Delayed Exercise Premium Methods for American Options under Cash Dividends

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Chapter 1

Introduction

Options are popularly traded in today's financial market. They are often connected to different underlying, such as listed stock, exchange indices or futures contacts. In this thesis, the stock option is discussed. There are two basic types of options, European and American. A European option is an option contract that can only be exercised on the expiration date. Futures contacts (i.e. options on commodities) are generally European style options. An American option is an option contact that can be exercised at any time between the date of purchase and expiration date. Most exchange-traded options are American style. Stock options are typically American style.

The famous Black-Scholes model is a fast and effective way to calculate the option price. An analytical solution for European options exists, however, for stock options which are American style, a numerical approach is necessary. In real markets, many companies pay dividends to the stock holder. The classical Black-Scholes model cannot deal with dividend payments, therefore we extend the model to include the cash dividends. We focus on the piecewise lognormal Model, i.e. we assume that the stock price shows a jump downwards at dividend date (equal to the cash dividend payments those dates) and follows a geometric Brownian motion in between those dates. The discrete dividend is either of known rate or of a known size.

The American put value can be decomposed into the corresponding European put price and the early exercise premium. This decomposition was derived by Carr, Jarrow and Myneni [3], Jacka [9] and Kim [13] if the underlying asset does not pay a discrete dividend. Although the solutions to this decomposition can only be found numerically, it provides some computational advantages. Meyer [16] was able to price the American put when the underlying asset pays a discrete dividends by using the method of lines. We show that the problem of pricing the American put with discrete dividends is equivalent to solving an optimal stopping problem. The optimal stopping problem gives rise to a free boundary problem. We show that there is a unique solution to this problem. We will focus rather on the theory then numerically calculation; nevertheless our results will be shown to correspond to the calculation done by Meyer.

In this thesis, the following issues are discussed. In chapter 2, the Black-Scholes model under the assumption that the underlying asset is paying a discrete dividend is introduced. Further, we do some basic option theory. In chapter 3, properties of the Snell envelope and American option are derived. In chapter 4, the analysis of pricing American option is transformed into a free boundary problem. Moreover, some numerical calculation are done.

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Chapter 2

The American Option Problem

In this chapter we will introduce some basic notation and assumptions. Our economy will follow the standard Black-Scholes model, except that the underlying asset is paying a discrete dividend.

In section 2.2, we will present roughly the theory of option pricing and introduce European and American options. The American option can be decomposed into the corresponding European one and the *early exercise premium*, which, as we will see, is crucial for pricing the American put. We are able duplicate the American option by a self-financing trading and consumption strategy. This leads us to the *Snell envelope* and the theory of optimal stopping.

This chapter is based on the work of Karatzas [10] and Myneni [18].

2.1 Black-Scholes Model

All economical activities will take place on a finite time horizon $[0, T]$. We have a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on which a standard Brownian $\widetilde{W}_t$ is given. We assume $\widetilde{W}_0 = 0$ almost surely. Define

$$\mathcal{F}_t^{\widetilde{W}} := \sigma\{\widetilde{W}_s; 0 \leq s \leq t\}, \quad \forall t \in [0, T] \tag{2.1}$$

to be the filtration generated by $\widetilde{W}$, and let $\mathcal{N}$ denote the $\mathbb{P}$-null subset of $\mathcal{F}_t^{\widetilde{W}}$. We shall use the augmented filtration

$$\mathcal{F}_t := \sigma(\mathcal{F}_t^{\widetilde{W}} \cup \mathcal{N}), \quad \forall t \in [0, T]. \tag{2.2}$$

This gives us the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, \mathbb{P})$, that we will work with. The filtration $\mathcal{F}_t$ satisfies the usual conditions since $\mathcal{F}_t^{\widetilde{W}}$ is right-continuous and $\mathcal{N} \in \mathcal{F}_0$. We have $\mathcal{F}_T = \mathcal{F}$.

As mentioned in the introduction the aim of this thesis is to find a pricing formula if the underlying asset is paying a discrete dividend. We will use the Black-Scholes model to price the options, see Black and Scholes [2]. Therefore we have the following assumptions:

Assumption 2.1.1: Perfect Bond and Stock Markets: There are no short sale restrictions, transactions costs, taxes or other frictions. Investors can trade bonds and stocks continuously in time.

Assumption 2.1.2: Market participants prefer more wealth to less. In addition, they have homogeneous knowledge of asset prices and symmetric information. That is, $\mathbb{P}$ and $\{\mathcal{F}_t\}_{0\leq t\leq T}$ are known to everyone.

Assumption 2.1.3: Constant Interest Rate: There is a saving account representing the time value of money which appreciates at a deterministic constant rate $r > 0$. Consequently, the value of the saving account at time t, B_t, solves the differential equation $dB_t = rB_t dt$, for all $t \in [0, T]$ with $B_0 = 1$, this gives us

$$B_t = e^{rt}, \quad \forall t \in [0, T]. \tag{2.3}$$

Assumption 2.1.4: Dividends: The underlying stock S is paying a discrete dividend at a *fixed* and *known* date $t_D \in (0, T)$. There will be either a *known* dividend payment of amount $D > 0$ or a *known* dividend rate $\rho \in (0, 1)$ at t_D.

Assumption 2.1.5: Geometric Brownian Motion: The stock price S moves between dividends dates according to the stochastic process $\{\widetilde{S}_t, \mathcal{F}_t; 0 \leq t \leq T\}$ defined by

$$d\widetilde{S}_t = \mu\widetilde{S}_t dt + \sigma\widetilde{S}_t d\widetilde{W}_t, \quad \forall t \in [0, T], \tag{2.4}$$

with expected rate of return $\mu > r$ and volatility $\sigma > 0$ and follows the jump condition

$$S_{t_D} = S_{t_{D-}} - \rho S_{t_{D-}} \text{ or } S_{t_D} = (S_{t_{D-}} - D)^+ \quad \text{a.s.} \tag{2.5}$$

with $S_{t_{D-}} := \lim_{t\uparrow t_D} S_t$ and $(S_{t_{D-}} - D)^+ := \max(S_{t_{D-}} - D, 0)$. It may happen that $S_{t_{D-}} < D$. To avoid that $S_{t_D} < 0$, we define the dividend to be $\min(D, S_{t_{D-}})$. This justifies the notation in (2.5).

We have for the case that a dividend at a rate ρ is paid

$$S_t := \begin{cases} \widetilde{S}_t, & \text{for } t < t_D, \\ \frac{(1-\rho)\widetilde{S}_{t_{D-}}}{\widetilde{S}_{t_D}}\widetilde{S}_t, & \text{for } t \geq t_D. \end{cases} \tag{2.6}$$

And if a payment of magnitude D is made

$$S_t := \begin{cases} \widetilde{S}_t, & \text{for } t < t_D, \\ \frac{(\widetilde{S}_{t_{D-}} - D)^+}{\widetilde{S}_{t_D}} \widetilde{S}_t, & \text{for } t \geq t_D. \end{cases} \tag{2.7}$$

Then S is continuous in time on $[0, t_D)$ and $[t_D, T]$. The initial value of S on $[t, T]$ is $x \in \mathbb{R}^+$, i.e. $S_t = x$ and we use the notation S_s^x for all $s \in [t, T]$, most of the time $t = 0$ and the initial value is $S_0 = x$.

Definition 2.1.6: The *cumulative dividend process* D_t is a RCLL (=right continuous with left limits, sometimes we will use the French acronym càdlàg), nondecreasing and adapted process satisfying:

(i) $D_t := \rho \widetilde{S}^x_{t_{D-}} \mathbb{1}_{\{t \geq t_D\}}$, if S is paying a dividend at a rate ρ.

(ii) $D_t := \min(D, \widetilde{S}^x_{t_{D-}}) \mathbb{1}_{\{t \geq t_D\}}$, if S is paying a magnitude D.

With this notation we do not need to distinguish between the two dividend cases. The ex-dividend stock process is given by

$$S_t^x := \frac{\widetilde{S}^x_{t_{D-}} - D_t}{\widetilde{S}^x_{t_D}} \widetilde{S}^x_t, \quad \forall t \in [t_D, T]. \tag{2.8}$$

A new probability measure is now introduced that is critical to everything that follows. Define the probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ by the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\bigg|_{\mathcal{F}} = \exp\bigg(-\frac{1}{2}\Big(\frac{\mu - r}{\sigma}\Big)^2 T - \Big(\frac{\mu - r}{\sigma}\Big)\widetilde{W}_T\bigg). \tag{2.9}$$

By Girsanov's theorem, the measure $\mathbb{Q}$ is the unique probability measure so that the process $\{S_t^x / B_t + D_t / B_{t_D}, \mathcal{F}_t; 0 \leq t \leq T\}$ is a martingale with respect to $\mathbb{Q}$, i.e. $S_t^x + \frac{D_t}{B_{t_D}} B_t$ is tradeable.

Moreover, the stock price process must evolve according to

$$dS_t^x = r S_t^x dt + \sigma S_t^x dW_t, \quad \forall t \in [0, T] \setminus \{t_D\}, \tag{2.10}$$

where

$$W_t := \widetilde{W}_t + \frac{\mu - r}{\sigma} t, \quad \forall t \in [0, T] \tag{2.11}$$

is a $\mathbb{Q}$-Brownian motion. On the whole interval we have

$$d\Big(S_t^x + D_t \frac{B_t}{B_{t_D}}\Big) = r\Big(S_t^x + D_t \frac{B_t}{B_{t_D}}\Big) dt + \sigma S_t^x dW_t, \quad \forall t \in [0, T]. \tag{2.12}$$

This means, we invest the cash received from the dividend immediately in the saving account.

2.2 Option Theory

Definition 2.2.1: A *trading strategy* or *portfolio* is a pair of predictable processes $(\phi, \psi) = (\phi_t, \psi_t)_{t\in[0,T]}$. We interpret ϕ_t as the number of risky assets held at time t, and ψ_t as the number of bonds. We assume that

$$\int_0^T \phi_t^2 dt + \int_0^T |\psi_t| dt < \infty, \quad \text{a.s.} \tag{2.13}$$

then the (stochastic) integral (2.15) is well-defined.

Definition 2.2.2: The *value process* associated with a portfolio (ϕ, ψ) is the process $V = (V_t)_{t\in[0,T]}$ defined by

$$V_t := \phi_t\Big(S_t^x + D_t \frac{B_t}{B_{t_D}}\Big) + \psi_t B_t, \tag{2.14}$$

with *initial value* $V_0 = \phi_0 S_0^x + \psi_0 B_0$. Again, we invest the dividend payments in the bank account. Clearly V is an adapted process.

Definition 2.2.3: A *consumption process* C is a right-continuous, nondecreasing, $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process satisfying $C_0 = 0$.

Definition 2.2.4: A *trading and consumption strategy* is a triple (ϕ, ψ, C), where (ϕ, ψ) is a trading strategy, C is a consumption process and the *self-financing condition* is satisfied,

$$\phi_t\Big(S_t^x + D_t \frac{B_t}{B_{t_D}}\Big) + \psi_t B_t = \phi_0 S_0^x + \psi_0 B_0 + \int_0^t \phi_u d\Big(S_u^x + D_u \frac{B_u}{B_{t_D}}\Big) + \int_0^t \psi_u dB_u - C_t, \quad \text{a.s.} \tag{2.15}$$

for all $t \in [0, T]$ with $C_0 = 0$. We transform (2.16) into the value equation and use (2.12)

$$\begin{aligned} V_t &= V_0 + \int_0^t \phi_u d\Big(S_u^x + D_u \frac{B_u}{B_{t_D}}\Big) + \int_0^t \psi_u dB_u - C_t \\ &= V_0 + \int_0^t \phi_u \Big(r\Big(S_u^x + D_u \frac{B_u}{B_{t_D}}\Big) du + \sigma S_u^x dW_u\Big) + \int_0^t \psi_u r B_u du - C_t \\ &= V_0 + \int_0^t r V_u du + \int_0^t \sigma \phi_u S_u^x dW_u - C_t, \quad \text{a.s.} \end{aligned} \tag{2.16}$$

We need the notion of a self-financing portfolio for hedging strategies. Such a trading and consumption strategy is created using some starting capital at time 0, and after time 0 the portfolio is only changed by replacing bonds by stocks or vice versa minus the amount consumed. No additional injection or withdrawals of money are allowed. To keep the portfolio self-financing we invest the amount of dividends earned in bonds.

Definition 2.2.5: On our self-financing trading strategies, we impose a mild restriction limiting the size of the position in the stock

$$\mathbb{E}_{\mathbb{Q}}\Big(\int_0^T \phi_t^2 S_t^2 dt\Big) < \infty. \tag{2.17}$$

These strategies are called *admissible* and the class of admissible trading and consumption strategies will be denoted by $\mathcal{A}$. The restriction (2.17) is needed to apply the martingale representation theorem, i.e. Theorem 2.2.12.

Definition 2.2.6: An *arbitrage* is a trading and consumption strategy $(\phi, \psi, C) \in \mathcal{A}$ satisfying $\phi_0 S_0^x + \psi_0 B_0 = 0$ and also satisfying for some $t > 0$,

$$\mathbb{Q}\Big(\phi_t\Big(S_t^x + D_t\frac{B_t}{B_{t_D}}\Big) + \psi_t B_t + C_t \geq 0\Big) = 1, \quad \mathbb{Q}\Big(\phi_t\Big(S_t^x + D_t\frac{B_t}{B_{t_D}}\Big) + \psi_t B_t + C_t > 0\Big) > 0. \tag{2.18}$$

We will use a slightly stronger argument to proof the later Theorem 2.2.14, namely

$$\phi_0 S_0^x + \psi_0 B_0 \leq 0, \quad \mathbb{Q}\Big(\phi_t\Big(S_t^x + D_t\frac{B_t}{B_{t_D}}\Big) + \psi_t B_t + C_t \geq 0\Big) = 1. \tag{2.19}$$

Definition 2.2.7: A *European option* is a financial instrument consisting of:

(i) An expiration date $T \in (0, \infty)$.

(ii) A terminal payoff $(S_T^x - K)^+$ for the *call* and $(K - S_T^x)^+$ for the *put*, where $K \geq 0$ is called the *exercise price* of the option.

Lemma 2.2.8: The *value of the European put* with expiration $T \in (0, \infty)$, initial asset price $S_0 = x > 0$ and dividend date $t_D \in (0, T)$ is

$$p_{t_D}(x, T) := \mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(K - S_T^x)^+\big). \tag{2.20}$$

Later on, we will see that it is important to write the dividend date t_D in the formula, since the dividend date will change if we move in time.

If we want to calculate the price of a European option for the case of a dividend rate ρ we get

$$p_{t_D}(x, T) = Ke^{-rT} N\left(\frac{\ln\big(\frac{K}{(1-\rho)x}\big) - \big(r - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) - (1-\rho)xN\left(\frac{\ln\big(\frac{K}{(1-\rho)x}\big) - \big(r + \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right), \tag{2.21}$$

where $N(x) = \int_0^x \frac{\exp(-z^2/2)}{\sqrt{2\pi}} dz$ is the standard normal density function. And if a magnitude D is paid

$$p_{t_D}(x,T) = e^{-rt_D}\Big(\int_D^\infty p(S-D,T-t_D)\xi(S,t_D;x)dS + K\int_0^D \xi(S,t_D;x)dS\Big), \quad (2.22)$$

where $p(x,T)$ is the ex-dividend European put with initial asset price x and expiration T and $\xi(S_{t_2}, t_2 - t_1; S_{t_1})$ is the transition density function of the asset price at t_2, given the asset price at t_1 is S_{t_1}.

PROOF. We notice that for the first part it is not important when the dividend will be paid, since the dividend is proportional to the stock price. Therefore we can set $t_D = 0$ and thus $S_0 = (1-\rho)x$. The classical Black-Scholes formula solves (2.21).

See Haug, Haug and Lewis [7] for the proof of the second part. But the basic idea is that after the dividend payment the option price reduces to the simple Black-Scholes price for a non-dividend paying stock. Since the dividend should not exceed the value of the stock at time t_D, we introduce a dividend policy. The policy sets the size of the dividend equal to the stock price, if the stock price is smaller than the dividend. □

Definition 2.2.9: An *American option* is a financial instrument consisting of:

(i) An expiration date $T \in (0,\infty]$.

(ii) The selection of an exercise time $\tau \in \mathcal{S}_0$.

(iii) A terminal payoff $(S_\tau^x - K)^+$ for the *call* and $(K - S_\tau^x)^+$ for the *put* at the exercise time.

Where $\mathcal{S}$ is the class of $\{\mathcal{F}_t\}$-stopping times with values in $[0,T]$. For any $s,t \in [0,T]$, we set $\mathcal{S}_{s,t} := \{\tau \in \mathcal{S}; s \le \tau \le t\}$ and $\mathcal{S}_s := \mathcal{S}_{s,T}$.

By (2.20), the European version will pay the amount $(K - S_T^x)^+$ at time T to the holder of the contract. If the contract is of American type, then the holder will obtain the amount $(K - S_t)^+$ if he exercises the contract at time t. The underwriter of the contract only has to maintain the discounted value of the payoff. The exercise time t does not have to be chosen a priori, i.e. at $t = 0$. It can be chosen on the basis of the information generated by the stock price process, and thus the holder will choose a random exercise time τ. The exercise time τ has to be chosen such that the decision on whether to exercise the contract at time t or not, depends only upon information generated by the stock S up to time t. This gives us the set of stopping times $\mathcal{S}_0$. As we do not know which τ will be used, the underwriter has to prepare for the worst possible case, and charge the maximum value (maximized over *all* possible stopping strategies), i.e.

Definition 2.2.10: The *value of the American put* with expiration $T \in (0, \infty]$, initial asset price $S_0 = x > 0$ and dividend date $t_D \in (0, T)$ is

$$P_{t_D}(x, T) := \sup_{\tau \in \mathcal{S}_0} \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau}(K - S^x_\tau)^+\big). \tag{2.23}$$

Remark 2.2.11 We notice that $T \in \mathcal{S}_0$ and thus

$$P_{t_D}(x, T) := \sup_{\tau \in \mathcal{S}_0} \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau}(K - S^x_\tau)^+\big) \geq \mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(K - S^x_T)^+\big) = p_{t_D}(x, T) \tag{2.24}$$

or $P_{t_D}(x, T) = p_{t_D}(x, T) + e(0)$, where $e(0)$ is the *early exercise premium*. The early exercise premium is crucial for the price of an American option, we will get back to this in chapter 4.

Theorem 2.2.12: Let $W = \{W_t, \mathcal{F}_t; 0 \leq t \leq T\}$ be a Brownian motion on $(\Omega, \mathcal{F}, \mathbb{Q})$. Then, for any square-integrable martingale $M = \{M_t, \mathcal{F}_t; 0 \leq t \leq T\}$ with RCLL paths, a.s., there exists a predictable process η with $\mathbb{E}_{\mathbb{Q}} \int_0^t \eta_s^2 ds < \infty$ for every $0 \leq t \leq T$, and

$$M_t = M_0 + \int_0^t \eta_s dW_s, \quad \forall 0 \leq t \leq T, \quad \text{a.s.} \tag{2.25}$$

PROOF. See Karatzas and Shreve [11], Theorem 3.4.15. □

Lemma 2.2.13: Let U be the Snell envelope that we will introduce in the next chapter. The process

$$V_t = e^{rt}U_t = e^{rt}\text{ess sup}_{\tau \in \mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau}(K - S^x_\tau)^+ \big| \mathcal{F}_t\big), \quad t \in [0, T], \quad \text{a.s.} \tag{2.26}$$

is a value process. That is there exists $(\phi, \psi, C) \in \mathcal{A}$ corresponding to (2.26), satisfying (2.16).

PROOF. We postpone the proof until the end of section 3.2.

Theorem 2.2.14: If V_0 is the initial value of the American put option, then

$$V_0 = U_0 \tag{2.27}$$

is necessary for no arbitrage.

PROOF. We postpone the proof until the end of section 3.2.

Chapter 3

Optimal Stopping

As we have seen in chapter 2, pricing American options is closely related to optimal stopping problems. The value of the American put is given by $P_{t_D}(x,T) = \sup_{\tau \in \mathcal{S}_0} \mathbb{E}_{\mathbb{Q}}(L_\tau)$, where $L_t = e^{-rt}(K - S_t^x)^+$ is a discontinuous RCLL process. In this chapter we will have a closer look at what the optimal stopping time is and prove how to hedge an American option on a dividend paying stock.

In section 3.1 we introduce the essential supremum, which is needed in the next section.

We define the Snell envelope of the process L in section 3.2. Under certain conditions there exists an optimal stopping time $\tau^\star$, such that $\sup_{\tau \in \mathcal{S}_0} \mathbb{E}_{\mathbb{Q}}(L_\tau) = \mathbb{E}_{\mathbb{Q}}(L_{\tau^\star})$ and we will characterize $\tau^\star$. 3.2. is an extension of Kühn [14], a more general treatment (but under the assumption that L is continuous) can be found in Karatzas and Shreve [12].

In section 3.3 we will show the relation between the American put and the Snell envelope and derive some basic properties of the put.

Battauz and Pratelli [1] also treat the optimal stopping problem for American option with discrete dividends. They stretch time and filtration to justify the stopping strategies. This is especially useful for the American call options.

3.1 Essential Supremum

Definition 3.1.1: Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{X}$ be a nonempty family of real-valued random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. A random variable $X^\star$ is the *essential supremum* of $\mathcal{X}$, denoted by $X^\star :=$ ess sup $\mathcal{X}$, if $X^\star$ satisfies:

(i) $X^\star \geq X$ a.s. $\forall X \in \mathcal{X}$.

(ii) If Y is a random variable satisfying $X \leq Y$ a.s. for all $X \in \mathcal{X}$, then $X^\star \leq Y$ a.s.

Since random variables are defined only up to $\mathbb{P}$-almost sure equality, it is in general not meaningful to speak of an "ω by ω" supremum, i.e. $\sup_{X \in \mathcal{X}} X(\omega)$. To see this, let $\Omega = [0,1]$, $\mathcal{X} = \{\mathbb{1}_{\{y\}}; y \in [0,1]\}$ and $\mathbb{P}$ is the Lebesgue measure on $[0,1]$. Then we have $\sup_{X \in \mathcal{X}} X(\omega) = 1, \forall \omega \in [0,1]$, but $\mathbb{P}(X = 0) = 1$ for every $X \in \mathcal{X}$.

Remark 3.1.2: The definition of the essential supremum will not change if we use an equivalent measure $\mathbb{Q}$ instead of $\mathbb{P}$.

Theorem 3.1.3: Let $\mathcal{X}$ be a nonempty family of real-valued random variables. Then $X^\star = \text{ess sup } \mathcal{X}$ exists and is almost surely unique (with values in $\mathbb{R} \cup \{\infty\}$). Furthermore, if $\mathcal{X}$ is closed under pairwise maximization, i.e., $X, Y \in \mathcal{X}$ implies $X \vee Y \in \mathcal{X}$, then there exists a nondecreasing sequence $\{Z_n\}_{n=1}^{\infty}$ of random variables in $\mathcal{X}$ satisfying $\lim_{n \to \infty} Z_n = X^\star$ a.s.

PROOF. See Karatzas and Shreve [12], Theorem A.3. □

3.2 The Optimal Stopping Problem

In this section we derive properties of random variables; many of the following results will be true only almost surely. We will denote as before this by *a.s.*, but we will not always repeat this notation in proofs.

Definition 3.2.1: A process $U = (U_t)_{t \in [0,T]}$ with càdlàg paths defined by

$$U_t := \text{ess sup}_{\tau \in \mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}(L_\tau | \mathcal{F}_t), \quad t \in [0,T]. \tag{3.1}$$

is called the *Snell envelope* of the process $L = (L_t)_{t \in [0,T]}$ with respect to $\mathbb{Q}$. We take the essential supremum of the set $\{\mathbb{E}_{\mathbb{Q}}(L_\tau | \mathcal{F}_t); \tau \in \mathcal{S}_t\}$ with respect to the σ-algebra $\mathcal{F}_t$, i.e. U_t is the smallest upper bound of the set of $\mathcal{F}_t$-measurable random variables.

Remark 3.2.2: In equation (3.1) we have constructed the Snell envelope U_t for $t \in [0,T]$. The Snell envelope can be generalized for $\{\mathcal{F}_t\}$-stopping times with value in $[0,T]$, i.e. $U_\sigma = \text{ess sup}_{\tau \in \mathcal{S}_\sigma} \mathbb{E}_{\mathbb{Q}}(L_\tau | \mathcal{F}_\sigma)$ with $\sigma \in \mathcal{S}_0$ is well defined. In this section we derive properties of U, but limit our attention to deterministic time $t \in [0,T]$. These properties are also valid for U_σ with $\sigma \in \mathcal{S}_0$, but the proofs become more difficult. For a more general treatment see Karatzas and Shreve [12]. To prove Theorem 3.2.10 and Theorem 3.2.11, we will use some properties of U_σ with $\sigma \in \mathcal{S}_0$.

Proposition 3.2.3: Let the process $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ be a supermartingale, and assume the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions. The following statements are equivalent:

(i) The process X has a càdlàg modification.

(ii) The function $t \mapsto \mathbb{E}(X_t)$ is càdlàg.

PROOF. See Karatzas and Shreve [12], Theorem 1.3.13. □

Lemma 3.2.4: For any $t \in [0,T]$ the family $\{\mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_t); \tau \in \mathcal{S}_t\}$ is closed under pairwise maximization. Furthermore, there exists a sequence $\{\tau_n\}_{n=1}^{\infty}$ of stopping times in $\mathcal{S}_t$ such that the sequence $\{\mathbb{E}_{\mathbb{Q}}(L_{\tau_n}|\mathcal{F}_t)\}_{n=1}^{\infty}$ is nondecreasing and

$$\text{ess sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_t) = \lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}}(L_{\tau_n}|\mathcal{F}_t), \quad \text{a.s.} \tag{3.2}$$

PROOF. Let $\tau_1, \tau_2 \in \mathcal{S}_t$ and set $A = \{\mathbb{E}_{\mathbb{Q}}(L_{\tau_1}|\mathcal{F}_t) \geq \mathbb{E}_{\mathbb{Q}}(L_{\tau_2}|\mathcal{F}_t)\}$ and $\tau_3 = \tau_1\mathbb{1}_A + \tau_2\mathbb{1}_{A^c}$. Since $A \in \mathcal{F}_t$, the random time τ_3 is a stopping time. In particular $\tau_3 \in \mathcal{S}_t$ and

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(L_{\tau_3}|\mathcal{F}_t) &= \mathbb{1}_A\mathbb{E}_{\mathbb{Q}}(L_{\tau_1}|\mathcal{F}_t) + \mathbb{1}_{A^c}\mathbb{E}_{\mathbb{Q}}(L_{\tau_2}|\mathcal{F}_t)\\ &= \mathbb{E}_{\mathbb{Q}}(L_{\tau_1}|\mathcal{F}_t) \vee \mathbb{E}_{\mathbb{Q}}(L_{\tau_2}|\mathcal{F}_t).\end{aligned} \tag{3.3}$$

By Theorem 3.1.3 there exists a nondecreasing sequence of random variables $\{\tau_n\}_{n=1}^{\infty} \in \mathcal{S}_t$ such that

$$\text{ess sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_t) = \lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}}(L_{\tau_n}|\mathcal{F}_t). \tag{3.4}$$

□

Theorem 3.2.5: Let L be a stochastic process with càdlàg paths and

$$\mathbb{E}_{\mathbb{Q}}(\sup_{t\in[0,T]} L_t) < \infty. \tag{3.5}$$

Then the Snell envelope U from (3.1) exists and satisfies the following properties:

(i) $U \geq L$ a.s.

(ii) U is a $\mathbb{Q}$-supermartingale.

(iii) We have for every right-continuous $\mathbb{Q}$-supermartingale $\widetilde{U} = (\widetilde{U}_t)_{t\in[0,T]}$

$$\widetilde{U} \geq L \Rightarrow \widetilde{U} \geq U \text{ a.s.}$$

Therefore U is the smallest RCLL supermartingale that dominates the reward process L. We notice that in our case $L_t = e^{-rt}(K - S_t^x)^+ \geq 0$, thus $U \geq 0$.

PROOF. First, we want to show that U is a supermartingale, i.e.

$$\mathbb{E}_{\mathbb{Q}}(U_t|\mathcal{F}_s) \leq U_s, \ \ \forall s \leq t, \tag{3.6}$$

from this we will see that U has a càdlàg modification and thus the Snell envelope exists.
By Lemma 3.2.4 there exists a sequence $\{\tau_n\}_{n=1}^{\infty} \in \mathcal{S}_t$ such that

$$\mathbb{E}_{\mathbb{Q}}(L_{\tau_n}|\mathcal{F}_t) \to \text{ess sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_t) = U_t. \tag{3.7}$$

Then we have by the monotone convergence theorem and Fatou's lemma for all $s \leq t$

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(U_t|\mathcal{F}_s) &= \mathbb{E}_{\mathbb{Q}}\big(\lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}}(L_{\tau_n}|\mathcal{F}_t)\big|\mathcal{F}_s\big)\\ &\leq \lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}(L_{\tau_n}|\mathcal{F}_t)\big|\mathcal{F}_s\big)\\ &= \lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}}(L_{\tau_n}|\mathcal{F}_s)\\ &\leq \text{ess sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_s)\\ &\leq \text{ess sup}_{\tau\in\mathcal{S}_s}\mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_s)\\ &= U_s.\end{aligned} \tag{3.8}$$

The second inequality is valid since for all τ_n, $\mathbb{E}_{\mathbb{Q}}(L_{\tau_n}|\mathcal{F}_s) \leq \text{ess sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_s)$, by definition of the essential supremum. We notice that $\mathcal{S}_t \subset \mathcal{S}_s$, this gives us the third inequality.
Next we have to show that there exists a modification of (3.1) with càdlàg paths. Since $\{U_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is a supermartingale, we have that $t \mapsto \mathbb{E}_{\mathbb{Q}}(U_t)$ is a nonincreasing function bounded below by 0. And therefore the limits from the right and from the left exists for every $t \in (0, T)$ as well as the one side limits at the boundaries. For $s = 0$ we have by equation (3.6)

$$\mathbb{E}_{\mathbb{Q}}(U_t) \leq \sup_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}(L_\tau). \tag{3.9}$$

On the other hand by definition of the Snell envelope $U_t \geq \mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_t)$ for all $\tau \in \mathcal{S}_t$. Next we use expectation and the tower property to see that

$$\mathbb{E}_{\mathbb{Q}}(U_t) \geq \mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_t)\big) = \mathbb{E}_{\mathbb{Q}}(L_\tau), \quad \forall\tau \in \mathcal{S}_t. \tag{3.10}$$

And therefore

$$\mathbb{E}_{\mathbb{Q}}(U_t) = \sup_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}(L_\tau). \tag{3.11}$$

We notice that $L_\tau \le \sup_{s\le u\le t} L_u$ for $\tau \in \mathcal{S}_{s,t}$. Taking the expectation and the supremum of $\tau \in \mathcal{S}_{s,t}$, gives us

$$\sup_{\tau\in\mathcal{S}_{s,t}} \mathbb{E}_{\mathbb{Q}}(L_\tau) \le \sup_{\tau\in\mathcal{S}_{s,t}} \mathbb{E}_{\mathbb{Q}}(\sup_{s\le u\le t} L_u) = \mathbb{E}_{\mathbb{Q}}(\sup_{s\le u\le t} L_u). \tag{3.12}$$

For $s \le t$ we have from (3.11)

$$|\mathbb{E}_{\mathbb{Q}}(U_s) - \mathbb{E}_{\mathbb{Q}}(U_t)| = |\sup_{\tau\in\mathcal{S}_s} \mathbb{E}_{\mathbb{Q}}(L_\tau) - \sup_{\sigma\in\mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}(L_\sigma)| = \sup_{\tau\in\mathcal{S}_s} \mathbb{E}_{\mathbb{Q}}(L_\tau) - \sup_{\sigma\in\mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}(L_\sigma).$$

The equation is equal to 0 if the τ has the supremum in $\mathcal{S}_t$. Therefore we focus on the case that $\sup_{\tau\in\mathcal{S}_{s,t}} \mathbb{E}_{\mathbb{Q}}(L_\tau) > \sup_{\sigma\in\mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}(L_\sigma)$. Since $t \in \mathcal{S}_t$ and by (3.12) we have

$$\begin{aligned}
|\mathbb{E}_{\mathbb{Q}}(U_s) - \mathbb{E}_{\mathbb{Q}}(U_t)| &= \sup_{\tau\in\mathcal{S}_{s,t}} \mathbb{E}_{\mathbb{Q}}(L_\tau) - \sup_{\sigma\in\mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}(L_\sigma) \\
&\le \sup_{\tau\in\mathcal{S}_{s,t}} \mathbb{E}_{\mathbb{Q}}(L_\tau) - \mathbb{E}_{\mathbb{Q}}(L_t) \\
&\le \mathbb{E}_{\mathbb{Q}}(\sup_{s\le u\le t} L_u) - \mathbb{E}_{\mathbb{Q}}(L_t).
\end{aligned} \tag{3.13}$$

Since L is right continuous and finite in expectation (3.5), we have that $t \mapsto \mathbb{E}_{\mathbb{Q}}(U_t)$ is right continuous and by Proposition 3.2.3 U has a càdlàg modification.

Proof of the properties: We already have proven that U is a supermartingale. By definition of the Snell envelope $U_t \ge L_t$ for all $t \in [0,T]$ since the processes U and L have right continuous sample paths we have $\mathbb{P}(U_t \ge L_t; \forall t \in [0,T]) = 1$. This proves property (i). Let $\widetilde{U}$ be supermartingale with $\widetilde{U} \ge L$ and $\tau \in \mathcal{S}_t$

$$\widetilde{U}_t \ge \mathbb{E}_{\mathbb{Q}}(\widetilde{U}_\tau|\mathcal{F}_t) \ge \mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_t). \tag{3.14}$$

This is true for all $\tau \in \mathcal{S}_t$ hence we have for all $t \in [0,T]$

$$\widetilde{U}_t \ge \text{ess sup}_{\tau\in\mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_t) = U_t. \tag{3.15}$$

The processes U and $\widetilde{U}$ are right-continuous and therefore $\mathbb{P}(\widetilde{U}_t \ge U_t; \forall t \in [0,T]) = 1$. □

Definition 3.2.6: A right-continuous process $\{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ is said to be *of class D* if the family $\{X_\tau\}_{\tau\in\mathcal{S}_0}$ is uniformly integrable (UI), i.e. for a given $\varepsilon > 0$ there exists a $K \in [0,\infty)$ such that $\sup_{\tau\in\mathcal{S}_0} \mathbb{E}(|X_\tau| \mathbb{1}_{\{|X_\tau|>K\}}) < \varepsilon$.

Proposition 3.2.7: Every martingale on a closed interval is of class D.

PROOF. Let $X = \{X_t, \mathcal{F}_t; 0 \le t \le T\}$ be a martingale. Then we have for all stopping times $\tau \in \mathcal{S}_0$, $\mathbb{E}(X_T|\mathcal{F}_\tau) = X_\tau$. We notice that $X_T \in \mathcal{L}^1$, therefore there exists a

$d \geq 0$ such that $\mathbb{E}(|X_T|\mathbb{1}_{\{|X_T|>d\}}) < \frac{\varepsilon}{2}$ for a given $\varepsilon > 0$. Define $K := \frac{2d}{\varepsilon}\mathbb{E}(|X_T|)$, then we have for all $\tau \in \mathcal{S}_0$

$$\begin{aligned}
\mathbb{E}(|X_\tau|\mathbb{1}_{\{|X_\tau|>K\}}) &\leq \mathbb{E}\big(\mathbb{E}(|X_T|\,|\mathcal{F}_\tau)\mathbb{1}_{\{|X_\tau|>K\}}\big) \\
&= \mathbb{E}\big(\mathbb{E}(|X_T|\mathbb{1}_{\{|X_\tau|>K\}}|\mathcal{F}_\tau)\big) \\
&= \mathbb{E}(|X_T|\mathbb{1}_{\{|X_\tau|>K\}}) \\
&= \mathbb{E}(|X_T|\mathbb{1}_{\{|X_\tau|>K\}}\mathbb{1}_{\{|X_T|>d\}}) + \mathbb{E}(|X_T|\mathbb{1}_{\{|X_\tau|>K\}}\mathbb{1}_{\{|X_T|\leq d\}}) \\
&\leq \mathbb{E}(|X_T|\mathbb{1}_{\{|X_T|>d\}}) + d\mathbb{P}(|X_\tau| > K) \\
&\leq \frac{\varepsilon}{2} + \frac{d}{K}\mathbb{E}(|X_\tau|) \\
&\leq \frac{\varepsilon}{2} + \frac{d}{K}\mathbb{E}(|X_T|) \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,
\end{aligned} \tag{3.16}$$

since $\mathbb{1}_{\{|X_\tau|>K\}}$ is $\mathcal{F}_\tau$-measurable and by Markov's inequality. The family $\{X_\tau\}_{\tau\in\mathcal{S}_0}$ is uniformly integrable and therefore of class D. □

Lemma 3.2.8: The Snell envelope U is of class D.

PROOF. Define $V_t := \mathbb{E}_{\mathbb{Q}}(\sup_{0\leq u\leq T} L_u|\mathcal{F}_t)$. V is a martingale on $[0,T]$. To see this, we have by the tower property for all $s \leq t$,

$$\mathbb{E}_{\mathbb{Q}}(V_t|\mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}(\sup_{0\leq u\leq T} L_u|\mathcal{F}_t)\big|\mathcal{F}_s\big) = \mathbb{E}_{\mathbb{Q}}(\sup_{0\leq u\leq T} L_u|\mathcal{F}_s) = V_s. \tag{3.17}$$

Furthermore $L_t \leq \sup_{0\leq u\leq T} L_u$ and

$$L_t = \mathbb{E}_{\mathbb{Q}}(L_t|\mathcal{F}_t) \leq \mathbb{E}_{\mathbb{Q}}(\sup_{0\leq u\leq T} L_u|\mathcal{F}_t) = V_t. \tag{3.18}$$

Since L and V have right-continuous paths we have $V \geq L$ by Theorem 3.2.5 (iii) U is the smallest $\mathbb{Q}$-supermartingale dominating L and thus $V \geq U$. By Proposition 3.2.7 the martingale $\{V_t, \mathcal{F}_t; 0 \leq t \leq T\}$ is of class D. Since $V \geq U$ the Snell envelope U is of class D. □

Definition 3.2.9: A stochastic process has *upper semi-continuous* sample paths if for every $t \in [0,T]$, $\limsup_{s\to t} X_s(\omega) \leq X_t(\omega)$ for all $\omega \in \Omega$.

Theorem 3.2.10: Define for all $\lambda \in (0,1)$ and $s \in [0,T]$ the stopping time

$$\tau_s^\lambda := \inf\{t \geq s; \lambda U_t \leq L_t\} \wedge T. \tag{3.19}$$

The following can be said about τ_s^λ:

(i) For all $\lambda \in (0,1)$ and $s \in [0,T]$ the stopped supermartingale $\{U_{t\wedge\tau_s^\lambda}, \mathcal{F}_t; s \le t \le T\}$ is a martingale.

(ii) τ_s^λ is a λ-optimal stopping time for all $s \in [0,T]$, i.e.

$$\mathbb{E}_{\mathbb{Q}}(L_{\tau_s^\lambda}|\mathcal{F}_s) \ge \lambda \cdot \text{ess sup}_{\tau\in\mathcal{S}_s} \mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_s) = \lambda U_s, \quad \text{a.s.} \tag{3.20}$$

(iii) If L has upper semi-continuous paths (since L is càdlàg this means that only positive jumps are allowed), then there exists an optimal stopping time

$$\tau_s^\star := \lim_{\lambda\uparrow 1} \tau_s^\lambda = \inf\{t \ge s; U_t = L_t\} \tag{3.21}$$

on $[s,T]$ such that

$$\mathbb{E}_{\mathbb{Q}}(L_{\tau_s^\star}|\mathcal{F}_s) = \text{ess sup}_{\tau\in\mathcal{S}_s} \mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_s) = U_s, \quad \text{a.s.} \tag{3.22}$$

PROOF. We notice that the process $(\tau_t^\lambda)_{t\in[0,T]}$ is right-continuous. This follows from the right continuity of U and L.
We first prove that, if $\{X_t, \mathcal{F}_t; s \le t \le T\}$ is a supermartingale and $\mathbb{E}(X_T|\mathcal{F}_t) = X_t$ for all $s \le t \le T$, then $\{X_t, \mathcal{F}_t; s \le t \le T\}$ is a martingale. We have by the tower property for all $s \le r \le t \le T$

$$X_r \ge \mathbb{E}(X_t|\mathcal{F}_r) \ge \mathbb{E}\big(\mathbb{E}(X_T|\mathcal{F}_t)\big|\mathcal{F}_r\big) = \mathbb{E}(X_T|\mathcal{F}_r) = X_r. \tag{3.23}$$

Thus $\mathbb{E}(X_t|\mathcal{F}_r) = X_r$ for all $s \le r \le t \le T$ and $\{X_t, \mathcal{F}_t; s \le t \le T\}$ is a martingale.

U is a supermartingale, the stopped process $U_{t\wedge\tau_s^\lambda}$ is a supermartingale as well. To prove (i) it is sufficient to show that

$$\mathbb{E}_{\mathbb{Q}}(U_{\tau_s^\lambda}|\mathcal{F}_s) = U_s, \quad \forall\lambda \in (0,1),\ s \in [0,T]. \tag{3.24}$$

Define the random variable $Z_t := \mathbb{E}_{\mathbb{Q}}(U_{\tau_t^\lambda}|\mathcal{F}_t)$, for all $t \in [0,T]$, then Z_t is a right-continuous supermartingale. We have by Theorem 3.2.5(ii) for all $t \ge s$

$$\begin{aligned}\mathbb{E}_{\mathbb{Q}}(Z_t|\mathcal{F}_s) &= \mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}(U_{\tau_t^\lambda}|\mathcal{F}_t)\big|\mathcal{F}_s\big) = \mathbb{E}_{\mathbb{Q}}(U_{\tau_t^\lambda}|\mathcal{F}_s) = \mathbb{E}_{\mathbb{Q}}\big(\mathbb{E}_{\mathbb{Q}}(U_{\tau_t^\lambda}|\mathcal{F}_{\tau_s^\lambda})\big|\mathcal{F}_s\big)\\ &\le \mathbb{E}_{\mathbb{Q}}(U_{\tau_s^\lambda}|\mathcal{F}_s) = Z_s.\end{aligned} \tag{3.25}$$

By Fatou's lemma and the right continuity of $(U_t)_{t\in[0,T]}$ and $(\tau_t^\lambda)_{t\in[0,T]}$ we also have

$$\lim_{s\downarrow t} \mathbb{E}_{\mathbb{Q}} Z_s = \lim_{s\downarrow t} \mathbb{E}_{\mathbb{Q}} U_{\tau_s^\lambda} = \mathbb{E}_{\mathbb{Q}}(\lim_{s\downarrow t} U_{\tau_s^\lambda}) \ge \mathbb{E}_{\mathbb{Q}} U_{\tau_t^\lambda} = \mathbb{E}_{\mathbb{Q}} Z_t. \tag{3.26}$$

Therefore, the mapping $t \mapsto \mathbb{E}_{\mathbb{Q}} Z_t$ is right-continuous and by Proposition 3.2.3, Z has a càdlàg modification.

We show next that Z dominates U. Consider the RCLL supermartingale $\lambda U + (1-\lambda)Z$. Fix $t \in [0,T]$. Then on $\{\tau_t^\lambda = t\}$ we have from Theorem 3.2.5(i)

$$\lambda U_t + (1-\lambda)Z_t = \lambda U_t + (1-\lambda)\mathbb{E}_{\mathbb{Q}}(U_t|\mathcal{F}_t) = U_t \geq L_t. \tag{3.27}$$

On the other hand, we have that $U \geq 0$ and the definition of τ_t^λ implies that on the event $\{\tau_t^\lambda > t\}$

$$\lambda U_t + (1-\lambda)Z_t = \lambda U_t \geq L_t. \tag{3.28}$$

The right continuity of U, L and Z allows us to conclude that the supermartingale $\lambda U + (1-\lambda)Z$ dominates L, and by Theorem 3.2.5(iii) dominates U as well. And

$$\lambda U + (1-\lambda)Z \geq U \Rightarrow Z \geq U. \tag{3.29}$$

We have by optimal sampling that $\mathbb{E}_{\mathbb{Q}}(U_{\tau_t^\lambda}|\mathcal{F}_t) \leq U_t$ combing this with (3.29) we have $\mathbb{E}_{\mathbb{Q}}(U_{\tau_t^\lambda}|\mathcal{F}_t) = U_t$ and hence $\{U_{t\wedge\tau_s^\lambda}, \mathcal{F}_t; s \leq t \leq T\}$ is a martingale.

To prove (ii), since the processes U and L have right-continuous sample paths the infimum and minimum agree therefore we have at τ_s^λ with $s \in [0,T]$

$$L_{\tau_s^\lambda} \geq \lambda U_{\tau_s^\lambda}. \tag{3.30}$$

Taking conditional expectation of (3.30) given $\mathcal{F}_s$ and using (i)

$$\mathbb{E}_{\mathbb{Q}}(L_{\tau_s^\lambda}|\mathcal{F}_s) \geq \lambda \cdot \mathbb{E}_{\mathbb{Q}}(U_{\tau_s^\lambda}|\mathcal{F}_s) = \lambda U_s, \quad \forall \lambda \in (0,1). \tag{3.31}$$

We notice that $L_{\tau_s^m} \leq L_{\tau_s^n}$, with $m \leq n$ and $m,n \in (0,1)$. If the process L would have a continuous sample path then $L_{\tau_s^\star} = \lim_{\lambda\uparrow 1} L_{\tau_s^\lambda}$, but we allow positive jumps and therefore $L_{\tau_s^\star} \geq \lim_{\lambda\uparrow 1} L_{\tau_s^\lambda}$. By the monotone convergence theorem and (ii) this gives us

$$\mathbb{E}_{\mathbb{Q}}(L_{\tau_s^\star}|\mathcal{F}_s) \geq \mathbb{E}_{\mathbb{Q}}(\lim_{\lambda\uparrow 1} L_{\tau_s^\lambda}|\mathcal{F}_s) = \lim_{\lambda\uparrow 1} \mathbb{E}_{\mathbb{Q}}(L_{\tau_s^\lambda}|\mathcal{F}_s) \geq U_s. \tag{3.32}$$

Since $\mathbb{E}_{\mathbb{Q}}(L_{\tau_s^\star}|\mathcal{F}_s) \leq U_s$ by definition of the Snell envelope, we have

$$\mathbb{E}(L_{\tau_s^\star}|\mathcal{F}_s) = \text{ess sup}_{\tau\in\mathcal{S}_s} \mathbb{E}_{\mathbb{Q}}(L_\tau|\mathcal{F}_s) \tag{3.33}$$

and $\tau_s^\star$ is therefore an optimal stopping time on $[s,T]$. □

Theorem 3.2.11: The Snell envelope U admits the Doob-Meyer decomposition

$$U = M - \Lambda. \tag{3.34}$$

Where $M = (M_t)_{t\in[0,T]}$ is a uniformly integrable RCLL martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, $\Lambda = (\Lambda_t)_{t\in[0,T]}$ is a right-continuous nondecreasing $\{\mathcal{F}_t\}_{t\in[0,T]}$-adapted process with $\Lambda_0 = 0$ and $\mathbb{E}_{\mathbb{Q}}(\Lambda_T) < \infty$. Moreover, if L has upper semi-continuous paths, then Λ is continuous.

PROOF. We have by Lemma 3.2.8 that U is of class D. By Theorem 1.4.10 from Karatzas and Shreve [11] M is a uniformly integrable RCLL martingale and Λ is integrable nondecreasing process with $\Lambda_0 = 0$.
We want to show that if L has upper semi-continuous paths, then U is regular, i.e. if $\{\sigma_n\}_{n=1}^{\infty}$ is a nondecreasing sequence of stopping times in $\mathcal{S}$ and $\sigma = \lim_{n\to\infty}\sigma_n$, then $\mathbb{E}_{\mathbb{Q}}(U_\sigma) = \lim_{n\to\infty}\mathbb{E}_{\mathbb{Q}}(U_{\sigma_n})$. Since U is a supermartingale we have $\mathbb{E}_{\mathbb{Q}}(U_\sigma) \le \mathbb{E}_{\mathbb{Q}}(U_{\sigma_n})$ for all n and therefore $\mathbb{E}_{\mathbb{Q}}(U_\sigma) \le \liminf_{n\to\infty}\mathbb{E}_{\mathbb{Q}}(U_{\sigma_n})$. And by Theorem 3.2.10, reverse Fatou, upper semi-continuity of L and Theorem 3.2.10 again, we have

$$\begin{aligned}\limsup_{n\to\infty}\mathbb{E}_{\mathbb{Q}}(U_{\sigma_n}) &= \limsup_{n\to\infty}\mathbb{E}_{\mathbb{Q}}(L_{\tau^\star_{\sigma_n}}) \le \mathbb{E}_{\mathbb{Q}}(\limsup_{n\to\infty} L_{\tau^\star_{\sigma_n}}) \\ &\le \mathbb{E}_{\mathbb{Q}}(L_{\tau^\star_\sigma}) = \mathbb{E}_{\mathbb{Q}}(U_\sigma). \end{aligned} \tag{3.35}$$

We can use reverse Fatou since $L \ge 0$ by definition and $U \ge L$. By Theorem 1.4.14 from Karatzas and Shreve [11] we have that Λ is continuous. □

Lemma 3.2.12: $\Lambda_t = 0$ for all $t \in [s, \tau^\star_s]$, where $\tau^\star_s$ is the optimal exercise boundary defined in Theorem 3.2.10(iii).

PROOF. By Theorem 3.2.10(i) we have that the stopped process $\{U_{t\wedge\tau^\star_s}, \mathcal{F}_t; s \le t \le T\}$ is a martingale. Thus $\Lambda = 0$ on $[s, \tau^\star_s]$.

Proof of Lemma 2.2.13 By Theorem 3.2.11 U admits the Meyer-Doob decomposition

$$U = M - \Lambda,$$

where M is a bounded martingale, since the put is bounded by K and Λ is a continuous, nondecreasing process with $\Lambda_0 = 0$. So,

$$d(e^{rt}U_t) = re^{-rt}U_t dt + e^{rt}dM_t - e^{rt}d\Lambda_t, \quad t \in [0, T]. \tag{3.36}$$

By Theorem 2.2.12 there exists a predictable process η with

$$dM_t = \eta_t dW_t \tag{3.37}$$

Therefore, with $\phi_t = e^{rt}\eta_t\sigma^{-1}(S^x_t)^{-1}$, $\psi_t = U_t - \eta_t\sigma^{-1}$ and $C_t = \int_0^t e^{-ru}d\Lambda_u$, it follows from (2.15) that $e^{rt}U_t$ is a value process. □

Proof of Theorem 2.2.14: Suppose that the actual market price of this option were $U_0 < V_0$. We are going sell one option and construct a portfolio with that money. Let (η, ξ) and c form the admissible investment and consumption process of Lemma 2.2.13. Consider the following trading and consumption strategy given by an stopping time

$\tau \in \mathcal{S}_0$ selected by the buyer

$$\phi_t = \begin{cases} \eta_t, & \text{for } t \le \tau, \\ 0, & \text{for } t > \tau, \end{cases}, \quad \psi_t = \begin{cases} \xi_t, & \text{for } t \le \tau, \\ \eta_\tau S^x_\tau / B_\tau + \xi_\tau, & \text{for } t > \tau, \end{cases} \quad C_t = c_{t\wedge\tau}.$$

We have from the dominating property of Snell envelope U, that $e^{rt}U_t \ge (K - S^x_t)^+$ for all $t \in [0,T]$ and therefore

$$\eta_\tau S^x_\tau + \xi_\tau B_\tau \ge (K - S^x_\tau)^+, \tag{3.38}$$

it follows that $\psi_\tau B_\tau \ge V_\tau$, and thus

$$\psi_T B_T = U_T \ge V_T. \tag{3.39}$$

But, by construction,

$$\phi_0 S^x_0 + \psi_0 B_0 = U_0 < V_0, \tag{3.40}$$

and therefore we have an arbitrage.

Similarly, suppose that $U_0 > V_0$. Now we are going to sell the portfolio and buy the option. Again, let (η, ξ) and c be the admissible investment and consumption process of (last theorem). Consider the following trading and consumption strategy given by an stopping time $\tau \in \mathcal{S}_0$ selected by the buyer and we will choose the optimal stopping time $\tau^\star$

$$\phi_t = \begin{cases} \eta_t, & \text{for } t \le \tau^\star, \\ 0, & \text{for } t > \tau^\star, \end{cases}, \quad \psi_t = \begin{cases} \xi_t, & \text{for } t \le \tau^\star, \\ \eta_{\tau^\star} S^x_{\tau^\star} / B_{\tau^\star} + \xi_{\tau^\star}, & \text{for } t > \tau^\star, \end{cases} \quad C_t = c_{t\wedge\tau^\star}.$$

But $c = 0$ on $[0, \tau^\star]$ by Lemma 3.2.11, and

$$\eta_{\tau^\star} S^x_{\tau^\star} + \xi_{\tau^\star} B_{\tau^\star} = (K - S^x_{\tau^\star})^+, \tag{3.41}$$

it follows that $\psi_{\tau^\star} B_{\tau^\star} = V_{\tau^\star}$, and thus

$$\psi_T B_T = U_T = V_T. \tag{3.42}$$

Again, by construction,

$$\phi_0 S^x_0 + \psi_0 B_0 = U_0 > V_0, \tag{3.43}$$

and there is an arbitrage. □

3.3 Properties of the American Put

For the American put, we have $L_t = e^{-rt}(K - S_t^x)^+$. We notice that S has a negative jump at the dividend date t_D and therefore $e^{-rt}(K - S_t^x)^+$ might have a positive jump at t_D and since S is right-continuous $e^{-rt}(K - S_t^x)^+$ is upper semi-continuous. Thus we can apply the theory obtained in 3.2 to price the American put. And we observe that this is independent of the way we model the dividend payment.

Proposition 3.3.1: The Snell envelope of L is given by

$$U_t = \text{ess sup}_{\tau \in \mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}(L_\tau | \mathcal{F}_t) = e^{-rt} P_{t_D - t}(S_t^x, T - t). \tag{3.44}$$

PROOF. Generally speaking, the process S_t^x has the Markov property and therefore also the Snell envelope U, which is the price process of the option. Thus the value of U at time t should be the same as a new option with maturity $T - t$ and dividend date $t_D - t$. To be able to compare these two, we have to discount back to time 0. But this argument does not hold, since the set stopping times $\mathcal{S}_t$ for $U_t = \text{ess sup}_{\tau \in \mathcal{S}_t} \mathbb{E}_{\mathbb{Q}}(L_\tau | \mathcal{F}_t)$ depends on the information in $\mathcal{F}_t$ and not of just $\sigma(S_t^x)$. Fortunately these additional stopping times do not change the optimal value.

A formal proof can be found in El Karoui, Lepeltier and Millet [4] Theorem 3.4. □

Remark 3.3.2: It can happen that in (3.44) $t_D - t \leq 0$ then there will be no upcoming dividend date and we write $P(S_t^x, T - t)$ instead of $P_{t_D - t}(S_t^x, T - t)$.

Remark 3.3.3: Proposition 3.3.1 gives us the connection between the Snell envelope and the American put as we move in time. Now there are two different ways of proving properties of the American put $P_{t_D}(x, T)$. We could use the right-hand side (3.44) for a given $t > 0$ and derive the properties of $P_{t_D - t}(x, T - t)$, a new option with starting point 0 and maturity $T - t$. This is quite common in the literature if the underlying asset is not paying a discrete dividend. We have some difficulties since in our case we would study put options with different dividend dates. Instead we move in time, i.e the information $\mathcal{F}_t$ is given, maturity is still T and we use the left-hand side of (3.44). This gives us a new definition of the initial stock price, see Assumption 2.1.5. Let $\mathcal{F}_t$ be given then S_s^x is the stock price and the initial value at time t is x, i.e. $S_t = x$. Basically, we will use the right-hand side of (3.44) if t is fixed and the Snell envelope if t is changing.

Definition 3.3.4: The *optimal exercise time* for the American put option with initial stock price $S_t = x$ with $t \in [0, T]$ is as a result of Theorem 3.2.10(iii) given by

$$\tau_t^x := \inf\{s \geq t; P_{t_D - s}(S_s^x, T - s) = (K - S_s^x)^+\}. \tag{3.45}$$

The proof of Theorem 3.2.10(i) shows that the stopped process

$$\{e^{-r(s \wedge \tau_t^x)} P_{t_D - (s \wedge \tau_t^x)}(S_{s \wedge \tau_t^x}^x, T - (s \wedge \tau_t^x)), \mathcal{F}_s; t \leq s \leq T\} \tag{3.46}$$

is a $\mathbb{Q}$-martingale.

Theorem 3.3.5: The Snell envelope U of the process $e^{-rt}(K-S_t^x)^+$ is continuous on $[0,T]$.

PROOF. By Theorem 3.2.10 we know that U admits the Doob-Meyer decomposition $U = M - \Lambda$. The process L is defined by $L_t = e^{-rt}(K - S_t^y)^+$. S jumps down at t_D, see Assumption 2.1.5, thus L is upper semi-continuous and Λ is continuous. To proves that U is continuous we have to show that the martingale M is continuous. M is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\in[0,T]}$, this filtration is defined by (2.2), i.e.

$$\mathcal{F}_t = \sigma(\mathcal{F}_t^{\widetilde{W}} \cup \mathcal{N}), \quad \forall t \in [0,T], \tag{3.47}$$

where $\mathcal{F}_t^{\widetilde{W}}$ is generated by the Brownian motion $\widetilde{W}$. As we know, martingales on Brownian filtration are always continuous. Therefore, we have that the Snell envelope U is continuous. □

This proof for a more general case can also be found in Hamadène and Lepeltier [6], Lemma 1.

Lemma 3.3.6: Let $T \in \mathbb{R}^+$ and $t_D \in (0,T)$ be given, then we have the following properties for the American put option with discrete dividend:

(i) The mapping $x \mapsto P_{t_D-t}(x, T-t)$ is convex and nonincreasing for all $t \in [0,T]$.

(ii) $K > P_{t_D-t}(x,T-t) > 0$ for all $(x,t) \in \mathbb{R}^+ \times [0,T)$ and $P_{t_D-t}(x,T-t) \geq (K-x)^+$ for all $(x,t) \in \mathbb{R}^+ \times [0,T]$.

(iii) $P_{t_D-t}(x,T-t)$ is Lipschitz continuous in x with constant 1 for all $t \in [0,T]$.

(iv) $(x,t) \mapsto P_{t_D-t}(x,T-t)$ is continuous on $\mathbb{R}^+ \times [0,t_D)$ and $\mathbb{R}^+ \times [t_D,T]$.

(v) At t_D, P satisfies the interface condition $\lim_{t\uparrow t_D} P_{t_D-t}(x,T-t) = P(x-D,T-t_D)$ and $\lim_{t\uparrow t_D} P_{t_D-t}(x,T-t) = P((1-\rho)x, T-t_D)$ respectively.

PROOF. We have by the definition of P, i.e. (2.23)

$$P_{t_D-t}(x,T-t) = \sup_{\tau\in\mathcal{S}_{0,T-t}} \mathbb{E}_{\mathbb{Q}}(e^{-r\tau}(K-S_\tau^x)^+) = \sup_{\tau\in\mathcal{S}_{0,T-t}} \mathbb{E}_{\mathbb{Q}}(e^{-r\tau}(K-xS_\tau^1)^+). \tag{3.48}$$

(i) The map $x \mapsto (K-\alpha x)^+$ is convex for every $\alpha \in \mathbb{R}^+$. By linearity of the expectation this is also true for $x \mapsto \mathbb{E}_{\mathbb{Q}}(e^{-r\tau}(K-xS_\tau^1)^+)$ for every $\tau \in \mathcal{S}_{0,T-t}$. The supremum of a convex function is convex again. The proof for the other property can be established by writing "nonincreasing" instead of "convex".

(ii) The value of the American put can not exceed it's maximal payoff which is $(K - S_t^x)^+ < K$. To prove positivity we have

$$\begin{aligned} P_{t_D-t}(x, T-t) &\geq \mathbb{E}_{\mathbb{Q}}(e^{-r(T-t)}(K - S_{T-t}^x)^+) = e^{-r(T-t)}(K - \mathbb{E}_{\mathbb{Q}}(S_{T-t}^x))\mathbb{Q}(S_{T-t}^x \leq K) \\ &\geq e^{-r(T-t)}\frac{K}{2}\mathbb{Q}\Big(S_{T-t}^x \leq \frac{K}{2}\Big) > 0, \quad \forall x > 0,\ t \in [0,T). \end{aligned} \tag{3.49}$$

The value of the put dominates the option intrinsic value since we can always take $\tau = 0$.

(iii) Fix $(x,t) \in \mathbb{R}^+ \times [0,T]$ and let τ^x be an optimal stopping time for $P_{t_D-t}(x, T-t)$, this means

$$\tau^x = \inf\{s \geq 0; P_{t_D-t-s}(S_s^x, T-t-s) = (K - S_s^x)^+\}. \tag{3.50}$$

Since $z_1^+ - z_2^+ \leq (z_1 - z_2)^+$ for any $z_1, z_2 \in \mathbb{R}$, we have for any $y \in \mathbb{R}^+$

$$\begin{aligned} P_{t_D-t}(x, T-t) - P_{t_D-t}(y, T-t) &\leq \mathbb{E}_{\mathbb{Q}}(e^{-r\tau_x}(K - S_{\tau_x}^x)^+) - \mathbb{E}_{\mathbb{Q}}(e^{-r\tau_x}(K - S_{\tau_x}^y)^+) \\ &\leq \mathbb{E}_{\mathbb{Q}}(e^{-r\tau_x}(yS_{\tau_x}^1 - xS_{\tau_x}^1)^+) \\ &\leq (y-x)^+\mathbb{E}_{\mathbb{Q}}(e^{-r\tau_x}S_{\tau_x}^1) \\ &\leq (y-x)^+ \leq |x-y|. \end{aligned} \tag{3.51}$$

The first inequality is valid since τ^x is not optimal for $P_{t_D-t}(y, T-t)$. We notice that $\{e^{-rs}S_s^1, \mathcal{F}_s; 0 \leq s \leq T-t\}$ is a $\mathbb{Q}$-supermartingale, due to the jump. This justifies the fourth inequality. Interchanging the roles of x and y gives us $|P_{t_D-t}(x, T-t) - P_{t_D-t}(y, T-t)| \leq |x-y|$, so $P_{t_D-t}(x, T-t)$ is Lipschitz continuous in x with constant 1.

(iv) Since $x \mapsto P_{t_D-t}(x, T-t)$ is Lipschitz continuous with a constant independent of $T-t$ it is sufficient to show that $t \mapsto P_{t_D-t}(x, T-t)$ is continuous on $[0, t_D)$ and $[t_D, T]$. By Theorem 3.3.5 $U_t = e^{-rt}P_{t_D-t}(S_t^x, T-t)$ is continuous. S_t is continuous on $[0, t_D)$ and $[t_D, T]$, thus P is continuous for all t in $[0, t_D)$ and $[t_D, T]$.

(v) The Snell envelope U is by Theorem 3.3.5 continuous. This means we have $\lim_{t\uparrow t_D} U_t = U_{t_D}$ and by (3.44)

$$\lim_{t\uparrow t_D} P_{t_D-t}(S_t^x, T-t) = P(S_{t_D}^x, T-t_D). \tag{3.52}$$

We have $S_{t_D}^x = \lim_{t\uparrow t_D} S_t^x - D$ for a dividend payment and $S_{t_D}^x = \lim_{t\uparrow t_D} S_t^x - \rho S_{t_D}^x$ for a dividend rate. Define $x := \lim_{t\uparrow t_D} S_t^x$ and insert these equations in (3.52), then we get the desired result. □

Definition 3.3.7: The domain $\{(x,t) \in \mathbb{R}^+ \times [0,T)\}$ can be divided into two regions. Let

$$\mathcal{C} := \{(x,t) \in \mathbb{R}^+ \times [0,T); P_{t_D-t}(x, T-t) > (K-x)^+\} \tag{3.53}$$

be the *continuation region* and

$$\mathcal{D} := \{(x,t) \in \mathbb{R}^+ \times [0,T); P_{t_D-t}(x,T-t) = (K-x)^+\} \tag{3.54}$$

the *stopping region*. By Lemma 3.3.6(ii) $P_{t_D-t}(x,T-t) \geq (K-x)^+$ for all $t \in [0,T]$ we have $\mathcal{C} \cup \mathcal{D} = \mathbb{R}^+ \times [0,T)$. We notice that $\mathcal{C} \cap \mathbb{R}^+ \times (0,t_D)$ and $\mathcal{C} \cap \mathbb{R}^+ \times (t_D,T)$ are open since $P_{t_D-t}(x,T-t)$ is continuous on $[0,t_D)$ and $[t_D,T]$.

Proposition 3.3.8: Let $\mathcal{C}_t$ be the t-section of the continuation region, i.e.

$$\mathcal{C}_t := \{x \in \mathbb{R}^+; (x,t) \in \mathcal{C}\} = \{x \in \mathbb{R}^+; P_{t_D-t}(x,T-t) > (K-x)^+\}, \quad \forall t \in [0,T). \tag{3.55}$$

Then, for all $t \in [0,T)$ there exists a number $c(t) \in [0,K)$ such that

$$\mathcal{C}_t = (c(t), \infty). \tag{3.56}$$

PROOF. First we want to show that for every $y > x$ with $x \in \mathcal{C}_t$ we have that $y \in \mathcal{C}_t$. Let $y > x$ with $x \in \mathcal{C}_t$ be given, then we have by Lemma 3.3.6(iii)

$$P_{t_D-t}(y,T-t) \geq P_{t_D-t}(x,T-t) + x - y > (K-x)^+ + x - y \geq K - y. \tag{3.57}$$

From Lemma 3.3.6(ii) we have that $P_{t_D-t}(y,T-t) > 0$ and therefore $P_{t_D-t}(y,T-t) > (K-y)^+$. This shows that $\mathcal{C}_t$ is unbounded from above. Define

$$c(t) = \inf\{x \in \mathbb{R}^+; P_{t_D-t}(x,T-t) > (K-x)^+\}. \tag{3.58}$$

To verify (3.56) we need to check that $c(t) \notin \mathcal{C}_t$. If $c(t) = 0$ we have $P_{t_D-t}(0,T-t) = K$. For $c(t) > 0$ we have by definition of $c(t)$ that $P_{t_D-t}(x,T-t) = (K-x)^+$ for all $x \in (0,c(t))$. By Lemma 3.3.6(iii) $P_{t_D-t}(x,T-t)$ is continuous and therefore $P_{t_D-t}(c(t),T-t) = (K-c(t))^+$. Lemma 3.3.6(ii) shows that $P_{t_D-t}(K,T-t) > 0 = (K-K)^+$, i.e. $K \in \mathcal{C}_t$ and since $\mathcal{C}_t$ is open it follows $c(t) < K$. □

Theorem 3.3.9: The optimal exercise boundary $c(t)$ is continuous in $[0,t_D)$ and $[t_D,T)$.

PROOF. Let $t \in (0,T) \setminus \{t_D\}$ be given, choose any $\eta > 0$ such that $(x,t) \in \mathcal{C}$ for all $x \in [c(t)+\eta, K]$. Since $\mathcal{C} \cap \mathbb{R}^+ \times (0,t_D)$ and $\mathcal{C} \cap \mathbb{R}^+ \times (t_D,T)$ are open, there exists a $\delta > 0$ such that all (x,s) with $x \in [c(t)+\eta, K]$ and $s \in (c(t)-\delta, c(t)+\delta)$ are in $\mathcal{C}$, nd $\mathcal{R} \subset \mathcal{C}$ for the closed rectangle $\mathcal{R} = [c(t)+\eta, K] \times [c(t)-\frac{\delta}{2}, c(t)+\frac{\delta}{2}]$. Therefore there exists an $\varepsilon > 0$, defined by

$$\varepsilon := \min_{(x,s)\in\mathcal{R}} P_{t_D-s}(x,T-s) - (K-x)^+. \tag{3.59}$$

Then we have for all $(x,s) \in \mathcal{R}$

$$P_{t_D-s}(x,T-s) - (K-x)^+ \geq \varepsilon. \tag{3.60}$$

Let $\{s_n\}_{n=1}^{\infty}$ be a sequence such that $\{s_n\}_{n=1}^{\infty} \in [c(t) - \frac{\delta}{2}, c(t) + \frac{\delta}{2}]$ and $\lim_{n\to\infty} s_n = t$, then

$$P_{t_D - s_n}(c(s_n) + \eta, T - s_n) - (K - (c(s_n) + \eta))^+ \geq \varepsilon, \quad \forall n \in \mathbb{N}. \tag{3.61}$$

By Lemma 3.3.6(iv) $P_{t_D-t}(x, T-t)$ is continuous for all $(x,t) \in \mathbb{R}^+ \times [0, t_D)$ and $\mathbb{R}^+ \times [t_D, T]$. Therefore we can take the limit inside and have

$$\begin{aligned} &\lim_{n\to\infty} P_{t_D - s_n}(c(s_n) + \eta, T - s_n) - (K - (c(s_n) + \eta))^+ \\ =&P_{t_D - t}(\lim_{n\to\infty} c(s_n) + \eta, T - t) - (K - (\lim_{n\to\infty} c(s_n) + \eta))^+ \geq \varepsilon > 0. \end{aligned} \tag{3.62}$$

This is true for all $\eta > 0$, implying

$$P_{t_D - t}(\lim_{n\to\infty} c(s_n), T - t) - (K - \lim_{n\to\infty} c(s_n))^+ > 0. \tag{3.63}$$

and hence $\lim_{n\to\infty} c(s_n) \in \mathcal{C}_t$, i.e. $\lim_{n\to\infty} c(s_n) \geq c(t)$ in $(0, t_D)$ and (t_D, T). At 0 and t_D may take the one-side limit using the same method as above (due to Lemma 3.3.6(iv)). Then $\lim_{n\to\infty} c(s_n) \geq c(t)$ in $[0, t_D)$ and $[t_D, T)$.

Now we want to show that $\lim_{n\to\infty} c(s_n) \leq c(t)$ for all t in $[0, T)$. Let $\{s_n\}_{n=1}^{\infty}$ be a sequence with $\lim_{n\to\infty} s_n = t$. By definition of the continuation region

$$(c(s_n), s_n) \notin \mathcal{C}, \quad \forall n \in \mathbb{N}. \tag{3.64}$$

Since $\mathcal{C} \cap \mathbb{R}^+ \times (0, t_D)$ and $\mathcal{C} \cap \mathbb{R}^+ \times (t_D, T)$ are open, we have

$$(\lim_{n\to\infty} c(s_n), t) \notin \mathcal{C} \tag{3.65}$$

and thus $\lim_{n\to\infty} c(s_n) \leq c(t)$ for all t in $[0, T)$. Combining the two results, we have that $\lim_{n\to\infty} c(s_n) = c(t)$ for all t in $[0, t_D)$ and $[t_D, T)$. We define $c(T) := \lim_{t\uparrow T} c(t)$. $\square$

Chapter 4

Free Boundary Formulation

We know by Proposition 3.3.1 and Theorem 3.2.11 that the American Put $P_{t_D}(x,T)$ admits a Doob-Meyer decomposition. The aim of this chapter is to derive this decomposition, which will give us an analytic valuation of the option price.

In section 4.1, we will show that the American option problem can be transformed to an initial boundary value problem on the closure of $\mathcal{C}$, this means that we assume that the optimal exercise boundary c is known. In case of a discrete dividend we actually have a representation of two initial boundary value problems as we will see later. We will show that P is the unique solution to a function $f \in C^{2,1}$, which solves the initial boundary value problem.

We will extend the system found in section 4.1 to the whole domain in such a way, that we are able to use Itô's rule. This extended system will be known as the free boundary problem, since the lower boundary d is unknown. The pair of function (P, c) is the unique solution to (f, d) solving the free boundary problem. By using Itô's rule we can calculate the Doob-Meyer decomposition and find a pricing formula for the American put based on c. The optimal exercise boundary c is the solution to an integral equation and can only be solved numerically.

In section 4.3 we will solve the integral equation using the trapezoidal rule and price the option.

The connection between optimal stopping and the free boundary problem for option pricing was first studied by McKean [15] and van Moerbeke [17]. Section 4.1 and section 4.2 is based on the work of Karatzas and Shreve [12] and Jacka [9] and section 4.3 on Huang, Subrahmanyam and Yu [8].

4.1 Initial Boundary Value Problem

We will use following notations for the derivatives of a given function $f(x,t) \in C^{2,1}$:

$$f_1(x,t) := \frac{\partial}{\partial x} f(x,t), \tag{4.1}$$

$$f_{11}(x,t) := \frac{\partial^2}{\partial x^2} f(x,t), \tag{4.2}$$

$$f_2(x,t) := \frac{\partial}{\partial t} f(x,t). \tag{4.3}$$

Let $\mathcal{L}$ be the operator defined by

$$\mathcal{L}f := \frac{1}{2}\sigma^2 x^2 f_{11} + rxf_1 - rf + f_2. \tag{4.4}$$

The following *initial boundary value problem* consists of finding a solution f of

$$\mathcal{L}f(x,T-t) = 0, \quad x \in \Omega, \quad t \in (0,T), \tag{4.5}$$

$$f(x,T-t) = g(x,T-t), \quad x \in \partial\Omega, \quad t \in [0,T), \tag{4.6}$$

$$f(x,0) = u(x), \quad x \in \bar{\Omega}, \tag{4.7}$$

with given continuous functions g, u. Ω is an open subset of $\mathbb{R}$, $\partial\Omega$ the boundary of Ω and $\bar{\Omega}$ the closure of Ω.

Lemma 4.1.1: Assume that $\mathcal{L}$ is uniformly parabolic on $\Omega \times (0,T)$, that the parameters are Hölder continuous on the closure of $\Omega \times (0,T)$, g, u are continuous on $\partial\Omega \times [0,T]$ and $\bar{\Omega}$ and $g = f$ on $\partial\Omega \times \{0\}$. Then there exists a unique solution f of the initial value problem (4.5) - (4.7). The solution has Hölder continuous derivatives f_{11}, f_1, f_2.

PROOF. See Theorem 6.3.6 of Friedman [5].

Remark 4.1.2: The operator $\mathcal{L}$ does not satisfy the conditions of Lemma 4.1.1, but we can make the change of variable $y = \ln x$ and introduce the new function $h(y, T-t) = f(x, T-t)$. We require that $x > 0$, which is case for our problem. This gives us $f_2(x,T-t) = h_2(y,T-t)$, $f_1(x,T-t) = \frac{1}{x}h_1(y,T-t)$ and $f_{11}(x,T-t) = \frac{1}{x^2}h_{11}(y,T-t) - \frac{1}{x^2}h_1(y,T-t)$. The initial boundary value problem (4.5)-(4.7) becomes

$$\mathcal{G}h(y,T-t) = 0, \quad e^y \in \Omega, \quad t \in (0,T), \tag{4.8}$$

$$h(y,T-t) = g(e^y,T-t), \quad e^y \in \partial\Omega, \quad t \in [0,T), \tag{4.9}$$

$$h(y,0) = u(e^y), \quad e^y \in \bar{\Omega}, \tag{4.10}$$

with $\mathcal{G}h := \frac{1}{2}\sigma^2 h_{11} + \big(r - \frac{1}{2}\sigma^2\big)h_1 - rh + h_2$. The parameters $\mathcal{G}$ are Hölder continuous since they are constant and $\mathcal{G}$ is uniformly parabolic since $\sigma^2 > 0$. Now, we may apply

Lemma 4.1.1 if g, u are continuous, therefore h_{11}, h_1, h_2 are continuous and by the backward transformation f_{11}, f_1, f_2 as well.

Theorem 4.1.3: Define $\mathcal{C}_{t_{D-}} := \mathcal{C} \cap \mathbb{R}^+ \times (0, t_D)$ and $\mathcal{C}_{t_{D+}} := \mathcal{C} \cap \mathbb{R}^+ \times (t_D, T)$. The optimal expected payoff function $P_{t_D - t}(x, T-t)$ with $(x,t) \in \bar{\mathcal{C}} \setminus [\lim_{t \uparrow t_D} c(t), c(t_D)) \times \{t_D\}$ is the unique solution of the initial boundary value problem:

$$\mathcal{L} f(x, T-t) = 0, \quad (x,t) \in \mathcal{C}_{t_{D-}} \cup \mathcal{C}_{t_{D+}}, \tag{4.11}$$

$$f(x, 0) = (K-x)^+, \quad x \in [c(T), \infty), \tag{4.12}$$

$$\lim_{t \uparrow t_D} f(x, T-t) = f((x-D)^+, T-t_D), \quad x \in [\lim_{t \uparrow t_D} c(t), \infty), \tag{4.13}$$

$$f(c(t), T-t) = (K - c(t))^+, \quad t \in [0, T), \tag{4.14}$$

$$\lim_{x \to \infty} f(x, T-t) = 0, \quad t \in [0, T). \tag{4.15}$$

In particular, the partial derivatives P_{11}, P_1 and P_2 exist and are continuous in $\mathcal{C}_{t_{D-}} \cup \mathcal{C}_{t_{D+}}$. Condition (4.13) will be replaced by $\lim_{t \uparrow t_D} f(x, T-t) = f((1-\rho)x, T-t_D)$ if the underlying asset is paying a dividend rate ρ at t_D.

PROOF. P satisfies the initial conditions (4.12) and (4.13), this follows from the definition of the American put and from Lemma 3.3.6(v). The boundary condition (4.14) follows from Proposition 3.3.8. We now prove (4.15), let $(x,t) \in (K, \infty) \times [0, T)$ be given, $x = S_t$ and τ_s^x is the optimal exercise time defined in (3.45), i.e.

$$\tau_t^x := \inf\{s \geq t; P_{t_D - s}(S_s^x, T-s) = (K - S_s^x)^+\}. \tag{4.16}$$

We notice $e^{-rt} P_{t_D - t}(x, T-t) = \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau_t^x}(K - S_{\tau_t^x}^x)^+ \big| \mathcal{F}_t\big)$. Define $\rho_t^x := \inf\{s \geq t; S_s^x \leq K\} \wedge T$, then $\tau_t^x \geq \rho_t^x$ on $\{\rho_t^x < T\}$ since $(K - S_s^x)^+ \in [0, K]$ for all $s \in [t, T)$ and $\tau_t^x = T$ on $\{\rho_t^x = T\}$. Then we have for all $(x,t) \in (K, \infty) \times [0, T)$

$$\begin{aligned} 0 \leq e^{-rt} P_{t_D - t}(x, T-t) &= \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau_t^x}(K - S_{\tau_t^x}^x)^+ \big| \mathcal{F}_t\big) \\ &\leq \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\rho_t^x < T\}} e^{-r\tau_t^x} K \big| \mathcal{F}_t\big) + \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\rho_t^x = T\}} e^{-rT}(K - S_T^x)^+ \big| \mathcal{F}_t\big) \\ &\leq K \cdot \mathbb{Q}\big(\rho_t^x < T \big| \mathcal{F}_t\big). \end{aligned} \tag{4.17}$$

The second inequality is valid since $S_T^x \geq K$ on the set $\{\rho_t^x = T\}$. And thus for all $t \in [0, T)$

$$\lim_{x \to \infty} P_{t_D - t}(x, T-t) \leq \lim_{x \to \infty} e^{rt} K \cdot \mathbb{Q}\big(\rho_t^x < T \big| \mathcal{F}_t\big) = 0. \tag{4.18}$$

Since the probability that $x = S_t$ will hit the boundary K on an finite time interval will decrease to 0 as x tends to infinity.

In order verify (4.11), we first prove that this equation is valid for all $(x,t) \in \mathcal{C}_{t_{D+}}$. Let us consider the initial boundary value problem

$$\mathcal{L} f(x, T-t) = 0, \quad (x,t) \in \mathcal{C}_{t_{D+}}, \tag{4.19}$$

$$f(x, T-t) = P(x, T-t), \quad \partial \mathcal{C}_{t_{D+}} \setminus (c(t_D), \infty) \times \{t_D\}. \tag{4.20}$$

By Lemma 3.3.6(iv) $f(x, T-t)$ is continuous on $\partial\mathcal{C}_{t_D+} \setminus (c(t_D), \infty) \times \{t_D\}$. The existence and uniqueness of a solution to this initial value problem follows from Lemma 4.1.1 and Remark 4.1.2. This gives us that $f(x, T-t) \in C^{2,1}$ on $\mathcal{C}_{t_D+}$. Now we want to show that f and P agree on $\mathcal{C}_{t_D+}$. Let $(x_0, t_0) \in \mathcal{C}_{t_D+}$ be given with $x_0 = S_{t_0}$. Consider the stopping time $\tau \in \mathcal{S}_{t_0,T}$ defined by

$$\tau := \inf\{s \geq t_0 : (S_s^{x_0}, s) \notin \mathcal{C}_{t_D+}\}. \tag{4.21}$$

We notice that $\tau \leq T$. Define the process

$$N_s := e^{-rs} f(S_s^{x_0}, T-s). \tag{4.22}$$

We have by definition of τ that $(S_{s\wedge\tau}^{x_0}, s\wedge\tau) \in \bar{\mathcal{C}}_{t_D+}$ for all $s \geq t_0$, thus we can use Itô's lemma and derive that the stopped process $\{N_{s\wedge\tau}, \mathcal{F}_s; t_0 \leq s \leq T\}$ is a $\mathbb{Q}$-martingale.

$$\begin{aligned} N_{s\wedge\tau} =& N_{t_0\wedge\tau} + \int_{t_0}^{s\wedge\tau} e^{-ru} f_1\big(S_u^{x_0}, T-u\big) dS_u^{x_0} + \int_{t_0}^{s\wedge\tau} \Big(e^{-ru} f_{11}\big(S_u^{x_0}, T-u\big) \frac{\sigma^2 (S_u^{x_0})^2}{2} \\ & - re^{-ru} f\big(S_u^{x_0}, T-u\big) + e^{-ru} f_2\big(S_u^{x_0}, T-u\big)\Big) du, \quad \forall s \in [t_0, T]. \end{aligned} \tag{4.23}$$

Under the probability measure $\mathbb{Q}$ we have $dS_u = rS_u du + \sigma S_u dW_u$ and by (4.11) $\mathcal{L}f(x, T-t) = 0$ for all $(x, t) \in \mathcal{C}$. And thus

$$\begin{aligned} N_{s\wedge\tau} &= N_{t_0\wedge\tau} + \int_{t_0}^{s\wedge\tau} e^{-ru} \sigma S_u^{x_0} f_1\big(S_u^{x_0}, T-u\big) dW_u \\ &= N_{t_0} + \int_{t_0}^{s} e^{-ru} \sigma S_u^{x_0} f_1\big(S_u^{x_0}, T-u\big) dW_u, \end{aligned} \tag{4.24}$$

since $t_0 \leq \tau$ a.s. To check that (4.24) is a $\mathbb{Q}$-martingale we notice, that the stochastic integral is a martingale under $\mathbb{Q}$ and therefore for all $q \geq s \geq t_0$

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}(N_{q\wedge\tau} | \mathcal{F}_s) &= N_{t_0} + \mathbb{E}_{\mathbb{Q}}\Big(\int_{t_0}^{q\wedge\tau} e^{-ru} \sigma S_u^{x_0} f_1\big(S_u^{x_0}, T-u\big) dW_u \Big| \mathcal{F}_s \Big) \\ &= N_{t_0} + \int_{t_0}^{s\wedge\tau} e^{-ru} \sigma S_u^{x_0} f_1\big(S_u^{x_0}, T-u\big) dW_u \\ &= N_{s\wedge\tau}. \end{aligned} \tag{4.25}$$

Since N is bounded by $\partial\mathcal{C}_{t_D+} \setminus (c(t_D), \infty) \times \{t_D\}$ we have $N_{t_0} = N_{t_0\wedge\tau} = \mathbb{E}_{\mathbb{Q}} N_{T\wedge\tau} = \mathbb{E}_{\mathbb{Q}} N_\tau$. We use optimal stopping time $\tau_{t_0}^{x_0}$ defined by (3.45), i.e.

$$\tau_{t_0}^{x_0} := \inf\{s \geq t_0 : P(S_s^{x_0}, T-s) = (K - S_s^{x_0})^+\}. \tag{4.26}$$

We have by (4.12) that $\tau_{t_0}^{x_0} \leq T$ a.s. and by (3.46) that the stopped process

$$\{e^{-r(s\wedge\tau_{t_0}^{x_0})} P(S_{s\wedge\tau_{t_0}^{x_0}}^{x_0}, T - (s\wedge\tau_{t_0}^{x_0})), \mathcal{F}_s; t_0 \leq s \leq T\} \tag{4.27}$$

is a bounded $\mathbb{Q}$-martingale. We notice that the two stopping times defined in (4.12) and (4.26) are the same. This gives us

$$
\begin{aligned}
e^{-rt_0} f(x_0, T-t_0) = N_{t_0} &= \mathbb{E}_{\mathbb{Q}}\big(N_\tau \big| \mathcal{F}_{t_0}\big) = \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau} P(S_\tau^{x_0}, T-\tau)\big|\mathcal{F}_{t_0}\big) \\
&= \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau_{t_0}^{x_0}} P(S_{\tau_{t_0}^{x_0}}^{x_0}, T-\tau_{t_0}^{x_0})\big|\mathcal{F}_{t_0}\big) = e^{-rt_0} P(x_0, T-t_0). \qquad (4.28)
\end{aligned}
$$

The third equality follow from the fact that f and P agree on $\partial\mathcal{C}_{t_{D+}} \setminus (c(t_D), \infty) \times \{t_D\}$. Thus f and P agree on $\bar{\mathcal{C}}_{t_{D+}}$ and hence P_{11}, P_1 and P_2 are defined and continuous and satisfy (4.11) for every point $(x,t) \in \mathcal{C}_{t_{D+}}$.

Now we show that (4.11) is satisfied for every $(x,t) \in \mathcal{C}_{t_{D-}}$. Therefore we define

$$
\widetilde{f}(x, T-t) = \begin{cases} f(x, T-t), & \text{for } (x,t) \in \bar{\mathcal{C}} \cap \mathbb{R}^+ \times [0, t_D), \\ \lim_{t\uparrow t_D} f(x, T-t), & \text{for } (x,t) \in \bar{\mathcal{C}} \cap \mathbb{R}^+ \times \{t_D\}, \end{cases} \qquad (4.29)
$$

and

$$
\widetilde{P}_{t_D-t}(x, T-t) = \begin{cases} P_{t_D-t}(x, T-t), & \text{for } (x,t) \in \bar{\mathcal{C}} \cap \mathbb{R}^+ \times [0, t_D), \\ \lim_{t\uparrow t_D} P_{t_D-t}(x, T-t), & \text{for } (x,t) \in \bar{\mathcal{C}} \cap \mathbb{R}^+ \times \{t_D\}, \end{cases} \qquad (4.30)
$$

Notice that $\widetilde{f}(x, T-t)$ and $\widetilde{P}(x, T-t)$ are defined on $\bar{\mathcal{C}}_{t_{D-}}$. The new initial boundary value problem is

$$
\mathcal{L}\widetilde{f}(x, T-t) = 0, \quad (x,t) \in \mathcal{C}_{t_{D-}}, \qquad (4.31)
$$

$$
\widetilde{f}(x, T-t) = \widetilde{P}_{t_D-t}(x, T-t) \quad (x,t) \in \partial\mathcal{C}_{t_{D-}} \setminus (c(0), \infty) \times \{0\}. \qquad (4.32)
$$

Now we are using the same chain of proof as in the first part. The boundary is continuous, therefore there exist by Lemma 4.1.1 and Remark 4.1.2 a unique solution of (4.31) - (4.32). This gives us that $\widetilde{f}(x, T-t) \in C^{2,1}$ on $\mathcal{C}_{t_{D-}}$. Let $(x_0, t_0) \in \mathcal{C}_{t_{D-}}$ be given with $x_0 = S_{t_0}$. Define the stopping time

$$
\widetilde{\tau} := \inf\{s \geq t_0 : (S_s^{x_0}, s) \notin \mathcal{C}_{t_{D-}}\}, \qquad (4.33)
$$

and the process

$$
M_s := e^{-rs} \widetilde{f}(S_s^{x_0}, T-s). \qquad (4.34)
$$

Using the steps (4.23) - (4.25), we can show that the stopped process $\{M_{s\wedge\widetilde{\tau}}, \mathcal{F}_s; t_0 \leq s \leq t_D\}$ is a $\mathbb{Q}$-martingale. The optimal stopping time is given by

$$
\sigma_{t_0}^{x_0} := \inf\{s \geq t_0 : \widetilde{P}_{t_D-s}(S_s^{x_0}, T-s) = (K - S_s^{x_0})^+\} \wedge t_D. \qquad (4.35)
$$

And by (3.46) we have that the stopped process

$$
\{e^{-r(s\wedge\sigma_{t_0}^{x_0})} \widetilde{P}_{t_D-(s\wedge\sigma_{t_0}^{x_0})}(S_{s\wedge\sigma_{t_0}^{x_0}}^{x_0}, T-(s\wedge\sigma_{t_0}^{x_0})), \mathcal{F}_s; t_0 \leq s \leq t_D\} \qquad (4.36)
$$

is a bounded $\mathbb{Q}$-martingale. Again, the two stopping times $\widetilde{\tau}$ and $\sigma_{t_0}^{x_0}$ the same. This gives us similar to (4.28)

$$\begin{aligned} e^{-rt_0}\widetilde{f}(x_0, T-t_0) &= M_{t_0} = \mathbb{E}_{\mathbb{Q}}\big(M_{\widetilde{\tau}}\big|\mathcal{F}_{t_0}\big) = \mathbb{E}_{\mathbb{Q}}\big(e^{-r\widetilde{\tau}}\widetilde{P}_{t_D-\widetilde{\tau}}(S_{\widetilde{\tau}}^{x_0}, T-\widetilde{\tau})\big|\mathcal{F}_{t_0}\big) \\ &= \mathbb{E}_{\mathbb{Q}}\big(e^{-r\sigma_{t_0}^{x_0}}\widetilde{P}_{t_D-\sigma_{t_0}^{x_0}}(S_{\sigma_{t_0}^{x_0}}^{x_0}, T-\sigma_{t_0}^{x_0})\big|\mathcal{F}_{t_0}\big) = e^{-rt_0}\widetilde{P}_{t_D-t_0}(x_0, T-t_0). \end{aligned} \tag{4.37}$$

And we have that $\widetilde{f}$ and $\widetilde{P}$ agree on $\mathcal{C}_{t_{D-}}$. Since $\widetilde{f} = f$ and $\widetilde{P} = P$ on $\mathcal{C}_{t_{D-}}$ we have that (4.11) is satisfied for every $(x,t) \in \mathcal{C}_{t_{D-}} \cup \mathcal{C}_{t_{D+}}$ and P_{11}, P_1 and P_2 are defined and continuous on $\mathcal{C}_{t_{D-}} \cup \mathcal{C}_{t_{D+}}$. We mentioned above that $P(x, T-t)$ is the unique solution on $\bar{\mathcal{C}}_{t_{D+}}$ and $\widetilde{P}_{t_D-t}(x, T-t)$ is the unique on $\bar{\mathcal{C}}_{t_{D-}}$. By definition $\widetilde{P}_{t_D-t}(x, T-t) = P_{t_D-t}(x, T-t)$ for all $(x,t) \in \bar{\mathcal{C}}_{t_{D-}} \setminus [c(t_D), \infty) \times \{t_D\}$. Since $\lim_{t\uparrow t_D} c(t) \leq c(t_D)$, see Theorem 3.3.9 we have that $P_{t_D-t}(x, T-t)$ is the unique solution of (4.11) - (4.15) for all $(x,t) \in \bar{\mathcal{C}}_{t_{D-}} \setminus [\lim_{t\uparrow t_D} c(t), \infty) \times \{t_D\} \cup \bar{\mathcal{C}}_{t_{D+}} = \bar{\mathcal{C}} \setminus [\lim_{t\uparrow t_D} c(t), c(t_D)) \times \{t_D\}$. □

We change the notation for the put, if we have partial derivatives, i.e. $P^{t_D-t}(x, T-t) := P_{t_D-t}(x, T-t)$ for all $(x,t) \in \mathbb{R}^+ \times [0,T]$, then we may write for partial derivatives $P_1^{t_D-t}(x, T-t) = \frac{\partial}{\partial x}P^{t_D-t}(x, T-t)$.

Lemma 4.1.4: $P_1^{t_D-t}(x, T-t)$ is continuous across the exercise boundary $c(t)$ for every $t \in [0,T)$. In particular, $P_1^{t_D-t}(c(t), T-t) = -1$.

PROOF. Define the stopping time

$$\tau_t^{x+\varepsilon} := \{s \geq t; S_s^{x+\varepsilon} \leq c(s)\} \wedge T, \tag{4.38}$$

for $\varepsilon \geq 0$. Now we choose a pair $(x,t) = (c(t), t)$ with $t \in [0,T)$ and notice that $\lim_{\varepsilon\downarrow 0}\tau_t^{c(t)+\varepsilon} = 0$ for all sample paths. This is true, since the sample paths are decreasing if $\varepsilon \downarrow 0$ and the initial point $(c(t)+\varepsilon, t)$ converges to the point $(c(t), t)$ and therefore $\tau_t^{c(t)} = 0$. We notice that by Proposition 3.3.8, $c(t) \in [0, K)$ for all $t \in [0,T)$. And we have the following inequality for $t < t_D$

$$\begin{aligned} P_{t_D-t}(c(t)+\varepsilon, T-t) =& \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau_t^{c(t)+\varepsilon}}(K - S_{\tau_t^{c(t)+\varepsilon}}^{c(t)+\varepsilon})^+\big|\mathcal{F}_t\big) \\ =& \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau_t^{c(t)+\varepsilon}}(K - S_{\tau_t^{c(t)+\varepsilon}}^{c(t)})^+\big|\mathcal{F}_t\big) \\ &- \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau_t^{c(t)+\varepsilon}}(K - S_{\tau_t^{c(t)+\varepsilon}}^{c(t)})^+ - e^{-r\tau_t^{c(t)+\varepsilon}}(K - S_{\tau_t^{c(t)+\varepsilon}}^{c(t)+\varepsilon})^+\big|\mathcal{F}_t\big) \\ \leq& P_{t_D-t}(c(t), T-t) \\ &- \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_t^{c(t)+\varepsilon} < t_D\}}e^{-r\tau_t^{c(t)+\varepsilon}}\big((K - S_{\tau_t^{c(t)+\varepsilon}}^{c(t)}) - (K - S_{\tau_t^{c(t)+\varepsilon}}^{c(t)+\varepsilon})\big)\big|\mathcal{F}_t\big) \\ &- \mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_t^{c(t)+\varepsilon} \geq t_D\}}e^{-r\tau_t^{c(t)+\varepsilon}}\big((K - S_{\tau_t^{c(t)+\varepsilon}}^{c(t)})^+ - (K - S_{\tau_t^{c(t)+\varepsilon}}^{c(t)+\varepsilon})^+\big)\big|\mathcal{F}_t\big) \end{aligned}$$

$$\begin{aligned}&\le P_{t_D-t}(c(t),T-t)-\varepsilon\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_t^{c(t)+\varepsilon}<t_D\}}e^{-r\tau_t^{c(t)+\varepsilon}}S^1_{\tau_t^{c(t)+\varepsilon}}\big|\mathcal{F}_t\big)\\&=P_{t_D-t}(c(t),T-t)-\varepsilon\mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau_t^{c(t)+\varepsilon}}S^1_{\tau_t^{c(t)+\varepsilon}}\big|\mathcal{F}_t\big)\\&\quad+\varepsilon\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_t^{c(t)+\varepsilon}\ge t_D\}}e^{-r\tau_t^{c(t)+\varepsilon}}S^1_{\tau_t^{c(t)+\varepsilon}}\big|\mathcal{F}_t\big),\end{aligned}\tag{4.39}$$

for $\varepsilon > 0$. To verify the second inequality, we notice that $S_s^{c(t)+\varepsilon} = S_s^{c(t)} + S_s^{\varepsilon}$ for all $s \ge t$ and $t_D \notin [t, s)$. But we used the condition that $\tau_t^{c(t)+\varepsilon} < t_D$ therefore we can use this argument. And it follows that

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}P_1^{t_D-t}(c(t)+\varepsilon,T-t)&=\lim_{\varepsilon\downarrow 0}\frac{P_{t_D-t}(c(t)+\varepsilon,T-t)-P_{t_D-t}(c(t),T-t)}{\varepsilon}\\&\le\lim_{\varepsilon\downarrow 0}\mathbb{E}_{\mathbb{Q}}\big(\mathbb{1}_{\{\tau_t^{c(t)+\varepsilon}\ge t_D\}}e^{-r\tau_t^{c(t)+\varepsilon}}S^1_{\tau_t^{c(t)+\varepsilon}}\big|\mathcal{F}_t\big)\\&\quad-\lim_{\varepsilon\downarrow 0}\mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau_t^{c(t)+\varepsilon}}S^1_{\tau_t^{c(t)+\varepsilon}}\big|\mathcal{F}_t\big)=-1.\end{aligned}\tag{4.40}$$

For $t \in [t_D, T]$, we distinguish between the sets $\{\tau_t^{c(t)+\varepsilon} < T\}$ and $\{\tau_t^{c(t)+\varepsilon} = T\}$. The proof is similar to (4.39). And thus $\lim_{\varepsilon\downarrow 0} P_1^{t_D-t}(c(t)+\varepsilon, T-t) \le -1$ for all $t \in [0, T)$.

By Lemma 3.3.6(i) $x \mapsto P_{t_D-t}(x, T-t)$ is convex and therefore $P_1^{t_D-t}(x, T-t) \le P_1^{t_D-t}(y, T-t)$ whenever $x \le y$. This gives us that $P_1^{t_D-t}(x, T-t) \ge -1$ on $\mathcal{C}$. And finally we have

$$\lim_{\varepsilon\downarrow 0} P_1^{t_D-t}(c(t)+\varepsilon, T-t) = -1. \tag{4.41}$$

□

4.2 Analytic Valuation of the American Put

Lemma 4.2.1 Let $f(x,t) \in C^{1,0}$ and piecewise $f(x,t) \in C^{2,1}$. Then we can apply Itô's Lemma.

PROOF. See Theorem 7.9 of Karaztas and Shreve [12] for the proof.

Theorem 4.2.2 The Snell envelope $U_t = e^{-rt}P_{t_D-t}(S_t^x, T-t)$ has the Doob-Meyer decomposition

$$U_t = M_t - \Lambda_t, \quad 0 \le t \le T, \tag{4.42}$$

where

$$M_t := P_{t_D}(S_0^x, T) + \int_0^t e^{-ru}\sigma S_u^x P_1^{t_D-t}(S_u^x, T-u)dW_u \tag{4.43}$$

is a $\mathbb{Q}$-martingale on $[0, T]$ and

$$\Lambda_t := rK \int_0^t e^{-ru} \mathbb{1}_{\{S_u^x \le c(u)\}} du \tag{4.44}$$

is a nondecreasing process.

PROOF. By Theorem 4.1.3 and Proposition 3.3.8 $P \in C^{2,1}$ piecewise for all (x, t) on $\mathbb{R}^+ \times [0, t_D)$ and $\mathbb{R}^+ \times [t_D, T]$, since $P_{t_D - t}(x, T - t) = (K - x)^+$ on $\mathcal{D}$. By Lemma 4.1.4 and Lemma 3.3.6(iv) $P \in C^{1,0}$ on $\mathbb{R}^+ \times [0, t_D)$ and $\mathbb{R}^+ \times [t_D, T]$. By the previous Lemma we can apply Itô's rule. Define

$$\widetilde{P}(x, T - t) = \begin{cases} P_{t_D - t}(x, T - t), & \text{for } t < t_D, \\ \lim_{t \uparrow t_D} P_{t_D - t}(x, T - t), & \text{for } t = t_D. \end{cases} \tag{4.45}$$

According to Itô's Lemma, we have for $\widetilde{P}(x, T - t)$ on $\mathbb{R}^+ \times [0, t_D]$

$$\begin{aligned} e^{-rt} \widetilde{P}(S_t^x, T - t) &= \widetilde{P}(S_0^x, T) + \int_0^t e^{-ru} \widetilde{P}_1(S_u^x, T - u) dS_u \\ &+ \int_0^t e^{-ru} \widetilde{P}_{11}(S_u^x, T - u) \frac{\sigma^2 (S_u^x)^2}{2} - r e^{-ru} \widetilde{P}(S_u^x, T - u) + e^{-ru} \widetilde{P}_2(S_u^x, T - u) du, \end{aligned} \tag{4.46}$$

for all $t \in [0, t_D]$. And $P(x, T - t)$ on $\mathbb{R}^+ \times [t_D, T]$

$$\begin{aligned} e^{-rt} P(S_t^x, T - t) &= e^{-rt_D} P(S_{t_D}^x, T - t_D) + \int_{t_D}^t e^{-ru} P_1(S_u^x, T - u) dS_u \\ &+ \int_{t_D}^t e^{-ru} P_{11}(S_u^x, T - u) \frac{\sigma^2 (S_u^x)^2}{2} - r e^{-ru} P(S_u^x, T - u) + e^{-ru} P_2(S_u^x, T - u) du, \end{aligned} \tag{4.47}$$

for all $t \in [t_D, T]$. Under the martingale measure $\mathbb{Q}$, we have $dS_t^x = rS_t^x dt + \sigma S_t^x dW_t$. And

$$P_{t_D - t}(S_t^x, T - t) = \mathbb{1}_{\{S_t^x \le c(t)\}} P_{t_D - t}(S_t^x, T - t) + \mathbb{1}_{\{S_t^x > c(t)\}} P_{t_D - t}(S_t^x, T - t).$$

This gives us for all $t \in [0, t_D]$

$$\begin{aligned}
e^{-rt}\widetilde{P}(S_t^x, T-t) =& \widetilde{P}(S_0^x, T-0) + \int_0^t e^{-ru}\widetilde{P}_1(S_u^x, T-u)(rS_u^x du + \sigma S_u^x dW_u) \\
&+ \int_0^t e^{-ru}\widetilde{P}_{11}(S_u^x, T-u)\frac{\sigma^2 (S_u^x)^2}{2} - re^{-ru}\widetilde{P}(S_u^x, T-u) \\
&+ e^{-ru}\widetilde{P}_2(S_u^x, T-u)du \\
=& \widetilde{P}(S_0^x, T) + \int_0^t e^{-ru}\mathcal{L}\widetilde{P}(S_u^x, T-u)du \\
&+ \int_0^t e^{-ru}\sigma S_u^x \widetilde{P}_1(S_u^x, T-u)dW_u \\
=& \widetilde{P}(S_0^x, T) + \int_0^t e^{-ru}\mathbb{1}_{\{S_t^x \le c(t)\}}\mathcal{L}\widetilde{P}(S_u^x, T-u)du \\
&+ \int_0^t e^{-ru}\mathbb{1}_{\{S_t^x > c(t)\}}\mathcal{L}\widetilde{P}(S_u^x, T-u)du \\
&+ \int_0^t e^{-ru}\sigma S_u^x \widetilde{P}_1(S_u^x, T-u)dW_u. \qquad (4.48)
\end{aligned}$$

The same can be done for (4.47), i.e.

$$\begin{aligned}
e^{-rt}P(S_t^x, T-t) = e^{-rt_D}P(S_{t_D}^x, T-t_D) + \int_{t_D}^t e^{-ru}\mathbb{1}_{\{S_t^x \le c(t)\}}\mathcal{L}P(S_u^x, T-u)du \\
+ \int_{t_D}^t e^{-ru}\mathbb{1}_{\{S_t^x > c(t)\}}\mathcal{L}P(S_u^x, T-u)du + \int_{t_D}^t e^{-ru}\sigma S_u^x P_1(S_u^x, T-u)dW_u,
\end{aligned} \qquad (4.49)$$

for all $t \in [t_D, T]$. Equation (4.48) gives us the decomposition of the theorem for $t \in [0, t_D)$, since by Theorem 4.1.3 $\int_0^t e^{-ru}\mathbb{1}_{\{S_u^x > c(u)\}}\mathcal{L}P_{t_D-u}(S_u^x, T-u)du = 0$. And we have $\mathbb{1}_{\{S_u^x \le c(u)\}}\mathcal{L}P_{t_D-u}(S_u^x, T-u) = \mathbb{1}_{\{S_u^x \le c(T-u)\}}rK$ since $P_{t_D-t}(x, T-t) = K-x$ on the stopping region $\mathcal{D}$ and $\widetilde{P}(x, T-t) = P_{t_D-t}(x, T-t)$ for all $(x,t) \in \mathbb{R}^+ \times [0, t_D)$. And thus

$$\begin{aligned}
e^{-rt}P_{t_D-t}(S_t^x, T-t) =& P_{t_D}(S_0^x, T) + \int_0^t e^{-ru}\sigma S_u^x P_1^{t_D-t}(S_u^x, T-u)dW_u \\
&- rK\int_0^t e^{-ru}\mathbb{1}_{\{S_u^x \le c(u)\}}du.
\end{aligned}$$

Now, let $t \in [t_D, T]$. We have by definition of the dynamics of the asset process that $S_{t_D}^x = (1-\rho)S_{t_{D-}}^x$ for the dividend rate and $S_{t_D}^x = (S_{t_{D-}}^x + D)^+$ for the dividend payment. Thus by Lemma 3.3.6(v)

$$\widetilde{P}(S_{t_D}^x, T-t_D) = \lim_{t \uparrow t_D} P_{t_D-t}(S_t^x, T-t) = P(S_{t_D}^x, T-t_D). \qquad (4.50)$$

Now we can combine the two equations (4.48) with $t = t_D$ and (4.49) with (4.50). And therefore

$$\begin{aligned}
e^{-rt}P(S_t^x, T-t) =& e^{-rt_D}\widetilde{P}(S_{t_D}^x, T-t_D) + \int_{t_D}^{t} e^{-ru}\mathbb{1}_{\{S_t^x \le c(t)\}}\mathcal{L}P(S_u^x, T-u)du \\
&+ \int_{t_D}^{t} e^{-ru}\mathbb{1}_{\{S_t^x > c(t)\}}\mathcal{L}P(S_u^x, T-u)du + \int_{t_D}^{t} e^{-ru}\sigma S_u^x P_1(S_u^x, T-u)dW_u \\
=& \widetilde{P}(S_0^x, T) + \int_0^{t_D} e^{-ru}\mathbb{1}_{\{S_t^x \le c(t)\}}\mathcal{L}\widetilde{P}(S_u^x, T-u)du \\
&+ \int_0^{t_D} e^{-ru}\mathbb{1}_{\{S_t^x > c(t)\}}\mathcal{L}\widetilde{P}(S_u^x, T-u)du \\
&+ \int_0^{t_D} e^{-ru}\sigma S_u^x \widetilde{P}_1(S_u^x, T-u)dW_u + \int_{t_D}^{t} e^{-ru}\mathbb{1}_{\{S_t^x \le c(t)\}}\mathcal{L}P(S_u^x, T-u)du \\
&+ \int_{t_D}^{t} e^{-ru}\mathbb{1}_{\{S_t^x > c(t)\}}\mathcal{L}P(S_u^x, T-u)du + \int_{t_D}^{t} e^{-ru}\sigma S_u^x P_1(S_u^x, T-u)dW_u \\
=& P_{t_D}(S_0^x, T) + \int_0^{t} e^{-ru}\sigma S_u^x P_1^{t_D-t}(S_u^x, T-u)dW_u \\
&- rK\int_0^{t} e^{-ru}\mathbb{1}_{\{S_u^x \le c(u)\}}du
\end{aligned} \tag{4.51}$$

since $\widetilde{P}(x, T-t) = P_{t_D-t}(x, T-t)$ for all $(x,t) \in \mathbb{R}^+ \times [0, t_D)$. Again, by Theorem 4.1.3 $\int_0^t e^{-ru}\mathbb{1}_{\{S_u^x > c(u)\}}\mathcal{L}P_{t_D-u}(S_u^x, T-u)du = 0$ and $\mathbb{1}_{\{S_u^x \le c(u)\}}\mathcal{L}P_{t_D-u}(S_u^x, T-u) = \mathbb{1}_{\{S_u^x \le c(T-u)\}}rK$. M is a $\mathbb{Q}$-martingale since $P_{t_D}(S_0^x, T)$ is constant and $\int_0^t e^{-ru}\sigma S_u^x P_1^{t_D-u}(S_u^x, T-u)dW_u$ is a $\mathbb{Q}$-martingale for all $t \in [0,T]$. Λ is nondecreasing since $e^{-ru}rK \ge 0$ and $\Lambda_0 = 0$. Which proves the theorem. □

Proposition 4.2.3 The Snell envelope $U_t = e^{-rt}P_{t_D-t}(S_t^x, T-t)$ admits the representation

$$U_t = \mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(K - S_T^x)^+ \big| \mathcal{F}_t\big) + \mathbb{E}_{\mathbb{Q}}(\Lambda_T - \Lambda_t|\mathcal{F}_t), \quad \forall t \in [0,T]. \tag{4.52}$$

PROOF. From Theorem 4.2.2 we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(K - S_T^x)^+ \big| \mathcal{F}_t\big) &= \mathbb{E}_{\mathbb{Q}}(U_T|\mathcal{F}_t) \\
&= \mathbb{E}_{\mathbb{Q}}(M_T|\mathcal{F}_t) + \mathbb{E}_{\mathbb{Q}}(\Lambda_T|\mathcal{F}_t) \\
&= M_t - \Lambda_t - \mathbb{E}_{\mathbb{Q}}(\Lambda_T - \Lambda_t|\mathcal{F}_t) \\
&= U_t - \mathbb{E}_{\mathbb{Q}}(\Lambda_T - \Lambda_t|\mathcal{F}_t), \quad t \in [0,T].
\end{aligned} \tag{4.53}$$

□

Consider the problem of finding a pair of functions $f : \mathbb{R}^+ \times [0,T] \mapsto \mathbb{R}^+$ and $d : [0,T] \mapsto [0,K]$ such that:

$$
\begin{aligned}
f \text{ is continuous on } \mathbb{R}^+ \times [0,T] \setminus \mathbb{R}^+ \times \{t_D\}, && (4.54)\\
d \text{ is continuous on } [0,t_D) \text{ and } [t_D,T] \text{ with } d(T) := \lim_{t\uparrow T} d(t), && (4.55)\\
\mathcal{L}f(x,T-t) =& 0, \quad (x,t) \in \mathcal{G} \setminus (d(t_D),\infty) \times \{t_D\}, & (4.56)\\
\text{with } \mathcal{G} := \{(x,t) \in \mathbb{R}^+ \times (0,T); x > d(t)\}, &&\\
f(x,T-t) =& (K-x)^+, \quad x \le d(t), \quad t \in [0,T), & (4.57)\\
f(x,T-t) \ge& (K-x)^+, \quad x \in [0,\infty), \quad t \in [0,T), & (4.58)\\
f(x,0) =& (K-x)^+, \quad x \in [0,\infty), & (4.59)\\
\lim_{t\uparrow t_D} f(x,T-t) =& f((x-D)^+, T-t_D), \quad x \in [0,\infty), & (4.60)\\
\lim_{x\to\infty} f(x,T-t) =& 0, \quad t \in [0,T), & (4.61)\\
\lim_{x\downarrow d(t)} f_1(x,T-t) =& -1, \quad t \in [0,T), & (4.62)
\end{aligned}
$$

where $\mathcal{L} := \frac{1}{2}\sigma^2 x^2 f_{11} + rxf_1 - rf + f_2$. Condition (4.60) will be replaced by $\lim_{t\uparrow t_D} f(x,T-t) = f((1-\rho)x, T-t_D)$ if the underlying asset is paying a dividend rate ρ at t_D.

Theorem 4.2.4: The pair of function $\big(P_{t_D-t}(x,T-t), c(t)\big)$ is the unique solution to the free boundary problem (4.54)-(4.62).

PROOF. We know from Lemma 3.3.6(iv), Definition 3.3.7, Proposition 3.3.8, Theorem 4.1.3 and Lemma 4.1.4 that $\big(P_{t_D-t}(x,T-t), c(t)\big)$ solves (4.56) - (4.64). Suppose $\big(f(x,T-t), d(t)\big)$ is any solution to (4.54) - (4.62). By (4.55), (4.56), (4.59), (4.60) and (4.61) we can apply Lemma 4.1.1. This and (4.57) gives us that $f \in C^{2,1}$ piecewise and by (4.54) and (4.62) $f \in C^{1,0}$ on $\mathbb{R}^+ \times [0,T] \setminus (d(t_D),\infty) \times \{t_D\}$. We can use a similar argument as in Theorem 4.2.2 to show that

$$e^{-rt} f(S_t^x, T-t) = M_t^f - \Lambda_t^f, \quad 0 \le t \le T, \tag{4.63}$$

where $M_t^f = f(S_0^x, T) + \int_0^t e^{-ru}\sigma S_u^x f_1(S_u^x, T-u)dW_u$ is a $\mathbb{Q}$-martingale and Λ is a nondecreasing process with $\Lambda_t^f = rK\int_0^t e^{-ru}\mathbb{1}_{\{S_u^x < d(u)\}}du \ge 0$. We notice that by (4.60), $e^{-rt}f(S_t^x, T-t)$ is a nonnegative $\mathbb{Q}$-supermartingale. Thus we can use the optional stopping theorem. Define the $\mathcal{S}_t$ stopping time

$$\tau_t^x = \inf\{s \ge t; S_s^x \le d(s)\} \wedge T. \tag{4.64}$$

We have by (4.57) and (4.59) $f(S^x_{\tau^x_t}, T-\tau^x_t) = (K - S^x_{\tau^x_t})^+$ and $\Lambda^f_{\tau^x_t} = 0$ with $S_t = x$. Choose a $(x,t) \in \mathbb{R}^+ \times [0,T]$ then we have at t

$$\begin{aligned}
e^{-rt}P_{t_D-t}(x,T-t) &= \text{ess sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau}(K-S^x_\tau)^+\big|\mathcal{F}_t\big) \\
&\geq \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau^x_t}(K-S^x_{\tau^x_t})^+\big|\mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau^x_t}f(S^x_{\tau^x_t}, T-\tau^x_t)\big|\mathcal{F}_t\big) \\
&= \mathbb{E}_{\mathbb{Q}}\big(M^f_{\tau^x_t} - \Lambda^f_{\tau^x_t}\big|\mathcal{F}_t\big) = \mathbb{E}_{\mathbb{Q}}\big(M^f_{\tau^x_t}\big|\mathcal{F}_t\big) \\
&= M^f_t \geq M^f_t - \Lambda^f_t = e^{-rt}f(x,T-t),
\end{aligned} \tag{4.65}$$

hence $P_{t_D-t}(x,T-t) \geq f(x,T-t)$. On the other hand, we have by (4.58) and the optional stopping theorem for supermartingales

$$\mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau}(K-S^x_\tau)^+\big|\mathcal{F}_t\big) \leq \mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau}f(S^x_\tau, T-\tau)\big|\mathcal{F}_t\big) \leq e^{-rt}f(S^x_t, T-t), \tag{4.66}$$

for any stopping time $\tau \in \mathcal{S}_t$. And therefore

$$e^{-rt}P_{t_D-t}(x,T-t) = \text{ess sup}_{\tau\in\mathcal{S}_t}\mathbb{E}_{\mathbb{Q}}\big(e^{-r\tau}(K-S^x_\tau)^+\big|\mathcal{F}_t\big) \leq e^{-rt}f(x,T-t). \tag{4.67}$$

This gives us that $P_{t_D-t}(y,T-t) = f(y,T-t)$ for all $(y,t) \in \mathbb{R}^+ \times [0,T]$.

Next we want to show that the function c and d are equal. We have by (4.57) and the equality of P and f for all $x \leq c(t)$ and $t \in [0,T)$

$$(K-x)^+ = P_{t_D-t}(x,T-t) = f(x,T-t). \tag{4.68}$$

And therefore $c(t) \leq d(t)$. The roles of P and f in this argument may be reversed to obtain $c(t) \geq d(t)$ for all $t \in [0,T)$. By (4.55), $c = d$ on $[0,T]$. □

Theorem 4.2.5: The price of an American option paying a discrete dividend rate ρ at $t_D \in (0,T)$ is given by

$$\begin{aligned}
P_{t_D}(x,T) =& Ke^{-rT}N\left(\frac{\ln\big(\frac{K}{(1-\rho)x}\big) - \big(r-\frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) - (1-\rho)xN\left(\frac{\ln\big(\frac{K}{(1-\rho)x}\big) - \big(r+\frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) \\
&+ rK\int_0^{t_D} e^{-ru}N\left(\frac{\ln\big(\frac{c(u)}{x}\big) - \big(r-\frac{\sigma^2}{2}\big)u}{\sigma\sqrt{u}}\right)du \\
&+ rK\int_{t_D}^{T} e^{-ru}N\left(\frac{\ln\big(\frac{c(u)}{(1-\rho)x}\big) - \big(r-\frac{\sigma^2}{2}\big)u}{\sigma\sqrt{u}}\right)du,
\end{aligned} \tag{4.69}$$

where $N(x) = \int_0^x \frac{\exp(-z^2/2)}{\sqrt{2\pi}}dz$ is the standard normal density function.

PROOF. We use Proposition 4.2.3 with $t = 0$, this gives us

$$\begin{aligned}
P_{t_D}(x,T) &= \mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(K-S^x_T)^+\big) + \mathbb{E}_{\mathbb{Q}}\big(rK\int_0^T e^{-ru}\mathbb{1}_{\{S^x_u\leq c(u)\}}du\big) \\
&= p_{t_D}(x,T) + e(0),
\end{aligned} \tag{4.70}$$

where $p_{t_D}(x,T)$ is the corresponding European put and

$$e(t) := \mathbb{E}_{\mathbb{Q}}\big(rK\int_t^T e^{-ru}\mathbb{1}_{\{S_u^x \le c(u)\}}du\big) \tag{4.71}$$

is the *early exercise premium*, introduced in Chapter 2. The value of $p_{t_D}(x,T)$ is according to (2.21),

$$p_{t_D}(x,T) = Ke^{-rT}N\left(\frac{\ln\big(\frac{K}{(1-\rho)x}\big) - \big(r - \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right) - (1-\rho)xN\left(\frac{\ln\big(\frac{K}{(1-\rho)x}\big) - \big(r + \frac{\sigma^2}{2}\big)T}{\sigma\sqrt{T}}\right). \tag{4.72}$$

The early exercise premium is the value of an income process that pays a rate rK whenever the stock price is in the region where the put should be exercised. The stock price process S was defined at (2.6), i.e.

$$S_t^x = \begin{cases} \widetilde{S}_t^x, & \text{for } t < t_D, \\ \frac{(1-\rho)\widetilde{S}_{t_D-}^x}{\widetilde{S}_{t_D}^x}\widetilde{S}_t^x, & \text{for } t \ge t_D, \end{cases} \tag{4.73}$$

where $\widetilde{S}_t^x$ is a geometric Brownian motion. Thus

$$\begin{aligned} e(0) =& \mathbb{E}_{\mathbb{Q}}\big(rK\int_0^{t_D} e^{-ru}\mathbb{1}_{\{S_u^x \le c(u)\}}du\big) + \mathbb{E}_{\mathbb{Q}}\big(rK\int_{t_D}^T e^{-ru}\mathbb{1}_{\{S_u^x \le c(u)\}}du\big) \\ =& rK\int_0^{t_D} e^{-ru}\mathbb{Q}\big(\widetilde{S}_u^x \le c(u)\big)du + rK\int_{t_D}^T e^{-ru}\mathbb{Q}\left(\frac{(1-\rho)\widetilde{S}_{t_D-}^x}{\widetilde{S}_{t_D}^x}\widetilde{S}_u^x \le c(u)\right)du \\ =& rK\int_0^{t_D} e^{-ru}N\left(\frac{\ln\big(\frac{c(u)}{x}\big) - \big(r - \frac{\sigma^2}{2}\big)u}{\sigma\sqrt{u}}\right)du \\ &+ rK\int_{t_D}^T e^{-ru}N\left(\frac{\ln\big(\frac{c(u)}{(1-\rho)x}\big) - \big(r - \frac{\sigma^2}{2}\big)u}{\sigma\sqrt{u}}\right)du, \end{aligned} \tag{4.74}$$

since $\widetilde{S}_{t_D-}^x = \widetilde{S}_{t_D}^x$. This proves the theorem. □

Theorem 4.2.6: The price of an American option paying a discrete dividend of magnitude D at $t_D \in (0,T)$ is given by

$$
\begin{aligned}
P_{t_D}(x,T) =& e^{-rt_D}\Big(\int_D^\infty p(S-D,T-t_D)\xi(S,t_D;x)dS + K\int_0^D \xi(S,t_D;x)dS\Big) \\
&+ rK\int_0^{t_D} e^{-ru}\mathbb{Q}\big(S_u^x \le c(u)\big)du \\
&+ rK\int_{t_D}^T e^{-ru}\int_D^\infty \mathbb{Q}\big(S_u - D \le c(u)\big)\xi(S,t_D;x)dS\,du \\
&+ rK\int_{t_D}^T e^{-ru}\int_0^D \xi(S,t_D;x)dS\,du,
\end{aligned}
\tag{4.75}
$$

where S_u for $u \in [t_D, T]$ is the asset price given that $S_{t_D} = S$.

PROOF. Again, by Proposition 4.2.3 with $t = 0$, this gives us

$$
\begin{aligned}
P_{t_D}(x,T) &= \mathbb{E}_{\mathbb{Q}}\big(e^{-rT}(K - S_T^x)^+\big) + \mathbb{E}_{\mathbb{Q}}\big(rK\int_0^T e^{-ru}\mathbb{1}_{\{S_u^x \le c(u)\}}du\big) \\
&= p_{t_D}(x,T) + e(0),
\end{aligned}
\tag{4.76}
$$

The European put is given by (2.22), i.e.

$$
p_{t_D}(x,T) = e^{-rt_D}\Big(\int_D^\infty p(S-D,T-t_D)\xi(S,t_D;x)dS + K\int_0^D \xi(S,t_D;x)dS\Big) \tag{4.77}
$$

The early exercise premium can be divided into the two intervals $[0,t_D)$ and $[t_D,T]$. On $[0,t_D)$ the stock price process S follows a geometric Brownian motion. On second interval we have to check whether the dividend D exceeds the stock price at t_D. If $S_{t_D} \le D$ then the early exercise premium is added on the whole interval $[t_D,T]$, otherwise we calculate the probability that $S_u < c(u)$ given that $S_{t_D} = S$. □

We notice that the unknown function c is needed to price the American put in Theorem 4.2.5 and 4.2.6. c can be calculated recursively using an integral equation.

Proposition 4.2.7: The optimal exercise boundary c satisfies the following integral equation,

$$\begin{aligned}K - c(t) = &Ke^{-r(T-t)}N\left(\frac{\ln\left(\frac{K}{(1-\rho)c(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\ &- (1-\rho)c(t)N\left(\frac{\ln\left(\frac{K}{(1-\rho)c(t)}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\ &+ rK\int_t^{t_D} e^{-r(u-t)}N\left(\frac{\ln\left(\frac{c(u)}{c(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(u-t)}{\sigma\sqrt{u-t}}\right)du\\ &+ rK\int_{t_D}^{T} e^{-r(u-t)}N\left(\frac{\ln\left(\frac{c(u)}{(1-\rho)c(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(u-t)}{\sigma\sqrt{u-t}}\right)du,\end{aligned} \tag{4.78}$$

if a dividend of rate ρ is paid at t_D. And

$$\begin{aligned}K - c(t) =&e^{-r(t_D-t)}\Big(\int_D^{\infty} p(S-D, T-t_D)\xi(S, t_D-t; c(t))dS + K\int_0^D \xi(S, t_D-t; c(t))dS\Big)\\ &+ rK\int_t^{t_D} e^{-r(u-t)}\mathbb{Q}\big(S_u^x \le c(u)\big)du\\ &+ rK\int_{t_D}^{T} e^{-r(u-t)}\int_D^{\infty}\mathbb{Q}\big(S_u - D \le c(u)\big)\xi(S, t_D-t; c(t))dS\,du\\ &+ rK\int_{t_D}^{T} e^{-r(u-t)}\int_0^D \xi(S, t_D-t; c(t))dS\,du,\end{aligned} \tag{4.79}$$

if a dividend of magnitude D is paid at t_D. We notice that (4.78) and (4.79) are valid for $t \in [0, t_D)$.

PROOF. We apply the boundary condition $P_{t_D-t}(c(t), T-t) = K - c(t)$ to the put price formula (4.69) or (4.75). □

For $t \in [t_D, T]$, we have the non-dividend model, and the integral equation becomes

$$\begin{aligned}K - c(t) = &Ke^{-r(T-t)}N\left(\frac{\ln\left(\frac{K}{c(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\ &- c(t)N\left(\frac{\ln\left(\frac{K}{c(t)}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)\\ &+ rK\int_t^{T} e^{-r(u-t)}N\left(\frac{\ln\left(\frac{c(u)}{c(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(u-t)}{\sigma\sqrt{u-t}}\right)du.\end{aligned} \tag{4.80}$$

Lemma 4.2.8: We have following limits for the optimal exercise boundary, $c(T) := \lim_{t\uparrow T} c(t) = K$ and $\lim_{t\uparrow t_D} c(t) = 0$.

PROOF. In order to investigate the behavior of c near expiration, we take the limit of equation (4.80) the integral vanishes since integral function is continuous and we have

$$\begin{aligned}\lim_{t\uparrow T} K - c(t) =& \lim_{t\uparrow T} Ke^{-r(T-t)} N\left(\frac{\ln\left(\frac{K}{c(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ &- \lim_{t\uparrow T} c(t) N\left(\frac{\ln\left(\frac{K}{c(t)}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\ =& KN\left(\lim_{t\uparrow T}\frac{\ln\left(\frac{K}{c(t)}\right)}{\sigma\sqrt{T-t}} - \lim_{t\uparrow T}\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{T-t}\right) \\ &- \lim_{t\uparrow T} c(t) N\left(\lim_{t\uparrow T}\frac{\ln\left(\frac{K}{c(t)}\right)}{\sigma\sqrt{T-t}} - \lim_{t\uparrow T}\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)\sqrt{T-t}\right) \\ =& KN\left(\lim_{t\uparrow T}\frac{\ln\left(\frac{K}{c(t)}\right)}{\sigma\sqrt{T-t}}\right) - \lim_{t\uparrow T} c(t) N\left(\lim_{t\uparrow T}\frac{\ln\left(\frac{K}{c(t)}\right)}{\sigma\sqrt{T-t}}\right). \end{aligned} \tag{4.81}$$

Now, define $x := \lim_{t\uparrow T}(\ln\left(\frac{K}{c(t)}\right))/(\sigma\sqrt{T-t})$. The limit exist, thus $x \in [-\infty, \infty]$ and therefore $N(x) \in [0, 1]$. And we get

$$\big(1 - N(x)\big)K = \big(1 - N(x)\big) \lim_{t\uparrow T} c(t). \tag{4.82}$$

And we have $c(T) := \lim_{t\uparrow T} c(t) = K$.

To prove the second part, we have by Lemma 3.3.6(v) and 3.3.6(i)

$$\lim_{t\uparrow t_D} P_{t_D-t}(x, T-t) = P((x-D)^+, T-t_D) > P(x, T-t_D) \geq (K-x)^+, \quad x \in \mathbb{R}^+. \tag{4.83}$$

In Assumption 2.1.5, we set $S_t^x = 0$ for all $t \in [t_D, T]$ if $\lim_{t\uparrow t_D} S_t^x < D$. Therefore we have for all x that $\lim_{t\uparrow t_D} P_{t_D-t}(x, T-t) > (K-x)^+$ and this gives us $\lim_{t\uparrow t_D} c(t) = 0$. In case of a dividend rate (4.83) becomes

$$\lim_{t\uparrow t_D} P_{t_D-t}(x, T-t) = P((1-\rho)x, T-t_D) > P(x, T-t_D) \geq (K-x)^+, \quad x \in \mathbb{R}^+, \tag{4.84}$$

and we also get $\lim_{t\uparrow t_D} c(t) = 0$. □

4.3 Numerical Solution of the Integral Equation

The integral equation can be solved with a recursive method. We approximate the integral by the trapezoidal rule. In this section we follow the paper by Huang, Subrahmanyan and Yu [8]. We focus on the case that the stock S is paying a dividend rate ρ

at t_D, the other case is similar. Let us divide the interval $[0, T]$ into n equally spaced subintervals t_i for $i = 0, 1, 2, \dots, n$, with endpoints $t_0 = 0$ and $t_n = T$. We denote the integral function and the put by

$$f(c(t), c(u), u-t, \rho) = rKe^{-r(u-t)}N\left(\frac{\ln\left(\frac{c(u)}{(1-\rho)c(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(u-t)}{\sigma\sqrt{u-t}}\right), \quad (4.85)$$

$$p(c(t), t, \rho) = Ke^{-r(T-t)}N\left(\frac{\ln\left(\frac{K}{(1-\rho)c(t)}\right) - \left(r - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right)$$
$$-(1-\rho)c(t)N\left(\frac{\ln\left(\frac{K}{(1-\rho)c(t)}\right) - \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}\right). \quad (4.86)$$

Let $t_1 = T$, the integral can be approximate by

$$\int_0^T f(c(0), c(u), u, 0)du \approx \frac{T}{2}\Big(f(c(0), c(0), 0, 0) + f(c(0), c(T), T, 0)\Big). \quad (4.87)$$

Now we have following non-linear algebraic equation

$$K - c(0) = p(c(0), 0, 0) + \frac{T}{2}\Big(f(c(0), c(0), 0, 0) + f(c(0), c(T), T, 0)\Big). \quad (4.88)$$

We have by Lemma 4.2.8 that $c(T) = K$. Then, the solution can be found by any root-finding method.
The general solution for $0 \leq k \leq n-1$ is

$$K - c(t_k) = p(c(t_k), t_k, 0) + \frac{T}{2n}\Big(f(c(t_k), c(t_k), 0, 0)$$
$$+ f(c(t_k), c(T), T - t_k, 0) + 2\sum_{i=1}^{n-k-1} f(c(t_k), c(t_{k+i}), t_{k+i} - t_k, 0)\Big), \quad (4.89)$$

with $\sum_{i=1}^{0} f = 0$. We can solve $c(t_k)$ recursively, moving k from $n-1$ to 0. Now let $t_0 = 0$, $t_{m+n} = T$ and $t_m = t_D$, at t_D a dividend of ρS_{t_D} will be paid. The optimal exercise boundary c is discontinuous at t_D, therefore we set $t_m = t_D$. The solution for the boundary for $k \geq m$ can be found by the method above. At time point t_{m-1} we are one step away from the dividend date. After the dividend the stock is reduced by $(1-\rho)S_{t_D}$ this gives us following equation

$$K - c(t_{m-1}) = p(c(t_{m-1}), t_{m-1}, \rho) + \frac{T}{2(m+n)}\Big(f(c(t_{m-1}), c(t_m), t_m - t_{m-1}, \rho)$$
$$+ f(c(t_{m-1}), c(T), T - t_{m-1}, \rho) + 2\sum_{i=1}^{n-1} f(c(t_{m-1}), c(t_{m+i}), t_{m+i} - t_{m-1}, \rho)\Big)$$
$$+ \frac{T}{2(m+n)}\Big(f(c(t_{m-1}), c(t_{m-1}), 0, 0) + f(c(t_{m-1}), c(t_m), t_m - t_{m-1}, 0)\Big). \quad (4.90)$$

The general solution for $0 \le k \le m-1$ is

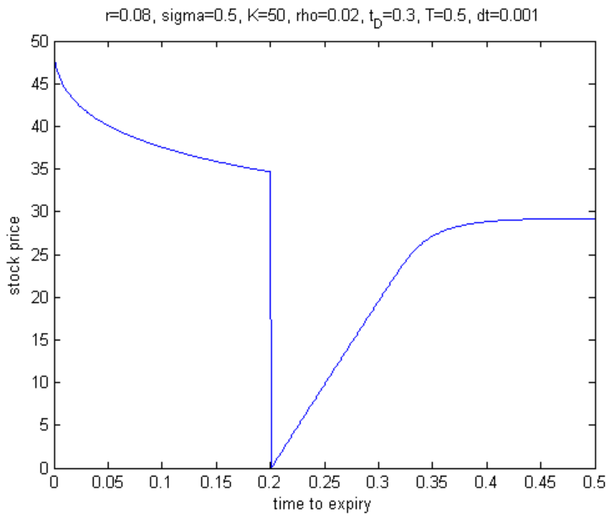
$$\begin{aligned} K - c(t_k) = p(c(t_k), t_k, \rho) &+ \frac{T}{2(m+n)}\Big(f(c(t_k), c(t_m), t_m - t_k, \rho) \\ &+ f(c(t_k), c(T), T - t_k, \rho) + 2\sum_{i=1}^{n-1} f(c(t_k), c(t_{m+i}), t_{m+i} - t_k, \rho)\Big) \\ &+ \frac{T}{2(m+n)}\Big(f(c(t_k), c(t_k), 0, 0) + f(c(t_k), c(t_m), t_m - t_k, 0) \\ &+ 2\sum_{j=1}^{m-k-1} f(c(t_k), c(t_{k+j}), t_{k+j} - t_k, 0)\Big). \end{aligned} \tag{4.91}$$

After the boundaries $c(k)$ have been calculated the price can be approximated by

$$\begin{aligned} P_{t_D}(x, T) = p(x, 0, \rho) &+ \frac{T}{2(m+n)}\Big(f(x, c(0), 0, 0) + f(x, c(t_m), t_m, 0) \\ &+ 2\sum_{j=1}^{m-1} f(x, c(t_j), t_j, 0) + f(x, c(t_m), t_m, \rho) + f(x, c(T), T, \rho) \\ &+ 2\sum_{i=1}^{n-1} f(x, c(t_{m+i}), t_{m+i}, \rho)\Big). \end{aligned} \tag{4.92}$$

The formula (4.92) also offers a simple way to calculate the hedge parameters. See Huang, Subrahmanyan and Yu [8].

This gives us the following optimal exercise boundary.



These numerical results correspond to values calculated by Meyer [16] using the method of lines for the Black-Scholes equation.

Chapter 5

Conclusion

To evaluate the price of American options on assets that pay a discrete dividend, we have analyzed the corresponding optimal stopping problem. And we were able to find the optimal stopping time, which turned out to be similar to the non-dividend case. Furthermore, we could decompose the American put into the European put and the early exercise premium. Even though the American put is discontinuous at t_D, the date the dividend is paid, we proved the interface condition, which glues the ”between dividend periods” together. Using the decomposition we could price the option.

To-do list:

Derive more properties of the optimal exercise boundary, such as maxima or inflection points. The optimal exercise boundary on an asset that is paying a discrete dividend seems to be more interesting than on non-dividend paying asset, for which the boundary is monotone.

A proof of Itô’s lemma for function which are in $C^{1,0}$ and piecewise $C^{2,1}$. The proof of lemma 4.2.1 is too shaky.

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