition we define for each object X a trace Tr_X. As long as standard conjugates are used the trace is independent of which standard conjugate is used.

Proposition 3.3.12. *Let* $\mathcal{C}$ *be a TC** *and* X *an object of* $\mathcal{C}$ *with standard conjugate* $(\overline{X}, r, \overline{r})$*. Define the trace map as*

$$Tr_X : End(X) \to \mathbb{C}, \quad s \mapsto r^* \circ id_{\overline{X}} \otimes s \circ r.$$

This map is well defined in the sense that it does not depend on the choice of standard conjugate. It satisfies the following two properties.

$$Tr_X(s \circ t) = Tr_Y(t \circ s), \quad \forall s : X \to Y,\ t : Y \to X,$$

$$Tr_{X \otimes Y}(s \otimes t) = Tr_X(s) Tr_Y(t), \quad \forall s : X \to X,\ t : Y \to Y.$$

A full proof can be found in Section 3 of Longo and Roberts [26]. This proof works as follows. First a scalar product for $s, t : X \to Y$ is defined by

$$\langle s, t \rangle = r^* \circ id_{\overline{X}} \otimes (s^* \circ t) \circ r$$

This product satisfies the properties $\langle s, s \rangle \geq 0$ and $\langle s, q \circ t \rangle = \langle q^* \circ s, t \rangle$ whenever the compositions are defined. It is subsequently shown in Lemma 3.7 that for standard conjugate this product in independent of the choice of the standard conjugate. This is the first claim of the proposition as we have that $Tr_X(s) = \langle id_X, s \rangle$. In this lemma it is also proven that the scalar product is tracial in the sense that $\langle s, t \rangle = \langle t^*, s^* \rangle$. The first property of the proposition is a consequence of this fact.

$$\begin{aligned} Tr_X(s \circ t) &= \langle id_X, s \circ t \rangle = \langle s^* \circ id_X, t \rangle = \langle s^*, t \rangle \\ &= \langle t^*, s \rangle = \langle t^* \circ id_Y, s \rangle = \langle id_Y, t \circ s \rangle = Tr_Y(t \circ s). \end{aligned}$$

The second property can easily be proven using the standard conjugate from Lemma 3.3.11. Using this lemma the proof amounts to writing out the left hand side.

$$\begin{aligned} Tr_{X \otimes Y}(s \otimes t) &= r'^* \circ id_{\overline{Y}} \otimes r^* \otimes id_Y \circ id_{\overline{Y}} \otimes \overline{X} \otimes s \otimes t) \circ id_{\overline{Y}} \otimes r \otimes id_Y \circ r' \\ &= r'^* \circ id_{\overline{Y}} \otimes r^* \circ id_{\overline{X}} \otimes s \circ r \otimes t \circ r' = Tr_X(s) Tr_Y(t) \end{aligned}$$

Proposition 3.3.13. *Let $\mathcal{C}$ be a TC* and X a object of $\mathcal{C}$. Define the dimension of X as $d(X) = Tr_X(id_X)$. For any standard conjugate $(\overline{X}, r, \overline{r})$ of X we have $d(X) = r^* \circ r$. The dimension is additive in the sense that $d(X \oplus Y) = d(X) + d(Y)$ and multiplicative in the sense $d(X \otimes Y) = d(X)d(Y)$. It satisfies the properties $d(\overline{X}) = d(X)$ and $d(X) \geq 1$ with $d(X) = 1$ implying that X is invertible ($X \otimes \overline{X} \cong \mathbf{1}$).*

Proof. Multiplicativity follows straight from Proposition 3.3.12. Regarding additivity, suppose that we have isometries $w_1 : X \to W \oplus Y$ and $w_2 : Y \to X \oplus Y$. Take for $X \oplus Y$ the standard conjugate $(\overline{X} \oplus \overline{Y}, r = \sum_i \overline{w}_i \otimes w_i \circ r_i, \overline{r} = \sum_i w_i \otimes \overline{w}_i \circ \overline{r})$.

$$d(X \oplus Y) = r^* \circ r = \sum_{i,j} r_i^*(\overline{w}_i^* \otimes w_i^*)(\overline{w}_j \otimes w_j) r_j = \sum_i r_i^* \circ r_i = d(X) + d(Y).$$

If $(\overline{X}, r, \overline{r})$ is a standard conjugate for X then $(X, \overline{r}, r)$ is a standard conjugate for $\overline{X}$. This proves that $d(\overline{X}) = d(X)$. Because the $*$-operation is positive, $d(X) > 0$. We know that $\mathbb{1}$ is in the direct sum of $\overline{X} \otimes X$. Combining this observation with the already proven properties provides us with $d(X)^2 \geq 1$ implying $d(X) \geq 1$. If $d(X) = 1$ then $\mathbb{1}$ is the only nonzero object in the direct sum, implying that X is invertible. □

3.4 Tannaka-Krein Duality

We are done with the categorical preparations and now get ready to state the Tannaka-Krein duality theorem. First a small recollection of what we did in the previous chapter. Starting out with a compact group G we studied the structure of the category of finite dimensional representations of G, the STC* category $Rep_f(G, \mathbb{C})$. The Tannaka group G_ω was constructed as the group of unitary monoidal natural transformations of the forgetful functor $\omega : Rep_f(G, \mathbb{C}) \to Vect_{\mathbb{C}}$ to itself. The Tannaka group was shown to be a compact group isomorphic to G. Stated differently, we found an equivalence of tensor $*$-categories $Rep_f(G_\omega, \mathbb{C}) \cong Rep_f(G, \mathbb{C})$. In this section we do something similar. We start out with a STC* $\mathcal{C}$ and a suitable functor $E : \mathcal{C} \to \mathcal{H}_f$ that reminds of the forgetful functor. Replacing $Vect_{\mathbb{C}}$ by $\mathcal{H}_f$ allows us to compare the $*$-operations of the categories. The unitary monoidal natural transformations of the functor E to itself form a compact group G_E. It will turn out that $\mathcal{C} \cong Rep_f(G_E, \mathbb{C})$ as symmetric tensor $*$-categories. The material in this section is based on Appendices B1 and B2 of Müger [30]. We start with defining a suitable functor η.

Definition 3.4.1. *Let $\mathcal{C}$ be a STC* category. A fiber functor for $\mathcal{C}$ is a faithful $\mathbb{C}$-linear tensor functor $E : \mathcal{C} \to Vect_{\mathbb{C}}$. A $*$-preserving fiber functor for $\mathcal{C}$ is a faithful functor $E : \mathcal{C} \to \mathcal{H}_f$ of tensor $*$-categories. E is called symmetric if for all objects X and Y of $\mathcal{C}$, $E(c_{X,Y}) \circ d_{X,Y} = d_{X,Y} \circ \Sigma_{E(X),E(Y)}$.*

The functor is thus called symmetric if it maps the symmetry of $\mathcal{C}$ to the flip symmetry of $\mathcal{H}_f$. If a STC* category $\mathcal{C}$ is equipped with a symmetric $*$-preserving fiber functor it is clear that $\mathcal{C}$ is equivalent to a tensor subcategory of $\mathcal{H}_f$. In this chapter we assume to have a symmetric $*$-preserving fiber functor for each STC* category under consideration. The next chapter is concerned with constructing a fiber functor for any given STC* category. The result of this investigation is Deligne's embedding theorem.

Lemma 3.4.2. *Let $\mathcal{C}$ be a STC* category and $E : \mathcal{C} \to \mathcal{H}_f$ a symmetric $*$-preserving fiber functor. The set $G_E \subset End(E)$ of unitary monoidal natural transformations of E to itself is a compact group.*

Proof. Composition provides group multiplication and the identity natural transformation acts as a unit. The identity transformation is clearly an element of G_E. Let $g \in G_E$ then g^{-1} is defined by $(g^{-1})_X = \overline{g_X}^t$. We should check that $g^{-1} \in G_E$. This amounts to showing that it is natural and monoidal. Naturality means that for any linear map $f : X \to Y$ in $\mathcal{C}$ we have that $E(f) \circ (g^{-1})_X = (g^{-1})_Y \circ E(f)$. Taking the adjoint of both sides of the equation $E(f) \circ g_X = g_Y \circ E(f)$ gives $E(f)^\dagger \circ (g^{-1})_Y = (g^{-1})_X \circ E(f)^\dagger$. Note that E is $*$-preserving so this is equivalent to the claim that $E(f^*) \circ (g^{-1})_Y = (g^{-1})_X \circ E(f^*)$ holds for every $f^* : Y \to X$. As the $*$-map is involutive this proves naturality and we move onto proving that g^{-1} is monoidal. We have maps $d_{X,Y} : E(X) \otimes E(Y) \to E(X \otimes Y)$ and $e : \mathbb{C} \to E(\mathbb{1}_\mathcal{C})$ that are unitary. Using this unitarity, taking the adjoint of the equality $(g_X \otimes g_Y) \circ d_{X,Y} = d_{X,Y} \circ g_{X\otimes Y}$ gives the desired equality $((g^{-1})_X \otimes (g^{-1})_Y) \circ d_{X,Y} = d_{X,Y} \circ (g^{-1})_{X\otimes Y}$. Similarly we deduce $e = (g^{-1})_\mathbb{1} \circ e$. This makes G_E into a group. Proving that it is a compact topological group proceeds in the same way as in Section 2.3 and Proposition 2.5.6. We identify G_E as a closed subset of $\prod_{X\in\mathcal{C}} U(E(X))$ where $U(E(X))$ are the unitary transformations $E(X) \to E(X)$. By Tychonov's theorem the space $\prod_{X\in\mathcal{C}} U(E(X))$ is compact and therefore G_E is also compact. The multiplication and inverse maps are easily checked to be continuous. □

Each Hilbert space $E(X)$ carries a unitary representation of G_E. We have the following continuous homomorphism of groups

$$\pi_X : G_E \to End(E(X)), \quad g \mapsto \pi_X(g) = g_X.$$

We use this fact to prove the following proposition.

Proposition 3.4.3. *Let $\mathcal{C}$ be a STC* category and $E : \mathcal{C} \to \mathcal{H}_f$ a symmetric $*$-preserving fiber functor. Then there exists a symmetric faithful tensor $*$-functor $F : \mathcal{C} \to Rep_f(G_\eta, \mathbb{C})$ such that $\omega \circ F = E$ where $\omega : Rep_f(G_E, \mathbb{C}) \to \mathcal{H}_f$ is the forgetful functor.*

Proof. Define for any object X of $\mathcal{C}$, $F(X) = (E(X), \pi_X)$ with π_X as above. Each arrow $s : X \to Y$ is mapped to $F(s) = E(s)$. We need to check that $E(s) : (E(X), \pi_X) \to (E(Y), \pi_Y)$ is an intertwining operator. Using that g is defined as a natural transformation we can easily see that

$$F(s) \circ \pi_X(g) = E(s) \circ g_X = g_Y \circ E(s) = \pi_Y(g) \circ F(s).$$

As F clearly respects composition of arrows it is a well-defined functor. The facts that F is faithful and $*$-preserving follow directly from the properties of E. The unitary maps $d^F_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$ and $e^F : \mathbb{C} \to F(\mathbb{1}_{\mathcal{C}})$ are borrowed from E. We define $d^F_{X,Y} = d^E_{X,Y}$ and $e^F = e^E$. These maps should be arrows in $Rep_f(G_E, \mathbb{C})$ so we need to check again if we have defined intertwining operators. For $d^F_{X,Y}$ this follows from the fact that g is a monoidal natural transformation as can be seen by

$$d^F_{X,Y} \circ (\pi_X(g) \otimes \pi_Y(g)) = d^E_{X,Y} \circ g_X \otimes g_Y = g_{X \otimes Y} \circ d^E_{X,Y} = \pi_{X \otimes Y}(g) \circ d^F_{X,Y}.$$

Because each g is monoidal it follows that $g_{\mathbb{1}} = id_{E(\mathbb{1})}$ must hold for each $g \in G_E$. The functor F therefore maps the tensor unit $\mathbb{1}$ of $\mathcal{C}$ to the trivial representation $(E(\mathbb{1}), \pi_{\mathbb{1}})$. Next we equip the tensor unit $\mathbb{C}$ of $\mathcal{H}_f$ with the trivial representation. If $\mathbb{C}$ happened to be in the image of E then it was already given the trivial representation so this raises no conflict. The map e^F is an intertwining operator. The rest of the claims are easy to check. □

The following proposition is needed to prove Tannaka-Krein duality. The proof of this proposition will occupy the following three sections. Although proving this proposition will be a lot of work the efforts are well rewarded. During the computations we will see the diversity of the possible fiber functors for a given STC* category. It turns out that each fiber functor would give the same group G_E up to an isomorphism. From this point onwards we presuppose that $\mathcal{C}$ is a essentially small category. This means that it is equivalent to a small category, i.e. a category of which the class of objects $Obj\mathcal{C}$ is a set. This makes it possible to define a set I as in the following proposition.

Proposition 3.4.4. *Define $\{X_i | i \in I\}$ to be a set of irreducible pairwise inequivalent objects of the STC* category $\mathcal{C}$ such that every object is isomorphic to a finite direct sum of elements of $\{X_i | i \in I\}$. Take $S \subset I$ to be any finite subset. Then the following holds for the closed linear span*

$$\overline{span}_{\mathbb{C}}\{\pi_{S_1}(g) \oplus ... \oplus \pi_{S_{|S|}}(g) | g \in G_E\} = \bigoplus_{s \in S} End(E(X_s)).$$

Modulo Proposition 3.4.4 we can state and prove Tannaka-Krein duality.

Theorem 3.4.5. *(Tannaka-Krein Duality) Let $\mathcal{C}$ be a STC* category and $E : \mathcal{C} \to \mathcal{H}_f$ a symmetric $*$-preserving fiber functor. Let G_E and $F : \mathcal{C} \to Rep_f(G_E, \mathbb{C})$ be as previously defined. Then F is an equivalence of symmetric tensor $*$-categories.*

Proof. Analogous to Proposition 2.6.13 it is true that F defines an equivalence of symmetric tensor $*$-categories if it is a symmetric tensor $*$-functor that is faithful, full and essentially surjective. Looking at Proposition 3.4.3 it remains to be checked that F is full and essentially surjective. We start with proving that F is full. Both categories $\mathcal{C}$ and $Rep_f(G_E, \mathbb{C})$ are semisimple, therefore it suffices to prove the following two claims. First we show that if $X \in Obj(\mathcal{C})$ is irreducible, then $F(X)$ is irreducible. Second, we show that if X and Y are inequivalent irreducible objects of $\mathcal{C}$, then the hom set $Hom_{Rep_f(G_E,\mathbb{C})}(F(X), F(Y)) = \{0\}$. The first claim follows from Proposition 3.4.4 if we take S to be a singleton. Every endomorphism $s : E(X) \to E(X)$ can be approximated by a representation $\pi_X(g) : E(X) \to E(X)$. It follows that $End(E(X)) = \mathbb{C}id$. The second claim also follows from Proposition 3.4.4, but in this case we take S to be a set with two elements. Take $s \in Hom_{Rep_f(G_E,\mathbb{C})}(F(X), F(Y))$. By definition of F we know that $s \in Hom_{\mathcal{H}_f}(E(X), E(Y))$ with the property that $s \circ \pi_X(g) = \pi_Y(g) \circ s$ holds for any $g \in G_E$. By Proposition 3.4.4, $s \circ u = v \circ s$ holds for any pair of endomorphism $u : E(X) \to E(X)$ and $v : E(Y) \to E(Y)$. Choosing $u = 0$ and $v = 1$ shows that $s = 0$. This proves that F is full, and we move onto proving that it is essentially surjective. The category $\mathcal{C}$ is equivalent to a full tensor subcategory of $Rep_f(G_E, \mathbb{C})$. The representations that constitute this tensor subcategory form a set that is closed in the sense of Section 2.6. If the subcategory separates points, then Lemma 2.6.9 immediatly proves essential surjectivity. Let $g \in G_E$ be nontrivial. Then there is an object $X \in Obj\mathcal{C}$ such that $g_X \neq id_{E(X)}$. This means that $\pi_X(g) \neq id_{E(X)}$, hence the representations $\{F(X)|X \in Obj\mathcal{C}\}$ separate the points of G_E. □

For a compact group G the category $Rep_f(G, \mathbb{C})$ is a STC* category and the forgetful functor $\omega : Rep_f(G, \mathbb{C}) \to \mathcal{H}_f$ is a symmetric $*$-preserving fiber functor. The group G_ω is just the Tannaka group and the fact that $F : Rep_f(G, \mathbb{C}) \to Rep_f(G_\omega, \mathbb{C})$ defines an equivalence of symmetric tensor $*$-categories is easy to see.

3.5 Algebras and Fiber Functors

The theory developed in this section will be important in more than one way. It will help prove Proposition 3.4.4, thereby completing the proof of Tannaka-Krein duality. It will also help in showing that the group constructed using Tannaka-Krein duality is unique up to an isomorphism. This is done by proving that for any two symmetric $*$-preserving fiber functors, there is a unitary monoidal natural isomorphism between them. Finally, in the next chapter the theory of this section will help in the construction of a $*$-preserving fiber functor. This section is based on Appendices B3 and B4 of Müger [30].

Take $\mathcal{C}$ to be a TC* category and let $E_i : \mathcal{C} \to \mathcal{H}_f$ with $i \in \{1, 2\}$ be $*$-preserving fiber functors. From this point onwards we are going to study the set $Nat(E_1, E_2)$ of natural transformations $E_1 \to E_2$. We will not do this directly. Rather, we construct an algebra $A(E_1, E_2)$, that acts as a predual for $Nat(E_1, E_2)$. More precisely, $Nat(E_1, E_2)$ is the algebraic dual space for $A(E_1, E_2)$. The space $A(E_1, E_2)$ can be made into a unital C*-algebra, (which is commutative when considering STC* categories and symmetric $*$-preserving fiber functors, as we will discuss in the next section) which will lead to interesting results.

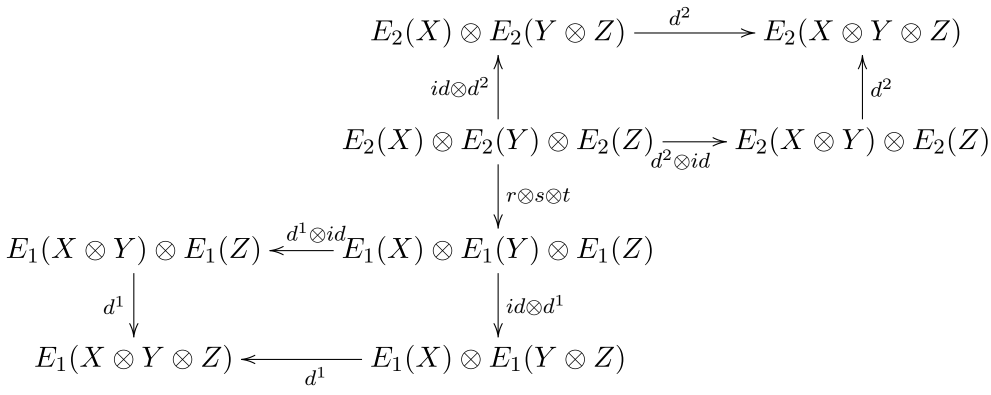
First let $\mathcal{C}$ to be a TC* category and let $E_i : \mathcal{C} \to Vect_{\mathbb{C}}$ with $i \in \{1, 2\}$ be fiber functors, not necessarily $*$-preserving. Define the $\mathbb{C}$-vector space

$$A_0(E_1, E_2) = \bigoplus_{X \in \mathcal{C}} Hom(E_2(X), E_1(X)).$$

Let $[X, s]$ denote the element of $A_0(E_1, E_2)$ that takes the value $s : E_2(X) \to E_1(X)$ in $X \in Obj\mathcal{C}$ and is 0 everywhere else. We can make $A_0(E_1, E_2)$ into an associative $\mathbb{C}$-algebra by defining

$$[X, s] \cdot [Y, t] = [X \otimes Y, d^1_{X,Y} \circ s \otimes t \circ (d^2_{X,Y})^{-1}]$$

where the maps $d^i_{X,Y} : E_i(X) \otimes E_i(Y) \to E_i(X \otimes Y)$ come from the tensor functors E_i. We need to check that this multiplication operation is associative. This means that $([X, r] \cdot [Y, s]) \cdot [Z, t] = [X, r] \cdot ([Y, s] \cdot [Z, t])$ holds for all $X, Y, Z \in Obj\mathcal{C}$ and possible choices of r, s and t. Recall the commutative diagrams belonging to a tensor functor given in Definition 3.1.3. Because we restricted our attention to strict monoidal categories, the maps α, λ and ρ are taken to be the identity maps. These diagrams provide us with the following commutative diagram.



Writing out $([X, r] \cdot [Y, s]) \cdot [Z, t]$ and $[X, r] \cdot ([Y, s] \cdot [Z, t])$ associativity follows directly from the fact that the above diagram is commutative. The maps $e^i : \mathbb{C} \to E_i(\mathbb{1})$ coming from the tensor functors can be used to define a multiplicative unit. This multiplicative unit is given by $[\mathbb{1}, e^1 \circ (e^2)^{-1}]$. The diagrams of Definition 3.1.3 give us

$$d^i_{X,\mathbb{1}} \circ id_{E_i(X)} \otimes e^i = id_{E_i(X)}, \quad d^i_{\mathbb{1},X} \circ e^i \otimes id_{E_i(X)} = id_{E_i(X)}$$

from which we can deduce that

$$[\mathbb{1}, e^1 \circ (e^2)^{-1}] \cdot [X, s] = [X, s] = [X, s] \cdot [\mathbb{1}, e^1 \circ (e^2)^{-1}].$$

We have thus shown that $A_0(E_1, E_2)$ is an associative algebra with multiplicative unit.

Lemma 3.5.1. *Let $\mathcal{C}$ be a TC* category and let $E_i : \mathcal{C} \to Vect_{\mathbb{C}}$, $i \in \{1, 2\}$ be fiber functors. Then the following subspace of $A_0(E_1, E_2)$ is a 2-sided ideal.*

$$I(E_1, E_2) = span_{\mathbb{C}}\{[X, a \circ E_2(s)] - [Y, E_1(s) \circ a] | s : X \to Y, a : E_2(Y) \to E_1(X)\}.$$

Proof. First note that by naturality of the d^i we have the following equalities

$$E_2(s) \otimes E_2(id_Z) \circ (d^2_{X,Z})^{-1} = (d^2_{Y,Z})^{-1} \circ E_2(s \otimes id_Z),$$

$$d^1_{Y,Z} \circ E_1(s) \otimes E_1(id_Z) = E_1(s \otimes id_Z) \circ d^1_{X,Z}.$$

Using these equalities we find

$$\begin{aligned}
&([X, a \circ E_2(s)] - [Y, E_1(s) \circ a]) \cdot [Z, t] \\
&= [X \otimes Z, d^1_{X,Z} \circ (a \circ E_2(s)) \otimes t \circ (d^2_{X,Z})^{-1}] \\
&\quad - [Y \otimes Z, d^1_{Y,Z} \circ (E_1(s) \circ a) \otimes t \circ (d^2_{Y,Z})^{-1}] \\
&= [X \otimes Z, d^1_{X,Z} \circ a \otimes t \circ (d^2_{Y,Z})^{-1} \circ E_2(s \otimes id_Z)] \\
&\quad - [Y \otimes Z, E_1(s \otimes id_Z) \circ d^1_{X,Z} \circ a \otimes t \circ (d^2_{Y,Z})^{-1}].
\end{aligned}$$

Defining $X' = X \otimes Z$, $Y' = Y \otimes Z$, $s' = s \otimes id_Z : X' \to Y'$ and

$$a' = d^1_{X,Z} \circ a \otimes t \circ (d^2_{Y,Z})^{-1} : E_2(Y') \to E_1(X')$$

the above identity becomes $[X', a' \circ E_2(s')] - [Y', E_1(s') \circ a']$, which is clearly an element of $I(E_1, E_2)$. Thus we have proven that $I(E_1, E_2)$ is a left ideal. Proving that it is a right ideal proceeds in the same way.

$$\begin{aligned}
&[Z, t] \cdot ([X, a \circ E_2(s)] - [Y, E_1(s) \circ a]) \\
&= [Z \otimes X, d^1_{Z,X} \circ t \otimes (a \circ E_2(s)) \circ (d^2_{Z,X})^{-1}] \\
&\quad - [Z \otimes Y, d^1_{Z,Y} \circ t \otimes (E_1(s) \circ a) \circ (d^2_{Z,Y})^{-1}] \\
&= [Z \otimes X, d^1_{Z,X} \circ t \otimes a \circ (d^2_{Z,Y})^{-1} \circ E_2(id_Z \otimes s)] \\
&\quad - [Z \otimes Y, E_1(id_Z \otimes s) \circ d^1_{Z,X} \circ t \otimes a \circ (d^2_{Z,Y})^{-1}] \\
&= [X'', a'' \circ E_2(s'')] - [Y'', E_1(s'') \circ a''] \in I(E_1, E_2).
\end{aligned}$$

□

Let $A(E_1,E_2)$ denote the quotient algebra $A_0(E_1,E_2)/I(E_1,E_2)$. At the risk of possible confusion we shall denote the images of $[X,s] \in A_0(E_1,E_2)$ under the canonical map

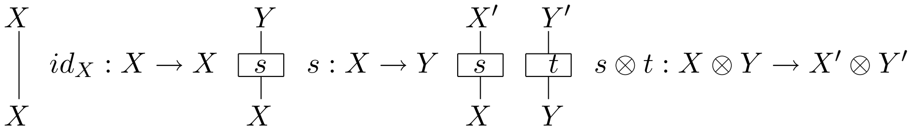
$$\gamma : A_0(E_1,E_2) \to A(E_1,E_2) = A_0(E_1,E_2)/I(E_1,E_2)$$

again by $[X,s]$. The algebra $A(E_1,E_2)$ can be thought of as generated by the elements $[X,s]$, subject to the relations

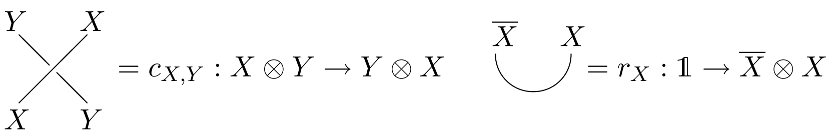
$$[X,\alpha s+\beta t] = \alpha[X,s]+\beta[X,t], \quad \alpha,\beta \in \mathbb{C}, \quad s,t : E_2(X) \to E_1(X),$$

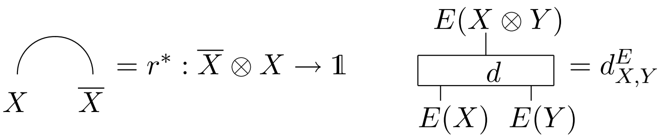
$$[X, a\circ E_2(s)] = [Y, E_1(s)\circ a], \quad s : X \to Y, \quad a : E_2(Y) \to E_1(X).$$

As promised we will show that the algebra $A(E_1,E_2)$ has additional structure, but in order to make the proofs more transparant we will introduce some new notation. We use the following graphical notation for arrows in a monoidal category.

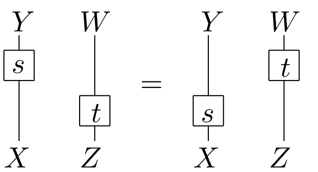


Identity arrows are denoted by vertical lines and arrows $s : X \to Y$ are denoted by a box. Tensor products of arrows are denoted by horizontal juxtaposition of the boxes. The composition of two arrows is written as the vertical juxtaposition of the two boxes. The arrows coming from braidings, tensor functors and conjugations are given by





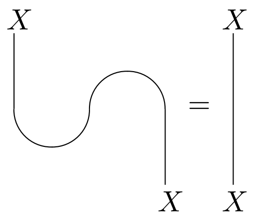
As an example consider the following trivial equality.



$$s\otimes id_W \circ id_X \otimes t = id_Y \otimes t \circ s \otimes id_Z$$

$$s : X \to Y, \quad t : Z \to W.$$

The next example is one of the conjugate equations.



$$id_X \otimes r^* \circ \overline{r} \otimes id_X = id_X$$

Using this notation we will prove the following proposition

Proposition 3.5.2. *Let $\mathcal{C}$ be a TC^* category and let $E_i : \mathcal{C} \to \mathcal{H}_f$, $i \in \{1, 2\}$ be $*$-preserving fiber functors. Then $A(E_1, E_2)$ has a positive $*$-operation. It is an antilinear and antimultiplicative involution such that $a^*a = 0$ implies that $a = 0$.*

Proof. For this proof we will pretend that the functors E_i are strict tensor functors. The proof for non-strict fiber functors works in the same way if we insert the maps $d^i_{X,Y}$ and e^i whenever needed to make the maps involved well defined. We start with defining a antilinear involution on $A_0(E_1, E_2)$. It is defined on the generators by

$$[X, s]^\star = [\overline{X}, t] = [\overline{X}, id_{E_1(\overline{X})} \otimes E_2(\overline{r}^*) \circ id_{E_1(\overline{X})} \otimes s^* \otimes id_{E_2(\overline{X})} \circ E_1(r) \otimes id_{E_2(\overline{X})}]$$

where $(\overline{X}, r, \overline{r})$ is a standard conjugate for X. Since we used standard conjugates $(\overline{X}, r, \overline{r})$ we should ask the question if this definition depends on the particular choice of conjugates. It turns out that it does, but the difference in $[X, s]^\star$ for different choices of standard conjugates is an element of the ideal $I(E_1, E_2)$. The canonical map

$$\gamma : A_0(E_1, E_2) \to A(E_1, E_2)$$

maps the difference to zero. In order to show this, pick another standard conjugate $(\overline{X}', r', \overline{r}')$ for X. Then there exists a unique unitary arrow $u : \overline{X} \to \overline{X}'$ such that $r' = u \otimes id_X \circ r$ (Section 3 in Longo and Roberts [26]). Consider $([X, s]^\star)' = [\overline{X}', t']$, where t' is expressed in terms of the primed conjugates.

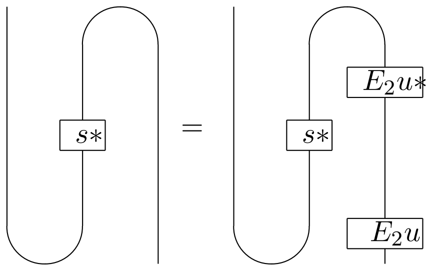
$$\begin{aligned}
&[\overline{X}, t] - [\overline{X}', t'] \\
&= [\overline{X}, id_{E_1(\overline{X})} \otimes E_2(\overline{r}^*) \circ id_{E_1(\overline{X})} \otimes s^* \otimes id_{E_2(\overline{X})} \circ E_1(r) \otimes id_{E_2(\overline{X})}] \\
&\quad - [\overline{X}', id_{E_1(\overline{X}')} \otimes E_2(\overline{r}^{*\prime}) \circ id_{E_1(\overline{X}')} \otimes s^* \otimes id_{E_2(\overline{X}')} \circ E_1(r') \otimes id_{E_2(\overline{X}')}]
\end{aligned}$$

We have the following identities

$$\begin{aligned}
&id_{E_1(\overline{X})} \otimes E_2(\overline{r}^*) \circ id_{E_1(\overline{X})} \otimes s^* \otimes id_{E_2(\overline{X})} \circ E_1(r) \otimes id_{E_2(\overline{X})} \\
&= id_{E_1(\overline{X})} \otimes E_2(\overline{r}'^*) \circ id_{E_1(\overline{X})} \otimes s^* \otimes id_{E_2(\overline{X}')} \circ E_1(r) \otimes id_{E_2(\overline{X}')} \circ E_2(u).
\end{aligned}$$

$$id_{E_1(\overline{X}')} \otimes E_2(\overline{r}'^*) \circ id_{E_1(\overline{X}')} \otimes s^* \otimes id_{E_2(\overline{X}')} \circ E_1(r') \otimes id_{E_2(\overline{X}')}$$
$$= E_1(u) \circ id_{E_1(\overline{X})} \otimes E_2(\overline{r}'^*) \circ id_{E_1(\overline{X})} \otimes s^* \otimes id_{E_2(\overline{X}')} \circ E_1(r) \otimes id_{E_2(\overline{X}')}.$$

These identities can easily be seen from the graphical representation. For instance the first equality is represented by



If we define $a : E_2(\overline{X}') \to E_1(\overline{X})$ by

$$a = id_{E_1(\overline{X})} \otimes E_2(\overline{r}'^*) \circ id_{E_1(\overline{X})} \otimes s^* \otimes id_{E_2(\overline{X}')} \circ E_1(r) \otimes id_{E_2(\overline{X}')},$$

we see that

$$[\overline{X}, t] - [\overline{X}', t'] = [\overline{X}, a \circ E_2(u)] - [\overline{X}', E_1(u) \circ a].$$

This is clearly an element of $I(E_1, E_2)$. We would like to define the $*$-operation on $A(E_1, E_2)$ as the arrow that makes the diagram

$$\begin{array}{ccc} A_0(E_1,E_2) & \xrightarrow{\gamma} & A(E_1,E_2) \\ \star \downarrow & & \vdots\, * \\ A_0(E_1,E_2) & \xrightarrow[\gamma]{} & A(E_1,E_2) \end{array}$$

commutative. This map is only well defined if the composition $\gamma \circ \star : A_0(E_1, E_2) \to A(E_1, E_2)$ maps the ideal $I(E_1, E_2)$ to zero. Choose $X, Y \in \mathcal{C}$, $s : X \to Y$, $a : E_2(Y) \to E_1(X)$ and conjugates $(\overline{X}, r_X, \overline{r}_X)$ and $(\overline{Y}, r_Y, \overline{r}_Y)$.

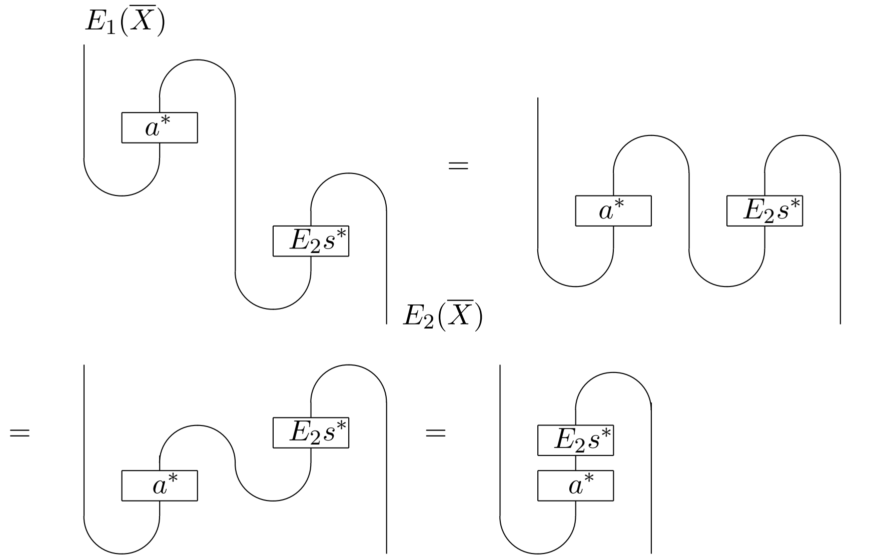
$$[X, a \circ E_2(s)]^* - [Y, E_1(s) \circ a]^*$$
$$= [\overline{X}, id_{E_1(\overline{X})} \otimes E_2(\overline{r}_X^*) \circ id_{E_1(\overline{X})} \otimes (a \circ E_2(s))^* \otimes id_{E_2(\overline{X})} \circ E_1(r_X) \otimes id_{E_2(\overline{X})}]$$
$$- [\overline{Y}, id_{E_1(\overline{Y})} \otimes E_2(\overline{r}_Y^*) \circ id_{E_1(\overline{Y})} \otimes (E_1(s) \circ a)^* \otimes id_{E_2(\overline{Y})} \circ E_1(r_Y) \otimes id_{E_2(\overline{Y})}]$$
$$= [\overline{X}, \tilde{a} \circ E_2(\tilde{s})] - [\overline{Y}, E_1(\tilde{s}) \circ \tilde{a}]$$

where $\tilde{a} : E_2(\overline{Y}) \to E_1(\overline{X})$ and $\tilde{s} : \overline{X} \to \overline{Y}$ are defined by

$$\tilde{a} = id_{E_1(\overline{X})} \otimes E_2(\overline{r}_X^*) \circ id_{E_1(\overline{X})} \otimes a^* \otimes id_{E_2(\overline{Y})} \circ E_1(r_X) \otimes id_{E_2(\overline{Y})}$$

$$\tilde{s} = id_{\overline{Y}} \otimes \overline{r}_X^* \circ id_{\overline{Y}} \otimes s^* \otimes id_{\overline{X}} \circ r_Y \otimes id_{\overline{X}}.$$

Again, diagrams present the easiest way to see the calculations. The calculation showing that $[X,t]^\star = [\overline{X}, \tilde{a} \circ E_2(\tilde{s})]$ is given below as an example.



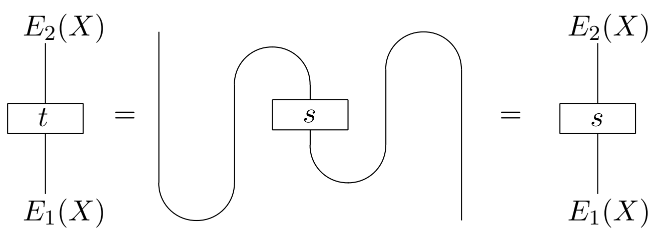
Note that in the last step the conjugate equations were used. Thusfar we have proven that the $*$-operation on $A(E_1, E_2)$ is well defined and does not depend on the choice of the standard conjugates that are used in the definition. It is easy to check that we have

$$[X, \lambda s]^* = \overline{\lambda}[X,s]^*, \quad [X, s+t] = [X,s] + [X,t], \quad \lambda \in \mathbb{C}.$$

It remains to be checked that the $*$-operation is involutive, anti-multiplicative and positive. By anti-multiplicative we mean that

$$([X,s] \cdot [Y,t])^* = [Y,t]^* \cdot [X,s]^*, \quad [X,s],[Y,t] \in A(E_1,E_2).$$

Checking this property is straightforward if one sticks to the standard conjugate for $X \otimes Y$ as given in Lemma 3.3.11. The next diagram shows that the $*$-operation is involutive if we write $([X,s]^*)^* = [X,t]$



Note that we used the conjugate equations twice. We end by proving that the $*$-involution is positive. Pick a $[X,s] \in A(E_1,E_2)$ and a conjugate

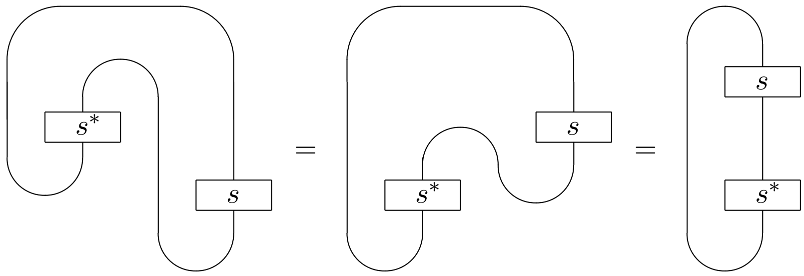
$(\overline{X}, r, \overline{r})$ for X. We have $[X,s]^* \cdot [X,s] = [\overline{X} \otimes X, t]$, where $t : E_2(\overline{X} \otimes X) \to E_1(\overline{X} \otimes X)$ is given by

$$t = d^1_{\overline{X} \otimes X} \circ id_{E_1(\overline{X})} \otimes E_2(\overline{r}^*) \circ id_{E_1(\overline{X})} \otimes s^* \otimes id_{E_2(\overline{X})} \circ E_1(r) \otimes id_{E_2(\overline{X})} \otimes s \circ (d^2_{\overline{X},X})^{-1}.$$

Using the arrows $r : \overline{X} \otimes X \to \mathbb{1}$ and $E_1(r^*) \circ t : E_2(\overline{X} \otimes X) \to E_1(\mathbb{1})$ we can see that

$$[\overline{X} \otimes X, t] = [\overline{X} \otimes X, E_1(r) \circ E_1(r^*) \circ t] = [\mathbb{1}, E_1(r^*) \circ t \circ E_2(r)].$$

The latter identity is equal to $[\mathbb{1}, E_1(r^*) \circ id_{E_1(\overline{X})} \otimes (s \circ s^*) \circ E_1(r)]$ as is shown in the diagram below.



Define $u = id_{E_1(\overline{X})} \otimes s^* \circ E_1(r)$, then the previous calculation gives us

$$[X,s]^* \cdot [X,s] = [\overline{X} \otimes X, t] = [\mathbb{1}, u^* u].$$

This is equal to 0 if and only if $u = 0$. Applying the conjugate equations once more reveals that this is equivalent with $s = 0$. This proves positivity, provided that every element of $A(E_1, E_2)$ can be written in the form $[X,s]$. Now consider an arbitrary element $\sum_i [X_i, s_i]$ of $A(E_1, E_2)$. Pick isometries $v_i : X_i \to X$ such that $\sum_i v_i \circ v_i^* = id_X$ and $X = \bigoplus_i X_i$.

$$[X_i, s_i] = [X_i, E_1(v_i^*) \circ E_1(v_i) \circ s_i] = [X, E_1(v_i) \circ s_i \circ E_2(v_i^*)].$$

Therefore every element of $A(E_1, E_2)$ is of the form $[X,s]$ as we can write

$$\sum_i [X_i, s_i] = [X, \sum_i E_1(v_i) \circ s_i \circ E_2(v_i^*)] = [X, t].$$

□

Proposition 3.5.3. *Let $\mathcal{C}$ be a TC* category and let $E_i : \mathcal{C} \to \mathcal{H}_f$, $i \in \{1,2\}$ be $*$-preserving fiber functors. Then there is a C^*-norm on $A(E_1, E_2)$ defined by*

$$\|a\| = \inf_{b, \gamma(b) = a} \sup_{X \in \mathcal{C}} \|b_X\|_X$$

where the infinum is taken over the representers $b \in A_0(E_1, E_2)$ of a and $\|\cdot\|_X$ denotes the C^*-norm for linear maps between the Hilbert spaces $E_2(X) \to E_1(X)$

Proof. In the standard introductory textbooks on functional analysis one finds the proof that on the space of bounded operators on a Hilbert space $\mathcal{B}(H)$, the operator norm defines a C^*-norm. That is, a submultiplicative norm $\|\cdot\|$, such that $\|A^*A\| = \|A\|^2$ holds for every $A \in H$. Exactly in the same way one can prove that the operator norm on $\mathcal{B}(H, K)$, the space of bounded linear maps between the Hilbert spaces $H \to K$ defines a C^*-norm. The only difference in the proof is that one has to label the inner products to keep track of which Hilbert space they refer to. Looking only at finite dimensional Hilbert spaces, and the spaces $E_1(X)$ and $E_2(X)$ in particular, this means that there is a C^*-norm $\|\cdot\|_X$ that we can use. The norm $\sup_{X\in\mathcal{C}} \|b_X\|_X$ on $A_0(E_1, E_2)$ is submultiplicative because the maps $d^i_{X,Y}$ are unitary. Define $u = d^1_{X,Y} \circ s \otimes t \circ (d^2_{X,Y})^{-1}$, then in $A_0(E_1, E_2)$

$$\|[X, s] \cdot [Y, t]\| = \|[X \otimes Y, u]\| = \|u\|_{X\otimes Y} \leq \|s\|_X \|t\|_Y = \|[X, s]\| \cdot \|[Y, t]\| \,.$$

We pass from $A_0(E_1, E_2)$ onto the quotient algebra by taking the infinum over all representers. Using the properties of the norm on $A_0(E_1, E_2)$, it is straightforward to check that this construction in general ensures that the map $\|\cdot\|$ on $A(E_1, E_2)$ has all the properties of a norm except maybe nondegeneracy. We need to check that $\|a\| = 0$ implies that $a = 0$. First note that $\|[X, s]\| = \|s\|_X$ and that in the proof of the previous proposition we showed that every element of $A(E_1, E_2)$ can be written as some $[X, s]$. This immediatly implies nondegeneracy but also shows that the submultiplicative property of the norm on $A_0(E_1, E_2)$ is carried over to $A(E_1, E_2)$. Using the notation of the proof of the previous proposition we can also see the C*-condition on the norm

$$\|[X, s]^*[X, s]\| = \|[\mathbb{1}, u * u]\| = \|u\|^2_{\mathbb{1}} = \|s\|^2_X = \|[X, s]\|^2 \,.$$

□

Definition 3.5.4. *Let $\mathcal{C}$ be a TC* and let $E_i : \mathcal{C} \to \mathcal{H}$, $i \in \{1, 2\}$ be $*$-preserving fiber functors. Then $\mathcal{A}(E_1, E_2)$ denotes the $\|\cdot\|$-completion of $A(E_1, E_2)$. This is a unital C^*-algebra.*

We are almost ready to establish a connection between $Nat(E_1, E_2)$ and $A(E_1, E_2)$. We only need one more proposition, which by the way is interesting in its own right.

Proposition 3.5.5. *Let $\mathcal{C}$ be a TC* category, $\mathcal{D}$ a strict tensor category and $E_1, E_2 : \mathcal{C} \to \mathcal{D}$ be strict tensor functors. Then every monoidal natural transformation is a natural isomorphism.*

Proof. Let $\alpha : E_1 \to E_2$ be a monoidal natural transformation. For each object X of $\mathcal{C}$ we construct an arrow $\beta_X : E_2(X) \to E_1(X)$ such that β_X is a two sided inverse for α_X. Let $(\overline{X}, r, \overline{r})$ be a conjugate for X. Define β_X by

$$\beta_X = (id_{E_1(X)} \otimes E_2(r^*)) \circ (id_{E_1(X)} \otimes \alpha_{\overline{X}} \otimes id_{E_2(X)}) \circ (E_1(\overline{r}) \otimes id_{E_2(X)}).$$

Because α is assumed to be monoidal we have the identities

$$E_1(r^*) = E_2(r^*) \circ (\alpha_{\overline{X}} \otimes \alpha_X), \quad (\alpha_X \otimes \alpha_{\overline{X}}) \circ E_1(\overline{r}) = E_2(\overline{r}).$$

These are used in the following computations which show that β_X is a two sided inverse for α_X.

$$\begin{aligned}\beta_X \circ \alpha_X &= id_{E_1(X)} \otimes E_2(r^*) \circ id_{E_1(X)} \otimes \alpha_{\overline{X}} \otimes id_{E_2(X)} \circ E_1(\overline{r}) \otimes id_{E_2(X)} \circ \alpha_X \\ &= id_{E_1(X)} \otimes E_2(r^*) \circ id_{E_1(X)} \otimes \alpha_{\overline{X}} \otimes \alpha_X \circ E_1(\overline{r}) \otimes id_{E_1(X)} \\ &= id_{E_1(X)} \otimes E_2(r^*) \circ E_2(\overline{r}) \otimes id_{E_1(X)} = id_{E_1(X)}.\end{aligned}$$

$$\begin{aligned}\alpha_X \circ \beta_X &= \alpha_X \circ id_{E_1(X)} \otimes E_2(r^*) \circ id_{E_1(X)} \otimes \alpha_{\overline{X}} \otimes id_{E_2(X)} \circ E_1(\overline{r}) \otimes id_{E_2(X)} \\ &= id_{E_2(X)} \otimes E_2(r^*) \circ \alpha_X \otimes \alpha_{\overline{X}} \otimes id_{E_2(X)} \circ E_1(\overline{r}) \otimes id_{E_2(X)} \\ &= id_{E_2(X)} \otimes E_2(r^*) \circ E_2(\overline{r}) \otimes id_{E_2(X)} = id_{E_2(X)}.\end{aligned}$$

□

Recall that for fiber functors $E_i : \mathcal{C} \to Vect_{\mathbb{C}}$ the natural transformations are given by

$$\begin{aligned}Nat(E_1, E_2) = \{(\alpha_X)_{X \in \mathcal{C}} \in \prod_{X \in \mathcal{C}} Hom(E_1(X), E_2(X))| \\ E_2(s) \circ \alpha_X = \alpha_Y \circ E_1(s), \forall s : X \to Y\}\end{aligned}$$

We can define a pairing between $A_0(E_1, E_2)$ and the space of natural transformations $Nat(E_1, E_2)$. Let α be an element of $Nat(E_1, E_2)$ and $a = \sum_{i=1}^n [X_i, a_i]$ be in $A_0(E_1, E_2)$, then define

$$\langle \alpha, a \rangle = \sum_{i=1}^n Tr_{E_1(X_i)}(a_i \circ \alpha_{X_i})$$

Proposition 3.5.6. *Let $\mathcal{C}$ be a TC* category and let $E_i : \mathcal{C} \to Vect_{\mathbb{C}}$, $i \in \{1, 2\}$ be fiber functors. Then the pairing given above descends to a pairing between $Nat(E_1, E_2)$ and the quotient algebra $A(E_1, E_2)$ such that $Nat(E_1, E_2) \cong A(E_1, E_2)^*$. Any element $a \in A(E_1, E_2)^*$ corresponds to a monoidal natural transforation if and only if a is multiplicative.*

Proof. Recall that $A_0(E_1, E_2)$ was defined as the direct sum of the vector spaces $Hom(E_2(X), E_1(X))$. The dual to this direct sum is the direct product vector space $\prod_{X\in\mathcal{C}} Hom(E_2(X), E_1(X))^*$. Note that the pairing

$$Hom(E_2(X), E_1(X)) \times Hom(E_1(X), E_2(X)) \quad (s,t) \mapsto Tr_{E_1(X)}(s \circ t)$$

is nondegenerate. This provides us with a nondegenerate pairing between $A_0(E_1, E_2)$ and $\prod_{X\in\mathcal{C}} Hom(E_1(X), E_2(X))$ from which we can conclude

$$A_0(E_1, E_2)^* \cong \prod_{X\in\mathcal{C}} Hom(E_1(X), E_2(X)).$$

As we know, the algebra $A(E_1, E_2)$ is the quotient of $A_0(E_1, E_2)$ with the 2-sided ideal $I(E_1, E_2)$. The dual space $A(E_1, E_2)^*$ corresponds precisely with those elements of $A_0(E_1, E_2)^*$ that are identically 0 on $I(E_1, E_2)$. Assume that $\alpha = (\alpha_X)_{X\in\mathcal{C}} \in \prod_{X\in\mathcal{C}} Hom(E_1(X), E_2(X))$ satisfies $\langle \alpha, a\rangle = 0$ for all $a \in I(E_1, E_2)$. This is equivalent to that statement

$$\langle \alpha, [X, b \circ E_2(s)] - [Y, E_1(s) \circ b]\rangle = 0, \quad s : X \to Y, b : E_2(Y) \to E_1(X).$$

By the definition of the pairing and using the cyclic property of the trace this is equivalent to

$$Tr_{E_1(X)}(b \circ E_2(s) \circ \alpha_X) = Tr_{E_1(X)}(b \circ \alpha_Y \circ E_1(s)),$$

holds for all $s : X \to Y$ and $b : E_2(Y) \to E_1(X)$. By nondegeneracy of the trace this can only hold for each $b : E_2(Y) \to E_1(X)$ if

$$E_2(s) \circ \alpha_X = \alpha_Y \circ E_1(s) \quad \forall s : X \to Y.$$

This proves that $Nat(E_1, E_2) \cong A(E_1, E_2)^*$. It remains to show that the monoidal natural transformations correspond to the multiplicative elements of $A(E_1, E_2)^*$. Let $\phi = \langle \alpha, \cdot\rangle \in A(E_1, E_2)^*$ be multiplicative. This means that

$$\phi([X, s] \cdot [Y, t]) = \phi([X, s]) \cdot \phi([Y, t]) \;\; \forall [X, s], [Y, t] \in A(E_1, E_2).$$

Equivalently

$$\langle \alpha, [X, s] \cdot [Y, t]\rangle = \langle \alpha, [X, s]\rangle \langle \alpha, [Y, t]\rangle \quad \forall [X, s], [Y, t] \in A(E_1, E_2).$$

Using the definition of the pairing and some properties of the trace we find

$$\begin{aligned} &Tr_{E_1(X\otimes Y)}(d^1_{X,Y}(s \otimes t)(d^2_{X,Y})^{-1}\alpha_{X\otimes Y}) \\ &\quad = Tr_{E_1(X)}(s \circ \alpha_X)Tr_{E_1(Y)}(t \circ \alpha_Y) \\ &\quad = Tr_{E_1(X)\otimes E_1(Y)}((s \circ \alpha_X) \otimes (t \circ \alpha_Y)) \\ &\quad = Tr_{E_1(X)\otimes E_1(Y)}((s \otimes t) \circ (\alpha_X \otimes \alpha_Y)). \end{aligned}$$

Recognizing that the left hand side is equal to

$$Tr_{E_1(X)\otimes E_1(Y)}((s\otimes t)\circ(d^2_{X,Y})^{-1}\circ\alpha_{X\otimes Y}\circ d^1_{X,Y}),$$

by repeated application of the cyclic property of the trace, nondegeneracy of the trace gives us that the above equeality can only hold for each $s\otimes t$ if

$$\alpha_{X\otimes Y}=d^2_{X,Y}\circ\alpha_X\otimes\alpha_Y\circ(d^1_{X,Y})^{-1},\ \ \forall X,Y\in Obj(\mathcal{C}).$$

The map ϕ should send the multiplicative unit $[\mathbb{1},e^1\circ(e^2)^{-1}]$ to 1. This is equivalent to $Tr_{E_2(\mathbb{1})}(\alpha_{\mathbb{1}}\circ e^1\circ(e^2)^{-1})=1$. This in turn is equivalent with $\alpha_{\mathbb{1}}\circ e^1=e^2$. This shows that $\alpha\in Nat(E_1,E_2)$ satisfies all demands in Definition 3.1.5, and is therefore monoidal. □

Proposition 3.5.7. *Let $\mathcal{C}$ be a TC* category and let $E_i:\mathcal{C}\to\mathcal{H}_f$, $i\in\{1,2\}$ be $*$-preserving fiber functors. Then a monoidal natural transformation $\alpha\in Nat(E_1,E_2)$ is unitary if and only if the corresponding $\phi\in A(E_1,E_2)^*$ is a character. This means that ϕ is linear, multiplicative and $*$-preserving in the sense that $\phi(a^*)=\overline{\phi}(a)$.*

Proof. We only need to check that ϕ is $*$-preserving. Pick $[X,s]\in A(E_1,E_2)$ and a monoidal $\alpha\in Nat(E_1,E_2)$. Using $\overline{Tr}(AB)=Tr(A^*B^*)$ we find

$$\phi([X,s])=Tr_{E_1(X)}(s\circ\alpha_X)\quad\overline{\phi}([X,s])=Tr_{E_2(X)}(s^*\circ\alpha^*_X).$$

$$\begin{aligned}
&\phi([X,s]^*)\\
&\ =\left\langle\alpha,[\overline{X},id_{E_1(\overline{X})}\otimes E_2(\overline{r}^*)\circ id_{E_1(\overline{X})}\otimes s^*\otimes id_{E_2(\overline{X})}\circ E_1(r)\otimes id_{E_2(\overline{X})}]\right\rangle\\
&\ =Tr_{E_1(\overline{X})}(id_{E_1(\overline{X})}\otimes E_2(\overline{r}^*)\circ id_{E_1(\overline{X})}\otimes s^*\otimes id_{E_2(\overline{X})}\circ E_1(r)\otimes id_{E_2(\overline{X})}\circ\alpha_{\overline{X}})\\
&\ =E_2(\overline{r}^*)\circ s^*\otimes\alpha_{\overline{X}}\circ E_1(\overline{r})=E_2(\overline{r}^*)\circ(\alpha_X\circ\alpha_X^{-1}\circ s^*)\otimes\alpha_{\overline{X}}\circ E_1(\overline{r})\\
&\ =E_2(\overline{r}^*)\circ\alpha_X\otimes\alpha_{\overline{X}}\circ(\alpha_X^{-1}\circ s^*)\otimes id_{E_1(\overline{X})}\circ E_1(\overline{r})\\
&\ =E_1(\overline{r}^*)\circ(\alpha_X^{-1}\circ s^*)\otimes id_{E_1(\overline{X})}\circ E_1(\overline{r})\\
&\ =Tr_{E_1(X)}(\alpha_X^{-1}\circ s^*)=Tr_{E_2(X)}(s^*\circ\alpha_X^{-1})
\end{aligned}$$

The second equality is by definition of the pairing. The third equality follows from the definition of the trace as in in Proposition 3.3.12. If the reader wants to verify that this definition of the trace coincides with the usual trace in $Vect_{\mathbb{C}}$ note that the trace is independent of the particular choice of standard conjugates and that taking the standard conjugates $r:\mathbb{C}\to X^\vee\otimes X$ and $\overline{r}:\mathbb{C}\to X\otimes X^\vee$ defined by $r(1)=\sum_i e_i^\vee\otimes e_i$ and $\overline{r}(1)=\sum_i e_i\otimes e_i^\vee$ leads to the usual notion of trace. In the fourth equality we used that α is invertible, which holds by Proposition 3.5.5. In the sixth equality we used that α is a monoidal natural transformation. The above calculation gives us

$$\phi([X,s]^*)=\overline{\phi}([X,s])\ \ \forall[X,s]\quad\Leftrightarrow\quad\alpha^*_X=\alpha_X^{-1}\ \ \forall X\in\mathcal{C}.$$

□

3.6 Uniqueness of Fiber Functors

Combining Proposition 3.5.6 and Proposition 3.5.7 tells us that for two $*$-preserving fiber functors $E_i : \mathcal{C} \to \mathcal{H}_f$, $i \in \{1,2\}$ there is a unitary monoidal natural isomorphism $\alpha : E_1 \to E_2$ if $A(E_1, E_2)^*$ contains a character. If the fiber functors are symmetric we can say more.

Proposition 3.6.1. *Let $\mathcal{C}$ be an STC* category and let $E_i : \mathcal{C} \to Vect_{\mathbb{C}}$, $i \in \{1,2\}$ be symmetric fiber functors. Then $A(E_1, E_2)$ is commutative.*

Proof. Suppose that

$$E_i(c_{X,Y}) \circ d^i_{X,Y} = d^i_{X,Y} \circ \Sigma_{E_i(X),E_i(Y)}.$$

Pick $[X,s]$,$[Y,t] \in A_0(E_1,E_2)$ with $s : E_2(X) \to E_1(X)$ and $t : E_2(Y) \to E_1(Y)$.

$$[X,s] \cdot [Y,t] = [X \otimes Y, d^1_{X,Y} \circ s \otimes t \circ (d^2_{X,Y})^{-1}].$$

$$\begin{aligned}
[Y,t] \cdot [X,s] &= [Y \otimes X, d^1_{Y,X} \circ t \otimes s \circ (d^2_{Y,X})^{-1}] \\
&= [Y \otimes X, d^1_{Y,X} \circ \Sigma_{E_1(X),E_1(Y)} \circ s \otimes t \circ \Sigma_{E_2(Y),E_2(X)} \circ (d^2_{Y,X})^{-1}] \\
&= [Y \otimes X, E_1(c_{X,Y}) \circ d^1_{X,Y} \circ s \otimes t \circ (d^2_{X,Y})^{-1} \circ E_2(c_{Y,X})]
\end{aligned}$$

If we define $X' = X \otimes Y$, $Y' = Y \otimes X$, $s' = c_{X,Y} : X' \to Y'$ and

$$a' = d^1_{X,Y} \circ s \otimes t \circ (d^2_{X,Y})^{-1} \circ E_2(c_{Y,X}) : E_2(Y') \to E_1(X')$$

we can see that

$$[X,s] \cdot [Y,t] = [X', a' \circ E_2(s')], \quad [Y,t] \cdot [X,s] = [Y', E_1(s') \circ a'].$$

The difference is clearly in $I(E_1, E_2)$. The canonical map γ maps the difference to 0. □

Thus the algebra $\mathcal{A}(E_1, E_2)$ from Definition 3.5.4 is a commutative unital C*-algebra. As we will show this implies that there are a lot of characters in $\mathcal{A}(E_1, E_2)^*$. More precisely, a compact Hausdorff space full of characters. Restricting the characters to $A(E_1, E_2)$, this gives characters in $A(E_1, E_2)^*$ proving the existence of a unitary monoidal natural isomorphism. In short, if we show that $\mathcal{A}(E_1, E_2)^*$ contains at least 1 character, then we have proven the following theorem.

Theorem 3.6.2. *Let $\mathcal{C}$ be a STC* category and let $E_1 : \mathcal{C} \to \mathcal{H}_f$, $E_2 : \mathcal{C} \to \mathcal{H}_f$ be $*$-preserving symmetric fiber functors. Then there exists a unitary monoidal natural transformation $\alpha : E_1 \to E_2$. In particular, the Tannaka groups G_{E_i} obtained from these fiber functors are isomorphic.*

The material in this section is based on Pedersen [32]. The only aim is to prove the next theorem, so the hurried reader may skip the rest of this section. This theorem is important because of two reasons. First of all it completes the proof of the uniqueness claim in Theorem 3.6.2 as it shows the existence of characters in $A(E_1, E_2)^*$. It also plays an important role in the proof of Proposition 3.4.4.

Let $\mathcal{A}$ be a commutative unital C*-algebra and $\mathcal{A}^*$ be its Banach space dual. Define

$$P(\mathcal{A}) = \{\phi \in \mathcal{A}^* | \phi(1) = 1, \|\phi\| \leq 1\},$$

$$X(\mathcal{A}) = \{\phi \in \mathcal{A}^* | \phi(1) = 1, \phi(ab) = \phi(a)\phi(b), \phi(a^*) = \overline{\phi}(a), \forall a, b \in \mathcal{A}\}.$$

$P(\mathcal{A})$ and $X(\mathcal{A})$ are equipped with the w*-topology. A net $(\phi_\lambda)_{\lambda\in\Lambda}$ converges with respect to this topology to ϕ if and only if for each $a \in \mathcal{A}$ the net $(\phi_\lambda(a))_{\lambda\in\Lambda}$ converges to $\phi(a)$.

Theorem 3.6.3. *Let $\mathcal{A}$ be a commutative unital C*-algebra and $\mathcal{A}^*$ its Banach space dual. Let $P(\mathcal{A})$ and $X(\mathcal{A})$ be as above, then*

1. *$X(\mathcal{A}) \subset P(\mathcal{A})$.*
2. *$X(\mathcal{A})$ is compact with respect to the w*-topology on $P(\mathcal{A})$.*
3. *The map $\mathcal{A} \to C(X(\mathcal{A}))$ given by $a \mapsto ev_a$, where $ev_a(\phi) = \phi(a)$, is an isomorphism of C*-algebras.*
4. *The convex hull of $X(\mathcal{A})$ is w*-dense in $P(\mathcal{A})$.*

We will prove these four claims one by one and explain the new terminology that is used. The first claim does not require much preparation.

Definition 3.6.4. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. The spectrum of a, $\sigma(a)$ is defined by*

$$\sigma(a) = \{\alpha \in \mathbb{C} | a - \alpha \textit{ is not invertible}\}.$$

The spectral radius of a, $r(a)$ is defined as $r(a) = sup\{|\alpha| \, | \alpha \in \sigma(a)\}$.

On multiple occasions we will need the following proposition. A proof of it can be found in Pedersen [32], Section 4.1.

Proposition 3.6.5. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$. Then the spectrum $\sigma(a)$ of a is a compact nonempty subset of $\mathbb{C}$ and the spectral radius of a is the limit of the convergent sequence $(\|a^n\|^{1/n})_{n\in\mathbb{N}}$.*

Corollary 3.6.6. *Let $\mathcal{A}$ be a unital Banach algebra and $a \in \mathcal{A}$ be self-adjoint, i.e. $a = a^*$. Then $\|a\| = r(a)$.*

Proof. If $a = a^*$, then $\|a^2\| = \|a^*a\| = \|a\|^2$. This implies that for each $n \in \mathbb{N}$, $\|a^{2^n}\| = \|a\|^{2^n}$. By the previous proposition

$$r(a) = \lim_{n\to\infty} \|a^n\|^{1/n} = \lim_{n\to\infty} \left\|a^{2^n}\right\|^{1/2^n} = \|a\|.$$

This proves the claim. □

We can prove the first claim of Theorem 3.6.3.

Proof. Let $x \in \mathcal{A}$ and $\phi \in X(\mathcal{A})$. First note that $\sigma(\phi(x)) \subset \sigma(x)$. Suppose that this does not hold and take a $\lambda \in \sigma(\phi(x))$, $\lambda \notin \sigma(x)$. This implies that $\phi(x)-\lambda = \phi(x-\lambda)$ is not invertible but $x-\lambda$ is invertible. Denote the inverse of $x-\lambda$ by b. As ϕ is a character we have $1 = \phi(1) = \phi(b(x-\lambda)) = \phi(b)\phi(x-\lambda)$ which is clearly a contradiction. This proves that $\sigma(\phi(x)) \subset \sigma(x)$. By definition of the spectral radius we have $r(\phi(x)) \leq r(x)$. Consequently

$$\|\phi(a)\|^2 = \|\phi(a^*a)\| = r(\phi(a^*a)) \leq r(a^*a) = \|a^*a\| = \|a\|^2.$$

Here we used the previous corollary for the self-adjoint operator a^*a. This proves the claim that $X(\mathcal{A}) \subset P(\mathcal{A})$. □

The second claim of Theorem 3.6.3 requires more preparation. We will work to proving Alaoglu's theorem which shows that the unit ball in $\mathcal{A}^*$ is w*-compact. By 3.6.3(1) the set $X(\mathcal{A})$ is a subset of $P(\mathcal{A})$, which is a closed subset of the unit ball. The set of characters $X(\mathcal{A})$ is closed with respect to the w*-topology and $P(\mathcal{A})$ is a compact Hausdorff space with respect to this topology. This implies that $X(\mathcal{A})$ is also w* compact. There is a lot of terminology in this section that was not used in the previous sections. For the sake of completeness, and at the risk of boring the reader, we give the necessary terminology in order to talk about w* topologies and convergence of nets.

Definition 3.6.7. *A subbasis S for a topology on a set X is a collection of subsets of X, whose union equals X. The topology generated by the subbasis S is defined to be the collection of all unions of finite intersections of S. Let $\mathcal{F}$ be a family of functions, $f : X \to Y_f$ where Y_f has a topology $\mathcal{T}_f$ and X is a set. We construct the coarsest topology that makes all the functions of $\mathcal{F}$ continuous. We take the topology that has as a subbasis $\{f^{-1}(A) | A \in \mathcal{T}_f, f \in \mathcal{F}\}$. This topology is called the initial topology induced by $\mathcal{F}$. Now take a normed space X. This space has a dual X^* and a corresponding pairing $X \times X^* \to \mathbb{C}$, $(x, \phi) \mapsto \langle x, \phi\rangle = \phi(x)$. The space X can be viewed as a separating space of functionals on X^*. By separating we mean that if $\phi, \psi \in X^*$ and $\phi \neq \psi$ then there exists an $x \in X$ such that $\langle x, \phi\rangle \neq \langle x, \psi\rangle$. The initial topology on X^* induced by this family of functionals $\langle x, \cdot\rangle$, $x \in X$ is called the w* topology.*

We will make use of nets in the proof of Alaoglu's theorem. The next two definitions explain the terminology surrounding nets.

Definition 3.6.8. *A net in a space X is a pair (Λ, i) where Λ is a directed set and i is a map from Λ into X. A directed set is a partially ordered set such that for each $\lambda_1, \lambda_2 \in \Lambda$ there is a $\lambda \in \Lambda$ such that $\lambda_1 \leq \lambda$ and $\lambda_2 \leq \lambda$. The standard notation for a net will be $(x_\lambda)_{\lambda \in \Lambda}$, where x_λ denotes $i(\lambda)$. A net can be thought of as a generalization of a sequence. A sequence is a net for which $\Lambda = \mathbb{N}$. A subnet of a net (Λ, i) in X is a net (M, j) in X together with a map $h : M \to \Lambda$ such that $j = i \circ h$ and such that for every $\lambda \in \Lambda$ there is a $\mu(\lambda) \in M$ with $h(\mu) \geq \lambda$ for every $\mu \geq \mu(\lambda)$.*

Definition 3.6.9. *A net $(x_\lambda)_{\lambda \in \Lambda}$ in X is said to be eventually in $Y \subset X$ if there is a $\lambda(Y) \in \Lambda$ such that $x_\lambda \in Y$ for every $\lambda \in \Lambda$ for which $\lambda \geq \lambda(Y)$. A net is frequently in Y if for each $\lambda \in \Lambda$ there is a $\mu \in \Lambda$ with $\mu \geq \lambda$ and $x_\mu \in Y$. A net $(x_\lambda)_{\lambda \in \Lambda}$ in a topological space X is said to converge to $x \in X$ if it is eventually in each neighborhood of x. We denote this by $x_\lambda \to x$. A point $x \in X$ is a accumulation point for a net $(x_\lambda)_{\lambda \in \Lambda}$ in X if the net is frequently in each neighborhood of x. A net $(x_\lambda)_{\lambda \in \Lambda}$ in X is called universal if for every subnet Y of X the net is either eventually in Y or eventually in X/Y. A universal net in a topological space will converge to every one of its accumulation points.*

We need three more results before proving Alaoglu's theorem. The first is a theorem about the existence of universal nets. We do not prove it here as we do not want to get too far away from Tannaka-Krein duality. The interested reader can find a proof in Section 1.3 of Pedersen [32].

Theorem 3.6.10. *Every net $(x_\lambda)_{\lambda \in \Lambda}$ in X has a universal subnet.*

Lemma 3.6.11. *Let X be a topological space, $Y \subset X$ be equipped with the subspace topology and $x \in X$. Then $x \in \overline{Y}$ if and only if for each neighborhood A of x, $A \cap Y \neq \emptyset$ holds.*

Proof. Suppose that $A \cap Y = \emptyset$ for a neighborhood A of x. Without loss of generality we can assume that A is open in X. This implies that X/A is closed and contains Y, yet it does not contain x. This contradicts the definition of the closure of a set. Conversely, if $x \notin \overline{Y}$, then $X/\overline{Y}$ is an open neighborhood of x that is disjoint from Y. □

Theorem 3.6.12. *The following conditions on a topological space X are equivalent:*

1. *Every open covering of X has a finite subcovering*
2. *If Δ is a system of closed subsets of X, such that no intersection of finitely many elements from Δ is empty, then the intersection of all elements in Δ is nonempty.*

3. Every net in X has an accumulation point.

4. Every universal net in X is convergent.

5. Every net in X has a convergent subnet.

Proof. (1)⇒(2). Suppose that as F ranges over Δ, we have that $\cap F = \emptyset$. In that case the system $\{X/F | F \in \Delta\}$ is an open covering of X. By the compactness of (1) there is a finite subcovering $\{X/F_j | 1 \leq j \leq n\}$. This in turn implies that $\cap_{j=1}^{n} F_j = \emptyset$. This is a contradiction.

(2)⇒(3). Let $(x_\lambda)_{\lambda\in\Lambda}$ be a net in X. Take $F_\lambda = \{x_\mu | \lambda \leq \mu\}^-$, where the bar at the end indicates that the closure should be taken. Given $\lambda_1, ..., \lambda_n$ in Λ there is a common majorant λ. From this we can conclude that $F_\lambda \subset F_{\lambda_k}$ holds for every $k \in \{1, ..., n\}$. In particular $\cap_{k=1}^{n} F_{\lambda_k} \neq \emptyset$. By (2) there exists an $x \in \cap_{\lambda\in\Lambda} F_\lambda$. By the previous lemma this implies that for every neighborhood A of x and every $\lambda \in \Lambda$ there are elements x_μ in A with $\mu \geq \lambda$. The net is frequently in each neighborhood of x, implying that x is a accumulation point for the net.

(3)⇒(4), (4)⇒(5) and (5)⇒(3) are easy so we complete the proof by showing that (3)⇒(1). Take U to be an open covering of X. Ordered by inclusion the finite subsets λ of U are a directed set. If no λ covers X there is, if we assume the axiom of choice, a net $(x_\lambda)_{\lambda\in\Lambda}$ such that for every λ

$$x_\lambda \in X/\bigcup_{A\in\lambda} A = \bigcap_{A\in\lambda} X/A.$$

By (3) this net has an accumulation point $x \in X$. For any given A in U and B a neighborhood of x there is therefore a λ such that $\{A\} \leq \lambda$ and $x_\lambda \in B$. In particular $(X/A) \cap B \neq \emptyset$. Since X/A is closed and B is arbitrary, we conclude that $x \in X/A$. As this holds for every $A \in U$ and U is a covering we have a contradiction. □

We can now state and prove Alaoglu's theorem. This theorem completes the proof of part (2) of Theorem 3.6.3.

Theorem 3.6.13. *(Alaoglu's Theorem) For each normed space X, the unit ball $B(X)$ of X^* is w* compact*

Proof. Pick a universal net $(\phi_\lambda)_{\lambda\in\Lambda}$ in $B(X)$. By the previous theorem it suffices to show that this net is convergent. For every $x \in X$ we know that $|\phi_\lambda(x)| \leq \|x\|$. The image net $(\phi_\lambda(x))_{\lambda\in\Lambda}$ is therefore contained in a compact subset of $\mathbb{C}$. The image net is also universal, as the reader can check. By the previous theorem, it converges to a number $\phi(x)$. Using straightforward computations with limits we have, for all $x, y \in X$ and $\alpha \in \mathbb{C}$

$$|\phi(x)| \leq \|x\|, \qquad \phi(x + \alpha y) = \phi(x) + \alpha\phi(y).$$

In this way we constructed a $\phi \in B(X)$ such that $\phi_\lambda \to \phi$ in the w* topology. As the universal net was arbitrary, this completes the proof. □

Proving the third claim in Theorem 3.6.3 requires us to explore the Gelfand transform. Let in the discussion that follows $GL(\mathcal{A})$ denote the invertible elements of the unital Banach algebra $\mathcal{A}$.

Lemma 3.6.14. *Let $\mathcal{A}$ be a unital C*-algebra. If $\mathcal{A}$ is a division ring, then $\mathcal{A} = \mathbb{C}$.*

Proof. If $\mathcal{A}$ is a division ring, then $GL(\mathcal{A}) = \mathcal{A}/\{0\}$. Pick $a \in \mathcal{A}$. By Proposition 3.6.5 the spectrum $\sigma(a)$ is nonempty, so there is a $\lambda \in \sigma(a)$. Hence $\lambda - a \notin GL(\mathcal{A})$. This can only be the case if $\lambda - a = 0$, or equivalently $a = \lambda$. □

Lemma 3.6.15. *Let $\mathcal{A}$ denote a unital Banach algebra and $a \in \mathcal{A}$ such that $\|a\| \leq 1$. Then $1 - a \in GL(\mathcal{A})$ and*

$$(1-a)^{-1} = \sum_{n=0}^{\infty} a^n.$$

Proof. By the submultiplicative property of the norm $\|a^n\| \leq \|a\|^n$ holds and the series $\sum_n a^n$ converges in $\mathcal{A}$ to an element b. As we have $ab = ba = b - 1$ it is easy to check that $b = (1-a)^{-1}$. □

Proposition 3.6.16. *Let $\mathcal{A}$ be a commutative unital Banach algebra. Then there is a bijective correspondence given by $\phi \leftrightarrow ker\phi$ between the set $X(\mathcal{A})$ of characters on $\mathcal{A}$ and the set $M(\mathcal{A})$ of maximal ideals in $\mathcal{A}$. Every $\phi \in X(\mathcal{A})$ is continuous and every $J \in M(\mathcal{A})$ is closed. Finally, we have for each $a \in \mathcal{A}$ that*

$$\sigma(a) = \{\langle a, \phi\rangle \,|\phi \in X(\mathcal{A})\}.$$

Proof. Take a proper ideal $J \subset \mathcal{A}$. The ideal can only be proper if $J \cap GL(\mathcal{A}) = \emptyset$. By Lemma 3.6.15 this can only be the case if $\|1 - a\| > 1$ for every $a \in J$. As a consequence the norm closure of J is still not equal to $\mathcal{A}$. As the closure of an ideal is also an ideal, we can conclude that each maximal ideal is closed.

Pick any $a \in \mathcal{A}/GL(\mathcal{A})$. Because $a \notin GL(\mathcal{A})$, we have that $1 \notin \mathcal{A}a$ so a is contained in some proper ideal $J(a)$. The set of ideals that contain a but not 1 is inductively ordered by inclusion. Applying Zorn's lemma we find that for each $a \in \mathcal{A}/GL(\mathcal{A})$ there is a maximal ideal such that a is contained in it.

Let $J \in M(\mathcal{A})$ and consider the quotient algebra $\mathcal{A}/J$. This algebra has no proper ideals. As such it is a division ring and can, by Lemma 3.6.14, be identified with $\mathbb{C}$. The quotient map $\phi : \mathcal{A} \to \mathcal{A}/J$ is an element of $X(\mathcal{A})$ and is continuous. Conversely, take $\phi \in X(\mathcal{A})$ and consider the ideal $ker(\phi)$. This ideal has co-dimension 1 and is therefore maximal. We already established that a maximal ideal is closed, so $ker(\phi)$ is closed and ϕ

continuous. Thus, we establish the bijective correspondence between $X(\mathcal{A})$ and $M(\mathcal{A})$.

Let $a \in \mathcal{A}$ and $\lambda \in \sigma(a)$. By definition of the spectrum $\lambda - a \notin GL(\mathcal{A})$. From the established bijection we can conclude that there is a $\phi \in X(\mathcal{A})$ such that $\langle \lambda - a, \phi \rangle = 0$, i.e. $\lambda = \langle a, \phi \rangle$. Conversely, let $\langle a, \phi \rangle = \lambda$ for some $\phi \in X(\mathcal{A})$. This is equivalent to $\lambda - a \in ker(\phi)$, from which it follows that $\lambda - a \notin GL(\mathcal{A})$ and $\lambda \in \sigma(a)$. □

Theorem 3.6.17. *(Gelfand Transform) Let $\mathcal{A}$ be a commutative unital Banach algebra. The set of characters $X(\mathcal{A})$ has a compact Hausdorff topology such that the map Γ, where we write $\Gamma(a) = \hat{a}$, defined by*

$$\Gamma(a)(\phi) = \hat{a}(\phi) = \langle a, \phi \rangle, \qquad a \in \mathcal{A}, \quad \phi \in X(\mathcal{A})$$

is a norm decreasing homomorphism of $\mathcal{A}$ onto a subalgebra of $C(X(\mathcal{A}))$ that separates points of $X(\mathcal{A})$. For every $a \in \mathcal{A}$ we have

$$\hat{a}(X(\mathcal{A})) = \sigma(a), \qquad \|\hat{a}\|_{sup} = r(a).$$

Proof. Not much in this theorem is new except for the definition of the Gelfand transform $\Gamma : \mathcal{A} \to C(X(\mathcal{A}))$. The rest of the claims follow from Theorem 3.6.3(2), Proposition 3.6.16 or are easy to check. □

Lemma 3.6.18. *Let $\mathcal{A}$ be a unital C*-algebra and $a \in \mathcal{A}$ a normal element. This means that $a^*a = aa^*$ holds. Then $r(a) = \|a\|$.*

Proof. The proof works in the same way as in Corollary 3.6.6. As $a \in \mathcal{A}$ is normal we have $(a^*a)^m = a^{*m}a^m$. Using th C*-property of the norm

$$\|a\|^{2^n} = \|a^*a\|^{2^{n-1}} = \left\|a^{*2^n}a^{2^n}\right\|^{1/2} = \left\|a^{2^n*}a^{2^n}\right\|^{1/2} = \left\|a^{2^n}\right\|.$$

Applying this calculation to Proposition 3.6.5 completes the proof. □

After this preparation we can prove the third claim of Theorem 3.6.3 in the shape of the next theorem.

Theorem 3.6.19. *Every commutative unital C*-algebra $\mathcal{A}$ is isometrically $*$-isomorphic to $C(X(\mathcal{A}))$, where $X(\mathcal{A})$ is the compact Hausdorff space of characters of $\mathcal{A}$.*

Proof. The C*-algebra is commutative, in particular each element is normal. Combining Lemma 3.6.18 with Theorem 3.6.17 shows that the Gelfand transform is an isometry. We need to show that the Gelfand transform is $*$-preserving and and bijective. Let $a \in \mathcal{A}$ be self adjoint, i.e. $a^* = a$. Then $\sigma(a) \subset \mathbb{R}$ (readers that want to see this proven are directed to Section 4.3 of Pedersen [32]). By Proposition 3.6.16, for any $\phi \in X(\mathcal{A})$ it follows that $\hat{a}(\phi) \in \mathbb{R}$. $\hat{a}$ is self adjoint in $C(X(\mathcal{A}))$. Every element in $a \in \mathcal{A}$ can be written as $a = b + ic$ with b and c self adjoint. Take $b = 1/2(a + a^*)$ and $c = 1/2 \cdot i(a^* - a)$. It follows that

$$\Gamma(a^*) = \Gamma(b - ic) = \Gamma(b) - i\Gamma(c) = \overline{\Gamma(b) + i\Gamma(c)} = \overline{\Gamma(b + ic)} = \overline{\Gamma(a)}.$$

The Gelfand transform is $*$-preserving. By Theorem 3.6.17 the image of the Gelfand transform separates points of $X(\mathcal{A})$. Applying the Stone-Weierstrass Theorem proves surjectivity. Injectivity is left to the reader. □

The following terminology is useful in proving claim (4) of Theorem 3.6.3.

Definition 3.6.20. *A face of a convex subset C of a vector space X is a nonempty, convex subset J of C with the property that $\lambda x + (1 - \lambda)y \in J$ with $\lambda \in (0, 1)$ and $x, y \in C$ implies that $x, y \in J$. An extreme point in C is a one-point face of C. The extremal boundary ∂C of C is the set of extreme points of C.*

The proof of the fourth claim of Theorem 3.6.3 rests on two theorems. Both will be stated without proof as this would take up too much space. The first of these theorems is the Krein-Millman theorem. Proofs of this theorem can be found in Pederson [32] Section 2.5 and Conway [8] Section V.7.

Theorem 3.6.21. *(Krein-Millman) Let X be a vector space, equipped with a topology induced by a separating space of functionals X^* on X. Then for every convex compact subset C of X the convex hull of the extremal boundary ∂C of C is dense in C.*

Using the notation of Theorem 3.6.3, the second theorem that we use shows that the characters $X(\mathcal{A})$ are the extremal points of $P(\mathcal{A})$. Applying the Krein-Millman theorem to the convex set $P(\mathcal{A})$ then directly proves that the finite convex combinations of characters are w* dense in $P(\mathcal{A})$. The second theorem is

Theorem 3.6.22. *Let X be a compact Hausdorff space and equip the Banach space $C(X)$, consisting of the continuous complex valued functions on X, with the supremum norm. Denote the dual space of $C(X)$ by $M(X)$ and define $P(X) = \{\mu \in M(X) | \mu(1) = 1,\ \|\mu\| \leq 1\}$ Then $P(X)$ is a convex w* compact set, whose extremal points are the Dirac measures δ_x, $x \in X$, given by $\delta_x(f) = f(x)$ for every $f \in C(X)$.*

Let us look at how this will help prove Theorem 3.6.3(4). By part (3) of 3.6.3 we can identify $\mathcal{A}$ with $C(X(\mathcal{A}))$. Part (2) of the theorem tells us that $X(\mathcal{A})$ is a compact Hausdorff space. Under this identification $M(X)$ corresponds with $\mathcal{A}^*$ and $P(X)$ with $P(\mathcal{A})$. Let $\phi \in X(\mathcal{A})$ and $a \in \mathcal{A}$, then the Dirac measures δ_ϕ, correspond to the characters $\delta_\phi(a) = \phi(a)$. This concludes the proof of Theorem 3.6.3. A proof of Theorem 3.6.22 can be found in Section 2.5 of Pedersen [32].

3.7 Tannaka-Krein Duality Completed

At the beginning of Section 3.5 we set out to prove Proposition 3.4.4. We will shortly finish that proof. More precisely, right after the next proposition, the proof of 3.4.4 will be completed. The next proposition can be found in Müger et.al. [29]

Proposition 3.7.1. *Let $\mathcal{D}$ be a semisimple $*$-category (for instance a TC* category), $\mathcal{K}$ a $*$-category and $F : \mathcal{D} \to \mathcal{K}$ a $*$-preserving functor. Let I denote a set of pairwise nonisomorphic irreducible elements of $\mathcal{D}$ such that each object of $\mathcal{D}$ is isomorphic to a finite direct sum of elements X_i with $i \in I$. Then there exists an isomorphism*

$$\psi_F : Nat(F) \to \prod_{i \in I} End(F(X_i))$$

of $*$-algebras.

Proof. Define a map $\psi_F : Nat(F) \to \prod_{i \in I} End(F(X_i))$ by $\psi_F(b) = \prod_{i \in I} b_{X_i}$. for all $b \in Nat(F)$. It is clear that ψ_F is a unital $*$-algebra homomorphism. We start by showing that it is injective. Suppose that $\psi_F(b) = 0$. Then $b_{X_i} = 0$ for each X_i. Take $X \in \mathcal{D}$ arbitrary. We must show that $b_X = 0$. First note that for any $i \in I$, $Hom(X_i, X)$ is a Hilbert space with the inner product defined as

$$\langle s, t \rangle \, id_{X_i} = t^* \circ s.$$

Also note that every $s : X_i \to X$ is a scalar multiple of a isometry. Let $s_{i_\alpha} : X_i \to X$, $i \in I$, $\alpha \in \{1, ..., dim(Hom(X_i, X))\}$ be an orthonormal basis with respect to this inner product satisfying $\sum_{i_\alpha} s_{i_\alpha} \circ s_{i_\alpha}^* = id_X$. Hence

$$b_X = b_X \circ F(id_X) = \sum_{i\alpha} b_X \circ F(s_{i_\alpha} \circ s_{i_\alpha}^*) = \sum_{i\alpha} F(s_{i_\alpha}) \circ b_{X_i} \circ F(s_{i_\alpha}^*) = 0.$$

Next we prove surjectivity of ψ_F. Given $(b_i \in EndF(X_i), i \in I)$ we need to construct a $b \in Nat(F)$ such that $\psi_F(b) = \prod_{i \in I} b_i$. Let $X \in \mathcal{D}$ be arbitrary. As before pick a orthonormal basis $s_{i_\alpha} : X_i \to X$ such that $\sum_{i_\alpha} s_{i_\alpha} \circ s_{i_\alpha}^* = id_X$. Define $b_X = \sum_{i\alpha} F(s_{i_\alpha}) \circ b_{X_i} \circ F(s_{i_\alpha}^*)$. We need to show that this construction of b yields a natural transformation. We need to show that for every $u : X \to Y$, we have $F(u) \circ b_X = b_Y \circ F(u)$. To this purpose pick orthogonal isometries $t_{j_\beta} : X_j \to Y$ such that $\sum_{j\beta} t_{j_\beta} \circ t_{j_\beta}^* = id_Y$. Define b_Y as $b_Y = \sum_{j\beta} F(t_{j_\beta}) \circ b_j \circ F(t_{j_\beta}^*)$. Then

$$\begin{aligned} F(u) \circ b_X &= \sum_{i\alpha} F(u \circ s_{i_\alpha}) \circ b_i \circ F(s_{i_\alpha}^*) \\ &= \sum_{i\alpha, j\beta} F(t_{j_\beta} \circ t_{j_\beta}^* \circ u \circ s_{i_\alpha}) \circ b_i \circ F(s_{i_\alpha}^*) \end{aligned}$$

The maps $t_{j_\beta}^* \circ u \circ s_{i_\alpha} : X_i \to X_j$ are 0 unless $i = j$ and if $i = j$ the maps are a scalar multiple of the identity map id_{X_i}. We use this is the next steps of the calculation.

$$F(u) \circ b_X = \sum_{i\alpha,\beta} F(t_{i_\beta} \circ (t_{i_\beta}^* \circ u \circ s_{i_\alpha})) \circ b_i \circ F(s_{i_\alpha}^*)$$
$$= \sum_{i\alpha,j\beta} F(t_{j_\beta}) \circ b_j \circ F(t_{j_\beta}^* \circ u \circ s_{i_\alpha} \circ s_{i_\alpha}^*) = b_Y \circ F(u)$$

This shows that b is natural and proves surjectivity. □

Finally we can prove Proposition 3.4.4 completing the proof of Tannaka-Krein duality for compact groups.

Proof. Let $\mathcal{C}$ be a STC* category and $E : \mathcal{C} \to \mathcal{H}_f$ be a $*$-preserving symmetric fiber functor. The category $\mathcal{C}$ is semisimple and assumed to be essentially small so we have a set I and a family of pairwise nonisomorphic irreducible objects $\{X_i | i \in I\}$ such that each object of $\mathcal{C}$ is isomorphic to a finite direct sum of elements of this family. Take $Nat(E)$ to be the space of natural transformations of E to itself. By the previous proposition we can associate to each $\alpha \in Nat(E)$ an element $(\alpha_i | i \in I) \in \prod_{i \in I} End(E(X_i))$ where $\alpha_i = \alpha_{X_i}$. In this way we obtain an isomorphism of vector spaces

$$\gamma : Nat(E) \to \prod_{i \in I} End(E(X_i)) \quad \alpha \mapsto (\alpha_i)_{i \in I}.$$

Consider the following linear map

$$\delta : \bigoplus_{i \in I} End(E(X_i)) \to \mathcal{A}(E) \quad (a_i)_{i \in I} \mapsto \sum_{i \in I} [X_i, a_i].$$

We know that every element of $A(E)$ can be written in the form $[X, s]$ and every $[X, s]$ can be expressed as a finite sum over $[X_i, s_i]$ with X_i irreducible. This implies that the map δ is surjective. When viewed as a map to $A_0(E) = A_0(E, E)$, δ is also injective. If $i \neq j$ then $Hom(X_i, X_j) = \{0\}$. Consequently, the image in $A_0(E)$ of δ has a trivial intersection with the ideal $I(E, E)$. The map δ is therefore also injective and an isomorphism of vector spaces. Next, pull the C*-norm on $A(E)$ back via δ to obtain the norm

$$\|(a_i)_{i \in I}\| = \sup_{i \in I} \|a_i\|_{End(E(X_i))}$$

on $\bigoplus_{i \in I} End(E(X_i))$. In this way we obtain a isomorphism of the norm closures.

$$\overline{\delta} : \overline{\bigoplus}_{i \in I} End(E(X_i))^{\|\cdot\|} \to \mathcal{A}(E).$$

Recall from Proposition 3.5.6 that we have a pairing $\langle\cdot,\cdot\rangle : Nat(E)\times\mathcal{A}(E)\to\mathbb{C}$. Using the isomorphisms γ and δ we find the following pairing

$$\langle .,.\rangle' : \prod_{i\in I} End(E(X_i)) \times \bigoplus_{i\in I} End(E(X_i)) \to \mathbb{C},$$

$$((\alpha_i)_{i\in I}, (a_i)_{i\in I}) \mapsto \sum_{i\in I} Tr_{E(X_i)}(\alpha_i \circ a_i).$$

This is the bilinear form such that $\langle\cdot,\delta(\cdot)\rangle = \langle\gamma(\cdot),\cdot\rangle'$ holds as maps $Nat(E)\times\bigoplus_{i\in I} End(E(X_i))\to\mathbb{C}$.

Take $\alpha\in Nat(E)$ such that $\gamma(\alpha)\in\prod_{i\in I} End(E(X_i))$ has only finitely many non zero components. In this case $\phi=\langle\alpha,\cdot\rangle$ surely extends to an element of $\mathcal{A}(E)^*$. By Theorem 3.6.3(4) every $\phi\in\mathcal{A}(E)^*$ is in the w* limit of a net $(\phi_\lambda)_{\lambda\in\Lambda}$ in the $\mathbb{C}$-span of the $*$-characters $X(\mathcal{A}(E))$ of $\mathcal{A}(E)$. Therefore every element $(\alpha_i)\in\bigoplus_{i\in I} E(X_i)$ can be w* approximated by a such a net $(\phi_\lambda)_{\lambda\in\Lambda}$

$$lim_{w*}\phi_\lambda = \left\langle\gamma^{-1}((\alpha_i)_{i\in I}),\cdot\right\rangle \in \mathcal{A}(E)^*.$$

Restrict each ϕ_λ to $A(E)$ to obtain a net in $Nat(E)$ that converges to $\gamma^{-1}((\alpha_i))$. We know from Proposition 3.5.6 and Proposition 3.5.7 that the characters $X(\mathcal{A}(E))$ correspond with unitary monoidal transformations of E. The elements of $X(\mathcal{A}(E))$ correspond to elements of G_E as defined in Section 3.4. The reader may verify that this correspondence $X(\mathcal{A}(E))\leftrightarrow G_E$ is a homeomorphism. The easiest way to do this is to compare the notions of convergence in both spaces. $X(\mathcal{A}(E))$ is equipped with the w* topology and convergence of nets is pointwise. By the definition of the product topology on G_E a net $(g_\lambda)_{\lambda\in\Lambda}$ converges if and only if the net $(g_{\lambda,X})_{\lambda\in\Lambda}$ converges for each $X\in\mathcal{C}$. These two notions of convergence coincide when looking at the correspondence $\phi\leftrightarrow g$ as defined in Proposition 3.5.6.

Restrict the discussion to a finite set $S\subset I$. Pick any $\alpha\in\bigoplus_{s\in S} E(X_s)$ and follow the previous discussion in order to conclude that α is in the closure of

$$span_{\mathbb{C}}\{\pi_{s_1}(g)\oplus ...\oplus\pi_{s_{|S|}}(g)|g\in G_E\}\}.$$

□

Having proved Tannaka-Krein duality for compact groups there are a number of directions in one can proceed. We take the following direction in the next chapter. For what STC* categories is it possible to construct a $*$-preserving symmetric fiber functors? We could also ask the following question. Can we pull of a Tannaka-Krein kind of construction using categories with less structure than STC* categories? Looking at the DHR analysis for lower dimensional spacetime, we could well use a duality theorem for BTC* categories.

The reader that wants to go beyond Tannaka-Krein duality for compact groups may find the refereces Joyal and Street [21] and Müger, Roberts and Tuset [29] very useful. These articles are about Tannaka duality for quantum groups rather than compact groups. A different related topic worth mentioning is Tannaka-Krein duality for affine group schemes. This algebraic version of Tannaka duality shows many similarities with Tannaka-Krein duality for compact groups. A lot of the techniques in this thesis are adaptations (and simplifications) of this algebraic Tannaka duality. The interested reader is refered to Deligne and Milne [10], Deligne [9], Bichon [4] and Saavedra Rivano [36] for material on the subject.

Chapter 4

Deligne's Embedding Theorem

In the previous chapter it was shown that for any STC* category with an embedding in the category $\mathcal{H}_f$ in the form of a symmetric $*$-preserving fiber functor $E : \mathcal{C} \to \mathcal{H}_f$, there exists a compact group G, unique up to isomorphism, such that $\mathcal{C}$ is equivalent to $Rep_f(G, \mathbb{C})$ as a symmetric tensor $*$-category. This chapter is concerned with the construction of such an embedding for a given STC* category. This investigation will lead to the proof of Deligne's Embedding theorem which shows to what extent we can realize a suitable embedding for any STC* category. The proof presented here is based on the proof given in Müger [30] appendices B6 through B11. The original proof of this theorem for STC* categories can be found in Doplicher and Roberts [11] [12]. In the article by Müger the embedding theorem is stated in the same generality as in the original work by Doplicher and Roberts. The proofs however, differ. Especially the more algebraic part starting in Section 4.5 differs from the approach of Doplicher and Roberts. Recall that at the end of the previous chapter it was pointed out that there is also an algebraic version of Tannaka-Krein duality. There is also an algebraic counterpart of the embedding theorem fior STC* categories. The approach in Müger [30] is greatly motivated by this algebraic version as he points out in Appendix B8 in [30]. Even the name Deligne's embedding theorem is taken from the algebraic version. Another, more superfacial, difference between Müger's approach and that of Doplicher and Roberts is the terminology which is used. First of all names like supergroups, as defined in Section 4.1, are not used. Secondly the categories under considerations are certain C*-categories. C*-categories are defined as follows.

Definition 4.0.2. *A C*-category is a $\mathbb{C}$-linear category with a positive $*$-operation such that each $Hom_{\mathcal{C}}(X, Y)$ is a Banach space, and*

$$\|t \circ s\|_{Hom(X,Z)} \leq \|s\|_{Hom(X,Y)} \|t\|_{Hom(Y,Z)}, \quad \forall s : X \to Y, t : Y \to Z,$$

$$\|s^* \circ s\|_{End(X)} = \|s\|^2_{Hom(X,Y)}, \quad \forall s : X \to Y.$$

A C-tensor category is a C*-category that is a tensor category for which* $\|s \otimes t\| \leq \|s\| \, \|t\|$ *holds.*

Thus each vector space $End(X)$ of a C*-category is a C*-algebra. The next proposition tells us that every TC* category can be viewed as a C*-tensor category.

Proposition 4.0.3. *Let* $\mathcal{C}$ *be a C*-tensor category with direct sums and an irreducible tensor unit. If* X *and* Y *are objects of* $\mathcal{C}$ *that have conjugates, then* $Hom_{\mathcal{C}}(X,Y)$ *is finite dimensional. Consequently a C*-tensor category with direct sums, conjugates, subobjects and a irreducible tensor unit is a TC*. Conversely, given a TC* category* $\mathcal{D}$*, there are unique norms on the spaces* $Hom_{\mathcal{D}}(X,Y)$ *rendering* $\mathcal{D}$ *a C*-tensor category.*

This proposition, as well as the proof that a C*-tensor category with direct sums, conjugates, subobjects and a irreducible tensor unit is a TC* category can be found in Müger [30]. The other half of the proof, going from a TC* category to a C*-tensor category, can be found in Müger [28]. This concludes our discussion of the differences between the paper by Müger and the original by Doplicher and Roberts.

We start in Section 4.1 with the definition of the twist on a BTC* category. We show that this twist provides an obstruction for the existence of an embedding $E : \mathcal{C} \to \mathcal{H}_f$ and how this obstruction can be overcome by making use of supergroups. Having taken care of the twist, the other sections only have to deal with even STC* categories. These are STC* categories that have a trivial twist. In Section 4.2 it is shown how to construct a symmetric $*$-preserving fiber functor from a symmetric fiber functor. At the end of this section we have reduced the search for a symmetric $*$-preserving fiber functor on a STC* category to the search for a symmetric fiber functor on an even STC*.

Before we can simplify this search any further, some more categorical preparations are in order. The last sections make use of the theory of abelian categories and commutative algebra in abelian tensor categories. Section 4.3 gives some background on abelian categories and Section 4.4 covers the basics on the used commutative categorical algebra. In Section 4.5, two important steps are taken. In the first step colimits are used to reduce the embedding problem to finding an embedding for finitely generated even STC* categories. Categories of this kind can be investigated using algebraic methods. The second step is reducing the problem to finding an abelian category $\hat{\mathcal{C}}$ that has the finitely generated even STC* $\mathcal{C}$ as a full subcategory and an absorbing monoid (Q, m, η) in $\hat{\mathcal{C}}$. Section 4.6 is a study of the permutation symmetry of finitely generated even STC* categories. The results of that section will help in Section 4.7 where we construct the desired absorbing monoid thereby completing the proof of Deligne's embedding theorem.

4.1 Let's Do The Twist

The material in this section is based on sections 3 and 4 of Longo and Roberts [26], appendices A4 and B2 of Müger [30] and another paper by Müger [28]. Let $\mathcal{C}$ be any STC* category. The aim of this chapter is to construct an embedding of $\mathcal{C}$ into the category $\mathcal{H}_f$ of finite dimensional Hilbert spaces. This embedding should be in the form of a symmetric $*$-preserving fiber functor such that $\mathcal{C}$ is, as a tensor $*$-category, equivalent to the category $Rep_f(G, \mathbb{C})$ of a compact group by Tannaka-Krein duality. It will turn out that this is not possible for arbitrary STC* categories. Only the socalled even STC* categories admit a symmetric $*$-preserving fiber functor into $\mathcal{H}_f$. The obstruction for the general case is in the form of the twist which we now define.

Definition 4.1.1. *Let $\mathcal{C}$ be a braided strict tensor category with conjugates. A twist for $\mathcal{C}$ is a natural transformation Θ of the identity functor to itself such that the following holds.*

$$\Theta(X \otimes Y) = \Theta(X) \otimes \Theta(Y) \circ c_{Y,X} \circ c_{X,Y}, \quad X, Y \in Obj\mathcal{C}.$$

The following diagram is commutative.

$$\begin{array}{ccc} \mathbb{1} & \xrightarrow{\ r\ } & \overline{X} \otimes X \\ {\scriptstyle r}\downarrow & & \downarrow{\scriptstyle id \otimes \Theta} \\ \overline{X} \otimes X & \xrightarrow[\Theta \otimes id]{} & \overline{X} \otimes X \end{array}$$

Here $(\overline{X}, r, \overline{r})$ denotes a standard conjugate for X. If $\mathcal{C}$ is a tensor $*$-category the maps $\Theta(X)$ are required to be unitary.

The fact that Θ is a natural transformation of the identity functor is equivalent to the statement that for each arrow $s : X \to Y$ the identity $s \circ \Theta(X) = \Theta(Y) \circ s$ holds.

In the proof of the following proposition it will be convenient to make use of the following mappings which are called standard left inverses[1] in Longo en Roberts [26]. Let X, Y and Z be objects of a BTC* category $\mathcal{C}$, and $(\overline{X}, r, \overline{r})$ a standard conjugate for X. Define the standard left inverse by

$$\phi^X_{Y,Z}(s) = r^* \otimes id_Z \circ id_{\overline{X}} \otimes s \circ r \otimes id_Y, \quad s : X \otimes Y \to X \otimes Z.$$

[1]The name has nothing to do with the definition of conjugates as an adjuntion.

Notice that for $s \in End(X)$ we have $Tr(s) = \phi^X_{\mathbb{1},\mathbb{1}}(s)$. Suppose that X and Y have left inverses ϕ^X and ϕ^Y. If we pick for $X \otimes Y$ the standard conjugate from Lemma 3.3.11, then the left inverses are related by

$$\phi^{X\otimes Y}_{X',Y'}(s) = \phi^Y_{X',Y'}(\phi^X_{Y\otimes X',Y\otimes Y'}(s)), \quad s : X \otimes Y \otimes X' \to X \otimes Y \otimes Y'.$$

Using the diagrams introduced in Section 3.5 this is a straightforward check.

Proposition 4.1.2. *Let $\mathcal{C}$ be a BTC* category. Fix for each X in $\mathcal{C}$ a standard conjugate $(\overline{X}, r, \overline{r})$ Define the arrows $\Theta(X) \in End(X)$ by*

$$\Theta(X) = r^* \otimes id_X \circ id_{\overline{X}} \otimes c_{X,X} \circ r \otimes id_X.$$

Then this map does not depend on the choice of standard conjugates and defines a twist for $\mathcal{C}$. This twist will be called the canonical twist.

Proof. Let's start by showing that the maps $\Theta(X)$ are well-defined, i.e. do not depend on the particular choice of standard conjugates. Let $(\overline{X}', r', \overline{r}')$ be another choice of standard conjugates for X. Then there exists a unitary arrow $u : \overline{X} \to \overline{X}'$ such that $r' = u \otimes id_X \circ r$. We have

$$\begin{aligned}\Theta'(X) &= r^* \otimes id_X \circ u^* \otimes id_{X\otimes X} \circ id_{\overline{X}'} \otimes c_{X,X} \circ u \otimes id_{X\otimes X} \circ r \otimes id_X \\ &= r^* \otimes id_X \circ (u^*u) \otimes id_{X\otimes X} \circ id_{\overline{X}} \otimes c_{X,X} \circ r \otimes id_X \\ &= r^* \otimes id_X \circ id_{\overline{X}} \otimes c_{X,X} \circ r \otimes id_X = \Theta(X).\end{aligned}$$

Take any arrow $s : X \to Y$. We need to show that $\Theta(Y) \circ s = s \circ \Theta(X)$. By definition of a standard conjugate there are irreducibles X_i, $\overline{X}_i$, isometries $w_i : X_i \to X$, $\overline{w}_i : \overline{X}_i \to \overline{X}$ with the properties $w_i^* \circ w_j = \delta_{ij} id_{X_i}$ and $\overline{w}_i^* \circ \overline{w}_j = \delta_{ij} id_{\overline{X}_i}$, and there are standard conjugates $r_i : \mathbb{1} \to \overline{X}_i \otimes X_i$ such that $r = \sum_i \overline{w}_i \otimes w_i \circ r_i$. Using these identities we can write $\Theta(X)$ as

$$\sum_i r_i^* \otimes id_X \circ id_{\overline{X}_i} \otimes w_i^* \otimes id_X \circ id_{\overline{X}_i} \otimes c_{X,X} \circ id_{\overline{X}_i} \otimes w_i \otimes id_X \circ r_i \otimes id_X.$$

From naturality of the braiding it follows that $w_i^* \circ \Theta(X) = \Theta(X_i) \circ w_i^*$ and $\Theta(X) \circ w_i = w_i \circ \Theta(X_i)$. The reader may find the diagrammatical method of the previous chapter helpful in showing this. Now consider $s : X \to Y$. For Y we can also find suitable isometries $v_k : Y_k \to Y$ where the Y_k are irreducible. We can write

$$s \circ \Theta(X) = \sum_{i,k} v_k \circ (v_k^* \circ s \circ w_i) \circ w_i^* \circ \Theta(X).$$

The map $v_k^* \circ s \circ w_i$ is always zero if X_i and Y_k are nonisomorphic. If X_i and Y_k are isomorphic we can consider them equal and the arrow is a scalar multiple of id_{X_i}. Consequently it commutes with $\Theta(X_i)$. Pulling $\Theta(X)$ all the way through the above expression shows that Θ is a natural transformation from the identity functor to itself.

We still need to show that the expressions in Definition 4.1.1 hold and that each $\Theta(X)$ is unitary. The previously defined standard left inverse will help us with the computations. The twist can be expressed in terms of the standard left inverse, $\Theta(X) = \phi_{X,X}(c_{X,X})$, where $c_{X,X}$ denotes the braiding of $\mathcal{C}$, $c_{X,Y} : X \otimes Y \to Y \otimes X$. In the following computations we use the following shorthand notation $c = c_{X_1 \otimes X_2, X_1 \otimes X_2}$ and $c_{ij} = c_{X_i,X_j}$. We also drop the subscripts for the left inverses. First note the following equation that can be obtained from the braid equations.

$$c = id_{X_1} \otimes c_{12} \otimes id_{X_2} \circ c_{11} \otimes id_{X_2 \otimes X_2} \circ id_{X_1 \otimes X_1} \otimes c_{22} \circ id_{X_1} \otimes c_{21} \otimes id_{X_2}.$$

From this we find

$$\begin{aligned}
\phi_1(c) &= c_{12} \otimes id_{X_2} \circ \phi_1(c_{11}) \otimes id_{X_2 \otimes X_2} \circ id_{X_1} \otimes c_{22} \circ c_{21} \otimes id_{X_2} \\
&= id_{X_2} \otimes \phi_1(c_{11}) \otimes id_{X_2} \circ c_{12} \otimes id_{X_2} \circ id_{X_1} \otimes c_{22} \circ c_{12} \otimes id_{X_2} \\
&= id_{X_2} \otimes \phi_1(c_{11}) \otimes id_{X_2} \circ c_{X_1 \otimes X_2, X_2} \circ c_{21} \otimes id_{X_2} \\
&= id_{X_2} \otimes \phi_1(c_{11}) \otimes id_{X_2} \circ id_{X_2} \otimes c_{21} \circ c_{X_2 \otimes X_1, X_2} \\
&= id_{X_2} \otimes \phi_1(c_{11}) \otimes id_{X_2} \circ id_{X_2} \otimes c_{21} \circ c_{22} \otimes id_{X_1} \circ id_{X_2} \otimes c_{12}
\end{aligned}$$

If we apply $\phi_2 = \phi^{X_2}$ to both sides of this equation we obtain

$$\phi_2 \phi_1(c) = \phi_1(c_{11}) \otimes \phi_2(c_{22}) \circ c_{12} \circ c_{12}.$$

Which is just $\Theta(X_1 \otimes X_2) = \Theta(X_1) \otimes \Theta(X_2) \circ c_{X_2,X_1} \circ c_{X_1,X_2}$. We move to unitarity of the twist. Recall that the braiding morphisms $c_{X,Y}$ are assumed to be unitary. Let $(\overline{X}, r, \overline{r})$ again be a standard conjugate for X. Then $c_{\overline{X},X} \circ r$ defines a conjugate for $\overline{X}$. It is easy to check this claim when X is irreducible. If X is not irreducible then we can write $r = \sum_i w_i \otimes \overline{w}_i \circ r_i$, where each i corresponds to an irreducible summand X_i of X. By naturality of the braiding we have

$$c_{\overline{X},X} \circ r = \sum_i w_i \otimes \overline{w}_i \circ c_{\overline{X}_i, X_i} \circ r_i,$$

from which it follows that $c_{\overline{X},X} \circ r$ is a conjugate for $\overline{X}$. In the next computation we show that if c is unitary and r is standard, then $c_{\overline{X},X} \circ r$ is standard by using the characterization of standardness given in Lemma 3.3.8. Let $s \in End(\overline{X})$, then

$$\begin{aligned}
r^* \circ c^*_{\overline{X},X} \circ id_X \otimes s \circ c_{\overline{X},X} \circ r &= r^* \circ c^*_{\overline{X},X} \circ c_{\overline{X},X} \circ s \otimes id_X \circ r \\
&= r^* \circ s \otimes id_X \circ r = \overline{r}^* \circ id_X \otimes s \circ \overline{r} \\
&= \overline{r}^* \circ c^*_{X,\overline{X}} \circ c_{X,\overline{X}} \circ id_X \otimes s \circ \overline{r} \\
&= \overline{r}^* \circ c^*_{X,\overline{X}} \circ s \otimes id_X \circ c_{X,\overline{X}} \circ \overline{r}
\end{aligned}$$

If $(\overline{X}, r, \overline{r})$ is a standard conjugate for X then $(X, \overline{r}, r)$ is a standard conjugate for $\overline{X}$. Therefore there exists a unique unitary arrow $t : X \to X$ such that $c_{\overline{X},X} \circ r = t^* \otimes id_{\overline{X}} \circ \overline{r}$. The following computation shows that t is equal to $\Theta(X)$ from which we can conclude that Θ is unitary.

$$\begin{aligned} t &= (\overline{r}^* \circ t \otimes id_{\overline{X}}) \otimes id_X \circ id_X \otimes r \\ &= (r^* \circ c^*_{\overline{X},X}) \otimes id_X \circ id_X \otimes r \\ &= r^* \otimes id_X \circ id_{\overline{X}} \otimes c_{X,X} \circ c^*_{\overline{X}\otimes X,X} \circ id_X \otimes r \\ &= r^* \otimes id_X \circ id_{\overline{X}} \otimes c_{X,X} \circ r \otimes id_X = \Theta(X) \end{aligned}$$

In the first equality we used the conjugate equations and in the third equality the braid equations. The last identity we need to check is $\Theta(\overline{X}) \otimes id_X \circ r = id_X \otimes \Theta(X) \circ r$. This is left to the reader as we will not use this identity. $\square$

From this point onward, if we talk about a twist on a BTC* category, then this is the canonical twist as defined in Proposition 4.1.2. If the BTC* category is a STC* category, this twist takes on a simpler form. The following lemma can easily be proven from the definition of the canonical twist given in Proposition 4.1.2. It can also be shown to hold for an arbitrary twist on a STC* category.

Corollary 4.1.3. *Let $\mathcal{C}$ be a STC* category with a twist Θ. Then for all objects X and Y in $\mathcal{C}$ we have $\Theta(X)^2 = id_X$ and $\Theta(X \otimes Y) = \Theta(X) \otimes \Theta(Y)$. If X is irreducible then $\Theta(X) = \omega(X) id_X$ where $\omega(X) \in \{-1, 1\}$. If Z is an irreducible summand of $X \otimes Y$ with X and Y irreducible, then $\omega(Z) = \omega(X)\omega(Y)$.*

Definition 4.1.4. *Let $\mathcal{C}$ be a STC* category and Θ the twist on $\mathcal{C}$. Then $\mathcal{C}$ is called even if for each object X in $\mathcal{C}$ we have $\Theta(X) = id_X$.*

Both categories $\mathcal{H}_f$ and $Rep_f(G, \mathbb{C})$ are examples of even STC* categories. This can be checked easily using the standard conjugates given straight after Definition 3.2.9. The next proposition helps to show that a STC* category must be even in order to admit a symmetric $*$-preserving fiber functor.

Proposition 4.1.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be BTC* categories and $E : \mathcal{C} \to \mathcal{D}$ a $*$-preserving braided tensor functor. This functor comes with unitary arrows $d^E_{X,Y} : E(X) \otimes E(Y) \to E(X \otimes Y)$ and $e^E : \mathbb{1}_{\mathcal{D}} \to E(\mathbb{1}_{\mathcal{C}})$. If $(\overline{X}, r, \overline{r})$ is a standard conjugate for X, then*

$$(E(\overline{X}), (d^E_{\overline{X},X})^{-1} \circ E(r) \circ e^E, (d^E_{X,\overline{X}})^{-1} \circ E(\overline{r}) \circ e^E),$$

is a standard conjugate for $E(X)$. In particular

$$d(E(X)) = d(X), \qquad \Theta(E(X)) = E(\Theta(X)), \quad \forall X \in Obj(\mathcal{C}),$$

where $d(X)$ is the dimension of X.

Proof. First we take on the case that X is irreducible and E is a strict tensor functor. Pick a standard conjugate $(\overline{X}, r, \overline{r})$. Using the conjugate equations it is easy to check that $(E(\overline{X}), E(r), E(\overline{r}))$ defines a conjugate for $E(X)$. If the functor E is full then this conjugate is immediately standard, but as we did not presuppose this standardness requires more work. Using the notation $\Theta(\overline{X}) = \omega_{\overline{X}} id_{\overline{X}}$, the proof of Proposition 4.1.2 shows us that $c_{\overline{X},X} \circ \overline{r} = \overline{\omega}_{\overline{X}} r$. Take $s \in End(E(X))$ arbitrary and compute

$$\begin{aligned}
&E(r^*) \circ id_{E(\overline{X})} \otimes s \circ E(r) \\
&= E(r^*) \circ c^*_{E(\overline{X}),E(X)} \circ c_{E(\overline{X}),E(X)} \circ id_{E(\overline{X})} \otimes s \circ E(r) \\
&= (c_{E(\overline{X}),E(X)} \circ E(r))^* \circ c_{E(\overline{X}),E(X)} \circ id_{E(\overline{X})} \otimes s \circ E(r) \\
&= (c_{E(\overline{X}),E(X)} \circ E(r))^* \circ s \otimes id_{E(\overline{X})} \circ c_{E(\overline{X}),E(X)} \circ E(r) \\
&= E(c_{\overline{X},X} \circ r)^* \circ s \otimes id_{E(\overline{X})} \circ E(c_{\overline{X},X} \circ r) \\
&= E(\overline{\omega}_{\overline{X}} \overline{r})^* \circ s \otimes id_{E(\overline{X})} \circ E(\overline{\omega}_{\overline{X}} \overline{r}) \\
&= E(\overline{r})^* \circ s \otimes id_{E(\overline{X})} \circ E(\overline{r})
\end{aligned}$$

This proves that $(E(\overline{X}), E(r), E(\overline{r}))$ defines a standard conjugate for $E(X)$ in the case that X is irreducible. Now let X be reducible. If $(\overline{X}, r, \overline{r})$ is a standard conjugate for X then there are isometries $v_i : X_i \to X$ and $w_i : \overline{X}_i \to \overline{X}$ such that all the X_i and $\overline{X}_i$ are irreducible and

$$r_i = w_i^* \otimes v_i^* \circ r, \quad \overline{r}_i = v_i^* \otimes w_i^* \circ \overline{r}$$

hold where $(\overline{X}_i, r_i, \overline{r}_i)$ define standard conjugates for X_i. We have already proven that each $(E(\overline{X}_i), E(r_i), E(\overline{r}_i))$ defines a standard conjugate for $E(X_i)$. By noting that

$$E(r) = E(\sum_i w_i \otimes v_i \circ r_i) = \sum_i E(w_i) \otimes E(v_i) \circ E(r_i)$$

and a similar expression for $E(\overline{r})$ we can see that $(E(\overline{X}), E(r), E(\overline{r}))$ is a standard conjugate for $E(X)$.

The proof works the same way for non-strict tensor functors, the main difference being that we have to insert the maps $d^E_{X,Y}$ and e^E at various places. The identities $d(E(X)) = d(X)$ and $\Theta(E(X)) = E(\Theta(X))$ follow directly as the twist and dimension are expressed in terms of conjugates. □

Proposition 4.1.6. *Let $\mathcal{C}$ be an STC* category and $E : \mathcal{C} \to \mathcal{H}_f$ a symmetric $*$-preserving fiber functor. Then $\mathcal{C}$ is even.*

Proof. By Proposition 4.1.5 we know that $E(\Theta(X)) = \Theta(E(X))$. This implies that $E(\Theta(X)) = id_{E(X)}$ because $\mathcal{H}_f$ is even. As we assumed that E is faithful, it follows that $\Theta(X) = id_X$. □

Being even is a necessary condition for admitting a symmetric $*$-preserving fiber functor, but is it a sufficient condition? The bulk of this chapter is used to prove the following theorem that claims that it is.

Theorem 4.1.7. *(Deligne's embedding theorem) Let $\mathcal{C}$ be an even STC* category. Then there exists a symmetric $*$-preserving fiber functor $E : \mathcal{C} \to \mathcal{H}_f$.*

If we combine this theorem with the Tannaka-Krein duality of the previous chapter we find

Theorem 4.1.8. *Let $\mathcal{C}$ be an even STC* category. Then there exists a compact group G, unique up to isomorphism, such that there is an equivalence $F : \mathcal{C} \to Rep_f(G, \mathbb{C})$ of STC* categories.*

The main motivation for studying the embedding theorem for STC* categories comes from algebraic quantum field theory. Looking at the discussion given in the Epilogue it is not desirable to restrict our attention to STC* categories that are even. If in AQFT, the Doplicher-Roberts reconstruction theorem is applied to an even STC* category (of 'physical' representations of the quasilocal algebra) one obtains a Hilbert space representation that only has bosonic superselection sectors. In order to get sufficient generality we also need to consider fermionic superselection sectors and therefore arbitrary STC* categories. For the remainder of this section we will develop terminology in order to formulate a version of the embedding theorem for arbitrary STC* categories. Assuming Theorem 4.1.7 we will subsequently prove this theorem.

Definition 4.1.9. *A (compact) supergroup is a pair (G, k), where G is a (compact) Hausdorff group and $k \in G$ is a central element of order two. An isomorphism of supergroups $\alpha : (G, k) \to (G', k')$ is an isomorphism of topological groups $\alpha : G \to G'$ such that $\alpha(k) = k'$.*

Note that this definition does not coincide with the usual definition of a supergroup as a group object in the category of supermanifolds.

Definition 4.1.10. *A (finite dimensional, unitary, continuous) representation of a compact supergroup (G, k) is just a (finite dimensional, unitary, continuous) representation (H, π) of G. Intertwiners, tensor product and direct sums of representations are defined as for topological groups. The category $Rep_f((G, k), \mathbb{C})$ is thus equivalent as a $*$-tensor category to $Rep_f(G, \mathbb{C})$.*

Next we define a symmetry on $Rep_f((G, k), \mathbb{C})$ that differs from that on $Rep_f(G, \mathbb{C})$. Let (H, π) be an irreducible representation of the supergroup (G, k). Because k is central Schur's lemma gives us that $\pi(k) = \lambda id_H$, where $\lambda \in \mathbb{C}$. Because $k^2 = e$, $\lambda \in \{-1, 1\}$. Thus k induces a $\mathbb{Z}_2$ grading on every irreducible representation of (G, k). Given an irreducible representation (H, π) we can decompose the representation space H as a direct sum $H_+ \oplus H_-$ where for each $v \in H_\pm$ we have $\pi(k)v = \pm v$. The direct sum is orthogonal with respect to the invariant inner product. The vectors in H_+ and H_- are called homogenous. If we take the direct sum of two irreducible representations H and H' there is again a $\mathbb{Z}_2$-grading. The homogenous spaces are given by $H_+ \oplus H'_+$ and $H_- \oplus H'_-$. By semisimplicity of the category, every representation is $\mathbb{Z}_2$-graded. The intertwiners preserve this grading.

Define the projector $P^\pi_\pm = \frac{1}{2}(id_H \pm \pi(k))$ on the homogenous subspaces of H. Define the symmetry on Σ_k by

$$\Sigma_k((H, \pi), (H', \pi')) = \Sigma(H, H')(\mathbb{1} - 2P^\pi_- \otimes P^{\pi'}_-)$$

where Σ denotes the flip symmetry of $Rep_f(G, \mathbb{C})$. If $v \in H$, $w \in H'$ are homogenous, then $\Sigma_k(v \otimes w) = -w \otimes v$ if $v \in H_-$ and $w \in H'_-$. In all other (homogenous) cases we have $\Sigma_k(v \otimes w) = w \otimes v$.

Definition 4.1.11. *Consider the supergroup with two elements $(\{e, k\}, k)$. For this supergroup the category $\mathcal{SH}_f = Rep_f((\{e, k\}, k), \mathbb{C})$ is called the category of super Hilbert spaces.*

The category of super Hilbert spaces is the category if finite-dimensional Hilbert spaces (the inner products are given by the invariant inner products on the representations) that have a $\mathbb{Z}_2$-grading that is respected by the linear maps that make up the arrows. This category is equipped with the symmetry Σ_k defined above. The symmetry on $\mathcal{SH}_f$ is defined in the same way as for the categories $Rep_f((G, K), \mathbb{C})$. Using this observation we can define a forgetful symmetric tensor functor $Rep_f((G, k), \mathbb{C}) \to \mathcal{SH}_f$.

Lemma 4.1.12. *Let Σ_k by defined on the category $Rep_f((G, k), \mathbb{C})$ as above. Then Σ_k defines a symmetry and $Rep_f((G, k), \mathbb{C})$ is a STC* category. For every object $(H, \pi) \in Rep_f((G, k), \mathbb{C})$ the twist $\Theta((H, \pi))$ is given by $\pi(k)$.*

The proof of this lemma is straighforward. In order to check that $\Theta((H, \pi)) = \pi(k)$ pick a homogenous basis for $H = H_+ \oplus H_-$ and look how $\Theta((H, \pi))$ acts on each basis element.

The following proposition follows from Tannaka-Krein duality and is the last result that we need in order to formulate the main theorem of this chapter.

Proposition 4.1.13. *Let G be a compact group and $Rep_f(G, \mathbb{C})$ the category of finite dimensional representations. Then the unitary monoidal natural isomorphisms of the identity functor on $Rep_f(G, \mathbb{C})$ to itself form an abelian group that is isomorphic to the center $Z(G)$ of G.*

Proof. Pick a $k \in Z(G)$ and let $(H, \pi) \in Rep_f(G, \mathbb{C})$ be irreducible and unitary. Then by Schur's lemma $\pi(k) = \omega_{(H,\pi)} id_H$ for some complex number $\omega_{(H,\pi)}$. Define for each irreducible representation (H, π) the arrow $\Theta((H, \pi)) = \omega_{(H,\pi)} id_H$. The collection of arrows $\{\Theta((H, \pi))\}$ can be extended uniquely to reducible objects in such a way that this gives a unitary monoidal natural isomorphism of the identity functor on $Rep_f(G, \mathbb{C})$. Conversely, take α to be a unitary monoidal natural isomorphism of the identity functor on $Rep_f(G, \mathbb{C})$. Let $\omega : Rep_f(G, \mathbb{C}) \to \mathcal{H}_f$ be the forgetful functor. Then $\beta_{(H,\pi)} = \omega(\alpha((H, \pi)))$ defines a unitary monoidal natural isomorphism of the forgetful functor to itself. By Tannaka-Krein duality there is a $g \in G$ such that $\pi(g) = \beta_{(H,\pi)}$ holds for every $(H, \pi) \in Rep_f(G, \mathbb{C})$. We know that for every irreducible representation $\pi(g)$ is a scalar multiple of the identity arrow. This implies that $g \in Z(G)$. □

Now we are in the position to state and prove the main theorem for STC* categories (assuming Theorem 4.1.7 which is non-trivial to prove). The rest of this chapter will be concerned with the construction of a symmetric $*$-preserving fiber functor on even STC* categories.

Theorem 4.1.14. *Let $\mathcal{C}$ be an STC* category. Then there exists a compact supergroup (G, k) which is unique up to an isomorphism of supergroups, and an equivalence $F : \mathcal{C} \to Rep_f((G, k), \mathbb{C})$ of symmetric tensor $*$-categories.*

In particular, if $\omega : Rep_f((G, k), \mathbb{C}) \to \mathcal{SH}_f$ is the forgetful functor, then the composition $E = \omega \circ F : \mathcal{C} \to \mathcal{SH}_f$ is a faithful symmetric $*$-preserving tensor functor into the STC category of super Hilbert spaces.*

Proof. We start by constructing an even STC* category $\tilde{\mathcal{C}}$ from $\mathcal{C}$, called the bosonization of $\mathcal{C}$. As a tensor $*$-category the category $\tilde{\mathcal{C}}$ is the same as $\mathcal{C}$. The difference is the symmetry on $\tilde{\mathcal{C}}$ which is defined as

$$\tilde{c}_{X,Y} = (-1)^{\frac{1}{4}(1-\Theta(X))(1-\Theta(Y))} c_{X,Y}$$

when X and Y in $\mathcal{C}$ are irreducible. For general (reducible) objects X and Y the symmetry $\tilde{c}_{X,Y}$ can be obtained by naturality of $\tilde{c}$ from the symmetry on the irreducible summands of X and Y. The category $\tilde{\mathcal{C}}$ is an even STC* category, so by Theorem 4.1.8 there is a compact group, unique up to isomorphism, and an equivalence $F : \tilde{\mathcal{C}} \to Rep_f(G, \mathbb{C})$ of STC* categories. The twist Θ on $\mathcal{C}$ defines a unitary monoidal natural isomorphism of the identity functor on $\tilde{\mathcal{C}}$. Using the equivalence $\tilde{\mathcal{C}} \cong Rep_f(G, \mathbb{C})$ the twist defines a unitary monoidal natural isomorphism on $Rep_f(G, \mathbb{C})$. By Proposition 4.1.13, this isomorphism corresponds to a unique central element $k \in G$. Furthermore, because Θ^2 is just the identity this element $k \in G$ is of order two, making (G, k) into a supergroup. The correspondence from 4.1.13 is given by $\Theta((H, \pi)) = \pi(k)$ up to the identificaion $\tilde{\mathcal{C}} \cong Rep_f(G, \mathbb{C})$. We want to show that $\mathcal{C} \cong Rep_f((G, k), \mathbb{C})$ as STC* categories. If we forget about the symmetries on both categories for the moment, this quivalence follows straight from $\tilde{\mathcal{C}} \cong Rep_f(G,\mathbb{C})$. If we look at the symmetries a straightforward check shows that the correspondence $\Theta((H,\pi)) = \pi(k)$ ensures that the symmetries of $\mathcal{C}$ and $Rep_f((G,k),\mathbb{C})$ coincide. This proves the desired equivalence.

Note that G is unique up to isomorphism and that for each G the element k is unique. Suppose that we pick another G' such that that $\mathcal{C} \cong Rep_f((G',k'),\mathbb{C})$. There is an isomorphism $\alpha : G \to G'$ and we want to show that $\alpha(k) = k'$. If this is the case, then α defines an isomorphism of supergroups. The isomorphism α defines an isomorphism $Rep_f(G,\mathbb{C}) \to Rep_f(G',\mathbb{C})$ by $(H,\pi) \mapsto (H, \pi \circ \alpha^{-1})$. Applying the above construction shows that for each representation in $Rep_f(G,\mathbb{C})$ we have $\pi(k) = \pi(\alpha^{-1}(k'))$. This can only hold for each representation if $k' = \alpha(k)$, proving uniqueness. □

The super Tannaka group G of $\mathcal{C}$ thus consists of unitary monoidal natural transformations of E to itself, where E is the fiber functor belonging to the bosonization of $\mathcal{C}$.

4.2 Making Symmetric Fiber Functors $*$-Preserving

In this section we follow the approach of appendix B6 of Müger [30]. The aim of this and the following sections is to prove that for each even STC* $\mathcal{C}$ category there exists a symmetric $*$-preserving fiber functor $\mathcal{C} \to \mathcal{H}_f$. The aim of this section is to prove the following theorem. This theorem states that it is sufficient to find a symmetric fiber functor as it can be made $*$-preserving.

Theorem 4.2.1. *Let $\mathcal{C}$ be an even STC* category and $E : \mathcal{C} \to Vect_{\mathbb{C}}$ a symmetric fiber functor. Then there exists a symmetric $*$-preserving fiber functor $\mathcal{C} \to \mathcal{H}_f$.*

Lemma 4.2.2. *Let $\mathcal{C}$ be a STC* category and $E : \mathcal{C} \to Vect_{\mathbb{C}}$ a symmetric fiber functor. Pick for each object X in $\mathcal{C}$ an inner product $\langle \cdot, \cdot \rangle_X^0$ on the finite dimensional vector space $E(X)$. Define the maps $X \mapsto E(X)$ and $s \mapsto E(s^*)^\dagger$ for any $X \in Obj(\mathcal{C})$ and any $s \in Hom_{\mathcal{C}}(X,Y)$. Here the adjoint $\dagger$ is taken relative to the previously defined inner products $\langle \cdot, \cdot \rangle_X^0$. These maps define a faithful functor $\tilde{E} : \mathcal{C} \to \mathcal{H}_f$. Defining $d^{\tilde{E}}_{X,Y} = ((d^E_{X,Y})^\dagger)^{-1}$ and $e^{\tilde{E}} = ((e^E)^\dagger)^{-1}$, this functor is a symmetric fiber functor.*

Proof. Proving that $\tilde{E}$ defines a faithful functor is easy. First observe that $\tilde{E}(id_X) = id_{E(X)} = id_{\tilde{E}(X)}$ and that $\tilde{E}$ respects composition of arrows as shown by

$$\tilde{E}(s \circ t) = E(t^* \circ s^*)^\dagger = (E(t^*)E(s^*))^\dagger = E(s^*)^\dagger E(t^*)^\dagger = \tilde{E}(s)\tilde{E}(t).$$

This shows that $\tilde{E}$ defines a functor. It is faithful because E is faithful and both $*$ and $\dagger$ define involutions. Proving that $\tilde{E}$ defines symmetric fiber functor is only a bit more work. First note that $\tilde{E}$ is $\mathbb{C}$-linear. That it is symmetric follows from

$$\begin{aligned}\tilde{E}(c_{X,Y}) \circ d^{\tilde{E}}_{X,Y} &= E(c^*_{X,Y})^\dagger \circ (d^{E\,\dagger}_{X,Y})^{-1} = ((d^E_{X,Y})^{-1} \circ E(c_{Y,X}))^\dagger \\ &= (\Sigma_{E(Y),E(X)} \circ (d^E_{X,Y})^{-1})^\dagger = d^{\tilde{E}}_{X,Y} \circ \Sigma_{E(X),E(Y)}.\end{aligned}$$

We still need to show that $\tilde{E}$ is a tensor functor. This amounts to showing that the diagrams in Definition 3.1.3 of a tensor functor are satisfied and that the arrows $d^{\tilde{E}}_{X,Y}$ constitute a natural transformation. Due to the fact that E is a tensor functor we know that

$$d^E_{X\otimes Y,Z} \circ (d^E_{X,Y} \otimes id_{E(Z)}) = d^E_{X,Y\otimes Z} \circ (id_{E(X)} \otimes d^E_{Y,Z}).$$

Taking the Hermitian adjoints of both sides yields

$$(d^E_{X,Y})^\dagger \otimes id_{E(Z)} \circ (d^E_{X\otimes Y,Z})^\dagger = id_{E(X)} \otimes (d^E_{Y,Z})^\dagger \circ (d^E_{X,Y\otimes Z})^\dagger.$$

Taking the inverse of both sides gives the first diagram for functor $\tilde{E}$, namely

$$d^{\tilde{E}}_{X\otimes Y,Z} \circ (d^{\tilde{E}}_{X,Y} \otimes id_{\tilde{E}(Z)}) = d^{\tilde{E}}_{X,Y\otimes Z} \circ (id_{\tilde{E}(X)} \otimes d^{\tilde{E}}_{Y,Z}).$$

The other two diagrams are obtained in the same way. Take the appropriate equality for E, take the adjoints and then the inverse. As E is a tensor functor we know that

$$E(s \otimes t) \circ d^E_{X,Y} = d^E_{X',Y'} \circ E(s) \otimes E(t), \quad \forall s : X \to X', \quad t : Y \to Y'.$$

This is equivalent to

$$E(s \otimes t)^\dagger \circ ((d^E_{X',Y'})^{-1})^\dagger = ((d^E_{X,Y})^{-1})^\dagger \circ (E(s) \otimes \eta(t))^\dagger$$

from which naturality of $\tilde{E}$ readily follows. □

Just like in Section 3.5 we can construct a unital associative $\mathbb{C}$-algebra $A(E, \tilde{E})$ from the symmetric fiber functors of the previous lemma. We can use Proposition 3.6.1 to show that $A(E, \tilde{E})$ is a commutative unital $\mathbb{C}$-algebra. However, we cannot use all the results from Section 3.5 because the fiber functors are not $*$-preserving. The algebra $A(E, \tilde{E})$ does not have the positive $*$-operation in the way of Proposition 3.5.2. The proof of that proposition relied on the fiber functors being $*$-preserving. Note that the fiber functors E and $\tilde{E}$ are not arbitrary fiber functors $\mathcal{C} \to \mathcal{H}_f$ but are related by $E(X) = \tilde{E}(X)$ and $E(s) = \tilde{E}(s^*)^\dagger$. These relations provide the means to turn $A(E, \tilde{E})$ into a $*$-algebra. The $*$-operation that we will define should not be confused with the operation as it is defined in the proof of Proposition 3.5.2.

Proposition 4.2.3. *Let $\mathcal{C}$ be an STC* category, $E : \mathcal{C} \to Vect_{\mathbb{C}}$ a symmetric fiber functor and $\tilde{E}$ be defined as in Lemma 4.2.2. Then $[X, s]^* = [X, s^\dagger]$, where $\dagger$ denotes the adjoint of $s \in End(E(X))$ with respect to the inner product $\langle \cdot, \cdot \rangle_X^0$, is well defined and defines a positive $*$-operation on $A(E, \tilde{E})$. With respect to this operation, the norm from Proposition 3.5.3 is a C*-norm on $A(E, \tilde{E})$.*

Proof. We start by defining the $*$-operation on $A_0(E, \tilde{E})$ so we don't have to worry that it is well defined. Define as in the lemma $[X, s]^* = [X, s^\dagger]$. This operation is clearly antilinear and involutive. The $*$-operation can only descend to a well-defined operation on $A(E, \tilde{E})$ if it maps the elements of the ideal $I(E, \tilde{E})$ to elements in this ideal. Take arrows $s : X \to Y$ and $a : \tilde{E}(Y) \to E(X)$, then

$$\begin{aligned}([X, a \circ \tilde{E}(s)] - [Y, E(s) \circ a])^* &= [X, a \circ E(s^*)^\dagger]^* - [Y, E(s) \circ a]^* \\ &= [X, E(s^*) \circ a^\dagger] - [Y, a^\dagger \circ E(s)^\dagger] \\ &= [X, E(s^*) \circ a^\dagger] - [Y, a^\dagger \circ \tilde{E}(s^*)]\end{aligned}$$

This is again an element of $I(E, \tilde{E})$, proving that the map is well-defined. Next we show that the operation is antimultiplicative (or multiplicative which is the same for a commutative algebra).

$$\begin{aligned}([X, s] \cdot [Y, t])^* &= [X \otimes Y, d^{\tilde{E}}_{X,Y} \circ s \otimes t \circ (d^E_{X,Y})^{-1}]^* \\ &= [X \otimes Y, ((d^E_{X,Y})^\dagger)^{-1} \circ s \otimes t \circ (d^E_{X,Y})^{-1}]^* \\ &= [X \otimes Y, ((d^E_{X,Y})^\dagger)^{-1} \circ s^* \otimes t^* \circ (d^E_{X,Y})^{-1}] = [X, s]^* \cdot [Y, t]^*\end{aligned}$$

We move onto positivity

$$[X, s] \cdot [X, s]^* = [X \otimes X, ((d^E_{X,X})^\dagger)^{-1} \circ s \otimes s^* \circ (d^E_{X,X})^{-1}] = 0.$$

This can only be the case if $s \otimes s^* = 0$. This implies that $s^*s \otimes s^*s = 0$ which in turn implies that $s^*s = 0$. By positivity of the $*$-operation on $\mathcal{C}$ we find that $s = 0$, proving the desired positivity.

Recall that in Section 3.7 we defined an isomorphism δ that in this setting gives us an isomorphism

$$\delta : \bigoplus_{i \in I} End(E(X_i)) \to A(E, \tilde{E})$$

such that $\|\delta((a_i)_{i \in I})\|_{A(E,\tilde{E})} = \sup_{i \in I} \|a_i\|_{End(E(X_i))}$. Note that $\delta((a_i))^* = \delta((a_i^\dagger))$. This immediately gives us the desired C* property of the norm on $A(E, \tilde{E})$; $\|a^*a\|_{A(E,\tilde{E})} = \|a\|^2_{A(E,\tilde{E})}$. □

The following conjecture can be found in Müger [30]. Alas, no complete proof is given there and the author of this thesis was unable to provide one thusfar.

Conjecture 4.2.4. *Let $\mathcal{C}$ be a STC* category, $E : \mathcal{C} \to Vect_{\mathbb{C}}$ a symmetric fiber functor and $\tilde{E}$ be as defined in Lemma 4.2.2. Then there exists a natural monoidal isomorphism $\alpha : E \to \tilde{E}$ whose components α_X are positive in sense that for each $s \in End(X)$, $s \neq 0$, we have $\langle s, \alpha_X s\rangle_X^0 > 0$.*

By the discussion in Section 3.6 it is clear that we have monoidal natural isomorphisms at our disposal. What needs to be proven is that at least 1 of these transformations can give rise to a positive map. Assuming the existence of a natural isomorphism α as in the previous conjecture allows us te define new inner products such that E is $*$-preserving with respect to these inner products. The following theorem proves this claim and completes the proof of Theorem 4.2.1.

Theorem 4.2.5. *Let $\mathcal{C}$ be an even STC* category and $E : \mathcal{C} \to Vect_{\mathbb{C}}$ a symmetric fiber functor. Then there are inner products $\langle\cdot,\cdot\rangle_X$ on each vector space $E(X)$ such that $E' : \mathcal{C} \to \mathcal{H}_f$, $E'(X) = (E(X), \langle\cdot,\cdot\rangle_X)$ defines a symmetric $*$-preserving fiber functor.*

Proof. As before, pick for each $X \in \mathcal{C}$ an inner product $\langle\cdot,\cdot\rangle_X^0$ on $E(X)$. For the 1 dimensional vector space $E(\mathbb{1})$ which is spanned by $e^E(1)$ the inner product is of the form

$$\left\langle ae^E(1), be^E(1)\right\rangle_{\mathbb{1}}^0 = a\bar{b}\left\langle e^E(1), e^E(1)\right\rangle_{\mathbb{1}}^0.$$

We choose $\left\langle e^E(1), e^E(1)\right\rangle_{\mathbb{1}}^0 = 1$. Define the symmetric fiber functor $\tilde{E}$ as in Lemma 4.2.2 and the monoidal natural isomorphism α as in Conjecture 4.2.4. We define the new inner products on the spaces $E(X)$ by

$$\langle u, v\rangle_X = \langle u, \alpha_X v\rangle_X^0, \quad u, v \in E(X).$$

It is because of the naturality of α combined with the choice of $\tilde{E}$ that ensures us that the $*$-operation is preserved. Take $s : X \to Y$, then

$$\begin{aligned}\langle u, E(s)v\rangle_Y &= \langle u, \alpha_Y E(s)v\rangle_Y^0 \\ &= \left\langle u, E(s^*)^\dagger \alpha_X v\right\rangle_Y^0 = \langle E(s^*)u, \alpha_X v\rangle_X^0 = \langle E(s^*)u, v\rangle_X.\end{aligned}$$

When we have verified that each $\langle\cdot,\cdot\rangle_X$ defines an inner product the above calculation immediately implies that $E(s^*) = E(s)^*$, where $*$ denotes conjugation with respect to the inner products $\langle\cdot,\cdot\rangle_X$. Positivity and nondegeneracy for $\langle\cdot,\cdot\rangle_X$ follow from Conjecture 4.2.4 combined with the fact that each α_X is invertible. The only nontrivial claim to check is symmetry. Let $\phi : A(E, \tilde{E}) \to \mathbb{C}$ be the $*$-character corresponding with α. If $a \in A(E, \tilde{E})$ then $\phi(a^*a) > 0$ if $a \neq 0$. Pulling ϕ back to $A_0(E, \tilde{E})$ we have for each $b \in A_0(E, \tilde{E})$ we have $\phi(b^*b) \geq 0$. Pick a basis for $E(X)$ and let α_{ij} denote the (i, j)-th element of α_X with respect to this basis. If $b = [X, \delta_{ij}]$, where δ_{ij} is the matrix that has 1 as the (i,j)-th element while all other elements are 0. Writing out $\phi(b^*b) \geq 0$ for this b shows that $\alpha_{ij}\alpha_{ji} \geq 0$ holds for each i and j. Let $z \in \mathbb{C}$ be any complex number with nonzero imaginary part and take $b = \delta_{ij} + z\delta_{kl}$. The identity $\phi(b^*b) \geq 0$ implies in particular that the imaginary part of $\phi(b^*b)$ is equal to zero corresponding to $\alpha_{ij}\alpha_{lk} = \alpha_{ji}\alpha_{kl}$. Taking $i = j$ this implies that each $\alpha_{kl} = \alpha_{lk}$ unless every $\alpha_{ii} = 0$ (the latter case can be avoided by taking a suitable basis for the invertible α_X). Combining these observations the matrix elements of α_X are real numbers (the square is either positive or 0) and the matrix is symmetric. This implies that each α_X is hermitian which is exactly what is needed to get the symmetry property $\langle u, v\rangle_X = \overline{\langle v, u\rangle}_X$. This completes the proof that each $(E(X), \langle\cdot,\cdot\rangle_X)$ is a finite dimensional Hilbert space. The only thing left to show is that the maps $d^E_{X,Y}$ and e^E are unitary. Combining the monoidality of α with the definition of the maps $d^{\tilde{E}}_{X,Y}$ gives us the following identity

$$\alpha_X \otimes \alpha_Y = (d^E_{X,Y})^\dagger \circ \alpha_{X\otimes Y} \circ d^E_{X,Y}.$$

Unitarity of the maps $d^E_{X,Y}$ now follows from the calculation

$$\begin{aligned}\left\langle d^E_{X\otimes Y}(u\otimes v), d^E_{X,Y}(w\otimes z)\right\rangle_{X\otimes Y} &= \left\langle d^E_{X,Y}(u\otimes v), \alpha_{X\otimes Y} \circ d^E_{X,Y}(w\otimes z)\right\rangle^0_{X\otimes Y} \\ &= \langle u\otimes v, \alpha_X \otimes \alpha_Y \circ w\otimes z\rangle^0_{X\otimes Y} \\ &= \langle u, w\rangle_X \langle v, z\rangle_Y = \langle u\otimes v, w\otimes z\rangle_{X\otimes Y}.\end{aligned}$$

There is still one special case left to check; is $d^E_{\mathbb{1},\mathbb{1}}$ unitary? Using the relation $d^E_{\mathbb{1},\mathbb{1}} \circ e^E(1) \otimes e^E(1) = e^E(1)$ this follows in a straightforward fashion.

By irreducibility $\alpha_{\mathbb{1}} = \lambda id_{E(\mathbb{1})}$. If we insert this into $\alpha_{\mathbb{1}} \otimes \alpha_{\mathbb{1}} = (d^E_{\mathbb{1},\mathbb{1}})^\dagger \circ \alpha_{\mathbb{1}} \circ d^E_{\mathbb{1},\mathbb{1}}$ we find that $\lambda^2 = \lambda$. Because of the fact that α is a natural isomorphism we find that $\lambda = 1$. Consequently $\langle\cdot,\cdot\rangle_{\mathbb{1}} = \langle\cdot,\cdot\rangle^0_{\mathbb{1}}$. The calculation

$$\left\langle e^E 1, e^E 1\right\rangle_{\mathbb{1}} = \left\langle e^E 1, e^E 1\right\rangle^0_{\mathbb{1}} = \langle 1, 1\rangle_{\mathbb{C}}.$$

proves the unitarity of $e^E : \mathbb{1} \to E(\mathbb{1})$. □

4.3 Intermission: Abelian Categories

The next steps in the proof of Deligne's embedding theorem require more knowledge of category theory. Sections 4.3, 4.4 and part of 4.5 give a minimal overview of the required theory. This section is concerned with the basics on abelian categories. The discussion here is based on Freyd [16], Chapter VIII of Mac Lane [27] and Appendix A5 of Müger [30]. Readers that have experience with abelian categories are adviced to skip this section.

In order to define what an abelian category is, we first need to define what monomorphism, epimorphism, kernels and cokernels are for arbitrary additive categories.

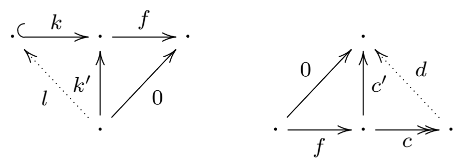
Definition 4.3.1. *Let $\mathcal{C}$ be an additive category. An arrow $s : X \to Y$ is called monic, or a monomorphism, if for each two arrows $t_i : Z_i \to X$, $i \in \{1, 2\}$ for which $s \circ t_1 = s \circ t_2$ holds we have that $t_1 = t_2$. An arrow $s : X \to Y$ is called epi, or an epimorphism, if for each two arrows $t_i : Y \to Z_i$, $i \in \{1, 2\}$ for which $t_1 \circ s = t_2 \circ s$ holds we have that $t_1 = t_2$.*

In the category $Vect_{\mathbb{C}}$ a linear map is monic in the above sense exactly when it is injective. Epi corresponds to surjectivity in this category. Note that in $Vect_{\mathbb{C}}$ any morphism is both monic and epi if and only if it is a isomorphism. This will turn out to hold for all abelian categories. The fact that any isomorphism is a monomorphism and a epimorphism holds more generally as we will now show. From the definition one can prove that if the composition $A \to B \to C$ is a monomorphism then the arrow $A \to B$ is a monomorphism. If the composition $A \to B \to C$ is epi, then $B \to C$ is epi. Let $s : A \to B$ be an isomorphism, then there is an arrow (the inverse) $t_1 : B \to A$ such that $t_1 \circ s$ is epi and an arrow $t_2 : B \to A$ such that $s \circ t_2$ is monic. Thus we conclude that every isomorphism is both an epimorphism and a monomorphism.

Definition 4.3.2. *Let $\mathcal{C}$ be an additive category and let $f : X \to Y$ be a morphism. A morphism $k : Z \to X$ is called a kernel for f if $f \circ k = 0$ and it satisfies the following universal property. Given any morphism $k' : Z' \to X$ such that $f \circ k' = 0$ then there is a unique morphism $l : Z' \to Z$ such that $k' = k \circ l$.*

A morphism $c : Y \to Z$ is called a cokernel for f if $c \circ f = 0$ and the following universal property is satisfied. Given any morphism $c' : Y \to Z'$ such that $c \circ f = 0$ there is a unique morphism $d : Z \to Z'$ such that $c' = d \circ c$.

The definition of kernels and cokernels is summarized in the following commutative diagrams. Here k and c are the kernel and cokernel of f respectively. By the way, there is no a priori reason that a morphism in an additive category has a kernel or a cokernel.



Notice the use of different arrows in the diagrams. We used $\cdot \hookrightarrow \cdot$ to denote a monomorphism (every kernel is monic as one can show from the definitions), $\cdot \twoheadrightarrow \cdot$ to denote an epimorphism (every cokernel is epi) and dotted arrows to denote the arrows that exist by virtue of the universal properties of kernels and cokernels.

Consider as an example the category $Vect_{\mathbb{C}}$ and a linear map $f : V \to W$. If we take for $Z = \{v \in V | f(v) = 0\}$ the vector space that is the kernel of

f as defined in basic linear algebra, then the inclusion $k : Z \hookrightarrow V$ defines a kernel in the sense of Definition 4.3.2. If we take $Z = W/Im(f)$, then the natural map $c : W \to Z$ defines a cokernel in the sense of Definition 4.2.2. We can now define abelian categories.

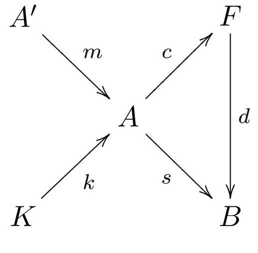
Definition 4.3.3. *Let $\mathcal{C}$ be an additive category. Then $\mathcal{C}$ is called an abelian category if the following conditions are satisfied*

1. *Every morphism has a kernel and a cokernel.*
2. *Every monomorphism is the kernel of some morphism.*
3. *Every epimorphism is the cokernel of some morphism.*

This is a good place to introduce the notion of duality in category theory. Loosely put; for a given a category $\mathcal{C}$ the opposite category $\mathcal{C}^0$ defined in Section 3.3 can function as a dual category. If f is a monomorphism in $\mathcal{C}$ then its image in $\mathcal{C}^0$ is an epimorphism. In the same way kernels and cokernels are dual notions. Some notions are selfdual, for example being an isomorphism. The main point here is that duality can sometimes save us a lot of work. First note that $\mathcal{C}$ is abelian if and only if $\mathcal{C}^0$ is abelian. For example; the dual statement of every monomorphism is the kernel of its cokernel is the statement that every epimorphism is the cokernel of its kernel. If we have proven a proposition for every abelian category, then we have also proven the dual proposition for every abelian opposite category. As $(\mathcal{C}^0)^0 = \mathcal{C}$ we have then proven both claims for every abelian category. A proper introduction to this notion can be found in Chapter II of Mac Lane [27].

Lemma 4.3.4. *Let $\mathcal{C}$ be an abelian category. Every monomorphism is the kernel of its cokernel and every epimorphism is the cokernel of its kernel.*

Proof. Looking at the above discussion it is sufficient to only prove the claim that every monomorphism is the kernel of its cokernel. Suppose that $m : A' \to A$ is a monomorphism. Because $\mathcal{C}$ is abelian, m is the kernel of some morphism $s : A \to B$. Let $c : A \to F$ be the cokernel of m. Let $k : K \to A$ be the kernel of c. We will repeatedly use the universal property of kernels and cokernels. First note that because $s \circ m = 0$ there is a unique $d : F \to B$ such that $k = d \circ c$. The last few remarks are summarized in the following commutative diagram.



We have $c \circ m = 0$, so by the universal property of k there is a unique $k' : A' \to K$ such that $m = k \circ k'$. Noting that $s \circ k = 0$ and using the universal property of m we find a unique morphism $d' : K \to A'$ such that $m = k \circ d'$. Using the morphisms k' and d' we can see that $m : A' \to A$ acts as a kernel for $c = Coker(m)$, i.e. has the universal property. □

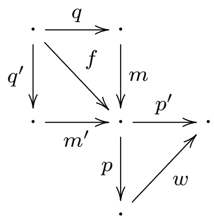
Lemma 4.3.5. *Let $\mathcal{C}$ be an abelian category. A morphism is an isomorphism if and only if it is a monomorphism and a epimorphism.*

Proof. The only if part is trivial. Suppose that $s : A \to B$ is both a monomorphism and a epimorphism. Because s is monic, the morphism $B \to 0$ defines a cokernel, where 0 denotes the zero object. The morphism $id_B : B \to B$ defines a kernel for this cokernel. By the previous lemma $s : A \to B$ also defines a kernel for $Coker(s)$. The universal property we can conclude that there is a morphism $t_1 : B \to A$ such that $id_B = s \circ t_1$. Note that $0 \to A$ is a kernel for $s : A \to B$. Both $id_A : A \to A$ and $s : A \to B$ define cokernels for $0 \to A$. This provides us with a map $t_2 : B \to A$ such that $id_A = t_2 \circ s$. We can conclude that $s : A \to B$ defines an isomorphism. □

In the proof of the following proposition we will use equalizers. Let $r, s : A \to B$ be two arrows. An equalizer for these arrows (called a difference kernel in Freyd) is a morphism $e : K \to A$ such that $re = se$ and the following universal property is satisfied. Let $t : X \to A$ be a morphism such that $rt = st$, then there is a unique morphism $d : X \to K$ such that $t = ed$. For our abelian categories, which are additive, equalizers always exist and the equalizer of f and g is just the kernel of $f - g$. The dual notion of an equalizer, the coequalizer will not be used in this text.

Proposition 4.3.6. *Let $\mathcal{C}$ be an abelian category and $f : X \to Y$ be any morphism. Then there exists a factorization $f = m \circ e$ where $e : X \to Z$ is epi and $m : Z \to Y$ is monic. Given another factorization $f = m' \circ e'$ with $e' : X \to Z'$ epi and $m' : Z' \to Y$ there exists an isomorphism $u : Z \to Z'$ such that $e' = u \circ e$ and $m = m' \circ u$.*

Proof. Define $m = ker(coker(f))$, then it is clear that m is monic and that there is a morphism q such that $f = m \circ q$. Suppose that we have another factorization $f = m'q'$ where m' is a monomorphism. By Lemma 4.3.4 we can write $m' = ker(p')$ where $p' = coker(m')$. Likewise take $p = coker(m) = coker(f)$. By definition of p' we have $p'm' = 0$ from which it follows that $p'f = p'm'q' = 0$. Because $p = coker(f)$ there exists a unique arrow w such that $p' = wp$. The reasoning thusfar is given in the following commutative diagram



The factorization $p' = wp$ shows us that $p'm = wpm = 0$. Therefore m factors through $m' = ker(p')$ as $m = m'u$. Consequently $m'q' = mq = m'uq$, so using the fact that m' is monic we find that $q' = uq$. Assume for a moment that q and q' are epimorphisms (which we still need to show). We know that u is monic because $m'u$ is monic and u is epic because uq is epic. By Lemma 4.3.5, u is an isomorphism. At this stage we need to show that q in the canonical factorization $f = mq$ is an epimorphism.

Take any two parallel arrows (both arrows have the same source and target) r and s such that $rq = sq$. In order to show that q is epi we need to show that $r = s$. Let e be the equalizer of r and s. By the definition of e the morphism q factors as $q = eq'$ for a unique arrow q'. Now $f = mq = meq' = m'q'$ where $m' = me$ is a monomorphism. As $\mathcal{C}$ is abelian, m' must be the kernel of some morphism. This implies that there is a unique arrow t such that $m = m't = met$. The morphism m is monic so we find $id = et$. The equalizer e has a right inverse. Now we can conclude from $re = se$ that $r = s$, proving that q is an epimorphism. □

Definition 4.3.7. *Let $\mathcal{C}$ be an abelian category and $f : X \to Y$ a morphism. The image of f is the monomorphism $m : Z \to Y$ in the factorization $f = me$ of the previous proposition. Note that the image is defined up to an isomorphism.*

Definition 4.3.8. *Let $\mathcal{C}$ be an abelian category. An object P of $\mathcal{C}$ is called projective*[2] *if, for any given epimorhism $p : A \to B$ and any morphism $b : P \to B$ there is a morphism $a : P \to A$ such that $b = p \circ a$.*

If an abelian category $\mathcal{C}$ also has a tensor structure we call $\mathcal{C}$ an abelian tensor category if the structures are compatible. In order to make this more precise, we first need some definitions.

Definition 4.3.9. *Let $\mathcal{C}$ be an abelian category, and X_i be objects of $\mathcal{C}$. Then a sequence ... $\to X_1 \to X_2 \to X_3 \to$ ... is called exact if for each index i the image of $X_{i-1} \to X_i$ is the kernel of $X_i \to X_{i+1}$. A right-exact sequence is an exact sequence of the form $X_1 \to X_2 \to X_3 \to 0$. A left-exact sequence is an exact sequence of the form $0 \to X_1 \to X_2 \to X_3$.*

[2]We could have defined a projective object as an object X such that the hom functor $Hom(X, \cdot) : \mathcal{C} \to \mathbf{Ab}$ is exact, where $\mathbf{Ab}$ denotes the category of abelian groups. We will not make use of this definition.

As an example consider the following short sequences. $0 \to X \to Y$ is exact if and only if $X \to Y$ is a monomorphism. $0 \to X \to Y \to Z$ is exact if and only if $X \to Y$ is the kernel of $Y \to Z$. $X \to Y \to Z \to 0$ is exact if and only if $Y \to Z$ is the cokernel of $X \to Y$.

Definition 4.3.10. *Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called exact if it carries exact sequences to exact sequences. That is, if $... \to X_1 \to X_2 \to X_3 \to ...$ is exact, then $... \to F(X_1) \to F(X_2) \to F(X_3) \to ...$ is also exact. A functor is called right-exact if it carries right-exact sequences into right-exact sequences. A functor is called left-exact if it carries left-exact sequences into left-exact sequences.*

Looking at the previous examples we can see that a right-exact functor preserves cokernels and a left-exact functor preserves kernels.

Definition 4.3.11. *Let $\mathcal{C}$ be an abelian category that is also a tensor category. Then we call $\mathcal{C}$ an abelian tensor category if $\otimes$ is biadditive on the hom-sets and the functors $\langle X \otimes - \rangle : \mathcal{C} \to \mathcal{C}$, $\langle - \otimes X \rangle : \mathcal{C} \to \mathcal{C}$ that tensor to the left and to the right with X respectively are both right-exact for any object X of $\mathcal{C}$.*

Consequently, if $p : Y \to Z$ is a cokernel for $s : X \to Y$, then $id_W \otimes p : W \otimes Y \to W \otimes Z$ defines a cokernel for $id_W \otimes s : W \otimes X \to W \otimes Y$, and $p \otimes id_W$ defines a cokernel for $s \otimes id_W$. The tensor functors preserve cokernels but not necessarily kernels. Left-exactness however, is not necessary for what follows. As further motivation for the definition, consider $R - Mod$ the concrete category of R-modules over a ring R. The tensor functor in this category is right-exact but generally not left-exact. See for example Lang [25] Chapter XVI for a discussion.

Any TC* category that has a zero object is an abelian tensor category. It is clear that such a category is additive. The fact that the category is abelian follows from semisimplicity of the category. But TC* categories are not the only abelian categories that are important for the proof of Deligne's embedding theorem as we will see in Section 4.5.

4.4 Intermission: Monoids and Modules

This section is mainly based on Appendix A.6 of Müger [30]. In this section we will only scratch the surface of the theory of commutative algebra in abelian symmetric tensor categories, giving only those results that we actually need. Towards the end of this section some results are left for the reader to work out. This is mainly because we want to get back to proving the embedding theorem as soon as possible. The aim is to give enough background material in this section such that the interested reader can work out the omitted details with some confidence.

A monoid is a set M with the following structure. There is a multiplication map $m : M \times M \to M$, $m(x, y) = xy$, which is associative in the sense that $(xy)z = x(yz)$ holds for all $x, y, z \in M$. There is a unit element $1 \in M$ such that $1x = x1 = x$ for each $x \in M$. If we let $\eta : \{1\} \to M$ denote the inclusion map, we can state the monoid axioms in terms of the functions m and η.

$$\begin{array}{ccc} M \times M \times M & \xrightarrow{m \times id} & M \times M \\ {\scriptstyle id \times m}\downarrow & & \downarrow{\scriptstyle m} \\ M \times M & \xrightarrow[m]{} & M \end{array} \qquad \begin{array}{ccccc} \{1\} \times M & \xrightarrow{\eta \times id} & M \times M & \xleftarrow{id \times \eta} & M \times \{1\} \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \rho} & \\ & & M & & \end{array}$$

Here we used the maps $\lambda : \{1\} \times M \to M$, $\lambda(1, x) = x$ and $\rho : M \times \{1\} \to M$, $\rho(x, 1) = x$. This definition of a monoid in terms of arrows serves as a motivation for the following definition.

Definition 4.4.1. *Let $(\mathcal{C}, \otimes, \mathbb{1}, \lambda, \rho, \alpha)$ be a tensor category. A monoid in $\mathcal{C}$ is a triple (Q, m, η), where Q is an object of $\mathcal{C}$, $m : Q \otimes Q \to Q$ and $\eta : \mathbb{1} \to Q$ are morphisms satisfying*

$$\begin{array}{ccccc} Q \otimes (Q \otimes Q) & \xrightarrow{\alpha} & (Q \otimes Q) \otimes Q & \xrightarrow{m \otimes id} & Q \otimes Q \\ {\scriptstyle id \otimes m}\downarrow & & & & \downarrow{\scriptstyle m} \\ Q \otimes Q & & \xrightarrow{\qquad m \qquad} & & Q \end{array}$$

$$\begin{array}{ccccc} \mathbb{1} \otimes Q & \xrightarrow{\eta \otimes id} & Q \otimes Q & \xleftarrow{id \otimes \eta} & Q \otimes \mathbb{1} \\ & {\scriptstyle \lambda}\searrow & \downarrow{\scriptstyle m} & \swarrow{\scriptstyle \rho} & \\ & & Q & & \end{array}$$

For strict tensor categories the diagrams simplify to

$$m \circ (m \otimes id_Q) = m \circ (id_Q \otimes m), \quad m \circ \eta \otimes id_Q = id_Q = m \circ id_Q \otimes \eta.$$

If the strict tensor category is braided then the monoid is called commutative if $m \circ c_{Q,Q} = m$.

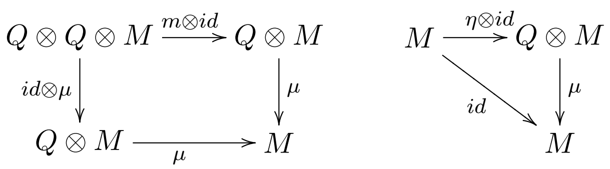
Lemma 4.4.2. *Let $\mathcal{C}$ be a symmetric strict tensor category with symmetry c and let (Q_1, m_1, η_1), (Q_2, m_2, η_2) be commutative monoids in $\mathcal{C}$. Then $(Q_1 \otimes Q_2, m_{Q_1 \otimes Q_2}, \eta_{Q_1 \otimes Q_2})$, where*

$$m_{Q_1 \otimes Q_2} = m_1 \otimes m_2 \circ id_{Q_1} \otimes c_{Q_2, Q_1} \otimes id_{Q_2}, \quad \eta_{Q_1 \otimes Q_2} = \eta_1 \otimes \eta_2,$$

defines a commutative monoid in $\mathcal{C}$. This monoid is called the direct product monoid

The proof of this lemma, which amounts to checking that all the axioms of a monoid are satisfied, is left to the reader. The properties follow from the corresponding properties of (Q_1, m_1, η_1) and (Q_2, m_2, η_2) and naturality of the symmetry c. As the direct product is strictly associative, we can define multiple products inductively. If, for two concrete monoids M_1 and M_2 we let $c_{M_1,M_2} : M_1 \times M_2 \to M_2 \times M_1$ be the flip map $c_{M_1,M_2}(x, y) = (y, x)$ then the above definition for the direct product monoid coincides with the standard defintion of a direct product monoid in the concrete case.

Definition 4.4.3. *Let $\mathcal{C}$ be a strict tensor category and (Q, m, η) a monoid in $\mathcal{C}$. A Q-Module in $\mathcal{C}$ is a pair (M, μ) where M is an object in $\mathcal{C}$ and $\mu : Q \otimes M \to M$ is a morphism such that the following diagrams are commutative*



A morphism $s : (M, \mu) \to (R, \rho)$ of Q-modules is a morphism $s \in Hom_{\mathcal{C}}(M, R)$ which is equivariant in the sense that $s \circ \mu = \rho \circ id_Q \otimes s$.

It is easily checked that the Q-modules in $\mathcal{C}$ together with their morphisms form a category, which we denote by $Q - Mod_{\mathcal{C}}$. It is also easy to verify that if $\mathcal{C}$ is k-linear, then $Q - Mod_{\mathcal{C}}$ is k-linear. Following the notation from Müger [30] we will denote the hom-sets in the category $Q - Mod_{\mathcal{C}}$ by $Hom_Q(\cdot, \cdot)$.

If (Q, m, η) is monoid in a strict tensor category $\mathcal{C}$, then it follows straight from the definitions that (Q, m) is a Q-module. Another simple construction of Q-modules is the following. Let (M, μ) be a Q-module. Then for any object X in $\mathcal{C}$ the pair $(M \otimes X, \mu \otimes id_X)$ defines a Q-module. Combining these two statements we find the free Q-module $(Q \otimes X, m \otimes id_X)$ for every object X of $\mathcal{C}$.

Suppose that $\mathcal{C}$ is a strict tensor category that has direct sums. If (M_1, μ_1) and (M_2, μ_2) are two Q-modules and $R \cong M_1 \oplus M_2$, then we can construct a Q-module (R, ρ). Suppose that $w_i : M_i \to R$ are the isometries corresponding to the direct sum decomposition of R into the M_i. We define

$$\rho = w_1 \circ \mu_1 \circ id_Q \otimes w_1^* + w_2 \circ \mu_2 \circ id_Q \otimes w_2^*.$$

$$\begin{aligned}
\rho \circ id_Q \otimes \rho &= \sum_{i,j} w_i \circ \mu_i \circ id_Q \otimes w_i^* \circ id_Q \otimes w_j \circ id_Q \otimes \mu_j \circ id_{Q\otimes Q} \otimes w_j^* \\
&= \sum_i w_i \circ \mu_i \circ id_Q \otimes \mu_i \circ id_{Q\otimes Q} \otimes w_i^* \\
&= \sum_i w_i \circ \mu_i \circ m \otimes id_{M_i} \circ id_{Q\otimes Q} \circ w_i^* \\
&= \sum_{i,j} w_i \circ \mu_i \circ id_Q \otimes w_i^* \circ id_Q \otimes w_j \circ m \otimes id_{M_i} \circ id_{Q\otimes Q} \otimes w_j^* \\
&= \rho \circ m \otimes id_R
\end{aligned}$$

$$\begin{aligned}
\rho \circ \eta \otimes id_R &= \sum_{i,j} w_i \circ \mu_i \circ id_Q \otimes w_i^* \circ id_Q \otimes w_j \circ \eta \otimes id_{M_j} \circ w_j^* \\
&= \sum_i w_i \circ \mu_i \circ \eta \otimes id_{M_i} \circ w_i^* = \sum_i w_i \circ w_i^* = id_R
\end{aligned}$$

This shows that (R, ρ) is a Q-module. If we inductively apply this construction to Q-module (Q, m), we can construct the n-fold direct sum $n \cdot (Q, m)$ of (Q, m) as a Q-module for any natural number n.

The next definition shows that with each abstract monoid (Q, m, η) in a (braided) strict tensor category $\mathcal{C}$ we can associate a concrete monoid Γ_Q in the category of (small) sets.

Definition 4.4.4. *Let $\mathcal{C}$ be a (braided) strict tensor category and $\mathbb{1}$ be a tensor unit. Suppose that (Q, m, η) is a (commutative) monoid in $\mathcal{C}$. Consider the set $\Gamma_Q = Hom_{\mathcal{C}}(\mathbb{1}, Q)$. Defining multiplication by $s \Box t = m \circ t \otimes s$ and taking $\eta : \mathbb{1} \to Q$ as a unit, Γ_Q becomes a (commutative) monoid.*

The claim that Γ_Q is a monoid follows directly from the fact that (Q, m, η) is a monoid. In the case that $\mathcal{C}$ is braided and (Q, m, η) commutative, commutativity of $(\Gamma_Q, \Box, \eta)$ follows from

$$s \Box t = m \circ t \otimes s = m \circ c_{Q,Q} \circ t \otimes s = m \circ s \otimes t = t \Box s.$$

Let again (Q, m, η) be a monoid in a strict tensor category $\mathcal{C}$. Then set $End_Q((Q, m))$ of Q-module endomorphisms is a concrete monoid where multiplication is given by composition and the identity map plays the role of identity element. The following lemma shows that this monoid is isomorphic to the previously defined Γ_Q.

Lemma 4.4.5. *Let $\mathcal{C}$ be a strict tensor category and (Q, m, η) a monoid in $\mathcal{C}$. Then the following maps define an isomorphism of monoids.*

$$\gamma : End_Q((Q, m)) \to (\Gamma_Q, \Box, \eta), \quad u \mapsto u \circ \eta,$$

$$\gamma^{-1} : (\Gamma_Q, \Box, \eta) \to End_Q((Q, m)), \quad s \mapsto m \circ id_Q \circ s.$$

Proof. We start by showing that γ and γ^{-1} are indeed 2-sided inverses of each other.

$$\gamma(\gamma^{-1}(s)) = m \circ id_Q \otimes s \circ \eta = m \circ \eta \otimes id_Q \circ s = s.$$

In the third equality we used one of the defining properties of a monoid.

$$\gamma^{-1}(\gamma(u)) = m \circ id_Q \otimes u \circ id_Q \otimes \eta = u \circ m \circ id_Q \otimes \eta = u.$$

In the second equality we used equivariance of $u \in End_Q((Q, m))$ and in the third equality we used one of the defining properties of a monoid. We have thus shown that γ defines a bijection. Note that $\gamma(id_Q) = \eta$ so we only need to show that γ preserves multiplication.

$$\begin{aligned}\gamma^{-1}(s) \circ \gamma^{-1}(t) &= m \circ (id_Q \otimes s) \circ m \circ (id_Q \otimes t) = m \circ m \otimes id_Q \circ id_Q \otimes t \otimes s \\ &= m \circ id_Q \otimes m \circ id_Q \otimes t \otimes s = m \circ id_Q \otimes (m \circ t \otimes s) \\ &= \gamma^{-1}(s \square t)\end{aligned}$$

If the second equality isn't clear, draw a diagram. The third equality is again a defining property of a monoid. This completes the proof that γ defines an ismorphism of monoids. $\square$

Note that if $\mathcal{C}$ in the previous lemma is k-linear, then the isomorphism of monoids turns into an isomorphism of k-algebras. Also note that if $\mathcal{C}$ is braided and the monoid (Q, m, η) is commutative, then $End_Q((Q, m))$ is a commutative monoid. From this point onwards we restrict our attention to abelian strict tensor categories.

Proposition 4.4.6. *Let $\mathcal{C}$ be an abelian strict tensor category and (Q, m, η) be a monoid in $\mathcal{C}$. Then the category $Q - Mod_{\mathcal{C}}$ is an abelian category.*

Proof. As there are a lot of things to check, this proof won't be short. The proof may help readers that have no experience with abelian categories to get some feeling for the subject. We start with showing that $Q - Mod_{\mathcal{C}}$ is an additive category. Let 0 denote the zero object of $\mathcal{C}$. We need to give this object the structure of a Q-module such that it acts as a zero object in $Q - Mod_{\mathcal{C}}$. There should exist a morphism $\mu : Q \otimes 0 \to 0$. By definition of 0 there is exactly one such morphism. We know that $(0, \mu)$ is a Q-module if it satisfies $\mu \circ id_Q \otimes \mu = \mu \circ m \otimes id_0$ and $\mu \circ \eta \otimes id_0 = id_0$. Again this holds by definition of the zero object. The zero object of $\mathcal{C}$ thus provides a suitable zero object for $Q - Mod_{\mathcal{C}}$.

Next we check that $Q - Mod_{\mathcal{C}}$ is pre-additive. This follows trivially from the fact that $\mathcal{C}$ is pre-additive. The last step in making $Q - Mod_{\mathcal{C}}$ additive is checking that it has biproducts. Suppose that (X, μ_X) and (Y, μ_Y) are two Q-modules. We would like to construct a biproduct Q-module (Z, μ_Z). As $\mathcal{C}$ is additive we can construct a biproduct object $Z \cong Z \oplus Y$. Let $i_1 : X \to Z$,

$i_2 : Y \to Z$, $p_1 : Z \to X$ and $p_2 : Z \to Y$ be the morphisms effecting the direct sum decomposition. Define $\mu_Z : Q \otimes Z \to Z$ by

$$\mu_Z = id_Q \otimes p_1 \circ \mu_X \circ i_1 + id_Q \otimes p_2 \circ \mu_Y \circ i_2.$$

We need to show that $\mu_Z \circ id_Q \otimes \mu_Z = \mu_Z \circ m \otimes id_Z$ and $\mu_Z \circ \eta \otimes id_Z = id_Z$ are satisfied, proving that (Z, μ_Z) is a Q-module. This is easy to check if we use $p_1 i_1 = id_X$, $p_2 i_2 = id_Y$ and the identity

$$m \otimes id_Z = id_Q \otimes i_1 \circ m \otimes id_X \circ id_{Q \otimes Q} \otimes p_1 + id_Q \otimes i_2 \circ m \otimes id_Y \circ id_{Q \otimes Q} \otimes p_2.$$

After checking this, we are not finished with additivity. The morphism i_k, p_k that provide the decomposition should be morphisms in $Q - Mod_{\mathcal{C}}$, not just morphisms in $\mathcal{C}$. For i_1 and p_1 this amount to the following demands

$$i_1 \circ \mu_X = \mu_Z \circ id_Q \otimes i_1, \qquad p_1 \circ \mu_Z = \mu_X \circ id_Q \otimes p_1.$$

Filling in the given expression for μ_Z and armed with the fact that $p_1 i_2 = 0 = p_2 i_1$ we find that these equations are indeed satisfied. The approach for proving equivariance of i_2 and p_2 is the same. Thus we have proven that $Q - Mod_{\mathcal{C}}$ is an additive category if $\mathcal{C}$ is additive. Consider a morphism of Q-modules $f : (X, \mu_X) \to (Y, \mu_Y)$. We need to show that this morphism has both a kernel and a cokernel in $Q - Mod_{\mathcal{C}}$. Let $Z \to X$ be the kernel of $f : X \to Y$ in $\mathcal{C}$. Such a map exists as $\mathcal{C}$ is abelian. We need to make Z into a Q-module. Define $k' : Q \otimes Z \to X$ by $k' = \mu_X \circ id_Q \otimes k$. Then, using equivariance of f, we find

$$f \circ k' = f \circ \mu_X \circ id_Q \otimes k = \mu_Y \circ id_Q \otimes f \circ id_Q \circ k = 0$$

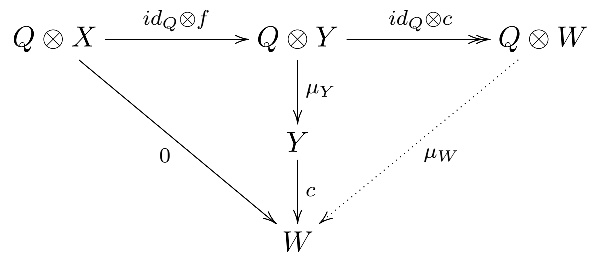
because $f \circ k = 0$. By the universal propery of k there is a unique map $\mu_Z : Q \otimes Z \to Z$ such that $k \circ \mu_Z = k' = \mu_X \circ id_Q \otimes k$. Consider the calculation

$$\begin{aligned} k \circ \mu_Z \circ id_Q \otimes \mu_Z &= \mu_X \circ id_Q \otimes k \circ id_Q \otimes \mu_Z = \mu_X \circ id_Q \otimes \mu_X \circ id_{Q \otimes Q} \otimes k \\ &= \mu_X \circ m \otimes id_X \circ id_{Q \otimes Q} \otimes k = \mu_X \circ id_Q \otimes k \circ m \otimes id_Z \\ &= k \circ \mu_Z \circ m \otimes id_Z. \end{aligned}$$

In the third equality we used that (X, μ_X) is a Q-module. Because k is monic it follows that $\mu_Z \circ id_Q \otimes \mu_Z = \mu_Z \circ m \otimes id_Z$. The identity $\mu_Z \circ \eta \otimes id_Z = id_Z$ follows from

$$\begin{aligned} k \circ \mu_Z \circ \eta \otimes id_Z &= \mu_X \circ id_Q \otimes k \circ \eta \otimes id_Z \\ &= \mu_X \circ \eta \otimes id_X \circ k = id_X \circ k = k. \end{aligned}$$

This proves that the source of the kernel Z is, in a natural way, a Q-module. Next notice that the equality $k \circ \mu_Z = \mu_X \circ id_Q \otimes k$ is exactly what we need to conclude that $k : (Z, \mu_Z) \to (X, \mu_X)$ is a Q-module morphism. We have proven that every morphism in $Q - Mod_{\mathcal{C}}$ has a kernel. Next we show that every morphism has a cokernel. Unfortunately we cannot simply apply to duality and prove it from the existence of kernels. Again consider a morphism $f : (X, \mu_X) \to (Y, \mu_Y)$ in $Q - Mod_{\mathcal{C}}$. As this is also a morphism in $\mathcal{C}$, which is an abelian category, there exists a cokernel $c : Y \to W$ in $\mathcal{C}$. Just like before we want to make W into a Q-module in a natural way, such that c becomes equivariant. Tensor product preserve cokernels thus we know that $id_Q \otimes c$ is a cokernel for $id_Q \otimes f$. Consider the following commutative diagram, where we made use of the equivariance of f



The universal property of $id_Q \otimes c$ provides us with a morphism $\mu_W : Q \otimes W \to W$ which satisfies $c \circ \mu_Y = \mu_W \circ id_Q \otimes c$. This equality shows that if (W, μ_W) is a Q-module, then $c : (Y, \mu_Y) \to (W, \mu_W)$ is a morphism in $Q - Mod_{\mathcal{C}}$. We will check $\mu_W \circ id_Q \otimes \mu_W = \mu_W \circ m \otimes id_W$, leaving the simpler identity $\mu_W \circ \eta \otimes id_W = id_W$ to the reader. If we note that $id_{Q \otimes Q} \otimes c$ is an epimorphism, then the desired identity follows from the next calculation.

$$\begin{aligned}
\mu_W \circ id_Q \otimes (\mu_W \circ id_Q \otimes c) &= \mu_W \circ id_Q \otimes c \circ id_Q \otimes \mu_Y = c \circ \mu_Y \circ id_Q \otimes \mu_Y \\
&= c \circ \mu_Y \circ m \otimes id_Y = \mu_W \circ id_Q \otimes c \circ m \otimes id_Y \\
&= \mu_W \circ m \otimes id_W \circ id_{Q \otimes Q} \otimes c
\end{aligned}$$

This proves that every morphism in $Q - Mod_{\mathcal{C}}$ has a cokernel in this category. We still need to check 2 properties. Every monomorphism should be the kernel of some morphism and every epimorphism should be the cokernel of some morphism. To a large extent the reasoning is the same as above so the last 2 properties are left for the reader. A few hints are in order. One can prove that a monomorphism/epimorphism in $Q - Mod_{\mathcal{C}}$ is also a monomorphism/epimorphism in $\mathcal{C}$. Secondly, Lemma 4.3.4 can provide a lot of help in finding Q-module structures for the relevant objects. □

Definition 4.4.7. *Let $\mathcal{C}$ be a abelian strict tensor category and (Q, m, η) a commutative monoid in $\mathcal{C}$. An ideal in this monoid is a monomorphism $j : (J, \mu_J) \hookrightarrow (Q, m)$ in $Q - Mod_{\mathcal{C}}$. The ideal is called proper if it is not an isomorphism. Let $j' : (J', \mu_{J'}) \hookrightarrow (Q, m)$ be another ideal, then we say that j is contained in j', denoted $j \prec j'$, if there is a monomorphism $i : (J, \mu_J) \hookrightarrow (J', \mu_{J'})$ in $Q - Mod_{\mathcal{C}}$ such that $j = j' \circ i$. A proper ideal j is called maximal if every proper ideal j' which contains j is isomorphic to j.*

It doesn't take a lot of imagination to guess what inspired these definitions. Using these definitions we can mimic constructions from abstract algebra. We will proceed with discussing only those constructions, which we will actually need. Recall that an essentially small category is a category that is equivalent to a small category. The isomorphism classes of objects thus form a set.

Lemma 4.4.8. *Let $\mathcal{C}$ be an essentially small abelian strict symmetric tensor category and (Q, m, η) be a commutative monoid in $\mathcal{C}$. Then every proper ideal $j : (J, \mu_J) \hookrightarrow (Q, m)$ is contained in some maximal ideal $\tilde{j} : (\tilde{J}, \mu_{\tilde{J}}) \hookrightarrow (Q, m)$.*

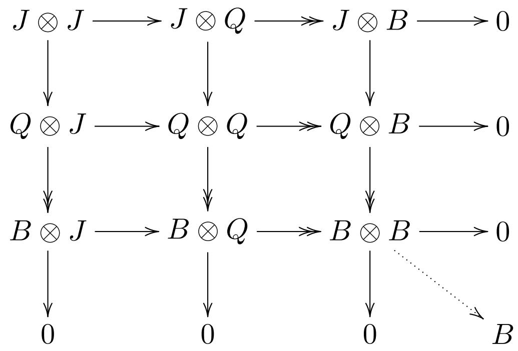
Proof. We call two ideals $j : (J, \mu_J) \hookrightarrow (Q, m)$ and $j' : (J', \mu_{J'}) \hookrightarrow (Q, m)$ isomorphic if there is an isomorphism $(J, \mu_J) \to (J', \mu_{J'})$. The category $\mathcal{C}$ is essentially small, so if we consider isomorphism classes of ideals, these classes will form a set. The relation $\prec$ defines a partial order on the isomorphism classes. We will only check if the relation is symmetric as reflexivity and transitivity are straightforward. Suppose that $j \prec j'$ and $j' \prec j$. Then there exist monomorphisms k and l such that $j' \circ k = j$ and $j \circ l = j'$. This implies that $j' \circ k \circ l = j'$ and $j \circ l \circ k = j$. Both j and j' are monic so this implies that k has l as a 2-sided inverse. j and j' are isomorphic and thus belong to the same class. Maximal ideals correspond to the maximal elements in this set of isomorphism classes. The lemma follows from applying Zorn's lemma to this partially ordered set where every chain of ideal classes clearly has an upper bound. □

Proposition 4.4.9. *Let $\mathcal{C}$ be an abelian symmetric strict tensor category, (Q, m, η) a commutative monoid in $\mathcal{C}$ and $j : (J, \mu_J) \hookrightarrow (Q, m)$ an ideal in (Q, m, η). Let $p : (Q, m) \twoheadrightarrow (B, \mu_B)$ be the cokernel of j. Then there exist unique morphisms $m_B : B \otimes B \to B$ and $\eta_B : \mathbb{1} \to B$ such that*

1. *(B, m_B, η_B) is a commutative monoid in $\mathcal{C}$,*
2. *$p \circ m = m_B \circ p \otimes p$,*
3. *$p \circ \eta = \eta_B$.*

The monoid (B, m_B, η_B) is called the quotient of (Q, m, η) by the ideal j. It is nontrivial (B is not a zero object) if and only if the ideal is proper.

Proof. We need to construct the morphisms m_B and η_B that make B into a commutative monoid. The morphism η_B is easy. It is defined by property 3, so we just take $\eta_B = p \circ \eta : \mathbb{1} \to Q \to B$. For m_B we consider the following commutative diagram.



The labels of the morphisms were left out of the diagram. Each arrow is the obvious one. The dotted arrow shows the arrow m_B that we wish to find. We will prove the existence of such a morphism by the technique known in category theory as diagram chasing. Finding an arrow $B \otimes B \to B$ in the above diagram is equivalent to finding an arrow $B \otimes Q \to B$ such that $B \otimes J \to B \otimes Q \to B$ is equal to the 0 morphism. This follows from the universal property of $B \otimes Q \twoheadrightarrow B \otimes B$. Next note that $B \otimes J \to B \otimes Q \to B$ is equal to 0 if and only if $Q \otimes J \twoheadrightarrow B \otimes J \to B \otimes Q \to B$ is equal to 0. We need migrate one step further into the diagram. Finding an arrow $B \otimes Q \to B$ is equivalent to finding an arrow $Q \otimes Q \to B$ such that $J \otimes Q \to Q \otimes Q \to B$ is equal to 0. If we take all this together we need to find a morphism $Q \otimes Q \to B$ such that $Q \otimes J \to Q \otimes Q \to B$ is 0 and $J \otimes Q \to Q \otimes Q \to B$ is 0. The obvious candidate is $p \circ m : Q \otimes Q \to Q \to B$. The demand that $Q \otimes J \to Q \otimes Q \to B$ is 0 means that $p \circ m \circ id_Q \otimes j$ is equal to 0. This holds as $p \circ m = \mu_B \circ id_Q \otimes p$ and $p \circ j = 0$. Recall that (Q, m, η) is commutative. This implies that $p \circ m = p \circ m \circ c_{Q,Q}$. From this we can conclude that also $J \otimes Q \to Q \otimes Q \to B$ is 0. As both properties are satisfied $p \circ m : Q \otimes Q \to B$ is indeed as suitable map. The diagram chasing method also automatically gives us the relation $p \circ m = m_B \circ p \otimes p$.

The only thing left to check is that (B, m_B, η_B) as defined above is indeed a commutative monoid. The relation $m_B \circ m_B \otimes id_B = m_B \circ id_B \otimes m_B$ follows from

$$\begin{aligned} m_B \circ m_B \otimes id_B \circ p \otimes p \otimes p &= m_B \circ id_B \otimes p \circ m_B \otimes id_Q \circ p \otimes p \otimes id_Q \\ &= m_B \circ id_B \otimes p \circ p \otimes id_B \circ m \otimes id_Q \\ &= p \circ m \circ m \otimes id_Q \\ &= p \circ m \circ id_Q \otimes m = m_B \circ p \otimes p \circ id_Q \otimes m \\ &= m_B \circ id_B \otimes m_B \circ p \otimes p \otimes p. \end{aligned}$$

The morphism $p \otimes p \otimes p$ is an epimorphism as it is the composition of only epimorphisms. Thus we have proven the first identity. The other identities can be proven in the same way. Straightforward checks reveal that

$$m_B \circ \eta_B \otimes id_B \circ p = p = m_B \circ id_B \otimes \eta_B \circ p,$$

$$m_B \circ c_{B,B} \circ p \otimes p = m_B \circ p \otimes p.$$

We have thus proven that (B, m_B, η_B) is a commutative monoid in $\mathcal{C}$.

Recall that $p : (Q, m) \to (B, \mu_B)$ was defined as the cokernel of j. From this it follows that B is not equal to the zero object if and only if j is not an epimorphism. We know that j is a monomorphism so this is the case if and only if j is not an isomorphism. In other words, if and only if j is a proper ideal. □

Lemma 4.4.10. *Let $\mathcal{C}$ be an abelian symmetric strict tensor category, let (Q, m, η) be a commutative monoid in $\mathcal{C}$ and $j : (J, \mu_J) \hookrightarrow (Q, m)$ an ideal in (Q, m, η). Let (B, m_B, η_B) be the the quotient of (Q, m, η) by the ideal j. Then the map $p_\Gamma : \Gamma_Q \to \Gamma_B$ defined by $s \mapsto p \circ s$ is a homomorphism of commutative algebras. The map is surjective if the tensor unit $\mathbb{1} \in \mathcal{C}$ is a projective object.*

Proof. Checking that the map is a homomorphism of commutative algebras is straightforward when using the previously proven relations $p \circ m = m_B \circ p \otimes p$ and $p \circ \eta = \eta_B$. Now suppose that $\mathbb{1}$ is a projective object. Then for each morphism $t : \mathbb{1} \to B$ there is a morphism $s : \mathbb{1} \to Q$ such that $s = p \circ t$. This shows that p_Γ is surjective. □

Proposition 4.4.11. *Let $\mathcal{C}$ be an essentially small abelian symmetric strict tensor category, (Q, m, η) a commutative monoid in $\mathcal{C}$ and $j : (J, \mu_J) \hookrightarrow (Q, m)$ an ideal in (Q, m, η). Let (B, m_B, η_B) be the the quotient of (Q, m, η) by the ideal j. Then there is a bijective correspondence between the equivalence classes of ideals in (B, m_B, η_B) and equivalence classes of ideals in (Q, m, η) that contain j. In particular, if j is a maximal ideal then all ideals in (B, m_B, η_B) are either 0 or isomorphic to (B, m_B).*

The proof of this proposition is left to the reader as a exercise. The construction is basically the same as in abstract algebra, at least when it is formulated in terms of functions instead of elements of sets. The next lemma will only be used at the very end of this chapter.

Lemma 4.4.12. *Let k be a field, $\mathcal{C}$ be an abelian k-linear symmetric strict tensor category and (Q, m, η) a commutative monoid in $\mathcal{C}$. If every ideal in (Q, m, η) is either 0 or isomorphic to (Q, m), then the commutative unital k-algebra $End_Q((Q, m))$ is a field.*

Proof. Suppose that $s \in End_Q((Q, m))$ and $s \neq 0$. We will show that s is an isomorphism. Recall that the image of s was defined as the monomorphism in the epic/monic factorization of Proposition 4.3.6. The image of s, $im(s)$

is an ideal in (Q, m, η). The demand that $s \neq 0$ gives $im(s) \neq 0$, therefore $im(s)$ must be an isomorphism. As $im(s)$ is an epimorphism, so is s. Next consider $ker(s)$, the kernel of s. Again this is an ideal in (Q, m, η). The kernel cannot be isomorphic to (Q, m) since $s \neq 0$. The kernel must be equal to 0, proving that s is a monomorphism. The morphism s is both monic and epic. In the abelian category $Q - Mod_{\mathcal{C}}$ this implies that s is an isomorphism. Every nonzero morphism is invertible making $End_Q((Q, m))$ into a field. This field has k as a subfield. □

Lemma 4.4.13. *Let $\mathcal{C}$ be an abelian symmetric strict tensor category and (Q, m, η) be a commutative monoid in $\mathcal{C}$. Then every epimorphism of Q-modules in $End_Q((Q, m))$ is an isomorphism.*

Proof. The category $Q - Mod_{\mathcal{C}}$ can be given, in a natural way the structure of a tensor category. The construction mimics that of abstract algebra. Suppose that (X, μ_X) and (Y, μ_Y) are Q-modules. With a slight abuse of notation (identifying a cokernel with the target of this morphism) we define the tensor product Q-module by

$$X \otimes_Q Y = coker(id_X \otimes \mu_Y - \mu_X \otimes id_Y \circ c_{X,Q} \otimes id_Y).$$

Using this definition $Q - Mod_{\mathcal{C}}$ becomes a tensor category where the tensor unit is given by (Q, m). Working out the details of this construction is left as an exercise for the reader.

Let $g \in End_Q((Q, m))$ be an epimorphism and $j : (J, \mu_J) \hookrightarrow (Q, m)$ an ideal in (Q, m, η). Using the fact that $Q - Mod_{\mathcal{C}}$ is a tensor category with tensor unit (Q, m) there exists an isomorphism of Q-modules $s : (J, \mu_J) \to (Q \otimes_Q J, \mu_{Q \otimes_Q J})$. Define $h \in End_Q((J, \mu_J))$ to be the composition

$$(J, \mu_J) \xrightarrow{s} (Q \otimes_Q J, \mu_{Q \otimes_Q J}) \xrightarrow{g \otimes id_J} (Q \otimes_Q J, \mu_{Q \otimes_Q J}) \xrightarrow{s^{-1}} (J, \mu_J).$$

The tensor product of Q-modules, $\otimes_Q$ is right-exact, thus $g \otimes id_J$ is an epimorphism. Recognizing $s = \lambda_J$ as a component of a natural transformation, we can use naturality of s to show that $j \circ h = g \circ j$. If we take for j the ideal $ker(g)$, then this becomes $j \circ h = 0$. The morphism h is an epimorphism thus $ker(g) = j = 0$. The epimorphism g turns out to be a monomorphism, proving that g is an isomorphism. □

4.5 Reduction to Finitely Generated Categories

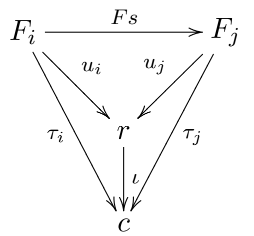
In this section we take 3 important steps towards proving Deligne's embedding theorem. We start off with the last categorical preparations, which are centered around the use of colimits. Subsequently we reduce the proof of Deligne's embedding theorem to the simpler setting of finitely generated STC* categories. In the last step we show that the proof of the embedding theorem is equivalent to finding an absorbing monoid in a suitable enlarged category. This will be explained shortly, but before doing so, we start with some more category theory.

Suppose that $\mathcal{C}$ and $\mathcal{J}$ are categories. We define the functor category $\mathcal{C}^{\mathcal{J}}$ as follows. The objects of this category are functors $\mathcal{J} \to \mathcal{C}$ and the arrows are natural transformations between those functors. The reader can check that this defines a category. Next define the diagonal functor

$$\Delta : \mathcal{C} \to \mathcal{C}^{\mathcal{J}}$$

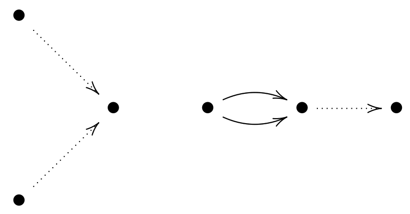
that sends each object $c \in \mathcal{C}$ to the constant functor Δc. This functor maps each object of $\mathcal{J}$ to c in $\mathcal{C}$ and maps each arrow of $\mathcal{J}$ to id_c. If $f : c \to c'$ is an arrow in $\mathcal{C}$, then Δf is the natural transformation $\Delta f : \Delta c \to \Delta c'$ that has the same value f for each object of $\mathcal{J}$. The category $\mathcal{J}$ acts as a index category and is considered to be small from this point onwards.

Definition 4.5.1. *Let $F : \mathcal{J} \to \mathcal{C}$ be a functor. A colimit or inductive limit for F is a universal arrow $\langle r, u \rangle$ from F to the diagonal functor Δ. This means the following. First of all there exists an object $r \in \mathcal{C}$, sometimes denoted $r = \lim_{\to} F$ and a natural transformation $u : F \to \Delta r$. Let F_i denote $F(i)$ for each object $i \in \mathcal{J}$. Then we have arrows $u_i : F_i \to r$ satisfying the property that if $s : i \to j$ is an arrow in $\mathcal{J}$, then $u_j \circ F(s) = u_i$. The pair $\langle r, u \rangle$ satisfies the following universal property. Suppose that c is an object of $\mathcal{C}$ and that $\tau : F \to \Delta c$ is a natural transformation. Then there exists a unique arrow $\iota : r \to c$ such that for each $i \in \mathcal{J}$ we have $\tau_i = \iota \circ u_i$. This implies that the following diagram, where $s : i \to j$ is an arrow in $\mathcal{J}$, commutes*



Notice that we did not guarantee the existence of a colimit for a given functor. We can only say that if it exists, then it is unique up to an isomorphism. The dual notion (obtained by reversing all arrows) of the inductive limit is the projective limit, or limit.

Definition 4.5.2. *Let $\mathcal{I}$ be a non-empty category. Then $\mathcal{I}$ is called a filtered category if the following 2 demands are satisfied. For any 2 objects $X, Y \in \mathcal{I}$ there is an object $Z \in \mathcal{I}$ and arrows $i : X \to Z$, $j : Y \to Z$. For any 2 arrows $u, v : X \to Y$ in $\mathcal{I}$ there is an arrow $w : Y \to Z$ such that $w \circ u = w \circ v$. In* *terms of diagrams this becomes*



Next we define the Ind-category of a category. This category, which has $\mathcal{C}$ as a full subcategory, is very helpful as it contains all (small) inductive limits coming from functors $\mathcal{I} \to \mathcal{C}$ where $\mathcal{I}$ is a small filtered category.

Definition 4.5.3. *Let $\mathcal{C}$ be a category. The ind-category of $\mathcal{C}$ $Ind(\mathcal{C})$ is the functor category whose objects are all functors $F : \mathcal{I} \to \mathcal{C}$, with $\mathcal{I}$ small filtered categories. For two objects $F : \mathcal{I} \to \mathcal{C}$, $G : \mathcal{J} \to \mathcal{C}$ the hom-set is defined as*

$$Hom_{Ind(\mathcal{C})}(F, G) = \varprojlim_{i \in \mathcal{I}} \varinjlim_{j \in \mathcal{J}} Hom_{\mathcal{C}}(F_i, G_j).$$

An element of $Hom_{Ind(\mathcal{C})}(F, G)$ thus consists of a family of arrows $f_{i,j} : F_i \to G_j$ in $\mathcal{C}$ such that for each $s : j \to j'$ in $\mathcal{J}$ and each $t : i \to i'$ in $\mathcal{I}$

$$G(s) \circ f_{i,j} = f_{i,j'}, \quad f_{i',j} \circ F(t) = f_{i,j}$$

holds. For each $i \in \mathcal{I}$ there is some arrow $f_{i,j}$ of $\mathcal{C}$. There need not be a morphism $f_{i,j}$ for every $j \in \mathcal{J}$. Composition of arrows in $Ind(\mathcal{C})$ is defined using the composition of arrows in $\mathcal{C}$. The reader should check that this leads to a well-defined morphism in $Ind(\mathcal{C})$. Let $\mathbf{1}$ denote the category with just 1 object and the identity arrow being the only arrow. Define a functor $F^X : \mathbf{1} \to \mathcal{C}$, mapping the object to an object X and the identity arrow of $\mathbf{1}$ to id_X. The correspondence $X \to F^X$ provides an embedding of $\mathcal{C}$ into $Ind(\mathcal{C})$.

The standard references on the subject of Ind-categories are SGA 4 [1] and Artin and Mazur [2]. Another text on the subject is the book by Kashiwara and Schapira [24]. In order to bridge the gap to these references we take a different view on $\mathcal{C}$ and $Ind(\mathcal{C})$. Both these categories can be viewed as subcategories of the category $\hat{\mathcal{C}}$. The category $\hat{\mathcal{C}}$ is the functor category that has functors $\mathcal{C}^0 \to Set$, from the opposite category of $\mathcal{C}$ to the category of small sets Set, as objects. The arrows are just the natural transformations between the functors. A functor $\mathcal{C}^0 \to Set$ is also called a presheaf[3].

[3]The reader that has encountered algebraic geometry will most likely have seen a different definition of a presheaf. Our definition may be seen as a generalization of this definition

An inductive limit for $F : \mathcal{I} \to \mathcal{C}$ can be seen as an element of $\hat{\mathcal{C}}$ defined as

$$\varinjlim_i F_i : \mathcal{C} \to Set, \quad X \to \varinjlim_i Hom_{\mathcal{C}}(X, F_i) = \varinjlim_i (h_{\mathcal{C}}(F_i)(X))$$

where $h_{\mathcal{C}}(F_i)$ denotes the contravariant hom-functor defined in Section 3.3. The Ind-objects in this context are those elements of $\hat{\mathcal{C}}$ that are isomorphic to an inductive limit $lim_{\to}F$ for some $F : \mathcal{I} \to \mathcal{C}$, where $\mathcal{I}$ is a small filtrant category. The Ind-category is the full subcategory of $\hat{\mathcal{C}}$ that has all the Ind-objects as its objects. The contravariant hom-functor defines a fully faithful functor[4] that embeds $\mathcal{C}$ into $\hat{\mathcal{C}}$

$$h_{\mathcal{C}} : \mathcal{C} \to \hat{\mathcal{C}}, \quad X \mapsto Hom_{\mathcal{C}}(\cdot, X).$$

Using this embedding it follows once more that $\mathcal{C}$ is a full subcategory of $Ind(\mathcal{C})$.

We can check from the definitions that if $\mathcal{C}$ is pre-additive or $\mathbb{C}$-linear, then so is $Ind(\mathcal{C})$. If $\mathcal{C}$ is a strict symmetric tensor category then so is $Ind(\mathcal{C})$. In order to see this we need to define the tensor product of 2 inductive systems $F : \mathcal{I} \to \mathcal{C}$ and $G : \mathcal{J} \to \mathcal{C}$. Take the product category $\mathcal{I} \times \mathcal{J}$, which is again filtered, and define

$$F \otimes G : \mathcal{I} \times \mathcal{J} \to \mathcal{C}, \quad (i, j) \mapsto F_i \otimes G_j.$$

Checking the rest of the claims is straightforward. The following theorem requires more work than the previous claims. For a proof the reader is referred to SGA 4 Expose I-8 [1].

Theorem 4.5.4. *Let $\mathcal{C}$ be a category and $Ind(\mathcal{C})$ be the corresponding ind-category of $\mathcal{C}$. Then $Ind(\mathcal{C})$ contains colimits for every small filtrant category $\mathcal{I}$. If $\mathcal{C}$ is an abelian category, then so is $Ind(\mathcal{C})$.*

The following conjecture comes from [30]. It is stated there as a lemma. The proof of this lemma makes use of the assumption that for an epimorphism in $Ind(\mathcal{C})$ the arrows $f_{i,j}$ are epimorphisms in $\mathcal{C}$ provided that $i \in \mathcal{I}$ is 'large' enough. The author of this paper has thusfar not proven this claim. A weaker version of this result is needed in the proof of Deligne's embedding theorem. This weaker claim is that the tensor unit of a TC* category $\mathcal{C}$ is projective as an object of $Ind(\mathcal{C})$. We will only need this claim at the very end.

Conjecture 4.5.5. *Let $\mathcal{C}$ be a TC* category and X an object of $\mathcal{C}$. Then X regarded as an object of $Ind(\mathcal{C})$ is projective.*

[4] The fact that this functor is full and faithful follows from Yoneda's lemma. See Mac Lane[27] for a treatment of this lemma

The direct sum of two projective objects in an abelian category is projective [16]. As any TC* category is semisimple this implies that it sufficies to show that every irreducible object of $\mathcal{C}$ is projective as an object of $Ind(\mathcal{C})$. To make this more explicit, assume that we have shown every irreducible object to be projective as an element of $Ind(\mathcal{C})$. Let X be a reducible object of $\mathcal{C}$. As such X is the direct sum of a finite number of irreducible objects X_i with a decomposition given by isometries $w_i : X_i \to X$. Let $p : A \to B$ be an epimorphism and $b : X \to B$ a morphism. Define $b_i = b \circ w_i : X_i \to B$. By assumption we find morphisms $\hat{b}_i : X_i \to A$ such that $p \circ \hat{b}_i = b_i$[5]. Define $\hat{b} = \sum_i \hat{b}_i \circ w_i^* : X \to A$. Then

$$p \circ \hat{b} = \sum_i p \circ \hat{b}_i \circ w_i^* = \sum_i b_i \circ w_i^* = \sum_{i,j} b \circ w_j \circ w_i^* = b$$

which proves projectivity of X.

Armed with enough category theory we proceed with the proof of Deligne's embedding theorem. The next step is to reduce the proof to the case of finitely generated STC* categories as these can be attacked using algebraic methods. The following discussion closely follows Müger Appendix B7 [30].

Definition 4.5.6. *Let $\mathcal{C}$ be an additive tensor category. Then $\mathcal{C}$ is called finitely generated if there exists an object Z in $\mathcal{C}$ such that every object X is a direct summand of some tensor power of Z, $X \prec Z^{\otimes n} = Z \otimes ... \otimes Z$, where $n \in \mathbb{N}_0$.*

We define for any object $X^{\otimes 0} = \mathbb{1}$ where $\mathbb{1}$ is the tensor unit. This helps to ensure that any finitely generated tensor category contains a tensor unit. The above definition may seem a bit strange. For example, there is only 1 generator where one would expect multiple, albeit a finite number of generators. Suppose that we would take generators $Z_1, ..., Z_k$. Defining $Z = Z_1 \oplus ... \oplus Z_k$ shows that the additive tensor category generated by Z contains objects of the kind $Z_1^{n_1} \otimes ... \otimes Z_k^{n_k}$, where $n_1, ..., n_k \in \mathbb{N}_0$. The category generated by Z is the same as the category generated by $Z_1, ..., Z_k$.

Lemma 4.5.7. *Let $\mathcal{C}$ be an essentially small STC* category. The finitely generated tensor subcategories of $\mathcal{C}$ that are STC* categories form an directed system. The category $\mathcal{C}$ is the inductive limit of this system.*

Proof. Let $\mathcal{C}_i$ be a finitely generated tensor subcategory of $\mathcal{C}$, then $\mathcal{C}_i$ inherits a lot of structure of $\mathcal{C}$. In order to let $\mathcal{C}_i$ be an STC* category we need to check that it carries a conjugate object for every object in this category. Conjugates can be included by choosing suitable generators. Let Z be the generator of $\mathcal{C}_i$ and $(\overline{Z}, r, \overline{r})$ a conjugate for Z in $\mathcal{C}$. Then the full tensor category generated by $Z \oplus \overline{Z}$ is an STC* that contains $\mathcal{C}_i$. We will only

[5] The definition of a projective object was given in 4.3.8.

consider finitely generated subcategories that have generators that admit conjugates to ensure that every subcategory under consideration is an STC* category.

As $\mathcal{C}$ is essentially small, the equivalence classes of full tensor subcategories form a set. This set is partially ordered by inclusion. Now suppose that we have two finitely generated tensor subcategories that are generated by objects Z_1 and Z_2. Then the full tensor subcategory generated by $Z_1 \oplus Z_2$ contains both, proving that the equivalence classes of finitely generated full tensor subcategories form a directed set. This provides us with a fully faithful tensor functor from the inductive limit of this system $lim_{\rightarrow}\mathcal{C}_i \rightarrow \mathcal{C}$. This functor defines an equivalence of tensor categories if it is shown to be essentially surjective. This is clearly the case as each object is in a finitely generated tensor subcategory, namely the one generated by it (and its conjugate). □

Suppose that we have proven Deligne's embedding theorem in the case of finitely generated even STC* categories. Then the previous lemma will help to prove the embedding theorem for general (essentially small even) STC* categories. More precisely, we will now prove Deligne's embedding theorem assuming the following theorem.

Theorem 4.5.8. *A finitely generated even STC* category $\mathcal{C}$ admits a symmetric fiber functor $E : \mathcal{C} \rightarrow Vect_{\mathbb{C}}$.*

Proof. (of Deligne's Embedding Theorem 4.1.7) By Lemma 4.5.7 we know that $\mathcal{C}$ is the inductive limit of finitely generated STC* categories $\mathcal{C}_i$. By Theorem 4.5.8. each such STC* category admits a symmetric fiber functor $E_i : \mathcal{C}_i \rightarrow Vect_{\mathbb{C}}$. By using Theorem 4.2.5 we can turn these fiber functors into symmetric $*$-preserving fiber functors $E_i : \mathcal{C}_i \rightarrow \mathcal{H}_f$. Tannaka-Krein duality then gives compact groups G_i and representations $\pi_{i,X}$ on the Hilbert spaces $E_i(X)$. We have equivalences of tensor $*$-categories

$$F_i : \mathcal{C}_i \rightarrow Rep_f(G_i, \mathbb{C}), \quad X \mapsto (E_i(X), \pi_{i,X}).$$

Suppose that $i \leq j$ holds, which implies that $\mathcal{C}_i$ is a subcategory of $\mathcal{C}_j$. Restricting E_j to $\mathcal{C}_i$ gives a symmetric $*$-preserving fiber functor $E_j \upharpoonright \mathcal{C}_i : \mathcal{C}_i \rightarrow \mathcal{H}_f$. By the uniqueness of fiber functors, Theorem 3.6.2, there exists a unitary monoidal natural isomorphism $\alpha^{i,j} : E_i \rightarrow E_j \upharpoonright \mathcal{C}_i$. Take an element $g \in G_j$. Then this is a unitary monoidal natural transformation $E_j \rightarrow E_j$. Let X be an object of $\mathcal{C}_i$ and define $h_X = (\alpha_X^{i,j})^* \circ g_X \circ \alpha^{i,j}$. This defines a unitary monoidal natural transformation $h : E_i \rightarrow E_i$, hence it is an element of G_i. This construction provides us with a map $\beta^{i,j} : G_j \rightarrow G_i$, which can be checked to be a continuous group homomorphism. Note that for each irreducible object X the morphism $\alpha_X^{i,j}$ is unique up to a phase. Looking at the definition of h_X we see that the map $\beta^{i,j}$ is independent of the choice of α. Take the inverse limit

$$G = \lim_{\substack{\longleftarrow \\ i\in\mathcal{I}}} = \{(g_i \in G_i)_{i\in\mathcal{I}} | \beta^{i,j}(g_j) = g_i,\ i \leq j\},$$

which provides us with an compact group G and surjective group homomorphisms $\gamma_i : G \to G_i$ for all the $i \in \mathcal{I}$. Using this group we can define a functor $F : \mathcal{C} \to Rep_f(G, \mathbb{C})$ as follows. Let X be an object $\mathcal{C}$. Pick an $i \in \mathcal{I}$ such that X is an object of $\mathcal{C}_i$. Define $F(X) = (E_i(X), \pi_{i,X} \circ \gamma_i)$. This definition depends of the choice of i, but thanks to the natural isomorphisms $\alpha^{i,j}$, the isomorphism class in $Rep_f(G, \mathbb{C})$ is independent of this choice. The functor F is full and faithful as it restricts to equivalences $\mathcal{C}_i \to Rep_f(G_i, \mathbb{C})$. Every finite dimensional representation of G factors through to a finite dimensional representation of some G_i, which implies that the functor is also essentially surjective, hence defines an equivalence of categories. As the functor preserves all the structure this is also an equivalence of tensor $*$-categories. □

As an aside, consider the special case that $\mathcal{C} = Rep_f(G, \mathbb{C})$ for some compact group G. Then the above reasoning shows that this category is the inverse limit of categories $Rep_f(G_i, \mathbb{C})$, that are finitely generated. It can be shown that for a compact group H, the category $Rep_f(H, \mathbb{C})$ is finitely generated if and only if H is a compact Lie group[6]. Using this observation we see that for any compact group G, $Rep_f(G, \mathbb{C})$ is the inverse limit of representation categories of compact Lie groups.

We have reduced the proof of Deligne's embedding theorem to giving a proof of Theorem 4.5.8. In the last step of this section we reduce the problem of proving Theorem 4.5.8 to finding a suitable monoid in a suitable category that contains $\mathcal{C}$. We follow Müger[30] Appendix B8 closely as he gives a detailed exposition of the construction.

Proposition 4.5.9. *Let $\mathcal{C}$ be a TC* category and $\hat{\mathcal{C}}$ be a $\mathbb{C}$-linear strict tensor category that has $\mathcal{C}$ as a full tensor subcategory. Let (Q, m, η) be a monoid in $\hat{\mathcal{C}}$ satisfying the following two properties.*

1. *$Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q) = \mathbb{C}\eta$.*
2. *For every object X in $\mathcal{C}$ there is a number $n(X) \in \mathbb{N}_0$ that is positive whenever X is not a zero object and an isomorphism of Q-modules $\alpha_X : (Q \otimes X, m \otimes id_X) \to n(X) \cdot (Q, m)$.*

Then the functor defined by

$$E : \mathcal{C} \to Vect_{\mathbb{C}},\ X \to Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q \otimes X),$$

[6]proving this claim requires a lot more representation theory than given in this thesis, so we will not attempt to do so.

$$E(s)\phi = id_Q \otimes s \circ \phi, \quad s : X \to Y, \quad \phi \in Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q \otimes X)$$

is a faithful tensor functor. The functor satisfies $dim_{\mathbb{C}} E(X) = n(X)$. If $\hat{\mathcal{C}}$ has a symmetry and (Q, m, η) is commutative with respect to this symmetry, then E is a symmetric tensor functor.

Proof. The claim that $dim_{\mathbb{C}} E(X) = n(X)$ is easy to verify

$$E(X) = Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q \otimes X) \cong n(X) Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q) \cong \mathbb{C}^{n(X)},$$

where we used both of the above properties of the monoid. Observe that $E(X)$ has a positive dimension whenever X is not a zero object. Looking at the definition of E and using this observation it is clear that the functor E is faithful. In order to make the functor into a tensor functor we need to define isomorphisms e and $d_{X,Y}$. Using $\mathbb{1}_{Vect_{\mathbb{C}}} = \mathbb{C}$ and $E(\mathbb{1}) = \mathbb{C}\eta$, we take the obvious isomorphism $e : \mathbb{C} \to \mathbb{C}\eta$ given by $1 \mapsto \eta$. For the $d_{X,Y}$ morphisms we take

$$d_{X,Y} : E(X) \otimes E(Y) \to E(X \otimes Y), \ \ \phi \otimes \psi \to m \otimes id_{X \otimes Y} \circ id_Q \otimes \phi \otimes id_Y \circ \psi.$$

Checking that these maps define a tensor functor is straightforward to a large extent if one keeps the properties of monoids given in Definition 4.4.1 in mind. With the exception of showing that the $d_{X,Y}$ maps are isomorphisms we leave proving that E is a tensor functor to the reader. In order to show that the $d_{X,Y}$ arrows are isomorphisms we define the following bilinear map where we write $\boxtimes$ for the tensor product in $Vect_{\mathbb{C}}$ in order to distinguish it from the tensor product $\otimes$ in $Q - Mod$.

$$\gamma_{X,Y} : Hom_Q(Q, Q \otimes X) \boxtimes Hom_Q(Q, Q \otimes Y) \to Hom_Q(Q, Q \otimes X \otimes Y),$$

$$s \boxtimes t \mapsto s \otimes id_Y \circ t$$

The $\gamma_{X,Y}$ maps are related to the $d_{X,Y}$ maps through the isomorphisms

$$\delta_X : Hom_Q(Q, Q \otimes X) \to Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q \otimes X),$$

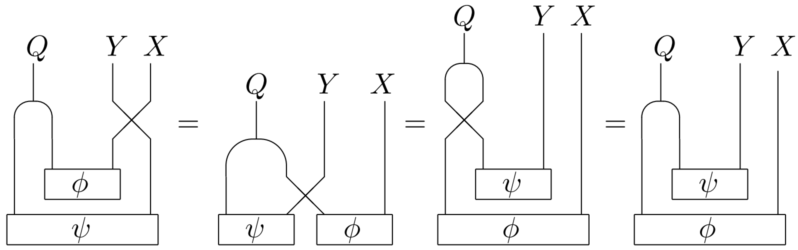
$$d_{X,Y} = \delta_{X,Y} \circ \gamma_{X,Y} \circ \delta_X^{-1} \boxtimes \delta_Y^{-1}.$$

The arrows $d_{X,Y}$ are isomorphisms when the morphisms $\gamma_{X,Y}$ are bijective for each object X and Y of $\mathcal{C}$. By property (2) of the proposition there exist Q-module morphisms $s_i : Q \to Q \otimes X$ and $s'_i : Q \otimes X \to Q$ where $i \in \{1, ..., n(X)\}$, such that $s'_j \circ s_i = \delta_{ij} id_Q$ and $\sum_i s_i \circ s'_i = id_{Q \otimes X}$. For Y there are similar maps t_i, t'_i. Define

$$\gamma_{i,j} = \gamma_{X,Y}(s_i \boxtimes t_j), \ \ \gamma'_{i,j} = t'_j \circ s'_i \boxtimes id_Y.$$

The relations $\gamma'_{i',j'} \circ \gamma_{i,j} = \delta_{i'i}\delta_{j'j} id_Q$ show that the $\gamma_{i,j}$ are all linearly independent. As both domain and codomain of $\gamma_{X,Y}$ have dimension $n(X)n(Y)$ it follows that $\gamma_{X,Y}$ is bijective proving that $d_{X,Y}$ is an isomorphism.

Assume that $\hat{\mathcal{C}}$ has a symmetry and that (Q, m, η) is commutative with respect to this symmetry. In order to show that E is symmetric we need to prove that $E(c_{X,Y}) \circ d_{X,Y} = d_{Y,X} \circ \Sigma_{E(X),E(Y)}$. This is shown in the following diagrams. The diagram at the left-hand side is $(E(c_{X,Y}) \circ d_{X,Y})(\phi \otimes \psi)$.



In the last step the commutativity of the monoid was used. The diagram at the right is $(d_{X,Y} \circ \Sigma_{E(X),E(Y)})(\phi \otimes \psi)$ proving that E is symmetric. □

If the steps in the calculation with the diagrams are unclear, the reader can verify the steps making use of naturality of the symmetry and the braid equations

$$\begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{c \otimes id} & Y \otimes X \otimes Z \\ & \searrow^{c} & \downarrow{\scriptstyle id \otimes c} \\ & & Y \otimes Z \otimes X \end{array} \qquad \begin{array}{ccc} X \otimes Y \otimes Z & \xrightarrow{id \otimes c} & X \otimes Z \otimes Y \\ & \searrow^{c} & \downarrow{\scriptstyle c \otimes id} \\ & & Z \otimes X \otimes Y \end{array}$$

These equations are just the hexagon axioms from Section 3.1 in the case of strict tensor categories. In the next two sections there will be more calculations using diagrams where symmetries play an important role. If a step is not immediately clear, it is not unlikely that it was obtained by (repeated) use of the braid equations and naturality of the symmetry.

Looking at the proposition the existence of a commutative monoid with the above properties in a suitable category $\hat{\mathcal{C}}$ is a sufficient condition for the construction of a symmetric fiber functor. But is it also a necessary condition? In Müger et. al [29] it is argued that this is the case. Given a tensor $*$-category that admits a $*$-preserving fiber functor $E : \mathcal{C} \to \mathcal{H}_f$. The category $\mathcal{C}$ is equivalent to the category of finite dimensional representations of some discrete algebraic quantum group. The category $\hat{\mathcal{C}}$ is taken to be the category of representations of any dimension of that quantum group. The left regular representation provides the desired monoid.

The previous proposition can be formulated in a more useful way for proving Deligne's embedding theorem for finitely generated even STC* categories.

Corollary 4.5.10. *Let $\mathcal{C}$ be a TC* category that is generated by the object Z and $\hat{\mathcal{C}}$ be a $\mathbb{C}$-linear strict tensor category that has $\mathcal{C}$ as a full tensor subcategory. Suppose that (Q, m, η) is a monoid in $\hat{\mathcal{C}}$ which satisfies*

1. $Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q) = \mathbb{C}\eta$*.*

2. There is a $d \in \mathbb{N}$ *and an isomorphism of* Q*-modules* $\alpha_Z : (Q \otimes Z, m \otimes id_Z) \to d \cdot (Q, m)$*.*

Then the functor $E : \mathcal{C} \to Vect_{\mathbb{C}}$*, given by* $X \mapsto Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q \otimes X)$ *is a fiber functor.*

Proof. If we can show that property (2) of Proposition 4.5.9 is satisfied, then the claim follows immediately. Let X be an object of $\mathcal{C}$. There is a natural number $n \in \mathbb{N}$ such that X is a direct summand of $Z^{\otimes n}$. This implies that there are morphisms $v : X \to Z^{\otimes n}$ and $w : Z^{\otimes n} \to X$ such that $w \circ v = id_X$. This provides us with morphisms of Q-modules $\tilde{v} = id_Q \otimes v : Q \otimes X \to Q \otimes Z^{\otimes n}$, $\tilde{w} = id_Q \otimes w : Q \otimes Z^{\otimes n} \to Q \otimes X$ satisfying $\tilde{w} \circ \tilde{v} = id_{Q \otimes X}$. The Q-module $(Q \otimes X, m \otimes id_X)$ is thus a direct summand of the Q-module $(Q \otimes Z^{\otimes n}, m \otimes id_Q)$. The latter is by (2) equivalent to the Q-module $d^n \cdot (Q, m)$.

Combining Definition 4.4.4, $\Gamma_Q = Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q)$ and Lemma 4.4.5 tells us that $End_Q((Q, m)) \cong Hom_{\hat{\mathcal{C}}}(\mathbb{1}, Q)$ as monoids. By Property (1) we find that $End_Q((Q, m)) \cong \mathbb{C}$, hence it is irreducible. Combining the findings thusfar we have that $(Q \otimes X, m \otimes id_X)$ is a direct summand of d^n copies of the irreducible Q-module (Q, m). It immediately follows that $(Q \otimes X, m \otimes id_X)$ is a direct sum of r copies of (Q, m), where $r \leq d^n$. This proves the claim. □

In order to complete the construction of the embedding for a finitely generated even STC* category $\mathcal{C}$ we thus need two things. First of all a suitable $\mathbb{C}$-linear strict tensor category that contains $\mathcal{C}$ as a full subcategory and secondly a commutative monoid in this category that has some special properties. The category $\hat{\mathcal{C}}$ will turn out to be $Ind(\mathcal{C})$. The commutative monoid will be constructed in Section 4.7.

4.6 Determinants and the Symmetric Algebra

In this section we explore certain representations of the symmetric groups in strict tensor categories. This will lead to the definitions of determinants and the symmetric algebra which are both of crucial importance in the construction of a monoid like in Corollary 4.5.10. The discussion here follows appendices B9 and B10 of Müger [30]. The material in this section prior to the construction of the symmetric algebra can also be found in Section 2 of Doplicher and Roberts [11].

We start by recalling that the symmetric group on n labels, P_n is generated by elements σ_i, $i \in \{1, ..., n-1\}$ that are subject to the relations

$$|i - j| \geq 2 \Rightarrow \sigma_i \sigma_j = \sigma_j \sigma_i, \quad \sigma_i^2 = 1, \quad \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}.$$

Also recall the signature map, defined as a homomorphism of groups

$$sgn : P_n \to \{-1, 1\}.$$

For a STC* category we can use the symmetry to define an action of the symmetric group P_n on n-fold tensor powers $X^{\otimes n}$ of objects X. In order to show this we need the following result.

Lemma 4.6.1. *(Yang-Baxter Equation) Let $\mathcal{C}$ be a symmetric strict tensor category with a symmetry c. Then c satisfies the equation*

$$id_Z \otimes c_{X,Y} \circ c_{X,Z} \otimes id_Y \circ id_X \otimes c_{Y \otimes Z} = c_{Y,Z} \otimes id_X \circ id_Y \otimes c_{X,Z} \circ c_{X,Y} \otimes id_Z.$$

Proof. The claim follows straight from

$$\begin{aligned} & id_Z \otimes c_{X,Y} \circ c_{X,Z} \otimes id_Y \circ id_X \otimes c_{Y \otimes Z} \\ & \quad = id_Z \otimes c_{X,Y} \circ c_{X \otimes Y,Z} = c_{Y \otimes X,Z} \circ c_{X,Y} \otimes id_Z \\ & \quad = c_{Y,Z} \otimes id_X \circ id_Y \otimes c_{X,Z} \circ c_{X,Y} \otimes id_Z. \end{aligned}$$

where we used the braid equations and naturality of the symmetry. □

Lemma 4.6.2. *Let $\mathcal{C}$ be an STC* category, X an object of $\mathcal{C}$ and $n \in \mathbb{N}$. Then the map*

$$\Pi_n^X : P_n \to End(X^{\otimes n}), \quad \sigma_i \mapsto id_{X^{\otimes i-1}} \otimes c_{X,X} \otimes id_{X^{\otimes n-i-1}}$$

defines a homomorphism of groups.

Proof. We need to check the following equations

$$|i-j| \geq 2 \Rightarrow \Pi_n^X(\sigma_i)\Pi_n^X(\sigma_j) = \Pi_n^X(\sigma_j)\Pi_n^X(\sigma_i), \quad \Pi_n^X(\sigma_i)^2 = id_{X^{\otimes n}},$$
$$\Pi_n^X(\sigma_i) \circ \Pi_n^X(\sigma_{i+1}) \circ \Pi_n^X(\sigma_i) = \Pi_n^X(\sigma_{i+1}) \circ \Pi_n^X(\sigma_i) \circ \Pi_n^X(\sigma_{i+1}).$$

The first relations are trivial and the last one follows straight from the Yang-Baxter equation. □

For each object X in a STC* category $\mathcal{C}$ and each $n \in \mathbb{N}_0$ we can define morphisms $S_n^X : X^{\otimes n} \to X^{\otimes n}$ and $A_n^X : X^{\otimes n} \to X^{\otimes n}$ as follows. For $n = 0$ we take $S_0^X = A_0^X = id_{\mathbb{1}}$. If $n \in \mathbb{N}$, then

$$S_n^X = \frac{1}{n!} \sum_{\sigma \in P_n} \Pi_n^X(\sigma), \quad A_n^X = \frac{1}{n!} \sum_{\sigma \in P_n} sgn(\sigma)\Pi_n^X(\sigma).$$

Lemma 4.6.3. *Let $\mathcal{C}$ be an STC* category, X an object of $\mathcal{C}$, and the maps S_n^X and A_n^X be defined as above. Then for any $n \in \mathbb{N}$ and any $\sigma \in P_n$ we have*

$$\Pi_n^X(\sigma) \circ S_n^X = S_n^X \circ \Pi_n^X(\sigma) = S_n^X,$$
$$\Pi_n^X(\sigma) \circ A_n^X = A_n^X \circ \Pi_n^X(\sigma) = sgn(\sigma)A_n^X.$$

Consequently, for every $n \in \mathbb{N}_0$ the morphisms S_n^X and A_n^X are orthogonal projections in $End(X^{\otimes n})$.

Proof.

$$\begin{aligned}\Pi_n^X(\sigma)\circ S_n^X &= \frac{1}{n!}\sum_{\sigma'\in P_n}\Pi_n^X(\sigma\sigma') = \frac{1}{n!}\sum_{\sigma'\in P_n}\Pi_n^X(\sigma') \\ &= \frac{1}{n!}\sum_{\sigma'\in P_n}\Pi_n^X(\sigma'\sigma) = S_n^X\circ\Pi_n^X(\sigma).\end{aligned}$$

Each $\sigma \in P_n$ can be expressed in terms of the generators $\sigma = \sigma_{i_1}\cdots\sigma_{i_r}$. The inverse is given by $\sigma^{-1} = \sigma_{i_r}\cdots\sigma_{i_1}$. Note that $sgn(\sigma) = sgn(\sigma^{-1})$.

$$\begin{aligned}\Pi_n^X(\sigma)\circ A_n^X &= \frac{1}{n!}\sum_{\sigma'\in P_n} sgn(\sigma')\Pi_n^X(\sigma\sigma') = \frac{1}{n!}\sum_{\sigma'\in P_n} sgn(\sigma^{-1}\sigma')\Pi_n^X(\sigma') \\ &= sgn(\sigma^{-1})A_n^X = sgn(\sigma)A_n^X.\end{aligned}$$ □

By definition each STC* category has a subobject for each projection.

Definition 4.6.4. *Let $\mathcal{C}$, X, S_n^X and A_n^X be as in the previous lemma. Denote the subobjects of $X^{\otimes n}$ corresponding to the projections S_n^X and A_n^X by $S_n(X)$ and $A_n(X)$ respectively.*

The next proposition is important as it will help us show that for each object X in an STC* category $\mathcal{C}$ the dimension $d(X)$, as defined in Section 3.3, is a nonnegative integer.

Proposition 4.6.5. *Let $\mathcal{C}$ be an even STC* category and X an object of $\mathcal{C}$. Then for any $n \in \mathbb{N}$ we have*

$$Tr_{X^{\otimes n}}A_n^X = \frac{1}{n!}d(X)(d(X)-1)(d(X)-2)\cdots(d(X)-n+1).$$

Proof. For $\sigma \in P_n$ let $\#\sigma$ denote the number of disjoint cycles in which σ can be decomposed. We can obtain the desired identity directly from the following 2 equations.

$$Tr_{X^{\otimes n}}\Pi_n^X(\sigma) = d(X)^{\#\sigma}, \quad \forall X\in\mathcal{C}, \sigma\in P_n.$$

$$\sum_{\sigma\in P_n} z^{\#\sigma} = z(z-1)\cdots(z-n+1), \quad \forall z\in\mathbb{C}, n\in\mathbb{N}.$$

The second equation can be obtained by induction over n so we will focus on the first. First consider the simple cases $\sigma = 1$ and $\sigma = \sigma_i$.

$$Tr_{X^{\otimes n}}\Pi_n^X(1) = Tr_{X^{\otimes n}}id_{X^{\otimes n}} = d(X)^n.$$

$$Tr_{X^{\otimes n}}\Pi_n^X(\sigma_i) = d(X)^{n-2}Tr_{X\otimes X}(\Theta(X)) = d(X)^{n-1}.$$

Here we used the properties of the trace, the definition of the dimension and, in the last step, the fact that $\mathcal{C}$ is even. Note that $\#1 = n$ and $\#\sigma_i = n-1$

so we indeed obtain the desired identities. For the general case we need a more systematic approach. Let $(\overline{X}, r, \overline{r})$ be a standard conjugate for X, then following Doplicher and Roberts [11] we introduce for each $n \in \mathbb{N}$ the map

$$\Phi_n : End(X^{\otimes n}) \to End(X^{\otimes n-1}),\ s \mapsto r^* \otimes id_{X^{\otimes n-1}} \circ id_{\overline{X}} \otimes s \circ r \otimes id_{X^{\otimes n-1}}.$$

Using Lemma 3.3.11, we see that

$$Tr_{X^{\otimes n}} \Pi_n^X(\sigma) = \Phi_1(\Phi_2(...(\Phi_n(\Pi_n^X(\sigma))))).$$

Let $\sigma \in P_n$ be arbitrary. Then either $\Pi_n^X(\sigma)$ does not permute the first two legs of $X^{\otimes n}$ or it can be written in the form $\Pi_n^X(\sigma) = \Pi_n^X(\sigma')\Pi_n^X(\sigma_1)\Pi_n^X(\sigma'')$ where both $\Pi_n^X(\sigma')$ and $\Pi_n^X(\sigma'')$ do not permute the first two legs of $X^{\otimes n}$. In the first case we find $\Phi_n(\Pi_n^X(\sigma)) = d(X)\Pi_{n-1}^X(\hat{\sigma})$, where $\hat{\sigma}$ is σ but seen as an element of P_{n-1}. For the second case we find

$$\Phi_n(\Pi_n^X(\sigma)) = \Pi_{n-1}^X(\hat{\sigma}') \circ \Theta(X) \otimes id_{X^{\otimes n-1}} \circ \Pi_{n-1}^X(\hat{\sigma}'') = \Pi_{n-1}^X(\hat{\sigma}'\hat{\sigma}'')$$

where in the last step we used that $\mathcal{C}$ is even. Continuing this procedure will yield the desired identity for $Tr_{X^{\otimes n}}\Pi_n^X(\sigma)$ proving the proposition. □

In the proof of the previous proposition the fact that $\mathcal{C}$ is even turned out to be very important. We shall use the previous proposition to show that the dimension of any object in a STC* category is a non-negative integer. This holds for any STC* category, not just the even ones.

Corollary 4.6.6. *Let $\mathcal{C}$ be a STC* category and X be an non-zero object of $\mathcal{C}$. Then $d(X) \in \mathbb{N}$.*

Proof. First assume that $\mathcal{C}$ is an even STC* category. For the subobject $A_n(X) \prec X^{\otimes n}$ there are morphisms $s : A_n(X) \to X^{\otimes n}$ and $s^* : X^{\otimes n} \to A_n(X)$ such that $s \circ s^* = A_n^X$ and $s^* \circ s = id_{A_n(X)}$. By the cyclic property of the trace we find

$$Tr_{X^{\otimes n}} A_n^X = Tr_{X^{\otimes n}}(s \circ s^*) = Tr_{A_n(X)}(s^* \circ s) = d(A_n(X)).$$

By positivity of the $*$-operation the right-hand side should be non-negative for every $n \in \mathbb{N}$. If we use the identity from the previous proposition for $Tr_{X^{\otimes n}} A_n^X$, then this can only be realized for every n if $d(X) \in \mathbb{N}_0$. By Proposition 3.3.13 $d(X) \geq 1$ so $d(X) = 0$ is excluded. This proves the claim for even STC* categories. Now let $\mathcal{C}$ be any STC* category. As in the proof of Theorem 4.1.14, we can consider the bosonization of $\mathcal{C}$ by changing the symmetry. This category is even so we have $d(X) \in \mathbb{N}$ for every X in the bozonization of $\mathcal{C}$. The dimension of an object is defined in terms of the conjugates and the $*$-operation. It is independent of the chosen symmetry on $\mathcal{C}$. Therefore, for each object X, $d(X)$ is the same in $\mathcal{C}$ and the bosonization of $\mathcal{C}$, proving the claim also in the non-even case. □

If we set $n = d(X)$ then Proposition 4.6.5 tells us that $A_{d(X)}(X)$ is irreducible because of Proposition 3.3.13 and

$$d(A_{d(X)}(X)) = Tr_{X^{\otimes d(X)}} A^X_{d(X)} = \frac{d(X)!}{d(X)!} = 1.$$

Definition 4.6.7. *Let $\mathcal{C}$ be an STC* category and X a non-zero object of $\mathcal{C}$. Then the isomorphism class of $A_{d(X)}(X)$ is called the determinant of X and is denoted by $det(X)$.*

in Doplicher and Roberts [11] objects X that have a determinant isomorphic to $\mathbb{1}$ are called special. In the next section the object X will often be the generator of a finitely generated even STC* category. As explained in the previous section we are interested in generators of the kind $Z \oplus \overline{Z}$. The next lemma proves that such an object is always special.

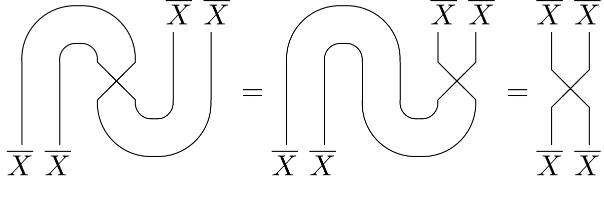
Lemma 4.6.8. *Let $\mathcal{C}$ be an STC* category, X and Y be objects of $\mathcal{C}$ and $(\overline{X}, r, \overline{r})$ be a standard conjugate for X. Then the determinant obeys*

1. *$det(\overline{X}) \cong \overline{det(X)}$,*
2. *$det(X \oplus Y) \cong det(X) \otimes det(Y)$,*
3. *$det(X \oplus \overline{X}) \cong \mathbb{1}$.*

Proof. Claim 3 follows from the first two claims and the fact that determinants are irreducible objects, so we restrict to proving the first two claims starting with 1. Using Lemma 3.3.11 we can construct a standard conjugate $(\overline{X}^{\otimes n}, r_n, \overline{r_n})$ for $X^{\otimes n}$ from the standard conjugate of X. Take $\sigma \in P_n$, which can be written as $\sigma = \sigma_{i_1} \cdots \sigma_{i_m}$. Using the notation $\sigma' = \sigma_{n-i_m} \cdots \sigma_{n-i_1}$ we can show that

$$\Pi^{\overline{X}}_n(\sigma') = r_n^* \otimes id_{\overline{X}^{\otimes n}} \circ id_{\overline{X}^{\otimes n}} \otimes \Pi^X_n(\sigma) \otimes id_{\overline{X}^{\otimes n}} \circ id_{\overline{X}^{\otimes n}} \otimes \overline{r_n}.$$

This identity can be checked straightforwardly using diagrams. The following identity, where the first step is based on the same arguments as in the proof of Proposition 4.5.9, may be helpful



Using the identity for $\Pi^{\overline{X}}_n(\sigma)$ and the fact $sgn(\sigma) = sgn(\sigma')$ we find

$$A^{\overline{X}}_n = r_n^* \otimes id_{\overline{X}^{\otimes n}} \circ id_{\overline{X}^{\otimes n}} \otimes A^X_n \otimes id_{\overline{X}^{\otimes n}} \circ id_{\overline{X}^{\otimes n}} \otimes \overline{r}_n.$$

Now claim 1 follows straight from Lemma 3.3.10 and we move on to the second claim. We use the shorthand notation $d_X = d(X)$ and $A^X = A^X_{d(X)}$. Take $Z \cong X \oplus Y$ and let $v : X \to Z$ and $w : Y \to Z$ be the isometries corresponding to the direct sum decomposition. It follows that $X^{\otimes d_X}$ is a subobject of $Z^{\otimes d_X}$. Analogously $Y^{\otimes d_Y}$ is a subobject of $Z^{\otimes d_Y}$. The object $det(X) \otimes det(Y)$ is a subobject of $X^{\otimes d_X} \otimes Y^{\otimes d_Y}$ corresponding to the projection $A^X \otimes A^Y$. It is isomorphic to a subobject of $Z^{\otimes(d_X+d_Y)} = Z^{\otimes d_Z}$ corresponding to the projector

$$B^Z = v^{\otimes d_X} \otimes w^{\otimes d_Y} \circ A^X \otimes A^Y \circ (v^*)^{\otimes d_X} \otimes (w^*)^{\otimes d_Y} : Z^{\otimes d_Z} \to Z^{\otimes d_Z}.$$

The object $det(X \oplus Y)$ is a subobject of $Z^{\otimes d_Z}$ corresponding to the projector A^Z. We will compare the projectors A^Z and B^Z. We can write B_Z as

$$\begin{aligned} B_X = & \frac{1}{d_X! d_Y!} \sum_{\sigma \in P_{d_X}} \sum_{\sigma' \in P_{d_Y}} sgn(\sigma) sgn(\sigma') v^{\otimes d_X} \otimes w^{\otimes d_Y} \circ \Pi^X_{d_X}(\sigma) \otimes \Pi^Y_{d_Y}(\sigma') \\ & \circ (v^*)^{\otimes d_X} \otimes (w^*)^{\otimes d_Y} \\ = & \frac{1}{d_X! d_Y!} \sum_{\sigma \in P_{d_X}} \sum_{\sigma' \in P_{d_Y}} sgn(\sigma) sgn(\sigma') \Pi^Z_{d_X}(\sigma) \otimes \Pi^Z_{d_Y}(\sigma') \\ & \circ (p_X)^{\otimes d_X} \otimes (p_Y)^{\otimes d_Y} \\ = & \frac{1}{d_X! d_Y!} \sum_{\sigma \in P_{d_X}} \sum_{\sigma' \in P_{d_Y}} sgn(\sigma \times \sigma') \Pi^Z_{d_Z}(\sigma \times \sigma') \circ (p_X)^{\otimes d_X} \otimes (p_Y)^{\otimes d_Y} \end{aligned}$$

where we used the naturality of the symmetry in the first equality. In the second equality we merely introduced some notation. We denote the juxtaposition of σ and σ' by $\sigma \times \sigma' \in P_{d_Z}$. For A^Z we have

$$\begin{aligned} A^Z &= \frac{1}{d_Z!} \sum_{\sigma \in P_{d_Z}} sgn(\sigma) \Pi^Z_{d_Z}(\sigma) \\ &= \frac{1}{d_Z!} \sum_{\sigma \in P_{d_Z}} sgn(\sigma) \Pi^Z_{d_Z}(\sigma) \circ (p_X + p_Y)^{\otimes(d_X+d_Y)}. \end{aligned}$$

At first sight this does not look like B^Z, but Proposition 4.6.5 comes to the rescue. Of the 2^{d_Z} terms into which the product $(p_X + p_Y)^{\otimes d_Z}$ can be decomposed only those terms that have exactly d_X factors p_X and d_Y factors p_Y give nonzero contributions. This is so because $A^X_n = 0$ whenever $n > d_X$ and $A^Y_n = 0$ whenever $n > d_Y$. There are $d_Z!/d_X!d_Y!$ terms that have d_X factors p_X and d_Y factors p_Y. After working out the signs for each term it becomes clear that A^Z is the same as B^Z. Consequently the corresponding subobjects $det(X) \otimes det(Y)$ and $det(X \oplus Y)$ are isomorphic, proving claim 2. □

For the moment we are done with the projectors A_n^X and determinants and we move onto the construction of the symetric algebra. In linear algebra the symmetric algebra over a vector space is constructed as a infinite direct sum of certain vector spaces. By definition every STC* category $\mathcal{C}$ has finite direct sums, but how do we deal with infinite direct sums? Here the category $Ind(\mathcal{C})$ introduced in the previous section will help us as we shall see shortly. Before defining symmetric algebra we first need the next lemma.

Again we start out with an arbitrary STC* category $\mathcal{C}$ and an object X of $\mathcal{C}$. For every $n \in \mathbb{N}$, take the subobject $S_n(X)$ of $X^{\otimes n}$ as defined in Definition 4.6.4. Pick for each $S_n(X)$ an isometry $u_n : S_n(X) \to X^{\otimes n}$ such that $u_n \circ u_n^* = S_n^X$. For $S_0(X) = \mathbb{1}$, define $u_0 = id_{\mathbb{1}} : S_0(X) \to X^{\otimes 0}$. Now for each $i, j \in \mathbb{N}_0$ define the morphisms

$$m_{i,j} : S_i(X) \otimes S_j(X) \to S_{i+j}(X), \quad m_{i,j} = u_{i+j}^* \circ u_i \otimes u_j.$$

Lemma 4.6.9. *Let, $\mathcal{C}$, X, $S_n(X)$ and $m_{i,j}$ be as above and let c be the symmetry of $\mathcal{C}$. Then for each $i, j, k \in \mathbb{N}_0$ we have*

$$m_{i+j,k} \circ m_{i,j} \otimes id_{S_k(X)} = m_{i,j+k} \circ id_{S_i(X)} \otimes m_{j,k},$$

$$m_{i,j} = m_{j,i} \circ c_{S_i(X),S_j(X)}, \quad m_{i,0} = m_{0,i} = id_{S_i(X)}.$$

Proof. The identity $m_{i,0} = m_{0,i} = id_{S_i(X)}$ is trivial. We start with showing that $m_{i,j} = m_{j,i} \circ c_{S_i(X),S_j(X)}$. Recall the identity $S_n^X \circ \Pi_n^X(\sigma) = S_n^X$ from Lemma 4.6.3, and let $\sigma \in P_{i+j}$ denote the permutation that exchanges the first i labels and the last j labels.

$$\begin{aligned} m_{j,i} \circ c_{S_i(X),S_j(X)} &= u_{i+j}^* \circ u_j \otimes u_i \circ c_{S_i(X),S_j(X)} = u_{i+j}^* \circ c_{X^{\otimes i},Y^{\otimes j}} \circ u_i \otimes u_j \\ &= u_{i+j}^* \circ \Pi_{i+j}^X(\sigma) \circ u_i \otimes u_j \\ &= u_{i+j}^* \circ S_{i+j}^X \circ \Pi_{i+j}^X(\sigma) \circ u_i \otimes u_j \\ &= u_{i+j}^* \circ S_{i+j}^X \circ u_i \otimes u_j = u_{i,j}^* \circ u_i \otimes u_j = m_{i,j}. \end{aligned}$$

In the second equality we used the naturality of the symmetry. In the fourth equality and sixth equality we used

$$u_{i+j}^* = id_{S_{i+j}(X)} \circ u_{i+j}^* = u_{i+j}^* \circ u_{i+j} \circ u_{i+j}^* = u_{i+j}^* \circ S_{i+j}^X.$$

We move onto proving $m_{i+j,k} \circ m_{i,j} \otimes id_{S_k(X)} = m_{i,j+k} \circ id_{S_i(X)} \otimes m_{j,k}$. The identity $S_n^X \circ \Pi_n^X(\sigma) = S_n^X$ will provide us with the following useful relations.

$$S_{i+j+k}^X \circ S_{i+j}^X \otimes id_{X^{\otimes k}} \circ S_i^X \otimes S_j^X \otimes id_{X^{\otimes k}} = S_{i+j+k}^X \circ S_{i+j}^X \otimes id_{X^{\otimes k}} = S_{i+j+k}^X,$$

$$S_{i+j+k}^X \circ id_{X^{\otimes i}} \otimes S_{j+k}^X \circ id_{X^{\otimes i}} \otimes S_j^X \otimes S_k^X = S_{i+j+k}^X \circ id_{X^{\otimes i}} \otimes S_{j+k}^X = S_{i+j+k}^X.$$

Next we take the composition of these identities with u_{i+j+k}^* on the left and $u_i \otimes u_j \otimes u_k$ on the right. We obtain the equality

$$u_{i+j+k}^* \circ S_{i+j}^X \otimes id_{X^{\otimes k}} \circ u_i \otimes u_j \otimes u_k = u_{i+j+k}^* \circ id_{X^{\otimes k}} \otimes S_{j+k}^X \circ u_i \otimes u_j \otimes u_k.$$

If we write $S^X_{i+j} = u_{i+j} \circ u^*_{i+j}$ and S^X_{j+k} similarly, then this equation is just the relation that we wanted to prove. □

In the previous section we defined for every STC* category $\mathcal{C}$, the Ind-category of this category $Ind(\mathcal{C})$. The category $Ind(\mathcal{C})$ is a $\mathbb{C}$-linear strict tensor category that contains $\mathcal{C}$ as a full strict tensor subcategory and it contains all small filtrant inductive limits of $\mathcal{C}$. We define the symmetric algebra over an object X in $\mathcal{C}$ as the object

$$S(X) = \lim_{n\to\infty} \bigoplus_{i=0}^{n} S_i(X)$$

of $Ind(\mathcal{C})$. Here we look at $\mathbb{N}_0$ as a small directed category where we have 1 arrow $i \to j$ only when $j \geq i$. Pick an $i \in \mathbb{N}_0$ and suppose that $j \geq i$. Then there is a monomorphism $S_i(X) \to \oplus_{k=0}^{j} S_k(X)$. This provides us with monomorphisms $v_n : S_n(X) \to S(X)$ for each $n \in \mathbb{N}_0$.

Proposition 4.6.10. *Let $\mathcal{C}$ be an STC* and X an object of $\mathcal{C}$. Then there exists a morphism $m_{S(X)} : S(X) \otimes S(X) \to S(X)$ such that*

$$m_{S(X)} \circ v_i \otimes v_j = v_{i+j} \circ m_{i,j} : S_i(X) \otimes S_j(X) \to S(X)$$

holds and $(S(X), m_{S(X)}, \eta_{S(X)})$, with $\eta_{S(X)} = v_0 : \mathbb{1} \to S(X)$, is a commutative monoid in $Ind(\mathcal{C})$.

Proof. Recall from Definition 4.5.3 that $m_{S(X)}$ should be an element of

$$\lim_{\overleftarrow{m}} \lim_{\overrightarrow{n}} Hom_{\mathcal{C}} \left(\bigoplus_{i,j=0}^{m} S_i(X) \otimes S_j(X), \bigoplus_{k=0}^{n} S_k(X) \right).$$

We can construct such a family of morphisms using the $m_{i,j}$ morphisms. If $n \geq 2l$ then, using direct sum decompositions and $m_{i,j}$ morphisms we get a morphism

$$M_{l,n} : \bigoplus_{i,j=0}^{l} S_i(X) \otimes S_j(X) \to \bigoplus_{k=0}^{n} S_k(X).$$

In order to be a morphism in $Ind(\mathcal{C})$ the family of morphisms $M_{i,j}$ should satisfy the relations given right after Definition 4.5.3. These are easily checked to be satisfied. We have thus defined a morphism $m_{S(X)} : S(X) \otimes S(X) \to S(X)$ in the strict tensor category $Ind(\mathcal{C})$. The identity $m_{S(X)} \circ v_i \otimes v_j = v_{i+j} \circ m_{i,j}$ is satisfied by the definition of $m_{S(X)}$. The fact that $(S(X), m_{S(X)}, \eta_{S(X)})$ defines a commutative monoid follows from the relations of Lemma 4.6.9. We leave working out the details to the reader. □

4.7 Construction of an Absorbing Monoid

The title of this section refers to the second property of the monoid from Proposition 4.5.9, which is called the absorbing property. In this last section we will construct a monoid as in Proposition 4.5.9 completing the proof of Deligne's embedding theorem. The discussion here follows Appendix B11 of Müger [30]. The construction is an adaptation of the work of Bichon [4]. The approach originally came from the work of Deligne [9].

Let $\mathcal{C}$ be a STC* category and X an object of $\mathcal{C}$ such that $det(X) \cong \mathbb{1}$. At the end of the previous section we saw that $(S(X), m_{S(X)}, \eta_{S(X)})$ defines a commutative monoid in $Ind(\mathcal{C})$, where $S(X)$ is the symmetric algebra over X. Let $d = d(X)$ be the dimension of X. Using Lemma 4.4.2 we define a commutative monoid (Q, m_Q, η_Q) as the d-fold direct product monoid

$$(Q, m_Q, \eta_Q) = (S(X), m_{S(X)}, \eta_{S(X)})^{\times d}.$$

Here $Q = X^{\otimes d}$ and m_Q and η_Q are inductively defined by Lemma 4.4.2. Because X has $det(X) \cong \mathbb{1}$ there exist morphisms $s : \mathbb{1} \to X^{\otimes d}$ and $s^* : X^{\otimes d} \to \mathbb{1}$ such that $s^* \circ s = id_{\mathbb{1}}$ and $s \circ s^* = A_d^X$. We will also use the monomorphisms $v_0 : \mathbb{1} \to S(X)$ and $v_1 : X \to S(X)$ that come with $S(X)$. In the upcoming computations we will make use of the following morphisms

$$f : \mathbb{1} \to Q, \quad f = v_1^{\otimes d} \circ s,$$

$$u_i : X \to Q, \quad u_i = v_0^{\otimes(i-1)} \otimes v_1 \otimes v_0^{\otimes(d-i)}, \quad i \in \{1, ..., d\},$$

$$t_i : X^{\otimes(d-1)} \to Q, \quad t_i = (-1)^{d-i} v_1^{\otimes(i-1)} \otimes v_0 \otimes v_1^{\otimes(d-i)}, \quad i \in \{1, ..., d\}.$$

Lemma 4.7.1. *Let m_Q, t_j, u_i, s and f be as above. Then*

$$m_Q \circ t_j \otimes u_i \circ s = \delta_{ij} f, \quad \forall i, j \in \{1, ..., d\}.$$

Proof. We begin with the case when $i = j$. We can prove $m_Q \circ t_i \otimes u_i \circ s = f$ if we have the following identity.

$$m_Q \circ t_i \otimes u_i \circ s = (-1)^{(d-i)} id_{S(X)^{(i-1)}} \otimes c_{S(X)^{\otimes(d-i)}, S(X)} \circ v_1^{\otimes d} \circ s.$$

The chore of proving this identity is left to the reader. It can be obtained diagrammaticaly from the fact that v_0 is the monoid unit $\eta_{S(X)}$, the definition of m_Q (Lemma 4.4.2), naturality of the symmetry c and some patience. From this identity we now prove the claim

$$\begin{aligned} m_Q \circ t_i \otimes u_i \circ s &= (-1)^{(d-i)} id_{S(X)^{(i-1)}} \otimes c_{S(X)^{\otimes(d-i)}, S(X)} \circ v_1^{\otimes d} \circ s \\ &= (-1)^{(d-i)} v_1^{\otimes d} \circ id_{X^{\otimes(i-1)}} \otimes c_{X^{\otimes(d-i)}, X} \circ s \\ &= v_1^{\otimes d} \circ s = f \end{aligned}$$

Here we used naturality of the symmetry in the second equality and for the third equality we used the antisymmetry of s:

$$\Pi_d^X(\sigma) \circ s = \Pi_d^X(\sigma) \circ A_d^X \circ s = sgn(\sigma) A_d^X \circ s = sgn(\sigma) s.$$

It remains to show that for $i \neq j$ we have $m_Q \circ t_j \otimes u_i \circ s = 0$. This can done in a straightforward fashion, but requires a lot of calculations. To give an idea of the proof we show the case that $j = d-1, i = d$. This case is not special as all other cases can be treated using the same arguments.

$$\begin{aligned}
&m_Q \circ t_{d-1} \otimes u_d \circ s \\
&\quad = id_{S(X)^{\otimes(d-1)}} \otimes m_{S(X)} \circ v_1^{\otimes(d-2)} \otimes v_0 \otimes v_1 \otimes v_1 \circ s \\
&\quad = id_{S(X)^{\otimes(d-1)}} \otimes (m_{S(X)} \circ c_{S(X),S(X)}) \circ v_1^{\otimes(d-2)} \otimes v_0 \otimes v_1 \otimes v_1 \circ s \\
&\quad = id_{S(X)^{\otimes(d-1)}} \otimes m_{S(X)} \circ v_1^{\otimes(d-2)} \otimes v_0 \otimes v_1 \otimes v_1 \circ id_{X^{\otimes(d-2)}} \otimes c_{X,X} \circ s \\
&\quad = (-1) id_{S(X)^{\otimes(d-1)}} \otimes m_{S(X)} \circ v_1^{\otimes(d-2)} \otimes v_0 \otimes v_1 \otimes v_1 \circ s \\
&\quad = (-1) m_Q \circ t_{d-1} \otimes u_d \circ s.
\end{aligned}$$

In the second equality we used that $m_{S(X)}$ comes from a commutative monoid. It follows directly that $m_Q \circ t_{d-1} \otimes u_d \circ s = 0$. □

From the monoid (Q, m_Q, η_Q) we can construct a monoid with the absorbing property by taking a suitable quotient. In order to prove this we need the following technical lemma.

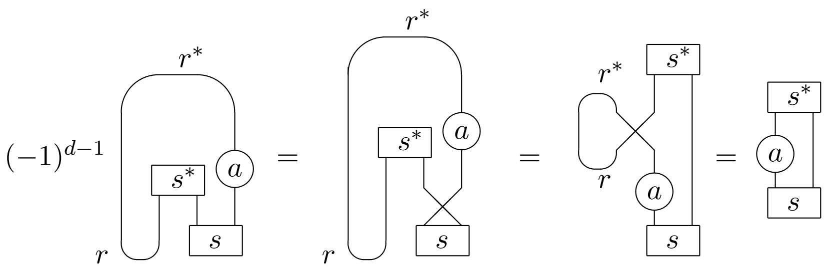
Lemma 4.7.2. *Let $\mathcal{C}$ be an even STC* category and X an object of $\mathcal{C}$ with $det(X) \cong \mathbb{1}$. Let $d = d(X)$ denote the dimension of X. Take $s : \mathbb{1} \to X^{\otimes d}$ an isometry such that $s \circ s^* = A_d^X$. Then*

$$s^* \otimes id_X \circ id_X \otimes s = \frac{(-1)^{d-1}}{d} id_X.$$

Proof. By non-degeneracy of the trace it suffices to prove

$$Tr_X(a \circ s^* \otimes id_X \circ id_X \otimes s) = (-1)^{d-1} d^{-1} Tr_X(a), \quad \forall a \in End_{\mathcal{C}}(X).$$

The following diagrams help us in proving this.



In the first equality we used the antisymmetry property of s, in the second equality the braid equations and naturality of the symmetry and in the last step the triviality of the twist. The diagrams tell us that

$$\begin{aligned} Tr_X(a \circ s^* \otimes id_X \circ id_X \otimes s) &= (-1)^{d-1} s^* \circ a \otimes id_{X^{\otimes(d-1)}} \circ s \\ &= (-1)^{d-1} Tr_{\mathbb{1}}(s^* \circ a \otimes id_{X^{\otimes(d-1)}} \circ s) \\ &= (-1)^{d-1} Tr_{X^{\otimes d}}(a \otimes id_{X^{\otimes(d-1)}} \circ A_d^X). \end{aligned}$$

In the last step we used the cyclic property of the trace. Proving the lemma now amounts to proving

$$Tr_{X^{\otimes d}}(a \otimes id_{X^{\otimes(d-1)}} \circ A_d^X) = d^{-1} Tr_X(a).$$

This identity can be derived using the techniques of the proof of Proposition 4.6.5. Using those methods one can derive

$$Tr_{X^{\otimes d}}(a \otimes id_{X^{\otimes(d-1)}} \circ \Pi_d^X(\sigma)) = d^{\#\sigma - 1} Tr_X(a)$$

from which the desired identity readily follows. □

Consider the following morphism $m_0 : Q \to Q$ in $Ind(\mathcal{C})$.

$$m_0 = m_Q \circ id_Q \otimes (f - \eta_Q) = m_Q \circ id_Q \otimes f - id_Q$$

where $f : \mathbb{1} \to Q$ is the morphism defined at the beginning of this section. This morphism is also a morphism of Q-modules $m_0 : (Q, m_Q) \to (Q, m_Q)$. Recall that the image of a morphism was defined in Section 4.4 as the monomorphism in the epic/monic factorization of a morphism. The image of m_0 thus defines an ideal $j_0 = im(m_0) : (J, \mu) \to (Q, m_Q)$ in the monoid (Q, m_Q, η_Q).

Proposition 4.7.3. *Let $\mathcal{C}$ be an even STC* category and X an object of $\mathcal{C}$ such that $det(X) \cong \mathbb{1}$. Let the commutative monoid (Q, m_Q, η_Q) be defined as in the beginning of this section and the ideal $j_0 : (J, \mu) \to (Q, m_Q)$ be defined as above. Suppose that $j' : (J', \mu') \to (Q, m)$ is a proper ideal in (Q, m_Q, η_Q) that contains j_0. Take (B, m_B, η_B) to be the quotient of (Q, m_Q, η_Q) by the ideal j'. Then there exists an isomorphism of B-modules*

$$(B \otimes X, m_B \otimes id_X) \cong d(X) \cdot (B, m_B).$$

Proof. Take f, u_i, t_i, s and s^* as in the beginning of this section. As in Proposition 4.4.9 we have, for the quotient (B, m_B, η_B) an epimorphism $coker j' = p : Q \to B$ that satisfies

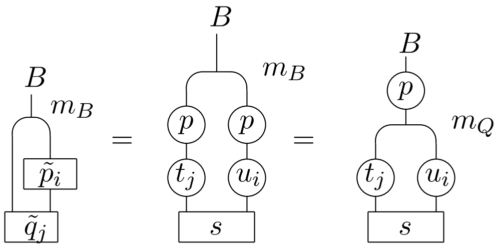
$$p \circ m_Q = m_B \circ p \otimes p, \quad p \circ f = p \circ \eta_Q = \eta_B.$$

The identity $p \circ f = p \circ \eta_Q$ or equivalently $p \circ (f - \eta_Q) = 0$ follows from the fact that j' contains j_0 and the definition of j_0 as the image of m_0. For each $i \in \{1, ..., d\}$ we define the following morphisms that help us show that $B \otimes X \cong d \cdot B$ as B-modules.

$$\tilde{q}_i : \mathbb{1} \to B \otimes X, \quad \tilde{q}_i = p \otimes id_X \circ t_i \otimes id_X \circ s,$$

$$\tilde{p}_i : X \to B, \quad \tilde{p}_i = p \circ u_i.$$

These morphisms satisfy the relations



$= \delta_{ij} p \circ f = \delta_{ij} \eta_B.$

In the third equality we used Lemma 4.7.1. Define the following morphisms

$$q_i : B \to B \otimes X, \quad q_i = m_B \otimes id_X \circ id_B \otimes \tilde{q}_i,$$

$$p_i : B \otimes X \to B, \quad p_i = m_B \circ id_B \otimes \tilde{p}_i.$$

Note that these morphisms are morphisms of B-modules. Not only that, the morphisms also provide the decomposition of $B \otimes X$ into d copies of B. In order to show this we calculate

$$\begin{aligned} p_i \circ q_i &= m_B \circ id_B \otimes \tilde{p}_i \circ m_B \otimes id_X \circ id_B \otimes \tilde{q}_j \\ &= m_B \circ m_B \otimes id_B \circ id_{B \otimes B} \otimes \tilde{p}_i \circ id_B \otimes \tilde{q}_j \\ &= m_B \circ id_B \otimes m_B \circ id_{B \otimes B} \otimes \tilde{p}_i \circ id_B \otimes \tilde{q}_j \\ &= m_B \circ id_B \otimes (m_B \circ id_B \otimes \tilde{p}_i \circ \tilde{q}_j) \\ &= m_B \circ id_B \otimes \delta_{ij} \eta_B = \delta_{ij} id_B \end{aligned}$$

In the third equality we used the associativity axiom for monoids and in the fifth equality we used the result of the previous diagrammatic calculation. The sixth equality is again an axiom of monoids. We also need to show that $\sum_i q_i \circ p_i = id_{B \otimes X}$. The following calculation turns out to be helpful to that end. The identities in the calculation are long and look complex so it may not be clear what is going on. If this is the case, draw diagrams and every step will become clear. In the second equality we use the associativity axiom of monoids. In the third equality we express the $\tilde{q}_i$ and $\tilde{p}_i$ morphisms in terms of the u_i, t_i, s and p morphisms. In the fifth equality we use $p \circ m_Q = m_B \circ p \otimes p$.

$$\sum_{i=1}^{d} q_i \circ p_i = \sum_{i=1}^{d} m_B \otimes id_X \circ id_B \otimes \tilde{q}_i \circ m_B \circ id_B \otimes \tilde{p}_i$$
$$= \sum_{i=0}^{d} m_B \otimes id_X \circ id_B \otimes m_B \otimes id_X \circ id_{B\otimes B} \otimes \tilde{q}_i \circ id_B \otimes \tilde{p}_i$$
$$= \sum_{i=0}^{d} (m_B \circ id_B \otimes m_B \circ id_{B\otimes B} \otimes (p \circ t_i)) \otimes id_X \circ id_B \otimes (id_B \otimes s \circ p \circ u_i)$$
$$= \sum_{i=0}^{d} (m_B \circ id_B \otimes m_B \circ id_B \otimes p \otimes p \circ id_B \otimes u_i \otimes t_i) \otimes id_X \circ id_{B\otimes X} \otimes s$$
$$= \sum_{i=0}^{d} (m_B \circ id_B \otimes p \circ id_B \otimes m_Q \circ id_B \otimes u_i \otimes t_i) \otimes id_X \circ id_{B\otimes X} \otimes s.$$

If we take the composition of the final identity with $\eta_B \otimes id_X$ then it becomes clear that $\sum_{i=1}^{d} q_i \circ p_i = id_{B\otimes X}$ holds if and only if

$$\sum_{i=1}^{d} p \otimes id_X \circ m_Q \otimes id_X \circ u_i \otimes t_i \otimes id_X \circ id_X \otimes s = \eta_B \otimes id_X.$$

Writing out the left-hand side we get

$$\sum_{i=1}^{d} (-1)^{d-i} (p \circ c_{S(X),S(X)^{\otimes(i-1)}} \otimes id_{S(X)^{\otimes(d-i)}} \circ v_1^{\otimes d}) \otimes id_X \circ id_X \otimes s$$
$$(p \circ v_1^{\otimes d}) \otimes id_X \circ \left(\sum_{i=1}^{d} (-1)^{d-i} c_{X,X^{\otimes(i-1)}} \otimes id_{X^{\otimes(d-i)}} \otimes id_X \circ id_X \otimes s \right).$$

Define $K_i = c_{X,X^{\otimes(i-1)}} \otimes id_{X^{\otimes(d-i)}} \otimes id_X \circ id_X \otimes s$, $i \in \{1, ..., d\}$. Then for every $j \in \{1, ..., d\}$ we have

$$\Pi_{d+1}^X(\sigma_j) \circ K_i = \begin{cases} K_{i-1} & \text{if } j = i-1 \\ K_{i+1} & \text{if } j = i \\ -K_i & \text{otherwise} \end{cases}$$

As a consequence $\sum_{i=1}^{d} (-1)^{d-i} K_i : X \to X^{\otimes(d+1)}$ is completely antisymmetric with respect to the first d legs. This implies that we can insert $A_d^X \otimes id_X$ straight after $\sum_{i=1}^{d} (-1)^{d-i} K_i$ without changing the identity. Using $A_d^X = s \circ s^*$ we find

$$
\begin{aligned}
&(p \circ v_1^{\otimes d}) \otimes id_X \circ (s \circ s^*) \otimes id_X \\
&\quad \circ \left(\sum_{i=1}^{d} (-1)^{d-i} c_{X,X^{\otimes(i-1)}} \otimes id_{X^{\otimes(d-i)}} \otimes id_X \circ id_X \otimes s \right) \\
&= (p \circ v_1^{\otimes d} \circ s) \otimes id_X \\
&\quad \circ \left(\sum_{i=1}^{d} (-1)^{d-i} s^* \otimes id_X \circ c_{X,X^{\otimes(i-1)}} \otimes id_{X^{\otimes(d-i)}} \otimes id_X \circ id_X \otimes s \right) \\
&= \eta_B \otimes id_X \circ \left(\sum_{i=1}^{d} (-1)^{d-i} (-1)^{i-1} s^* \otimes id_X \circ id_X \otimes s \right) \\
&= \eta_B \otimes id_X \circ d(-1)^{d-1} s^* \otimes id_X \circ id_X \otimes s = \eta_B \otimes id_X \circ id_X
\end{aligned}
$$

In the second equality we used $p \circ v^{\otimes d} \circ s = p \circ f = \eta_B$ and $s^* \circ c_{X,X^{\otimes(i-1)}} \otimes id_{X^{\otimes(d-i)}} = (-1)^{i-1} s^*$. The last step follows from Lemma 4.7.2. After these messy calculations we have thus shown that $\sum_{i=1}^{d} q_i \circ p_i = id_{B \otimes X}$, completing the proof. □

As long as we can find proper ideals in (Q, m_Q, η_Q) that contain $j_0 = im(m_0)$ we can construct commutative monoids in $Ind(\mathcal{C})$ with the absorbing property. This raises the following question. Is there a proper ideal containing j_0, such that the constructed monoid has the property that $Hom_{Ind(\mathcal{C})}(\mathbb{1}, B)$ is equal to $\mathbb{C}\eta_B$? If this is the case, then Corollary 4.5.10 provides us with a symmetric fiber functor $\xi : \mathcal{C} \to Vect_{\mathbb{C}}$. The next two lemmas help in answering this question.

Lemma 4.7.4. *Let $\mathcal{C}$ be an STC* category, X an object of $\mathcal{C}$ that has $det(X) \cong \mathbb{1}$ and the commutative monoid (Q, m_Q, η_Q) be defined as in the beginning of this section. Then the dimension of the commutative $\mathbb{C}$-algebra $\Gamma_Q = Hom(\mathbb{1}, Q)$ is at most countable and the algebra has a $\mathbb{N}_0$ grading.*

Proof. Recall that the object Q is the d-fold tensor product of the symmetric algebra over some object X of $\mathcal{C}$. The symmetric algebra is the inductive limit $lim_{n\to\infty} \oplus_k^n S_k(X)$ with $S_k(X)$ certain subobjects of X. We thus have

$$
\begin{aligned}
\Gamma_Q &= Hom_{Ind(\mathcal{C})}(\mathbb{1}, S(X)^{\otimes d}) \\
&= \lim_{n\to\infty} \bigoplus_{i_1,\ldots,i_d=0}^{n} Hom_{\mathcal{C}}(\mathbb{1}, S_{i_1}(X) \otimes \cdots \otimes S_{i_d}(X)) \\
&= \bigoplus_{i_1,\ldots,i_d \geq 0} Hom_{\mathcal{C}}(\mathbb{1}, S_{i_1}(X) \otimes \cdots \otimes S_{i_d}(X)).
\end{aligned}
$$

By definition of a STC* category each term in the direct sum is a finite dimensional $\mathbb{C}$-vector space. The countable direct sum thus has at most a countable dimension.

As for the $\mathbb{N}_0$-grading, recall that for $s, t \in \Gamma_Q$ we have $s \Box t = m_Q \circ t \otimes s$. Here m_Q is constructed from the maps $m_{ij} : S_i(X) \otimes S_j(X) \to S_{i+j}(X)$ with $i, j \geq 0$. The algebra Γ_Q gets it's $\mathbb{N}_0$-grading from these m_{ij} maps as the reader can straightforwardly check. □

Lemma 4.7.5. *Let K be a field extension of $\mathbb{C}$ such that $[K : \mathbb{C}] = dim_{\mathbb{C}} K$ is at most countable. Then $K = \mathbb{C}$*

Proof. Suppose that there exists an $x \in K$ that is transcedental over $\mathbb{C}$. We will show that the elements of the set $\{(x + a)^{-1} | a \in \mathbb{C}\}$ are all linearly independent in K. If that is the case, then the fact that $\mathbb{C}$ is uncountable contradicts the assumption that K is at most countable over $\mathbb{C}$ and there is no such transcedental $x \in K$. This implies that $K/\mathbb{C}$ is algebraic. As $\mathbb{C}$ is algebraically closed it follows that $K = \mathbb{C}$ and we are done.

In order to see that $\{(x + a)^{-1} | a \in \mathbb{C}\} \subset K$ consists of only linearly independent elements, suppose that $\sum_{i=1}^{N} b_i(x + a_i)^{-1}$, where $b_i, a_i \in \mathbb{C}$ and the a_i are pairwise different. Multiply the equation with the product $\prod_i (x + a_i)$ to obtain

$$\sum_{i=1}^{N} b_i \prod_{j \neq i} (x + a_j) = \sum_{i=0}^{N-1} c_i x^i = 0$$

where the coefficients c_i can be expressed in terms of the numbers b_j and a_k. By assumption x is transcedental thus for every $i \in \{0, ..., N - 1\}$ we have $c_i = 0$. This in turn provides us with N linear equations

$$\sum_{i=1}^{N} M_{ki} b_i = 0, \quad M_{ki} = \sum_{s \subset \{1,...,N\} - \{i\}, \#S = k-1} \prod_{s \in S} a_s.$$

Using elementary row operations the matrix (M_{ki}) can be transformed into a Vandermonde matrix (V_{ki}) where $V_{ki} = a_i^{k-1}$. The determinant of such a matrix is nonzero by Vandermonde's formula $det(V_{ik}) = \prod_{i<j} (a_j - a_i)$. It follows that for all $i \in \{1, ..., N\}$, $b_i = 0$, proving the claim. □

Using the previous lemma's and Proposition 4.7.3 we can finally finish the construction of a monoid in $Ind(\mathcal{C})$ that provides a symmetric fiber functor.

Theorem 4.7.6. *Let $\mathcal{C}$ be an even STC* category and X an object of $\mathcal{C}$ such that $det(X) \cong \mathbb{1}$. Then there exists a commutative monoid (B, m_B, η_B) in $Ind(\mathcal{C})$ such that $Hom_{Ind(\mathcal{C})}(\mathbb{1}, B) = \mathbb{C}\eta_B$ and there is an isomorphism $B \otimes X \cong d(X)B$ of B-modules.*

Proof. Take as before the commutative monoid (Q, m_Q, η_Q) and the ideal $j_0 = im(m_0)$. If we want to make use of Proposition 4.7.3, we should first check that the ideal j_0 is proper. Suppose that j_0 is an isomorphism. Then in particular it is an epimorphism and consequently m_0 is an epimorphism. By Lemma 4.4.13 $m_0 : (Q, m_Q) \to (Q, m_Q)$ defines an isomorphism of Q-modules. Consequently the map $\Gamma_Q \to \Gamma_Q$ given by $s \mapsto s\Box(f - \eta_Q)$ defines an isomorphism. There should by some $t : \mathbb{1} \to Q$ such that $\eta_Q = t\Box(f - \eta_Q)$. Noting that Γ_Q is $\mathbb{N}_0$-graded and that $f - \eta_Q$ is not in the degree 0 part of Γ_Q there can be no such t and we have a contradiction. The ideal j_0 is not an isomorphism and thus defines a proper ideal.

By Lemma 4.4.8 there is a maximal ideal j that contains j_0. Take (B, m_B, η_B) to be the commutative monoid that is the quotient of (Q, m_Q, η_Q) by the maximal ideal j. Applying Proposition 4.7.3 gives us the isomorphism $B \otimes X \cong d(X)B$ of B-modules. Proposition 4.4.11 tells that the quotient monoid contains no proper non-zero ideals because j is maximal. By Lemma 4.4.12 the commutative $\mathbb{C}$-algebra $End_B((B, m_B))$ is a field extension of $\mathbb{C}$. Lemma 4.4.5 shows that $End_B((B, m_B))$ is isomorphic as a $\mathbb{C}$-algebra to Γ_B. Recall that every object of $\mathcal{C}$ is projective as an object of $Ind(\mathcal{C})$ (Conjecture 4.5.5). Applying this to $\mathbb{1}$, Lemma 4.4.10 tells us that the morphism $p_\Gamma : \Gamma_Q \to \Gamma_B$ of this lemma is surjective. We know from Lemma 4.7.4 that Γ_Q has at most a countable dimension. Surjectivity of p_Γ implies that Γ_B has at most a countable dimension. As Γ_B is a field extension of $\mathbb{C}$ that has at most a countable dimension, Lemma 4.7.5 shows that $\Gamma_B = Hom_{Ind(\mathcal{C})}(\mathbb{1}, B) = \mathbb{C}\eta_B$. □

Now suppose that $\mathcal{C}$ is an even STC* category that is finitely generated by an object $X = Z \oplus \overline{Z}$. Then $det(X) \cong \mathbb{1}$. Applying both Theorem 4.7.6 and Corollary 4.5.10 gives us a symmetric fiber functor $E : \mathcal{C} \to Vect_{\mathbb{C}}$. This concludes the proof of Deligne's embedding theorem.

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