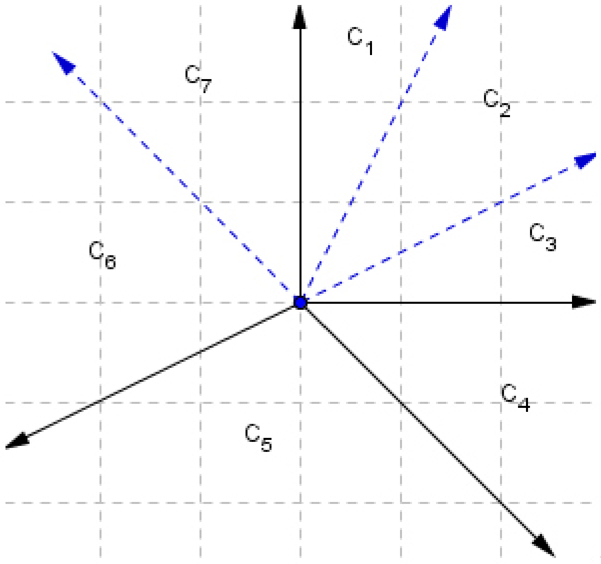
Toric Varieties over the Secondary fan and the Gröbner fan



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Chapter 1

Introduction

The paper *The complex geometry of the spherical pendulum* by Beukers and Cushman (see [5]) describes the phase space of the complexified, spherical pendulum as a $\mathbb{C}^*$-bundle over a family of elliptic curves. The aim of this thesis is to give an interpretation of this family of elliptic curves in the context of toric geometry. For this we look at curves of the form

$$f(x,y) := c_1x^{-1}y^{-1} + c_2y^{-1} + c_3x^{-1} + c_4 + c_5xy = 0, \tag{1.1}$$

for $x, y \in \mathbb{C}^*$ and with coefficients $c_1, ..., c_5 \in \mathbb{C}^*$.

By introducing homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$ we will see that the family of curves in (1.1) can be represented as a 2-parameter family of elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. A major drawback to this approach is that all these curves intersect each other in two fixed points. Even after (repeatedly) blowing-up these singularities we still can not find a smooth parametrization for the family of elliptic curves.

Therefore we search for another model to describe the family of elliptic curves. For this we define a matrix

$$\mathcal{A} := \begin{pmatrix} -1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}, \tag{1.2}$$

whose columns correspond to the exponents of the individual terms of the family of elliptic curves in (1.1). A natural way to look at these curves is by defining a map corresponding to the columns of $\mathcal{A}$

$$\psi : (\mathbb{C}^*)^2 \to \mathbb{P}^4, \text{ defined by } (x,y) \mapsto [x^{-1}y^{-1} : y^{-1} : x^{-1} : 1 : xy].$$

The Zarisky closure of the image of ψ has the structure of a toric variety, which we denote by $\mathbb{X}_{\mathcal{A}}$. This toric variety $\mathbb{X}_{\mathcal{A}} \subset \mathbb{P}^4$ intersected with a general hyperplane

$$\mathcal{H}_{\mathbf{c}} = \left\{ [x_1 : \dots : x_5] \in \mathbb{P}^4 \,\middle|\, \sum_{j=1}^{5} c_j x_j = 0,\ [c_1 : \dots : c_5] \in \mathbb{P}^{4\vee} \right\}$$

describes the set of solutions $f(x,y) = 0$.

Next state that for coefficients $\mathbf{c}, \mathbf{c}'$, two elliptic curves $f_{\mathbf{c}}(x,y)$ and $f_{\mathbf{c}'}(x,y)$ in (1.1) are isomorphic if and only if there exist $\lambda, \mu, \nu \in \mathbb{C}^*$ such that

$$\nu f_{\mathbf{c}}(\lambda x, \mu y) = f_{\mathbf{c}'}(x,y).$$

Via the map ψ this gives rise to a torus action of $(\mathbb{C}^*)^2$ on $\mathbb{P}^4$. Because we vary the coefficients, we view them as elements of the dual space $\mathbb{P}^{4\vee}$. Therefore we can view the family of elliptic curves as elements of $\mathbb{P}^4 \times \mathbb{P}^{4\vee}$. Via the dual action on $\mathbb{P}^{4\vee}$ we see that the torus $(\mathbb{C}^*)^2$ acts on the product $\mathbb{P}^4 \times \mathbb{P}^{4\vee}$. Now define an incidence relation $\mathcal{I} = \{(p, \mathcal{H}) \in \mathbb{P}^4 \times \mathbb{P}^{4\vee} | p \in \mathcal{H}\}$ so that we can view the family of elliptic curves (up to isomorphism) as

$$\left((\mathbb{X}_{\mathcal{A}} \times \mathbb{P}^{4\vee}) \cap \mathcal{I}\right) \big/ (\mathbb{C}^*)^2 \hookrightarrow \left(\mathbb{P}^4 \times \mathbb{P}^{4\vee}\right) \big/ (\mathbb{C}^*)^2 \to \mathbb{P}^{4\vee} \big/ (\mathbb{C}^*)^2.$$

The parameter space $\mathbb{P}^{4\vee} \big/ (\mathbb{C}^*)^2$ is an intuitive choice, but spaces like these have some pathological properties and one has to look for better models which compactify the space $(\mathbb{C}^*)^5 \big/ (\mathbb{C}^*)^3$. Geometric invariant theory (GIT) constructs such a model by looking at the action of the torus $(\mathbb{C}^*)^3$ on the polynomial ring $\mathbb{C}[c_1, ..., c_5]$ induced from the action of this torus on $\mathbb{P}^{4\vee}$. We will show that the monomials transforming according to the character $\lambda\mu\nu^5$ for $(\lambda, \mu, \nu) \in (\mathbb{C}^*)^3$ give rise to a map

$$\mathbb{C}^5 \xrightarrow{\iota} \mathbb{P}^5.$$

The GIT model for the quotient space is the closure of the image of ι, which is constant on the orbits of the $(\mathbb{C}^*)^2$-action. We will show that this model corresponds to the toric variety constructed from the Secondary fan.

The aim of this thesis is to get a model for the quotient space over which there is a universal family of (elliptic) curves.

We will introduce toric ideals $I_{\mathcal{A}} \subset k[\mathbf{x}]$, where $\mathcal{A} \subset \mathbb{Z}^d$ is a finite set and k a field. These toric ideals are precisely the prime ideals generated by binomials. Moreover, for the finite set $\mathcal{A}$ in (1.2) we will show that $\mathbb{X}_{\mathcal{A}}$ is the variety defined by the toric ideal $I_{\mathcal{A}}$.

The key ingredient in solving the moduli problem is the Gröbner fan for the toric ideal $I_{\mathcal{A}}$: For this define an equivalence relation by stating that two weight vectors $\omega, \omega' \in \mathbb{R}^5$ are equivalent if and only if $\mathrm{In}_\omega(I_{\mathcal{A}}) = \mathrm{In}_{\omega'}(I_{\mathcal{A}})$, where $\mathrm{In}_\omega(I_{\mathcal{A}})$ is the initial ideal of $I_{\mathcal{A}}$ with respect to ω. The equivalence classes under this equivalence relation are cones, whose union is the Gröbner fan for $I_{\mathcal{A}}$. Each of these cones (only finitely many) correspond to a reduced Gröbner basis for $I_{\mathcal{A}}$.

Next we will introduce new coordinates $y_j = c_j x_j,\ c_j \neq 0,\ 1 \leq j \leq 5$ on $\mathbb{P}^4$. Using these new coordinates we will see that we fix our hyperplane and replace the toric ideal $I_{\mathcal{A}}$ by a family of ideals $I_{\mathcal{A},\mathbf{c}}$, depending on $\mathbf{c}$. It turns out that this family of ideals extends well over the toric variety associated with the Gröbner fan. In this way we can view the toric variety of the Gröbner fan as a moduli space for our problem and the family of curves associated with the ideals $I_{\mathcal{A},\mathbf{c}}$ as the universal family over this moduli space.

Chapter 2

Fans in toric geometry

This chapter covers only those parts of Toric Geometry which are relevant for the upcoming chapters. The subjects will be covered very briefly throughout this chapter, for more background information look at [9] chapters 1 and 2, [4] and [18] chapter 13.

Definition 2.1. Let $I \subseteq \mathbb{C}[\mathbf{x}] = \mathbb{C}[x_1, ..., x_n]$ be an ideal, then I gives rise to the affine variety $\mathcal{V}(I) := \{p \in \mathbb{C}^n \,|\, f(p) = 0,\ \forall f \in I\}$. The other way around, an affine variety $\mathcal{V} \subseteq \mathbb{C}^n$ gives rise to the ideal $I(\mathcal{V}) = \{f \in \mathbb{C}[\mathbf{x}] \,|\, f(p) = 0,\ \forall p \in \mathcal{V}\}$. We introduce the coordinate ring associated to an affine variety $\mathcal{V}$ as $\mathbb{C}[\mathcal{V}] := {}^{\mathbb{C}[\mathbf{x}]}\!/\!_{I(\mathcal{V})}$. Note that there is a one-to-one relation between the points of $\mathcal{V}$ and the maximal ideals of $\mathbb{C}[\mathcal{V}]$.

Definition 2.2. Besides the topology induced from $\mathbb{C}^n$ we can define another topology on an affine variety $\mathcal{V} \subset \mathbb{C}^n$, which is called the Zariski topology. The Zariski closed sets are the subvarieties of $\mathcal{V}$ and the Zariski open sets are their complements. Given a subset $W \subseteq \mathcal{V}$ its closure $\overline{W}$ in the Zariski topology (called the Zariski closure) is defined as the smallest subvariety of $\mathcal{V}$ containing W.

Definition 2.3. A toric variety is an irreducible variety $\mathbb{X}$ such that $(\mathbb{C}^*)^n$ is a Zariski open subset of $\mathbb{X}$ and the torus action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on $\mathbb{X}$.

Example 2.4. The easiest (and trivial) example of a toric variety is the complex torus $(\mathbb{C}^*)^n$ itself. For another example define the map

$$\psi : (\mathbb{C}^*)^n \to \mathbb{P}^n \text{ by } (z_1, ..., z_n) \mapsto [1 : z_1 : ... : z_n].$$

This way we obtain

$$(\mathbb{C}^*)^n \cong \mathbb{P}^n \setminus \mathcal{V}\left(\prod_{j=0}^{n} x_j\right).$$

As $\mathbb{P}^n \setminus \mathcal{V}\left(\prod_{j=0}^{n} x_j\right)$ is an open set in the Zariski topology we see that $(\mathbb{C}^*)^n$ is a Zariski open subset of $\mathbb{P}^n$. Next define the action

$$(\mathbb{C}^*)^n \times \mathbb{P}^n \to \mathbb{P}^n \text{ by } (z_1, ..., z_n) \star [x_0 : ... : x_n] = [x_0 : z_1 x_1 : ... : z_n x_n]$$

to see that the action of $(\mathbb{C}^*)^n$ on itself extends to an action of $(\mathbb{C}^*)^n$ on $\mathbb{P}^n$, making $\mathbb{P}^n$ a toric variety by definition 2.3.

Definition 2.5. A lattice $\mathbb{L}$ is a free abelian group of finite rank. By picking a $\mathbb{Z}$-basis for $\mathbb{L}$ we can identify it with $\mathbb{Z}^n$. Define the dual lattice to $\mathbb{L}$ as $\mathbb{L}^\vee := \mathrm{Hom}_{\mathbb{Z}}(\mathbb{L}, \mathbb{Z})$. These lattices give rise to a natural pairing $\langle \cdot, \cdot \rangle : \mathbb{L}^\vee \times \mathbb{L} \to \mathbb{Z}$.

Definition 2.6. Given a lattice $\mathbb{L}$ we define the corresponding torus $\mathbb{T}_{\mathbb{L}} := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{C}^*$ (notice that in case $\mathbb{L} \cong \mathbb{Z}^n$ then $\mathbb{T}_{\mathbb{L}} = \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{C}^* \cong (\mathbb{C}^*)^n$). Each $\mathbf{u} \in \mathbb{L}$ gives rise to a group homomorphism

$$\lambda^{\mathbf{u}} : \mathbb{C}^* \to \mathbb{T}_{\mathbb{L}} \text{ defined by } \lambda^{\mathbf{u}}(z) = \mathbf{u} \otimes z.$$

The image of such a group homomorphism is called a 1-parameter subgroup of $\mathbb{T}_{\mathbb{L}}$. Let $\mathbb{L}^\vee$ be the dual lattice to $\mathbb{L}$, then each $\mathbf{m} \in \mathbb{L}^\vee$ gives rise to a group homomorphism

$$\chi^{\mathbf{m}} : \mathbb{T}_{\mathbb{L}} \to \mathbb{C}^* \text{ defined by } \chi^{\mathbf{m}}\left(\sum_{j=1}^{l} u_j \otimes z_j\right) = \prod_{j=1}^{l} z_j^{\langle \mathbf{m}, u_j \rangle}.$$

These group homomorphisms are called the characters of $\mathbb{T}_{\mathbb{L}}$.

Example 2.7. Let $\mathbb{L} \cong \mathbb{Z}^n$, then $\mathbb{T}_{\mathbb{L}} \cong (\mathbb{C}^*)^n$ and $\mathbb{L}^\vee \cong \mathbb{Z}^{n\vee}$ (note that $\cdot^\vee$ indicates we look at the dual set). A 1-parameter subgroup of $\mathbb{T}_{\mathbb{L}}$ is given by the image of a group homomorphism of the form $\lambda^{\mathbf{u}} : \mathbb{C}^* \to \mathbb{T}_{\mathbb{L}}$ defined by $\lambda^{\mathbf{u}}(z) = (z^{u_1}, ..., z^{u_n})$ for $\mathbf{u} = (u_1, ..., u_n) \in \mathbb{Z}^n$. The characters of $\mathbb{T}_{\mathbb{L}}$ are given by group homomorphisms $\chi^{\mathbf{m}} : \mathbb{T}_{\mathbb{L}} \to \mathbb{C}^*$ defined by $\chi^{\mathbf{m}}(z_1, ..., z_n) = \prod_{j=1}^{n} z_j^{m_j}$ for $\mathbf{m} = (m_1, ..., m_n) \in \mathbb{Z}^{n\vee}$. The natural pairing $\langle \cdot, \cdot \rangle : \mathbb{L}^\vee \times \mathbb{L} \to \mathbb{Z}$ becomes a normal inner product in this case.

Definition 2.8. Given a lattice $\mathbb{L}$ and its dual lattice $\mathbb{L}^\vee$ we define the associated real vector spaces as $\mathbb{L}_{\mathbb{R}} := \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{R}$ and $\mathbb{L}_{\mathbb{R}}^\vee := \mathbb{L}^\vee \otimes_{\mathbb{Z}} \mathbb{R}$. Notice that if $\mathbb{L} \cong \mathbb{Z}^n$, then $\mathbb{L}_{\mathbb{R}} = \mathbb{Z}^n \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^n$ and $\mathbb{L}_{\mathbb{R}}^\vee \cong \mathbb{R}^{n\vee}$.

Example 2.9. In this example we show how 1-parameter subgroups lead to a compactification of a complex torus (a general approach is given in [9] §2.3). Set $\mathbb{L} = \mathbb{Z}^2$, so $\mathbb{L}_{\mathbb{R}} \cong \mathbb{R}^2$. Each vector $\mathbf{u} = (u_1, u_2)$ in the lattice $\mathbb{L}$ gives rise to a map $\lambda^{\mathbf{u}} : \mathbb{C}^* \to \mathbb{P}^1 \times \mathbb{P}^1$ defined by $z \mapsto ([1 : z^{u_1}], [1 : z^{u_2}])$. As

$\mathbb{P}^1 \times \mathbb{P}^1$ is a compact space, the limit $\lim_{z\to 0} \lambda^{\mathbf{u}}(z)$ exists. As we are using homogeneous coordinates we see that

$$\begin{aligned}\lim_{z\to 0} \lambda^{\mathbf{u}}(z) &= \lim_{z\to 0} \left([1 : z^{u_1}], [1 : z^{u_2}]\right) = \lim_{z\to 0} \left(\left[z^{-u_1} : 1\right], \left[z^{-u_2} : 1\right]\right) \\ &= \lim_{z\to 0} \left([1 : z^{u_1}], \left[z^{-u_2} : 1\right]\right) = \lim_{z\to 0} \left(\left[z^{-u_1} : 1\right], [1 : z^{u_2}]\right).\end{aligned}$$

This way we can compute the limit for all $\mathbf{u} \in \mathbb{L}$, i.e.

$$\lim_{z\to 0} \lambda^{\mathbf{u}}(z) = \begin{cases} ([1:0], [1:0]) & u_1, u_2 > 0 \\ ([1:1], [1:0]) & u_1 = 0, u_2 > 0 \\ ([1:1], [1:1]) & u_1 = u_2 = 0 \\ ([0:1], [0:1]) & u_1 < 0, u_2 < 0 \\ ([0:1], [1:0]) & u_1 < 0, u_2 > 0 \\ ([1:0], [0:1]) & u_1 > 0, u_2 < 0 \\ ([0:1], [1:1]) & u_1 < 0, u_2 = 0 \\ ([1:1], [0:1]) & u_1 = 0, u_2 < 0 \\ ([1:0], [1:1]) & u_1 > 0, u_2 = 0 \end{cases}$$

So $\mathbb{L}_{\mathbb{R}}$ is decomposed into nine disjoint regions: The four quadrants (without axes and origin), the positive and negative part of the u_1 and u_2 axis and the origin. We will see below that this is precisely the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.10. Let $\mathbb{L}$ be a lattice with dual lattice $\mathbb{L}^\vee$, then $\mathcal{C} \subset \mathbb{L}_{\mathbb{R}}$ is a convex, rational, polyhedral cone if there is a non-empty, finite subset $S \subset \mathbb{L}$ such that

$$\mathcal{C} = \left\{ \sum_{\mathbf{u}\in S} t_{\mathbf{u}}\mathbf{u} \;\middle|\; t_{\mathbf{u}} \geq 0 \right\} \subset \mathbb{L}_{\mathbb{R}}.$$

We define the dual cone $\mathcal{C}^\vee := \left\{ \mathbf{m} \in \mathbb{L}^\vee_{\mathbb{R}} \;\middle|\; \langle \mathbf{m}, \mathbf{n}\rangle \geq 0,\ \forall \mathbf{n} \in \mathcal{C} \right\} \subset \mathbb{L}^\vee_{\mathbb{R}}$.

Definition 2.11. Fix some $\mathbf{m} \in \mathbb{L}^\vee_{\mathbb{R}}$, let $\mathcal{H}_{\mathbf{m}}$ be the hyperplane determined by $\mathbf{m}$, i.e.

$$\mathcal{H}_{\mathbf{m}} := \{\mathbf{u} \in \mathbb{L}_{\mathbb{R}} \,|\langle \mathbf{m}, \mathbf{n}\rangle = 0\}.$$

We can regard $\mathcal{H}_{\mathbf{m}}$ as the intersection of the closed half-spaces

$$\mathcal{H}^{\geq}_{\mathbf{m}} = \{\mathbf{u} \in \mathbb{L}_{\mathbb{R}} \,|\langle \mathbf{m}, \mathbf{u}\rangle \geq 0\} \text{ and } \mathcal{H}^{\leq}_{\mathbf{m}} = \{\mathbf{u} \in \mathbb{L}_{\mathbb{R}} \,|\langle \mathbf{m}, \mathbf{u}\rangle \leq 0\}.$$

We say that $\mathcal{H}_{\mathbf{m}}$ is a supporting hyperplane for a cone $\mathcal{C}$ if $C \cap \mathcal{H}_{\mathbf{m}} \neq \emptyset$ and either $\mathcal{C} \subset \mathcal{H}^{\geq}_{\mathbf{m}}$ or $\mathcal{C} \subset \mathcal{H}^{\leq}_{\mathbf{m}}$. For $\mathbf{m} \in \mathbb{L}^\vee_{\mathbb{R}}$ we will also use the notation $\mathbf{m}^\perp$ for the hyperplane $\mathcal{H}_{\mathbf{m}}$.

Definition 2.12. A face $F \subseteq \mathcal{C}$ is the intersection of $\mathcal{C}$ with any supporting hyperplane, i.e.

$$F := \mathcal{C} \cap \mathbf{m}^\perp = \{\mathbf{n} \in \mathcal{C} \,|\langle \mathbf{m}, \mathbf{n}\rangle = 0\},$$

for some $\mathbf{m} \in \mathcal{C}^\vee$. Note that $\mathcal{C}$ is a face of itself, because $\mathbf{0} \in \mathcal{C}^\vee$ and $\mathbf{0}^\perp = \mathbb{L}_\mathbb{R}$. The other faces of a cone are called proper faces. The dimension of a face F is the dimension of the linear subspace of $\mathbb{L}_\mathbb{R}$ spanned by the set of vectors in F. The codimension 1 faces are called the facets of a cone.

Proposition 2.13. ([9], §1.2)

1. Every face of a convex polyhedral cone in $\mathbb{L}_\mathbb{R}$ is itself a convex polyhedral cone in $\mathbb{L}_\mathbb{R}$.
2. If F_1, F_2 are two faces of a convex polyhedral cone $\mathcal{C} \subset \mathbb{L}_\mathbb{R}$, then $F_1 \cap F_2$ is also a face of $\mathcal{C}$.
3. A face of a face of $\mathcal{C}$ is again a face of $\mathcal{C}$.

Proof.

1. A face F of $\mathcal{C}$ is of the form $F = \mathcal{C} \cap \mathbf{m}^\perp = \{\mathbf{v} \in \mathcal{C} \,|\, \langle \mathbf{m}, \mathbf{v} \rangle = 0\}$ for some $\mathbf{m} \in \mathcal{C}^\vee$. So F is generated by vectors $\mathbf{v}_j$ in a generating set for $\mathcal{C}$ such that $\langle \mathbf{m}, \mathbf{v}_j \rangle = 0$.
2. Let $F_1 = \mathcal{C} \cap \mathbf{m}_1^\perp$ and $F_2 = \mathcal{C} \cap \mathbf{m}_2^\perp$ for $\mathbf{m}_1, \mathbf{m}_2 \in \mathcal{C}^\vee$. Define $\mathbf{m} := \frac{\mathbf{m}_1}{2} + \frac{\mathbf{m}_2}{2}$, then $\mathbf{m} \in \mathcal{C}^\vee$. Moreover for any $\mathbf{a} \in \mathcal{C}$ we have
$$\mathbf{ma} = 0 \Leftrightarrow \mathbf{m}_1\mathbf{a} = 0 \text{ and } \mathbf{m}_2\mathbf{a} = 0 \Leftrightarrow \mathbf{a} \in F_1 \cap F_2.$$
3. Let $F_1 = \mathcal{C} \cap \mathbf{m}_1^\perp$ and $F_2 = F_1 \cap \mathbf{m}_2^\perp$ for $\mathbf{m}_1 \in \mathcal{C}^\vee$ and $\mathbf{m}_2 \in F_1^\vee$. Then for $p > 0$ large enough $\mathbf{m}_2 + p\mathbf{m}_1 \in \mathcal{C}^\vee$, implying that $F_2 = \mathcal{C} \cap (\mathbf{m}_2 + p\mathbf{m}_1)^\perp$, as required.

□

Proposition 2.14. (Gordon's lemma, [9], §1.2) Let $\mathcal{C} \subset \mathbb{L}_\mathbb{R}$ be a convex, rational, polyhedral cone and define $S_{\mathcal{C}^\vee} := \mathcal{C}^\vee \cap \mathbb{L}^\vee$, then $S_{\mathcal{C}^\vee}$ is a finitely generated semigroup under addition with identity $\mathbf{0} \in S_{\mathcal{C}^\vee}$.

Proof. Obviously $S_{\mathcal{C}^\vee}$ is a semigroup with $\mathbf{0}$ as identity. Let $\mathbf{u}_1, ..., \mathbf{u}_s \in \mathcal{C}^\vee \cap \mathbb{L}^\vee$ be a set of generators for $\mathcal{C}^\vee \subset \mathbb{L}_\mathbb{R}^\vee$. Define the compact set $K := \{\sum_{j=1}^s t_j\mathbf{u}_j \,|0 \leq t_j < 1\}$. As $\mathbb{L}^\vee$ is a lattice the set $K \cap \mathbb{L}^\vee$ is finite. Next let $\mathbf{u} \in S_{\mathcal{C}^\vee}$, then there exist $r_j \geq 0$ such that $\mathbf{u} = \sum_{j=1}^s r_j\mathbf{u}_j$. For each j we can write $r_j = \lfloor r_j \rfloor + t_j$, where $\lfloor \cdot \rfloor$ is the floor function and $0 \leq t_j \leq 1$. So $\mathbf{u} = \sum_{j=1}^s \lfloor r_j \rfloor \mathbf{u}_j + \sum_{j=1}^s t_j\mathbf{u}_j$ with $\mathbf{u}_j \in \mathcal{C}^\vee \cap \mathbb{L}^\vee$ for all j and $\sum_{j=1}^s t_j\mathbf{u}_j \in K \cap \mathbb{L}^\vee$ by construction. As $\mathbf{u}$ is chosen arbitrarily, the result follows. □

Definition 2.15. To each $S_{\mathcal{C}^\vee}$ we can associate a semigroup algebra $\mathbb{C}[S_{\mathcal{C}^\vee}]$, which as a $\mathbb{C}$-vector space admits $S_{\mathcal{C}^\vee}$ as a basis. So elements of $\mathbb{C}[S_{\mathcal{C}^\vee}]$ are of the form $\sum_{\mathbf{m} \in S_{\mathcal{C}^\vee}} a_\mathbf{m}\chi^\mathbf{m}$, with only finitely many $a_\mathbf{m}$ nonzero. Multiplication in $\mathbb{C}[S_{\mathcal{C}^\vee}]$ is determined by addition in $S_{\mathcal{C}^\vee}$, i.e. $\chi^\mathbf{m} \cdot \chi^{\mathbf{m}'} = \chi^{\mathbf{m}+\mathbf{m}'}$. By Gordon's lemma we obtain that $\mathbb{C}[S_{\mathcal{C}^\vee}]$ is finitely generated.

Definition 2.16. Let $\mathcal{C}$ be a rational, strongly convex, polyhedral cone and $S_{\mathcal{C}^\vee}$ the corresponding semigroup. Then we define the affine toric variety

$$U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C}) := \mathrm{Hom}_{\mathrm{sg}}\left(S_{\mathcal{C}^\vee}, \mathbb{C}^\times\right).$$

Normally we view charts as ring homomorphisms from a commutative ring to a field. This does not violate with the definition given here as we can view $\mathbb{Z}[S_{\mathcal{C}}^\vee]$ as a commutative, unitary ring, implying that $U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C}) = \mathrm{Hom}_{\mathrm{ring}}\left(\mathbb{Z}\left[S_{\mathcal{C}^\vee}\right], \mathbb{C}\right) = \mathrm{Hom}_{\mathrm{sg}}\left(S_{\mathcal{C}^\vee}, \mathbb{C}^\times\right)$.

Example 2.17. Let $\{e_j\}_{j=1}^n$ be a basis for a lattice $\mathbb{L} \cong \mathbb{Z}^n$, and let $\left\{e_j^\vee\right\}_{j=1}^n$ be the dual basis, which is a basis for $\mathbb{L}^\vee$. Then for the cone $\{0\}$ we see that $S_{\{0\}^\vee} := \{0\}^\vee \cap \mathbb{L}^\vee = \mathbb{L}^\vee$. As a semigroup $\mathbb{L}^\vee$ is generated by $\left\{\pm e_j^\vee\right\}_{j=1}^n$. Let $x_i := \chi^{e_j^\vee}$ $(1 \leq j \leq n)$, be the coordinates for the semigroup algebra $\mathbb{C}\left[S_{\{0\}^\vee}\right]$. This implies $\mathbb{C}\left[S_{\{0\}^\vee}\right] = \mathbb{C}\left[x_1^{\pm 1}, ..., x_n^{\pm 1}\right]$ is the ring of Laurent polynomials. Then

$$U_{\mathbb{Z}\left[S_{\{0\}^\vee}\right]}(\mathbb{C}) = \mathrm{Hom}_{\mathrm{sg}}\left(S_{\{0\}^\vee}, \mathbb{C}^\times\right) = (\mathbb{C}^*)^n \cong \mathbb{L} \otimes_{\mathbb{Z}} \mathbb{C}^* = \mathbb{T}_{\mathbb{L}}.$$

For every rational, polyhedral, strongly convex cone $\mathcal{C} \supset \{0\}$ we know that $\mathcal{C}^\vee \subset \{0\}^\vee = \mathbb{L}_{\mathbb{R}}^\vee$, hence $S_{\mathcal{C}^\vee} \subset S_{\{0\}^\vee}$ and $\mathbb{C}\left[S_{\mathcal{C}^\vee}\right] \subset \mathbb{C}\left[S_{\{0\}^\vee}\right]$. This implies that

$$\mathbb{T}_{\mathbb{L}} = U_{\mathbb{Z}\left[S_{\{0\}^\vee}\right]}(\mathbb{C}) \subset U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C}).$$

We can conclude that all charts over a rational, polyhedral, strongly convex cone containing the origin, contain the complex torus $\mathbb{T}_{\mathbb{L}}$.

The upcoming definitions give rise to more general toric varieties obtained by gluing together affine toric varieties.

Definition 2.18. Let $\mathbb{L}$ be a lattice. A fan $\mathcal{F}$ in $\mathbb{L}_{\mathbb{R}}$ is a collection of rational, polyhedral, strongly convex cones $\mathcal{C} \subset \mathbb{L}_{\mathbb{R}}$ such that

1. Each face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$.

2. If $\mathcal{C}_1, \mathcal{C}_2 \in \mathcal{F}$ are two cones, then $\mathcal{C}_1 \cap \mathcal{C}_2$ is a face of each.

Definition 2.19. Let $\mathcal{V}$ be an irreducible variety and $f \in \mathbb{C}[\mathcal{V}]$ nonzero, then the localization of $\mathbb{C}[\mathcal{V}]$ at f is defined as

$$\mathbb{C}[\mathcal{V}]_f := \left\{ \frac{g}{f^p} \in \mathbb{C}[\mathcal{V}] \,\middle|\, g \in \mathbb{C}[\mathcal{V}],\ p \geq 0 \right\}.$$

Let $\mathcal{F} \subset \mathbb{L}_{\mathbb{R}}$ be a fan, $\mathcal{C} \in \mathcal{F}$ a cone and $F \subset \mathcal{C}$ a face of $\mathcal{C}$. As F is a cone as well by proposition 2.13 we see that

$$F \subset \mathcal{C} \Rightarrow \mathcal{C}^\vee \subset F^\vee \Rightarrow S_{\mathcal{C}^\vee} \subset S_{F^\vee} \Rightarrow \mathbb{C}\left[S_{\mathcal{C}^\vee}\right] \subset \mathbb{C}\left[S_{F^\vee}\right].$$

This last inclusion gives rise to a map $\varphi : U_{\mathbb{Z}[S_{F^\vee}]}(\mathbb{C}) \to U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C})$. We will take a closer look at this map: As F is a face of $\mathcal{C}$ we can write $F = \mathcal{C} \cap \mathbf{m}^\perp$ for some $\mathbf{m}$ in $\mathcal{C}^\vee$. Furthermore we can take $\mathbf{m} \in \mathbb{L}^\vee$ since $\mathcal{C}$ is rational. Choose any $\mathbf{b} \in S_{F^\vee}$, then $\mathbf{b} + p\mathbf{m} \in \mathcal{C}^\vee$ for large $p > 0$. Choosing this p integral implies $\mathbf{b} \in S_{\mathcal{C}^\vee} - \mathbf{m}\mathbb{Z}_{\geq 0}$. We conclude that $S_{F^\vee} = S_{\mathcal{C}^\vee} - \mathbf{m}\mathbb{Z}_{\geq 0}$. So we see that $\mathbb{C}[S_{\mathcal{C}^\vee}] \subset \mathbb{C}[S_{\mathcal{C}^\vee}]_{\chi^{\mathbf{m}}} = \mathbb{C}[S_{F^\vee}]$. Therefore can we write each of the basis elements of $\mathbb{C}[S_{F^\vee}]$ as $\chi^{\mathbf{n}-p\mathbf{m}} = \frac{\chi^{\mathbf{n}}}{(\chi^{\mathbf{m}})^p}$ for $\mathbf{n} \in S_{\mathcal{C}^\vee}$ (using definition 2.19). So we see that $U_{\mathbb{Z}[S_{F^\vee}]}(\mathbb{C})$ is isomorphic to the Zariski open set $U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C})$ where $\chi^{\mathbf{m}} \neq 0$.

This information gives rise to the gluing of two affine toric varieties: Let $\mathcal{C}, \mathcal{C}' \in \mathcal{F} \subset \mathbb{L}_\mathbb{R}$ be two cones sharing a common face $\mathcal{C} \cap \mathcal{C}'$. As we can regard $\mathcal{C} \cap \mathcal{C}'$ as a cone in $\mathcal{F}$ (by proposition 2.13) contained in both $\mathcal{C}$ and $\mathcal{C}'$, we get by the previous paragraph open immersions

$$\begin{aligned} &U_{\mathbb{Z}\left[S_{(\mathcal{C}\cap\mathcal{C}')^\vee}\right]}(\mathbb{C}) \xrightarrow{\psi_1} U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C}), \text{ and} \\ &U_{\mathbb{Z}\left[S_{(\mathcal{C}\cap\mathcal{C}')^\vee}\right]}(\mathbb{C}) \xrightarrow{\psi_2} U_{\mathbb{Z}[S_{\mathcal{C}'^\vee}]}(\mathbb{C}). \end{aligned}$$

Then we have an isomorphism (a gluing map) $g_{\mathcal{C}\cap\mathcal{C}'} : \mathrm{Im}(\psi_1) \xrightarrow{\cong} \mathrm{Im}(\psi_2)$. So the collection $\left\{U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C}), \mathrm{Im}(\psi_1), g_{\mathcal{C}\cap\mathcal{C}'}\right\}$ gives the gluing data between two affine toric varieties. This leads to the following definition:

Definition 2.20. Let $\mathcal{F} \subset \mathbb{L}_\mathbb{R}$ be a fan, let $\mathcal{C}, \mathcal{C}' \in \mathcal{F}$ be two cones and fix any $\alpha \in U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C})$ and $\beta \in U_{\mathbb{Z}[S_{\mathcal{C}'^\vee}]}(\mathbb{C})$. Define an equivalence relation

$$\alpha \sim \beta \iff \alpha \in U_{\mathbb{Z}\left[S_{(\mathcal{C}\cap\mathcal{C}')^\vee}\right]}(\mathbb{C}) \text{ and } \beta = g_{\mathcal{C}\cap\mathcal{C}'}(\alpha).$$

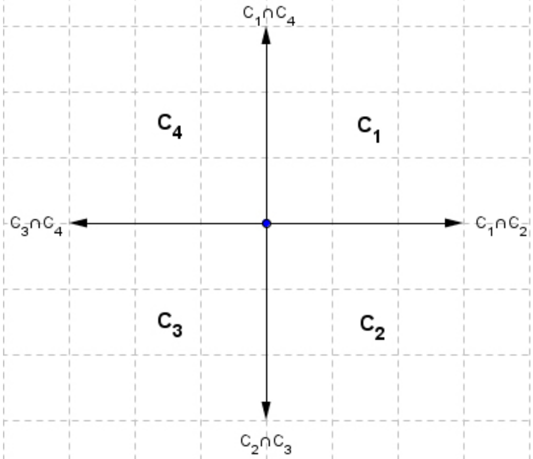
We define the variety over the fan $\mathcal{F}$ to be the set

$$\mathbb{X}_\mathcal{F} := \coprod_{\mathcal{C}\in\mathcal{F}} U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C}) \Big/ \sim.$$

Proposition 2.21. Let $\mathcal{F}$ be a fan, then the set $\mathbb{X}_\mathcal{F}$ as defined in definition 2.20 is a toric variety.

Proof. Every cone $\mathcal{C} \in \mathcal{F}$ gives rise to an affine open subset $U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C}) \subset \mathbb{X}_\mathcal{F}$. We have seen that the cone $\{0\} \in \mathcal{F}$ corresponds to the open subset $\mathbb{T}_\mathbb{L} \subset \mathbb{X}_\mathcal{F}$. As $\{0\}$ is a face of every cone $\mathcal{C} \in \mathcal{F}$, $\mathbb{T}_\mathbb{L} \subset U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C})$. So $\mathbb{T}_\mathbb{L}$ is a Zarisky dense subset of $\mathbb{X}_\mathcal{F}$ and of $U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C})$ for every cone $\mathcal{C} \in \mathcal{F}$, implying that $\mathbb{X}_\mathcal{F}$ is irreducible. As $\mathbb{T}_\mathbb{L}$ acts on each $U_{\mathbb{Z}[S_{\mathcal{C}^\vee}]}(\mathbb{C})$ and the gluing data is $\mathbb{T}_\mathbb{L}$-equivariant we see that the action of $\mathbb{T}_\mathbb{L}$ on itself extends to an action of $\mathbb{T}_\mathbb{L}$ on $\mathbb{X}_\mathcal{F}$. By definition 2.3 we see that $\mathbb{X}_\mathcal{F}$ is a toric variety. $\square$

Example 2.22. Define the following fan $\mathcal{F}$ in $\mathbb{R}^2$:



The fan $\mathcal{F}$ consists of the following cones (and their faces):

$$\begin{aligned}
\mathcal{C}_1 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{R}_{\geq 0} \\
\mathcal{C}_2 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{R}_{\geq 0} \\
\mathcal{C}_3 &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\mathbb{R}_{\geq 0} \\
\mathcal{C}_4 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\mathbb{R}_{\geq 0}.
\end{aligned}$$

Let $\mathbf{g}_j, \mathbf{h}_j$ be the generators of cone $\mathcal{C}_j$ for $1 \leq j \leq 4$. Then the semigroups over these cones equal $S_{\mathcal{C}_j^\vee} = \mathbf{g}_j^\vee \mathbb{Z}_{\geq 0} + \mathbf{h}_j^\vee \mathbb{Z}_{\geq 0}$ for all j. This yields the following affine toric varieties:

$$\begin{aligned}
U_{\mathbb{Z}\left[S_{\mathcal{C}_j^\vee}\right]}(\mathbb{C}) = \mathrm{Hom}_{\mathrm{sg}}\left(S_{\mathcal{C}_j^\vee}, \mathbb{C}^\times\right) &\xrightarrow{\cong} \mathbb{C}^2 \\
\beta &\longmapsto \left(\beta\left(\mathbf{g}_j^\vee\right), \beta\left(\mathbf{h}_j^\vee\right)\right).
\end{aligned}$$

Next we look at the gluing between two adjacent, affine toric varieties. For example, we find the following gluing map φ between the toric varieties over $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\begin{tikzcd}
\{(x_1,y_1) \in \mathbb{C}\times\mathbb{C} | x_1 \neq 0\} \arrow[rr, hook] \arrow[dd, "\varphi(x_1,y_1)=(x_1^{-1},y_1)"'] \arrow[dr, leftrightarrow, "\cong"] & & \mathbb{C}\times\mathbb{C} \cong U_{\mathbb{Z}[S_{\mathcal{C}_1^\vee}]}(\mathbb{C}) \\
& U_{\mathbb{Z}[S_{(\mathcal{C}_1\cap\mathcal{C}_2)^\vee}]}(\mathbb{C}) \arrow[ur, hook, "\psi_1"] \arrow[dr, hook, "\psi_2"] \arrow[dl, leftrightarrow, "\cong"] & \\
\{(x_2,y_2) \in \mathbb{C}\times\mathbb{C} | x_2 \neq 0\} \arrow[rr, hook] & & \mathbb{C}\times\mathbb{C} \cong U_{\mathbb{Z}[S_{\mathcal{C}_2^\vee}]}(\mathbb{C})
\end{tikzcd}$$

Let $([u_1 : u_2], [v_1 : v_2])$ be the homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$, then we find in a similar way as above the following gluing isomorphisms:

$$\begin{tikzcd}
U_{\mathbb{Z}\left[S_{C_1^\vee}\right]}(\mathbb{C}) \arrow[d, leftrightarrow, "\cong"] & U_{\mathbb{Z}\left[S_{C_2^\vee}\right]}(\mathbb{C}) \arrow[d, leftrightarrow, "\cong"] \\
\left\{ \left(\frac{u_1}{u_2}, \frac{v_1}{v_2}\right) \in \mathbb{C}^2 \middle| \; u_2 \neq 0,\; v_2 \neq 0\right\} \arrow[r, leftrightarrow, "(x^{-1},y)"] \arrow[d, leftrightarrow, "(x,y^{-1})"] & \left\{ \left(\frac{u_2}{u_1}, \frac{v_1}{v_2}\right) \in \mathbb{C}^2 \middle| \; u_1 \neq 0,\; v_2 \neq 0\right\} \arrow[d, leftrightarrow, "(x,y^{-1})"] \\
\left\{ \left(\frac{u_1}{u_2}, \frac{v_2}{v_1}\right) \in \mathbb{C}^2 \middle| \; u_2 \neq 0,\; v_1 \neq 0\right\} \arrow[r, leftrightarrow, "(x^{-1},y)"] \arrow[d, leftrightarrow, "\cong"'] & \left\{ \left(\frac{u_2}{u_1}, \frac{v_2}{v_1}\right) \in \mathbb{C}^2 \middle| \; u_1 \neq 0,\; v_1 \neq 0\right\} \arrow[d, leftrightarrow, "\cong"'] \\
U_{\mathbb{Z}\left[S_{C_4^\vee}\right]}(\mathbb{C}) & U_{\mathbb{Z}\left[S_{C_3^\vee}\right]}(\mathbb{C})
\end{tikzcd}$$

So the affine, toric varieties $U_{\mathbb{Z}\left[S_{C_j^\vee}\right]}(\mathbb{C})$ for $1 \leq j \leq 4$ are precisely the affine charts for $\mathbb{P}^1 \times \mathbb{P}^1$ implying that $\mathbb{X}_{\mathcal{F}} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Definition 2.23. Let $\mathbb{L} \cong \mathbb{Z}^n$ be a lattice and let $\mathbb{L}^\vee$ be the dual lattice. We define a lattice polytope $P \subset \mathbb{L}^\vee_{\mathbb{R}} \cong \mathbb{R}^n$ to be the convex hull of a finite subset of points in $\mathbb{L}^\vee$. Let $F \subset P$ be a facet and $\mathbf{n}_F$ be the primitive, outer normal vector to this facet. Then we can also define P as an intersection of closed half-spaces: Each facet $F \subset P$ has a supporting hyperplane defined by $\langle \mathbf{m}, \mathbf{n}_F \rangle = b_F$ for some $b_F \in \mathbb{Z}$. Then we can write

$$P = \bigcap_{F \subset P \text{ a facet}} \left\{ \mathbf{m} \in \mathbb{L}^\vee_{\mathbb{R}} \,|\, \langle \mathbf{m}, \mathbf{n}_F \rangle \leq b_F \in \mathbb{Z} \right\}.$$

Proposition 2.24. Let $\mathbb{L}$ be a lattice, P a lattice polytope and $F \subset P$ a face. Let $\mathcal{N}_P(F)$ be the cone generated by the facet (outer) normals for all facets of P containing F. Then $\mathcal{N}_P(F)$ is a strongly convex, rational polyhedral cone in $\mathbb{L}_{\mathbb{R}}$ (called the normal cone for P at F) and the set $\mathcal{F}_P := \{\mathcal{N}_P(F) \mid F \subset P \text{ a face}\}$ is a fan, called the outer normal fan of P.

Example 2.25. Let $\mathbb{L} \cong \mathbb{Z}^2$ and let P be the unit square with vertices $A := \left\{(0,0)^t, (1,0)^t, (0,1)^t, (1,1)^t\right\}$ (so P is a lattice polytope w.r.t. $\mathbb{L}$). The 2-dimensional, outer normal cones $\mathcal{N}_P(F)$ (as defined in proposition 2.24) are the cones defined by the vertices $F \in A \subset P$. These are precisely the cones given in example 2.22. Therefore we conclude that the toric variety corresponding to the outer normal fan $\mathcal{F}_P$ equals $\mathbb{X}_{\mathcal{F}_P} \cong \mathbb{P}^1 \times \mathbb{P}^1$.

Proposition 2.26. Let $G \subset \mathrm{GL}\,(n, \mathbb{C})$ be a finite subgroup, then G acts on $\mathbb{C}^n$ (by matrix-vector multiplication) and on the polynomial ring $\mathbb{C}[\mathbf{x}]$ (by defining $g \star f(\mathbf{x}) = f(g\mathbf{x})$ for any $f \in \mathbb{C}[\mathbf{x}]$ and $g \in G$). So we can define the ring of invariant polynomials $\mathbb{C}[\mathbf{x}]^G := \{f \in \mathbb{C}[\mathbf{x}] \,|\, g \star f = f\} \subset \mathbb{C}[\mathbf{x}]$ and it makes sense to denote by ${}^{\mathbb{C}^n}\!/_{G}$ the set of G-orbits. Furthermore, there is a bijection ${}^{\mathbb{C}^n}\!/_{G} \cong \mathrm{Spec}\left(\mathbb{C}[\mathbf{x}]^G\right)$.

Proof. The proof is explained in full detail in [2] ch. 7. □

Definition 2.27. Let $\mathbb{L}$ be a lattice and let $\mathcal{F}$ be a fan in $\mathbb{L}_{\mathbb{R}}$. Let $\mathbf{u} \in \mathbb{L}$ be a lattice point in $\mathbb{L}$, then we can subdivide $\mathcal{F}$ by replacing the cones $\mathcal{C} \in \mathcal{F}$ containing $\mathbf{u}$ by the joins (sums) of its faces with the ray through point $\mathbf{u}$, the cones in $\mathcal{F}$ not containing $\mathbf{u}$ are left unchanged. The result is a new fan $\mathcal{F}'$ in $\mathbb{L}_{\mathbb{R}}$, which is a refinement of the original fan $\mathcal{F}$. As the fans have the same support (i.e. the union of their cones coincide), the mapping $\mathbb{X}_{\mathcal{F}'} \to \mathbb{X}_{\mathcal{F}}$ is proper and birational.

Proposition 2.28. ([9], §2.3) For any toric variety $\mathbb{X}_{\mathcal{F}}$ there is a refinement $\mathcal{F}'$ of $\mathcal{F}$ such that the equivariant map $\mathbb{X}_{\mathcal{F}'} \to \mathbb{X}_{\mathcal{F}}$ is a resolution of singularities.

Finally we present a different approach to toric varieties (following [18], chapter 13):

Let $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\} \subset \mathbb{Z}^d$ be a finite set. Let $\mathcal{A} = \{\mathbf{a}_1, ..., \mathbf{a}_n\} \subset \mathbb{Z}^{d+1}$ be the finite set with elements $\mathbf{a}_j := (\mathbf{b}_j, 1)^t$ for all $1 \leq j \leq n$. This way we have homogenized the elements in $\mathcal{B}$ and we define the ideal $I_{\mathcal{A}}$ as the homogeneous, toric ideal

$$\left\langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \,\middle|\, \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \mathbb{Z}^n \text{ s.t. } \sum_{j=1}^{n} u_j \mathbf{a}_j = 0 \right\rangle,$$

where $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ is such that $\mathbf{u}^+$ and $\mathbf{u}^-$ have nonnegative entries and disjoint support. For more details on toric ideals see section 4.2. Define a map

$$\psi : (\mathbb{C}^*)^d \to \mathbb{P}^{n-1} \text{ given by } \mathbf{t} \mapsto [\mathbf{t}^{\mathbf{a}_1} : ... : \mathbf{t}^{\mathbf{a}_n}].$$

Then $\mathbb{X}_{\mathcal{A}}$ is the projective, toric variety defined as the Zarisky closure $\overline{\mathrm{Im}(\psi)}$, or equivalently as the variety $\mathcal{V}(I_{\mathcal{A}})$.

Example 2.29. Define the matrix

$$\mathcal{A} = \{\mathbf{a}_1, ..., \mathbf{a}_4\} := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix},$$

which corresponds to the unit square in $\mathbb{R}^2$ embedded at height 1 in $\mathbb{R}^3$. Define a map from the (3-dimensional) complex torus to $\mathbb{P}^3$:

$$\psi : (\mathbb{C}^*)^3 \to \mathbb{P}^3, \text{ given by, } \mathbf{t} \longmapsto [\mathbf{t}^{\mathbf{a}_1} : \mathbf{t}^{\mathbf{a}_2} : \mathbf{t}^{\mathbf{a}_3} : \mathbf{t}^{\mathbf{a}_4}].$$

Then the toric variety from $\mathcal{A}$ is given by the Zarisky closure

$$\mathbb{X}_{\mathcal{A}} = \overline{\mathrm{Im}(\psi)} = \left\{[x_1 : ... : x_4] \in \mathbb{P}^3 \,|x_1x_3 = x_2x_4 \right\}.$$

Equivalently, consider the toric ideal

$$I_{\mathcal{A}} := \left\langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \,\middle|\, \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \mathbb{Z}^n \text{ s.t. } \sum_{j=1}^{n} u_j \mathbf{a}_j = 0 \right\rangle = \langle x_1x_3 - x_2x_4 \rangle .$$

Observe that $\mathbb{X}_{\mathcal{A}} = \mathcal{V}(I_{\mathcal{A}})$. How does this projective, toric variety relate to the projective, toric variety $\mathbb{X}_{\mathcal{F}}$ over the fan $\mathcal{F}$ in example 2.22? For this consider the Segre embedding (see [10] for the details):

$$\phi : \mathbb{P}^1 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^3, \text{ given by } ([u : v], [w : z]) \to [uw : uz : vz : vw] .$$

Via this embedding we see that

$$\mathbb{X}_{\mathcal{F}} \cong \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathrm{Im}(\phi) = \left\{[x_1 : x_2 : x_3 : x_4] \in \mathbb{P}^3 \,|x_1x_3 = x_2x_4 \right\} \cong \mathbb{X}_{\mathcal{A}},$$

so these toric varieties are isomorphic.

Chapter 3

The Secondary fan in a moduli problem

As mentioned in the introduction; the paper *The complex geometry of the spherical pendulum* by Beukers and Cushman (see [5]) describes the phase space of the complexified, spherical pendulum as a $\mathbb{C}^*$-bundle over a family of elliptic curves. An interpretation of this family of elliptic curves in the context of toric geometry will be studied in this chapter.

3.1 A family of elliptic curves

We will begin by introducing Newton polytopes and show that we can describe the family of elliptic curves mentioned in [5] by the vertices of a Newton polytope.

Definition 3.1. Let k be a field, $f \in k\left[x_1^{\pm 1}, ..., x_n^{\pm 1}\right]$ a Laurent polynomial over k, and write $f = \sum_{\mathbf{a} \in \mathbb{Z}^n} c_{\mathbf{a}} \mathbf{x}^{\mathbf{a}}$. Then the *Newton polytope* of f is defined as

$$\operatorname{New}(f) = \operatorname{Conv}\left(\{\mathbf{a} \in \mathbb{Z}^n : c_{\mathbf{a}} \neq 0\}\right).$$

Example 3.2. Let $\mathbb{L} \cong \mathbb{Z}^2$ be a lattice, set $k = \mathbb{C}$, and consider the following Newton polytope defined on $\mathbb{L}$:

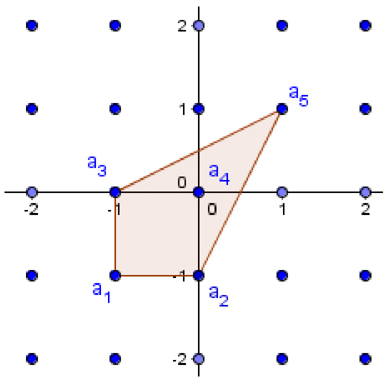


Figure 3.1: A Newton polytope.

The vertices of this polytope correspond to a finite set

$$\mathcal{A} := \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\},$$

given by

$$\mathbf{a}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \ \mathbf{a}_2 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad \mathbf{a}_3 = \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \ \mathbf{a}_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad \mathbf{a}_5 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Let $f \in \mathbb{C}\left[x^{\pm 1}, y^{\pm 1}\right]$ be the Laurent polynomial corresponding to the Newton polytope of example 3.2, defined by the elements of $\mathcal{A}$, then

$$f(x,y) = c_1 x^{-1} y^{-1} + c_2 y^{-1} + c_3 x^{-1} + c_4 + c_5 xy,$$

with coefficients $c_1, ..., c_5 \in \mathbb{C}^*$.

We can view these elliptic curves as a 2-parameter family in $\mathbb{P}^1 \times \mathbb{P}^1$:

Lemma 3.3. The set of solutions to the family of Laurent polynomials given by

$$f(x,y) = c_1 x^{-1} y^{-1} + c_2 y^{-1} + c_3 x^{-1} + c_4 + c_5 xy = 0 \tag{3.1}$$

for $x, y \in \mathbb{C}^*$ corresponds to a 2-parameter family of elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$.

Proof. For this we introduce homogeneous coordinates; substitute $x = \frac{x_1}{x_2}$ and $y = \frac{y_1}{y_2}$ in (3.1) to get a family of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ parametrized by the coefficients $c_1, ..., c_5 \in \mathbb{C}^*$, given by the equation

$$c_1 x_2^2 y_2^2 + c_2 x_1 x_2 y_2^2 + c_3 x_2^2 y_1 y_2 + c_4 x_1 x_2 y_1 y_2 + c_5 x_1^2 y_1^2 = 0.$$

Notice that in this case we use $([x_1 : x_2], [y_1, y_2])$ as the homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$. Next rescale the coordinates by

$$([x_1 : x_2], [y_1 : y_2]) \mapsto ([\mu x_1 : x_2], [\nu y_1 : y_2]), \quad \mu, \nu \in \mathbb{C}^*,$$

this yields

$$c_1 x_2^2 y_2^2 + \mu c_2 x_1 x_2 y_2^2 + \nu c_3 x_2^2 y_1 y_2 + \mu\nu c_4 x_1 x_2 y_1 y_2 + \mu^2 \nu^2 c_5 x_1^2 y_1^2 = 0.$$

Multiply the equation on both sides with $\lambda \in \mathbb{C}^*$ to obtain

$$\lambda c_1 x_2^2 y_2^2 + \lambda\mu c_2 x_1 x_2 y_2^2 + \lambda\nu c_3 x_2^2 y_1 y_2 + \lambda\mu\nu c_4 x_1 x_2 y_1 y_2 + \lambda\mu^2\nu^2 c_5 x_1^2 y_1^2 = 0.$$

If we set $\lambda = c_1^{-1}, \mu = c_1 c_2^{-1}$ and $\nu = c_1 c_3^{-1}$ in the equation above we see that

$$\begin{aligned} &x_2^2 y_2^2 + x_1 x_2 y_2^2 + x_2^2 y_1 y_2 + c_1 c_2^{-1} c_3^{-1} c_4 x_1 x_2 y_1 y_2 + c_1^3 c_2^{-2} c_3^{-2} c_5 x_1^2 y_1^2 = \\ &x_2^2 y_2^2 + x_1 x_2 y_2^2 + x_2^2 y_1 y_2 + v_1 x_1 x_2 y_1 y_2 + v_2 x_1^2 y_1^2 = 0, \end{aligned}$$

with the two parameters defined by $v_1 := c_1 c_2^{-1} c_3^{-1} c_4$ and $v_2 := c_1^3 c_2^{-2} c_3^{-2} c_5$. So indeed we obtain a 2-parameter family of curves in $\mathbb{P}^1 \times \mathbb{P}^1$. As it is a degree $(2,2)$ family of curves in $\mathbb{P}^1 \times \mathbb{P}^1$ it is a family of elliptic curves, which finishes the proof. □

Clearly, all curves pass through the points $p = ([0:1],[1:0])$ and $q = ([1:0],[0:1])$ in $\mathbb{P}^1 \times \mathbb{P}^1$. One can show that (repeatedly) blowing up the points p and q (in order to resolve the degeneracy in v_1 and v_2) only works for the parameter v_2. So looking at these elliptic curves as a 2-paramater family in $\mathbb{P}^1 \times \mathbb{P}^1$ does not yield a smooth family of elliptic curves parametrized by $(v_1, v_2) \in (\mathbb{C}^*)^2$. Therefore we look at this family of elliptic curves in a different toric variety than $\mathbb{P}^1 \times \mathbb{P}^1$ in the next section.

3.2 The toric variety from the Newton polytope

3.4. Recall matrix $\mathcal{A} = \begin{pmatrix} -1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}$ from example 3.2, whose columns correspond to the vertices of the Newton polytope of the elliptic curves $f(x,y)$. A natural way to look at the family of elliptic curves is by defining a map from the 2-dimensional torus to $\mathbb{P}^4$ given by

$$\begin{aligned} \psi : \quad (\mathbb{C}^*)^2 &\to \mathbb{P}^4, \\ (x,y) &\mapsto \left[x^{-1}y^{-1} : y^{-1} : x^{-1} : 1 : xy\right]. \end{aligned} \tag{3.2}$$

By taking the Zarisky closure of the image of ψ we define a new toric variety $\mathbb{X}_{\mathcal{A}} := \overline{\operatorname{Im}(\psi)}$. This way we have compactified the torus $(\mathbb{C}^*)^2$ in $\mathbb{P}^4$ (by adding four lines to the torus, as we will see in paragraph 3.7).

By varying the coefficients we can regard $f(x,y)$ as a *family* of elliptic curves. Therefore we will regard the coefficients $c_1, ..., c_5$ from now on as elements of a dual projective space, i.e. $[c_1 : ... : c_5] \in \mathbb{P}^{4\vee}$. Look at the set of solutions of

$$f(x,y) = c_1 x^{-1} y^{-1} + c_2 y^{-1} + c_3 x^{-1} + c_4 + c_5 xy = 0 \tag{3.3}$$

for some fixed choice of the coefficients $[c_1 : ... : c_5] \in \mathbb{P}^{4\vee}$. As the exponents of the Laurent monomials in (3.3) correspond to the lattice points of the Newton polytope New(f), we see that the set of solutions of (3.3) is equal to the intersection of a general hyperplane $\mathcal{H}_\mathbf{c} \subset \mathbb{P}^4$ defined by

$$\mathcal{H}_\mathbf{c} := \left\{ [x_1 : ... : x_5] \in \mathbb{P}^4 \;\middle|\; \sum_{j=1}^{5} c_j x_j = 0 \right\}$$

with the toric variety $\mathbb{X}_\mathcal{A}$ (for more details see paragraph 3.8).

We shall derive a much more explicit description of the toric variety $\mathbb{X}_\mathcal{A}$ in the following lemma:

Lemma 3.5. The toric variety $\mathbb{X}_\mathcal{A}$ corresponding to the Newton polytope in example 3.2 is given by

$$\mathbb{X}_\mathcal{A} = \left\{ [x_1 : ... : x_5] \in \mathbb{P}^4 \,\middle|\, x_1x_4 = x_2x_3 \text{ and } x_1x_5 = x_4^2 \right\}.$$

Proof. Start by homogenizing matrix $\mathcal{A}$ by embedding the Newton polytope from example 3.2 at height one in $\mathbb{R}^3$: For this redefine the vectors in the set $\mathcal{A}$ from example 3.2 by

$$\mathcal{A} := \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\},$$

where

$$\mathbf{a}_1 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}, \mathbf{a}_2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}, \mathbf{a}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{a}_4 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \mathbf{a}_5 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

Let $[x_1 : ... : x_5]$ be the coordinates on $\mathbb{P}^4$. Define a lattice in $\mathbb{Z}^5$:

$$\mathbb{L} := \left\{ \mathbf{u} \in \mathbb{Z}^5 \,\middle|\, \sum_{j=1}^{5} u_j \mathbf{a}_j = \mathbf{0} \right\}.$$

We can write each $\mathbf{u} \in \mathbb{L}$ as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ with $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{N}^5$ having disjoint support (this decomposition of $\mathbf{u}$ is unique). So we see that

$$\mathbb{X}_\mathcal{A} = \left\{ [x_1 : ... : x_5] \in \mathbb{P}^4 \,\middle|\, \prod_{j=1}^{5} x_j^{u_j^+} - \prod_{j=1}^{5} x_j^{u_j^-} = 0,\ \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \mathbb{L} \right\}.$$

We begin by retrieving a $\mathbb{Z}$-basis for the lattice $\mathbb{L}$: Take $\mathbf{u} \in \mathbb{L}$ arbitrary, then $\mathbf{u}$ is such that

$$\begin{cases} -u_1 - u_3 + u_5 & = & 0 \\ -u_1 - u_2 + u_5 & = & 0 \\ u_1 + u_2 + u_3 + u_4 + u_5 & = & 0 \end{cases}$$

From the second equation we see immedeatly $u_5 = u_1 + u_2$, from the first and second equation we conclude $u_2 = u_3$ and then from equation three and the previous results we see directly that $u_4 = -2u_1 - 3u_2$. So we obtain a $\mathbb{Z}$-basis

$$\mathbf{u} = (u_1, u_2, u_2, -2u_1 - 3u_2, u_1 + u_2)^t .$$

So the vectors $\mathbf{u}$ in the kernel of $\mathcal{A}$ form a lattice in $\mathbb{Z}^5$ given by

$$\mathbb{L} = \left\langle (1,0,0,-2,1)^t , (0,1,1,-3,1)^t \right\rangle = \left\langle (1,0,0,-2,1)^t , (1,-1,-1,1,0)^t \right\rangle .$$

The equations in the above definition of $\mathbb{X}_{\mathcal{A}}$ give rise to an ideal called a toric ideal (denoted by $I_{\mathcal{A}}$):

$$I_{\mathcal{A}} := \left\langle \prod_{j=1}^{5} x_j^{u_j^+} - \prod_{j=1}^{5} x_j^{u_j^-} \,\middle|\, \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \mathbb{L} \right\rangle .$$

Toric ideals will be explained in detail in section 4.2. We claim that $I_{\mathcal{A}} = \left\langle x_1x_4 - x_2x_3,\ x_1x_5 - x_4^2 \right\rangle$:
If $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I_{\mathcal{A}}$, then there exist $n, m \in \mathbb{Z}$ such that

$$\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- = n(1,0,0,-2,1) + m(1,-1,-1,1,0).$$

If $m = 0$ then

$$\begin{aligned} & \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \\ = \ & (x_1x_5)^{|n|} - x_4^{|n|} \\ = \ & (x_1x_5 - x_4^2)\left((x_1x_5)^{|n|-1} + (x_1x_5)^{|n|-2}x_4 + \ldots + x_1x_5x_4^{2|n|-2} + x_4^{2|n|-1}\right) . \end{aligned}$$

So $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ is divisible by $x_1x_5 - x_4^2$, which implies it is an element of $\left\langle x_1x_4 - x_2x_3,\ x_1x_5 - x_4^2 \right\rangle$.

If $m \neq 0$, then $\mathbf{u}^+ = (\cdot, -m, -m, \cdot, \cdot)$ if $m < 0$ or $\mathbf{u}^- = (\cdot, -m, -m, \cdot, \cdot)$ if $m > 0$. So for $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ either $\mathbf{x}^{\mathbf{u}^+}$ or $\mathbf{x}^{\mathbf{u}^-}$ is divisible by x_2x_3. Say $\mathbf{x}^{\mathbf{u}^+}$ is divisible by x_2x_3, then we can write

$$\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} = \frac{\mathbf{x}^{\mathbf{u}^+}}{x_2x_3}(x_2x_3 - x_1x_4) + \frac{\mathbf{x}^{\mathbf{u}^+}x_1x_4}{x_2x_3} - \mathbf{x}^{\mathbf{u}^-} .$$

In the binomial $\frac{\mathbf{x}^{\mathbf{u}^+}x_1x_4}{x_2x_3} - \mathbf{x}^{\mathbf{u}^-} \in I_{\mathcal{A}}$ we increased m by one w.r.t. $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$. So continue dividing out terms x_2x_3 in the same fashion until $m = 0$. This way we obtain that $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in \left\langle x_1x_4 - x_2x_3,\ x_1x_5 - x_4^2 \right\rangle$ for $m \neq 0$ as well.

So we obtain an explicit description of the projective, toric variety, namely

$$\mathbb{X}_{\mathcal{A}} = \mathcal{V}\left(I_{\mathcal{A}}\right) = \left\{ [x_1 : \ldots : x_5] \in \mathbb{P}^4 \,\middle|\, x_1x_4 = x_2x_3, x_1x_5 = x_4^2 \right\} .$$

□

Remark 3.6. The approach we took in the previous proof to find a finite number of generators for the toric ideal $I_{\mathcal{A}}$ can not be applied in general. To find a finite set of generators for an arbitrary toric ideal we need the notion of Gröbner bases, which will be defined in chapter 4.

3.7. In this paragraph we will compute the orbits of $\mathbb{X}_{\mathcal{A}}$: Obviously the interior

$$\left\{[x_1 : ... : x_5] \in \mathbb{P}^4 \left| x_1x_4 = x_2x_3,\ x_1x_5 = x_4^2,\ x_1, ..., x_5 \neq 0 \right.\right\}$$

is an orbit of $\mathbb{X}_{\mathcal{A}}$. This orbit has all non-zero coefficients and is 2-dimensional, so it is isomorphic to the complex torus $(\mathbb{C}^*)^2$, where the isomorphism is given by the map ψ in (3.2). If $x_4 \neq 0$, then $x_1, x_2, x_3, x_5 \neq 0$ and we obtain the orbit isomorphic to $(\mathbb{C}^*)^2$ given above. Next suppose $x_4 = 0$, then we see that $x_2x_3 = x_1x_5 = 0$, hence the other orbits of $\mathbb{X}_{\mathcal{A}}$ are four lines and four points:

$$\begin{aligned}
L_{12} &= \left\{[x_1 : x_2 : 0 : 0 : 0] \in \mathbb{P}^4, x_1, x_2 \neq 0\right\} \\
L_{13} &= \left\{[x_1 : 0 : x_3 : 0 : 0] \in \mathbb{P}^4, x_1, x_3 \neq 0\right\} \\
L_{25} &= \left\{[0 : x_2 : 0 : 0 : x_5] \in \mathbb{P}^4, x_2, x_5 \neq 0\right\} \\
L_{35} &= \left\{[0 : 0 : x_3 : 0 : x_5] \in \mathbb{P}^4, x_3, x_5 \neq 0\right\} \\
P_i &= \left\{[x_1 : ... : x_5] \in \mathbb{P}^4 \left| x_i \neq 0, x_j = 0\ \forall j \neq i \right.\right\}, i \in \{1, 2, 3, 5\}.
\end{aligned}$$

Because

$$L_{ij} \cup P_i \cup P_j \cong \mathbb{P}^1$$

for all possible combinations of i, j, the union of the other 8 orbits corresponds to the union of four projective lines. Thus, in this example we have compactified the complex torus $(\mathbb{C}^*)^2$ in $\mathbb{X}_{\mathcal{A}}$ by adding four projective lines to it, as illustrated in the following figure:

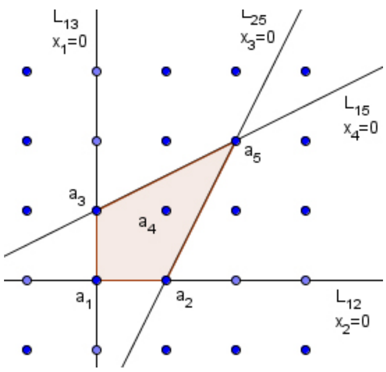


Figure 3.2: The compactification of $(\mathbb{C}^*)^2$ in $\mathbb{X}_{\mathcal{A}}$.

3.8. From paragraph 3.4 we know that the solutions to (3.3) are given by intersecting a general hyperplane in $\mathbb{P}^4$ with the image of the map

$$\psi : (\mathbb{C}^*)^2 \to \mathbb{P}^4, \ (x,y) \mapsto \left[x^{-1}y^{-1} : y^{-1} : x^{-1} : 1 : xy\right].$$

From the previous paragraph we know that the interior of $\mathbb{X}_\mathcal{A} \subset \mathbb{P}^4$ (denoted by $\mathbb{X}_\mathcal{A}^{\text{int}}$) is isomorphic to the complex torus $(\mathbb{C}^*)^2$. So we see that the set of solutions to (3.3) can also be given by intersecting $\mathbb{X}_\mathcal{A}^{\text{int}}$ with $\mathcal{H}_\mathbf{c}$ for some $\mathbf{c} \in \mathbb{P}^{4\vee}$. We can make this more explicit:

Let $x_1, ..., x_5 \neq 0$ and rewrite the relations between the homogeneous coordinates in $\mathbb{X}_\mathcal{A}$, i.e. $x_1x_4 = x_2x_3$ and $x_1x_5 = x_4^2$, as

$$\frac{x_1}{x_4} \cdot \frac{x_5}{x_4} = 1, \ \frac{x_1}{x_4} = \frac{x_2}{x_4} \cdot \frac{x_3}{x_4}. \tag{3.4}$$

This construction allows us to return to affine coordinates $\xi_1, \xi_2, \xi_3, \xi_5 \in \mathbb{C}^*$, by defining

$$\xi_1 := \frac{x_1}{x_4}, \ \xi_2 := \frac{x_2}{x_4}, \ \xi_3 := \frac{x_5}{x_4}, \ \xi_5 := \frac{x_5}{x_4}.$$

This way the relations in (3.4) rewrite to

$$\xi_1\xi_5 = 1, \ \xi_2\xi_3\xi_1^{-1} = 1. \tag{3.5}$$

Next set $\xi_2 = y^{-1}$ and $\xi_3 = x^{-1}$ for $x, y \in \mathbb{C}^*$. Using the relations in (3.5) we now see that

$$\xi_1 = x^{-1}y^{-1}, \ \xi_5 = xy.$$

For an element in the intersection $\mathcal{H}_\mathbf{c} \cap \mathbb{X}_\mathcal{A}^{\text{int}}$ we now have

$$\begin{aligned} & c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5x_5 = 0 \\ \Leftrightarrow \quad & c_1\frac{x_1}{x_4} + c_2\frac{x_2}{x_4} + c_3\frac{x_3}{x_4} + c_4 + c_5\frac{x_5}{x_4} = 0. \end{aligned}$$

This implies

$$\begin{aligned} & c_1\frac{x_1}{x_4} + c_2\frac{x_2}{x_4} + c_3\frac{x_3}{x_4} + c_4 + c_5\frac{x_5}{x_4} \\ = \quad & c_1\xi_1 + c_2\xi_2 + c_3\xi_3 + c_4 + c_5\xi_5 \\ = \quad & c_1x^{-1}y^{-1} + c_2y^{-1} + c_3x^{-1} + c_4 + c_5xy \\ = \quad & f(x,y) \\ = \quad & 0. \end{aligned}$$

So indeed we see that $\mathbb{X}_\mathcal{A}^{\text{int}} \cap \mathcal{H}_\mathbf{c}$ corresponds to the set of solutions of (3.3).

3.9. In this paragraph we will introduce a torus action on the curves f and construct a new model for the family of elliptic curves. We start by looking at elliptic curves isomorphic to

$$f_\mathbf{c}(x,y) = c_1x^{-1}y^{-1} + c_2y^{-1} + c_3x^{-1} + c_4 + c_5xy = 0, \tag{3.6}$$

and state that two such curves are isomorphic if and only if

$$f_{\mathbf{c}}(x,y) = \nu f_{\mathbf{c}'}(\lambda x, \mu y),$$

for some $(\lambda, \mu, \nu) \in (\mathbb{C}^*)^3$. So we look at curves of the form

$$c_1\lambda^{-1}\mu^{-1}x^{-1}y^{-1} + c_2\mu^{-1}y^{-1} + c_3\lambda^{-1}x^{-1} + c_4 + c_5\lambda\mu xy = 0.$$

Then under the map (3.2) we obtain an induced action of $(\mathbb{C}^*)^2$ on $\mathbb{P}^4$ given by

$$(\lambda, \mu) \star [x_1 : ... : x_5] = \left[\lambda^{-1}\mu^{-1}x_1 : \mu^{-1}x_2 : \lambda^{-1}x_3 : x_4 : \lambda\mu x_5\right]. \tag{3.7}$$

This is precisely how the action of $(\mathbb{C}^*)^2$ on itself extends to an action of $(\mathbb{C}^*)^2$ on the toric variety $\mathbb{X}_{\mathcal{A}}$. Next let the coefficients defining the elliptic curve vary, these coefficients give rise to a family of hyperplanes $\mathcal{H}_{\mathbf{c}}$ for $\mathbf{c} \in \mathbb{P}^{4\vee}$. Define an incidence relation on $\mathbb{P}^4 \times \mathbb{P}^{4\vee}$ by $\mathcal{I} = \{(p, \mathcal{H}) \,|p \in \mathcal{H}\}$. This way the family of elliptic curves can be given by

$$\left(\mathbb{X}_{\mathcal{A}} \times \mathbb{P}^{4\vee}\right) \cap \mathcal{I} =$$
$$\left\{ \left([x_1 : ... : x_5], [c_1 : ... : c_5]\right) \in \mathbb{P}^4 \times \mathbb{P}^{4\vee} \,\middle|\, x_1x_4 = x_2x_3,\ x_1x_5 = x_4^2,\ \sum_{j=1}^{5} c_jx_j = 0 \right\}.$$

As $(\mathbb{C}^*)^2$ acts on $\mathbb{P}^4$, it also acts on the dual space $\mathbb{P}^{4\vee}$, namely by the dual action

$$(\lambda, \mu) \star [c_1 : ... : c_5] = \left[\lambda\mu c_1 : \mu c_2 : \lambda c_3 : c_4 : \lambda^{-1}\mu^{-1}c_5\right].$$

Combining the action of $(\mathbb{C}^*)^2$ on $\mathbb{P}^4$ and $\mathbb{P}^{4\vee}$ yields an action of $(\mathbb{C}^*)^2$ on the product $\mathbb{P}^4 \times \mathbb{P}^{4\vee}$. Notice that the relations between the coefficients in the space $\left(\mathbb{X}_{\mathcal{A}} \times \mathbb{P}^{4\vee}\right) \cap \mathcal{I}$ are invariant under the $(\mathbb{C}^*)^2$ torus action. So the space $\left(\mathbb{X}_{\mathcal{A}} \times \mathbb{P}^{4\vee}\right) \cap \mathcal{I}$ is invariant under the $(\mathbb{C}^*)^2$-action. This yields a restriction of the projection $\left(\mathbb{P}^4 \times \mathbb{P}^{4\vee}\right)/(\mathbb{C}^*)^2 \to \mathbb{P}^{4\vee}/(\mathbb{C}^*)^2$ to this space, i.e.

$$\begin{tikzcd}
\left(\left(\mathbb{X}_{\mathcal{A}} \times \mathbb{P}^{4\vee}\right) \cap \mathcal{I}\right)/(\mathbb{C}^*)^2 \arrow[r, hook] & \left(\mathbb{P}^4 \times \mathbb{P}^{4\vee}\right)/(\mathbb{C}^*)^2 \arrow[d, "\varphi"] \\
& \mathbb{P}^{4\vee}/(\mathbb{C}^*)^2
\end{tikzcd}$$

The isomorphism classes of the family of elliptic curves, denoted by

$$\left(\left(\mathbb{X}_{\mathcal{A}} \times \mathbb{P}^{4\vee}\right) \cap \mathcal{I}\right)/(\mathbb{C}^*)^2$$

in the above model, is what we want to get a clearer picture of. In this model we view $\mathbb{P}^{4\vee} \diagup_{(\mathbb{C}^*)^2}$ as the parameter or moduli space of the two-parameter family of curves in $\left(\left(\mathbb{X}_{\mathcal{A}} \times \mathbb{P}^{4\vee}\right) \cap \mathcal{I}\right) \diagup_{(\mathbb{C}^*)^2}$. Note that by construction (more precisely, by varying the coefficients and viewing them as elements of the dual space $\mathbb{P}^{4\vee}$) these elliptic curves are 'nicely ordered', i.e. they do not intersect each other. Therefore this model looks like a good approach to get a clear description of the family of elliptic curves. However the parameter space $\mathbb{P}^{4\vee} \diagup_{(\mathbb{C}^*)^2}$ is an intuitive choice, but spaces like these have some pathological properties and one has to look for better models which compactify the space $(\mathbb{C}^*)^5 \diagup_{(\mathbb{C}^*)^3}$. Geometric invariant theory (GIT) constructs such a model, as we will see in the next section.

3.3 Torus actions and the secondary polytope

3.10. By varying the coefficients $c_1, ..., c_5 \in \mathbb{C}^*$ defining the elliptic curve of the form (3.6), we look at a family of elliptic curves. This defined an action of $(\mathbb{C}^*)^2$ on the coefficients in the dual space $\mathbb{P}^{4\vee}$. By (3.7) there is an induced action of the complex torus $(\mathbb{C}^*)^3 = \left\{(\lambda, \mu, \nu) \in (\mathbb{C}^*)^3\right\}$ on the coefficients $c_1, ..., c_5 \in \mathbb{C}^*$ (in affine coordinates), given by

$$(c_1, c_2, c_3, c_4, c_5) \mapsto \left(\nu\lambda\mu c_1, \nu\mu c_2, \nu\lambda c_3, \nu c_4, \nu\lambda^{-1}\mu^{-1}c_5\right)$$

This gives rise to an action of $(\mathbb{C}^*)^3$ on the polynomials in the ring $\mathbb{C}[c_1, ..., c_5]$, which is completely defined by the following $(\mathbb{C}^*)^3$-action on the monomials in $\mathbb{C}[c_1, ..., c_5]$:

$$\begin{aligned} (\mathbb{C}^*)^3 \times \mathbb{C}[c_1, ..., c_5] &\to \mathbb{C}[c_1, ..., c_5], \\ (\lambda, \mu, \nu) \star \left(\prod_{j=1}^{5} c_j^{n_j}\right) &= \lambda^{n_1+n_3-n_5}\mu^{n_1+n_2-n_5}\nu^{\sum_{j=1}^{5} n_j} \prod_{j=1}^{5} c_j^{n_j}, \end{aligned} \tag{3.8}$$

where the exponents $n_1, ..., n_5$ are nonnegative integers.

3.11. In this paragraph we construct the invariant monomials under the torus action of the previous paragraph. Remark that a monomial $\prod_{j=1}^{5} c_j^{n_j}$ for some $n_1, ..., n_5 \in \mathbb{Z}_{\geq 0}$ is invariant under the torus action if and only if

$$(\lambda, \mu, \nu) \star \prod_{j=1}^{5} c_j^{n_j} = \prod_{j=1}^{5} c_j^{n_j},$$

This is equal to

$$\lambda^{n_1+n_3-n_5}\mu^{n_1+n_2-n_5}\nu^{\sum_{j=1}^{5} n_j} \prod_{j=1}^{5} c_j^{n_j} = \prod_{j=1}^{5} c_j^{n_j}.$$

So we see that the exponents of the invariant monomials correspond with the solutions $n_1, ..., n_5 \geq 0$ of the system of linear equations:

$$\begin{cases} n_1 + n_3 - n_5 &= 0 \\ n_1 + n_2 - n_5 &= 0 \\ n_1 + n_2 + n_3 + n_4 + n_5 &= 0 \end{cases}$$

Solving this system of equations we see that the exponents of the invariant monomials are vectors (with only non-negative entries) corresponding to the rank 2 lattice

$$\mathbb{L} := \langle (-1, 1, 1, -1, 0), (1, 0, 0, -2, 1) \rangle \subset \mathbb{Z}^5.$$

Notice that the elements in this lattice represent the two reduced relations between the vertices of the Newton polytope (embedded in $\mathbb{R}^3$).

From this we also see that the exponents corresponding to the invariant monomials can be given by

$$(n_1, ..., n_5) = (k_1, k_2)\, B := (k_1, k_2) \begin{pmatrix} -1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & -2 & 1 \end{pmatrix},$$

for any $(k_1, k_2) \in \mathbb{Z}^2$ such that $(n_1, ..., n_5) \geq \mathbf{0}$. Note that the convex hull of the columns of matrix B corresponds to a polytope, which looks like:

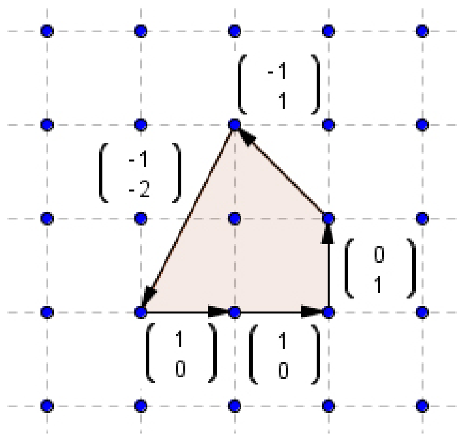


Figure 3.3: The polytope corresponding to the columns of matrix B.

This polytope is called the secondary polytope, see definition 3.15. Every edge of the polytope corresponds to one of the exponents n_i $(1 \leq i \leq 5)$. By fixing the center point of the polytope as the origin, every edge of the polytope also corresponds to a line given by an equation of the form $ax + by = 1,\ a, b \in \mathbb{Z}$. Observe that the monomial corresponding to the vector $(n_1, ..., n_5) = (1, ..., 1)$ is $\prod_{j=1}^{5} c_j$, which under the torus action equals

$\lambda\mu\nu^5 \prod_{j=1}^5 c_j$.

From definition 2.6 we see that examples of characters are group homomorphisms

$$\chi^{\mathbf{m}} : (\mathbb{C}^*)^3 \to \mathbb{C}^*, \text{ defined by } \chi^{\mathbf{m}}(z_1, z_2, z_3) = \prod_{j=1}^{3} z_j^{m_j}, \ \mathbf{m} \in \mathbb{Z}^3.$$

So in this case we see that for $\mathbf{m} = (1, 1, 5)$ we obtain the character

$$\chi^{(1,1,5)}(\lambda, \mu, \nu) = \lambda\mu\nu^5.$$

So we can remark that the monomial $\prod_{j=1}^5 c_j$ transforms according to the character $\lambda\mu\nu^5$ under the induced torus action defined on $\mathbb{C}[c_1, ..., c_5]$. In the upcoming paragraphs we will compute all monomials in $\mathbb{C}[c_1, ..., c_5]$ having this same constant under the torus action.

3.12. A monomial transforms according to some character under the $(\mathbb{C}^*)^3$ action on $\mathbb{C}\,[c_1, ..., c_5]$ if there are fixed constants $\alpha, \beta, \gamma \in \mathbb{Z}$ such that

$$(\lambda, \mu, \nu) \star \prod_{j=1}^{5} c_j^{n_j} = \nu^\alpha \mu^\beta \lambda^\gamma \prod_{j=1}^{5} c_j^{n_j}.$$

For this let $n_1, ..., n_5 \in \mathbb{Z}_{\geq 0}$ and let $\prod_{j=1}^5 c_j^{n_j} \in \mathbb{C}\,[c_1, ..., c_5]$ be an arbitrary monomial. In the previous paragraph we described the torus action, i.e.:

$$(\lambda, \mu, \nu) \star \left(\prod_{j=1}^{5} c_j^{n_j} \right) = \lambda^{n_1+n_3-n_5} \mu^{n_1+n_2-n_5} \nu^{\sum_{j=1}^5 n_j} \prod_{j=1}^{5} c_j^{n_j}.$$

From this we see that an arbitrary monomial in $\mathbb{C}[c_1, ..., c_5]$ transforms according to some character if and only if for some $(\alpha, \beta, \gamma)^t$

$$\mathcal{D}\mathbf{n} := \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} n_1 \\ n_2 \\ n_3 \\ n_4 \\ n_5 \end{pmatrix} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}.$$

As the torus action on the monomials is induced from the action on the vertices of the Newton polytope, we see that the columns of matrix $\mathcal{D}$ correspond to the same Newton polytope mirrored in the origin. So between the vertices of the polytope corresponding to the columns of $\mathcal{D}$ we have the same relations as before, which can be represented by the rows of the following matrix:

$$\mathcal{B} := \begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -2 & 1 \end{pmatrix}.$$

So by construction we now obtain that $\mathcal{B}$ is such that $\mathcal{D}\mathcal{B}^t = \mathbf{0}$.

Notice that a particular solution to $\mathcal{D}\mathbf{n} = (\alpha, \beta, \gamma)^t$ can be given by $\hat{\mathbf{n}} = (\alpha, \beta - \alpha, 0, \gamma - \beta, 0)^t$. Hence all solutions $\mathbf{n}$ to $\mathcal{D}\mathbf{n} = (\alpha, \beta, \gamma)^t$ can be given by

$$\begin{aligned} \mathcal{D}\mathbf{n} = \mathcal{D}\hat{\mathbf{n}}, \quad &\iff \quad \mathcal{D}\left(\mathbf{n} - \hat{\mathbf{n}}\right) = \mathbf{0} \\ &\iff \quad \left(\mathbf{n} - \hat{\mathbf{n}}\right) \in \operatorname{ColSpace}\left(\mathcal{B}^t\right) \\ &\iff \quad \mathbf{n} - \hat{\mathbf{n}} = \mathcal{B}^t\omega = \mathcal{B}^t \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}. \end{aligned}$$

Let $\mathbf{b}_i$ $(1 \leq i \leq 5)$ be the columns of $\mathcal{B}$, then the last argument is equal to

$$\begin{pmatrix} \mathbf{b}_1 \cdot \omega \\ \mathbf{b}_2 \cdot \omega \\ \mathbf{b}_3 \cdot \omega \\ \mathbf{b}_4 \cdot \omega \\ \mathbf{b}_5 \cdot \omega \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta - \alpha \\ 0 \\ \gamma - \beta \\ 0 \end{pmatrix} = \mathbf{n} \geq \mathbf{0},$$

which yields the following system of inequalities

$$\begin{cases} \omega_1 + \omega_2 & \geq & -\alpha \\ -\omega_1 & \geq & \alpha - \beta \\ -\omega_1 & \geq & 0 \\ \omega_1 - 2\omega_2 & \geq & \beta - \gamma \\ \omega_2 & \geq & 0. \end{cases}$$

Geometrically these inequalities can enclose a polytope, where the case $(\alpha, \beta, \gamma) = (1, 1, 5)$ (see the previous paragraph) is illustrated in the following figure:

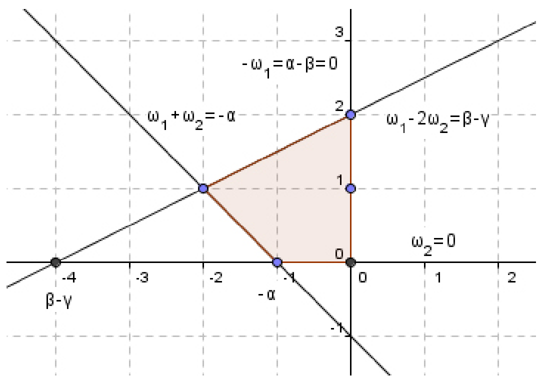


Figure 3.4: The inequalities define a polytope, compare this to the polytope in figure 3.3.

3.13. The Secondary polytope can also be obtained by looking at the triangulations of the Newton polytope (see figure 3.2), this construction can be found in [11]. We begin by defining the Secondary polytope:

Let $\mathcal{A} = \{\mathbf{a}_j\}_{j\in J}$ be a configuration of points in $\mathbb{R}^d$ and let $\omega : \mathcal{A} \to \mathbb{R}^{d+1}$ be a function. Let $\mathcal{A}^\omega \subset \mathbb{R}^{d+1}$ be the lifted point configuration, i.e. $\mathcal{A}^\omega = \left\{ \begin{pmatrix} \mathbf{a}_j \\ \omega_j \end{pmatrix} \right\}_{j\in J}$. Then ω induces a triangulation (i.e. a subdivision in simplices) of $\mathcal{A}$ consisting of the projection of the lower faces of the convex hull $\mathrm{Conv}(\mathcal{A}^\omega)$. Triangulations obtained by this construction are called regular. See section 3.4 for more details on this construction.

Definition 3.14. The volume of a d-dimensional simplex with vertex set $\{\mathbf{a}_j\}_{j\in J}$ is defined as

$$\mathrm{Vol}\left(\{\mathbf{a}_j\}_{j\in J}\right) := \left|\det\left(\{\mathbf{a}_j\}_{j\in J}\right)\right|.$$

Definition 3.15. Let $e_j, j = 1, ..., d$ be the standard basis vectors on $\mathbb{R}^d$, and let $\mathcal{T}$ be a regular triangulation of $\mathcal{A}$, then define the GKZ-vector

$$\phi_{\mathcal{A}}(\mathcal{T}) := \sum_{j\in J} \sum_{C\in\mathcal{T},\ \mathrm{s.t.}\ j\in C} \mathrm{Vol}(C)\cdot e_j \in \mathbb{R}^J.$$

The Secondary polytope $\mathrm{Sec}(\mathcal{A})$ is defined as the convex hull

$$\mathrm{Sec}(\mathcal{A}) := \mathrm{Conv}\left(\phi_{\mathcal{A}}(\mathcal{T})\,|\mathcal{T} \text{ a regular triangulation of } \mathcal{A}\right).$$

Definition 3.16. A regular triangulation of a polytope defined over a set of points $\mathcal{A}$ is said to be *unimodular* if all d-dimensional simplices in the triangulation have volume 1.

Using this construction we compute the Secondary polytope of the configuration of points

$$\mathcal{A} := \begin{pmatrix} -1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 \end{pmatrix}$$

corresponding to the Newton polytope in figure 3.2. From the definition of the Secondary polytope, we see that we only have to look at the regular triangulations of the Newton polytope. These triangulations are given in the figure below (notice that triangulations 3 and 4 are unimodular):

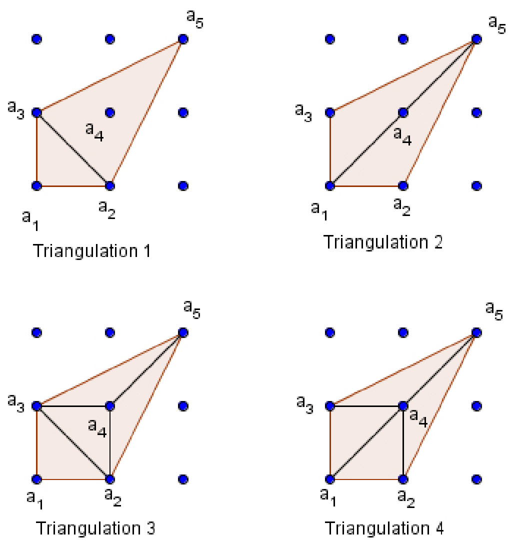


Figure 3.5: The regular triangulations of the Newton polytope.

All regular triangulations of the Newton polytope indicated in above figure are given by:

$$\begin{aligned}\Delta_1 &= \{123, 235\}, \Delta_2 = \{125, 135\},\\ \Delta_3 &= \{123, 234, 245, 345\}, \Delta_4 = \{124, 134, 345, 245\}.\end{aligned}$$

We compute the corresponding GKZ-vectors using definition 3.15:

$$\begin{aligned}\phi_{\mathcal{A}}(\Delta_1) &= (1,4,4,0,3), \phi_{\mathcal{A}}(\Delta_2) = (4,2,2,0,4),\\ \phi_{\mathcal{A}}(\Delta_3) &= (1,3,3,3,2), \phi_{\mathcal{A}}(\Delta_4) = (2,2,2,4,2).\end{aligned}$$

From these vectors we see that there is an integer vector between $\phi_{\mathcal{A}}(\Delta_2)$ and $\phi_{\mathcal{A}}(\Delta_4)$, namely the vector $\mathbf{a} := \frac{1}{2}(\phi_{\mathcal{A}}(\Delta_2) + \phi_{\mathcal{A}}(\Delta_4)) = (3,2,2,2,3)$. Next observe that $\phi_{\mathcal{A}}(\Delta_1)$ and $\mathbf{a}$ also have an integer vector between them, namely $\mathbf{b} := (2,3,3,1,3)$. Finally observe that $\phi_{\mathcal{A}}(\Delta_1)+\phi_{\mathcal{A}}(\Delta_2) = \phi_{\mathcal{A}}(\Delta_3)+\mathbf{b}$. Up to symmetries these vectors can only be represented in the following configuration:

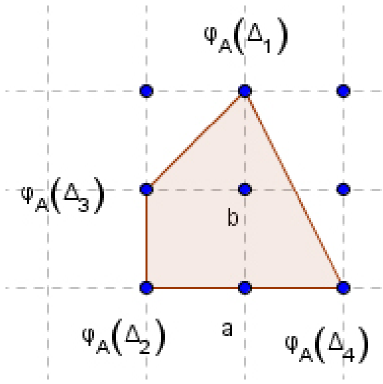


Figure 3.6: The construction of the Secondary polytope.

And we see that this is precisely the same polytope as the one obtained in paragraph 3.12. So we can conclude that this polytope is indeed a Secondary polytope.

3.17. We embed the Secondary polytope of the previous paragraph in $\mathbb{R}^3$ at height 1, this way the vertices of the Secondary polytope correspond to the columns of the homogenized matrix

$$\hat{\mathcal{A}} := \{\hat{\mathbf{a}}_1, \hat{\mathbf{a}}_2, \hat{\mathbf{a}}_3, \hat{\mathbf{a}}_4, \hat{\mathbf{a}}_5, \hat{\mathbf{a}}_6\} := \begin{pmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Lemma. The toric variety $\mathbb{X}_{\hat{\mathcal{A}}}$ corresponding to the Secondary polytope is given by

$$\mathbb{X}_{\hat{\mathcal{A}}} = \left\{[w_1 : ... : w_6] \in \mathbb{P}^5 \,\middle|\, w_2w_4 = w_3^2,\ w_3w_6 = w_1^2,\ w_1w_3 = w_2w_5, w_4w_6 = w_1w_5, w_1w_4 = w_3w_5 \right\}.$$

Proof. (Sketch) In the proof of lemma 3.1 we created a work-around for the use of Gröbner bases, which is not possible in this proof. Let $[w_1 : ... : w_6]$ be the coordinates on $\mathbb{P}^5$. Define a lattice in $\mathbb{Z}^6$:

$$\mathbb{L} := \left\{ \mathbf{u} \in \mathbb{Z}^6 \,\middle|\, \sum_{j=1}^{6} u_j \hat{\mathbf{a}}_j = \mathbf{0} \right\}.$$

We can write each $\mathbf{u} \in \mathbb{L}$ as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$ with $\mathbf{u}^+, \mathbf{u}^- \in \mathbb{N}^6$ having disjoint support (this decomposition of $\mathbf{u}$ is unique). So we see that

$$\mathbb{X}_{\hat{\mathcal{A}}} = \left\{ [w_1 : ... : w_6] \in \mathbb{P}^5 \,\middle|\, \prod_{j=1}^{6} w_j^{u_j^+} - \prod_{j=1}^{6} w_j^{u_j^-} = 0,\ \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \mathbb{L} \right\}.$$

The equations in the above definition of $\mathbb{X}_{\mathcal{A}}$ give rise to a toric ideal (denoted by $I_{\hat{\mathcal{A}}}$):

$$I_{\hat{\mathcal{A}}} := \left\langle \prod_{j=1}^{6} w_j^{u_j^+} - \prod_{j=1}^{6} w_j^{u_j^-} \,\middle|\, \mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \mathbb{L} \right\rangle.$$

By the Hilbert basis theorem we know there is a finite set of generators for this toric ideal. Such a finite set of generators can be found by computing a Gröbner basis for this ideal. The definition of a Gröbner basis can be found in chapter 4, the only property we need here is that it is a finite set of generators for an ideal. Such a Gröbner basis $\mathcal{G}$ for $I_{\hat{\mathcal{A}}}$ can be computed using algorithm 4.5 in [18], we find

$$\begin{aligned} \mathcal{G} = \quad & \{w_2w_5w_6 - w_1^3, w_3w_6 - w_1^2, w_1w_3 - w_2w_5, \\ & \qquad w_4w_6 - w_1w_5, w_1w_4 - w_3w_5, w_2w_4 - w_3^2\}. \end{aligned}$$

As $w_2w_5w_6 - w_1^3 = w_1(w_3w_6 - w_1^2) - w_6(w_1w_3 - w_2w_5)$ and $\mathcal{G}$ is a basis for $I_{\hat{\mathcal{A}}}$ we can write

$$\begin{aligned} I_{\hat{\mathcal{A}}} = \quad & \langle w_3w_6 - w_1^2, w_1w_3 - w_2w_5, w_4w_6 - w_1w_5, \\ & \qquad w_1w_4 - w_3w_5, w_2w_4 - w_3^2\rangle. \end{aligned}$$

So we obtain an explicit description of the projective, toric variety, namely

$$\begin{aligned} \mathbb{X}_{\hat{\mathcal{A}}} = \mathcal{V}\left(I_{\hat{\mathcal{A}}}\right) = \quad & \{[w_1 : ... : w_6] \in \mathbb{P}^5 | w_3w_6 = w_1^2, w_1w_3 = w_2w_5, \\ & \qquad w_4w_6 = w_1w_5, w_1w_4 = w_3w_5, w_2w_4 = w_3^2\}. \end{aligned}$$

$\square$

3.18. We have seen in paragraph 3.12 that the exponents $n_1, ..., n_5$ corresponding to the monomials in $\mathbb{C}[c_1, ..., c_5]$ transforming according to the character $\lambda\mu\nu^5$ fulfill the equation

$$\mathcal{D}\mathbf{n} = \begin{pmatrix} 1 & 0 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} (n_1, ..., n_5)^t = (1, 1, 5)^t.$$

In other words, those vectors $(n_1, ..., n_5) \geq 0$ such that

$$\left\{ \begin{array}{rcl} n_1 + n_3 - n_5 & = & 1 \\ n_1 + n_2 - n_5 & = & 1 \\ n_1 + n_2 + n_3 + n_4 + n_5 & = & 5 \end{array} \right.$$

Solving this system of equations implies that the monomials transforming according to the $\lambda\mu\nu^5$-character under the torus action are of the form

$$c_1^{1-n_3+n_5} c_2^{n_3} c_3^{n_3} c_4^{4-n_3-2n_5} c_5^{n_5},$$

where each exponent is greater than or equal to 0. This gives only a finite list of such 'invariant' monomials, by trial and error we easily find that all exponents of the 'invariant' monomials can be given by the following list:

$$\begin{array}{lcl} (n_3, n_5) = (1,1) & \Longrightarrow & (1,1,1,1,1)\,, \\ (n_3, n_5) = (0,2) & \Longrightarrow & (3,0,0,0,2)\,, \\ (n_3, n_5) = (0,1) & \Longrightarrow & (2,0,0,2,1)\,, \\ (n_3, n_5) = (0,0) & \Longrightarrow & (1,0,0,4,0)\,, \\ (n_3, n_5) = (1,0) & \Longrightarrow & (0,1,1,3,0)\,, \\ (n_3, n_5) = (2,1) & \Longrightarrow & (0,2,2,0,1)\,. \end{array}$$

This corresponds to the following list of monomials:

$$\begin{array}{rcl} w_1 & = & c_1c_2c_3c_4c_5, \\ w_2 & = & c_1^3c_5^2, \\ w_3 & = & c_1^2c_4^2c_5, \\ w_4 & = & c_1c_4^4, \\ w_5 & = & c_2c_3c_4^3, \\ w_6 & = & c_2^2c_3^2c_5. \end{array}$$

Let $[w_1 : ... : w_6]$ be the homogeneous coordinates on $\mathbb{P}^5$. Notice that the same relations as in the toric variety $\mathbb{X}_{\hat{\mathcal{A}}}$ in the previous paragraph also hold between these monomials ω_j, so they correspond to the vertices of the Secondary polytope. These 'invariant' monomials give rise to an embedding

$$\mathbb{C}^5 \setminus Z \hookrightarrow \mathbb{P}^5,\ (c_1, ..., c_5) \longmapsto [w_1 : ... : w_6]\,,$$

where

$$\begin{aligned} Z := \quad & \left\{(c_1, ..., c_5) \in \mathbb{C}^5 | c_1 = c_2 = 0\right\} \cup \left\{(c_1, ..., c_5) \in \mathbb{C}^5 | c_1 = c_3 = 0\right\} \\ & \cup \left\{(c_1, ..., c_5) \in \mathbb{C}^5 | c_4 = c_5 = 0\right\}. \end{aligned}$$

This gives rise to an embedding

$$\iota : {}^{\left(\mathbb{P}^{4\vee} \setminus Z\right)}\!\big/_{(\mathbb{C}^*)^2} \hookrightarrow \mathbb{P}^5.$$

The parameter space ${}^{\mathbb{P}^{4\vee}}\!\big/_{(\mathbb{C}^*)^2}$ is an intuitive choice, but spaces like these have some pathological properties and one has to look for better models which compactify the space ${}^{(\mathbb{C}^*)^5}\!\big/_{(\mathbb{C}^*)^3}$. By taking the Zarisky closure of the image of ι we obtain a toric variety in $\mathbb{P}^5$. Finding equations for this toric variety corresponds to finding <u>all</u> monomial relations between $w_1, ..., w_6$. So we see that the toric variety $\overline{\mathrm{Im}(\iota)}$ is equal to the toric variety from the

Secondary fan (i.e. the outer normal fan to the Secondary polytope).

In terms of the moduli problem we have found the embedding ι:

$$\begin{tikzcd}
\left(\left(\mathbb{X}_{\mathcal{A}} \times \mathbb{P}^{4\vee}\right) \cap \mathcal{I}\right) /\!\!/ (\mathbb{C}^*)^2 \arrow[r, hook, dotted] \arrow[d, hook] & ? \arrow[dd, dotted] \\
\left(\mathbb{P}^4 \times \mathbb{P}^{4\vee}\right) /\!\!/ (\mathbb{C}^*)^2 \arrow[d, "\varphi"] & \\
\mathbb{P}^{4\vee} /\!\!/ (\mathbb{C}^*)^2 \arrow[r, hook, "\iota"] & \mathbb{P}^5
\end{tikzcd}$$

It remains an open question if we can identify the universal family of curves with a toric variety, embedding it in some projective space over $\mathbb{P}^5$, as indicated with a question mark in the figure above.

3.4 Regular triangulations and the Secondary fan

In this subsection we define the Secondary fan using regular triangulations of point configurations in $\mathbb{R}^d$. One way of viewing regular triangulations is by lifting the original point configuration up one dimension and projecting down in a special way as described in the following definition:

Definition 3.19. A point configuration in $\mathbb{R}^d$ with label set I is a map $I \to \mathbb{R}^d$. A subdivision of a point configuration $\mathcal{A} = (\mathbf{a}_1, ..., \mathbf{a}_n) \in \mathbb{R}^d$ is called regular if it can be obtained by the following construction:

- Pick a lifting function $\omega : \mathcal{A} \to \mathbb{R}$ and define the lifted point configuration by $\mathcal{A}^\omega := \begin{pmatrix} \mathbf{a}_1 & \cdots & \mathbf{a}_n \\ \omega_1 & \cdots & \omega_n \end{pmatrix} \in \mathbb{R}^{d+1}$.
- Project the 'lower faces' of Conv$(\mathcal{A}^\omega)$ on the first d coordinates. By a lower face of Conv$(\mathcal{A}^\omega)$ we mean a face where the outward-pointing normal vector has a negative last coordinate.

If the lower faces of Conv$(\mathcal{A}^\omega)$ are simplices, their projections on the first d coordinates is called a regular triangulation of $\mathcal{A}$.

One can easily verify that every point configuration contains at least one regular triangulation.

We will illustrate the construction in definition 3.19 in the following two examples:

Example 3.20. For the point configuration

$$\mathcal{A} = (\mathbf{a}_1, ..., \mathbf{a}_5) := \begin{pmatrix} -1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

corresponding to the Newton polytope we can restrict ourselves in construction 3.19 without loss of generality to lifting functions ω where $\omega = (\omega_1, 0, 0, \omega_4, 0)$ for $\omega_1, \omega_4 \in \mathbb{R}$. This way we easily find all 9 regular subdivisions of Conv($\mathcal{A}$), notice that 4 of them are regular triangulations. In figure 3.7 we see the regular subdivisions corresponding to the lifting functions ω.

We will introduce some notation for (regular) triangulations and subdivisions of point configurations:

Definition 3.21. Let $\mathcal{A} \subset \mathbb{R}^d$ be an n-point configuration. By Δ we denote any regular triangulation of $\mathcal{A}$. By $\mathcal{S}_\omega(\mathcal{A})$ we denote a regular subdivision obtained from the construction in 3.19 using $\mathcal{A}$ and lifting function ω. By Δ_ω we denote a regular triangulation obtained from the construction in 3.19 using $\mathcal{A}$ and lifting function ω.

Given an n-point configuration $\mathcal{A} \subset \mathbb{R}^d$, many lifting functions $\omega \in \mathbb{R}^n$ will give rise to the same regular subdivision $\mathcal{S}_\omega(\mathcal{A})$. The collection of those $\omega \in \mathbb{R}^n$ giving the same regular subdivision $\mathcal{S}_\omega(\mathcal{A})$ using construction 3.19 form a cone, called the Secondary cone:

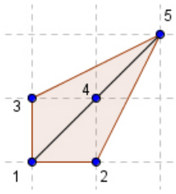
Definition 3.22. Let $\mathcal{A} \subset \mathbb{R}^d$ be an n-point configuration. If Δ and Δ' are two polyhedral subdivisions, then we write $\Delta \preceq \Delta'$ if Δ refines Δ'. For every polyhedral subdivision Δ we define the closed Secondary cone $\overline{\mathcal{C}_\Delta} := \{\omega \in \mathbb{R}^n \,|\Delta \preceq \mathcal{S}_\omega(\mathcal{A})\}$. We can also define the open Secondary cones as $\mathcal{C}_\Delta := \{\omega \in \mathbb{R}^n | \Delta = \Delta_\omega\}$.

Proposition 3.23. ([6], cor. 5.2.8) The closed/open Secondary cone $\mathcal{C}_\Delta$ defined in definition 3.22 is a (closed/open) convex cone.

The Secondary fan will be the collection of Secondary cones and is defined in the following proposition:

Proposition 3.24. (Gel'fand, Kapranov, Zelevinsky, 1984) Let $\mathcal{A} \subset \mathbb{R}^d$ be an n-point configuration and let Δ be a regular polyhedral subdivision. We define the Secondary fan as $\bigcup_\Delta \mathcal{C}_\Delta$, which is a fan where the full-dimensional cones correspond to regular triangulations of Conv$(\mathcal{A})$.

Example 3.25. Let $\mathcal{A} \subset \mathbb{R}^3$ be the matrix whose columns are the vertices of the Newton polytope. Using example 3.20 we see that the vectors $\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5$ generate the regular subdivisions of Conv($\mathcal{A}$).



Consider the triangulation $\Delta = \{125, 135\}$ depicted on the left. Using definition 3.22 we compute the corresponding Secondary cone

$$\begin{aligned} & \overline{\mathcal{C}_{\Delta}} \\ = & \left\{(\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\Delta \preceq \mathcal{S}_{\omega}(\mathcal{A})\right\} \\ = & \left\{(\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\omega_1 \leq 0,\ \omega_4 \geq 2\omega_1\right\}. \end{aligned}$$

Using the other three regular triangulations we find in a similar way all four of the Secondary cones. These four closed cones together generate the following 2-dimensional Secondary fan:

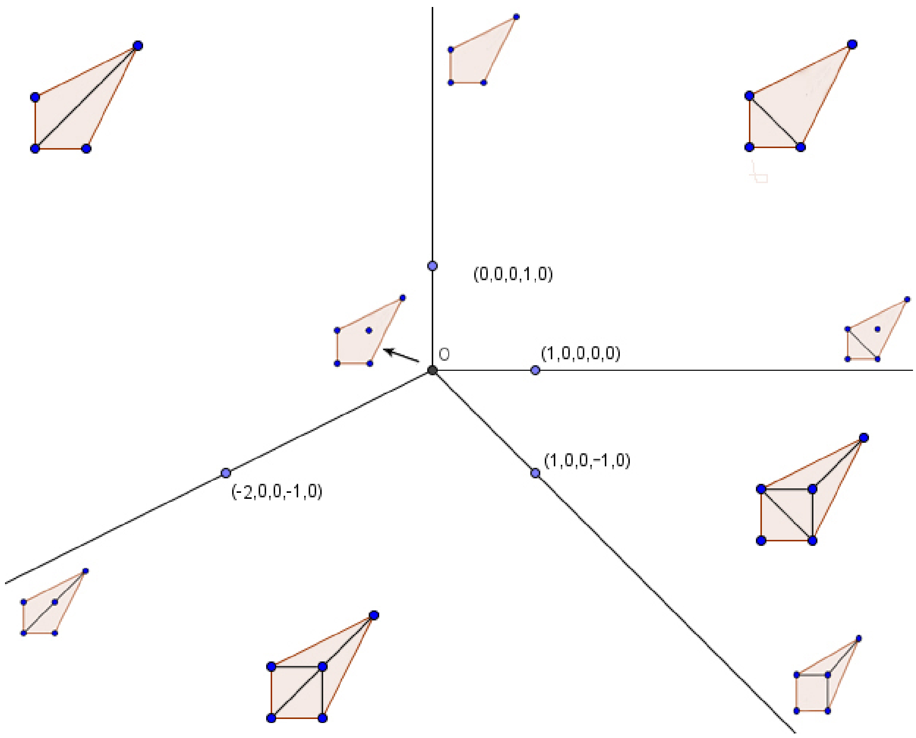


Figure 3.7: The Secondary fan for $\mathcal{A}$.

Notice that the regular triangulations correspond to the 2-dimensional cones and that the other regular subdivisions correspond to the 1-dimensional rays of the fan and to the origin.

3.5 A (not so) different method to obtain the Secondary fan

In this subsection we explain another construction of the Secondary fan following article [16], §4. Let $\mathcal{A} = \{\mathbf{a}_1, ..., \mathbf{a}_n\} \subset \mathbb{Z}^d$ be a subset which generates $\mathbb{Z}^d$ as an abelian group. Let $h : \mathbb{Z}^d \to \mathbb{Z}$ be a group homomorphism such that $h(\mathbf{a}_j) = 1$ for all $1 \leq j \leq n$. Define the lattice

$$\mathbb{L} := \left\{ \ell = (\ell_1, ..., \ell_n) \in \mathbb{Z}^n \,\middle|\, \sum_{j=1}^{n} \ell_j \mathbf{a}_j = 0 \right\}.$$

Using h we obtain $\sum_{j=1}^{n} \ell_j = 0$ for each $\ell \in \mathbb{L}$. The input data is the short exact sequence

$$0 \to \mathbb{L} \to \mathbb{Z}^n \to \mathbb{Z}^d \to 0. \tag{3.9}$$

As before write $\mathbb{L}_{\mathbb{R}}^{\vee} := \mathbb{L}^{\vee} \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathrm{Hom}\,(\mathbb{L}, \mathbb{R})$ and similarly $\mathbb{Z}_{\mathbb{R}}^{d\vee} := \mathbb{Z}^d \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^d$ and $\mathbb{Z}_{\mathbb{R}}^{n\vee} \cong \mathbb{R}^n$. Then the $\mathbb{R}$-dual of the short exact sequence in (3.9) is the short exact sequence

$$0 \to \mathbb{R}^d \to \mathbb{R}^n \xrightarrow{\pi} \mathbb{L}_{\mathbb{R}}^{\vee} \to 0. \tag{3.10}$$

Define a subset $\mathcal{P} := \{(x_1, ..., x_n) \in \mathbb{R}^n \,|x_j \geq 0,\ \forall j\} \subset \mathbb{R}^n$ and define a restriction on the map π in (3.10): $\hat{\pi} := \pi|_{\mathcal{P}}$, which is a surjective map as well.

Example 3.26. Define the matrix

$$\mathcal{A} = \{\mathbf{a}_1, ..., \mathbf{a}_5\} := \begin{pmatrix} -1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

corresponding to the Newton polytope used in all previous examples. Then $\mathbb{L} = \mathbb{Z}(1, -1, -1, 1, 0) \oplus \mathbb{Z}(1, 0, 0, -2, 1)$ and so $\pi : \mathbb{R}^5 \to \mathbb{R}^2$ is defined as $\pi(x_1, ..., x_5) = \begin{pmatrix} x_1 - x_2 - x_3 + x_4 \\ x_1 - 2x_4 + x_5 \end{pmatrix}$.

Let $\mathbf{b}_1, ..., \mathbf{b}_n \in \mathbb{L}_{\mathbb{R}}^{\vee}$ be the images of the standard basis vectors of $\mathbb{R}^5$ under the map π. Then for an arbitrary $\xi \in \mathbb{L}_{\mathbb{R}}^{\vee}$ we obtain

$$(x_1, ..., x_n) \in \hat{\pi}^{-1}(\xi) \Longleftrightarrow \xi = \sum_{j=1}^{n} x_j \mathbf{b}_j \text{ and } x_j \geq 0,\ \forall j.$$

Hence for each $\xi \in \mathbb{L}_{\mathbb{R}}^{\vee}$ the fiber $\hat{\pi}^{-1}(\xi)$ is a convex polyhedron.

Lemma 3.27. ([16], §4, lemma 3) The vector $\mathbf{v} = (v_1, ..., v_n) \in \mathcal{P}$ is a vertex of $\hat{\pi}^{-1}(\xi)$ if and only if $\xi = \sum_{j=1}^{n} v_j \mathbf{b}_j$ and the vectors $\mathbf{b}_j$ with $v_j \neq 0$ are $\mathbb{R}$-linearly independent.

For a vertex $\mathbf{v}$ of $\hat{\pi}^{-1}(\xi)$ we define the subset $I_{\mathbf{v}} := \{j \,|\, v_j = 0\} \subseteq \{1, ..., n\}$. This way each $\xi \in \mathbb{L}_{\mathbb{R}}^{\vee}$ yields a list of subsets of $\{1, ..., n\}$, i.e. $T_\xi := \left\{ I_{\mathbf{v}} \,\middle|\, \mathbf{v} \text{ vertex of } \hat{\pi}^{-1}(\xi) \right\}$. By lemma 3.27 we obtain another (more constructive) description of T_ξ:

Corollary 3.28. ([16], §4, cor. 1) A subset $I \subseteq \{1, ..., n\}$ is on the list T_ξ if and only if the vectors $\mathbf{b}_j$ with $j \notin I$ are $\mathbb{R}$-linearly independent and $\xi = \sum_{j \in \{1,...,n\} \setminus I} \tau_j \mathbf{b}_j$ and $\tau_j \in \mathbb{R}_{>0}$ for all j.

Define an equivalence relation on $\mathbb{L}_{\mathbb{R}}^{\vee}$ by $\xi \sim \xi' \Leftrightarrow T_\xi = T_{\xi'}$. Then by corollary 3.28 the equivalence class containing ξ is the strongly, convex, polyhedral cone $\mathcal{C} := \bigcap_{I \in T_\xi} (\text{positive span of } \{\mathbf{b}_j\}_{j \notin I})$.

Definition 3.29. The lists T_ξ give the triangulations of the polygon $\mathrm{Conv}(\mathbf{a}_1, ..., \mathbf{a}_n)$ and the collection of cones $\mathcal{C}$ is called the Secondary fan of $\mathcal{A}$.

Example 3.30. We continue the previous example of this subsection. Let $\{e_j\}_{j \in \{1,...,5\}}$ be the standard basis of $\mathbb{R}^5$. Using $\mathbf{b}_j := \pi(e_j)$ we obtain $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_5\} = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -2 & 1 \end{pmatrix}$, which is the matrix whose rows are precisely the generators of $\mathbb{L}_{\mathbb{R}}^{\vee}$.

Next we use corollary 3.28 to compute the lists T_ξ: Start with $\xi = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$. The only $\mathbb{R}$-linearly independent collections of vectors $\mathbf{b}_j$ contain at most two elements. The only subsets of $\mathbf{b}_j$ fulfilling the other claim in the corollary are $\{\mathbf{b}_2, \mathbf{b}_4\}$ and $\{\mathbf{b}_3, \mathbf{b}_4\}$ as $\xi = \frac{3}{2}\mathbf{b}_i + \frac{1}{2}\mathbf{b}_4$ for $i \in \{2, 3\}$. So we conclude $T_\xi = \{\{1, 3, 5\}, \{1, 2, 5\}\}$ and $\mathcal{C} = \tau_1 \mathbf{b}_2 + \tau_2 \mathbf{b}_4$ with $\tau_1, \tau_2 \geq 0$.

Similarly we find the other Secondary cones yielding the Secondary fan as depicted below:

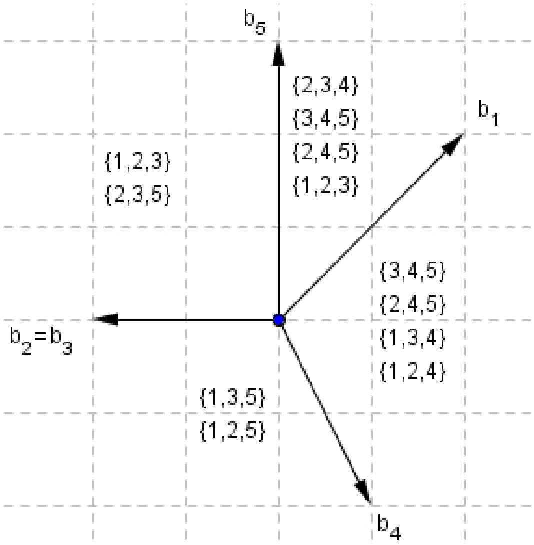


Figure 3.8: The Secondary fan for $\mathcal{A}$, the lists (regular triangulations) T_ξ are given for every $\xi \in \mathbb{L}_{\mathbb{R}}^{\vee}$.

Note that this fan is indeed the same fan as the one in figure 3.7 (up to rotation).

There is also a parallel construction of the Secondary fan using polyhedra:

Lemma 3.31. Define the matrix $\mathcal{B} = \{\mathbf{b}_1, ..., \mathbf{b}_n\}$, where $\mathbf{b}_j$ is the image of the standard basis vector e_j of $\mathbb{R}^n$ under the map π. Let $\xi \in \mathbb{L}_{\mathbb{R}}^{\vee}$ and fix a $\mathbf{y} \in \mathbb{R}^n$ such that $\mathcal{B}\mathbf{y} = \xi$. Then we obtain the convex polyhedron $\hat{\pi}^{-1}(\xi) \cap \mathcal{P} = \left\{\mathbf{z} \in \mathbb{R}^d \,|\mathbf{z} \cdot \mathbf{a}_j \geq -y_j, \ \forall j\right\}$.

Proof. Look at vectors $\mathbf{x} \in \mathcal{P}$ such that $\mathcal{B}\mathbf{x} = \xi$. All solutions to this inhomogeneous system of linear equations are given by $\mathcal{B}(\mathbf{x} - \mathbf{y}) = \mathbf{0}$. So by construction

$$\mathbf{x} - \mathbf{y} \in \ker(\mathcal{B}) = \text{colspace}\left(\mathcal{A}^t\right) = \text{rowspace}\left(\mathcal{A}\right).$$

Therefore there is a $\mathbf{z} \in \mathbb{R}^d$ such that $\mathbf{x} - \mathbf{y} = \mathbf{z}\mathcal{A}$. As $x_j \geq 0, \ \forall j$, the result follows. $\square$

Example 3.32. As in the previous example set

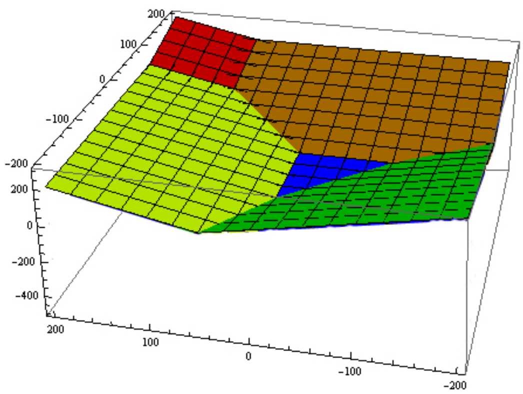
$$\mathcal{B} := \{\mathbf{b}_1, ..., \mathbf{b}_5\} = \begin{pmatrix} 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & 0 & -2 & 1 \end{pmatrix}.$$

A particular solution $\mathbf{x}_0$ to $\mathcal{B}\mathbf{x} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ is given by $\mathbf{x}_0 = (\xi_1, 0, 0, 0, \xi_2 - \xi_1)^t$.

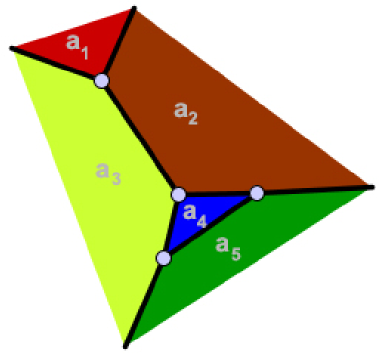
Let $\mathbf{z} \in \text{Rowspace}(\mathcal{A})$ be such that $\mathbf{z} \cdot \mathbf{a}_j \geq -(x_0)_j$, $\forall j$. This yields the following system of inequalities:

$$\begin{cases} z_3 & \geq & -\xi_1 + z_1 + z_2 \\ z_3 & \geq & z_2 \\ z_3 & \geq & z_1 \\ z_3 & \geq & 0 \\ z_3 & \geq & \xi_1 - \xi_2 - z_1 - z_2 \end{cases}$$

Choosing for example $\xi = \begin{pmatrix} 100 \\ 200 \end{pmatrix}$ results in the following convex polyhedron (only the relevant piece of the boundary of the polyhedron is drawn):



The polyhedron $\mathcal{P}$ corresponding to the choice $\xi = (100, 200)^t$.



Top view of the faces of the polyhedron, presenting the inward pointing normal vectors $\mathbf{a}_j$ to each face.

Each vector $\mathbf{a}_j$ is by construction an inward pointing normal vector to one of the faces of $\mathcal{P}$. In each vertex of $\mathcal{P}$, list the indices of the vectors $\mathbf{a}_j$ corresponding to faces meeting there. An easy check shows that the polytope $\mathcal{P}$ gives rise to the list $\{\{1,2,3\},\{2,3,4\},\{2,4,5\},\{3,4,5\}\}$, which corresponds exactly to the list presented in the cone containing ξ in figure 3.8. So by varying ξ we obtain the same Secondary fan with this construction. Notice the similarity between this construction and the construction of the Secondary fan given in a previous subsection by projecting down the lower faces of the polyhedron to obtain regular triangulations.

3.6 The toric variety from the Secondary fan

By looking at monomials transforming according to a specific character under the torus action on $\mathbb{C}[c_1, ..., c_5]$ we found an embedding of the parameter space $\left(\mathbb{P}^{4\vee} \setminus Z\right) / (\mathbb{C}^*)^2$ for a suitably chosen set Z into $\mathbb{P}^5$ (see section 3.3 for the details). By taking the Zarisky closure of this embedding we gave the moduli space the structure of a toric variety. This variety corresponds to the outer normal fan of the Secondary polytope, which is the toric variety for the Secondary fan. In this section we will see that the charts in the toric variety for the Secondary fan can be used to give explicit equations for the 2-parameter family of elliptic curves in the $\mathbb{P}^1 \times \mathbb{P}^1$ model. From now on we will denote the Secondary polytope by Δ_S and the Secondary fan by $\mathcal{F}_{\Delta_S}$

3.33. By taking outward pointing normal vectors to the edges of the Secondary polytope we obtain the following outer normal fan, which is called the Secondary fan:

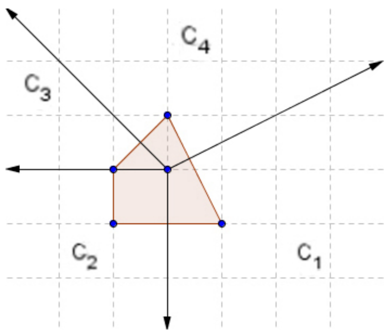


Figure 3.9: The Secondary fan of Δ_S.

This fan consistes of nine cones, where the full-dimensional cones are defined as:

$$\begin{aligned}
\mathcal{C}_1 &= \begin{pmatrix} 2 \\ 1 \end{pmatrix} \mathbb{R}_{\geq 0} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} \mathbb{R}_{\geq 0} \\
\mathcal{C}_2 &= \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathbb{R}_{\geq 0} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathbb{R}_{\geq 0} \\
\mathcal{C}_3 &= \begin{pmatrix} -1 \\ 0 \end{pmatrix} \mathbb{R}_{\geq 0} + \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathbb{R}_{\geq 0} \\
\mathcal{C}_4 &= \begin{pmatrix} -1 \\ 1 \end{pmatrix} \mathbb{R}_{\geq 0} + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \mathbb{R}_{\geq 0}
\end{aligned}$$

The five other cones are $\mathcal{C}_i \cap \mathcal{C}_j$ for $(i,j) \in \{(1,2),(2,3),(3,4),(1,4)\}$ and $\mathcal{C}_0 := (0,0)^t$ (these are faces of the full-dimensional cones and are therefore cones by proposition 2.13). Remember that the semigroup $S_{\mathcal{C}_i^\vee}$ over the dual cone $\mathcal{C}_i^\vee$ is defined as

$$S_{\mathcal{C}_i^\vee} = \left\{ \mathbf{b} \in \mathbb{Z}^{2\vee} \,|\mathbf{bv} \geq 0,\ \forall \mathbf{v} \in \mathcal{C}_i \right\}.$$

Using this definition we find the following semigroups (corresponding to the full-dimensional cones):

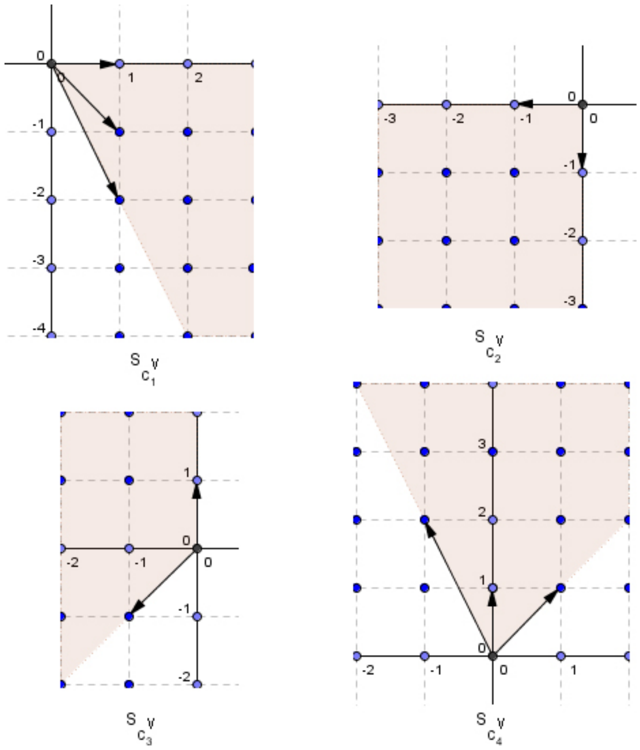


Figure 3.10: The semigroups corresponding to the cones $\mathcal{C}_1, ..., \mathcal{C}_4$.

These semigroups can be described in terms of their generators:

$$\begin{aligned}
S_{\mathcal{C}_1^\vee} &= \begin{pmatrix} 1 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}\mathbb{Z}_{\geq 0} \\
S_{\mathcal{C}_2^\vee} &= \begin{pmatrix} 0 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\mathbb{Z}_{\geq 0} \\
S_{\mathcal{C}_3^\vee} &= \begin{pmatrix} -1 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{Z}_{\geq 0} \\
S_{\mathcal{C}_4^\vee} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} -1 \\ 2 \end{pmatrix}\mathbb{Z}_{\geq 0}
\end{aligned}$$

This yields the following charts (affine, toric varieties):

$$\begin{aligned}
\mathbb{X}_1 := U_{\mathbb{Z}\left[S_{\mathcal{C}_1^\vee}\right]}(\mathbb{C}) = \quad & \mathrm{Hom}_{\mathrm{sg}}\left(S_{\mathcal{C}_1^\vee}, \mathbb{C}^\times\right) && \cong \left\{(x_1, y_1, z_1) \in \mathbb{C}^3 \,\middle|\, x_1^2 = y_1 z_1\right\} \\
& \beta && \mapsto (\beta(1,-1), \beta(1,0), \beta(1,-2))
\end{aligned}$$

$$\begin{aligned}
\mathbb{X}_2 := U_{\mathbb{Z}\left[S_{\mathcal{C}_2^\vee}\right]}(\mathbb{C}) = \quad & \mathrm{Hom}_{\mathrm{sg}}\left(S_{\mathcal{C}_2^\vee}, \mathbb{C}^\times\right) && \cong \mathbb{C}^2 \\
& \beta && \mapsto (\beta(0,-1), \beta(-1,0))
\end{aligned}$$

$$\begin{aligned} \mathbb{X}_3 := U_{\mathbb{Z}\left[S_{\mathcal{C}_3^\vee}\right]}(\mathbb{C}) = \quad \mathrm{Hom}_{\mathrm{sg}}\left(S_{\mathcal{C}_3^\vee}, \mathbb{C}^\times\right) &\cong \mathbb{C}^2 \\ \beta &\mapsto (\beta(-1,-1), \beta(0,1)) \end{aligned}$$

$$\begin{aligned} \mathbb{X}_4 := U_{\mathbb{Z}\left[S_{\mathcal{C}_4^\vee}\right]}(\mathbb{C}) = \quad \mathrm{Hom}_{\mathrm{sg}}\left(S_{\mathcal{C}_4^\vee}, \mathbb{C}^\times\right) &\cong \left\{(x_4, y_4, z_4) \in \mathbb{C}^3 \,\middle|\, x_4^3 = y_4 z_4\right\} \\ \beta &\mapsto (\beta(0,1), \beta(1,1), \beta(-1,2)) \end{aligned}$$

3.34. We glue the affine varieties $\mathbb{X}_1, ..., \mathbb{X}_4$ along the charts over the semigroups $S_{(\mathcal{C}_i \cap \mathcal{C}_j)^\vee}$ for $(i,j) \in \{(1,2),(2,3),(3,4),(1,4)\}$. For example, we find the following gluing between the toric varieties over $\mathcal{C}_1$ and $\mathcal{C}_2$:

$$\begin{tikzcd}
\{(x_1,y_1,z_1) \in \mathbb{C}^3 | x_1^2 = y_1 z_1, y_1 \neq 0\} \arrow[rr, hook] \arrow[dd, "{\varphi(x_1,y_1,z_1)=(x_1y_1^{-1},y_1^{-1})}"'] \arrow[dr, leftrightarrow, "\cong"] & & U_{\mathbb{Z}[S_{\mathcal{C}_1^\vee}]}(\mathbb{C}) \\
& U_{\mathbb{Z}[S_{(\mathcal{C}_1\cap\mathcal{C}_2)^\vee}]}(\mathbb{C}) \arrow[ur, hook, "\psi_1"] \arrow[dr, hook, "\psi_2"] \arrow[dl, leftrightarrow, "\cong"] & \\
\{(x_2,y_2) \in \mathbb{C}\times\mathbb{C} | y_2 \neq 0\} \arrow[rr, hook] & & U_{\mathbb{Z}[S_{\mathcal{C}_2^\vee}]}(\mathbb{C})
\end{tikzcd}$$

where $\varphi : U_{\mathbb{Z}\left[S_{\mathcal{C}_1^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{\mathcal{C}_2^\vee}\right]}(\mathbb{C})$ is defined by

$$\begin{aligned} (x_1,y_1,z_1) := (\beta(1,-1), \beta(1,0), \beta(1,-2)) \longmapsto (x_2,y_2) \quad &:= \quad (\beta(0,-1), \beta(-1,0)) \\ &= \quad \left(x_1 y_1^{-1}, y_1^{-1}\right). \end{aligned}$$

Similarly we find the following gluing isomorphisms between the affine toric varieties:

$$\begin{tikzcd}
\mathbb{X}_1 \arrow[r, "{(x_2,y_2)=\left(x_1y_1^{-1},y_1^{-1}\right),\ y_1\neq0}"] & \mathbb{X}_2 \arrow[d, "{(x_3,y_3)=\left(x_2y_2^{-1},x_2^{-1}\right),\ x_2\neq0, y_2\neq0}"] \\
\mathbb{X}_4 \arrow[u, "{(x_1,y_1,z_1)=\left(x_4z_4^{-1},x_4^2z_4^{-1},z_4^{-1}\right),\ z_4\neq0}"] & \mathbb{X}_3 \arrow[l, "{(x_4,y_4,z_4)=\left(y_3,x_3^{-1},x_3y_3^3\right),\ x_3\neq0}"]
\end{tikzcd}$$

and the inverse maps are given by

$$\begin{tikzcd}
\mathbb{X}_1 \arrow[d, "{(x_4,y_4,z_4)=\left(x_1^{-1}y_1,x_1^{-1}y_1^2,x_1^{-2}y_1\right),\ x_1\neq0}"'] & \mathbb{X}_2 \arrow[l, "{(x_1,y_1,z_1)=\left(x_2y_2^{-1},y_2^{-1},x_2^2y_2^{-1}\right),\ y_2\neq0}"] \\
\mathbb{X}_4 \arrow[r, "{(x_3,y_3)=\left(y_4^{-1},x_4\right),\ y_4\neq0}"] & \mathbb{X}_3 \arrow[u, "{(x_2,y_2)=\left(y_3^{-1},x_3y_3\right),\ y_3\neq0}"']
\end{tikzcd}$$

The affine toric varieties $\mathbb{X}_j$ $1 \leq j \leq 4$ and the gluing isomorphisms together form a projective, toric variety $\mathbb{X}_{\mathrm{SF}}$, corresponding to the Secondary fan $\mathcal{F}_{\Delta_S}$.

3.7 A resolution of singularities

From the previous section we see that the cones $\mathcal{C}_1$ and $\mathcal{C}_4$ in the Secondary fan give rise to singular affine toric varieties. In order to resolve the singularities in the toric variety from the Secondary fan we refine these two cones in the fan. The following lemma gives an easy method to resolve the singularities:

Lemma 3.35. ([9], exercise in §2.6) Let $\mathbb{L} \cong \mathbb{Z}^2$ be a lattice and $\mathcal{C} \subset \mathbb{R}^2$ a two-dimensional cone. A unique minimal resolution of singularities is obtained by inserting rays through the vertices on the edge of the polygon P, where $P = \mathrm{Conv}\,((\mathcal{C} \cap \mathbb{L}) \setminus 0)$. The rays inserted in above construction are exactly those through the vertices on the boundary of the polygon P.

3.36. Using lemma 3.35 we find the following refinement of the Secondary fan:

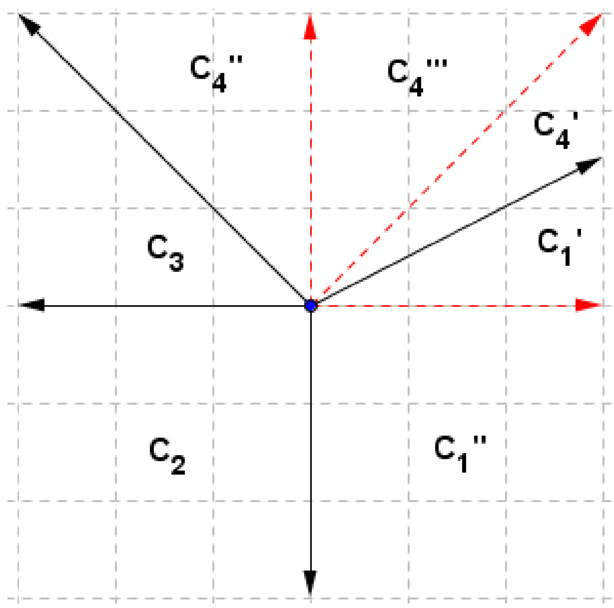


Figure 3.11: Resolving the singularities in the Secondary fan: The red, dashed arrows indicate the subdivisions.

This fan consists of five new cones (replacing the cones $\mathcal{C}_1$ and $\mathcal{C}_4$ of the

Secondary fan):

$$\begin{aligned}
\mathcal{C}_1' &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{R}_{\geq 0} \quad \mathcal{C}_1'' = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\mathbb{R}_{\geq 0} \\
\mathcal{C}_4' &= \begin{pmatrix} 2 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} \quad \mathcal{C}_4'' = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} \\
\mathcal{C}_4''' &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0}.
\end{aligned}$$

This yields the following new semigroups:

$$\begin{aligned}
S_{\mathcal{C}_1'^\vee} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 1 \\ -2 \end{pmatrix}\mathbb{Z}_{\geq 0} \\
S_{\mathcal{C}_1''^\vee} &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0} \\
S_{\mathcal{C}_4'^\vee} &= \begin{pmatrix} -1 \\ 2 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0} \\
S_{\mathcal{C}_4''^\vee} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\mathbb{Z}_{\geq 0} \\
S_{\mathcal{C}_4'''^\vee} &= \begin{pmatrix} -1 \\ 1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{Z}_{\geq 0}.
\end{aligned}$$

Finally, this yields the following new affine, toric varieties:

$$U_{\mathbb{Z}\left[S_{\mathcal{C}_1'^\vee}\right]}(\mathbb{C}) \cong U_{\mathbb{Z}\left[S_{\mathcal{C}_1''^\vee}\right]}(\mathbb{C}) \cong U_{\mathbb{Z}\left[S_{\mathcal{C}_4'^\vee}\right]}(\mathbb{C}) \cong U_{\mathbb{Z}\left[S_{\mathcal{C}_4''^\vee}\right]}(\mathbb{C}) \cong U_{\mathbb{Z}\left[S_{\mathcal{C}_4'''^\vee}\right]}(\mathbb{C}) \cong \mathbb{C}^2.$$

All these affine varieties are non-singular. So we see that the identity map on the lattice $\mathbb{L} \cong \mathbb{Z}^2$ induces a birational, proper morphism from the projective, toric variety from the refined Secondary fan to the projective, toric variety from the Secondary fan. This morphism between projective, toric varieties is completely defined by giving the maps between the affine charts:

$$
\begin{aligned}
&\pi_1' : U_{\mathbb{Z}\left[S_{c_1'^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{c_1^\vee}\right]}(\mathbb{C})\\
&\left(x_1', y_1'\right) = (\beta(0,1), \beta(1,-2)) \longmapsto\\
&\qquad (x_1, y_1, z_1) = (\beta(1,-1), \beta(1,0), \beta(1,-2)) = \left(x_1' y_1', x_1'^2 y_1', y_1'\right)\\
&\pi_1'' : U_{\mathbb{Z}\left[S_{c_1''^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{c_1^\vee}\right]}(\mathbb{C})\\
&\left(x_1'', y_1''\right) = (\beta(1,0), \beta(0,-1)) \longmapsto\\
&\qquad (x_1, y_1, z_1) = (\beta(1,-1), \beta(1,0), \beta(1,-2)) = \left(x_1'' y_1'', x_1'', x_1'' y_1''^2\right)\\
&\pi_2 = \mathrm{id}|_{U_{\mathbb{Z}\left[S_{c_2^\vee}\right]}(\mathbb{C})}\\
&\pi_3 = \mathrm{id}|_{U_{\mathbb{Z}\left[S_{c_3^\vee}\right]}(\mathbb{C})}\\
&\pi_4' : U_{\mathbb{Z}\left[S_{c_4'^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{c_4^\vee}\right]}(\mathbb{C})\\
&\left(x_4', y_4'\right) = (\beta(-1,2), \beta(1,-1)) \longmapsto\\
&\qquad (x_4, y_4, z_4) = (\beta(0,1), \beta(1,1), \beta(-1,2)) = \left(x_4' y_4', x_4'^2 y_4'^3, x_4'\right)\\
&\pi_4'' : U_{\mathbb{Z}\left[S_{c_4''^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{c_4^\vee}\right]}(\mathbb{C})\\
&\left(x_4'', y_4''\right) = (\beta(1,1), \beta(-1,0)) \longmapsto\\
&\qquad (x_4, y_4, z_4) = (\beta(0,1), \beta(1,1), \beta(-1,2)) = \left(x_4'' y_4'', x_4'', x_4''^2 y_4''^3\right)\\
&\pi_4''' : U_{\mathbb{Z}\left[S_{c_4'''^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{c_4^\vee}\right]}(\mathbb{C})\\
&\left(x_4''', y_4'''\right) = (\beta(-1,1), \beta(1,0)) \longmapsto\\
&\qquad (x_4, y_4, z_4) = (\beta(0,1), \beta(1,1), \beta(-1,2)) = \left(x_4''' y_4''', x_4''' y_4'''^2, x_4'''^2 y_4'''\right)
\end{aligned}
$$

This way we obtain a resolution of the singularities for the Secondary fan.

3.8 Describing the family of curves explicitly

In this section we will give an explicit description of the 2-paramater family of elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ above each of the affine toric varieties in the Secondary fan. For the cones in the Secondary fan corresponding to singular, affine toric varieties we will see that the parameters in the description of the family of elliptic curves will contain roots. To get rid of these roots we will use the fan which is the resolution of singularities in the Secondary fan (computed in the previous section) to give a description of the family of elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ where the parameters do not contain any roots.

Recall the list of exponent vectors $\mathbf{n}_j$ corresponding to the 'invariant' monomials w_j found at the end of section 3.3:

$$\begin{array}{rclll} \mathbf{n}_1 &=& (1,1,1,1,1), && w_1 = c_1c_2c_3c_4c_5 \\ \mathbf{n}_2 &=& (3,0,0,0,2), && w_2 = c_1^3c_5^2 \\ \mathbf{n}_3 &=& (2,0,0,2,1), && w_3 = c_1^2c_4^2c_5 \\ \mathbf{n}_4 &=& (1,0,0,4,0), && w_4 = c_1c_4^4 \\ \mathbf{n}_5 &=& (0,1,1,3,0), && w_5 = c_2c_3c_4^3 \\ \mathbf{n}_6 &=& (0,2,2,0,1), && w_6 = c_2^2c_3^2c_5 \end{array}$$

Also recall the cones in the Secondary fan, drawn as the outer normal cones to the Secondary polytope:

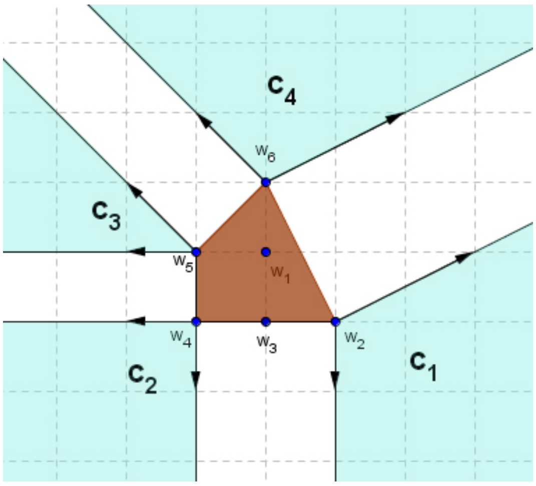


Figure 3.12: The Secondary polytope, displaying the monomials $w_1, ..., w_6$ and parts of the outer normal cones of this polytope.

In lemma 3.3 we described the family of elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ parametrized by $c_1, ..., c_5 \in \mathbb{C}^*$ by the equation

$$x_2^2y_2^2 + x_1x_2y_2^2 + x_2^2y_1y_2 + v_1x_1x_2y_1y_2 + v_2x_1^2y_1^2 = 0,$$

where $v_1 := c_1c_2^{-1}c_3^{-1}c_4$ and $v_2 = c_1^3c_2^{-2}c_3^{-2}c_5$. We will present a method to describe this family of elliptic curves over the charts $U_{\mathbb{Z}[S_{\mathcal{C}_j^\vee}]}(\mathbb{C}), 1 \leq j \leq 4$ in the toric variety for the secondary fan:

Notice that $\mathcal{C}_1^\vee$ is generated by $\mathbf{n}_6 - \mathbf{n}_2$ and $\mathbf{n}_3 - \mathbf{n}_2$. Corresponding to these difference vectors are the monomials $\frac{w_6}{w_2} = c_1^{-3}c_2^2c_3^2c_5^{-1} =: p_1$ and $\frac{w_3}{w_2} = c_1^{-1}c_4^2c_5^{-1} =: q_1$. Then one easily checks that $v_1 = (p_1q_1)^{-\frac{1}{2}}$ and $v_2 = p_1^{-1}$. So we see that the family of elliptic curves over the chart

$U_{\mathbb{Z}[S_{\mathcal{C}_1^\vee}]}(\mathbb{C})$ can be described by the equation

$$x_2^2y_2^2 + x_1x_2y_2^2 + x_2^2y_1y_2 + (p_1q_1)^{-\frac{1}{2}}x_1x_2y_1y_2 + p_1^{-1}x_1^2y_1^2 = 0. \qquad (3.11)$$

As $U_{\mathbb{Z}[S_{\mathcal{C}_1^\vee}]}(\mathbb{C})$ is a singular, affine toric variety the parameters of the family of elliptic curves contain roots. Below we will use the refinement of the Secondary fan to give a description of the curves above $U_{\mathbb{Z}[S_{\mathcal{C}_1^\vee}]}(\mathbb{C})$ where the parameters do not contain roots.

Next observe that $\mathcal{C}_2^\vee$ is generated by $\mathbf{n}_3 - \mathbf{n}_4$ and $\mathbf{n}_5 - \mathbf{n}_4$. The corresponding monomials are $\frac{w_3}{w_4} = c_1c_4^{-2}c_5 =: p_2$ and $\frac{w_5}{w_4} = c_1^{-1}c_2c_3c_4^{-1} =: q_2$. Then $v_1 = q_2^{-1}$ and $v_2 = p_2q_2^{-2}$. So the family of elliptic curves over the chart $U_{\mathbb{Z}[S_{\mathcal{C}_2^\vee}]}(\mathbb{C})$ can be described by the equation

$$x_2^2y_2^2 + x_1x_2y_2^2 + x_2^2y_1y_2 + q_2^{-1}x_1x_2y_1y_2 + p_2q_2^{-2}x_1^2y_1^2 = 0. \qquad (3.12)$$

Clearing denominators by multiplying both sides by q_2^2 and afterwards rescaling by $x_2' := q_2x_2$ gives a family of elliptic curves over $U_{\mathbb{Z}[S_{\mathcal{C}_2^\vee}]}(\mathbb{C})$ isomorphic to (3.12):

$$x_2'^2y_2^2 + x_1x_2'y_2^2 + x_2'^2y_1y_2 + x_1x_2'y_1y_2 + p_2x_1^2y_1^2 = 0.$$

Similarly observe that $\mathcal{C}_3^\vee$ is generated by $\mathbf{n}_6 - \mathbf{n}_5$ and $\mathbf{n}_4 - \mathbf{n}_5$. The corresponding monomials are $\frac{w_6}{w_5} = c_2c_3c_4^{-3}c_5 =: p_3$ and $\frac{w_4}{w_5} = c_1c_2^{-1}c_3^{-1}c_4 =: q_3$. Then $v_1 = q_3$ and $v_2 = p_3q_3^3$. So the family of elliptic curves over the chart $U_{\mathbb{Z}[S_{\mathcal{C}_3^\vee}]}(\mathbb{C})$ can be described by the equation

$$x_2^2y_2^2 + x_1x_2y_2^2 + x_2^2y_1y_2 + q_3x_1x_2y_1y_2 + p_3q_3^3x_1^2y_1^2 = 0.$$

Finally observe that $\mathcal{C}_4^\vee$ is generated by $\mathbf{n}_2-\mathbf{n}_6$ and $\mathbf{n}_5-\mathbf{n}_6$. The corresponding monomials are $\frac{w_2}{w_6} = c_1^3c_2^{-2}c_3^{-2}c_5 =: p_4$ and $\frac{w_5}{w_6} = c_2^{-1}c_3^{-1}c_4^3c_5^{-1} =: q_4$. Then $v_1 = (p_4q_4)^{\frac{1}{3}}$ and $v_2 = p_4$. So the family of elliptic curves over the chart $U_{\mathbb{Z}[S_{\mathcal{C}_4^\vee}]}(\mathbb{C})$ can be described by the equation

$$x_2^2y_2^2 + x_1x_2y_2^2 + x_2^2y_1y_2 + (p_4q_4)^{\frac{1}{3}}x_1x_2y_1y_2 + p_4x_1^2y_1^2 = 0.$$

As $U_{\mathbb{Z}[S_{\mathcal{C}_4^\vee}]}(\mathbb{C})$ is a singular, affine toric variety we see that the parameters contain roots.

Now subdivide the cone $\mathcal{C}_1$ in the Secondary fan according to the resolution of singularities of the Secondary fan as presented in the previous section:

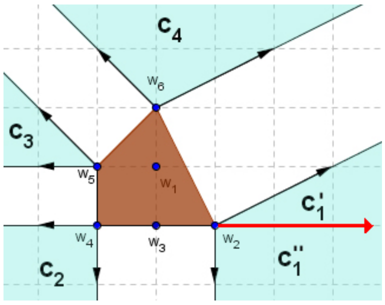


Figure 3.13: The Secondary polytope, displaying the invariant monomials and parts of the outer normal cones of this polytope. The red arrow divides $\mathcal{C}_1$ in two new cones according to the resolution of singularities of the Secondary fan in figure 3.11.

We will show that the curves over $U_{\mathbb{Z}[S_{\mathcal{C}_1'^\vee}]}(\mathbb{C})$ and $U_{\mathbb{Z}[S_{\mathcal{C}_1''^\vee}]}(\mathbb{C})$ are such that the parameters do not contain roots:

Begin by introducing a new lattice point, $\mathbf{n}_7 := \mathbf{n}_2 + \mathbf{n}_3 - \mathbf{n}_1$. Notice that $\mathcal{C}_1'^\vee$ is generated by $\mathbf{n}_6 - \mathbf{n}_2$ and $\mathbf{n}_7 - \mathbf{n}_2$. Corresponding to these difference vectors are the monomials $\frac{w_6}{w_2} = c_1^{-3}c_2^2c_3^2c_5^{-1} =: p_1'$ and $\frac{w_7}{w_2} = c_1c_2^{-1}c_3^{-1}c_4 =: q_1'$. Then one easily checks that $v_1 = q_1'$ and $v_2 = (p_1')^{-1}$. So we see that the family of elliptic curves over the chart $U_{\mathbb{Z}[S_{\mathcal{C}_1'^\vee}]}(\mathbb{C})$ can be described by the equation

$$x_2^2y_2^2 + x_1x_2y_2^2 + x_2^2y_1y_2 + q_1'x_1x_2y_1y_2 + p_1'^{-1}x_1^2y_1^2 = 0.$$

Clearing denominators we obtain

$$p_1'x_2^2y_2^2 + p_1'x_1x_2y_2^2 + p_1'x_2^2y_1y_2 + p_1'q_1'x_1x_2y_1y_2 + x_1^2y_1^2 = 0.$$

For the second cone we also introduce a new lattice point, $\mathbf{n}_8 := \mathbf{n}_1 + \mathbf{n}_2 - \mathbf{n}_3$. Notice that $\mathcal{C}_1''^\vee$ is generated by $\mathbf{n}_8 - \mathbf{n}_2$ and $\mathbf{n}_3 - \mathbf{n}_2$. Corresponding to these difference vectors are the monomials $\frac{w_8}{w_2} = c_1^{-1}c_2c_3c_4^{-1} =: p_1''$ and $\frac{w_3}{w_2} = c_1^{-1}c_4^2c_5^{-1} =: q_1''$. Then one easily checks that $v_1 = (p_1'')^{-1}$ and $v_2 = (p_1'')^{-2}(q_1'')^{-1}$. So we see that the family of elliptic curves over the chart $U_{\mathbb{Z}[S_{\mathcal{C}_1''^\vee}]}(\mathbb{C})$ can be described by the equation

$$x_2^2y_2^2 + x_1x_2y_2^2 + x_2^2y_1y_2 + (p_1'')^{-1}x_1x_2y_1y_2 + (p_1'')^{-2}(q_1'')^{-1}x_1^2y_1^2 = 0.$$

Clearing denominators we obtain

$$(p_1'')^2q_1''x_2^2y_2^2 + (p_1'')^2q_1''x_1x_2y_2^2 + (p_1'')^2q_1''x_2^2y_1y_2 + p_1''q_1''x_1x_2y_1y_2 + x_1^2y_1^2 = 0.$$

Subdividing the cone $\mathcal{C}_4$ using the resolution of singularities of the Secondary fan, also gives a description of the 2-parameter family of curves in

$\mathbb{P}^1 \times \mathbb{P}^1$ where the parameters do not contain any roots. This computation is completely similar to the one above and will therefore be omitted here.

Chapter 4

Computing a Gröbner Fan

4.1 Introduction

In this chapter we give a short introduction to the theory of Gröbner bases (following [1] and [2]) and toric ideals (following [18] and [3]). We define the Gröbner fan of an (homogeneous) ideal as discussed in [18], and compute an explicit Gröbner fan. This Gröbner fan refines the Secondary fan, as will be proven in theorem 4.69. We will give an interpretation of the toric variety from this Gröbner fan (using [9]). Most proofs, especially in [18], are presented rather short and compact, in this chapter I tried to explain them in more detail than given in the literature.

In the first subsection of this chapter we will study (toric) ideals of polynomial rings $k[x_1, ..., x_n]$ with k a field. For these ideals we recall the definition of a term order and the corresponding (reduced) Gröbner basis, a special kind of basis for an ideal. These bases are used for example for the ideal membership problem (given a finite set of generators for an ideal, i.e. $I = \langle f_1, ..., f_s \rangle$, does a given element f belong to this ideal I?), or to solve some systems of polynomial equations (in more variables) (see [2] ch. 2, §8 for more information). In my Bachelor thesis I tried to answer these questions, in this Master thesis I will look at another construction using Gröbner bases, namely that of a Gröbner fan. This construction will be much related to toric geometry and it will turn out to be a nice connection between the Gröbner bases of my Bachelor thesis and my initial aim to write a thesis on a topic in toric geometry.

In this chapter we will not be focused on single Gröbner bases for ideals (as I was during my Bachelor thesis), but on the set of all reduced Gröbner bases over an ideal. This set (called a Universal Gröbner basis) will turn out to be finite. By varying (in an efficient way) over all term orders we can use the Buchberger algorithm to compute all of these reduced Gröbner bases. Each of these reduced Gröbner bases will give rise to a specific cone, called a Gröbner cone. This finite set of cones together forms a fan, which is defined as the Gröbner fan of the ideal.

In this chapter we will restrict ourselves to a special type of ideals, namely ideals which are prime, homogeneous and generated by binomials. We will show that a prime ideal generated by binomials is equivalent to a toric ideal, which is the zero-set of a toric variety (not necessarily normal). The part on toric ideals will be explained in more detail than covered in [18] chapter 4. The homogeneity claim on an ideal will turn out to make sure that the Gröbner fan over such an ideal is a complete fan.

The other fan construction in toric geometry we have seen in the previous chapter is the Secondary fan, which is constructed out of all regular triangulations of a given polytope. This complete fan is the outer normal fan of the Secondary polytope (constructed by taking outer normal vectors to the edges of the polytope). As the Gröbner fans we study in this thesis are complete fans as well, we can talk about polytopes for which the outer normal fan is precisely this Gröbner fan. Such polytopes are called state polytopes. It will be proven that for our specific choice of ideals, the Gröbner fan and the Secondary fan are related in the sense that the Gröbner fan refines the Secondary fan. In terms of polytopes this is equivalent to saying that the Secondary polytope is a Minkowski summand of the state polytope.

4.2 Gröbner bases over toric ideals

In this section we will define toric ideals and look at some of its properties.

Definition 4.1. For a field k and a semigroup S we define the semigroup ring $k[S]$ to be the set of all functions $f : S \to k$ with finite support. Addition is defined as $(f+g)(\sigma) = f(\sigma) + g(\sigma)$ for $f, g \in k[S], \sigma \in S$. Multiplication is defined as the convolution

$$\star : k[S]\times k[S] \to k[S], \quad (g\star f)(\sigma) = \sum_{\lambda+\mu=\sigma} f(\lambda)g(\mu), \quad \text{for } f, g \in k[S],\ \lambda, \mu, \sigma \in S.$$

As $f, g \in k[S]$ have finite support, we see that $\sum_{\lambda+\mu=\sigma} f(\lambda)g(\mu)$ is a finite sum, and therefore $g \star f$ has finite support. This means that $f \star g \in k[S]$ and the convolution $\star$ is a well-defined map. The 0-element of $k[S]$ is the 0-function and the 1-element is the function mapping $0 \in S$ to $1 \in k$ and $\sigma \in S \setminus \{0\}$ to $0 \in k$.

Proposition 4.2. Let S, S' be semigroups and let $\pi : S \to S'$ be a semigroup homomorphism. Then π induces a map $\tilde{\pi} : k[S] \to k[S']$ defined by

$\tilde{\pi}(f)(\sigma) = \sum_{\pi(\lambda)=\sigma} f(\lambda)$, for $f \in k[S], \lambda \in S, \sigma \in S'$. This map $\tilde{\pi}$ is an algebra homomorphism.

Proof. As $k[S]$ and $k[S']$ are algebras over k (a trivial check) and as for every $f \in k[S]$, $\tilde{\pi}(f)$ has finite support we see that $\tilde{\pi}$ is a well-defined map between algebras.

Next we check that $\tilde{\pi}$ is a homomorphism between algebras: Choose $f, g \in k[S],\ \lambda, \mu, \nu \in S,\ \sigma \in S',\ c \in k$. It is a trivial computation that $\tilde{\pi}(c \star f) = c \star \tilde{\pi}(f)$, $\tilde{\pi}(f+g) = \tilde{\pi}(f) + \tilde{\pi}(g)$ and $\tilde{\pi}(1) = 1$. The only non-trivial part of this proof is checking that $\tilde{\pi}(f \star g) = \tilde{\pi}(f) \star \tilde{\pi}(g)$:

$$\begin{aligned}
\tilde{\pi}(f \star g)(\sigma) &= \sum_{\pi(\lambda)=\sigma} (f \star g)(\lambda) \\
&= \sum_{\pi(\lambda)=\sigma} \sum_{\mu+\nu=\lambda} f(\mu)g(\nu) \\
&= \sum_{\pi(\mu+\nu)=\sigma} f(\mu)g(\nu) \\
&= \sum_{u+v=\sigma} \sum_{\pi(\mu)=u,\ \pi(\nu)=v} f(\mu)g(\nu) \\
&= \sum_{u+v=\sigma} \left(\sum_{\pi(\mu)=u} f(\mu) \right) \left(\sum_{\pi(\nu)=v} g(\nu) \right) \\
&= \sum_{u+v=\sigma} \tilde{\pi}(f)(u)\tilde{\pi}(g)(v) \\
&= \left(\tilde{\pi}(f) \star \tilde{\pi}(g)\right)(\sigma).
\end{aligned}$$

□

Proposition 4.3. Let $n \in \mathbb{N}$, then $\mathbb{N}^n$ is a semigroup and $k[\mathbb{N}^n] \cong k[\mathbf{x}] = k[x_1, ..., x_n]$.

Proof. It is trivial that $\mathbb{N}^n$ is a semigroup. The second statement follows from the isomorphism

$$f \in k[\mathbb{N}^n] \longmapsto \sum_{\mathbf{m} \in \mathbb{N}^n} f(\mathbf{m})\mathbf{x}^{\mathbf{m}} \in k[\mathbf{x}].$$

□

Corollary 4.4. Let $d \in \mathbb{N}$, and define the Laurent polynomial ring $k\left[\mathbf{t}^{\pm 1}\right] = k\left[t_1, ..., t_d, t_1^{-1}, ..., t_d^{-1}\right]$. By choosing a basis for $\mathbb{Z}^d$ we obtain the isomorphism $k\left[\mathbf{t}^{\pm 1}\right] \cong k\left[\mathbb{Z}^d\right]$.

The next definition can be found in [18], ch. 4:

Definition 4.5. Let k be a field, and let $\mathcal{A} = \{\mathbf{a}_1, ..., \mathbf{a}_n\} \subset \mathbb{Z}^d$ be a finite set. Identify each vector $\mathbf{a}_i \in \mathcal{A}$ with a monomial $\mathbf{t}^{\mathbf{a}_i}$ in the Laurent polynomial ring $k\left[\mathbf{t}^{\pm 1}\right]$. Define a semigroup homomorphism

$$\pi : \mathbb{N}^n \to \mathbb{Z}^d, \quad \mathbf{u} \mapsto u_1\mathbf{a}_1 + ... + u_n\mathbf{a}_n.$$

Using proposition 4.2, proposition 4.3 and corollary 4.4 we can lift this map π to the following homomorphism of semigroup algebras:

$$\hat{\pi} : k[\mathbf{x}] \to k\left[\mathbf{t}^{\pm 1}\right], \quad x_i \mapsto \mathbf{t}^{\mathbf{a}_i}.$$

Indeed,

$$\begin{aligned}
\hat{\pi}\left(\sum_{\mathbf{u}\in\mathbb{N}^n} f(\mathbf{u})\mathbf{x}^{\mathbf{u}}\right) &= \sum_{\mathbf{u}\in\mathbb{N}^n} f(\mathbf{u})\mathbf{t}^{\pi(\mathbf{u})} \\
&= \sum_{\mathbf{v}\in\mathbb{Z}^d}\left(\sum_{\mathbf{u}\text{ s.t. }\pi(\mathbf{u})=\mathbf{v}} f(\mathbf{u})\right)\mathbf{t}^{\mathbf{v}} \\
&= \sum_{\mathbf{v}\in\mathbb{Z}^d} \left(\hat{\pi}(f)\right)(\mathbf{v})\mathbf{t}^{\mathbf{v}}.
\end{aligned}$$

Define $I_{\mathcal{A}} := \ker(\hat{\pi})$, such an ideal is called the *toric ideal of* $\mathcal{A}$.

Notice the following nice property of toric ideals:

Lemma 4.6. A toric ideal $I_{\mathcal{A}}$ is a prime ideal in $k[\mathbf{x}]$.

Proof. Obviously $k[\mathbb{N}^n] = k[\mathbf{x}]$ is a domain. We will show that $k\left[\mathbb{Z}^d\right] = k\left[\mathbf{t}^{\pm 1}\right]$ is also a domain. Let $f, g \in k\left[\mathbf{t}^{\pm 1}\right]$ be such that $fg = 0$. Let $c \in k\left[\mathbb{N}^d\right] = k[\mathbf{t}]$ be a monomial such that $cf, cg \in k\left[\mathbb{N}^d\right]$. As $fg = 0$ also $(cf)(cg) = 0$, hence $cf = 0$ or $cg = 0$ as $k[\mathbf{t}]$ is a domain. Then either $f = 0$ or $g = 0$, so $k\left[\mathbf{t}^{\pm 1}\right]$ is a domain as well.

Next take arbitrary polynomials $\alpha, \beta \in I_{\mathcal{A}}$, then as $\hat{\pi}$ is a homomorphism we obtain $\hat{\pi}(\alpha) \cdot \hat{\pi}(\beta) = \hat{\pi}(\alpha \cdot \beta) = 0$. As $k\left[\mathbf{t}^{\pm 1}\right]$ is a domain either $\hat{\pi}(\alpha) = 0$ or $\hat{\pi}(\beta) = 0$. Hence either α or β is in $\ker(\hat{\pi}) = I_{\mathcal{A}}$. $\square$

As the toric ideal $I_{\mathcal{A}} = \ker(\hat{\pi})$ is a prime ideal, its affine variety $\mathcal{V}(I_{\mathcal{A}})$ of zeroes in k^n is irreducible. Next we prove that a toric ideal is generated by binomials.

Let $\mathcal{A} = \{\mathbf{a}_1, ..., \mathbf{a}_n\} \subset \mathbb{Z}^d$. Let $e_1, ..., e_n$ be the standard basis of $\mathbb{Z}^n$ and define a map $\psi : \mathbb{Z}^n \to \mathbb{Z}^d$ given by $\psi(e_j) = \mathbf{a}_j$. Define the lattice $\mathbb{L} := \ker(\psi) = \left\{\mathbf{u} \in \mathbb{Z}^n \,\middle|\, \sum_{j=1}^n u_j\mathbf{a}_j = 0\right\}$. Also define an equivalence relation $\sim_{\mathbb{L}}$ on $\mathbb{N}^n$ by $\mathbf{u} \sim_{\mathbb{L}} \mathbf{v}$ if and only if $\mathbf{u} - \mathbf{v} \in \mathbb{L}$. The equivalence classes form a semigroup $S := {}^{\mathbb{N}^n}\!/\!_{\sim_{\mathbb{L}}}$. Observe that $S = \operatorname{im}(\pi)$ with $\pi : \mathbb{N}^n \to \mathbb{Z}^d$ the homomorphism given in definition 4.5. In other words S is the sub-semigroup generated by the elements of $\mathcal{A}$. To lattice $\mathbb{L}$ we associate the ideal $I_{\mathbb{L}} := \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \,|\, \mathbf{u}, \mathbf{v} \in \mathbb{N}^n \text{ such that } \mathbf{u} - \mathbf{v} \in \mathbb{L} \rangle \subseteq k[\mathbf{x}]$. In the next proposition we will show that $I_{\mathbb{L}}$ coincides with the toric ideal $I_{\mathcal{A}}$.

Proposition 4.7. There is an isomorphism $k[S] \cong {}^{k[\mathbf{x}]}\!/\!_{I_{\mathbb{L}}}$.

Proof. We can view $k[S]$ as the k-algebra generated by $\mathbf{t}^{\mathbf{a}_1}, ..., \mathbf{t}^{\mathbf{a}_n}$ subject to the relations $\prod_{l=1}^{r} \mathbf{t}^{\mathbf{a}_{i_l}} = \prod_{m=1}^{s} \mathbf{t}^{\mathbf{a}_{j_m}}$ whenever $\sum_{l=1}^{r} \mathbf{a}_{i_l} = \sum_{m=1}^{s} \mathbf{a}_{j_m}$ in $\mathcal{A}$. The kernel of $\hat{\pi} : k[\mathbf{x}] \to k[S]$ sending x_j to $\mathbf{t}^{\mathbf{a}_j}$ obviously contains the ideal generated by binomials $\left(\prod_{l=1}^{r} x_{i_l}^{\mathbf{a}_{i_l}}\right) - \left(\prod_{m=1}^{s} x_{j_m}^{\mathbf{a}_{j_m}}\right)$ satisfying $\sum_{l=1}^{r} \mathbf{a}_{i_l} = \sum_{m=1}^{s} \mathbf{a}_{j_m}$ in $\mathcal{A}$, which equals the toric ideal $I_{\mathbb{L}}$. The thing to prove is whether or not there can be more relations. For this consider for each element $\mathbf{a} \in \mathcal{A}$ the vector space $V_{\mathbf{a}}$, whose basis consists of the monomials $\mathbf{x}^{\mathbf{u}}$ mapping to $\mathbf{t}^{\mathbf{a}}$. The image of $V_{\mathbf{a}}$ in the quotient ${}^{k[x]}\!/\!_{I_{\mathbb{L}}}$ has dimension 1 over k, since the image of the basis vectors of $V_{\mathbf{a}}$ are equal in the quotient (note that we assumed $\mathbf{a} \in S$, for the dimension of $V_{\mathbf{a}}$ is zero otherwise). Let $f = \sum_{\mathbf{u}} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}}$ be a polynomial in $k[\mathbf{x}]$. Then $f \in \ker(\hat{\pi})$ if and only if $\sum_{\mathbf{u}} c_{\mathbf{u}} = 0$. Now fix a $\mathbf{u}_0$ such that $\pi(\mathbf{u}_0) = \mathbf{a}$, then $c_{\mathbf{u}_0} = \sum_{\mathbf{u} \neq \mathbf{u}_0} -c_{\mathbf{u}}$, hence

$$f = c_{\mathbf{u}_0}\mathbf{x}^{\mathbf{u}_0} + \sum_{\mathbf{u}\neq\mathbf{u}_0} c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} = \sum_{\mathbf{u}\neq\mathbf{u}_0} -c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}_0} + \sum_{\mathbf{u}\neq\mathbf{u}_0} c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} = \sum_{\mathbf{u}\neq\mathbf{u}_0} c_{\mathbf{u}}\left(\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{u}_0}\right).$$

Now write $f = \sum_{\mathbf{u}} c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} = \sum_{\mathbf{a}} \sum_{\mathbf{u},\ \pi(\mathbf{u})=\mathbf{a}} c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}$. Then $f \in \ker(\hat{\pi})$ if and only if $\forall \mathbf{a}$, $\sum_{\mathbf{u},\ \pi(\mathbf{u})=\mathbf{a}} c_{\mathbf{u}} = 0$. Then by the previous we see that the kernel of $\hat{\pi}$ is spanned as a k-vector space by the binomials $\{\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \,|\, \mathbf{u} - \mathbf{v} \in \mathbb{L}\}$. Hence the canonical map $k[S] \to {}^{k[x]}\!/\!_{I_{\mathbb{L}}}$ is an isomorphism of vector spaces graded by S, and therefore it is an isomorphism of k-algebras, as required. $\square$

Next we present the more involved approach of Sturmfels to the above proposition, this requires the definition of a term order and initial ideals:

Definition 4.8. Let k be a field, a term order on $k[\mathbf{x}]$ is a relation $\prec$ on the set of monomials $\mathbf{x}^{\mathbf{u}}$ in $k[\mathbf{x}]$ satisfying:

- $\prec$ is a total ordering.
- $\prec$ is compatible with multiplication in $k[\mathbf{x}]$ in the sense that if $\mathbf{x}^{\mathbf{u}} \prec \mathbf{x}^{\mathbf{v}}$, then for any monomial $\mathbf{x}^{\mathbf{w}}$ also $\mathbf{x}^{\mathbf{u}+\mathbf{w}} \prec \mathbf{x}^{\mathbf{v}+\mathbf{w}}$.
- $\prec$ is a well-ordering, meaning that every nonempty collection of monomials has a largest element w.r.t. $\prec$.

Remark 4.9. Throughout this thesis we use the convention in [18] that every nonempty collection of monomials has a *largest* element w.r.t. $\prec$, other books such as [1] and [2] use an equivalent *minimality* condition.

Definition 4.10. Given a term order $\prec$, every polynomial $f \in k[\mathbf{x}]$ has a unique initial term denoted by $\mathrm{In}_\prec(f) := \max_\prec \{m \,|m \in f \text{ a monomial}\}$. For an ideal I we define its initial ideal $\mathrm{In}_\prec(I)$ as the ideal generated by the initial monomials, i.e. $\mathrm{In}_\prec(I) := \langle \mathrm{In}_\prec(f)|\, f \in I\rangle$. So $\mathrm{In}_\prec(I)$ is an example of a monomial ideal, i.e. an ideal generated by monomials.

Proposition 4.11. ([18], lemma 4.1) The toric ideal $I_\mathcal{A}$ is spanned as a k-vector space by the set

$$\mathcal{B} := \{\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} |\, \mathbf{u}, \mathbf{v} \in \mathbb{N}^n, \text{ with } \pi(\mathbf{u}) = \pi(\mathbf{v})\}.$$

Proof. We prove both inclusions, " $\supseteq$ ": Notice that

$$\begin{aligned} \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in \mathcal{B} &\iff \pi(\mathbf{u}) = \pi(\mathbf{v}), \mathbf{u}, \mathbf{v} \in \mathbb{N} \\ &\iff u_1\mathbf{a}_1 + \ldots + u_n\mathbf{a}_n = v_1\mathbf{a}_1 + \ldots + v_n\mathbf{a}_n \\ &\iff \mathbf{t}^{u_1\mathbf{a}_1 + \ldots + u_n\mathbf{a}_n} - \mathbf{t}^{v_1\mathbf{a}_1 + \ldots + v_n\mathbf{a}_n} = 0 \\ &\iff \hat{\pi}\left(\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}\right) = 0 \\ &\iff \mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v} \in I_\mathcal{A}. \end{aligned}$$

" $\subseteq$ ": We will show by contradiction that each $f \in I_\mathcal{A}$ can be written as a k-linear combination of binomials in $\mathcal{B}$. For this fix an arbitrary term order $\prec$ on $k[\mathbf{x}]$ and let $f \in I_\mathcal{A}$ be a polynomial that can *not* be written as a k-linear combination of elements in $\mathcal{B}$ and such that the monomial $\mathbf{x}^\mathbf{u} := \mathrm{In}_\prec(f)$ is minimal in f w.r.t. $\prec$. As $f \in I_\mathcal{A} = \ker(\hat{\pi})$ we know that the term $\mathbf{t}^{\pi(\mathbf{u})} = \hat{\pi}(\mathbf{x}^\mathbf{u})$ cancels against at least one other term of the form $\mathbf{t}^{\pi(\mathbf{v})} = \hat{\pi}(\mathbf{x}^\mathbf{v})$, with $\mathbf{x}^\mathbf{v}$ a monomial in f. This implies that $\hat{\pi}(\mathbf{x}^\mathbf{u}) - \hat{\pi}(\mathbf{x}^\mathbf{v}) = \hat{\pi}(\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}) = 0$. As $\mathbf{x}^\mathbf{v} \neq \mathbf{x}^\mathbf{u} = \mathrm{In}_\prec(f)$, we see that $\mathbf{x}^\mathbf{v} \prec \mathbf{x}^\mathbf{u}$. As f can not be written as a k-linear combination of elements in $\mathcal{B}$, neither can $g := f - \mathbf{x}^\mathbf{u} + \mathbf{x}^\mathbf{v}$. Notice that $g \in I_\mathcal{A}$ as $\hat{\pi}(f - \mathbf{x}^\mathbf{u} + \mathbf{x}^\mathbf{v}) = \hat{\pi}(f) - \hat{\pi}(\mathbf{x}^\mathbf{u} - \mathbf{x}^\mathbf{v}) = 0$. But we know that $\mathrm{In}_\prec(g) \prec \mathbf{x}^\mathbf{u} = \mathrm{In}_\prec(f)$, contradicting the minimality of $\mathrm{In}_\prec(f)$ w.r.t. $\prec$ in our choice of f. $\square$

We can rephrase the previous lemma by introducing pure binomials:

Definition 4.12. ([13], definition 6.1.2) We can write a vector $\mathbf{u} = (u_1, \ldots, u_n) \in \mathbb{Z}^n$ uniquely as $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^-$, where $\mathbf{u}^+$ and $\mathbf{u}^-$ are coordinate wise non-negative and have disjoint support. Binomials $\beta_\mathbf{u} := \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ corresponding to such a vector $\mathbf{u}$ are called pure.

Corollary 4.13. ([18], corollary 4.3) The toric ideal $I_\mathcal{A}$ coincides with the ideal

$$\langle \beta_\mathbf{u} \mid \mathbf{u} \in \ker(\pi) \rangle.$$

Example 4.14. Consider the matrix corresponding to our Newton polytope:

$$\mathcal{A} := \begin{pmatrix} -1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

We computed in lemma 3.5 that

$$I_{\mathcal{A}} = \left\langle x_1x_5 - x_4^2, x_1x_4 - x_2x_3 \right\rangle$$

is the binomial toric ideal for $\mathcal{A}$ (consisting of pure binomials). Notice the correspondence between the exponent vectors of the pure binomials in $I_{\mathcal{A}}$ and the reduced relations between the vectors in $\mathcal{A}$.

The algebraic geometry computer package Macaulay2 (see [14] for the freeware package) can compute toric ideals over matrices, although it will not always minimize the number of generators for the ideal (see [7], chapter 'Algorithms for the Hilbert Schemes' for a detailed description of the implementation of the toric ideal algorithm). Another computer package to compute toric ideals is Singular (see [15]).

By lemma 4.6 and proposition 4.11 we see that a toric ideal is prime and generated by binomials. We will now prove the opposite direction, i.e. that for a prime ideal $I \subset k[\mathbf{x}]$ generated by binomials we can find a set $\mathcal{A}$ such that the toric ideal $I_{\mathcal{A}}$ coincides with I, see theorem 4.21 below. Reduced Gröbner bases are the main objects in this proof (and in this whole chapter) and defined as follows:

Definition 4.15. A finite subset $\mathcal{G}_\prec(I) \subset I$ is a Gröbner basis for I w.r.t. a term order $\prec$ if

$$\mathrm{In}_\prec(I) = \langle \mathrm{In}_\prec(g) | \, g \in \mathcal{G}_\prec(I) \rangle \, .$$

A Gröbner basis $\mathcal{G}_\prec(I) = \{g_1, ..., g_s\}$ is called *minimal* if for every $g_j \in \mathcal{G}_\prec(I)$ the set $\mathcal{G}_\prec(I) \backslash \{g_j\}$ is not a Gröbner basis. And it is called *reduced* if for any $g_j \in \mathcal{G}$ no term of g_j lies in $\langle \mathrm{In}_\prec(g_1), ..., \mathrm{In}_\prec(g_{j-1}), \mathrm{In}_\prec(g_{j+1}), ..., \mathrm{In}_\prec(g_s)\rangle$, and for each $g_j \in \mathcal{G}_\prec(I)$ the coefficient of the initial term of g_j is 1. Remark that reduced Gröbner bases are unique w.r.t. a term order $\prec$ (see [2], ch. 2, §7).

Gröbner bases have the nice property that they can be computed using a simple algorithm, called the Buchberger algorithm (named after its inventor B. Buchberger). This algorithm uses S-polynomials; these polynomials are constructed in such a way that they cancel initial terms of two input-polynomials.

Definition 4.16. Fix a term order $\prec$ and let $f_i, f_j \in k[\mathbf{x}]$ be two polynomials. Let $\mathbf{x}^{\mathbf{u}}$ be the monomial

$$\mathrm{lcm}\left(\frac{\mathrm{In}_\prec(f_i)}{c_i}, \frac{\mathrm{In}_\prec(f_j)}{c_j}\right),$$

where c_i (resp. c_j) is the coefficient of the initial term of f_i (resp. f_j). We define the S-polynomial of f_i and f_j to be

$$S(f_i, f_j) := \frac{\mathbf{x}^{\mathbf{u}} f_i}{\mathrm{In}_{\prec}(f_i)} - \frac{\mathbf{x}^{\mathbf{u}} f_j}{\mathrm{In}_{\prec}(f_j)}.$$

Let $\mathcal{G} \subset k[\mathbf{x}]$ be a non-empty and finite subset, we denote by $\overline{S(f_i, f_j)}^{\mathcal{G}}$ the remainder of $S(f_i, f_j)$ on division by $\mathcal{G}$ w.r.t a term order $\prec$ using the division algorithm in $k[\mathbf{x}]$ (see for example [2], Ch.2, §3, Thm. 3).

Note that the Buchberger algorithm computes a Gröbner basis given a term order and a non-empty set of generators for an ideal. A (non-optimal) implementation of the Buchberger algorithm is given by ([2] p. 16):

Algorithm 4.17. (Buchberger's algorithm)
Input: Generators $f_1, ..., f_s$ for an ideal I and a term order $\prec$.
Output: A Gröbner basis $\mathcal{G}$ for I.
Algorithm:

```
G ← {f_1, ..., f_s}
repeat
  G' ← G
  for each pair {g_i, g_j}, g_i ≠ g_j in G' do
    if S := overline(S(g_i, g_j))^{G'} ≠ 0 then
      G ← G ∪ {S}
    end if
  end for
until G = G'
```

Example 4.18. We compute a Gröbner basis using the Buchberger algorithm for the ideal $I_{\mathcal{A}} = \langle f_1, f_2 \rangle$ with $f_1 = x_1x_5 - x_4^2$ and $f_2 = x_1x_4 - x_2x_3$ as in example 4.14. For this fix the lexicographic term order $\prec_{lex}$ with $x_4 \succ x_1 \succ x_3 \succ x_2 \succ x_5$. Set $\mathcal{G} = \mathcal{G}' = \{f_1, f_2\}$. Notice that for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$ we have $\mathbf{x}^{\mathbf{u}} \prec_{lex} \mathbf{x}^{\mathbf{v}}$ if the leftmost non-zero coordinate in $\mathbf{u} - \mathbf{v}$ is positive. We compute

$$S(f_1, f_2) = \frac{x_1x_4^2}{-x_4^2}\left(x_1x_5 - x_4^2\right) - \frac{x_1x_4^2}{x_1x_4}\left(x_1x_4 - x_2x_3\right) = x_2x_3x_4 - x_1^2x_5 =: f_3.$$

As $\overline{S(f_1, f_2)}^{\mathcal{G}'} \neq 0$ we add f_3 to our set $\mathcal{G}'$. For the same reason we also add $S(f_2, f_3) = x_1^3x_5 - x_2^2x_3^2$ to $\mathcal{G}'$. Next we compute that $\overline{S(f_i, f_j)}^{\mathcal{G}'} = 0$ for all distinct

$$f_i, f_j \in \mathcal{G}' = \left\{x_1x_5 - x_4^2, x_1x_4 - x_2x_3, x_2x_3x_4 - x_1^2x_5, x_1^3x_5 - x_2^2x_3^2\right\}.$$

So $\mathcal{G} := \mathcal{G}'$ is a Gröbner basis for $I_{\mathcal{A}}$, in fact we easily compute that $\mathcal{G}$ is even a *reduced* Gröbner basis for $I_{\mathcal{A}}$.

Notice that the Gröbner basis of the specific toric ideal $I_{\mathcal{A}}$ in example 4.18 consists solely of binomials. This fact holds in general:

Lemma 4.19. Let $I_{\mathcal{A}} \subset k[\mathbf{x}]$ be a toric ideal, $\prec$ a term order and $\mathcal{G}_{\prec}(I_{\mathcal{A}})$ a reduced Gröbner basis. Then $\mathcal{G}_{\prec}(I_{\mathcal{A}})$ consists solely of pure binomials.

Proof. If $I_{\mathcal{A}} \subset k[\mathbf{x}]$ is a toric ideal, then by the Hilbert Basis theorem (cf. [1], p.4) and proposition 4.11 there exists a finite number of binomials generating $I_{\mathcal{A}}$. Apply the Buchberger algorithm to these generators. Obviously the S-pair of two binomials is again a binomial, implying that the Gröbner basis for $I_{\mathcal{A}}$ consists solely of binomials.
Next we show that the binomials in $\mathcal{G}_{\prec}(I_{\mathcal{A}})$ are pure: Let $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in \mathcal{G}_{\prec}(I_{\mathcal{A}})$, $\mathbf{u}, \mathbf{v} \in \mathbb{N}^n$. If this binomial is not pure, then $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$ have a common factor x_i such that $x_i\left(\frac{\mathbf{x}^{\mathbf{u}}}{x_i} - \frac{\mathbf{x}^{\mathbf{v}}}{x_i}\right) \in I_{\mathcal{A}}$. As $x_i \notin I_{\mathcal{A}}$ and $I_{\mathcal{A}}$ is a prime ideal we obtain that $\frac{\mathbf{x}^{\mathbf{u}}}{x_i} - \frac{\mathbf{x}^{\mathbf{v}}}{x_i} \in I_{\mathcal{A}}$. But $\mathrm{In}_{\prec}\left(\frac{\mathbf{x}^{\mathbf{u}}}{x_i} - \frac{\mathbf{x}^{\mathbf{v}}}{x_i}\right) \in \mathrm{In}_{\prec}(I_{\mathcal{A}})$ and it divides either $\mathbf{x}^{\mathbf{u}}$ or $\mathbf{x}^{\mathbf{v}}$ (non-trivially), implying that $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}}$ is not an element of the *reduced* Gröbner basis $\mathcal{G}_{\prec}(I_{\mathcal{A}})$, yielding a contradiction. □

Observe the following list of properties of ideals generated by binomials:

Lemma 4.20. Let $I \subset k[\mathbf{x}]$ be an ideal, then the following statements hold:

1. If I is generated by binomials and $f \in I$, then the coefficients of f sum to 0.

2. If I is generated by binomials then I does not contain any monomials.

3. If I is a prime ideal generated by binomials, then I is generated by pure binomials.

4. For $\mathbf{v} \in \mathbb{Z}^n$ such that $\beta_{\mathbf{v}} = \mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-} \in I$, also $\beta_{-\mathbf{v}} \in I$.

5. If I is a prime ideal generated by binomials and $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$ are such that $\beta_{\mathbf{u}}, \beta_{\mathbf{v}} \in I$, then also $\beta_{\mathbf{v}-\mathbf{u}} \in I$ and $\beta_{\mathbf{v}+\mathbf{u}} \in I$. Moreover for any $c \in \mathbb{N}$ also $\beta_{c\mathbf{u}} \in I$.

6. If I is a prime ideal generated by binomials and $\mathbf{u} \in \mathbb{Z}^n$ is such that $\beta_{c\mathbf{u}} \in I$ for some $c \in \mathbb{N} \setminus \{0\}$, then also $\beta_{\mathbf{u}} \in I$.

7. Assume I is a prime ideal generated by binomials and $\beta_{\mathbf{u}_i} \in I$ for all $\mathbf{u}_1, ..., \mathbf{u}_d \in \mathbb{Z}^n$. If we choose $a_1, ..., a_d \in \mathbb{Q}$ such that $\delta := \sum_{i=1}^d a_i \mathbf{u}_i \in \mathbb{Z}^n$, then $\beta_\delta \in I$.

Proof. 1. Trivial: We can write every polynomial in I as a finite product of binomials, where each binomial consists of a term with a $+1$ coefficient and one with a -1 coefficient. So in the expansion of this product the sum of the coefficients is zero.

2. Trivial: If $c\mathbf{x}^{\mathbf{u}} \in I$ is a monomial then $c \neq 0$, which contradicts part 1.

3. Let $\mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \in I,\ \mathbf{u}, \mathbf{v} \in \mathbb{N}^n$. If this binomial is not pure, then $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$ have a common factor $\mathbf{x}^{\mathbf{w}},\ \mathbf{w} \in \mathbb{N}^n$ such that $\mathbf{x}^{\mathbf{w}}\left(\frac{\mathbf{x}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{w}}} - \frac{\mathbf{x}^{\mathbf{v}}}{\mathbf{x}^{\mathbf{w}}}\right) \in I$. As I does not contain any monomials according to part 2 we see that $\mathbf{x}^{\mathbf{w}} \notin I$. As I is prime we see that the pure binomial $\frac{\mathbf{x}^{\mathbf{u}}}{\mathbf{x}^{\mathbf{w}}} - \frac{\mathbf{x}^{\mathbf{v}}}{\mathbf{x}^{\mathbf{w}}} \in I$.

4. Trivial: $\beta_{-\mathbf{u}} = \mathbf{x}^{\mathbf{u}^-} - \mathbf{x}^{\mathbf{u}^+} = -\beta_{\mathbf{u}} \in I$.

5. Fix a term order $\prec$ such that $\mathbf{x}^{\mathbf{u}^-} \prec \mathbf{x}^{\mathbf{u}^+}$ and $\mathbf{x}^{\mathbf{v}^-} \prec \mathbf{x}^{\mathbf{v}^+}$ and define $\mathbf{x}^{\alpha} := \operatorname{lcm}\left(\mathbf{x}^{\mathbf{u}^+}, \mathbf{x}^{\mathbf{v}^+}\right)$. Also define the monomial $\mathbf{x}^{\mathbf{w}} := \gcd\left(\mathbf{x}^{\alpha-\mathbf{u}}, \mathbf{x}^{\alpha-\mathbf{v}}\right)$. Then $\alpha - \mathbf{u} - \mathbf{w}$ and $\alpha - \mathbf{v} - \mathbf{w}$ are non-negative and have disjoint support. Using this we find:

$$
\begin{aligned}
S\left(\beta_{\mathbf{v}}, \beta_{\mathbf{u}}\right) &= \frac{\mathbf{x}^{\alpha}}{\mathrm{In}_{\prec}\left(\beta_{\mathbf{v}}\right)}\beta_{\mathbf{v}} - \frac{\mathbf{x}^{\alpha}}{\mathrm{In}_{\prec}\left(\beta_{\mathbf{u}}\right)}\beta_{\mathbf{u}} \\
&= \frac{\mathbf{x}^{\alpha}}{\mathbf{x}^{\mathbf{v}^+}}\left(\mathbf{x}^{\mathbf{v}^+} - \mathbf{x}^{\mathbf{v}^-}\right) - \frac{\mathbf{x}^{\alpha}}{\mathbf{x}^{\mathbf{u}^+}}\left(\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}\right) \\
&= \mathbf{x}^{\alpha} - \mathbf{x}^{\alpha+\mathbf{v}^- - \mathbf{v}^+} - \mathbf{x}^{\alpha} + \mathbf{x}^{\alpha+\mathbf{u}^- - \mathbf{u}^+} \\
&= \mathbf{x}^{\alpha-\mathbf{u}} - \mathbf{x}^{\alpha-\mathbf{v}} \\
&= \mathbf{x}^{\mathbf{w}}\left(\mathbf{x}^{\alpha-\mathbf{u}-\mathbf{w}} - \mathbf{x}^{\alpha-\mathbf{v}-\mathbf{w}}\right) \\
&= \mathbf{x}^{\mathbf{w}}\beta_{\alpha-\mathbf{u}-\mathbf{w}-(\alpha-\mathbf{v}-\mathbf{w})} \\
&= \mathbf{x}^{\mathbf{w}}\beta_{\mathbf{v}-\mathbf{u}} \in I.
\end{aligned}
$$

As $\mathbf{x}^{\mathbf{w}} \notin I$ by part 2 and I is prime we see that $\beta_{\mathbf{v}-\mathbf{u}} \in I$ as required. As $\mathbf{u} \in \mathbb{Z}^n$ was arbitrary, we have also proven that $\beta_{\mathbf{v}-(-\mathbf{u})} = \beta_{\mathbf{v}+\mathbf{u}} \in I$. Using induction on $c \in \mathbb{N}$, part 4 and the fact that $\beta_{\mathbf{v}+\mathbf{u}} \in I$ we see that $\beta_{c\mathbf{u}} \in I$ for all $c \in \mathbb{N}$.

6. Case 1: Suppose that $\mathrm{char}(k) = p > 0$ and $p \nmid c$, then expand $\beta_{c\mathbf{u}}$ as

$$
\begin{aligned}
\beta_{c\mathbf{u}} &= \mathbf{x}^{c\mathbf{u}^+} - \mathbf{x}^{c\mathbf{u}^-} \\
&= \left(\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}\right)\left(\mathbf{x}^{(c-1)\mathbf{u}^+} + \mathbf{x}^{(c-2)\mathbf{u}^+}\mathbf{x}^{\mathbf{u}^-} + \ldots + \mathbf{x}^{(c-1)\mathbf{u}^-}\right) \in I.
\end{aligned}
$$

By our assumption on $\mathrm{char}(k)$ and part 1 the second factor of this expansion is not in I. As I is a prime ideal we see that $\beta_{\mathbf{u}} = \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} \in I$ as required.

Case 2: Next suppose $\mathrm{char}(k) = p > 0$ and $c = p^s c'$ with $p \nmid c'$. Then we can write

$$
\beta_{c\mathbf{u}} = \mathbf{x}^{p^s c' \mathbf{u}^+} - \mathbf{x}^{p^s c' \mathbf{u}^-} = \left(\mathbf{x}^{c'\mathbf{u}^+} - \mathbf{x}^{c'\mathbf{u}^-}\right)^{p^s} = \left(\beta_{c'\mathbf{u}}\right)^{p^s}.
$$

As I is a prime ideal we see that $\beta_{c'\mathbf{u}} \in I$. Then by the first case of this part of the proof we obtain $\beta_{\mathbf{u}} \in I$.

7. Write $a_j = \frac{p_j}{q_j}$ with $p_j, q_j \in \mathbb{Z}$ for all $1 \leq j \leq d$ and define $a := \prod_{j=1}^{d} q_j$. Then $\mathbf{v} = a \sum_{j=1}^{d} a_j \mathbf{u}_j \in \mathbb{Z}^n$ and $\beta_{\mathbf{v}} \in I$ using part 5. By assumption $\frac{1}{a}\mathbf{v} \in \mathbb{Z}$, so by part 6 also $\beta_\delta = \beta_{\frac{1}{a}\mathbf{v}} \in I$.

□

The next theorem is the main result about toric ideals, it identifies toric ideals with prime ideals generated by binomials.

Theorem 4.21. Let $I \subset k[\mathbf{x}]$ be an ideal. Then I is a toric ideal if and only if I is a prime ideal generated by binomials.

Proof. " $\Rightarrow$ ": This follows from lemma 4.6 and proposition 4.11.
" $\Leftarrow$ ": Suppose $I \subset k[\mathbf{x}]$ is a prime ideal generated by binomials. By lemma 4.20 part 3 and the Hilbert Basis theorem we can assume that I is generated by finitely many *pure* binomials, i.e. $I = \langle \beta_{\mathbf{v}_1}, ..., \beta_{\mathbf{v}_l} \rangle$ with $\mathbf{v}_1, ..., \mathbf{v}_l \in \mathbb{Z}^n$. What we have to do is to construct a homomorphism $\hat{\pi} : k[\mathbf{x}] \to k[\mathbf{t}^{\pm 1}]$ such that $I = \ker(\hat{\pi})$. Also $\hat{\pi}$ must be induced by a semigroup homomorphism $\pi : \mathbb{N}^n \to \mathbb{Z}^d$ for some $d \in \mathbb{N}$.
The construction: Fix an $m \in \mathbb{N}$ such that the subset $\{\mathbf{u}_1, ..., \mathbf{u}_m\} \subset \{\mathbf{v}_1, ..., \mathbf{v}_l\}$ is a $\mathbb{Q}$-basis for $\operatorname{span}_{\mathbb{Q}}(\mathbf{v}_1, ..., \mathbf{v}_l)$. Extend this to a basis $\{\mathbf{u}_1, ..., \mathbf{u}_n\}$ for $\mathbb{Q}^n$. Let $\{e_i\}_{i=1}^{d}$ be the standard basis of $\mathbb{Q}^d$ ($d = n - m$) and define the linear map

$$\psi : \mathbb{Q}^n \to \mathbb{Q}^d \quad \text{by} \quad \begin{cases} \mathbf{u}_i \mapsto 0 & \text{if} \quad i \in \{1, ..., m\} \\ \mathbf{u}_{m+i} \mapsto e_i & \text{if} \quad i \in \{1, ..., n-m\} \end{cases}$$

We can view $\psi = (\psi)_{i,j}$ ($i \in \{1, ..., d\}$, $j \in \{1, ..., n\}$) as a $d \times n$-matrix with entries in $\mathbb{Q}$. Write each entry in this matrix as $\frac{p_{i,j}}{q_{i,j}}$ with $p_{i,j}, q_{i,j} \in \mathbb{Z}$ and define $q := \prod_{i=1}^{d} \prod_{j=1}^{n} q_{i,j}$. Then the matrix $\pi := q\psi$ is a $d \times n$-matrix with coefficients in $\mathbb{Z}$. So we can view π as a linear map from $\mathbb{N}^n$ to $\mathbb{Z}^d$. As in definition 4.5 the map π lifts to a homomorphism $\hat{\pi} : k[\mathbf{x}] \to k\left[\mathbf{t}^{\pm 1}\right]$ of semigroup algebras.
Next we show that $\ker(\hat{\pi}) = \langle \beta_{\mathbf{v}_1}, ..., \beta_{\mathbf{v}_l} \rangle$:
" $\supseteq$ ": As π is a homomorphism and $\mathbf{v}_i \in \ker(\pi)$ for all i, we obtain that $\beta_{\mathbf{v}_i} \in \ker(\hat{\pi})$.
" $\subseteq$ ": We know that $\ker(\hat{\pi})$ is a toric ideal by definition 4.5. So it is generated by pure binomials using lemma 4.19. Notice that $\beta_{\mathbf{w}} \in \ker(\hat{\pi})$ if and only if $\pi(\mathbf{w}) = 0$ (see the proof of proposition 4.11). So if we show that if $\pi(\mathbf{w}) = 0$ then $\beta_{\mathbf{w}} \in I$ we are done: By construction $\ker(\pi) = \langle \mathbf{u}_1, ..., \mathbf{u}_m \rangle$, hence we can write $\mathbf{w} = \sum_{j=1}^{m} a_j \mathbf{u}_j$ with $a_j \in \mathbb{Q}$. As $\beta_{\mathbf{u}_j} \in I$ for all $1 \leq j \leq m$ we see that $\beta_{\mathbf{w}} \in I$ using lemma 4.20 part 7. □

4.3 Homogeneous polynomials and weight vectors

We will only define the Gröbner fan for homogeneous ideals $I \subset k[\mathbf{x}]$, as for homogeneous ideals the Gröbner fan will turn out to be complete. This section is an introduction to homogeneous polynomials and the use of weight vectors. In some of the upcoming proofs in this chapter we will use these weight vectors.

Definition 4.22. A grading of the monomials of $k[\mathbf{x}]$ by an abelian group G is a map called degree; for a monomial $\mathbf{x}^{\mathbf{u}} \in k[\mathbf{x}]$ we define $\deg(\mathbf{x}^{\mathbf{u}}) = \sum_{j=1}^{n} u_j g_j$.

Definition 4.23. A polynomial $f \in k[\mathbf{x}]$ is homogeneous under the grading in definition 4.22 if all monomials in f have the same degree. An ideal $I \subset k[\mathbf{x}]$ is homogeneous if I is generated by homogeneous polynomials under the given grading, or equivalently, if every polynomial in I is a sum of homogeneous polynomials. These two definitions are equivalent by the upcoming proposition 4.25.

Definition 4.24. The grading on the monomials in $k[\mathbf{x}]$ we will use is by $G = \mathbb{Z}$, where $\deg(\mathbf{x}^{\mathbf{u}}) = \sum_{j=1}^{n} u_j d_j,\ d_j \in \mathbb{Z}$. We will call from now on $\mathbf{d} = (d_1, ..., d_n) \in \mathbb{Z}$ the grading. Let $f = \sum_{\mathbf{u} \in \mathbb{N}^n} c_{\mathbf{u}} \mathbf{x}^{\mathbf{u}} \in k[\mathbf{x}]$ be homogeneous with respect to a grading $\mathbf{d} \in \mathbb{Z}^n$. Then $\mathbf{u} \cdot \mathbf{d}$ is constant for all $c_{\mathbf{u}} \neq 0$ and we will call $\mathbf{u} \cdot \mathbf{d}$ the $\mathbf{d}$-degree of f (notation $\deg_{\mathbf{d}}(f)$).

Proposition 4.25. Let $I \subset k[\mathbf{x}]$ be a homogeneous ideal, $\mathbf{d} = (d_1, ..., d_n) \in \mathbb{N}^n$ a grading. Then as a vector space we can write $I = \bigoplus_{m \in \mathbb{N}} I_m$, where I_m is the vector space $\{f \in I \,|\deg_{\mathbf{d}}(f) = m\} \cup \{0\}$.

Proof. Let $f \in I$ be a polynomial. As I is homogeneous and finitely generated by the Hilbert Basis theorem, we can write $f = \sum_{j=1}^{m} h_j g_j$ with $h_j \in k[\mathbf{x}]$ and $g_j \in I$ homogeneous w.r.t. $\mathbf{d}$. By splitting h_j into monomials we can write f uniquely as $f = \sum_{j=1}^{m'} g'_j$, with $g'_j \in I_j$. $\square$

In this subsection we will restrict ourselves to Gröbner fans of homogeneous ideals $I \subset k[\mathbf{x}]$ (with k a field) with respect to a positive grading $\mathbf{d} = (d_1, ..., d_n) \in \mathbb{N}^n_{>0}$ (following [18]). Gröbner fans of homogeneous ideals will turn out to be complete fans, while Gröbner fans of other ideals might be incomplete (see [8] for a definition and construction of Gröbner fans over arbitrary ideals in $k[\mathbf{x}]$).

Definition 4.26. We can generalize the construction of initial ideals, by defining so called *weight vectors* $\omega \in \mathbb{R}^n$: Fix some $\omega = (\omega_1, ..., \omega_n) \in \mathbb{R}^n$. For any polynomial $f = \sum_{\mathbf{u}} c_{\mathbf{u}} x^{\mathbf{u}}$ we define the initial form $\mathrm{in}_{\omega}(f)$ to be the sum of all terms $c_{\mathbf{u}} x^{\mathbf{u}}$ such that the inner product $\omega \cdot \mathbf{u}$ is maximal. For an ideal we define the initial ideal to be the ideal generated by all initial forms, i.e.

$$\mathrm{In}_\omega(I) := \langle \mathrm{In}_\omega(f) \,|\, f \in I \rangle .$$

A weight vector $\omega \in \mathbb{R}^n$ is said to be generic for an ideal I if $\mathrm{In}_\omega(I)$ is a monomial ideal, i.e. an ideal generated by monomials. Such a generic ω is said to represent a term order $\prec$ if $\mathrm{In}_\prec(I) = \mathrm{In}_\omega(I)$.

Definition 4.27. If $\omega \in \mathbb{R}^n_{\geq 0}$ and $\prec$ is a term order, we can define an induced term order $\prec_\omega$ as follows:

$$\mathbf{a} \prec_\omega \mathbf{b}, \quad \mathbf{a}, \mathbf{b} \in \mathbb{N}^n \iff \begin{array}{l} \omega \cdot \mathbf{a} < \omega \cdot \mathbf{b} \text{ or} \\ \omega \cdot \mathbf{a} = \omega \cdot \mathbf{b} \text{ and } \mathbf{a} \prec \mathbf{b}. \end{array}$$

In other words if ω can not make a distinction between two terms, use the term order $\prec$ as a 'tie-breaker'.

Example 4.28. Define $f_1 := x_1x_2x_3 + x_3^6 + 2x_2^2x_3^2$, $f_2 := x_2^5 + x_1^2x_2 + x_1x_2^3x_3 \in k[x_1, x_2, x_3]$, $\omega = (3, 2, 1) \in \mathbb{R}^3$. Let $I = \langle f_1, f_2 \rangle \subset k[x_1, x_2, x_3]$ be an ideal and let $\prec$ be the lexicographic term order. We compute

$$\begin{array}{rcl}
\mathrm{In}_\prec(f_1) &=& x_1x_2x_3 \\
\mathrm{In}_\omega(f_1) &=& f_1 \\
\mathrm{In}_{\prec_\omega}(f_1) &=& x_1x_2x_3 \\
\mathrm{In}_\prec(f_2) &=& x_1^2x_2 \\
\mathrm{In}_\omega(f_2) &=& x_2^5 + x_1x_2^3x_3 \\
\mathrm{In}_{\prec_\omega}(f_2) &=& x_1x_2^3x_3 \\
\mathrm{In}_\prec(I) &=& \left\langle x_1x_2x_3, x_1^2x_2 \right\rangle \\
\mathrm{In}_\omega(I) &=& \langle f_1, \mathrm{In}_\omega(f_2) \rangle \\
\mathrm{In}_{\prec_\omega}(I) &=& \langle x_1x_2x_3 \rangle .
\end{array}$$

We give some properties of an initial ideal (w.r.t. some weight vector) for homogeneous ideals. These will be useful later on:

Proposition 4.29. Let $I \subset k[\mathbf{x}]$ be a homogeneous ideal w.r.t. a grading $\mathbf{d} \in \mathbb{Z}^n$, and let $\omega \in \mathbb{R}^n$ be a weight vector. Then for all $\lambda \in \mathbb{R}$ we have $\mathrm{In}_\omega(I) = \mathrm{In}_{\omega+\lambda\mathbf{d}}(I)$.

Proof. We prove both inclusions: " $\subseteq$ " : Choose some $f \in I$ and split f in $\mathbf{d}$-homogeneous components using proposition 4.25, i.e. we can write $f = \sum_{j=0}^m f_j$ with $f_j \in I_j$ and $I = \bigoplus_{j\in\mathbb{N}} I_j$. During this decomposition no terms of f cancel, so we can write $\mathrm{In}_\omega(f) = \sum_{j\in J} \mathrm{In}_\omega(f_j)$ for some suitable index set $J \subset \mathbb{N}$. As these f_j are $\mathbf{d}$-homogeneous we obtain $\mathrm{In}_\omega(f_j) = \mathrm{In}_{\omega+\lambda\mathbf{d}}(f_j) \in \mathrm{In}_{\omega+\lambda\mathbf{d}}(I)$, hence $\mathrm{In}_\omega(f) \in \mathrm{In}_{\omega+\lambda\mathbf{d}}(I)$, as required. " $\supseteq$ " : Define $\omega' := \omega + \lambda\mathbf{d}$, then by the previous inclusion we have

$\mathrm{In}_{\omega'-\lambda\mathbf{d}}(I) \subseteq \mathrm{In}_{\omega'}(I)$ for all $\lambda \in \mathbb{R}$. This is equivalent to $\mathrm{In}_{\omega'+\lambda\mathbf{d}}(I) \subseteq \mathrm{In}_{\omega'}(I)$. As $\omega \in \mathbb{R}^n$ was arbitrary, we can choose $\omega' = \omega$ to obtain the desired result. □

Corollary 4.30. Let $I \subset k[\mathbf{x}]$ be a homogeneous ideal with respect to a grading $\mathbf{d} \in \mathbb{N}^n_{>0}$, and let $\omega \in \mathbb{R}^n$ be a weight vector. Then we can find a $\omega' \in \mathbb{R}^n_{>0}$ such that $\mathrm{In}_{\omega'}(I) = \mathrm{In}_{\omega}(I)$.

Proof. Trivial: As $\mathbf{d}$ is positive we can find a $\lambda \in \mathbb{R}$ such that $\omega' := \omega + \lambda\mathbf{d}$ is positive. The result now follows from proposition 4.29. □

4.4 Finitely many reduced Gröbner bases

In this section we will prove that given a homogeneous, toric ideal there exist only finitely many reduced Gröbner bases. The proof consists of two parts: First we proof the rather surprising fact that for every ideal $I \subset k[\mathbf{x}]$ the set of monomial initial ideals (obtained by varying over all term orders) is finite. Secondly we establish for a homogeneous, toric ideal $I_{\mathcal{A}}$ a bijection between the set of monomial initial ideals and the set of reduced Gröbner bases.

Following [1], Ch. 8, §4:

Definition 4.31. Let $I \subset k[\mathbf{x}]$ be an ideal, then we define the set of monomial initial ideals of I as

$$\mathrm{Mon}(I) := \{\mathrm{In}_{\prec}(I) \,|\, \prec \text{ a term order}\}.$$

We also define the set of reduced Gröbner bases of I as

$$\mathcal{G}_{\mathrm{red}}(I) = \{\mathcal{G}_{\prec}(I) |\, \prec \text{ a term order}\}.$$

Standard monomials play an important role in the proof of theorem 4.34, for this proof we need the following definition and lemma:

Definition 4.32. Let $I \subset k[\mathbf{x}]$ be an ideal and $\prec$ a term order. A standard monomial is a monomial in the set $\mathrm{SM}_{\prec}(I) := \{\mathbf{x}^{\mathbf{u}} \,|\, \mathbf{u} \in \mathbb{N}^n,\ \mathbf{x}^{\mathbf{u}} \notin \mathrm{In}_{\prec}(I)\}$.

Lemma 4.33. ([18], prop. 1.1) The standard monomials form a k-vector space basis for $k[\mathbf{x}]/I$.

Proof. We have to show that we can write $k[\mathbf{x}]$ as the direct sum $I \oplus \mathrm{Span}_k(\mathrm{SM}_{\prec}(I))$. The division algorithm writes each $f \in k[\mathbf{x}]$ as $f = f_1 + f_2$ with $f_1 \in I$ and $f_2 \in \mathrm{Span}_k(\mathrm{SM}_{\prec}(I))$ (for details see [2], Ch. 2, §3, theorem 3). For uniqueness assume there exist $f_1, g_1 \in I$ and $f_2, g_2 \in \mathrm{Span}_k(\mathrm{SM}_{\prec}(I))$ such that $f_1 + f_2 = g_1 + g_2$. As $f_1 - g_1 \in I$, also $f_2 - g_2 \in I$. Suppose $f_2 \neq g_2$, i.e. $f_2 - g_2 \neq 0$, then $\mathrm{In}_{\prec}(f_2 - g_2) \in \mathrm{In}_{\prec}(I)$. On the other hand, $f_2 - g_2 \in \mathrm{Span}_k(\mathrm{SM}_{\prec}(I))$, and hence $\mathrm{In}_{\prec}(f_2 - g_2) \in \mathrm{SM}_{\prec}(I)$, which is a contradiction. So $f_2 = g_2$ and then also $f_1 = g_1$ as required. □

Theorem 4.34. ([18], thm. 1.2) For every ideal $I \subset k[\mathbf{x}]$ the set $\mathrm{Mon}\,(I)$ is finite.

Proof. We will prove this by contradiction, so suppose that $\mathrm{Mon}(I)$ is an infinite set. For each monomial ideal M in $\mathrm{Mon}(I)$ let $\prec_M$ be any one particular term order such that $M = \mathrm{In}_{\prec_M}(I)$. Let Σ be the collection of term orders $\{\prec_M \,| M \in \mathrm{Mon}(I)\}$. As we assumed that $\mathrm{Mon}(I)$ is an infinite set, Σ is an infinite set as well.

By the Hilbert basis theorem we can write $I = \langle f_1, ..., f_s\rangle$ for $f_1, ..., f_s \in k[\mathbf{x}]$. Each of these f_j contains only a finite number of terms, so by a pigeonhole principle argument, there exists an infinte subset $\Sigma_1 \subset \Sigma$ such that for all $j \leq s$, $\mathrm{In}_\prec(f_j)$ is the same for all term orders $\prec$ in Σ_1.

Take any term order $\prec$ in Σ_1 and write $M_1 = \langle \mathrm{In}_\prec(f_1), ..., \mathrm{In}_\prec(f_s)\rangle$. If $F := \{f_1, ..., f_s\}$ were a Gröbner basis for I w.r.t. some term order $\prec_1$ in Σ_1, then we claim that F would be a Gröbner basis for I w.r.t. every term order $\prec$ in Σ_1: To prove this, let $\prec$ be any term order in Σ_1 other that $\prec_1$, and let $f \in I$ be arbitrary. Divide f by F using $\prec$ to obtain

$$f = c_1 f_1 + ... + c_s f_s + r, \tag{4.1}$$

where no term of r is divisible by any of the $\mathrm{In}_\prec(f_j)$. However, both $\prec$ and $\prec_1$ are in Σ_1, so $\mathrm{In}_\prec(f_j) = \mathrm{In}_{\prec_1}(f_j)$ for all j. Since $r = f - \sum_{j=1}^{s} c_j f_j$ is in I and F is a Gröbner basis for I w.r.t. $\prec_1$, this implies $r = 0$. Since (4.1) was obtained using the division algorithm, we obtain that $\mathrm{In}_\prec(f)$ is divisible by $\mathrm{In}_\prec(f_j)$. Therefore F is also a Gröbner basis for I w.r.t. $\prec$, which proves the claim.

However this cannot be the case since the original set of term orders $\Sigma \supset \Sigma_1$ was chosen so that the monomial ideals $\mathrm{In}_\prec(I)$ for $\prec$ in Σ were all distinct. Therefore, given any $\prec_1$ in Σ_1 there must be some $f_{s+1} \in I$ such that $\mathrm{In}_{\prec_1}(f_{s+1}) \notin \langle \mathrm{In}_{\prec_1}(f_1), ..., \mathrm{In}_{\prec_1}(f_s)\rangle = M_1$. Replacing f_{s+1} by its remainder on division by $f_1, ..., f_s$ we may assume in fact that no term in f_{s+1} is divisible by any of the generators for M_1.

Next apply the pigeonhole principle again to obtain an infinite subset $\Sigma_2 \subset \Sigma_1$ such that the initial terms of $f_1, ..., f_{s+1}$ are the same for all $\prec$ in Σ_2. Define $M_2 := \langle \mathrm{In}_\prec(f_1), ..., \mathrm{In}_\prec(f_{s+1})\rangle$ for all $\prec$ in Σ_2, and notice that $M_1 \subset M_2$. By the same argument as before we see that $\{f_1, ..., f_{s+1}\}$ can not be a Gröbner basis w.r.t. any of the term orders in Σ_2. So by fixing some $\prec_2$ in Σ_2, we find a polynomial $f_{s+2} \in I$ such that no term of f_{s+2} is divisible by any of the generators for $M_2 = \langle \mathrm{In}_{\prec_2}(f_1), ..., \mathrm{In}_{\prec_2}(f_{s+1})\rangle$.

Continuing in the same way, we end up with a descending chain of infinite subsets $\Sigma \supset \Sigma_1 \supset \Sigma_2 \supset \Sigma_3 \supset \cdots$, and an infinite, strictly ascending chain of monomial ideals $M_1 \subsetneq M_2 \subsetneq M_3 \subsetneq \cdots$. This contradicts the ascending chain condition in $k[\mathbf{x}]$, hence $\mathrm{Mon}(I)$ is a finite set. $\square$

Let k be a field with $\mathrm{char}(k) \neq 2$. If $I_{\mathcal{A}} \subset k[\mathbf{x}]$ is a toric ideal and is homogeneous w.r.t. a positive grading $\mathbf{d} \in \mathbb{N}^n_{>0}$, then we will show there is a bijection between the set $\mathrm{Mon}\,(I_{\mathcal{A}})$ and the set $\mathcal{G}_{\mathrm{red}}\,(I_{\mathcal{A}})$. By theorem 4.34 we then see there exist only finitely many reduced Gröbner bases for a homogeneous toric ideal. We will also see that the number of reduced Gröbner bases coincides with the number of Gröbner cones forming the Gröbner fan. So theorem 4.34 makes sure that for every homogeneous (w.r.t. to a positive grading $\mathbf{d} \in \mathbb{N}^n_{>0}$), toric ideal $I_{\mathcal{A}} \subset k[\mathbf{x}]$, the Gröbner fan of $I_{\mathcal{A}}$ is constructed or formed by only finitely many cones.

Before we can establish this bijection we need a few lemmas:

Proposition 4.35. ([18], prop. 1.8) Let $\omega \in \mathbb{R}^n_{\geq 0}$, $\prec$ a term order, and let $I \subset k[\mathbf{x}]$ be an ideal, then $\mathrm{In}_{\prec}\left(\mathrm{In}_{\omega}\left(I\right)\right) = \mathrm{In}_{\prec_{\omega}}\left(I\right)$.

Proof. " $\subseteq$ ": Let $\mathcal{G}$ be a Gröbner basis of I w.r.t. $\prec_{\omega}$, and define the set $\mathcal{G}' = \{\mathrm{In}_{\omega}(g) | g \in \mathcal{G}\}$. Using the definition of a Gröbner basis we obtain

$$\mathrm{In}_{\prec_{\omega}}(I) = \langle \mathrm{In}_{\prec_{\omega}}(\mathcal{G})\rangle = \left\langle \mathrm{In}_{\prec}(\mathcal{G}')\right\rangle \subseteq \mathrm{In}_{\prec}\left(\mathrm{In}_{\omega}\left(I\right)\right).$$

" $\supseteq$ ": Choose $f \in \mathrm{In}_{\omega}(I)$. As $\mathrm{In}_{\prec}(\mathrm{In}_{\omega}(I)) = \langle \mathrm{In}_{\prec}(p) | p \in \mathrm{In}_{\omega}(I)\rangle$, we can write $f = \sum_{j=1}^{l} g_j \mathrm{In}_{\omega}(h_j)$ for $g_j \in k[\mathbf{x}]$ and $h_j \in I$. We can rewrite f as $f = \sum_{j=1}^{m} \mathrm{In}_{\omega}\left(f_j\right)$ with $f_j \in I$ for all j. Let $s \leq m$ and choose $\{j_1, ..., j_s\}$ such that $\{f_{j_1}, ..., f_{j_s}\} \subset \{f_1, ..., f_m\}$ is the subset of polynomials such that $\mathrm{In}_{\omega}(f_{j_1}), ..., \mathrm{In}_{\omega}(f_{j_s})$ have the same ω-degree as $\mathrm{In}_{\prec}\left(f\right)$ (notice that this subset is non-empty). Then we obtain the equalities:

$$\begin{aligned} \mathrm{In}_{\prec}\left(f\right) &= \mathrm{In}_{\prec}\left(\sum_{j=1}^{m} \mathrm{In}_{\omega}\left(f_j\right)\right) = \mathrm{In}_{\prec}\left(\sum_{i=1}^{s} \mathrm{In}_{\omega}\left(f_{j_i}\right)\right) \\ &= \mathrm{In}_{\prec}\left(\mathrm{In}_{\omega}\left(\sum_{i=1}^{s} f_{j_i}\right)\right) = \mathrm{In}_{\prec_{\omega}}\left(\sum_{i=1}^{s} f_{j_i}\right). \end{aligned}$$

So we obtain that $\mathrm{In}_{\prec}\left(f\right) = \mathrm{In}_{\prec_{\omega}}\left(\sum_{i=1}^{s} f_{j_i}\right) \in \mathrm{In}_{\prec_{\omega}}\left(I\right)$ as $\sum_{i=1}^{s} f_{j_i} \in I$. Then $\mathrm{In}_{\prec}\left(f\right) \in \mathrm{In}_{\prec_{\omega}}\left(I\right)$, implying that $\mathrm{In}_{\prec}(\mathrm{In}_{\omega}(I)) \subset \mathrm{In}_{\prec_{\omega}}(I)$. $\square$

Corollary 4.36. ([18], cor. 1.9) Let $\omega \in \mathbb{R}^n_{\geq 0}$, let $\prec$ be a term order, let $I \subset k[\mathbf{x}]$ be an ideal and let $\mathcal{G}$ be a Gröbner basis of I w.r.t. $\prec_{\omega}$, then $\mathcal{G}' = \{\mathrm{In}_{\omega}(g) | g \in \mathcal{G}\}$ is a Gröbner basis for $\mathrm{In}_{\omega}(I)$ w.r.t. $\prec$.

Proof. We use the same notation as in the proof of proposition 4.35. By this proposition we obtain

$$\mathrm{In}_{\prec}\left(\mathrm{In}_{\omega}\left(I\right)\right)=\left\langle \mathrm{In}_{\prec}\left(\mathcal{G}'\right)\right\rangle =\left\langle \mathrm{In}_{\prec_{\omega}}\left(\mathcal{G}\right)\right\rangle =\mathrm{In}_{\prec_{\omega}}\left(I\right),$$

as required. □

Lemma 4.37. Let $I\subset k[\mathbf{x}]$ be an ideal and let $\prec_1$ and $\prec_2$ be two term orders. If $\mathrm{In}_{\prec_1}(I)\subseteq \mathrm{In}_{\prec_2}(I)$ then $\mathrm{In}_{\prec_1}(I)=\mathrm{In}_{\prec_2}(I)$.

Proof. As $\mathrm{In}_{\prec_1}(I)\subseteq \mathrm{In}_{\prec_2}(I)$ we obtain $\mathrm{SM}_{\prec_2}(I)\subseteq \mathrm{SM}_{\prec_1}(I)$. By lemma 4.33 we see that $\mathrm{SM}_{\prec_1}(I)$ and $\mathrm{SM}_{\prec_2}(I)$ are both vector space bases of $k[\mathbf{x}]/I$. This implies $\mathrm{SM}_{\prec_1}(I)=\mathrm{SM}_{\prec_2}(I)$ and the result follows. □

Lemma 4.38. Let $I\subset k[\mathbf{x}]$ be an ideal and let $\prec_1$ and $\prec_2$ be two arbitrary term orders. Let $\mathcal{G}_{\prec_1}(I)$ and $\mathcal{G}_{\prec_2}(I)$ be two reduced Gröbner bases. Suppose that $\mathrm{In}_{\prec_1}(g)=\mathrm{In}_{\prec_2}(g)$ for all $g\in\mathcal{G}_{\prec_1}(I)$, then $\mathcal{G}_{\prec_1}(I)=\mathcal{G}_{\prec_2}(I)$.

Proof. By our assumption and by the definition of a Gröbner basis we obtain

$$\mathrm{In}_{\prec_1}(I)=\left\langle \mathrm{In}_{\prec_1}\left(\mathcal{G}_{\prec_1}(I)\right)\right\rangle =\left\langle \mathrm{In}_{\prec_2}\left(\mathcal{G}_{\prec_1}(I)\right)\right\rangle \subseteq \mathrm{In}_{\prec_2}(I).$$

Using lemma 4.37 we obtain $\mathrm{In}_{\prec_1}(I)=\mathrm{In}_{\prec_2}(I)$, which implies $\left\langle \mathrm{In}_{\prec_2}\left(\mathcal{G}_{\prec_1}(I)\right)\right\rangle =\mathrm{In}_{\prec_2}(I)$. In other words, $\mathcal{G}_{\prec_1}(I)$ is a reduced Gröbner basis for I w.r.t. $\prec_2$. So $\mathcal{G}_{\prec_1}(I)=\mathcal{G}_{\prec_2}(I)$ by the uniqueness of reduced Gröbner bases. □

Theorem 4.39. ([1] Ch.8 §4, Exercise 4) Let k be a field with $\mathrm{char}(k)\neq 2$. Let $I_{\mathcal{A}}\subset k[\mathbf{x}]$ be a homogeneous (w.r.t. a positive grading $\mathbf{d}\in\mathbb{N}^n$), toric ideal. Then there exists a bijective map

$$\psi:\mathcal{G}_{\mathrm{red}}\left(I_{\mathcal{A}}\right)\to \mathrm{Mon}\left(I_{\mathcal{A}}\right)\quad \text{defined by}\quad \mathcal{G}_{\prec}\left(I_{\mathcal{A}}\right)\overset{\psi}{\mapsto}\mathrm{In}_{\prec}\left(I_{\mathcal{A}}\right),$$

for all term orders $\prec$.

Proof. Choose a term order $\prec$. As I is a toric ideal lemma 4.19 implies that $\mathcal{G}_{\prec}(I_{\mathcal{A}})=\{\mathbf{x}^{\mathbf{u}_1}-\mathbf{x}^{\mathbf{v}_1},...,\mathbf{x}^{\mathbf{u}_n}-\mathbf{x}^{\mathbf{v}_n}\}$, i.e. it is generated by pure binomials. Furthermore
$\mathrm{In}_{\prec}(\mathbf{x}^{\mathbf{u}_j}-\mathbf{x}^{\mathbf{v}_j})=\mathbf{x}^{\mathbf{u}_j}$ for all $1\leq j\leq n$ as $\mathcal{G}_{\prec}(I_{\mathcal{A}})$ is reduced (recall that the coefficient of the initial term of a reduced Gröbner basis is 1, which is not equal to -1 as we assumed that $\mathrm{char}(k)\neq 2$). So we obtain

$$\psi\left(\mathcal{G}_{\prec}\left(I_{\mathcal{A}}\right)\right)=\psi\left(\{\mathbf{x}^{\mathbf{u}_1}-\mathbf{x}^{\mathbf{v}_1},...,\mathbf{x}^{\mathbf{u}_n}-\mathbf{x}^{\mathbf{v}_n}\}\right)=\langle \mathbf{x}^{\mathbf{u}_1},...,\mathbf{x}^{\mathbf{u}_n}\rangle\in \mathrm{Mon}_{\prec}(I).$$

Hence ψ is a well-defined map. We will prove the injectivity and surjectivity of ψ:
Let $\prec$ be a term order, and look at monomial initial ideals $\mathrm{In}_{\omega}(I_{\mathcal{A}})$ for $\omega\in\mathbb{R}^n$. As $I_{\mathcal{A}}$ is homogeneous we can restrict $\omega\in\mathbb{R}^n_{\geq 0}$ using corollary 4.30. This

implies that $\prec_\omega$ is a term order using definition 4.27. By proposition 4.35 we now obtain $\mathrm{In}_\omega(I_\mathcal{A}) = \mathrm{In}_\prec(\mathrm{In}_\omega(I_\mathcal{A})) = \mathrm{In}_{\prec_\omega}(I_\mathcal{A})$. So for every $\mathrm{In}_\omega(I)$ there is a reduced Gröbner basis, namely $\mathcal{G}_{\prec_\omega}(I_\mathcal{A})$, such that $\psi(\mathcal{G}_{\prec_\omega}(I_\mathcal{A})) = \mathrm{In}_{\prec_\omega}(I_\mathcal{A}) = \mathrm{In}_\omega(I_\mathcal{A})$, implying that ψ is a surjective map.
Next assume that $\prec_1$ and $\prec_2$ are two term orders such that $\psi(\mathcal{G}_{\prec_1}(I_\mathcal{A})) = \psi(\mathcal{G}_{\prec_2}(I_\mathcal{A}))$, i.e. $\mathrm{In}_{\prec_1}(I_\mathcal{A}) = \mathrm{In}_{\prec_2}(I_\mathcal{A})$. Choose some $g \in \mathcal{G}_{\prec_1}(I_\mathcal{A})$. As $\mathcal{G}_{\prec_1}(I_\mathcal{A})$ is a reduced Gröbner basis only one of the monomials in g, namely $\mathrm{In}_{\prec_1}(g)$, is in $\mathrm{In}_{\prec_1}(I_\mathcal{A}) = \mathrm{In}_{\prec_2}(I_\mathcal{A})$ (this follows directly from the definition of a reduced Gröbner basis). On the other hand, as $g \in I_\mathcal{A}$ also $\mathrm{In}_{\prec_2}(g) \in \mathrm{In}_{\prec_2}(I_\mathcal{A}) = \mathrm{In}_{\prec_1}(I_\mathcal{A})$. So we can conclude that $\mathrm{In}_{\prec_1}(g) = \mathrm{In}_{\prec_2}(g)$. Then by lemma 4.38 also $\mathcal{G}_{\prec_1}(I_\mathcal{A}) = \mathcal{G}_{\prec_2}(I_\mathcal{A})$, implying that ψ is an injective map. □

4.5 Gröbner cones

In this subsection we define the Gröbner cones for a toric ideal $I_\mathcal{A}$, following [18].

Definition 4.40. Fix some ideal $I \subset k[\mathbf{x}]$ and define an equivalence relation $\sim$ between two weight vectors $\omega, \nu \in \mathbb{R}^n$ by defining $\omega \sim \nu$ if and only if $\mathrm{In}_\omega(I) = \mathrm{In}_\nu(I)$. The equivalence classes under this relation will be denoted by $C_\omega(I)$. The closure of such an equivalence class is called a Gröbner cone (see corollary 4.42).

Proposition 4.41. Let $\prec$ be a term order and $I \subset k[\mathbf{x}]$ an ideal. Choose some $\omega \in \mathbb{R}^n_{\geq 0}$ and let $\mathcal{G}_{\prec_\omega}(I)$ be the reduced Gröbner basis of I w.r.t $\prec_\omega$. Then

$$\begin{aligned} \mathcal{J} &:= \left\{\nu \in \mathbb{R}^n_{\geq 0} \,|\mathrm{In}_\nu(I) = \mathrm{In}_\omega(I)\right\} \\ &= \left\{\nu \in \mathbb{R}^n_{\geq 0} \,|\mathrm{In}_\nu(g) = \mathrm{In}_\omega(g) \;\; \forall g \in \mathcal{G}_{\prec_\omega}(I)\right\} =: \mathcal{K}. \end{aligned}$$

Proof. We will prove $\mathcal{J} = \mathcal{K}$ by showing both inclusions, starting with " $\subseteq$ ": Choose a $\nu \in \mathcal{J}$, then $\mathrm{In}_\nu(I) = \mathrm{In}_\omega(I)$, which implies (using proposition 4.35)

$$\mathrm{In}_{\prec_\nu}(I) = \mathrm{In}_\prec(\mathrm{In}_\nu(I)) = \mathrm{In}_\prec(\mathrm{In}_\omega(I)) = \mathrm{In}_{\prec_\omega}(I).$$

Choose an element g in the reduced Gröbner basis $\mathcal{G}_{\prec_\omega}(I)$. As $\mathcal{G}_{\prec_\omega}(I)$ is reduced only one term of g (namely $\mathrm{In}_{\prec_\omega}(g)$) belongs to $\mathrm{In}_{\prec_\omega}(I) = \mathrm{In}_{\prec_\nu}(I)$. Furthermore $\mathrm{In}_{\prec_\nu}(g) \in \mathrm{In}_{\prec_\nu}(I)$ as $g \in I$. Hence $\mathrm{In}_{\prec_\omega}(g) = \mathrm{In}_{\prec_\nu}(g)$. As g was chosen arbitrarily we obtain $\mathcal{G}_{\prec_\omega}(I) = \mathcal{G}_{\prec_\nu}(I)$ using lemma 4.38. Next fix a $g \in \mathcal{G}_{\prec_\omega}(I) = \mathcal{G}_{\prec_\nu}(I)$. We can write $\mathrm{In}_\nu(g) = \mathrm{In}_{\prec_\nu}(g) + h$ and $\mathrm{In}_\omega(g) = \mathrm{In}_{\prec_\omega}(g) + h'$ with h, h' k-linear combinations of standard monomials. Observe that

$$\begin{aligned} h - h' &= \mathrm{In}_\nu(g) - \mathrm{In}_{\prec_\nu}(g) - \mathrm{In}_\omega(g) + \mathrm{In}_{\prec_\omega}(g) \\ &= \mathrm{In}_\nu(g) - \mathrm{In}_\omega(g) \in \mathrm{In}_\nu(I) = \mathrm{In}_\omega(I). \end{aligned}$$

If $h-h' \neq 0$, then by proposition 4.35 $\mathrm{In}_{\prec}(h-h') \in \mathrm{In}_{\prec}(\mathrm{In}_{\nu}(I)) = \mathrm{In}_{\prec_\nu}(I)$, which contradicts the choice of h and of h'. So $h-h'=0$, hence $\mathrm{In}_{\nu}(g) = \mathrm{In}_{\omega}(g)$. This implies $\nu \in \mathcal{K}$, as required.
Next we prove the other inclusion " $\supseteq$ ": Choose a $\nu \in \mathcal{K}$, then by proposition 4.35

$$\mathrm{In}_{\prec_\nu}(g) = \mathrm{In}_{\prec}(\mathrm{In}_{\nu}(g)) = \mathrm{In}_{\prec}(\mathrm{In}_{\omega}(g)) = \mathrm{In}_{\prec_\omega}(g),$$

for all $g \in \mathcal{G}_{\prec_\omega}(I)$. This means that $\mathcal{G}_{\prec_\nu}(I) = \mathcal{G}_{\prec_\omega}(I)$ by lemma 4.38. Using proposition 4.36 twice we obtain

$$\begin{aligned}\mathcal{G}_{\prec}(\mathrm{In}_{\nu}(I)) &= \{\mathrm{In}_{\nu}(g) \,|\, g \in \mathcal{G}_{\prec_\nu}(I)\} \\ &= \{\mathrm{In}_{\omega}(g) \,|\, g \in \mathcal{G}_{\prec_\omega}(I)\} \\ &= \mathcal{G}_{\prec}(\mathrm{In}_{\omega}(I)).\end{aligned}$$

This implies $\mathrm{In}_{\nu}(I) = \mathrm{In}_{\omega}(I)$, so $\nu \in \mathcal{J}$. □

Suppose that $I \subset k[\mathbf{x}]$ is a $\mathbf{d}$-homogeneous ideal, where the grading $\mathbf{d}$ is positive. Using corollary 4.30 we see that we do not have to restrict ourselves to positive vectors in proposition 4.41, but can take arbitrary vectors in $\mathbb{R}^n$. As a direct consequence of proposition 4.41 we have the following corollary:

Corollary 4.42. Let $I \subset k[\mathbf{x}]$ be an ideal. Choose some $\omega \in \mathbb{R}^n$ and let $C_\omega(I)$ be the equivalence class containing ω, then the closure of $C_\omega(I)$ denoted by $\overline{C_\omega(I)}$ is a closed, convex polyhedral cone in $\mathbb{R}^n$.

Proof. Let $\prec$ be a term order, and for $\omega \in \mathbb{R}^n_{\geq 0}$ let $\mathcal{G}_{\prec_\omega}(I)$ be the reduced Gröbner basis for I w.r.t. $\prec_\omega$. Using proposition 4.41 we see that

$$\begin{aligned}C_\omega(I) &= \{\nu \in \mathbb{R}^n_{\geq 0} | \mathrm{In}_\nu(I) = \mathrm{In}_\omega(I)\} \\ &= \{\nu \in \mathbb{R}^n_{\geq 0} | \mathrm{In}_\nu(g) = \mathrm{In}_\omega(g),\ \forall g \in \mathcal{G}_{\prec_\omega}(I)\}\end{aligned}$$

In other words $C_\omega(I)$ can be described by a finite set of equations and a finite set of inequalities: The equations are of the form $\nu \cdot \mathbf{u} = \nu \cdot \mathbf{v}$, where $\mathbf{x}^{\mathbf{u}}$ and $\mathbf{x}^{\mathbf{v}}$ run over the terms in $\mathrm{In}_\omega(g)$ for $g \in \mathcal{G}_{\prec_\omega}(I)$. The inequalities are of the form $\nu \cdot \mathbf{u} > \nu \cdot \mathbf{w}$, where $\mathbf{x}^{\mathbf{u}}$ runs over the terms in $\mathrm{In}_\omega(g)$ and $\mathbf{x}^{\mathbf{w}}$ runs over the terms in $g \setminus \mathrm{In}_\omega(g)$ for $g \in \mathcal{G}_{\prec_\omega}(I)$. As an equivalence class is never empty, we obtain that $\overline{C_\omega(I)}$ is a closed, convex, polyhedral cone in $\mathbb{R}^n$. □

4.6 The Gröbner fan and State polytope

In this section we will define the Gröbner fan of a homogeneous ideal $I \subset k[\mathbf{x}]$. Such a fan will be complete (as I is homogeneous) and consequently there exists a polytope (called the State polytope) whose outer normal fan is the Gröbner fan.

Definition 4.43. An open halfspace is a set of points $\mathcal{H}^{<}_{\mathbf{d},\delta} := \{\mathbf{x} \in \mathbb{R}^n \,|\mathbf{d} \cdot \mathbf{x} < \delta\}$ or $\mathcal{H}^{>}_{\mathbf{d},\delta} := \{\mathbf{x} \in \mathbb{R}^n \,|\mathbf{d} \cdot \mathbf{x} > \delta\}$, where $\mathbf{d} \in \mathbb{R}^n \setminus \{0\}$ and $\delta \in \mathbb{R}$. Similarly we can define $\mathcal{H}^{\leq}_{\mathbf{d},\delta} := \{\mathbf{x} \in \mathbb{R}^n \,|\mathbf{d} \cdot \mathbf{x} \leq \delta\}$ and $\mathcal{H}^{\geq}_{\mathbf{d},\delta} := \{\mathbf{x} \in \mathbb{R}^n \,|\mathbf{d} \cdot \mathbf{x} \geq \delta\}$. We define a hyperplane as

$$\mathcal{H}_{\mathbf{d},\delta} := \mathcal{H}^{\geq}_{\mathbf{d},\delta} \cap \mathcal{H}^{\leq}_{\mathbf{d},\delta} = \{\mathbf{x} \in \mathbb{R}^n \,|\mathbf{d} \cdot \mathbf{x} = \delta\}.$$

Definition 4.44. A polyhedron P is a finite intersection of closed half-spaces in $\mathbb{R}^n$, i.e. $P = \bigcap_{j=1}^m \mathcal{H}^{\geq}_{\mathbf{d}_j,\delta_j}$. A bounded polyhedron is called a polytope. If $\delta_j = 0,\ \forall j$ then P is called a (polyhedral) cone.

Proposition 4.45. Every polytope can be written as the convex hull of a finite set of points, called the vertices of the polytope.

Definition 4.46. Let P be a polyhedron, then $P \cap \mathcal{H}_{\mathbf{d},\delta}$ is called a face of P if $P \subseteq \mathcal{H}^{\geq}_{\mathbf{d},\delta}$.

Another definition of a face is by maximizing some linear functional:

Definition 4.47. Let $P \subset \mathbb{R}^n$ be a polyhedron, $\omega \in \mathbb{R}^n$. A face of P is defined as

$$\mathrm{face}_\omega(P) := \{\mathbf{u} \in P \,|\omega \cdot \mathbf{u} \geq \omega \cdot \mathbf{v}, \forall \mathbf{v} \in P\}.$$

This definition is equivalent to the previous definition by the following proposition:

Proposition 4.48. Let $P \subset \mathbb{R}^n$ be a polyhedron, then $\mathrm{face}_\omega(P)$ is a face of P for all $\omega \in \mathbb{R}^n$. And if $F \subseteq P$ is a non-empty face of P, then there exists a $\omega \in \mathbb{R}^n$ such that $F = \mathrm{face}_\omega(P)$.

Proof. Assume $\omega \neq \mathbf{0}$ and $\mathrm{face}_\omega(P) \neq \emptyset$ (these two cases are trivial). Then ω attains its maximum m on P. As ω is constant on $\mathrm{face}_\omega(P)$ we see that $\mathrm{face}_\omega(P) \subseteq P \cap \mathcal{H}_{\omega,m} = P \cap \mathcal{H}_{-\omega,-m}$. As ω attains its maximum m on $\mathcal{H}_{\omega,m}$ we obtain $\mathrm{face}_\omega(P) \supseteq P \cap \mathcal{H}_{\omega,m} = P \cap \mathcal{H}_{-\omega,-m}$. So we see that $\mathrm{face}_\omega(P) = P \cap \mathcal{H}_{\omega,m} = P \cap \mathcal{H}_{-\omega,-m}$. As $P \subset \mathcal{H}^{\geq}_{-\omega,-m}$ we conclude that $\mathrm{face}_\omega(P)$ is a face of P.

Let $F \subseteq P$ be a non-empty face, and assume that $F \neq P$ (if $F = P$ then $F = \mathrm{face}_{\mathbf{0}}(P)$). Then there exist ω and m such that $F = P \cap \mathcal{H}_{\omega,m}$ and $P \subseteq \mathcal{H}^{\geq}_{\omega,m}$. As $F \neq \emptyset$ we know that $-\omega$ attains its maximum $-m$ on P. Both $\mathrm{face}_{-\omega}(P)$ and $F = P \cap \mathcal{H}_{\omega,m}$ are precisely the elements attaining this maximum. $\square$

Definition 4.49. If $P \subset \mathbb{R}^n$ is a polyhedron and F a face of P, then the *normal cone* of F at P is

$$\mathcal{N}_P(F) = \{\omega \in \mathbb{R}^n \,|\mathrm{face}_\omega(P) = F\}.$$

Compare this to proposition 2.24.

Definition 4.50. The Minkowski sum of two polyhedra P_1 and P_2 is defined as

$$P_1 \oplus P_2 := \{p_1 + p_2 \,|p_1 \in P_1, p_2 \in P_2\}.$$

Example 4.51. An example of Minkowski addition: Let $u_1, ..., u_5, v_1, ..., v_6 \in \mathbb{C}^*$ and define $f(x, y) := u_1x^{-1}y^{-1} + u_2y^{-1} + u_3x^{-1} + u_4 + u_5xy$ for $x, y \in \mathbb{C}^*$. Note that $f(x, y)$ corresponds to the Newton polytope of this thesis. Also define $g(x, y) := v_1 + v_2xy^{-1} + v_3y^{-1} + v_4x^{-1}y^{-1} + v_5x^{-1} + v_6y$ for $x, y \in \mathbb{C}^*$. Then $g(x, y)$ corresponds to the Secondary polytope of this thesis. We can write

$$f \cdot g = \sum_{i,j \in \{-2,...,2\} \subset \mathbb{Z}} w_{i,j}x^iy^j,$$

where $w_{i,j} = 0$ if

$$(i, j) \in \{(-2, 0), (-2, 1), (-2, 2), (-1, 2), (0, 2), (2, 2), (2, 1), (2, -1), (2, -2)\},$$

and $w_{i,j}$ is non-zero otherwise. The Newton polytope over this product is:

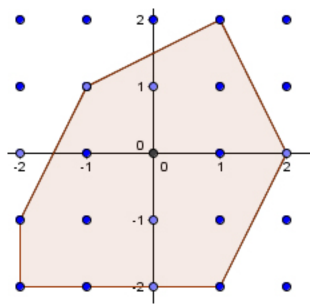


Figure 4.1: The polytope New$(f \cdot g)$.

We perform a Minkowski addition:

$$\begin{aligned} & \text{New}\,(f) \oplus \text{New}\,(g) \\ = \; & \text{Conv}\,((-1, -1), (0, -1), (-1, 0), (1, 1)) \\ \oplus \; & \text{Conv}\,((1, -1), (0, -1), (-1, -1), (-1, 0), (0, 1)) \\ = \; & \text{Conv}\,((1, 2), (2, 0), (1, -2), (-2, -2), (-2, -1), (-1, 1)), \end{aligned}$$

This convex hull corresponds precisely to the polytope pictured above. So we can conclude that New$(f \cdot g) =$ New$(f) \oplus$ New(g). Remark that this relation holds for any two polynomials $f, g \in k[\mathbf{x}]$ (see [18], lemma 2.2).

The next proposition gives a geometric reformulation of the definition of Gröbner cones:

Proposition 4.52. ([18], page 13) Let k be a field, let $I \subset k[\mathbf{x}]$ be an ideal, choose a term order $\prec$ and choose any $\omega \in \mathbb{R}^n$. Let $\mathcal{G}_{\prec}(I)$ be the reduced Gröbner basis for this term order and let $\mathcal{C}_{\omega}(I)$ be an open Gröbner cone, i.e.

$$\mathcal{C}_{\omega}(I) := \left\{\omega' \in \mathbb{R}^n \,|\mathrm{In}_{\omega'}(g) = \mathrm{In}_{\omega}(g), \text{for all } g \in \mathcal{G}_{\prec}(I)\right\}.$$

Define the polytope $Q := \mathrm{New}\left(\prod_{g \in \mathcal{G}_{\prec}(I)} g\right) = \bigoplus_{g \in \mathcal{G}_{\prec}(I)} \mathrm{New}(g)$, then we can write

$$C_{\omega}(I) = \mathcal{N}_Q\left(\mathrm{face}_{\omega}(Q)\right).$$

Definition 4.53. A complex $\mathcal{F}$ is a finite collection of polyhedra in $\mathbb{R}^n$ such that the following two conditions are satisfied:

1. If $P \in \mathcal{F}$ and F is a face of P, then $F \in \mathcal{F}$.
2. If $P_1, P_2 \in \mathcal{F}$, then $P_1 \cap P_2$ is a face of both P_1 and P_2.

We call a complex consisting solely of cones a *fan*, and say that a fan is complete if $\mathrm{supp}(\mathcal{F}) := \cup_{\mathcal{C} \in \mathcal{F}} \mathcal{C} = \mathbb{R}^n$.

Definition 4.54. Let $I \subset k[\mathbf{x}]$ be a homogeneous ideal. We define the Gröbner fan $\mathrm{GF}(I)$ to be the following set of closed cones:

$$\mathrm{GF}(I) = \left\{\overline{C_{\omega}(I)}, \omega \in \mathbb{R}^n\right\}.$$

Proposition 4.55. ([18], prop. 2.4) The Gröbner fan $\mathrm{GF}(I)$ is a fan.

Theorem 4.56. ([18], thm. 2.5) Let $I \subset k[\mathbf{x}]$ be a homogeneous ideal, then there exists a polytope $\mathrm{State}(I) \subset \mathbb{R}^n$ whose normal fan $\mathcal{F}_{\mathrm{State}(I)}$ coincides with $\mathrm{GF}(I)$.

We describe the construction of the *State polytope* $\mathrm{State}(I)$ for a homogeneous ideal $I \subset k[\mathbf{x}]$: Let I_d be the vector space of homogeneous polynomials of degree d in I. Define

$$\sum \mathrm{In}_{\prec}(I)_d := \sum_{\substack{\mathbf{u} \in \mathbb{N}^n \text{ s.t. } \deg(\mathbf{x}^{\mathbf{u}}) = d, \\ \text{and } \mathbf{x}^{\mathbf{u}} \in \mathrm{In}_{\prec}(I)}} \mathbf{u}.$$

Next define

$$\mathrm{State}(I)_d := \mathrm{Conv}\left\{\sum \mathrm{In}_{\prec}(I)_d \,\middle|\, \prec \text{ any term order}\right\}.$$

Let D be the largest degree of any element in the union of all reduced Gröbner bases of I, then the state polytope is defined as the following Minkowski sum:

$$\mathrm{State}(I) := \bigoplus_{d=1}^{D} \mathrm{State}(I)_d.$$

Easy examples of Gröbner fans can be constructed by the following proposition:

Proposition 4.57. Let k be a field and $f \in k[\mathbf{x}]$ a homogeneous polynomial. Let $I \subseteq k[\mathbf{x}]$ be the principal ideal $I := \langle f \rangle$. Then the Newton polytope $\mathrm{New}(f)$ is a state polytope for I.

Proof. Notice that $\mathcal{G}_{\prec}(I) = \{f\}$ for all term orders $\prec$. So we obtain by proposition 4.52 that $\mathcal{C}_{\omega}(I) = \mathcal{N}_{\mathrm{New}(f)}\left(\mathrm{face}_{\omega}\left(\mathrm{New}(f)\right)\right)$. In other words the Gröbner cones are the normal cones of the Newton polytope $\mathrm{New}(f)$, as required. □

Example 4.58. Proposition 4.57 allows us to compute some easy Gröbner fans: Define $f(x,y) := u_1x^{-1}y^{-1} + u_2y^{-1} + u_3x^{-1} + u_4 + u_5xy \in \mathbb{C}\left[x^{\pm 1}, y^{\pm 1}\right]$ and define $g(x,y) = v_1x^{-1}y^{-1} + v_2y^{-1} + v_3xy^{-1} + v_4x^{-1} + v_5 + v_6y \in \mathbb{C}\left[x^{\pm 1}, y^{\pm 1}\right]$. Then by proposition 4.57 we obtain that $\mathrm{GF}\left(\langle f \rangle\right) = \mathcal{F}_{\mathrm{New}(f)}$ (this is the outer normal fan to the polytope in figure 3.2) and $\mathrm{GF}\left(\langle g \rangle\right) = \mathcal{F}_{\mathrm{New}(g)}$ (this is precisely the Secondary fan in figure 3.7).

4.7 The Gröbner fan refines the Secondary fan

In this section we prove that the Gröbner fan of a toric ideal refines the Secondary fan of this ideal. Chapter 8 of [18] gives a proof using techniques from linear programming (see [18] thm. 8.3). In article [17] we find a different proof, which we will present here. The difference between both proofs depends on what definition of an initial complex is used.

Definition 4.59. A simplicial complex Δ is a finite collection of simplices in $\mathbb{R}^n$ such that

- If $\mathcal{T} \in \Delta$ and F is a face of $\mathcal{T}$, then $F \in \Delta$.
- If $\mathcal{T}_1, \mathcal{T}_2 \in \Delta$, then $\mathcal{T}_1 \cap \mathcal{T}_2$ is a face of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Definition 4.60. Let $I \subset k[\mathbf{x}]$ be an ideal, $\omega \in \mathbb{R}^n$ sufficiently generic. The initial complex of I w.r.t. ω, denoted by $\Delta_{\omega}(I)$, is a simplicial complex on the vertex set $\{1, 2, ..., n\}$ defined by the following rule: $F \subset \{1, ..., n\}$ is a face of $\Delta_{\omega}(I)$ if there does not exist a $f \in I$ such that $\mathrm{In}_{\omega}(f)$ uses only the variables $\{x_i \,|\, i \in F\}$.

Definition 4.61. Let R be a commutative ring, $I \subset R$ an ideal, then the radical of I is defined as

$$\sqrt{I} := \{r \in R \,|\, r^n \in I \text{ for some } n \in \mathbb{Z}_{>0}\}.$$

We will mostly be interested in radicals of monomial ideals, this radical can be interpreted as removing all powers of the monomials in the ideal.

Definition 4.62. A monomial $\mathbf{x}^{\mathbf{u}} \in k[\mathbf{x}]$ is squarefree if $u_j \in \{0,1\}$ for all $1 \leq j \leq n$.

Definition 4.63. A non-face of a simplicial complex Δ is a subset of the vertices of Δ that is not a face itself. A non-face is called minimal if all its proper subsets are faces of Δ.

Definition 4.64. Let the subset $F \subseteq \{1, ..., n\}$ be identified with its square-free vector in $\mathbf{u} = (u_1, ..., u_n) \in \{0,1\}^n$ which has entry $u_j = 1$ if $j \in F$ and $u_j = 0$ if $j \notin F$. So we can write $\mathbf{x}^F := \prod_{j \in F} \mathbf{x}_j$. The Stanley-Reisner ideal of the simplicial complex Δ is defined as the square-free monomial ideal $I_\Delta := \langle \mathbf{x}^F \,|F \notin \Delta \rangle$, i.e. generated by monomials corresponding to non-faces F of Δ.

Proposition 4.65. Let $I \subset k[\mathbf{x}]$ be an ideal, $\omega \in \mathbb{R}^n$ sufficiently generic and Δ a simplicial complex. Then the following definitions of an initial complex $\Delta_\omega(I)$ are equivalent:

1. $\Delta_\omega(I)$ is the simplicial complex Δ whose Stanley-Reisner ideal equals the radical $\sqrt{\mathrm{In}_\omega(I)}$.

2. $\Delta_\omega(I)$ consists of those faces $F \subseteq \{1, ..., n\}$ for which there is no polynomial $f \in I$ whose initial monomial $\mathrm{In}_\omega(f)$ uses only the variables $\{x_j \,|j \in F\}$.

The surprising fact is that for any toric ideal $I_\mathcal{A} \subset k[\mathbf{x}]$ and any $\omega \in \mathbb{R}^n$ sufficiently generic, the initial complex $\Delta_\omega(I)$ equals the regular triangulation Δ_ω. This will be proven in the upcoming theorem (following [17]).

Lemma 4.66. Let Δ_ω be a regular triangulation of $\mathcal{A}$, $\sigma = \{i_1, ..., i_k\}$ a face of Δ_ω and $\tau = \{j_1, ..., j_l\}$ a non-face of Δ_ω such that

$$\mathrm{relint}(\mathrm{Conv}(\tau)) \cap \mathrm{relint}(\mathrm{Conv}(\sigma)) \neq \emptyset,$$

with 'relint' I mean the relative interior of the convex hull. Choose $\lambda_1, ..., \lambda_k, \mu_1, ..., \mu_l \in \mathbb{Q}_{>0}$ such that $\sum_{r=1}^k \lambda_r \mathbf{a}_{i_r} = \sum_{s=1}^l \mu_s \mathbf{a}_{j_s}$ and $\sum_{r=1}^k \lambda_r = \sum_{s=1}^l \mu_s = 1$. Then $\sum_{j=1}^k \omega_j \lambda_j < \sum_{j=1}^l \omega_j \mu_j$.

Proof. The facets of Δ_ω are precisely the lower facets of $\mathrm{Conv}(\mathcal{A}^\omega)$, so σ lifts to a lower face of $\mathrm{Conv}(\mathcal{A}^\omega)$ where for each $i \in \sigma$, $\mathbf{a}_i \in \mathcal{A}$ is lifted to height ω_i. So the convex combination $\sum_{r=1}^k \lambda_r \mathbf{a}_{i_r}$ will be lifted to height $\sum_{r \in \sigma} \lambda_r \omega_r$ since the lifted points indexed by σ span an affine plane. But the convex hull of the lifted points indexed by τ do not form a lower face of $\mathrm{Conv}(\mathcal{A}^\omega)$, implying that $\sum_{j=1}^l \omega_j \mu_j > \sum_{j=1}^k \omega_j \lambda_j$, as required. $\square$

Proposition 4.67. Let $I_\mathcal{A}$ be a toric ideal, $\omega \in \mathbb{R}^n$ sufficiently generic. Then $\sqrt{\mathrm{In}_\omega(I_\mathcal{A})}$ equals the Stanley-Reisner ideal $(I_\mathcal{A})_{\Delta_\omega}$ of the regular triangulation Δ_ω of $\mathcal{A}$.

Proof. "$\subseteq$": Let $\tau = \{j_1, ..., j_l\}$ be a non-face of Δ_ω, i.e. $\prod_{s=1}^{l} x_{j_s} \in (I_\mathcal{A})_{\Delta_\omega}$. Then there is a face $\sigma = \{i_1, ..., i_k\}$ of Δ_ω such that

$$\mathrm{relint}(\mathrm{Conv}(\mathbf{a}_{i_1}, ..., \mathbf{a}_{i_k})) \cap \mathrm{relint}(\mathrm{Conv}(\mathbf{a}_{j_1}, ..., \mathbf{a}_{j_l})) \neq \emptyset.$$

So pick $\tilde{\lambda}_1, ..., \tilde{\lambda}_k, \tilde{\mu}_1, ..., \tilde{\mu}_l \in \mathbb{Q}_{>0}$ with $\sum_{j=1}^{k} \tilde{\lambda}_j = \sum_{j=1}^{l} \tilde{\mu}_j = 1$ such that $\sum_{r=1}^{k} \tilde{\lambda}_r \mathbf{a}_{i_r} = \sum_{s=1}^{l} \tilde{\mu}_s \mathbf{a}_{j_s}$. By clearing denominators we obtain $\lambda, \mu \in \mathbb{N}^n$ such that

$$\mathcal{A}\lambda = \sum_{r=1}^{k} \lambda_r \mathbf{a}_{i_r} = \sum_{s=1}^{l} \mu_s \mathbf{a}_{j_s} = \mathcal{A}\mu.$$

Hence $\mathcal{A}(\lambda - \mu) = \mathbf{0}$ and so $\pm(\lambda - \mu) \in \ker_\mathbb{Z}(\mathcal{A})$, i.e. $\pm(\mathbf{x}^\lambda - \mathbf{x}^\mu) \in I_\mathcal{A}$. By lemma 4.66 we know that $\omega \cdot \lambda < \omega \cdot \mu$, so $\mathrm{In}_\omega(\mathbf{x}^\lambda - \mathbf{x}^\mu) = -\mathbf{x}^\mu$. Therefore $\mathbf{x}^\mu \in \mathrm{In}_\omega(I_\mathcal{A})$ and then automatically $\mathbf{x}^\mu = \prod_{s=1}^{l} x_{j_s} \in \sqrt{\mathrm{In}_\omega(I_\mathcal{A})}$. As τ was an arbitrary non-face of Δ_ω we obtain $(I_\mathcal{A})_{\Delta_\omega} \subseteq \sqrt{\mathrm{In}_\omega(I_\mathcal{A})}$.
"$\supseteq$": We prove this by contradiction, suppose $\sqrt{\mathrm{In}_\omega(I_\mathcal{A})}$ is not contained in $(I_\mathcal{A})_{\Delta_\omega}$. Then there is a monomial $\mathbf{x}^\mu \in \mathrm{In}_\omega(I_\mathcal{A})$ with $\mathrm{supp}(\mathbf{x}^\mu)$ a face of Δ_ω. As $I_\mathcal{A}$ is generated as a k-vector space by binomials $\mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-}$ with $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \ker_\mathbb{Z}(\mathcal{A})$, there is a binomial $\mathbf{x}^\mu - \mathbf{x}^\lambda \in I_\mathcal{A}$ with $\mathrm{In}_\omega(\mathbf{x}^\mu - \mathbf{x}^\lambda) = \mathbf{x}^\mu$ and $\mathrm{supp}(\mathbf{x}^\mu)$ a face of Δ_ω. Therefore $\omega \cdot \mu > \omega \cdot \lambda$, and $\mathcal{A}\lambda = \mathcal{A}\mu$ which means that $\mathrm{supp}(\mathbf{x}^\lambda)$ is a non-face of Δ_ω (otherwise we have two faces intersecting in their relative interiors). This contradicts lemma 4.66, so $\sqrt{\mathrm{In}_\omega(I_\mathcal{A})} \subseteq (I_\mathcal{A})_{\Delta_\omega}$. □

Theorem 4.68. ([17], thm. 3.1) Let $I_\mathcal{A}$ be a toric ideal, $\omega \in \mathbb{R}^n$ sufficiently generic. Then the initial complex $\Delta_\omega(I_\mathcal{A})$ is the regular triangulation Δ_ω of $\mathcal{A}$.

Proof. This follows directly from proposition 4.67 by observing that $\Delta_\omega(I)$ is the simplicial complex Δ such that $(I_\mathcal{A})_{\Delta_\omega} = \sqrt{\mathrm{In}_\omega(I_\mathcal{A})}$. □

Corollary 4.69. Let $\mathcal{A} \subset \mathbb{Z}^d$ be an n-point configuration. The Gröbner fan of a toric ideal $I_\mathcal{A}$ is a refinement of the Secondary fan for this ideal.

Proof. Suppose that $\omega, \omega' \in \mathbb{R}^n$ are sufficiently generic and such that $\mathrm{In}_\omega(I_\mathcal{A}) = \mathrm{In}_{\omega'}(I_\mathcal{A})$. Then we have the following implications:

$$\begin{aligned} \mathrm{In}_\omega(I_\mathcal{A}) = \mathrm{In}_{\omega'}(I_\mathcal{A}) \quad &\Rightarrow \quad \sqrt{\mathrm{In}_\omega(I_\mathcal{A})} = \sqrt{\mathrm{In}_{\omega'}(I_\mathcal{A})} \\ &\Rightarrow_{\text{thm. 4.68}} \quad I_{\Delta_\omega} = I_{\Delta_{\omega'}} \\ &\Rightarrow \quad \Delta_\omega = \Delta_{\omega'}. \end{aligned}$$

□

Remark 4.70. Let $I_\mathcal{A} \subset k[\mathbf{x}]$ be a toric ideal. If for generic $\omega, \omega' \in \mathbb{R}^n$ we have $\mathrm{In}_\omega(I_\mathcal{A}) = \mathrm{In}_{\omega'}(I_\mathcal{A})$, then automatically $\sqrt{\mathrm{In}_\omega(I_\mathcal{A})} = \sqrt{\mathrm{In}_{\omega'}(I_\mathcal{A})}$. The reverse implication is only true if the initial ideals $\mathrm{In}_\omega(I_\mathcal{A})$ and $\mathrm{In}_{\omega'}(I_\mathcal{A})$ are square-free. This leads to a correspondence (in some literature called the Sturmfels' correspondence), which states that if all initial ideals of a toric ideal $I_\mathcal{A}$ are square-free then the Gröbner fan and the Secondary fan coincide (see [18], remark 8.10 and prop. 8.15).

4.8 Computing an explicit Gröbner fan

In this subsection we compute an explicit Gröbner fan of a homogeneous ideal using the Gröbner walk algorithm. The Gröbner fan of a homogeneous ideal $I \subset \mathbb{R}^n$ has support equal to $\mathbb{R}^n$, i.e. the Gröbner cones span $\mathbb{R}^n$.

Example 4.71. We start with the ideal $I_\mathcal{A} = \left\langle x_1x_5 - x_4^2, x_1x_4 - x_2x_3 \right\rangle$. In example 4.18 we used the Buchberger algorithm to find a reduced Gröbner basis

$$\mathcal{G}_1 := \mathcal{G}_{\prec_{lex}}(I) = \left\{x_4^2 - x_1x_5, x_1x_4 - x_2x_3, x_2x_3x_4 - x_1^2x_5, x_1^3x_5 - x_2^2x_3^2\right\}$$

for the lexicographic term order with $x_4 \succ x_1 \succ x_3 \succ x_2 \succ x_5$.

Note that the first terms in each of the binomials in $\mathcal{G}_1$ are the initial terms w.r.t. the ordering chosen. Gröbner basis with marked initial terms (some books underline the initial terms) are called marked Gröbner bases. All Gröbner bases presented in this section consist of binomials and will thus be marked by writing the initial term of a binomial with a $+1$ coefficient.

As the Gröbner fan refines the Secondary fan, we know that without loss of generality we can assume that each vector $\omega \in \mathrm{GF}(I)$ is of the form $\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5$. This way the Gröbner cone corresponding to the reduced Gröbner basis $\mathcal{G}_1$ can be given by:

$$\begin{aligned} \overline{\mathcal{C}}_1 &= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\omega_1 \leq 2\omega_4, \omega_1 + \omega_4 \geq 0, \omega_4 \geq 2\omega_1, 3\omega_1 \geq 0\right\} \\ &= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\omega_4 \geq 2\omega_1, \omega_1 \geq 0\right\}. \end{aligned}$$

This Gröbner cone $\mathcal{C}_1$ looks like:

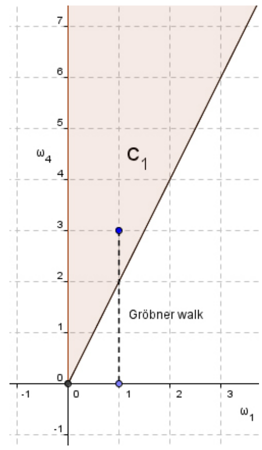


Figure 4.2: Gröbner cone $\overline{\mathcal{C}}_1$ and the Gröbner walk we perform below.

There exists a Gröbner basis conversion algorithm, known as the Gröbner walk to convert one reduced Gröbner basis into another one. Using this algorithm we can compute every reduced Gröbner basis corresponding to a homogeneous ideal in $k[\mathbf{x}]$ and henceforth find all the Gröbner cones in the Gröbner fan for this ideal.

Algorithm 4.72. The Gröbner walk algorithm: Let I be a homogeneous ideal, let $\mathcal{C}_1, \mathcal{C}_2$ be two adjacent Gröbner cones in the Gröbner fan. Choose $\omega_1 \in \mathcal{C}_1, \omega_2 \in \mathcal{C}_2$ and $\omega \in \overline{\mathcal{C}_1} \cap \overline{\mathcal{C}_2}$.
Input: A reduced Gröbner basis $\mathcal{G}_{\omega_1}(I)$
Output: A reduced Gröbner basis $\mathcal{G}_{\omega_2}(I)$
Algorithm:
1) $\mathcal{H}_1 \leftarrow \mathrm{In}_\omega(\mathcal{G}_{\omega_1}(I)) = \{\mathrm{In}_\omega(g) \,|g \in \mathcal{G}_{\omega_1}(I)\}$
2) Compute the reduced Gröbner basis $\mathcal{H}_2 \leftarrow \mathcal{G}_{\omega_2}(\mathrm{In}_\omega(I)) = \mathcal{G}_{\omega_2}(\langle \mathcal{H}_1 \rangle)$
3) $\mathcal{G}'_{\omega_2}(I) \leftarrow \emptyset$
for each $h \in \mathcal{H}_2$ **do**
4.1) Use the division algorithm to reduce h to zero modulo $\mathcal{H}_1$ w.r.t. ω_1. This way we can write $h = \sum_{g \in \mathcal{G}_{\omega_1}(I)} p_g \cdot \mathrm{In}_\omega(g)$
4.2) $\mathcal{G}'_{\omega_2}(I) \leftarrow \mathcal{G}'_{\omega_2}(I) \cup \left\{\sum_{g \in \mathcal{G}_{\omega_1}(I)} p_g \cdot g\right\}$
end for
5) Transform the minimal Gröbner basis $\mathcal{G}'_{\omega_2}(I)$ into the reduced Gröbner basis $\mathcal{G}_{\omega_2}(I)$

Proof. See [18], subroutine 3.7 for the correctness of this algorithm. □

Example 4.73. Let $I_\mathcal{A} = \left\langle x_1x_5 - x_4^2, x_1x_4 - x_2x_3 \right\rangle$ be the toric ideal of example 4.71. In this example we perform a Gröbner walk along the line depicted in figure 4.71. Let $\omega_1 := (1,0,0,3,0) \in \mathcal{C}_1, \omega = (1,0,0,2,0) \in \overline{\mathcal{C}_1} \setminus$

$\partial\mathcal{C}_1$ ($\partial\mathcal{C}_1$ is the boundary of $\mathcal{C}_1$) and set $\omega_2 := \omega - \epsilon\omega_1 = (1-\epsilon, 0, 0, 2-3\epsilon, 0)$ for some $\epsilon > 0$ very small. From example 4.71 we know that

$$\mathcal{G}_1 := \mathcal{G}_{\omega_1}(I) = \left\{x_1x_4 - x_2x_3, x_2x_3x_4 - x_1^2x_5, x_1^3x_5 - x_2^2x_3^2, x_4^2 - x_1x_5\right\}.$$

According to the algorithm we obtain

$$\mathcal{H}_1 = \mathrm{In}_\omega(\mathcal{G}_1) = \left\{x_1x_4, x_2x_3x_4 - x_1^2x_5, x_1^3x_5, x_4^2\right\}.$$

We apply the Buchberger algorithm on $\langle\mathcal{H}_1\rangle =: \langle f_1, f_2, f_3, f_4\rangle$: We compute $S(f_1, f_2) = x_2x_3x_4^2 = x_2x_3f_4$, $S(f_2, f_3) = -x_2x_3f_1$, $S(f_2, f_4) = -x_2x_3x_4f_4$. For the other pairs f_i, f_j we have $S(f_i, f_j) = 0$. So $\mathcal{H}_2$ contains at most all generators of $\langle\mathcal{H}_1\rangle$. Also observe that $\mathrm{In}_{\omega_2}(f_3) = x_1^3x_5 = -x_1\mathrm{In}_{\omega_2}(f_2)$, hence we can leave out f_3. So $\mathcal{H}_2 := \left\{x_1x_4, x_2x_3x_4 - x_1^2x_5, x_4^2\right\}$ is the reduced Gröbner basis of $\langle\mathcal{H}_1\rangle$ w.r.t. ω_2.

Next set $\mathcal{G}_2' = \emptyset$ and choose $h = x_1x_4$. Then we can write $h = \sum_{g\in\mathcal{G}_1} p_g \cdot \mathrm{In}_\omega(g) = x_1x_4$, where $p_g = 1$ if $g = x_1x_4 - x_2x_3$ and $p_g = 0$ otherwise. So we conclude that $\sum_{g\in\mathcal{G}_1} p_g \cdot g = x_1x_4 - x_2x_3$ can be added to $\mathcal{G}_2'$. Continuing in the same manner we find the reduced Gröbner basis $\mathcal{G}_2 = \mathcal{G}_2' = \left\{x_1x_4 - x_2x_3, x_1^2x_5 - x_2x_3x_4, x_4^2 - x_1x_5\right\}$.

Example 4.74. The Gröbner cone corresponding to the reduced Gröbner basis $\mathcal{G}_2$ can be given by:

$$\begin{aligned}
\overline{\mathcal{C}}_2 &= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\omega_1 + \omega_4 \geq 0, 2\omega_1 \geq \omega_4, 2\omega_4 \geq \omega_1\right\} \\
&= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|2\omega_1 \geq \omega_4, 2\omega_4 \geq \omega_1\right\}.
\end{aligned}$$

Using the Gröbner walk algorithm we also find the other reduced Gröbner bases and Gröbner cones:

$$\begin{aligned}
\mathcal{G}_3 &= \left\{x_1x_4 - x_2x_3, x_1x_5 - x_4^2, x_4^3 - x_2x_3x_5\right\}, \\
\overline{\mathcal{C}}_3 &= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|2\omega_1 \geq \omega_4, \omega_4 \geq 0\right\} \\
\mathcal{G}_4 &= \left\{x_1x_4 - x_2x_3, x_1x_5 - x_4^2, x_2x_3x_5 - x_4^3\right\}, \\
\overline{\mathcal{C}}_4 &= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\omega_1 + \omega_4 \geq 0, \omega_4 \leq 0\right\} \\
\mathcal{G}_5 &= \left\{x_2x_3 - x_1x_4, x_1x_5 - x_4^2\right\}, \\
\overline{\mathcal{C}}_5 &= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\omega_1 + \omega_4 \leq 0, \omega_1 \geq 2\omega_4\right\} \\
\mathcal{G}_6 &= \left\{x_2x_3 - x_1x_4, x_4^2 - x_1x_5\right\}, \\
\overline{\mathcal{C}}_6 &= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\omega_1 + \omega_4 \leq 0, 2\omega_4 \geq \omega_1\right\} \\
\mathcal{G}_7 &= \left\{x_1x_4 - x_2x_3, x_2x_3x_4 - x_1^2x_5, x_2^2x_3^2 - x_1^3x_5, x_4^2 - x_1x_5\right\}, \\
\overline{\mathcal{C}}_7 &= \left\{\omega = (\omega_1, 0, 0, \omega_4, 0) \in \mathbb{R}^5 \,|\omega_1 + \omega_4 \geq 0, \omega_1 \leq 0\right\}.
\end{aligned}$$

Together they form the complete Gröbner fan of the ideal $I_\mathcal{A} = \left\langle x_1x_5 - x_4^2, x_1x_4 - x_2x_3\right\rangle$. This complete Gröbner fan is depicted in the figure below (notice that it refines the Secondary fan, which is indicated by the black-colored lines in the picture):

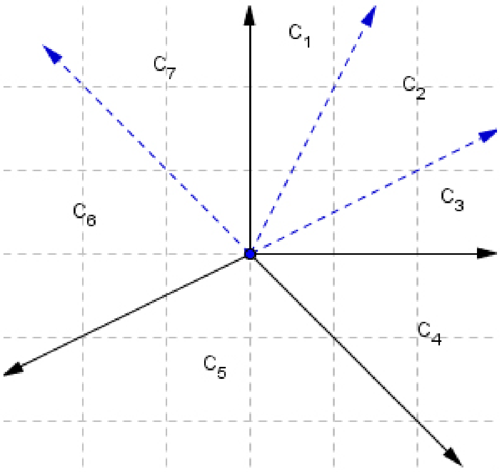
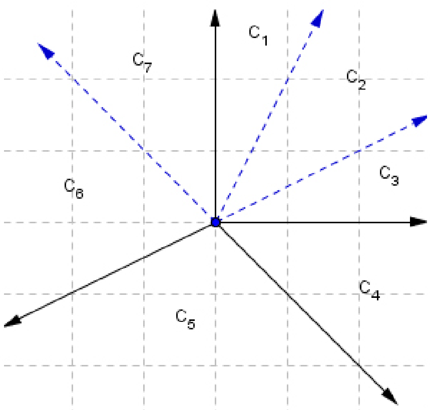


Figure 4.3: The Gröbner fan in a hyperplane where $\omega_2 = \omega_3 = \omega_5 = 0$ and where ω_1 is along the horizontal axis, and ω_4 along the vertical axis.

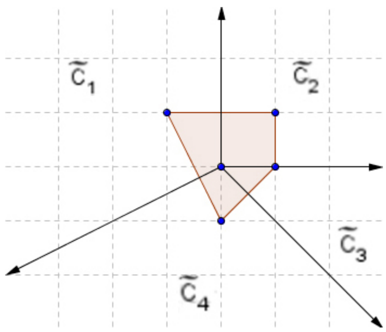
Over time several computer packages have been developed to compute the Gröbner fan, or equivalently all reduced Gröbner bases, over an ideal, for example the package Cats (see [12]).

4.9 The toric variety from an explicit Gröbner fan

In the previous sections we have found the Secondary fan and the Gröbner fan for the point configuration $\mathcal{A} = \begin{pmatrix} -1 & 0 & -1 & 0 & 1 \\ -1 & -1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$. In this section we compare the projective, toric variety from this Secondary fan $\mathbb{X}_{\mathrm{SF}}$ with the projective, toric variety from the Gröbner fan $\mathbb{X}_{\mathrm{GF}}$. Notice that the Gröbner fan is a refinement of the Secondary fan.



The Gröbner fan.



The Secondary fan $\mathcal{F}_\Delta$.

We see that the Gröbner fan consists of the following full-dimensional cones:

$$\begin{aligned}
\mathcal{C}_1 &= \begin{pmatrix} 1 \\ 2 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0}, \qquad \mathcal{C}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 1 \\ 2 \end{pmatrix}\mathbb{R}_{\geq 0}, \\
\mathcal{C}_3 &= \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 2 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} \qquad \mathcal{C}_4 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{R}_{\geq 0}, \\
\mathcal{C}_5 &= \begin{pmatrix} -2 \\ -1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}\mathbb{R}_{\geq 0}, \quad \mathcal{C}_6 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} -2 \\ -1 \end{pmatrix}\mathbb{R}_{\geq 0}, \\
\mathcal{C}_7 &= \begin{pmatrix} -1 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{R}_{\geq 0}.
\end{aligned}$$

Let $\mathbf{u}_j, \mathbf{v}_j$ ($1 \leq j \leq 7$) be the generators of cone $\mathcal{C}_j$ and compute $\det(\mathbf{u}_j, \mathbf{v}_j)$. We obtain that $|\det(\mathbf{u}_j, \mathbf{v}_j)| = 1$ if $j \in \{1, 3, 4, 7\}$, which implies that these cones $\mathcal{C}_j$ are non-singular. For $j \in \{2, 5, 6\}$ we compute $\det(\mathbf{u}_j, \mathbf{v}_j) = 3$ and so these cones are singular. Lemma 3.35 gives an easy method to resolve the singularities by adding rays to a fan. By applying this lemma we see immediately from the picture of the Gröbner fan that we have to add four extra rays to the Gröbner fan to resolve all singularities. The singular cones give rise to the following semigroups:

$$\begin{aligned}
S_{\mathcal{C}_2^\vee} &= \begin{pmatrix} -1 \\ 2 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 2 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\mathbb{Z}_{\geq 0}, \\
S_{\mathcal{C}_5^\vee} &= \begin{pmatrix} 1 \\ -2 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} 0 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0}, \\
S_{\mathcal{C}_6^\vee} &= \begin{pmatrix} -1 \\ 2 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} -1 \\ -1 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} -1 \\ 0 \end{pmatrix}\mathbb{Z}_{\geq 0} + \begin{pmatrix} -1 \\ 1 \end{pmatrix}\mathbb{Z}_{\geq 0}.
\end{aligned}$$

This yields the following affine toric varieties:

$$\begin{aligned}
U_{\mathbb{Z}\left[S_{\mathcal{C}_j^\vee}\right]}(\mathbb{C}) &\cong \mathbb{C}^2 \;\text{ if } j \in \{1, 3, 4, 7\} \\
U_{\mathbb{Z}\left[S_{\mathcal{C}_2^\vee}\right]}(\mathbb{C}) &= \left\{(z_1, ..., z_4) \in \mathbb{C}^4 \,\middle|\, z_1 z_3 = z_2^2,\; z_2 z_4 = z_3^2\right\} \\
U_{\mathbb{Z}\left[S_{\mathcal{C}_5^\vee}\right]}(\mathbb{C}) &= \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \,\middle|\, z_1 z_2 = z_3^3\right\} \\
U_{\mathbb{Z}\left[S_{\mathcal{C}_6^\vee}\right]}(\mathbb{C}) &= \left\{(z_1, ..., z_4) \in \mathbb{C}^4 \,\middle|\, z_2 z_4 = z_3^2,\; z_1 z_3 = z_4^2\right\}.
\end{aligned}$$

Next we try to establish a map between the Gröbner fan $GF(I_{\mathcal{A}})$ and the Secondary fan $\mathcal{F}_\Delta$. We obtain the following maps between the affine, toric varieties of the two fans

$$\pi_1 : U_{\mathbb{Z}\left[S_{\mathcal{C}_1^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{\tilde{\mathcal{C}}_2^\vee}\right]}(\mathbb{C})$$
$$(x_1, y_1) = (\beta(-2,1), \beta(1,0)) \longmapsto (\tilde{x}_2, \tilde{y}_2) = (\beta(0,1), \beta(1,0)) = (x_1 y_1^2, y_1)$$
$$\pi_2 : U_{\mathbb{Z}\left[S_{\mathcal{C}_2^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{\tilde{\mathcal{C}}_2^\vee}\right]}(\mathbb{C})$$
$$(x_2, y_2, z_2, w_2) = (\beta(-1,2), \beta(2,-1), \beta(1,0), \beta(0,1)) \longmapsto (\tilde{x}_2, \tilde{y}_2) = (\beta(0,1), \beta(1,0)) = (w_2, z_2)$$
$$\pi_3 : U_{\mathbb{Z}\left[S_{\mathcal{C}_3^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{\tilde{\mathcal{C}}_2^\vee}\right]}(\mathbb{C})$$
$$(x_3, y_3) = (\beta(0,1), \beta(1,-2)) \longmapsto (\tilde{x}_2, \tilde{y}_2) = (\beta(0,1), \beta(1,0)) = (x_3, x_3^2 y_3)$$
$$\pi_4 : U_{\mathbb{Z}\left[S_{\mathcal{C}_4^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{\tilde{\mathcal{C}}_3^\vee}\right]}(\mathbb{C})$$
$$\pi_4 = \mathrm{id}|_{U_{\mathbb{Z}\left[S_{\mathcal{C}_4^\vee}\right]}(\mathbb{C})}$$
$$\pi_5 : U_{\mathbb{Z}\left[S_{\mathcal{C}_5^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{\tilde{\mathcal{C}}_4^\vee}\right]}(\mathbb{C})$$
$$\pi_5 = \mathrm{id}|_{U_{\mathbb{Z}\left[S_{\mathcal{C}_5^\vee}\right]}(\mathbb{C})}.$$
$$\pi_6 : U_{\mathbb{Z}\left[S_{\mathcal{C}_6^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{\tilde{\mathcal{C}}_1^\vee}\right]}(\mathbb{C})$$
$$(x_6, y_6, z_6, w_6) = (\beta(-1,2), \beta(-1,-1), \beta(-1,0), \beta(-1,1)) \longmapsto (\tilde{x}_1, \tilde{y}_1, \tilde{z}_1) = (\beta(-1,1), \beta(-1,0), \beta(-1,2)) = (w_6, z_6, x_6)$$
$$\pi_7 : U_{\mathbb{Z}\left[S_{\mathcal{C}_7^\vee}\right]}(\mathbb{C}) \to U_{\mathbb{Z}\left[S_{\tilde{\mathcal{C}}_1^\vee}\right]}(\mathbb{C})$$
$$(x_7, y_7) = (\beta(0,1), \beta(1,-2)) \longmapsto (\tilde{x}_1, \tilde{y}_1, \tilde{z}_1) = (\beta(-1,1), \beta(-1,0), \beta(-1,2)) = (x_7^2 y_7, x_7, x_7^3 y_7^2)$$

These affine charts together form a map $\mathbb{X}_{\mathrm{GF}} \to \mathbb{X}_{\mathrm{SF}}$ between the toric varieties from the Gröbner fan $GF(I_{\mathcal{A}})$ and the Secondary fan $\mathcal{F}_\Delta$. Using this map we can also give explicit equations for the 2-parameter family of elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ for the charts in $\mathbb{X}_{\mathrm{GF}}$, similar to what we did for the charts in $\mathbb{X}_{\mathrm{SF}}$ in section 3.8.

Chapter 5

Solving the moduli problem

In this chapter we will use the Gröbner fan to extract an explicit description of the family of curves within the moduli problem. The Gröbner fan will turn out to be the key ingredient in solving the moduli problem.

Recall that on $\mathbb{P}^4$ we had coordinates $[x_1 : ... : x_5]$ and that we assumed the coefficients $[c_1 : ... : c_5]$ to be elements of the dual space $\mathbb{P}^{4\vee}$. Now introduce new coordinates $[y_1 : ... : y_5]$ on $\mathbb{P}^4$ such that $y_j = c_j x_j$, $c_j \neq 0$ for all $1 \leq j \leq 5$. Then the hyperplane

$$\mathcal{H}_{\mathbf{c}} = \left\{ [x_1 : ... : x_5] \,\middle|\, \sum_{j=1}^{5} c_j x_j = 0,\ \mathbf{c} = [c_1 : ... : c_5] \in \mathbb{P}^{4\vee} \right\}$$

changes under this coordinate change to $\sum_{j=1}^{5} y_j = 0$.

Furthermore as we assume that $c_j \neq 0$ for all j we can write $x_j = y_j c_j^{-1}$. The equations in the toric variety $\mathbb{X}_{\mathcal{A}}$ are given by

$$\mathbf{x}^{\mathbf{u}^+} = \prod_{j=1}^{5} x_j^{u_j^+} = \prod_{j=1}^{5} x_j^{u_j^-} = \mathbf{x}^{\mathbf{u}^-},$$

where $\mathbf{u} = \mathbf{u}^+ - \mathbf{u}^- \in \mathbb{L} = \{\mathbf{u} \in \mathbb{Z}^5 \,|\, \sum_{j=1}^{5} u_j \mathbf{a}_j = 0\}$. By our new coordinates this is equal to equations in $\mathbb{P}^4 \times \mathbb{P}^{4\vee}$ of the form

$$\mathbf{y}^{\mathbf{u}^+}\mathbf{c}^{-\mathbf{u}^+} = \mathbf{y}^{\mathbf{u}^-}\mathbf{c}^{-\mathbf{u}^-},$$

i.e.

$$\mathbf{c}^{\mathbf{u}^-}\mathbf{y}^{\mathbf{u}^+} - \mathbf{c}^{\mathbf{u}^+}\mathbf{y}^{\mathbf{u}^-} = 0. \tag{5.1}$$

To describe the universal family of elliptic curves, we look for each $\mathbf{c}$ at equations of the form $\mathbf{c}^{\mathbf{u}^-}\mathbf{y}^{\mathbf{u}^+} - \mathbf{c}^{\mathbf{u}^+}\mathbf{y}^{\mathbf{u}^-} = 0$ intersected with a fixed hyperplane

$\sum_{j=1}^{5} y_j = 0$. Therefore instead of varying the hyperplanes $\mathcal{H}_{\mathbf{c}}$ as before we now vary the ideal

$$\left\langle \mathbf{c}^{\mathbf{u}^-}\mathbf{y}^{\mathbf{u}^+} - \mathbf{c}^{\mathbf{u}^+}\mathbf{y}^{\mathbf{u}^-} \right\rangle.$$

Note that for $c_j \neq 0,\ \forall j$, this ideal gives the closure of the orbit through $[c_1 : ... : c_5] \in \mathbb{P}^{4\vee}$.

Furthermore we can also describe what happens if in the Gröbner fan model for the quotient space $\mathbf{c}$ approaches the boundary for this: Such an approach of the boundary of a moduli space can be described by the 1-parameter subgroups for $(\mathbb{C}^*)^2 \cong {(\mathbb{C}^*)^5}/{(\mathbb{C}^*)^3}$, i.e. the image of group homomorphisms of the form

$\lambda^{\omega} : \mathbb{C}^* \to {(\mathbb{C}^*)^5}/{(\mathbb{C}^*)^3}$, defined as $\lambda^{\omega}(t) = (t^{\omega_1}, ..., t^{\omega_5}) = (c_1, ..., c_5)$, $\omega \in \mathbb{R}^5$.

So we can rewrite the curves in (5.1) as $t^{\omega\cdot\mathbf{u}^-}\mathbf{y}^{\mathbf{u}^+} - t^{\omega\cdot\mathbf{u}^+}\mathbf{y}^{\mathbf{u}^-} = 0$, which is equal to

$$\mathbf{y}^{\mathbf{u}^+} - t^{\omega\cdot\mathbf{u}}\mathbf{y}^{\mathbf{u}^-} = 0. \tag{5.2}$$

By choosing ω in a cone of the Gröbner fan and letting $t \to 0$ we find a natural extension for the equations of the elliptic curves to the boundary of the moduli space. Let us illustrate this with a concrete example:

Example 5.1. Recall from figure 4.74 the Gröbner cone

$$\overline{\mathcal{C}}_1 = \{(w_1, 0, 0, w_4, 0) \in \mathbb{R}^5 | w_4 \geq 2w_1, w_1 \geq 0\}$$

and the corresponding Gröbner basis

$$\mathcal{G}_1 = \{x_4^2 - x_1x_5, x_1x_4 - x_2x_3, x_2x_3x_4 - x_1^2x_5, x_1^3x_5 - x_2^2x_3^2\}.$$

Under the coordinate change $y_j = c_jx_j$ with $c_j \neq 0$, $1 \leq j \leq 5$ we can rewrite this Gröbner basis as

$$\{y_4^2 - c_1^{-1}c_4^2c_5^{-1}y_1y_5, y_1y_4 - c_1c_2^{-1}c_3^{-1}c_4y_2y_3, y_2y_3y_4 - c_1^{-2}c_2c_3c_4c_5^{-1}y_1^2y_5, y_1^3y_5 - c_1^3c_2^{-2}c_3^{-2}c_5y_2^2y_3^2\}.$$

Now choose $\omega \in \overline{\mathcal{C}}_1$, then $\omega = (w_1, 0, 0, w_4, 0) \in \mathbb{R}^5$ such that $w_4 \geq 2w_1$ and $w_1 \geq 0$. Then for this ω we obtain (by looking at the 1-parameter subgroups) the following equations for the 2-parameter family of elliptic curves within the model:

$$\begin{aligned} y_4^2 - t^{-w_1+2w_4}y_1y_5 &= 0, \\ y_1y_4 - t^{w_1+w_4}y_2y_3 &= 0, \\ y_2y_3y_4 - t^{-2w_1+w_4}y_1^2y_5 &= 0, \\ y_1^3y_5 - t^{3w_1}y_2^2y_3^2 &= 0. \end{aligned}$$

If we choose ω in the interior of $\overline{\mathcal{C}}_1$, then all exponents for t are strictly greater than 0. So for $t \to 0$ we only keep the initial terms of the binomials. If ω is chosen on the boundary of $\overline{\mathcal{C}}_1$ then for $t \to 0$ one also finds the initial terms except for one equation, which still contains a binomial.

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