String topology operations

with three postscripts:

Constructing higher string operations using radial slit configurations

The string topology structure of the Lie groups $SU(n)$, $U(n)$, $Sp(n)$, G_2 and F_4

An elementary proof of the string topology structure of compact oriented surfaces

A.P.M. Kupers
3021009

ABSTRACT. String topology studies the topology of the free loop space of a manifold and related spaces. The merging and splitting of these strings endows the homology of these spaces with a rich structure. In this thesis we give a construction of these string operations in the more general case where one also allows open strings with endpoints restricted to submanifolds, known as branes. By doing this we solve Godin's conjecture A about the existence of string operations in this general case. Finally, we discuss results on explicitly determining the structure coming from these operations.

In three postscripts, we extend this work: (1) we calculate the string topology structure of some Lie groups, (2) of compact oriented surfaces and (3) give an alternative simpler construction of these operations based on Bödigheimer's radial slit configuration model of the moduli space of Riemann surfaces with boundary.

Contents

Preface

This is the master thesis written for my degree in Mathematical Sciences. It was written in 2010 and 2011 under the supervision of prof. dr. Ieke Moerdijk and dr. Urs Schreiber.

The first motivation for writing my thesis about string topology was that I wanted to learn about a topic that would introduce me to many different parts of algebraic topology and thus allow me to pick up valuable tools along the way. The construction of string operations certainly has allowed me to do this; the subjects that one needs to know include operads and props, spectra and parametrized spectra, mapping class groups and moduli spaces of Riemann surfaces, classifying spaces of categories, and the differential geometry of mapping spaces. It was worthwhile and interesting to master these tools and to take the time to write down the relevant parts for this thesis.

The second motivation for writing a thesis about this topic was to prove a new and interesting result. A proof of Godin's conjecture A [**God07**] seemed tractable at the time and, indeed, this thesis proves it and derives some of its consequences. The problem with this proof is that to do it rigorously, one needs to check a lot of compatibility conditions, which takes time to do and more time to write down in a comprehensible and instructive manner. Urs' many comments and questions certainly helped in my attempts to do this.

The current length of the thesis is the result of balancing these two forces which compel one to write a very long thesis, with the need to finish a thesis at some point. For readers that are not familiar with string topology, I have tried to include enough details such that they do not need to read secondary literature all the time. On the other hand, for more experienced readers, I have tried to make clear distinctions between the main arguments and the details and background information. The concrete implementation of this is by including separate sections containing technical details and separate appendices containing background information.

This thesis wouldn't have been possible without the support of several people. First of all, I would like to thank Ieke Moerdijk for suggesting this topic and supervising the thesis, but most importantly for providing a lot of support in getting to know the mathematical community and finding a PhD position. Secondly, I would like to thank – in no particular order – Andre Henriques, Urs Schreiber, Richard Hepworth and Veronique Godin for helping with some of the mathematical aspects of this thesis. Thirdly, I would like to thank – again in no particular order – Kate Poirier, Ulrike Tillmann, Christoper Douglas, Ralph Cohen, Soren Galatius, Kevin Costello and Carl-Friedrich Bödigheimer for taking the time to discuss various aspects of this thesis with me. Finally, I would like to thank all my mathematical and non-mathematical friends and my family for the fun times we had during the writing of this thesis, in particular Tjerk, Thessa, Quirine, Johan and Gijs. But most importantly, this thesis wouldn't have been possible without the support of Tessa; bedankt dat ik dagen vol grafiekjes mocht tekenen en gestresst mocht zijn.

Sander Kupers, 2011

CHAPTER 1

Introduction

1. History and motivation

We will discuss the history of string topology, while paying attention to the reasons that people were interested in it. The main player in all of this is the free loop space LM. A rough sketch of relations between it and various fields in mathemathics is given in figure 1.1.

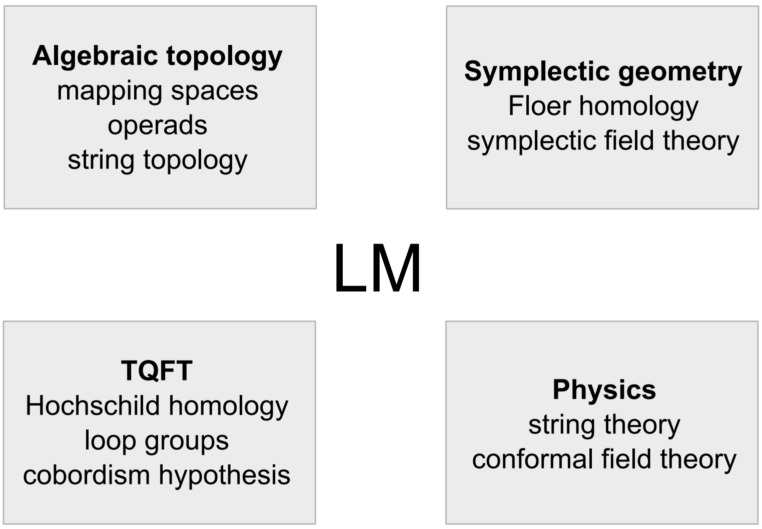


FIGURE 1.1. This diagram gives an overview of topics related to the free loop space LM.

There are many reasons why people in the 90's were interested in the spaces LM of continuous loops in a compact oriented manifold M. First of all, the study of iterated based loop spaces had paid off tremendously, because it required the introduction of the general theory of operads and their algebras and the important examples of the little disks operads. It is easy to see why it was tempting to figure out whether there are similar structures related to free loop spaces. This might explain why algebraic topologists were interested in LM.

On the other hand, there had been a convergence of results related to free loop spaces in other subjects. For example, loop groups play an important role in conformal field theory and the Hochschild homology of the A_∞ algebra of cochains on a manifold is isomorphic to the chains on the free loop space. For references in this direction see the references of [**Vai07**].

Another example of related results is in the direction of symplectic geometry. One can define the degree zero string operations using symplectic geometry and there are conjectures relating string topology and symplectic field theory. A good starting point for this and other interesting connections string topology has to other fields is the little book by Cohen and Voronov [**CV06**].

Our discussion of string topology below focusses on the construction of string topology that we will use in this thesis, as used by Godin [**God07**]. There are related but alternative constructions using the theory of Hilbert manifolds [**Cha03**] and stratifolds [**Mei09**].

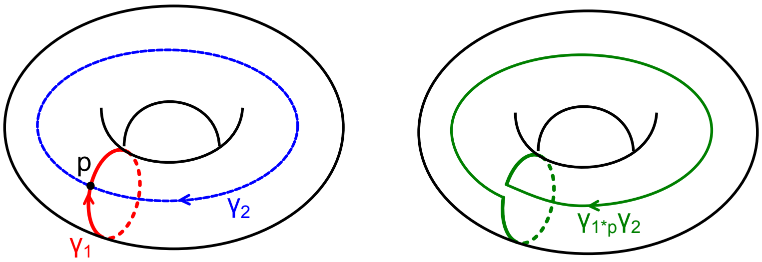


FIGURE 1.2. The left hand side shows two representatives of isotopy classes of closed curves on a surface of genus 1 and the right hand side their Goldman bracket. We have chosen to depict a slightly different representative than $\gamma_1 *_p \gamma_2$ itself in its isotopy class, to make the picture clearer.

1.1. The Goldman bracket and the Chas-Sullivan operations. But it seems that the inspiration for string topology came from low-dimensional geometric topology. In the foundational article on string topology by Chas and Sullivan [**CS99**] they state that they discovered the first string topology operations while working on the Goldman bracket in the geometry of surfaces.

Although classically Lie algebras are defined over a field, it is possible to define them over $\mathbb{Z}$ as well. To see this, simply note that the Lie operad can be defined in the category of abelian groups and then a Lie algebra over $\mathbb{Z}$ is an algebra over the Lie operad in abelian groups. The Goldman bracket is a Lie bracket on the free abelian group with generators given by the isotopy classes of closed curves in a compact oriented surface Σ_g [**Gol86**]. Here a closed curve is simply a continuous map $S^1 \to \Sigma_g$ and hence has a base point. The Goldman bracket is defined as follows: if $[\gamma_1]$ and $[\gamma_2]$ are two such isotopy classes, then we can find transversally intersecting representatives γ_1 and γ_2. Then their bracket is given by

$$\{[\gamma_1], [\gamma_2]\} = \sum_{p \in \gamma_1 \cap \gamma_2} \mathrm{sign}(p)[\gamma_1 *_p \gamma_2]$$

where $\mathrm{sign}(p)$ is the sign of the intersection, which is $+1$ is $(T_p\gamma_1, T_p\gamma_2)$ is an oriented basis of $T_p\Sigma_g$ and -1 otherwise, and $\gamma_1 *_p \gamma_2$ is the closed curve obtained by starting by moving along γ_1 from the base point until one meets p, then traversing γ_2 to eventually return to p and continue along γ_1 until one is back at the base point. See figure 1.2.

Chas and Sullivan noticed that the free abelian group with generators the closed curves is naturally isomorphic with $H_0(L\Sigma_g)$, the free abelian group with basis the connected components of $L\Sigma_g$. They were interested in the question what happened if one tried to do something similar to the Goldman bracket for homology groups of higher degree. The reason they were interested in this is that the definition of the Goldman bracket uses the smooth structure when it chooses transversally intersecting representatives. This meant that there was a chance that similar operations might be able to distinguish different smooth structures that exist on the same topological manifold if its dimension is greater than or equal to 4.

What Chas and Sullivan found was a product on all homology of LM and a derivation of this product which could be used to recover the Goldman bracket. We will first describe the string product. If $a \in H_i(LM)$ and $b \in H_j(LM)$ then one can find representing i- and j-chains α and β. There is a natural map $\mathrm{ev} : LM \to M$ given by evaluation at the base point. If one applies this map to α and β one gets a i-chain $\mathrm{ev}(\alpha)$ and $\mathrm{ev}(\beta)$ on M. The transveral intersection of $\mathrm{ev}(\alpha)$ and $\mathrm{ev}(\beta)$ will be $((i+j)-d)$-chain if M is d-dimensional and we denote this chain suggestively by $\mathrm{ev}(\delta)$. To make this well-defined, $\mathrm{ev}(\alpha)$ and $\mathrm{ev}(\beta)$ have to be deformed within their homology classes in such a way that they intersect transversally. One can lift these deformations to α and β. Indeed, it is the chain obtained by applying ev to a $((i+j)-d)$-chain δ in LM. This chain δ is given by putting above $x \in \mathrm{ev}(\delta)$ the loop obtained by concatenating α_x and β_x at their common base point x. See figure 1.3. We define the string product $\cdot$ as follows:

$$a \cdot b = [\alpha] \cdot [\beta] := [\delta]$$

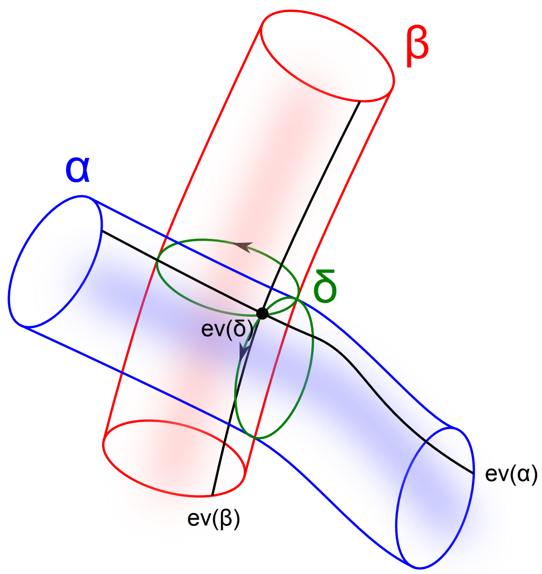


FIGURE 1.3. The Chas-Sullivan product for two 1-chains α and β.

Theorem 1.4 (Chas-Sullivan)**.** *This construction gives a well-defined product*

$$\cdot : H_*(LM) \otimes H_*(LM) \to H_{*-d}(LM)$$

with which we mean that is associative and graded-commutative.

To recover the Goldman bracket, they introduced another string topology operation which is now called the BV-operator. To define this operator, we use the rotation map ρ:

$$\begin{aligned} \rho : S^1 \times LM &\to LM \\ (\theta, \gamma) &\mapsto (t \mapsto \gamma(\theta + t)) \end{aligned}$$

Then the BV-operator $\Delta : H_*(LM) \to H_{*+1}(LM)$ is given by $a \mapsto \rho_*([S^1] \times a)$. In other words, Δ takes an i-chain and makes it into a $(i+1)$-chain by adding all rotations of the loops in the i-chain as well. We claim that the Goldman bracket $\{[\gamma_1], [\gamma_2]\}$ can be recovered using the string product and the BV-operator as follows.

Lemma 1.5. *The Goldman bracket $\{[\gamma_1], [\gamma_2]\}$ is given by the string product $\Delta([\gamma_1]) \cdot \Delta([\gamma_2])$, where we identify the free abelian group on the isotopy classes of closed curves on Σ_g with $H_0(L\Sigma_g)$.*

PROOF. To see this, pick representatives γ_1 and γ_2 of $[\gamma_1]$ and $[\gamma_2]$ respectively, which intersect transversally. Then note that $\Delta([\gamma_1])$ and $\Delta([\gamma_2])$ are representated by the 1-chains $\theta \mapsto (t \mapsto \gamma_1(\theta + t))$ and $\theta \mapsto (t \mapsto \gamma_2(\theta + t))$ on $L\Sigma_g$. Thus the 1-chains $\mathrm{ev}(\Delta([\gamma_1]))$ and $\mathrm{ev}(\Delta([\gamma_2]))$ on M are given by $\theta \mapsto \gamma_1(\theta)$ and $\theta \mapsto \gamma_2(\theta)$ and their intersection is exactly $\gamma_1 \cap \gamma_2$. Above each of the points p in this set we exactly put the concatenation $\gamma_1 *_p \gamma_2$. □

For more results about the string topology of surfaces, see the results of Vaintrob [**Vai07**] described in section 6.3.1.

Chas-Sullivan also described derivative operations. The most important of these a Lie bracket which measures the amount to which the BV-operator, shown in proposition 6.92 to be a second order derivation of the string product, fails to be a first order derivation. We define it in definition 6.93. In retrospect, Chas-Sullivan found the basic degree zero string operations, i.e. the string product, and the basic degree one string operations, the BV-operator.

1.2. The Cohen-Jones construction. Cohen-Jones [**CJ02**] found a different definition of these operations. They were motivated by two things. First of all, although the Chas-Sullivan definition of the BV-operator is purely homotopy-theoretical, the string product uses transverse intersections. It seems reasonable that a purely homotopy-theoretical definition exists and then the machinery of homology would absorb much of the difficulties with the arguments using transverse intersection. Secondly, a homotopy-theoretical definition opens up a line of attack to the problem of showing whether the string product distinguishes smooth structures.

Their definition of the string product is indeed less involved than Chas and Sullivans. It is equal to the one we call the closed coproduct in section 3.1.1. For convenience of the reader and to show that the Cohen-Jones construction is pretty straightforward, we explain it here as well.

Let $8 = S^1 \vee S^1$ be the wedge sum of two circles. Then we have a "fold" map $j : S^1 \to 8$ given by traversing the two circles in order and a "inclusion" map $i : S^1 \sqcup S^1 \to 8$ given by including both circles. We thus get a diagram

$$S^1 \sqcup S^1 \xrightarrow{i} 8 \xleftarrow{j} S^1$$

where we will map into the d-dimensional compact oriented manifold M. We can extend the diagram obtained this way by adding evaluations at the base point. The result is the following diagram

$$\begin{tikzcd} LM \times LM \arrow[d, "\mathrm{ev}\times\mathrm{ev}"'] & M^8 \arrow[l, "M^i"'] \arrow[r, "M^j"] \arrow[d, "\mathrm{ev}"] & LM \\ M \times M & M \arrow[l, "\nabla"'] & \end{tikzcd}$$

where $M^8 := \mathrm{Map}(8, M)$, $\mathrm{ev} \times \mathrm{ev} : LM \times LM \to M \times M$ evaluates both circles at the base point $1 \in S^1$, $\mathrm{ev} : Map(8, M) \to M$ evaluates at the common base point of the two circles and $\nabla : M \to M \times M$ is the diagonal map.

To get the string product, we want to apply to the top line. Then M^j induces a map in homology which points in the right direction, but M^i doesn't. Therefore, we want to apply an umkehr map construction to M^i.

This would have been easy if both $LM \times LM$ and M^8 were finite-dimensional manifolds, but clearly they are not (as long as M is not 0-dimensional). Intuitively this is the case because these spaces of maps have infinitely many degrees of freedom; the value at any point can be modified. However, the trick is to note that the standard construction of umkehr maps still works if one is dealing with a finite codimension embedding of infinite-dimensional manifolds. Indeed, $LM \times LM$ and M^8 can in fact be seen as infinite-dimensional manifolds. Furthermore, M^i is a finite codimension embedding because the square is a pullback square such that the bottom row is a finite codimension embedding of finite-dimensional manifolds.

It turns out that this is enough to copy the construction of umkehr maps from the finite-dimensional case. To do this construction, we first note that the normal bundle for M^i is the pullback of the normal bundle for ∇. Let's denote the latter by ν and the former by $\mathrm{ev}^*(\nu)$. Then there exists a tubular neighborhood $f : \mathrm{ev}^*(\nu) \to LM \times LM$. Now we can construct a Thom collapse $\bar{f} : LM \times LM \to LM \times LM/(LM \times LM \backslash \mathrm{ev}^*(\nu)) := \mathrm{Thom}(\mathrm{ev}^*(\nu))$. The so-called umkehr map $(M^i)^!$ is then the following composition.

$$H_*(LM \times LM) \xrightarrow{\bar{f}_*} H_*(\mathrm{Thom}(\mathrm{ev}^*(\nu))) \xrightarrow{\mathrm{Thom}} H_{*-d}(M^8)$$

The string product is then the composition $(M^j)_*(M^i)^! : H_*(LM) \otimes H_*(LM) \to H_{*-d}(LM)$.

Theorem 1.6 (Cohen-Jones)**.** *The string product as defined by*

$$(M^j)_*(M^i)^! : H_*(LM) \otimes H_*(LM) \to H_{*-d}(LM)$$

coincides with the Chas-Sullivan string product.

Using this they proved that the hopes of Chas-Sullivan weren't justified: the string product is invariant under oriented homotopies of d-dimensional manifolds and therefore the string product doesn't depend on the smooth structure.

However as a consequence of their constructions calculations in string topology became easier: see e.g. [**Tam07**], [**Tam08b**], [**Tam08c**], [**Hep10**], [**Hep09**], [**CJJ03**]. Secondly, it allowed the generalisation of the construction of the string operations leading to the work of Godin.

1.3. Godin's work. After working with Cohen on a partial extension of the string operations, Godin found a general method to construct all previously known string operations and many new higher degree ones [**God07**].

In hindsight one can motivate Godin's construction by combining two observations. The first is that an ordinary TQFT is based on the (symmetric monoidal) category of 2-dimensional oriented cobordisms which has as objects the 1-dimensional compact oriented manifolds and as the morphisms the isomorphism classes of cobordisms. A TQFT is then a symmetric monoidal functor from this category to another symmetric monoidal category. For more about this, see section 3.2 of chapter 6. By direct calculation Tamanoi proved in [**Tam07**] that, up to some signs, string topology has the structure of a partial TQFT: a TQFT with positive boundary.

The other observation comes from Lurie's work on the cobordism hypothesis [**Lur09**]. This discusses fully extended TQFT's, which are obtained by adding higher and lower morphisms to the cobordisms in a suitable way. To include the higher morphisms, the discrete morphism $[\Sigma]$ in the category of cobordisms should be replaced by the classifying space $B\mathrm{Diff}_+(\Sigma)$ of the group of orientation and boundary preserving diffeomorphisms, or equivalently a classifying space $B\Gamma_\Sigma$ of the corresponding mapping class groups. To include the lower morphisms, one needs to look not just at ordinary cobordisms but at the more general open-closed cobordisms. See chapter 2 for these various definitions.

From work by Costello [**Cos06b**] one can conjecture that string topology should similarly be extendable to a fully extended partial TQFT. This is essentially what Godin has done: extending the string operations to the open-closed cobordisms and the non-trivial morphism spaces.

To be precise, Godin constructs string operations for each open-closed cobordism Σ with r incoming circles, s incoming intervals, m outgoing circles and n outgoing intervals:

$$H_*(B\Gamma_\Sigma;\mathcal{L}^{\otimes d}) \otimes H_*(LM)^{\otimes r} \otimes H_*(PM)^{\otimes s} \to H_*(LM)^{\otimes m} \otimes H_*(PM)^{\otimes n}$$

These operations fit together into the structure of d-dimensional HCFT with positive boundary, see definition 2.27.

2. Main theorems

This thesis is a natural continuation of Godin's progamme. By using open-closed cobordisms one is looking at fully-extended TQFT's with a single object and then the natural extension is to use multiple objects. The corresponding notion of cobordism with a $\mathcal{B}$-labelled cobordism, with $\mathcal{B}$ a set of labels. There is a corresponding notion of HCFT as well, called a HCFT with set of branes $\mathcal{B}$. In this terminology, Godin has shown that $\{H_*(LM), H_*(PM)\}$ are a d-dimensional HCFT with positive boundary and set of branes $\{M\}$.

Conjecture A in [**God07**] says one should be able to extend this to any set of branes $\mathcal{B}$ consisting of compact oriented submanifolds of M: there are string operations making the graded vector spaces $\{H_*(LM), H_*(P_M(A,B))_{A,B\in\mathcal{B}}\}$ into a d-dimensional HCFT with positive boundary and set of branes $\mathcal{B}$. This is our main theorem 2.29, formulated with rational coefficients to prevent problems with the Künneth theorem:

Theorem. *Let M be an oriented compact manifold of dimension d and $\mathcal{B} = \{A, B, \ldots\}$ a collection of oriented compact submanifolds. Then the set $(H_*(LM;\mathbb{Q}), \{H_*(P_M(A,B);\mathbb{Q})\})$ can be given the structure of a d-dimensional HCFT with positive boundary condition and set of branes $\mathcal{B}$, such that the operations on the closed cobordisms coincide with the standard string operations.*

We aim to give a complete and self-contained proof of this, which concerns chapters 2, 3, 4 and 5 of the thesis. In chapter 6 we discuss what the string operations actually do. The result can be summarized as the following (meta-)theorem:

Theorem. *All known string operations are described by the slogans:*

(1) Degree zero operations are TQFT operations.

(2) *The degree one operations are TQFT operations together with a BV-operator.*
(3) *Operations from the stable range are zero.*
(4) *The genus zero operations are the BV-operad.*

Finally, in chapter 7 we give some ideas in which directions one could take string topology in the future.

CHAPTER 2

HCFT's, with and without branes

In this chapter we introduce the notion of open-closed cobordisms, which are fundamental in the description of string topology. We then use mapping class groups to define a prop in topological space related to these cobordisms. Algebras over the homology of this prop will be called HCFT's.

1. Preliminaries

In this section we will give some preliminary definitions of the types of spaces we will be working with. We start with the spaces, with as we need more structure on our spaces to construct well-controlled umkehr maps, we will also consider semisimplicial complexes.

1.1. Topological spaces. First of all, we fix our category of topological spaces.

Convention 2.1. With topological spaces, i.e. Top, we mean the category of compactly generated weakly Hausdorff spaces.

The reason for this is the better behaviour of these spaces under categorical constructions. To be precise, it is cartesian closed. For example this implies the product and mapping space constructions are adjoint in this category, which is not the case for ordinary topological spaces.

Often we will use labels on spaces to give boundary conditions on maps from such a space to a target space. For this we introduce the general notion of a $\mathcal{B}$-labelled space. The $\mathcal{B}$ stands for *branes*.

Definition 2.2. Let $\mathcal{B}$ be a set. Then a $\mathcal{B}$-labelled space X is a topological space X with a function $b : X \to \mathcal{B} \sqcup \{\emptyset\}$, such that $b^{-1}(\beta)$ is closed for all $\beta \in \mathcal{B}$.

A morphism of a $\mathcal{B}$-labelled spaces is a continuous map which commutes with the label maps. This gives a category $\mathsf{Top}_{\mathcal{B}}$ of $\mathcal{B}$-labelled spaces.

1.2. Semisimplicial complexes. To have control over our construction of tubular neighborhood in chapter 4, we need to have specific choices of coordinates on our spaces. Coordinates which are particularly suited for this construction are those which are locally similar to the standard coordinates on a simplex $\Delta^n = \{(t_0, \ldots, t_n) \in \mathbb{R}^n_{\geq 0} | \sum_{i=0}^n t_i = 1\}$. There are face maps $\delta_i : \Delta^{n-1} \to \Delta^n$ for $0 \leq i \leq n$ given by adding an i'th coordinate equal to zero. We call a space with such given coordinates a semisimplicial complex, a notion we define more precisely below:

Definition 2.3. A semisimplicial complex is space X with a given homeomorphism to a space obtained from glueing sets X_i of i-dimensional simplices (faces) for $0 \leq i \leq n$ using boundary maps $\partial_j : X_i \to X_{i-1}$ for $i \geq 1$ and $0 \leq j \leq i$:

$$X \cong \left(\coprod_{n \in \mathbb{N} \sqcup \{0\}} X_n \times \Delta^n \right) / \sim$$

where $\sim$ identifies $(\partial_j(x), t)$ with $(x, \delta_j(t))$. The image of a Δ^n is called a n-simplex.

A morphism of semisimplicial complexes is a continuous map which sends each simplex linearly and homeomorphically onto another simplex of the same dimension. This gives a category SSComp.

Note that morphisms of semisimplicial complexes cannot collapse simplices, just identify several of the same dimension. There is a forgetful functor $\mathsf{SSComp} \to \mathsf{Top}$, which simply forgets the homeomorphism to a space glued from simplices.

Convention 2.4. Henceforth, semisimplicial complex will be abbreviated to complex if there is no danger of confusion.

Because we will only be able to deal with finitely many points and collapses which identify only points, we introduce a restricted category which only contains the nice complexes and well-behaved maps.

Definition 2.5. A 0-finite complex is one such that the set X_0 of 0-simplices is finite. A 0-regular morphism of complexes is a morphism which is morphisms which are bijective on all i-simplices for $i > 0$.

We let SSComp_0 be the subcategory of SSComp with as objects the 0-finite complexes and as morphisms the 0-regular morphisms between 0-finite complexes.

We call a complex n-dimensional if there is a cell of highest dimension n. If a complex is n-dimensional for some $n \in \mathbb{N}$, it is called finite dimensional. If no such cell exists, a complex is called infinite dimensional.

We will now discuss some examples. It is well-known that any smooth manifold M of dimension m admits a piecewise linear triangulation, hence admits the structure of a m-dimensional complex. If the manifold is compact, then it has the structure of 0-finite complex. But not every complex admits smooth structure.

Example 2.6. A simple example of complex which does not necessarily admit a smooth structure is given by the geometric realisation of a graph, as in section 1.1. There a precise definition of a graph is given, but for the moment just think of a graph as a set of vertices with some edges between them. For any graph Γ the space $|\Gamma|$ is given by taking a point for each vertex and an interval for each edge, glueing these to get a space. This geometric realisation $|\Gamma|$ is naturally a 0-finite complex of dimension 1, with 0-simplices equal to the vertices and 1-simplices equal to the edges.

Recall that we defined a simplex to the image of a closed standard simplex Δ^n. This means that when we assign a label to a simplex in a complex, its face automatically get the same label. The closure of $\alpha \in X_i$ is the set of all elements of $\beta \in X_j$ for $i \leq j$ which can be obtained from α by applying zero, one or more boundary maps.

Definition 2.7. Let $\mathcal{B}$ be a set. A $\mathcal{B}$-labelled complex is a finite dimensional complex together with a maps $b_i : X_i \to \mathcal{B} \cup \{\emptyset\}$ such that if $b_i(\alpha) = \beta \in \mathcal{B}$, then each simplex in the closure of α must also be assigned the label β.

A morphism of $\mathcal{B}$-labelled complexes is a morphism of complexes which commutes with the labelling. This gives a category $\mathsf{SSComp}_{\mathcal{B}}$ of $\mathcal{B}$-labelled complexes. By restricting to objects whose underlying complex is 0-finite and morphisms of complexes which are 0-regular, we obtain the category $\mathsf{SSComp}_{0,\mathcal{B}}$.

Note that there is a forgetful functor $\mathsf{SSComp}_{\mathcal{B}} \to \mathsf{SSComp}$ which forgets about the labelling, and similarly there is a forgetful functor $\mathsf{SSComp}_{\mathcal{B}} \to \mathsf{Top}_{\mathcal{B}}$ which forgets about the homeomorphism with the space glued from simplices. The same holds for the regular ($\mathcal{B}$-labelled) complexes.

2. Open-closed cobordisms

2.1. The notion of cobordism. The notion of cobordism is one that has proven useful in mathematics. It forms the basis of several generalized cohomology theories and categories of various types of cobordisms are an active topic of study. We will only be concerned with 2-dimensional cobordisms. We will think of cobordisms as modelling the worldsheet of interacting strings in certain quantum field theories. This should make the definitions more intuitive and add some motivation for considering cobordisms.

The simplest case of a worldsheet should just be part of the worldsheet of a closed string moving in a space-time. If no interactions occur, this gives a cylinder. However, we want to model interacting strings and therefore we want to have different circles, merging and splitting into different circles. However, since we don't want to work with an ambient space (although the deepest results about cobordisms, e.g. those by Galatius-Madsen-Tillmann-Weiss [**GMTW10**] and Lurie [**Lur09**] do use ambient spaces) we are going to think of these worldsheets as abstract manifolds. The actual string moving about and merging or splitting should then be a map of such an abstract cobordism to the ambient space.

2.2. Cobordisms. For cobordisms, we are moving from topological spaces or semisimplicial complexes to smooth manifolds, possibly with boundary. The smooth structure is not strictly speaking necessary for considering cobordisms in two dimensions, because 2-dimensional topological manifolds have a unique smooth structure up to diffeomorphism. It is however a very useful tool when considering these cobordisms, for example allowing one to use techniques from differential geometry. Also see remark [**Lur09**, remark 2.4.30].

In this setting, a (closed) cobordism in its most basic form is a 2-dimensional smooth compact manifold Σ with parametrised boundary divided into incoming and outgoing boundary. This means we equip Σ with a partitioning of $\partial\Sigma$ into the incoming boundary $\partial\Sigma_{in}$ and outgoing boundary $\partial\Sigma_{out}$, both of which consists of a union of connected components of $\partial\Sigma$, and diffeomorphisms $\rho_{in} : \coprod S^1 \to \partial\Sigma_{in}$ and $\rho_{out} : \coprod S^1 \to \partial\Sigma_{in}$. In this cases Σ is said to be a cobordism between the incoming and outgoing boundary, both of which are 1-dimensional compact manifolds.

One can glue a cobordism Σ' to a cobordism Σ if the outgoing boundary of Σ is diffeomorphic to the incoming boundary of Σ'. This is done by identifying their boundary using the maps ρ'_{in} and ρ_{out}. We denote the glueing by $\Sigma' \circ \Sigma$. The glueing gives us the problem of giving the smooth structure near the point of glueing. In two dimensions this is not a real problem, since there is a unique smooth structure on each topological 2-dimensional manifold (with boundary). Equivalently, one can demand that the cobordisms have a collar near the boundary, i.e. have a neighborhood which looks like a cylinder near the boundary. Then the smooth structure extends directly over the locus where the cobordisms are glued.

In the general case, an n-dimensional cobordism is an n-dimensional manifold X with boundary ∂X partitioned into an incoming and outgoing boundary and both of these come equipped with a diffeomorphisms to an $(n-1)$-dimensional manifold.

Often, one requires more structure on the cobordisms. For example, we will only be working with oriented cobordisms. In this case Σ is required to be orientable and equipped with a chosen orientation. There are induced orientations on the boundary. On the incoming boundary this is given by saying that $v \in T_x\partial\Sigma$ points in positive direction if the following condition holds: we take the ordered pair of first v and then a normal vector pointing inside the surface, then if we use a collar structure to identify these vectors with elements of $T_{\tilde{x}}\Sigma$ for a nearby $\tilde{x}$ the pair should be an oriented basis. For the outcoming boundary we use the normal vector pointing away from the surface.

Of course S^1 also has a canonical orientation coming from its standard embedding in the plane: $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ with positive direction in counterclockwise direction. The diffeomorphisms ρ_{in} and ρ_{out} should be compatible with these orientations, in the sense that the ρ_{in} should be

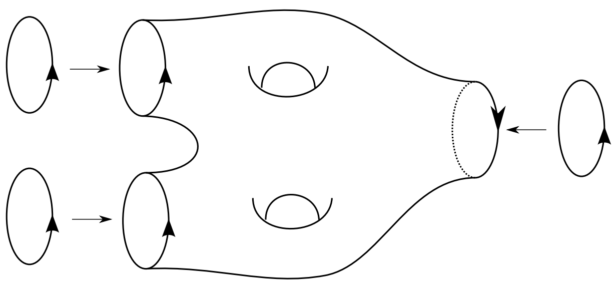


FIGURE 2.8. A cobordism with incoming boundary $S^1 \sqcup S^1$ on the left and outgoing boundary S^1 on the right. For clarity, the parametrisations and orientations of the boundary components are drawn. Notice that indeed the outgoing boundary parametrisation is orientation-reversing.

orientation preserving and ρ_{out} should be orientation-reversing. These conditions guarantee that the glued cobordisms will have an orientation as well.

Convention 2.9. Until further notice, in this and the following chapters all cobordisms are 2-dimensional and oriented. When drawing a cobordism, the incoming boundary will be on the left and the outgoing boundary on the right.

Remark 2.10. The step from cobordism to oriented cobordism can be generalized quite a bit. Firstly, one can consider G-cobordisms, where G a topological group with an injective continuous homomorphism $G \to GL(n)$. A G-cobordism is then a cobordism with a given refinement of transition functions of tangent bundle to G.

An slight generalization is when the cobordism M comes equipped with a continuous map $f : M \to BG$ and a isomorphism of vector bundles $TM \cong f^*\xi$, where $\xi \to BG$ is the canonical vector bundle $\mathbb{R}^n \times_G EG$ using a continuous homomorphism $G \to GL(n)$, not necessarily injective. For $G = \{1\}$, this is known as a framed cobordism, in other words a cobordism with a framing. For $G = SL(n)$ with the standard embedding, we get back our oriented cobordism and for $G = GL(n)$ with the identity map to $GL(n)$ we get back ordinary cobordism. There is a unique choice up to homotopy of a metric, so without loss of generality we could have replcaed $GL(n)$ with $O(n)$ and $SL(n)$ with $SO(n)$.

However, one need not even restrict to groups. Let X be a space with a n-dimensional vector bundle ζ, with $n \geq \dim M$. Then a (X, ζ)-cobordism is a cobordism with a map $f : M \to X$ and a isomorphism of vector bundles $TM \oplus \mathbb{R}^{n-\dim M} \cong f^*\zeta$. Lurie's proof of the cobordism hypothesis in particular allows one to classify topological field theories coming from these cobordisms [**Lur09**, theorem 2.4.18].

2.3. Open-closed cobordisms. In string topology we will in particular get operations coming from the moduli space of these oriented cobordisms. However, we can define operations for an even larger class of cobordisms. These are the so-called open-closed operations.

The idea of open-closed cobordisms is that instead of having a cobordism between two 1-dimensional manifolds, we look at cobordisms between two 1-dimensional manifolds with boundary. This means that not only can the incoming and outgoing boundary consist of a disjoint union of circles, but they can also contain intervals. In terms of worldsheets, our closed string can evolve into an open string and vice-versa. The open strings can merge and split as well.

Definition 2.11. A *open-closed cobordism* is a 2-dimensional orientable manifold Σ with boundary and corners on the boundary.[1] The orientation is fixed. There are subsets $\partial_f\Sigma$, the free boundary, and $\partial_s\Sigma$, the special boundary, of the boundary $\partial\Sigma$ with the following three properties:

(1) both $\partial_f\Sigma$ and $\partial_s\Sigma$ are 1-dimensional manifolds with boundary,
(2) their union $\partial_f\Sigma \cup \partial_s\Sigma$ is the full boundary $\partial\Sigma$,
(3) the intersection of the free and special boundary coincides with their boundaries, $\partial\partial_f\Sigma = \partial\partial_s\Sigma = \partial_f\Sigma \cap \partial_s\Sigma$ and with the subspace corners, denoted Corners(Σ).

The special boundary $\partial_s\Sigma$ is a disjoint union of an incoming and outgoing boundary $\partial_{in}\Sigma$ and $\partial_{out}\Sigma$, both of which are disjoint union of connected components of $\partial_s\Sigma$. Finally, these are parametrised, which means that there are given diffeomorphims $\rho_{in} : (\coprod S^1) \sqcup (\coprod I) \to \partial_{in}\Sigma$ and $\rho_{out} : (\coprod S^1) \sqcup (\coprod I) \to \partial_{out}\Sigma$. Note that this implies a labelling of these boundaries components, but we suppressed this to make the notation less cumbersome.

We have given an example of an open-closed cobordism in figure 2.12. Open-closed cobordisms can be glued in the same way as closed cobordisms, using the ρ_{in} and ρ_{out} maps. The implicit labelling is used to choose which circles and intervals to glue.

[1] A 2-dimensional manifold with boundary and corners on the boundary is a space which looks locally like $\mathbb{R}^2$, $\mathbb{R} \times \mathbb{R}_{\geq 0}$ or $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ and where transitions functions are required to extend to diffeomorphisms of a neighborhood of their domain and codomain. The latter reduces to the normal notion of diffeomorphisms when both charts of are the type $\mathbb{R}^2$. If the charts are of different type, then it means that the boundary is nice, which in this case means that it allows for the connection of a collar.

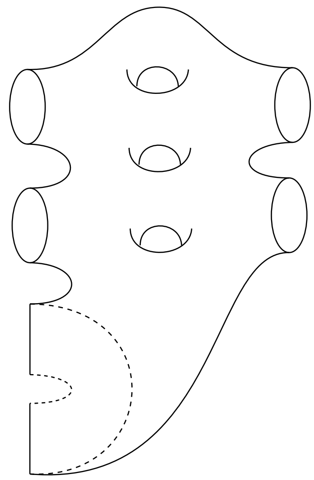


FIGURE 2.12. An open-closed cobordism between $S^1 \sqcup S^1 \sqcup I \sqcup I$ and $S^1 \sqcup S^1$. The free boundary is dotted.

There is an operation ∂ from open-closed cobordism to 1-dimensional cobordisms. This maps a cobordism to its free boundary, considered as a cobordism from the boundary of $\partial_{in}\Sigma$ to the boundary of $\partial_{out}\Sigma$. This operation is compatible with glueing. In fact, it can be made into a functor from the open-closed cobordism category to the oriented 1-dimensional cobordism category, if we define these correctly by adding identities and making composition strictly associative.

2.4. $\mathcal{B}$-labelled cobordisms. There is one final generalization to be made. In some physical theories the endpoints of open strings are confined to certain submanifolds known as branes. In terms of our abstract cobordisms, this means that each component of $\partial_f\Sigma$ should come equipped with the label of the brane we want it land on. This leads us to the following definitions.

Definition 2.13. Let $\mathcal{B}$ be a set. A *$\mathcal{B}$-labelled cobordism* is an open-closed cobordism with locally constant map $b : \partial_f\Sigma \to \mathcal{B}$.

In particular, a $\mathcal{B}$-labelled cobordism is a $\mathcal{B}$-labelled space, such that all points which have a non-empty label are in the free boundary.

It is easy to see there is a forgetful map from $\mathcal{B}$-labelled cobordisms to open-closed cobordism. We say that two $\mathcal{B}$-labelled cobordisms Σ and Σ' can be glued if the underlying open-closed cobordism can be glued in such a way that the labels on the endpoints of the outgoing boundary of Σ and the labels on the incoming boundary of Σ' coincide. It should be clear what the labelling on $\Sigma' \circ \Sigma$ should be: there is only one locally constant map $b : \partial_f(\Sigma' \circ \Sigma) \to \mathcal{B}$ which coincides with the original labellings of Σ', Σ on Σ', Σ respectively.

We should think of a $\mathcal{B}$-labelled cobordisms as a cobordism between $\mathcal{B}$-labelled 1-dimensional manifolds. In this case with a $\mathcal{B}$-labelled 1-dimensional manifold S we mean a 1-dimensional manifold with boundary and a locally-constant function $\partial S \to \mathcal{B}$. Note that because ∂S is 0-dimensional, in fact any function will be locally constant. In particular we conclude that this is a particular type of $\mathcal{B}$-labelled space.

Note we have the following isomorphism classes of $\mathcal{B}$-labelled 1-dimensional manifolds: the unit circle and unit intervals with label $\beta \in \mathcal{B}$ at 0 and label $\beta' \in \mathcal{B}$ at 1 for all possible pairs $(\beta, \beta') \in \mathcal{B} \times \mathcal{B}$. One would like to make these cobordisms into a category. There are some problems with associativity and identities, but these go away after we pass to isomorphism classes. An isomorphism class of a cobordism is given by the relation of diffeomorphism relative to the incoming and outgoing boundary, which only sends boundary with label $\beta \in \mathcal{B}$ to boundary with label $\beta \in \mathcal{B}$.

Definition 2.14. Let $\mathsf{Bord}^{\pi_0}_{\mathcal{B}}$ be the category with objects the subset of $\mathbb{Z}_{\geq 0} \times (\mathbb{Z}_{\geq 0})^{\mathcal{B}\times\mathcal{B}}$ consisting of those functions with only finitely many non-zero components.

The morphisms from $(n, \{n_{\beta,\beta'}\})$ to $(n', \{n'_{\beta,\beta'}\})$ is the set of isomorphism classes of $\mathcal{B}$-labelled cobordisms Σ such that $\partial_{in}\Sigma$ is diffeomorphic to n circles and $n_{\beta,\beta'}$ intervals with boundary labels β and β' and $\partial_{out}\Sigma$ is diffeomorphic to n' circles and $n'_{\beta,\beta'}$ intervals with boundary labels β and β'. Composition is induced by the composition of cobordisms.

Note that pointwise addition on objects and the disjoint union $\sqcup$ of surfaces on morphisms gives $\mathsf{Bord}^{\pi_0}_{\mathcal{B}}$ the structure of a symmetric monoidal category. The unit is the isomorphism class of the empty cobordism.

3. Mapping class groups

An important invariant of a space is its space of self-homeomorphisms. The mapping class groups tries to capture this information while at the same time making it computable, by looking at the connected components of the this space. It is also intricately linked to cobordisms, e.g. [**Til97**]. The properties of this group are discussed in section 2.

3.1. The various definitions of mapping class groups. Let Σ be a closed oriented 2-dimensional manifold. Then the group of orientation-preserving self-homeomorphisms of Σ, denoted $\mathrm{Homeo}^+(\Sigma)$ forms a topological group if we use the compact-open topology on $\mathrm{Map}(\Sigma, \Sigma)$ and the subspace topology on $\mathrm{Homeo}^+(\Sigma) \subset \mathrm{Map}(\Sigma, \Sigma)$. This implies that $\pi_0(\mathrm{Homeo}^+(\Sigma))$ is a group, whose identity is given by the component of the identity map of Σ.

Definition 2.15. Let Σ be a closed oriented 2-dimensional manifold. The *mapping class group* Γ_Σ of a closed oriented 2-dimensional manifold is given by $\pi_0(\mathrm{Homeo}^+(\Sigma))$.

Remark 2.16. Alternatively, we can define the mapping class group as $\mathrm{Homeo}^+(\Sigma)/\mathrm{Homeo}^+_0(\Sigma)$, $\pi_0(\mathrm{Diff}^+(\Sigma))$, $\mathrm{Homeo}^+(\Sigma)$/homotopy, $\mathrm{Diff}^+(\Sigma)$/isotopy, where $\mathrm{Homeo}_0(\Sigma)$ is the connected component of the identity, $\mathrm{Diff}^+(\Sigma)$ is the space of self-diffeomorphisms in the C^∞-topology. Furthermore, in this spaces the diffeomorphisms, homotopies or smooth isotopies are required to preserve the orientation. This definition might seem to give problems for those surfaces where homotopy is not the same as isotopy: the disk, the annulus, the once-punctured sphere and the twice-punctured sphere. However, there homotopy is the same as isotopy for orientation-preserving homeomorphisms. Hence the definitions still coincide. A reference for these remarks is [**FM10**, section 1.4, section 2.1].

Remark 2.17. This is not the only notation for the mapping class groups. One often encounters $\mathrm{Mod}(\Sigma)$, $\mathrm{Map}(\Sigma)$, $\mathrm{MCG}(\Sigma)$ and $\mathcal{M}(\Sigma)$.

The notion of mapping class group extends to manifold with boundary, marked points and even to labelled boundaries. We say that a map $f : X \to X$ preserves a set $S \subset X$ pointwise if $f(s) = s$ for all $s \in S$. It preserves S as a set if $f(S) = S$.

- Let Σ be a closed oriented 2-dimensional manifold with marked points $\vec{x}$, then we will let $\mathrm{Homeo}^+(\Sigma, \vec{x})$ denote the subgroup of $\mathrm{Homeo}^+(\Sigma)$ which preserves the marked points pointwise.
- Similarly, if Σ has a boundary $\partial\Sigma$, then $\mathrm{Homeo}^+(\Sigma, \partial\Sigma)$ denotes the subgroup of $\mathrm{Homeo}^+(\Sigma)$ which preserve the boundary pointwise.
- Let Σ be a manifold with boundary $\partial\Sigma$ and corners on the boundary and a fixed subset $\partial_s\Sigma$ of the boundary $\partial\Sigma$ which consists of 1-dimensional submanifolds with boundary. Then the complement of $\partial_s\Sigma$ is denoted by $\partial_f\Sigma$ and its closure consists of 1-dimensional submanifolds with boundary as well. In this case $\mathrm{Homeo}^+(\Sigma, \partial\Sigma, \partial_s\Sigma)$ denotes the subgroup of $\mathrm{Homeo}^+(\Sigma)$ of homeomorphisms which preserve the special boundary $\partial_s\Sigma$ pointwise and the free boundary $\partial_f\Sigma$ as a set.
- Finally, suppose we are in the previous situation but we are given a $\mathcal{B}$-labelling, i.e. a locally constant map $b : \partial_f\Sigma \to \mathcal{B}$ for some set $\mathcal{B}$. Then $\mathrm{Homeo}^+(\Sigma, \partial\Sigma, \partial_s\Sigma, \mathcal{B})$ denotes the subgroup of $\mathrm{Homeo}^+(\Sigma)$ which preserve the special boundary $\partial_s\Sigma$ pointwise and the free boundary $\partial_f\Sigma$ as a set, but only permutes components of $\partial_f\Sigma$ with the same labels.

Note that each definition gets more general, encompassing the earlier definitions. Getting mapping class groups from these definitions is nothing but applying π_0.

Definition 2.18. We define the following variations of the *mapping class group* for different types of surfaces.

- Let Σ be a manifold with marked points $\vec{x}$, then the group $\Gamma_\Sigma^{\vec{x}}$ is given by $\pi_0(\mathrm{Homeo}^+(\Sigma,\vec{x}))$.
- For a surface with boundary, the group $\Gamma_\Sigma^{\partial\Sigma}$ is given by $\pi_0(\mathrm{Homeo}^+(\Sigma,\partial\Sigma))$.
- For a surface with boundary and corners on the boundary, the group $\Gamma_{\Sigma,\partial\Sigma}^{\partial_s\Sigma}$ is given by $\pi_0(\mathrm{Homeo}^+(\Sigma,\partial\Sigma,\partial_s\Sigma))$.
- Finally, for a surface with boundary and corners on the boundary and $\mathcal{B}$-labels on the free boundary, the group $\Gamma_{\Sigma,\partial\Sigma}^{\partial_s\Sigma,\mathcal{B}}$ is given by $\pi_0(\mathrm{Homeo}^+(\Sigma,\partial\Sigma,\partial_s\Sigma,\mathcal{B}))$.

Remark 2.19. Again, we could replace the homeomorphisms by diffeomorphisms and π_0 by taking homotopy classes of homeomorphisms or oriented isotopy classes classes of diffeomorphisms. These give isomorphic groups.

Remark 2.20. What we call $\Gamma_\Sigma^{\vec{x}}$ is sometimes also called the pure mapping class group and denoted $P\Gamma_\Sigma$. In those cases the name mapping class group is used for π_0 of those self-homeomorphisms that preserve $\vec{x}$ as a set. If Σ is connected, this alternative definition of mapping class group can be obtained as an extension of the symmetric group Σ_n by the pure mapping class group:

$$0 \to P\Gamma_\Sigma^{\vec{x}} \to \Gamma_\Sigma^{\vec{x}} \to \Sigma_n \to 0$$

We describe some homomorphisms between mapping class groups given by the geometry of surfaces. We just treat the most general case, because the other cases are similar. The disjoint union $\Sigma' \sqcup \Sigma$ of surfaces Σ' and Σ gives us a homomorphism between mapping class groups

$$\Gamma(\sqcup) : \Gamma_{\Sigma,\partial\Sigma}^{\partial_s\Sigma,\mathcal{B}} \times \Gamma_{\Sigma',\partial\Sigma'}^{\partial_s\Sigma',\mathcal{B}} \to \Gamma_{\Sigma'\sqcup\Sigma,\partial(\Sigma'\sqcup\Sigma)}^{\partial_s(\Sigma'\sqcup\Sigma),\mathcal{B}}$$

for the simple reason that if we are given a self-homeomorphism of Σ and a self-homeomorphism of Σ', then we can make a self-homeomorphism of $\Sigma'\sqcup\Sigma$ by applying each of the self-homeomorphisms to their corresponding component.

The next thing to note is that the glueing of surfaces induces homomorphisms between mapping class groups. Let Σ and Σ' be surfaces with boundary and corners on the boundary and $\mathcal{B}$-labels on the free boundary which can be glued. Let $\chi : \Sigma \coprod \Sigma' \to \Sigma' \circ \Sigma$ be the map which glues the two surfaces along the relevant parts of their boundary. This induces a map

$$\Gamma(\chi) : \Gamma_{\Sigma,\partial\Sigma}^{\partial_s\Sigma,\mathcal{B}} \times \Gamma_{\Sigma',\partial\Sigma'}^{\partial_s\Sigma',\mathcal{B}} \to \Gamma_{\Sigma'\circ\Sigma,\partial(\Sigma'\circ\Sigma)}^{\partial_s(\Sigma'\circ\Sigma),\mathcal{B}}$$

since a pair of homeomorphisms $f' : \Sigma' \to \Sigma'$ and $f : \Sigma \to \Sigma$ which preserves the boundary pointwise can be made into a single homeomorphism $f' \circ f$ of $\Sigma' \circ \Sigma$ by setting $f' \circ f$ to equal to f' on Σ' and f on Σ. There is no ambiguity at the glued boundary.

In both these constructions passing to connected components doesn't give an problem, since both of these constructions are continuous with respect to the compact-open and product topology and hence sends the connected components in the product to connected components.

Convention 2.21. If it is clear from the context which type of mapping class group we use, we don't use the decorated Γ, but instead denote the mapping class group with Γ_Σ.

3.2. A nice cobordism prop. To give the link between the mapping class groups and cobordisms, remark that any $\mathcal{B}$-labelled cobordism is a manifold with boundary and corners on the boundary with special boundary $\partial_s\Sigma$ the disjoint union of the incoming and outgoing boundary and labelling on the free boundary. In particular, we have defined Γ_Σ for a cobordism. Since the incoming and outgoing boundary are preserved pointwise, the composition of cobordisms induces a homomorphism of mapping class groups as earlier.

We now define a topological prop coming from the mapping class groups, which generalizes the cobordism prop whille at the same time avoiding difficulties making some of the diagrams commute strictly. These difficulties were previously solved by using isomorphism classes of cobordisms instead of the cobordisms themselves. The reason for generalizing to this prop is given in section 1. To define this prop we use the functorial classifying space construction, see proposition B.37.

The disjoint union of the classifying spaces of the mapping class groups for all isomorphism classes of $\mathcal{B}$-labelled cobordisms will be the total morphism space of the topological category underlying our topological prop.

Definition 2.22. Let $\mathsf{Bord}_{\mathcal{B}}$ be the category enriched in Top with objects $\mathbb{Z}_{\geq 0} \times (\mathbb{Z}_{\geq 0})^{\mathcal{B}\times\mathcal{B}}$ such that only finitely many integers are non-zero. Such an object represents an isomorphism class of $\mathcal{B}$-labelled 1-dimensional manifolds consisting of a number of circles and a number of intervals with labelled boundaries.

The space of morphisms from $(n, \{n_{\beta,\beta'}\})$ to $(n', \{n'_{\beta,\beta'}\})$ is the disjoint union $\coprod_{[\Sigma]} B\Gamma_\Sigma$ where the disjoint union runs over all isomorphism classes of $\mathcal{B}$-labelled cobordisms Σ such that $\partial_{in}\Sigma$ is diffeomorphic to n circles and $n_{\beta,\beta'}$ intervals with boundary labels β and β' and $\partial_{out}\Sigma$ is diffeomorphic to n' circles and $n'_{\beta,\beta'}$ intervals with boundary labels β and β'. The composition is induced by the maps $B\Gamma_\Sigma \times B\Gamma_{\Sigma'} \to B\Gamma_{\Sigma'\circ\Sigma}$ given by functoriality of the classifying space construction.

Definition 2.23. The disjoint union of surfaces induces a coproduct map $\Gamma(\sqcup)$ of mapping class groups which in turn induces a symmetric monoidal structure on $\mathsf{Bord}_{\mathcal{B}}$, given by pointwise addition on objects and the induced map on morphisms. The identity is the empty cobordism. This gives $\mathsf{Bord}_{\mathcal{B}}$ the structure of a prop.

Note that applying π_0 to the hom-spaces gives us exactly the category $\mathsf{Bord}_{\mathcal{B}}^{\pi_0}$ back again. It is easy to see that the induced symmetric monoidal structure after applying π_0 is the same as the original symmetric monoidal structure on $\mathsf{Bord}_{\mathcal{B}}^{\pi_0}$. Something which will be useful later, when we deal with homological conformal field theories, is that we could just as well have replaced π_0 with H_0, since H_0 and the free abelian group on π_0 of a space are canonically isomorphic for sufficiently nice spaces (e.g. locally pathconnected).

We need a partial subprop of this prop for string topology. This has to do with the fact we can't construct operations for cobordisms with only incoming boundary on a connected component.

Definition 2.24. The *positive boundary prop*, denoted by $\mathsf{Bord}_{\mathcal{B}}^{+}$, is the full partial subprop of $\mathsf{Bord}_{\mathcal{B}}$ obtained by restricting to those $\mathcal{B}$-labelled cobordisms such that every connected component has non-empty outgoing boundary.

Remark 2.25. This is not the usual definition of positive boundary. The usual definition as in [**God07**] is that every component has a non-empty $\mathcal{B}$-labelled or outgoing boundary. However, if we do not impose our more restrictive positive boundary condition, we get the wrong components in our geometric realisation later on. See remark 3.65.

4. Homological conformal field theories

4.1. HCFT's. In the appendix we show that a topological prop induces a prop in graded vector spaces after applying any generalized cohomology theory for which we can get the Künneth theorem to hold. In particular, this applies to $H_*(-, k)$ with k any field. We are mostly interested in the case $k = \mathbb{Q}$.

Definition 2.26. A homological conformal field theory of dimension 0 and branes $\mathcal{B}$ is a symmetric monoidal functor $H_*(\mathsf{Bord}_{\mathcal{B}}; \mathbb{Q}) \to \mathsf{GrAbgrps}$. A HCFT of dimension 0 with positive boundary condition and branes $\mathcal{B}$ is a symmetric monoidal functor $H_*(\mathsf{Bord}_{\mathcal{B}}^{+}; \mathbb{Q})$.

Definition 2.27. Let $\mathcal{B}$ be a set with fixed element $M \in \mathcal{B}$. Let $\mathcal{L}_{\mathcal{B}}^{M}$ be the local system over $\mathsf{Bord}_{\mathcal{B}}$ defined in section 6 of chapter 5. A homological conformal field theory of dimension d and branes $\mathcal{B}$ is a symmetric monoidal functor $H_*(\mathsf{Bord}_{\mathcal{B}}; \mathcal{L}_{\mathcal{B}}^{M}) \to \mathsf{GrAbgrps}$. A HCFT of dimension d with positive boundary condition and branes $\mathcal{B}$ is a symmetric monoidal functor $H_*(\mathsf{Bord}_{\mathcal{B}}^{+}; \mathcal{L}_{\mathcal{B}}^{M}) \to \mathsf{GrAbgrps}$.

In particular $\mathcal{L}$ has the property that it trivializible as a virtual bundle of dimension $-\chi(\Sigma)d$ over the components of $\mathsf{Bord}_{\mathcal{B}}$ with labels only $\{M\}$ and at most one boundary component not containing an incoming or outgoing part.

Remark 2.28. To recover Godin's definition of a d-dimensional HCFT with positive boundary set $\mathcal{B} = \{M\}$ in the previous definition.

4.2. String topology as a HCFT. Let M be a manifold and A and B be submanifolds. Then LM denotes the space $\mathrm{Map}(S^1, M)$ of continuous loops in M and $P_M(A, B)$ denotes the subspace of the space $\mathrm{Map}(I, M)$ of paths p such that $p(0) \in A$ and $p(1) \in B$.

String topology is concerned with the homology of these spaces. The central claim of this thesis is the following:

Theorem 2.29. *Let M be an oriented compact manifold of dimension d and $\mathcal{B} = \{A, B, \dots\}$ a collection of oriented compact submanifolds. Then the set $(H_*(LM;\mathbb{Q}), \{H_*(P_M(A,B);\mathbb{Q})\})$ can be given the structure of a d-dimensional HCFT with positive boundary condition and set of branes $\mathcal{B}$. In other words, we will construct operations:*

$$\begin{array}{c} H_*(B\Gamma_\Sigma; \mathcal{L}^M_{\mathcal{B}}) \otimes H_*(LM;\mathbb{Q})^{\otimes r} \otimes \bigotimes_{A,B\in\mathcal{B}} H_*(P_M(A,B);\mathbb{Q})^{\otimes r_{A,B}} \\ \Big\downarrow {\scriptstyle \daleth_*} \\ H_*(LM;\mathbb{Q})^{\otimes s} \otimes \bigotimes_{A,B\in\mathcal{B}} H_*(P_M(A,B);\mathbb{Q})^{\otimes s_{A,B}} \end{array}$$

where r is the number of incoming boundary circles, $r_{A,B}$ is the umber of incoming boundary intervals with labels $A, B \in \mathcal{B}$, s is the number of outgoing boundary circles and $s_{A,B}$ is the number of outgoing boundary intervals with labels $A, B \in \mathcal{B}$.

Proof. The maps $\daleth_*$ are defined in definition 5.44. The compatibility with disjoint union is theorem 5.72 and the compatibility with glueing of cobordisms is theorem 5.75. These theorems also contain the exact formulation of what this compatibility means, but it boils down to the fact that indeed the $\daleth_*$ are the structure maps of a HCFT. $\square$

CHAPTER 3

Fat graphs

In the previous chapter we discussed HCFT's, which are algebras for a prop coming from the homology of the mapping class groups for cobordisms. Two-dimensional smooth cobordisms can be modelled - and must be modelled in our construction of the string operations, see chapter 5 - by fat graphs. After that, we describe the link between fat graphs and mapping class groups, which will be important for our definition of HCFT's and to be able to calculate explicitly the string operations later on. In chapter 7 we will see how other types of graphs can be used to give string-like operations in other situations.

1. Graphs and fat graphs

It is easy to see that if one cuts a small disk out of a torus, then the resulting surface with boundary deformation retracts onto a wedge sum of two circles, $S^1 \vee S^1$. This is exactly the space we saw in the Cohen-Jones construction of the string product 1.6. More generally, for any surface with boundary one can find a 1-dimensional subcomplex which sits inside it as a deformation retract. We want to represent surfaces with boundary by these 1-dimensional subcomplex as a way to define umkehr maps for cobordisms. To do this, we discuss graphs and fat graphs. We will start by getting our definitions straight.

1.1. Graphs. We use a slightly non-standard definition of graphs. Instead of seeing a graph as a collection of vertices and edges, we would rather see it as a collection of vertices and half edges with a pairing which tells you which pairs of half edges form an edge. This definition of a graph is equivalent to the ordinary definition, but will make it easier to write down the notion of a fat graph, where we have to demand a structure on all half edges attached to a vertex.

Definition 3.1. A *graph* consists of is a quadruple (V, H, s, i) of a finite set V of vertices and a finite set H of half edges, together with a map $s : H \to V$ assigning to an half edge its source and an involution $i : H \to H$ without fixed points, which pairs a half edge with its other half edge. We call the map s the source map and the map i the edge pairing.

The cycles of i are by definition of cardinality two and are known as edges. They form a set E.

Remark 3.2. The demand that V and H are finite is only for our convenience. There is no reason not to look at infinite graphs if you are interested in graph theory. However, we can not construct string operations for most infinite graphs, so we're not interested in them. To be precise, the techniques of our construction works as long as we only need to construct umkehr maps for pullbacks of finite-dimensional embeddings. So for example we could still construct an operation for fat graphs with a infinite incoming part and only finitely many additional vertices and edges.

Similarly one can define a theory of graphs with open half edges, i.e. graphs which admit cycles of i that have cardinality 1. These cycles are usually called leaves. Graphs with leaves play a important role in the theory of operads.

For the notion of a morphism of graphs, it is easier to work with s and i extended to $V \sqcup H$.

Definition 3.3. Let $\Gamma = (V, H, s, i)$ be a graph. The extended map $\tilde{s} : V \sqcup H \to V \sqcup H$ is given by extending s with the identity to V. Similarly, the extended map $\tilde{i} : V \sqcup H \to V \sqcup H$ is given by extending i with the identity to V.

Every graph gives us a 1-dimensional complex with 1-simplices the edges and 0-simplices the vertices, where the glueing maps are provided by the source maps. To be precise, we have the following definition of the geometric realisation of a graph:

Definition 3.4. Let Γ be a graph. The *geometric realisation* $|\Gamma|$ is defined to the following space:

$$|\Gamma| = \left((\coprod_{v\in V} *) \sqcup (\coprod_{h\in H} [0, \frac{1}{2}]) \right) / \sim$$

where $\sim$ is the equivalence relation identifying $\frac{1}{2} \in [0, \frac{1}{2}]_h$ with $\frac{1}{2} \in [0, \frac{1}{2}]_{i(h)}$ and $0 \in [0, \frac{1}{2}]_h$ with $*_{s(h)}$. It is naturally a 1-dimensional complex, with 0-simplices the vertices and 1-simplices the edges.

There are different notions of morphisms of graphs. We want to use two different notions of morphism of graphs:

(1) Maps of graphs are those morphisms which induce maps between the geometric realisations by mapping vertices to vertices and possibly collapsing some edges.
(2) Simple maps of graphs not only induce maps of geometric realisation, but this map should be a simple homotopy equivalence as well. Combinatorially, they are maps of graphs which only collapse embedded trees.

The reason that we want to use these two different notions is that the first is necessary to describe some of the constructions that change the homtopy type, like separating edges from vertices, while the second is useful when describing fat graphs.

Definition 3.5. A *map of graphs* $f : \Gamma = (V, H, s, i) \to \Gamma' = (V', H', s', i')$ is a map $f : V \sqcup H \to V \sqcup H$ such that $f \circ \tilde{s} = \tilde{s}' \circ f$ and $f \circ \tilde{i} = \tilde{i}' \circ f$.

Note that since $\tilde{s}$ has image in V and $\tilde{s}'$ has image in V', such maps must map vertices to vertices. However, half edges can be mapped to either vertices or half edges. The second condition $f \circ \tilde{i} = \tilde{i}' \circ f$ implies that if a half edge is sent to a vertex, then its other half will be sent to the same vertex. More generally, this condition implies that f induces a map from edges of Γ to edges and vertices of Γ', collapsing some edges to a vertex. The collection of maps of graphs has the following property:

Lemma 3.6. *For each graph Γ, the identity map is a map of graphs. The composite of two map of graphs is again a map of graphs and this composition is associative.*

Proof. The identity map extends to the identity $V \sqcup H \to V \sqcup H$. The conditions of commuting with $\tilde{s}$ and $\tilde{i}$ are therefore always satisfied.

Let $f : \Gamma_0 \to \Gamma_1$ and $g : \Gamma_1 \to \Gamma_2$ be two maps of graphs. Then $\tilde{s}_2 \circ f \circ g = f \circ \tilde{s}_1 \circ g = f \circ g \circ \tilde{s}_0$ and similarly for the extended edge pairing maps. Associativity is trivial. □

This allows us to define our first category of graphs.

Definition 3.7. The category of Graph has as objects graphs and as morphisms maps of graphs.

Finally, we prove the motivating lemma for this notion of morphism of graphs.

Lemma 3.8. *A map of graphs $f : \Gamma \to \Gamma'$ induces a map $|f|$ between $|\Gamma|$ and $|\Gamma'|$.*

Proof. We want to define a map $\tilde{f}$:

$$\tilde{f} : \left((\coprod_{v\in V} *) \sqcup (\coprod_{h\in H} [0, \frac{1}{2}]) \right) / \sim \to \left((\coprod_{v'\in V'} *) \sqcup (\coprod_{h'\in H'} [0, \frac{1}{2}]) \right) / \sim$$

This map is defined as follows: if f maps v to v', $\tilde{f}$ maps $*_v$ to $*_{v'}$, if f maps h to v', $\tilde{f}$ maps $[0, \frac{1}{2}]_h$ to $*_{v'}$ and if f maps h to h', $\tilde{f}$ maps $[0, \frac{1}{2}]_h$ to $[0, \frac{1}{2}]_{h'}$ identically.

Since f commutes with the extended s and i maps, this map is compatible with the equivalence relation $\sim$, hence induces a continuous map $|f| : |\Gamma| \to |\Gamma'|$. □

Corollary 3.9. *The geometric realisation of graphs is a functor $|-| : \mathsf{Graph} \to \mathsf{SSComp}$.*

PROOF. A quick look at the definition of the induced map shows that $|g \circ f| = |g| \circ |f|$ and that $|id| = id_{|\Gamma|}$. □

With these properties of maps of graphs established, we can now define our notion of a simple map of graphs.

Definition 3.10. A *simple map of graphs* $f : \Gamma \to \Gamma'$ is a map of graphs with the following properties:

(1) it is surjective,
(2) each half-edge in Γ' is the image of only one half-edge in Γ,
(3) it induces a homotopy equivalence on geometric realisation. Equivalently, we can demand that f has the property that $f^{-1}(v)$ is a connected tree for each vertex $v \in V$.

We start by noting that this indeed is a good notion of morphism.

Lemma 3.11. *The identity map is a simple map of graphs. The composite of two simple maps is a simple map and this composition is associative.*

PROOF. Using lemma 3.6 it suffices to show that the identity is surjective, injective on H, induces a homotopy equivalence and has $f^{-1}(v)$ a tree for each vertex. The first two properties and the last property are trivial; the identity is a bijection. The third property is proven by noting that $|id| = id_{|\Gamma|}$, which is in fact a homeomorphism.

To see that the composition of simple maps is a simple map, we simply check all the properies. We first note that a composition of surjections is a surjection and that if both simple maps only map a single half-edge to each half-edge then their composition will as well. Finally, one can note that a composition of homotopy equivalences is a homotopy equivalence or, equivalently , that if in a tree each vertex is blown up to a tree, the result is again a tree. Associativity follows from lemma 3.6. □

This gives us enough information to define our second category of graphs.

Definition 3.12. The category SGraph has as objects graphs and as morphisms the simple maps of graphs.

There is an inclusion functor $\mathsf{SGraph} \to \mathsf{Graph}$ because every simple map of graphs is a map of graphs. This means that the geometric realisation induces a functor $| - | : \mathsf{SGraph} \to \mathsf{SSComp}$.

1.2. Constructions on graphs.

We will now discuss several constructions that can be done on graphs. In particular, we are interested in constructions that behave well with respect to geometric realisation. We start with a natural notion of a subgraph of a graph.

Definition 3.13. A *subgraph* Γ' of a graph Γ is a triple (V', H', s', i') such that $V' \subset V$, $H' \subset H$, $s' = s|_{H'}$ and $i' = i|_{H'}$ such that s' has image in $V'$and i' has image in H'.

We also want to be able to take the complement of a subgraph. Taking the set-theoretic complement gives an incorrect result, however; some half-edges are no longer connected to a vertex. To solve this, we just naively add these missing vertices again.

Definition 3.14. Let Γ' be a subgraph of Γ, then $\Gamma\backslash\Gamma'$ is the subgraph of Γ obtained by removing the half-edges of Γ' and those vertices of Γ' which have no half-edge in $H\backslash H'$ leading into them. The edge pairing and the source map are the restrictiond of those of Γ.

It is clear that $|\Gamma\backslash\Gamma'|$ and $|\Gamma'|$ are both subspaces of $|\Gamma|$, their union is $|\Gamma|$ and they intersect only in their common vertices, i.e. the ones that we added to make $\Gamma\backslash\Gamma'$ a graph.

Next, we fix a notation for the functor that removes all half-edges, leaving just the vertices. That this does not restrict to a functor $\mathrm{v} : \mathsf{SGraph} \to \mathsf{SGraph}$ because the induced morphism can change the homotopy type, i.e. the number of vertices.

Definition 3.15. The functor $\mathrm{v} : \mathsf{Graph} \to \mathsf{Graph}$ is the functor which sends a graph (V, H, s, i) to the graph $(V, \emptyset, \emptyset, \emptyset)$. It sends a morphism to the map on vertices.

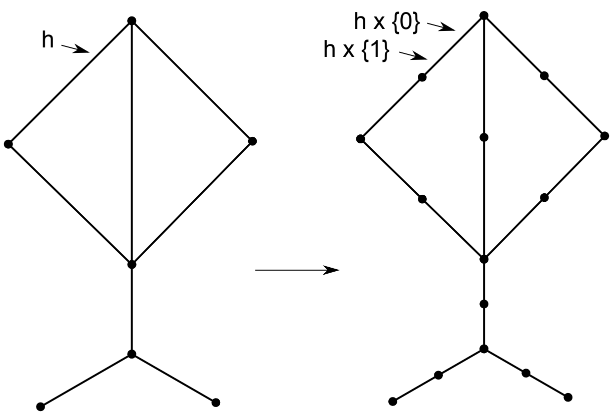


FIGURE 3.17. The effect of applying the subdivision functor sd to a graph.

We also fix a notation for the subdivision functor, with splits each edge into two separate edges. It does restrict to a functor sd : $\mathsf{SGraph} \to \mathsf{SGraph}$.

Definition 3.16. There is a functor sd : $\mathsf{Graph} \to \mathsf{Graph}$ which subdivides every edge. It sends a graph (V, H, s, i) to the graph $(V \sqcup E, H \times \{0,1\}, s', i')$ where $s'(h \times \{0\}) = s(h) \in V$, $s'(h \times \{1\}) = [h] \in E$, where $[h] \in E$ is the edge to which h belongs, and $i'(h \times \{j\}) = h \times \{1-j\}$. On morphisms it is the induced map, i.e. if a morphism collapses an edge then the induced morphisms collapses both parts of the subdivision of this edge.

Although the definition of the subdivision may seem a bit abstruse, there is in fact nothing going on except canonically adding a new vertex for each edge and doubling the half edges. Look at figure 3.17 for a picture.

Next, we fix a notation for the separation functor, which removes the attachment of the edges to the vertices. It does not restrict to a functor sep : $\mathsf{SGraph} \to \mathsf{SGraph}$ because a morphism can change the homotopy type, again by changing the number of connected components. See figure 3.19 for an example of what this functor does on a graph.

Definition 3.18. There is a functor sep : $\mathsf{Graph} \to \mathsf{Graph}$ which removes the attachment of each edge to the vertices. It sends the graph (V, H, s, i) to the graph $(V \sqcup E \times \{0,1\}, E \times \{0,1\}, s', i')$ where $s'(e \times \{i\}) = e \times \{i\}$ and $i'(e \times \{j\}) = e \times \{1-j\}$. On morphisms it is the induced morphism, i.e. if a morphism collapses an edge then the induced morphisms removes the component of this edge and identifies the two vertices.

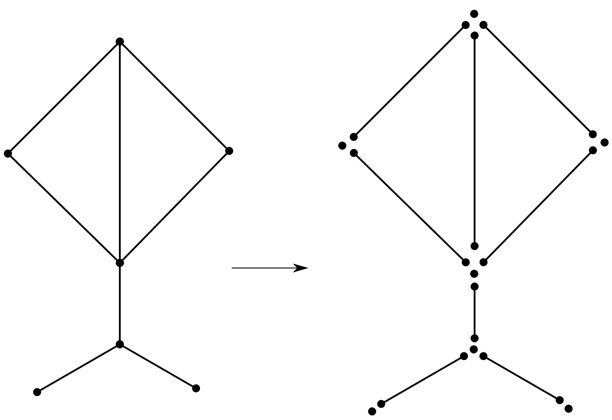


FIGURE 3.19. The effect of applying the separation functor sep to a graph.

Finally, we fix a notation for a functor half-way between separation and subdivision. It is unfortunate to have to introduce this, but this functor appears naturally in the construction of the string operations and cannot be written as a composition of our other functors. It is given by applying the separation functor and then splitting each half-edge into two edges which are no longer connected. Note that it doesn't retrict to a functor hsep : $\mathsf{SGraph} \to \mathsf{SGraph}$. See figure 3.21 for an example of this functor applied to a graph.

Definition 3.20. There is a half-edge separation functor hsep : $\mathsf{Graph} \to \mathsf{Graph}$ which sends the graph (V, H, s, i) to the graph $(V \sqcup H \times \{0,1\}, H \times \{0,1\}, s', i')$ where $s'(h \times \{i\}) = h \times \{i\}$ and $i'(h \times \{j\}) = h \times \{1-j\}$. On morphisms it is the induced morphism, i.e. if a morphism collapses an half-edge then the induced morphisms removes the component of this half-edge and identifies the two vertices.

1.3. $\mathcal{B}$-labelled graphs. We will be interested not only in ordinary graphs, but in graphs with labels encoding boundary conditions on maps from the geometric realisation of the graph to a fixed target space. To do this, we define what it means for a graph to be labelled by a set $\mathcal{B}$.

Definition 3.22. A $\mathcal{B}$-labelled graph is a graph (V, H, s, i) with a function $b : V \sqcup H \to \mathcal{B} \sqcup \{\emptyset\}$ such that the equation $b \circ \tilde{i} = b$ holds, $\tilde{s}(b^{-1}(\mathcal{B})) \subset b^{-1}(\mathcal{B})$ and when we restrict to $b^{-1}(\mathcal{B})$ the equation $b \circ \tilde{s} = b$ holds. Here $\tilde{i}$ and $\tilde{s}$ are the extended involution and source map respectively.

A (simple) map of $\mathcal{B}$-labelled graphs is a (simple) map of graphs commuting with the labelling functions. This gives categories $\mathsf{Graph}_{\mathcal{B}}$ and $\mathsf{SGraph}_{\mathcal{B}}$ of $\mathcal{B}$-labelled graphs.

In particular, the definition is tailored to make sure that the geometric realisation of a graph is a $\mathcal{B}$-labelled complex.

Proposition 3.23. *The geometric realisation* $|-|$ *is a functor* $\mathsf{Graph}_{\mathcal{B}} \to \mathsf{SSComp}_{\mathcal{B}}$ *if we define the labelling on* $|\Gamma|$ *by saying that a point gets the label of the vertex or edge in which it lies. Furthermore the image consists only of* 0*-finite complexes.*

Proof. The equations $b \circ \tilde{i} = b$ and $b \circ \tilde{s} = b$ on $b^{-1}(\mathcal{B})$ make sure that this is well-defined, as edges have a single label and each vertex can only get a single non-empty label. Furthermore, it is easy to see that the second equation implies that boundary of a simplex with non-empty label has the same label again.

That the complexes obtained this way are 0-finite is a direct consequence of the fact that a graph only has finitely many vertices in our definition. $\square$

It is clear that the constructions on graphs given earlier extend naturally to $\mathcal{B}$-labelled graphs by keeping track of the labels correctly. We will not define these functors explictly.

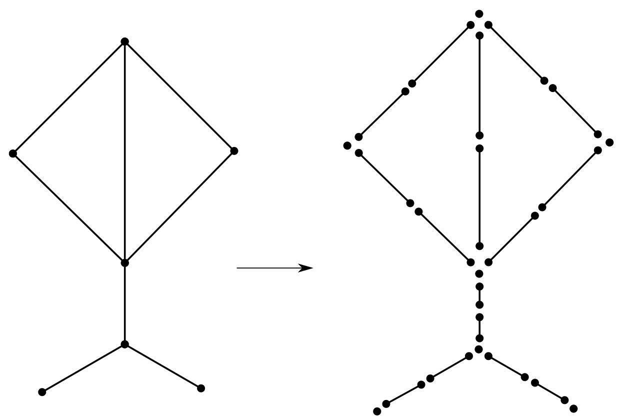


Figure 3.21. The effect of applying the half-edge separation functor hsep to a graph.

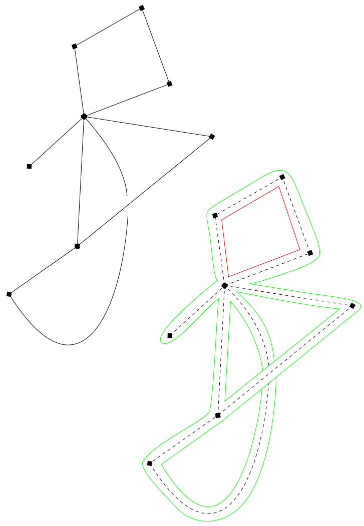


FIGURE 3.27. A fat graph and its corresponding surface. Note that the surface has two boundary components.

1.4. Fat graphs. A certain type of graph will represent a surface with boundary. Intuitively, we want to slightly thicken the edges of our graphs to finite width and the vertices to disks of finite size. Then we say this is the surface the graph represents. However, to make this construction well-defined we need additional data at the edge, i.e. we need to know in which order to attach the edges of our graph to the vertices. This can be encoded by a cyclic permutation of the half edges attached to a vertex. This leads us to the definition of a fat graph.

Definition 3.24. A *fat graph* is a graph $\Gamma = (V, H, s, i)$ together with a permutation $\sigma : H \to H$ such that its cycles correspond to the sets $s^{-1}(v)$ for $v \in V$. We demand that H is non-empty.

Convention 3.25. When we draw a fat graph, the cyclic ordering of the half edges attached to a vertex will be the cyclic ordering induced from the positive orientation of the plane.

Remark 3.26. A fat graph is also known as a ribbon graph, e.g. in [**Kon92**], [**Cos06a**].

We now make precise how to construct a surface with boundary from a fat graph. This makes rigorous the construction sketched above. Consider a finite set X with cyclic permutation σ, then we define a standard polygon with edges labelled by X.

- **Case $\#X \geq 3$:** If X contains three or more elements, we let $D_{X,\sigma}$ be the regular polygon in $\mathbb{R}^2$ with $\#X$ sides of length 1 and characteristic maps $b_x : I \to \partial D_{X,\sigma}$ for $x \in X$ such that b_x preserves orientation, i.e. 1 is counterclockwise from 0, and counterclockwise from the side b_x is the side $b_{\sigma(x)}$.
- **Case $\#X = 2$:** If X contains two elements $D_{X,\sigma}$ is a square I^2 with two characteristic maps b_x preserving orientation going to opposite edges.
- **Case $\#X = 1$:** If X contains a single element, $D_{X,\sigma}$ is a interval I with obvious characteristic maps $b_x : I \to I$ equal to the identity map.

The associated surface is given by glueing edges correctly to these standard polygons.

Definition 3.28. Let Γ be a fat graph. The *surface associated to* Γ, denoted Σ_Γ, is the space given by

$$\Sigma_\Gamma = \left((\coprod_{v \in V} D_{s^{-1}(v), \sigma|_{s^{-1}(v)}}) \sqcup (H \times [0, \frac{1}{2}] \times [0,1]) \right) / \sim$$

where $\sim$ is the equivalence relation which identifies $\{h\} \times \{\frac{1}{2}\} \times [0,1]$ with $\{i(h)\} \times \{\frac{1}{2}\} \times [0,1]$ using the map $t \mapsto 1 - t$ of $[0,1]$ and $\{h\} \times \{0\} \times [0,1]$ with an edge of $D_{s^{-1}(s(h)), \sigma|_{s^{-1}(s(h))}}$ using b_h.

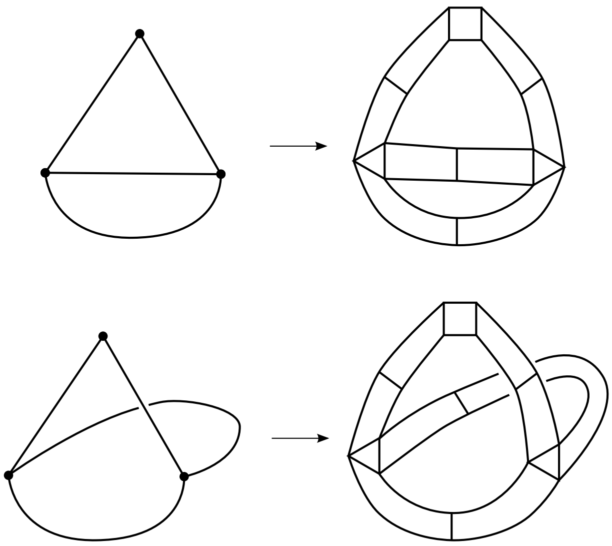


FIGURE 3.29. The construction of definition 3.28 for two similar fat graphs.

This has a natural smooth structure coming from the smooth structure of the intervals and the standard polygons. We give Σ_Γ the orientation such that at each vertex the disk is positively oriented.

An example of two fat graphs and their corresponding surfaces is given in figure 3.29. Note that by first glueing the two half edges, we can also construct Σ_Γ as

$$\Sigma_\Gamma = \left((\coprod_{v \in V} D_{s^{-1}(v), \sigma|_{s^{-1}(v)}}) \sqcup (E \times [0,1] \times [0,1]) \right) / \sim$$

Also note that Σ_Γ by construction deformation retracts onto Γ. We therefore sometimes call Γ the spline of Σ_Γ. From the construction we see that Σ_Γ will have at least one boundary component, but there might be more. We want to identify these boundary components in the combinatorics of fat graphs.

Lemma 3.30. *The boundary components of Σ_Γ can be identified with the cycles of $\sigma \circ i$ or equivalently $i \circ \sigma$.*

PROOF. We recommend drawing a fat graph and its corresponding surface to illuminate this proof. Let h be a half edge. Then we denote the two sides of $\{h\} \times [0, \frac{1}{2}] \times [0,1]$ by h_0 and h_1. We look at the boundary component of which h_0 forms a subset. h_0 is connected to $i(h)_0$ which is in turn connected to $\sigma(i(h))_1$. Using this we obtain that the cycle of $\sigma \circ i$ containing h contains a half edge in a side of each thickened edge $\{e\} \times [0,1] \times [0,1]$ which is a subset of the boundary component containing h_0. Since the entire boundary component is a union of such sides, we obtain a bijective correspondence between boundary components and cycles of $\sigma \circ i$.

For $i \circ \sigma$ it suffices to note that conjugation with i preserves the cycle structure: hence there is bijection between the cycles of $i \circ \sigma$ and the cycles of $i \circ i \circ \sigma i = \sigma \circ i$, where we used that $i = i^{-1}$. □

Giving the boundary cycles is in fact equivalent to given the cyclic ordering at the vertices. This is a consequence of the fact that i is an involution, which implies that σ can be reconstructed from $\sigma \circ i$ by precomposing with i.

Remark 3.31. Note that each connected component of Σ_Γ always has at least one boundary component. This is consistent with the fact that string topology is a HCFT with positive boundary.

Now that we have a definition of fat graphs, we want to know what a morphism of fat graphs is. In the case of graphs, we choose our simple morphisms such that they preserve the homotopy type of the geometric realisation. We do something similar for fat graphs: we want those morphisms that induce a map between the surfaces corresponding to a fat graphs and such that they preserve the homeomorphism type of these surfaces.

Definition 3.32. A *morphism of fat graphs* $f : \Gamma \to \Gamma'$ is a simple morphism of graphs which preserves the boundary cycles, i.e. we require that f doesn't collapse any boundary circles and that $\sigma' \circ i'$ is obtained from $\sigma \circ i$ by removing all collapsed half edges from the cycles and then applying f.

The last condition can be made more concrete, which is useful for the proofs.

Remark 3.33. The latter condition, in the context of simple morphisms of graphs, can be rewritten as follows.

First note that for each $h' \in H'$ then there is a unique $h \in f^{-1}(h')$ which is not collapsed by f. The condition is then equivalent to requiring that $\sigma' \circ i'(h') = f((\sigma \circ i)^k h)$ where k is the smallest integer such that $(\sigma \circ i)^k h$ is not collapsed by f.

Lemma 3.34. *The identity is a morphism of fat graphs and the composite of two morphisms of fat graphs is again a morphism of fat graphs. Furthermore, this composition is associative.*

Proof. See lemma 3.11 for the properties of simple morphisms. With this lemma, we only need to worry about possible collapse of the boundary components. Since the identity doesn't collapse any edges it automatically preserves the boundary components.

We already know that a composition of simple morphisms of graphs is again a simple morphism of graphs. It is clear that if both $g : \Gamma_1 \to \Gamma_2$ and $f : \Gamma_0 \to \Gamma_1$ don't collapse boundary cycles, then $g \circ f$ won't either. For the condition on the boundary cycles, we use the reprashing of the condition in remark 3.33. There is a unique $h_0 \in (g \circ f)^{-1}(h_2)$ which is not collapsed by $g \circ f$, given by the unique $h_0 \in f^{-1}(h_1)$ which is not collapsed by f where h_1 is the unique element of $g^{-1}(h_2)$ not collapsed by g. Then $\sigma_2 \circ i_2(h_2) = g((\sigma_1 \circ i_1)^{k_1} h_1)$ and $\sigma_1 \circ i_1(h_1) = f((\sigma_0 \circ i_0)^{k_0} h_0)$ for some k_1 and k_0. Now note that $(\sigma_1 \circ i_1)^{k_1}(h_1) = f((\sigma_0 \circ i_0)^{k_1+k_0} h_0)$. Therefore it suffices to note that the smallest k for which $(\sigma_0 \circ i_0)^k(h_0)$ is not collapsed is equal to $k_1 + k_0$. But this is clear, as every edge which is collapse has to be collapsed either by g or f.

Associativity follows from the associativity of the composition of morphisms of graphs. □

This has given us enough information to define a category of fat graphs.

Definition 3.35. Fat is the category with as objects the fat graphs and as morphisms the morphisms of fat graphs.

1.5. The connected components of Fat. We will see that Fat naturally splits into disjoint subcategories, indexed by the genus and the number of boundary components. We already know how to compute the number of boundary components from the combinatorial data of a fat graph, so it suffices to show how to compute the genus of Σ_Γ from the combinatorial data of a fat graph Γ.

Lemma 3.36. *The genus of Σ_Γ is given by $g = \frac{1}{2}(-\#V + \#E - \#\{\textit{boundary components}\} + 2)$.*

Proof. By attaching a two-cell to $|\Gamma|$ for each boundary component we get a compact oriented surface whose genus coincides with the genus of Σ_Γ. The complex for cellular homology over $\mathbb{R}$ has dimension $\#V$ in degree 0, $\#E$ in degree 1 and $\#\{\text{boundary components}\}$ in degree 2. Hence the Euler characteristic χ is given by $\#V - \#E + \#\{\text{boundary components}\}$. From the equation $\chi = 2 - 2g$ we derive the formula for g. □

Remark 3.37. Note that in figure 3.29 the surface with boundary on the top has three boundary components and genus 0, while the bottom one has one boundary component and genus 1.

We can now prove the statement that a morphism of fat graphs preserves the homeomorphism type of the surface.

Lemma 3.38. *If $f : \Gamma \to \Gamma'$ is a morphism of fat graphs, then Σ_Γ is a surface homeomorphic to $\Sigma_{\Gamma'}$.*

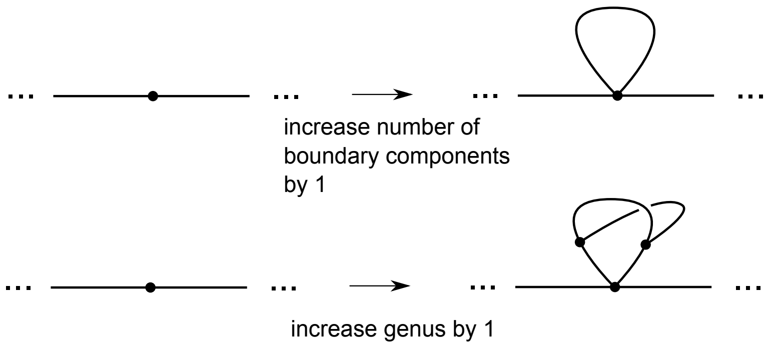


FIGURE 3.42. Two moves which can be used to obtain a fat graph for each homeomorphism type of surface.

PROOF. We will use the classification of connected compact oriented surfaces with boundary. These are uniquely determined up to homeomorphism by the genus g and the number of boundary components n. Since the morphisms of fat graphs must preserve the boundary cycles, the number of boundary cycles must the same for Γ and Γ'.

We use that the formula for g given in lemma 3.36. Since the number of boundary components is constant, it suffices to prove the difference of the edges and the vertices is constant under morphisms of fat graphs. But this is easy to see form the fact that we only allow collapses of trees. $\square$

Remark 3.39. There is a alternative proof of the previous statement. First note that the part of the surface with boundary coming from a fat graph which corresponds to a tree is homeomorphic to a disk. Also, we can identify the part of the surface corresponding to a vertex with a disk. Thus collapsing a tree doesn't change the homeomorphism type of the corresponding surface with boundary.

But does every surface of genus g with n boundary components arise as the surface corresponding to a fat graph? The answer to this question is yes. A more advanced case is treated by theorem 3.56, but we now give a simple construction for the case of ordinary fat graphs.

Lemma 3.40. *There exists a graph Γ such that Σ_Γ is a surface of genus g with n boundary components.*

PROOF. The underlying graph of Γ has one vertex $*$ and $4g+2n$ half edges $\{1,\dots,2g+n,\bar{1},\dots,\overline{2g+n}\}$. The source map s has constant value $*$ and $i(j)=\bar{j}$ determines i uniquely. We set $i\circ\sigma$ as follows, clearly giving $n+1$ boundary components (note that $1,2,\dots,n$ don't appear in this cycle):

$$i\circ\sigma = (\bar{1}\,\bar{2}\cdots\bar{n}\,n+1\,\overline{n+2}\,n+3\cdots\overline{2g+n}\,n\,\overline{n+1}\cdots 2g+n)$$

As a consequence σ should be:

$$\sigma = (1\,\bar{1}\,2\,\bar{2}\cdots n\,\bar{n}\,\overline{n+1}\,\overline{n+2}\cdots n+1\cdots 2g+n)$$

The genus of the surface Σ_Γ is $\frac{1}{2}(-1+(2g+n)-(n+1)+2)=g$ using lemma 3.36. $\square$

Remark 3.41. There is an alternative proof of the last lemma. For this proof one notices that there are two moves, which either increase the number of boundary components or the genus by 1. These are depicted in figure 3.42. Starting linear fat graph one can obtain all values for number of boundary components and the genus by repeatedly applying these moves.

Corollary 3.43. *The category Fat^c of connected fat graphs is a disjoint union of categories $\mathsf{Fat}_{g,n}$, which are the full subcategories of Fat on the connected fat graphs which realize to a surface of genus g with n boundary components.*

Proof. Lemma 3.38 implies that $\mathsf{Fat}_{g,n}$ is a disjoint union of connected components and lemma 3.40 shows that these components are non-empty. We need to show that it is connected. This is a consequence of theorem 3.59, which we will prove later. □

For future purposes, we will describe the fat graphs in the components $\mathsf{Fat}_{0,1}$ and $\mathsf{Fat}_{0,2}$ explicitly.

Lemma 3.44. *A fat graph Γ can have a corresponding surface Σ_Γ homeomorphic to a disk if and only if it is a tree. A fat graph Γ can have a corresponding surface Σ_Γ homeomorphic to an annulus if and only if it consists of a single cycle with trees attached.*

Proof. In both cases it is clear that the fat graphs described have the correct type of corresponding surface associated to it. So it suffices to prove the converse. In this proof, we use graphs with open edges. These are half-edge without source.

Suppose that Γ is not a tree. Let $\Gamma' \subset \Gamma$ be a maximal disjoint union of cycles in Γ. Then the set-theoretic complement of Γ' in Γ is a disjoint union of open trees. An open tree with k open leaves has $\#E - \#V = k - 1 \geq 0$ and a cycle has $\#E - \#V = 0$.

If Σ_Γ is a disk, then from the formula in lemma 3.36 we obtain that $\#E - \#V = -1$. From the earlier discussion this can only occur if it is a single tree without open leaves, because Γ has to be connected.

If Σ_Γ is an annulus, then we first note that $\#E - \#V = 0$. Γ can't be a tree, because then Σ_Γ would be a disk. Therefore there is at least one cycle and each cycle contributes zero to $\#E - \#V$ and each tree contributes $k - 1 \geq 0$ to $\#E - \#V$. Since Γ is connected, if there is more than one cycle then there must be a open tree connecting those, and this tree must contribute a positive number to $\#E - \#V$. We conclude that there is exactly one cycle. Furthermore, each open tree must have $k - 1 = 0$, which means that it attached to the cycle at exactly one vertex. Thus our description of such graphs as a cycle with attached trees is correct. □

1.6. Constructions on fat graphs. We now discuss some useful functors from fat graphs to graphs. The first is the forgetful functor, which forgets the cyclic ordering at each vertex. We denote it by S for spline. This name was chosen because $|S(\Gamma)|$ is the spline of Σ_Γ.

Definition 3.45. The functor $S : \mathsf{Fat} \to \mathsf{Graphs}$ is the forgetful functor on objects and morphisms.

Since any morphism of fat graphs is in particular a morphisms of graphs, this is indeed a functor. Secondly, there is a functor from the category of fat graphs to the category of graphs, given by a assigning to a fat graphs its boundary graph.

Definition 3.47. The functor $\partial : \mathsf{Fat} \to \mathsf{Graphs}$ assigns a fat graph $\Gamma = (V, H, s, i, \sigma)$ the graph $\partial\Gamma$ with vertices $\coprod_{v \in V} \left(\coprod_{h \in s^{-1}(v)} \{v_h\}\right)$ and half edges $\coprod_{h \in H} \{h_0, h_1\}$. The source map is given by $s(h_0) = v_h$ and $s(h_1) = v_{\sigma(h)}$. The involution is given by $i(h_0) = i(h)_1$ and $i(h_1) = i(h)_0$.

A morphism of fat graphs assigns a morphism of graphs by the induced map on the indexing set of the vertices and the edges. The functor ∂ is called the boundary graph functor.

It is geometrically clear that the boundary of a surface with boundary maps to the spline. This is made precise in the following proposition.

Proposition 3.48. *There is a natural transformation $s : \partial \dot{\longrightarrow} S$ given on a graph $\Gamma = (V, H, s, i)$ by the morphism of graphs which sends acts on vertices by $v_h \mapsto s(h)$ and on half-edges by $h_j \mapsto h$. This natural transformation is called the collapse map.*

The reason we call ∂ the boundary graph functor is that its geometric realisation is the boundary of $|\Gamma|$.

Proposition 3.49. *The geometric realisation $|\partial\Gamma|$ can be identified by the boundary $\partial|\Gamma|$ of the geometric realisation of Γ.*

Proof. There is a homeomorphism induced by the following map on the component composing $|\partial\Gamma|$. The vertex $*_{v_h}$ maps to the corner $b_h(0)$ of $D_{s^{-1}(v), \sigma|_{s^{-1}(v)}}$, the interval $[0, \frac{1}{2}]_{h_0}$ maps to $\{h\} \times [0, \frac{1}{2}] \times \{0\}$ and the interval $[0, \frac{1}{2}]_{h_1}$ maps to $\{h\} \times [0, \frac{1}{2}] \times \{1\}$. □

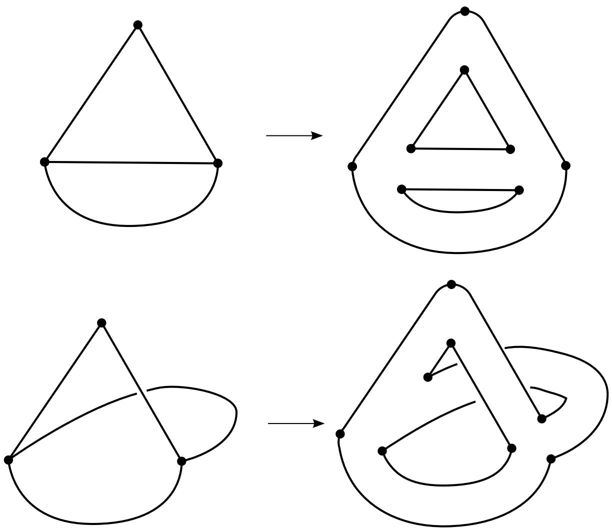


FIGURE 3.46. The boundary functor ∂ applied to two similar fat graph. The first has three boundary components while the second has a single one.

1.7. $\mathcal{B}$-labelled fat graphs. If fat graphs are to model open-closed cobordisms, we should be able to find a suitable notion of decorated fat graphs to model $\mathcal{B}$-labelled cobordisms. We choose a model first introduced by Ramirez in [**Ram06**].

Definition 3.50. A $\mathcal{B}$-labelled fat graph is a fat graph Γ with subgraph $\partial_f\Gamma$ of $\partial\Gamma$ together with a locally constant map $b : \partial_f\Gamma \to \mathcal{B}$.

A morphism of $\mathcal{B}$-labelled fat graphs $\Gamma \to \Gamma'$ is pair of morphisms of fat graphs $f : \Gamma \to \Gamma'$ and a morphisms of graphs $f_f : \partial_f\Gamma \to \partial_f\Gamma'$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{B} \arrow[d, equal] & \partial_f\Gamma \arrow[l, "b"'] \arrow[r, "i"] \arrow[d, "f_f"'] & \partial\Gamma \arrow[d, "\partial(f)"] \\
\mathcal{B} & \partial_f\Gamma' \arrow[l, "b'"] \arrow[r, "i'"'] & \partial\Gamma'
\end{tikzcd}$$

This defines a category $\mathsf{Fat}_{\mathcal{B}}$ of $\mathcal{B}$-labelled fat graphs.

Definition 3.51. If a $\mathcal{B}$-labelled fat graph Γ satisfies the condition that the composition of $i : \partial_f\Gamma \to \partial\Gamma$ with boundary collapse map $s : \partial\Gamma \to \Gamma$ of proposition 3.48 is a embedding, then it is called an nice $\mathcal{B}$-labelled fat graph. We denote the full subcategory of $\mathsf{Fat}_{\mathcal{B}}$ of $\mathcal{B}$-labelled on these nice fat graphs by $\mathsf{Fat}^n_{\mathcal{B}}$.

Of course, we also want to keep track of the incoming and outgoing boundary.

Definition 3.52. An open-closed $\mathcal{B}$-labelled fat graph is a $\mathcal{B}$-labelled fat graph with a partition of $\partial\Gamma\backslash\partial_f\Gamma$ into an ordered disjoint union of graphs $\partial_{in}\Gamma$ and $\partial_{out}\Gamma$ and a marked "starting vertex" for each component of $\partial\Gamma\backslash\partial_f\Gamma$. The graphs $\partial_{in}\Gamma$ and $\partial_{out}\Gamma$ will be $\mathcal{B}$-labelled, with labels appearing on vertices coming from $\partial_f\Gamma$.

A morphism of open-closed $\mathcal{B}$-labelled fat graphs $f : \Gamma \to \Gamma'$ is a morphism of $\mathcal{B}$-labelled fat graphs such that the map $\partial(f)$ sends $\partial_{in}\Gamma$ to $\partial_{in}\Gamma'$, preserving the ordering, and $\partial_{out}\Gamma$ to $\partial_{out}\Gamma'$, preserving the ordering, and sends starting vertices to starting vertices. This defines a category $\mathsf{Fat}^{oc}_{\mathcal{B}}$.

These fats graphs have the property that their geometric realisation can be given the labellings and parametrizations of boundary components to produce $\mathcal{B}$-labelled cobordism.

Definition 3.53. The cobordism realisation $|\Gamma|_{oc}$ of a open-closed $\mathcal{B}$-labelled fat graph is Σ_Γ as a manifold with boundary and corners on the boundary, where we identify the boundary $\partial|\Gamma|$ with

$|\partial\Gamma|$. The free boundary $\partial_f|\Gamma|$ is then $|\partial_f\Gamma| \subset \partial|\Gamma|$ and the incoming and outgoing boundaries are $|\partial_{in}\Gamma| \subset \partial|\Gamma|$ and $|\partial_{out}\Gamma| \subset \partial|\Gamma|$. The ordering of the incoming and outgoing boundary graphs gives us an ordering on the boundary components. $|\Gamma|_{oc}$ inherits an orientation from Σ_Γ and this together with the choice of starting vertex gives us a canonical parametrization of these incoming and outgoing boundary components.

It is easy to see that if there exists a morphism of open-closed $\mathcal{B}$-labelled fat graphs between Γ and Γ', then both the corresponding $\mathcal{B}$-labelled cobordisms will representent the same isomorphism class of $\mathcal{B}$-labelled cobordisms.

Because we will need to work with just the graphs for our umkehr maps, not their associated boundary graphs, it is inconvenient if an edge or vertex gets two labels. To solve this, we have to resort to nice graphs. First note that if the underlying $\mathcal{B}$-labelled fat graph of a open-closed $\mathcal{B}$-labelled fat graph is nice, each vertex and half-edge of Γ gets a labelling.

Definition 3.54. We say a open-closed $\mathcal{B}$-labelled fat graph is nice if the underlying $\mathcal{B}$-labelled graph is nice and the composition of $\partial_{in}\Gamma \to \partial\Gamma$ with $\partial\Gamma \to \Gamma$ is injective on half-edges and sends unlabelled edges to unlabelled edges. Furthermore, we require that if $\partial_{in}\Gamma$ maps to a labelled vertex, then one of the preimages of this vertex lies in $\partial_f\Gamma \cap \partial_{in}\Gamma$ (since the graph is nice, it is at most one). It is called admissible if the composition of $\partial_{in}\Gamma \to \partial\Gamma$ with $\partial\Gamma \to \Gamma$ is injective on vertices as well.

This way, we get a categories $\mathsf{Fat}^{oc,n}_{\mathcal{B}}$, $\mathsf{Fat}^{oc,a}_{\mathcal{B}}$ of nice, resp. admissible, open-closed $\mathcal{B}$-labelled fat graphs.

Remark 3.55. We will now describe some open-closed $\mathcal{B}$-labelled graphs that are nice and some that aren't.

- A graph is not nice when an edge has labels in $\mathcal{B}$ on both sides.
- A graph is not nice when an edge has a label in $\mathcal{B}$ on one side and an incoming label on the other.
- A graph is not nice when a vertex has incoming label on boundary component and label in $\mathcal{B}$ on another.
- A graph can still be nice when a vertex has incoming and label in $\mathcal{B}$ on the same boundary component. Of course, all other boundary component must then be outgoing.
- A graph can still be nice when a vertex is labelled incoming from multiple boundary components. If we only look at admissible graphs, this is not allowed. Only one boundary component of a vertex can carry a different label than outgoing.

We must be sure that we do not lose any information by considering only nice open-closed $\mathcal{B}$-labelled fat graphs. A starting point to showing this is the following important result by Ramirez [**Ram06**, lemma 9].

Theorem 3.56. *For each isomorphism class of $\mathcal{B}$-labelled cobordism there exists a nice open-closed $\mathcal{B}$-labelled graph Γ such that $|\Gamma|_{oc}$ is a representative of this isomorphism class.*

Sketch of proof. The proof works by first reducing open-closed $\mathcal{B}$-labelled graphs to a certain normal form and then applying an algorithm which resolves all bad edges and vertices, i.e. edges or vertices which can more than one label applied to them. $\square$

However, this theorem will also follow from theorem 3.69, as long as one assumes that there is some not necessarily nice open-closed $\mathcal{B}$-labelled graph representing an isomorphism class. We will also show that there one can represent each class with an admissible graph in the course of proving theorem 3.69.

2. The relation between fat graphs and mapping class groups

In the previous section we discovered a relation between fat graphs and cobordisms: fat graphs correspond to splines of cobordisms. In this section we will extend this to a relation between the category of fat graphs and mapping class groups. The idea behind these theorems is that the classifying space of mapping class group represents the moduli space of Riemann surfaces with boundary up to homotopy. Hence if we agree to pass to homology or some other generalized cohomology theory later on, we can replace moduli spaces with the geometric realisation of the category of fat graphs. We have the following goal in mind: using fat graphs to construct operations parametrised by homology classes of moduli spaces of Riemann surfaces with boundary.

2.1. Base case: fat graphs. We start with the fundamental theorem underlying this theory. The following theorem has been proven in different guises in the last twenty years, for example by Strebel [**Str84**], Penner [**Pen87**], Kontsevich [**Kon92**], Igusa [**Igu02**] and Costello [**Cos06a**]. Let Fat_3^c denote the full subcategory of connected fat graphs for which every vertex has valence greater than or equal to 3. We furthermore assume that the boundary components of the fat graphs are labelled and the morphisms respect these labels.

Theorem 3.57 (Costello, Kontsevich, Igusa, Penner, Strebel)**.** *There is a rational homology equivalence*

$$|\mathsf{Fat}_3^c| \simeq_{\mathbb{Q}} \coprod_{[\Sigma]} B\Gamma_\Sigma$$

where the disjoint union runs over all isomorphism classes of orientable closed 2-dimensional manifolds with $n \geq 1$ marked points except $(g,n) = (0,1), (0,2)$.

Sketch of proof. We sketch Strebel's construction using conformal geometry [**Str84**], as explained by Kontsevich in [**Kon92**]. There are other proofs: Penner gives a construction using hyperbolic geometry and Costello using an orbicell decomposition of the moduli space. Figure 3.58 illustrates the first part of the proof.

A quadratic differential on a Riemann surface S is a holomorphic section of $(T^*S)^{\otimes 2}$. This means that it can locally by written as $\phi(z)dz^2$ for a holomorphic function ϕ. A horizontal trajectory of a non-zero quadratic differential is a curve along which $\phi(z)$ attains real positive values. These trajectory are either closed or non-closed. The closed trajectory come in families and these families decompose the complement of the non-closed trajectories into annuli or punctured disks. In a single annulus or punctured disks all closed trajectories have the same length.

A Jenkens-Strebel differential is one such that the non-closed trajectories have measure zero. The fundamental theorem of the theory of these differentials is that for each choice of n marked points $\{x_1, \ldots, x_n\}$ in S, where $n > 0$ and $n > \chi(S)$ and each choice of n choice of n positive real numbers $\{r_1, \ldots, r_n\}$, there is a unique Jenkens-Strebel differential on $S \backslash \{x_1, \ldots, x_n\}$ which partitions $S \backslash \{x_1, \ldots, x_n\}$ into a n punctured disks in which the closed trajectories have lengths $r_1, \ldots, r_n$ respectively, together with non-closed trajectories and zeroes of the differential.

The union of the zeroes with the non-closed trajectories is a graph, which becomes a fat graph because it comes embedded into a Riemann surface. The boundary components of this fat graph are labelled by the labels of the marked point. In fact, this is a metric fat graph, as all edges come equipped with a length. If $\mathcal{M}_{g,n}$ is the moduli space of Riemann surfaces, then we get a function $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \to Fat_{g,n}$, where $Fat_{g,n}$ is the set of equivalence classes of metric fat graphs which realise to a surface of genus g with n labelled boundary components. This is bijective, using a construction similar to our construction of a topological surface from a fat graph, which needs to be adapted to the conformal category.

In fact, if one gives $Fat_{g,n}$ the correct topology orbispace structure, then the map $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \to Fat_{g,n}$ is an homotopy equivalence of orbispaces. One then shows that with this topology $Fat_{g,n}$ is homotopy equivalent as an orbispace to $|(\mathsf{Fat}_3^c)_{g,n}|$ with orbispace structure coming from the action of the self-isomorphisms of the fat graphs (this is essentially [**Kon92**, appendix 3]). Here $(\mathsf{Fat}_3^c)_{g,n}$ is the subcategory of Fat_3^c of fat graphs whose corresponding surface has genus g and n boundary components. Finally, $\mathcal{M}_{g,n}$ is obtained by modding out a contractible space (the Teichmüller space) by the mapping class group. This action is properly discontinuously but has finite stabilizers. From this we conclude

$$B\Gamma_{g,n} \simeq_{\mathbb{Q}} \mathcal{M}_{g,n} \times \mathbb{R}^n_{>0} \simeq Fat_{g,n} \simeq |(\mathsf{Fat}_3^c)_{g,n}|$$

where $\simeq_{\mathbb{Q}}$ denotes a rational homology equivalence.

The action of the symmetric group Σ_n on $\mathcal{M}_{g,n} \times \mathbb{R}^n_{>0}$ permuting the labels of the marked points is equivalent to the permutation of the labels of boundary components of a fat graphs.

Finally, we look at the restrictions imposed by the demand that $n > 0$ and $n > \chi(S)$. This demand implies, using the formula $\chi(S) = 2 - 2g$ that $n \geq 1$ if $g \geq 1$ and $n \geq 3$ is $g = 0$. This rules out exactly $(g, n) = (0, 1), (0, 2)$. In the other direction, since any surface constructed from a fat graph has at least one boundary components and n is the number of boundary components of the fat graph, so this first demand is always satisfied. Secondly $\chi(S) - n$ is the number of vertices minus the number of edges, like in lemma 3.36. Hence the demand that $n > \chi(S)$ translates to the demand on fat graphs that the number of vertices is strictly smaller than the number of edges. This is automatically guaranteed if each vertex has valence 3 or greater. $\square$

This theorem is not as strong as one would like it to be. However, in the following statements we can assume that it really says that there is a homotopy equivalence

$$|\mathsf{Fat}^c_3| \simeq \coprod_{[\Sigma]} B\Gamma_\Sigma$$

where the disjoint union runs over all isomorphism classes of orientable closed 2-dimensional manifolds with $n \geq 1$ marked points except $(g, n) = (0, 1), (0, 2)$. This is because if we would have worked with the moduli space $\mathcal{M}^n_g$ of Riemann surfaces of genus g with n boundary circles, then the mapping class has trivial stabilizers on Teichmüller space and the previous argument can be adapted to give a homotopy equivalence $\mathcal{M}^n_g \simeq B\Gamma_{g,n}$. Indeed, in the following sections we will only use Riemann surfaces with at least one boundary component. However, this was not the situation to which Strebel's theory applies. We could modify the proof and make everything in this chapter precise, but that would lead us to far astray into the territory of orbifolds and automorphisms of Riemann surfaces.

In conclusion, to be completely rigorous one would need to replace each homotopy equivalence with a classifying space that appears in the following sections with a rational homology equivalence. It is a lucky consequence of our choice to work with rational coefficients that this doesn't matter, because we only using the homology of the classifying space of the mapping class group with rational coefficients in our construction.

It is interesting to see what happens when we allow vertices of other valences as well. There is the generalisation by Godin, which allows for vertices of valence one and two. This proves was sketched by Godin in the beginning of theorem 3 [**God07**, theorem 3] and we fill in some of the details, especially concerning the two exceptional components.

Theorem 3.59. *Let Fat^c denote the full subcategory of connected fat graphs. Then $|\mathsf{Fat}^c| \simeq BU(1) \sqcup BU(1) \sqcup \coprod_{[\Sigma]} B\Gamma_\Sigma$ where the disjoint union runs over all isomorphism classes of orientable closed 2-dimensional manifolds with $n \geq 1$ marked points except $(g, n) = (0, 1), (0, 2)$. These exceptions correspond exactly to the two copies $BU(1)$.*

Proof. We need to show that allowing vertices of valence ≤ 2 doesn't change the homotopy type of the geometric realisation of the components of Fat^c with of fat graphs with number of boundary components n and genus g satisfying $(g, n) \neq (0, 1), (0, 2)$ and that the geometric realisations of the remaining components have the correct homotopy type.

We first prove we can add vertices of valence 2 and 1 to the components with $(g, n) \neq (0, 1), (0, 2)$. To do this, we introduce two intermediate categories $\mathsf{Fat}^c_{2,3}$ and $\mathsf{Fat}^c_{1,3}$. The category $\mathsf{Fat}^c_{2,3}$ is the full subcategory of Fat^c of connected fat graphs where each vertex has valence greater than or equal to 2 and the number of edges is stricly greater than the number of vertices. The category $\mathsf{Fat}^c_{1,3}$ is the full subcategory of Fat^c of connected fat graphs where each vertex has valence greater than or equal to 1 and the number of edges is strictly greater than the number of vertices.

Because morphisms only collapse subtrees, the condition that the number of edges is strictly greater than the number of vertices is preserved by morphisms. Thus these are indeed full subcategories.

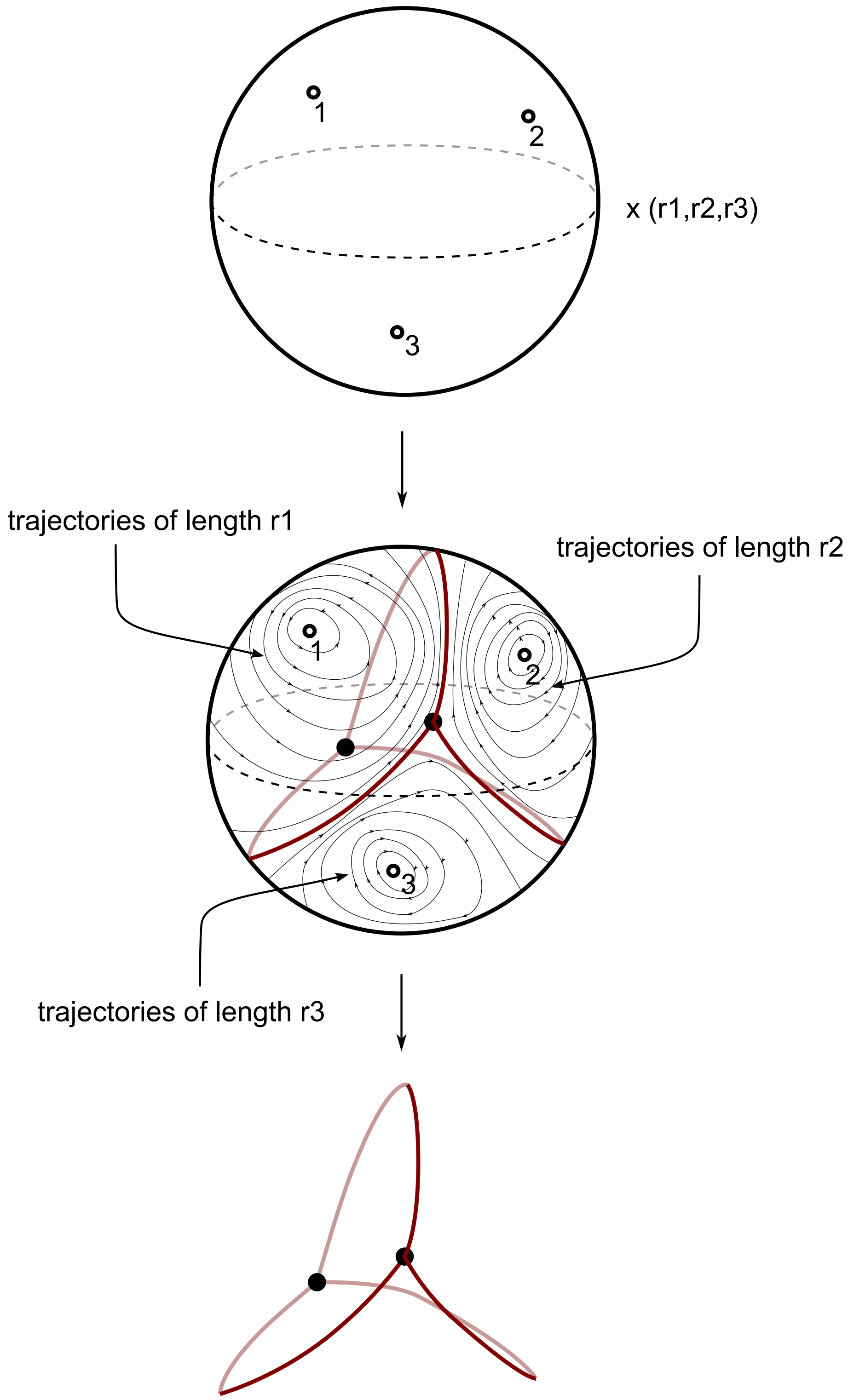


FIGURE 3.58. A summary of theorem 3.59. From a surface with marked points we construct a fat graph with the non-closed trajectories as edges and the zeroes as vertices.

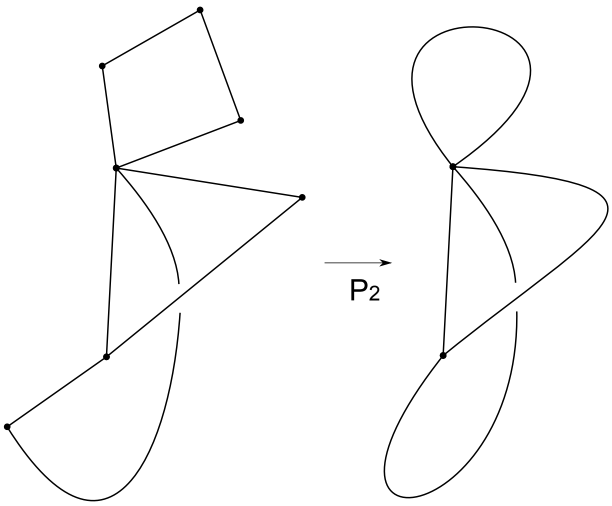


FIGURE 3.60. The effect of the functor P_2 on a fat graph.

From now on we drop the superscript c's, to make the notation slightly less cumbersome.

$|\mathsf{Fat}_{2,3}| \simeq |\mathsf{Fat}_3|$**:** We construct a functor $P_2 : \mathsf{Fat}_{2,3} \to \mathsf{Fat}_3$ and show this is a homotopy equivalence by investigating the fibers. While doing this for a fiber P_2/Γ, we define an auxiliary category $|P_2 \wr \Gamma| \simeq |P_2/\Gamma|$.

The functor P_2 acts on objects by sending a fat graph to the fat graph obtained by collapsing the open tree consisting of vertices of valence 2 and half-edges leading into them and pasting the remaining two half-edges together. Since the difference of the number of edges and the number of vertices remains the same, this is well-defined because it means that the collapse can not cause the formation of isolated vertices or vertices of valence 1 nor can it collapse the entire graph. See figure 3.60.

Let's make this precise. Let $T = (V_T, H_T, s_T, i_T, \sigma_T)$ be the open tree of 2-valent edges and $\Gamma = (V, E, s, i, \sigma)$ the original fat graph. Then the functor P_2 is defined on an object as $P_2(\Gamma) = (V \backslash V_T, H \backslash H_T, s|_{H\backslash H_T}, i', \sigma|_{H\backslash E_T})$, where we just need to say what i' is: it is given by the i on all edges except the ones whose partner is removed. If h, h' both don't have a partner and were connected by a removed open tree, we set $i(h) = h'$. This is well-defined, because a tree with only vertices of valence 2 is linear.

On morphisms P_2 acts by sending a morphism f to the following morphism: if an edge is the result of collapsing an open tree of valence 2, then we collapse this edge if and only if both parts of it were collapsed. To be precise, this map collapses an edge consisting of half-edges $\{h, h'\} \subset H\backslash H_T$ if and only if f collapse the original edges h and h' are part of.

It suffices to show that $|P_2|$ is a homotopy equivalence. For this we use Quillen's theorem A, here theorem B.38. The comma category P_2/Γ is the category of all graphs Γ' which admit a map $f' : P_2(\Gamma') \to \Gamma$. We will show that this category is contractible.

Let $P_2 \wr \Gamma$ be the full subcategory of P_2/Γ on the objects (Γ', f') such that f' is an isomorphism. We give a functor $Q_2 : P_2/\Gamma \to P_2 \wr \Gamma$ which induces a homotopy equivalence. Q_2 is given on an object (Γ', f') by collapsing all edges which are collapsed by f', thereby obtaining a fat graph $\tilde{\Gamma}'$. Note that $\Gamma' \to \tilde{\Gamma}'$ is a morphism of fat graphs. Then f' induces an isomorphism $\tilde{f}' : \tilde{\Gamma}' \to \Gamma$ and any morphism $(\Gamma', f') \to (\Gamma'', f'')$ induces an isomorphism of $\tilde{\Gamma}'$ with $\tilde{\Gamma}''$. Note that when we consider $P_2 \wr \Gamma$ as a subcategory of P_2/Γ, Q_2 is a retraction of P_2/Γ onto $P_2 \wr \Gamma$. To show this it suffices to show that there is a natural transformation from the identity functor of P_2/Γ to Q_2. But this is given by the collapse morphism $\Gamma' \to \tilde{\Gamma}'$. We conclude that $|P_2 \wr \Gamma| \simeq |P_2/\Gamma|$ using corollary B.27.

Finally, we show that $|P_2 \wr \Gamma|$ is contractible. To do this, we note that (Γ, id) is terminal in $P_2 \wr \Gamma$: the unique map from (Γ', f') to (Γ, id) is given by f'. Now we apply corollary B.29.

$|\mathsf{Fat}_{1,3}| \simeq |\mathsf{Fat}_{2,3}|$: There are functors $I_1 : \mathsf{Fat}_{2,3} \to \mathsf{Fat}_{1,3}$ and $P_1 : \mathsf{Fat}_{1,3} \to \mathsf{Fat}_{2,3}$. The first functor is the inclusion. The second functor, P_1, acts on objects by sending a fat graph to the fat graph with each vertex of valence 1 and the edge leading into it removed, repeating this process until no vertices of valence 1 remain. Note that this procedure is well defined, since the difference of the number of edges and vertices remains the same. This prevents the formation of isolated vertices.

On morphisms it is the induced morphism after finishing this collapsing procedure: the morphism induced by f just collapse all those edges which f collapses and weren't already collapsed by P_1. Note that the collapse map $\Gamma \to P_1(\Gamma)$ is a morphism of fat graphs, in contrast with any hypothetical map $\Gamma \to P_2(\Gamma)$.

The $P_1 \circ I_1$ is the identity functor of $\mathsf{Fat}_{2,3}$, so it suffices to prove that $|I_1 \circ P_1|$ is a homotopy equivalence. To show this, we produce a natural transformation between the identity functor and $I_1 \circ P_1$. This is given on Γ by the collapse map $\Gamma \to P_1(\Gamma)$.

Now for the other connected components. These are the geometric realisations of the following two full subcategories. The category Fat_T is the full subcategory of connected fat graphs which are trees. The category Fat_S is the full subcategory of connected fat graphs such that the underlying graph contains a single cycle. That these are disjoint connected components in the geometric realisation of Fat follows from lemma 3.44.

We claim that both categories are a $BU(1)$, being related to the category of cyclic sets, see remark 3.61. We will calculate the homotopy type of the geometric realisation of a similar category Z as a intermediate step.

$|\mathsf{Z}|$ is a $BU(1)$: To we compute the homotopy type of $|\mathsf{Fat}_T|$ and $|\mathsf{Fat}_S|$, we consider the topological category Z_{par} with objects finite sets $A = \{a_1, \ldots, a_n\}$ together with an injective map $\phi_A : A \to S^1$. The objects are topologized as $\coprod_{n \in \mathbb{N}} (S^1)^n$. A morphism is a surjective map $p : A \to B$ together with a map $\pi : \mathrm{im}(\phi_A) \to \mathrm{im}(\phi_B)$ which preserves the cyclic ordering coming from S^1. These are topologized as subspaces of the mapping spaces $(S^1)^n \to (S^1)^m$. We claim the geometric realisation of this category is contractible. To see this, consider the constant functor $C_1 : \mathsf{Z}_{par} \to \mathsf{Z}_{par}$ with value the singleton set $\{*\}$ which is mapped to 1. The collapse map $(A, \phi_A) \to (*, 1)$ is a natural transformation from the identity functor to C_1. This proves that Z_{par} is contractible. Now note that Z_{par} admits a free action from the topological group $U(1)$: $\alpha \in U(1)$ mapped to the functor which sends (A, ϕ_A) to $(A, \alpha \circ \phi_A)$ similarly on morphisms.

We conclude that the geometric realisation of $\mathsf{Z}_{par}/U(1)$ is a $BU(1) \simeq K(\mathbb{Z}, 2)$. Let's take a closer look at $\mathsf{Z}_{par}/U(1)$. The space of objects consists of a disjoint union of components Z_n, each of which consists of the unique cyclically ordered set with n elements up to ismorphism with a non-zero distance between each elements and all these distances must sum to 2π. In particular $Z_n \simeq \mathrm{int}(\Delta^{n-1})$ and is contractible. In fact, we can describe a retraction of $\mathsf{Z}_{par}/U(1)$ onto a subcategory Z with discrete objects space and morphism space. This category has space of objects consisting the cyclically ordered sets consisting of for example n elements, with distance $\frac{2\pi}{n}$ between each two elements. The morphisms are surjective order-preserving maps.

The category $\mathsf{Z}_{par}/U(1)$ coherently homotopes onto Z using the following homotopy on the components of the space of objects:

$$((d_1, \ldots, d_n), t) \mapsto \left((1-t)d_1 + t(\frac{2\pi}{n}), \ldots, (1-t)d_n + t(\frac{2\pi}{n}) \right)$$

On the morphism space, the homotopy is given by applying two such homotopies on the source and target of a morphism. Note that because the distances are uniquely determined by the set, we might as well forget about them.

Hence we conclude that $|\mathsf{Z}| \simeq |\mathsf{Z}_{par}/U(1)| \simeq BU(1)$.

$|\mathsf{Fat}_T|$ **is a** $BU(1)$**:** We first show that Fat_T retracts onto the subcategory $\mathsf{Fat}_\star$ consisting of trees with the shape of a star: there are n vertices of valence one and one vertex of valence n.

To apply corollary B.27 it suffices to give a functor $Q : \mathsf{Fat}_T \to \mathsf{Fat}_T$ such that $Q(\mathsf{Fat}_T) \subset \mathsf{Fat}_\star$ and $Q|_{\mathsf{Fat}_\star} = id$, together with a natural transformation $id \dot{\longrightarrow} Q$. For this, we let Q be the functor which collapses all internal edges, i.e. all edges that are not connected to a vertex of valence one. On morphisms Q is the induced map. Since the collapse map $\Gamma \to Q(\Gamma)$ is a morphism in Fat_T, we get the natural transformation as well.

Finally, we show that $|\mathsf{Fat}_\star|$ is a $BU(1)$. Let $i : \mathsf{Fat}_\star \to \mathsf{Z}$ be the functor which maps a star-like graph to its set of edges with cyclic ordering coming from the cyclic ordering at the center vertex. The distance between the edges is of course $\frac{2\pi}{\#E}$, otherwise it wouldn't land in Z. This is an isomorphism of categories. Since an isomorphism of categories is in particular an equivalence of categories, using corollary B.26 we conclude that $|\mathsf{Fat}_T| \simeq |\mathsf{Fat}_\star| \simeq |\mathsf{Z}| \simeq BU(1)$.

$|\mathsf{Fat}_S|$ **is a** $BU(1)$**:** Let $\mathsf{Fat}_\bigcirc$ the full subcategory of fat graphs which are a subdivided circle, then we claim that $\mathsf{Fat}_\bigcirc$ is homotopy equivalent to Fat_S after passing to geometric realisation.

If Γ contains a single cycle Ω, then $T = \Gamma\backslash\Omega$ is a tree and $\Gamma\backslash T$ is a subdivided circle. This construction is functorial by letting it work on morphisms by assigning the induced morphism. Thus we have just constructed a functor $P_\bigcirc : \mathsf{Fat}_S \to \mathsf{Fat}_\bigcirc$. We also have the inclusion functor $I_\bigcirc : \mathsf{Fat}_\bigcirc \to \mathsf{Fat}_S$. It is clear that $P_\bigcirc \circ I_\bigcirc$ is the identity functor on $\mathsf{Fat}_\bigcirc$ and we have a natural transformation from the identity functor on Fat_S to $I_\bigcirc \circ P_\bigcirc$ given by collapsing the trees. Hence $|\mathsf{Fat}_S| \simeq |\mathsf{Fat}_\bigcirc|$.

Finally, we show that $|\mathsf{Fat}_\bigcirc|$ is a $BU(1)$. We claim that it is possible to put an orientation on these graphs in such a way that the morphisms respect the orientation. Let Γ_n denote the standard graph with vertices $\{1, \dots, n\}$ and an edge between i and $i+1$, modulo n. We give this the orientation that i to $i+1$ is in positive direction. Since any morphism of fat graphs must preserve the boundary components and therefore can't switch them, the morphisms between the Γ_n's must preserve these. If we now extend the orientation to all graphs using the fact that the Γ_n are a skeleton, we have given exactly such an orientation.

Now note that there is a isomorphism of categories $j : \mathsf{Fat}_\bigcirc \to \mathsf{Z}$ mapping a graph to its edges with cyclic ordering coming from the ordering of the edges with the chosen orientation. As before, it now follows that $|\mathsf{Fat}_S| \simeq |\mathsf{Fat}_\bigcirc| \simeq |\mathsf{Z}| \simeq BU(1)$.

□

Remark 3.61. The dual of Z, denoted $\mathcal{Z}$, is the the category of finite sets with oriented cyclic ordering and morphisms the inclusions of degree 1, which is the category of cyclic sets in [**Igu04**]. By corollary B.30, $\mathcal{Z}$ is a $BU(1)$ and hence homotopy equivalent to the model of $K(\mathbb{Z}, 2)$ of given by $\mathbb{C}P^\infty$. Hence our proof has [**Igu04**, theorem 1.3] as a corollary.

Note that Igusa has already shown that the geometric realisation of Fat_S is a $BU(1)$ in [**Igu02**, lemma 8.1.13].

Let $\mathsf{Fat}_{g,n}$ denote the full subcategory of those fat graphs which realize to a surface with genus g and n boundary components, where $(g,n) \neq (0,1), (0,2)$. These graphs are necessarily connected and already appeared in lemma 3.43. In the next section we will need to pass to covers corresponding to subgroups of Σ_n and hence we need to look closer at the action of this group on $B\Gamma_{g,n}$ and $|\mathsf{Fat}_{g,n}|$.

Corollary 3.62. *The homotopy equivalence $B\Gamma_{g,n} \simeq |\mathsf{Fat}_{g,n}|$ is Σ_n-equivariant, where the action on the left hand side is by permuting the marked points and the action on the right hand side is by permuting the boundary components.*

Proof. In the proof of theorem 3.57 we saw action of Σ_n on $|(\mathsf{Fat}_3)_{g,n}|$ corresponding to the permutation of the marked points indeed is the permutation of the boundary components.

Since our construction of the homotopy equivalence $|\mathsf{Fat}_{g,n}| \simeq |(\mathsf{Fat}_3)_{g,n}|$ for $g = (0,1), (0,2)$ always respected the boundary components, if we make the Σ_n act $|\mathsf{Fat}_{g,n}|$ by permutation of the boundary components, this homotopy equivalence will be Σ_n-equivariant. □

2.2. $\mathcal{B}$-labelled cobordism graphs. We will now use the previous theorem to prove that the geometric realisation of the category $\mathcal{B}$-labelled cobordism graphs is homotopy equivalent to the classifying space of the mapping class groups of the corresponding cobordisms. We basically need to take care of two things: some boundary components – the fully free ones with the same label – can be permuted, and the incoming and outgoing boundary now have starting vertices. The first means going to some cover, the second adding circle bundles.

Theorem 3.63. *Let $(\mathsf{Fat}^{oc}_{\mathcal{B}})^c$ denote the full subcategory of connected $\mathcal{B}$-labelled cobordism graphs satisfying the positive boundary condition. Then $|(\mathsf{Fat}^{oc}_{\mathcal{B}})^c| \simeq \coprod_{[\Sigma]} B\Gamma_\Sigma$ where the disjoint union runs over all isomorphism classes of connected $\mathcal{B}$-cobordisms with ordered incoming and outgoing boundary components and satisfying the positive boundary condition.*

Proof. We will work over each component separately. For this we need to know that every $\mathcal{B}$-cobordism with positive boundary can be represented by a $\mathcal{B}$-labelled fat graphs. But this is a consequence of theorem 3.56.

The proof will then proceed as follows: we first consider the components of those graphs where the corresponding surface is not homeomorphic to the disk or the annulus. Let Σ be a fixed $\mathcal{B}$-labelled cobordism and $\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)$ be the full subcategory of $\mathcal{B}$-labelled cobordism graphs whose corresponding cobordisms $|\Gamma|_{oc}$ are isomorphic to Σ. We show that we can write $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ and $B\Gamma_\Sigma$ as circle bundles over homotopy equivalent base spaces $|\mathsf{Fat}^{f}_{\mathcal{B}}(\Sigma)|$ and $B\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$. We will define these later in the proof. Finally we prove that the homotopy equivalence of these base spaces lifts to the total spaces of the circle bundles.

After that we have to consider the cases of the disk and the annulus individually. We must do this because some of the short exact sequences do not apply in those cases.

Defining $\Gamma_{\tilde{\Sigma}}$ and $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$: Fix a $\mathcal{B}$-cobordism Σ and collapse each boundary component to marked point, obtaining a surface $\tilde{\Sigma}$ with marked points. We label each marked point with an element of $\{1, 2, \dots, n\}$, but also partly remember the labels from $\mathcal{B}$ via a map $b : \{1, 2, \dots, n\} \to \mathcal{B} \sqcup \{\text{special}\}$. This map b is given as follows: a marked point is given the special label if the boundary component has a non-free part and it is marked with the label $\beta \in \mathcal{B}$ if the boundary component is purely free with label β (and it is case the boundary component necessarily has a constant label β).

We let $\Gamma_{\tilde{\Sigma}}$ be the mapping class group of orientation-preserving homeomorphisms which fix the marked points pointwise. We let $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$ be the mapping class group of orientation-preserving homeomorphisms of $\tilde{\Sigma}$ which preserve special marked points pointwise and is allowed to permute the other marked points if they have the same label.

It is clear that $\Gamma_{\tilde{\Sigma}}$ is a subgroup of $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$ and in fact is the kernel of the map from $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$ to the permutations of the marked points with label from $\mathcal{B}$. Thus, we have a short exact sequence:

$$0 \to \Gamma_{\tilde{\Sigma}} \to \Gamma^{\mathcal{B}}_{\tilde{\Sigma}} \to \prod_{b \in \mathcal{B}} \Sigma(b^{-1}(\beta)) \to 0$$

This induces a fibration

$$\prod_{\beta \in \mathcal{B}} \Sigma(b^{-1}(\beta)) \to B\Gamma_{\tilde{\Sigma}} \to B\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$$

Defining $\mathsf{Fat}(\tilde{\Sigma})$ and $\mathsf{Fat}^{f}_{\mathcal{B}}(\Sigma)$: Let $\mathsf{Fat}(\Sigma)$ be the category of connected fat graphs whose geometric realisation is homeomorphic to Σ together with ordering on all boundary components and a map b from purely free boundary components to $\mathcal{B}$. This category is isomorphic to the category $\mathsf{Fat}(\tilde{\Sigma})$ of fat graphs whose geometric realisation is homeomorphic to $\tilde{\Sigma}$ after collapsing the boundary components, replacing them with marked points, together with ordering on all marked points and labels on some. In all cases the morphisms should preserve the ordering and marking.

Let $\mathsf{Fat}^f_{\mathcal{B}}(\Sigma)$ be the category of $\mathcal{B}$-labelled fat graphs whose geometric realisation is homeomorphic to Σ with ordering on the boundary components whose boundary component contains a non-free part and no ordering on the completely free boundary components. We also require a map b from purely free boundary components to $\mathcal{B}$.

There is a functor $\mathsf{Fat}(\tilde{\Sigma}) \to \mathsf{Fat}^f_{\mathcal{B}}(\Sigma)$ which sends a fat graphs Γ to the $\mathcal{B}$-labelled fat graph which labels each purely free boundary component according to b and forgets the ordering on these purely free boundary components. We investigate the fiber with the goal of applying Quillen's theorem B, here theorem B.40.

The fiber of the induced map over Γ on geometric realisation deformation retracts onto the geometric realisation of the category of fat graphs in $\mathsf{Fat}(\tilde{\Sigma})$ which are isomorphic to Γ after forgetting the ordering of the $\mathcal{B}$-labelled points. It is then easy to see that this category is equivalent to the group $\prod_{\beta \in \mathcal{B}} \Sigma(b^{-1}(\beta))$ considered as a category with one object. These correspond to all the different orderings of the purely free boundary components of Γ that are no longer distuingished after going to $\mathsf{Fat}^f_{\mathcal{B}}(\Sigma)$.

It is clear that a map in $\mathsf{Fat}^f_{\mathcal{B}}$ induces an isomorphism of fibers, and hence by applying Quillen's theorem B we get a fibration:

$$\prod_{b \in \mathcal{B}} \Sigma(b^{-1}(\beta)) \to |\mathsf{Fat}(\tilde{\Sigma})| \to |\mathsf{Fat}^f_{\mathcal{B}}(\Sigma)|$$

$B\Gamma^{\mathcal{B}}_{\tilde{\Sigma}} \simeq |\mathsf{Fat}^f_{\mathcal{B}}(\Sigma)|$: Recall that we saw two fibrations with fiber $\prod_{b \in \mathcal{B}} \Sigma(b^{-1}(\beta))$. Our goal is to prove that the base spaces are homotopy equivalent and we have a homotopy equivalence $B\Gamma_{\tilde{\Sigma}} \to |\mathsf{Fat}(\tilde{\Sigma})|$, because the positive boundary condition guarantees that the number of marked points is greater than or equal to 1.

The key is now to note that this homotopy equivalence is equivariant with respect to the action $\prod_{\beta \in \mathcal{B}} \Sigma(b^{-1}(\beta))$ as a consequence of corollary 3.62. We conclude that $|\mathsf{Fat}^f_{\mathcal{B}}(\Sigma)|$ is homotopy equivalent to $B\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$ since the homotopy descends.

Let d denote the number of special boundary components of Σ. The next step is to prove we have two fibrations $(S^1)^d \to |\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)| \to |\mathsf{Fat}^f_{\mathcal{B}}(\Sigma)|$ and $(S^1)^d \to B\Gamma_\Sigma \to B\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$, where Γ_Σ is the mapping class group of Σ as a $\mathcal{B}$-labelled surface with boundary and corners on the boundary.

Γ_Σ is a circle bundle over $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$: Let's discuss the second fibration first. It comes from a short exact sequence, induced by the map $\Gamma_\Sigma \to \Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$. This map is given by gluing a cap with marked point to each boundary component. If the boundary component is labelled totally with a label β, we give the marked point the label β. Otherwise, we give it the special label.

We must first check this is well-defined: that is, we must check we indeed land in $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$. The only issue is that some special marked points might be permuted. From the demand that an element of Γ_Σ must preserve the incoming and outgoing boundary pointwise and the purely free boundary components with the same label as a set, we see that if a boundary component contains both a labelled part and a incoming or outgoing part, it cannot be mapped to another boundary component by Γ_Σ. This implies that the Γ_Σ indeed maps into $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$.

Since the mapping class group is generate by Dehn twists, it is surjective. This is because any Dehn twist in $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$ can be made to have support disjoint from the marked points up to isotopy. We can then cut out small disks around the marked points to get an element of Γ_Σ is mapped to our element of $\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$.

However, because this construction uses the capping of boundary compoents, we can inductively apply the second Birman short exact sequence, here proposition 6.11. This allows us to show that the kernel corresponds to Dehn twists about each special boundary, because these are fixed pointwise. This is clear for completely incoming or outgoing boundary components. But also if a boundary component only contains an incoming or outgoing part, it must be fixed because up to isotopy we can pick representatives self-homeomorphism for elements of Γ_Σ which fix the rest of that boundary pointwise as well.

We don't get Dehn twists around non-special boundaries, since we do not require these to be fixed pointwise. Thus any Dehn twist parallel to a non-special boundary can be isotoped away by rotating the boundary.

In the end, we obtain a second short exact sequence:

$$0 \to (\mathbb{Z})^d \to \Gamma_\Sigma \to \Gamma^{\mathcal{B}}_{\tilde{\Sigma}} \to 0$$

Now we simply apply B to this short exact sequence to get a fibration:

$$(S^1)^d \to B\Gamma_\Sigma \to B\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$$

$|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is a circle bunde over $|\mathsf{Fat}^{f}_{\mathcal{B}}(\Sigma)|$: We will prove that $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is a circle bunde over $|\mathsf{Fat}^{f}_{\mathcal{B}}(\Sigma)|$ using Quillen's theorem B, here theorem B.40. Let $F : \mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma) \to \mathsf{Fat}^{f}_{\mathcal{B}}(\Sigma)$ be the functor which forgets the labelling by $\mathcal{B}$ on any boundary components containing a non-free part and forgets the starting vertices as well.

Then we consider the category F/Γ for any object Γ of $\mathsf{Fat}^{f}_{\mathcal{B}}(\Sigma)$. We claim that $|F/\Gamma|$ is homotopy equivalent to $(S^1)^d$. First of all, note that we have a retraction onto the subcategory $(F/\Gamma)_p$ of pairs Γ' and $f' : F(\Gamma') \to \Gamma$ where each boundary component of Γ' that is not purely free has labelling from $\mathcal{B}$ only on single vertices.

To prove this, let $P : F/\Gamma \to F/\Gamma$ be the functor given on objects (Γ', f') as follows. If Γ' contains a string of edges and vertices with a single label from $\mathcal{B}$ which is not a cycle, that boundary component must contain an incoming or outgoing part as well. We unlabel each edge and keeps only the first vertex in counterclockwise orientation from the first starting vertices. The morphism $f' : F(\Gamma') \to \Gamma$ remain the same, except that it now collapses what originally was an unlabelled edge instead of a labelled one. On morphisms P is given by the same morphism of underlying fat graphs, except it now sometimes collapse an unlabelled edge instead a labelled one. To prove that there is that F/Γ retracts onto the subcategory $(F/\Gamma)_p$, we define P_i for $0 \leq i \leq d$ to be the functor where the construction of P has been applied to all boundary components with label $j \leq i$. Here P_0 is simply the identity functor and $P_n = P$. We will give a converging functor system functor P_i to P_{i+1}, as in definition B.32.

Fix $0 \leq i \leq d$. Let T be the functor which, if the i'th boundary component is not purely free, for each string of $\mathcal{B}$-labelled edges doubles the first one in counterclockwise direction and only puts the label on the second of these two copies. If the doubled edge had a label on the other side, this label is extended to both copies. The map to Γ is given by collapsing the first of the two copies and then applying f'. On morphisms T collapses both copies if the original was collapsed by a morphism. Let Q be the functor which, for the i'th boundary component, for each string of labelled edges delabels the first one in counterclockwise direction. The map to Γ is simply f' and on morphisms it is the induced map. There is a natural transformation α from T to the identity given by collapsing the new unlabelled edge and a natural transformation β from T to Q by collapsing the new labelled edge.

We claim that there is a converging system functor with $F_{2n+1} = Q^n$, $F_{2n} = TQ^{n-1}$, $\eta_{2n+1} = \beta \circ Q^n$ and $\eta_{2n} = \alpha \circ Q^{n-1}$. This stabilizes after a number of steps equal to twice the maximum of the lengths of the strings of labelled edges in the i'th boundary component. On morphisms it behaves well becauses collapses can only decrease this number. By proposition B.34, there is a sequence of d homotopies between identity functor on F/Γ and P given by the converging functor systems:

$$|id| = |P_0| \xrightarrow{\sim} |P_1| \xrightarrow{\sim} |P_2| \xrightarrow{\sim} \dots \xrightarrow{\sim} |P_n| = |P|$$

Since P is the identity on $(F/\Gamma)_p$ and the composition of the above homotopies gives a homotopy from $|id|$ to $|P|$, we conclude that $|(F/\Gamma)_p|$ is a retract of $|F/\Gamma|$ and hence both are homotopy equivalent.

So to figure out the homotopy type of the fiber, it suffices to look at $(F/\Gamma)_p$. We will make it even smaller. Let $(F/\Gamma)_q$ be the full subcategory of $(F/\Gamma)_p$ of fat graphs with the correct marked points and labels such that if $f' : F(\Gamma') \to \Gamma$ collapses a edge, each

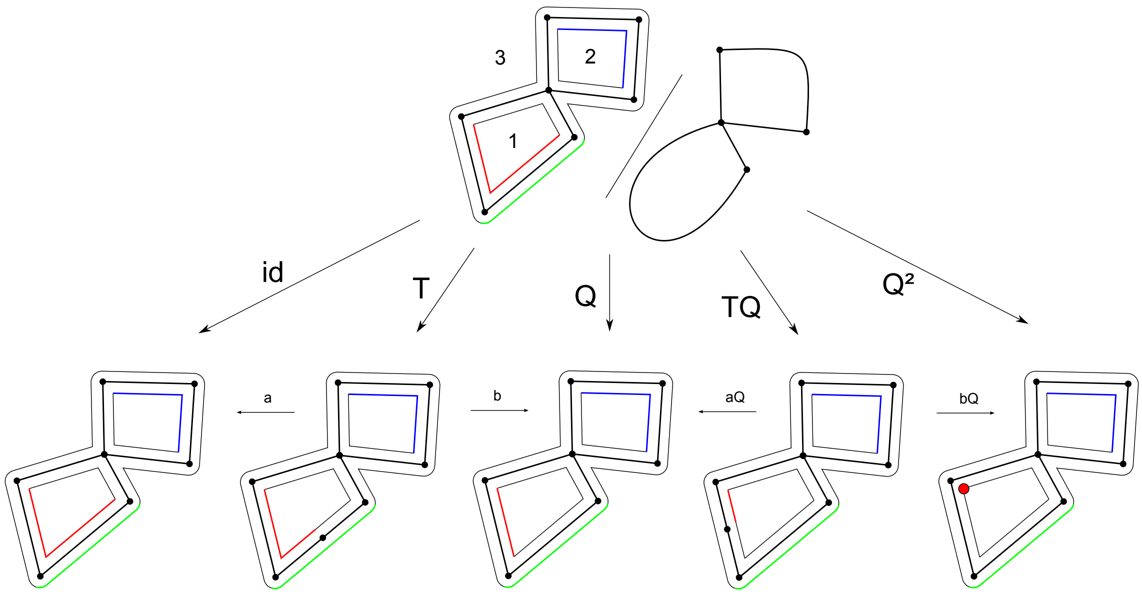


FIGURE 3.64. The converging functor system from id to P_1 at a single object.

of the endpoints is either a starting vertex or a $\mathcal{B}$-labelled point. There is a retraction of $(F/\Gamma)_p$ to $(F/\Gamma)_q$ given by a functor R. This functor sends an object $(\Gamma', f' : F(\Gamma') \to \Gamma)$ to the fat graph where all edges are collapsed which are collapsed by f', except those having as endpoints a starting vertex or a $\mathcal{B}$-labelled points. This collapse is a morphism in $(F/\Gamma)_p$ and hence we get a natural transformation from the identity functor to R.

Finally, we claim that $|(F/\Gamma)_q|$ is homotopy equivalent to a product of circles. First we let $\partial_i\Gamma$ denote the relevant special boundary components of Γ and N_i the number of marked or labelled points in $\partial_i\Gamma$. Fix a bijection between the marked points and labelled points of $\partial_i\Gamma$ and $\{1, \dots, N_i\}$ which preserves the cyclical ordering. Let $(F/\Gamma)_\mathbb{R}$ be the topological category with object space consisting of the order preserving maps ϕ_i from $A_i = \{1, \dots, N_i\}$ with standard transitive permutation to the universal cover of $|\partial_i\Gamma|$, i.e. $i \mapsto i+1$, where we assume the positive orientation on $|\partial_i\Gamma|$. The objects are parametrised as a subset of the mapping space $\prod_i (\mathbb{R})^{N_i}$. The morphisms are the maps $\pi : \mathrm{im}(\phi_i) \to \mathrm{im}(\phi_i')$ such that $\pi \circ \phi_i = \phi_i'$ and if $\phi_i(j)$ lands on a lift of a vertex of Γ, $\phi_i'(j)$ must land on one as well. The morphisms are topologized as subsets of the mapping space $\prod_i (\mathbb{R})^{N_i} \to \prod_i (\mathbb{R})^{N_i}$. Note that $(F/\Gamma)_\mathbb{R}$ admits a free continuous $\mathbb{Z}$-action by the deck transformations. Hence $|(F/\Gamma)_\mathbb{R}/\mathbb{Z}|$ is a $B\mathbb{Z}$. Note that $(F/\Gamma)_\mathbb{R}/\mathbb{Z}$ is isomorphic as a topological category to $(F/\Gamma)_{S^1}$. This category is defined in the same way as $(F/\Gamma)_\mathbb{R}$ but with $\mathbb{R}$ replaced by $S^1 \cong |\partial_i\Gamma|$.

We claim that there is a coherent homotopy of $(F/\Gamma)_{S^1}$ onto the image of $(F/\Gamma)_q$ under the functor which sends a object of $(F/\Gamma)_q$ to the map $\phi_i(j)$ given sending a j to the vertex of i if none of the edges leading into this vertex is collapsed and if both edges leading into vertex are collapsed, the image of j evenly distributing the marked or labelled points of the interior edges of the a preimage of an edge in Γ over the edge in Γ. The coherent homotopy is given by evening out the distance between different points on an edge.

Finally, we check that any map $\phi : F/\Gamma \to F/\Gamma'$ induces a homotopy equivalence. But this is clear because introducing a vertex for any collapsed an edge gives a homotopy equivalence between $(F/\Gamma')_{S^1}$ and $(F/\Gamma)_{S^1}$.

These circle bundles are the same up to homotopy: Thus, we have constructed two fibrations $(S^1)^d \to |\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)| \to |\mathsf{Fat}^{f}_{\mathcal{B}}(\Sigma)|$ and $(S^1)^d \to B\Gamma_\Sigma \to B\Gamma^{\mathcal{B}}_{\tilde{\Sigma}}$. Furthermore, we know that $|\mathsf{Fat}^f_{\mathcal{B}}(\Sigma)| \simeq B\Gamma^{\mathcal{B}}_{\Sigma}$. The final question is whether these two circle bundles are the same up to homotopy. If this is true, it follows that $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)| \simeq B\Gamma_\Sigma$ and we are done.

The two circle bundles are the same if the action of π_1 on the fiber is the same. By definition of classifying spaces, the π_1 of both sides is $\Gamma^{\mathcal{B}}_{\Sigma}$. In the case $(S^1)^d \to |\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)| \to |\mathsf{Fat}^f_{\mathcal{B}}(\Sigma)|$ these act by permuting the fibers of $(S^1)^d$ according to $\Sigma(b^{-1}(\beta))$ and the same holds for the other case. Hence we are done.

Next, we consider the two special cases: those graphs corresponding to cobordisms homeomorphic to a disk or an annulus.

The disk: Let Γ be a $\mathcal{B}$-labelled cobordism graph whose corresponding cobordism $\Sigma = |\Gamma|_{oc}$ is homeomorphic to a disk. Then the underlying fat graph of Γ must be a tree by lemma 3.44. This means we will be looking at the component Fat_T.

By the positive boundary condition, Σ must have a special boundary, and then $\Gamma_\Sigma = 0$ since the entire boundary must be fixed pointwise.

Let's prove that $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is contractible. To prove this, note that the argument which says that $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is a circle bundle over $|\mathsf{Fat}_T|$ still holds. But which circle bundle is it? If we label one vertex in each star-like graph, then the category $\mathsf{Fat}^{marked}_{\star}$ becomes contractible upon geometric realisation. We conclude that $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is contractible and hence $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is the canonical circle bundle $EU(1)$ over $BU(1)$.

The annulus: Let Γ be a $\mathcal{B}$-labelled cobordism graph whose corresponding cobordism $\Sigma = |\Gamma|_{oc}$ is homeomorphic to an annulus. Using lemma 3.44 we identify that this graph must be a subdivided circle with trees attached. Therefore we will be looking at covers of Fat_S.

There are two options. By the positive boundary condition, Σ must have either one or two special boundary components. In the latter case the two special boundary components can't be switched, even if their labels would allow it, because we have numbered the boundary components. Therefore, in the first case $\Gamma_\Sigma = 0$, in the second case $\Gamma_\Sigma = \mathbb{Z}$, see lemma 6.13.

Let's start with the first case, then we must show that $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is contractible. But this will be a circle bundle over $|\mathsf{Fat}_S|$ and by the same reasoning as before, it is the canonical bundle $EU(1) \to BU(1)$. This implies that $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is contractible.

For the second case, we must show that $|\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)|$ is a $B\mathbb{Z}$. By the same reasoning as before, it is a $S^1 \times S^1$-bundle over $|\mathsf{Fat}_S|$. In fact, it is $(S^1 \times S^1) \times_{U(1)} EU(1) \to BU(1)$. Using the long exact sequence of homotopy groups, we obtain that $(S^1 \times S^1) \times_{U(1)} EU(1)$ is a $B\mathbb{Z}$ and we are done.

□

Remark 3.65. From the last theorem comes our motivation to change the positive boundary condition. Note that if we have a cobordism Σ homeomorphic to the disk which has no special boundary, then $\mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma)$ is isomorphic to Fat_T by forgetting the labelling and we know that $|\mathsf{Fat}_T|$ is a $BU(1)$ from theorem 3.59. This is not contractible, which it should be because the mapping class group of the disk of self-homeomorphisms which do not fix the boundary is the trivial group.

2.3. Nice $\mathcal{B}$-labelled cobordism graphs. In chapter 5, we will construct the string operations, but it will turn out that we are only able to construct these operations for certain fat graphs. The fat graph must be nice, i.e. the edges of the incoming boundary must be disjoint and edges or vertices can't be labelled from both sides at the same time. We will therefore show that imposing these conditions does not change the homotopy type of the category.

The next result was rather difficult to prove. The problem is that it is hard to find a functorial way of making a $\mathcal{B}$-labelled cobordism graph nice. However, in several steps it is possible.

Remark 3.66. There exists an alternative proof, which works by replacing the starting point of the proofs of the previous section, the category of fat graphs with all valences ≥ 3, with a category of admissible graphs. To do this, one compares it directly to the classifying space of the mapping class groups. See the proof of [**God07**, theorem 4] for a proof using that technique.

Now we prove a proposition relating nice graphs and admissible graphs. In combination with the next theorem, this gives direct proof of a theorem of Godin [**God07**, theorem 4]: see corollary 3.70. It will also be used in an induction step in the proof of theorem 3.69.

Proposition 3.67. *Let* $(\mathsf{Fat}^{oc,a}_{\mathcal{B}})^c$ *denote the full subcategory of connected admissible* $\mathcal{B}$*-labelled cobordism graphs satisfying the positive boundary condition. Then* $|(\mathsf{Fat}^{oc,a}_{\mathcal{B}})^c| \simeq |(\mathsf{Fat}^{oc,n}_{\mathcal{B}})^c|$.

Proof. It suffices to show that the inclusion $I : (\mathsf{Fat}^{oc,a}_{\mathcal{B}})^c \hookrightarrow (\mathsf{Fat}^{oc,n}_{\mathcal{B}})^c$ induces a homotopy equivalence upon geometric realistion. The difference between nice graphs and admissible graphs is that in the former case we still allow that the collapse map $c : \partial\Gamma \to \Gamma$ is not injective on the vertices of the incoming boundary, while in the latter case it must injective. Therefore, the proof amounts to showing that we can functorially resolve these problems that nice fat graphs can have at the vertices of incoming boundary components.

To do this, it suffices to give a functor $R : (\mathsf{Fat}^{oc,n}_{\mathcal{B}})^c \to (\mathsf{Fat}^{oc,a}_{\mathcal{B}})^c$ such that we have $R \circ I = id$ and there exists a natural transformation $\epsilon : I \circ R \to id$. Let's define the functor R. We start by describing its value on a connected nice $\mathcal{B}$-labelled cobordism graph Γ. To do this, we let A denote the set of vertices of $\partial\Gamma$ which lie in the interior of $\partial_{in}\Gamma$ and B denote the set of vertices which lie at the boundary of $\partial_{in}\Gamma$.

We first describe the construction in words, then give it in formulas. However, the best is probably to look at figure 3.68. To an element $a \in A$ we are given the following associated data: (i) the vertex $c(a) \in \Gamma$ to which it collapses, (ii) the two half-edges h_1, h_2 attached to a in $\partial_{in}\Gamma$, (iii) their images $c(h_1), c(h_2)$ under the collapse map $c : \partial\Gamma \to \Gamma$ in Γ. We add a new vertex a to Γ. Then we disattach $c(h_1), c(h_2)$ from $c(a)$ and attach them instead to a. Finally we put an edge between a and $c(a)$ at the point in the cyclic order at $c(a)$ where $c(h_1), c(h_2)$ were originally. In other words, we add in an additional edge which can be collapsed to give back the original fat graph again. The labels and marked points extend in the obvious manner. The functor R does this for all $a \in A$ at the same time.

For an element $b \in B$ we similarly have the following data: (i) the vertex $c(b)$ in Γ to which it collapses, (ii) the half-edge h_1 with incoming label attached to b and the half-edge h_2 with other label, (iii) their images $c(h_1), c(h_2)$ in Γ. Again we add a new vertex b to Γ. $c(h_1)$ is disattached and attached to b instead. Finally, we put an edge between $c(b)$ and b which comes at the place of $c(h_1)$ in the cyclic ordering. Again labels and marked points are clear and R does this for all $b \in B$ at the same time.

Because Γ was nice to start with, this is well-defined, as no edge in Γ can be in the image of two incoming boundary edges under c.

We now give the precise definition. Let (V, H, s, i, σ) be the fat graph underlying a nice $\mathcal{B}$-labelled cobordism graph Γ. Let A, B be as above. Then the underlying fat graph of $R(\Gamma)$ is given by $V = V \sqcup A \sqcup B$, $H = H \sqcup A \times \{0,1\} \sqcup B \times \{0,1\}$. The new s is given by the original one on those half-edges which aren't in the image of the incoming boundary under the collapse map. If h is the image of $\tilde{h}$ in $\partial_{in}\Gamma$ and $\tilde{h}$ is attached to a or b, then $s(h) = a$. Finally we set $s(a_0) = s(a)$, $s(a_1) = a$, $s(b_0) = s(b)$ and $s(b_1) = b$. The new i is the original one on H, $i(a_j) = a_{1-j}$, $i(b_j) = b_{j-1}$ for $j = 0, 1$. The new σ is given as follows. In cycle corresponding to $v \in V$, we replace a pair of edges now attached to a by the half-edge a_0 and an edge now attached to b by the half-edge b_0. The cycle at a is given by first a_1 and then the two edges in the original order. There is only one choice for a cycle at b, since it has length 2. The edges in $\partial\Gamma$ corresponding to the new edges are considered as parts of $\partial_f\Gamma$ and are given the only possible color. The marked points stay where they are.

On morphisms, R is given by collapsing the edges that were originally collapsed. Clearly, this makes R respect composition and identity maps. The only thing we should check it that for $f : \Gamma_1 \to \Gamma_2$, the codomain of $R(f)$ indeed is $R(\Gamma_2)$. Because every morphism is a composition of collapses of single edges, it suffices to show this for a collapse of a single edge. A few sketches should convince one that this for a single edge this indeed holds.

Finally, we describe ϵ: it just collapses the edges that were newly added by R. Clearly this restores the $\mathcal{B}$-labelled cobordism graphs to their original state and it is compatible with morphisms. □

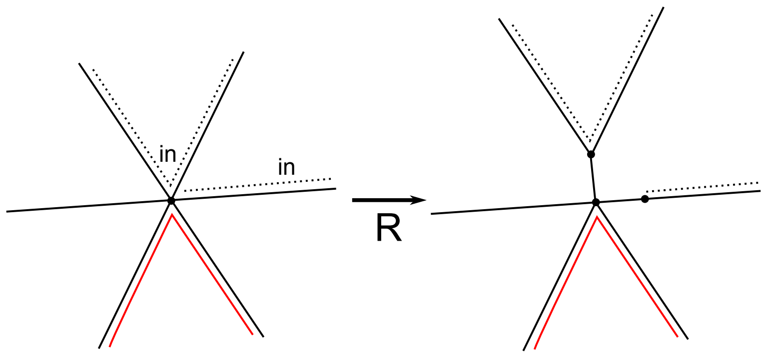


FIGURE 3.68. The value R around a vertex. The incoming boundary is dotted and the labelled boundary is red.

Finally, we prove the result that the geometric realisation of the category of nice graphs has the same homotopy type as the geometric realisation of the full category of $\mathcal{B}$-labelled cobordism graphs, discussed in theorem 3.63.

Theorem 3.69. *Let* $(\mathsf{Fat}_{\mathcal{B}}^{oc,n})^c$ *denote the full subcategory of connected nice* $\mathcal{B}$*-labelled cobordism graphs satisfying the positive boundary condition. Then* $|\mathsf{Fat}_{\mathcal{B}}^{oc,n}| \simeq \coprod_{[\Sigma]} B\Gamma_\Sigma$ *where the disjoint union runs over all isomorphism classes of connected* $\mathcal{B}$*-cobordisms with ordered incoming and outgoing boundary components with positive boundary.*

PROOF. The proof will proceed as follows. We start by saying that we call a boundary component special if it contains no outgoing part. We will first prove that due to the positive boundary condition we can seperate all special boundary components.

The semi-nice graphs $\mathsf{Fat}_{\mathcal{B},v}^{oc,sn}(\Sigma)$: . In the proof of theorem 3.63 we saw that without loss of generality, we can assume that if a boundary component is not fully free then all labels from $\mathcal{B}$ consist are localized to a single vertex. Denote the category of such $\mathcal{B}$-labelled cobordism graphs with corresponding cobordism isomorphic to Σ by $|\mathsf{Fat}_{\mathcal{B},v}^{oc}(\Sigma)|$. The previous remarks show that

$$|\mathsf{Fat}_{\mathcal{B},v}^{oc}(\Sigma)| \simeq |\mathsf{Fat}_{\mathcal{B}}^{oc}(\Sigma)|$$

Furthermore, we claim that we can sure make sure that all of these labels and the starting vertices of the incoming and outgoing boundary components are on the end of a leaf. This of course may bring us further from niceness, but it will prove useful later on. We denote by $\mathsf{Fat}_{\mathcal{B},v}^{oc,sn}(\Sigma)$ the category of graphs having these properties and having correspond $\mathcal{B}$-labelled cobordism isomorphic to Σ. We call these the semi-nice graphs. We will now show that

$$|\mathsf{Fat}_{\mathcal{B},v}^{oc,sn}(\Sigma)| \simeq |\mathsf{Fat}_{\mathcal{B},v}^{oc}(\Sigma)|$$

To do this, we note that $\mathsf{Fat}_{\mathcal{B},v}^{oc,sn}(\Sigma)$ can be considered as a subcategory of $\mathsf{Fat}_{\mathcal{B},v}^{oc}(\Sigma)$ and hence it suffices to give a functor $P : \mathsf{Fat}_{\mathcal{B},v}^{oc}(\Sigma) \to \mathsf{Fat}_{\mathcal{B},v}^{oc}(\Sigma)$ which has image in $\mathsf{Fat}_{\mathcal{B},v}^{oc,sn}(\Sigma)$ and is the identity on $\mathsf{Fat}_{\mathcal{B},v}^{oc,sn}(\Sigma)$ together with a natural transformation from P to the identity functor.

P is given simply by adding a new edge for each starting vertex or vertex with label from $\mathcal{B}$. This edge will be a leaf and the starting vertex or vertex with label from $\mathcal{B}$ is put on at the end point of this leaf. This is compatible with morphisms. The natural transformation from P to the identity is given by collapsing the newly added edges again.

Seperating the special boundary: We consider a new type of "cobordism": instead of having incoming and outgoing boundary, it has special and outgoing boundary. All our earlier definitions for cobordisms and fat graphs representing cobordisms carry over, as well as one replaces each mention of incoming boudary by special boundary. For clarity, the corresponding $\mathcal{B}$-labelled cobordism graphs will be called special $\mathcal{B}$-labelled cobordism graphs.

We create a new $\mathcal{B}$-labelled cobordism $\tilde{\Sigma}$ with the same underlying manifold with boundary, but other labels. Suppose that there is no boundary component which contains no incoming boundary, then by the positive boundary condition there exists a first incoming boundary part which is not completely incoming. We relabel each of the incoming boundary parts in this boundary component outgoing, but remember which of outgoing boundary component these are. The other boundary components we modify as follows: if an other boundary component contains an incoming part, all incoming and $\mathcal{B}$-labelled parts become special boundary with the parametrisation induced by that of the first incoming part on it.

The proof of theorem 3.63 tells us that $\Gamma_{\tilde{\Sigma}} \cong \Gamma_{\Sigma}$ and that this homotopy equivalence is induced on the level of cobordism graphs by a functor $U : \mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma) \to \mathsf{Fat}^{oc}_{\mathcal{B}}(\tilde{\Sigma})$. This functor is produced in a similar way to the way we modified Σ to obtain $\tilde{\Sigma}$: if there is no boundary component which contains no incoming boundary, then on the first incoming boundary part which is not completely incoming we make all incoming boundary part outgoing. On each other boundary component which contains an incoming part we make all incoming and $\mathcal{B}$-labelled parts special and we remember only the starting vertex of the first incoming boundary part.

We will prove that $|\mathsf{Fat}^{oc,n}_{\mathcal{B}}(\tilde{\Sigma})| \simeq |\mathsf{Fat}^{oc}_{\mathcal{B}}(\tilde{\Sigma})|$ and we will do this using an induction over the number k of fully special boundary components for $k < d$, where d is the total number of boundary components. By the positive boundary condition, this is always the case and we need it to start the induction.

Let's do the case $k = 0$. In that case we start with a special $\mathcal{B}$-labelled graph without fully special boundary components: then collapsing all edges which are labelled on both sides by special boundary results in a nice special $\mathcal{B}$-labelled graph. The fact that each boundary component contains an outgoing part implies that we do not collapse boundary components.

This collapse is given by a functor Q which canonically comes with a natural transformation from the identity functor to Q. We conclude that if $\tilde{\Sigma}$ is such that $k = 0$, we have

$$|\mathsf{Fat}^{oc,n}_{\mathcal{B}}(\tilde{\Sigma})| \simeq |\mathsf{Fat}^{oc}_{\mathcal{B}}(\tilde{\Sigma})| \simeq B\Gamma_{\tilde{\Sigma}}$$

For the induction, suppose that we have k special boundary components. By the previous results, we can restrict to the homotopy equivalent subcategory of semi-nice graphs $\mathsf{Fat}^{oc,sn}_{\mathcal{B},v}(\tilde{\Sigma})$.

If we collapse the fully special boundary component which has last boundary label, which we denote by $\partial_{bad}\Sigma$, we obtain a special $\mathcal{B}$-labelled graph with $(k-1)$ fully special boundary components. Because we work with semi-nice graphs, we don't lose starting vertices or labels from $\mathcal{B}$. In fact, the resulting graph is easily seen to be again semi-nice. So, more precisely, we define functor

$$C : \mathsf{Fat}^{oc,sn}_{\mathcal{B},v}(\tilde{\Sigma}) \to \mathsf{Fat}^{oc,sn}_{\mathcal{B},v}(\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma})$$

which collapses the special boundary component $\partial_{bad}\tilde{\Sigma}$ and labels the resulting point with a special label. Similarly, there is a surface $\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma}$ is obtained by glueing a disk with a marked point with special label to the special boundary componet $\partial_{bad}\tilde{\Sigma}$ of $\tilde{\Sigma}$. This capping operation produces a fibration:

$$S^1 \to B\Gamma_{\tilde{\Sigma}} \to B\Gamma_{\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma}}$$

It therefore suffices to show that $|C|$ is fibration with fiber homotopy equivalent to S^1 and the same monodromy. The fiber consists of all ways to blow up a vertex to a circle: this indeed has the homotopy type of S^1. We conclude that we have a fibration:

$$S^1 \to |\mathsf{Fat}^{oc,sn}_{\mathcal{B},v}(\tilde{\Sigma})| \to |\mathsf{Fat}^{oc,sn}_{\mathcal{B},v}(\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma})|$$

We know that we have a homotopy equivalence of base spaces

$$|\mathsf{Fat}^{oc,sn}_{\mathcal{B},v}(\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma})| \simeq B\Gamma_{\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma}}$$

which implies that there is a homotopy equivalence of total spaces because monodromy is also the same. Using the induction hypothesis we can replace the base space of semi-nice graphs $|\mathsf{Fat}^{oc,sn}_{\mathcal{B},v}(\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma})|$ by the nice graphs $|\mathsf{Fat}^{oc,n}_{\mathcal{B},v}(\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma})|$ and then by admissiable graphs $|\mathsf{Fat}^{oc,a}_{\mathcal{B},v}(\tilde{\Sigma}/\partial_{bad}\tilde{\Sigma})|$ using proposition 3.67. When we restrict the total space of the fibration to this space, we still have a $B\Gamma_{\tilde{\Sigma}}$.

However, because we are now working with admissible graph we know that there is at least one edge attached to the vertex with special label that is not incoming. By only opening there, we get the same homotopy type for the fiber, the resulting special $\mathcal{B}$-labelled graph will be nice. This proves the induction step:

$$|\mathsf{Fat}^{oc,n}_{\mathcal{B}}(\tilde{\Sigma})| \simeq B\Gamma_{\tilde{\Sigma}}$$

We conclude that $|\mathsf{Fat}^{oc,n}_{\mathcal{B}}(\tilde{\Sigma})| \simeq B\Gamma_{\tilde{\Sigma}}$ for every $\mathcal{B}$-labelled cobordism with outgoing and special boundary.

Fixing the labels from $\mathcal{B}$: Finally, we need to show we can go from the cobordism with outgoing and special boundary $\tilde{\Sigma}$ back to the original $\mathcal{B}$-labelled cobordism Σ. First of all, we note that using techniques similar to those of theorem 3.63 one can proof that the fiber of U is contractible.

So, we just need to be careful and remember that there are two things we need to take care of: (i) the labels from $\mathcal{B}$ that we made special, (ii) possibly the incoming boundary parts that we made outgoing. To solve the first problem, we simply note that we can restrict to admissible special $\mathcal{B}$-labelled graphs $\mathsf{Fat}^{oc,a}_{\mathcal{B}}(\tilde{\Sigma})$ without changing the homotopy type using proposition 3.67.

This is useful for the following reason: if we start with a admissible special $\mathcal{B}$-labelled graph then any way to change back the special boundary to incoming and $\mathcal{B}$-labelled parts almost gives an admissible $\mathcal{B}$-labelled cobordism graph. To be precise, it is admissible except for the second problem. We conclude over any point of $\mathsf{Fat}^{oc,a}_{\mathcal{B}}(\tilde{\Sigma})$ the fiber consists of $\mathcal{B}$-labelled cobordism graphs which are admissible except for the possible incoming parts that we made outgoing. But this problem can be resolved by adding a long linear graph to that boundary component and moving everything that is incoming or $\mathcal{B}$-labelled on there. This is possible because there is a outgoing part on that boundary component. The result indeed is an admissible $\mathcal{B}$-labelled graph and thus we have proven

$$|\mathsf{Fat}^{oc,a}_{\mathcal{B}}(\Sigma)| \simeq |\mathsf{Fat}^{oc,a}_{\mathcal{B}}(\tilde{\Sigma})| \simeq B\Gamma_{\Sigma}$$

By applying proposition 3.67 again, we obtain $|\mathsf{Fat}^{oc,n}_{\mathcal{B}}(\Sigma)| \simeq B\Gamma_{\Sigma}$ and thus we have proven the theorem.

□

In the proof of the previous theorem, the following was already noted.

Corollary 3.70. *We have a homotopy equivalence $|\mathsf{Fat}^{oc,a}_{\mathcal{B}}| \simeq \coprod_{[\Sigma]} B\Gamma_{\Sigma}$ where the disjoint union runs over all isomorphism classes of connected $\mathcal{B}$-cobordisms with ordered incoming and outgoing boundary components satisfying the positive boundary condition.*

3. Modelling glueing and disjoint union using $\mathcal{B}$-labelled cobordism graphs

In the previous section we described several models for $B\Gamma_{\Sigma}$ where Σ is a $\mathcal{B}$-labelled cobordism. In section 3.2 we saw that these spaces from a prop with composition induced by composition χ of cobordisms and addition induced by disjoint union $\sqcup$. In this section we will describe up to model these maps on the level of $\mathcal{B}$-labelled cobordism graphs.

3.1. Disjoint union using $\mathcal{B}$-labelled cobordism graphs. We start with modelling the disjoint union map

$$B\Gamma(\sqcup) : B\Gamma_\Sigma \times B\Gamma_{\Sigma'} \to B\Gamma_{\Sigma \sqcup \Sigma'}$$

This is very simple. It is clear that if Γ is a $\mathcal{B}$-labelled cobordism graph such that $|\Gamma|_{oc} \cong \Sigma$ and Γ' is one such that $|\Gamma'|_{oc} \cong \Sigma'$, then their disjoint union $\Gamma \sqcup \Gamma'$ has corresponding $\mathcal{B}$-labelled cobordism isomorphic to $\Sigma \sqcup \Sigma'$. The only thing which we need to worry about is the ordering of the incoming and outgoing boundary components: those of Σ' are put after those of Σ in the ordering of the boundary components of $\Sigma \sqcup \Sigma'$.

Hence we if we do the same with the ordering of the incoming and outgoing boundary of $\Gamma \sqcup \Gamma'$, we get a functor

$$\sqcup : \mathsf{Fat}_{\mathcal{B}}^{oc} \times \mathsf{Fat}_{\mathcal{B}}^{oc} \to \mathsf{Fat}_{\mathcal{B}}^{oc}$$

such that $|\sqcup| = B\Gamma(\sqcup)$. Note that $\sqcup$ maps the nice graphs to nice graphs and similarly for the admissible graphs.

3.2. Glueing using $\mathcal{B}$-labelled cobordism graphs. Now we get to the more formidable task of modelling the glueing of $\mathcal{B}$-labelled cobordisms on the level of graphs. That is, we want to give a functor between categories of $\mathcal{B}$-labelled graphs which induces the map

$$B\Gamma(\chi) : B\Gamma_\Sigma \times B\Gamma_{\Sigma'} \to B\Gamma_{\Sigma' \circ \Sigma}$$

where Σ and Σ' are composable $\mathcal{B}$-labelled cobordisms.

It turns out to be very useful to restrict to admissible graphs, because in these graphs we have seperated the incoming parts of the boundary. Although I have though about different constructions, Godin's seems the most useful. The reason for this is that it makes checking that the operations are compatible with compositions relatively easy in comparison with other constructions.

The idea of the construction is as follows: we restrict to a homotopy equivalent category $\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$ of $\mathsf{Fat}_{\mathcal{B}}^{oc,a}(\Sigma) \times \mathsf{Fat}_{\mathcal{B}}^{oc,a}(\Sigma')$, which we call the category of glueable $\mathcal{B}$-labelled graphs, and produce a glueing map $\chi : \mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma) \to \mathsf{Fat}_{\mathcal{B}}^{oc}(\Sigma' \circ \Sigma)$.

Definition 3.71. The category $\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$ has as objects the objects of $\mathsf{Fat}_{\mathcal{B}}^{oc,a}(\Sigma) \times \mathsf{Fat}_{\mathcal{B}}^{oc,a}(\Sigma')$, i.e. pairs of admissible $\mathcal{B}$-labelled cobordism graphs (Γ, Γ'), together with an isomorphism of graphs $\varphi : \partial_{out}\Gamma \to \partial_{in}\Gamma'$ which preserves the boundary ordering, sends starting vertices to starting vertices and reverses the induced orientation.

The morphisms of $\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$ are those pairs of morphisms compatible with these isomorphisms of the boundary: if an edge belonging to the i'th outgoing boundary of Γ is collapsed, then the corresponding edge of the i'th incoming boundary of Γ' must be collapsed as well.

Proposition 3.72. *We have a homotopy equivalence* $|\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)| \simeq |\mathsf{Fat}_{\mathcal{B}}^{oc,a}(\Sigma)| \times |\mathsf{Fat}_{\mathcal{B}}^{oc,a}(\Sigma')|$.

Proof. Suppose that Σ has d outgoing boundary parts and hence that Σ' has d incoming boundary parts.

We want to apply Quillen's theorem A. We can not apply it directly to I, because it is not surjective. To do this, we introduce an intermediate category $\mathsf{Fat}_{\mathrm{coll}}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$. This category has as objects the objects of $\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$ with for each $1 \leq i \leq d$ non-empty subgraphs C_i and C_i' of the i'th components of $\partial_{out}\Gamma$ and $\partial_{in}\Gamma'$ respectively. These are allowed to have multiple components, some of which are a single vertex. The idea is that these subgraphs should be considered as collapsed already, so we demand that if C_i and C_i' are collapsed the result is still a $\mathcal{B}$-labelled cobordism graph. The morphisms are those of $\mathsf{Fat}_{\mathrm{coll}}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$ additionally respecting these subsets: the morphisms should map the C_i and C_i' of the codomain surjectively those of the domain. Note if an edge between two components of C_i or C_i' is collapsed, these components are merged and hence the morphisms need not respect the homotopy type of the C_i and C_i'.

We define functors P, Q fitting in the following diagram:

$$\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma) \xleftarrow{P} \mathsf{Fat}_{\mathrm{coll}}^{\mathrm{glue}}(\Sigma' \circ \Sigma) \xrightarrow{Q} \mathsf{Fat}_{\mathcal{B}}^{oc,a}(\Sigma) \times \mathsf{Fat}_{\mathcal{B}}^{oc,a}(\Sigma')$$

Let $P : \mathsf{Fat}_{\mathrm{coll}}^{\mathrm{glue}}(\Sigma' \circ \Sigma) \to \mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$ be the functor which for each $1 \leq i \leq d$ collapses the edges of $C_i \cap \varphi^{-1}(C_i')$ in Γ and the edges of $C_i' \cap \varphi(C_i)$ in Γ'. We claim that it induces a homotopy equivalence upon geometric realisation. To see this, we want apply Quillen's theorem A and hence we look at the comma category $P/(\Gamma, \Gamma', \varphi)$. We will prove that this fiber has contractible geometric realisation and in particular, that it retracts onto the single object where each C_i or C_i' consists of the starting vertex of that boundary part. To prove this, one can use the same technique we used in theorem 3.63 to prove that one can assume that in those boundary components that are not purely free, the labels from $\mathcal{B}$ are localized at single vertices. This technique should be applied to the C_i and C_i'.

Now let $Q : \mathsf{Fat}^{\text{glue}}_{\text{coll}}(\Sigma' \circ \Sigma) \to \mathsf{Fat}^{oc,a}_{\mathcal{B}}(\Sigma) \times \mathsf{Fat}^{oc,a}_{\mathcal{B}}(\Sigma')$ be the functor that collapses the C_i in Γ and the C_i' in Γ'. This is easily seen to be surjective on objects and we claim that the comma category $Q/(\Gamma, \Gamma')$ has contractible geometric realisation. The fiber consists of all ways to add new edges to the outgoing boundary of Γ and the incoming boundary of Γ' such that we get to two $\mathcal{B}$-labelled graphs such that the outgoing boundary of the first is isomorphic to the incoming boundary of the second. We claim that the fiber retracts onto the object where all edges are added after the starting vertices. Again, the technique of theorem 3.63 provides a way to produce such a retraction. $\square$

Remark 3.73. The technique of theorem 3.63 uses converging functor systems to move around the collapsed parts C_i and C_i', which in the case will be particularly involved. Alternatively, one can use metric graphs to do this, which should give an alternative proof of this proposition, similar to [**God07**, lemma 3].

For later use we describe a direct functor between the categories $\mathsf{Fat}^{\text{glue}}(\Sigma' \circ \Sigma)$ and $\mathsf{Fat}^{oc,a}_{\mathcal{B}}(\Sigma) \times \mathsf{Fat}^{oc,a}_{\mathcal{B}}(\Sigma')$ which induces the homotopy equivalence of the previous proposition. We will use this when we prove that the string operations are compatible with composition.

Lemma 3.74. *The functor $I : \mathsf{Fat}^{\text{glue}}(\Sigma' \circ \Sigma) \to \mathsf{Fat}^{oc,a}_{\mathcal{B}}(\Sigma) \times \mathsf{Fat}^{oc,a}_{\mathcal{B}}(\Sigma')$ which forgets about the isomorphism between the outgoing boundary of the first and the incoming boundary of the second graph induces a homotopy equivalence.*

PROOF. It is the composition $Q \circ J$, where Q is as in the previous proof and J includes a pair with isomorphism into $\mathsf{Fat}^{\text{glue}}_{\text{coll}}(\Sigma' \circ \Sigma)$ by selecting the starting vertices as subgraphs to be collapsed. Then $P \circ J = id$ and P induces a homotopy equivalence upon geometric realisation, so J induces a homotopy equivalence as well. Q was shown to induce a homotopy equivalence upon geometric realisation in the previous proof. We conclude that $|I| = |Q| \circ |J|$ is a homotopy equivalence. $\square$

Now that we have established that the category $\mathsf{Fat}^{\text{glue}}(\Sigma' \circ \Sigma)$ is a model for the category of pairs of graphs corresponding to Σ and Σ', we will describe the glueing functor χ. We start by defining it on the objects. The idea is that $\Gamma' \circ \Gamma$ is obtained by glueing Γ into Γ' by identifying the incoming boundary of Γ' with the outgoing boundary of Γ.

Definition 3.75. Let Γ and Γ' be $\mathcal{B}$-labelled cobordism graphs, together with an isomorphism $\varphi : \partial_{out}\Gamma \to \partial_{in}\Gamma'$ of graphs which respect the ordering, the starting vertex and reverses orientation. Then we will now describe a $\mathcal{B}$-labelled fat graph $\Gamma'' = \Gamma' \circ \Gamma$.

We start by describing its underlying fat graph. Let (V, H, s, i, σ) and $(V', H', i', s', \sigma')$ be the underlying fat graphs of Γ and Γ' To describe the edges and half-edges, we introduce an equivalence relation on $V \sqcup V'$ and $H \sqcup H'$. We say that $v \sim v'$ if $\varphi(v) = v'$ and $h \sim h'$ if $\varphi(h) = h'$. Then we let the set of vertices of $\Gamma' \circ \Gamma$ be $V'' = (V \sqcup V')/\sim$ and the set of half-edges be $H'' = (H \sqcup H')/\sim$. Since φ respects the source map and edge pairing, the source map s'' and edge pairing i'' of $\Gamma' \circ \Gamma$ are simply the induced maps under the equivalence relation.

Finally, we need to describe the cyclic order σ'' at the vertices. The cycle at a vertex $v'' \in V''$ which is not of the type $v' = \phi(v)$ is simply given by the original cycle of σ or σ', depending on whether $v'' \in V$ or $v'' \in V'$. If v'' is of the type $v' = \phi(v)$ then there are two half-edges $h_1, h_2 \in H$ attached to v such that $\partial_{out}\Gamma$ maps to those half-edges under the collapse map s of proposition 3.48. Let σ_v by the cyclic obtained from $\sigma|_{s^{-1}(v)}$ by removing these. We get the cycle of σ'' at $v' = \phi(v)$ by inserting this σ_v into $\sigma'|_{s^{-1}(\phi(v))}$ between $\phi(h_1)$ and $\phi(h_2)$.

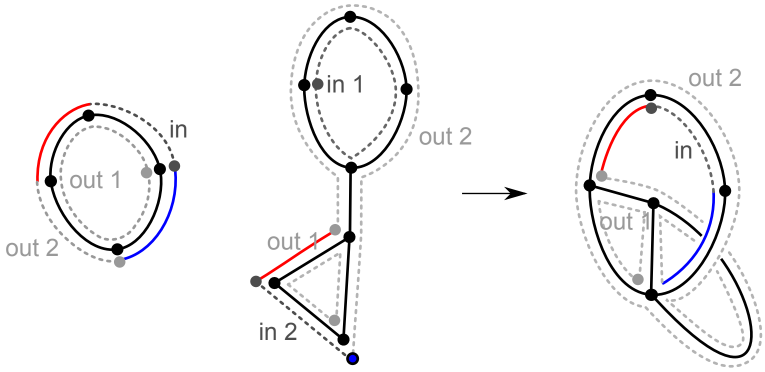


FIGURE 3.76. Two composable $\mathcal{B}$-labelled cobordism graphs and their composition. The composition should be thought of as obtained by glueing out_1 to in_1 and out_2 to in_2. We don't draw φ, because it is unique.

Next we describe the structure we need to add to make this a $\mathcal{B}$-labelled cobordism graph. The incoming boundary with its ordering and starting vertices is that of Γ, the outgoing boundary that of Γ'. The labels from $\mathcal{B}$ are the obvious ones coming from Γ or Γ'.

See figure 3.76 for an example of a composition of two $\mathcal{B}$-labelled cobordism graphs.

If $(f, g) : (\Gamma_1, \Gamma_1', \varphi_1) \to (\Gamma_2, \Gamma_2', \varphi_2)$ is a morphism of $\mathsf{Fat}^{\text{glue}}(\Sigma' \circ \Sigma)$, then it is easy to see that it induces a morphism $g \circ f : \Gamma_1' \circ \Gamma_1 \to \Gamma_2' \circ \Gamma_2$, because it is compatible with the isomorphisms φ and φ'.

Definition 3.77. The glueing functor $\chi : \mathsf{Fat}^{\text{glue}}(\Sigma' \circ \Sigma) \to \mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma' \circ \Sigma)$ is given on objects by sending $(\Gamma, \Gamma', \varphi)$ to $\Gamma' \circ \Gamma$ and on morphisms by sending $(f, g) : (\Gamma_1, \Gamma_1', \varphi_1) \to (\Gamma_2, \Gamma_2', \varphi_2)$ to $g \circ f : \Gamma_1' \circ \Gamma_1 \to \Gamma_2' \circ \Gamma_2$.

From the construction, it is easy to see that χ indeed sends $\mathcal{B}$-labelled cobordism graph representing a composable pair of morphisms Σ and Σ' to a $\mathcal{B}$-labelled cobordism graph representing $\Sigma' \circ \Sigma$. It is slightly harder to see but still true that it does the same for elements of the mapping class groups. We can therefore conclude that we have found a functor

$$\chi : \mathsf{Fat}^{\text{glue}}(\Sigma' \circ \Sigma) \to \mathsf{Fat}^{oc}_{\mathcal{B}}(\Sigma' \circ \Sigma)$$

which satisfies the property that $|\chi|$ is equal to $B\Gamma(\chi)$ up to homotopy.

CHAPTER 4

Umkehr maps

Generalized homology and cohomology theories are by definition functorial. This means that to each continuous map $f : X \to Y$ a homology theory E_* assigns a map $f_* : E_*(X) \to E_*(Y)$, in a way that respects composition and identitiy: homology is covariant. Similarly, a cohomology theory E^* assigns to each continuous map $f : X \to Y$ a map $f^* : E^*(Y) \to E^*(X)$: it is contravariant.

These induced maps are an important part of algebraic topology and depend only on the homotopy class of f. They are often easy to construct. For example, if a singular homology class is represented by a singular simplex $a : \Delta^n \to X$, then $f_*(a)$ is represented by the composition $f \circ a : \Delta^n \to Y$.

However, in certain circumstances one can construct 'wrong-way' maps. These are confusingly also known as pushfowards, but we prefer the term umkehr map. For a homology theory E_*, an umkehr map induced by a map $f : X \to Y$ is a map $f^! : E_*(Y) \to E_{*-n}(X)$ for some $n \in \mathbb{Z}$. For a cohomology theory E^*, it is a map $f_! : E^*(X) \to E^{*-n}(Y)$. It is easy to see why these maps are called umkehr maps or wrong-way maps: the direction of the induced maps is exactly the opposite of the direction that the functoriality condition of the (co)homology theory prescribes. We make the direction clear in our convention for the sub- and superscripts. A subscript, i.e. f_* and $f_!$, indicate covariance. A superscript, i.e. f^* and $f^!$, indicates contravariance.

In this chapter we provide the necessary background to construct umkehr maps in homology and cohomology. These constructions are by nature quite involved, using differential geometry and stable homotopy theory, which also makes the derivation of the properties of umkehr maps non-trivial. The fact that these constructions will be diffuclt can also be inferred from the fact that the most basic definition of umkehr maps only works for oriented manifolds, which have a lot of structure.

1. Tubular neighborhoods and Thom spectra

1.1. Tubular neighborhoods. In this subsection we look at a notion of "simple" neighborhood of a submanifold N of a manifold M. These neighborhoods look like the normal bundle of N in M and are such that M sits into this neighborhood in the same way as the zero section sits in the vector bundle. They will be a crucial component of our construction of umkehr maps.

1.1.1. *Definition and basic properties.* In this section, let M be a manifold of dimension m and N be a submanifold of M of dimension n. Let $i : N \hookrightarrow M$ denote the embedding of N as a submanifold in M. We start with some preliminary definitions of the vector bundles playing a role in this situation.

Definition 4.1. The restriction $i^*(TM)$ is denoted by $TM|_N$ and is called the tangent bundle to M at N. The normal bundle ν_i is by definition the vector bundle $TM|_N/TN$ over N. We will usually drop the i from ν_i, when it is clear from the context which normal bundle we mean.

Generally, it is not the case that there is a canonical identification $TN \oplus \nu \cong TM|_N$. However, there is such an identification in two special situations. The first is when M is a vector bundle V over N. Then TM is isomorphic to the vector bundle $\pi_*TN \oplus \pi_*V$ on V, where $\pi : V \to N$ is the projection and $TM|_N$ is isomorphic to the vector bundle $TN \oplus V$ on TN. Therefore we have that $\nu = TM|_N/TN$ is canonically isomorphic to V itself and this induces a canonical isomorphism $TN \oplus \nu \cong TM|_N$.

Another way to get a canonical isomorphism is to introduce a metric. This will also explain why we call ν the normal bundle. If M comes equipped with a Riemannian metric, then we have an identification of ν_i with $TN^\perp$, which is given by $TN_x^\perp = \{v \in (TM|_N)_x | \langle v, n\rangle = 0 \text{ for all } n \in TN_x\}$. This makes it clear that ν should be considered as the vector bundle of normal vectors to N in M.

Definition 4.2. A tubular neighborhood for an embedding $i : N \to M$ is an embedding $f : \nu \to N$ with the following properties:

i **is zero-section:** f restricted to the zero-section, which is canonically identifiable with N, is i:

$$\begin{tikzcd} \nu \arrow[dr, "f"] & \\ N \arrow[u, "0"] \arrow[r, "i"] & M \end{tikzcd}$$

derivative is identity at zero-section: the composition of the isomorphism $TN \oplus \nu \cong T\nu|_N$ with the restriction $df|_N : T\nu|_N \to TM|_N$ and the projection $TM|_N \to \nu$ is the projection on the second component $TN \oplus \nu \to \nu$.

Since f is an embedding, ν and $f(\nu)$ look the same near the zero section as topological spaces. The second condition guarantees using the implicit function theorem that locally near the zero section they are also the same as smooth manifolds. This rules out some pathological examples.

Example 4.3. Consider the embedding $\mathbb{R} \to \mathbb{R}^2$ given by $x \mapsto (x, 0)$. Then consider the map $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ given by $(x, y) \mapsto (x, g(x, y))$. We write $T(\mathbb{R} \times \mathbb{R})$ and $T(\mathbb{R}^2)$ as a pair of points followed by a pair of components of the tangent vector. Then df sends (x, y, x', y') to $(x, g(x, y), x', \partial_1 g(x, y)x' + \partial_2 g(x, y)y')$. Then the composition of the natural isomorphism with df and projection is given by $(x, 0, x', y') \mapsto (x, \partial_1 g(x, 0)x' + \partial_2 g(x, 0)y')$. From this we see that for this to coincide with the projection $(x, 0, x', y') \mapsto (x, y')$, we must have $\partial_1 g(x, 0) = 0$ and $\partial_2 g(x, 0) = 1$.

In particular taking $g(x, y) = y$ suffices. The conditions rule out $g(x, y) = y^3$, which has the property that f is not locally a diffeomorphism near the zero section.

Sometimes one may come across an embedding $f : \nu \to N$ which is the embedding i restricted to the zero section, but for which the composition $TN \oplus \nu \to T\nu|_N \xrightarrow{df} TM|_N \to \nu$ is not equal to the projection on the second component. If this composition is a projection composed with an vector bundle isomorphism $\nu \to \nu$, then it is clear that f can be made into a tubular neighborhoodby precomposition of f with a fiberwise linear transformation. Therefore the demand that the total derivative is the identity at the zero section is not that restrictive.

The most important question about tubular neighborhoods is of course whether they exist. We can answer this question in the affirmative; the following is [**Hir76**, theorem 5.2].

Theorem 4.4. *Every embedding admits a tubular neighborhood.*

Sketch of proof. One first proves the existence for embeddings into $\mathbb{R}^n$ using the inverse function theorem. For a general embeddings $N \hookrightarrow M$ one embeds M into an Euclidean space using Whitney's embedding theorem and uses the previously result. □

1.1.2. *The space of tubular neighborhoods.* Once one knows that a certain object exists, one is often interested in finding out how unique it is. We will do this now for tubular neighborhoods: clearly these are not unique, but they are unique up to homotopy in a sense to be made precise.

First recall that if we have two manifolds N, M, the C^∞ topology on the set $\mathrm{Map}(N, M)$ is given at the following set of seminorms indexed by $n \in \mathbb{Z}_{\geq 0}$. It is finer than the compact open topology.

Definition 4.5. Let $\mathrm{Tub}(i)$ be the set of tubular neighborhoods for the embedding $i : N \to M$. We give it a topology as a subspace of $\mathrm{Map}(\nu, M)$ in the C^∞ topology.

We want to have control over our choice of tubular neighborhoods. The following theorem and its proof are essentially [**God07**, proposition 31] and give a concise answer to this question.

Theorem 4.6. *Let and M, N be compact manifolds, then the space of tubular neighborhoods* $\mathrm{Tub}(i)$ *is contractible for each embedding $i : N \to M$.*

PROOF. We already know that at least one tubular neighborhood exists. Fix one such tubular neighborhood f_0. Let $\mathrm{Tub}(i, f_0)$ denote the subspace consisting of those tubular neighborhoods f such that $f(\nu) \subset f_0(\nu)$. The proof will proceed in two steps. We first show that $\mathrm{Tub}(i)$ retracts onto $\mathrm{Tub}(i, f_0)$ and then show that $\mathrm{Tub}(i, f_0)$ is retracts onto the point f_0.

$\mathrm{Tub}(i)$ **retracts onto** $\mathrm{Tub}(i, f_0)$**:** Put a smooth norm $||-||$ on ν. With a smooth norm we mean a norm such that $||v||$ depends smoothly on v. We let for $v \in \nu$ and $f \in \mathrm{Tub}(i)$, $||v||_f$ denote $d(f(v), i(\pi(v)))$. We define a function $|f| : N \to (0, \infty)$ as follows:

$$|f|(n) = \max\{1, \inf\{||v|| \,|v \in \pi^{-1}(n) \subset \nu \text{ and } f(v) \notin f_0(\nu)\}\}$$

This function depends smoothly on f. Because N is compact, we get a non-zero number if we define $|f| = \frac{1}{2}\inf\{|f|(n)\,|n \in N\}$.

Consider the homotopy $H : \mathrm{Tub}(i) \times I \to \mathrm{Tub}(i)$:

$$H(f,t)(v) = \begin{cases} f\left((1-t)v + t\frac{|f|}{1+||v||}v\right) & \text{if } v \notin \text{zero-section} \\ f(v) & \text{if } v \in \text{zero-section} \end{cases}$$

First note that $H(f, 0) = f$ and that $H(f, t)$ sends the zero-section to N for all t and f. Furthermore, we claim that $H(f, t)$ for fixed f and t is smooth embedding. To prove this, we must show that it is injective and has injective derivative. For the first, we first note we only need to look at each line in ν seperately, because the homotopy $t \mapsto (1-t)v + t\frac{|f_0|}{1+|f|}v$ preserves the lines and f is injective on the set of lines. Fix a vector $v_0 \neq 0$, then since f is injective, it suffices to show that for fixed t and f, the map

$$\lambda \mapsto (1-t)\lambda v_0 + t\frac{|f|}{1+|\lambda|||v_0||}\lambda v_0$$

is injective. Because both terms are strictly increasing in norm, it is clearly injective. To show that the derivative of $H(f, t)$ is injective, we can simply use the chain rule and note that the total derivative of the map $v \mapsto (1-t)v + t\frac{|f|}{1+||v||}v$ at v_0 is equal to

$$w \mapsto (1-t)w + t\frac{|f|}{1+||v||}w - t\frac{|f|}{(1+||v_0||)^2}\frac{\langle v_0, w\rangle}{||v_0||}v_0$$

Because the two terms are linearly indepedent if w is not a multiple of v_0, the only possible problem is when w is a multiple of v_0. In that case it is easy to check that it is injective.

We therefore have shown that $H(f, t)$ is an element of $\mathrm{Tub}(i)$. It depends continuously on both f and t, when is a homotopy between $H(f, 0)$, which is the identity and $H(f, 1)$. We claim is that $H(f, 1) \in \mathrm{Tub}(i, f_0)$ for all f. To prove this, it suffices to note that the term inside f always has norm smaller than the smallest vector in this fiber which gets mapped outside $f_0(\nu)$.

$\mathrm{Tub}(i, f_0)$ **is retracts onto the point** f_0**:** In the second step, we again construct an explicit homotopy. Note the function $G : \mathrm{Tub}(i, f_0) \to \mathrm{Aut}(\nu)$ given by $f \mapsto f_0^{-1} \circ f$ is continuous. Then consider $H_2 : \mathrm{Tub}(i, f_0) \times [0, 1] \to \mathrm{Tub}(i, f_0)$ given by

$$H_2(f,t)(v) = \begin{cases} f_0(\frac{1}{1-t}G(f)((1-t)v)) & \text{if } t > 0 \\ f_0(v) & \text{if } t = 1 \end{cases}$$

This is clearly continuous in f and t and $H_2(f, 0) = f$ and $H_2(f, 1) = f_0$.

∎

1.2. Compatibility of tubular neighborhoods. Many constructions one can do on smooth maps return an embedding when applied to embeddings. By the theorem on the existence of tubular neighborhoods applied to the result of such an construction, we see that we find a tubular neighborhood for the result. However, sometimes we want some more compatibility with the construction. We will discuss several examples of this now: products and compositions. The reasons for the looking at these constructions will become clearer when we get to the Thom spaces.

1.2.1. *Product of tubular neighborhoods.* In this section we discuss how to create a tubular neighborhood for a product of embeddings. Of course, a product of embeddings is an embedding itself and hence tubular neighborhoods for products of embeddings exist by theorem 4.4. However, sometimes we want this tubular neighborhood to be compatible with the structure of our products.

Let $i_1 : N_1 \to M_1$ and $i_2 : N_2 \to M_2$ be a pair of embeddings. Wewill show how construct a tubular neighborhood for the product of embeddings $i_1 \times i_2 : N_1 \times N_2 \to M_1 \times M_2$ out of tubular neighborhoods f_1 and f_2 for i_1 and i_2 respectively.

$$\begin{tikzcd} N_1 \arrow[d, "i_1"] & N_2 \arrow[d, "i_2"] & \Rightarrow & N_1 \times N_2 \arrow[d, "i_1 \times i_2"] \\ M_1 & M_2 & & M_1 \times M_2 \end{tikzcd}$$

We have already remarked following definition 4.1 that the isomorphism $T(M_1 \times M_2) \cong TM_1 \oplus TM_2$ is natural with map giving by the product of the the derivatives of the projections $\pi_1 : M_1 \times M_2 \to M_1$ and $\pi_2 : M_1 \times M_2 \to M_2$ to the components of the products. Similarly $T(N_1 \times N_2) \cong TN_1 \oplus TN_2$. Under these isomorphisms the quotient $\nu = T(M_1 \times M_2)|_{N_1 \times N_2}/T(N_1 \times N_2)$ can naturally be identified with $(TM_1|_{N_1}/TN_1) \oplus (TM_2|_{N_2}/TN_2)$. Hence we naturally have $\nu_{i_1 \times i_2} \cong \nu_{i_1} \oplus \nu_{i_2}$, where the latter is homeomorphic as a space to $\nu_{i_1} \times \nu_{i_2}$. We summarize this as:

$$\begin{aligned} &T(M_1 \times M_2) \cong TM_1 \oplus TM_2 \qquad \text{and} \qquad T(N_1 \times N_2) \cong TN_1 \oplus TN_2 \qquad \text{imply} \\ &\nu_{i_1 \times i_2} = T(M_1 \times M_2)|_{N_1 \times N_2}/T(N_1 \times N_2) \cong (TM_1|_{N_1}/TN_1) \oplus (TM_2|_{N_2}/TN_2) = \nu_{i_1} \oplus \nu_{i_2} \end{aligned}$$

Using these isomorphisms, we can define the tubular neighorbood of $f : N_1 \times N_2 \to M_1 \times M_2$ to be the composition of the isomorphism of vector bundles $\nu_{i_1 \times i_2} \cong \nu_{i_1} \oplus \nu_{i_2}$ with $f_1 \times f_2$. We now prove that this is in fact a tubular neighborhood.

Proposition 4.7. *The map $f_1 \times f_2$ is a tubular neighborhood for the embedding $i_1 \times i_2$.*

PROOF. Firstly, it is clear that $f_1 \times f_2$ is an embedding: it is injective because both component maps are injective and the derivative is injective because the derivative is the direct sum of the derivatives df_1 and df_2, both of which are injective on their respective components.

Secondly, note that $f_1 \times f_2$ restricted to the zero section is the same as the product of the restriction of f_1 and f_2 and hence equal to the embedding $i_1 \times i_2$.

Finally, under the natural isomorphisms $T(N_1 \times N_2) \oplus \nu_{i_1 \times i_2} \cong TN_1 \oplus TN_2 \oplus \nu_{i_1} \oplus \nu_{i_2}$ and $\nu_{i_1 \times i_2} \cong \nu_{i_1} \oplus \nu_{i_2}$, the composition of the isomorphism $T(N_1 \times N_2) \oplus \nu_{i_1 \times i_2} \cong (T\nu_{i_1 \times i_2})|_{N_1 \times N_2}$ with the map $d(f_1 \times f_2) : (T\nu_{i_1 \times i_2})|_{N_1 \times N_2} \to T(M_1 \times M_2)|_{N_1 \times N_2}$ and the projection $T(M_1 \times M_2)|_{N_1 \times N_2} \to \nu_{i_1 \times i_2}$ coincides with the composition of relevant maps for f_1 and f_2. Hence it is indeed the projection on the second component. $\square$

We now discuss the compatibility property we have in mind for tubular neighborhoods for products of embeddings.

Definition 4.8. A tubular neighborhood f for a product of embeddings $i_1 \times i_2 : N_1 \times N_2 \to M_1 \times M_2$ is said to be compatible with the product structure if it is a product of tubular neighborhoods f_1 for i_1 and f_2 for i_2.

Our earlier construction shows that these exist tubular neighborhoods compatible with the product structure. So these tubular neighborhoods from a subspace $\mathrm{Tub}^{\times}(i_1 \times i_2)$ of $\mathrm{Tub}(i_1 \times i_2)$. As before, we would like it to be contractible because that means heuristically that the choice of a tubular neighborhood compatible with the product structure does not matter up to homotopy. This is indeed the case, as proven in the following corollary.

Corollary 4.9. *The space* $\mathrm{Tub}^{\times}(i_1 \times i_2)$ *of tubular neighborhoods compatible with the product structure is a non-empty contractible subspace of* $\mathrm{Tub}(i_1 \times i_2)$.

PROOF. Our previous construction gives an injective and continuous map $\mathrm{Tub}(i_1) \times \mathrm{Tub}(i_2) \to \mathrm{Tub}(i_1 \times i_2)$ onto the space $\mathrm{Tub}^{\times}(i_1 \times i_2)$. A quick inspection of the topologies shows that this map is a homeomorphism onto its image. Since the two components of the space $\mathrm{Tub}(i_1) \times \mathrm{Tub}(i_2)$ are contractible, the result follows. □

1.2.2. *Composition and tubular neighborhoods.* Next, we discuss to what extent it is possible to find a tubular neighborhood for a composition of embeddings which is compatible with the composition in some sense.

To do this, consider the following situation: $j : L \to N$ is an embedding of L as a submanifold of N and $i : N \to M$ is an embedding of N as a submanifold of M.

$$\begin{tikzcd} L \arrow[dd, "j"'] \arrow[dr, "i \circ j"] & \\ & M \\ N \arrow[ur, "i"'] & \end{tikzcd}$$

Of course a tubular neighborhood each of these three maps individually exists. For i and j this is simply theorem 4.4 and for $i \circ j$ it is a consequence of this theorem and the fact that a composition of embeddings is an embedding. But we want certain compatibility conditions to hold. A first guess might be that given tubular neighorboods f_j and f_i for j and i one wants to construct a composition tubular neighborhood $f_{i \circ j}$ for $i \circ j$. However, this is not possible uniquely. It is only possible up to a contractible space of choices.

What we'll do instead is the following: suppose that we are given a tubular neighborhoods f_i and $f_{i \circ j}$, then we will describe how to get maps which make the following diagram commute:

$$\begin{tikzcd} \nu_{i \circ j} \arrow[r, "\cong"] \arrow[drr, "f_{i \circ j}"'] & \nu_i|_L \oplus \nu_j \arrow[r, "f_\lambda"] & \nu_i \arrow[d, "f_i"] \\ & & M \end{tikzcd}$$

Let's make this more precise. Consider the normal bundle $\nu_{i \circ j}$ over L of the embeddings $i \circ j$. This is non-canonically isomorphic to $\nu_i|_L \oplus \nu_j$, because the normal bundle $\nu_i|_L \oplus \nu_j$ is canonically the normal bundle of the embedding $\iota_\lambda : L \hookrightarrow \nu_i$ as zero section. However, the tubular neighborhood $f_{i \circ j}$ with its derivative does give such a canonical identification. So, the real question is to find a map $f_\lambda : \nu_i|_L \oplus \nu_j \to M$, which is necessarily a tubular neighborhood.

It is certainly not true that for each $f_{i \circ j}$ we can find such a f_λ. The problem might be that the image of f_i is too small in the sense that $\mathrm{im}(f_{i \circ j}) \backslash \mathrm{im}(f_i) \neq \emptyset$, because in that case we can never reach some points in the image of $f_{i \circ j}$. However, this turns out to be only obstruction. See figure 4.11 for a picture which makes this clear geometrically.

Lemma 4.10. *Suppose that* $\mathrm{im}(f_{i \circ j}) \subset \mathrm{im}(f_i)$, *then there exists a unique* f_λ *which makes the diagram commute.*

PROOF. The fact that $f_{i \circ j}$ and f_i are injective completely determines a candidate for f_λ. It thus suffices to check that this candidate indeed is a tubular neighborhood. The map f_λ composed with an embedding is an embedding, hence it was an embedding itself and similarly we obtain that f_λ has the other properties of a tubular neighborhood. □

We can now formulate the compatibility conditions we want.

Definition 4.12. A triple $f_{i \circ j}$, f_i and f_λ making the diagram commute is called compatible with the composition $i \circ j$.

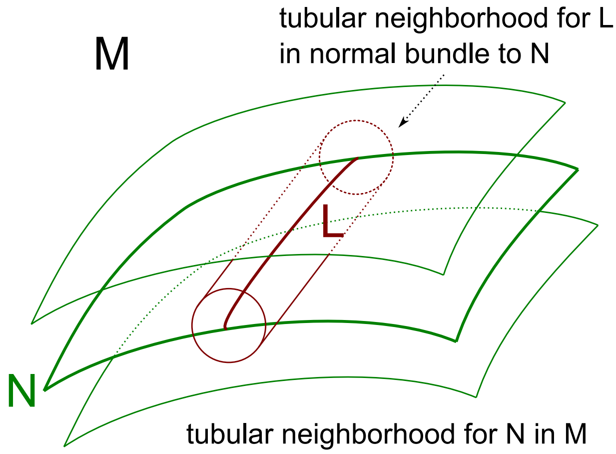


FIGURE 4.11. This figures shows how a tubular neighborhoods for N in M and for L in the normal bundle ν_i to N gives a compatible tubular neighborhood for L in M.

The previous remarks show that that the compatible triples form a non-empty subspace $\mathrm{Tub}^\circ(i \circ j)$ of $\mathrm{Tub}(i \circ j) \times \mathrm{Tub}(i) \times \mathrm{Tub}(\iota_\lambda)$. We would like this subspace to be contractible, which is indeed the case.

Corollary 4.13. *The subspace* $\mathrm{Tub}^\circ(i \circ j)$ *of tubular neighborhoods compatible with the composition* $i \circ j$ *is a non-empty contractible space.*

PROOF. Implicit in our discussion is a homeomorphism of $\mathrm{Tub}^\circ(i \circ j)$ with the space $\mathrm{Tub}(i) \times \mathrm{Tub}(L \hookrightarrow \nu_i|_L \oplus \nu_j)$. □

However, for our purposes another point of view is more fruitful. There is a projection map $p : \mathrm{Tub}^\circ(i \circ j) \to \mathrm{Tub}(i)$. This is a surjection, because given a point f_i of $\mathrm{Tub}(i)$ we can precompose with any element of $\mathrm{Tub}(\iota_\lambda)$ to obtain a element of $\mathrm{Tub}(i \circ j)$ which forms a compatible triple mapping to f_i under p. The fiber of this map over a point $f_i \in \mathrm{Tub}(i)$ consists of all liftings of f_i to a compatible triple $f_{i\circ j}$, f_i, and f_λ.

Lemma 4.14. *The fiber* $p^{-1}(f_i)$ *is a non-empty contractible space and in fact* $\mathrm{Tub}^\circ(i \circ j)$ *is a fiber bundle over* $\mathrm{Tub}(i)$ *with contractible fiber. In other words, compatible liftings exist and are essentially unique.*

PROOF. The fiber is simply homeomorphic to $\mathrm{Tub}(\iota_\lambda)$. □

1.3. Thom spaces and the Thom isomorphism. Suppose that one has a vector bundle over a space X. Since the projection onto X is a homotopy equivalence – simply use the action of scalar multiplication to give a homotopy between the identity map and the projection onto X – we have that the homology of the total space is isomorphic to the homology of the base space. One can then ask what happens if one adds a point at infinity in each of the fibers of the vector bundle and collapses all of these together to a single point. The resulting space is known as a Thom space and for oriented finite-dimensional vector bundles we have the Thom isomorphism theorem describing the relation between the homology of the Thom space and the homology of the base space. In this section we take a close look at these Thom spaces, the Thom isomorphism and their properties.

To make sure that all our constructions are well-behaved we will assume that our spaces X are manifolds or more generally CW complexes.

1.3.1. *Vector bundles and virtual vector bundles.* In this section we describe the categories of vector bundles and virtual vector bundles, which provide the right context for the definition of Thom spaces, Thom spectra, the Thom isomorphism and in the end, Thom collapse maps.

We assume that the reader knows what a vector bundle is, but here is a short reminder: a vector bundle is a bundle over a space X which is locally trivial and isomorphic to $U \times \mathbb{R}^n$, such that the transition functions are linear maps on each fiber.

An important construction on vector bundles is that of pullback: for any continuous map $f : X \to Y$ and vector bundle μ over Y we can create a new a vector bundle $f^*\mu$, called the pullback of μ along f. It is given by $\{(x, v) \in X \times \mu | f(x) = \pi(v)\}$ with projection X given by sending (x, v) to x. We will use this pullback construction in our definition of a category of vector bundles well-suited for working with Thom spaces and the Thom isomorphism.

Definition 4.15. The category VectB has as objects pairs (X, μ) of a paracompact space X and a vector bundle μ this space and as morphisms from (X, μ) to (Y, λ) the pairs (f, ψ) of a map of spaces $f : X \to Y$ and an isomorphism $\psi : f^*\lambda \to \mu$ of vector bundles over X.

The composition of two morphisms $(f, \psi) : \mu \to \lambda$ and $(g, \phi) : \lambda \to \nu$, where μ, λ, ν are vector bundles over X, Y, Z respectively, is given by the composition of isomorphisms:

$$f^*g^*\nu \xrightarrow{f^*\phi} f^*\lambda \xrightarrow{\psi} \mu$$

where $f^*\phi$ is the isomorphism obtained by pullback. To be precise, it is given by $(x, (y, v)) \mapsto (x, \phi(y, v))$.

An important property of vector bundles is that if the base space is nice enough, i.e. paracompact, then every d-dimensional vector bundle is obtained up to isomorphism as the pullback $f^*\xi$ of the canonical bundle ξ over the classifying space $BGL(d)$. Furthermore, this map f is unique up to homotopy. The space $BO(d) \subset BGL(d)$ is slightly nicer and has the same homotopy type. It is the classifying space of d-dimensional vector bundles with a metric. However, since the space of possible metrics on a vector bundle over a paracompact base is contractible, this choice of metric does not really matter.

A virtual bundle should be seen as a formal difference of vector bundles. The definition is therefore quite easy.

Definition 4.16. A virtual vector bundle $\mu_+ - \mu_-$ over a paracompact space X consists of a triple of (X, μ_+, μ_-) of a space X and two vector bundles μ_+ and μ_- over X.

However, we want to choose our morphisms in such a way that indeed virtual vector bundles behave like formal differences. In other words, $\mu_+ \oplus \mathbb{R} - \mu_- \oplus \mathbb{R}$ should be isomorphic to $\mu_+ - \mu_-$.

Definition 4.17. The category VirtB has as objects virtual vector bundles and as morphisms from $\mu_+ - \mu_-$ to $\lambda_+ - \lambda_-$, virtual bundles over X and Y respectively, we have quadruples $(f, \theta, \phi_+, \phi_-)$ of a map $f : X \to Y$, a vector bundle θ over X and two isomorphisms

$$\phi_+ : f^*\lambda_+ \oplus \theta \to \mu_+$$
$$\phi_- : f^*\lambda_- \oplus \theta \to \mu_-$$

The composition of two morphisms $(f, \theta, \phi_+, \phi_-)$ and $(g, \beta, \psi_+, \psi_-)$, morphisms $\mu_+ - \mu_- \to \lambda_+ - \lambda-$ and $\lambda_+ - \lambda- \to \nu_+ - \nu_-$ respectively, is the isomorphism $(g \circ f, f^*\beta \oplus \theta, \varphi_+, \varphi_-)$ where φ_+, φ_- are given by the compositions

$$\varphi_+ : (g \circ f)^*\nu_+ \oplus f^*\beta \oplus \theta \longrightarrow f^*(g^*\nu_+ \oplus \beta) \oplus \theta \overset{f^*\psi_+ \oplus id}{\longrightarrow} f^*\lambda_+ \oplus \theta \xrightarrow{\phi_+} \mu_+$$

$$\varphi_- : (g \circ f)^*\nu_- \oplus f^*\beta \oplus \theta \longrightarrow f^*(g^*\nu_- \oplus \beta) \oplus \theta \overset{f^*\psi_- \oplus id}{\longrightarrow} f^*\lambda_- \oplus \theta \xrightarrow{\phi_-} \mu_-$$

Note that there is a inclusion functor $\mathsf{VectB} \to \mathsf{VirtB}$ where an object μ is sent to $\mu - 0$ and a morphism (f, ϕ) is sent to $(f, 0, \phi, id_0)$.

1.3.2. *Thom spaces of vector bundles and basic properties.* We will now define the Thom space of a vector bundle. This is the model case of the more advanced constructions we will do later, which we will do in detail to get some geometric intuition for these constructions. We first define the Thom space for a vector bundle over a nice compact space, for which this definition is easier, and after that describe the general definition for non-compact spaces.

Definition 4.18. Let λ be a vector bundle of dimension ≥ 1 over a compact locally compact Hausdorff space X. Then $\text{Thom}(\lambda) = \dot{\lambda}$, the one point compactification of λ.

Why locally compact Hausdorff? This is the right condition to demand when one wants the one point compactification to exist. To get a feeling for this definition we look at some examples. Suppose that X is a point and λ the necessarily trivial one dimensional vector bundle $\mathbb{R}$, then the Thom space is simply the circle. The next example looks at a simple 1-dimensional vector bundle.

Example 4.19. Suppose we start with one of the simplest vector bundles over one of the simplest spaces: a trivial 1-dimensional vector bundle over the circle. Then the Thom space of this vector bundle is homeomorphic to the space obtained by taking a circle, a point and two cylinders $S^1 \times [0,1]$, and attaching one boundary component of each of the two cylinder to the circle and collapsing the other boundary component to the point. This is homeomorphic to the $\Sigma(S^1_+)$, the reduced suspension of S^1 with a disjoint basepoint added.

However, in the case of a non-compact space taking the one-point compactification will not only compactify the fibers of the vector bundle, but also the space itself. In general we do not want to modify our base space and therefore we have to take more care with our compactification if the base space is not compact. This leads us to the following definition.

Definition 4.20. Let λ be a vector bundle of dimension $d \geq 1$ over a non-compact space X. Then $\text{Thom}(\lambda)$ is the space obtained by taking the one point compactification of each fiber of λ, giving a bundle $T(\lambda)$ of spheres of dimension d with a section $\infty(\lambda)$, and collapsing this section to a point:

$$\text{Thom}(\lambda) = T(\lambda)/\infty(\lambda)$$

One case remains to be defined: that of a 0-dimensional vector bundle. In this case we simply add a disjoint basepoint.

Definition 4.21. Let λ be a vector bundle of dimension 0 over a space X. Then we define $\text{Thom}(\lambda)$ to be X_+, i.e. X with a disjoint basepoint added.

So, we have just seen two definition for the Thom space of a vector bundle over a compact locally compact Hausdorff space. It should be clear from the definitions that these coincide. Also note that in both cases $\text{Thom}(\lambda)$ comes with a canonical basepoint ∞, either as the new point added by the one point compactifiction or as the image of $\infty(\lambda)$ under the quotient map. In other words, $\text{Thom}(\lambda)$ naturally lives in the category Top_* of pointed topological spaces. If we do any construction with Thom spaces which uses a basepoint we mean this basepoint, unless mentioned otherwise.

The definition of a Thom space is natural in the sense that it is a functor from $\mathsf{VectB} \to \mathsf{Top}_+$. In fact, the reason for the specific choice of morphisms in VectB is to make this a functor. In the next proposition we define its value on morphisms and check that it is indeed a functor.

Proposition 4.22. *The Thom space construction is a functor* $\text{Thom} : \mathsf{VectB} \to \mathsf{Top}_+$.

Proof. As stated before, we have already defined its value on objects, so it suffices to define it on morphisms. If $(f,\psi) : \mu \to \lambda$ is an morphism in VectB, then this induces a map of Thom spaces $\text{Thom}(f,\psi) : \text{Thom}(\mu) \to \text{Thom}(\lambda)$ as the composition of two maps $\text{Thom}(\psi)$ and $\text{Thom}(f)$, which we will define now.

To do this, note that $\text{Thom}(\mu)$ is the union of μ and ∞ as a set. We define $\text{Thom}(\psi)|_\mu = \psi^{-1}$ and $\text{Thom}(\psi)(\infty) = \infty$. We define $\text{Thom}(f)|_{f^*\mu}(x,v) = v$ and $\text{Thom}(f)(\infty) = \infty$. Therefore we have $\text{Thom}(f,\psi) = \text{Thom}(f) \circ \text{Thom}(\psi)$.

We should check that this construction satisfies the properties $\text{Thom}(id, id_\mu) = id_{\text{Thom}(\mu)}$ and $\text{Thom}((g,\phi) \circ (f,\psi)) = \text{Thom}(g,\phi) \circ \text{Thom}(f,\psi)$. The first property is then easy to check, as both ψ and f are then the identity maps and therefore $\text{Thom}(id_\mu)$ is the identity on μ and ∞ separately.

For the second property, it is clear that both maps send ∞ to ∞. The point $v \in \mu$ is mapped to $\phi^{-1}(v) = (x, v')$, to v' to $\psi^{-1}(v') = (y, v'')$ to v'' under the composition $\text{Thom}(g,\phi) \circ \text{Thom}(f,\psi)$. Under $\text{Thom}((g,\phi) \circ (f,\psi))$ it is mapped to $(f^*\phi \circ \psi^{-1})(v) = (x, v'')$ and thent o v''. □

Furthermore, the Thom space construction has several other properties which will be useful. The first one describes compatibility with products and direct sums.

Proposition 4.23. *Let μ be a vector bundle over X and λ a vector bundle over Y. Let $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$ be the projections. Then $\pi_1^*(\mu) \oplus \pi_2^*(\lambda)$ is a vector bundle over $X \times Y$ and $\mathrm{Thom}(\pi_1^*(\mu) \oplus \pi_2^*(\lambda))$ is homeomorphic to $\mathrm{Thom}(\mu) \wedge \mathrm{Thom}(\lambda)$.*

Furthermore, if $X = Y$, then there is a natural map $\mathrm{Thom}(\mu \oplus \lambda) \to \mathrm{Thom}(\mu) \wedge \mathrm{Thom}(\lambda)$.

Proof. For the first statement we produce a homeomorphism $\phi : \mathrm{Thom}(\mu \oplus \lambda) \to \mathrm{Thom}(\mu) \wedge \mathrm{Thom}(\lambda)$. To define this homeomorphism, we note that $\mathrm{Thom}(\mu) \wedge \mathrm{Thom}(\lambda)$ has three types of point: points of $\mu \oplus \lambda$ and a point at ∞, obtained by collapsing $\mathrm{Thom}(\mu) \times \{\infty\} \cup \{\infty\} \times \mathrm{Thom}(\lambda)$. The map ϕ is simply the identity on $\mu \oplus \lambda$ and sends ∞ to ∞.

Let's now prove the second statement. If $X = Y$ and we consider the diagonal map $\Delta : X \to X \times X$, then the vector bundle $\mu \oplus \lambda$ is the pullback $\Delta^*(\pi_1^*(\mu) \oplus \pi_2^*(\lambda))$. By naturality, this induces a map of Thom spaces. □

Remark 4.24. The previous proof allows us to describe the Thom space of a direct sum of vector bundles. Using the notation of that proof, we have that the pullback of sphere bundle of $\pi_1^*(\mu) \oplus \pi_2^*(\lambda)$ along Δ is exactly the sphere bundle of $\mu \oplus \lambda$. But the pullback of the sphere bundle of $\pi_1^*(\mu) \oplus \pi_2^*(\lambda)$ has points exactly equal to the pair of vectors lying about the same point together with the section at infinity. In other words, the Thom space $\mathrm{Thom}(\mu \oplus \lambda)$ is homeomorphic to the subspace of $\mathrm{Thom}(\mu) \wedge \mathrm{Thom}(\lambda)$ consisting of all (v, w) such that $\pi_1(v) = \pi_2(w)$ together with $\{\infty\}$.

Next we look at the way the Thom space acts on trivial vector bundles or vector bundles containing a trivial summand. This will play a role when defining Thom spectra.

Proposition 4.25. *Let $\underline{\mathbb{R}}^n \to X$ be a trivial vector bundle, then $\mathrm{Thom}(\underline{\mathbb{R}}^n)$ is homeomorphic to $S^n \wedge X_+$. Furthermore $\mathrm{Thom}(\underline{\mathbb{R}}^n \oplus \mu) = S^n \wedge \mathrm{Thom}(\mu)$.*

Proof. For the first statement, it suffices to note that the one point compactification of each fiber is S^n. The section at ∞ can then be identified with $\{1\} \times X$.

For the second statement, we use the description of the Thom space of a direct sum. It is homeomorphic to the subspace of $\mathrm{Thom}(\mathbb{R}^n) \wedge \mathrm{Thom}(\mu) = S^n \wedge X_+ \wedge \mathrm{Thom}(\mu)$ consisting of $\{\infty\}$ together with all (r, x, w) such that $x = \pi_2(w)$. This is exactly $S^n \wedge \mathrm{Thom}(\mu)$. □

1.3.3. *Thom spectra.* We would like to extend the definition of a Thom space to virtual bundles. To do this, we will introduce in this section Thom spectra associated to virtual bundles. However, we start by defining the Thom spectrum of a vector bundle. For background information on spectra see section 1 of appendix A.

For a vector bundle μ over a locally compact Hausdorff space X, we get an entire sequence of vector bundles given by taking the direct sum with a trivial vector bundle: these are given by $\mathbb{R}^n \oplus \mu$ for $n \geq 0$. We can take the Thom spaces of each of these vector bundles and thereby get a sequence of spaces $\mathrm{Thom}(\mathbb{R}^n \oplus \mu)$. Using proposition 4.25, we see that $\mathrm{Thom}(\mathbb{R}^n \oplus \mu) \cong \Sigma^n \mathrm{Thom}(\mu)$, where Σ denotes the reduced suspension as in definition A.2. We can put all these spaces together in a single object to form a spectrum.

Definition 4.26. Let μ be a vector bundle of dimension d. Then we have a Thom spectrum X^μ given by

$$(X^\mu)_n := \mathrm{Thom}(\mathbb{R}^n \oplus \mu)$$

where $\sigma_n : \Sigma\mathrm{Thom}(\mathbb{R}^n \oplus \mu) \to \mathrm{Thom}(\mathbb{R}^{n+1} \oplus \mu)$ the homeomorphism indicated in proposition 4.25.

Remark 4.27. In the literature there is an alternative grading

$$(X^\mu)'_n := \begin{cases} \mathrm{Thom}(\mathbb{R}^{n-d} \oplus \mu) & \text{if } n \geq d \\ * & \text{otherwise} \end{cases}$$

This has the property that the Thom isomorphism no longer includes a dimension shift: $H_*((X^\mu)'; \mathbb{Z}) \cong H_*(X; \mathbb{Z})$ if μ is oriented.

We will show that this gives a functor from VectB to $\mathsf{Spectra}$, which restricts to the Thom space functor when we restrict to the zero'th level.

Proposition 4.28. *The Thom spectrum construction is a functor* $\mathsf{VectB} \to \mathsf{Spectra}$*. Furthermore, there is a natural isomorphism of spectra* $X^\mu \wedge Y^\lambda \to (X \times Y)^{\pi_1^*(\mu) \oplus \pi_2^*(\lambda)}$.

Proof. Consider a morphism $(f, \psi) : \mu \to \lambda$, then $\psi \oplus id_{\mathbb{R}^n} : f^*\lambda \oplus \mathbb{R}^n \to \mu \oplus \mathbb{R}^n$ be the direct sum of ψ and the identity map. It suffices to check that that the maps $\mathrm{Thom}(f, \psi \oplus id_{\mathbb{R}^n})$ are compatible with the suspension maps, but this is clear because it just corresponds to suspending the map f. Therefore, the second statement is a consequence of proposition 4.23. ☐

In fact, if $X = BSO(d)$ or $X = BU(d)$ and the vector bundle is the canonical one over these classifying spaces, we obtain spectra $MSO(d)$ and $MU(d)$ whose limits as $d \to \infty$ represent oriented or complex cobordism respectively. These play an important role in algebraic topology. For example, their homotopy groups give the classification of manifolds up to cobordism. For us, they will be interesting as universal cases for the Thom isomorphism.

The Thom spectra allow us to generalize the construction of a Thom space to virtual vector bundles.

Definition 4.29. Let λ be a vector bundle of finite dimension, then there exists a vector bundle ν over X such that $\lambda \oplus \nu$ is isomorphic to a trivial vector bundle $\mathbb{R}^n$. The spectrum $X^{-\lambda}$ is given by $\Sigma^{-n} X^\nu$, the n-fold desuspension of X^ν. Furthermore, for a virtual bundle $\mu - \lambda$, we define $X^{\mu - \lambda} = \Sigma^{-n} X^{\mu \oplus \nu}$.

Unfortunately, this doesn't define a functor from VirtB to $\mathsf{Spectra}$ because construction depends on a choice of ν. What we do have is a natural weak equivalence between the Thom spectra for two different choices of ν. Alternatively, one can work with manifolds embedded in some Euclidean space, e.g. as in [**GMTW10**].

Lemma 4.30. *Let ν_1 and ν_2 be two vector bundles such that $\lambda \oplus \nu_1$ and $\lambda \oplus \nu_2$ are isomorphic to trivial vector bundles $\mathbb{R}^{n_1}$ and $\mathbb{R}^{n_2}$ respectively. Then there is a canonical weak equivalence* $\Sigma^{-n_1} X^{\mu \oplus \nu_1} \to \Sigma^{-n_2} X^{\mu \oplus \nu_2}$.

Proof. Consider the vector bundle isomorphisms $\mu \oplus \nu_1 \oplus \mathbb{R}^{n_2} \cong \mathbb{R}^{n_1 + n_2} \cong \mu \oplus \nu_2 \oplus \mathbb{R}^{n_1}$. This induces isomorphisms of spectra:

$$S^{n_2} \wedge X^{\mu \oplus \nu_1} \cong S^{n_1 + n_2} \wedge X_+ \cong S^{n_1} \wedge X^{\mu \oplus \nu_2}$$

and using the natural weak equivalence between the cotensoring and the suspension, we obtain natural weak equivalence $\Sigma^{n_2} X^{\mu \oplus \nu_1} \cong \Sigma^{n_1} X^{\mu \oplus \nu_2}$. We can now desuspend the map $(n_1 + n_2)$ times to obtain the desired canonical weak equivalence. ☐

This is enough to guarantee that we get a functor from the category of virtual bundles to the homotopy category of spectra. We already had it on objects and iIn the next proposition, we define the value on morphisms and then shows that it respects identities and composition.

Proposition 4.31. *The construction of a Thom spectrum is a functor* $\mathsf{VirtB} \to \mathsf{HSpectra}$.

Proof. Let $(f, \theta, \phi_+, \phi_-) : \mu_+ - \mu_- \to \lambda_+ - \lambda_-$ be a morphism in VirtB. Let ν be a vector bundle on Y, $\lambda_- \oplus \nu \cong \mathbb{R}^n$. Then we know that μ_- is isomorphic to $f^*\lambda_- \oplus \theta$ and hence if η is a vector bundle such that $\theta \oplus \eta \cong \mathbb{R}^l$, then $f^*\nu \oplus \eta$ has the property that $\mu_- \oplus f^*\nu \oplus \eta \cong \mathbb{R}^{n+l}$. Hence we can take $X^{\mu_+ - \mu_-}$ and $Y^{\lambda_+ - \lambda_-}$ to be $\Sigma^{-n-l} X^{\mu_+ \oplus f^*\nu \oplus \eta}$ and $\Sigma^{-n} Y^{\lambda_+ \oplus \nu}$ respectively. The previous lemma shows that up to weak equivalence these spectra are independent of the choice of ν and η.

Consider the isomorphism of vector bundles

$$f^*(\lambda_+ \oplus \nu \oplus \mathbb{R}^l) \to f^*\lambda_+ \oplus f^*\nu \oplus \mathbb{R}^l \to f^*\lambda_+ \oplus \theta \oplus f^*\nu \oplus \eta \to \mu_+ \oplus f^*\nu \oplus \eta$$

By the fact that the construction of a Thom spectrum is functorial on VectB, we obtain an map of Thom spectra $X^{\mu_+ \oplus f^*\nu \oplus \eta} \to S^l \wedge Y^{\lambda_+ \oplus \nu}$, the latter of which is naturally weakly equivalent to $\Sigma^l Y^{\lambda_+ \oplus \nu}$. Desuspending $(n + l)$ times gives the required morphism of spectra.

We should show it sends identities to identites and respects composition. The identity morphism of $X^{\mu_+-\mu_-}$ is $(id_X, 0, id_{\mu_+}, id_{\mu_-})$. If we take the $\eta = 0$ in the previous construction, we clearly get the identity map.

Finally, let $(f, \theta, \phi_+, \phi_-) : \mu_+ - \mu_- \to \lambda_+ - \lambda_-$ and $(g, \beta, \psi_+, \psi_-) : \lambda_+ - \lambda_- \to \nu_+ - \nu_-$ be two morphisms of virtual bundles. Then the construction of the morphism of spectra corresponding to $(g, \beta, \psi_+, \psi_-)$ can be as before, using η such that $\beta \oplus \eta \cong \mathbb{R}^l$ and ν such that $\nu_- \oplus \nu \cong \mathbb{R}^n$. We know that $\lambda_- \oplus f^*\nu \oplus \eta \cong \mathbb{R}^{n+l}$. Let υ be a vector bundle such that $\theta \oplus \upsilon = \mathbb{R}^k$, then because $f^*\lambda_- \oplus \theta \cong \mu_+$, we know that $\mu_- \oplus f^*g^*\nu \oplus f^*\eta \oplus \upsilon \cong \mathbb{R}^{n+l+k}$. Therefore in the homotopy category our composition is represented by the composition of morphisms of spectra

$$\Sigma^{-n-k-l} X^{\mu_+ \oplus f^*g^*\nu \oplus f^*\eta \oplus \upsilon} \to \Sigma^{-n-l} Y^{\lambda_+ \oplus f^*\nu \oplus \eta} \to \Sigma^{-n} Z^{\nu_+ \oplus \nu}$$

But now note that for the construction of the morphism associated to the composition we can take the complement to ν_- to be ν and the complement to $f^*\beta \oplus \theta$ to be $f^*\eta \oplus \upsilon$. This shows that up to homotopy the construction of a Thom spectrum respects composition. $\square$

Note that if we restrict to the subcategory VirtB_0 of VirtB of those virtual bundle where we have a given isomorphism of the negative parts with a trivial vector bundle, then the construction of a Thom spectrum is in fact an honest functor from VirtB_0 to $\mathsf{Spectra}$.

1.3.4. *Orientations and Thom classes.* Now we will do some preparations necessary for defining the Thom isomorphism, by explaining the relation between orientations for vector bundles and Thom classes.

Let μ be a vector bundle of dimension d over a space X. Then for each $x \in X$ we have a map $\iota_x : S^d \to \mathrm{Thom}(\mu)$. This induces a map of Thom spectra $(S^d)^{\iota_x^*\mu} \to X^\mu$, where the first is weakly equivalent to $\Sigma^d \mathbb{S}^d$.

Definition 4.32. A Thom class u_μ for a vector bundle μ of dimension d is a cohomology class $H^d(\mathrm{Thom}(\mu); \mathbb{Z})$ such that $\iota_x^* u$ is a generator of $H^d(S^d; \mathbb{Z})$ for all $x \in X$.

Proposition 4.33. *The following three pieces data are equivalent:*

(1) an orientation of a vector bundle μ of dimension d on X,

(2) a Thom class $u_\mu \in H^d(\mathrm{Thom}(\mu), \mathbb{Z})$,

(3) a homotopy class of maps $X^\mu \to \Sigma^d H\mathbb{Z}$ of spectra such that the composition

$$\Sigma^d \mathbb{S} \xrightarrow{\iota_x} X^\mu \to \Sigma^d H\mathbb{Z}$$

is a generator of $[\Sigma^d \mathbb{S}, \Sigma^d H\mathbb{Z}] \cong \mathbb{Z}$ for all $x \in X$.

(4) a homotopy class of maps $X \to BSO(d)$ of spaces classifying the vector bundle μ,

(5) a homotopy class of maps $X^\mu \to MSO(d)$ of spectra classifying the vector bundle μ.

Proof. **1 ⇔ 2:** Let $p : \mu \to X$ denote the projection map. An orientation is a consistent choice of oriented basis in the fibers $p^{-1}(x)$. Each such choice is equivalent to a choice of a generator of the d'th homology group $p^{-1}(x) \cup \{\infty\}$, which is isomorphic to $H^d(p^{-1}(x), \infty; \mathbb{Z})$. The data that this choice of oriented basis is consistent, is equivalent to the fact that these generators patch together to a homology class in $H^d(T(\mu), \infty(\mu); \mathbb{Z}) \cong H^n(\mathrm{Thom}(\mu); \mathbb{Z})$. By definition this class gives a generator of d'th homology group of the fiber, hence is a Thom class.

2 ⇔ 3: By definition of homology groups in terms of a spectra, a class in $H^d(\mathrm{Thom}(\mu); \mathbb{Z})$ is the same as a class in $H^d(X^\mu; \mathbb{Z})$, which is the same as a homotopy class of maps $X^\mu \to \Sigma^d H\mathbb{Z}$. The condition on this map of spectra given in the proposition is nothing but a direct translation of the condition in the definition of the Thom class.

1 ⇔ 4: A d-dimensional vector bundle with orientation is up to isomorphism the same as a homotopy class of a map $f : X \to BSO(d)$. The equivalence goes through the relation $\mu \cong f^*\xi$, where ξ is the canonical bundle $ESO(d) \times_{SO(d)} \mathbb{R}^n$ over $BSO(d)$.

4 ⇔ 5: Any map of spaces $X \to BSO(d)$ induces a map $X^\mu \to MSO(d) = BSO(d)^\xi$ of spectra, where $\mu = f^*\xi$ is an oriented vector bundle. One can find back f again by restricting to $X \subset (X^\mu)_0$, homotoping the image away from $\infty \in (BSO(d)^\xi)_0$ and projecting down to $BSO(d)$.

□

We want to make the link between data (2), (3) and (4), (5) more clear. This is done in the following corollary.

Corollary 4.34. *If $f : X \to BSO(d)$ is a map and $u_\xi \in H^d(\mathrm{Thom}(\xi); \mathbb{Z})$ is a Thom class for the universal bundle, then $f^* u_\xi \in H^d(\mathrm{Thom}(\mu); \mathbb{Z})$ is a Thom class for $f^*\xi$. The homotopy class of maps of spectra for this Thom class is given by the composition $X^\mu \to MSO(d) \to \Sigma^d H\mathbb{Z}$, where the last map corresponds to u_ξ.*

Proof. The Thom class $u_\xi \in H^d(\mathrm{Thom}(\xi); \mathbb{Z})$ corresponds to a map $MSO(d) \to \Sigma^d H\mathbb{Z}$. The pullback $f^* u_\xi \in H^d(\mathrm{Thom}(f^*\xi); \mathbb{Z})$ corresponds to the composition $X^\mu \to MSO(d) \to \Sigma^d H\mathbb{Z}$. To check that this is a Thom class, it suffices to note that for an inclusion of a compactified fiber $\iota_x : \Sigma^d \mathbb{S} \to X^\mu$ the following diagram commutes:

$$\begin{tikzcd}
\Sigma^d \mathbb{S} \arrow[r, "\iota_x"] \arrow[dr, "\iota_{f(x)}"'] & X^\mu \arrow[d] \arrow[dr] & \\
& MSO(d) \arrow[r] & \Sigma^d H\mathbb{Z}
\end{tikzcd}$$

and hence the composition $\Sigma^d \mathbb{S} \xrightarrow{\iota_x} X^\mu \to \Sigma^d H\mathbb{Z}$ represents a generator if and only if $\Sigma^d \mathbb{S} \xrightarrow{\iota_{f(x)}} MSO(d) \to \Sigma^d H\mathbb{Z}$ is. □

In this section we discuss orientations over $\mathbb{Z}$. One can make analoguous definition with any ring R in the place of $\mathbb{Z}$. For example, if $R = \mathbb{Z}_2$ then it will turn out that every vector bundle is oriented.

1.3.5. *The Thom isomorphism.* Now that we know what orientations are and that oriented vector bundles have a Thom class, we will prove that capping with the Thom class induces an isomorphism $H_*(X^\mu; \mathbb{Z}) \to H_{*-d}(X; \mathbb{Z})$. For this we use the following generalization of the Serre spectral sequence.

Proposition 4.35. *Let $E \to B$ and $E' \to B$ be fibrations with fibers F and F' respectively and R a ring. Furthermore, suppose that $i : E' \to E$ is a cofibration over identity map on B. Then there exist a multiplicative Serre spectral sequence*

$$H^*(B, \mathcal{H}^*(F, F'; R)) \Rightarrow H^*(E, E'; R)$$

and a dual Serre spectral

$$H_*(B, \mathcal{H}_*(F, F'; R)) \Rightarrow H_*(E, E'; R)$$

We first look at the universal case of the canonical vector bundle ξ over the space $BSO(n)$.

Proposition 4.36. *There is an isomorphism $H^*(\mathrm{Thom}(\xi)) \to H^{*-d}(BSO(d))$ induced by capping with a Thom class u_ξ. There is also an isomorphism $H_*(\mathrm{Thom}(\xi)) \to H_{*-d}(BSO(d))$ induced by capping with u_ξ.*

Proof. Take $B = BSO(d)$, $E = T(\xi)$ the fiberwise one-point compactification of μ, a vector bundle of dimension n, $E' = \infty(\xi)$ the section at infinity and $i : E' \to E$ the inclusion map. We let $R = \mathbb{Z}$ Because μ is oriented, the local system $\mathcal{H}_*(F, F')$ is trivial and hence isomorphic to the constant local system $H_*(F, F')$. Now note that $F = S^n$ and $F' = pt$, and hence $H_*(F, F')$ is $\mathbb{Z}$ in degree n and 0 in all other degrees. Hence the E^2-page of the spectral sequence is localized in a row at vertical degree n. The same holds for the cohomology spectral sequence.

Hence both spectral sequences collapse at the E^2-page and we obtain that $H_*(T(\xi), \infty(\xi)) \cong H_{*-d}(BSO(d))$ and $H^*(T(\xi), \infty(\xi)) \to H^{*-d}(BSO(d); R)$. Now note that by collapsing the section $\infty(\mu)$ at infinity we have that $H_*(T(\xi), \infty(\xi)) \cong H_*(\mathrm{Thom}(\xi))$ and $H^*(T(\xi), \infty(\xi)) \cong H^*(\mathrm{Thom}(\xi))$ and we are done.

Using the multiplicativity of the cohomology spectral sequences, it follows automatically that the isomorphism is induced by cupping with a generator $u_\xi \in H^d(\mathrm{Thom}(\xi); \mathbb{Z})$ of the group at at $E_2^{d0} \cong \mathbb{Z}$. By using the duality of the cohomology spectral differentials with the homology spectral sequence ones, we see that capping with a generating cohomology class u_ξ of the cohomology spectral sequence induces this isomorphism.

To show that u_ξ is a Thom class we use the inclusion $p^{-1}(x)\cup\infty \to \mathrm{Thom}(\xi)$, where $p^{-1}(x)\cup\infty$ has base space $\{x\}$, and naturality. It is clear that $p^{-1}(x)\cup\infty$ is a S^d and hence the E_2-page is consists of a single $\mathbb{Z}$ at $(d,0)$. A generator of this induces a generator of the fiber. But by naturality this can be chosen to be the pullback of u_ξ. □

This proof also works with a general ring R in place of $\mathbb{Z}$. The other case that occurs often is $\mathbb{Z}_2$, because then every vector bundle is orientable. We can translate our results about the Thom isomorphism in homology to spectra using a variation of the homology Whitehead theorem.

Corollary 4.37. *The Thom class u_ξ induces a weak equivalence of spectra*

$$MSO(d)\wedge H\mathbb{Z} \to \Sigma^\infty BSO(d)_+ \wedge \Sigma^d H\mathbb{Z}$$

Proof. The map $\alpha \mapsto \alpha \cap u_\xi$ is known to be an isomorphism in homology by the previous proposition. By lemma A.32 it is induced by the following map of spectra:

$$\begin{aligned} MSO(d)\wedge H\mathbb{Z} &\to \Sigma^\infty BSO(d)_+ \wedge MSO(d)\wedge H\mathbb{Z} \\ &\to \Sigma^\infty BSO(d)_+ \wedge \Sigma^d H\mathbb{Z}\wedge H\mathbb{Z} \to \Sigma^\infty BSO(d)_+ \wedge \Sigma^d H\mathbb{Z} \end{aligned}$$

where the first map is induced by the isomorphism $\xi \oplus \mathbb{R}^0 \cong \xi$, of vector bundles, the second map corresponds to the Thom class and the third map uses the ring spectrum structure on $H\mathbb{Z}$. Because all these spectra are bounded below CW spectra, the homology Whitehead theorem for spectra applies. Here it is theorem A.29. □

We now prove that this universal case in fact gives us all Thom isomorphisms.

Theorem 4.38. *If μ is an oriented vector bundle of dimension d, then there is a weak equivalence $X^\mu \wedge H\mathbb{Z} \to \Sigma^\infty X_+ \wedge \Sigma^d H\mathbb{Z}$ of spectra, corresponding to taking the cap product with a Thom class u_μ.*

Proof. Fix a Thom class u_ξ for the canonical bundle over $BSO(d)$ and hence an weak equivalence $MSO(d)\wedge H\mathbb{Z} \to \Sigma^\infty BSO(d)_+ \wedge \Sigma^d H\mathbb{Z}$. Let $f : X \to BSO(d)$ be a map classifying μ. The taking the cap product with $u_\mu = f^* u_\xi$ is induced by a map of spectra, where the map $X^\mu \to MSO(d)$ corresponds to f:

$$X^\mu \wedge H\mathbb{Z} \to \Sigma^\infty X_+ \wedge X^\mu \wedge H\mathbb{Z} \to \Sigma^\infty X_+ \wedge MSO(d)\wedge H\mathbb{Z} \to \Sigma^\infty X_+ \wedge \Sigma^d H\mathbb{Z}$$

where the last map is given by the weak equivalence in the previous corollary. But looking at the spectral sequence for $(T(\mu),\infty(\mu))$ and using naturality, we see that capping with f^*u_ξ is be an isomorphism and hence this map a weak equivalence of spectra by applying the homology Whitehead theorem A.29. That the pullback of u_ξ is a Thom class was already proven in corollary 4.34. □

Corollary 4.39. *If μ is an oriented vector bundle over X of dimension d, there is an isomorphism $H_*(\mathrm{Thom}(\mu)) \to H_{*-d}(X)$ or equivalently an isomorphism $H_*(X^\mu) \to H_{*-d}(\Sigma^\infty X_+)$.*

Proof. Simply apply homology to the previous weak equivalence of spectra to obtain the second statement, which is equivalent to the first statement. □

The second of these two equivalent statements is in fact the one that will generalize to virtual bundles.

Corollary 4.40. *If $\mu_+ - \mu_-$ is an oriented virtual bundle over X of dimension d, then there is an weak equivalence $X^{\mu-\lambda}\wedge H\mathbb{Z} \to X_+ \wedge \Sigma^d H\mathbb{Z}$ of spectra.*

Proof. Let μ_+ be of dimension m and μ_- of dimension l. Suppose that $\mu_- \oplus \nu = \mathbb{R}^n$. Write $X^{\mu_+ - \mu_-}$ as $\Sigma^{-n} X^{\mu_+\oplus\nu}$ and apply previous theorem to get

$$\Sigma^{-n} X^{\mu_+\oplus\nu} \wedge H\mathbb{Z} \to \Sigma^{-n} X_+ \wedge \Sigma^{k+n-l} H\mathbb{Z} \to X_+ \wedge \Sigma^{k-l} H\mathbb{Z}$$

where both morphisms are weak equivalences. Hence this map is a weak equivalence itself. □

Corollary 4.41. *If $\mu_+ - \mu_-$ is an oriented virtual bundle over X of virtual dimension d, there is an isomorphism $H_*(X^{\mu_+ - \mu_-}) \to H_{*-d}(\Sigma^\infty X_+)$.*

Proof. Simply apply homology to the weak equivalence in the previous corollary. $\square$

Remark 4.42. We have proven the Thom isomorphism for spectra using spectral sequences and the homology Whitehead theorem for spectra. There is an alternative proof which gives more insight when generalizing to unorientable vector bundles. This uses parametrized spectra as explained about in appendix 1. The idea is to start with the sphere bundle $S(\mu)$ over X. We can take its fiberwise suspension to obtain a parametrized spectrum $\Sigma^\infty_X S(\mu)_+$. Fiberwise smash with $H\mathbb{Z}$, then in each fiber we have a spectrum $\Sigma^\infty_+ S^d \wedge H\mathbb{Z} \simeq \Sigma^d H\mathbb{Z}$. If μ was oriented, then the latter gives a trivial parametrized spectrum with fiber $H\mathbb{Z}$ over X. Thus there is a fiberwise weak equivalence $\Sigma^\infty_X S(\mu)_+ \to \Sigma^d \mathbb{HZ}$ over X. Passing to homotopy classes of sections we again obtain the Thom isomorphism in homology.

1.3.6. *The Thom isomorphism for unoriented vector bundles.* In our construction of the string operations we will unfortunately come across vector bundles which are not orientable. In this section we describe how parametrized spectra can be used to give an analogue of the Thom isomorphism theorem. The idea is to generalize the construction above using parametrized spectra to unorientable vector bundles.

However, we first give a proof which is a corollary of the results in the last section. To apply these results we need to make our vector bundle oriented. There are two equivalent approaches to doing this, essentially corresponding to the different models for local systems as explained in 2 of appendix A.

(1) Take a cover of your space such that the vector bundle becomes orientable, apply the Thom isomorphism theorem and take a quotient by the deck transformations again.
(2) Tensor your vector bundle with a line bundle such that the vector bundle becomes orientable, apply the Thom isomorphism theorem and after absorb the line bundle in homology by taking local coefficients.

Suppose that μ is d-dimensional, then the vector bundle one should use is the orientation bundle $\mathbb{L} = \Lambda^d \mu$ and the double cover $\tilde{X}$ is that coming from the two connected components of the complement of the zero section of $\Lambda^d \mu$. Let's prove that these indeed make the vector bundle orientable.

Lemma 4.43. *We have that*

(1) the vector bundle $\mu \otimes \mathbb{L}$ over X is orientable
(2) that the pullback of μ to $\tilde{X}$ is orientable.

Proof. Both claims are almost trivial. For the first, an ordered set of basis vectors $v_i \otimes \lambda_i$ for a fiber of μ is said to be orientable if $v_1 \wedge \cdots \wedge v_d$ lies in the same component of the complement of the zero section of $\Lambda^d \mu$ as $\lambda_1 \cdot \ldots \cdot \lambda_d$. This gives a consistent notion of orientation.

For the second, note that the fiber of $\tilde{X}$ over a point of x exactly contains the information for an orientation for μ. Thus a consistent notion of orientation for μ over $\tilde{X}$ is not hard to obtain. $\square$

We can now already define the Thom isomorphism in the case of a double cover. Let's smash $H\mathbb{Z}$ against the Thom spectrum $\tilde{X}^{\tilde{\mu}}$ of the pullback vector bundle $\tilde{\mu}$ of μ over $\tilde{X}$. Then we get a map

$$\tilde{X}^{\tilde{\mu}} \wedge_{\tilde{X}} \mathbb{HZ} \to \Sigma^d \mathbb{HZ}$$

by applying the Thom isomorphism. Both sides can be seen as a trivial parametrized spectrum. In that case there is a $\mathbb{Z}_2$ action on both sides coming coming from the deck transformation of the double cover. The Thom isomorphism is clearly equivariant for this. Taking the quotient we thus get a map

$$X^\mu \wedge_X \mathbb{HZ} \to \Sigma^d \mathbb{HZ}_{\mathbb{L}^{-1}}$$

where the former is a trivial parametrized spectrum but the latter is not. This could be seen a implementation of the Thom isomorphism for unorientable vector bundles. However, we'd rather twist the domain instead of the codomain. This can be done by using the associated bundle construction, as explained in appendix 1. The result is a map

$$X^{\mu} \wedge_X \mathbb{HZ}_{\mathbb{L}} \to \Sigma^d \mathbb{HZ}$$

where $\mathcal{L}$ is local system representing a grading-preserving local system. Alternatively, by moving the d-fold suspension to the right hand side, one may consider this as a map

$$X^{\mu} \wedge_X \mathbb{HZ}_{\mathcal{L}} \to \mathbb{HZ}$$

where $\mathcal{L}$ is a full local system.

This formulation is also what one arrives at if one using the orientation vector bundle method. After tensoring with the $\Lambda^d \mu$, we get a Thom isomorphism:

$$X^{\mu \otimes \Lambda^d \mu} \wedge_X \mathbb{HZ} \to \Sigma^d \mathbb{HZ}$$

but the $X^{\mu \otimes \Lambda^d \mu} \cong X^{\mu} \wedge_X X^{\Lambda^d \mu}$. But the latter term naturally maps to a bundle with fiber $gl_1(H\mathbb{Z})$, hence we can twist $H\mathbb{Z}$ by it. The result is a map

$$X^{\mu} \wedge_X \mathbb{HZ}_{\mathbb{L}} \to \Sigma^{-d} \mathbb{HZ}$$

We will not prove this, but one can show that these two constructions give the same result.

Remark 4.44. These constructions make more sense in light of the proof of the Thom isomorphism as a map between parametrized spectra. Recall that for oriented vector bundles this was a fiberwise weak equivalence $\Sigma^{\infty}_X S(\mu)_+ \wedge_X H\mathbb{Z} \to \Sigma^d \mathbb{HZ}$ where the latter is a trivial parametrized spectrum. Exactly the same construction works for unorientable vector bundles with the modification that $\Sigma^d \mathbb{HZ}$ should be replaced with the non-trivial parametrized spectrum $\Sigma^d \mathbb{HZ}_{\mathcal{L}^{-1}}$. This is the proof of the Thom isomorphism as given in [**MS06**].

In contrast to the Thom isomorphism for oriented vector bundles, we are now dealing with an additional local system $\mathcal{L}$. What does it remember about the vector bundle μ? The previous construction in fact shows that it remembers the determinant bundle $\det(\mu)$, which is the graded line bundle Λ^d of degree d.

All of these constructions generalize to virtual bundles $\mu_+ - \mu_-$. In that case the twist again depends only on the determinant bundle, which we will define in full generality now.

Definition 4.45. Let $\mu = \mu_+ - \mu_-$ be a virtual bundle with components of dimension d_+ and d_- respectively. Then the determinant bundles is the graded line bundle given by the 1-dimensional vector bundle

$$\det(\mu) = \Lambda^{d_+} \mu_+ \otimes \Lambda^{d_-} \mu_-^*$$

in degree $d_+ - d_-$.

One can find more about determinants of virtual bundles and graded line bundles in section 6.1.

2. Umkehr map for oriented manifolds

In the previous section we described all the technical tools needed to define umkehr maps for homology as described in the introduction to this chapter.

2.1. The Thom collapse map. Finally, we will describe a method to obtain maps from a space to a Thom space. We will assume that our spaces are compact manifolds and we will construct such maps from tubular neighborhoods of embeddings.

Let N, M be compact manifolds of dimension n and m respectively and let $i : N \to M$ be a embedding with normal bundle ν. Let $f : \nu \to M$ be a tubular neighborhood.

Definition 4.46. The Thom collapse map associated to i is the composition

$$\bar{i} : M \to M/(M - f(\nu)) \to \mathrm{Thom}(\nu)$$

where the map is the quotient map and the second is a homeomorphism.

Using the contractibility of the space of tubular neighborhoods if N is compact, we can prove the following result.

Proposition 4.47. *Let N be compact, then any two Thom collapse maps associated to i are homotopic.*

Proof. Use that the space of tubular neighborhoods is contractible, hence path-connected to construct a Thom collapse map for $M \times I \to \mathrm{Thom}(\nu)$ which restricts to the two Thom collapse on $M \times \{0\}$ and $M \times \{1\}$. This gives the required homotopy. $\square$

Of course, the contractibility also tells us that every two homotopies are themselves homotopic, etc. Note that by looking at suspensions of the Thom collapse map, we can also regard it as a morphism $\Sigma^\infty M_+ \to N^\nu$ of spectra.

2.2. Umkehr maps. We are now finally able to define the umkehr maps in singular homology and we will also investigate their dependence on certain choices. The context is as follows: let M be a compact oriented manifold of dimension m and N be a compact oriented manifold of the dimension n. Furthermore, suppose that we are given an embedding $i : N \hookrightarrow M$. This is an embedding of codimension $d = m - n$. To define the umkehr map we need to fix an convention on the orientation of the normal bundle ν. We say that the fiber of ν at x is positively oriented if the orientations of $\nu_x \oplus TN_x$ and TM_x coincide.

Theorem 4.48. *For each tubular neighborhood $f : \nu \to M$ there exists a umkehr map $i^!(f) : H_*(M) \to H_{*-d}(N)$. This map is independent of the choice of tubular neighborhood.*

Proof. Define $i^!$ to be composition of the map $\bar{i}_* : H_*(M) \to H_*(\mathrm{Thom}(\nu))$ induced in homology by the Thom collapse map with the Thom isomorphism $H_*(\mathrm{Thom}(\nu)) \to H_{*-d}(N)$ with the orientation of ν as defined above.

The choice of orientation uniquely determines the Thom class, so the only choice is that of a tubular neighborhood. However, any two tubular neighborhoods induce homotopic Thom collapse maps, which therefore give the same map in homology. $\square$

Definition 4.49. Since there always exists a tubular neighborhood and the umkehr map $i^!(f)$ is in fact independent of the choice of f. For any choice of tubular neighborhood, we call this map the umkehr map and denote it by $i^!$.

This theorem did not use the language of spectra. However, remark that alternatively we could have defined $i^!$ to be composition $H_*(\Sigma^\infty M_+) \to H_*(N^\nu) \to H_{*-d}(\Sigma^\infty N_+)$. Spectra only become necessary when we are dealing with virtual bundles, i.e. in the case that instead of an embedding $i : N \hookrightarrow M$, we have an embedding $i : N \to \mu$ where μ is a vector bundle over M.

Suppose that μ has dimension m. Then note that we get a morphism $\Sigma^\infty \mu_+ \to N^\nu$ which is such that sufficiently far away from the zero section in μ points are sent to ∞. This means that this morphism induces a morphism $M^\mu \to N^\nu$. Let η be a vector bundle over M such that $\mu \oplus \eta \cong \mathbb{R}^n$. Then the morphism $M^\mu \to N^\nu$ induces a morphism $M^{\mu\oplus\eta} \to N^{\nu\oplus i^*\eta}$ up to weak equivalence.

Anyway, this map is naturally the same up to weak equivalence as a morphism of spectra $\Sigma^\infty M_+ \to \Sigma^{-n} N^{\nu\oplus i^*\eta}$. But now note that the latter spectrum is exactly the Thom spectrum associated to the virtual bundle $\nu - i^*\mu$. Therefore, the following theorem follows:

Theorem 4.50. *For each tubular neighborhood $f : \nu \to \mu$ there exists a umkehr map $i^! : H_*(M) \to H_{*+m-d}(N)$, which is independent of the choice of tubular neighborhood.*

Proof. Let $i^!$ to be the composition of $H_*(M) \cong H_*(\Sigma^\infty M) \to H_*(N^{\nu - i^*\mu})$ with the Thom isomorphism for virtual bundles. $\square$

2.3. Correspondences and push-pull constructions. The previous construction of umkehr maps for oriented manifolds actually allows one to extend the functoriality of homology from the category Mfd^{or} of oriented manifolds to the category $\mathsf{Corr}(\mathsf{Mfd}^{or})$ of correspondences of oriented manifolds.

Definition 4.51. Let C be a category, then the category $\mathsf{Corr}(\mathsf{C})$ of correspondences in C is category which has as the set of a finite sequences of morphisms in C

$$X \leftarrow X_1 \rightarrow X_2 \leftarrow \ldots \rightarrow Y$$

modulo to the equivalence relation which replaces an invertible arrow with its inverse, reverses the direction composes with the arrows next to it. A morphism from $X \leftarrow X_1 \rightarrow X_2 \leftarrow \ldots \rightarrow Y$ to $X' \leftarrow X_1' \rightarrow X_2' \leftarrow \ldots \rightarrow Y'$ only exists if $X = X'$ is then given by an equivalence of class of finite sequences $Y \leftarrow Y_1' \rightarrow Y_2' \leftarrow \ldots \rightarrow Y'$ such that its concatenation with $X \leftarrow X_1 \rightarrow X_2 \leftarrow \ldots \rightarrow Y$ is $X \leftarrow X_1' \rightarrow X_2' \leftarrow \ldots \rightarrow Y'$. Therefore, think of morphisms as commutative triangles.

Composition of morphisms is by concatenation. The identity morphism of an object which has Y as codomain of the final morphism is given by the equivalence class of

$$Y \overset{id_Y}{\leftarrow} Y \overset{id_Y}{\rightarrow} Y$$

Note that C embeds into $\mathsf{Corr}(\mathsf{C})$ by sending an object X in C to the class of the equivalence class of the sequence $X \overset{id_X}{\leftarrow} X \overset{id_X}{\rightarrow} X$ in $\mathsf{Corr}(\mathsf{C})$ and sending a morphism $f : X \rightarrow Y$ to the equivalence class of

$$X \overset{id_X}{\leftarrow} X \overset{f}{\rightarrow} Y$$

The reason for the relative difficulty of the definition of this category is that push-pull constructions like the umkehr map construction give arrows whose codomain doesn't just depend on the codomain of the final morphism in a sequence but on the entire sequence.

A strict push-pull extension is an extension of a functor $F : \mathsf{C} \rightarrow \mathsf{D}$ to a functor $\bar{F} : \mathsf{Corr}(\mathsf{C}) \rightarrow \mathsf{D}$. The previous results of this section imply the following proposition.

Proposition 4.52. *The umkehr map construction gives a strict push-pull extension of the homology functor* $H_*(-;\mathbb{Z}) : \mathsf{Mfd}^{or} \rightarrow \mathsf{GrAbgrps}$ *to* $\mathsf{Corr}(\mathsf{Mfd}^{or})$.

PROOF. Use the ordinary induced map in homology for arrows pointing rightwards and the umkehr map construction for arrows pointing leftwards. □

For this reason the construction of umkehr maps is sometimes known as a push-pull construction. We would like to define something similar for slightly more general phenomena: it usually happens that some of the objects need to be slightly modified to make the push-pull extension well defined and thus $\bar{F}$ doesn't strictly extend F, but only up to a natural isomorphism or weak equivalence.

Definition 4.53. A push-pull extension for a functor $F : \mathsf{C} \rightarrow \mathsf{D}$ is a functor $\bar{F} : \mathsf{Corr}(\mathsf{C}) \rightarrow \mathsf{D}$ together with a natural isomorphism η from the functor $\bar{F}|_{\mathsf{C}}$ to the functor F on the subcategory C of $\mathsf{Corr}(\mathsf{C})$.

If D is a category with a notion of homotopy like a model category, then a homotopy push-pull extension for a functor $F : \mathsf{C} \rightarrow \mathsf{D}$ is such a extension where η is only required to be a natural weak equivalence.

The reason for this definition is of course that this is exactly what happens when we stay on the level of spectra instead of passing to homology.

Corollary 4.54. *The umkehr map construction gives a homotopy push-pull construction for suspension functor* $\Sigma^\infty : \mathsf{Mfd}^{or} \rightarrow \mathsf{Spectra}$ *to* $\mathsf{Corr}(\mathsf{Mfd}^{or})$.

PROOF. Again use the standard construction for the arrows pointing rightwards and the umkehr map construction for arrows pointing leftwards, but remember to keep track of the umkehr data by adding this to the spectrum in your domain. □

3. Umkehr maps for spaces of maps into oriented manifolds

In the previous section we saw how to define umkehr maps for embeddings of manifolds. The main goal of this section is to define umkehr maps in the more advanced case of a space of maps into a oriented manifold.

It is known that such an umkehr map exists in the more general case of an embedding of "infinite dimensional manifolds" of finite codimension. In this case, heuristically the normal bundle is finite dimensional and we can still apply the reasoning of the last section. To make this precise it turns out that one needs to look at Hurewicz fibrations over oriented manifolds. This is not the path that we will pursue here.

The reason for this is that for the constructions in later chapters we do not just need the existence of the umkehr map, but we need to have a lot of control over these constructions on the level of spectra as well. To get this we use the technique of propagating flows to obtain a well-controlled construction of umkehr maps for the spaces of maps into an oriented manifold.

3.1. Umkehr maps for mapping spaces using propagating flows. As said before, in general an umkehr map can be constructed for pullbacks of an embedding of oriented finite-dimensional manifolds along a Hurewicz fibration but this construction is difficult to control. It turns out that in our special case, we can use the theory of differential equations to construct tubular neighborhoods. This is known as the technique of propagating flows [**God07**, section 3.1], which refers to [**Sta05**] as the source of this idea.

3.1.1. *Motivating the notion of propagating flows.* We will now motivate propagating flows from the perspective of string topology. This means that we will not be proving the statements, but merely try to give the intuition for this construction. Later on, we will make the statements precise.

The simplest case in which this construction works is the following. Suppose that we have a graph or fat graph Γ and a oriented compact manifold M. We look at the space M^Γ of continuous maps from $|\Gamma|$ to M with the compact open topology. If $v(\Gamma)$ denotes the graph obtained by removing all edges but keeping the vertices, then we have a map $\mathrm{ev} : M^\Gamma \to M^{v(\Gamma)}$ by precomposition with the inclusion $|v(\Gamma)| \to |\Gamma|$. Now suppose that we have a map $s : \Gamma' \to \Gamma$ of graphs which identifies certain vertices, then this induces a map $\sigma = M^s : M^\Gamma \to M^{\Gamma'}$ and furthermore we get a pullback diagram:

$$\begin{array}{ccc} M^\Gamma & \xrightarrow{\sigma} & M^{\Gamma'} \\ {\scriptstyle \mathrm{ev}}\downarrow & & \downarrow{\scriptstyle \mathrm{ev}'} \\ M^{v(\Gamma)} & \xrightarrow[\sigma_v]{} & M^{v(\Gamma')} \end{array}$$

where σ_v is the restriction of σ the vertices. It is know that ev' is a Hurewicz fibration and it is clear that $\sigma_v : M^{v(\Gamma)} \to M^{v(\Gamma')}$ is an embedding of compact oriented finite-dimensional manifolds. Hence there exists an umkehr map $\sigma^!$ in homology.

However, to get a well-controlled umkehr map, we want to make this construction more explicit. To be precise, if ν denotes the normal bundle for the embedding σ_v and $f : \nu \to M^{v(\Gamma')}$ is a tubular neighborhood, then we want to construct a homeomorphism between $\mathrm{ev}^*\nu$ and $(\mathrm{ev}')^{-1}(f(\nu))$. Let for each vertex $v \in v(\Gamma')$ the map $f_v : \nu \to M$ denote the v component of f.

The idea is as follows. A tubular neighborhood is nothing but a well-defined way to move in your space along the normal direction. Therefore, we want to move our maps $|\Gamma'| \to M$ in normal directions. We already know how to move the values of these maps at the vertices in normal directions, because the tubular neighborhood f tells us how to do this. Therefore, it suffices to extend these instructions to all points on the interior of edges. Now, on the interior of the edge these maps are essentially unconstrained and hence this boils down to smoothly extending the instructions to a small neighborhood of the vertices. We will do this using the flow of a differential equation.

We will construct a map $\phi : \mathrm{ev}^*\nu \to M^\Gamma$ which is a homeomorphism onto its image. To do this we must for each map $g \in \sigma(M^\Gamma)$ find a way to flow a $g(x) \in g(|\Gamma|)$ for a point $x \in |\Gamma|$ near a vertex v along a "normal direction" in M, in a way that depends smoothly on the point x and the normal direction $n \in \nu|_{\mathrm{ev}(f)}$. In particular, we first look at the case where x is equal to a vertex v. If $n \in \nu|_{\mathrm{ev}(g)}$, then we should have a path $\gamma : [0,1] \to M$ such that $\gamma(0) = g(v)$ and $\gamma(1) = f_v(n)$. Furthermore, it makes sense to try and demand that this path γ stays within the image $f(\nu)$ of the tubular neighborhood.

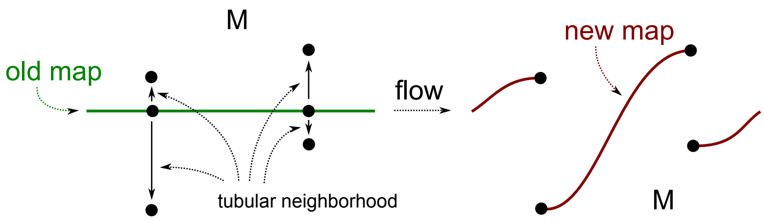


FIGURE 4.55. A linear graph is mapped into a manifold M. We want to map the graph cut along the two vertices into the manifold. An element from the normal bundle tells us through the tubular neighborhood where to map the vertices and the propagating flow interpolates smoothly.

If we want γ to arise from a flow, then there should be a vector field $Z_{n,v}$ on M such that γ solves the differential equation $\frac{d}{dt}\gamma(t) = Z_{n,v}(\gamma(t))$ with initial condition $\gamma(0) = g(v)$. Because we want γ to stay within $f(\nu)$ and we only need to move points close to v apart, we can try to pick to pick the vector field $Z_{n,v}$ to have in $f(\nu)$. Using the tubular neighborhood, we can then assume that it comes from a compactly supported vector field $\mathcal{Z}_{n,v}$ on ν. Furthermore, we want this flow to move only in normal direction, so $\mathcal{Z}_{n,v}$ should be a vertical vector field: if $\pi : \nu \to M^{v(\Gamma)}$ denotes the projection, then $T\pi(\mathcal{Z}_n) = 0$.

So the question is whether it is possible to write down a compactly supported vertical vector field $\mathcal{Z}_{n,v}$ on ν such that it indeed has solutions like γ. The answer is yes and can be found as follows. We first note that similarly to the fact that the tangent bundle $T\mathbb{R}^n$ is isomorphic to $\mathbb{R}^n \times \mathbb{R}^n$, it is true that $T\nu$ is naturally isomorphic to $\pi^*\nu$, where $\pi : \nu \to M^{v(\Gamma)}$ is the projection of the normal bundle to it base.

In the normal bundle ν, the path γ should go from $\mathrm{ev}(g)$ in the zero section to n_v, the vector in $\nu|_{\mathrm{ev}(g)}$ where only the v-component of n is non-zero. So we should essentially look for a smooth family $\mathcal{Z}_{n,v}$ of compactly supported vector fields on ν such that the flow along $\mathcal{Z}_{n,v}$ after time 1 of the point $\mathrm{ev}(g)$ on the 0-section is n_v. We see that this condition is satisfied if we take $\mathcal{Z}_{n,v}$ any compactly supported vector field such that on the line segment between the zero-section $\mathrm{ev}(g)$ and n_v in the fiber $\nu|_{\mathrm{ev}(g)} = \pi^{-1}(\mathrm{ev}(g))$ it is n_v.

Thus we have now given a way to determine the value of $\phi(n, g) : |\Gamma| \to M$ on the vertices of the graph not by directly using the tubular neighborhood, but using the flow of a vector field. The idea is then to use the fact that all edges of a graphs have a natural parametrization to smoothly flow points on the interior of the edges as well. This parametrization is given by the fact that each edge is composed of a single interval $[0, 1]$. More precisely, it consists of two intervals $[0, \frac{1}{2}]$ glued, but if as long as our constructions are symmetric under the maps $s \mapsto 1 - s$, we can identity the edge in $|\Gamma|$ with $[0, 1]$.

So let e be an edge between vertices v_1, v_2. Let (e, s) for $s \in (0, 1)$ be a point in the interior of the edge in $|\Gamma|$. Since the vector fields $\mathcal{Z}_{n,v}$ are compactly supported, we can extend them by zero to obtain vector fields $Z_{n,v}$ on M. Then we define $\phi(n, g)(e, t)$ to be the value of ξ at $t = 1$, where ξ is the solution to the differential equation

$$\frac{d\xi}{dt} = sZ_{n,v_1}(\xi(t)) + (1 - s)Z_{n,v_2}(\xi(t))$$

with the initial condition $\xi(0) = g(e, s)$. See figure 4.55 for an illustration of this idea.

In the rest of this section we will make this construction precise and investigate to what extent it depends on the choices of tubular neighborhood and the vector fields $\mathcal{Z}$.

3.1.2. *The space of propagating flows.* We start by fixing a notation for the set of compactly supported vertical vector fields on a vector bundle over a manifold that appear in our construction. We also topologize this set in a natural way.

Definition 4.56. Let λ be a vector bundle on a manifold M, then we denote by $\chi_{v,c}(\lambda)$ the space of compactly supported vertical vector fields on λ. This is given the subspace topology of the space of smooth sections of the tangent bundle $T\lambda$ with the C^∞-topology.

Recall in our earlier discussion of the motivation behind this construction we said that we wanted compactly supported vertical vector fields that are constant on certain line segments. This leads us to the following definition of a propagating flow.

Definition 4.57. Let λ be a vector bundle over manifold M. Then a *propagating flow* for λ is a continuous map $\mathcal{Z} : \lambda \to \chi_{v,c}(\lambda)$ with the property that for all $v \in \lambda$ and for each point p on the line segment between $\pi(v)$ and v in the fiber $\pi^{-1}(v)$ we have that $\mathcal{Z}(v)(p) = v$.

We let $\mathcal{P}(\lambda)$ be the space of propagating flows of λ, with the subspace topology of the compact-open of maps from λ to $\chi_{v,c}(\lambda)$.

Note that the condition that the vector fields are vertical implies that they cover the identity, i.e. that for each vector field $\mathcal{Z}(v)$ with corresponding flow $\mathfrak{X}_v : \mathbb{R} \to \mathrm{Diff}(M)$ satifies $\pi \circ \mathfrak{X}_v(t)(w) = \pi(w)$ for all $t \in \mathbb{R}$. This implicitly uses that these flows are defined for all times $t \in \mathbb{R}$, which is proposition 4.62.

We want to investigate the existence and the uniqueness of these propagating flows. It turns out that they always exist, but they are not unique. However, we have the next best thing: there is a contractible spaces of choices and thus they are unique up to homotopy. The following two results about the existence and uniqueness of propagating flows appear in [**God07**, proposition 3.4, lemma 3.5].

Proposition 4.58. *The space $\mathcal{P}(\lambda)$ is non-empty, i.e. for each vector bundle λ over a manifold M there exists a propagating flow.*

Proof. We will construct one locally and then paste it together using a partition of unity.

Let $\mathbb{R}^k \times \mathbb{R}^m$ be the trivial k-dimensional vector bundle over $\mathbb{R}^m$. Then we need to construct a continuous map $Z : \mathbb{R}^k \times \mathbb{R}^m \to \chi_{v,c}(\mathbb{R}^k \times \mathbb{R}^m)$ satisfying the condition defining a propagating flow: in this case the condition is that we must have $Z((x_1, x_2))(tx_1, x_2) = x_1$ for all $t \in [0, 1]$.

To construct such Z, we first need to define a function η to control the vector field in the direction of the fibers. Let $\eta : \mathbb{R}^2_{\geq 0} \to [0, 1]$ be smooth function such that

$$\eta(a, b) = \begin{cases} 0 & \text{if } a + 2 < b \\ 1 & \text{if } b \leq a \text{ or } a \leq 1 \end{cases}$$

Furthermore, we need to control the vector field in the direction along $\mathbb{R}^m$. To do this we let $\rho : \mathbb{R}_{\geq 0} \to [0, 1]$ be a compactly supported smooth function such that

$$\rho(a) = \begin{cases} 1 & \text{if } a \leq 1 \\ 0 & \text{if } a > 2 \end{cases}$$

Then we define the vector field $Z((x_1, x_2))$ by

$$Z((x_1, x_2))(y_1, y_2) = \rho(||y_2 - x_2||^2)\eta(||x_1||^2, ||y_1||^2)x_1$$

It is clear that this vector field is vertical and depends smoothly on (y_1, y_2). Due to ρ and η, the support of this vector field is contained in the set $\{(y_1, y_2) |\, ||y_2 - x_2||^2 \leq 2 \text{ and } ||y_1||^2 \leq ||x_1||^2 + 2\}$ and hence must be compact. It satisfies the condition that $Z((x_1, x_2))(tx_1, x_2) = x_1$ for $t \in [0, 1]$ because $\rho(0) = 1$ and $\eta(a, b) = 1$ is $b \leq a$. Finally, we should check that Z is continuous, but this is clear as it depends smoothly on (x_1, x_2). This concludes our local construction.

We now want to patch such local propagating flows together. We now work in the context of a vector bundle λ of dimension k over a manifold M of dimension m. There is a locally finite open cover $\{U_i\}_{i \in I}$ such that there are charts $\phi_i : U_i \to \mathbb{R}^m$ and λ trivializes using $\varphi_i : \lambda|_{U_i} \to \mathbb{R}^k \times U_i$. We define $\psi_i = (id \times \phi_i) \circ \varphi_i$. Let $\{\eta_i\}_{i \in I}$ be a squared partition of unity subordinate to this cover, i.e. the η_i are locally finite and $\sum_{i \in I} \eta_i^2$ is 1 on the open neighborhood $\bigcup_{i \in I} U_i \times U_i$ in $M \times M$.

Then we define a propagating flow $\mathcal{Z}$ to assign to $v \in \lambda$ the vector field

$$\mathcal{Z}(v)(w) = \sum_{i \in I} \eta_i(\pi(v))\eta_i(\pi(w))\psi_i^{-1} \left(Z(\psi_i(v))(\psi_i(w))\right)$$

where we see that all terms which are not defined are multiplied with zero anyway, so this gives no problems. We claim that $\mathcal{Z}(v)$ is a compactly supported smooth vector field. The smoothness is clear and for compact support, it suffices to note that only finite many terms are non-zero for fixed v and these all have compact support. It is a vertical vector field because Z is and ψ_i preserves vertical vector fields. Similarly, it satisfies the condition for a propagating flow, because ψ_i preserves the line between $\pi(v)$ and v. To show that $\mathcal{Z}$ is continuous, we can restrict to the case of a single chart and then it boils down to the fact that a finite sum of continuous functions is continuous. ☐

Now that we know that propagating flows exist, we show that they are essentially unique.

Proposition 4.59. *The space $\mathcal{P}(\lambda)$ is contractible.*

PROOF. We first note that the pointwise addition of vector fields makes $\chi_{v,c}(\lambda)$ into a topological vector space. We claim that $\mathcal{P}(\lambda)$ is a convex subspace of this space. To do this, we note that if $\mathcal{Z}$ and $\mathcal{Z}'$ are propagating flows, then their convex combination $t\mathcal{Z} + (1-t)\mathcal{Z}'$ for $t \in [0,1]$ is a map assigning compactly supported vertical fields to elements v of λ which in a point p on the line segment between $\pi(v)$ and v has value $t\mathcal{Z}(v,p) + (1-t)\mathcal{Z}'(v,p) = tv + (1-t)v = v$. It is well-known that any convex space is contractible. ☐

3.1.3. *The general setup for spaces of maps into manifold.* We will now define a slightly more general setup than the simple case of maps from an unlabelled graph into a manifold, where a propagating flow can still be used to produce a umkehr map. We use the terminology from section 1.2 of chapter 2.

Definition 4.60. Let X be an object of SSComp and M a manifold. Then we let M^X be the space of continuous maps $g : X \to M$ with the compact open topology.

Let $\mathcal{B}$ be a set of proper submanifolds of M and let Y be an object of $\mathsf{SSComp}_{\mathcal{B}}$, where we identify the label $\emptyset$ with the label M. Then we let M^Y be the space of continuous maps $g : Y \to M$ with the property that if $b(y) = B$, then $g(y) \in B$. We give this the subspace topology coming from the inclusion $M^Y \hookrightarrow M^{U(Y)}$. Here $U(Y)$ is the object of SSComp underlying Y (in other words $U : \mathsf{SSComp}_{\mathcal{B}} \to \mathsf{SSComp}$ is the forgetful functor).

For $X \in \mathsf{SSComp}_{\mathcal{B}}$, there exists a continuous map $\mathrm{ev} : M^X \to M^{X_0}$, given by evaluating a $f \in M^X$ at the 0-simplices. Note that the codomain may seem a bit mysterious, but is in fact nothing but a product of submanifolds of M and in fact it is naturally homeomorphic to $\prod_{\alpha \in X_0} b(\alpha)$.

In the case of a 0-regular morphism, these evaluation maps detect essentially all the information about the morphism. This is made precise by the following lemma.

Lemma 4.61. *Let $f : X \to Y$ be a 0-regular morphism and $f_0 : X_0 \to Y_0$ be its restriction to the subspace of 0-simplices. Then the following diagram commutes and is a pullback diagram:*

$$\begin{tikzcd} M^Y \arrow[r, "M^f"] \arrow[d, "\mathrm{ev}_Y"'] & M^X \arrow[d, "\mathrm{ev}_X"] \\ M^{Y_0} \arrow[r, "M^{f_0}"'] & M^{X_0} \end{tikzcd}$$

Furthermore, $M^{f_0} : M^{Y_0} \to M^{X_0}$ is the inclusion given by a product of diagonals; one for every 0-simplex α of Y_0, going from $b(\alpha)$ to $\prod_{\beta \in f_0^{-1}(\alpha)} b(\beta)$.

The previous definition in particular makes sense for objects X of $\mathsf{SSComp}_{0,\mathcal{B}}$ and compact oriented manifolds M. In that case we have that each evaluation map has a compact oriented manifold as it target. Furthermore we see that M^{f_0} is an embedding of finite dimensional manifolds with normal bundle ν and that there is a tubular neighborhood upstairs for M^f. In the next subsection we will use the technique of propagating flows to explicitly construct such a tubular neighborhood given a tubular neighborhood for M^{f_0} and a propagating flow.

3.1.4. *Constructing tubular neighborhoods using propagating flows.* We will now prove the main theorem about tubular neighborhoods for mapping spaces, theorem 4.63. In the proof of the theorem we want to flow along compactly supported smooth vector fields for times $t = 1, -1$, so it is necessary that at least for this interval the flows exist and are unique. This will be a consequence of the general theory as given in the next proposition.

Proposition 4.62. *Let V_n be a compactly supported smooth vector field on a manifold M depending smoothly on a parameter $n \in N$ for N a manifold. Then V_n has a unique flow ϕ_t, defined for all $t \in \mathbb{R}$, and this flow is in fact given by a continuous map $\mathbb{R} \times N \to \mathrm{Diffeo}(M)$ which is a 1-parameter group of diffeomorphisms when restricted to a fixed $n \in N$.*

Proof. The local existence, uniqueness and smooth dependence on initial conditions is a local property and thus is a consequence of the Picard-Lindelöf theorem which says that solutions ordinary differential equations in $\mathbb{R}^n$ of the type $\frac{dv}{dt} = f(v(t))$ locally exist, are unique and depend smoothly on the initial conditions, if f is smooth. But in this case this is implied by smoothness of our vector field.

To get from local existence to global existence, we use a theorem that the flow of a compactly supported vector field is complete. This is true because we can restrict to the support of V, which is compact and every the flow of a vector field on a compact manifold is always complete. All of this implies the existence of the flow for all $t \in \mathbb{R}$. □

For the following theorem, we just need the existence of a propagating flow. However, we always want that our tubular neighborhood is unique up to homotopy. This is guaranteed by the contractibility of the space of propagating flows.

Theorem 4.63. *Let $f : X \to Y$ be a morphism in $\mathsf{SSComp}_{0,\mathcal{B}}$. Let $g : \nu \to M^{X_0}$ be a tubular neighborhood and $\mathcal{Z} \in \mathcal{P}(\nu)$ be a propagating flow. Then we can explicitly construct a homeomorphism $\phi = \phi_{g,\mathcal{Z}} : (\mathrm{ev}_Y)^*\nu \to (\mathrm{ev}_X)^{-1}(g(\nu))$.*

$$\begin{tikzcd}
(\mathrm{ev}_Y)^*\nu \arrow[rr, dotted, "\phi"] \arrow[dr] \arrow[dd] & & (\mathrm{ev}_X)^{-1}(g(\nu)) \arrow[dr] \arrow[dd] & \\
& M^Y \arrow[rr] \arrow[dd, "\mathrm{ev}_Y"'] & & M^X \arrow[dd, "\mathrm{ev}_X"] \\
\nu \arrow[rr, "g"] \arrow[dr] & & g(\nu) \arrow[dr] & \\
& M^{Y_0} \arrow[rr] & & M^{X_0}
\end{tikzcd}$$

Proof. We will construct the flow that we want to use, then construct a candidate for the tubular neighborhood and show that it works by constructing an inverse.

Constructing the flow $\mathfrak{X}$: Without loss of generality we can replace X and Y with their corresponding complexes. Furthermore, because the morphism f is 0-regular, we can identify all i-simplices of Y for $i \geq 1$ with a unique i-simplex of X. Therefore, every point of X can uniquely be given by either an element of X_0 or by an element of X_i for $i \geq 1$ and a $\vec{s} \in \Delta^i$. Similarly every point of Y can uniquely be given by either an element of Y_0 or by an element of X_i for $i \geq 1$ and a $\vec{s} \in \Delta^i$.

Extend $\mathcal{Z}$ to a map $Z : \nu \to \chi_c(M^{X_0})$ using the tubular neighborhood g and extension by 0 outside $\mathrm{im}(g)$, which is well-defined because the vector fields have compact support. For $\alpha \in X_0$, let $Z_\alpha : \nu \to \chi_c(b(\alpha))$ be the vector field obtained by taking only the α-component of Z. We can pick an arbitrary extension of Z_α to a map $\nu \to \chi_c(M)$.

We define a compactly supported vector field on M depending smoothly on $\nu \times X$, in other words a continuous map $\mathfrak{Z} : \nu \times X \to \chi_c(M)$. A point of X can uniquely be written as either $\beta \in X_0$ or a pair $(\beta, \vec{s})$ with $\beta \in X_i$ for $i \geq 1$ and $\vec{s} \in \dot{\Delta}^i$. Write $\beta_0, \ldots, \beta_i$ for the $i+1$ vertices of the i-simplex β and write $\vec{s} = (s_0, \ldots, s_i)$ with $s_0 + \ldots + s_i = 1$. Then we define $\mathfrak{Z}$ by:

$$\mathfrak{Z}(v, x) = \begin{cases} Z_\beta(v) & \text{if } x = \beta \in X_0 \\ \sum_{j=0}^{i} s_j Z_{\beta_j}(v) & \text{if } x = (\beta, \vec{s}) \in X_i \times \dot{\Delta}^i \end{cases}$$

By proposition 4.62, $\mathfrak{Z}$ has a well-defined flow $\mathfrak{X} : \nu \times X \times \mathbb{R} \to \mathrm{Diffeo}(M)$, a continuous 1-parameter group of diffeomorphisms depending smoothly on parameters in ν and X.

Constructing the tubular neighborhood ϕ: We can write an element of $(\mathrm{ev}_Y)^*\nu$ as a pair (v, h), where $v \in \nu|_{\mathrm{ev}_Y(h)}$ and $h \in M^Y$. Then we define $\phi(v, h) : X \to M$ by

$$\phi(v, h)(x) = \mathfrak{X}(v, x, 1)(h(f(x)))$$

Note that if $b(x) \in \mathcal{B}$, then $\phi(v, h)(x) \in b(x)$ is well, because the flow was constructed from a vector field parallel to the submanifold $b(x)$. Therefore we obtain a function $(\mathrm{ev}_X)_*\nu \to M^X$. To show that this is a continuous function we can equivalently show that $(\mathrm{ev}_X)_*\nu \times X \to M$ is continuous. But this is a consequence of the fact that the maps $((v, h), x) \mapsto (v, x, h(f(x)))$ and $(v, x, h(f(x))) \mapsto \mathfrak{X}(v, x, 1)(h(f(x)))$ are continuous.

We haven't used the defining property of a propagating flow yet. This implies that $\mathrm{ev}_X(\phi(v, h)) = g(v)$. This implies that $\mathrm{im}(\phi) \subset \mathrm{ev}_X^{-1}(g(\nu))$.

Constructing the inverse of ϕ: Next we must show that ϕ maps $(\mathrm{ev}_Y)_*\nu$ homeomorphically onto $\mathrm{ev}_X^{-1}(g(\nu))$. To do this, we construct the inverse explicitly. First note that if $h' \in \mathrm{ev}_X^{-1}(g(\nu))$, then $g^{-1}(\mathrm{ev}_X(h'))$ is a well-defined element of ν. We define $\psi(h')$ to be the function $Y \to M$ given by

$$\psi(h')(y) = \mathfrak{X}(g^{-1}(\mathrm{ev}_X(h')), f^{-1}(y), -1)(h'(f^{-1}(y)))$$

A priori this is a function $X \to M$, since f^{-1} is multivalued on Y_0. However, we obtain that for $\beta \in X_0$, $t \mapsto \mathfrak{X}(g^{-1}(\mathrm{ev}_X(h')), \beta, -t)(h'(\beta))$ for $t \in [0, 1]$ is exactly a path between $h'(\beta)$ and a point in $\mathrm{im}((f_0)_*)$. Thus $\psi(h')$ is indeed a well-defined function $Y \to M$. By the same argument as before ψ is continuous.

We claim the inverse of ϕ can be constructed as follows:

$$\phi^{-1}(h') = (g^{-1}(\mathrm{ev}_X(h')), \psi(h'))$$

Clearly this is a continuous map, so now it suffices to show that that $\phi^{-1} \circ \phi = id$ and $\phi \circ \phi^{-1} = id$. But this follows from the group property of the flow and the defining property of a propagating flow. Let's calculate: $\phi^{-1} \circ \phi(v, h)$ has a ν-component $g^{-1}(\mathrm{ev}_X(\phi(v, h)))$, which is equal to v by the propagating flow property, and hence a M^Y-component

$$y \mapsto \mathfrak{X}(v, f^{-1}(y), -1)(\mathfrak{X}(v, x, 1)(h(y)))$$

which is equal to $h(y)$ by the group property of the flow. Therefore $\phi^{-1} \circ \phi = id$.

For the other direction we see that $\phi \circ \phi^{-1}(h')$ is given by

$$x \mapsto \mathfrak{X}(g^{-1}(\mathrm{ev}_X(h')), x, 1)(\mathfrak{X}(g^{-1}(\mathrm{ev}_X(h')), x, -1)(h'(x)))$$

which is easily seen to be equal to h' by the group property of the flow. We conclude that $\phi^{-1} \circ \phi = id$.

$\square$

Remark 4.64. We can be a bit more explicit about the tubular neighborhood ϕ, connecting it to the movitation given earlier and the presentation in the proof of [**God07**, proposition 3.5]. Recall that as in the proof, we can write an element of $(\mathrm{ev}_Y)^*\nu$ as a pair (v, h), where $v \in \nu|_{\mathrm{ev}_Y(h)}$ and $h \in M^Y$. Then we could alternatively define $\phi(v, h)$ as follows.

On a point $\beta \in X_0 \subset X$, then we can either define $\phi(v, h)(\beta) = f(v_\beta)$ or say that it is given by $z(\beta, 1)$, where $z(\beta, t)$ is the unique solution to the differential equation

$$\frac{dz(\beta, t)}{dt} = Z_\beta(z(\beta, t))$$

with initial condition $z(\beta, 0) = h(\beta)$.

Now we similarly define $\phi(v,h)$ on a point $(\beta,\vec{s})$ with $\beta \in X_i$ for $i \geq 1$ and $\vec{s} \in \dot{\Delta}^i$. Write $\beta_0,\ldots,\beta_i$ for the $i+1$ vertices of the i-simplex β and write $\vec{s} = (s_0,\ldots,s_i)$ with $s_0+\ldots+s_i = 1$. Then $\phi(v,h)(\beta,\vec{s})$ is $z((\beta,\vec{s}),1)$ with $z((\beta,\vec{s}),t)$ the unique solution to the differential equation:

$$\frac{dz((\beta,\vec{s}),t)}{dt} = \sum_{j=0}^{i} s_j Z_{\beta_j}(z((\beta,\vec{s}),t))$$

with initial condition $z((\beta,\vec{s}),0) = h((\beta,\vec{s}))$. Since the solutions of these first order ordinary differential equations coincide with the flow lines for the vector fields $\mathfrak{Z}$, we see that this formulation is equivalent. However, it is less useful for showing that all maps are continuous.

3.2. Thom collapse map and umkehr map for spaces of maps into oriented manifolds. Now that we have given an explicit method to construct a tubular neighborhood, we can proceed and define an associated Thom collapse map as before.

The context is as before. We start with an oriented manifold M and a set $\mathcal{B}$ of oriented submanifolds of M. Let $f : X \to Y$ be a 0-regular map of 0-finite $\mathcal{B}$-labelled complexes, then lemma 4.61 tells us that the induced map $f_* : M^Y \to M^X$ is obtained as the pullback of the restriction $(f_0)_* : M^{Y_0} \to M^{X_0}$ along the evaluation maps and that $(f_0)_*$ is an embedding of finite dimensional manifolds, which have induced orientations from the orientations of M and the submanifolds in $\mathcal{B}$. Thus, we can canonically orient the normal bundle of this embedding by saying that ν_x is positively oriented if the orientations of $\nu_x \oplus TN_x$ and TM_x coincide.

We know from theorem 4.63 that f_* admits a tubular neighborhood $g : (ev_Y)^*\nu \to M^X$.

Definition 4.65. The Thom collapse map $\bar{f}_* : M^X \to \mathrm{Thom}((ev_Y)^*\nu)$ for f_* is the map obtained as the composition

$$M^X \to M^X/(M^X - g((ev_Y)^*\nu)) \to \mathrm{Thom}((ev_Y)^*\nu)$$

and this is equivalent to a map of spectra $\Sigma^\infty M^X \to (M^Y)^{(ev_Y)^*\nu}$.

If we use the techniques of propagating flows to construct the tubular neighborhood g, then we see that this map only depends on a choice of element of $\mathrm{Tub}((f_0)_*) \times \mathcal{P}(\nu)$ and this dependence is continuous. Now suppose that M and each element of $\mathcal{B}$ is compact. Then using the fact that this space is then known to be contractible, hence path-connected, we obtain the following proposition.

Proposition 4.66. *If M and each element of $\mathcal{B}$ are compact, then any two Thom collapse maps are homotopic.*

To now construct the umkehr map, it suffices to note that the Thom isomorphism also applies to this situation because $(\mathrm{ev}_Y)^*\nu$ is an oriented finite dimensional vector bundle. If $d = \dim M^{X_0} - \dim M^{Y_0}$, by applying homology we therefore obtain.

Theorem 4.67. *Let M and each element of $\mathcal{B}$ be compact oriented manifolds and $f : X \to Y$ be a 0-regular map of 0-finite complexes. Given tubular neighborhoods for $M^{f_0} : M^{Y_0} \hookrightarrow M^{X_0}$ and a propagating flow on the normal bundle ν of M^{f_0}, there is an umkehr map $(M^f)^! : H_*(M^X) \to H_{*-d}(M^Y)$. This map is in fact independent of the choice of tubular neighborhood and propagating flow.*

We can again put the previous results into the context of correspondences and push-pull constructions, as described in section 2.3 of this chapter.

Proposition 4.68. *The umkehr map construction gives a strict push-pull extension of the functor $H_*(M^-;\mathbb{Z}) : \mathsf{SSComp}_{\mathcal{B},0} \to \mathsf{GrAbgrps}$ to $\mathsf{Corr}(\mathsf{SSComp}_{\mathcal{B},0})$ and a homotopy push-pull extension of the functor $\Sigma^\infty(M^-)_+ : \mathsf{SSComp}_{\mathcal{B},0} \to \mathsf{Spectra}$.*

4. Properties of umkehr maps

What have concluded in the last two sections is that in certain situations we can define umkehr maps on the level of Thom spaces or more generally spectra. These frameworks are equivalent as long as we are not working with virtual bundles.

Two cases will in particular be important to us and in our proofs we will need to use certain properties of umkehr maps in these cases. We will assume that we have fixed universal Thom classes. Let's start by summarizing the results of the previous section for the two special cases. We will treat the Thom space and Thom spectrum cases in parallel where possible.

(1) Let M, N be oriented manifolds and N compact. Suppose that $\iota : N \to M$ is an embedding with normal bundle ν. Then we have umkehr maps on the level of Thom spaces or spectra:
$$\iota^! : M \to \mathrm{Thom}(\nu) \qquad \text{or} \qquad \iota^! : \Sigma^\infty M_+ \to N^\nu$$
parametrized continuously by the space $\mathrm{Tub}(\iota)$, in the sense that each of the previous maps is a restriction to a point of the first factor of
$$\iota^! : \mathrm{Tub}(\iota) \times M \to \mathrm{Thom}(\nu) \qquad \text{or} \qquad \iota^! : \mathrm{Tub}(\iota)_+ \wedge \Sigma^\infty M_+ \to N^\nu$$

(2) Let M be a compact oriented manifold and $f : X \to Y$ be a 0-regular map of $\mathcal{B}$-labelled 0-finite complexes. Let ν be the normal bundle of the associated embedding $M^{f_0} : M^{Y_0} \to M^{X_0}$. Then we have umkehr maps on the level of spectra
$$(M^f)^! : M^X \to \mathrm{Thom}(\mathcal{N}) \qquad \text{or} \qquad (M^f)^! : \Sigma^\infty(M^X)_+ \to (M^Y)^{\mathcal{N}}$$
where from now on we will write $\mathcal{N} = (ev_Y)^*\nu$. The maps are parametrized continuously by the space $\mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu)$, in the sense that each of the previous maps is a restriction of
$$(M^f)^! : \mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu) \times M^X \to \mathrm{Thom}(\mathcal{N})$$
$$\text{or}$$
$$(M^f)^! : (\mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu))_+ \wedge \Sigma^\infty(M^X)_+ \to (M^Y)^{\mathcal{N}}$$

In this section we use our results on compatible tubular neighborhoods to get results about compatible umkehr maps. Recall the results from section 1.2. We will now give the motivation for those results: these results are exactly those that can translated to the framework of Thom spaces and Thom spectra to give nice lemma's about choosing the data necessary to construct a umkehr map in such a way that it will be compatible with certain structures. We will be interested in the following types of compatibility:

(1) Adding vector bundles in cases (1) and (2).
(2) Compatibility with products in case (1) and disjoint union in case (2).
(3) Compatibility with composition in cases (1) and (2).
(4) Simplifying subdivided simplices in case (2).
(5) Collapsing a 1-simplex in the special case (2) where the complexes are one-dimensional.

4.1. Adding vector bundles. In this subsection we give some results about the naturality when we add a vector bundle to the source and/or the target. This can only be formulated in the language of Thom spectra, because it involves virtual bundles.

Proposition 4.69. *Let W be an n-dimensional vector space with a given identification with $\mathbb{R}^n$. Suppose that we have an embedding $\iota : N \to M \times W$ and associated umkehr map $\iota^! : \mathrm{Tub}(\iota)_+ \wedge \Sigma^\infty(M \times W)_+ \to N^\nu$. Then there is a canonical map*
$$\tilde{\iota}^! : \mathrm{Tub}(\iota)_+ \wedge \Sigma^\infty M_+ \to N^{\nu - W}$$

Proof. This map is given by noting that the map $\mathrm{Tub}(\iota)_+ \wedge \Sigma^\infty(M \times W)_+ \to N^\nu$ induces a map
$$\mathrm{Tub}(\iota)_+ \wedge \Sigma^n \Sigma^\infty M_+ \to N^\nu$$
by factoring over the spectrum $\mathrm{Tub}(\iota)_+ \wedge \Sigma^\infty(M \times \dot{W})$, where $\dot{W}$ is the one-point compactification, and desuspending by n. Because W is trivial, we do not need to pick a complementary bundle and $N^{\nu - W}$ is equal to $\Sigma^{-n} N^\nu$. ∎

With the same reasoning applied to a simpler situation, we obtain the following result which sharpens the functoriality of the Thom spectrum functor. Recall that originally, when the negative part of the virtual bundle was allowed to be non-trivial, the Thom spectrum could only be defined up to weak equivalence because one had to pick a complementary bundle.

Lemma 4.70. *Let $f : X \to Y$ be any map of spaces and $\mu = \mu_+ - \mu_-$ a virtual bundle over Y such that μ_- is trivial with a given identification with $\mathbb{R}^n$. Then there is a unique canonical representative of the induced map of Thom spectra $Y^{f^*\mu} \to X^\mu$.*

Proof. Again we only need to note that because μ_- is trivial, there is no need to pick a complementary bundle. □

We now look at the question what happens when we have a map going into M^μ and a map out of $\Sigma^\infty M_+$. The question is whether these can be composed and on what choices this depends. The answer will turns out to be that they can be composed, but one needs to pick a connection to identify the fiber of the pullback of μ over a point v in ν with the fiber of $\iota^*\mu$ over $\pi(v)$. For a vector bundle μ, let $\mathcal{C}(\mu)$ denote the space of connections in the topology coming from their identification as a subspace of the space of sections of the vector bundle $Hom(TM \otimes \mu, \mu)$.

Proposition 4.71.
1. *Let $\iota : N \to M$ be an embedding as before with normal bundle ν and μ be a vector bundle over M. Then there is a canonical map*
$$\tilde{\iota}^! : (\mathrm{Tub}(\iota) \times \mathcal{C}(\nu))_+ \wedge M^\mu \to N^{\iota^*\mu \oplus \nu}$$
2. *Let $f : X \to Y$ be a 0-regular map as before and μ a vector bundle over M^X. Then there is a canonical map*
$$(M^f)^! : (\mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu) \times \mathcal{C}(\nu))_+ \wedge (M^X)^\mu \to (M^Y)^{(M^f)^*\mu \oplus \mathcal{N}}$$

Proof. Let's do the first case first. It is clear that a tubular neighborhood $f : \nu \to M$ induces a tubular neighborhood $F : f^*\mu \to \mu$. The problem is that we need to fix an isomorphism of $f^*\mu$ with $\iota^*\mu \oplus \nu$. This requires a choice of connection. The second case is similar. □

This may lead one to worry whether including the choice of connection might disturb the contractiblity of the space of choices. However, this is not the case, as the space of connections is a convex topological space and hence we have the following corollary.

Corollary 4.72. *For any vector bundle μ over a manifold M, the space $\mathcal{C}(\mu)$ of connections is a non-empty contractible space.*

Proof. The contractibility follows by convexity. The non-emptiness is a consequence of the existence of connections on vector bundles. This is done by constructing a connection in local coordinates and then patching these local constructions together using a partition of unity. Here one needs to check that a certain cohomological obstruction vanishes, which can be arranged. □

Remark 4.73. The results of this section extend to the cases of products and compositions given below. Just as one replaces the spaces of tubular neighborhoods and propagating flows with compatible ones, the connections should be chosen compatible as well. This is not difficult and some more details can be found in remarks in the relevant sections.

4.2. Umkehr maps compatible with products. The results about tubular neighborhoods compatible with products in section 1.2 of this chapter directly imply the following lemma if one translates them to Thom spaces. It solves the question of finding tubular neighborhoods compatible with the products for situation (1).

Lemma 4.74. *Let $i_1 : N_1 \to M_1$ and $i_2 : N_2 \to M_2$ be a pair of embeddings. There is a contractible subspace $\mathrm{Tub}^\times(i_1 \times i_2)$ of $\mathrm{Tub}(i_1 \times i_2)$ with maps $p_j : \mathrm{Tub}^\times(i_1 \times i_2) \to \mathrm{Tub}(i_j)$ such* *that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Tub}(i_1)\times M_1 \arrow[d] & \mathrm{Tub}^{\times}(i_1\times i_2)\times M_1\times M_2 \arrow[l, "p_1\times\pi_1"'] \arrow[r, "p_2\times\pi_2"] \arrow[d] & \mathrm{Tub}(i_2)\times M_2 \arrow[d] \\
\mathrm{Thom}(\nu_1) & \mathrm{Thom}(\pi_1^*(\nu_1)\times\pi_2^*(\nu_2))\cong \mathrm{Thom}(\nu_1)\wedge\mathrm{Thom}(\nu_2) \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & \mathrm{Thom}(\nu_2)
\end{tikzcd}$$

where the vertical arrows are given by the Thom collapse construction.

This can be translated to the language of spectra, given as the first part of proposition 4.78. There is a corresponding lemma in the framework of Thom spaces and translation of it to Thom spectra for situation (2), where now the product of mapping spaces corresponds to the disjoint union of 0-finite complexes. This is because there is a homeomorphism $M^{X_1\sqcup X_2}\cong M^{X_1}\times M^{X_2}$. This homeomorphism is functorial in the following sense: let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two 0-regular maps of $\mathcal{B}$-labelled 0-finite complexes. Then $f_1\sqcup f_2 : X_1\sqcup X_2 \to Y_2\sqcup Y_2$ is again a 0-regular map between $\mathcal{B}$-lablled 0-finite complexes and the map $M^{f_1\sqcup f_2}$ is sent to $M^{f_1}\times M^{f_2}$ under the homeomorphisms.

We want to find umkehr map data for $M^{f_1\sqcup f_2}$ which allows a commutative diagram with the restrictions M^{f_1} and M^{f_2} to the two components of the disjoint union. To write it down we need to do some more work: to be precise, we need to prove properties for propagating flows similar to those of tubular neighborhoods.

In general, let $i_1 : N_1 \to M_1$ and $i_2 : N_2 \to M_2$ with normal bundles ν_1 and ν_2. Firstly, similarly to the space $\mathrm{Tub}^{\times}(i_1\times i_2)$ we want to define a space $\mathcal{P}^{\times}(\pi_1^*\nu_1\oplus\pi_2^*\nu_2)$ to be the space of propagating flows which arise as a product of propagating flows on ν_1 and ν_2. Let's make this precise.

Definition 4.75. A propagating flow in $\mathcal{P}(\pi_1^*\nu_1\oplus\pi_2^*\nu_2)$ is said to be compatible with the product structure if its restriction to $\pi_j^*\nu_j$ for $j=1,2$ has image in those compactly supported vector field such that only the components in the $\pi_j^*(\nu_j)$-directions are non-zero and these components are constant along fibers of the projection to N_i.

We denote the space of propagating flows compatible with the product structure by $\mathcal{P}^{\times}(\pi_1^*\nu_1\oplus\pi_2^*\nu_2)$. Note that there are maps $p_j : \mathcal{P}^{\times}(\pi_1^*\nu_1\oplus\pi_2^*\nu_2)\to\mathcal{P}(\nu_j)$ for $j=1,2$ and conversely a map $\iota : \mathcal{P}(\nu_1)\times\mathcal{P}(\nu_2)\to\mathcal{P}^{\times}(\pi_1^*\nu_1\oplus\pi_2^*\nu_2)$.

Note that this space consists exactly of those propagating flows whose restrictions to the components are propagating flows themselves. Again, the homotopy type of this space is easy to determine.

Lemma 4.76. *The space $\mathcal{P}^{\times}(\pi_1^*\nu_1\oplus\pi_2^*\nu_2)$ is a non-empty and contractible subspace of $\mathcal{P}(\pi_1^*\nu_1\oplus\pi_2^*\nu_2)$.*

PROOF. It is easily seen to be contractible because it is convex. To show that it is non-empty, one takes elements $\mathcal{Z}_1$ and $\mathcal{Z}_2$ of $\mathcal{P}(\nu_1)$ and $\mathcal{P}(\nu_2)$ respectively and creates a $\mathcal{Z} : \nu_1\oplus\nu_2 \to \chi_{c,v}(\nu_1\oplus\nu_2)$ given by $(v,w)\mapsto\rho(||v||^2,||w||^2)\mathcal{Z}_1(v)\oplus\mathcal{Z}_2(w)$. Here ρ is a suitable bump function to guarantee that the vector fields are compactly supported and $||-||$ is a norm coming from some metric on the vector bundles. □

We can now formulate a proposition which gives the general statement about finding umkehr data compatible with the product structure for situation (2) in the framework of Thom spectra.

Lemma 4.77. *Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two 0-regular maps of $\mathcal{B}$-labelled 0-regular complexes. There is a contractible space $\mathrm{Tub}^{\times}\times\mathcal{P}^{\times} := \mathrm{Tub}^{\times}(M^{(f_1)_0}\times M^{(f_2)_0})\times\mathcal{P}^{\times}(\pi_1^*\nu_1\oplus\pi_2^*\nu_2)$ with maps p_j to $\mathrm{Tub}(M^{(f_j)_0})\times\mathcal{P}(\nu_j)$ such that the following diagram commutes for $j=1,2$:*

$$\begin{tikzcd}
\mathrm{Tub}^{\times}\times\mathcal{P}^{\times}\times M^{X_1\sqcup X_2} \arrow[r, "p_j\times\pi_j"] \arrow[d, "(M^{f_1\sqcup f_2})^!"] & \mathrm{Tub}(M^{(f_j)_0})\times\mathcal{P}(\nu_j)\times M^{X_j} \arrow[d, "(M^{f_j})^!"] \\
\mathrm{Thom}(\pi_1^*(\mathcal{N}_1)\times\pi_2^*(\mathcal{N}_2)) \arrow[r, "\pi_j"'] & \mathrm{Thom}(\mathcal{N}_j)
\end{tikzcd}$$

Proof. Say $j = 1$. The lower map is given by the composition $\mathrm{Thom}(\pi_1^*(\mathcal{N}_1) \times \pi_2^*(\mathcal{N}_2)) \cong \mathrm{Thom}(\mathcal{N}_1) \wedge \mathrm{Thom}(\mathcal{N}_2) \to \mathrm{Thom}(\mathcal{N}_1)$ of the homeomorphism of proposition 4.23 with the projection on the first component. Therefore we see the map π_2 preserves only those maps that were flowed using a component of zero in the $\mathcal{N}_2$, because of our demands on propagating flows and tubular neighborhoods compatible with the product structures. Hence the top-right and left-bottom compositions are equal. $\square$

Finally, we give the translation in terms of Thom spectra.

Proposition 4.78. *(1) Let $i_1 : N_1 \to M_1$ and $i_2 : N_2 \to M_2$ be a pair of embeddings. There is a contractible subspace $\mathrm{Tub}^\times(i_1 \times i_2)$ of $\mathrm{Tub}(i_1 \times i_2)$ with maps $p_j : \mathrm{Tub}^\times(i_1 \times i_2) \to \mathrm{Tub}(i_j)$ such that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Tub}(i_1)_+ \wedge \Sigma^\infty(M_1)_+ \arrow[d, "i_1^!"'] & \mathrm{Tub}^\times(i_1 \times i_2)_+ \wedge \Sigma^\infty(M_1 \times M_2)_+ \arrow[l, "p_1\times\pi_1"'] \arrow[r, "p_2\times\pi_2"] \arrow[d, "(i_1\times i_2)^!"'] & \mathrm{Tub}(i_2)_+ \wedge \Sigma^\infty(M_2)_+ \arrow[d, "i_2^!"] \\
N_1^{\nu_1} & (N_1 \times N_2)^{\pi_1^*(\nu_1)\times\pi_2^*(\nu_2)} \cong N_1^{\nu_1} \wedge N_2^{\nu_2} \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & N_2^{\nu_2}
\end{tikzcd}$$

(2) Let $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ be two 0-regular maps of $\mathcal{B}$-labelled 0-regular complexes. There is a contractible space $\mathrm{Tub}^\times \times \mathcal{P}^\times := \mathrm{Tub}^\times(M^{(f_1)_0} \times M^{(f_2)_0}) \times \mathcal{P}^\times(\pi_1^*\nu_1 \oplus \pi_2^*\nu_2)$ with maps p_j to $\mathrm{Tub}(M^{(f_j)_0}) \times \mathcal{P}(\nu_j)$ such that the following diagram commutes for $j = 1, 2$:

$$\begin{tikzcd}
(\mathrm{Tub}^\times \times \mathcal{P}^\times)_+ \wedge \Sigma^\infty(M^{X_1 \sqcup X_2})_+ \arrow[r, "p_j\times\pi_j"] \arrow[d, "(M^{f_1\sqcup f_2})^!"] & (\mathrm{Tub}(M^{(f_j)_0}) \times \mathcal{P}(\nu_j))_+ \wedge \Sigma^\infty(M_j)_+ \arrow[d, "(M^{f_j})^!"] \\
(M^{Y_1 \sqcup Y_2})^{\pi_1^*(\mathcal{N}_1)\times\pi_2^*(\mathcal{N}_2)} \cong (M^{Y_1})^{\mathcal{N}_1} \wedge (M^{Y_2})^{\mathcal{N}_2} \arrow[r, "\pi_j"'] & (M^{Y_j})^{\mathcal{N}_j}
\end{tikzcd}$$

We will later use this proposition to prove that higher string topology operations are compatible with disjoint unions.

Remark 4.79. To get the results of section 4.1 to hold for these umkehr maps compatible with the product, replace $\mathcal{C}(\nu)$ with the contractible non-empty subspace $\mathcal{C}^\times(\pi_1^*\nu_1 \oplus \pi_2^*\nu_2)$ consisting of those connections whose restriction to $\pi_j^*\nu_j$ for $j = 1, 2$ is again a connection which doesn't depend on the other coordiante.

4.3. Umkehr maps compatible with composition. We want to construct umkehr maps which are compatible with composition in a sense to be made precise. To do this we first note that the results in section 1.2 of this chapter about tubular neighborhoods compatible with composition imply the following lemma in the framework of Thom spaces.

Let's recall the notation first. Let $j : L \to N$ and $i : N \to M$ be a pair of embeddings and $\iota_\lambda : L \to \nu_i$ be the composition of j with the zero-section. Then there is a contractible subspace $\mathrm{Tub}^\circ(i \circ j)$ of $\mathrm{Tub}(i \circ j) \times \mathrm{Tub}(i) \times \mathrm{Tub}(\iota_\lambda)$ together with obvious projection maps $p_{i\circ j} : \mathrm{Tub}^\circ(i \circ j) \to \mathrm{Tub}(i \circ j)$, $p_i : \mathrm{Tub}^\circ(i \circ j) \to \mathrm{Tub}(i)$ and $p_\lambda : \mathrm{Tub}^\circ(i \circ j) \to \mathrm{Tub}(\iota_\lambda)$.

Lemma 4.80. *The umkehr map construction makes the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Tub}^\circ(i \circ j) \times M \arrow[r, "id\times i^!\circ(p_i\times id)"] \arrow[rrd, "(i\circ j)^!\circ(p_{i\circ j}\times id)"'] & \mathrm{Tub}^\circ(i \circ j) \times \mathrm{Thom}(\nu_i) \arrow[r, "\iota_\lambda^!\circ(p_\lambda\times id)"] & \mathrm{Thom}(\nu_i|_L \oplus \nu_j) \arrow[d, "\cong"] \\
 & & \mathrm{Thom}(\nu_{i\circ j})
\end{tikzcd}$$

We can also translate the results about the liftings of tubular neighborhoods into this context. The result we want is that data for an umkehr map for i lifts essentially uniquely to data of a compatible umkehr map of $i \circ j$. The diagram given this situation is as follows.

$$\begin{tikzcd}
L \arrow[d, "j"'] \arrow[dr, "i\circ j"] & \\
N \arrow[r, "i"'] & M
\end{tikzcd}$$

We want compatibility in the sense of that the umkehr map construction makes the following diagram commute:

$$\begin{tikzcd}
\mathrm{Thom}(\nu_{i\circ j}) & \\
\mathrm{Thom}(\nu_i|_L \oplus \nu_j) \arrow[u, "\cong"] & \\
\mathrm{Thom}(\nu_i) \arrow[u, "\iota_\lambda^!"] & M \arrow[l, "i^!"] \arrow[uul, "(i\circ j)^!"']
\end{tikzcd}$$

Furthermore, these liftings are parametrized by $\mathrm{Tub}(\iota_\lambda)$ and hence form a bundle with contractible fiber over $\mathrm{Tub}(i)$. This solves the problem of finding compatible umkehr maps for situation (1).

Now we want to do something similar for situation (2). Again, this will require some investigation of compatible propagating flows. To do this, we use the notation given above and do something similar as we did when we looked at tubular neighborhoods compatible with composition. In formulating this definition, we will use that our propagating flows consists of vertical vector fields, hence cover the identity and therefore can be restricted to submanifolds, in this case $\mathcal{Z}_i$ to L.

Definition 4.81. Suppose that an isomorphism $\nu_{i\circ j} \cong \nu_i|_L \oplus \nu_j$ is given, for example induced by tubular neighborhoods. A triple $(\mathcal{Z}_{i\circ j}, \mathcal{Z}_i, \mathcal{Z}_j)$ of propagating flow in $\mathcal{P}(\nu_{i\circ j}) \times \mathcal{P}(\nu_i) \times \mathcal{P}(\nu_j)$ is said to be compatible with the composition if the sum of $\mathcal{Z}_j$ with the restriction of $\mathcal{Z}_i$ to $\nu_i|_L$ coincides with $\mathcal{Z}_{i\circ j}$ under the isomorphism $\nu_{i\circ j} \cong \nu_i|_L \oplus \nu_j$.

We denote the space of propagating flows compatible with the composition by $\mathcal{P}^\circ(\nu_{i\circ j})$. It comes with natural maps $p_{i\circ j} : \mathcal{P}^\circ(\nu_{i\circ j}) \to \mathcal{P}(\nu_{i\circ j})$, $p_i : \mathcal{P}^\circ(\nu_{i\circ j}) \to \mathcal{P}(\nu_j)$ and $p_j : \mathcal{P}^\circ(\nu_{i\circ j}) \to \mathcal{P}(\nu_j)$.

Essentially, a point in the space of propagating flows compatible with composition is given by a propagating flow $i \circ j$ and a way to extend it to ν_i.

Lemma 4.82. *The space $\mathcal{P}^\circ(\nu_{i\circ j})$ is a non-empty contractible subspace of $\mathcal{P}(\nu_{i\circ j})$ and in fact forms a bundle with contractible fiber over $\mathcal{P}(\nu_i)$*

Proof. It is homeomorphic to $\mathcal{P}(\nu_i) \times \mathcal{P}(\nu_j)$ □

Note that although our space of compatible propagating flows depends on a choice of tubular neighborhood, it depends continuously on this choice. This implies the following lemma:

Lemma 4.83. *Let $\mathrm{Tub}^\circ \times \mathcal{P}^\circ \subset \mathrm{Tub}^\circ(i\circ j) \times \mathcal{P}(\nu_{i\circ j}) \times \mathcal{P}(\nu_i) \times \mathcal{P}(\nu_i|_L \oplus \nu_j)$ be the subspace consisting of compatible tubular neighborhoods with their compatible propagating flows. Then $\mathrm{Tub}^\circ \times \mathcal{P}^\circ$ is contractible and forms a bundle with contractible fiber over $\mathrm{Tub}(i) \times \mathcal{P}(\nu_i)$.*

This is the correct notion of compatible propagating flow to use when talking about umkehr maps compatible with composition. Let's start with proving the existence of compatible umkehr maps in situation (2) in the framework of Thom spaces.

Lemma 4.84. *Let $f : X \to Y$ and $g : Y \to Z$ be a pair of 0-regular maps between 0-finite complexes. Then the umkehr construction gives maps making the following diagram commute:*

$$\begin{tikzcd}
(\mathrm{Tub}^\circ \times \mathcal{P}^\circ) \times M^X \arrow[r] \arrow[drr] & (\mathrm{Tub}^\circ \times \mathcal{P}^\circ) \times \mathrm{Thom}(\mathcal{N}_f) \arrow[r] & \mathrm{Thom}(\mathcal{N}_f|_{M^Z} \oplus \mathcal{N}_g) \arrow[d, "\cong"] \\
& & \mathrm{Thom}(\mathcal{N}_{g\circ f})
\end{tikzcd}$$

PROOF. What is the tubular neighborhood for space from $\mathcal{N}_f|_{M^Z} \oplus \mathcal{N}_g \to \mathcal{N}_f$? To get it, we sent a point (w, v) over $h \in M^Z$ to the vector w over the point $\phi(v)(h)$, the propagating flow construction applied to h and v. This makes sense because the compatibility of the tubular neighborhoods means that fibers of $\mathcal{N}_f$ over $\phi(v, h)$ is naturally the same as the fiber $\mathcal{N}_f|_{M^Z}$ in which w lies.

We claim that the tubular neighborhoods upstairs induced by the tubular neighborhoods downstairs and the propagating flow are compatible. This is true because the flows of $\mathcal{Z}_i$ and $\mathcal{Z}_j$ commute and hence doing them at the same time or at different times give the same result. Now the claim has been proven, the proof continuous as in case (1). $\square$

We can now give the translation to Thom spectra.

Proposition 4.85. *(1) Let $j : L \to N$ and $i : N \to M$ be a pair of embeddings. There is a contractible subspace $\mathrm{Tub}^\circ(i \circ j)$ of $\mathrm{Tub}(i \circ j)$ together with maps making the following diagram commute:*

$$\begin{tikzcd}
\mathrm{Tub}^\circ(i\circ j)_+ \wedge \Sigma^\infty M_+ \arrow[r] \arrow[drr] & \mathrm{Tub}^\circ(i\circ j)_+ \wedge N^{\nu_i} \arrow[r] & L^{\nu_i|_L \oplus \nu_j} \arrow[d, "\cong"] \\
& & L^{\nu_{i\circ j}}
\end{tikzcd}$$

(2) Let $f : X \to Y$ and $g : Y \to Z$ be a pair of 0-regular maps between 0-finite complexes. There is a contractible space $\mathrm{Tub}^\circ \times \mathcal{P}^\circ$ with maps making the following diagram commute:

$$\begin{tikzcd}
(\mathrm{Tub}^\circ \times \mathcal{P}^\circ)_+ \wedge \Sigma^\infty (M^X)_+ \arrow[r] \arrow[drr] & (\mathrm{Tub}^\circ \times \mathcal{P}^\circ)_+ \wedge (M^Y)^{\mathcal{N}_f} \arrow[r] & (M^Z)^{\mathcal{N}_f|_{M^Z} \oplus \mathcal{N}_g} \arrow[d, "\cong"] \\
& & (M^Z)^{\mathcal{N}_{g\circ f}}
\end{tikzcd}$$

We can translate the result of the proposition in terms of lifting. It is in this form that we will use our knowledge of compatible umkehr maps to construct the string operations.

Corollary 4.86. *Suppose one has a diagram of the form*

$$\begin{tikzcd}
Y \arrow[r, "g"] & Z \\
X \arrow[u, "f"] \arrow[ur, "h"'] &
\end{tikzcd}$$

where all maps are 0-regular. Then given a subspace $A \subset \mathrm{Tub}(M^{h_0}) \times \mathcal{P}(\nu)$ one can find a subspace $\tilde{A} \subset \mathrm{Tub}^\circ(M^{f_0} \circ M^{g_0}) \times \mathcal{P}^\circ(\nu)$ of compatible liftings for umkehr data with a natural homotopy equivalence $\tilde{A} \to A$.

4.4. Simplifying complexes. Next we describe how to create umkehr maps compatible with the following situation. Let Δ^n be a standard n-simplex and $\mathrm{nsd}(\Delta^n)$ be the non-standard subdivision obtained by adding simplices from each of the edges of a face to the barycentre. We assume that all the simplices in the complement of the image of interior of Δ^n in $\mathrm{nsd}(\Delta^n)$ are labelled with a single label $b \in \mathcal{B}$.

If an n-dimensional $\mathcal{B}$-labelled complex X contains a subcomplex of the form $\mathrm{nsd}(\Delta^n)$, we can replace this subcomplex with Δ^n without essentially alternating the space. Let's denote this

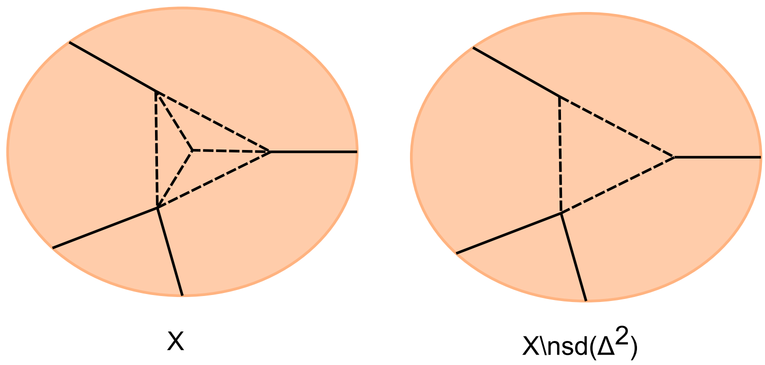


Figure 4.87. A two-dimensional complex X with an embedded non-standard subdivided simplex and the altered complex $X\backslash\mathrm{nsd}(\Delta^2)$. The modified part is dashed.

altered space by $X\backslash\mathrm{nsd}(\Delta^n)$, see figure 4.87. This situation often happens for the case $n=1$, in which case we want to remove subdivisions of an edge.

For umkehr maps, we are interested in situations of the form:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ \mathrm{nsd}(\Delta^n) \arrow[u] \arrow[r, "\cong"'] & \mathrm{nsd}(\Delta^n) \arrow[u] \end{tikzcd}$$

where $f : X \to Y$ is a 0-regular map of 0-finite complexes and the inclusions of the subdivided simplices is as before. Note that we have a diagram of spaces:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"] \\ X\backslash\mathrm{nsd}(\Delta^n) \arrow[r, "\tilde{f}"'] & Y\backslash\mathrm{nsd}(\Delta^n) \end{tikzcd}$$

We want to figure out how umkehr maps for the top and bottom rows relate to each other.

Proposition 4.88. *There exist natural homotopy equivalences* $\mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu) \to \mathrm{Tub}(M^{\tilde{f}_0}) \times \mathcal{P}(\tilde{\nu})$ *and* $(M^Y)^{\mathcal{N}_f} \to (M^{Y\backslash\mathrm{nsd}(\Delta^n)})^{\mathcal{N}_{\tilde{f}}}$ *making the following diagram commute:*

$$\begin{tikzcd} (\mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu))_+ \wedge \Sigma^\infty (M^X)_+ \arrow[r, "(M^f)^!"] \arrow[d, "\simeq"'] & (M^Y)^{\mathcal{N}_f} \arrow[d, "\simeq"] \\ (\mathrm{Tub}(M^{\tilde{f}_0}) \times \mathcal{P}(\tilde{\nu}))_+ \wedge \Sigma^\infty (M^{X\backslash\mathrm{nsd}(\Delta^n)})_+ \arrow[r, "(M^{\tilde{f}})^!"'] & (M^{Y\backslash\mathrm{nsd}(\Delta^n)})^{\mathcal{N}_{\tilde{f}}} \end{tikzcd}$$

More generally, one can do this for any finite number of subdivided n-simplices at the same time.

Proof. We first look at the underlying diagram of embeddings of finite-dimensional manifolds. For simplicity, we assume that the label $b \in \mathcal{B}$ is M, but the same argument works for other labels. To write down this diagram, let $\tilde{X}_0$ denote the 0-simplices of $X\backslash\mathrm{nsd}(\Delta^n)$ and c the barycenter of Δ^n, i.e. the additional 0-simplex in $\mathrm{nsd}(\Delta^n) \subset X$. Then $X_0 = \tilde{X}_0 \sqcup c$. There is similar notation

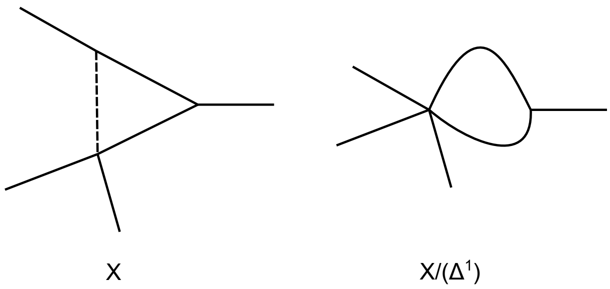


FIGURE 4.90. A collapse of a 1-simplex. This collapsed simplex is dashed on the left hand side.

for Y and its 0-simplices. Then we have the following diagram:

$$\begin{tikzcd} M^{Y_0} = M^{\tilde{Y}_0} \times M \arrow[r, "M^{\tilde{f}_0} \times id"] \arrow[d, "\pi_1"'] & M^{\tilde{X}_0} \times M = M^{X_0} \arrow[d, "\pi_1"] \\ M^{\tilde{Y}_0} \arrow[r, "M^{\tilde{f}_0}"'] & M^{\tilde{X}_0} \end{tikzcd}$$

This also allows us to identify naturally ν with $\tilde{\nu}$. Now our arguments about products of tubular neighborhoods and propagating flows gives the desired homotopy equivalence of umkehr data $\mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu) \to \mathrm{Tub}(M^{\tilde{f}_0}) \times \mathcal{P}(\mathcal{N}_f)$. It is in fact a homeomorphism.

However, the two constructed umkehr map are different for the top and bottom row. They differ only in the interior of Δ^n. In the top row of the diagram in the statement of the proposition the values of the maps to M at the barycentre are not changed by the flow, but in the bottom row they are. However, one can clearly see that there is a natural homotopy of the flow parameters which gives a homotopy between the two umkehr maps. This gives the desired homotopy equivalence $(M^Y)^{\mathcal{N}_f} \to (M^{Y \backslash \mathrm{nsd}(\Delta^n)})^{\mathcal{N}_{\tilde{f}}}$. □

We again want to put this proposition in the context of liftings for convenient use in the construction of compatible umkehr maps.

Corollary 4.89. *Suppose one has a diagram of spaces of the form*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\cong"'] & Y \arrow[d, "\cong"] \\ X\backslash \mathrm{nsd}(\Delta^n) \arrow[r, "\tilde{f}"'] & Y\backslash \mathrm{nsd}(\Delta^n) \end{tikzcd}$$

where the maps are induced by a 0*-regular map* f *between* 0*-finite complexes. Then given a subspace* $A \subset \mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu)$ *one can find a subspace* $\tilde{A} \subset \mathrm{Tub}^\circ(M^{\tilde{f}_0}) \times \mathcal{P}^\circ(\tilde{\nu})$ *of compatible liftings for umkehr data such that there is a natural homotopy equivalence* $\tilde{A} \to A$.

4.5. Collapsing a 1-simplex. Finally, we explain how to do one very particular construction: collapsing a 1-simplex in a 1-dimensional complex. We only do this for the one-dimensional case because the higher-dimensional analogues are much harder to formulate.

The idea is that up to homotopy having a 1-simplex with two boundary 0-simplices or just a single 0-simplex doesn't matter, so it also shouldn't matter for the construction of umkehr maps. We will show that this true up to a contractible space of choices.

We first fix some notation: let X be a 1-dimensional $\mathcal{B}$-labelled complex and Δ^1 a subcomplex which is either entirely labelled or has a single vertex labelled. Then X/Δ^1 is the 1-dimensional $\mathcal{B}$-labelled complex obtained by collapsing the image of Δ^1 consisting of two vertices and a single edge to a single vertex. See figure 4.90.

Let X and Y be of this type, then any 0-regular map $f : X \to Y$ which sends $\Delta^1 \subset X$ to $\Delta^1 \subset Y$ induces a map $\bar{f} : X/\Delta^1 \to Y/\Delta^1$. This map will be 0-regular if f was. Let ν_0 and ν_1 be those parts of the normal bundle of the finite-dimensional embedding M^{f_0} for 0 and 1 vertex of Δ^1 respectively, and $\nu_{0,1}$ the part of the normal bundle of the embedding $M^{\bar{f}_0}$ for the image of Δ^1.

Lemma 4.91. *Given a splitting $\nu_{0,1} \to \nu_1 \oplus \nu_2$, tubular neighborhoods f_1 and f_2 for ν_1 and ν_2 and propagating flows $\mathcal{Z}_1$ and $\mathcal{Z}_2$, there exists a contractible subspace* $\mathrm{Tub}^c(M^{\bar{f}_0}) \times \mathcal{P}^c(\bar{\nu})$ *of* $\mathrm{Tub}^c(M^{\bar{f}_0}) \times \mathcal{P}^c(\bar{\nu})$ *which makes the following diagram commute:*

$$\begin{tikzcd}
(\mathrm{Tub}^c(M^{\bar{f}_0}) \times \mathcal{P}^c(\bar{\nu}))_+ \wedge \Sigma^\infty(M^{X/\Delta^1})_+ \arrow[r, "(M^{\bar{f}})^!"] \arrow[d] & (M^{Y/\Delta^1})^{\bar{\mathcal{N}}} \arrow[d] \\
\Sigma^\infty(M^X)_+ \arrow[r, "(M^f)^!"'] & (M^Y)^{\mathcal{N}}
\end{tikzcd}$$

In fact, the spaces in the construction of the previous form a bundle over the space $\mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu) \times \mathrm{Split}$ and thus the lemma can be sharpened to the following proposition:

Proposition 4.92. *There exists a contractible subspace* $\mathrm{Tub}^c(M^{\bar{f}_0}) \times \mathcal{P}^c(\bar{\nu})$ *of* $\mathrm{Tub}^c(M^{\bar{f}_0}) \times \mathcal{P}^c(\bar{\nu}) \times \mathrm{Split}$ *with a map to* $\mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu) \times \mathrm{Split}$ *which is a homotopy equivalence, such that the following diagram commutes:*

$$\begin{tikzcd}
(\mathrm{Tub}^c(M^{\bar{f}_0}) \times \mathcal{P}^c(\bar{\nu}))_+ \wedge \Sigma^\infty(M^{X/\Delta^1})_+ \arrow[r, "(M^{\bar{f}})^!"] \arrow[d] & (M^{Y/\Delta^1})^{\bar{\mathcal{N}}} \arrow[d] \\
(\mathrm{Tub}^c(M^{\bar{f}_0}) \times \mathcal{P}^c(\bar{\nu}) \times \mathrm{Split})_+ \wedge \Sigma^\infty(M^X)_+ \arrow[r, "(M^f)^!"'] & (M^Y)^{\mathcal{N}}
\end{tikzcd}$$

Finally, when rephrased in terms of liftings, we get:

Corollary 4.93. *Suppose one has a diagram of the form*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\
X/\Delta^1 \arrow[r, "\bar{f}"'] & Y/\Delta^1
\end{tikzcd}$$

where f is 0-regular. Then given a subspace $A \subset \mathrm{Tub}(M^{f_0}) \times \mathcal{P}(\nu) \times \mathrm{Split}$ one can find a subspace $\tilde{A} \subset \mathrm{Tub}^c(M^{\tilde{f}_0}) \times \mathcal{P}^c(\tilde{\nu})$ of compatible liftings for umkehr data such that $\tilde{A} \simeq A$.

CHAPTER 5

Construction of HCFT operations

This chapter is the heart of the thesis. In it we complete the technically complicated task of constructing the HCFT operations for string topology using the umkehr maps defined in the previous chapter. The basic setup is the same as the one used by Veronique Godin in [**God07**], with modifications to allow multiple branes and a bit more exposition.

Convention 5.1. In this chapter, we fix a compact oriented manifold M and a set of branes $\mathcal{B}$ of compact oriented submanifolds A of M. The category of nice $\mathcal{B}$-labelled cobordism graphs with set of branes $\mathcal{B}$, $\mathsf{Fat}^{oc,n}_{\mathcal{B}}$ will be denoted by Fn to shorten the notation.

1. Overview of the construction

The idea of the proof is as follows:

(1) We first explain how to construct an umkehr map from the incoming to the outgoing boundary for a single nice $\mathcal{B}$-labelled cobordism graph Γ. This will be a map of spectra which depends continuously on a contractible space $\mathrm{Umk}(\Gamma)$ of choices for umkehr data. The result is a map of spectra:

$$\urcorner : \mathrm{Umk}(\Gamma)_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma})_+ \to (M^{\partial_{out}\Gamma})^{\kappa_\Gamma}$$

where κ_Γ is a virtual bundle to be defined in the proof of theorem 5.6.

The composition of the map $\urcorner$ with the Thom isomorphism is the *string operation corresponding to a single graph* and can in fact be seen as a degree zero string operation.

(2) Construct functors Split : $\nabla\mathsf{Fn} \to \mathsf{Top}$ and Umk : $\nabla\mathsf{Fn}_/ \to \mathsf{Top}$, containing information about splittings and umkehr data. Here $\nabla\mathsf{Fn}$ is a category of simplices in Fn. Use these to construct a supporting category $\overline{\nabla\mathsf{Fn}_/}$ which contains the addtional information of splittings and umkehr data necessary to patch the virtual bundles κ_Γ and the maps $\urcorner$ together. As a result we can define functors $\Sigma^\infty(M^{\partial_{in}-})_+ : \overline{\nabla\mathsf{Fn}_/} \to \mathsf{Spectra}$ and $(M^{\partial_{out}-})^\kappa : \overline{\nabla\mathsf{Fn}_/} \to \mathsf{Spectra}$ such that one can construct a map of spectra

$$\sqsupset : \mathrm{hocolim}\,\Sigma^\infty(M^{\partial_{in}-})_+ \to \mathrm{hocolim}(M^{\partial_{out}-})^\kappa$$

implementing the umkehr maps in family form.

(3) The previous step requires a complicated technical proposition, the construction of spaces of compatible choices for umkehr maps. This will have its own section, which depends heavily on the results on compatible umkehr maps in the previous chapter.

(4) Simplify the domain and codomain of the map $\sqsupset$ by showing that $\mathrm{hocolim}\,\Sigma^\infty M^{\partial_{in}-}$ is weakly equivalent to the suspension spectrum of $\coprod_{[\Sigma]}\left(|\mathsf{Fn}_\Sigma| \times M^{\partial_{in}\Sigma}\right)$ and $\mathrm{hocolim}\,\mathrm{Thom}(\kappa)$ is weakly equivalent to the ind-Thom spectrum associated to an ind-virtual bundle $\bar{\kappa}$ over the space $\coprod_{[\Sigma]}\left(|\mathsf{Fn}_\Sigma| \times M^{\partial_{out}\Sigma}\right)$.

Then we compose $\sqsupset$ with the Thom isomorphism to define a map $\daleth$ which gives the *string operations in family form.* The induced map in homology $\daleth_*$ will be the string operations of theorem 2.29:

$$\daleth_* : H_*(B\Gamma_\Sigma; \mathcal{L}^M_{\mathcal{B}}) \otimes H_*(M^{\partial_{in}\Sigma}, \mathbb{Q}) \to H_*(M^{\partial_{out}\Sigma}; \mathbb{Q})$$

where the particular implementation of the local system $\mathcal{L}^M_{\mathcal{B}}$ is defined in the next step.

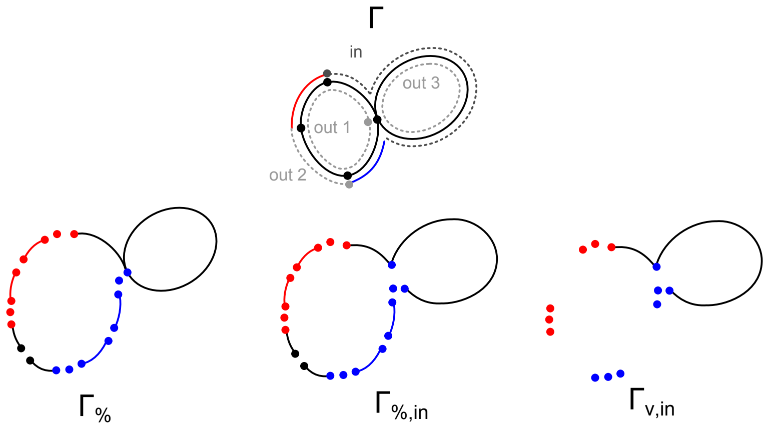


FIGURE 5.3. The graphs $\Gamma_{\div}$, $\Gamma_{\div,in}$ and $\Gamma_{v,in}$ for a $\mathcal{B}$-labelled cobordism graph Γ.

(5) We then investigate orientability issues coming from the virtual bundle κ. To be precise, we define the local system $\mathcal{L}^M_{\mathcal{B}}$ by comparing κ to other virtual bundles which have the same determinant graded line bundle. $\mathcal{L}^M_{\mathcal{B}}$ will arise as the determinant of a homological virtual bundle $\bar{\eta}$ over $B\Gamma_{\Sigma}$.
(6) Show that these constructions are compatible with disjoint union of cobordisms by first showing how to get compatibility for the map $\daleth$ for a single $\mathcal{B}$-labelled cobordism graph and explaining how to extend these techniques to the maps $\beth$ and $\gimel$.
(7) Show that these constructions are compatible with composition of cobordisms. This is a similar, but more involved, version of the argument in the previous step.

Convention 5.2. Because being concerned about orientations will make an already complicated argument even more difficult to follow, we will put off any discussion of orientations until section 6 of this chapter. This is step 5 of our list.

2. Operation for a single nice $\mathcal{B}$-labelled cobordism graph

We will now construct the umkehr map associated to a single nice $\mathcal{B}$-labelled cobordism graph. For examples of specific operations obtained by this construction, see section 3.1. Recall that we have a d-dimensional compact oriented manifold M and set of branes $\mathcal{B}$.

2.1. Breaking up $\mathcal{B}$-labelled graphs. We start by recalling that for a $\mathcal{B}$-labelled graph, the geometric realisation $|\Gamma|$ can naturally be seen as a 0-finite 1-dimensional $\mathcal{B}$-labelled complex, see proposition 3.23. This means that we can define the mapping space M^Γ as $M^{|\Gamma|}$ in the sense of definition 4.60: if Y a $\mathcal{B}$-labelled complex, then we let M^Y be the space of continuous maps $g : Y \to M$ with the property that if $b(y) = A \in \mathcal{B}$ for a point $y \in Y$, then $g(y) \in A$. This is given the subspace topology from the compact-open topology on the space of all continuous maps from the space Y to M.

We begin with the definitions of graphs involved in the construction. These graph give us a way to break up the graph into smaller pieces such that we can construct an umkehr map from the incoming to the outgoing boundary and this can be made compatible with morphisms later on. One can find examples of these graphs in figure 5.3, in figure 5.7 and in figure 5.35 in section 4.2 of this chapter.

Definition 5.4. Let Γ be a nice $\mathcal{B}$-labelled cobordism graph, then we define the following associated $\mathcal{B}$-labelled graphs. We will use the boundary collapse map s from proposition 3.48 which sends the boundary graph $\partial\Gamma$ of a $\mathcal{B}$-labelled cobordism graph Γ to the underlying ordinary graph of Γ.

- $\Gamma_{\div}$ is given by
$$\mathrm{hsep}(\Gamma \backslash s(\partial_{in}\Gamma)) \sqcup s(\partial_{in}\Gamma)$$

More concretely, it is the $\mathcal{B}$-labelled graph composed of the disjoint union of the following three types of graphs: (i) $s(\partial_{in}\Gamma)$, (ii) a linear graph with one edge e_h and two vertices $v_{h,0}, v_{h,1}$ for each half-edge h of $\Gamma\backslash s(\partial_{in}\Gamma)$ and (iii) the original vertices v of $\Gamma\backslash s(\partial_{in}\Gamma)$. The vertices v retain their original label and for the linear graph the labels are as follows: the vertices $v_{h,0}$ are given the same label as $s(h)$, the source vertex of the original half-edge h, while the new half-edges of the linear graph and the vertex $v_{h,1}$ are given the same label as the original half-edge.

- $\Gamma_{\div,in}$ is given by
$$\mathrm{hsep}(\Gamma\backslash s(\partial_{in}\Gamma)) \sqcup \partial_{in}\Gamma$$
This is the $\mathcal{B}$-labelled graph composed of the disjoint union of graphs $\partial_{in}\Gamma$ and the types of graph (ii) and (iii) defined previously. Alternatively, it can defined as $\Gamma_\div$ with $s(\partial_{in}\Gamma)$ replaced by $\partial_{in}\Gamma$.
- $\Gamma_{v,in}$ is given by
$$v(\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma))) \sqcup \partial_{in}\Gamma$$
Here one should consider $v(\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma)))$ as the $\mathcal{B}$-labelled graph obtained from $\Gamma\backslash s(\partial_{in}\Gamma)$ by creating a vertex for each half edge h and labelling this with the label of $s(h)$.

There is a number of natural maps between geometric realisations of these graphs: these are denoted $s_\div : |\Gamma_\div| \to |\Gamma|$, $s_{in} : |\Gamma_{\div,in}| \to |\Gamma_\div|$ and $s_v : |\Gamma_{\div,in}| \to |\Gamma_{v,in}|$. We describe them below. All will turn out to be 0-regular maps between 0-finite $\mathcal{B}$-labelled complexes obtained by the geometric realisations of $\mathcal{B}$-labelled graphs.

Definition 5.5.

- $s_\div : |\Gamma_\div| \to |\Gamma|$ is given by mapping the 0-simplices $*_v$ to themselves and the 0-simplices $*_{v_{h,0}}$ for $h \in s^{-1}(v)$ to $*_v$. It also identifies $[0,1]_{e_h}$ and $[0,1]_{e_{i(h)}}$ along $*_{v_{h,1}}$ and $*_{v_{i(h),1}}$. Note that $s_\div$ does not come from a morphism of $\mathcal{B}$-labelled graphs, as one of these must have picked on the edges to be collapsed, instead of glueing them together.
- $s_{in} : |\Gamma_{\div,in}| \to |\Gamma_\div|$ is given by $id_{\mathrm{hsep}(\Gamma\backslash s(\partial_{in}\Gamma))} \sqcup |s|$, where
$$s : \partial_{in}\Gamma \to s(\partial_{in}\Gamma)$$
is the collapse map.
- $s_v : |\Gamma_{\div,in}| \to |\Gamma_{v,in}|$ is given by $f \sqcup id_{\partial_{in}\Gamma}$, where
$$f : |\mathrm{hsep}(\Gamma\backslash s(\partial_{in}\Gamma))| \to |v(\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma)))|$$
is the map which collapse the edges $[0,1]_{e_h}$ onto the point $*_{v_{h,0}}$, which we can identify with the vertex of $\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma))$ corresponding to the half-edge h. Note that this behaves well with respect to the labellings.

2.2. Motivation and a Q&A. The next theorem is fundamental to the construction of higher string operations. The rough idea is that we want to apply our umkehr map construction to the correspondence
$$\partial_{in}\Gamma \overset{i_{in}}{\hookrightarrow} \Gamma_{v,in} \xleftarrow{s_v} \Gamma_{\div,in} \xrightarrow{s_{in}} \Gamma_\div \xrightarrow{s_\div} \Gamma \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma) \xleftarrow{s} \partial_{out}\Gamma$$

This is almost a correspondence in $\mathsf{Corr}(\mathsf{SSComp}_{\mathcal{B},0})$, except for i_{in}, s_v, i_{out} and s. In the next theorem we will essentially modify the homotopy push-pull extension of $\Sigma^\infty(M^-)_+$ obtained by the umkehr map construction to deal with these problematic maps. We comment on the reasons that the problems with these maps can be solved.

(1) The underlying reason why this is possible for s_v, i_{out} and s is that, after passing to mapping spaces, these maps point in the correct direction. Hence they don't really have to be 0-regular or surjective at all.
(2) The reason this is possible for i_{in} is that this map only creates points, which is a simple enough task such that our construction can handle it. The idea is to thicken the codomain using some Euclidean spaces.

We will now answer a few questions about the choice of this correspondence.

What are we trying to model: Naively, one would try write the simpler correspondence

$$\partial_{in}\Gamma \hookrightarrow \Gamma \hookleftarrow \partial_{out}\Gamma$$

and try to push-pull along this. However, the technology to do this is not yet available. Essentially all we can do using umkehr maps is: (1) creating new isolated points in the domain of the mapping spaces, (2) identify a finite number of points in the domain. To push-pull along $\partial_{in}\Gamma \hookrightarrow \Gamma$ one would presumably need to create the complement of the incoming boundary. However if the complement contains a cycle, there is no way to do that by simply creating single points without breaking up the graph.

Why do we consider a correspondence like this: Our correspondence essentially gives of map spectra a factorisation of the push-pull construction for the naive correspondence into maps for which we can define the umkehr maps. Because any good definition of umkehr maps should be functorial, this implies that if one could construct the umkehr map for the naive correspondence then the umkehr there should be equal to our composition.

If one doesn't believe these ideas, then there is a more cynical motivation: this corerspondence works. To be precise, the generalized push-pull construction applied to this correspondence will give all degree zero string operations, like the string product and coproduct, as we will prove in section 3.1 of chapter 6. It will also be the basis of the maps $\beth$ in theorem 5.31 and $\gimel$ in definition 5.43 which will induce the higher degree string operations of definition 5.44.

Why specifically this correspondence: There are other correspondences that produce the degree zero string operations. Heuristically, because any good one should be a factorisation of the naive correspondence, it should give the same operations as our correspondence. So the choice shouldn't really matter.

The reason for choosing this particular one is that it is the simplest way to make a correspondence in a natural way where all maps that, after going to mapping spaces, point in the wrong direction will either have come from 0-regular maps or have only minor problems as explained above.

2.3. The construction of $\daleth$. In the next theorem we will finally. construct the operation $\daleth$, the degree zero string operations for a single $\mathcal{B}$-labelled cobordism graph.

In it we use vector space W. This is an Euclidean space which admits an embedding $\iota : M \hookrightarrow W$ with normal bundle ν. The embeddings $A \hookrightarrow M$ induce an embeddings $\iota_A : A \hookrightarrow W$ with normal bundle ν_A for every brane A. These conditions are exactly such that W can serve as a thickening of some spaces involved in the constructions so that we can define umkehr maps for maps into them. In particular, we will use the space W as space to "create new vertices". Also see the discussion of the map ρ_{in} in the proof of the theorem.

Theorem 5.6. *Suppose that we have fixed an Euclidean space W with an embedding $M \hookrightarrow W$, tubular neighborhoods $\mathrm{Tub}(\iota) \times \prod_{A\in\mathcal{B}} \mathrm{Tub}(\iota_A)$ for the embeddings of M and branes $A \in \mathcal{B}$ into W and a choice of inner product for W in $\mathcal{I}(W)$.*

For every nice $\mathcal{B}$-labelled cobordism graph Γ, there exists a contractible space $\mathrm{Umk}(\Gamma)$ and a virtual bundle κ_Γ, which both will be described in the proof of the theorem, such that we have a map of spectra

$$\daleth : \mathrm{Umk}(\Gamma)_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma})_+ \to (M^{\partial_{out}\Gamma})^{\kappa_\Gamma}$$

This map depends continuously on the contractible space of choices made in the beginning of the statement of this theorem. Furthermore, if we fix a point $x \in \mathrm{Umk}(\Gamma)$, then we obtain a map $\daleth_*(x) : H_*(M^{\partial_{in}\Gamma}) \to H_*(M^{\partial_{out}\Gamma})$ in homology which is independent of x and the choices made in the beginning, and therefore denoted by $\daleth_*$.

Proof. Consider the following sequence of maps of $\mathcal{B}$-labelled graphs (see figure 5.7):

$$\partial_{in}\Gamma \overset{i_{in}}{\hookrightarrow} \Gamma_{v,in} \xleftarrow{s_v} \Gamma_{\div,in} \xrightarrow{s_{in}} \Gamma_{\div} \xrightarrow{s_{\div}} \Gamma \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma) \xleftarrow{s} \partial_{out}\Gamma \tag{1}$$

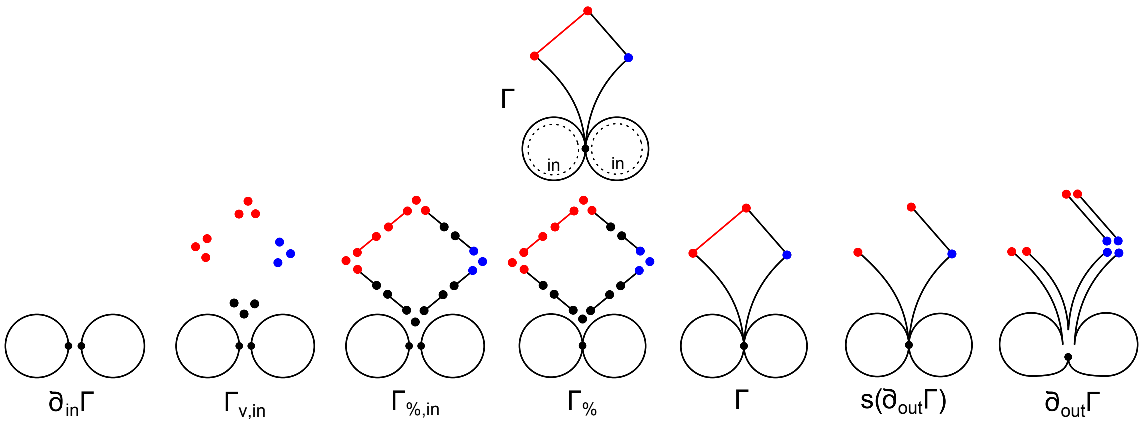


FIGURE 5.7. This figure exhibits the $\mathcal{B}$-labelled graphs used at different stages in the construction of theorem 5.6. We haven't indicated the outgoing boundary of Γ to stop the figure from being too cluttered.

This induces a diagram of spaces as follows, where as above we have dropped the $|-|$ denoting geometric realisation of graphs to limit the amount of notation:

$$M^{\partial_{in}\Gamma} \times W^{v(\Gamma\backslash s(\partial_{in}\Gamma))} \xleftarrow{\rho_{in}} M^{\Gamma_{v,in}} \xrightarrow{\sigma_v} M^{\Gamma_{\div,in}} \xleftarrow{\sigma_{in}} M^{\Gamma_\div} \xleftarrow{\sigma_\div} M^{\Gamma} \xrightarrow{r_{out}} M^{s(\partial_{out}\Gamma)} \xrightarrow{\sigma} M^{\partial_{out}\Gamma}$$

We will construct umkehr maps for all arrows pointing in the wrong direction, in other words, those pointing leftwards. We will pay special attention to the choices made at each point.

The map ρ_{in}: The map ρ_{in} is induced by the embeddings $\iota : M \hookrightarrow W$ and $\iota_A : A \hookrightarrow W$ for each brane.

This can be seen by noting that $M^{\Gamma_{v,in}}$ is a product of terms of the following form: (i) $M^{\partial_{in}\Gamma}$, (ii) terms A for each vertex of $\Gamma\backslash s(\partial_{in}\Gamma)$ with label A, (iii) terms M for each unlabelled vertex of $\Gamma\backslash s(\partial_{in}\Gamma)$, (iv) terms A for each half-edge of $\Gamma\backslash s(\partial_{in}\Gamma)$ ending on a vertex with label A, and (v) terms M for half-edges of $\Gamma\backslash s(\partial_{in}\Gamma)$ ending on an unlabelled vertex. Thus, if one has tubular neighborhoods for the embeddings $\iota : M \hookrightarrow W$ and $\iota_A : A \hookrightarrow M$ with $A \in \mathcal{B}$, we get a tubular neighborhood for the embedding

$$M^{v(\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma)))} \hookrightarrow W^{v(\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma)))}$$

with a normal bundle which we denote by ν. For convenience, we will denote the Euclidean space $W^{v(\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma)))}$ by $\bar{W}$. Using the results of chapter 4 the Thom collapse applied to the tubular neighborhood induces a map of spectra

$$R'_{in} : \Sigma^\infty(M^{\partial_{in}\Gamma} \times \bar{W})_+ \to (M^{\Gamma_{v,in}})^\nu$$

Because of proposition 4.69 we know that this is the same as a map – instead of a homotopy class of maps, because W and hence $\bar{W}$ is Euclidean – of spectra

$$R_{in} : \Sigma^\infty(M^{\partial_{in}\Gamma})_+ \to (M^{\Gamma_{v,in}})^{\nu-\bar{W}}$$

The map σ_v: This map $M^{\Gamma_{v,in}} \to M^{\Gamma_{\div,in}}$ is induced by the map $s_v : |\Gamma_{\div,in}| \to |\Gamma_{v,in}|$.

We have already seen that $M^{\Gamma_{v,in}}$ can be written given as a product of smaller spaces. The space $M^{\Gamma_{\div,in}}$ admits a similar decomposition and is in fact a product over the same indexing set. The terms from the incoming and the vertices will be the same, but the terms corresponding to half-edges of $\Gamma\backslash s(\partial_{in}\Gamma)$ are defined. We get following: (i) $M^{\partial_{in}\Gamma}$, (ii) terms A for each vertex of $\Gamma\backslash s(\partial_{in}\Gamma)$ with label A, (iii) terms M for each unlabelled vertex of $\Gamma\backslash s(\partial_{in}\Gamma)$, (iv) for a half-edge labelled completely by A we get a term $P_A(A, A)$, (v) for an unlabelled half-edge ending on a vertex with label A we get a term $P_M(A, M)$ and (vi) for an unlabelled edge we get a term $P_M(M, M)$.

Since the path spaces $P_A(A, A)$, $P_M(A, M)$ and $P_M(M, M)$ are homotopy equivalent to A, A and M respectively, the map σ_v is a homotopy equivalence.

If we want to use σ_v to obtain a map of spectra wecan apply the suspension functor Σ^∞:

$$S'_{in} : \Sigma^\infty(M^{\Gamma_{v,in}})_+ \to \Sigma^\infty(M^{\Gamma_{\div,in}})_+$$

This is a weak equivalence because σ_v is a homotopy equivalence. However, to compose with the previous map we want to use this map to get a map of Thom spectra. The result is a weak equivalence

$$S_{in} : (M^{\Gamma_{v,in}})^{\nu-\bar{W}} \to (M^{\Gamma_{\div,in}})^{\mathcal{N}-\bar{W}}$$

where $\mathcal{N} = (\sigma_v)^*\nu$ is the pullback of the normal bundle. That this map exists and is a weak equivalence is a consequence of the fact that the construction of Thom spectra is a functor from the category of virtual bundles. Using lemma 4.70, we furthermore see that it is unique as a map, not just defined up to weak equivalence.

The map σ_{in}: This map is an embedding of finite codimension induced by the map $s_{in} : \Gamma_{\div,in} \to \Gamma_{\div}$. To see that it has finite codimension, note that it can be written as a pullback over the finite dimensional embedding $\tau_{in} : M^{s(v(\partial_{in}\Gamma))} \to M^{v(\partial_{in}\Gamma)}$ induced by the collapse $v(\partial_{in}\Gamma) \to s(v(\partial_{in}\Gamma))$. This is shown in the following diagram:

$$\begin{tikzcd}
M^{\Gamma_\div} \arrow[r, "\sigma_{in}"] \arrow[d, "\mathrm{ev}_\div"'] & M^{\Gamma_{\div,in}} \arrow[d, "\mathrm{ev}_{\div,in}"] \\
M^{s(v(\partial_{in}\Gamma))} \arrow[r, "\tau_{in}"'] & M^{v(\partial_{in}\Gamma)}
\end{tikzcd}$$

where $\mathrm{ev}_\div$ evaluates a map $\Gamma_\div \to M$ at the the vertices of $s(\partial_{in}\Gamma)$ and $\mathrm{ev}_{\div,in}$ evaluates a map $\Gamma_{\div,in} \to M$ at the vertices of $\partial_{in}\Gamma$.

We denote the normal bundle of τ_{in} by ν_{in}. Using the results of the previous chapter we conclude to determine a tubular neighborhood for the normal bundle $\mathcal{N}_{in} = (\mathrm{ev}_\div)^*\nu_{in}$ upstairs, it suffices to give a tubular neighborhood for τ_{in} and a propagating flow on ν_{in} In other words, to get a tubular neighborhood for σ_{in} we have to choose a point in $\mathrm{Tub}(\tau_{in}) \times \mathcal{P}(\nu_{in})$. Each such choice gives a map of spectra

$$S''_{in}(x) : \Sigma^\infty(M^{\Gamma_{\div,in}})_+ \to (M^{\Gamma_\div})^{\mathcal{N}_{in}}$$

Because the tubular neighborhood for $\mathcal{N}_{in}$ depends continuously on the space of umkehr data $\mathrm{Tub}(\tau_{in}) \times \mathcal{P}(\nu_{in})$, we see that we can combine all the previous maps into a single map of spectra:

$$S'_{in} : (\mathrm{Tub}(\tau_{in}) \times \mathcal{P}(\nu_{in}))_+ \wedge \Sigma^\infty(M^{\Gamma_{\div,in}})_+ \to (M^{\Gamma_\div})^{\mathcal{N}_{in}}$$

To compose with the earlier map, we note that a map $\Sigma^\infty(M^{\Gamma_{\div,in}}) \to (M^{\Gamma_\div})^{\mathcal{N}_{in}}$ induces a canonical map $(M^{\Gamma_{\div,in}})^{\mathcal{N}-\bar{W}} \to (M^{\Gamma_\div})^{\mathcal{N}_{in}\oplus(\sigma_{in})^*\mathcal{N}-\bar{W}}$ after picking a connection in $\mathcal{C}(\nu_{in})$, as in proposition 4.71. Hence we get a map of spectra:

$$\begin{aligned} S_{in} :&(\mathrm{Tub}(\nu_{in}) \times \mathcal{P}(\nu_{in}) \times \mathcal{C}(\tau_{in}))_+ \wedge (M^{\Gamma_{\div,in}})^{\mathcal{N}-\bar{W}} \\ &\to (M^{\Gamma_\div})^{\mathcal{N}_{in}\oplus(\sigma_{in})^*\mathcal{N}-\bar{W}} \end{aligned}$$

The map $\sigma_\div$: This map is an embedding of finite codimension induced by the map $s_\div : \Gamma_\div \to \Gamma$. Note that this map can be written as the following pullback of the finite dimensional embedding $\tau_\div : M^{v(\Gamma)} \to M^{v(\Gamma_\div)}$ induced by the collapse $v(\Gamma_\div) \to v(\Gamma)$.

$$\begin{tikzcd}
M^{\Gamma} \arrow[r, "\sigma_\div"] \arrow[d, "\mathrm{ev}"'] & M^{\Gamma_\div} \arrow[d, "\mathrm{ev}_\div"] \\
M^{v(\Gamma)} \arrow[r, "\tau_\div"'] & M^{v(\Gamma_\div)}
\end{tikzcd}$$

where $\mathrm{ev}_\div$ is as before and ev evaluates a map $\Gamma \to M$ at its vertices. We denote the normal bundle of $\tau_\div$ by $\nu_\div$.

Thus it suffices to give a tubular neighborhood for $\tau_\div$ and a propagating flow on $\nu_\div$ to obtain a tubular neighborhood upstairs for $\mathcal{N}_\div = (\mathrm{ev})^*\nu_\div$. This is the same as choosing a point in the space of umkehr data $\mathrm{Tub}(\tau_\div) \times \mathcal{P}(\nu_\div)$. As before we have a map of spectra:

$$S'_\div : (\mathrm{Tub}(\tau_\div) \times \mathcal{P}(\nu_\div))_+ \wedge \Sigma^\infty(M^{\Gamma_\div}) \to (M^\Gamma)^{\mathcal{N}_\div}$$

and this induces a map of spectra which can be composed with the previous map:

$$\begin{aligned} S_\div :&(\mathrm{Tub}(\tau_\div) \times \mathcal{P}(\nu_\div) \times \mathcal{C}_{\nu_\div})_+ \wedge (M^{\Gamma_\div})^{\mathcal{N}_{in} \oplus (\sigma_{in})^*\mathcal{N} - \bar{W}} \\ &\to (M^\Gamma)^{\mathcal{N}_\div \oplus (\sigma_\div)^*\mathcal{N}_{in} \oplus (\sigma_{in} \circ \sigma_\div)^*\mathcal{N} - \bar{W}} \end{aligned}$$

The map r_{out}: This map is given by precomposing each element of M^Γ with the geometric realisation of the inclusion $i_{out} : s(\partial_{out}\Gamma) \to \Gamma$. This is a continuous map, hence induces a map of spectra upon suspension:

$$R'_{out} : \Sigma^\infty(M^\Gamma) \to \Sigma^\infty(M^{s(\partial_{out}\Gamma)})$$

By simply suspending with the correct bundles, we get a map of spectra

$$\begin{aligned} R_{out} :&(M^\Gamma)^{\mathcal{N}_\div \oplus (\sigma_\div)^*\mathcal{N}_{in} \oplus (\sigma_{in} \circ \sigma_\div)^*\mathcal{N} - \bar{W}} \\ &\to (M^{s(\partial_{out}\Gamma)})^{(r_{out})^*(\mathcal{N}_\div \oplus (\sigma_\div)^*\mathcal{N}_{in} \oplus (\sigma_{in} \circ \sigma_\div)^*\mathcal{N}) - \bar{W}} \end{aligned}$$

The map σ: This map is given by precomposing each element of $M^{s(\partial_{out}\Gamma)}$ with the collapse map $\sigma : \partial_{out}\Gamma \to s(\partial_{out}\Gamma)$. As above, this induces a map of spectra:

$$\begin{aligned} S :&(M^{s(\partial_{out}\Gamma)})^{(r_{out})^*(\mathcal{N}_\div \oplus (\sigma_\div)^*\mathcal{N}_{in} \oplus (\sigma_{in} \circ \sigma_\div)^*\mathcal{N}) - \bar{W}} \\ &\to (M^{\partial_{out}\Gamma})^{(r_{out} \circ \sigma)^*(\mathcal{N}_\div \oplus (\sigma_\div)^*\mathcal{N}_{in} \oplus (\sigma_{in} \circ \sigma_\div)^*\mathcal{N}) - \bar{W}} \end{aligned}$$

We can now define a contractible space of choices for our umkehr map on the level of spectra.

$$\mathrm{Umk}(\Gamma) = \mathrm{Tub}(\tau_\div) \times \mathcal{P}(\nu_\div) \times \mathcal{C}(\nu_\div) \times \mathrm{Tub}(\tau_{in}) \times \mathcal{P}(\nu_{in}) \times \mathcal{C}(\nu_{in})$$

We call this space the *space of umkehr data* for the correspondence 1, because it contains all the information necessary to define the umkehr maps between the mapping spaces unambiguously. Finally, we can compose all earlier defined maps to get an intermediate map of spectra which is our candidate for the map in the statement of the theorem:

$$\urcorner' : \mathrm{Umk}(\Gamma)_+ \wedge \Sigma^\infty M^{\partial_{in}\Gamma} \to (M^{\partial_{out}\Gamma})^{(r_{out} \circ \sigma)^*(\mathcal{N}_\div \oplus (\sigma_\div)^*\mathcal{N}_{in} \oplus (\sigma_{in} \circ \sigma_\div)^*\mathcal{N}) - \bar{W}}$$

However, clearly the vector bundle appearing on the right hand side is very complicated. We want to simplify it.

To do this, we introduce some notation. Let V be the set of all vertices of Γ, iV the vertices of $\partial_{in}\Gamma$, siV the vertices of $s(\partial_{in}\Gamma)$, eV the vertices in $\Gamma \backslash s(\partial_{in}\Gamma)$, H the half-edges of Γ, eH the half-edges of $\Gamma \backslash s(\partial_{in}\Gamma)$, E the edges of Γ and eE the edges of $\Gamma \backslash s(\partial_{in}\Gamma)$. Note that V, iV, siV, eV, E and eE naturally come equipped with a map b to $\mathcal{B}$. For the half edges, we also have a map b to $\mathcal{B}$ be sending a half-edge to the label of the vertex to which it is attached.

Recall that ν_M is the normal bundle for $\iota : M \to W$ and for each $A \in \mathcal{B}$ with ν_A we mean the normal bundle to $\iota_A : A \to W$. We have that ν is isomorphic to

$$\nu \cong \bigoplus_{v \in eV} \nu_{b(v)}^{\mathrm{val}(v)+1}$$

where as before if v or h is unlabelled, b is taken to assign the value M and $\mathrm{val} : eV \to \mathbb{N}$ denotes the valence of v in $\Gamma \backslash s(\partial_{in}\Gamma)$. The valence is the number of half-edges attached to a vertex. This isomorphisms holds because in $\Gamma_{v,in}$ we create a new vertex for each extra vertex and extra half-edge. These can be indexed by the extra vertex counted with multiplicity of their valence (for the half-edge) plus one (for the vertex itself).

From this we see that $(\sigma_{in} \circ \sigma_{div})^* \mathcal{N}$ is isomorphic to the vector bundle κ obtained as the following pullback

$$\begin{tikzcd} \kappa \arrow[r] \arrow[d] & \bigoplus_{v \in eV} \nu_{b(v)}^{\mathrm{val}(v)+1} \arrow[d] \\ M^\Gamma \arrow[r] & \prod_{v \in eV} b(v)^{\mathrm{val}(v)+1} \end{tikzcd}$$

where the bottom map is the composition of evaluation at the vertices of $\Gamma \backslash s(\partial_{in}\Gamma)$ with diagonal maps and the right map is the product of projection maps.

For τ_{in}, we can similarly note that it is isomorphic as $\bigoplus_{v \in siV} Tb(v)^{\mathrm{coval}(v)}$ where $\mathrm{coval} : siV \to \mathbb{N} \sqcup \{0\}$ denotes the covalence, i.e. the number of vertices in iV which are mapped to v by the collapse map minus 1. From this we see that $(\sigma_{div})^* \mathcal{N}_{in}$ is isomorphic to the vector bundle κ_{in} obtained as the following pullback

$$\begin{tikzcd} \kappa_{in} \arrow[r] \arrow[d] & \bigoplus_{v \in siV} Tb(v)^{\mathrm{coval}(v)} \arrow[d] \\ M^\Gamma \arrow[r] & \prod_{v \in siV} b(v)^{\mathrm{coval}(v)} \end{tikzcd}$$

where the bottom map is a composition of evaluation at the vertices of $s(\partial_{in}\Gamma)$ with diagonal maps.

Finally, we note that $\tau_{\div}$ is given by $\bigoplus_{v \in eV} Tb(v)^{\mathrm{val}(v)} \oplus \bigoplus_{e \in eE} Tb(e)$. The first term comes from collapsing additional edges and the incoming graph together at vertices. The second term from glueing the two half edges. This step is made possible because $\Gamma_{v,in}$, $\Gamma_{\div,in}$ and $\Gamma_{\div}$ included copies the vertices of $\Gamma \backslash s(\partial_{in}\Gamma)$ in addition to the half-edges. Includnig these vertices makes the isomorphism canonical: the vectors in the fiber of the normal bundle are equivalences classes modulo the diagonal and we can pick a canonical representative by setting the component in the v direction to be zero.

We conclude that the vector bundle $\mathcal{N}_{\div}$ is isomorphic to the vector bundle $\kappa_{\div}$ obtained as the following pullback

$$\begin{tikzcd} \kappa_{\div} \arrow[r] \arrow[d] & \bigoplus_{v \in eV} Tb(v)^{\mathrm{val}(v)} \oplus \bigoplus_{e \in eE} Tb(e) \arrow[d] \\ M^\Gamma \arrow[r] & \prod_{v \in eV} b(v)^{\mathrm{val}(v)} \times \prod_{e \in eE} b(e) \end{tikzcd}$$

where the bottom map is given the product of the composition of valuation at the vertices of $\Gamma \backslash s(\partial_{in}\Gamma)$ with diagonal maps and evaluation at the midway points of the half edges of $\Gamma \backslash s(\partial_{in}\Gamma)$.

Thus, to simplify the vector bundle appearing, we need to simplify $\kappa_{\div} \oplus \kappa_{in} \oplus \kappa$. The trick is that at the vertices copies of $Tb(v)$ and $\nu_{b(v)}$ meet. To be precise: a vertex $v \in eV$ contributes $\mathrm{val}(v)+1$ copies of $\nu_{b(v)}$ from κ and $\mathrm{val}(v)$ copies of $Tb(v)$. We want to cancel the $Tb(v)$ component of a half-edge against its $\nu_{b(v)}$ component. However, this cancelling is not natural unless we make a final choice: we pick an inner product on W. This gives us a method to canonically identify a sum of TA and ν_A with W. The space of inner products $\mathcal{I}(W)$ is convex as a subspace of the space of linear maps $W \otimes W^* \to \mathbb{R}$ in the obvious topology and hence contractible, so there is no danger in adding it.

Let κ_Γ^+ be the vector bundle constructed as the following pullback

$$\begin{tikzcd} \kappa_\Gamma^+ \arrow[r] \arrow[d] & \bigoplus_{v \in eV} \nu_{b(v)} \oplus \bigoplus_{e \in eE} Tb(e) \oplus \bigoplus_{v \in siV} Tb(v)^{\mathrm{coval}(v)} \arrow[d] \\ M^\Gamma \arrow[r] & \prod_{v \in eV} b(v) \times \prod_{e \in eE} b(e) \times \prod_{v \in siV} b(v)^{\mathrm{coval(v)}} \end{tikzcd}$$

In the previous discussion, we have seen that we can canonically cancel some copies of W appearing in the positive part of the virtual bundle against some in the negative part $\bar{W} =$

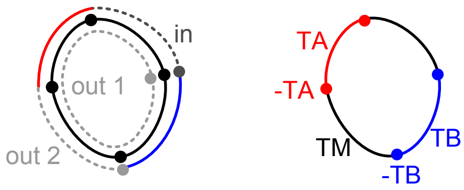


FIGURE 5.11. An example of an admissible $\mathcal{B}$-labelled cobordism graph together with the distribution of the virtual bundle around it.

$W^{v(\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma)))}$ to obtain the following isomorphism of virtual bundles over M^Γ:

$$\kappa_{\div} \oplus \kappa_{in} \oplus \kappa - W^{v(\mathrm{sep}(\Gamma\backslash s(\partial_{in}\Gamma)))} \cong \kappa_\Gamma^+ - W^{v(\Gamma\backslash s(\partial_{in}\Gamma))}$$

Hence if we set $\kappa_\Gamma^- = W^{v(\Gamma\backslash s(\partial_{in}\Gamma))}$ then we define a virtual bundle κ_Γ by $\kappa_\Gamma^+ - \kappa_\Gamma^-$. The previous remarks show that it is canonically isomorphic to the more complicated virtual bundle we had earlier. Furthermore, because we have only cancelled trivial bundles, the map with the more complicated virtual bundle induces a map – uniquely defined, not just up to homotopy – of spectra

$$\daleth : \mathrm{Umk}(\Gamma)_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma})_+ \to (M^{\partial_{out}\Gamma})^{\kappa_\Gamma}$$

This is the map in the statement of the theorem. □

Remark 5.8. The first lines of the theorem about W and tubular neighborhoods for the embeddings of M and $A \in \mathcal{B}$ into W will be replaced by convention 5.13 from the next section onwards.

Remark 5.9. We noted before that the construction of the map R_{in} has a space of choices given the umkehr data $\mathrm{Tub}(\iota) \times \prod_{A\in\mathcal{B}} \mathrm{Tub}(\iota_A)$. The continuous dependence of the Thom collapse map on the tubular neighborhoods implies that we could have included this space of choices to obtain a single map of spectra:

$$\tilde{R}_{in} : (\mathrm{Tub}(\iota) \times \prod_{A\in\mathcal{B}} \mathrm{Tub}(\iota_A))_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma}) \to (M^{\Gamma_{v,in}})^{\nu-\bar{W}}$$

We have not done this because for later proofs about the compatibility of this construction with disjoint union and glueing of $\mathcal{B}$-labelled graphs it turns out to be useful to fix these tubular neighborhoods beforehand.

Definition 5.10. The map $\daleth_*$ is called the *degree zero string operation* associated to the $\mathcal{B}$-labelled graph Γ.

We suggest the following way to think about the virtual bundle κ_Γ in the case that Γ is admissible: it is obtained by putting a $Tb(e)$ at the center of every edge e not in $s(\partial_{in}\Gamma)$ and a $\nu_{b(v)} - W$ or equivalently $-Tb(v)$ at every vertex v not in $s(\partial_{in}\Gamma)$. As before, if e or v has no label b assigns the value M. See figure 5.11.

This way of distributing the virtual bundle over the graph has the important property that it behaves well with respect to morphisms of open-closed $\mathcal{B}$-labelled graphs: if an edge e with $b(e) = A$ collapses, the TA of the edge combines with the two factors $\nu_A - W$ of the vertices to give the following isomorphism

$$\nu_A \oplus TA \oplus \nu_A - W \oplus W \cong \nu_A \oplus W - W \oplus W \cong \nu_A - W$$

Remark 5.12. In the proof of theorem 5.6, we saw that the map in homology can factorized as follows:

$$\daleth_* = \sigma_* \circ (r_{out})_* \circ \sigma_{\div}^! \circ \sigma_{in}^! \circ (\sigma_v)_* \circ \rho_{in}^!$$

We can say quite a bit about these maps. Because σ_v comes from collapsing edges to a point and the path spaces $P_A(A, A)$, $P_M(A, M)$ and $P_M(M, M)$ are homotopy equivalent to A, A and M respectively, the induced map $(\sigma_v)_*$ in homology is an isomorphism. Furthermore, if Γ is admissible then $\sigma_{in}^!$ is an isomorphism as well. We will see other ways to simplify this composition when we calculate the basic degree zero string operations in chapter 6.

3. The construction of the map between homotopy colimits

In this section we construct a map of spectra

$$\beth : \underset{\overline{\nabla \mathsf{Fn}_/}}{\operatorname{hocolim}} \Sigma^\infty M^{\partial_{in}-} \to \underset{\overline{\nabla \mathsf{Fn}_/}}{\operatorname{hocolim}} (M^{\partial_{out}-})^\kappa \tag{2}$$

and of course we will define the domain and codomain of this map.

Convention 5.13. For the remainder of this chapter, we fix an Euclidean space W with an embedding $M \hookrightarrow W$, tubular neighborhoods $\mathrm{Tub}(\iota) \times \prod_{A \in \mathcal{B}} \mathrm{Tub}(\iota_A)$ for the embeddings of M and branes $A \in \mathcal{B}$ into W and a choice of inner product for W in $\mathcal{I}(W)$.

All the constructions in this section depend continuously on these choices and could be included in our spaces of umkehr data. However, fixing these from the start simplifies the proofs of compatibility with disjoint union and composition later on.

3.1. Heuristics for the setup of the construction. We will start by explaining the idea behind the construction of $\beth$ and the motivation behind the choices made during the construction. In the rest of this section we will make this construction precise except for a hard technical proposition, whose proof will be section 4 of this chapter.

In the previous section we saw how to construct the degree zero string operation $\daleth$ associated to a single nice $\mathcal{B}$-labelled cobordism graph. The higher string operations arise when one asks the question whether we can construct these operations in families. With a family of operations, we heuristically mean a set of operations for a set of nice $\mathcal{B}$-labelled cobordism graphs related by morphisms, which are compatible with these morphisms. The universal family will naturally be parametrized by a space which is homotopy equivalent to $|\mathsf{Fn}|$ and exactly because it is universal, it suffices to construct this one universal family operation to get all family operations.

What we need to do first is put the mapping spaces that appear as domain and codomain of $\daleth$ in a family form as well. For example, to put space of maps $M^{\partial_{in}\Gamma}$ from the incoming boundary to M in a family form, one needs to glue all values of the functor $M^{\partial_{in}-} : \mathsf{Fn} \to \mathsf{Top}$ over the space $|\mathsf{Fn}|$. This can be achieved by taking the homotopy colimit $\operatorname{hocolim} M^{\partial_{in}-}$ over the diagram Fn. There is a nice explicit description of this homotopy colimit as a fat geometric realisation.

The suspension functor Σ^∞ commutes with homotopy colimits, so the homotopy colimit $\operatorname{hocolim} \Sigma^\infty (M^{\partial_{in}-})_+$ is easy to understand. For the codomain of the map $\daleth$ in theorem 5.6 we have to be more careful, but for the moment we simply need to note that homotopy colimits of spectra exist and hence we can make sense of $\operatorname{hocolim}(M^{\partial_{out}-})^\kappa$ as a spectrum as well. It will be a some kind of limit of Thom spectra which we call an ind-Thom spectrum.

In the previous discussion we have forgotten about the spaces of umkehr data Umk. These should be twisted into the diagram defining the homotopy colimits by replacing Fn by the topological category $\mathsf{Fn} \int \mathrm{Umk}$. This would be possible if only Umk was a functor.

Now we come to a crucial point. It turns out that if $\Gamma' \to \Gamma$ collapses some edges, then elements of $\mathrm{Umk}(\Gamma')$ can not naively be used to construct elements of $\mathrm{Umk}(\Gamma)$. We conclude that in the current setup Umk can not be made into a functor. Naively, one might try to solve this by just modifying Umk appropriately, but then one runs into the trouble that there are many morphisms into and out of Γ', which means the set of extra data and restrictions one needs to take into account quickly becomes intractable.

The trick is to replace Fn by its category of simplices $\nabla\mathsf{Fn}$, which has as objects sequences $(\Gamma_n \to \ldots \to \Gamma_0)$ of composable morphisms of Fn and as morphisms the compositions of the boundary map of the nerve of Fn. The geometric realisation of $\nabla\mathsf{Fn}$ is homotopy equivalent to Fn so we could replace Fn with $\nabla\mathsf{Fn}$ if we wanted to. To get a feeling for why it is a good choice, consider $\nabla\mathsf{Fn}$ as the category which knows about Γ_n but also about a finite set of possible futures for it. In other words, using this category we can "look into the future" because one already knows what graph one will map to some point.

Using this knowledge we will define a functor $\mathrm{Umk} : \nabla\mathsf{Fn} \to \mathsf{Top}$ which extends the definition of Umk to objects with sequences of length larger than 1. It essentially includes umkehr data for all graphs in a composable sequence together with data to make it functorial. We have to be a bit careful here because we haven't found all restrictions on our spaces of choices for *compatible* umkehr maps yet.

The additional source of restrictions on Umk is that in the end we not just want a map for each simplex in $\nabla\mathsf{Fn}$, but we want to define a map of spectra $\sqsupset$ from $\operatorname{hocolim}\Sigma^\infty M^{\partial_{in}-}$ to $\operatorname{hocolim}(M^{\partial_{out}-})^\kappa$. We need to choose our umkehr data such that $\urcorner$ makes sense over each of these simplices, but also that these maps coincide along the glueings of the boundary. To do this, it turns out that Fn needs to contain data of splittings of certain linear maps and gets replaced by $\mathsf{Fn}_/$. On this topological category we can define a functor $\mathrm{Umk} : \nabla\mathsf{Fn}_/ \to \mathsf{Top}$.

For this functor Umk, the Grothendieck construction $\nabla\mathsf{Fn}_/ \int \mathrm{Umk}$ makes sense as a topological category. Our homotopy colimits will be over this category. It turns out that we can assume that our homotopy colimits are glued from pieces of the form $\Delta^n \times X$ for some space X depending on simplices in the nerve of Fn and the functors we are taking the homotopy colimit over. Then the space X of course includes the mapping space $M^{\partial_{in}-}$, but also contains the subspace of the space of all umkehr data which makes $\urcorner$ compatible with the glueings. These spaces have the property that we can not only define the map $\urcorner$ for each simplex but these glue as well to obtain the map $\sqsupset$!

3.2. The supporting category $\overline{\nabla\mathsf{Fn}_/}$. Previously, we claimed that Fn needs to be replaced by a topological category which contains the correct data to construct the map $\sqsupset$. This will be done in three stages:

(1) Replace Fn by the simplex category $\nabla\mathsf{Fn}$.
(2) Replace $\nabla\mathsf{Fn}$ by the topological category $\nabla\mathsf{Fn}_/$ which contains data about splittings.
(3) Interweave the data of the functor Umk into $\nabla\mathsf{Fn}_/$ to obtain $\overline{\nabla\mathsf{Fn}_/}$.

Note that this description differs slightly from the recipe given earlier where we said that first the splittings needed to be introduced, i.e. Fn replaced by $\mathsf{Fn}_/$ and only then we should pass to the simplex category. Both recipes give the same category and Godin [**God07**, section 3.2] in fact introduces splittings first, but we find our recipe more convenient.

3.2.1. *The category* $\nabla\mathsf{Fn}$. Recall that for a category C we can define the simplex category $\nabla\mathsf{C}$ with objects the k-simplices of the nerve $N\mathsf{C}$ for $k \geq 0$ and morphism the maps induced by the boundary maps. Corollary B.31 tells us that $|\nabla\mathsf{C}| \simeq |\mathsf{C}|$.

If we apply this construction for Fn we obtain the category $\nabla\mathsf{Fn}$ with a geometric realisation homotopy equivalent to $|\mathsf{Fn}|$. The objects of $\nabla\mathsf{Fn}$ are sequences of objects with composable morphisms $(\Gamma_n \to \ldots \to \Gamma_0)$ and such an object admits a morphism to another object $(\Gamma_{i_k} \to \ldots \to \Gamma_{i_0})$ with $0 \leq i_0 < \ldots < i_k \leq n$ if and only if the katter sequence is obtained by removing or composing the morphisms in the original sequence.

To be precise, if $(\Gamma_n \to \ldots \to \Gamma_0)$ has morphisms $f_i : \Gamma_{i+1} \to \Gamma_i$ for $0 \leq i \leq n-1$, then we define f_i^j for $j > i$ to be the composition $f_{j-1} \circ f_{j-2} \circ \ldots \circ f_i$. In particular we conclude that there is morphism from $(\Gamma_n \to \ldots \to \Gamma_0)$ to $(\Gamma_{i_k} \to \ldots \to \Gamma_{i_0})$ then the latter must have morphisms $\Gamma_{i_{j+1}} \to \Gamma_{i_j}$ given by $\phi_{i_j}^{i_{j+1}}$. We conclude that if there is a morphism between two sequences, then it is unique and given by removing or composing composable morphisms.

Finally, we fix some notation for the objects and morphsism of $\nabla\mathsf{Fn}$.

Convention 5.14. An object of $\nabla\mathsf{Fn}$ will generally be denoted σ and a morphism $\phi : \sigma \to \tilde{\sigma}$.

3.2.2. *The category* $\mathsf{Fn}_/$. To be able to glue the Thom spectra appearing on the right hand side of equation 2, it turns that we need to compare the virtual vector bundles κ_Γ for different graphs Γ. In particular for a morphism $\Gamma' \to \Gamma$, we want to encode the data of isomorphisms of vector bundles over M^Γ

$$\kappa^-_{\Gamma'}|_{M^\Gamma} \cong \kappa^-_\Gamma \oplus \frac{\kappa^-_{\Gamma'}|_{M^\Gamma}}{\kappa^-_\Gamma} \qquad \text{and} \qquad \kappa^+_{\Gamma'}|_{M^\Gamma} \cong \kappa^+_\Gamma \oplus \frac{\kappa^-_{\Gamma'}|_{M^\Gamma}}{\kappa^-_\Gamma}$$

Note that the restriction of $\kappa^\pm_{\Gamma'}$ makes sense because we have a map $M^\Gamma \to M^{\Gamma'}$ along which we can pull it back and this map is in fact an inclusion.

We will look at the isomorphism on the left first, because κ^-_Γ is simple and hence this is the easier one. Recall that the negative part of κ_Γ is given by the trivial vector bundle $W^{v(\Gamma\setminus s(\partial_{in}\Gamma))}$.

This can be written in a simpler fashion: as before, we define eV be to the set of vertices in $\Gamma \backslash s(\partial_{in}\Gamma)$ and call these the extra vertices. It is clearly true that

$$W^{v(\Gamma\backslash s(\partial_{in}\Gamma))} = W^{eV} \cong W \otimes_{\mathbb{R}} \mathbb{R}^{eV}$$

Let $f : \Gamma' \to \Gamma$ be a morphism in Fn. This includes a map $q : eV' \to eV$ between the sets of the extra vertices. Thus we have an induced linear embedding $d : \mathbb{R}^{eV} \to \mathbb{R}^{eV'}$. This can be written as a product of diagonal maps $d_v : \mathbb{R} \to \mathbb{R}^{q^{-1}(v)}$ for each element of $v \in eV$. A section of d_v is a map $\alpha_v : \mathbb{R}^{q^{-1}(v)} \to \mathbb{R}$ such that $\alpha_v \circ d_v = id_{\mathbb{R}}$. Note that if we let p_v be the projection map $p_v : \mathbb{R}^{q^{-1}(v)} \to \mathbb{R}^{q^{-1}(v)}/\mathrm{im}(d_v)$, then we get a short exact sequence:

$$0 \longrightarrow \mathbb{R} \xrightarrow{d_v} \mathbb{R}^{q^{-1}(v)} \xrightarrow{p_v} \mathbb{R}^{q^{-1}(v)}/\mathrm{im}(d_v) \longrightarrow 0$$

The section α_v is equivalent to a section $\beta_v : \mathbb{R}^{q^{-1}(v)}/\mathrm{im}(d_v) \to \mathbb{R}^{q^{-1}(v)}$ by the demanding that the following equation holds

$$d_v \circ \alpha_v + \beta_v \circ p_v = id_{\mathbb{R}^{q^{-1}(v)}}$$

because d_v in injective and p_v surjective. We only need to check that $\alpha_v \circ d_v = id$ if and only if $p_v \circ \beta_v = id$ under the condition that $d_v \circ \alpha_v + \beta_v \circ p_v = id$. But this is easy to see. We call the linear map α_v a splitting and β_v the corresponding dual splitting.

Definition 5.15. The space $\mathrm{Split}(f)$ consists of sequences of elements $\alpha_v \in \mathrm{Hom}_{\mathsf{Vect}}(\mathbb{R}^{q^{-1}(v)}, \mathbb{R})$ for all $v \in eV$ satisfying the condition $\alpha_v \circ d_v = id$ and having the subspace topology of the usual topology on the sets of linear maps between two finite-dimensional vector spaces.

We want to check that this space has the correct properties. To do this, we first take a closer look at κ^+_Γ and $\kappa^+_{\Gamma'}$. The former is constructed as the following pullback, and the latter is of course similar:

$$\begin{array}{ccc}
\kappa^+_\Gamma & \longrightarrow & \bigoplus_{v\in eV} \nu_{b(v)} \oplus \bigoplus_{e\in eH} Tb(e) \oplus \bigoplus_{v\in siV} Tb(v)^{\mathrm{coval}(v)} \\
\downarrow & & \downarrow \\
M^\Gamma & \longrightarrow & \prod_{v\in eV} b(v) \times \prod_{e\in eE} b(e) \times \prod_{v\in siV} b(v)^{\mathrm{coval(v)}}
\end{array}$$

We will compare these pullbacks for two graphs Γ and Γ'. Under the bottom row, there is a map $M^\Gamma \to M^{\Gamma'}$ given by precomposition with the geometric realisation of $\Gamma' \to \Gamma$ for the left space and a map

$$\prod_{v\in eV} b(v) \times \prod_{e\in eH} b(e) \times \prod_{v\in siV} b(v)^{\mathrm{coval(v)}} \to \prod_{v'\in eV'} b(v') \times \prod_{e'\in eH'} b(e') \times \prod_{v'\in siV'} b(v')^{\mathrm{coval(v')}}$$

for the right space, given by mapping the component of a vertex by a diagonal map to its preimage, the component of an edge to the component of its preimage and given by the identity on the component of the collapsed incoming vertices. The latter makes sense because if two incoming vertices of this collapsed type would be identified, then the number of boundary components changes. These two maps are compatible with the two maps on the bottom row of the pullback diagram. There is a corresponding map between the vector bundles over the right space

$$\bigoplus_{v\in eV} \nu_{b(v)} \oplus \bigoplus_{e\in eE} Tb(e) \oplus \bigoplus_{v\in siV} Tb(v)^{\mathrm{coval}(v)} \to \bigoplus_{v'\in eV'} \nu_{b(v')} \oplus \bigoplus_{e'\in eE'} Tb(e') \oplus \bigoplus_{v'\in siV'} Tb(v')^{\mathrm{coval}(v')}$$

This induces a map $\kappa^+_\Gamma \to \kappa^+_{\Gamma'}$ of the pullback vector bundles.

If we now look at the quotient bundle $\kappa^+_{\Gamma'}/\kappa^+_\Gamma$ over M^Γ, then we see that the only components which are quotiented out are those coming from the collapsed edges. To be precise, we can only collapse subtrees that are fully, i.e. including endpoints, labelled by a single label, say A. Thus each single collapsed edge contributes $\nu_A \oplus TA \oplus \nu_A$, while there is a map ν_A into it diagonally, etc. We see that the quotient can therefore naturally be identified with a number of copies of the trivial bundle W equal to the number of collapsed edges and this is exactly $\frac{\kappa^-_{\Gamma'}}{\kappa^-_\Gamma}$ by the previous remark.

Proposition 5.16. *A point in* $\mathrm{Split}(f)$ *gives unique isomorphisms*

$$\kappa^-_{\Gamma'}|_{M^\Gamma} \cong \kappa^-_\Gamma \oplus \frac{\kappa^-_{\Gamma'}|_{M^\Gamma}}{\kappa^-_\Gamma} \qquad \textit{and} \qquad \kappa^+_{\Gamma'}|_{M^\Gamma} \cong \kappa^+_\Gamma \oplus \frac{\kappa^-_{\Gamma'}|_{M^\Gamma}}{\kappa^-_\Gamma}$$

Proof. It suffices to produce maps

$$\phi^- : \kappa^-_\Gamma \oplus \frac{\kappa^-_{\Gamma'}|_{M^\Gamma}}{\kappa^-_\Gamma} \to \kappa^-_{\Gamma'}|_{M^\Gamma} \qquad \text{and} \qquad \phi^+ : \kappa^+_\Gamma \oplus \frac{\kappa^-_{\Gamma'}|_{M^\Gamma}}{\kappa^-_\Gamma} \to \kappa^+_{\Gamma'}|_{M^\Gamma}$$

which are isomorphisms on each fiber. These will be continuous as well, they are then isomorphisms of vector bundles.

Let α be a splitting and β the corresponding dual splitting. Then the map ϕ^- is given by $id_W \otimes d_v \oplus id_W \otimes \beta$. Note here that $id_w \otimes d_v$ coincides with the natural embedding $\kappa^-_\Gamma \hookrightarrow \kappa^-_{\Gamma'}|_{M^\Gamma}$. The map ϕ^+ is given by composing $\iota \oplus id_W \otimes \beta$ with the isomorphism explained previously, where ι is the natural embedding $\iota : \kappa^+_\Gamma \hookrightarrow \kappa^+_{\Gamma'}|_{M^\Gamma}$ induced by diagonal maps. By the previous remarks both ϕ_+ and ϕ_- are isomorphisms on each fiber. □

But furthermore, this construction can be extended to a functor $\nabla\mathsf{Fn} \to \mathsf{Top}$.

Proposition 5.17. *The map which assigns to an object* $(\Gamma_n \to \ldots \to \Gamma_0)$ *the space* $\coprod_{0 \leq i \leq n-1} \mathrm{Split}(f_i)$ *can be made into a functor* $\mathrm{Split} : \nabla\mathsf{Fn} \to \mathsf{Top}$..

Proof. It suffices to give Split on morphisms and check that this behaves well with respect to identities and composition. If a morphism is removed, we just project away that component of $\mathrm{Split}(\Gamma_n \to \ldots \to \Gamma_0)$. If two arrows f_i and f_{i-1} are composed, then we define a map $\mathrm{Split}(f_i) \times \mathrm{Split}(f_{i-1}) \to \mathrm{Split}(f_i \circ f_{i-1})$ as follows. First note that for the composition $(q_{\Gamma_{i+2} \to \Gamma_i})^{-1}(v)$ for $v \in eV_i$ is given by $\coprod_{v' \in (q_{\Gamma_{i+1} \to \Gamma_i})^{-1}(v)} (q_{\Gamma_{i+2} \to \Gamma_{i+1}})^{-1}(v')$. The new d'_v will be given by $\left(\prod_{v' \in (q_{\Gamma_{i+1} \to \Gamma_i})^{-1}(v)} d_{v'}\right) \circ d_v$. We set the new α'_v to be equal to

$$\alpha'_v = \alpha_v \circ \left(\prod_{v' \in (q_{\Gamma_{i+1} \to \Gamma_i})^{-1}(v)} \alpha_{v'} \right)$$

This is continuous because the topology on these spaces of maps is compatible with composition. It is clear that this definition extends to compositions of more than two morphisms as well, and that this is compatible with composition of morphisms in $\nabla\mathsf{Fn}$, because all that happens is that linear maps are put together in products and composed. Finally, when applied to an identity morphism in $\nabla\mathsf{Fn}$, nothing happens because no composable sequences are discarded nor are any composed. □

We will now interweave the splitting data into our category $\nabla\mathsf{Fn}$ using the Grothendieck construction.

Definition 5.18. We define $\nabla\mathsf{Fn}_/$ to be the topological category $\nabla\mathsf{Fn} \int \mathrm{Split}$.

Remark 5.19. One may ask why we choose to topologize the splittings, because this causes quite some complications later on. The reason is that if we didn't, we would change the homotopy type of our category and we would no longer be constructing a HCFT.

More concretely, the object space of $\nabla\mathsf{Fn}_/$ consists of a disjoint union of $\mathrm{Split}(\sigma)$ over all objects of $\nabla\mathsf{Fn}$ and the morphism space is given by a disjoint union

$$\coprod_{\sigma \in \mathrm{Ob}(\nabla\mathsf{Fn})} \mathrm{Split}(\sigma) \times \{\phi \in \mathrm{Hom}(\nabla\mathsf{Fn}) | s(\phi) = \sigma\}$$

Hence each component consists of pairs (ϕ, x) where $\phi : \sigma \to \sigma'$ and $x \in \mathrm{Split}(\sigma)$. We set $s(\phi, x) = x$ and $t(\phi, x) = \mathrm{Split}(\phi)(x)$. The identity map 1 sends x to (id, x).

Note that the projection map $\mathrm{Split}(\Gamma_n \to \ldots \to \Gamma_0) \to *$ induces a natural transformation between Split and the functor c_* with constant value a point $*$. This induces a topological functor between $\nabla\mathsf{Fn} \int \mathrm{Split}$ and $\nabla\mathsf{Fn} \int c_* \cong \nabla\mathsf{Fn}$. This we call the forgetful functor.

Proposition 5.20. *The value of the functor* Split *on every object is a contractible space and hence the forgetful functor* $\nabla\mathsf{Fn}_/ \to \nabla\mathsf{Fn}$ *induces a homotopy equivalence upon geometric realisation.*

PROOF. Each $\mathrm{Split}(f)$ is non-empty and convex, therefore contractible. The second statement follows by applying topological Quillen A. $\square$

Convention 5.21. Generally, we will denote a splitting by x. An object of $\nabla\mathsf{Fn}_/$ will be denoted (σ, x), a morphism by $\psi = (\phi, x)$.

3.2.3. *The category* $\overline{\nabla\mathsf{Fn}_/}$. In this section we extend in a similar way as before the category $\mathsf{Fn}_/$ by including the information necessary to construct the umkehr maps. This means that we will also see the exact properties that this data needs to have.

We first recall that in appendix A we discussed the smash product $\wedge$, but also the tensoring of $\mathsf{Spectra}$ over Top which we denoted by the same symbol $\wedge$. The latter appears in the next proposition. In this proposition we will also come across the category $\nabla\sigma$ of subsimplices of a simplex σ in $\nabla\mathsf{Fn}$. If σ consists of n composable morphisms, then $|\nabla\sigma| \cong \Delta^n$.

We will need a proposition about the existence of functor given compatible umkehr data, whose proof will be the next section of this chapter. Together with this data comes an operation $\daleth^\sigma$ for each simplex $\sigma \in \mathrm{Ob}(\nabla\mathsf{Fn})$. The idea is that the $\daleth^\sigma$ will be the restrictions of $\daleth$ to a simplex. The third condition of the functor Umk, called "compatibility of the operations" of the proposition will allow us to glue these. In the context of topological categories, the correct notion of a functor is a space with action. This is described in appendix B, but the reader should just think of it as a functor.

Remark 5.22. In describing the conditions that Umk must satisfy, we use some functors will only be defined following the proposition. Their definition does not use Umk, but we thought it was better to finish the construction of $\overline{\nabla\mathsf{Fn}_/}$ in one go.

Proposition 5.23. *There exists a space* Umk *with action of* $\nabla\mathsf{Fn}_/$ *satisfying the following properties.*

Contractibility: *The object space* $\mathrm{Umk}(\mathrm{Ob}(\nabla\mathsf{Fn})_/)$ *is a disjoint union of non-empty contractible components* $\mathrm{Umk}(\sigma)$ *indexed by* $\sigma \in \mathrm{Ob}(\nabla\mathsf{Fn})$.

Existence of operations: *For each component of the object space given by* σ, *where* $\sigma = (\Gamma_n \to \ldots \to \Gamma_0)$ *there is an operation:*

$$\daleth^\sigma : (|\nabla\sigma| \times \mathrm{Umk}(\sigma))_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma_0})_+ \to (M^{\partial_{out}\Gamma_0})^{\kappa_\sigma} \tag{3}$$

which is equal to the operation $\urcorner$ *of a single nice* $\mathcal{B}$*-labelled cobordism graph up to a difference in virtual bundles in the codomain.*

Compatibility of operations: *Let* ψ *be a morphism between the components of* σ *and* $\tilde\sigma$, *where* σ *is as above and* $\tilde\sigma = (\Gamma_{i_k} \to \ldots \Gamma_{i_0})$, *note that* $|\mathcal{N}\tilde\sigma|$ *includes into* $|\mathcal{N}\sigma|$. *Then we have a commutative diagram:*

$$\begin{tikzcd}
(|\nabla\tilde\sigma| \times \mathrm{Umk}(\sigma))_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma_0})_+ \arrow[r, "\daleth^\sigma|_{|\nabla\tilde\sigma|}"] \arrow[d] & (M^{\partial_{out}\Gamma_0})^{\kappa_\sigma} \arrow[d] \\
(|\nabla\tilde\sigma| \times \mathrm{Umk}(\tilde\sigma))_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma_{i_0}})_+ \arrow[r, "\daleth^{\tilde\sigma}"'] & (M^{\partial_{out}\Gamma_{i_0}})^{\kappa_{\tilde\sigma}}
\end{tikzcd} \tag{4}$$

where the vertical maps will be explained later, but are purely functorial.

Definition 5.24. We define $\overline{\nabla\mathsf{Fn}_/}$ to be the topological category $\nabla\mathsf{Fn}_/ \int \mathrm{Umk}$.

More concretely, the object space of $\overline{\nabla\mathsf{Fn}_/}$ consists of a disjoint union of components $\mathrm{Umk}(\sigma)$, which contain the information of $\mathrm{Split}(\sigma)$ using the map $p : \mathrm{Umk}(\sigma) \to \mathrm{Split}(\sigma)$ which is part of the data of a topological functor with domain $\nabla\mathsf{Fn}_/$.

The morphism space of $\overline{\nabla \mathsf{Fn}_{/}}$ is given by the following pullbacks of diagram

$$\begin{tikzcd}
\mathrm{Umk}(\mathrm{Ob}(\nabla \mathsf{Fn}_{/})) \times_{\mathrm{Split}} \mathrm{Hom}(\nabla \mathsf{Fn}_{/}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathrm{Hom}(\nabla \mathsf{Fn}_{/}) \arrow[d, "s"] \\
\mathrm{Umk}(\mathrm{Ob}(\nabla \mathsf{Fn}_{/})) \arrow[r, "p"'] & \mathrm{Ob}(\nabla \mathsf{Fn}_{/})
\end{tikzcd}$$

Using our convention on the notation of elements in $\mathrm{Hom}(\nabla \mathsf{Fn}_{/})$ and $\mathrm{Ob}(\nabla \mathsf{Fn}_{/})$, we get the following equivalent description of $\mathrm{Hom}(\overline{\nabla \mathsf{Fn}_{/}})$:

$$\coprod_{\sigma \in \nabla \mathsf{Fn}} \mathrm{Umk}(\sigma) \times \{(\phi, x) \in \mathrm{Hom}(\nabla \mathsf{Fn}) | s(\phi, x) = (\sigma, x)\}$$

The source and target maps are given by $s(y, \phi, x) = y$ and $t(y, \phi, x) = \mathrm{Umk}(\phi, x)(y)$. The composition $(y', \phi', x') \circ (y, \phi, x)$ exists if ϕ' and ϕ are composable, i.e. $\phi : \sigma \to \sigma'$ and $\phi' : \sigma' \to \sigma''$ in $\nabla \mathsf{Fn}$, and if in addition we have that $x' = \mathrm{Split}(\phi)(x)$ and $y' = \mathrm{Umk}(\phi, x)(y)$. The identity map 1 sends y to $(y, id, p(y))$.

We have a forgetful functor $\overline{\nabla \mathsf{Fn}_{/}} \to \nabla \mathsf{Fn}$ which induces a homotopy equivalence upon geometric realisation. This functor factors as a composition of two forgetful functors $\overline{\nabla \mathsf{Fn}_{/}} \to \nabla \mathsf{Fn}_{/} \to \nabla \mathsf{Fn}$. The second of these we have already seen, the first is induced by the projection of $\mathrm{Umk}(\sigma)$ to $\mathrm{Split}(\sigma)$.

3.3. The functors $\Sigma^\infty(M^{\partial_{in}-})_+$ and $\Sigma^\infty(M^{\partial_{out}-})_+$. We will now look at the functors which associate to a simplex with umkehr data the suspension of space of maps to M from the incoming, respectively outgoing boundary of the last graph in the simplex.

To be precise, the functor $\Sigma^\infty(M^{\partial_{in}-})_+ : \overline{\nabla \mathsf{Fn}_{/}} \to \mathsf{Spectra}$ is defined be the suspension of the composition of the forgetful functor $\overline{\nabla \mathsf{Fn}_{/}} \to \nabla \mathsf{Fn}$ with the ordinary functor $M^{\partial_{in}-}$: this ordinary functor $\nabla \mathsf{Fn} \to \mathsf{Top}$ sends an object $\sigma = (\Gamma_n \to \dots \Gamma_0)$ of $\nabla \mathsf{Fn}$ to the space $M^{\partial_{in}\Gamma_0}$ and a morphism $\phi : \sigma \to \tilde{\sigma}$, where σ is as before and $\tilde{\sigma} = (\Gamma_{i_k} \to \dots \to \Gamma_{i_0})$, to the continuous map $M^{\partial_{in} f_0^{i_0}} : M^{\partial_{in}\Gamma_0} \to M^{\partial_{in}\Gamma_{i_0}}$ given by precomposition with the geometric realisation of the composition of maps of $\mathcal{B}$-labelled cobordism graphs $f_0^{i_0} : \Gamma_{i_0} \to \Gamma_0$.

Similarly, one can replace the incoming boundary with the outgoing boundary or the full graph to get the following definitions.

Definition 5.25. There are functors M^-, $M^{\partial_{in}-}$ and $M^{\partial_{out}-}$ from $\overline{\nabla \mathsf{Fn}_{/}} \to \mathsf{Top}$ all given on objects as follows: it sends a simplex $(\Gamma_n \to \dots \Gamma_0)$ to the mapping space of continuous maps to M from the relevant part of Γ_0, the last graph in the simplex.

By suspending each of the spaces we will obtain the functors into spectra which appear in equation 3.

Definition 5.26. There are functors $\Sigma^\infty(M^-)_+$, $\Sigma^\infty(M^{\partial_{in}-})_+$ and $\Sigma^\infty(M^{\partial_{out}-})_+$ from $\overline{\nabla \mathsf{Fn}_{/}}$ to $\mathsf{Spectra}$ all given on objects by the suspension of the mapping space of continuous maps to M from the relevant part of last graph in a simplex.

3.4. The functors κ and $(M^{\partial_{out}-})^\kappa$. In this section we construct the functor $\kappa : \overline{\nabla \mathsf{Fn}_{/}} \to \overline{\mathsf{VirtB}}$ which will extend the virtual bundle κ_Γ appearing in theorem 5.6. Because there is a forgetful functor $\overline{\nabla \mathsf{Fn}_{/}} \to \nabla \mathsf{Fn}_{/}$, it suffices to define the functor on the latter category and set κ equal to the composition of this functor with the forgetful functor.

We will first define a functor of virtual bundles κ' over M^- and by pull back define the functor κ of virtual bundles over $M^{\partial_{out}-}$.

There is one problem though, because we haven't defined the topological category $\overline{\mathsf{VirtB}}$ yet. The reason VirtB no longer suffices is that we made our space of splittings continuous. We will construct $\overline{\mathsf{VirtB}}$ as a category enriched in topological spaces, which is a particular type of topological category where the object space is discrete.

Definition 5.27. The category $\overline{\mathsf{VirtB}}$ has the same objects as the category VirtB in definition 4.17. The morphism spaces will have the sets of morphsism of VirtB as underlying set. However, in $\overline{\mathsf{VirtB}}$ the set $\mathrm{Hom}_{\mathsf{VirtB}}(\kappa, \lambda)$ will be topologized as follows: a morphism $(f, \theta, \phi_+, \phi_-)$ is considered as an element of the connected component $GL(f^*\lambda_+ \oplus \theta, \mu_+) \times GL(f^*\lambda_- \oplus \theta, \mu_-) \ni (\phi_+, \phi_-)$.

This is the correct codomain for our functor κ'.

Proposition 5.28. *There is a functor $\kappa' : \nabla\mathsf{Fat}_/ \to \overline{\mathsf{VirtB}}$ defined on objects by sending the component over $\sigma = (\Gamma_n \to \ldots \to \Gamma_0)$ to the virtual bundle $(M^{f_0^n})^*\kappa_{\Gamma_n}$ over M^{Γ_0}, where $f_0^n : \Gamma_n \to \Gamma_0$ is the composition $\Gamma_n \to \ldots \to \Gamma_0$ and κ_{Γ_n} is the virtual bundle as defined in theorem 5.6.*

Proof. It suffices to define κ' on morphisms and to show that it respects composition and identities. Recall that the maps $M^{f_i^j} : M^{\Gamma_0} \to M^{\Gamma_{i_0}}$ are induced by precomposition with the geometric realisation of the composition of morphisms $f_i^j : \Gamma_j \to \Gamma_i$.

We first do the construction for one morphism and then show it is continuous. Let $\psi = (\phi, x) : (\sigma, x) \to (\tilde{\sigma}, \mathrm{Split}(\phi)(x))$ be a morphism, and suppose that σ is as in the statement of the proposition and $\tilde{\sigma} = (\Gamma_{i_k} \to \ldots \to \Gamma_{i_0})$. Then we define $\kappa'(\psi) = (f, \theta, \phi_+, \phi_-)$ as follows, where $f = M^{f_0^{i_0}} : M^{\Gamma_0} \to M^{\Gamma_{i_0}}$ and $\phi_\pm$ will be isomorphisms $(M^{f_0^{i_k}})^*\kappa^\pm_{\Gamma_{i_k}} \oplus \theta \cong (M^{f_0^n})^*\kappa^\pm_{\Gamma_n}$.

Note that the discussion before our definition of the splitting spaces implies that there is a canonical map $\kappa^-_{\Gamma_{i_0}} \to (M^{f_{i_k}^n})^*\kappa^-_{\Gamma_n}$ given the diagonal map for collapses vertices. The bundle θ will be given by the quotient bundle by pulling back this quotient even further to M^{Γ_0}:

$$\theta = \frac{(M^{f_0^n})^*\kappa^-_{\Gamma_n}}{(M^{f_0^{i_k}})^*\kappa^-_{\Gamma_{i_k}}}$$

The element $\mathrm{Split}(\phi)(x)$ particularly includes as a component an element α of $\mathrm{Split}(f_{i_k}^n)$ and by proposition 5.16 this gives natural isomorphisms of vector bundles over $M^{\Gamma_{i_k}}$,

$$(M^{f_{i_k}^n})^*\kappa^-_{\Gamma_n} \cong \kappa^-_{\Gamma_{i_k}} \oplus \frac{(M^{f_{i_k}^n})^*\kappa^-_{\Gamma n}}{\kappa^-_{\Gamma_{i_k}}} \qquad \text{and} \qquad (M^{f_{i_k}^n})^*\kappa^+_{\Gamma_n} \cong \kappa^+_{\Gamma_{i_k}} \oplus \frac{(M^{f_{i_k}^n})^*\kappa^-_{\Gamma n}}{\kappa^-_{\Gamma_{i_k}}}$$

If we pull this back to M^{Γ_0} using $M^{f_0^{i_k}}$ to obtain isomorphisms of vector bundles

$$\phi_- : (M^{f_0^n})^*\kappa^-_{\Gamma_n} \cong (M^{f_0^{i_k}})^*\kappa^-_{\Gamma_{i_k}} \oplus \frac{(M^{f_0^n})^*\kappa^-_{\Gamma n}}{(M^{f_0^{i_k}})^*\kappa^-_{\Gamma_{i_k}}}$$

$$\phi_+ : (M^{f_0^n})^*\kappa^+_{\Gamma_n} \cong (M^{f_0^{i_k}})^*\kappa^+_{\Gamma_{i_k}} \oplus \frac{(M^{f_0^n})^*\kappa^-_{\Gamma n}}{(M^{f_0^{i_k}})^*\kappa^-_{\Gamma_{i_k}}}$$

To show that this association is continuous, it suffices to note in proposition 5.16 the maps ϕ^+ and ϕ^- were constructed as a direct sum of a constant natural map and a tensor product $id_W \otimes \beta$, where β depends continuously on the point $\alpha \in \mathrm{Split}(f_{i_k}^n)$.

Finally, we show that this construction is compatible with identities and composition. For the former note that if $n = i_k$, which in particular holds for the identity, then $\theta = 0$ and f, ϕ^+ and ϕ^- are the identity map. The latter, we remark that Split is functorial. □

Definition 5.29. The virtual bundle functor $\kappa : \overline{\nabla\mathsf{Fn}}_/ \to \overline{\mathsf{VirtB}}$ is given on objects and morphisms by the pullback along the natural maps $M^{\partial_{out}-} \to M^-$ of the value of κ' on objects and morphisms.

Finally, it is now easy to define the Thom spectrum functor $(M^{\partial_{out}-})^\kappa$ that appears on the right hand side of 2.

Definition 5.30. The functor $(M^{\partial_{out}-})^\kappa : \overline{\nabla\mathsf{Fn}_/} \to \mathsf{Spectra}$ is given by the composition of κ with the Thom spectrum functor.

Note that as the notation suggests, over the component of σ $(M^{\partial_{out}-})^\kappa$ assigns the Thom spectrum of a virtual bundle over the mapping space obtained by applying the functor $M^{\partial_{out}-}$ to σ.

3.5. Construction the map $\beth$ of spectra. Now that all terms in equation 3 make sense, we will try to glue all that $\beth^\sigma$ together to obtain a map $\beth$, which will be a map of spectra between homotopy colimits:

$$\beth : \underset{\nabla \mathsf{Fat}_/}{\operatorname{hocolim}} \Sigma^\infty (M^{\partial_{in}-})_+ \to \underset{\nabla \mathsf{Fat}_/}{\operatorname{hocolim}} (M^{\partial_{out}-})^\kappa$$

The trick is to note that both sides can be modelled by glueing suspensions of fat geometric realisations of a simplicial spectrum. To be precise, by using the appropriate modifications of definition B.56 for the case of spectra we have that

$$\underset{\nabla \mathsf{Fat}_/}{\operatorname{hocolim}} (\Sigma^\infty M^{\partial_{in}-})_+ = \left(\coprod_{n \in \mathbb{N} \sqcup \{0\}} \coprod_{\sigma_n \to \ldots \to \sigma_0 \in \mathcal{N}_n \nabla \mathsf{Fat}} (\Delta^n \times \operatorname{Umk}(\sigma_n))_+ \wedge \Sigma^\infty (M^{\partial_{in} \Gamma_0})_+ \right) / \sim_{\text{fat}}$$

$$\underset{\nabla \mathsf{Fat}_/}{\operatorname{hocolim}} (M^{\partial_{out}-})^\kappa = \left(\coprod_{n \in \mathbb{N} \sqcup \{0\}} \coprod_{\sigma_n \to \ldots \to \sigma_0 \in \mathcal{N}_n \nabla \mathsf{Fat}} (\Delta^n \times \operatorname{Umk}(\sigma_n))_+ \wedge (M^{\partial_{out} \Gamma_0})^\kappa \right) / \sim_{\text{fat}}$$

Because we work with the fat geometric realisation, $\sim_{\text{fat}}$ only identifies using the boundary maps. Note that we thus disregard the degenerary maps that are part of the structure of the simplicial spectrum.

Theorem 5.31. *There is a map of spectra*

$$\beth : \underset{\nabla \mathsf{Fat}_/}{\operatorname{hocolim}} \Sigma^\infty (M^{\partial_{in}-})_+ \to \underset{\nabla \mathsf{Fat}_/}{\operatorname{hocolim}} (M^{\partial_{out}-})^\kappa$$

which restricts to $\beth^\sigma|_{\Delta^n_{\sigma_n \to \ldots \sigma_0}}$ *on each simplex.*

Proof. We will first define the map $\beth^\tau$ for an index $\tau = \sigma_n \to \ldots \to \sigma_0$. It is a map from the component $(\Delta^n_\tau \times \operatorname{Umk}(\sigma_n))_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma_0})_+$ of the left homotopy colimit to the component $(\Delta^n_\tau \times \operatorname{Umk}(\sigma_n))_+ \wedge (M^{\partial_{in}\Gamma_0})^\kappa$ on the right homotopy colimit. We define it by taking the product of the identity map on $(\Delta^n \times \operatorname{Umk}(\sigma_n))_+$ with the map obtained by restricting $\beth^{\sigma_n} : (|\nabla \sigma_n| \times \operatorname{Umk}(\sigma)_+) \wedge \Sigma^\infty (M^{\partial_{in}\Gamma_0})_+ \to (M^{\partial_{out}\Gamma_0})^\kappa$ of proposition 5.23 to the geometric n-simplex $\Delta^n_\tau \hookrightarrow |\nabla \sigma_n|$.

It suffices to show that these maps glue along boundaries, this means that for each $0 \leq m \leq n$, we must show that a certain diagram commutes. For $0 \leq m \leq n-1$ we should have that the following diagram commutes:

$$\begin{tikzcd}
(\partial_m \Delta^n_\tau \times \operatorname{Umk}(\sigma_n))_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma_0})_+ \arrow[r, "\beth^\tau|_{\Delta^{n-1}_{\partial_m \tau}}"] \arrow[d, "\cong"'] & (\partial_m \Delta^n_\tau \times \operatorname{Umk}(\sigma_n))_+ \wedge (M^{\partial_{out}\Gamma_0})^\kappa \arrow[d, "\cong"] \\
(\Delta^{n-1}_{\partial_m \tau} \times \operatorname{Umk}(\sigma_n))_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma_0})_+ \arrow[r, "\beth^{\partial_m \tau}"'] & (\Delta^{n-1}_{\partial_m \tau} \times \operatorname{Umk}(\sigma_n))_+ \wedge (M^{\partial_{out}\Gamma_0})^\kappa
\end{tikzcd}$$

But this commutes because both the top and the bottom maps are made from restrictions of the same map $\beth^{\sigma_n}$.

Next we look at the case $m = n$. Suppose that $\sigma_{n-1} = (\Gamma_{i_k} \to \ldots \to \Gamma_{i_0})$. Then the following diagram should commute:

$$\begin{tikzcd}
(\partial_n \Delta^n_\tau \times \operatorname{Umk}(\sigma_n))_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma_0})_+ \arrow[r, "\beth^\tau|_{\Delta^{n-1}_{\partial_n \tau}}"] \arrow[d] & (\partial_m \Delta^n_\tau \times \operatorname{Umk}(\sigma_n))_+ \wedge (M^{\partial_{out}\Gamma_0})^\kappa \arrow[d] \\
(\Delta^{n-1}_{\partial_n \tau} \times \operatorname{Umk}(\sigma_{n-1}))_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma_{i_0}})_+ \arrow[r, "\beth^{\partial_n \tau}"'] & (\Delta^{n-1}_{\partial_n \tau} \times \operatorname{Umk}(\sigma_{n-1}))_+ \wedge (M^{\partial_{out}\Gamma_{i_0}})^\kappa
\end{tikzcd}$$

where the vertical maps are the natural functorial ones as given by the functors $M^{\partial_{in}-}$, $M^{\partial_{out}-}$, κ and the action of $\nabla \mathsf{Fn}_/$ on $\operatorname{Umk}(\sigma_n)$.

This diagram is commutative because the horizontal maps are obtained by restricting those of the the commutative diagram 4 in proposition 5.23 to subsimplices of $|\nabla \tilde{\sigma}|$. $\square$

4. Compatible umkehr maps, or the proof of proposition 5.23

In this section we will prove the main technical ingredient of the construction, proposition 5.23. It turns out that to prove the properties of Umk that are listed in that proposition, it is fact easiest to produce spaces umk which have slightly more refined properties.

In our proof of proposition 5.23 we will work our way backwards. This means that we first state a proposition, show how to derive proposition 5.23 from it, and then prove the proposition etc. We think that this allows the reader to best grasp the structure of the proof.

We recommend that the reader skips this proof upon reading the thesis for the first time. The proof can be summarized as: repeat the proof of 5.6 for the multiple graphs in a simplex at the same time, while making sure that the operations constructed very similarly to $\urcorner$ are compatible along the morphisms of the simplex.

4.1. Compatible umkehr maps on the level of spaces. The idea is to first reduce the proof from spectra to spaces. We did the other direction in our construction of the umkehr map $\urcorner$ for a single $\mathcal{B}$-labelled cobordism graph: there we first had a map of spaces and made in into a map of spectra. The reason for this was to simply the virtual bundles, but apart from that nothing required us. In this case it turns out to be easier to return to spectra, so that's what the next proposition will do.

Proposition 5.32. *There exists a space* Umk *with action of* $\nabla\mathsf{Fn}_/$ *which has the following properties:*

Contractibility: *The space* $\mathrm{Umk}(\mathrm{Ob}(\nabla\mathsf{Fn}_/))$ *is a disjoint union of non-empty contractible components* $\mathrm{Umk}(\sigma)$ *for* $\sigma \in \mathrm{Ob}(\nabla\mathsf{Fn})$.

Existence of operations: *For each component of the object space given by* $\sigma = (\Gamma_n \to \ldots \to \Gamma_0)$ *there is an operation:*

$$\urcorner^\sigma : |\nabla\sigma| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_0} \times \kappa_\sigma^- \to \mathrm{Thom}(\kappa_\sigma^+)$$

where $\kappa_\sigma^+ = ((M^{f_0^n})^* \kappa_{\Gamma_n}^+)|_{M^{\partial_{out}\Gamma_0}}$ *and* $\kappa_\sigma^- = ((M^{f_0^n})^* \kappa_{\Gamma_n}^-)|_{M^{\partial_{out}\Gamma_0}}$ *are both vector bundles over* $M^{\partial_{out}\Gamma_0}$*, the latter of which is trivial.*

Compatibility of operations: *For each morphism* $\psi : \sigma \to \tilde{\sigma}$*, the following diagram commutes:*

$$\begin{tikzcd}
|\nabla\tilde{\sigma}| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_0} \times \kappa_\sigma^+ \arrow[r, "\urcorner^\sigma|_{|\nabla\tilde{\sigma}|}"] \arrow[d] & \mathrm{Thom}(\kappa_\sigma^+) \arrow[d] \\
|\nabla\tilde{\sigma}| \times \mathrm{Umk}(\tilde{\sigma}) \times M^{\partial_{in}\Gamma_{i_0}} \times \left(\kappa_{\tilde{\sigma}}^- \oplus \frac{\kappa_\sigma^-|_{M^{\partial_{in}\Gamma_{i_0}}}}{\kappa_{\tilde{\sigma}}^-}\right) \arrow[r, "\urcorner^{\tilde{\sigma}} \wedge id"'] & \mathrm{Thom}\left(\kappa_{\tilde{\sigma}}^- \oplus \frac{\kappa_\sigma^-|_{M^{\Gamma_{i_0}}}}{\kappa_{\tilde{\sigma}}^-}\right)
\end{tikzcd} \tag{5}$$

Assuming this, we will prove proposition 5.23 by an inspection of the way that the desuspension with parts of the virtual bundle obtained using the splitting data works.

Proof of proposition 5.23. The space with action Umk of proposition 5.23 is the same as the one in 5.32. This automatically implies the first property about the contractibility of its connected components.

The maps $\sqsupset^\sigma$ are obtained from $\urcorner^\sigma$ be desuspending using the trivial vector bundle κ_σ^-. This desuspension gives a unique map using lemma 4.70, because κ_σ^- is trivial. This proves the second property about the existence of operations and hence it suffices to prove that these operations are compatible in the sense of the third property in proposition 5.23.

We recall the notation $\sigma = (\Gamma_n \to \ldots \to \Gamma_0)$ and $\tilde{\sigma} = (\Gamma_{i_k} \to \ldots \to \Gamma_{i_0})$. The two horizontal maps in the diagram 4 desuspend to:

$$|\nabla\tilde{\sigma}| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_0} \times \kappa_{\tilde{\sigma}}^- \to \mathrm{Thom}(\kappa_{\tilde{\sigma}}^+)$$

$$|\nabla\tilde{\sigma}| \times \mathrm{Umk}(\tilde{\sigma}) \times M^{\partial_{in}\Gamma_{i_0}} \times \kappa_\sigma^- \to \mathrm{Thom}(\kappa_\sigma^+)$$

Recall that by definition for the virtual bundle κ we have

$$\begin{aligned}
\kappa_\sigma^+ &= ((M^{f_0^n})^*\kappa_{\Gamma_n}^+)|_{M^{\partial_{out}\Gamma_0}} & \kappa_{\tilde\sigma}^+ &= ((M^{f_{i_0}^{i_k}})^*\kappa_{\Gamma_{i_k}}^+)|_{M^{\partial_{out}\Gamma_{i_0}}}\\
\kappa_\sigma^- &= ((M^{f_0^n})^*\kappa_{\Gamma_n}^-)|_{M^{\partial_{out}\Gamma_0}} & \kappa_{\tilde\sigma}^- &= ((M^{f_{i_0}^{i_k}})^*\kappa_{\Gamma_{i_k}}^-)|_{M^{\partial_{out}\Gamma_{i_0}}}
\end{aligned}$$

Furthermore, as proven in proposition 5.28 the splitting data in $\mathrm{Split}(\phi)$ included in $\nabla\mathsf{Fn}_/$ gives us isomorphisms

$$\phi_+ : (M^{f_0^n})^*\kappa_{\Gamma_n}^+ \cong (M^{f_0^{i_k}})^*\kappa_{\Gamma_{i_k}}^+ \oplus \frac{(M^{f_0^n})^*\kappa_{\Gamma n}^-}{(M^{f_0^{i_k}})^*\kappa_{\Gamma_{i_k}}^-}$$

$$\phi_- : (M^{f_0^n})^*\kappa_{\Gamma_n}^- \cong (M^{f_0^{i_k}})^*\kappa_{\Gamma_{i_k}}^- \oplus \frac{(M^{f_0^n})^*\kappa_{\Gamma n}^-}{(M^{f_0^{i_k}})^*\kappa_{\Gamma_{i_k}}^-}$$

By pulling back to the outgoing boundary, we hence obtain isomorphisms:

$$\tilde\phi_+ : \kappa_\sigma^+ \cong (M^{\partial_{out}f_0^{i_0}})^*\kappa_{\tilde\sigma}^+ \oplus \frac{\kappa_\sigma^-}{(M^{\partial_{out}f_0^{i_0}})^*\kappa_{\tilde\sigma}^-}$$

$$\tilde\phi_- : \kappa_\sigma^- \cong (M^{\partial_{out}f_0^{i_0}})^*\kappa_{\tilde\sigma}^- \oplus \frac{\kappa_\sigma^-}{(M^{\partial_{out}f_0^{i_0}})^*\kappa_{\tilde\sigma}^-}$$

Therefore, it suffices to show that the following diagram commutes, where $m = \dim \frac{(M^{f_{i_0}^n})^*\kappa_{\Gamma n}^-}{\kappa_{\tilde\sigma}^-}$:

(6)

$$\begin{tikzcd}
{|\nabla\tilde\sigma| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_0} \times \kappa_\sigma^-} \arrow[rr, "\urcorner^\sigma|_{|\nabla\tilde\sigma|}"] \arrow[d, "\mathrm{Thom}(\tilde\phi_-)"', "\cong"] & & \mathrm{Thom}(\kappa_\sigma^+) \arrow[d, "\cong"', "\mathrm{Thom}(\tilde\phi_+)"] \\
{\begin{array}{c}|\nabla\tilde\sigma| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_0}\\ \times \left((M^{\partial_{out}f_0^{i_0}})^*\kappa_{\tilde\sigma}^- \oplus \frac{\kappa_\sigma^-}{(M^{\partial_{out}f_0^{i_0}})^*\kappa_{\tilde\sigma}^-}\right)\end{array}} \arrow[d, "id\times\mathrm{Umk}(\psi)\times M^{f_0^{i_0}}"'] & & \mathrm{Thom}\left((M^{\partial_{out}f_0^{i_0}})^*\kappa_{\tilde\sigma}^+ \oplus \frac{\kappa_\sigma^-}{(M^{\partial_{out}f_0^{i_0}})^*\kappa_{\tilde\sigma}^-}\right) \arrow[d, "\mathrm{Thom}(M^{f_0^{i_0}})"] \\
{\begin{array}{c}|\nabla\tilde\sigma| \times \mathrm{Umk}(\tilde\sigma) \times M^{\partial_{in}\Gamma_{i_0}}\\ \times \left(\kappa_{\tilde\sigma}^- \oplus \frac{(M^{f_{i_0}^n})^*\kappa_{\Gamma n}^-}{\kappa_{\tilde\sigma}^-}\right)\end{array}} \arrow[rr, "\urcorner^{\tilde\sigma}\wedge id"] \arrow[d, "\cong"'] & & \mathrm{Thom}\left(\kappa_{\tilde\sigma}^+ \oplus \frac{(M^{f_{i_0}^n})^*\kappa_{\Gamma n}^-}{\kappa_{\tilde\sigma}^-}\right) \arrow[d, "\cong"', "\mathrm{Thom}(\phi_+)"] \\
{|\nabla\tilde\sigma| \times \mathrm{Umk}(\tilde\sigma) \times \Sigma^m(M^{\partial_{in}\Gamma_{i_0}} \times \kappa_{\Gamma_{i_k}}^-)} \arrow[rr, "\Sigma^m\urcorner^{\tilde\sigma}"] & & \Sigma^m\mathrm{Thom}(\kappa_{\Gamma_{i_k}}^+)
\end{tikzcd}$$

But the bottom square commutes because we know that $\frac{(M^{f_{i_0}^n})^*\kappa_{\Gamma n}^-}{\kappa_{\tilde\sigma}^-}$ is trivial of dimension l and the upper square commutes due to the properties of Umk. □

It will turn out to be more convenient to let $\mathrm{Umk}(\sigma)$ contain operations for any subsimplex as well. Essentially this happens because a "compatible" set of umkehr data admits umkehr data for subsimplices as well. This will make the diagram 5 split into two commutative squares. We will therefore state a more refined version of proposition 5.32 in order to prove it.

Proposition 5.33. *There exists a space* Umk *with action of* $\nabla\mathsf{Fn}_/$ *which has the following properties:*

- **Contractibility:** *The space* $\mathrm{Umk}(\mathrm{Ob}(\nabla\mathsf{Fn}_/))$ *is a disjoint union of non-empty contractible components* $\mathrm{Umk}(\sigma)$ *for* $\sigma \in \mathrm{Ob}(\nabla\mathsf{Fn})$.

Existence of operations: *For each component of the object space given by* $\sigma = (\Gamma_n \to \ldots \to \Gamma_0)$ *and each subsimplex* $\tilde{\sigma} = (\Gamma_{i_k} \to \ldots \to \Gamma_{i_0})$ *of* σ *there are operations* $\urcorner_r^{\tilde{\sigma}\subset\sigma}$ *for* $0 \le r \le i_0$*, which are compatible in the sense that the following diagram commutes:*

$$\begin{tikzcd}
|\nabla\tilde{\sigma}| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_0} \times (M^{f_0^{i_0}})^*\kappa_{\tilde{\sigma}}^- \arrow[r, "\urcorner_0^{\tilde{\sigma}\subset\sigma}"] \arrow[d, "M^{\partial_{in} f_0^1}"'] & \mathrm{Thom}((M^{f_0^{i_0}})^*\kappa_{\tilde{\sigma}}^+) \arrow[d, "\mathrm{Thom}(M^{f_0^1})"] \\
|\nabla\tilde{\sigma}| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_1} \times (M^{f_1^{i_0}})^*\kappa_{\tilde{\sigma}}^- \arrow[r, "\urcorner_1^{\tilde{\sigma}\subset\sigma}"] \arrow[d, "M^{\partial_{in} f_1^2}"'] & \mathrm{Thom}((M^{f_1^{i_0}})^*\kappa_{\tilde{\sigma}}^+) \arrow[d, "\mathrm{Thom}(M^{f_1^2})"] \\
\vdots \arrow[d, "M^{\partial_{in} f_{i_0-1}^{i_0}}"'] & \vdots \arrow[d, "\mathrm{Thom}(M^{f_{i_0-1}^{i_0}})"] \\
|\nabla\tilde{\sigma}| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_{i_0}} \times \kappa_{\tilde{\sigma}}^- \arrow[r, "\urcorner_{i_0}^{\tilde{\sigma}\subset\sigma}"] & \mathrm{Thom}(\kappa_{\tilde{\sigma}}^+)
\end{tikzcd}$$

where $\kappa_{\tilde{\sigma}}^+ = ((M^{f_{i_k}^{i_0}})^*\kappa_{\Gamma_{i_k}}^+)|_{M^{\partial_{out}\Gamma_{i_0}}}$ *and* $\kappa_{\tilde{\sigma}}^- = ((M^{f_{i_k}^{i_0}})^*\kappa_{\Gamma_{i_k}}^-)|_{M^{\partial_{out}\Gamma_{i_0}}}$ *are both vector bundles over* $M^{\Gamma_{\partial_{out} i_0}}$*, the latter of which is trivial.*

Associativity in first component: *Note that the* $\urcorner^{\tilde{\sigma}\subset\sigma}$ *depend on both* $\tilde{\sigma}$ *and* σ*. We will demand certain conditions on this dependence, starting with the first index. For each composable pair of morphisms* $\psi : \sigma \to \dot{\sigma}$ *and* $\dot{\psi} : \dot{\sigma} \to \ddot{\sigma}$*, where* $\ddot{\sigma} = (\Gamma_{i_{j_l}} \to \ldots \to \Gamma_{i_{j_0}})$*, the following diagram commutes for* $0 \le r \le i_0$*:*

$$\begin{tikzcd}
|\nabla\ddot{\sigma}| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_r} \times \left(\kappa_{\ddot{\sigma}}^- \oplus \frac{\kappa_{\dot{\sigma}}^-}{\kappa_{\ddot{\sigma}}^-}\right)|_{(M^{\partial_{in}\Gamma_r})} \arrow[r, "\urcorner_r^{\ddot{\sigma}\subset\sigma}\wedge id"] \arrow[d] & \mathrm{Thom}\left(\kappa_{\ddot{\sigma}}^+ \oplus \frac{\kappa_{\dot{\sigma}}^-}{\kappa_{\ddot{\sigma}}^-}\right)|_{(M^{\partial_{out}\Gamma_r})} \arrow[d] \\
|\nabla\dot{\sigma}| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_r} \times \kappa_{\dot{\sigma}}^- \arrow[r, "\urcorner_r^{\dot{\sigma}\subset\sigma}"'] & \mathrm{Thom}(\kappa_{\dot{\sigma}}^+)
\end{tikzcd} \tag{7}$$

where the vertical arrows are given by $|\nabla\dot{\psi}|$*, which is the inclusion of a subsimpex, and the splitting data.*

Associativity in second component: *We now state the condition on the dependence on the second component. Let* $\psi : \sigma \to \tilde{\sigma}$ *and* $\tilde{\psi} : \tilde{\sigma} \to \dot{\sigma}$ *be a pair of composable morphisms. Then for each* $i_0 \le r \le i_{j_0}$ *the following diagrams commutes:*

$$\begin{tikzcd}
|\nabla\dot{\sigma}| \times \mathrm{Umk}(\sigma) \times M^{\partial_{in}\Gamma_r} \times \kappa_{\dot{\sigma}}^- \arrow[r, "\urcorner_r^{\dot{\sigma}\subset\sigma}"] \arrow[d, "id\times\mathrm{Umk}(\psi)\times id"'] & \mathrm{Thom}((M^{f_r^{i_{j_0}}})^*\kappa_{\dot{\sigma}}^+) \arrow[d, equal] \\
|\nabla\dot{\sigma}| \times \mathrm{Umk}(\tilde{\sigma}) \times M^{\partial_{in}\Gamma_r} \times \kappa_{\dot{\sigma}}^- \arrow[r, "\urcorner_r^{\dot{\sigma}\subset\tilde{\sigma}}"'] & \mathrm{Thom}((M^{f_r^{i_{j_0}}})^*\kappa_{\dot{\sigma}}^+)
\end{tikzcd}$$

To prove proposition 5.32 from this proposition is easy.

Proof of proposition 5.32. One can take $\urcorner^{\sigma}$ be equal to $\urcorner_0^{\sigma\subset\sigma}$. The commutative diagram 5 is then obtained by noting that the associativity conditions imply that $\urcorner^{\sigma}|_{|\nabla\tilde{\sigma}|}$ is given by $\urcorner_0^{\tilde{\sigma}\subset\sigma} \wedge id$ after using the splitting data.

Thus the commutative diagram is obtained by stacking four commutative squares on top of each other.

(1) The first one comes from the splitting data, i.e. similar to the top of diagram 6. To be precise, one notes that after restricting to $|\nabla\ddot{\sigma}|$ both vertical arrows in the associativity in the first component diagram for $\sigma' = \dot{\sigma}' = \sigma$, $\ddot{\sigma}' = \tilde{\sigma}$, $\psi' = id$, $\dot{\psi}' = \psi$ and $r = 0$ (where we have put a prime to all the data in the associativity diagram for clarity), become isomorphisms. Thus we can split the bundle appearing.

(2) The second one comes from commutative diagram in the existence of the operations, for σ and $\tilde{\sigma}$ as given and $r = i_0$, i.e. the outer square. This square should be wedged by the trivial bundle to match with the previous commutative square.
(3) The third one is given by the associativity in the first component for $\sigma' = \dot{\sigma}' = \sigma$, $\ddot{\sigma}' = \tilde{\sigma}$, $\psi' = id$, $\dot{\psi}' = \psi$ and $r = i_0$ (where we have put a prime to all the data in the associativity diagram for clarity).
(4) The fourth and final square is the obtained by associativity in the second component for $\sigma' = \sigma$, $\tilde{\sigma}' = \dot{\sigma}' = \tilde{\sigma}$ and $r = i_0$.

□

So, it suffices to construct a space Umk with action of $\nabla\mathsf{Fn}_{/}$ satisfying the properties of proposition 5.33. This will be done in the next subsection: we will construct Umk as a disjoint union of homotopy limits of functors from the subsimplex category of a simplex in $\nabla\mathsf{Fn}$.

4.2. Constructing compatible umkehr maps for each subsimplex. In this section we will start with a lemma in which we define the functors whose homotopy limits we will take and only after that will we state and prove the properties which lead to a proof of proposition 5.33.

Recall that $\nabla\sigma$ is the category of subsimplices of a simplex σ, which in particular is a subcategory of $\nabla\mathsf{Fn}$. We define $\nabla\sigma_{/}$ to be the topological category $\nabla\sigma \int \mathrm{Split}|_{\nabla\sigma}$, where the functor $\mathrm{Split} : \nabla\mathsf{Fn} \to \mathsf{Top}$ was defined in proposition 5.17.

Lemma 5.34. *For each $\sigma \in \mathrm{Ob}(\nabla\mathsf{Fn})$ there exists a space umk^σ with action of $\nabla\sigma_{/}$ with the following properties:*

Contractiblity: *The space $\mathrm{umk}^\sigma(\mathrm{Ob}(\nabla\sigma))$ is a disjoint union of contractible components $\mathrm{umk}^\sigma(\tilde{\sigma})$ with $\tilde{\sigma}$ a subsimplex of σ.*

Existence of operations: *For each subsimplex $\tilde{\sigma} = (\Gamma_{i_k} \to \ldots \to \Gamma_{i_0})$ of σ there are operations $\urcorner_r^{\tilde{\sigma}\subset\sigma}$ for $0 \le r \le i_0$, which are compatible in the sense that the following diagram commutes:*

$$
\begin{tikzcd}
\mathrm{umk}^\sigma(\tilde{\sigma}) \times M^{\partial_{in}\Gamma_0} \times (M^{f_0^{i_0}})^*\kappa_{\tilde{\sigma}}^- \arrow[r, "\urcorner_0^{\tilde{\sigma}\subset\sigma}"] \arrow[d, "M^{\partial_{in} f_0^1}"'] & \mathrm{Thom}((M^{f_0^{i_0}})^*\kappa_{\tilde{\sigma}}^+) \arrow[d, "\mathrm{Thom}(M^{f_0^1})"] \\
\mathrm{umk}^\sigma(\tilde{\sigma}) \times M^{\partial_{in}\Gamma_1} \times (M^{f_1^{i_0}})^*\kappa_{\tilde{\sigma}}^- \arrow[r, "\urcorner_1^{\tilde{\sigma}\subset\sigma}"] \arrow[d, "M^{\partial_{in} f_1^2}"'] & \mathrm{Thom}((M^{f_1^{i_0}})^*\kappa_{\tilde{\sigma}}^+) \arrow[d, "\mathrm{Thom}(M^{f_1^2})"] \\
\vdots \arrow[d, "M^{\partial_{in} f_{i_0-1}^{i_0}}"'] & \vdots \arrow[d, "\mathrm{Thom}(M^{f_{i_0-1}^{i_0}})"] \\
\mathrm{umk}^\sigma(\tilde{\sigma}) \times M^{\partial_{in}\Gamma_{i_0}} \times \kappa_{\tilde{\sigma}}^- \arrow[r, "\urcorner_{i_0}^{\tilde{\sigma}\subset\sigma}"] & \mathrm{Thom}(\kappa_{\tilde{\sigma}}^+)
\end{tikzcd}
\tag{8}
$$

where $\kappa_{\tilde{\sigma}}^+ = ((M^{f_{i_k}^{i_0}})^*\kappa_{\Gamma_{i_k}}^+)|_{M^{\partial_{out}\Gamma_{i_0}}}$ and $\kappa_{\tilde{\sigma}}^- = ((M^{f_{i_k}^{i_0}})^*\kappa_{\Gamma_{i_k}}^-)|_{M^{\partial_{out}\Gamma_{i_0}}}$ are both vector bundles over $M^{\Gamma_{\partial_{out} i_0}}$, the latter of which is trivial.

Associativity in first component: *For each pair of subsimplicies $\ddot{\sigma} \subset \dot{\sigma} \subset \sigma$, where $\ddot{\sigma} = (\Gamma_{i_{j_l}} \to \ldots \to \Gamma_{i_{j_0}})$, the following diagram commutes for $0 \le r \le i_0$:*

$$
\begin{tikzcd}
\mathrm{umk}^\sigma(\ddot{\sigma}) \times M^{\partial_{in}\Gamma_r} \times \left(\kappa_{\ddot{\sigma}}^- \oplus \frac{\kappa_{\dot{\sigma}}^-}{\kappa_{\ddot{\sigma}}^+}\right)|_{(M^{\partial_{in}\Gamma_r})} \arrow[r, "\urcorner_r^{\ddot{\sigma}\subset\sigma}\wedge id"] \arrow[d] & \mathrm{Thom}\left(\kappa_{\ddot{\sigma}}^+ \oplus \frac{\kappa_{\dot{\sigma}}^+}{\kappa_{\ddot{\sigma}}^+}\right)|_{(M^{\partial_{out}\Gamma_r})} \arrow[d] \\
\mathrm{umk}^\sigma(\dot{\sigma}) \times M^{\partial_{in}\Gamma_r} \times \kappa_{\dot{\sigma}}^- \arrow[r, "\urcorner_r^{\dot{\sigma}\subset\sigma}"'] & \mathrm{Thom}(\kappa_{\dot{\sigma}}^+)
\end{tikzcd}
\tag{9}
$$

where the vertical arrows are given by the splitting data.

Proof. During the proof, we will define the spaces $\mathrm{umk}^{\sigma}(\tilde{\sigma})$. To do this, we need to consider the diagram of maps of spaces for which we need to construct umkehr maps for all maps pointing in the wrong direction in figure 5.76, which can be found at the end of this chapter. This figure arises by considering the correspondence similar that for the construction of $\daleth$ as in theorem 5.6 for each row of diagram 8. In fact, they only differ in the choice of virtual bundle.

Recall convention 5.13: we have fixed W and tubular neighborhoods for $M \hookrightarrow W$ and $A \hookrightarrow W$ for all $A \in \mathcal{B}$ already. It suffices to pick additional umkehr data to produce compatible umkehr maps for each column of maps pointing in the wrong direction individually. Pick the data as in the proof of theorem 5.6 for the bottom line. This gives one a space umk_1 of umkehr data for level 1: a product of spaces of tubular neighborhoods, propagating flows and connections. Then $\daleth_0^{\tilde{\sigma}\subset\sigma}$ is simply given by a construction very similar to $\daleth$; only the virtual bundles differ slightly.

Using the lifting propositions for compaible umkehr data we pick lifts for each square such that the maps in the bottom square of diagram 8 exist and these squares commute. To be precise, we need to provide lifts in three situations, starting from the right:

(1) The first lift is for the following square of $\mathcal{B}$-labelled complexes, which induces a square of mapping spaces

$$\begin{tikzcd} \Gamma_{1,\div} \arrow[r] \arrow[d] & \Gamma_1 \arrow[d] \\ \Gamma_{0,\div} \arrow[r] & \Gamma_0 \end{tikzcd}$$

Now note that that this diagram can be factored as the following diagram of 0-finite $\mathcal{B}$-labelled complexes:

(10)
$$\begin{tikzcd} \Gamma_{1,\div} \arrow[r] \arrow[d] & \Gamma_1 \arrow[d, equal] \\ \Gamma_{1,\div,nsd} \arrow[r] \arrow[d] & \Gamma_1 \arrow[d] \\ \Gamma_{1,\div,exp} \arrow[r] \arrow[d] & \Gamma_{0,exp} \arrow[d] \\ \Gamma_{0,\div} \arrow[r] & \Gamma_0 \end{tikzcd}$$

where $\Gamma_{1,\div,nsd}$ is the graph where we have attached the half-edges and vertices that are collapsed already, but they are still subdivided simplices. In $\Gamma_{1,\div,\mathrm{exp}}$ and $\Gamma_{0,\mathrm{exp}}$ we have removed this, ending up with complexes homotopy equivalent to $\Gamma_{0,\div}$ and Γ_0 with some 1-simplices expanded. See figure 5.35 for an example of all these graphs.

We can lift umkehr data in the top square due to corollary 4.86, that we can lift umkehr data to the middle square is corollary 4.89 and that we can lift it for the bottom square is corollary 4.93.

(2) The second lift is for the following square of $\mathcal{B}$-labelled complexes, which also induces a square of mapping spaces

$$\begin{tikzcd} \Gamma_{1,\div,in} \arrow[r] \arrow[d] & \Gamma_{1,\div} \arrow[d] \\ \Gamma_{0,\div,in} \arrow[r] & \Gamma_{0,\div} \end{tikzcd}$$

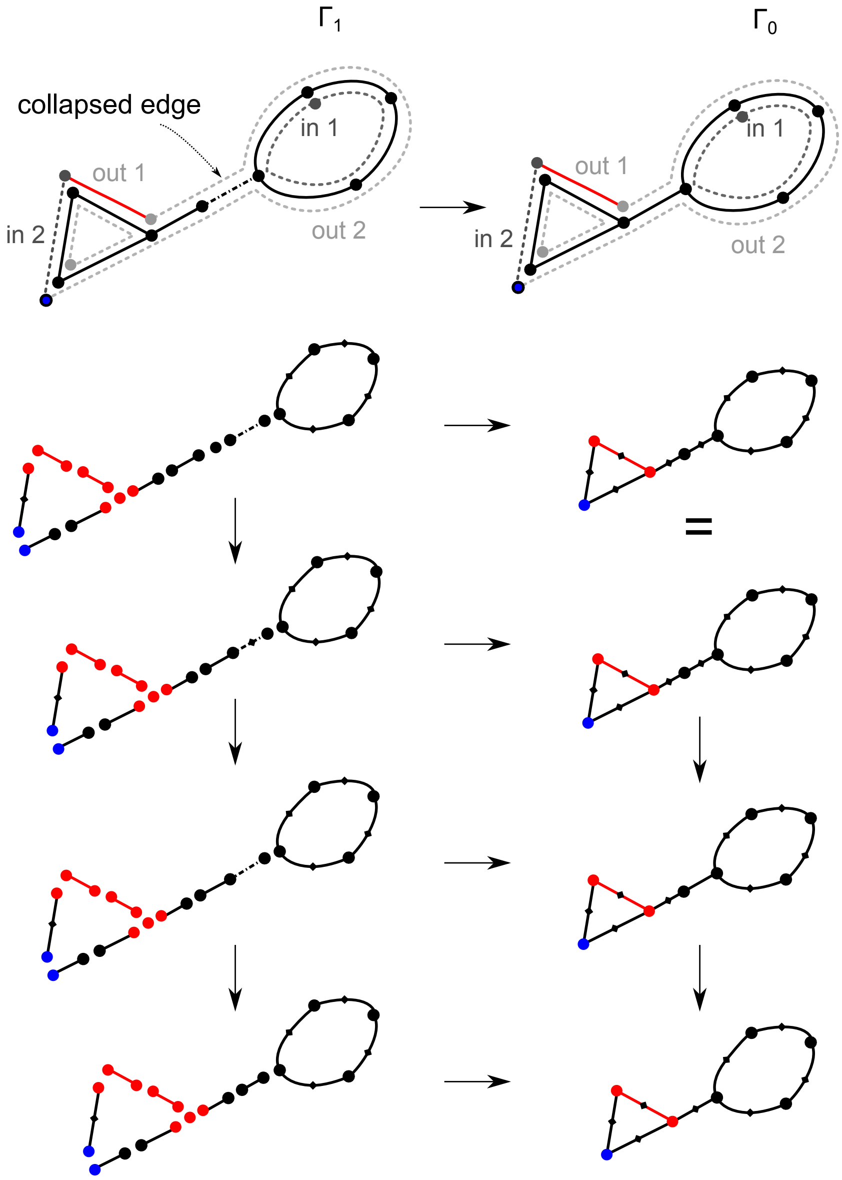


FIGURE 5.35. An example of the diagram 10 for a specific choice of Γ_1 and Γ_0. We have marked the center of the edges to be able to show the difference between the second and third row.

This diagram can be factored in a similar fashion as:

$$\begin{tikzcd}
\Gamma_{1,\div,in} \arrow[r] \arrow[d] & \Gamma_{1,\div} \arrow[d, equal] \\
\Gamma_{1,\div,in,nsd} \arrow[r] \arrow[d] & \Gamma_{1,\div} \arrow[d] \\
\Gamma_{1,\div,in,exp} \arrow[r] \arrow[d] & \Gamma_{0,\div,exp} \arrow[d] \\
\Gamma_{0,\div,in} \arrow[r] & \Gamma_{0,\div}
\end{tikzcd}$$

where $\Gamma_{1,\div,in,nsd}$ is the graph where we have attached the already collapsed edges in the incoming boundary, in $\Gamma_{1,\div,in,exp}$ we forget that these edges are subdivided $\mathcal{B}$-labelled complexes at the place where the half-edges meet and in $\Gamma_{0,\div,exp}$ we have expanded the collapsed edges of the incoming boundary.

Again we can use the lifting corollaries to produce compatible umkehr data: that we can lift umkehr data in the top square is corollary 4.86, that we can lift it to the middle square is corollary 4.89 and that we can lift for the bottom square is corollary 4.93.

(3) The third is for the following square of mapping spaces, which is not induced by a square of $\mathcal{B}$-labelled complexes:

$$\begin{tikzcd} M^{\partial_{in}\Gamma_1} \times W^{v(\Gamma_1 \backslash s(\partial_{in}\Gamma_1))} \arrow[d] & M^{(\Gamma_1)_{v,in}} \arrow[l] \arrow[d] \\ M^{\partial_{in}\Gamma_0} \times W^{v(\Gamma_0 \backslash s(\partial_{in}\Gamma_0))} & M^{(\Gamma_0)_{v,in}} \arrow[l] \end{tikzcd}$$

Now note that for the umkehr map, we are only concerned with a part of this diagram. Let eV_0 and eV_1 denote the vertices in $\Gamma_0 \backslash s(\partial_{in}(\Gamma_0))$ and $\Gamma_1 \backslash s(\partial_{in}(\Gamma_1))$ respectively, considered as $\mathcal{B}$-labelled complexes with only 0-simplices. Then we only need to look at:

$$\begin{tikzcd} W^{v(\Gamma_1 \backslash s(\partial_{in}\Gamma_1))} \arrow[d] & M^{eV_1} \arrow[l] \arrow[d] \\ W^{v(\Gamma_0 \backslash s(\partial_{in}\Gamma_0))} & M^{eV_0} \arrow[l] \end{tikzcd}$$

There is no need to lift umkehr data, because we have tubular neighborhoods for the embedding of each copy of M or $A \in \mathcal{B}$ into W available. Because these will automatically be the same for each level, the corresponding umkehr maps will automatically be compatible.

So we end up with a fiber bundle over umk_1 with contractible fiber, hence have total space contractible. The umkehr data is this space of course gives us the operations $\daleth_0^{\tilde{\sigma}\subset\sigma}$ for level 1, but also compatible operations $\daleth_1^{\tilde{\sigma}\subset\sigma}$ for level 2.

By induction, after $i_0 + 1$ steps, one has a space umk_{i_0+1} which contains compatible umkehr data for all levels and is contractible. This proves the first two properties.

The associativity in the first component is rather trivial. A subsimplex $\ddot{\sigma} \subset \dot{\sigma}$ induces a map $\mathrm{umk}^\sigma(\ddot{\sigma}) \to \mathrm{umk}^\sigma(\dot{\sigma})$ which is simply a projection of umkehr data using the splitting maps, which truncates the amounts of lifts from umkehr data for Γ_0. For $\ddot{\sigma}$ we lifted i_{j_0} times, but for $\dot{\sigma}$ only i_0 times. $\square$

The next lemma will lead to the associativity in the second component.

Lemma 5.36. *For each simplex given by $\sigma = (\Gamma_n \to \ldots \to \Gamma_0)$ and each subsimplex $\tilde{\sigma} = (\Gamma_{i_k} \to \ldots \to \Gamma_{i_0})$ of σ, there is a natural transformation*

$$\theta^{\tilde{\sigma}\subset\sigma} : \mathrm{umk}^\sigma \to \mathrm{umk}^{\tilde{\sigma}}$$

which is compatible with the operations in the sense that for each $i_0 \leq r \leq i_{j_0}$ the following diagrams commutes:

$$\begin{tikzcd} \mathrm{umk}^\sigma(\sigma) \times M^{\partial_{in}\Gamma_r} \times \kappa_{\dot{\sigma}}^- \arrow[r, "\daleth_r^{\dot{\sigma}\subset\sigma}"] \arrow[d, "\theta^{\tilde{\sigma}\subset\sigma}\times id"'] & \mathrm{Thom}((M^{\partial_{out} f_r^{i_{j_0}}})^* \kappa_{\dot{\sigma}}^+) \arrow[d, equal] \\ \mathrm{umk}^{\tilde{\sigma}}(\dot{\sigma}) \times M^{\partial_{in}\Gamma_r} \times \kappa_{\dot{\sigma}}^- \arrow[r, "\daleth_r^{\dot{\sigma}\subset\tilde{\sigma}}"'] & \mathrm{Thom}((M^{\partial_{out} f_r^{i_{j_0}}})^* \kappa_{\dot{\sigma}}^+) \end{tikzcd}$$

Proof. Again it is not difficult to construct the natural transformation. Suppose that $\dot{\sigma} \subset \tilde{\sigma} \subset \sigma$ and $\dot{\sigma} = (\Gamma_{i_{j_l}} \to \ldots \to \Gamma_{i_{j_0}})$. Then $\mathrm{umk}^\sigma(\dot{\sigma})$ contains umkehr data for compatible umkehr maps from Γ_0 to $\Gamma_{i_{j_0}}$, while $\mathrm{umk}^\sigma(\dot{\sigma})$ contains such data for compatible umkehr maps from Γ_{i_0} to $\Gamma_{i_{j_0}}$. We define θ to be given by forgetting about the umkehr data for Γ_0 to Γ_{i_0-1}. That way we obtain a map $\theta^{\tilde\sigma\subset\sigma}(\dot\sigma) : \mathrm{umk}^{\sigma}(\dot\sigma) \to \mathrm{umk}^{\tilde\sigma}(\dot\sigma)$. Combining these maps for all connected components gives $\theta^{\tilde\sigma\subset\sigma}$.

The commutativity of the diagram in the statement of the lemma is an obvious consequence of our construction of $\theta^{\tilde\sigma\subset\sigma}$. $\square$

The next proposition finally proves proposition 5.33.

Proposition 5.37. *The space* Umk *given by*

$$\mathrm{Umk} = \coprod_{\sigma\in\mathrm{Ob}(\nabla\mathsf{Fn})} \mathrm{holim}_{\nabla\sigma_/}\,\mathrm{umk}^{\sigma}$$

satisfies the properties of proposition 5.33.

Proof. By definition B.58 of the homotopy limit, a point in $\mathrm{Umk}(\sigma)$ consists of a space of maps $|\nabla\tilde\sigma| \to \mathrm{umk}^{\sigma}(\tilde\sigma)$ for subsimplices $\tilde\sigma\subset\sigma$. The operations $\urcorner_r^{\tilde\sigma\subset\sigma}$ are given by evaluating the maps $|\nabla\tilde\sigma| \to \mathrm{umk}^{\sigma}(\tilde\sigma)$ on the point of $|\nabla\tilde\sigma|$ and then applying the construction of the umkehr maps. This proves the existence of the operations. The associativity in first component is then an immediate consequence of lemma's 5.34.

The action of $\nabla\mathsf{Fn}_/$ is induced by the natural transformations θ of the previous lemma: they give a map

$$\mathrm{holim}\theta^{\tilde\sigma\subset\sigma} : \mathrm{holim}_{\nabla\sigma_/}\,\mathrm{umk}^{\sigma} \to \mathrm{holim}_{\nabla\tilde\sigma_/}\,\mathrm{umk}^{\tilde\sigma}$$

The associativity in the second component is then a consequence of the commutative diagram in lemma 5.36. Finally, the contracibility of each component $\mathrm{Umk}(\sigma)$ is consequence of the fact that a homotopy limit of a contractible diagram of contractible spaces is contractible. Clearly each of the spaces is contractible and the diagram $\nabla\sigma_/$ has geometric realisation homotopy equivalent to $|\nabla\sigma|$, which is homeomorphic to a standard simplex. $\square$

5. Identifying the spectra

In this section we will simplify the domain and codomain of $\beth$. We start with the following theorem about those homotopy colimits, which is a relatively easy combination of standard theorems about homotopy colimits and a calculation of the homotopy type of the geometric realisation of a category.

Theorem 5.38.
1. *For the homotopy colimit over* Fn*, or equivalently* $\nabla\mathsf{Fn}$ *or* $\overline{\nabla\mathsf{Fn}_/}$*, we have*

$$\mathrm{hocolim}\,\Sigma^\infty(M^{\partial_{in}-})_+ \simeq \coprod_{[\Sigma]} B\Gamma_\Sigma \wedge \Sigma^\infty(M^{\partial_{in}\Sigma})_+$$

where the coproduct is over all isomorphism classes of $\mathcal{B}$-labelled open-closed cobordisms.

2. *Similarly, for the homotopy colimit over* Fn*, or equivalently* $\nabla\mathsf{Fn}$ *or* $\overline{\nabla\mathsf{Fn}_/}$*, we have*

$$\mathrm{hocolim}\,\Sigma^\infty(M^{\partial_{out}-})_+ \simeq \coprod_{[\Sigma]} B\Gamma_\Sigma \wedge \Sigma^\infty(M^{\partial_{out}\Sigma})_+$$

where again the coproduct is over all isomorphism classes of $\mathcal{B}$-labelled open-closed cobordisms.

Proof. We only prove the first claim, as the second is completely analogous.

We begin by noting that indeed, the homotopy colimit is the same whether one works with Fn, $\nabla\mathsf{Fn}$ or $\overline{\nabla\mathsf{Fn}_/}$. This is because the functor $\Sigma^\infty(M^{\partial_{in}-})_+$ with its various choices for domains can be written as:

$$\overline{\nabla\mathsf{Fn}_/} \longrightarrow \nabla\mathsf{Fn} \longrightarrow \mathsf{Fn} \xrightarrow{\Sigma^\infty(M^{\partial_{in}-})_+} \mathsf{Spectra}$$

where the first two maps are the forgetful functors. We first prove that one can take $\nabla\mathsf{Fn}$ instead of Fn without changing the homotopy type. To do this, we simply apply theorem B.55, which essentially says that the homotopy colimit over $G\circ F$ is weakly homotopy equivalent to the homotopy colimit over G if the comma category of F is contractible. The condition that the comma category of F is contractible is also known as the statement that F is homotopy cofinal. We have already seen that this condition holds for the forgetful functor $F : \nabla\mathsf{Fn} \to \mathsf{Fn}$ and hence this result proves that one can use $\nabla\mathsf{Fn}$ without changing the homotopy type of the homotopy colimit.

To replace $\nabla\mathsf{Fn}$ with $\overline{\nabla\mathsf{Fn}}_/$, one can use similar results, but it is also easy to see the result using an explicit model of the homotopy colimit over such topological categories. Recall that we have the following model for the homotopy colimit

$$\operatorname*{hocolim}_{\overline{\nabla\mathsf{Fat}}_/}(\Sigma^\infty M^{\partial_{in}-})_+ = \left(\coprod_{n\in\mathbb{N}\sqcup\{0\}}\ \coprod_{\sigma_n\to\ldots\to\sigma_0\in\mathcal{N}_n\nabla\mathsf{Fat}}(\Delta^n\times\mathrm{Umk}(\sigma_n))_+\wedge\Sigma^\infty(M^{\partial_{in}\Gamma_0})_+\right)/\sim$$

where $\sim$ is the equivalence relation for the fat geometric realisation. In other words, the homotopy colimit is the fat geometric realisation of the simplicial spectrum whose n'th degree is

$$\coprod_{\sigma_n\to\ldots\to\sigma_0\in\mathcal{N}_n\nabla\mathsf{Fat}}(\Delta^n\times\mathrm{Umk}(\sigma_n))_+\wedge\Sigma^\infty(M^{\partial_{in}\Gamma_0})_+$$

Because the spaces Umk are contractible, the map from these to a point is a homotopy equivalence. This implies that the induced map from the previous space to

$$\coprod_{\sigma_n\to\ldots\to\sigma_0\in\mathcal{N}_n\nabla\mathsf{Fat}}\Delta^n_+\wedge\Sigma^\infty(M^{\partial_{in}\Gamma_0})_+$$

is a homotopy equivalence. Proposition B.47 give conditions under which the induced map on fat geometric realisation is a weak equivalence. We conclude that the homotopy colimit over $\nabla\mathsf{Fn}$ and $\overline{\nabla\mathsf{Fn}}_/$ are the same.

We claim that there is as homotopy equivalence as follows

$$\operatorname*{hocolim}_{\nabla\mathsf{Fn}}\Sigma^\infty(M^{\partial_{in}-})_+\simeq\coprod_{[\Sigma]}|\nabla\mathsf{Fn}_\Sigma|_+\wedge\Sigma^\infty(M^{\partial_{in}\Sigma})_+$$

where Fn_Σ is the subcategory of the category Fn of nice $\mathcal{B}$-labelled cobordism graphs whose geometric realisation as cobordism is isomorphic to Σ. We have seen in chapter 3 that $|\mathsf{Fn}_\Sigma|$ is homotopy equivalent to the classifying space $B\Gamma_\Sigma$. Furthermore we know that $|\nabla\mathsf{Fn}_\Sigma|\simeq|\mathsf{Fn}_\Sigma|$. Hence to prove the theorem, it therefore suffices to prove the claimed homotopy equivalence.

To prove this, we note that the functor $\Sigma^\infty(M^{\partial_{in}-})_+$, restricted to the subcategory Fn_Σ factors as follows:

$$\mathsf{Fn}_\Sigma\xrightarrow{\partial_{in}}\mathsf{Fn}_{\partial_{in}\Sigma}\xrightarrow{\Sigma^\infty(M^-)_+}\mathsf{Spectra}$$

where $\mathsf{Fn}_{\partial_{in}\Sigma}$ is the category of $\mathcal{B}$-labelled graphs with a starting vertex and orientation on each component, such that the realisation is homeomorphic to the incoming boundary of the $\mathcal{B}$-labelled cobordism Σ. In particular, each component has a label.

We claim that the geometric realisation of $\mathsf{Fn}_{\partial_{in}\Sigma}$ is contractible. This is easy in comparison to theorems we proved in chapter 3 about geometric realisations of categories of different types of graphs. We will give a functor F with a natural transformation to the identity functor on $\mathsf{Fn}_{\partial_{in}\Sigma}$ and to the constant functor with value the object G_0: this is the object for which each component has one edge and hence is either a circle or an interval with endpoints labelled with elements of $\mathcal{B}$.

The functor F adds a single new edge after the marked point on each component. This is well-defined because the components are oriented. The natural transformation to the identity functor collapses this edge again. The natural transformation to the constant functor collapses all other edges. As before, this implies that upon geometric realisation the identity is homotopy equivalent to the map to a single point and hence the geometric realisation $|\mathsf{Fn}_{\partial_{in}\Sigma}|$ is contractible.

Note that these natural transformations also induce natural maps $\Sigma^\infty(M^G)_+$ to $\Sigma^\infty(M^{G_0})_+$, which is in turn isomorphic to $\Sigma^\infty(M^{\partial_{in}\Sigma})_+$.

We therefore obtain a space with the same homotopy type if instead of taking the homotopy hocolimit of the functor $\Sigma^\infty(M^{\partial_{in}-})_+$ over $\nabla\mathsf{Fn}$, we take the homotopy colimit of the constant functor $\Sigma^\infty(M^{\partial_{in}\Sigma})_+$ over $\nabla\mathsf{Fn}$. But this homotopy colimit is easily seen to be equal to

$$\operatorname*{hocolim}_{\mathsf{Fn}}\Sigma^\infty(M^{\partial_{in}\Sigma})_+\simeq\left(\operatorname*{hocolim}_{\mathsf{Fn}}*\right)\wedge\Sigma^\infty(M^{\partial_{in}\Sigma})_+\simeq|\nabla\mathsf{Fn}|_+\wedge\Sigma^\infty(M^{\partial_{in}\Sigma})_+$$

This proves the claim and thus we have proven the theorem. □

Remark 5.39. We could have used ordinary geometric realisation instead of fat geometric realisation as a model for the homotopy colimits. This has nicer formal properties, but a levelwise weak equivalence of simplicial spaces only induces a weak equivalence of the geometric realisation if they are proper. In this case, both the simplicial spaces that would be used are proper.

They are proper because they are good: for the latter this is trivial, as each degeneracy is a product of an identity map with the inclusion of a face $\Delta^{n-1} \hookrightarrow \Delta^n$ and product of closed cofibrations is a closed cofibration. For the former simplicial space one needs to investigate the spaces of the functor Umk a bit or maybe modified slightly. The conditions of proposition B.44 would then be be satisfied.

We therefore have identified the domain of the higher string operations. For the codomain of $\beth$, we need to investigate the homotopy colimit of the Thom spectrum $(M^{\partial_{out}-})^{\kappa}$ over $\overline{\mathsf{Fn}}_{/}$. This homotopy colimit is itself almost a Thom spectrum, but the problem is that the dimensions of the positive and negative part of the virtual vector bundle go to infinity. The result is something we call a ind-virtual bundle. Heuristically, this is a homotopy colimit of virtual bundles, but most importantly it has the property that the Thom isomorphism theorem on the level of spectra holds.

Definition 5.40. A ind-virtual bundle is by definition a triple (X, B, p), where X should be a space of the form

$$X = \operatorname*{hocolim}_{\mathsf{C}} \kappa$$

where $\kappa : \mathsf{C} \to \overline{\mathsf{VirtB}}$ is a functor. We say X is an ind-virtual bundle over the space $B = \operatorname{hocolim} \pi \circ \kappa$ where $\pi : \overline{\mathsf{VirtB}} \to \mathsf{Top}$ is the projection functor to the base of a virtual bundle. The map $p : X \to B$ should be the one induced by π.

To an ind-virtual bundle we can associate an ind-Thom spectrum as homotopy colimit of the composition of κ with the Thom spectrum functor:

$$\operatorname*{hocolim}_{\mathsf{C}} (\pi \circ \kappa)^{\kappa}$$

We call a pro-virtual bundle oriented if each virtual bundle $\kappa(C)$ is oriented and this orientation is preserved by the morphisms $\kappa(f)$. If the dimension of the virtual bundles $\kappa(C)$ is constant d, then the pro-virtual bundle is said to be of dimension d

Proposition 5.41. *An ind-Thom spectrum associated to an oriented ind-virtual bundle of dimension d has a Thom isomorphism. To be precise, after fixing universal Thom classes, we have weak equivalences of spectra*

$$\left(\operatorname*{hocolim}_{\mathsf{C}} (\pi \circ \kappa)^{\kappa}\right) \wedge H\mathbb{Z} \simeq \operatorname*{hocolim}_{\mathsf{C}} \left((\pi \circ \kappa)^{\kappa} \wedge H\mathbb{Z}\right) \simeq \Sigma^{\infty}(B)_{+} \wedge \Sigma^{d} H\mathbb{Z}$$

Proof. The first weak equivalence is a consequence of the fact that smash products preserve homotopy colimits in nice models of spectra. These nice models in fact are monoidal model categories: the smash product of a two cofibrations is a cofibration which is trivial if one of them is. See for example [**HSS99**, Corollary 5.3.8.]. Thus smashing with $H\mathbb{Z}$ preserves cofibrant replacement and we can take it inside the homotopy colimit.

For the second weak equivalence, we use that we have a natural weak equivalence

$$(\pi \circ \kappa)^{\kappa}(C) \wedge H\mathbb{Z} \simeq \Sigma^{\infty}(\pi \circ \kappa(C))_{+} \wedge \Sigma^{d} H\mathbb{Z}$$

using the Thom isomorphism theorem. Now take $\Sigma^d H\mathbb{Z}$ outside of the homotopy colimit again and we are done. □

However, we also need a variation in the case where κ is not oriented and may have different dimensions over different connected components. To take care of this, let $\mathcal{L} = \det(\kappa)$ be the local system associated the the determinant graded line bundle of κ over $\pi \circ \kappa$. Alternatively, we can make a graded-preserving local system $\mathbb{L}$ by extracting the grading shift.

Corollary 5.42. *Consider an ind-Thom spectrum associated to an ind-virtual bundle κ of dimension d. After fixing universal Thom classes, we have a weak equivalence of parametrized spectra*

$$\left(\operatorname*{hocolim}_{\mathsf{C}}\,(\pi\circ\kappa)^{\kappa}\right)\wedge_{\mathbb{L}} H\mathbb{Z}\simeq\Sigma^{\infty}(B)_{+}\wedge\Sigma^{d}H\mathbb{Z}$$

Alternatively, one can absorb the d-fold suspension into the local system $\mathbb{L}$ to get a local system $\mathcal{L}$ for which there is weak equivalence of parametrized spectra

$$\left(\operatorname*{hocolim}_{\mathsf{C}}\,(\pi\circ\kappa)^{\kappa}\right)\wedge_{\mathcal{L}} H\mathbb{Z}\simeq\Sigma^{\infty}(B)_{+}\wedge H\mathbb{Z}$$

Now we can look at the map $\gimel$ induced by $\beth$ after applying the Thom isomorphism and the map $\gimel_*$ induced by $\gimel$ in homology.

Definition 5.43. The map $\gimel$ is the composition:

$$\begin{tikzcd}
\left(\bigvee_{[\Sigma]}(B\Gamma_{\Sigma})_{+}\wedge\Sigma^{\infty}(M^{\partial_{in}\Sigma})_{+}\right)\wedge_{\mathcal{L}} H\mathbb{Z} \arrow[r, "\simeq"] \arrow[dddd, dotted, "\gimel"'] & \operatorname*{hocolim}_{\overline{\nabla\mathsf{Fn}}_{/}}\Sigma^{\infty}(M^{\partial_{in}-})_{+}\wedge_{\mathcal{L}} H\mathbb{Z} \arrow[d, "\beth\wedge_{\mathcal{L}}H\mathbb{Z}"] \\
 & \left(\operatorname*{hocolim}_{\overline{\nabla\mathsf{Fn}}_{/}}(M^{\partial_{out}-})^{\kappa}\right)\wedge_{\mathcal{L}} H\mathbb{Z} \arrow[d, "\simeq"] \\
 & \operatorname*{hocolim}_{\overline{\nabla\mathsf{Fn}}_{/}}\Sigma^{\infty}(M^{\partial_{out}-})_{+}\wedge H\mathbb{Z} \arrow[d, "\simeq"] \\
 & \left(\coprod_{[\Sigma]}B\Gamma_{\Sigma}\wedge\Sigma^{\infty}(M^{\partial_{out}\Sigma})_{+}\right)\wedge H\mathbb{Z} \arrow[d, phantom] \\
\Sigma^{\infty}(M^{\partial_{out}\Sigma})_{+}\wedge H\mathbb{Z} & \arrow[l] {}
\end{tikzcd}$$

where the middle map on the right hand side is the generalized Thom isomorphism. In the next section we will take a closer look at the local system $\mathcal{L}=\mathcal{L}_{\mathcal{B}}^{M}$, see definition 5.64.

This map $\gimel$ consists of the string operations in family form. To obtain the HCFT operations we only need to pass to homology. We could go to homology with integer coefficients, but then we don't have the Künneth theorem to split the codomain. Therefore we decide to replace $\mathbb{Z}$ with $\mathbb{Q}$. This is done by smashing $H\mathbb{Z}$ with $H\mathbb{Q}$ over $H\mathbb{Z}$.

Definition 5.44. The string operations in homology

$$\begin{tikzcd}
H_*(B\Gamma_{\Sigma};\mathcal{L}_{\mathcal{B}}^{M})\otimes H_*(LM;\mathbb{Q})^{\otimes r}\otimes\bigotimes_{A,B\in\mathcal{B}}H_*(P_M(A,B);\mathbb{Q})^{\otimes r_{A,B}} \arrow[d, "\gimel_*"'] \\
H_*(LM;\mathbb{Q})^{\otimes s}\otimes\bigotimes_{A,B\in\mathcal{B}}H_*(P_M(A,B);\mathbb{Q})^{\otimes s_{A,B}}
\end{tikzcd}$$

are given by the maps induced in homology by $\gimel$.

Remark 5.45. Although a priori, the local system $\mathcal{L}_{\mathcal{B}}^{M}$ lives above the entire spectrum $\bigvee_{[\Sigma]}(B\Gamma_{\Sigma})_{+}\wedge\Sigma^{\infty}(M^{\partial_{in}\Sigma})_{+}$, in the next section we show that it is actually obtained as a pullback of a local system $\det(\bar{\xi})$ over $\coprod_{[\Sigma]}B\Gamma_{\Sigma}$ and hence we are justified in writing the domain as $\gimel$ as in definition 5.44 as above.

6. Orientations and the local system $\mathcal{L}_{\mathcal{B}}^{M}$

In this section we will finally investigate how to deal with the orientations of the virtual bundles involved in the construction of the string operations. While doing this, we will give a simple definition of the local system $\mathcal{L}_{\mathcal{B}}^{M}$ of that appeared in the construction of $\gimel$ in the previous section and show how it simplifies in some situations. However, we start by looking a bit at the theory of determinants of virtual vector bundles.

6.1. Properties of determinants. Recall that the Thom isomorphism for an unoriented virtual bundle system depends on a local system. This local system, or equivalently the corresponding twist of homology, only depends on the determinant graded line bundle of κ, as defined in definition 4.45. Because in the next section we want to prove that certain determinant graded line bundles are isomorphic, we start by discussing some properties of determinants of virtual bundles. Because the determinant construction is fiberwise, it suffices to look at virtual vector spaces. Virtual vector spaces are the local model for virtual vector bundles and hence are defined as follows:

Definition 5.46. A virtual vector space over k is a pair of finite-dimensional vector spaces $V_+ - V_-$ over a field k. A morphism of virtual vector spaces from $V_+ - V_-$ to $W_+ - W_-$ is a triple (ϕ_+, ϕ_-, θ) of a vector space θ and two isomorphisms of vector spaces:

$$\phi_+ : V_+ \oplus \theta \to W_+$$
$$\phi_- : V_- \oplus \theta \to W_-$$

This gives a category VirtV_k of virtual spaces over a field k.

The analogue of fiber of a graded line bundle is a graded line.

Definition 5.47. A *graded line* is a one-dimensional vector space L together with an integer $d \in \mathbb{Z}$ called the degree.

A morphism of graded lines is an isomorphism of one-dimensional vector spaces which respects the degree. This gives a category GrL_k of graded lines over k.

Definition 5.48. The *determinant* of a virtual vector space $V = V_+ - V_-$ with components of dimension d_+ and d_- respectively, is the graded line

$$\det(V) = \Lambda^{d_+} V_+ \wedge \Lambda^{d_-} V_-^*$$

with degree $d_+ - d_-$.

Lemma 5.49. *The determinant is a functor* $\det : \mathsf{VirtV}_k \to \mathsf{GrL}_k$.

Proof. We show that a morphism $f = (\phi_+, \phi_-, \theta) : V_+ - V_- \to W_+ - W_-$ induces a natural morphism $\det(f)$ between the graded lines of the domain and codomain. The dimensions of V and W, which are the degrees of the corresponding graded lines, are clearly the same. Therefore it suffices to prove that there is a natural isomorphism between the one-dimensional vector spaces of their determinants. To do this, we fix the notation that θ is of dimension d. The coevaluation gives us a natural isomorphism

$$\det(V) \cong \det(V) \otimes \Lambda^d \theta \otimes \Lambda^d \theta^*$$

Then the idea is note that there is a natural isomorphism

$$\Lambda^{d_+ + d}(V_+ \oplus \theta) \cong \Lambda^{d_+} V_+ \otimes \Lambda^d \theta$$

and similarly for the negative part and hence there is natural isomorphism

$$\det(V) \otimes \Lambda^d \theta \otimes \Lambda^d \theta^* \cong \Lambda^{d_+ + d}(V_+ \oplus \theta) \otimes \Lambda^{d_- + d}(V_-^* \oplus \theta^*)$$

Now use ϕ_+ and ϕ_- to map the right hand side isomorphically onto $\det(W) = \Lambda^{d_+ + d}(W_+) \otimes \Lambda^{d_- + d}(W_-^*)$. □

Suppose that $k = \mathbb{R}$. Because all the isomorphisms used are continuous with respect to the natural topologies on the spaces of linear maps, this lemma proves a corresponding statement for virtual bundles.

Corollary 5.50. *The construction of the graded line bundle* $\det(\mu)$ *associated to a (real) virtual bundle* μ *gives a functor* $\det : \mathsf{VirtB} \to \mathsf{GrLb}$*, where the latter is the category of graded (real) line bundles.*

Now that we have proven that the construction of a graded line bundle actually makes sense, we look at some of its properties. A direct sum of two virtual vector spaces is simply given by the direct sum of their positive and negative parts. The tensor product of two graded lines is the given by the tensor product of the corresponding one-dimensional vector space with degree equal to the sum of the degrees.

Proposition 5.51. *There is a natural isomorphism*

$$\det(V \oplus W) \cong \det(V) \otimes \det(W)$$

Proof. If V_+ and V_- are of dimension d_+ and d_- respectively and W_+ and W_- of dimension e_+ and e_-, then we know that we have natural isomorphisms

$$\Lambda^{d_+ + e_+}(V_+ \oplus W_+) \cong \Lambda^{d_+} V_+ \otimes \Lambda^{e_+} W_+ \qquad \Lambda^{d_- + e_-}(V_- \oplus W_-)^* \cong \Lambda^{d_-} V_-^* \otimes \Lambda^{e_-} W_-^*$$

Taking the tensor product of these two isomorphisms together with a reordering of the terms gives the isomorphism in the statement of this proposition. For the degree it suffices to note that $(d_+ + e_+) - (d_- + e_-) = (d_+ - d_-) + (e_+ - e_-)$. ☐

The previous proposition can be generalized to the following more advanced statement, using the fact that short exact sequences of vector spaces always split.

Proposition 5.52. *Suppose that we have a short exact sequence of virtual vector spaces*

$$0 \to U \to V \to W \to 0$$

then there is a natural isomorphism:

$$\det(V) \cong \det(U) \otimes \det(W)$$

As before, these two properties have corollaries for determinant graded line bundles of virtual bundles

Corollary 5.53. *Let μ and λ be virtual bundles, then there is a natural isomorphism of determinant graded line bundles*

$$\det(\mu \oplus \lambda) \cong \det(\mu) \otimes \det(\lambda)$$

More generally, if κ is a third virtual bundle such that there is a short exact sequence

$$0 \to \kappa \to \mu \to \lambda \to 0$$

there is a natural isomorphism of determinant graded line bundles

$$\det(\mu) \cong \det(\kappa) \otimes \det(\lambda)$$

If we pick an orientation of a vector space V of dimension d, then $\Lambda^d V$ is canonically isomorphic to $\mathbb{R}$. This can be used to prove the following proposition.

Proposition 5.54. *Suppose that V is an oriented vector space of dimension d, then there is a natural isomorphism*

$$\det(V \otimes W) \cong \det(W)^{\otimes d}$$

Proof. Let $e_1, \ldots, e_d$ be an oriented basis of V, then the isomorphism from $\det(V \otimes W)$ to $\det(W)^{\otimes d}$ is given by mapping $v_1 \wedge \ldots \wedge v_n^* \wedge e_1 \wedge \ldots \wedge e_d$. This is independent of the choice of oriented basis. ☐

As before, we obtain a corresponding corollary for virtual bundles.

Corollary 5.55. *Let μ be an oriented vector bundle of dimension d, then there is a natural isomorphism*

$$\det(\mu \otimes \lambda) \cong \det(\lambda)^{\otimes d}$$

6.2. Choosing orientations for κ. We first recall the definition of the ind-virtual bundles κ. It is the homotopy colimit of a functor $\kappa : \overline{\nabla \mathsf{Fn}_{/}}$ given by in proposition 5.28. For a simplex $(\Gamma_n \to \ldots \to \Gamma_0)$ this is defined as the pullback $((M^{f_0^n})_* \kappa_{\Gamma_n})|_{M^{\partial_{out}\Gamma_0}}$ of a virtual bundle κ_{Γ_n} over M^{Γ_n} to $M^{\partial_{out}\Gamma_0}$.

The negative part of this virtual bundle is easy; it is the trivial bundle $W^{v(\Gamma_n \backslash s(\partial_{in}\Gamma_n))}$. For the positive part we need some notation. Let siV the vertices of $s(\partial_{in}\Gamma_n)$, eV the vertices in $\Gamma_n \backslash s(\partial_{in}\Gamma_n)$ and eE the edges of $\Gamma_n \backslash s(\partial_{in}\Gamma_n)$. These come equipped with a map b to $\mathcal{B}$. The map $\mathrm{val} : eV \to \mathbb{N}$ denotes the valence of a vertex in $\Gamma_n \backslash s(\partial_{in}\Gamma_n)$ and the map $\mathrm{coval} : siV \to \mathbb{N} \sqcup \{0\}$ denotes the number of vertices in iV which are mapped to a vertex by the collapse map, minues 1.

The virtual bundle κ_{Γ_n} then has positive part

$$\begin{tikzcd}
\kappa_{\Gamma_n}^{+} \arrow[r] \arrow[d] & \bigoplus_{v \in eV} \nu_{b(v)} \oplus \bigoplus_{e \in eE} Tb(e) \oplus \bigoplus_{v \in siV} Tb(v)^{\mathrm{coval}(v)} \arrow[d] \\
M^{\Gamma_n} \arrow[r] & \prod_{v \in eV} b(v) \times \prod_{e \in eE} b(e) \times \prod_{v \in siV} b(v)^{\mathrm{coval(v)}}
\end{tikzcd}$$

The first thing we note is we might as well work over the geometric realisation of the subcategory $\overline{\nabla \mathsf{Fn}}^{a}_{/}$ of admissible graphs and always assume that $siV = \emptyset$.

Unfortunately, there doesn't seem to be a canonical choice of ordering the terms W in the negative part or each of the term in the positive part. This makes sense: there is no reason to believe that κ is an orientable virtual bundle.

However, to get some grip on the graded line bundle $\det(\kappa)$, we will replace κ by a different virtual bundle whose orientability is easy to investigate. The trick is note that the local system and corresponding twist only depend on the determinant bundle of κ, as defined in definition 4.45. That means that if we can find other virtual bundles with isomorphic determinant bundle, we can investigate their orientations instead.

We will give an easier virtual bundle $\bar{\kappa}$ to replace κ. This was the one already hinted at before in section 2 of this chapter, when we talked about distributing the virtual bundle over the graph. See figure 5.11. The idea behind it is that as a virtual bundle $\nu_A - W$ is isomorphic to $-TA$. Thus we are tempted to replace all terms $\nu_A - W$ by $-TA$.

Definition 5.56. The modified virtual bundle $\bar{\kappa} : \overline{\nabla \mathsf{Fn}}^{a}_{/} \to \overline{\mathsf{VirtB}}$ assigns to an object $(\Gamma_n \to \ldots \to \Gamma_0)$ the virtual bundle over $M^{\partial_{out}\Gamma_0}$ obtained by pulling back the following virtual bundle over M^{Γ_n}: it has positive part given by

$$\begin{tikzcd}
\bar{\kappa}_{\Gamma_n}^{+} \arrow[r] \arrow[d] & \bigoplus_{e \in eE} Tb(e) \arrow[d] \\
M^{\Gamma_n} \arrow[r] & \prod_{e \in eE} b(e)
\end{tikzcd}$$

and negative part given by

$$\begin{tikzcd}
\bar{\kappa}_{\Gamma_n}^{-} \arrow[r] \arrow[d] & \bigoplus_{v \in eV} Tb(v) \arrow[d] \\
M^{\Gamma_n} \arrow[r] & \prod_{v \in eV} b(v)
\end{tikzcd}$$

Lemma 5.57. *We have an isomorphism of graded line bundles*

$$\det(\kappa) \cong \det(\bar{\kappa})$$

PROOF. Simply use the isomorphisms of virtual bundles $\nu_M - W \cong -TM$ and $\nu_A - W \cong -TA$ for branes $A \in \mathcal{B}$ a number of times and note that the determinant bundle construction is compatible with morphisms of virtual bundles according to the corollary 5.50. $\square$

Because we have assumed M and $A \in \mathcal{B}$ to be oriented we have canonical orientations for TM and TA for each $A \in \mathcal{B}$. We now claim that we can obtain the same determinant bundle as for $\bar{\kappa}$ by pulling back a certain determinant bundle $\det(\lambda)$ over the base space $|\overline{\nabla\mathsf{Fn}}^a_/|$.

To write the virtual bundles that we will use to define $\det(\lambda)$, note that for the positive part and the negative part we have isomorphisms

$$\bigoplus_{v\in eE} Tb(v) \cong \left(TM \otimes \bigoplus_{\text{unlabelled } e\in eE} \mathbb{R}\right) \oplus \bigoplus_{A\in\mathcal{B}} \left(TA \otimes \bigoplus_{e\in b^{-1}(\beta)\subset eE} \mathbb{R}\right)$$

$$\bigoplus_{v\in eV} Tb(v) \cong \left(TM \otimes \bigoplus_{\text{unlabelled } v\in eV} \mathbb{R}\right) \oplus \bigoplus_{A\in\mathcal{B}} \left(TA \otimes \bigoplus_{v\in b^{-1}(\beta)\subset eV} \mathbb{R}\right)$$

Let $a = \dim(A)$ for all $a \in \mathcal{B}$ and $d = \dim(M)$. Because TA is oriented for the determinent over each simplex it doesn't make a difference whether we use TA or $\mathbb{R}^a$. Similarly, we can replace TM with $\mathbb{R}^d$ in the determinant. Using this we can find basic virtual bundles over $|\overline{\nabla\mathsf{Fn}}^a_/|$, whose determinant bundles can used to produce $\det(\lambda)$. We start by defining these basic virtual bundles.

Definition 5.58. For each $A \in \mathcal{B}$, the basic virtual bundle $\lambda_A : \overline{\nabla\mathsf{Fn}}^a_/ \to \overline{\mathsf{VirtB}}$ assigns to an object $(\Gamma_n \to \ldots \to \Gamma_0)$ the virtual bundle over a point with positive part and negative parts

$$\lambda_A^+ = \bigoplus_{e\in b^{-1}(\beta)\subset eE} \mathbb{R} \qquad \lambda_A^- = \bigoplus_{v\in b^{-1}(\beta)\subset eV} \mathbb{R}$$

The basic virtual bundle $\lambda_M : \overline{\nabla\mathsf{Fn}}^a_/ \to \overline{\mathsf{VirtB}}$ assigns to an object $(\Gamma_n \to \ldots \to \Gamma_0)$ the virtual bundle over a point with positive part and negative parts

$$\lambda_M^+ = \bigoplus_{\text{unlabelled } e\in eE} \mathbb{R} \qquad \lambda_M^- = \bigoplus_{\text{unlabelled } v\in eV} \mathbb{R}$$

To define the virtual bundle λ, we recall the tensor products of graded line bundles. For μ a graded line bundle of degree k, the tensor product $\mu^{\otimes d}$ is given by the d-fold tensor product of the underlying line bundle, now put in degree $d \cdot k$.

Lemma 5.59. *The pullback of the graded line bundle* $\det(\lambda) := \det(\lambda_M)^{\otimes d} \otimes \bigotimes_{A\in\mathcal{B}} \det(\lambda_A)^{\otimes a}$ *to* M^{Γ_0} *satisfies*

$$\det(\lambda) \cong \det(\bar{\kappa})$$

Proof. If we apply corollaries 5.55 and 5.53 to the right hand sides of the following isomorphisms

$$\bigoplus_{v\in eE} Tb(v) \cong \left(TM \otimes \bigoplus_{\text{unlabelled } e\in eE} \mathbb{R}\right) \oplus \bigoplus_{A\in\mathcal{B}} \left(TA \otimes \bigoplus_{e\in b^{-1}(\beta)\subset eE} \mathbb{R}\right)$$

$$\bigoplus_{v\in eV} Tb(v) \cong \left(TM \otimes \bigoplus_{\text{unlabelled } v\in eV} \mathbb{R}\right) \oplus \bigoplus_{A\in\mathcal{B}} \left(TA \otimes \bigoplus_{v\in b^{-1}(\beta)\subset eV} \mathbb{R}\right)$$

then we get respectively

$$\det(\bigoplus_{\text{unlabelled } e\in eE} \mathbb{R})^{\otimes d} \otimes \bigotimes_{A\in\mathcal{B}} \det(\bigoplus_{e\in b^{-1}(\beta)\subset eE} \mathbb{R})^{\otimes d}$$

$$\det(\bigoplus_{\text{unlabelled } v\in eV} \mathbb{R})^{\otimes d} \otimes \bigotimes_{A\in\mathcal{B}} \det(\bigoplus_{v\in b^{-1}(\beta)\subset eV} \mathbb{R})^{\otimes d}$$

Tensoring the dual of the second with the first exactly gives $\det(\lambda)$. □

We will use λ to find a nice homological virtual bundle. This will allow us to do some actual calculations later on. Let A_{Γ_n} denote the part of the geometric realisation of Γ_n labelled by A and $\partial_{in} A_{\Gamma_n}$ the part of $\partial_{in}\Gamma_n$ labelled by A.

Definition 5.60. The basic homological graph virtual bundle $\xi_A : \overline{\nabla \mathsf{Fn}}^a_/ \to \overline{\mathsf{VirtB}}$ assigns to an object $(\Gamma_n \to \ldots \to \Gamma_0)$ the virtual bundle over a point with positive part and negative part given by

$$\xi_A^+ = H_0(A_{\Gamma_n}, \partial_{in} A_{\Gamma_n}; \mathbb{R}) \qquad \xi_A^- = H_1(A_{\Gamma_n}, \partial_{in} A_{\Gamma_n}; \mathbb{R})$$

We also define a virtual bundle $\xi_M : \overline{\nabla \mathsf{Fn}}^a_/ \to \overline{\mathsf{VirtB}}$ which assigns to an object $(\Gamma_n \to \ldots \to \Gamma_0)$ the virtual bundle over a point with positive and negative parts given by

$$\xi_M^+ = H_0(\Gamma_n, \partial_{in}\Gamma_n \cup \bigcup_{A \in \mathcal{B}} \partial_{in} A_{\Gamma_n}; \mathbb{R}) \qquad \xi_M^+ = H_1(\Gamma_n, \partial_{in}\Gamma_n \cup \bigcup_{A \in \mathcal{B}} \partial_{in} A_{\Gamma_n}; \mathbb{R})$$

The full homological graph determinant bundle is given by

$$\det(\xi) = \det(\xi_M)^{\otimes d} \otimes \bigotimes_{A \in \mathcal{B}} \det(\xi_A)^{\otimes a}$$

Lemma 5.61. *We have an isomorphism of graded line bundles*

$$\det(\lambda) \cong \det(\xi)$$

Proof. We note that λ_A^+ and λ_A^- are the zeroth and first terms of the two-term cellular chain complex for the relative homology groups $H_*(A_{\Gamma_n}; \partial_{in}\Gamma_n)$. By corollary 5.50 in the determinant one can cancel the kernel of the differential d of this chain complex in λ_A^- against the image of d in λ_A^+. This proves that $\det(\lambda_A) \cong \det(\xi_A)$. A similar argument holds for M and hence the isomorphism of the statement holds. □

Finally we will compare this homological graph determinant bundle to a homological cobordism determinant bundle $\bar{\xi}$ over $B\Gamma_\Sigma$. This allows us obtain various situations in which the local system is easy to understand.

To define it, note that there is a $\mathcal{B}$-labelled cobordism Σ represented by Γ_n and hence by Γ_i for $0 \leq i \leq n-1$ as well. We let A_Σ denote the part of the boundary of Σ labelled by A and $\partial_{in} A_\Sigma$ the part of $\partial_{in}\Sigma$ labelled by A. Let $\partial_s \Sigma$ be the union of the incoming and $\mathcal{B}$-labelled boundary. Then we have an isomorphism

$$H_*(A_{\Gamma_n}, \partial_{in} A_{\Gamma_n}; \mathbb{R}) \cong H_*(A_\Sigma, \partial_{in} A_\Sigma l \mathbb{R})$$

For the brane M something special is going on, because the interior of the cobordism is of course also labelled M. In this case we obtain an isomorphism

$$H_*(\Gamma_n, \partial_{in}\Gamma_n \cup \bigcup_{A \in \mathcal{B}} \partial_{in} A_{\Gamma_n}; \mathbb{R}) \cong H_*(\Sigma, \partial_s \Sigma; \mathbb{R})$$

Definition 5.62. The homological cobordism virtual bundle $\bar{\xi}_A$ over $B\Gamma_\Sigma$ is given by taking the quotient by Γ_Σ of the virtual bundle, where degree 0 is positive and degree 1 negative,

$$H_*(A_\Sigma; \partial_{in} A_\Sigma)$$

over $E\Gamma_\Sigma$, thereby obtaining a virtual bundle over $B\Gamma_\Sigma$. The homological cobordism virtual bundle $\bar{\xi}_M$ over $B\Gamma_\Sigma$ is similarly obtained from

$$H_*(\Sigma, \partial_s \Sigma)$$

Lemma 5.63. *The homotopy equivalence $B\Gamma_\Sigma \simeq |\overline{\nabla \mathsf{Fn}}^a_/(\Sigma)|$ pulls ξ_A back to $\bar{\xi}_A$ for all $A \in \mathcal{B}$ and ξ_M back to $\bar{\xi}_M$.*

Proof. The geometric realisation of the admissible $\mathcal{B}$-labelled cobordism graphs gives a model for the universal cobordism bundle over $B\Gamma_\Sigma$. Both ξ_A and $\bar{\xi}_A$ are obtained by taking the homology of natural subbundle of this bundle. □

Definition 5.64. The *local system* $\mathcal{L}^M_{\mathcal{B}}$ is the one corresponding to the determinant graded line bundle $\det(\bar{\xi})$ over $B\Gamma_\Sigma$ defined as follows:

$$\det(\bar{\xi}) = \det(\bar{\xi}_M)^{\otimes d} \otimes \bigotimes_{A \in \mathcal{B}} \det(\bar{\xi}_A)^{\otimes a}$$

This is the local system that we have used in our statement of theorem 2.29 on the existence of higher string operations for a general set of branes $\mathcal{B}$. However, we have to be careful that this system is compatible with the prop operations.

6.3. Compatibility of $\mathcal{L}^M_{\mathcal{B}}$ with the prop operations. We will prove that the local system $\mathcal{L}^M_{\mathcal{B}}$ is compatible with the operations of the prop $\mathsf{Bord}^+_{\mathcal{B}}$. This is necessary to make sure that our operations compose well.

Remark 5.65. Compare this to the results of the next subsection, in particular 5.67, tells us that over many components $B\Gamma_\Sigma$ the local system is trivializable. However, it is not possible to do this in a way that is compatible with the prop operations.

Equivalently, we can check that the replacement $\bar{\xi}$ we to define the local system corresponding to κ is compatible with these operations. This is done in the following proposition:

Proposition 5.66. *The determinant graded line bundles $\bar{\xi}$ extends the prop structure of $\mathsf{Bord}^+_{\mathcal{B}}$ to a prop in the category of spaces with local systems. To be precise, we must prove that there exist a natural isomorphism*

$$\det(\bar{\xi}_\Sigma) \otimes \det(\bar{\xi}_{\Sigma'}) \cong (B\Gamma(\chi))^* \det(\bar{\xi}_{\Sigma' \circ \Sigma})$$

over $B\Gamma_\Sigma \times B\Gamma_{\Sigma'}$, where $B\Gamma(\chi) : B\Gamma_\Sigma \times B\Gamma_{\Sigma'} \to B\Gamma_{\Sigma' \circ \Sigma'}$ is the map of classifying spaces induced by glueing of cobordisms.

Proof. Consider the long exact sequence associated to the triple $(\Sigma' \circ \Sigma, \Sigma, \partial_{in}\Sigma)$. This has non-zero part

$$\begin{aligned} &H_1(\Sigma, \partial_{in}\Sigma; \mathbb{R}) \to H_1(\Sigma' \circ \Sigma, \partial_{in}\Sigma; \mathbb{R}) \to H_1(\Sigma' \circ \Sigma, \Sigma; \mathbb{R}) \\ &\qquad \to H_0(\Sigma, \partial_{in}\Sigma; \mathbb{R}) \to H_0(\Sigma' \circ \Sigma, \partial_{in}\Sigma; \mathbb{R}) \to H_0(\Sigma' \circ \Sigma, \Sigma; \mathbb{R}) \end{aligned}$$

where in the two rows the last terms are naturally isomorphic to $H_1(\Sigma', \partial_{in}\Sigma'; \mathbb{R})$ and $H_0(\Sigma', \partial_{in}\Sigma'; \mathbb{R})$ respectively using the excision axiom for homology. By correctly cancelling the kernels and cokernels of the maps in the short exact sequence, one obtains the natural isomorphism in the statement of this proposition. ☐

6.4. Simplifying the local system $\mathcal{L}^M_{\mathcal{B}}$. We will now describe a lemma which will help to handle the local system. To be precise, we will describe when $\mathcal{L}^M_{\mathcal{B}}$ is trivializable, i.e. consists of nothing but a grading shift after a choice of trivialization of the determinant line bundle.

Proposition 5.67. *If Σ has no more than one purely free boundary component of each label on each connected component, then $\mathcal{L}^M_{\mathcal{B}}$ is trivializable over Σ.*

Proof. The determinant line bundle is trivializable if and only if its holonomy is trivial around each loop. But under the conditions of the lemma, without loss of generality we can assume that each representative of a class in the mapping class group preserves each boundary component.

Now consider the non-zero part of the long exact sequence of the pair $(\Sigma, \partial_s\Sigma)$:

$$H_1(\partial_s\Sigma; \mathbb{R}) \to H_1(\Sigma; \mathbb{R}) \to H_1(\Sigma, \partial_s\Sigma; \mathbb{R}) \to H_0(\partial_s\Sigma; \mathbb{R}) \to H_0(\Sigma; \mathbb{R}) \to H_0(\Sigma, \partial_s\Sigma; \mathbb{R})$$

where all groups have compatible actions of the mapping class groups. This means that we can express the determinant of $\bar{\eta}$ is terms of the determinants of the powers of the other groups $H_1(\partial_s\Sigma; \mathbb{R})$, $H_1(\Sigma; \mathbb{R})$, $H_0(\partial_s\Sigma; \mathbb{R})$ and $H_1(\Sigma; \mathbb{R})$ that appear here. By the previous remarks the action must be trivial on all except $H_1(\Sigma; \mathbb{R})$. But the action does preserving an orientation put on $H_1(\Sigma)$, because the mappign class group acts via the symplectic group and the action of the symplectic group preserves the orientation. A similar argument works for $H_*(A_\Sigma, \partial_{in}A_\Sigma; \mathbb{R})$ and this proves the statement. ☐

However, note that it is impossible to trivialize the local systems over all components at the same time in a way that is compatible with the composition of cobordisms. A consequence of this that two compositions which lead to a cobordism Σ will give two ways to trivialize the determinant line bundle, which may differ in sign.

However, this problem does not occur if we assume that M and all branes that appear are of even dimension.

Proposition 5.68. *Let M be even-dimensional and let $S = \{\Sigma_1, \Sigma_2, \ldots\}$ be a set of $\mathcal{B}$-labelled cobordisms such that all branes $A \in \mathcal{B}$ that appear are even-dimensional as well. Then we can coherently trivialize the local system for all $\mathcal{B}$-labelled cobordisms in S.*

PROOF. For this one uses the model $\bar{\kappa}$ for the virtual bundle. If all bundles that appear are even-dimensional, then the order in which they appear doesn't matter for the orientation. Thus we have canonically fixed an orientation if we have fixed orientations for TM and for each TA. This orientation is compatible with composition. $\square$

7. Disjoint union of cobordisms

In this section we prove that the maps $\beth$ of theorem 5.31, and hence the maps $\gimel$ of definition 5.43 and the operations they induce in homology, can made compatible with the disjoint union of $\mathcal{B}$-labelled cobordisms. The disjoint union of $\mathcal{B}$-labelled cobordisms is described in section 3.2 of chapter 2 and is modelled by the disjoint union of $\mathcal{B}$-labelled cobordism graphs as described in section 3.1 of chapter 3.

7.1. Compatibility for a disjoint union of a single pair of $\mathcal{B}$-labelled cobordism graphs.

We start by showing that the construction of $\daleth$ in theorem 5.6 for a single pair of nice $\mathcal{B}$-labelled cobordism graphs Γ and Γ' is compatible with disjoint union in the sense of proposition 5.69. We first need to describe the maps that will appear.

7.1.1. *The functors $\Sigma^\infty(M^{\partial_{in}-})_+$ and $\Sigma^\infty(M^{\partial_{out}-})_+$.* Recall that we had the functor $\sqcup : \mathsf{Fn} \to \mathsf{Fn}$ encoding the disjoint union of cobordisms on the level of graphs. The functors $M^{\partial_{in}-} : \mathsf{Fn} \to \mathsf{Top}$ and $M^{\partial_{out}-} : \mathsf{Fn} \to \mathsf{Top}$ are compatible with this functor in the sense that there are canonical natural transformations

$$M^{\partial_{in}-} \times M^{\partial_{in}-} \overset{M^{\partial_{in}\sqcup}}{\Longrightarrow} M^{\partial_{in}-} \circ \sqcup : \mathsf{Fn} \times \mathsf{Fn} \to \mathsf{Top}$$
$$M^{\partial_{out}-} \times M^{\partial_{out}-} \overset{M^{\partial_{out}\sqcup}}{\Longrightarrow} M^{\partial_{out}-} \circ \sqcup : \mathsf{Fn} \times \mathsf{Fn} \to \mathsf{Top}$$

These are simply given by the natural homeomorphisms $M^{\partial_{in}\Gamma} \times M^{\partial_{in}\Gamma'} \cong M^{\partial_{in}(\Gamma \sqcup \Gamma')}$ and $M^{\partial_{in}\Gamma} \times M^{\partial_{in}\Gamma'} \cong M^{\partial_{in}(\Gamma \sqcup \Gamma')}$. Composing with the suspension functors gives the required natural transformations on the level of spectra.

7.1.2. *The virtual bundle κ.* Next we discuss the virtual bundles. We claim that there is a natural isomorphism between the virtual bundle $\kappa_{\Gamma \sqcup \Gamma'}$ over $M^{\partial_{out}(\Gamma \sqcup \Gamma')}$ and the pull back of the virtual bundle $\pi_1^*\kappa_\Gamma \oplus \pi_2^*\kappa_{\Gamma'}$ over $M^{\partial_{out}\Gamma} \times M^{\partial_{out}\Gamma'}$ along the homeomorphism mentioned above.

This is clear upon inspection of the definition of the virtual bundle κ. Both its positive and negative part are given by direct sums over vertices or edges and hence are compatible with disjoint union. The result is an isomorphism we denote by $\kappa_\sqcup$. Together with $M^{\partial_{out}\sqcup}$ this induces a map

$$(M^{\partial_{out}\sqcup})^{\kappa_\sqcup} : (M^{\partial_{out}\Gamma} \times M^{\partial_{out}\Gamma'})^{\pi_1^*\kappa_\Gamma \oplus \pi_2^*\kappa_{\Gamma'}} \to (M^{\partial_{out}(\Gamma \sqcup \Gamma')})^{\kappa_{\Gamma \sqcup \Gamma'}}$$

between Thom spectra, which is in fact an isomorphism.

7.1.3. *Compatibility for a single pair.* We can now state and prove the proposition that our construction of the map $\daleth$ modelling a degree zero string operation is compatible with disjoint unions.

Proposition 5.69. *Suppose that we have fixed an Euclidean space W with an embedding $M \hookrightarrow W$, tubular neighborhoods $\mathrm{Tub}(\iota) \times \prod_{A \in \mathcal{B}} \mathrm{Tub}(\iota_A)$ for the embeddings of M and branes $A \in \mathcal{B}$ into W and a choice of inner product in $\mathcal{I}(W)$. This is data necessary to define the spaces of umkehr data and the maps $\daleth$ in the following diagram.*

There is a natural map $\mathrm{Umk}(\sqcup) : \mathrm{Umk}(\Gamma) \times \mathrm{Umk}(\Gamma') \to \mathrm{Umk}(\Gamma \sqcup \Gamma')$ *such that the following diagram commutes:*

$$\begin{tikzcd}
\mathrm{Umk}(\Gamma)_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma})_+ \wedge \mathrm{Umk}(\Gamma')_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma'})_+ \arrow[r, "\daleth(\Gamma)\wedge\daleth(\Gamma')"] \arrow[d, "\cong"'] & (M^{\partial_{out}\Gamma})^{\kappa_\Gamma} \wedge (M^{\partial_{out}\Gamma'})^{\kappa_{\Gamma'}} \arrow[d, "\cong"] \\
(\mathrm{Umk}(\Gamma) \times \mathrm{Umk}(\Gamma'))_+ \wedge \Sigma^\infty (M^{\partial_{in}\Gamma} \times M^{\partial_{in}\Gamma'})_+ \arrow[d, "\mathrm{Umk}(\sqcup)\wedge\Sigma^\infty M^{\partial_{in}\sqcup}"'] & (M^{\partial_{out}\Gamma} \times M^{\partial_{out}\Gamma'})^{\pi_1^*\kappa_\Gamma \oplus \pi_2^*\kappa_{\Gamma'}} \arrow[d, "(M^{\partial_{out}\sqcup})^{\kappa_\sqcup}"] \\
\mathrm{Umk}(\Gamma \sqcup \Gamma') \wedge \Sigma^\infty (M^{\partial_{in}(\Gamma\sqcup\Gamma')})_+ \arrow[r, "\daleth(\Gamma\sqcup\Gamma')"'] & (M^{\partial_{out}(\Gamma\sqcup\Gamma')})^{\kappa_{\Gamma\sqcup\Gamma'}}
\end{tikzcd}$$

All the maps in this diagram depend continuously on the choices made in the beginning of the statement of this proposition.

Proof. Recall that $\mathrm{Umk}(\Gamma)$ is a product of spaces of tubular neighborhoods Tub's, spaces of propagating flows $\mathcal{P}$, connections $\mathcal{C}$ and a space of inner products $\mathcal{I}$:

$$\mathrm{Umk}(\Gamma) = \mathrm{Tub}(\tau_\div) \times \mathcal{P}(\nu_\div) \times \mathcal{C}(\nu_\div) \times \mathrm{Tub}(\tau_{in}) \times \mathcal{P}(\nu_{in}) \times \mathcal{C}(\nu_{in})$$

where $\tau_\div$ and τ_{in} where defined in the proof of theorem 5.6. Similarly $\mathrm{Umk}(\Gamma')$ can be written as follows:

$$\mathrm{Umk}(\Gamma') = \mathrm{Tub}(\tau'_\div) \times \mathcal{P}(\nu'_\div) \times \mathcal{C}(\nu'_\div) \times \mathrm{Tub}(\tau'_{in}) \times \mathcal{P}(\nu'_{in}) \times \mathcal{C}(\nu'_{in})$$

Now we note that the vector bundle $\tau''_\div$ for $\Gamma \sqcup \Gamma'$ is given by $\pi_1^*\tau_\div \oplus \pi_2^*\tau'_\div$ and similar for τ_{in}. Hence the product of tubular neighborhoods, propagating flows and connections give us maps $\mathrm{Tub}(\tau_\div) \times \mathrm{Tub}(\tau'_\div) \to \mathrm{Tub}(\tau''_\div)$, etc., which induce a map $\mathrm{Umk}(\Gamma) \times \mathrm{Umk}(\Gamma') \to \mathrm{Umk}(\Gamma \sqcup \Gamma')$. This is the map $\mathrm{Umk}(\sqcup)$ meant in the statement of the proposition.

The commutativity of the diagram can now be inferred from proposition 4.78. To do this, first note $\mathrm{Umk}(\sqcup)$ has an image in $\mathrm{Umk}(\Gamma \sqcup \Gamma')$ which lies inside the product of the subspace of the type $\mathrm{Tub}^\times$, $\mathcal{P}^\times$ and $\mathcal{C}^\times$, i.e. inside the subspace of umkehr data for $\Gamma \sqcup \Gamma'$ compatible with the product. Furthermore one sees the maps p and p' back from this subspace to the spaces of umkehr data $\mathrm{Umk}(\Gamma)$ and $\mathrm{Umk}(\Gamma')$ for Γ and Γ' respectively (called p_j in proposition 4.78) satisfy $(p \circ p') \circ \mathrm{Umk}(\sqcup) = id$. Hence combining the two commutative diagrams of proposition 4.78 for each of the three times that we construct an umkehr map gives the commutative diagram in the statement of the proposition. $\square$

A consequence of the previous proposition is that the degree zero string operations are compatible with disjoint union. In the next section we look at the higher degree string operations as well, by considering the entire map $\beth$.

7.2. Compatibility of $\beth$, $\gimel$ and the operations with disjoint union. We now extend the previous ideas to the family version of the string operations. This is slightly more complicated than the compatibility for a single nice $\mathcal{B}$-labelled graph, because we need to introduce an intermediate category $\nabla(\mathsf{Fn} \times \mathsf{Fn})$ with functors as follows

$$\nabla\mathsf{Fn} \times \nabla\mathsf{Fn} \overset{i}{\hookleftarrow} \nabla(\mathsf{Fn} \times \mathsf{Fn}) \overset{\nabla\sqcup}{\to} \nabla\mathsf{Fn}$$

where $\nabla\sqcup$ is induced by $\sqcup$ due to the naturality of the construction of the category of simplices and i is the inclusion functor. The statement that $|\nabla\mathsf{C}| \simeq |\mathsf{C}|$ and the compatibility of geometric realisation with products implies that $|i|$ is a homotopy equivalence. We have already seen how to thicken the left and right categories to $\overline{\nabla\mathsf{Fn}}_/ \times \overline{\nabla\mathsf{Fn}}_/$ and $\overline{\nabla\mathsf{Fn}}_/$ respectively. We will now show how to do this for the middle category such that there are induced functors $\overline{i}$ and $\overline{\nabla\sqcup}$ such that the following diagram commutes:

$$\begin{tikzcd}
\overline{\nabla\mathsf{Fn}}_/ \times \overline{\nabla\mathsf{Fn}}_/ \arrow[d] & \overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/ \arrow[l, "\bar{i}"'] \arrow[r, "\overline{\nabla\sqcup}"] \arrow[d] & \overline{\nabla\mathsf{Fn}}_/ \arrow[d] \\
\nabla\mathsf{Fn}\times\nabla\mathsf{Fn} & \nabla(\mathsf{Fn}\times\mathsf{Fn}) \arrow[l, "i"] \arrow[r, "\nabla\sqcup"'] & \nabla\mathsf{Fn}
\end{tikzcd} \tag{11}$$

where the vertical arrows are the forgetful functors which forget the umkehr and splitting data.

We will construct operation $\beth_m$ for the middle category and show that it is compatible with the constructions for the left and right category. This implies compatibility of $\beth$ with disjoint union as $\bar{i}$ is a homotopy equivalence. The compatibility for $\daleth$ and the induced map in homology are then formal consequences.

7.2.1. *The category $\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/$.* To construct this category we need to interweave splitting data using a functor $\mathrm{Split}_m : \nabla(\mathsf{Fn}\times\mathsf{Fn}) \to \mathsf{Top}$ and then umkehr data using a functor $\mathrm{Umk}_m : \nabla(\mathsf{Fn}\times\mathsf{Fn})_/ \to \mathsf{Top}$.

We simply define the splitting data functor by setting $\mathrm{Split}_m = (\mathrm{Split}\times\mathrm{Split})\circ i$ and define using the Grothendieck construction

$$\nabla(\mathsf{Fn}\times\mathsf{Fn})_/ = \nabla(\mathsf{Fn}\times\mathsf{Fn}) \int \mathrm{Split}_m$$

Note that the canonically this definition implies that i extends to a functor $i_/ : \nabla(\mathsf{Fn}\times\mathsf{Fn})_/ \to \nabla\mathsf{Fn}_/ \times \nabla\mathsf{Fn}_/$. Then we define Umk_m as $(\mathrm{Umk}\times\mathrm{Umk})\circ i_/$ and set

$$\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/ = \nabla(\mathsf{Fn}\times\mathsf{Fn})_/ \int \mathrm{Umk}_m$$

As before, this choice for Umk_m implies that there is a canonical extension of $i_/$ to $\bar{i}$ which makes the left square of diagram 11 commute.

7.2.2. *The operation $\beth_m$ and compatibility with the left map $\beth\times\beth$.* The properties of the components Umk, which appear twice in the definition of Umk_m imply the existence of two sets of operations. To write them down, we first need to look at the functors involved in the domain and codomain of these operations. We define $M_m^{\partial_{in}-}$, $M_m^{\partial_{out}-}$ and κ_m by taking the composition of $\bar{i}$ with the product of the corresponding functors on $\overline{\nabla\mathsf{Fn}}_/ \times \overline{\nabla\mathsf{Fn}}_/$.

To be precise, if $\sigma\times\sigma' = ((\Gamma_n,\Gamma'_n)\to\ldots\to(\Gamma_0,\Gamma'_0))$ is a simplex of $\nabla(\mathsf{Fn}\times\mathsf{Fn})$ then we get two operation

$$\begin{aligned}
\beth^{\sigma} &: (|\nabla\sigma|\times\mathrm{Umk}(\sigma))_+ \wedge \Sigma^\infty_+(M^{\partial_{in}\Gamma_0}) \to (M^{\partial_{out}\Gamma_0})^{\kappa_\sigma}\\
\beth^{\sigma'} &: (|\nabla\sigma'|\times\mathrm{Umk}(\sigma'))_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma'_0})_+ \to (M^{\partial_{out}\Gamma'_0})^{\kappa_{\sigma'}}
\end{aligned}$$

We use these to create a single map $\beth^{\sigma\times\sigma'}$ as follows. First note that $|\nabla\sigma|$ and $|\nabla\sigma'|$ are naturally isomorphic to Δ^n. The codomain of $\beth_m^{\sigma\times\sigma'}$ will be $(\Delta^n\times\mathrm{Umk}(\sigma)\times\mathrm{Umk}(\sigma'))_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma_0})_+ \wedge \Sigma^\infty(M^{\partial_{in}\Gamma_0})_+$ and the domain $(M^{\partial_{out}\Gamma_0})^{\kappa_\sigma} \wedge (M^{\partial_{out}\Gamma'_0})^{\kappa_{\sigma'}}$. The operation is given by the composition of $\beth^\sigma\wedge\beth^{\sigma'}$ with the map which is induced by the diagonal $\Delta^n \to \Delta^N\times\Delta^n \cong |\nabla\sigma|\times|\nabla\sigma'|$ and reordering the terms. The result is as follows

$$\beth_m^{\sigma\times\sigma'} : (\Delta^n\times\mathrm{Umk}(\sigma)\times\mathrm{Umk}(\sigma'))_+\wedge\Sigma^\infty(M^{\partial_{in}\Gamma_0})_+\wedge\Sigma^\infty(M^{\partial_{in}\Gamma'_0})_+ \to (M^{\partial_{out}\Gamma_0})^{\kappa_\sigma}\wedge(M^{\partial_{out}\Gamma'_0})^{\kappa_{\sigma'}}$$

The properties of Umk imply that these operations $\beth^{\sigma\times\sigma'}$ glue to give the operation β_m on the homotopy colimits.

$$\beth_m : \operatorname*{hocolim}_{\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/} \Sigma^\infty(M^{\partial_{in}\pi_1-})_+\wedge\Sigma^\infty(M^{\partial_{in}\pi_2-})_+ \to \operatorname*{hocolim}_{\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/} (M^{\partial_{out}\pi_1-})^\kappa\wedge(M^{\partial_{out}\pi_2-})^\kappa$$

where $\pi_i : \overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/ \to \nabla\mathsf{Fn}$ for $i=1,2$ is the projection on the first or second simplex.

We want to get the codomain and domain of the map $\beth\times\beth$. We start by noting the functor $\bar{i}$ induces weak equivalences of homotopy colimits which we denote as follows:

$$\bar{i}^1_* : \operatorname*{hocolim}_{\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/} \Sigma^\infty(M^{\partial_{in}\pi_1-})_+\wedge\Sigma^\infty(M^{\partial_{in}\pi_2-})_+ \to \operatorname*{hocolim}_{\overline{\nabla\mathsf{Fn}}_/\times\overline{\nabla\mathsf{Fn}}_/} \Sigma^\infty(M^{\partial_{in}\pi_1-})_+\wedge\Sigma^\infty(M^{\partial_{in}\pi_2-})_+$$

$$\bar{i}^2_* : \underset{\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/}{\operatorname{hocolim}} (M^{\partial_{out}\pi_1 -})^\kappa \wedge (M^{\partial_{out}\pi_2 -})^\kappa \to \underset{\overline{\nabla\mathsf{Fn}}_/\times\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} (M^{\partial_{out}\pi_1 -})^\kappa \wedge (M^{\partial_{out}\pi_2 -})^\kappa$$

If we model homotopy colimits by the fat geometric realisation of a simplicial space, then these maps will induced by a map of simplicial spaces which is an inclusion of a simplicial subspace. The homotopy colimit over a product of categories of two functors, each of which depends only on one of the categories, can be split. The fact that the smash product is a left adjoint to an internal hom in a nice model of spectra in particular implies that we have weak equivalences

$$\bar{j}^1_* : \underset{\overline{\nabla\mathsf{Fn}}_/\times\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} \Sigma^\infty(M^{\partial_{in}\pi_1 -})_+ \wedge \Sigma^\infty(M^{\partial_{in}\pi_2 -})_+ \to (\underset{\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} \Sigma^\infty(M^{\partial_{in} -})_+) \wedge (\underset{\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} \Sigma^\infty(M^{\partial_{in} -})_+)$$

$$\bar{j}^2_* : \underset{\overline{\nabla\mathsf{Fn}}_/\times\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} (M^{\partial_{out}\pi_1 -})^\kappa \wedge (M^{\partial_{out}\pi_2 -})^\kappa \to (\underset{\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} (M^{\partial_{out} -})^\kappa) \wedge (\underset{\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} (M^{\partial_{out} -})^\kappa)$$

This allows us to state the compatibility between the left hand side $\beth \wedge \beth$ and the middle $\beth_m$ of diagram 11.

Proposition 5.70. *The following diagram commutes*

$$\begin{tikzcd}
\underset{\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/}{\operatorname{hocolim}} \Sigma^\infty(M^{\partial_{in}\pi_1 -})_+ \wedge \Sigma^\infty(M^{\partial_{in}\pi_2 -})_+ \arrow[r, "\beth_m"] \arrow[d, "\bar{j}^1_*\circ\bar{i}^1_*"', "\simeq"] & \underset{\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/}{\operatorname{hocolim}} (M^{\partial_{out}\pi_1 -})^\kappa \wedge (M^{\partial_{out}\pi_2 -})^\kappa \arrow[d, "\bar{j}^2_*\circ\bar{i}^2_*", "\simeq"'] \\
(\underset{\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} \Sigma^\infty(M^{\partial_{in} -})_+) \wedge (\underset{\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} \Sigma^\infty(M^{\partial_{in} -})_+) \arrow[r, "\beth\wedge\beth"'] & (\underset{\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} (M^{\partial_{out} -})^\kappa) \wedge (\underset{\overline{\nabla\mathsf{Fn}}_/}{\operatorname{hocolim}} (M^{\partial_{out} -})^\kappa)
\end{tikzcd}$$

Proof. If we use fat geometric realisation of simplicial spaces as a model for homotopy colimits, it suffices prove that the diagram commutes for each individual simplex. But this follows directly from our definition of $\beth_m^{\sigma\times\sigma'}$ as $\beth^\sigma \wedge \beth^{\sigma'}$. $\square$

7.2.3. *The compatibility of $\beth_m$ with the right map $\beth$.* We now want to show that $\beth_m$ is also compatible with the map $\beth$ for the right hand side of diagram 11. To do this we first need to describe $\overline{\nabla\sqcup}$, which requires describing natural transformations

$$\mathrm{Split}\times\mathrm{Split} \overset{\mathrm{Split}(\sqcup)}{\Longrightarrow} \mathrm{Split}\circ\sqcup : \nabla(\mathsf{Fn}\times\mathsf{Fn}) \to \mathsf{Top}$$

$$\mathrm{Umk}\times\mathrm{Umk} \overset{\mathrm{Umk}(\sqcup)}{\Longrightarrow} \mathrm{Umk}\circ\sqcup : \nabla(\mathsf{Fn}\times\mathsf{Fn})_/ \to \mathsf{Top}$$

The first natural transformation is easy to write down once one recalls the definition of Split, found in definition 5.15. It is a product of subspaces of the topologized space of linear maps $\mathrm{Hom}_{\mathsf{Vect}}(\mathbb{R}^{q^{-1}(v)}, \mathbb{R})$, where v runs over all vertices of the graph. This implies that there is a homeomorphism $\mathrm{Split}(\Sigma)\times\mathrm{Split}(\Sigma) \cong \mathrm{Split}(\Sigma\times\Sigma')$ and this homeomorphism gives the desired natural transformation.

To define $\mathrm{Umk}(\sqcup)$ we simply note that we can take the products of the umkehr data, as in the proof of proposition 5.69, for each level of lifts separately. This is an inclusion which has image inside the subspace of umkehr data compatible with the product structure. The end result is a functor

$$\overline{\nabla\sqcup} : \overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/ \to \overline{\nabla\mathsf{Fn}}_/$$

which makes the right square of diagram 11 commute.

We now address the question of writing down natural maps between the homotopy colimits. Note that there are natural transformations for the functors defining the codomain and domain of the operations induced the natural inclusion functor $\overline{\nabla\sqcup} : \overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/ \to \overline{\nabla\mathsf{Fn}}_/$. Those for $M^{\partial_{in}-}$ and $M^{\partial_{out}-}$ are induced by the natural transformations $M^{\partial_{in}\sqcup}$ and $M^{\partial_{out}\sqcup}$ of the previous section by forgetting the additional data. The suspension of the former will be more important for us:

$$\Sigma^\infty(M^{\partial_{in}\pi_1 -})_+ \wedge \Sigma^\infty(M^{\partial_{in}\pi_1 -})_+ \overset{\Sigma^\infty M^{\partial_{in}\sqcup}}{\Longrightarrow} \Sigma^\infty(M^{\partial_{in} -})_+ \circ \overline{\nabla\sqcup} : \overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_/ \to \mathsf{Spectra}$$

For κ we recall that we have an isomorphism between the virtual bundle $\kappa_{\sigma\sqcup\sigma'}$ over $M^{\partial_{out}(\Gamma_0\sqcup\Gamma_0')}$ and the pullback of the virtual bundle $\pi_1^*\kappa_\sigma \oplus \pi_2^*\kappa_{\sigma'}$ over $M^{\partial_{out}\Gamma_0}\times M^{\partial_{out}\Gamma_0'}$ along the homeomorphism $M^{\partial_{out}(\Gamma_0\sqcup\Gamma_0')} \cong M^{\partial_{out}\sigma}\times M^{\partial_{out}\sigma}$ used in the natural transformation $M^{\partial_{out}\sqcup}$ and an isomorphism of Thom spectra $(M^{\partial_{out}\Gamma} \times M^{\partial_{out}\Gamma'})^{\pi_1^*\kappa_\sigma \oplus \pi_2^*\kappa_{\sigma'}} \cong (M^{\partial_{out}\Gamma})^{\kappa_\sigma} \wedge (M^{\partial_{out}\Gamma'})^{\kappa_{\sigma'}}$. Using the splitting data the composition of the latter with the isomorphism of Thom spectra induced by the former gives a natural transformation

$$(M^{\partial_{out}\pi_1 -})^\kappa \wedge (M^{\partial_{out}\pi_1 -})^\kappa \overset{(M^{\partial_{out}\sqcup})^{\kappa_\sqcup}}{\Longrightarrow} (M^{\partial_{out}-})^\kappa \circ \circ\overline{\nabla\sqcup} : \overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_{/} \to \mathsf{Spectra}$$

Because these natural transformations induce maps between the homotopy colimits, in fact strictly if we use a fat geometric realisation model, all maps in the diagram in the following proposition are well-defined.

Proposition 5.71. *The following diagram commutes*

$$\begin{array}{ccc} \underset{\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_{/}}{\mathrm{hocolim}}\, \Sigma^\infty(M^{\partial_{in}\pi_1 -})_+ \wedge \Sigma^\infty(M^{\partial_{in}\pi_1 -})_+ & \xrightarrow{\beth_m} & \underset{\overline{\nabla(\mathsf{Fn}\times\mathsf{Fn})}_{/}}{\mathrm{hocolim}}\, (M^{\partial_{out}\pi_1 -})^\kappa \wedge (M^{\partial_{out}\pi_1 -})^\kappa \\ \downarrow & & \downarrow \\ \underset{\overline{\nabla\mathsf{Fn}}_{/}}{\mathrm{hocolim}}\, \Sigma^\infty(M^{\partial_{in}-})_+ & \xrightarrow[\beth]{} & \underset{\overline{\nabla\mathsf{Fn}}_{/}}{\mathrm{hocolim}}\, (M^{\partial_{out}-})^\kappa \end{array}$$

Proof. Again it suffices to prove the commutativity of the operations in the diagram over each individual simplex in the fat geometric realisations which give the homotopy colimit. Using our definition of Umk($\sqcup$), this is a direct consequence of proposition 5.69 because over each simplex $\beth$ is simply a restriction of $\daleth$. $\square$

Putting the commutative diagrams of the two propositions in this section on top of each other then gives the following commutative diagram

$$\begin{array}{ccc} (\underset{\overline{\nabla\mathsf{Fn}}_{/}}{\mathrm{hocolim}}\, \Sigma^\infty(M^{\partial_{in}-})_+) \wedge (\underset{\overline{\nabla\mathsf{Fn}}_{/}}{\mathrm{hocolim}}\, \Sigma^\infty(M^{\partial_{in}-})_+) & \xrightarrow{\beth\wedge\beth} & (\underset{\overline{\nabla\mathsf{Fn}}_{/}}{\mathrm{hocolim}}\, (M^{\partial_{out}-})^\kappa) \wedge (\underset{\overline{\nabla\mathsf{Fn}}_{/}}{\mathrm{hocolim}}\, (M^{\partial_{out}-})^\kappa) \\ \downarrow & & \downarrow \\ \underset{\overline{\nabla\mathsf{Fn}}_{/}}{\mathrm{hocolim}}\, \Sigma^\infty(M^{\partial_{in}-})_+ & \xrightarrow[\beth]{} & \underset{\overline{\nabla\mathsf{Fn}}_{/}}{\mathrm{hocolim}}\, (M^{\partial_{out}-})^\kappa \end{array}$$

which exactly expresses the compatibility of $\beth$ with disjoint union. This is the family version of proposition 5.69.

7.2.4. *The compatibility of* $\gimel$ *and the string operations with disjoint union.* Finally for the string operations, we still need to apply the Thom isomorphism to the two spectra in the columns of the previous commutative diagram after smashing $H\mathbb{Z}$ into each of them. This is possible, because the local system $\mathcal{L}_{\mathcal{B}}^M$ is clearly compatible with disjoint union. Again it suffices to check the this for each individual simplex in the homotopy colimits but over those we are simply dealing with an isomorphism of virtual bundles.

So indeed we can conclude that $\gimel$ is compatible with disjoint union. After passing to homology with rational coefficients, we obtain the following theorem.

Theorem 5.72. *The string operations* $\gimel_*$ *are compatible in the sense that the following diagram commutes:*

$$\begin{array}{ccc} H_*(B\Gamma_\Sigma;\mathcal{L}_{\mathcal{B}}^M)\otimes H_*(M^{\partial_{in}\Sigma};\mathbb{Q})\otimes H_*(B\Gamma_{\Sigma'};\mathcal{L}_{\mathcal{B}}^M)\otimes H_*(M^{\partial_{in}\Sigma'};\mathbb{Q}) & \xrightarrow{\gimel_*\otimes\gimel_*} & H_*(M^{\partial_{out}\Sigma};\mathbb{Q})\otimes H_*(M^{\partial_{out}\Sigma'};\mathbb{Q}) \\ \cong\downarrow & & \downarrow\cong \\ H_*(B\Gamma_{\Sigma\sqcup\Sigma'};\mathcal{L}_{\mathcal{B}}^M)\otimes H_*(M^{\partial_{in}(\Sigma\sqcup\Sigma')};\mathbb{Q}) & \xrightarrow[\gimel_*]{} & H_*(M^{\partial_{out}(\Sigma\sqcup\Sigma')};\mathbb{Q}) \end{array}$$

where we have shortened the large tensor product over the homology of all incoming and outgoing boundary components, for example

$$H_*(M^{\partial_{in}\Sigma};\mathbb{Q}) := H_*(LM;\mathbb{Q})^{\otimes r} \otimes \bigotimes_{A,B\in\mathcal{B}} H_*(P_M(A,B);\mathbb{Q})^{\otimes r_{A,B}}$$

The isomorphisms should be clear upon inspection of the vertical maps and are a nice piece of additional information.

8. Glueing of cobordisms

In this final section of the chapter about the construction of string operations, we show that the maps $\beth$, and hence $\gimel$ and the operations it induces in homology, can be made compatible with the glueing of $\mathcal{B}$-labelled cobordisms. This is essentially a more involved version of the argument that we used to prove compatibility with disjoint unions. Again we will show compatibility of the operations with the glueing of a pair of graphs and then we will prove the general case.

Recall from definitions 3.71 and 3.75 that the functor modelling the glueing of cobordisms Σ and Σ' on the level of $\mathcal{B}$-labelled cobordism graphs is only defined on a smaller category $\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$ of pairs of glueable admissible $\mathcal{B}$-labelled cobordism graphs. Let's recall the definition of $\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$. Its objects are pairs of graphs (Γ, Γ') together with an isomorphism between the outgoing boundary $\partial_{out}\Gamma$ and the incoming boundary $\partial_{in}\Gamma'$, which preserves the ordering of the boundary components, sends starting vertices to starting vertices and reserves the orientations. Its morphisms are pairs of morphisms compatible with these isomorphisms of parts of the boundary.

This category has geometric realisation homotopy equivalent to the geometric realisation of $\mathsf{Fn}(\Sigma) \times \mathsf{Fn}(\Sigma')$, so there is no loss in restricting to it. This homotopy equivalence is induced by the inclusion functor $I : \mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma) \to \mathsf{Fn}(\Sigma) \times \mathsf{Fn}(\Sigma')$ of lemma 3.74.

If we shorten the notation $\mathsf{Fat}^{\mathrm{glue}}(\Sigma' \circ \Sigma)$ to $\mathsf{Fg}(\Sigma' \circ \Sigma)$, then the previous can be summarized by the diagram:

$$\mathsf{Fn}(\Sigma) \times \mathsf{Fn}(\Sigma') \xleftarrow{I} \mathsf{Fg}(\Sigma' \circ \Sigma) \xrightarrow{\chi} \mathsf{Fn}(\Sigma)$$

8.1. Compatibility with glueing for a single pair of graphs. In this section we will compare what happens to a diagram

$$(\Gamma, \Gamma') \longleftarrow (\Gamma, \Gamma', \phi) \longrightarrow \Gamma' \circ \Gamma$$

of graphs in the previous diagram

$$\mathsf{Fn}(\Sigma) \times \mathsf{Fn}(\Sigma') \xleftarrow{I} \mathsf{Fg}(\Sigma' \circ \Sigma) \xrightarrow{\chi} \mathsf{Fn}(\Sigma)$$

when we try to apply the construction of $\daleth$ twice in succession to the left and middle graphs and once to the right glued graph. Of course, in general on $\mathsf{Fn}(\Sigma) \times \mathsf{Fn}(\Sigma')$ we do not know how to compose two such maps; that's exactly why the isomorphism ϕ between the incoming boundary of the second graph and outgoing boundary of the first graph was introduced. However, we do want to check that two constructions of $\daleth$ for the middle individually coincide with those on the left.

We start by looking at the maps involved in stating the compatibility diagram. We do this with a bit more generality so that we can use some of these results in the compatibility for $\beth$ as well.

8.1.1. *The functors $M^{\partial_{in}-}$ and $M^{\partial_{out}-}$.* We will define the analogues of the incoming and outgoing boundary functors $M^{\partial_{in}-}$ and $M^{\partial_{out}-}$ on $\mathsf{Fn}(\Sigma) \times \mathsf{Fn}(\Sigma')$, $\mathsf{Fg}(\Sigma' \circ \Sigma)$ and $\mathsf{Fn}(\Sigma)$ and give natural transformations between them. Intuitively, in all three cases we just want to use the incoming boundary of the first graph.

The idea is simple if one recall that we already know how to define the functor $M^{\partial_{in}-}$ on a category of the form $\mathsf{Fn}(\Sigma)$. It is simply the restriction of the functor $M^{\partial_{in}-}$ on Fn, and this functor sends a graph Γ to the mapping space $M^{\partial_{in}\Gamma}$. This takes care of the category $\mathsf{Fn}(\Sigma)$ on the right: we can just take this restriction $M^{\partial_{in}-}$ of the ordinary incoming boundary functor.

Let $\pi_1 : \mathsf{Fn}(\Sigma) \times \mathsf{Fn}(\Sigma') \to \mathsf{Fn}(\Sigma)$ be the projection on the first component. For this category we will use $M^{\partial_{in}-} \circ \pi_1$. Finally, for $\mathsf{Fg}(\Sigma' \circ \Sigma)$ we will use $M^{\partial_{in}-} \circ \pi_1 \circ I$.

The functor $M^{\partial_{in}-} \circ \pi_1 \circ I$ and the composition of $M^{\partial_{in}-} \circ \pi_1$ with I clearly coincide, so no natural transformation is need for the left of the diagram. For the right of the diagram we do need one:

$$M^{\partial_{in}-} \circ \pi_1 \circ I \overset{M^{\partial_{in}\chi}}{\Longrightarrow} M^{\partial_{in}-} \circ \chi : \mathsf{Fg}(\Sigma' \circ \Sigma) \to \mathsf{Top}$$

It is given by noting that incoming boundary of a graph $\Gamma' \circ \Gamma$ in $\mathsf{Fn}(\Sigma' \circ \Sigma)$ is naturally isomorphic to the incoming boundary of Γ and hence $M^{\partial_{in}-} \circ \pi_1 \circ I(\Gamma, \Gamma')$ and $M^{\partial_{in}-} \circ \chi(\Gamma, \Gamma')$ are naturally homeomorphic topological spaces.

Exactly the same reasoning works for the outgoing boundary, except one needs to use π_2 instead of π_1. In particular, we have a natural transformation

$$M^{\partial_{out}-} \circ \pi_2 \circ I \overset{M^{\partial_{out}\chi}}{\Longrightarrow} M^{\partial_{out}-} \circ \chi : \mathsf{Fg}(\Sigma' \circ \Sigma) \to \mathsf{Top}$$

If we compose all of these with the suspension functor, we get functors and natural transformations of the incoming and outgoing boundary on the level of spectra.

8.1.2. *The virtual bundle κ.* Next we need to discuss the virtual bundles a bit. Three virtual bundles play a role when one tries to construct the operations. The left and middle one are the same and are equal to $\kappa_{\Gamma'} \oplus p^*\phi_*\kappa_\Gamma$, where p^* is the correct composition of pullbacks along maps to move κ_Γ from $M^{\partial_{in}\Gamma'}$ to $M^{\partial_{out}\Gamma'}$ and ϕ_* is the pushforward along the homeomorphism $M^{\partial_{out}\Gamma} \cong M^{\partial_{in}\Gamma'}$ induced by ϕ. On the right we will have the virtual bundle $\kappa_{\Gamma' \circ \Gamma}$.

We claim that that pullback along $M^{\partial_{out}\chi}$ of $\kappa_{\Gamma'\circ\Gamma}$ is naturally isomorphic to $\kappa_{\Gamma'} \oplus p^*\phi_*\kappa_\Gamma$. To see this, recall that κ_Γ can be thought of as being obtained as follows: we call the extra vertices and edges those that are not in $s(\partial_{in}\Gamma)$. The κ_Γ is obtained by putting a copy of $-TA$ at each extra vertex with label A, a copy of $-TM$ at each unlabelled extra vertex, a copy of TA at the middle of each extra edge with label A and a copy of TM at the middle of each unlabelled extra edge.

The extra edges and vertices of $\Gamma' \circ \Gamma$ are exactly given as the union of the extra edges and vertices of Γ and Γ'. This explains the natural isomorphism between the two virtual bundles. We denote it by κ_χ.

8.1.3. *Compatibility of $\urcorner$ with glueing.* We can now state the compatibility of the construction of $\urcorner$ with composition.

Proposition 5.73. *There is a contractible space* $\mathrm{Umk}^{\mathrm{glue}}(\Gamma' \circ \Gamma)$ *with natural maps*

$$\mathrm{Umk}(\chi) : \mathrm{Umk}^{\mathrm{glue}}(\Gamma' \circ \Gamma) \to \mathrm{Umk}(\Gamma' \circ \Gamma)$$
$$p_1 : \mathrm{Umk}^{\mathrm{glue}}(\Gamma' \circ \Gamma) \to \mathrm{Umk}(\Gamma)$$
$$p_2 : \mathrm{Umk}^{\mathrm{glue}}(\Gamma' \circ \Gamma) \to \mathrm{Umk}(\Gamma')$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Umk}^{\mathrm{glue}}(\Gamma' \circ \Gamma)_+ \Sigma^\infty(M^{\partial_{in}\Gamma})_+ \arrow[d, "\urcorner(\Gamma)"'] \arrow[rr, "\mathrm{Umk}(\chi) \wedge id"] & & \mathrm{Umk}(\Gamma' \circ \Gamma)_+ \wedge \Sigma^\infty(M^{\partial_{in}(\Gamma' \circ \Gamma)})_+ \arrow[dd, "\urcorner(\Gamma' \circ \Gamma)"] \\
\mathrm{Umk}(\Gamma')_+ \wedge (M^{\partial_{out}\Gamma})^{\kappa_\Gamma} \arrow[d, "(\urcorner(\Gamma'))^{\kappa_\Gamma}"'] & & \\
(M^{\partial_{out}\Gamma'})^{\kappa_{\Gamma'} \oplus p^*\phi_*\kappa_\Gamma} \arrow[rr, "\cong", "(M^{\partial\chi})^{\kappa_\chi}"'] & & (M^{\partial_{out}(\Gamma' \circ \Gamma)})^{\kappa_{\Gamma' \circ \Gamma}}
\end{tikzcd}$$

As suggested by the notation, the maps $\urcorner(\Gamma)$ and $\urcorner(\Gamma')$ coincide with the ones constructed in theorem 5.6 for the individual graphs.

Proof. We will just define the vertical maps such that the second statement of the proposition holds. While comparing them, we will find $\mathrm{Umk}^{\mathrm{glue}}(\Gamma' \circ \Gamma)$ and $\mathrm{Umk}(\gamma)$.

For convenience let's desuspend the negative parts of the virtual bundles. Then the map on the left is obtained by the applying the same construction as in theorem 5.6 but now to a larger diagram of graphs

$$\partial_{in}\Gamma \overset{i_{in}}{\hookrightarrow} \Gamma_{v,in} \xleftarrow{s_v} \Gamma_\div \xrightarrow{s_\div} \Gamma \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma) \xleftarrow{s} \partial_{out}\Gamma$$
$$\xrightarrow[\cong]{\phi} \partial_{in}\Gamma' \overset{i_{in}}{\hookrightarrow} \Gamma'_{v,in} \xleftarrow{s_v} \Gamma'_\div \xrightarrow{s_\div} \Gamma' \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma') \xleftarrow{s} \partial_{out}\Gamma'$$

where we have used that our graphs are admissible and therefore $s(\partial_{in}\Gamma)$ coincides with $\partial_{in}\Gamma$ and similarly for Γ'. The corresponding diagram of mapping spaces is

$$\begin{aligned} M^{\partial_{in}\Gamma} \times W^{v(\Gamma'\circ\Gamma\backslash s(\partial_{in}\Gamma))} &\xleftarrow{\rho_{in}} M^{\Gamma_{v,in}} \times W^{v(\Gamma'\backslash s(\partial_{in}\Gamma'))} \xleftarrow{\sigma_{in}} M^{\Gamma_\div} \times W^{v(\Gamma'\backslash s(\partial_{in}\Gamma'))} \\ &\xleftarrow{\sigma_\div} M^{\Gamma} \times W^{v(\Gamma'\backslash s(\partial_{in}\Gamma'))} \xrightarrow{r_{out}} M^{s(\partial_{out}\Gamma)} \times W^{v(\Gamma'\backslash s(\partial_{in}\Gamma'))} \\ &\xrightarrow{\sigma} M^{\partial_{out}\Gamma} \times W^{v(\Gamma'\backslash s(\partial_{in}\Gamma'))} \xrightarrow{M^\phi} M^{\partial_{in}\Gamma} \times W^{v(\Gamma'\backslash s(\partial_{in}\Gamma'))} \\ &\xleftarrow{\rho_{in}} M^{\Gamma_{v,in}} \xrightarrow{\sigma_v} M^{\Gamma_\div} \xleftarrow{\sigma_\div} M^{\Gamma} \xrightarrow{r_{out}} M^{s(\partial_{out}\Gamma)} \xrightarrow{\sigma} M^{\partial_{out}\Gamma} \end{aligned}$$

where have dropped the identity maps on W in the notation of the maps on the first three lines.

On the right the map comes from the following diagram:

$$\partial_{in}\Gamma = \partial_{in}(\Gamma'\circ\Gamma) \overset{i_{in}}{\hookrightarrow} (\Gamma'\circ\Gamma)_{v,in} \xleftarrow{s_v} (\Gamma'\circ\Gamma)_\div \xrightarrow{s_\div} \Gamma'\circ\Gamma \overset{i_{out}}{\hookleftarrow} s(\partial_{out}(\Gamma'\circ\Gamma)) \xleftarrow{s} \partial_{out}(\Gamma'\circ\Gamma) = \partial_{out}\Gamma'$$

with corresponding diagram of mapping spaces as in the proof of theorem 5.6. We want to show that we can rearrange the first diagram of graphs in a way compatible with the umkehr maps to obtain the second diagram of graphs. Let's explain how to do this.

Moving the additional vertices in $\Gamma'_{v,in}$: We want the vertices corresponding to the extra vertices and half-edges of Γ' to already be there in $\Gamma_{v,in}$. Because the tubular neighborhoods for the embedding of M and $A \in \mathcal{B}$ into W are fixed, this requires no additional data of tubular neighborhoods. However, because we are carrying around larger vector bundles on our Thom spectra, we need to include the data of connections on the additional parts of these vector bundles for the umkehr maps.

This will be the first part of the data that needs to be included in $\mathrm{Umk}^{\mathrm{glue}}(\Gamma'\circ\Gamma)$. Let's denote in $\mathcal{C}(\Gamma'\circ\Gamma)$.

Expanding these additional vertices: It should be clear that it doesn't matter whether we expand the additional vertices of $(\Gamma'\circ\Gamma)_{v,in}$ all at the same time or whether we first do those from Γ and then those from Γ'. After this and the previous step, the diagram of graphs is of the form

$$\begin{aligned} \partial_{in}\Gamma &\hookrightarrow (\Gamma'\circ\Gamma)_{v,in} \longleftarrow \Gamma_\div \sqcup \mathrm{hsep}(\Gamma'\backslash s(\partial_{in}\Gamma')) \longrightarrow \Gamma \sqcup \mathrm{hsep}(\Gamma'\backslash s(\partial_{in}\Gamma')) \\ &\hookleftarrow s(\partial_{out}\Gamma) \sqcup \mathrm{hsep}(\Gamma'\backslash s(\partial_{in}\Gamma')) \longleftarrow \partial_{out}\Gamma \sqcup \mathrm{hsep}(\Gamma'\backslash s(\partial_{in}\Gamma')) \\ &\longrightarrow \Gamma'_\div \xrightarrow{s_\div} \Gamma' \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma') \xleftarrow{s} \partial_{out}\Gamma' \end{aligned}$$

Also note that $\Gamma_\div \sqcup \mathrm{hsep}(\Gamma'\backslash s(\partial_{in}\Gamma')) = (\Gamma'\circ\Gamma)_\div$.

Glueing both divided graphs at the same time: Now comes the most difficult part, where we will argue that the two steps in which the separated edges and vertices of $(\Gamma'\circ\Gamma)_\div$ are glued – first those of $\Gamma_\div$ and then those of $\Gamma'_\div$ – can be combined into a single step.

This is a consequence of corollary 4.86 about umkehr data compatible with composition. The reason for is that while glueings first happened in a two steps, we now want it to happen in a single step. This corollary tells us that we can find a contractible space of compatible umkehr data $\mathrm{Umk}^\circ(\Gamma'\circ\Gamma)$ with maps to p_1, p_2 to $\mathrm{Umk}(\Gamma)$ and $\mathrm{Umk}(\Gamma')$ respectively, such that if we use this umkehr data as input for the construction of $\urcorner$, our umkehr maps will commute. The space $\mathrm{Umk}^\circ(\Gamma'\circ\Gamma)$ is a subspace of $\mathrm{Umk}(\Gamma'\circ\Gamma)$.

The diagram of graphs is now equal to the second diagram of graphs and hence we are done.

We saw that if we take $\mathrm{Umk}^{\mathrm{glue}}(\Gamma'\circ\Gamma)$ to be the product of $\mathrm{Umk}^\circ(\Gamma'\circ\Gamma)$ and $\mathcal{C}$, then this umkehr data and connections first of all give an inclusion $\mathrm{Umk}(\chi)$ to $\mathrm{Umk}(\Gamma'\circ\Gamma)$, but secondly maps p_1 and p_2 to $\mathrm{Umk}(\Gamma)$ and $\mathrm{Umk}(\Gamma')$. This concludes the proof. □

This proposition tells us that degree zero string operations are compatible with the glueing of cobordisms. Together with the previous section about compatibility with disjoint union, this is enough prove that string topology gives $H_*(LM;\mathbb{Q})$ the structure of a TQFT with positive boundary up to signs that we discuss in chapter 6.

8.2. Compatibility for $\sqsupset$, $\gimel$ and the induced maps in homology with glueing of cobordisms. Our next goal is extend the previous argument to the map $\sqsupset$. As before, the compatibility of $\gimel$ and the string operations $\gimel_*$ with glueing will be formal consequences of this.

8.2.1. *The category $\overline{\nabla(\mathsf{Fg}(\Sigma' \circ \Sigma))}_/$.* We first need to describe what umkehr data we want to add to $\mathsf{Fg}(\Sigma' \circ \Sigma)$ to get the lift in the middle of the diagram

$$\begin{tikzcd}
\overline{\nabla\mathsf{Fn}}_/ \times \overline{\nabla\mathsf{Fn}}_/ \arrow[d] & \overline{\nabla(\mathsf{Fg}(\Sigma' \circ \Sigma))}_/ \arrow[l, "\bar{I}"'] \arrow[r, "\overline{\nabla\chi}"] \arrow[d] & \overline{\nabla\mathsf{Fn}}_/ \arrow[d] \\
\nabla\mathsf{Fn} \times \nabla\mathsf{Fn} & \nabla(\mathsf{Fg}(\Sigma' \circ \Sigma)) \arrow[l, "I"] \arrow[r, "\nabla\chi"'] & \nabla\mathsf{Fn}
\end{tikzcd} \tag{12}$$

Because this is very similar to the case of compatibility of $\sqsupset$ with disjoint union, will skip the description of the functors in this diagram and focues on the umkehr data that we add in the middle: again we want to define $\nabla(\mathsf{Fg}(\Sigma' \circ \Sigma))_/ = \nabla(\mathsf{Fg}(\Sigma' \circ \Sigma)) \int \mathrm{Split}^{\mathrm{glue}}$ and $\overline{\nabla(\mathsf{Fg}(\Sigma' \circ \Sigma))}_/ = \nabla(\mathsf{Fg}(\Sigma' \circ \Sigma))_/ \int \mathrm{Umk}^{\mathrm{glue}}$ for certain functors $\mathrm{Split}^{\mathrm{glue}}$ and $\mathrm{Umk}^{\mathrm{glue}}$.

We define $\mathrm{Split}^{\mathrm{glue}}$ to be the subspace of $\mathrm{Split} \times \mathrm{Split} \circ I$ of splittings for each individual graph, which are compatible with the isomorphism between the incoming boundary of the second and the outgoing boundary of the first. To make this precise, recall that in definition 5.15 we defined $\mathrm{Split}(\sigma)$ to a product of subspaces of the topologized space of linear maps $\mathrm{Hom}_{\mathsf{Vect}}(\mathbb{R}^{q^{-1}(v)}, \mathbb{R})$ for vertices v in graphs in the simplex σ and q depends on the morphisms in the simplex. We want that if ϕ maps v to v', then the corresponding elements of $\mathrm{Hom}_{\mathsf{Vect}}(\mathbb{R}^{q^{-1}(v)}, \mathbb{R})$ and $\mathrm{Hom}_{\mathsf{Vect}}(\mathbb{R}^{q^{-1}(v')}, \mathbb{R})$ are equal.

For $\mathrm{Umk}^{\mathrm{glue}}$ we can be rather concise. Motivated by our previous discussion of the compatibility of $\daleth$ with glueing, we will do the same construction as in section 4 of this chapter except for two minor things: (1) in the definition of umk we start at level zero not with all umkehr data, but just that subspace of umkehr data compatible with the compositions of the map first attaching $(\Gamma_0)_\div$ and then $(\Gamma_0')_\div$ in $(\Gamma_0 \sqcup \Gamma_0')_\div$ and (2) we only lift at similarly compatible data in higher levels.

The forgetful functor and projection onto the first element of a simplex allow us to extend the incoming and outgoing boundary functors from the bottom row of diagram 12 to the top row. Similarly, after including the splitting data we can define the virtual bundle functors there will be natural transformations between these using the same argument as for a single pair of glueable graphs.

8.2.2. *The compatibility of $\sqsupset$ with glueing.* The arguments about the existence of maps between homotopy colimits then are similar to the arguments in the previous section about compatibility with disjoint. Using those arguments we can write down the diagram expressing compatibility of $\sqsupset$ with glueing of cobordisms.

Proposition 5.74. *The following diagram commutes*

$$\begin{tikzcd}
\underset{\overline{\nabla(\mathsf{Fg}(\Sigma' \circ \Sigma))}_/}{\mathrm{hocolim}} \Sigma^\infty(M^{\partial_{in}} \circ \pi_1 \circ I)_+ \arrow[r, "M^{\partial_{in}\chi} \circ \overline{\nabla\chi}"] \arrow[d, "\sqsupset"'] & \underset{\overline{\nabla\mathsf{Fn}}_/}{\mathrm{hocolim}}\, \Sigma^\infty(M^{\partial_{in}-})_+ \arrow[dd, "\sqsupset"] \\
\underset{\overline{\nabla(\mathsf{Fg}(\Sigma' \circ \Sigma))}_/}{\mathrm{hocolim}} (M^{\partial_{out}-} \circ \pi_1 \circ I)^\kappa \arrow[d, "(\sqsupset)^\kappa"'] & \\
\underset{\overline{\nabla(\mathsf{Fg}(\Sigma' \circ \Sigma))}_/}{\mathrm{hocolim}} (M^{\partial_{out}-} \circ \pi_2 \circ I)^{\kappa \oplus \phi_*\kappa} \arrow[r, "\cong", "(M^{\partial\chi})^{\kappa\chi} \circ \overline{\nabla\chi}"'] & \underset{\overline{\nabla\mathsf{Fn}}_/}{\mathrm{hocolim}}\, (M^{\partial_{out}-})^\kappa
\end{tikzcd}$$

where the two maps $\sqsupset$ on the left hand coincide with the restriction of those constructed for each individual graph.

PROOF. Again it suffices to prove that the operations are compatible over each simplex in the fat geometric realisation. But this is a consequence of proposition 5.73 because of our choice of $\mathrm{Umk}^{\mathrm{glue}}$ of suitably compatible umkehr data to make the operations $\daleth$ over each simplex compatible. $\square$

8.2.3. *The compatibility of $\gimel$ and the string operations with glueing.* Finally we derive the compatibility of $\gimel$ and the string operations $\gimel_*$ with glueing of cobordisms. To do this, we will use the previous proposition. The isomorphism of ind-Thom spectra on the bottom row can be composed with the Thom isomorphism, because the local system is compatible with glueing. The consequence of this is the compatibility of $\gimel$ with glueing, as expressed by the following diagram, where we already used our identifications of the homotopy colimits from section 5 of this chapter.

$$\begin{tikzcd}
\big((B\Gamma_{\Sigma'} \times B\Gamma_{\Sigma})_+ \wedge \Sigma^\infty(M^{\partial_{in}\Sigma})_+\big) \wedge_{\mathcal{L}^M_{\mathcal{B}}} H\mathbb{Z} \arrow[r] \arrow[d, "\gimel"'] & \big((B\Gamma_{\chi(\Sigma,\Sigma')})_+ \wedge \Sigma^\infty(M^{\partial_{in}\Sigma})_+\big) \wedge_{\mathcal{L}^M_{\mathcal{B}}} H\mathbb{Z} \arrow[dd, "\gimel"] \\
\Big((B\Gamma_{\Sigma'})_+ \wedge \Sigma^\infty(M^{\partial_{in}\Sigma'})_+\Big) \wedge_{\mathcal{L}^M_{\mathcal{B}}} H\mathbb{Z} \arrow[d, "\gimel"'] & \\
\Sigma^\infty(M^{\partial_{out}\Sigma'})_+ \wedge H\mathbb{Z} \arrow[r] & \Sigma^\infty(M^{\partial_{out}\Sigma'})_+ \wedge H\mathbb{Z}
\end{tikzcd}$$

Passing to homology in this diagram, we finally obtain the compatibility of the string operations $\gimel_*$ with glueing of cobordisms.

Theorem 5.75. *The string operations $\gimel_*$ are compatible with glueing in the sense that the following diagram commutes:*

$$\begin{tikzcd}
H_*(B\Gamma_{\Sigma'};\mathcal{L}^M_{\mathcal{B}}) \otimes H_*(B\Gamma_{\Sigma};\mathcal{L}^M_{\mathcal{B}}) \otimes H_*(M^{\partial_{in}\Sigma};\mathbb{Q}) \arrow[r, "\gimel_*"] \arrow[d] & H_*(B\Gamma_{\Sigma'};\mathcal{L}^M_{\mathcal{B}}) \otimes H_*(M^{\partial_{in}\Sigma'};\mathbb{Q}) \arrow[d, "\gimel_*"] \\
H_*(B\Gamma_{\Sigma\sqcup\Sigma'};\mathcal{L}^M_{\mathcal{B}}) \otimes H_*(M^{\partial_{in}(\Sigma\sqcup\Sigma')};\mathbb{Q}) \arrow[r, "\gimel_*"'] & H_*(M^{\partial_{out}(\Sigma\sqcup\Sigma')};\mathbb{Q})
\end{tikzcd}$$

where we have shortened the large tensor product over the homology of all incoming and outgoing boundary components, for example

$$H_*(M^{\partial_{in}\Sigma};\mathbb{Q}) := H_*(LM;\mathbb{Q})^{\otimes r} \otimes \bigotimes_{A,B\in\mathcal{B}} H_*(P_M(A,B);\mathbb{Q})^{\otimes r_{A,B}}$$

Note that the left vertical arrow is an isomorphism if there are no more than one fully free boundary components for each label in $\mathcal{B}$.

$$
\begin{tikzcd}
M^{\partial_{in}(\Gamma_0)} \times W^{v((\Gamma_0)\backslash s(\partial_{in}(\Gamma_0)))} \arrow[d] & M^{(\Gamma_0)_{v,in}} \arrow[l] \arrow[r] \arrow[d] & M^{(\Gamma_0)_{\div,in}} \arrow[d] & M^{(\Gamma_0)_{\div}} \arrow[l] \arrow[d] & M^{(\Gamma_0)} \arrow[l] \arrow[r] \arrow[d] & M^{s(\partial_{out}(\Gamma_0))} \arrow[r] \arrow[d] & M^{\partial_{out}(\Gamma_0)} \arrow[d] \\
M^{\partial_{in}(\Gamma_1)} \times W^{v((\Gamma_1)\backslash s(\partial_{in}(\Gamma_1)))} \arrow[d] & M^{(\Gamma_1)_{v,in}} \arrow[l] \arrow[r] \arrow[d] & M^{(\Gamma_1)_{\div,in}} \arrow[d] & M^{(\Gamma_1)_{\div}} \arrow[l] \arrow[d] & M^{(\Gamma_1)} \arrow[l] \arrow[r] \arrow[d] & M^{s(\partial_{out}(\Gamma_1))} \arrow[r] \arrow[d] & M^{\partial_{out}(\Gamma_1)} \arrow[d] \\
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] \\
M^{\partial_{in}(\Gamma_{i_0})} \times W^{v((\Gamma_{i_0})\backslash s(\partial_{in}(\Gamma_{i_0})))} & M^{(\Gamma_{i_0})_{v,in}} \arrow[l] \arrow[r] & M^{(\Gamma_{i_0})_{\div,in}} & M^{(\Gamma_{i_0})_{\div}} \arrow[l] & M^{(\Gamma_{i_0})} \arrow[l] \arrow[r] & M^{s(\partial_{out}(\Gamma_{i_0}))} \arrow[r] & M^{\partial_{out}(\Gamma_{i_0})}
\end{tikzcd}
$$

FIGURE 5.76. The diagram of maps involved in the construction of compatible umkehr maps of lemma 5.34.

CHAPTER 6

String topology HCFT operations

1. Overview of known operations

In this chapter we will discuss the operations that we get out of the previous construction and the algebraic structures they put on the homology of the free loop space LM and the restricted path spaces $P_M(A,B)$.

These operations depend on the homology groups $H_*(B\Gamma_\Sigma, \mathcal{L}_{\mathcal{B}}^M)$, that in turn depend on several parameters: the homological degree k, the genus g and the numbers of incoming boundary circles, outgoing boundary circles and labelled intervals for each possible pair of labels. At the moment we only know the explicitly operations for a certain range of these parameters. There are two reasons for this. The first is that the homology groups the operations depend on are hardly known and therefore we don't know whether there even exist operations for a certain choice of parameters. Secondly, even in cases when the homology is known there are often no explicits cycles of $\mathcal{B}$-labelled cobordism graphs to efficiently calculate the operations explicitly.

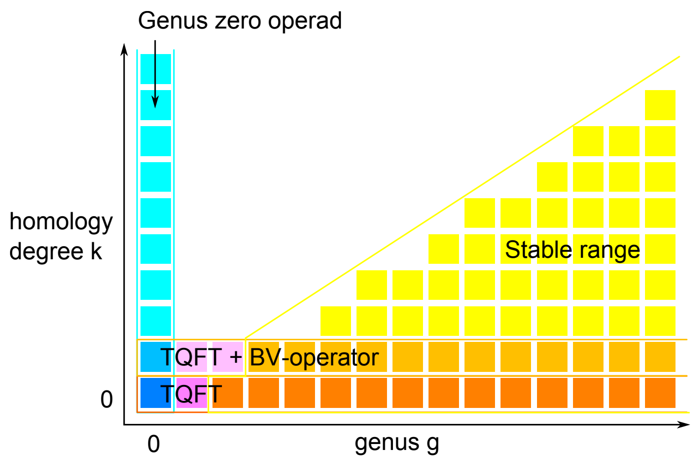


FIGURE 6.1. This figure shows our current knowledge of the higher string operations in a simplified manner. It does not show the number of incoming and outgoing circles and labelled intervals, but only the homological degree and the genus.

In this chapter, we will first discuss some basic properties of mapping class groups, which are necessary for the computations in the later sections. These later sections discuss the four different structures listed in the figure 6.1: the TQFT structure in homological degree 0, the TQFT structure with BV-operator in homological degree 1, the vanishing of the higher string operations in the stable range and the operad structure in genus zero.

2. Basic facts about mapping class groups

We now state some basic facts about the mapping class group. A great source for this is the forthcoming book by Farb and Margalit [**FM10**]. Because all results about mapping class group of cobordisms are built from those about mapping class groups about surfaces with marked points and/or boundaries, we start by introducing special notation for the mapping class group of the surface $\Sigma_{g,n}^r$ which is obtained as a connected sum of g tori with n marked points and r boundary components. This group is denoted $\Gamma_{g,n}^r$.

2.1. Dehn twists. To describe certain – and in sense to be made precise, all – elements of the mapping class groups we have to introduce the notion of a Dehn twist. These are twists done around a curve in the surface. To make this precise, we first recall a certain notion of curve on a surface. Let Σ be a surface, then a closed curve on Σ is a map $\gamma : S^1 \to \Sigma$. This curve is called simple if the map γ is a homeomorphism onto its image. If Σ has a fixed smooth structure, then we call γ a smooth curve if the map γ is smooth.

A regular neighborhood of a simple closed curve is a open neighborhood $U \subset \Sigma$ containing γ together with a homeomorphism $\phi : I \times S^1 \to U$ such that its restriction $\phi|_{\{\frac{1}{2}\}\times S^1} : \{\frac{1}{2}\} \times S^1 \cong S^1 \to \Sigma$ is equal to γ. By the tubular neighborhood theorem 4.4, every smooth simple closed curve admits a regular neighborhood.

Next we define our prototype of a twist. If we write the annulus as a cylinder $S^1 \times I$, then there is a homeomorphism T fixing the boundary given by $(\theta, t) \mapsto (\theta + 2\pi t, t)$. It is the map obtained by holding an end of the cylinder in each hand and twisting the cylinder 360 degrees.

Definition 6.2. Let γ be a simple closed curve on a surface Σ with a regular neighborhood (U, ϕ), then the Dehn twist around γ is the homeomorphism T_γ of Σ given by:

$$T_\gamma(x) = \begin{cases} \phi \circ T \circ \phi^{-1}(x) & \text{if } x \in U \\ \mathrm{id} & \text{if } x \in U^c \end{cases}$$

This definition is independent of the choice of regular neighborhood, because if (U, ϕ) and (V, φ) are both regular neighborhoods of γ, one can construct isotopies moving (U, ϕ) and (V, ϕ) into a regular neighborhood (W, ψ), where $W \subset U \cap V$.

In fact any simple closed curve γ admits a regular neighborhood, not just every smooth simple closed curve. For the proof we begin with a statement about approximation of continuous curves by smooth curves: Every closed curve γ is homotopy equivalent to a smooth simple closed curve $\tilde{\gamma}$. This can be done by first using a small convolution in local coordinates to obtain local replacement curves, glueing these local replacements together and again using a small enough convolution near the nodes where we glue the local replacements.

This result can be improved when we look at simple closed curves using the statement that every two homotopy equivalent simple closed curves are isotopy equivalent [**FM10**, proposition 1.10]. To prove this we just need to take more care when working locally. The result is the following lemma.

Lemma 6.3. *For each simple closed curve γ we can find a smooth simple closed curve $\tilde{\gamma}$ isotopic by an isotopy F to our original curve γ.*

Using extension of isotopies [**FM10**, proposition 1.11], this gives us a way to use the regular neighborhood of the smooth simple closed curve as a regular neighborhood for our simple closed curve. The precise statement about extension of isotopies is the following.

Proposition 6.4. *If $F : S^1 \times I \to \Sigma$ is an isotopy of closed simple curves, then there is an isotopy $\tilde{F} : \Sigma \times I \to \Sigma$ such that $\tilde{F}|_{\Sigma\times\{0\}}$ is the identity and $\tilde{F}|_{F(S^1\times\{0\})\times I} = F$.*

In particular if F denotes the original isotopy between γ and $\tilde{\gamma}$, then we can extend F to get an isotopy $\tilde{F} : \Sigma \times I \to \Sigma$ of the entire surface. Therefore, if (U, ϕ) is a regular neighborhood for the simple closed curve $\gamma = F(S^1 \times \{0\})$ then $(\tilde{F}(U \times \{1\}), \tilde{F}(\phi \times \{1\}))$ is a regular neighorbood for $\tilde{\gamma} = F(S^1 \times \{1\})$. This is summarized by the following lemma.

Lemma 6.5. *Let γ be a simple closed curve, then γ admits a regular neighborhood.*

As a consequence of these approximation and extension theorems we obtain the following result.

Lemma 6.6. *A γ is a simpe closed curve, we can define the Dehn twist T_γ. Furthermore, if two simple closed curves γ, η are homotopic, the Dehn twists T_γ and T_η represent the same element in the mapping class group.*

Proof. The first statement is an obvious consequence of the existence of regular neighborhoods.

For the second, first pick a regular neighborhood (U, ϕ) for γ. Replace the homotopy between γ and η by an isotopy F and extend this isotopy to an isotopy $\tilde{F}$ of the surface. Applying the Dehn twist construction to each curve $F(S^1 \times \{t\})$ with the regular neighborhood $(\tilde{F}(U \times \{t\}), \tilde{F}(\phi \times \{t\}))$ we get an isotopy between the Dehn twists T_γ and T_η. □

We will use the notation T_γ for the the element in the mapping class group represented by T_γ as well. The last lemma implies that this element is well-defined.

It is well known that if we have two curves then these can be changed by a homotopy to intersect transversely. This can be proven by working in local coordinates, patching and a small convolution near the glueing nodes. If we have a given orientation, then we can define the oriented intersection number of two transversely intersecting simple closed curves γ, η as the sum over all intersection points with a sign: $+1$ if the rotation going from tangent vector to γ at an intersection point to the tangent vector to η is going in positive direction and -1 if it is going in negative direction. This number only depends on the homotopy classes of γ and η (it can in fact be determined homologically) and therefore we can extend it to non-transversely intersecting simple closed curves. We denote this intersection number by $i(\gamma, \eta)$.

Since Dehn twists around two different curves interact at each intersection point of the curves, one expect that the commutation relations depend on the intersection number. In fact, the cases of intersection number 0 and 1 are easy to see geometrically.

Proposition 6.7. *Let γ and η be simple closed curves. If $i(\gamma, \eta) = 0$ then $T_\gamma T_\eta = T_\gamma T_\eta$. If $i(\gamma, \eta) = 1$ then $T_\gamma T_\eta T_\gamma = T_\eta T_\gamma T_\eta$.*

These will be important for calculations because we can figure out when two elements coming from Dehn twists commute: if the isotopy classes of simply closed curves defining them have representatives which are disjoint. This will especially be useful in the light of theorem 6.16, which says that Dehn twists generate the mapping class group.

2.2. Birman's short exact sequences. We often want to know how the mapping class groups change when we modify our surfaces. The most valuable tools to do this are two important short exact sequences. These are related to the modifications of the surface given by adding marked points and collapsing a boundary component to a marked point respectively.

2.2.1. *Adding a marked point.* The first short exact sequence is for adding a marked point. To explain it we first fix some notation. Let $\vec{x}$ be a set of points of a surface Σ and y be an point disjunct from $\vec{x}$. There is a forgetful map

$$F : \mathrm{Homeo}^+(\Sigma, \vec{x} \cup y) \to \mathrm{Homeo}^+(\Sigma, \vec{x})$$

which forgets that an orientation-preserving self-homeomorphism fixes y. This induces a surjective upon passing to π_0, since for any element of $\mathrm{Homeo}(\Sigma, \vec{x})$ we can find an isotopy to a self-homeomorphism which fixes y in addition to the points $\vec{x}$.

There is a second "pushing" map

$$P : \pi_1(\Sigma, y) \to \mathrm{Homeo}(\Sigma, \vec{x} \cup y)$$

which is given as follows: for any homotopy class in $\pi_1(\Sigma, y)$ we can find a representative γ with a regular neighborhood disjoint from $\vec{x}$. Let γ_+ and γ_- be the loops in Σ obtained as the image of $\frac{1}{4} \times S^1$ and $\frac{3}{4} \times S^1$ respectively in the identification of the regular neighborhood with $I \times S^1$. Note that the restriction of the regular neighborhood of γ to $[\frac{1}{8}, \frac{3}{8}] \times S^1$ and $[\frac{5}{8}, \frac{7}{8}] \times S^1$ gives disjoint union regular neighborhoods for γ_+ and γ_- which are disjoint from y and $\vec{x}$. Then these regular neighborhoods gives allows us to define $P(\gamma) = T_{\gamma_+} T_{\gamma_-}^{-1}$, which is an orientation-preserving self-homeomorphism that fixes y and $\vec{x}$. Note that the order of the Dehn twists does not matter, since these are Dehn twists along disjoint curves, which commute by proposition 6.7.

The sequence will be a consequence of the following theorem [**FM10**].

Theorem 6.8. *There is a fiber bundle* $\mathrm{Homeo}^+(\Sigma, \vec{x} \cup y) \longrightarrow \mathrm{Homeo}^+(\Sigma, \vec{x}) \xrightarrow{ev_y} \Sigma$.

By applying the long exact sequence of homotopy groups we get an exact sequence which is a priori a long exact sequence, but in many cases reduces to a short exact sequence.

Corollary 6.9 (Birman). *The following sequence is exact*

$$0 \longrightarrow \pi_1(\Sigma^r_{g,n}, y) \xrightarrow{P} \Gamma^r_{g,n+1} \xrightarrow{F} \Gamma^r_{g,n} \to 0$$

except in the cases $(g,n,r) = (1,0,0),(0,0,1),(0,0,2),(0,1,1),(1,2,0),(0,0,0)$ *(the sphere, the sphere with one marked point, the disk, the annulus, the disk with one marked point, the punctured sphere with two marked points, and the torus).*

Sketch of proof. Apply the long exact sequence of homotopy groups to the fiber bundle and use that the fact that if $\pi_1(\Sigma^r_{g,n})$ is non-abelian, then $\pi_1(\mathrm{Homeo}^+(\Sigma^r_{g,n})) = 0$. The abelian fundamental groups exactly give the list of exceptions in the statement of the corollary. Finally, one needs to identity P and F as the maps in the long exact sequence. □

Remark 6.10. For the exceptions mentioned in the theorem, there is still a long exact sequence which can be useful for some calculations.

2.2.2. *Collapsing a boundary component.* The second short exact sequence is for collapsing a boundary component to a marked point and has a similar proof. We first fix some notation. Let $C : \Gamma^{r+1}_{g,n-1} \to \Gamma^r_{g,n}$ be the map which glues a punctured disk to a boundary component and extends the self-homeomorphismh as the identity to that punctured disk. Again one can construct a fiber bundle, although in this case the construction is more intricate. The consequence is the following proposition.

Proposition 6.11 (Birman). *The following sequence is exact*

$$0 \longrightarrow \mathbb{Z} \xrightarrow{i} \Gamma^{r+1}_{g,n} \xrightarrow{C} \Gamma^r_{g,n+1} \to 0$$

except in the cases $(g,n+1,r) = (1,0,0),(0,0,1),(0,0,2),(0,1,1),(1,2,0),(0,0,0)$ *(the sphere, the sphere with one marked point, the disk, the annulus, the disk with one marked point, the punctured sphere with two marked points, and the torus)*

Remark 6.12. Again there is a long exact sequence for the exceptions, which allows for some additional calculations

2.3. Generators of the mapping class groups. In this section we will discuss a basic result on the generators of the mapping class group. This will be necessary to prove our classification of degree 1 operations. We start with our first examples of mapping class groups.

Lemma 6.13. *The mapping class groups for the disk, the disk with one marked point, the sphere and the sphere with one marked point,* $\Gamma^1_{0,0}$, $\Gamma^1_{0,1}$, $\Gamma^0_{0,0}$ *and* $\Gamma^0_{0,1}$ *respectively, are zero. The mapping class of the annulus (cylinder),* $\Gamma^2_{0,0}$ *is isomorphic to* $\mathbb{Z}$.

Proof. These are lemma 2.1 and the following remark and proposition 2.4 of [**FM10**]. □

Remark 6.14. Later, in section 6.1 of this chapter we will calculate the mapping class groups of spheres with disks removed. They will be pure ribbon braid groups PRB_n.

In fact, we can describe the mapping class group of the annulus explicitly using Dehn twists:

Lemma 6.15. *The mapping class of the annulus is generated by a Dehn twist around a closed curve parallel to one of the boundary components.*

The question naturally arises how much of the mapping class group is generated by Dehn twists for an arbitrary surface. A simple closed curve is called non-separating if removing it from the surface doesn't make the surface fall apart into different connected components.

Theorem 6.16 (Dehn-Lickorish). $\Gamma^r_{g,n}$ *is generated by finitely many Dehn twists around non-separating curves and* r *Dehn twists around the boundary components.*

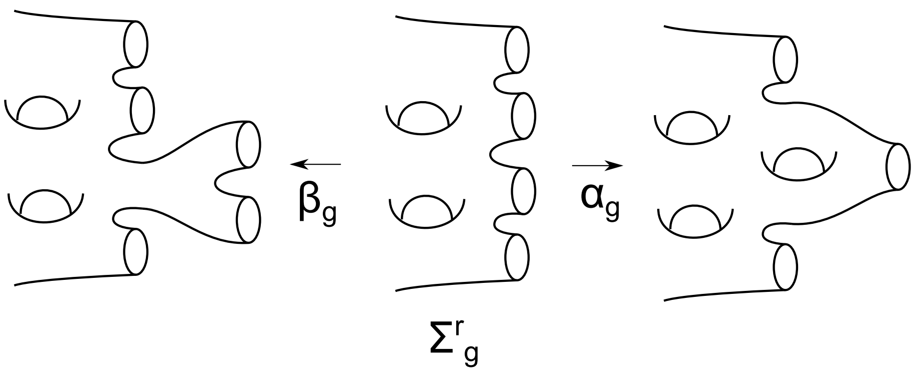


FIGURE 6.19. The maps of surfaces involved in the homological stability of mapping class groups.

PROOF. This is theorem 5.1 of [**FM10**]. One first proves this for surfaces without boundary components. Thus the first goal is to prove that the mapping class group $\Gamma_{g,n}$ is generated by finitely many Dehn twists. The cases $g = 0, 1$ are treated separately and the case $g \geq 2$ is done by first applying induction on the genus and then applying induction on the number of marked points.

The idea of the induction on g is that the connectedness of a complex of curves allows one to prove that any element of the pure mapping class groups can be multiplied with a product of Dehn twists to stabilize a non-separating closed curve. Then one cuts along this curve and the genus is reduced by one, which then allows the induction hypothesis to kick in.

After that, the induction over n is done using the first Birman exact sequence, the induction of r by the second Birman exact sequence. □

Remark 6.17. Recall that our definition of the mapping class group is sometimes known as the pure mapping class groups, because we do not allow permutations of the punctures of the boundary components. To give the analogue of Dehn-Lickorish theorem for the mapping class group which is allowed to permute these, one must add some permutations to the list of generators.

The previous remark also points us to the correct generalisation of the previous theorem for the generators of the mapping class group of a $\mathcal{B}$-labelled cobordism.

Corollary 6.18. *Let Σ be a $\mathcal{B}$-labelled cobordism then the mapping class group Γ_σ is generated by finitely many Dehn twists around non-separating curves, Dehn twists around the not purely free boundary components and the permutations of the purely free boundary components with the same label.*

2.4. Stability of mapping class groups. Finally, we discuss results on the stability of mapping class groups which we will use in our proof that the string topology operations corresponding to classes of relatively low degree with respect to the genus of the cobordism will vanish. A good reference is the survey by Wahl [**Wah11**].

We start by considering the mapping class groups Γ_g^r of a surface Σ_g^r of genus g with r boundary components. The statement of Harer stability uses the following two maps given by glueing pair of pants to boundary components. They are depicted in figure 6.19

Definition 6.20. Let Σ_0^3 be the pair of pants, then there is a map $\alpha_g : \Gamma_g^r \to \Gamma_{g+1}^{r-1}$ given by glueing a pair of pants along two of the boundary components of Σ_g^r and putting the identity self-homeomorphism on the pair of pants:

$$\alpha_g = \Gamma(\chi)(-, id_{\Sigma_0^3}) : \Gamma_g^r \to \Gamma_{g+1}^{r-1}$$

Similarly, there is a map $\beta_g : \Gamma_g^r \to \Gamma_g^{r+1}$ given by glueing a pair of pants along a single boundary component of Σ_g^r:

$$\beta_g = \Gamma(\chi)(-, id_{\Sigma_0^3}) : \Gamma_g^r \to \Gamma_g^{r+1}$$

Homological stability is the statement that these maps induce isomorphisms in homology in an increasing range of degrees as g goes to infinity. The current best bounds on these stable ranges are as follows:

Theorem 6.21. *The map $(\alpha_g)_* : H_*(B\Gamma^r_g;\mathbb{Z}) \to H_*(B\Gamma^{r-1}_{g+1};\mathbb{Z})$ is surjective for $* \leq \frac{2}{3}g + \frac{1}{3}$ and an isomorphism for $* \leq \frac{2}{3}g - \frac{2}{3}$.*

The map $(\beta_g)_* : H_*(B\Gamma^r_g;\mathbb{Z}) \to H_*(B\Gamma^{r+1}_g;\mathbb{Z})$ is injective and an isomorphism for $* \leq \frac{2}{3}g$.

Remark 6.22. These isomorphisms are compatible with the action induced by permutation of the labels of the boundary components.

We will use this to prove homological stability for mapping class groups of $\mathcal{B}$-labelled cobordisms. Let Σ be a $\mathcal{B}$-labelled cobordisms with two or more incoming and outgoing boundary circles. Then there are maps α_g and β_g, which will only glue along incoming or outgoing boundary circles, but not along boundary components which contain labels from $\mathcal{B}$.

Lemma 6.23. *The induced maps $(\alpha_g)_*$ and $(\beta_g)_*$ for mapping class groups of $\mathcal{B}$-labelled cobordisms are injective, surjective or isomorphisms in the same range as in theorem 6.21.*

Proof. Let Σ be a $\mathcal{B}$-labelled cobordism. Then we have two short exact sequences relating the mapping class group of Σ as a $\mathcal{B}$-labelled cobordism to the mapping class groups that appear in the previous theorem.

For convenience, we denoted the mapping class group of Σ as a $\mathcal{B}$-labelled cobordism by $\Gamma^{\mathcal{B}}_\Sigma$, its mapping class group when we no longer allow permutations of free boundary components by $\Gamma^{\mathcal{B},l}_\Sigma$ and its mapping class group when we require that each boundary component is pointwise fixed by Γ_Σ. Suppose that Σ has a genus g and r boundary components, then $\Gamma_\Sigma = \Gamma^r_g$. Let $d < r$ denote the number of purely free boundary components.

The two short exact sequences are:

$$0 \to \mathbb{Z}^d \to \Gamma^r_g \to \Gamma^{\mathcal{B},l}_\Sigma \to 0$$

$$0 \to \Gamma^{\mathcal{B},l}_\Sigma \to \Gamma^{\mathcal{B}}_\Sigma \to \prod_{\beta \in \mathcal{B}} \Sigma(b^{-1}(\beta)) \to 0$$

It is important to note that the first one is split because the Dehn twists around the d purely free boundary component can be taken disjoint from the other generators. Thus we obtain that $B\Gamma^r_g = (S^1)^d \times B\Gamma^{\mathcal{B},l}_\Sigma$. It now suffices to see that the maps induced by α_g by β_g on the left hand side are given on the right hand side by maps which are the identity on the circles. Hence homological stability for the left hand side implies homological stability for the right hand side, with the same ranges.

For the second short exact sequence we get a spectral sequences for the homology of the classifying spaces given by the following fibration:

$$B\Gamma^{\mathcal{B},l}_\Sigma \to B\Gamma^{\mathcal{B}}_\Sigma \to \prod_{\beta \in \mathcal{B}} B\Sigma(b^{-1}(\beta))$$

To be precise, the associated spectral sequence is $E^2_{pq} = H_p(B\prod_{\beta\in\mathcal{B}} \Sigma(b^{-1}(\beta)); \mathcal{H}_q(B\Gamma^{\mathcal{B},l}_\Sigma;\mathbb{Z})) \Rightarrow H_{p+q}(B\Gamma^{\mathcal{B}}_\Sigma)$ and by naturality α_g and β_g induce maps of spectral sequences. The homology of the base space doesn't change and that of the fiber is stable, in fact as a module over the fundamental group of the base. Because the spectral sequence collapses at the E^2-page, we get stability for the homology of the total space with the same ranges. $\square$

3. The basic degree zero genus zero operations and TQFT structures

In the section we will look at the simplest string topology operations, which unsuprisingly were also the first ones to be found by Chas and Sullivan [**CS99**]. These are the operations that come from the degree zero homology the relevant mapping class groups.

We start by writing down the basic operations of this type in terms of elementary operations in algebraic topology. After that we show that indeed these operations determine all degree zero

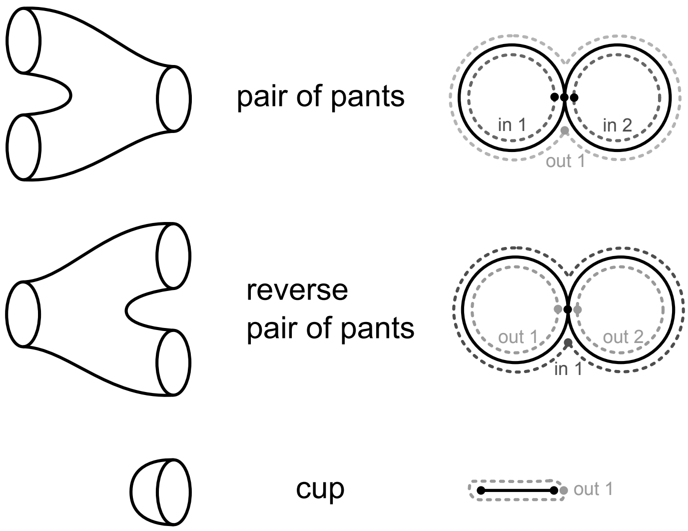


FIGURE 6.24. The three isomorphism classes of cobordisms which will give the basic closed operations. They are listed with their name and a simple $\mathcal{B}$-labelled cobordism graph whose corresponding cobordism is a representative of that isomorphism class.

operations, by looking at the classification of TQFT's. During this we find several relations between the basic operations and we will look at some of the consequences of these.

3.1. The product, coproduct and unit maps. We will start by extracting elementary constructions of some basic string topology operations from the complicated and pretty abstract operations we have defined in chapter 4.

The procedure for this calculation is as follows: to calculate a degree zero operations for a generator $a_0 \in H_0(B\Gamma_\Sigma;\mathbb{Q})$ one can take the string operation for any single graph Γ corresponding to a vertex in the connected component. One of the improvements of our treatment over Godin's, is that by allowing nice instead of admissable fat graphs, we can use the most simple graphs to calculate explicitly the degree zero operations.

Because we work with the operation over a vertex of the geometric realistion, it is clear we are never bothered much by the local system: after trivializing it, it will just contribute a suitable dimension shift. Because there is no canonical choice of trivialization, this will cause some signs to appear when we try to comparing different ways to decompose the operation coming from a $\mathcal{B}$-labelled cobordism into basic operations.

3.1.1. *The closed operations.* We will start with the original string operations, which we call the closed operations. We discuss the three $\mathcal{B}$-labelled open-closed cobordisms listed in figure 6.24 together with the nice $\mathcal{B}$-labelled graphs that we will use to compute them. The resulting operations are described below.

Closed product: We start by describing the operation coming from the generator of 0'th homology group $H_0(B\Gamma_{\text{pair of pants}};\mathbb{Q})$, which can be represented by the class of the vertex corresponding to the first $\mathcal{B}$-labelled cobordism graph in figure 6.24. We denote this graph with Γ. For the convenience of the reader, we will spell out the complete construction of the operation in this case. We advise to take a look at section 2 of chapter 5 again for the general construction. We need to consider the diagram of graphs:

$$\partial_{in}\Gamma \xhookrightarrow{i_{in}} \Gamma_{v,in} \xleftarrow{s_v} \Gamma_{\div,in} \xrightarrow{s_{in}} \Gamma_\div \xrightarrow{s_\div} \Gamma \xhookleftarrow{i_{out}} s(\partial_{out}\Gamma) \xleftarrow{s} \partial_{out}\Gamma$$

In this case i_{in}, s_v, $s_\div$ and i_{out} are isomorphisms, because no edges need to be added to the incoming boundary. Thus the diagram reduces to

$$\partial_{in}\Gamma \longrightarrow \Gamma \longleftarrow \partial_{out}\Gamma$$

This diagram is depicted in figure 6.25. It is easily seen that upon geometric realisation this diagram is the same as the diagram of spaces

$$S^1 \sqcup S^1 \xrightarrow{i} S^1 \vee S^1 \xleftarrow{j} S^1$$

where i is the map which glues the base points of the circles together and j is the pinch map. The space $S^1 \vee S^1$ is called the figure eight and is sometimes denoted by 8 in the literature, e.g. [**CV06**]. This induces the following diagram of mapping spaces

$$LM \times LM \xleftarrow{M^i} M^{S^1 \vee S^1} \xrightarrow{M^j} LM$$

to which we apply homology with rational coefficients. If we apply the umkehr construction to the first map and use the ordinary induced map in homology for the second map, we get the string topology operation corresponding to Γ. To see that as expected an umkehr construction for the first map is possible, we remark that it is part of a pullback diagram whose bottom row contains a embedding of finite dimensional manifolds:

$$\begin{tikzcd} M^{S^1 \vee S^1} \arrow[r, "M^i"] \arrow[d, "ev"'] & LM \times LM \arrow[d, "ev \times ev"] \\ M \arrow[r, "\Delta"'] & M \times M \end{tikzcd}$$

Thus, we conclude that there is a string operation μ_c, which we will call the *closed product* for reasons to be made clear in the next section, given by

$$\mu_c : H_*(LM; \mathbb{Q}) \otimes H_*(LM; \mathbb{Q}) \to H_{*-d}(LM; \mathbb{Q})$$
$$x \otimes y \mapsto (M^j)_*(M^i)^!(x \otimes y)$$

Closed coproduct: Next is the construction of an operation that is very similar to the closed product and whose construction we therefore do not discuss in the same level of detail. This operation is the operation corresponding to the generator of the zero'th homology group $H_0(B\Gamma_{\text{reverse pair of pants}}; \mathbb{Q})$. Again we decide to use the nice $\mathcal{B}$-labelled cobordism graph listed in figure 6.24: it is the second one listed and we denote it by Γ. In this case the diagram again reduces to

$$\partial_{in}\Gamma \longrightarrow \Gamma \longleftarrow \partial_{out}\Gamma$$

which is clearly seen to be the same upon geometric realisation as

$$S^1 \xrightarrow{j} S^1 \vee S^1 \xleftarrow{i} S^1 \vee S^1$$

where as before j is the pinch map and i the map which collapses the two basepoints. The resulting diagram for mapping spaces is

$$LM \xleftarrow{M^j} M^{S^1 \vee S^1} \xrightarrow{M^i} LM \times LM$$

If we apply homology to the entire diagram, then the second arrow is fine and gives an ordinary induced map in homology pointing in the right directoin, but we need to apply the umkehr map construction for the second one. The result will be a string operation Δ_c, which we will call the *closed coproduct*:

$$\Delta_c : H_*(LM; \mathbb{Q}) \to H_{*-d}(LM \times LM; \mathbb{Q})$$
$$x \mapsto (M^i)_*(M^j)^!(x)$$

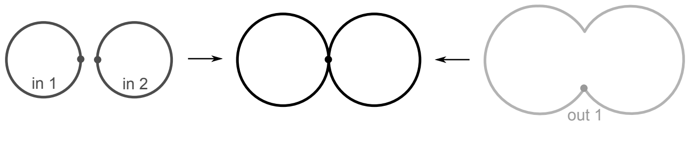


FIGURE 6.25. The diagram of graphs involved in the construction of the closed product.

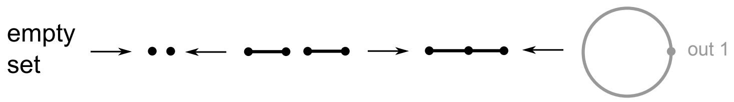


FIGURE 6.26. The graphs involved in the construction of the string operation corresponding to the cup.

Closed unit: Finally we construct a very simple operation which will serve the function of a unit for the closed product. It is the operation corresponding to the generator of $H_0(B\Gamma_{\text{cup}};\mathbb{Q})$. As before we pick a nice graph, which in this case is the third one listed in figure 6.24. We need to simplify the diagram of graphs:

$$\partial_{in}\Gamma \overset{i_{in}}{\hookrightarrow} \Gamma_{v,in} \overset{s_v}{\longleftarrow} \Gamma_{\div,in} \overset{s_{in}}{\longrightarrow} \Gamma_{\div} \overset{s_{\div}}{\longrightarrow} \Gamma \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma) \overset{s}{\longleftarrow} \partial_{out}\Gamma$$

We note that $\partial_{in}\Gamma = \emptyset$, which also implies that s_{in} is the identity map. Furthermore, s_v is a homotopy equivalence mapping an interval to a point, $s_{\div}$ glues the two intervals and i_{out} maps a circle to an interval. Thus in this case the diagram reduces to the following one, depicted in figure 6.26:

$$\emptyset \longrightarrow \Gamma_{v,in} \longleftarrow \Gamma_{\div} \longrightarrow \Gamma \longleftarrow \partial_{out}\Gamma$$

Let $*$ denote a point and I and interval. With the previous remarks it is easy to see that upon geometric realisation this diagram is the same as

$$\emptyset \longrightarrow * \sqcup * \longleftarrow I \sqcup I \longrightarrow I \longleftarrow S^1$$

Up to homotopy we can replace I with $*$ and reduce the diagram to a $\emptyset \longrightarrow * \sqcup * \longrightarrow * \longleftarrow S^1$, where we denote the last map by p. This means that the resulting diagram in mapping spaces for the construction of the string operation is

$$W \times W \overset{\iota\times\iota}{\hookleftarrow} M \times M \overset{\Delta}{\hookleftarrow} M \overset{M^p}{\longrightarrow} LM$$

where $\iota : M \to W$ is a embedding in an Euclidean space. Remark that another way to see the map M^p is as the map which sends a point $x \in M$ to the loop c_x with constant value x. We want to apply homology to the previous diagram and then should apply the umkehr map construction to $\iota \times \iota$ and Δ. After that we should apply the map in homology induced by M^p. Our first guess is therefore that η_c is given by $(M^p)_*\Delta^!(\iota\times\iota)^!$.

It is well-known that $(\iota \times \iota) : H_*(W;\mathbb{Q}) \to H_{*+2d}(M;\mathbb{Q})$ sends the only non-trivial class – the generator of $H_0(W,\mathbb{Q})$ – to the fundamental class $[M] \otimes [M] \in H_d(M;\mathbb{Q})$. The umkehr map $\Delta^!$ is the intersection product, which is dual to the cup-product under Poincaré duality. Hence $[M]$ is the unit for the intersection product and $\Delta^!$ maps $[M]\otimes[M]$ to $[M]$. Thus we obtain a string operation η_c that we will call the *closed unit* as follows:

$$\begin{aligned} \eta_c : \mathbb{Q} &\to H_d(LM;\mathbb{Q}) \\ 1 &\mapsto (M^p)_*([M]) \end{aligned}$$

where of course $\mathbb{Q}$ is considered as lying in degree 0.

Remark 6.27. Note that we actually had to make choices in these constructions to get the umkehr maps, like tubular neighborhoods and propagating flow. However, as we have made clear previously, these choices come in contractible families on which the construction depends continuously, so they all induce the same map in homology.

In the construction of unit we saw a general phenomena: creating two edges with the same labels everywhere and glueing them is the same as creating a single edge. This is seen by a simple generalization of the proof used there. It can be made precise using our discussion of compatible umkehr maps in chapter 4. Therefore, we can conclude we could have used the diagram in 6.28 instead to arrive at our construction of the closed unit more quickly.

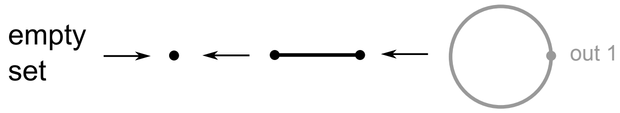


FIGURE 6.28. A simpler diagram of graphs which also gives the string operation corresponding to the cup.

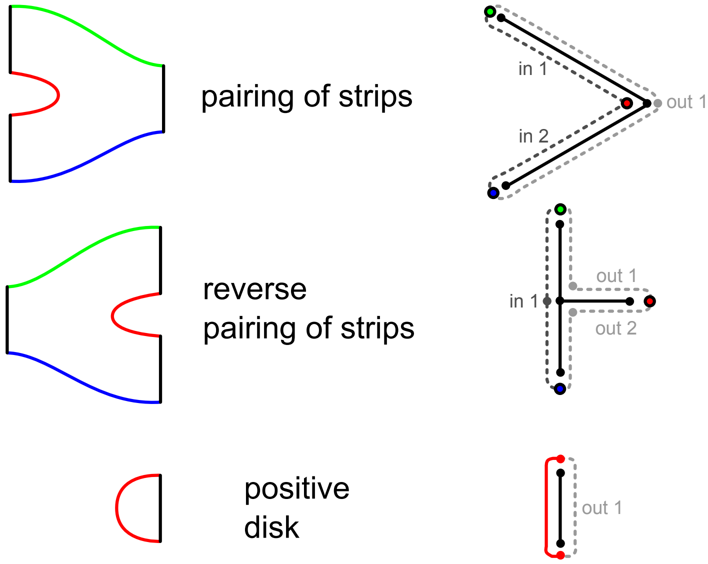


FIGURE 6.29. This figure shows the three isomorphism classes of basic $\mathcal{B}$-labelled cobordisms which will give the open operations. The colours denote the labels in $\mathcal{B}$. They are listed with their name and a nice $\mathcal{B}$-labelled cobordism graph whose corresponding cobordism is a representative of the isomorphism class.

3.1.2. *The open operations.* Next we describe with the string operations coming from the open part, i.e. the part where all incoming and outgoing boundary components are intervals labelled by pairs of label $A, B \in \mathcal{B}$. We will determine explicit constructions of several basic operations using the techniques of the last section. The basic operations we consider are listed in figure 6.29. Because they had to be nice graphs, some of these are not as simple as one might naively guess.

Open product: We will start by describing the operation coming from the generator of the homology group $H_0(B\Gamma_{\text{pairing}}; \mathbb{Q})$, which is represented by the first graph in figure 6.29. We denote this graph by Γ. The next step is to simplify the diagram of graphs:

$$\partial_{in}\Gamma \overset{i_{in}}{\hookrightarrow} \Gamma_{v,in} \overset{s_v}{\longleftarrow} \Gamma_{\div,in} \overset{s_{in}}{\longrightarrow} \Gamma_{\div} \overset{s_{\div}}{\longrightarrow} \Gamma \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma) \overset{s}{\longleftarrow} \partial_{out}\Gamma$$

There are no extra edges or vertices added, so i_{in}, s_v and $s_{\div}$ are isomorphisms. Note that i_{out} is also an isomorphism, but although the map s is an isomorphism of the underlying graphs, it is not an isomorphism of $\mathcal{B}$-labelled graphs, because the $\partial_{out}\Gamma$ is lacking the red label. Thus the diagram reduces to the following, depicted in figure 6.30:

$$\partial_{in}\Gamma \xrightarrow{i} \Gamma \xleftarrow{j} \partial_{out}\Gamma$$

We associate to the green label a submanifold A, to red a submanifold B and to blue a submanifold C. If we go to the associated diagram of mapping spaces we therefore obtain the diagram:

$$P_M(A,B) \times P_M(B,C) \xleftarrow{M^i} P_M(A,B) \times_B P_M(B,C) \xleftarrow{M^j} P_M(A,C)$$

where as before $P_M(A,B)$ denotes the space of paths γ in M such that $\gamma(0) \in A$ and $\gamma(1) \in B$ and $P_M(A,B) \times_B P_M(B,C)$ is the space of pair of paths whose endpoints in

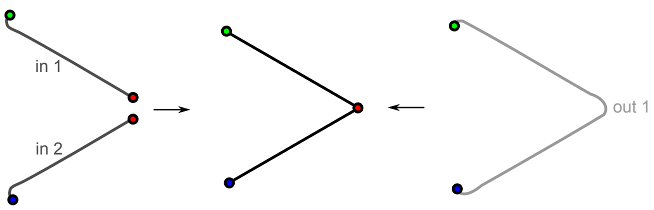


FIGURE 6.30. The diagram of graphs used in the construction of the open product operation.

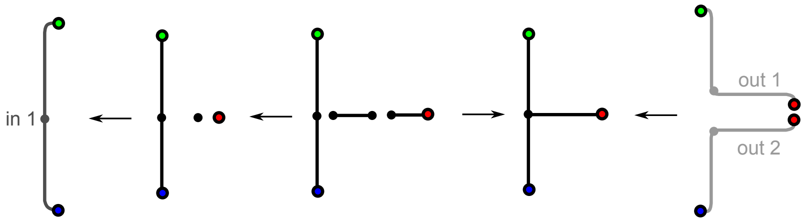


FIGURE 6.31. The diagram of graphs involved in the construction of the open coproduct operation.

B coincide. The maps M^i and M^j are simply the inclusions. The former obtained by projecting to both components and the latter obtained by glueing the two paths together at their endpoints, which coincide at a point in B.

If we apply homology to this diagram, we should construct an umkehr map for M^i. This is done as in general and is possible because we can indeed write it as part of a pullback diagram over a embedding of finite-dimensional manifolds:

$$\begin{array}{ccc} P_M(A,B) \times_B P_M(B,C) & \xrightarrow{M^i} & P_M(A,B) \times P_M(B,C) \\ \Big\downarrow{\scriptstyle \mathrm{ev}_{\frac{1}{2}}} & & \Big\downarrow{\scriptstyle \mathrm{ev}_1 \times \mathrm{ev}_0} \\ B & \xrightarrow[\Delta]{} & B \times B \end{array}$$

where $\mathrm{ev}_{\frac{1}{2}}$ is the map which evaluates at 1 of the first or equivalently 0 of the second path, ev_0 and ev_1 evaluate a path at 0, resp. 1, and Δ is the diagonal map.

We conclude that if the dimension of B is b, then we get a string operation μ_{abc}, which we called the *open product*, given by

$$\begin{aligned} \mu_{abc} : H_*(P_M(A,B);\mathbb{Q}) \otimes H_*(P_M(B,C);\mathbb{Q}) &\to H_{*-b}(P_M(A,C);\mathbb{Q}) \\ x \otimes y &\mapsto (M^j)_*(M^i)^!(x \otimes y) \end{aligned}$$

Open coproduct: Next up is the open coproduct, which will be quite a bit more complicated than our earlier examples because we are now adding an additional edges. This makes it more difficult to simplify the diagrams of graphs. We will be considering the operation corrsponding to the generator of $H_0(B\Gamma_{\text{reverse pairing}};\mathbb{Q})$, as represented by the second graph of figure 6.29.

In this case there is not much that we can simplify in the diagram

$$\partial_{in}\Gamma \overset{i_{in}}{\hookrightarrow} \Gamma_{v,in} \xleftarrow{s_v} \Gamma_{\div,in} \xrightarrow{s_{in}} \Gamma_{\div} \xrightarrow{s_\div} \Gamma \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma) \xleftarrow{s} \partial_{out}\Gamma$$

We only have that s_{in} and i_{out} are isomorphisms, but are therefore still left with the following diagram, as depicted in figure 6.31:

$$\partial_{in}\Gamma \overset{i_{in}}{\hookrightarrow} \Gamma_{v,in} \xleftarrow{s_v} \Gamma_{\div,in} \xrightarrow{s_\div} \Gamma \xleftarrow{i_{out}} \partial_{out}\Gamma$$

By recalling that the property $\Delta^! \circ (\iota \times \iota)^! = \iota^!$ for Δ the diagonal and $\iota : M \to W$ an embedding, we can simplify the diagram slightly and remove the totally unlabelled added edge. Similarly, we can up to homotopy replace the added edge with single red vertex by a red vertex. Then s_v is an isomorphism and we get the following diagram after going to mapping spaces:

$$P_M(A,C) \times W \xleftarrow{M^{i_{in}}} P_M(A,C) \times B \xleftarrow{M^{s_\div}} P_M(A,B) \times_B P_M(A,C) \xrightarrow{M^{i_{out}}} P_M(A,B) \times P_M(A,C)$$

Once we go to homology, we want to construct umkehr maps for the first two maps of this diagram. The result is a candidate for Δ_{abc} given by $(M^{i_{out}})_* \circ (M^{s_\div})^! \circ (M^{i_{in}})^!$. To simplify this, we will take a closer look at the first two maps. The map $M^{i_{in}}$ is nothing but $id \times \iota_B$ and we know that for a suitable choice of umkehr data $(id \times \iota_B)^! = id \otimes (\iota_B)^!$. Hence $x \in H_*(P_M(A,C);\mathbb{Q})$ is mapped to $x \otimes [B]$ by $(M^{i_{in}})^!$. The second map fits in a pullback diagram

$$\begin{tikzcd} P_M(A,B) \times_B P_M(A,C) \arrow[r, "M^{s_\div}"] \arrow[d, "\mathrm{ev}_{\frac{1}{2}}"'] & P_M(A,C) \times B \arrow[d, "\mathrm{ev}_{\frac{1}{2}} \times id"] \\ B \arrow[r, "j \times id"'] & M \times B \end{tikzcd}$$

where $j : B \to M$ is the inclusion. But exactly the same pullback is obtained by dropping B from the right column. This means that if $M^s : P_M(A,B) \times_B P_M(A,C) \to P_M(A,C)$ is the inclusion, then we can write Δ_{abc} as our construction applied to the diagram

$$P_M(A,C) \xleftarrow{M^s} P_M(A,B) \times_B P_M(A,C) \xrightarrow{M^{i_{out}}} P_M(A,B) \times P_M(A,C)$$

The heuristic behind this is that we are essentially introducing $[B]$ and dropping it again and this is the same as doing nothing. This makes sense because $[B]$ acts as a unit for the intersection product as coming from B and we are lifting this intersection product to the mapping spaces. Note that it is also consistent with the dimension shifts; for let b be the dimension of B, then the umkehr map associated with the B in the right column shifts the degree by $-d$ to which we should add the degree b of $[B]$ and the umkehr map for M^s has dimension shift $-(d-b)$.

We conclude that we get a string operation Δ_{abc}, which we called the *open coproduct*, given by

$$\begin{aligned} \Delta_{abc} : H_*(P_M(A,C);\mathbb{Q}) &\to H_{*-(d-b)}(P_M(A,B) \times P_M(B,C);\mathbb{Q}) \\ x &\mapsto (M^{i_{out}})_*(M^s)^!(x) \end{aligned}$$

Open unit: Finally, we advance to the open unit. This is operation coming from the generator of $H_0(B\Gamma_{\text{positive disk}};\mathbb{Q})$ and represented by the third nice $\mathcal{B}$-labelled cobordism graph of figure 6.29.

The construction will be very similar to the construction of the closed unit and therefore we won't provide a picture of the graphs involved. Like in that case, the diagram simplifies to

$$\emptyset \longrightarrow \Gamma_{v,in} \longleftarrow \Gamma_\div \longrightarrow \Gamma \longleftarrow \partial_{out}\Gamma$$

Now we note that using the same trick as before, we can replace the two components of $\Gamma_{v,in}$ and $\Gamma_\div$ by a single one and replace the interval of $\Gamma_\div$ by a homotopy equivalent point. The corresponding diagram of mapping spaces for the construction will then be:

$$W \xleftarrow{\iota_B} B \xrightarrow{M^{i_{out}}} P_M(B,B)$$

where the map $\iota_B : B \to W$ is an embedding into an Euclidean space and $M^{i_{out}}$ is simply the map $B \to P_B(M,M)$ which sends a point $x \in B$ to the path c_x with constant value x. If we apply homology, then the umkehr map $(\iota_B)^!$ sends the only non-trivial class to $[B]$. If b denotes the dimension of b, we therefore obtain a string operation η_b that we

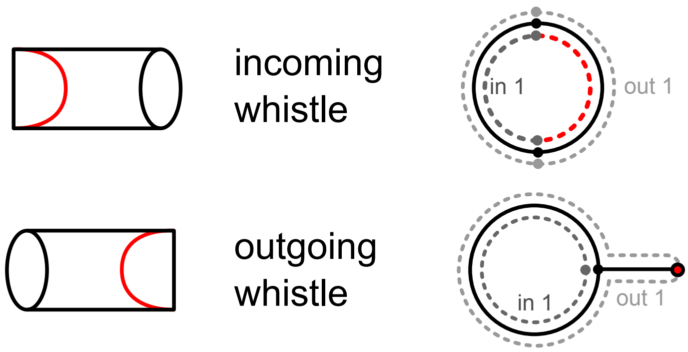


FIGURE 6.33. The incoming and outgoing whistle cobordisms, together with nice $\mathcal{B}$-labelled cobordism graphs whose corresponding cobordism give isomorphic $\mathcal{B}$-labelled cobordisms.

will call the *open unit*:

$$\eta_b : \mathbb{Q} \to H_b(P_M(B,B);\mathbb{Q})$$
$$x \mapsto (M^{i_{out}})_*([B])$$

Remark 6.32. Again, the contractibility of the spaces of choices involved in the construction guarantee that these operations are independent of any choices made for the umkehr maps.

3.1.3. *The whistles.* Finally, we get to two basic operations that are less obvious. These come from $\mathcal{B}$-labelled cobordisms called whistles and allow the open and closed parts to interact. These cobordisms and nice $\mathcal{B}$-labelled cobordism graphs representing them are listed in figure 6.33.

Incoming whistle: We will begin with the incoming whistle. This is the first cobordism listed in figure 6.33 and the $\mathcal{B}$-labelled cobordism graph Γ next to it represents the generator of $H_0(B\Gamma_{\text{incoming whistle}};\mathbb{Q})$. Again we consider the diagram of graphs given by

$$\partial_{in}\Gamma \xhookrightarrow{i_{in}} \Gamma_{v,in} \xleftarrow{s_v} \Gamma_{\div,in} \xrightarrow{s_{in}} \Gamma_{\div} \xrightarrow{s_{\div}} \Gamma \xhookleftarrow{i_{out}} s(\partial_{out}\Gamma) \xleftarrow{s} \partial_{out}\Gamma$$

This can simplified by noting the following. Firstly, the maps s_{in} and s are isomorphisms. Secondly, because in the construction we are adding two new intervals with the same label, we can assume it is a single interval and up to a homotopy a point making s_v into an isomorphism. This means that we have the following diagram after going to mapping spaces:

$$P_M(B,B) \times W \xleftarrow{M^{i_{in}}} P_M(B,B) \times B \xleftarrow{M^{s_{\div}}} P_M(B,B) \times_B B \xrightarrow{M^{i_{out}}} LM$$

where $P_M(B,B) \times_B B$ is the space of paths in M both of whose endpoints in B coincide; in other words they are loops whose basepoint is restricted to lie in B. As in the construction of the open coproduct we can simply as follows. The introduction of a component B by an umkehr map which occurs only to remove it again later using a second umkehr map can be removed to get the following simpler diagram:

$$P_M(B,B) \xleftarrow{M^{s}} P_M(B,B) \times_B B \xrightarrow{M^{i_{out}}} LM$$

where the map M^s fits into the following pullback diagram covering an embedding of finite-dimensional manifolds:

$$\begin{tikzcd}
P_M(B,B) \times_B B \arrow[r, "M^s"] \arrow[d, "\mathrm{ev}"'] & P_M(B,B) \arrow[d, "\mathrm{ev}_0 \times \mathrm{ev}_1"] \\
B \arrow[r, "\Delta"'] & B \times B
\end{tikzcd}$$

where ev evaluates a path of $P_M(B,B) \times_B B$ at either of its endpoints and ev_0 and ev_1 evaluate a path in $P_M(B,B)$ at 0, resp. 1. Therefore, if b denotes the dimension of B we obtain a string operation ι_b that we will call the *incoming whistle* operation:

$$\begin{aligned} \iota_b : H_*(P_M(B,B);\mathbb{Q}) &\to H_{*-b}(LM;\mathbb{Q}) \\ x &\mapsto (M^{i_{out}})_*(M^s)^!(x) \end{aligned}$$

Outgoing whistle: Finally, we conclude with the outgoing whistle operation. This is the second cobordism listed in figure 6.33 and the last basic operation we need to discuss. As before, the nice $\mathcal{B}$-labelled cobordism graph Γ next it to represents the generator of $H_0(B\Gamma_{\text{outg. whistle}};\mathbb{Q})$. In the diagram of graphs given by

$$\partial_{in}\Gamma \overset{i_{in}}{\hookrightarrow} \Gamma_{v,in} \overset{s_v}{\longleftarrow} \Gamma_{\div,in} \overset{s_{in}}{\longrightarrow} \Gamma_{\div} \overset{s_{\div}}{\longrightarrow} \Gamma \overset{i_{out}}{\hookleftarrow} s(\partial_{out}\Gamma) \overset{s}{\longleftarrow} \partial_{out}\Gamma$$

we have that s_{in} is an isomorphism. Furthermore, as in the construction of the open coproduct and the incoming whistle, we can replace the creation of the red vertex and a black vertex by the creation of a single red vertex. The diagram of mappig spaces then becomes

$$LM \overset{M^s}{\longleftarrow} P_M(B,B) \times_B B \overset{M^{i_{out}}}{\longrightarrow} P_M(B,B)$$

where M^s fits into the following pullback diagram over an embedding of finite-dimensional manifolds:

$$\begin{tikzcd} P_M(B,B) \times_B B \arrow[r, "M^s"] \arrow[d, "ev"'] & LM \arrow[d, "ev"] \\ B \arrow[r] & M \end{tikzcd}$$

We conclude that if the dimension of B is b, then we get a string operation e_b, which we called the outgoing whistle operation, given by

$$\begin{aligned} e_b : H_*(LM;\mathbb{Q}) &\to H_{*-(d-b)}(P_M(B,B);\mathbb{Q}) \\ x &\mapsto (M^{i_{out}})_*(M^s)^!(x) \end{aligned}$$

Remark 6.34. Again, the contractibility of the spaces of choices involved in the construction guarantee that these operations are independent of any choices made for the umkehr maps.

3.2. Classification of TQFT's. Our next goal is to describe the properties of the operations defined in the previous section. This will turn out to involve the study of topological quantum fields theories, usually shortened to TQFT's.

In chapter 2 we defined cobordisms and more generally open-closed and $\mathcal{B}$-labelled cobordisms. One can always see cobordisms as parametrizing certain algebraic structures because they form (partial) prop's. In this section we make tose algebraic structures precise by looking at the link between cobordisms, topological quantum field theories and Frobenius algebras. Good references about this topic include [**Bar05**] and [**Koc04**]. Our motivation for discussing these results is that we will use them to describe the structure of the degree zero operations of string topology.

In this section we will describe these algebraic structures in terms of generators and relations for several cases, each more advanced than the previous one. The simplest case is that closed TQFT's and positive boundary closed TQFT's. After that we do open-closed TQFT's with a single boundary label, followed by open-closed with a set of boundary labels $\mathcal{B}$.

3.2.1. *Closed TQFT's.* We start with the simplest case, that of closed cobordisms. This will be the base case for our other proofs.

One can make the 2-dimensional oriented cobordisms into a category Bord^{π_0} as follows: its objects are non-negative numbers $n \in \mathbb{Z}_{\geq 0}$, representing the 1-dimensional manifold consisting of a disjoint union of n circles. Its morphisms are isomorphism classes of cobordisms, where a isomorphism of cobordisms is taken to be a diffeomorphism $f : \Sigma \to \Sigma'$ such that $f \circ \rho_{in} = \rho'_{in}$ and $f \circ \rho_{out} = \rho'_{out}$. The identity is a cylinder with one incoming and one outgoing circle. The fact that we work with isomorphism classes of cobordisms makes sure that composition is associative and the identitity is in fact an identity. Otherwise this would only be true up to coherent homotopy.

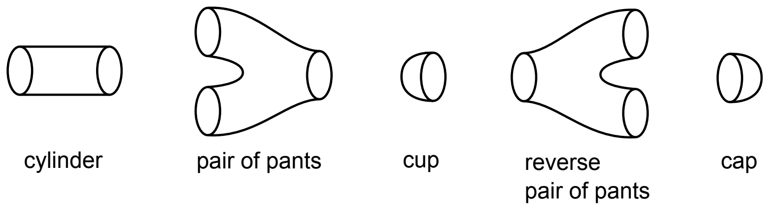


FIGURE 6.37. Cobordisms inducing the distinguished operations of a TQFT.

Incorporating this data leads one to a definition of a cobordism $(\infty, 1)$-category [**Lur09**], see chapter 7.

Remark 6.35. The reason for the π_0 in the notation Bord^{π_0} is that this category is obtained from a topological cobordism prop Bord which has classifying spaces of mapping class groups as morphism spaces by applying the π_0-functor.

This category has been studied quite a bit because it encodes the structure of 2-dimensional topological quantum field theories. Note that addition of non-negative numbers and disjoint union of cobordisms makes Bord^{π_0} into a symmetric monoidal category and in fact a prop.

Definition 6.36. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A C-*valued topological quantum field theory* (TQFT) is a symmetric monoidal functor $F : \mathsf{Bord}^{\pi_0} \to \mathsf{C}$. A morphism of C-valued TQFT's is a natural transformation. This gives us a category TQFT_C of C-valued TQFT's.

Because the functor must be symmetric monoidal, a TQFT is determined on objects by its value on 1, i.e. its value on the circle. As consequence, a TQFT consists of an object $X \in Ob(\mathsf{C})$ together with operations $F(\Sigma) : X^{\otimes n} \to X^{\otimes m}$ for each isomorphism class Σ of cobordisms, in this case with n incoming boundary circles and m outgoing boundary circles.

There are several distinguished operations, which we can write down directly and which are basic in the sense that they will be shown to be the building blocks of all other operations. See figure 6.37 for a drawing of the corresponding cobordisms.

The cylinder: The cylinder is cobordism from a single incoming circle to a single outgoing circle, and induces the identity of the object 1 in Bord^{π_0}. By functoriality, we conclude that F sends the cylinder to the identity of X.

The pair of pants: The pair of pants induces an operation $\mu : X^{\otimes 2} \to X$. If one draws the right picture, it is easy to see that this operation is commutative. Picture the pair of pants morphism as going from left to right, with first input circle on top, the second input circle at the bottom. Then there is a diffeomorphism between this pair of pants and the one with the first input circle at the bottom and the second one on top. It is given by rotation of 180 degrees in the horizontal axis. Similarly one sees that this operation is associative.

We conclude that the following diagrams commute:

$$\begin{tikzcd} X^{\otimes 3} \arrow[r, "id\otimes\mu"] \arrow[d, "\mu\otimes id"'] & X^{\otimes 2} \arrow[d, "\mu"] \\ X^{\otimes 2} \arrow[r, "\mu"'] & X \end{tikzcd} \qquad \begin{tikzcd} X^{\otimes 2} \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\ X^{\otimes 2} \arrow[r, "\mu"'] & X \end{tikzcd}$$

where τ is the symmetry. These exactly the properties of a product and therefore we call the operation induced by the pair of pants the *product*. We denote it by μ.

The reverse pair of pants: The reverse pair of pants is similar to the pair of pants, except we now have one incoming boundary circle and two outgoing boundary circles. If we use the same reasoning as for the pair of pants, we obtain dually that it induces a cocommutative and coassociative *coproduct* $\Delta : X \to X^{\otimes 2}$, i.e. the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X^{\otimes 2} \arrow[d, "id\otimes\Delta"] \\ X^{\otimes 2} \arrow[r, "\Delta\otimes id"'] & X^{\otimes 3} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\Delta"'] \arrow[dr, "\Delta"] & \\ X^{\otimes 2} \arrow[r, "\tau"'] & X^{\otimes 2} \end{tikzcd}$$

The cup: The cup is a disk with one outgoing boundary circle and an empty incoming boundary. Therefore it is a map $\eta : 1 \to X$. By looking at the pictures, we see that when we attach a disk to one of the incoming boundary circles of the pair of pants, the resulting surface is diffeomorphic to the cylinder. Therefore we conclude that η is a *unit* for the product μ, i.e. the following diagrams commute:

$$\begin{tikzcd} 1 \otimes X \arrow[d, "\eta\otimes id"'] & X \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] & X \otimes 1 \arrow[d, "id\otimes\eta"] \\ X^{\otimes 2} \arrow[r, "\mu"'] & X & X^{\otimes 2} \arrow[l, "\mu"] \end{tikzcd}$$

The cap: The cap is a disk with one incoming boundary circle. Dually, it induces a *counit* $\epsilon : X \to 1$ for the coproduct Δ, i.e. the following diagrams commute:

$$\begin{tikzcd} X^{\otimes 2} \arrow[d, "\epsilon\otimes id"'] & X \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & X^{\otimes 2} \arrow[d, "id\otimes\epsilon"] \\ 1 \otimes X \arrow[r, "\cong"'] & X & X \otimes 1 \arrow[l, "\cong"] \end{tikzcd}$$

What remains is to resolve the interaction between these basic operations, in particular between the product and the coproduct. By drawing pictures of the surfaces and using them to find diffeomorphisms, one obtains the following result known as Frobenius compatibility:

Lemma 6.38. *The following maps $X^{\otimes 2} \to X^{\otimes 2}$ are equal:*

$$(\mu \otimes id) \circ (id \otimes \Delta) = \Delta \circ \mu = (id \otimes \mu) \circ (\Delta \otimes id)$$

PROOF. Consider figure 6.39. □

This proves that X is a Frobenius object in C, the definition of which is given below:

Definition 6.40. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A *Frobenius object* in C is an object $X \in Ob(\mathsf{C})$ together with an associative product $\mu : X^{\otimes 2} \to X$, a coassociative coproduct

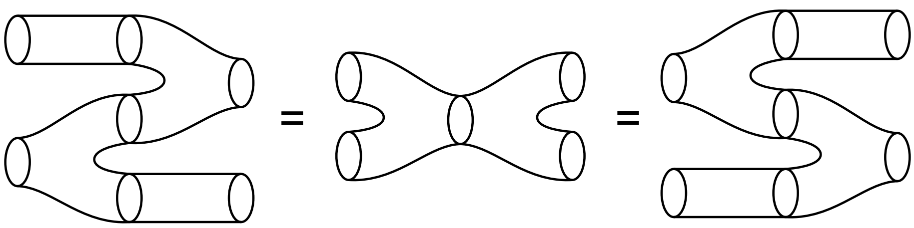


FIGURE 6.39. Three diffeomorphic cobordisms, which imply Frobenius compatibility for the TQFT operations.

$\Delta : X \otimes X^{\otimes 2}$, a unit $\eta : 1 \to X$ and a counit $\epsilon : X \to 1$ which make following diagrams commute:

$$\begin{tikzcd} X^{\otimes 3} \arrow[r, "id\otimes\mu"] \arrow[d, "\mu\otimes id"'] & X^{\otimes 2} \arrow[d, "\mu"] \\ X^{\otimes 2} \arrow[r, "\mu"'] & X \end{tikzcd} \qquad \begin{tikzcd} X^{\otimes 2} \arrow[rr, "id\otimes\Delta"] \arrow[dr, "\mu"] \arrow[dd, "\Delta\otimes id"'] & & X^{\otimes 3} \arrow[dd, "\mu\otimes id"] \\ & X \arrow[dr, "\Delta"] & \\ X^{\otimes 3} \arrow[rr, "id\otimes\mu"'] & & X^{\otimes 2} \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X^{\otimes 2} \arrow[d, "id\otimes\Delta"] \\ X^{\otimes 2} \arrow[r, "\Delta\otimes id"'] & X^{\otimes 3} \end{tikzcd}$$

$$\begin{tikzcd} 1\otimes X \arrow[d, "\eta\otimes id"'] & X \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] & X\otimes 1 \arrow[d, "id\otimes\eta"] \\ X^{\otimes 2} \arrow[r, "\mu"'] & X & X^{\otimes 2} \arrow[l, "\mu"] \end{tikzcd} \qquad \begin{tikzcd} X^{\otimes 2} \arrow[d, "\epsilon\otimes id"'] & X \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, equal] & X^{\otimes 2} \arrow[d, "id\otimes\epsilon"] \\ 1\otimes X \arrow[r, "\cong"'] & X & X\otimes 1 \arrow[l, "\cong"] \end{tikzcd}$$

A Frobenius object is *commutative* if, in addition to the axioms of a Frobenius object, the product is commutative and the coproduct cocommutative. This means that the following diagrams should commute.

$$\begin{tikzcd} X^{\otimes 2} \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\ X^{\otimes 2} \arrow[r, "\mu"'] & X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[d, "\Delta"'] \arrow[dr, "\Delta"] & \\ X^{\otimes 2} \arrow[r, "\tau"'] & X^{\otimes 2} \end{tikzcd}$$

A Frobenius object is called *symmetric* if, in addition to the axioms of a Frobenis object, the unit and counit are invariant under the twist. This means that the following diagrams should commute:

$$\begin{tikzcd} X^{\otimes 2} \arrow[r, "\mu"] \arrow[d, "\tau"'] & X \arrow[dr, "\epsilon"] & \\ X^{\otimes 2} \arrow[r, "\mu"'] & X \arrow[r, "\epsilon"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\eta"] \arrow[dr, "\eta"'] & X \arrow[r, "\Delta"] & X^{\otimes 2} \arrow[d, "\tau"] \\ & X \arrow[r, "\Delta"'] & X^{\otimes 2} \end{tikzcd}$$

A morphism $f : X \to Y$ of Frobenius objects is a morphism $f : X \to Y$ in C compatible wit the product, coproduct, unit and counit. Let Frob_C denote the category of Frobenius objects in C. The category $\mathsf{CommFrob}_\mathsf{C}$ is the full subcategory on the commutative Frobenius objects and the category $\mathsf{SymmFrob}_\mathsf{C}$ is the full subcategory on the symmetric Frobenius objects.

Remark 6.41. Every commutative Frobenius algebra is symmetric and every commutative or symmetric Frobenius algebra is an ordinary Frobenius algebra.

Remark 6.42. There are several alternative, but equivalent definitions of a Frobenius object. First of all, a Frobenius object can be described as an object X which is simultaneously an algebra and coalgebra object such that the Frobenius compatibility relation holds. One can also use the pairing $\epsilon \circ \mu$ and copairing $\Delta \circ \eta$ to replace one of the μ or Δ.

Although the pictures of two-dimensional cobordisms may give another impression, in fact we are dealing with abstract cobordisms and for those the product is commutative and the coproduct is cocommutative.

Corollary 6.43. *A TQFT F with values in C endows $X = F(1)$ with the structure of a commutative Frobenius object.*

The main question of the classification of TQFT's is which objects in a symmetric monoidal category can be the image of 1 under a symmetric monoidal functor $F : \mathsf{Bord}^{\pi_0} \to \mathsf{C}$. Certainly one must be able to find morphisms for the product, coproduct, unit and counit which satisfies the relation given by above. Therefore boils down to the question whether there are any additional building blocks or relations to ones just given. In two dimensions this question can be solved by relatively elementary means. The main ingredient is the following lemma:

Lemma 6.45. *Every connected cobordism Σ can be decomposed into cups, caps, cylinders, pairs of pants and reverse pairs of pants. This decomposition can be brought into a unique normal form $\Sigma_{\text{out}} \circ \Sigma_{\text{genus}} \circ \Sigma_{\text{in}}$ with the components of the following form (see figure 6.44):*

- *the incoming part Σ_{in} is a cup, a cylinder or a composition of pair of pants, and in the last case first the incoming circles labelled 1 and 2 are attached to a pair of pants, then the outgoing circle of this with the incoming circle labelled 3, etc.,*
- *the genus-creating part Σ_{genus} is a cylinder or a composition of genus units, which are cobordisms made by attaching the two outgoing circles of a pair of pants to the two incoming circles of a reverse pair of pants (see figure 6.46),*
- *and the outgoing part Σ_{out} is a cap, a cylinder or a composition of reverse pair of pants, and in the last case first the outgoing circles labelled 1 and 2 are attached to a reverse pair of pants, then the incoming circle of this with the outgoing circle labelled 3, etc.*

To bring it into this normal form, one only needs the following moves:

- *isotopies,*
- *removing a cylinder,*
- *cancelling a cup or cap using the unit or counit properties,*
- *moving a pair of pants past a reverse pair of pants using Frobenius compatibility,*
- *exchanging pair of pants using associativity or exchanging reverse pair of pants using coassociativity.*

Proof. One first use Morse theory of surfaces with boundary to show that there exists at least one decomposition into basic components, i.e. unit, counit, pair of pants and reverse pair of pants. In particular one can find a Morse function $f : \Sigma \to [0,1]$ such that $f^{-1}(\{0\}) = \partial_{in}\Sigma$, $f^{-1}(\{1\}) = \partial_{out}\Sigma$, the critical points are disjoint and have image in $[0,1]$ disjoint from $\{0,1\}$. A cup corresponds to a critical point of index 0, a cap to one of index 2 and a pair of pants or reverse pair of pants to one of index 1. This proves the first part of the statement about the existence of a decomposition.

The second statement is that we can bring this into a normal form using only moves in the statement of the lemma. Note that we already know we can use these moves. To prove this statement about the normal form, we assign a directed graph to a decomposition of our surface as obtained by our Morse function. This directed graph is given as follows. There are three types of vertices: there is

(1) a vertex for each incoming boundary circle, labelled by $-$ and the label of the incoming boundary circle,
(2) a vertex for each outgoing boundary circle, labelled by $+$ and the label of the outgoing boundary circle and
(3) an unlabelled vertex for each component of the decomposition.

Similarly, there are three types of directed edges. There is

(1) a directed edge from an incoming vertex to a component vertex if that incoming circle is the boundary of the respective component,

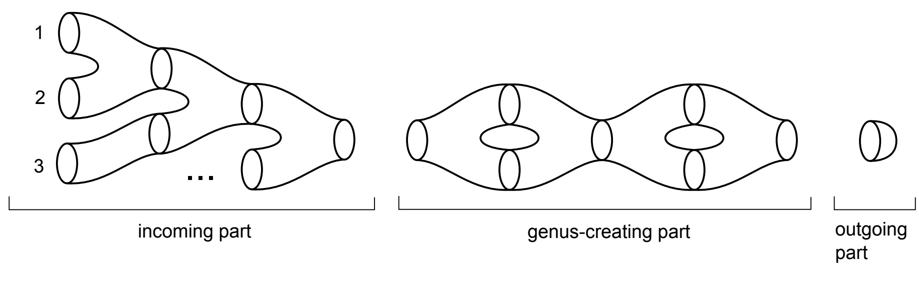


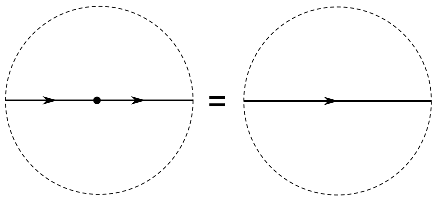
Figure 6.44. The normal form of a cobordism in the case of $n \geq 3$ incoming boundary circles and no outgoing boundary circles.

(2) a directed edge from a component vertex to an outgoing vertex if that component vertex has the respective outgoing circle as boundary component and
(3) a directed edge from one component vertex to another for each outgoing boundary component of the first attached to an incoming boundary component of the second.

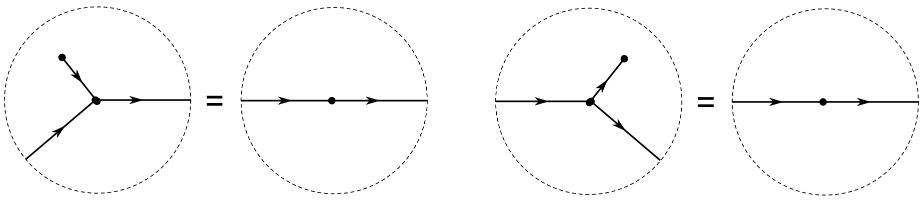
Note that we can assume that this graph only has vertices of valence 1, 2 or 3 and that each vertex with valence greater than 1 has at least one incoming and outgoing edge. Furthermore, we can assume the graph contains no directed cycles. We call a graph of this type a directed 3-graph. Because our surface is assumed connected, this graph will be connected as well.

The invariance under isotopies implies that we don't lose any information by looking at these directed 3-graphs. The other moves in the lemma then correspond to the following operations on graphs:

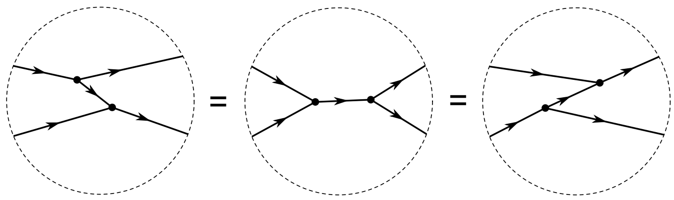
- Removing cylinder corresponds to removing an unlabelled vertex with exactly one incoming and one outgoing edge. In terms of graphs, this corresponds to the following move:



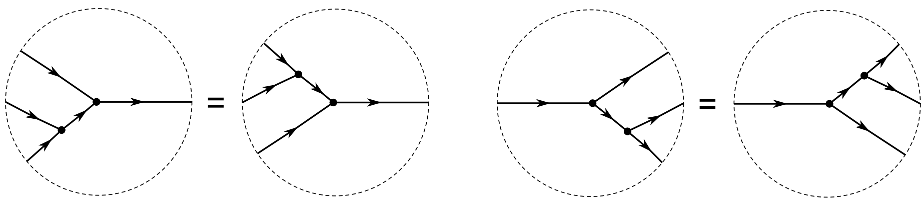
- Cancelling cups and caps means that we can remove a vertex of valence one and its corresponding edge, as long as this edge is not connected to a vertex labelled $+$ or $-$ respectively and we don't end up with a vertex of valence 2 with two incoming or outgoing edges. In terms of graphs, this corresponds to the following pair of moves:



- Moving a pair of pants past a reverse pair of pants using Frobenius compatibility corresponds to the following move:



- Exchanging two pair of pants using associativity or exchanging reverse pair of pants using coassociativity correspond to the following two moves:



Now we do induction on the genus of the underlying graph, which is given by $\hat{g} = \#E - \#V$. This number is preserved by all the moves and can easily be seen to be greater than or equal to -1.

If $\hat{g} = -1$, then we will now show the underlying graph can not contain any cycle. For suppose that there is a cycle, then we can pick a maximal set of disjoint cycles and removing these leaves

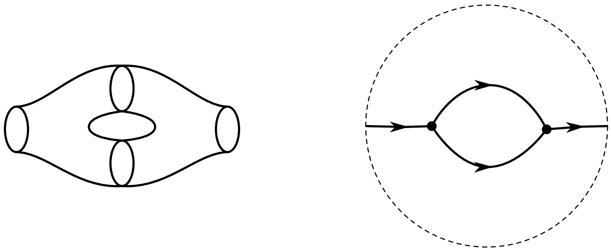


FIGURE 6.46. The genus unit, both as cobordism and the corresponding directed 3-graph.

behind a forest of half-open or open trees. Each cycle has the same number of vertices as edges, hence the remaining forest must have one more vertex than edges. There are two types of trees: half-open trees, which have the same number of edges as vertices, and open ones, which have one more edge than vertex. We obtain a contradiction with $\hat{g} = 0$ as follows: each cycle contributed 0 to the genus, each half-open trees contributes 0 and each open trees 1. This means that the genus is strictly greater than -1 while on the other hand we assumed it was equal to -1.

We conclude that if $\hat{g} = 0$ the directed 3-graph is a directed tree and this must necessarily be of the normal form already, except for using the associativity and coassociativity to bring incoming and outgoing vertices in the correct order.

For the induction step, assume that the statement is true for directed 3-graphs of genus $\hat{g}$. We will prove it for directed 3-graphs of genus $\hat{g} + 1$. To do this, we use a special type of subgraph called a genus unit. This corresponds to a reverse pair of pants and a pair of pants attached along two circles.

We first claim that the moves always allow us to create a genus unit. The graph must contain a cycle. Let n be the length of the cycle, with which we mean the number of vertices in the cycle. We will do an induction over the length of the cycle. The first case is $n = 2$, in which case we already have a genus unit. Suppose we can prove it for cycles of length n and we have a cycle of length $n + 1$. If there is a vertex of valence 2 with one incoming and one outgoing edge we can remove it using the first move and we are left with a cycle of length n. So let's assume that there are no vertices of valence 2 with one incoming and outgoing edge in the cycle. Pick an orientation on the cycle. Walking along it, we either move in the same direction as the edges of the graph or in the opposite direction. Since a directed 3-graph has no directed cycles, there is at least one vertex where the direction of the edges in the cycle changes from incoming to outgoing.

Pick one of these vertices. Then the cycle must go through either two outgoing edges of this vertex or two incoming ones. We will assume it goes through two outgoing edges, but the proof is similar if it goes through two incoming edges. There are two cases: either one of the edges is attached to a vertex of valence 3 with two incoming edges or both edges are attached to vertices of valence 3 with two outgoing edges. In the first case we can apply Frobenius compatibility to reduce the length of the cycle with 1, in the second case we can use associativity to do this.

Hence the moves allow us to create a genus unit. The genus unit can move freely past other vertices using the moves, so we can in fact remove it for the moment and insert it again later at any point in the graph. If we remove it, a graph of genus $\hat{g}$ remains. This can be brough into normal form. Now we simply insert the genus unit somewhere in the genus-creating part of the graph and we have brought our graph into normal form. This concludes the proof. $\square$

This result tells us that if we know the unit, counit, product and coproduct we know the entire TQFT and that the only relations are those given in the definition of a Frobenius object. Therefore it implies that there is a bijective corresponce between TQFT's with values in C and commutative Frobenius objects in C. We can make this correspondence more precise using our earlier definitions. Recall that there exists a category TQFT_C of TQFT's with values in C and a category $\mathsf{CommFrob}_\mathsf{C}$ of commutative Frobenius object in C. Then the previous observations imply

Corollary 6.47. *The category* $\mathsf{CommFrob}_\mathsf{C}$ *is equivalent to the category* TQFT_C.

In fact, this result can be sharpened. One can show that Bord^{π_0} is a free symmetric monoidal category on a single commutative Frobenius object and then the previous theorem is a simple consequence of the universal property of free symmetric monoidal categories.

3.2.2. *Closed TQFT's with positive boundary.* Although string topology has operations coming from the pair of pants, cup and reverse pair of pants, there is no operation for the cap. This is due to the positive boundary condition. One may ask whether this could solved, i.e. whether we just have to discover a cap. The answer is no, because the existence of the cap allows one to use the product and the cap to produce a non-degenerate pairing on C. In graded vector spaces, this forces finite-dimensionality. However, for most compact oriented manifolds M the homology of the free loop space $H_*(LM;\mathbb{Q})$ has non-zero elements in arbitrarily high degrees. In particular, Bott's theorem [**Bot58**] tells us this is the case when M is a Lie group G. In this case we have that $H_*(LG;\mathbb{Z})$ is a free commutative algebra generated by finitely many generators of even degree. We conclude that a general construction of an operation corresponding to cap cannot exist in string topology.

So what is string topology then? In particular, from degree zero operations we get a TQFT with positive boundary. Let $\mathsf{Bord}^{\pi_0,+}$ be the partial sub-prop of Bord^{π_0} of those morphisms with non-zero outgoing boundary components. In particular, this a symmetric monoidal category.

Definition 6.48. Let $(\mathsf{C},\otimes)$ be a symmetric monoidal category. A C-valued topological quantum field theory with positive boundary is a symmetric monoidal functor $F : \mathsf{Bord}^{\pi_0,+} \to \mathsf{C}$. Let $\mathsf{TQFT}^+_{\mathsf{C}}$ denote the category of TQFT's with positive boundary.

We can directly use the previous results of the previous section to prove a classification result for TQFT's with positive boundary, as long as we remember to discard anything having to do with the cap.

Definition 6.49. Let $(\mathsf{C},\otimes)$ be a symmetric monoidal category. A positive Frobenius object in C is an object $X \in Ob(\mathsf{C})$ together with an associative product $\mu : X^{\otimes 2} \to X$, a coassociative coproduct $\Delta : X \otimes X^{\otimes 2}$ and a unit $\eta : 1 \to X$ which make following diagrams commute:

$$\begin{tikzcd} X^{\otimes 3} \arrow[r, "id\otimes\mu"] \arrow[d, "\mu\otimes id"'] & X^{\otimes 2} \arrow[d, "\mu"] \\ X^{\otimes 2} \arrow[r, "\mu"'] & X \end{tikzcd}
\qquad
\begin{tikzcd} X^{\otimes 2} \arrow[rr, "id\otimes\Delta"] \arrow[dr, "\mu"] \arrow[dd, "\Delta\otimes id"'] & & X^{\otimes 3} \arrow[dd, "\mu\otimes id"] \\ & X \arrow[dr, "\Delta"] & \\ X^{\otimes 3} \arrow[rr, "id\otimes\mu"'] & & X^{\otimes 2} \end{tikzcd}
\qquad
\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & X^{\otimes 2} \arrow[d, "id\otimes\Delta"] \\ X^{\otimes 2} \arrow[r, "\Delta\otimes id"'] & X^{\otimes 3} \end{tikzcd}$$

$$\begin{tikzcd} 1\otimes X \arrow[d, "\eta\otimes id"'] & X \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] & X\otimes 1 \arrow[d, "id\otimes\eta"] \\ X^{\otimes 2} \arrow[r, "\mu"'] & X & X^{\otimes 2} \arrow[l, "\mu"] \end{tikzcd}$$

A positive Frobenius object is commutative if, in addition, the product is commutative and the coproduct cocommutative. This means that the following diagrams should commute.

$$\begin{tikzcd} X^{\otimes 2} \arrow[d, "\tau"'] \arrow[dr, "\mu"] & \\ X^{\otimes 2} \arrow[r, "\mu"'] & X \end{tikzcd}
\qquad
\begin{tikzcd} X \arrow[d, "\Delta"'] \arrow[dr, "\Delta"] & \\ X^{\otimes 2} \arrow[r, "\tau"'] & X^{\otimes 2} \end{tikzcd}$$

A morphism $f : X \to Y$ of positive Frobenius objects is a morphism $f : X \to Y$ in C compatible with the product, coproduct and unit maps. This gives us a category $\mathsf{Frob}^+_{\mathsf{C}}$ of positive Frobenius object in C. Let $\mathsf{CommFrob}^+_{\mathsf{C}}$ denote the full category on the commutative positive Frobenius objects in C.

Now we can simply modify the result of corollary 6.47 appropriately to obtain the following result.

Corollary 6.50. *The category* $\mathsf{CommFrob}^+_{\mathsf{C}}$ *is equivalent to the category* $\mathsf{TQFT}^+_{\mathsf{C}}$*.*

3.3. The classification of open-closed TQFT's. However, in the case of string topology we have seen that the incoming boundary does not consist solely of circles. There are also intervals with prescribed boundary labels. We will therefore classify these open-closed TQFT's. To this, we first try to figure out what structures we get on the object corresponding to an interval boundary with two equal labels on the boundary points.

3.3.1. *The classification of open and open-closed TQFT's.* To do this, we consider open-closed TQFT's. These are defined in a way similar to ordinary TQFT's, except that the boundary components can now also be oriented intervals in addition to circles. To make this precise, we define the category $\mathsf{Bord}^{\pi_0}_{oc}$. This category has as objects the pairs of non-negative integers $(n, m) \in \mathbb{Z}^2_{\geq 0}$, representing the 1-dimensional comapct oriented manifold with boundary consisting of a disjoint union of n circles and m oriented intervals. The morphisms from (n, m) to (n', m') are isomorphism classes of oriented open-closed cobordisms with n circles and m intervals as incoming boundary and n' circles and m' intervals as outgoing boundary. This category is symmetric monoidal by addition of numbers on objects and disjoint union of isomorphism classes of cobordisms on morphisms. To define an open TQFT we simply apply the same construction as before.

Definition 6.51. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A C-valued open topological quantum field theory is a symmetric monoidal functor $F : \mathsf{Bord}^{\pi_0}_{oc} \to \mathsf{C}$. A morphism of an open TQFT's is a natural transformation and together these form a category $\mathsf{TQFT}^{oc}_{\mathsf{C}}$ of open TQFT's with values in C.

Again one can try to classify these by finding elementary operations, determining their relations and proving a result about normal forms to show that these generators and relations capture all information. This is considerably harder than in the case of closed TQFT's. A full proof can be found in [**LP08**]. We will only describe the results.

Note that the value of an open-closed TQFT on objects is completely determined by its value on the circle and the interval. Let's denote these objects of C by C and S respectively. The C stands for commutative and/or circle, the S for symmetric and/or strip.

First we give a list of candidates for elementary operations. These will be operations from tensor products of C and S to other tensor products of C and S. Figure 6.52 illustrates these cobordisms.

The cylinder, pair of pants, cup, reverse pair of pants and cap: Of course an open-closed TQFT should at least contain all the information of a closed TQFT, so these operations have the same properties as before, except with X replaced by C. We will denote them by $id_C, \mu_C, \eta_C, \Delta_C, \epsilon_C$.

The strip: The strip is the cobordism from a single incoming interval to a single outgoing one. It corresponds to the identity of the object 1 and by functoriality is corresponds to the identity morphism of S.

The pairing of strips: The pairing of strips is a cobordism from two incoming intervals to a single outgoing one. It looks similar to a pair of pants and in fact induces a *product* μ_S on S. There is however one major difference: we cannot switch the entries by a diffeomorphism and hence this product is not commutative, only associative. This can easily seen by finding a diffeomorphism between the two corresponding cobordisms. This means that the following diagram commutes:

$$\begin{tikzcd} S^{\otimes 3} \arrow[r, "id\otimes\mu_S"] \arrow[d, "\mu_S\otimes id"'] & S^{\otimes 2} \arrow[d, "\mu_S"] \\ S^{\otimes 2} \arrow[r, "\mu_S"'] & S \end{tikzcd}$$

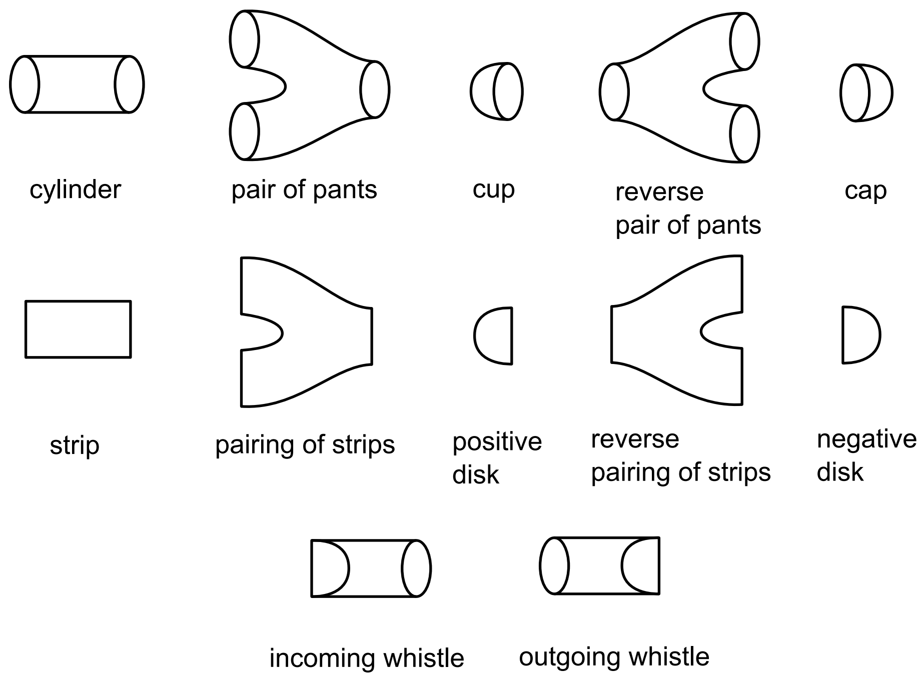


FIGURE 6.52. The cobordisms corresponding to the elementary operations of an open-closed TQFT.

The reverse pairing of strips: The cobordism with one outgoing strip and two incoming strips is dual to the previous cobordism. It looks similar to the reverse pair of pants and similarly induces a *coproduct* Δ_S on S, which is not cocommutative, but coassociative. This means that following diagram commutes:

$$\begin{tikzcd} S \arrow[r, "\Delta_S"] \arrow[d, "\Delta_S"'] & S^{\otimes 2} \arrow[d, "id\otimes\Delta_S"] \\ S^{\otimes 2} \arrow[r, "\Delta_S\otimes id"'] & S^{\otimes 3} \end{tikzcd}$$

It is clear that as before, we have a Frobenius compatibility relation between the product and the coproduct, i.e. the following should commute:

$$\begin{tikzcd} S^{\otimes 2} \arrow[rr, "id\otimes\Delta_S"] \arrow[dr, "\mu_S"] \arrow[dd, "\Delta_S\otimes id"'] & & S^{\otimes 3} \arrow[dd, "\mu_S\otimes id"] \\ & S \arrow[dr, "\Delta_S"] & \\ S^{\otimes 3} \arrow[rr, "id\otimes\mu_S"'] & & S^{\otimes 2} \end{tikzcd}$$

The positive disk: The positive disk is the cobordism with one outgoing boundary interval and a free boundary interval. The operation it induces acts as a *unit* η_S for the product μ_S, similarly to how η_C acts a unit of μ_C. It has one additional property, however. If we precompose a reverse pairing of strips with the positive disk, we are allowed to switch the outputs, by rotating the positive disk. This means that in addition to the diagram expressing that η_S acts a unit, we get a diagram similarly one of those expressing the symmetry of a Frobenius algebra

$$\begin{tikzcd}
1 \otimes S \arrow[d, "\eta_S \otimes id"'] & S \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, equal] & S \otimes 1 \arrow[d, "id \otimes \eta_S"] \\
S^{\otimes 2} \arrow[r, "\mu_S"'] & S & S^{\otimes 2} \arrow[l, "S", "\mu"']
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \arrow[r, "\eta_S"] \arrow[dr, "\eta_S"'] & S \arrow[r, "\Delta_S"] & S^{\otimes 2} \arrow[d, "\tau"] \\
& S \arrow[r, "\Delta_S"'] & S^{\otimes 2}
\end{tikzcd}$$

The negative disk: The negative disk is dual to the positive disk, having one outgoing boundary interval and a free boundary interval. The operation ϵ_S is a *counit* for Δ_S and similarly to the positive disk, it symmetrizes a pairing of strips and therefore induces a symmetry diagram dual to the one for the positive disk.

$$\begin{tikzcd}
S^{\otimes 2} \arrow[d, "\epsilon_S \otimes id"'] & S \arrow[l, "\Delta_S"'] \arrow[r, "\Delta_S"] \arrow[d, equal] & S^{\otimes 2} \arrow[d, "id \otimes \epsilon_S"] \\
1 \otimes S \arrow[r, "\cong"'] & S & S \otimes 1 \arrow[l, "\cong"]
\end{tikzcd}
\qquad
\begin{tikzcd}
S^{\otimes 2} \arrow[r, "\mu_S"] \arrow[d, "\tau"'] & S \arrow[dr, "\epsilon_S"] & \\
S^{\otimes 2} \arrow[r, "\mu_S"'] & S \arrow[r, "\epsilon_S"'] & 1
\end{tikzcd}$$

The incoming whistle: Until now we haven't seen any interaction of the "open" and "closed" operations. However, it is clearly possible let them interact. The first example takes a cylinder and keeps the outgoing boundary the same, a circle. However, we modify the incoming boundary and only take half of the circle to be incoming. The rest of the circle is a free boundary. This cobordism gives us a morphism $\iota : S \to C$. In fact, we claim that this is an homomorphism of coalgebras. We will show this later in lemma 6.57.

The outgoing whistle: The outgoing whistle is dual to the incoming whistle. Instead of modifying the incoming boundary of the cylinder, we modify the outgoing boundary. The outgoing boundary will be an interval and we have a free boundary component. This gives us a morphism $e : C \to S$. This will be a homomorphism of algebras.

We will first look at the structure that an open-closed TQFT gives on S. We see that the operations described above endow S with the structure of a symmetric Frobenius object. This can be proven using TQFT's by defining the notion of an open TQFT. In particular the restriction of open-closed TQFT to the full subcategory on those objects consisting solely of intervals should give an open TQFT.

Let $\mathsf{Bord}^{\pi_0}_{open}$ be the category with objects non-negative integers $n \in \mathbb{Z}_{\geq 0}$ and as morphisms from n to n' the isomorphism classes of open-closed cobordisms with incoming boundary n intervals and outgoing boundary n' intervals. This can be given a symmetric monoidal structure by defining it to be addition on objects and disjoint union of isomorphism classes of cobordisms on morphisms.

Definition 6.53. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. Then a *open TQFT with values in* C is a symmetric monoidal functor $F : \mathsf{Bord}^{\pi_0}_{open} \to \mathsf{C}$. A morphism of open TQFT's is a natural transformation and this gives us a category $\mathsf{TQFT}^{open}_{\mathsf{C}}$ of open TQFT's.

From the generators and relations described above one can see that an open TQFT should almost be the same as a closed TQFT, except that the product and coproduct are not commutative, but symmetric. To prove this, we need to modify our lemma about standard decompositions of cobordisms.

Lemma 6.54. *Every connected open cobordism Σ can be decomposed into positive disks, negative disks, strips, pairings of strips and reverse pairings of strips. This decomposition can be brought into a unique normal form $\Sigma_{\text{out}}\sigma\Sigma_{\text{genus}} \circ \Sigma_{\text{in}}$ where the components are of the following form:*

- Σ_{in} *is a positive disk, a strip or a composition of pairings of strips, and in the last case we first pair the incoming intervals labelled 1 and 2, then pair the output of this with the incoming interval labelled 3, etc.,*
- Σ_{genus} *is a composition of cobordisms of genus units,*
- *and* Σ_{out} *is a positive disk, a strip or a composition of reverse pairings of strips, and in the last case the intervals labelled 1 and 2 are first attached and the input of this if attached to the outgoing interval with label 3, etc.*

To bring it into this normal form, one only needs the following moves:

- *isotopies,*
- *removing a strip,*
- *cancelling a positive or negative disk using the unit or counit properties,*
- *moving a pairing of strips past a reverse pairing of strips using the Frobenius compatibility,*
- *exchanging pairing of strips using associativity or exchanging reverse pairing of strips using coassociativity.*

PROOF. One can either refer to the proof of [**LP08**] or use the following tricks. Let $\bar{\Sigma}$ be the cobordisms with opposite orientation. Then, $\bar{\Sigma} \cup_{\partial_f \Sigma} \Sigma$ becomes a closed cobordism with boundary and involution i. One now does exactly the same proof, but equivariantly with respect to this involution. One needs to introduce one additional move in this equivariant setting: one may remove a twist in the fixed points of the involution using a Dehn twist in a cylinder. □

We can now make the statement of the classification of open TQFT's precise in a way as for closed TQFT's. To do this we define the category of symmetric Frobenius objects in C.

Definition 6.55. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category, then $\mathsf{SymmFrob}_\mathsf{C}$ is the full subcategory of Frob_C on the symmetric Frobenius objects.

Corollary 6.56. *A open TQFT F with values in C gives $S = F(1)$ the structure of a symmetric Frobenius object. The category $\mathsf{Frob}_\mathsf{C}^{open}$ is equivalent to the category $\mathsf{SymmFrob}_\mathsf{C}$.*

Now that we've identified the structure that an open-closed TQFT gives on the image of a interval and a circle seperately, we need to figure out how these interact. In particular, we need to look at the operations ι and e induced by the incoming and outgoing whistle respectively. We can find candidates for the relations between ι, e and the other operations by looking at specific diffeomorphisms.

Lemma 6.57. *The morphism ι is a coalgebra homomorphism, i.e. the following equations hold:*

$$\Delta_C \circ \iota = (\iota \otimes \iota) \circ \Delta_S \qquad \epsilon_C \circ \iota = \epsilon_S$$

The morphism e is dual to ι:

$$\epsilon_C \circ \mu_C \circ (id_C \otimes \iota) = \epsilon_S \circ \mu_S \circ (e \otimes id_S)$$

As a consequence, the morphism e is an algebra homomorphism, i.e. the following equations hold:

$$e \circ \mu_C = \mu_S \circ (e \otimes e) \qquad e \circ \eta_C = \eta_S$$

PROOF. See figure 6.58. □

Lemma 6.59. *The following relation, called the "knowledge about the center", holds:*

$$\mu_S \circ (e \otimes id_S) = \mu_S \circ (id_S \otimes e) \circ \tau$$

PROOF. See figure 6.60. □

Lemma 6.61. *The following relation, called the "Cardy condition", holds:*

$$e \circ \iota = \mu_S \circ \tau \circ \Delta_S$$

PROOF. See figure 6.62. □

These are in fact all the relations and to state this result, we define the notion of a knowledgeable Frobenius object, as introduced in [**LP08**].

Definition 6.63. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A *knowledgeable Frobenius object in* C, is a quadruple (C, S, ι, e) of a commutative Frobenius object C in C, a symmetric Frobenius

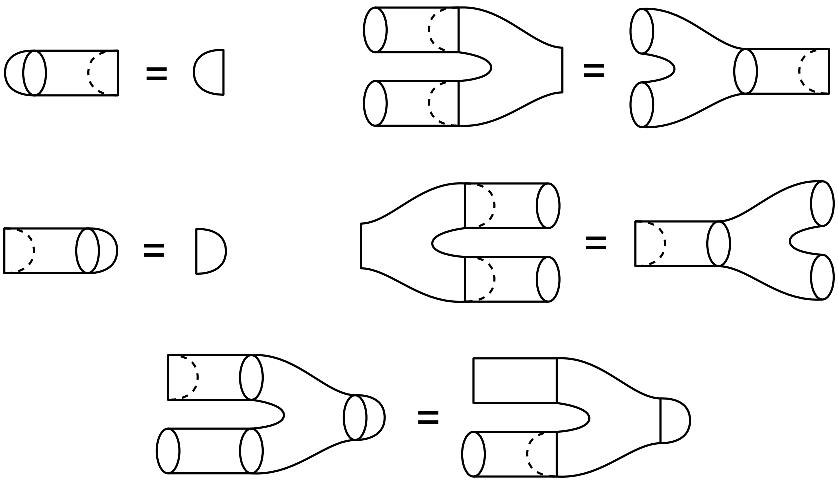


FIGURE 6.58. Diffeomorphisms showing that ι and e are coalgebra, resp. algebra homomorphisms and dual to each other.

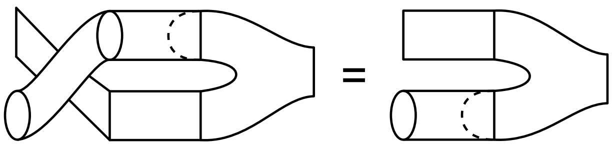


FIGURE 6.60. A diffeomorphism showing that the "knowledge about the center" relation holds.

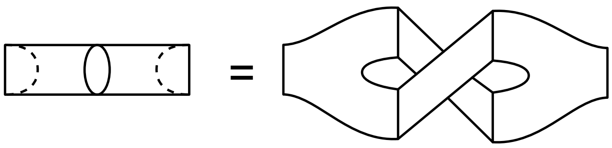


FIGURE 6.62. A diffeomorphism illustrating the Cardy condition for open-closed TQFT's.

object S in C, a coalgebra homomorphism $\iota : S \to C$ and a dual algebra homomorphism $e : C \to S$, such that the following diagrams commute:

$$\begin{tikzcd} C \otimes S \arrow[r, "e \otimes id_S"] \arrow[d, "\tau"'] & S^{\otimes 2} \arrow[dr, "\mu_S"] & \\ S \otimes C \arrow[r, "id_S \otimes e"'] & S^{\otimes 2} \arrow[r, "\mu_S"'] & S \end{tikzcd} \qquad \begin{tikzcd} S \arrow[r, "\iota"] \arrow[d, "\Delta_S"'] & C \arrow[dr, "e"] & \\ S^{\otimes 2} \arrow[r, "\tau"'] & S^{\otimes 2} \arrow[r, "\mu_S"'] & S \end{tikzcd}$$

A morphism of knowledgeable Frobenius objects $f : (C_X, S_X, \iota_X, e_X) \to (C_Y, S_Y, \iota_Y, e_Y)$ is a pair $f_C : C_X \to C_Y$, $f_S : S_X \to X_Y$ of morphisms of commutative, resp. symmetric Frobenius object in C, which are compatible with the algebra and coalgebra homomorphisms. This gives us a category $\mathsf{KFrob}_{\mathsf{C}}$ of knowledgeable Frobenius objects in C.

Then the main result of [**LP08**] is that the list of generators and relations for an open-closed TQFT that we have given is complete.

Theorem 6.64. *An open-closed TQFT F with values in C endows the pair $C = F(\text{circle})$ and $S = F(\text{interval})$ with the structure of a knowledgeable Frobenius object. Furthermore, the category $\mathsf{KFrob}_{\mathsf{C}}$ is equivalent to the category $\mathsf{TQFT}^{\mathrm{oc}}_{\mathsf{C}}$.*

SKETCH OF PROOF. The proof proceeds in exactly the same way as our other proof. Use a form of Morse theory for surfaces with boundary using tangent cones, to prove a decomposition into the standard components. To prove that the resulting operation is well-defined, one gives a method to bring every open-closed cobordism into a standard form. It is this last step which becomes quite involved in the proof of this theorem. ☐

3.3.2. *The classification of open-closed TQFT's with set of branes $\mathcal{B}$.* If we allow different labels of the edges of an interval, we get more cobordisms. Again, we make our definition of the category of cobordisms precise. It is the essentially the same definition as in chapter 2. The category $\mathsf{Bord}^{\pi_0}_{\mathcal{B}}$ has objects the sequences $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}^{\mathcal{B}\times\mathcal{B}}$ with finitely many entries non-zero. The morphisms from $(n, \{n_{b,b'}\})$ to $(m, \{m_{b,b'}\})$ are the isomorphisms classes of $\mathcal{B}$-labelled cobordisms with n incoming boundary circles, $n_{b,b'}$ incoming boundary intervals with boundary labels b, b', m outgoing boundary circles and $m_{b,b'}$ outgoing boundary intervals with boundary labels b, b'. This is a symmetric monoidal category by addition of integers and disjoint union of cobordisms.

Definition 6.65. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category, then a $\mathcal{B}$-labelled open-closed TQFT is a symmetric monoidal functor $F : \mathsf{Bord}^{\pi_0}_{\mathcal{B}} \to \mathsf{C}$. A morphism of such TQFT's is a natural transformation and together these form a category $\mathsf{TQFT}^{oc,\mathcal{B}}_{\mathsf{C}}$.

Although the category of cobordisms has become more complicated, the classification of open-closed TQFT's still works as long as we keep track of the boundary labels. The result is a generalization of a knowledgeable Frobenius algebra object, where the symmetric Frobenius object coming from the boundary intervals gets replaced by a large number of objects indexed by pairs of boundary labels.

Definition 6.66. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A *$\mathcal{B}$-labelled knowledgeable Frobenius object in* C, is a commutative Frobenius object $(C, \mu_C, \Delta_C, \epsilon_C, \eta_C)$ together with objects $\{S_{ab}\}_{a,b\in\mathcal{B}}$ of C and morphisms $\{\mu_{abc} : S_{ab} \otimes S_{bc} \to S_{ac}\}_{a,b,c\in\mathcal{B}}$, $\{\Delta_{abc} : S_{ac} \to S_{ab} \otimes S_{bc}\}_{a,b,c\in\mathcal{B}}$, $\{\epsilon_a : S_{aa} \to 1\}_{a\in\mathcal{B}}$, $\{\eta_a : 1 \to S_{aa}\}_{a\in\mathcal{B}}$, $\{i_a : S_{aa} \to C\}_{a\in\mathcal{B}}$ and $\{e_a : C \to S_{aa}\}_{a\in\mathcal{B}}$. These must satisfy the following relations:

Associativity: For all $a, b, c, d \in \mathcal{B}$ we have

$$\mu_{acd} \circ (\mu_{abc} \otimes id_{S_{cd}}) = \mu_{abd} \circ (id_{S_{ab}} \otimes \mu_{bcd}) : S_{ab} \otimes S_{bc} \otimes S_{cd} \to S_{ad}$$

Coassociativity: For all $a, b, c, d \in \mathcal{B}$ we have

$$(\Delta_{abc} \otimes id_{S_{cd}}) \circ \Delta_{acd} = (id_{S_{ab}} \otimes \Delta_{bcd}) \circ \Delta_{abd} : S_{ad} \to S_{ab} \otimes S_{bc} \otimes S_{cd}$$

Unit: For all $a, b \in \mathcal{B}$ we have

$$\mu_{aab} \circ (\eta_a \otimes id_{S_{ab}}) = id_{S_{ab}} : S_{ab} \to S_{ab}$$

$$\mu_{abb} \circ (id_{S_{ab}} \otimes \eta_b) = id_{S_{ab}} : S_{ab} \to S_{ab}$$

Counit: For all $a, b \in \mathcal{B}$ we have

$$(\epsilon_a \otimes id_{S_{ab}}) \circ \Delta_{aab} = id_{S_{ab}} : S_{ab} \to S_{ab}$$

$$(id_{S_{ab}} \otimes \epsilon_b) \circ \Delta_{abb} = id_{S_{ab}} : S_{ab} \to S_{ab}$$

Frobenius compatibility: For all $a, b, c, d \in \mathcal{B}$ we have:

$$(id_{S_{ad}} \otimes \mu_{dbc}) \circ (\Delta_{adb} \otimes id_{S_{bc}}) = \Delta_{adc} \circ \mu_{abc}$$
$$= (\mu_{abd} \otimes id_{S_{dc}}) \circ (id_{ab} \otimes \Delta_{bdc}) : S_{ab} \otimes S_{bc} \to S_{ad} \otimes S_{dc}$$

Symmetry: For all $a, b \in \mathcal{B}$ we have

$$\epsilon_a \circ \mu_{aba} = \epsilon_b \circ \mu_{bab} \circ \tau : S_{ab} \otimes S_{ba} \to 1$$

$$\Delta_{aba} \circ \eta_a = \tau \circ \Delta_{bab} \circ \eta_b : 1 \to S_{ab} \otimes S_{ba}$$

Homomorphism properties: For all $a \in \mathcal{B}$ we have:

$$\Delta_C \circ \iota_a = (\iota_a \otimes \iota_a) \circ \Delta_{S_{aa}} : S_{aa} \to C \qquad \epsilon_C \circ \iota_a = \epsilon_a : S_{aa} \to 1$$

$$\epsilon_C \circ \mu_C \circ (id_C \otimes \iota_a) = \epsilon_a \circ \mu_{aaa} \circ (e_a \otimes id_{S_{aa}}) : C \otimes S_{aa} \to 1$$

$$e_a \circ \mu_C = \mu_{aaa} \circ (e_a \otimes e_a) : C \otimes C \to S_{aa} \qquad e_a \circ \epsilon_C = \epsilon_a : 1 \to S_{aa}$$

Knowledge about center: For all $a, b \in \mathcal{B}$ we have:

$$\mu_{aab} \circ (e_{aa} \otimes id_{S_{ab}}) = \mu_{abb} \circ (id_{S_{ab}} \otimes e_{bb}) \circ \tau : C \otimes S_{ab} \to S_{ab}$$

Cardy condition: For all $a, b \in \mathcal{B}$ we have:

$$e_a \circ \iota_a = \mu_{aa} \circ \tau \circ \Delta_{aa} : S_{aa} \to S_{aa}$$

Theorem 6.67. *An $\mathcal{B}$-labelled open-closed TQFT F with values in C gives the set $C = F(circle)$, $S_{ab} = F(interval\ with\ labels\ a, b)$ the structure of a $\mathcal{B}$-labelled knowledgeable Frobenius object. Furthermore, the category $\mathsf{TQFT}_{\mathsf{C}}^{\mathrm{oc},\mathcal{B}}$ is equivalent to the category $\mathsf{KFrob}_{\mathsf{C}}^{\mathcal{B}}$.*

3.3.3. *The classification of positive open-closed TQFT's with set of branes $\mathcal{B}$.* Finally, we describe the result which is most relevant to the construction of this thesis. It is also the most complicated one. The positive boundary condition tells us that each connected component must have a non-empty outgoing boundary. This means that the cap and the negative disks are ruled out. This amounts to dropping the counits and changing the axioms referring to them. The next definition is phrased slightly akwardly to avoid repeating the entire list of axioms of a $\mathcal{B}$-labelled knowledgeable Frobenius object

Definition 6.68. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A *positive $\mathcal{B}$-labelled knowledgeable Frobenius object* in C is a $\mathcal{B}$-labelled knowledgeable Frobenius object without the counits ϵ and ϵ_{aa} or axioms referring to these.

A morphism of *positive $\mathcal{B}$-labelled knowledgeable Frobenius object* is a set of morphisms compatible with all the structure maps. This gives a category $\mathsf{KFrob}_{\mathsf{C}}^{\mathcal{B},+}$.

From the previous results, it is clear that a $\mathcal{B}$-labelled open-closed TQFT with positive boundary gives us such a positive $\mathcal{B}$-labeled knowledgeable Frobenius object. A more precise version of stating this is the following corollary.

Corollary 6.69. *The category $\mathsf{KFrob}_{\mathsf{C}}^{\mathcal{B},+}$ is equivalent to the category $\mathsf{TQFT}_{\mathsf{C}}^{\mathcal{B},+}$. The equivalence is given by sending a TQFT F to $C = F(S^1)$ and $S_{ab} = F(interval\ with\ labels\ a,b)$.*

3.4. The closed case. We will now describe what the $\mathcal{B}$-labelled closed TQFT with positive boundary structure on $H_*(LM)$ is like. In other words, we restrict our attention to the string operations coming from those cobordism Σ with no free boundary.

To do this, we must look at the way the local system interferes with our classification. This turns out not to be that difficult: we recall from proposition 5.67 that the local system $\mathcal{L}_{\mathcal{B}}^M$ is in fact trivializable in this case. This is the case exactly because in the closed case we have no free boundary.

To get operations depending on the zero'th homology with local coefficients, one needs to to trivialize the virtual bundle determining the local system $\mathcal{L}_{\mathcal{B}}^M$ by picking bases of the homology groups that appear in our definition of the virtual bundle $\bar{\eta}$. This gives a problem: two compositions of cobordisms may be isomorphic, but our choice of trivializations of the virtual bundles may not agree. This means that signs will appear and hence we are not able to prove that string topology is a "reall" TQFT with positive boundary, but only an analogue of a TQFT with positive boundary with additonal signs in the axioms.

However, if we just pick trivializations we make sense of the individual operations: for each closed cobordism Σ the generator of $H_0(B\Gamma_\Sigma; \mathbb{Q})$ gives us an operation from $H_*(LM; \mathbb{Q})^{\otimes r}$ to $H_*(LM; \mathbb{Q})^{\otimes s}$ which shifts the dimension by $\chi(\Sigma)d$. These operations satisfy the relations of a commutative Frobenius algebra with positive boundary up to some sign and grading. This deserves a slogan reflected in figure 6.1:

Degree zero operations are TQFT operations.

In next section we will figure out what these signs are and how to fix them to get the conventions in the literature.

3.4.1. *Determining the signs.* As said before, the HCFT structure with the local system $\mathcal{L}_{\mathcal{B}}^{M}$ in theory takes care of all of the signs. However, to do calculations and use the classification of closed TQFT's with positive boundary, we want to use the explicit representatives of the closed operations coming from classes in the ordinary rational homology of the mapping class groups, not homology with local coefficients. Indeed, for those obtained in section 3.1 of this chapter we have implicitly chosen such trivializations and therefore signs might appear when we compose them.

We will determine the signs that appear in the relations between the closed product μ_c, the closed coproduct Δ_c and the closed unit η_c both to demonstrate the technique and to pin down our variation of a TQFT with positive boundary with signs. We start with the commutativity relations. For this we let T denote the twist induced by the cobordism consisting of two cylinders which interchange the first and second boundary components.

Lemma 6.70. *The closed product is commutative up to a sign in the following sense:*

$$\mu_c(a \otimes b) = (-1)^d \mu_c \circ T(a \otimes b) = (-1)^d \mu_c(b \otimes a)$$

PROOF. Let $\Sigma_0^{2,1}$ denote the pair of pants cobordism with two incoming and one outgoing boundary circle. Then the local system $\mathcal{L}_{\mathcal{B}}^{M}$ is completely determined as the d'th power of the determinant of the virtual bundle $H_0(\Sigma_0^{2,1}, \partial_{in}\Sigma_0^{2,1}; \mathbb{Q}) - H_1(\Sigma_0^{2,1}, \partial_{in}\Sigma_0^{2,1}; \mathbb{R}) \cong 0 - \mathbb{R}$. The generator x of H_1 is represented by a path from the first to the second incoming boundary circle. Then the twist T applied to x sends it to $-x$. Thus the sign of the determinant changes by $(-1)^d$. □

Lemma 6.71. *The closed coproduct is cocommutative up to a sign in the following sense:*

$$\Delta_c(a) = (-1)^d T \circ \Delta_c(a)$$

PROOF. Now let $\Sigma_0^{1,2}$ denote the reverse pair of pants cobordism with one incoming and two outgoing boundary circles. Then the local system is the d'th power of the determinant of the virtual bundle $H_0(\Sigma_0^{1,2}, \partial_{in}\Sigma_0^{1,2}; \mathbb{R}) - H_1(\Sigma_0^{1,2}, \partial_{in}\Sigma_0^{1,2}; \mathbb{R}) \cong 0 - \mathbb{R}$ as well, where the generator y of H_1 is given by the difference of the loop around the first outgoing boundary circle and the loop around the second outgoing boundary circle. Then T sends y to $-y$ and hence the sign is $(-1)^d$ again. □

Next we get to the associativity and coassociativity.

Lemma 6.72. *The closed product is associative up to a sign in the following sense:*

$$\mu_c(a \otimes \mu_c(b \otimes c)) = (-1)^d \mu_c(\mu_c(a \otimes b) \otimes c)$$

PROOF. Let $\Sigma_0^{3,1}$ be the genus zero surface with three incoming and one outgoing boundary circles. We have that $H_0(\Sigma_0^{3,1}, \partial_{in}\Sigma_0^{3,1}; \mathbb{R}) - H_1(\Sigma_0^{3,1}, \partial_{in}\Sigma_0^{3,1}; \mathbb{R}) = 0 - \mathbb{R}^2$, where the latter can be chosen to be generated by x_{12} and x_{23}, the paths from the first to the second and from the second to the third incoming boundary circle respectively.

In the left hand side tensor product of the two generators x for $H_1(\Sigma_0^{3,1}, \partial_{in}\Sigma_0^{3,1}; \mathbb{Q})$ used before is mapped to $x_{12} \wedge x_{23}$ in the determinant bundle, while in the left hand side it is mapped to $x_{13} \wedge x_{23}$. Now use that $x_{13} = -x_{12} - x_{23}$ to see that the latter is equal to $-x_{12} \wedge x_{23}$. We conclude that a sign $(-1)^d$ appears. □

Lemma 6.73. *The closed coproduct is strictly coassociative:*

$$(\Delta_c \otimes id) \circ \Delta_c(a) = (id \otimes \Delta_c) \circ \Delta_c(a)$$

PROOF. Again we have that $H_0(\Sigma_0^{1,3}, \partial_{in}\Sigma_0^{1,3}; \mathbb{R}) - H_1(\Sigma_0^{1,3}, \partial_{in}\Sigma_0^{1,3}; \mathbb{R}) = 0 - \mathbb{R}^2$, where the latter can be chosen to be generated by x_1, x_2, x_3, the loops around the first, second and third outgoing boundary circles subject to the relation $x_3 = -x_1 - x_2$.

In the left hand side, the two generators x are sent to $(x_1 - x_2) \wedge (x_1 + x_2 - x_3) = 2x_1 \wedge x_3 - 2x_2 \wedge x_3$. On the right hand side they are sent to $(x_2 - x_3) \wedge (x_1 - x_2 - x_3) = 2x_2 \wedge x_1 - 2x_3 \wedge x_1$. These two elements can easily be seen to be equal using the relation. □

The unit is next. The following is a relatively easy consequence of the previous lemmas and will not be proven.

Lemma 6.74. *The closed unit $\eta_c(1)$ is a left unit, but only a right unit up to sign.*

Finally, we do the Frobenius compatibility.

Lemma 6.75. *The correct signs for the Frobenius compatibility relation are:*

$$(\mu \otimes id) \circ (id \otimes \Delta) = (-1)^d \Delta \circ \mu = (id \otimes \mu) \circ (\Delta \otimes id)$$

Proof. Let $\Sigma_0^{2,2}$ be the genus zero surface with two incoming and two outgoing boundary circles. Then we have an isomorphism of virtual bundles $H_0(\Sigma_0^{2,2}, \partial_{in}\Sigma_0^{2,2}; \mathbb{R}) - H_1(\Sigma_0^{2,2}, \partial_{in}\Sigma_0^{2,2}; \mathbb{R}) = 0 - \mathbb{R}^2$. The latter is generated by x, the path from the first incoming to the second incoming boundary circle, and y, the difference of the loop around the first and second outgoing boundary circle.

Clearly, the middle maps the generators of the local system of its components to $x \wedge y$. The first and third differ from this by a minus sign. □

One thing to take away from this discussion, as expected from proposition 5.68, is that if M is even-dimensional then the degree-zero string operations endow the vector space $H_*(LM; \mathbb{Q})$ with the structure of a TQFT with positive boundary. However, if M is of odd dimension, then signs appear. In this next section we will show to fix part of these signs in accordance with the literature.

3.4.2. *Fixing the signs.* The previous results have some pretty nasty signs. There is a way to fix some them in a way that makes our results fit with the literature. Unfortunately, it is not that clear what it means for the signs to fit with the literature, because it is safe to say that almost every paper in string topology – including this thesis – has some sign errors. The only author I trust concerning signs is Tamanoi and his convention is the one that we will use. In particular see [**Tam08a**].

Definition 6.76. Define the modified closed product $\bar{\mu}_c$ to be

$$\bar{\mu}_c(a \otimes b) = (-1)^{|a|(d-|a|)} \mu_c(a \otimes d)$$

This fixes the product in such a way that it becomes a graded-commutative associative product on the shifted homology $\mathbb{H}_*(LM; \mathbb{Q}) := H_{*+d}(LM; \mathbb{Q})$.

Lemma 6.77. *The modified closed product $\bar{\mu}_c$ satisfies:*

$$\bar{\mu}_c(a \otimes b) = (-1)^{(|a|-d)(|b|-d)} \bar{\mu}_c(b \otimes a)$$

$$\bar{\mu}_c(a \otimes \bar{\mu}_c(b \otimes c)) = \bar{\mu}_c(\bar{\mu}_c(a \otimes b) \otimes c)$$

The closed unit $\eta_c(1)$ is both a left and a right unit for the modified closed product.

Of course, as a result the signs in the Frobenius compatibility relation change. This is to be expected; in the end we are just shuffling around signs and there doesn't seem be a way to make all signs behave nicely.

Remark 6.78. The modified closed product $\bar{\mu}_c$ is what is called the string product in the literature. We will also denote it as $\cdot : \mathbb{H}_*(LM; \mathbb{Q}) \otimes \mathbb{H}_*(LM; \mathbb{Q}) \to \mathbb{H}_*(LM; \mathbb{Q})$.

3.4.3. *The string coproduct is almost trivial.* All this is of course very interesting, but becomes even more interesting when one sees that one can use it to derive some nice results about the operations. For example, using a bit ingenuity one can derive an easy formula for the closed coproduct from this, which we will do now. In the following sections we will see many other applications of the HCFT structures.

To get a nice formula for the coproduct, we will use Frobenius compatibility together with the following direct calculation, due to Tamanoi [**Tam07**]. To state it, recall that the connected components of LM are in bijection with the conjugacy classes of $\pi_1(M)$. It is also useful to reread the definitions of the basic closed degree zero string operations earlier this chapter.

Lemma 6.79. *Suppose that M is connected, then the closed coproduct Δ_c applied to the closed unit $\eta_c(1)$ is given by $\chi(M)[c_0] \otimes [c_0]$, where $[c_0] \in H_0(LM; \mathbb{Q})$ represents the connected component L_cM of constant loops and hence $[c_0] \otimes [c_0]$ the connected component of $LM \times LM$ corresponding to the pairs of loops that are both constant.*

Sketch of proof. One easily sees that $\Delta_c(\eta_c(1))$ is given by some multiple of $[c_0] \otimes [c_0]$ because it must be in degree zero and can be seen to lie in the connected components of pairs of loops that are both constant. One can then find the multiple by evaluating at the base point. The naturality of the cap product allows one to show that the multiplying is given by cap product of the Euler class $e(M)$ of M with the fundamental class $[M]$. This is known to be $\chi(M)$ as it is the number of zeroes of a generic vector field, which shown to be equal to the Euler characteristic using the Gauss-Bonnet theorem. □

For full details of the previous proof see [**Tam08a**]. The following calculation is then purely formal using the HCFT properties.

Proposition 6.80. *The closed coproduct Δ_c vanishes on everything except classes containing a non-zero multiple of the closed unit. The closed unit $\eta_c(1)$ is sent to $\chi(M)[c_0] \otimes [c_0]$ as in the previous lemma.*

Proof. Let $a \in H_*(LM)$ not be a multiple of the closed unit. Then we can write it as a product $a \cdot \eta_c(1)$ or $\eta_c(1) \cdot a$, where we use $\cdot$ instead of μ_c for the closed product.

By Frobenius compatibility we have equations up to sign

$$\begin{aligned}\Delta_c(a) &= \Delta_c(a \cdot \eta_c(1)) = a \cdot \Delta_c(\eta_c(1)) = \chi(M)(a \cdot [c_0]) \otimes [c_0] \\ &= \Delta_c(\eta_c(1) \cdot a) = \Delta_c(\eta_c(1)) \cdot a = \chi(M)[c_0] \otimes (a \cdot [c_0])\end{aligned}$$

From the first line we obtain that $\Delta_c(a) \in H_{*-d}(LM;\mathbb{Q}) \otimes H_0(LM;\mathbb{Q})$ and from the second line that $\Delta_c(a) \in H_0(LM;\mathbb{Q}) \otimes H_{*-d}(LM;\mathbb{Q})$. Thus $\Delta_c(a) = 0$ unless a lies in degree d. In that case we must have $a \cdot [c_0] = [c_0]$, which only happens if $a = \eta_c(1)$. This proves the proposition. □

Heuristically, this proposition says that the string coproduct is almost completely trivial. This implies that there is no hope if one wants to extend the degree zero string operations to a full TQFT. In a full TQFT the product and coproduct are dual to each other and one can't have that the product is rich but the coproduct is almost trivial.

3.5. The open-closed case. From the previous results of this section we see that the degree zero operations for the open-closed cobordisms give the vector spaces $H_*(LM;\mathbb{Q})$ and $\{H_*(P_M(A,B);\mathbb{Q})\}_{A,B\in\mathcal{B}}$ the structure of an open-closed TQFT with set of branes $\mathcal{B}$ and positive boundary condition, up to some signs.

Although it is possible to figure out exactly what exactly these signs and grading shifts are, we do not feel that this would be enlightening for the reader. We will however spend a bit of time looking at the open coproduct to show that it has analogous properties to the closed coproduct.

Lemma 6.81. *Suppose that the brane A is connected, then the open coprodct Δ_{aaa} applied to the open unit η_a satisfies:*

$$\Delta_{aaa} \circ \eta_a(1) = \chi(A)[a_0] \otimes [a_0]$$

where $[a_0] \in H_*(P_M(A,A);\mathbb{Q})$ represents the connected component of constant paths.

This is done by repeating the proof for the closed case. Similarly, the proof for the closed case also gives us the following proposition:

Proposition 6.82. *The open coproduct Δ_{aaa} vanishes on everything except classes that are a non-zero multiple of $\eta_a(1)$, which is sent to $\chi(A)[a_0] \otimes [a_0]$.*

4. The basic higher degree genus zero operation: the BV-operator Δ

In this section we will define the basic higher degree genus zero operation. This is the BV-operator Δ, which is of degree 1, and it remains the only known non-zero string operation of non-zero degree which can not be written as a composition of a BV-operator with operations of lower degree.

The BV-operator should be thought of as the string operation corresponding to a Dehn twist. This statement will be made precise in this section. After that we will investigate some of its properties and use the fact that mapping class groups are generated by Dehn twists to classify

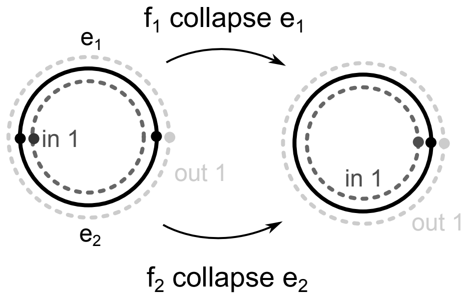


FIGURE 6.84. A pair of morphisms representing a loop in $|\mathsf{Fat}_{\mathcal{B}}^{oc,n}(\Sigma_{\mathrm{cyl}})| \simeq B\Gamma_{\mathrm{cyl}} \simeq S^1$.

all degree 1 operations. In section 6 of this chapter we will see more of the interaction of the BV-operator with the degree zero operations.

4.1. The BV-operator. Our goal will be to extract the BV-operator from our general definition of the string operations and look at some of its properties. After that we define a derived operation known as the string bracket.

4.1.1. *The BV-operator.* The BV-operator is very easy to define, but more difficult to extract from our definition of the string operations. We begin by giving the easy definition and then show how to obtain this description from the general definition of the operations.

There is an action of the circle on LM given by rotating the loops:

$$\begin{aligned}\rho : S^1 \times LM &\to LM\\ (\theta, \gamma) &\mapsto (t \mapsto \gamma(t+\theta))\end{aligned}$$

Definition 6.83. The BV-operator $\Delta : H_*(LM;\mathbb{Q}) \to H_{*+1}(LM;\mathbb{Q})$ is given by

$$\Delta(a) = \rho_*([S^1] \times a)$$

The idea is that the BV-operator arises by rotating the parametrisations of the incoming and outgoing boundary of the cylinder a single time with respect to each other. This might be difficult to see from our definition of $B\Gamma_{\mathrm{cyl}}$.

Recall though that we have proven in theorem 3.63 that the category of $\mathcal{B}$-labelled cobordism graphs whose corresponding cobordism is a cylinder with one incoming and outgoing boundary, has the homotopy type of a circle. We first need to find a nice way to represent the generator of $H_1(B\Gamma_{\mathrm{cyl}})$ in terms of $\mathcal{B}$-labelled cobordism graphs.

Lemma 6.85. *The loop representing the generator is given by the pair of morphisms depicted in figure 6.84.*

Now that we have found a nice representative, we can compute the corresponding string operation. We do not need to worry about the local system, because no collapses happen and the local system is therefore trivial without dimension shift.

Proposition 6.86. *The operation corresponding to the generator of $H_1(B\Gamma_{\mathrm{cyl}})$ is given by the BV-operator.*

PROOF. How can the construction of the map $\daleth_*$ in homology from the map of spectra $\daleth$ be concrete in this case? Because there is no local sytem nor any shift, we can simply restrict the zero'th component of the spectra. Then given a closed singular 1-simplex $S^1 \to B\Gamma_{\mathrm{cyl}}$ the corresponding string operation is obtained by looking at corresponding map $\bar{\daleth} : S^1 \times LM \to LM$ and evaluating the induced map in homology on $[S^1]$.

In this case we need to find a way to write our two singular 1-simplices as a closed 1-simplex. The idea is that the second of the two morphisms $\Delta^1 \to |\mathsf{Fat}_{\mathcal{B}}^{oc,n}(\Sigma_{\mathrm{cyl}})|$ can be reversed to give a map $S^1 \to |\mathsf{Fat}_{\mathcal{B}}^{oc,n}(\Sigma_{\mathrm{cyl}})|$.

The corresponding string operation is easy to determine, because it doesn't do anything but change the relative parametrisations of the boundaries. To write it down, we first give the homotopy $H : I \times S^1 \to S^1$, where we see S^1 as $[0, 2\pi]$ with identified end points. H is given by:

$$H(s,\theta) = \begin{cases} 1 & \text{if } s \leq \frac{1}{2} \text{ and } \theta \leq 2\pi(\frac{1}{2} - s) \\ \frac{2\pi}{2\pi(\frac{1}{2}+s)}\theta & \text{if } s \leq \frac{1}{2} \text{ and } \theta \geq 2\pi(\frac{1}{2} - s) \\ 1 & \text{if } t \geq \frac{1}{2} \text{ and } \theta \geq 2\pi(\frac{3}{2} - s) \\ \frac{2\pi}{2\pi(\frac{3}{2}-s)}\theta & \text{if } s \geq \frac{1}{2} \text{ and } \theta \leq 2\pi(\frac{3}{2} - s) \end{cases}$$

The string operation map $\bar{\gimel} : S^1 \times LM \to LM$ then given on a loop $\gamma \in LM$ by sending it to the 1-parameter family of loops $\bar{\gimel}(s,\gamma)(\theta) = \gamma(H(\frac{s}{2\pi}, \theta))$. The trick is now to note that $\bar{\gimel}$ is homotopic to ρ. What does H do? It starts with half of the circle contracted, which it expands until $s = \frac{1}{2}$. After this it contracts the other half of the circle until it is completely contracted at $s = 1$. This is homotopic to a rotation of the circle, as a quick sketch shows.

We conclude that $\bar{\gimel}$ and ρ induce the same maps in homology and hence the string operations $\bar{\gimel}_*([S^1] \times a)$ is equal to $\rho_*([S^1] \times a) = \Delta(a)$. □

4.1.2. *The BV-operator, Dehn twists and a classification of all degree 1 operations.* We will now place the BV-operator in the context of mapping class groups. As stated earlier, the mapping class group of the cylinder is $\mathbb{Z}$, generated by a Dehn twist with respect to a simple closed curve going around the cylinder once. The BV-operator corresponds to the string topology operation which comes from the generator of the first homology of this mapping class group. This first homology group is the abelianisation of the already abelian mapping class group of the cylinder and hence we could say that the BV-operator directly corresponds to this Dehn twist. The leads us to the following slogan:

BV-operators are Dehn twists.

Let's see what this means using the results of section 2.1 of this chapter. Obviously there are more Dehn twists in mapping class groups than just the one in the mapping class group of the cylinder. However, consider a Dehn twist T_γ around a simple closed curve γ in a cobordism Σ. We know that this curve γ has a regular neighborhood U homeomorphic to the cylinder, such that the Dehn twist can be considered as localized to this regular neighborhood.

This implies that the isotopy class self-homeomorphism T_γ of Σ has a representative given by the identity homeomorphism on $\Sigma \backslash U$ and a simple Dehn twist around a cylinder on U. Suppose that γ is non-separating, then we can find other simple closed curves $\gamma_1, \dots, \gamma_n$ such that the set $\{\gamma, \gamma_1, \dots, \gamma_n\}$ of simple closed curves is separating. Let $U_1, \dots, U_n$ be regular neighborhoods for $\gamma_1, \dots, \gamma_n$. Then we can write the cobordism Σ as a composition:

$$\Sigma = \Sigma_2 \circ \left(U \sqcup \coprod_{i=1}^{n} U_i \right) \circ \Sigma_1$$

but more important we can write the class corresponding to T_γ in $H_1(B\Gamma_\Sigma; \mathbb{Q})$ as being the image of the class in

$$H_0(B\Gamma_{\Sigma_2}; \mathbb{Q}) \otimes \left(H_1(B\Gamma_{\text{cyl}}; \mathbb{Q}) \otimes \bigotimes_{i=1}^{n} H_0(B\Gamma_{\text{cyl}}; \mathbb{Q}) \right) \otimes H_0(B\Gamma_{\Sigma_1}; \mathbb{Q})$$

given by the tensor product of generators. Hence the string operation corresponding to T_γ is nothing but a composition of degree zero string operations with a BV-operator. To be precise it is given by the composition

$$\gimel_*([c_{\Sigma_2}]) \circ (\Delta \otimes id) \circ \gimel_*([c_{\Sigma_1}])$$

where $\gimel_*([c_{\Sigma_i}])$ for $i = 1, 2$ denotes the degree zero string operation coming from the generator of the zero'th homology group of $B\Gamma_{\Sigma_i}$. This argument can be generalized to the following theorem:

Theorem 6.87. *Every degree one string operations can be written as a sum of compositions of degree zero string operations with a BV-operator.*

Proof. Because the mapping class group is generated by Dehn twists, for all open-closed $\mathcal{B}$-labelled cobordisms Σ we have that $H_1(B\Gamma_\Sigma;\mathbb{Q})$ has a basis of Dehn twists on Σ. Hence any degree one string operation can be written as a sum of string operations corresponding to Dehn twists. Applying the previous argument to each of these Dehn twists separately, the theorem follows. $\square$

The previous theorem is an important consequence of the idea behind our slogan and deserves to made into a slogan itself, reflected in figure 6.1:

The degree one operations are TQFT operations together with a BV-operator.

In the next section will also see how the BV-identity can naturally be deduced using similar ideas. We will sharpen the previous result in the section on vanishing results.

4.1.3. *The BV-identity from the Lantern relation.* We are now interested in the properties of Δ. Because the homology of $B\Gamma_{\mathrm{cyl}}$ is concentrated in degrees 0 and 1, the composition Δ^2 must be zero because there is no non-zero class in $H_2(B\Gamma_{\mathrm{cyl}})$ to which it could correspond.

This makes one suspect that Δ might behave like a derivation with respect to the string product. In the next example we take a small look at derivations.

Example 6.88. Derivations come in different orders. In the ungraded case, a first-order derivation is a linear map d which satisfies $d^2 = 0$ and $d(ab) = d(a)\cdot b + a\cdot d(b)$. Think of the derivative $\frac{d}{dx}$ working on products of functions. This also suggests a way to generalize the defining equations of a first-order derivation to higher order derivations. The second derivative $D := \frac{d^2}{dx^2}$ doesn't behave well with respect to products with two terms, giving the following expression:

$$\frac{d^2}{dx^2}(fg) = \frac{d^2 f}{dx^2}g + f\frac{d^2 g}{dx^2} + 2\frac{df}{dx}\frac{dg}{dx}$$

This is not expressible in terms of D again. However, D does behave well with respect to triple products:

$$\begin{aligned}\frac{d^2}{dx^2}(fgh) &= \frac{d^2 f}{dx^2}gh + f\frac{d^2 g}{dx^2}h + fg\frac{d^2 h}{dx^2} + 2\frac{df}{dx}\frac{dg}{dx}h + 2\frac{df}{dx}g\frac{dh}{dx} + 2f\frac{dg}{dx}\frac{dh}{dx}\\ &= \frac{d^2}{dx^2}(fg)h + \frac{d^2}{dx^2}(fh)g + f\frac{d^2}{dx^2}(gh) - \frac{d^2 f}{dx^2}gh - f\frac{d^2 g}{dx^2}h - fg\frac{d^2 h}{dx^2}\end{aligned}$$

This suggests the following definition for a second-order derivation in the ungraded case. It is a map D satisfying $D^2 = 0$ and the identity

$$D(abc) = D(ab)c + D(ac)b + aD(bc) - D(a)bc - aD(b)c - abD(c)$$

In general, n'th-order derivations are defined to be linear maps that square to zero and satisfy an identity as above for n-ary products.

We claim that the BV-operator is such a second-order derivation with respect to the string product in a graded sense. To prove this, we need to look at the lantern relation in the mapping class groups. The lantern relation is a relation between certain Dehn twists in the sphere with four disks removed [**FM10**, section 6.1].

Proposition 6.90. *The Dehn twists around the simple closed curves $a_1,\dots,a_4,x,y,z$ depicted in figure 6.89 satisfy the relation*

$$T_{a_1}T_{a_2}T_{a_3}T_{a_4} = T_xT_yT_z$$

in the mapping class group Γ_0^4 of the sphere with four circles removed. This is called the lantern relation.

Remark 6.91. Note that the ordering the a_i's in the left hand side of the Latern relation doesn't matter, because they are be represented by simple closed curves which are pairwise disjoint. Also note that in the right hand side, only the cyclic ordering matters.

The mapping class group Γ_0^4 is isomorphic to the mapping class group of the sphere with four disks removed, considered as a $\mathcal{B}$-labelled cobordism with three circles incoming and one outgoing. Using the ideas of the previous section, we see that the lantern relation should give a relation between BV-operators applied to different classes. The following proposition makes this precise.

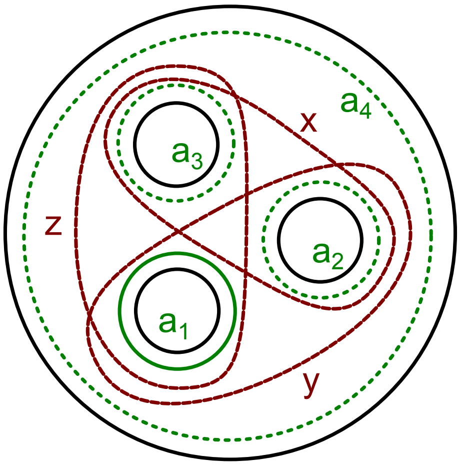


FIGURE 6.89. The curves that play a role in the lantern relation.

Proposition 6.92. *The BV-operator* Δ *satisfies* $\Delta^2 = 0$ *and the BV-identity up to sign with respect to the closed product. In terms of the modified closed product the BV-identity is the following one:*

$$\Delta(a \cdot b \cdot c) = \Delta(a \cdot b) \cdot c + (-1)^{|a|} a \cdot \Delta(b \cdot c) + (-1)^{|a||b|+|b|} b \cdot \Delta(a \cdot c)$$
$$- \Delta(a) \cdot b \cdot c - (-1)^{|a|} a \cdot \Delta(b) \cdot c - (-1)^{|a|+|b|} a \cdot b \cdot \Delta(c)$$

PROOF. The claim that $\Delta^2 = 0$ has already been proven, so let's get the to the BV-identity.

In the sphere with four disks removed, considered as a $\mathcal{B}$-labelled cobordism with three circles incoming and one outgoing, we call a_4 the curve around the outgoing component. The lantern relation implies that the sum of the operations corresponding to the T_{a_i} is the same as the sum of the operations corresponding to T_x, T_y and T_z. The operation corresponding to the Dehn twist T_{a_4} gives the term $\Delta(a \cdot b \cdot c)$ in the BV-identity, the T_{a_i} for $i = 1, 2, 3$ give the last three terms of the right hand side and T_x, T_y and T_z the first three terms of the right hand side.

The only problem are the signs. These can be determined by looking at the differences between the trivialisations of the local system and our formula for the modified closed product. ☐

Similarly there is a reverse BV-identity for the coproduct, by taking the lantern relation for the sphere with four circles, considered as $\mathcal{B}$-labelled cobordism with one incoming boundary circles and three outgoing ones. This doesn't seem to be nearly as useful as the BV-identity, mainly because the coproduct is pretty boring, see 6.80.

4.1.4. *The string bracket.* The string bracket is operation which can be written in terms of the BV-operator and the string product. We will spend some time on it for two reasons: firstly because it is a very natural operation to consider and secondly because it will appear in section 6 of this chapter. The string bracket measures the extent to which the BV-operator fails to be a derivation.

Definition 6.93. The string bracket $[-,-] : H_*(LM, \mathbb{Q}) \otimes H_*(LM; \mathbb{Q}) \to H_{*-d+1}(LM; \mathbb{Q})$ is given by

$$[a, b] = \Delta(a \cdot b) - \Delta(a) \cdot b - (-1)^{|a|} a \cdot \Delta(b)$$

It gives the shifted homology of the loop space LM the structure of a 2-Poisson algebra, see theorem 6.128.

5. Tamanoi's vanishing results and the stable range

In this section we combine our results from the previous sections to prove a result of Tamanoi about the vanishing string topology operations coming from homology classes in the image of the stabilizing map and some other nice vanishing results.

5.1. Vanishing results. We start with a nice selection of vanishing results. That is, we will describe a number of compositions of string operations which induce the zero map in homology. These all depend on the fact that string coproduct is almost trivial, in the sense of proposition 6.80. Apart from this, the proofs consist of manipulations using the HCFT structure. Note that this differs from Tamanoi's original proofs, which depended on explicit calculations.

We start by looking at the interaction of the BV-operator with the coproduct.

Lemma 6.94. *The following two compositions are zero:*

$$(\Delta \otimes id) \circ \Delta_c = 0 = (id \otimes \Delta) \circ \Delta_c$$

PROOF. It is suffices to recall proposition 6.80 together with the fact that the action of S^1 on the constant loop c_0 is trivial. □

We now give our promised sharpening of theorem 6.87.

Proposition 6.95. *If a homology class in $H_1(B\Gamma_\Sigma;\mathbb{Q})$ can be represented by a Dehn twist along a simple closed curve γ which is either (1) non-separating, (2) separates Σ into two components which both contain at least one original outgoing boundary circle, then the corresponding string operation vanishes.*

PROOF. Let U denote a regular neighborhood of γ. In both cases we can find other simple closed curves $\gamma_1, \ldots, \gamma_n$ and η with regular neighborhoods $U_1, \ldots, U_n$ and V such that our cobordism can be written as a composition:

$$\Sigma = \Sigma_2 \circ \left((U \sqcup U_1) \circ \Sigma_0^{1,2} \sqcup \coprod_{i=2}^{n} U_i \right) \Sigma_1$$

where $\Sigma_0^{1,2}$ is a reverse pair of pants with the curves γ and γ_1 as outgoing boundary and the curve η as incoming boundary. This can be proven by cutting Σ along γ and applying our results on normal forms of cobordisms from section 3.2 to the resulting surface(s).

The string operation dorresponding to this decomposition includes the composition of a BV-operator and the closed coproduct and by the previous lemma should therefore be zero. □

There is a similar statement about the vanishing of degree one operations in corollary 6.100, obtained using the homological stability of the mapping class groups. Finally, we will look at the interaction between the closed product and coproduct. It should come as no suprise that there is a vanishing result to be found there. This will be the basis of the results in the next section.

Proposition 6.96. *The following composition is zero:*

$$\mu_c \circ \Delta_c = 0$$

PROOF. The image of Δ_c is $\mathbb{Q}[c_0] \otimes [c_0]$ according to proposition 6.80. The product $\mu_c([c_0] \otimes [c_0]) = [c_0] \cdot [c_0]$ must be zero, because it lies in $H_{-d}(LM;\mathbb{Q})$ which must be the vector space $\{0\}$ by definition. □

This tells us about vanishing of closed degree zero operations.

Corollary 6.97. *Let Σ be a connected $\mathcal{B}$-labelled cobordism, then if the genus of Σ is greater than or equal to* 1*, the degree zero corresponding string operations vanishes.*

PROOF. From the classification of TQFT's, we know in normal form at least one genus unit appears. This must give the zero operation according to the previous proposition. □

Remark 6.98. Tamanoi gives large list of other vanishing compositions of operations in [**Tam08c**].

5.2. String operations from the stable range are zero. In this section we will combine the vanishing result given in proposition 6.96 with the stability as reflected in lemma 6.23. From this we will derive that all operations coming from homology classes in the stable range are zero. For this we will use theorem 6.21 in the guise of lemma 6.23 to show that in a certain range an operation can be replaced by the composition of that operation with a closed product and coproduct.

Proposition 6.99. *Let Ψ be an operation of degree k coming from a connected $\mathcal{B}$-labelled cobordism of genus g having at least one incoming or outgoing boundary circle. Then Ψ is identically zero if $k \leq \frac{2}{3}g + \frac{1}{3}$.*

Proof. In the stated range both $(\alpha_g)_*$ and $(\beta_g)_*$ are isomorphisms. This means that Ψ is the same operation as the composition of Ψ with the operations obtained by glueing first a pair of pants along a single boundary circle and then a pair of pants along the two boundary circles of the previous one.

This means that if there is an incoming boundary circle then we can write

$$\Psi = \Psi \circ (\mu_c \circ \Delta_c \otimes id)$$

which must be zero due to proposition 6.96. If there is an outgoing boundary circle then similarly we can write $\Psi = (\mu_c \circ \Delta_c \otimes id) \circ \Psi = 0$. □

This deserves a slogan which appears in figure 6.1:

Operations from the stable range are zero.

Corollary 6.100. *Operations of degree 1 are zero for connected $\mathcal{B}$-labelled cobordism of genus $g \geq 1$.*

Proof. For $g \geq 1$ the bound of the previous proposition is satisfied for degree 1 operations. □

6. The genus zero operad

In this section we discuss the operations coming from genus zero cobordisms by looking at the operad obtained by restricting the prop to those cobordisms having a single outgoing boundary circle and no free boundary. This leads us to the well-known BV-algebra structure in string topology.

6.1. An operad in the moduli space of genus zero Riemann surfaces. Let D_n for $n \geq 1$ be a n-punctured disk. Then we claim that the mapping class group Γ_{Σ_n} is isomorphic to PB_n, the pure braid group on n-strands. We start with the definition of the pure braid group group. Although there are definitions in terms of generators and relations, the simplest one is geometric.

Definition 6.101. Let $C(\mathbb{C}, n)$ be the configuration space of n-ordered points in $\mathbb{C} \cong \mathbb{R}^2$. Let ζ be $e^{2\pi i/n}$ and $\xi = \frac{1}{2}(1, \zeta, \zeta^2, \ldots, \zeta^{n-1})$. Then $PB_n = \pi_1(C(\mathbb{C}, 2), \xi)$.

Remark 6.102. The choice of basepoint ξ is not important, because any other choice will give an isomorphic group. However, this choice turns out to be a convenient one for comparison with the mapping class group.

One can find generators and relations for PB_n, but for us it suffices to note that using Seifert-van Kampen we can conclude PB_n is generated by homotopy classes of based loops γ_{ij} for $1 \leq i < j \leq n$. These loops can be obtained by letting the i'th point circle around the j'th point in counterclockwise direction.

Let's get back to the mapping class group. The n-punctured disk D_n can concretely be represented by $D \backslash \xi$, where $D = \{z \in \mathbb{C} | \, |z| \leq 1\}$ is the unit disk. There exists a natural map $F : \Gamma_{D_n} \to PB_n$. To define it, consider the isotopy $\phi_t : \mathbb{C} \to \mathbb{C}$ for $t \in [0, 1]$ given by $x \mapsto (1 - \frac{1}{2}t)x$. We define F by:

$$F(g)(t) = \phi_{1-t}^{-1} g \phi_t(\xi)$$

where g is extended by the identity outside the disk. This extension is well-defined because g is the identity on the boundary. Because g fixes the boundary and the punctures, we note that this is indeed a loop. An isotopy of orientation-preserving self-homeomorphisms induces a homotopy of loops, so the homotopy class of $F(g)$ is well-defined.

Proposition 6.103. *$F : \Gamma_{D_n} \to PB_n$ is an isomorphism of groups for $n \geq 1$.*

Proof. It is clearly a homomorphism, because the concatenation $F(g) * F(g')$ is homotopic to $F(g'g)$ by the homotopy

$$s \mapsto (\phi_{1-t}g'\phi_s t(\xi)) * (\phi_{1-st}g\phi_t(\xi))$$

for $s \in [0, 1]$.

To show that it is injective, let $L_1, \ldots, L_n$ be the isotopy classes of paths produced by $\phi_t(\xi)$ for $t \in [0, 1]$. Then we note that from $F(g)$ we can reconstruct the image of the $L_1, \ldots, L_n$ under g up to isotopy. Because $D\backslash\{L_1, \ldots, L_n\}$ is simply connected and connected, this determines g completely by Alexander's principle.

To see that F is surjective, it suffices to see that the classes γ_{ij} are in the image of F. But these classes are produced by the Dehn twist along a loop around the i'th and j'th point. □

As a corollary, we see that because PB_n is generated by the loops γ_{ij}, Γ_{D_n} is generated by the Dehn twists around loops around the i'th and j'th point which we denote by S_{ij}.

Now we blow up the punctures of D_n to circles, thereby obtaining a surface with boundary Σ^n. It is a disk with n disks removed or equivalently a sphere with $n + 1$ disks removed. This blowing up only adds Dehn twists along these boundary components to the mapping class group, as proven in the next proposition. To pin down the exact group, we make the following definition.

Definition 6.104. PRB_n, the pure ribbon braid group, is the product $PB_n \times \mathbb{Z}^n$.

Remark 6.105. This definition makes more sense in the context of the total braid group B_n, defined by $\pi_1(C(\mathbb{R}^2, n)/\Sigma_n, [\xi])$, where the symmetric group Σ_n acts on $C(\mathbb{R}^2, n)$ by permuting the component of a configuration. It is clear by covering space theory that PB_n is the kernel of the homomorphism $B_n \to \Sigma_n$ given by the monodromy. Hence we get a short exact sequence of groups:

$$0 \to PB_n \to B_n \to \Sigma_n \to 0$$

We note that $C(\mathbb{R}^2, n)$ is homotopy equivalent to the configuration space of embedded disks in $\mathbb{R}^2$. If we add a framing of this disks, the fundamental group of the space obtained this way is known as the ribbon braid group RB_n. It can be seen to be isomorphic to the semidirect product $B_n \rtimes \mathbb{Z}^n$, where the semidirect product is given by the action of $\Sigma(n)$ on $\mathbb{Z}^n$. Again we have a map $RB_n \to \Sigma_n$ and PRB_n is in fact the kernel of this map.

Proposition 6.106. *Γ_{Σ^n} is isomorphic to PRB_n.*

Proof. By the exact sequence of proposition 6.11 blowing up the points to disk gives a short exact sequence of groups

$$0 \to \mathbb{Z}^n \to \Gamma\Sigma^n \to \Gamma_{D_n} \to 0$$

where the $\mathbb{Z}^n$ map to the product of Dehn twists around the boundary components. It therefore suffices to show that these Dehn twists commute with the S_{ij}. But this is clear, because the simple closed curves defining the Dehn twists along S_{ij} and the boundary components can be taken to be disjoint. □

The prop structure of $\mathsf{Bord}^+_{\mathcal{B}}$ by general theory restricts to an operad when we just look at the spaces with n incoming boundary circles for $n \geq 0$ and 1 outgoing boundary circles. Furthermore, the spaces $\mathsf{Bord}^+_{\mathcal{B}}(n, 1)$ obtained this way are a disjoint union of classifying space $B\Gamma_\Sigma$ for isomorphism classes Σ of $\mathcal{B}$-labelled cobordisms with the correct incoming and outgoing boundary. In particular, it is a disjoint union of spaces $(\mathsf{Bord}^+_{\mathcal{B}}(n, 1))_{g,m,r}$ of surfaces with m connected components whose individual values of the genus sum to g and r purely free boundary components.

The composition in the prop is given by glueing of surfaces. Because a glueing of connected components is connected, the spaces with $m = 1$ are stable under composition. Because genus is additive, the spaces with $g = 0$ are stable. Finally, because glueing cobordisms with no free boundary gives again a cobordism with no free boundary, the spaces with $r = 0$ are stable. Therefore, we can make the following defininition.

Definition 6.107. The operad $\mathcal{O}_0$ is the operad in Top given by $\mathcal{O}_0(n) = (\mathsf{Bord}^+_{\mathcal{B}}(n,1))_{0,1,0}$ with composition induced by the prop. We call it the topological genus zero operad.

The operad $\mathcal{H}^M_{\mathcal{B},0}$ is the operad in $\mathsf{GrVect}_{\mathbb{Q}}$ given by $\mathcal{H}^M_{\mathcal{B},0}(n) = H_*((\mathsf{Bord}^+_{\mathcal{B}}(n,1))_{0,1,0}; \mathcal{L}^M_{\mathcal{B}})$ with composition induced by the prop. We call it the homological genus zero operad for set of branes $\mathcal{B}$.

As an intermediate we define the operad $\mathcal{H}^0_0$ given by $\mathcal{H}^0_0(n) = H_*((\mathsf{Bord}^+_{\mathcal{B}}(n,1))_{0,1,0}; \mathbb{Q})$. We call this operad the degree zero homological genus zero operad.

We will see that these operads control essentially all string topology operations coming from genus zero surfaces with no free boundary. In the next section, we describe the specific structure this operad gives on $H_*(LM;\mathbb{Q})$; that is, we will calculate the degree 0 homological genus zero operad.

6.2. The degree 0 homological genus zero operad. The first step would of course be to calculate the homology of $\mathcal{O}_0$ and the maps induced by the operad structure to get a full description of $\mathcal{H}^0_0$. To do this, we shall will relate the $\mathcal{O}_0(n)$ to the framed little disks operad and exhibit its homology.

To do this, we will first define the little disks operad and first show it is homotopy equivalent to $\mathcal{O}_0$ as an operad. This is not as easy as it seems: a direct proof would require explicitly exhibiting maps compatible with the operad structure. Hence we take a detour through a recognition theorem. This reduces the proof to showing that the framed little disks operad is a $K\mathcal{E}_{RB}$-operad, i.e. the operad structure lifts to a ribbon braided operad structure on the universal cover, which is contractible and has free action of the ribbon braid groups. We also think this is of independent interest, because as far as we know this theory doesn't appear in the literature.

6.2.1. *Recognition theorems for topological operads.* To state and prove our recognition theorem, we generalize operads to allow different group actions than the symmetric groups. Sequences of groups that could give the group actions for these generalized operads will be called equivariance systems.

Definition 6.108. A pre-equivariance system $\mathcal{E}$ is a non-symmetric operad in the category Set, i.e. an operad without any group actions or equivariance conditions. We denote its components by E_n and the composition by χ. We require that each E_n is a group with a group homomorphism $p : E_n \to \Sigma_n$ and the composition is a group homomorphism $\chi : E_n \ltimes \prod_{i=1}^n E_{k_i} \to E_{\sum_{i=1}^n k_i}$, where E_n acts via Σ_n the symmetric group on the components of the product. There are two maps that we want to give a special name:

Strand splitting map: This map is given applying

$$\chi : E_n \ltimes \prod_{i=1}^n E_{k_i} \to E_{\sum_{i=1}^n k_i}$$

to $(\sigma, (e_{k_1}, \ldots, e_{k_n}))$, where e_{k_i} denotes the identity element of E_{k_i}. We denote it by $\alpha : E_n \to E_{\sum_{i=1}^n k_i}$ and it is a group homomorphism

Disjoint union map: This map is given applying

$$\chi : E_n \ltimes \prod_{i=1}^n E_{k_i} \to E_{\sum_{i=1}^n k_i}$$

to $(e_n, (\sigma_1, \ldots, \sigma_n))$. We denote it by $\beta : \prod_{i=1}^n E_{k_i} \to E_{\sum_{i=1}^n k_i}$ and it is a group homomorphism as well.

A morphism of pre-equivariance systems is a morphism of non-symmetric operads in Set such that each of the components is a group homomorphism. This gives a category PEqSys of equivariance systems.

Remark 6.109. Unfortunately, there doesn't seem to a simpler way to write down this definition. At first one might naively hope that a pre-equivariance is a non-symmetric operad in the category Grp of groups, but for the major examples the composition maps χ are not homomorphisms.

We will now treat an example of a pre-equivariance system which we will need to continue with our definitions.

Example 6.110. There is one very important pre-equivariance system, which we denote by $\mathcal{E}_\Sigma$. It is the pre-equivariance system of symmetric groups and has the symmetric group Σ_n has n'th component. The non-symmetric operad structure is given by defining $\chi(\sigma, (\tau_1, \ldots, \tau_n))$ for $\sigma \in \Sigma_n$ and $\sigma_i \in \Sigma_{k_i}$ as follows:

- Let $\alpha(\sigma)$ be the block permutation associated to σ on the set $\{1, \ldots, \sum_{i=1}^n k_i\}$ given by making it permute the blocks $\{\sum_{i=1}^j k_i + 1, \ldots, \sum_{i=1}^{j+1} k_i\}$.
- Let $\beta(\tau_1, \ldots, \tau_n)$ be the permutation of $\{1, \ldots, \sum_{i=1}^n k_i\}$ given by putting the individual permutations side-by-side such that τ_j acts on the set $\{\sum_{i=1}^{j-1} k_i + 1, \ldots, \sum_{i=1}^j\}$.

We then define the operad maps by

$$\chi(\sigma, (\tau_1, \ldots, \tau_n)) = \alpha(\sigma)\beta(\tau_1, \ldots, \tau_n)$$

The notion was chosen suggestively: one can check that indeed the α and β defined here are the strand splitting and disjoint union maps. Hopefully the drawings in example 6.113 make clear why these maps were named in this way.

Definition 6.111. An equivariance system is a pre-equivariance system $\mathcal{E}$ together with a morphism $p : \mathcal{E} \to \mathcal{E}_\Sigma$ of pre-equivariance systems. The maps $p : E_n \to \Sigma_n$ are called the permutation maps and should coincide with those given in the definition of a pre-equivariance system.

A morphism of equivariance systems is a morphism in the category $\mathsf{PEqSys}/\mathcal{E}_\Sigma$. Hence we obtain a category EqSys of equivariance systems.

Remark 6.112. There is no reason to limit oneself to equivariance systems in Set, like we have done. The definition can be amended for group objects in C instead of groups. The only thing one needs to think about is what happens to $\mathcal{E}_\Sigma$, but if C admits arbitrary coproducts, one can make sense of the symmetric groups internally in C by making $\coprod_{n!} 1$ into a group object.

Example 6.113. There are several important examples of the equivariance systems. We will later see that these give types of operads that appear naturally.

Trivial equivariance system: We can define an equivariance system $\mathcal{E}_*$ with each component the trivial group. Because this has one element in each component there is only one choice for the operad structure. The map to $\mathcal{E}_\Sigma$ are uniquely defined by the fact that all maps must be homomorphisms: the single element of the n'th component should be mapped to the identity of Σ_n.

Symmetric groups: Of course the identity homomorphisms $\Sigma_n \to \Sigma_n$ give a map of pre-equivariance systems from $\mathcal{E}_\Sigma$ to $\mathcal{E}_\Sigma$. It shows that the symmetric groups indeed form an equivariance system.

We want to spend a bit of time explaining how to visualize all the data of an equivariance system. To each permutation $\sigma \in \Sigma_n$ we can assign a drawing given by putting n ordered points at the top, n ordered points at the bottom and drawing a line between a point i at the top and a point j at the bottom if $\sigma(i) = j$. The composition is then giving by putting one of these drawings on top of another one.

The strand splitting maps α are given by splitting each line in the drawing with a number of parallel lines. The disjoint union maps β are given by putting the drawings horizontally next to each other.

Braid groups: Heuristically, each way to refine the permutation groups should give an equivariance system. The first example of this is given by the braid groups B_n. Together they form an equivariance system $\mathcal{E}_B$ with n'th component the group B_n.

Each element of a braid group can be visualized by a braid, with strands going from points at the top to points at the bottom and then composition is simply attaching the bottom of one braid to the top of another. The strand splitting maps α are given by literally splitting each strand into a number of parallel strands and the disjoint union maps β are given by putting a number of braids next to each other. Finally, the permutation map $p : B_n \to \Sigma_n$ is the induced permutation of the strands of the braid obtained by following each strand from top to bottom.

Ribbon braid groups: The equivariance system associated to the ribbon braid groups is similar to the one associated to the braid group. Recall that the ribbon braid groups are given by $RB_n = B_n \ltimes \mathbb{Z}^n$ and should be visualized as coming from braids where each strand is given a finite thickness and hence is a ribbon: the $\mathbb{Z}$'s keep track of the number of twists in each ribbon. Using the exactly the same definitions for the composition, α, β and p as for the braid groups now gives the ribbon braid groups the structure of an equivariance system. We denote it $\mathcal{E}_{RB}$.

Note that by forgetting the thickness of the ribbons or, equivalently, projecting away the factor $\mathbb{Z}^n$, we obtain a map $RB_n \to B_n$. These maps together from a map morphism $\mathcal{E}_{RB} \to \mathcal{E}_B$.

Next we will define the notion of an operad over an equivariance system $\mathcal{E}$. To do this, recall the definition of an operad as given in definition B.74. There we first defined symmetric sequences, which will be replaced by $\mathcal{E}$-sequences in this section.

Definition 6.114. Let C be a category, then a $\mathcal{E}$-sequence P is a set of functors $E_n \to \mathsf{C}$, where the group E_n is considered as a category with a single object. We denote the image of this single object as $P(n)$.

A morphism of $\mathcal{E}$-sequence is a set of natural transformations and this defines a category $\mathcal{E}\mathsf{Seq}$ of $\mathcal{E}$-sequences.

Using this, we can define the notion of a $\mathcal{E}$-operad by introducing a composition which satisfies the correct axioms.

Definition 6.115. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category, then a $\mathcal{E}$-operad is a $\mathcal{E}$-sequence $\mathcal{O}$ together with a set of composition maps $\chi : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \ldots \otimes \mathcal{O}(k_n) \to \mathcal{O}(\sum_{i=1}^n k_i)$ and a unit map $u : 1 \to \mathcal{O}(1)$ satisfying the following axioms:

Associativity and unit: These are the same as for an ordinary operad, see definition B.74.

$\mathcal{E}$-Equivariance: The composition maps χ should be equivariant in the sense that the following two diagrams commute for all $\sigma \in E_n$ and $(\tau_1, \ldots, \tau_n) \in \prod_{i=1}^n E_{k_i}$:

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \ldots \otimes \mathcal{O}(k_n) \arrow[r, "\sigma\otimes p(\sigma)"] \arrow[d, "\chi"'] & \mathcal{O}(n) \otimes \mathcal{O}(k_{\sigma(1)}) \otimes \ldots \otimes \mathcal{O}(k_{\sigma(n)}) \arrow[d, "\chi"] \\
\mathcal{O}(\sum_{i=1}^n k_i) \arrow[r, "\alpha(\sigma)"'] & \mathcal{O}(\sum_{i=1}^n k_i)
\end{tikzcd}$$

where $p(\sigma) \in \Sigma_n$ acts by the symmetry of the symmetric monoidal category.

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \mathcal{O}(k_n) \arrow[r, "id\otimes\tau_1\otimes\ldots\otimes\tau_n"] \arrow[d, "\chi"'] & \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \mathcal{O}(k_n) \arrow[d, "\chi"] \\
\mathcal{O}(\sum_{i=1}^n k_i) \arrow[r, "\beta(\tau_1,\ldots,\tau_n)"'] & \mathcal{O}(\sum_{i=1}^n k_i)
\end{tikzcd}$$

Remark 6.116. We conclude that equivariance systems over $\mathcal{E}_\Sigma$ give operads with different equivariance conditions. There are generalizations of this construction. For example, if C is not symmetric monoidal but braided monoidal, then it naturally supports a notion of braided operad. Then alternative equivariance conditions for braided operads can be constructed by considering equivariance systems over $\mathcal{E}_B$.

For each of the equivariance systems described above, we will give the corresponding notion of operad.

Example 6.117. The trivial equivariance system imposes no equivariance conditions at all. In this case we thus recover the notion of a non-symmetric operad. For the equivariance system $\mathcal{E}_\Sigma$, we simply get the usual equivariance conditions using the symmetric groups. In other words, we recover the notion of an ordinary (symmetric) operad.

The operads corresponding to the equivariance systems $\mathcal{E}_B$ and $\mathcal{E}_{RB}$ were considered by Fiedorowicz [**Fie99**] and Wahl [**Wah01**] respectively. They are called braided and ribbon braided operads and are closely related to the little disks and framed little disks operads to be considered later in this section.

Operads for different equivariance systems are related to each other if there exist maps between the equivariance systems. Clearly, if $\mathcal{E} \to \mathcal{F}$ is any morphism of equivariance systems, this allows us to see a $\mathcal{F}$-operad as a $\mathcal{E}$-operad.

More interestingly, if $\mathcal{E} \to \mathcal{F}$ is an epimorphism, i.e. surjective in each degree, then we can make a $\mathcal{F}$-operad $\mathcal{O}/K$ out of a $\mathcal{E}$-operad $\mathcal{O}$. Consider the kernels K_n of the maps $E_n \to F_n$. Then $E_n/K_n \cong F_n$ and therefore if we take the quotient of $\mathcal{O}(n)$ by K_n we get a space on which the original E_n-action induces an action of F_n. The equivariance of the composition means that there is an induced $\mathcal{F}$-operad structure on these spaces. The resulting operad is $\mathcal{O}/K$.

There is a simple method to make a $\mathcal{E}$-operad in topological spaces. From the standard method to construct product-preserving classifying space functor $B : \mathsf{Grp} \to \mathsf{Top}$, one also gets an associated product-preserving universal cover functor $E : \mathsf{Grp} \to \mathsf{Top}$ which produces a contractible space with free G-action for each group G.

Thus, using this functor one can make a topological operad $E\mathcal{E}$. It has the property that each component is contractible and the E_n-action is free. We give a special name to the class of $\mathcal{E}$-operads with those properties.

Definition 6.118. A $\mathcal{E}^\infty$-operad is a topological $\mathcal{E}$-operad such that the n'th component is contractible and has free E_n-action.

There is a simple relation between the non-symmetric operad $B\mathcal{E}$ and the $\mathcal{E}^\infty$-operad $E\mathcal{E}$: the latter is the universal cover of the other and if we use the canonical morphism of equivariance systems $\mathcal{E} \to \mathcal{E}_*$ to make a non-symmetric operad out of $E\mathcal{E}$ we exactly get $B\mathcal{E}$ back again.

Definition 6.119. Let $\mathcal{E} \to \mathcal{F}$ be an epimorphism of equivariance systems. Then a $\mathcal{E}$-operad structure of $\mathcal{O}$ is said to extend a $\mathcal{F}$-operad structure on $\mathcal{P}$ to a $\mathcal{E}$-operad structure if there is an isomorphism of operads between $\mathcal{O}/K$ and $\mathcal{P}$.

We can rephrase the previous remarks as saying that the $\mathcal{E}_*$-operad structure on $B\mathcal{E}$ can be extended to a $\mathcal{E}$-operad structure on its universal cover $E\mathcal{E}$. This universal cover operad has the property that each component is contractible with free E_n-action. We introduce a name for a generalisation of this property.

Definition 6.120. Let $\mathcal{E} \to \mathcal{F}$ be an epimorphism of equivariance systems. An $\mathcal{F}$-operad $\mathcal{O}$ is said to be a $K\mathcal{E}$-operad if its universal cover $\tilde{\mathcal{O}}$ has a $\mathcal{E}$-operad structure extending the $\mathcal{F}$-operad structure such that $\tilde{\mathcal{O}}$ is a $\mathcal{E}^\infty$-operad.

The next theorem will show that this property exactly characterizes operads which are homotopy equivalent to a cover of $B\mathcal{E}$ as an operad.

Theorem 6.121. *Let $\mathcal{E} \to \mathcal{F}$ be an epimorphism of equivariance systems and $\mathcal{O}$ be a $\mathcal{F}$-operad. $\mathcal{O}$ is a $K\mathcal{E}$-operad if and only if $E\mathcal{E}/K$ and $\mathcal{O}$ are weakly equivalent, i.e. there is a zigzag of weak homotopy equivalences between them.*

Proof. **($\Rightarrow$):** As before, let K_n denote the kernel of the map $E_n \to F_n$. Then $E_n/K_n \cong F_n$ and hence the quotient of the n'th component of a $\mathcal{E}$-operad by F_n gives a $\mathcal{F}$-operad. If we apply this to $E\mathcal{E}$ and $\tilde{\mathcal{O}}$ we quotient out all deck transformations, hence get $E\mathcal{E}/K$

and the original $\mathcal{F}$-operad $\mathcal{O}$. Now consider the following diagram of operads

$$\begin{tikzcd}
\tilde{\mathcal{O}} \arrow[d] & \tilde{\mathcal{O}} \times E\mathcal{E} \arrow[l, "\pi_1"', "\simeq"] \arrow[r, "\pi_2", "\simeq"'] \arrow[d] & E\mathcal{E} \arrow[d] \\
\mathcal{O} = \tilde{\mathcal{O}}/K & \tilde{\mathcal{O}} \times_K E\mathcal{E} \arrow[l, "\simeq"'] \arrow[r, "\simeq"'] & E\mathcal{E}/K
\end{tikzcd}$$

where the top row consists of $\mathcal{E}$-operads and the bottom row of $\mathcal{F}$-operads. Both maps on the top row are homotopy equivalences, because they are maps between contractible spaces. Because they are E_n-equivariant and hence K_n-equivariant, the induced maps on the bottom rows are also homotopy equivalences.

($\Leftarrow$): For the converse, the Boardman-Vogt cofibrant resolution of topological operads gives us a homotopy equivalence $W(E\mathcal{E}/K) \to \mathcal{O}$. Note that inside $E\mathcal{E}/K(n)$ there is a canonical basepoint e/K_n by taking $e \in E_n$ the identity. These basepoints form a non-symmetric operad and by functoriality of W we get maps $W(e/K) \to W(E\mathcal{E}/K) \to \mathcal{O}$, where $W(e/K)(n)$ is contractible for all $n \in \mathbb{N}$. We will use these to produce basepoints for lift of the operad structure.

In the following diagram we have to construct a lift $\tilde{\gamma}$:

$$\begin{tikzcd}
\tilde{\mathcal{O}}(n) \times \tilde{\mathcal{O}}(k_1) \times \ldots \times \tilde{\mathcal{O}}(k_n) \arrow[r, dotted, "\tilde{\gamma}"] \arrow[d] & \tilde{\mathcal{O}}(\sum_{i=1}^n k_i) \arrow[d] \\
\mathcal{O}(n) \times \mathcal{O}(k_1) \times \ldots \times \mathcal{O}(k_n) \arrow[r, "\gamma"'] & \mathcal{O}(\sum_{i=1}^n k_i)
\end{tikzcd}$$

To do this, we pick for each $n \in \mathbb{N}$ one lift $\tilde{W}(e/K)(n)$ of $W(e/K)(n) \subset \mathcal{O}(n)$ to $\tilde{\mathcal{O}}(n)$ and demand that $\tilde{\gamma}$ sends $\tilde{W}(e/K)(n) \times \tilde{W}(e/K)(k_1) \times \ldots \times \tilde{W}(e/K)(k_n)$ to $\tilde{W}(e/K)(\sum_{i=1}^n k_i)$. This completely defines $\tilde{\gamma}$. It is not hard to see that the associativity axiom holds due to the uniqueness of lifts after specifying basepoints which should be mapped to each other: the map $\tilde{\gamma} \circ (\tilde{\gamma})^{\times n}$ is a map lifting $\gamma \circ \gamma^{\times n}$ which maps the basepoints to the correct basepoints. Hence it must coincides with the lift of $\gamma \circ \gamma^{\times n}$ that the construction produces.

To construct the action of E_n on $\tilde{\mathcal{O}}(n)$ we use a similar method. For $g \in E_n$, consider the path α_g in $\mathcal{O}(n)$ from any point x_0 in the image of $W(e/K)(n)$ in $\mathcal{O}(n)$ to $(gK)x_0$. This has a unique lift $\tilde{\alpha}$ to $\tilde{\mathcal{O}}(n)$ such that the lift starts at a point $\tilde{x}_0 \in \tilde{W}(e/K)(n)$. Then g acts on $\tilde{\mathcal{O}}(n)$ by the unique lift of the map $(gK) \circ \pi : \tilde{\mathcal{O}}(n) \to \mathcal{O}(n)$ which sends $\tilde{x}_0$ to $\tilde{\alpha}(1)$.

$\square$

The Salvatore-Wahl recognition theorem and Fiedorowicz recognition theorems are special cases of the previous theorem. For example, if $\mathcal{F} = \mathcal{E}_\Sigma$ and $\mathcal{E} = \mathcal{E}_B$ with the canonical map to $\mathcal{E}_\Sigma$, then this theorem implies the following: any $K\mathcal{E}_B$-opeard has a zigzag of weak equivalences to the classifying space operad $B(PB_n)$ which is homotopy equivalent to $E\mathcal{E}_B/K$. By showing that the little disks operad $\mathcal{D}_2$ is $K\mathcal{E}_B$, which we will do in proposition 6.127, Fiedorowicz proved a characterization of operads which admit a zigzag of weak equivalences with the little disks operad.

Corollary 6.122 (Fiedorowicz)**.** *An operad $\mathcal{O}$ is weakly equivalent to $\mathcal{D}_2$ if and only if it is a $K\mathcal{E}_B$-operad.*

By applying the same reasoning to $\mathcal{F} = \mathcal{E}_\Sigma$ and $\mathcal{E} = \mathcal{E}_{RB}$ and showing the framed little disks operad $f\mathcal{D}_2$ is $K\mathcal{E}_{RB}$, which we will do in corollary 6.137, one obtains a similar characterization of operads which admit a zigzag of weak equivalence with the framed little disks operad.

Corollary 6.123 (Salavatore-Wahl)**.** *An operad $\mathcal{O}$ is weakly equivalent to $f\mathcal{D}_2$ if and only if it is a $K\mathcal{E}_{RB}$-operad.*

We need a slight variation of our the Salvatore-Wahl theorem, because for us the main operad of interests is not $f\mathcal{D}_2$, but $\mathcal{O}_0$.

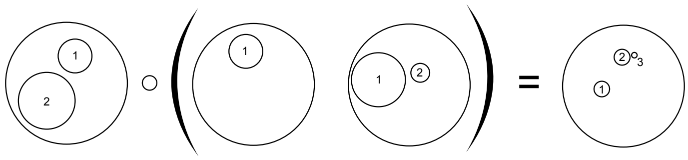


FIGURE 6.126. An example of a composition $\chi : \mathcal{D}_2(2) \otimes \mathcal{D}_2(1) \otimes \mathcal{D}_2(2) \to \mathcal{D}_2(3)$.

Corollary 6.124. *An operad $\mathcal{P}$ is weakly equivalent to the genus zero operad $\mathcal{O}_0$ if and only if it is a $K\mathcal{E}_{RB}$-operad.*

6.2.2. *Little disks and its homology.* Using corollary 6.124 we conclude that to calculate the homology of the degree BV-operad $\mathcal{H}_0^0$ it suffices to find an $K\mathcal{E}_{RB}$-operad, i.e. an operad with contractible universal cover which it is itself a $\mathcal{E}_{RB}$-operad with free action of the ribbon braid groups. Preferably this should have a homology which is not that hard to compute. Our choice for such an operad will be the operad of framed little disks.

For the remainder of this section we will first define the little disks operad and show it is is a $K\mathcal{E}_B$-operad. After that we will compute its homology operad. Then we introduce the framings to obtain the framed little disks operad, show it is a $K\mathcal{E}_{RB}$-operad and compute its homology operad. In the end, we will find out what the string topology operations coming from genus zero are. Let D^2 denote the unit disk.

Definition 6.125. The little disks operad $\mathcal{D}_2$ has as n'th component the space $\mathcal{D}_2(n)$ of embeddings $\coprod_{i=1}^n D_i^2 \hookrightarrow D^2$ where each individual component is a composition of scaling and translation. This space has the subspace topology coming from the compact-open topology on the space of all maps from $\coprod_{i=1}^n D_i^2$ to D^2.

The composition maps γ are given by composition of the embeddings and the symmetric groups act by permuting the labels of the disks.

We have drawn several elements of the little disks operad and their composition in figure 6.126. There is nothing special in this definition about the fact that the disks are 2-dimensional. The analogous construction for arbitrary d gives an operad known as $\mathcal{D}_d$, the little d-disks operad.

Proposition 6.127. *The little disks operad $\mathcal{D}_2$ is a $K\mathcal{E}_B$-operad.*

PROOF. By shrinking the disks, it is easy to see that $\mathcal{D}_2(n)$ is homotopy equivalent to the configuration space $C(D^2, n)$ of n numbered points in the open disk D^2.

To show that its universal cover is contractible, it suffices to prove that $\pi_k(C(D^2, n))$ vanishes for $k \geq 2$. We will do a bit more and prove that $\pi_k(C(D^2 \backslash \{x_1, \ldots, x_l\}, n))$ vanishes for $k \geq 2$, where $x_1, \ldots, x_l$ are l distinct points. Since $C(D^2 \backslash \{x_1, \ldots, x_l\}, 1) \simeq \vee_l S^1$, the case $n = 1$ is clear. Let's do an induction over n. Consider the fibration

$$C(D^2 \backslash \{x_1, \ldots, x_l, x_{l+1}\}, n-1) \to C(D^2 \backslash \{x_1, \ldots, x_l\}, n) \to D^2 \backslash \{x_1, \ldots, x_l\}$$

The π_k for $k \geq 2$ of the middle term lies between $\pi_k(C(D^2 \backslash \{x_1, \ldots, x_l, x_{l+1}\}, n-1))$, which is zero by the induction hypothesis, and $\pi_k(D^2 \backslash \{x_1, \ldots, x_l\})$, which is zero since this space is homotopy equivalent to a bouquet of circles.

Finally, we need to describe a lifting of the operad structure to a braided operad structure on the universal cover. To do this, we use essentially the same techniques to lift operad structures as used in the proof of the general recognition theorem. In other words, we pick certain basepoints in the universal cover and create the operad maps by picking those lifts to the universal covers which send basepoints to basepoints.

To do this, consider the connected contractible subspace $D_0(n)$ in $\mathcal{D}_2(n)$ obtained by making disks out of n segments on the equator and labelling these from left to right. This lifts to a disjoint union of contractible subspaces in the universal cover $\widetilde{\mathcal{D}}_2(n)$. For each $n \in \mathbb{N}$, pick one of these components $\tilde{D}_0(n)$. We define the lifted composition map $\tilde{\gamma}$ by saying it is in the unique map in the diagram

$$\begin{tikzcd} \tilde{\mathcal{D}}_2(n) \times \tilde{\mathcal{D}}_2(k_1) \times \dots \tilde{\mathcal{D}}_2(k_n) \arrow[r, dotted, "\tilde{\gamma}"] \arrow[d] & \tilde{\mathcal{D}}_2(\sum_{i=1}^n k_1) \arrow[d] \\ \mathcal{D}^2(n) \times \mathcal{D}_2(k_1) \times \dots \times \mathcal{D}_2(k_n) \arrow[r, "\gamma"'] & \mathcal{D}_2(\sum_{i=1}^n (k_i)) \end{tikzcd}$$

which sends $\tilde{D}_0(n) \times \tilde{D}_0(k_1) \times \ldots \times \tilde{D}_0(k_n)$ to $\tilde{D}_0(\sum_{i=1}^n k_i)$. Because $\tilde{D}_0(n)$ is connected and contractible, this defines $\tilde{\gamma}$ completely and associativity holds as in the proof of the general recognition theorem.

To construct the action of B_n on $\tilde{\mathcal{O}}(n)$ we lift certain loops. For any point $x_0 \in D_0(n)$, let α_i be the loop in $\mathcal{D}^2(n)$ which rotates the i'th and $(i+1)$'th intervals amongst each other. This path lifts to a path $\tilde{\alpha}_i$ in $\tilde{\mathcal{D}}_2(n)$ if we specify that it starts at the unique lift $\tilde{x}_0 \in \tilde{D}_0(n)$. We let σ_i act on $\tilde{\mathcal{D}}_2(n)$ as follows. It is the unique lift in the diagram

$$\begin{tikzcd} \tilde{\mathcal{D}}_2(n) \arrow[r, dotted, "\sigma_i"] \arrow[d] & \tilde{\mathcal{D}}_2(n) \arrow[d] \\ \mathcal{D}_2(n) \arrow[r, "p(\sigma_i)"'] & \mathcal{D}_2(n) \end{tikzcd}$$

which sends $\tilde{x}_0$ to $\tilde{\alpha}_i(1)$. That this action is free is a consequence of the fact that the action of the symmetric group on $\mathcal{D}_2(n)$ is free and that the loops α_i exactly generate the fundamental group, which by definition is PB_n. □

We will not give a complete proof of the homology of this operad, because take would take too much space. Instead, we refer to a well-written exposition by Sinha [**Sin10**]. There one finds the following result:

Theorem 6.128. *The homology operad associated to $\mathcal{D}_2$ is the (2-)Poisson operad. This is the operad whose algebras are Poisson algebras with a product $\cdot$ of degree 0 and a bracket $[-,-]$ of degree 1. These satisfy the following relations:*

Graded commutativity: $a \cdot b = (-1)^{|a||b|} b \cdot a$,
Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
Derivation: *the bracket is a derivation of the product:* $[a, b \cdot c] = (-1)^{|a||b|} b \cdot [a, c] + [a, b] \cdot c$,
Anti-symmetry: *we have that* $[a, b] = (-1)^{|a||b|}[b, a]$,
Jacobi identity: *and* $[[a, b], c] + [[b, c], a] + [[c, a], b] = 0$.

Sketch of proof. We use again that each of the components of the little disks operad $\mathcal{D}_2$ is homotopy equivalent to a configuration space. We will start by producing explicit cycles in this configuration space. A solar system configuration cycle is a map $(S^1)^k \to C(D^2, n)$ of the following form: one fixes a planar binary tree with k internal vertices, a root and n labelled leaves and lengths $\epsilon_{v_1}, \ldots, \epsilon_{v_k} \in [0, 1)$ which become a lot smaller each step further away from the root. Then one produces the solar system configuration cycle by mapping $(\theta_1, \ldots, \theta_k)$ to an element $(x_1, \ldots, x_n)$ of the configuration space with i'th component x_i given as follows: x_i a sum over all vertices on the path from the i'th leaf to the root of terms $\pm\epsilon_{v_t}(\sin(\theta_i), \cos(\theta_i))$ where the choice of $\pm$ depends on whether the path goes into the vertex from the left or right.

By applying the induced map in homology to the fundamental class of $(S^1)^k$, one finds candidates for classes in $H_k(C(D^2, n))$. In particular, the class in degree 1 coming from the obvious solar system configuration $S^1 \to C(D^2, 2)$ will give the bracket. Then one shows that this bracket satisfies the antisymmetry and Jacobi relations.

Similarly, by mapping a configuration to angles between certain pairs of elements, one obtains many maps $C(D^2, n) \to (S^1)^k$ and the image of the dual of the fundamental class under the induced map in cohomology gives candidates for classes in $H^k(C(D^2, n))$. It is then not such a difficult task to check that these cohomology classes pair perfectly with the explicitly constructed homology classes.

After that a spectral sequence argument using the Serre spectral sequence applied to the fibration we used to show that $\mathcal{D}^2$ is a $K\mathcal{E}_B$-operad can be used to show by induction that all of $H_k(C(D^2,n))$ and $H^k(C(D^2,n))$ is obtained this way. To find the operad structure, one uses the Fulton-MacPherson compactification which is homotopy equivalent to the little disks operad. Here the radii in the solar systems can be taken infinitely small, which make it easy to calculate what happens when one applies an operad composition map to a number of homology classes. $\square$

Remark 6.129. The reason this is called the 2-Poisson operad has to do this the degree of the bracket and the signs. If one replaces the little disks with little d-spheres, the homology of the resulting operad $\mathcal{D}^d$ will be the d-Poisson operad. Sinha's article describes this general case.

6.2.3. *Framed little disks and the homology operad $\mathcal{H}_0^0$.* Our next goal is to explain how to modify the little disks operad to obtain a $\mathcal{E}_{RB}$ operad. This will be the framed little disks operad $f\mathcal{D}_2$. After that we compute its homology.

This will give us the structure that comes from genus zero string topology operations if the target manifold is 0-dimensional, because we didn't take into account the local system. However, these calculations are of independent interest, because the framed little disks operad $f\mathcal{D}_2$ for example also acts on $\Omega^2 X$ for any topological space. This gives an additional structure on the homology of double based loop spaces.

The framed little disks operad is a special case of a semi-direct product operad. This is a general construction of a new operad from an operad which allows for a group action compatible with the operad structure. We discuss semi-direct product operads in the framework of $\mathcal{E}$-operads.

Definition 6.130. An topological $\mathcal{E}$-operad $\mathcal{O}$ is said to carry a compatible action of a topological group G if each $\mathcal{O}(n)$ admits an G-action λ. The composition should be equivariant for these actions, in the sense that the following diagram commutes:

$$\begin{tikzcd}
G \times \mathcal{O}(n) \times \mathcal{O}(k_1) \times \ldots \times \mathcal{O}(k_n) \arrow[r, "id \times \chi"] \arrow[d, "\tau \circ (\Delta_{n+1} \times id^{\times(n+1)})"'] & G \times \mathcal{O}(\sum_{i=1}^n k_i) \arrow[dd, "\lambda"] \\
G \times \mathcal{O}(n) \times G \times \mathcal{O}(k_1) \times \ldots \times G \times \mathcal{O}(k_n) \arrow[d, "\lambda^{\times(n+1)}"'] & \\
\mathcal{O}(n) \times \mathcal{O}(k_1) \times \ldots \times \mathcal{O}(k_n) \arrow[r, "\chi"'] & \mathcal{O}(\sum_{i=1}^n k_i)
\end{tikzcd}$$

where Δ_{n+1} is the $(n+1)$-fold diagonal map $G \to G^{n+1}$ and τ is the appropiate permutation of the factors of the product.

There is a more concise way to describe these compatibility conditions. This uses our previous remark that equivariance systems can be made internal to the category one is working in if it has enough coproducts. The category Top satisfies this condition.

From a topological group G and an equivariance system $\mathcal{E}$ one can create a new equivariance system $\mathcal{E} \ltimes G$. It is given by setting $(\mathcal{E} \ltimes G)_n = E_n \ltimes G^n$, where the semi-direct product comes from the action of E_n on G^n via $p : E_n \to \Sigma_n$ acting on G^n by permuting the factors. The strand-splitting maps is given by diagonals, the disjoint maps by the canonical maps and the map to $\mathcal{E}_\Sigma$ by the composition of the projection $\mathcal{E} \ltimes G \to \mathcal{E}$ with the map $\mathcal{E} \to \mathcal{E}_\Sigma$.

Then a $\mathcal{E}$-operad with compatible action of G is the same as a $(\mathcal{E} \ltimes G)$-operad. From an $\mathcal{E}$-operad $\mathcal{O}$ with compatible action of G we can construct a new $\mathcal{E}$-operad $\mathcal{O} \rtimes G$. The idea is to take as spaces $\mathcal{O}(n) \times G^n$ and use the action to extend the original operad structure to these spaces.

Definition 6.131. Let $\mathcal{O}$ be a topological $\mathcal{E}$-operad with compatible G-action. The semi-direct product operad $\mathcal{O} \rtimes G$ is given by spaces $(\mathcal{O} \rtimes G)(n) = \mathcal{O}(n) \times G^n$ with the groups E_n action acting by the original action of $\mathcal{O}(n)$ and by the action of the symmetric action via $p : E_n \to \Sigma_n$

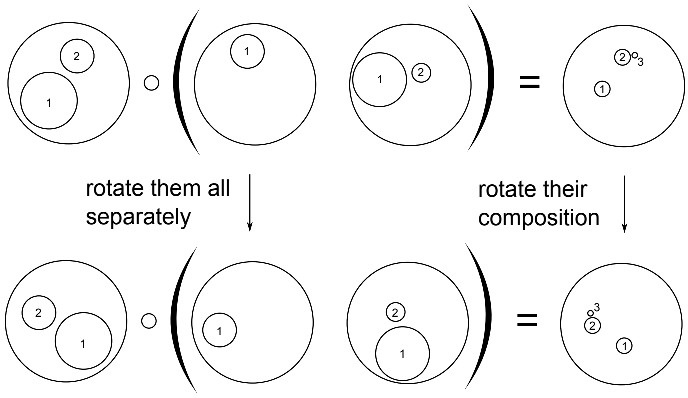


FIGURE 6.133. To show that the action of a rotation of the little disks is compatible, we need to show that rotating each element individually and then composing gives the same result as rotating the composition of these elements. This figure makes it plausible that this is indeed the case.

on G^n. The composition maps are given as follows

$$\chi : (\mathcal{O}(n) \times G^n) \times (\mathcal{O}(k_1) \times G^{k_1}) \times \ldots \times (\mathcal{O}(k_n) \times G^{k_n}) \to \mathcal{O}(\sum_{i=1}^{n} k_1) \times G^{\sum_{i=1}^{n} k_1}$$

$$((x, g_1, \ldots, g_n), (x_1, g_1^1, \ldots, g_{k_1}^1), \ldots, (x_n, g_1^n, \ldots, g_{k_n}^n)) \mapsto (\chi(x, g_1 x_1, \ldots, g_n x_n), (g_1 g_1^1, \ldots, g_n g_{k_n}^n))$$

Note that the semi-direct product $\mathcal{E}$-operad $\mathcal{O} \rtimes G$ always has a map of $\mathcal{E}$-operads to $\mathcal{O}$ gives by projection on the first component. However, there is also a map $\mathcal{O} \to \mathcal{O} \rtimes G$ given by putting identities in all components G. This implies that algebras for $\mathcal{O}$ are in particular algebras for $\mathcal{O} \rtimes G$, but vice-versa also algebras for $\mathcal{O} \rtimes G$ are algebras for $\mathcal{O}$. Heuristically, the first map adds operations coming from G which act trivially in the algebras, while the second map forgets about these operations from G.

Remark 6.132. A nice mnemonic is the formula:

$$\mathcal{F} \circlearrowright \mathcal{O} \circlearrowleft G \Leftrightarrow (\mathcal{F} \ltimes G) \circlearrowleft \mathcal{O} \Leftrightarrow \mathcal{F} \circlearrowright (\mathcal{O} \ltimes G)$$

We want to apply this construction to the ordinary operad $\mathcal{O} = \mathcal{D}_2$ and $G = S^1$. Our first claim is of course that the little disks operad $\mathcal{D}_2$ admits a compatible action of S^1. This is the action of S^1 given by rotation of the configuration of the disks. In figure 6.133 we show that it is a compatible action.

Definition 6.134. The framed little disks $f\mathcal{D}_2$ is the semi-direct product operad $\mathcal{D}_2 \rtimes S^1$. Alternatively, one can describe $f\mathcal{D}_2(n)$ as the space of embeddings $\coprod_{i=1}^n D_i^2 \hookrightarrow D^2$ such that each individual embedding is a composition of scaling, translation and rotation. This is given the subspace topology, composition is composition of embeddings and the symmetric group permutes the labels.

To describe the homotopy type of $f\mathcal{D}_2$, we note that $S^1 = B\mathbb{Z}$ and prove the following proposition. To do this, we first sketch the general context. The notion of compatible action is not limited to topological spaces, but also works in the category of groups, Grps. So let A act compatible on an equivariance system $\mathcal{E}$. Then there exists a semi-direct product operad $\mathcal{E} \rtimes A$, which is an equivariance system by the composition $p : \mathcal{E} \rtimes A \to \mathcal{E} \to \mathcal{E}_\Sigma$ of the projection on the first component with the original map $p : \mathcal{E} \to \mathcal{E}_\Sigma$. Recall that for each (possibly topological) abelian group A, the classifying space BA is a topological group.

Proposition 6.135. *If $\mathcal{O}$ is an $\mathcal{F}$-operad with compatible BA-action which is a $K\mathcal{E}$-operad, then $\mathcal{O} \rtimes BA$ is an $\mathcal{F}$-operad which is a $K(\mathcal{E} \ltimes A)$-operad.*

Proof. It is clear that each space $(\mathcal{O} \rtimes BA)(n)$ will have a contractible universal cover $\tilde{\mathcal{O}}(n) \times EA^n$ with free action of $E_n \times A^n$.

The $\mathcal{E} \rtimes A$-operad structure on the universal cover is given by lifting the operad actions of $\mathcal{O} \rtimes BA$ as before. To do this, we pick a basepoint of EA and use the basepoints in $\tilde{\mathcal{O}}(n) \times EA^n$ given by the product of the basepoint of $\tilde{\mathcal{O}}(n)$ with the basepoints of the EA. This allows us to lift the composition map. Similarly, we lift the original actions of F_n to actions of $E_n \times A^n$. The direct description of the composition of the semi-direct product operad implies that equivariance holds only if A acts on $E_n \times A^n$ in the composition. $\square$

Remark 6.136. The second mnemonic for working with these kinds of objects is:

$$\mathcal{F} \circlearrowleft K\mathcal{E} \circlearrowleft BA \Leftrightarrow \mathcal{F} \circlearrowleft (K\mathcal{E} \rtimes BA) \Leftrightarrow \mathcal{F} \circlearrowleft K(\mathcal{E} \ltimes A)$$

In the previous proposition, one can construct the compatible action of A on $\mathcal{E}$ directly. It is given by simply applying π_1 to the compatible action of BA. It is then easy to check that a compatible action of S^1 on $\mathcal{D}_2$ gives an equivariace system $\mathcal{E}_B \rtimes \mathbb{Z}$ which is isomorphic to $\mathcal{E}_{RB}$. As a consequence we therefore have the following corollary:

Corollary 6.137. *The framed little disks operad $f\mathcal{D}_2$ is a $K\mathcal{E}_{RB}$-operad.*

Our recognition theorems then imply that indeed up to homotopy $\mathcal{O}_0$ is modelled by $f\mathcal{D}_2$.

Corollary 6.138. *The genus zero operad $\mathcal{O}_0$ is weakly equivalent to the framed little disks operad $f\mathcal{D}_2$.*

This means that to calculate the homology of the operad $\mathcal{O}_0$, it suffices to calculate the homology of the framed little disks operad. This is not that hard given the homology of the little disks operad, because the homology of S^1 is easy.

Theorem 6.139. *The homology operad associated to $f\mathcal{D}_2$ is the (2-)Batalin-Vilkovisky operad whose algebras are BV-algebras with a product $\cdot$ and a map Δ of degree 1. These satisfy the following relations:*

Graded commutativity: *We have that $a \cdot b = (-1)^{|a||b|} b \cdot a$.*
Associativity: $a \cdot (b \cdot c) = (a \cdot b) \cdot c$.
Δ is second order derivation: *The map Δ satisfies $\Delta^2 = 0$ and the BV-identity:*

$$\begin{aligned}\Delta(a \cdot b \cdot c) = \Delta(a \cdot b) \cdot c + (-1)^{|a|} a \cdot \Delta(b \cdot c) + (-1)^{|a||b|+|b|} b \cdot \Delta(a \cdot c)\\ - \Delta(a) \cdot b \cdot c - (-1)^{|a|} a \cdot \Delta(b) \cdot c - (-1)^{|a|+|b|} a \cdot b \cdot \Delta(c)\end{aligned}$$

One can recover the bracket from the relation:

$$(-1)^{|a|}[a, b] = \Delta(a \cdot b) - \Delta(a) \cdot b - (-1)^{|a|} a \cdot \Delta(b)$$

Proof. Because $f\mathcal{D}_2(n) \cong \mathcal{D}_2(n) \times (S^1)^n$, we get new classes in the homology coming from $H_*((S^1)^n; \mathbb{Z}) \cong \Lambda_{\mathbb{Z}}[a_1, \ldots, a_n]$ where $|a_i| = 1$. In particular note that $\mathcal{D}_2(1)$ is contractible, hence $H_*(f\mathcal{D}_2(1); \mathbb{Z}) = \Lambda_{\mathbb{Z}}[a]$. We shall denote the operation of degree 1 on algebras induced a by Δ. Since $\chi(a, a) = 0 \in H_*(f\mathcal{D}_2(1); \mathbb{Z})$ for dimensional reasons, we have that $\Delta^2 = 0$.

We have that $x \otimes \prod_{j=1}^k a_i^{\epsilon_i} \in H_*(f\mathcal{D}_2(n); \mathbb{Z})$, with $\epsilon_i \in \{0, 1\}$, can be obtained from the composition map and $H_*(f\mathcal{D}_2(1); \mathbb{Z})$ as follows:

$$x \otimes \prod_{j=1}^{n} a_i^{\epsilon_i} = \chi_*(x, a_1^\epsilon, \ldots, a_n^\epsilon)$$

This implies that it suffices to calculate the interaction of Δ with the product and bracket operation to get to know the entire homology operad structure. We denote a point in $f\mathcal{D}_2(2)$ by $(x_1, \epsilon_1, \phi_1, x_2, \epsilon_2, \phi_2)$, i.e. we give the center of the disk, its radius and the angle of rotation. Let X be a sphere with four disks removed: it thus has boundary circles $S_1^1, \ldots, S_4^1$. Let $\theta_1 : U_i \to S^1$ be a local coordinate around the i'th boundary circle that coincides with the boundary parametrization at the boundary circle. Now consider the following map:

$$f : X \to f\mathcal{D}_2(2)$$
$$x \mapsto \begin{cases} ((\frac{1}{2},0),\frac{1}{4},\theta_1,(-\frac{1}{2},0),\frac{1}{4},0) & \text{if } x \in U_1 \\ ((\frac{1}{2},0),\frac{1}{4},0,(-\frac{1}{2},0),\frac{1}{4},\theta_2) & \text{if } x \in U_2 \\ ((\frac{1}{2}\cos(\theta_3),0),\frac{1}{4},0,(\frac{1}{2}\sin(\theta_3),0),\frac{1}{4},0) & \text{if } x \in U_3 \\ ((\frac{1}{2}\cos(-\theta_4),0),\frac{1}{4},-\theta_4,(\frac{1}{2}\sin(-\theta_4),0),\frac{1}{4},-\theta_4) & \text{if } x \in U_4 \\ \text{smooth interpolation} & \text{otherwise} \end{cases}$$

Then the $f_*([X])$ exhibits $f_*([\partial X])$ as a boundary. But the latter class is represented by the linear combination of operations $\Delta(a \cdot b) - \Delta(a) \cdot b - (-1)^{|a|} a \cdot \Delta(b) - (-1)^{|a|}[a,b]$, which must hence be zero. Hence, the bracket can be written completely in terms of the BV-operator and the product, and therefore does not have to listed as a basic operation.

Now the BV-relation follows fromm the fact that the bracket is a derivation. We put $b = b \cdot c$ and consider the following expression

$$0 = [a, b \cdot c] - (-1)^{|a||b|} b \cdot [a,c] + [a,b] \cdot c$$

Each of the terms can be written as:

$$[a, b \cdot c] = (-1)^{|a|}\Delta(a \cdot b \cdot c) - (-1)^{|a|}\Delta(a) \cdot b \cdot c - a \cdot \Delta(b \cdot c)$$
$$-(-1)^{|a||b|} b \cdot [a,c] = -(-1)^{|a||b|} b \cdot \Delta(a \cdot c)(-1)^{|a||b|} b \cdot \Delta(a) \cdot c(-1)^{|a||b|+|a|} b \cdot a \cdot \Delta(c)$$
$$[a,b] \cdot c = (-1)^{|a|}\Delta(a \cdot b) \cdot c - (-1)^{|a|}\Delta(a) \cdot b \cdot c - a \cdot \Delta(b) \cdot c$$

Adding all these gives the BV-identity multiplied $(-1)^{|a|}$ after using graded commutativity to simplify the expression. Thus we have shown that there is a map from the BV-operad to the homology operad of $f\mathcal{D}_2$.

To show that they are equal, we remark that although we have see exactly how the product and bracket interact with the BV-operator, we haven't shown that we can drop the axiom that the bracket should be a Poisson bracket. To do this, we note that BV-relation implies that if one tries to define a bracket using the formula, this will be a Poisson bracket. This is quite a tedious calculation, hence we won't do this. ☐

Remark 6.140. There is an analogue of the framed little disks operad for higher dimensions. This operad, called the framed little d-disks operad $f\mathcal{D}_d$ is the semi-direct product $\mathcal{D}_d \rtimes SO(d)$. Its homology describes so-called d-BV algebras. For more about this, see [**CV06**].

Remark 6.141. There is another $K\mathcal{E}_{RB}$-operad known as the cactus operad. It is used in the book by Cohen and Voronov [**CV06**] to produce the BV-algebra structure on the homology of the free loop space instead of our more abstract mapping class group operad $\mathcal{O}_0$.

6.3. The homological genus zero operads for set of branes $\mathcal{B}$. Now we get to the operad coming from a d-dimensional HCFT: the operad $\mathcal{H}^M_{\mathcal{B},0}$ given by $\mathcal{H}^M_{\mathcal{B},0}(n) = H_*((\mathsf{Bord}^+_{\mathcal{B}}(n,1))_{0,1,0}; \mathcal{L}^M_{\mathcal{B}})$. This means dealing with the local system $\mathcal{L}^M_{\mathcal{B}}$ coming from the virtual bundle κ. Finally, we discuss the examples in the literature.

From the previous section it is clear that up to signs and grading shifts the homology of the free loop space is a BV-algebra. Hence the closed product satisfies the BV-identity up to sign, as we already saw in proposition 6.92. The question that remains is which signs and grading shifts appear.

Proposition 6.142. *If one uses the modified closed product as in definition 6.76 for the product and Δ for the BV-operator, then $\mathbb{H}_*(LM;\mathbb{Q}) = H_{*+d}(LM;\mathbb{Q})$ is a BV-algebra and the bracket coincides with the string bracket as in definition 6.93.*

Proof. We already know that the grading shift will be by $-d$ for the closed product and $+1$ for the BV-operator. Any modification of signs in the closed product give another way to make

$\mathbb{H}_*(LM;\mathbb{Q})$ into a BV-algebra up to signs, so it suffices to find a modification which has the correct signs.

For this it suffices to note that lemma 6.77 and 6.92 show that the modifications made in the modified closed product satisfy the correct signs to make $\mathbb{H}_*(LM;\mathbb{Q}) = H_{*+d}(LM;\mathbb{Q})$ into a BV-algebra. □

Definition 6.143. If a graded vector space V_* is such that V_{*+d} is a BV-algebra, then we call V a degree d BV-algebra.

So, the calculation of genus zero string operations results in the following theorem.

Theorem 6.144. *The homology $H_*(LM;\mathbb{Q})$ of the free loop space is a degree d BV-algebra.*

This theorem gives us the final slogan from figure 6.1:

The genus zero operations are the BV-operad.

6.3.1. *Examples.* Finally, having determined that the homology of LM, where M is a d-dimensional manifold carries the structure of a degree d BV-algebra, we look at some examples. None of these examples is easy to calculate, so we'll just list the results and give references to the literature. The hope is that the reader gets some idea of the complexity of the structure that string topology gives on the homology of the free loop space.

Spheres: Spheres are among the basic building blocks of algebraic topology. It is therefore interesting to see what structure the genus zero part of string topology gives on the homology of LS^n. It turns out that we don't need to switch to rational coefficients here to have all instances of Künneth theorem that we need.

The string topology structure of the spheres was first determined completely in [**Men09**]. The result is the following:

Theorem 6.145. *Let $n \geq 1$, then we have that as a ring with the string product $H_{*+n}(LS^n;\mathbb{Q}) \cong \mathbb{Q}[u_{n-1}] \otimes \Lambda_{\mathbb{Q}}[a_{-n}]$, where $|u_{n-1}| = n-1$ and $|a_{-n}| = -n$. The BV-operator is given by:*

$$\Delta(u_{n-1}^k \otimes a_{-n}) = k u_{n-1}^{k-1} \otimes 1$$
$$\Delta(u_{n-1}^k \otimes 1) = 0$$

Lie groups: A lot is known about the string topology structure on the homology of LG for a G a compact Lie group, not only the genus zero part. The following two facts simplify a lot of the operations and make them calculatable by elementary means: the first is that $LG \cong G \times \Omega G$ and the second is that G is parallellizable, hence has Euler characteristic 0.

The main reference is [**Hep10**], which looks at the family of Lie groups $SO(n)$ and some special cases like $S^3 \cong SU(2)$ and describe a method to calculate the BV-algebra structure coming from string topology. One can find results about other string operations and calculations of the BV-algebra structure for the families $SU(n)$, $U(n)$, $Sp(n)$ and the exceptional Lie groups G_2 and F_4 in [**Kup10**].

Projective spaces: A second class of spaces for which the string topology structure is known is the class of projective spaces. Although the string product was determined earlier using a spectral sequence argument, the full BV-algebra structure was given in [**Hep09**]. That article gives a nice description of a purely topological method to calculate the string topology structure, but the results are too long to quote here.

Surfaces: Vaintrob has shown how to calculate the string topology BV-algebra structure on $H_*(L\Sigma)$, where Σ is a compact oriented surface in [**Vai07**]. We give a more elementary calculation in [**Kup11b**]. He proves a theorem comparing $H_*(LX)$ with $HH^*(\mathbb{Z}[\pi_1(X)])$ in the case that X is a manifold which is also an Eilenberg-Mac Lane space $K(\pi_1(X),1)$. In particular, this is the case for compact oriented surfaces: the surface of genus g has as only non-trivial homotopy group

$$\pi_1(\Sigma_g) \cong \langle \alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g \rangle / ([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1)$$

The result is the following theorem, which uses the Goldman bracket described in section 1.1.

Theorem 6.146. *We have that*

$$H_*(L\Sigma;\mathbb{Q}) = \begin{cases} H_0(L\Sigma;\mathbb{Q}) & \text{if } * = 0 \\ H_1(\Sigma;\mathbb{Q}) \oplus H_0(L\Sigma;\mathbb{Q})/\mathbb{Q}\gamma_0 & \text{if } * = 1 \\ \mathbb{Q} & \text{if } * = 2 \\ 0 & \text{otherwise} \end{cases}$$

The only non-zero string products are on H_1 and are given by

$$(a,\gamma)\cdot(a',\gamma') = (a\odot a')\gamma_0 + (a\odot\gamma')\gamma' + (a'\odot\gamma)\gamma + \{\gamma,\gamma\}$$

where $\odot$ is the intersection product and $\{-,-\}$ is the Goldman bracket. The BV-operator is non-trivial only on H_0 and is there equal to the projection $H_0(L\Sigma;\mathbb{Q}) \to H_0(L\Sigma;\mathbb{Q})/\mathbb{Q}\gamma_0$.

CHAPTER 7

Perspectives on string topology

In this chapter we will discuss different ways to look at string topology. In the first section we look more closely at the role of fat graphs in our construction and the scope this gives for generalisation.

In the second section we look at two other methods to construct the string operations and try to explain the advantages and disadvantages of these methods in comparison to the one sketched in the previous chapters.

1. Other categories of graphs

In some sense the string topology operations are not that dependent on the fat graphs and their relation to moduli of Riemann surfaces. First of all, we will explain how one can extend the string operations to Klein surfaces and then we will speculate about other spaces of graphs that might support string operations.

Let's explain the first remark: what we mean is that for each simplex in $|\mathsf{Fat}_{\mathcal{B}}^{oc,n}|$ the construction of the string operations doesn't really depend on the fat graph structure. It does need to know the incoming and outgoing parts and the labels from $\mathcal{B}$, but not the cyclic permutation at each vertex. These only start to play a role when one wants to glue all the operations together over the geometric realisation into one map. By using other categories of graphs, one can alter the construction of string operations as follows: still do the same construction over each simplex, but glue these together in a different way. We will give a simple example of this and then sketch a more general theory.

1.1. Klein surfaces and Moebius graphs. Every orientable 2-dimensional manifold with boundary can be given the structure of a Riemann surface with boundary and the rational homology of the mapping class groups in fact calculates the rational homology of the moduli space of such Riemann surfaces with boundary.

One may ask what happens when we drop the orientability assumption. Then Riemann surfaces are replaced by the more general notion of a Klein surface. A Riemann surface is locally modelled on $\mathbb{C}$ and has biholomorphic maps as transition functions, i.e. holomorphic maps with a holomorphic inverse. A Klein surface is also locally modelled on $\mathbb{C}$, but the transition functions are now allowed to be both holomorphic maps with holomorphic inverse and antiholomorphic maps with antiholomorphic inverse. Examples of Klein surfaces without boundary include the real projective plane $\mathbb{R}P^2$ and the Klein bottle $\mathbb{K}$ and there is a classification of connected Klein surfaces with boundary: there are the orientable ones, indexed by the genus and the number of boundary components and the unorientable ones, indexed the number of crosscaps and the number of boundary components.

Like for Riemann surfaces with boundary, we can make a moduli space of Klein surfaces with boundary. Again, its rational homology is computed by the rational homology of the corresponding mapping class group, which now no longer cares about preserving the orientations. This becomes interesting once we note that there is a graph model of this classifying space. To obtain it, we need to replace the fat graphs by Moebius graphs. Let's define these, following Braun [**Bra10**].

We first consider a generalisation of fat graphs: bicoloured fat graphs. A bicoloured fat graph is a fat graph (V, H, s, i, σ) with a map $c : H \to \{0, 1\}$ colouring each half-edge with a colour. Morphisms of bicoloured fat graphs are morphisms of the underlying fat graphs which preserve the colouring.

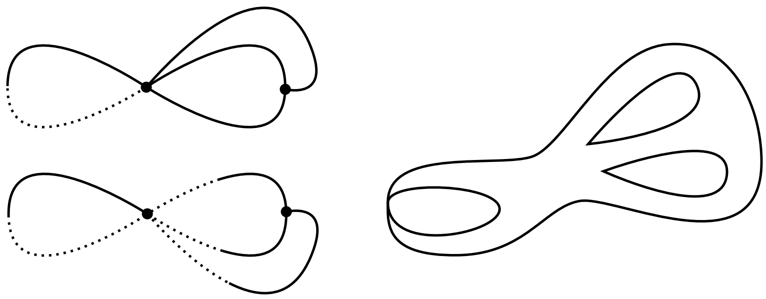


FIGURE 7.1. Two representative bicoloured fat graphs of the same Moebius graph and the corresponding unorientable Klein surface with boundary. The left part of the Klein surface is supposed to represent a twisted ribbon, as in a Moebius band.

There is an equivalence relation $\sim$ on these graphs. We saw that two bicoloured fat graphs are equivalent if they are related by a finite number of the following moves: given a vertex $v \in V$, switch the colours of all the attached half-edges and reverse the cyclic ordering of these half-edges. See figure 7.1.

Definition 7.2. A Moebius graph is an equivalence class of bicoloured fat graphs under the equivalence relation $\sim$. A morphism between two Moebius graph is an equivalence class of morphisms of bicoloured fat graphs between two representatives. This gives a category Moeb of Moebius graphs.

There is the following theorem, which is very similar to theorem 3.57, which was one of the starting points for our construction of the string operations.

Theorem 7.3. *Let* Moeb^c *be the category of connected Moebius graphs. Then we have a rational homology equivalence:*

$$|\mathsf{Moeb}^c| \simeq_{\mathbb{Q}} \coprod_{[\Sigma]} B\Gamma_\Sigma$$

where the disjoint union is over all isomorphism classes of connected Klein surfaces except for a finite number of exceptional cases.

It doesn't seem difficult to modify the definition of a nice $\mathcal{B}$-labelled cobordism graph to this setting and hence get a definition of a nice $\mathcal{B}$-labelled cobordism Moebius graph. The idea is note that a Moebius graph has a canonical realisation as a Klein surface with boundary and hence we can use the same techniques to label parts of the boundary incoming, outgoing or with an element of $\mathcal{B}$. The realisation of a Moebius graph is constructed similarly to the surface with boundary associated to a fat graph, but when two edges with different colours are paired, we don't put a strip but a twisted strip in the corresponding Klein surfaces. For example, a single vertex with a two half-edges of different colour attached it to gives a Moebius band.

An inspection of the construction of the string operations in chapter 5 suggests that similarly it should be possible to construct a map of spectra:

$$\beth_K : \underset{\overline{\mathsf{Moeb}}^{n,oc}_{\mathcal{B}}}{\operatorname{hocolim}} M^{\partial_{in}-} \to \underset{\overline{\mathsf{Moeb}}^{n,oc}_{\mathcal{B}}}{\operatorname{hocolim}} (M^{\partial_{out}-})^{\kappa}$$

for some virtual bundle κ and where $\overline{\mathsf{Moeb}}^{n,oc}_{\mathcal{B}}$ is a modification of $\mathsf{Moeb}^{n,oc}_{\mathcal{B}}$ which includes compatible umkehr data. This map $\beth_K$ can be composed with a Thom isomorphism after smashing with $H\mathbb{Z}$ and this induces string-like operations upon passing to homology. These operatoins should be compatible with disjoint union and composition of not necesssarily orientable cobordisms. There is a corresponding notion of a HKFT, a homological Klein field theory.

Conjecture 7.4. *Let* M *be a* d*-dimensional compact oriented manifold and* $\mathcal{B}$ *be a set of compact oriented submanifolds of* M*. Then the set* $(H_*(LM;\mathbb{Q}), \{H_*(P_M(A,B);\mathbb{Q})\})$ *can be give the structure of a* d*-dimensional HKFT with positive boundary condition and set of branes* $\mathcal{B}$*.*

If we just look at the connected components of $|\mathsf{Moeb}_{\mathcal{B}}^{n,oc}|$ corresponding to orientable Klein surfaces with boundary, then we see that this is a finite cover of $|\mathsf{Fat}_{\mathcal{B}}^{n,oc}|$ corresponding to the choices of orientation for each connected component. Thus the previous conjecture implies that $(H_*(LM;\mathbb{Q}), \{H_*(P_M(A,B);\mathbb{Q})\})$ is a d-dimensional HCFT with positive boundary condition and set of branes $\mathcal{B}$.

1.2. Graph spaces. In fact, fat graphs and Moebius graphs are special exmaples of $\mathcal{C}$-labelled graphs for a cyclic operad $\mathcal{C}$. A $\mathcal{C}$-labelled graph is a graph $\Gamma = (V, H, s, i)$ with an element of $\mathcal{C}(s^{-1}(v))$ for each vertex. For example, fat graphs are Ass-graphs, where Ass is the associative cyclic operad. Similarly, Moebius graphs are Ass^h-graphs, where Ass^h is the associative operad with involution.

We believe that it is possible to define string operations for each type of $\mathcal{C}$-graphs for which is it possible to give a natural boundary. This seems to be not only the case for Ass and Ass^h, but all cyclic versions of the framed little disks operads. We will denote these operads by E_k for $k \in \{1, 2, \ldots, \infty\}$. Note that $E_1 \simeq \mathsf{Ass}$, because up to a homotopy an embedding of a number of intervals into a circle is the same as given a permutation up to cyclic reordering.

There are two particular cases we want to remark upon. The first is E_2, which turns out to be related to 3-manifolds. A theorem of Giansiracusa [**Gia10**] can be rephrased as saying that E_2-graphs model 3-handlebodies and there we should get string operations indexed by the homology of mapping class groups of 3-handlebodies as well.

The second case is $E_\infty \simeq \mathsf{Comm}$. The reason for looking at this case is the following: it is the simplest of the E_k and in some sense the universal one, because the construction of the string operations doesn't care about the data at the vertex and this operad puts no new data there. Let's make this precise: a map $\mathcal{C} \to \mathcal{D}$ of cyclic operads induces a map between $\mathcal{C}$-graphs and $\mathcal{D}$-graphs. Because every E_k maps into E_∞, we can pull back the string operations for E_∞ to E_k. We believe E_∞ is universal in the sense that the string operations for any cyclic operad are obtained as a pull back in this way. In particular, we make the following conjecture:

Conjecture 7.5. *String operations can be constructed over E_∞-graphs and the pull back of these string operations to E_1-graphs gives the string operations for fat graphs.*

The hope is that the induced map in homology from the geometric realisation of the category of E_1-graphs to that of E_∞-graphs will tells us something about the string operations. For example, we believe it can be used to derive Tamanoi's result of the vanishing of string operations associated to stable homology classes.

2. Other methods of constructing string operations

In this section we discuss two alternative methods for constructing the string operations. The first is closely related to the approach explained earlier in this thesis, but uses a different model for the classifying space of the mapping class group. The second approach differs on a very fundamental level: instead of constructing the operations and noticing that they form something like a topological field theory, one classifies all objects that could carry the structure of such a field theory and tries to find an object that naturally gives string topology.

2.1. Radial slit configurations. As the reader may have noticed, Godin's construction is hard. One first needs to do quite a bit preparatory work about $\mathcal{B}$-labelled cobordism graphs. Although the idea of constructing the operations one simplex at a time and glueing these together is not that hard, one needs a lot of technical lemma's to actually pull it off.

One gets something back in return: the technique is flexible and can easily be modified to include multiple branes – like we did in this thesis – or to use other category of graphs. If one is not interested in this flexibility, but just wants to get to the operations as quickly as possible, there exists an alternative construction.

This alternative construction is called the radial slit model and a full account is given in [**Kup11a**]. The backbone of this alternative construction is the replacement of the geometric realisation of the category of $\mathcal{B}$-labelled cobordism graphs with a different model for the classifying

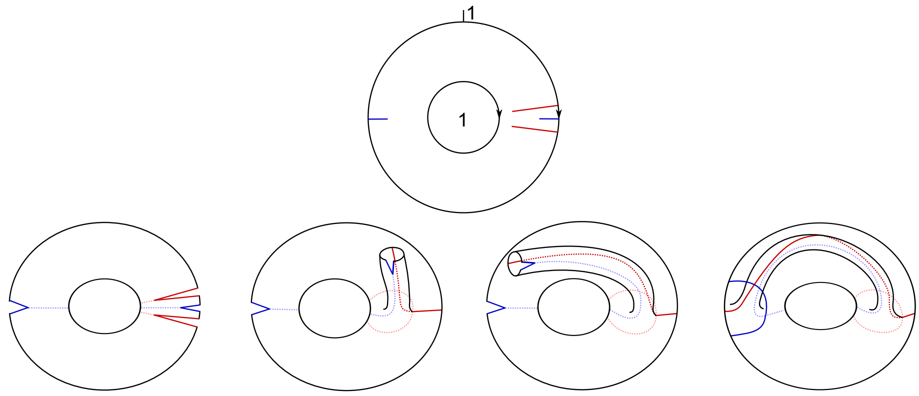


FIGURE 7.6. A genus 1 surface with a single incoming and outgoing boundary circle can be obtained by glueing four cuts in a single annulus.

space of the mapping class group: Bödigheimer's radial slit configuration space [**Böd06**]. The idea behind this space is that one can obtain every Riemann surface with at least one incoming and outgoing boundary component by glueing a certain number of annuli with cuts together. See figure 7.6 for an example.

Essentially, Bödigheimer's radial slit configuration space contains the all possible choices for making cuts on annuli and glueing these together to a surface. These spaces turns out to carry natural families of fat graphs. For each of these one can construct a string operation very similar to the one constructed in theorem 5.6 for single nice $\mathcal{B}$-labelled cobordism graph. The great advantage of this model is then the following: the umkehr maps appearing in these constructions are significantly easier and more uniform that those in theorem 5.6. This dramatically reduces the amount of technical lemma's that one needs.

One advantage of the radial slit model is therefore that it is significantly simpler than Godin's construction. A second advantage is that there is a very natural candidate of a compactification to which one might be able to extend the string operations. This would give new relations between string operations and possibly depend on the smooth structure. Related work was done by Poirier [**Poi10**].

The major disadvantage of this model is it can only model the closed operations. It therefore lacks all the operations coming from open-closed cobordisms, let alone those from $\mathcal{B}$-labelled cobordisms.

2.2. Costello and Lurie's universal construction. In this section we describe a completely different method to construct the string operations. We start with a warning: the details of this approach still have to filled in and this means that some of the following definitions and statements might need to be amended. It was first conjectured in [**Cos06b**], which also provides the general idea of the construction, and made more precise in [**Lur09**], as a consequence of the proof of the cobordism hypothesis.

There are important shifts in perspective from Godin's construction of the string operations. The first is that these constructions work on the chain level: this means that the operations constructed this way will be between copies of $C_*(LM;\mathbb{Q})$ and related chain complexes. The second is that one starts by constructing only the open operations and then uses some universal construction to obtain the closed part.

I do not know of a proof that shows that Godin's construction and this constructoin give the same operations. Presumably this follows from universality, but I do not know how to prove that Godin's construction is universal. The reason for comparatively many words on this construction is that it really sheds a new light of string topology, but also that it shows how string topology was and still is a motivating example in the exciting new field of topological field theories.

2.2.1. *A chain level approach to string topology.* We begin in this subsection and the next how Costello's proposes to construct the string operations. In the two following these subsections we get to Lurie's refinements.

Their constructions aim at a more refined construction than this thesis. While we only look at the homology of the loop space and path spaces and want operations between tensor products of these, they want to use the singular or cellular chains or cochains of these spaces. Think of it as the difference between wanting a associative algebra structure on homology or a A_∞-structure on the original complex.

The motivation for this more refined construction lies in links with Floer homology, where one is also interested in chains. Furthermore, the (co)chains have the property that they preserve enough information of the rational homotopy type on one's space to be able to do geometric constructions, while on the other hand being linear, which makes many tools available.

Convention 7.7. We will work with a field of characteristic 0. This way we do not need to worry when we use orbispaces instead of spaces.

The notion of a HCFT is replaced by that a TCFT when one is working on the chain level. We describe the most general case of a set of branes $\mathcal{B}$. The definition of a TCFT with branes $\mathcal{B}$ is obtained essentially by replacing in the definition of HCFT all homology with singular or cellular chains, but now everything only has to commute up to natural quasi-isomorphism. We will thus be working with the prop with components $C_*(\mathcal{M}_\Sigma; \mathcal{L}^{\otimes d})$ where $\mathcal{M}_\Sigma$ denotes some model for moduli space of $\mathcal{B}$-labelled cobordisms of type Σ and a $\mathcal{L}$ is $\mathbb{Q}$-local system.

Definition 7.8. A d-dimensional TCFT is a chain complex A_* which is an algebra up to quasi-isomorphism over the prop $C_*(\mathcal{M}_\Sigma; \mathbb{Q})$, i.e. a symmetric monoidal functor up to quasi-isomorphism $C_*(\mathcal{M}_\Sigma; \mathbb{Q}) \to \mathsf{Ch}$.

Thus, if we have a single brane M, we want to get operations of the following type in the end:

$$C_*(\mathcal{M}_\Sigma; \mathcal{L}^{\otimes d}) \otimes C_*(LM; \mathbb{Q})^{\otimes m} \otimes C_*(M; \mathbb{Q})^{\otimes n} \to C_*(LM; \mathbb{Q})^{\otimes r} \otimes C_*(LM; \mathbb{Q})^{\otimes s}$$

where Σ has m, resp. n, incoming boundary circles and intervals and r, resp. s, outgoing boundary circles and intervals.

However, recall that we need to worry about the positive boundary condition. This condition is implemented for TCFT in the same way as for HCFT: we restrict the TCFT to those cobordisms which have the property that every connected component has non-empty outgoing boundary.

Remark 7.9. In Costello and Lurie's work they are looking at TCFT's or more general field theories with negative boundary. However, by taking a suitable notion of opposite category one can move between the two types of boundary conditions.

2.2.2. *Universal TCFT's.* Costello describes in [**Cos06b**] a very interesting construction of TCFT's, originally motivated by Gromov-Witten theory. To describe it, we introduce some notation. TCFT's use moduli spaces $\mathcal{M}_\Sigma$ of open-closed cobordisms. A d-dimensional TCFT additionally uses the local system $\mathcal{L}^{\otimes d}$ over these moduli spaces. One can restrict attention to those cobordisms which only have intervals as incoming and outgoing boundary: we denote these $O\mathcal{M}_\Sigma$ and we have an embedding $\iota : O\mathcal{M} \to \mathcal{M}$. Clearly the local systems restrict as well and the resulting structures are called a d-dimensional open TCFT's.

The idea is that for each open TCFT there exists a homotopy universal extension to an open-closed TCFT. The idea of this construction is pretty easy: given an open TCFT A_* we can create a TCFT $\iota^! A_*$ by homotopy Kan extension.

$$\begin{tikzcd} O\mathcal{M} \arrow[r, "A_*"] \arrow[d, "\iota"'] & \mathsf{Ch} \\ \mathcal{M} \arrow[ur, dotted, "\iota^! A_*"'] & \end{tikzcd}$$

There are two issues with this: first of all, one needs to develop homotopy theory of symmetric monoidal functors. Costello does this in an ad hoc fashion, but recents results on model categories of ∞-operads allow one to make this precise.

More importantly, this is only useful if we can construct open TCFT's. It turns out that this is a lot easier than construction an open-closed TCFT and furthermore, we can calculate what the universal open-closed TCFT will assign to the circle. The important input for these results is the dual ribbon orbicell decomposition of the moduli space of Riemann surfaces, described in [**Cos06b**] and in more detail in [**Cos06a**]. This gives an explicit generators and relations description of cellular chain complexes of the open moduli spaces. One therefore obtains an explicit characterization of open TCFT's, which is the analogue of the statement that commutative Frobenius algebras give ordinary TQFT's. Recall that an A_∞-category is a weaker version of the notion of a linear category: the composition need not be associative but only associative up to coherent homotopies.

Definition 7.10. A d-dimensional positively dualizable A_∞-category on a set of branes $\mathcal{B}$ is an A_∞-category $\mathcal{A}$ together with cotraces $k[d] \to Hom(X,X)$ satisfying the following properties

Branes: The object of $\mathcal{A}$ are indexed by $\mathcal{B}$.

Non-degeneracy and symmetry: The cotrace is non-degenerate and symmetric.

Using the previously described generators and relations for the cellular chains on moduli space of Riemann surfaces, Costello was able to prove the following theorem.

Theorem 7.11. *The category of d-dimensional open TCFT's on a set of branes $\mathcal{B}$ is equivalent to the category of d-dimensional positively dualizable A_∞-categories on a set of branes $\mathcal{B}$.*

In particular, a d-dimensional open TCFT is given by a d-dimensional positively dualizable A_∞-category.

Furthermore, Costello was able able to describe exactly what cells with which boundary maps one has to add to $\mathcal{OM}$ to get to $\mathcal{M}$. This allows one to compute that the homotopy universal TCFT associated to an open TCFT assigns to the circle.

Theorem 7.12. *Let A_* be a open TCFT with associated positively dualizable A_∞-category $\mathcal{A}$, then the homotopy universal TCFT $\iota^! A_*$ assigns to the circle a complex quasi-isomorphic to the Hochschild homology chain complex $HC_*(\mathcal{A})$ of $\mathcal{A}$.*

It should be the case that the $\mathbb{R}$-valued chain $C_*(P_M(A,B);\mathbb{R})$ of the restricted path spaces play the role of the positively dualizable A_∞-category associated to the open part of string topology. Cohen, Blumberg and Teleman sketch a partial proof of this result [**BCT09**].

2.2.3. *The cobordism hypothesis in general.* In chapter 5 we saw that 2-dimensional oriented TQFT's give rise to Frobenius algebras and the previous section we found similar statement for TCFT's. The obvious question is what the resulting structures on the state spaces of other TQFT's are, but of higher dimensional TQFT's or TQFT's with different structures. This leads naturally to the world of higher category theory.

The cobordism hypothesis – now the Lurie-Hopkins theorem – answers this question in the case of extended TQFT's. What is an extended TQFT? To answer, it is useful to go back to the motivation of TQFT's. In the physics literature field theories are often demanded to be fully local, i.e. to only depend on data in a neighborhood of a point. This means that one should be able to cut up your space-time, i.e. the manifold, into small pieces, do calculations there and glue the results together.

Recall that a classical n-dimensional TQFT assigns data to $(n-1)$-manifolds and n-dimensional cobordisms between $(n-1)$-dimensional manifolds. The classical axioms of a TQFT thus do not fully reflect this locality, because only codimension 1 cuttings are allowed. One way we therefore want to extended our definition is to also assign data to manifolds of lower codimension, i.e. extend *downwards.*

On the other hand, category theory has taught us that it is often useful to keep track of isomorphisms instead of identifying isomorphic objects. Thus two diffeomorphic n-manifolds need not be assigned the same data, but only isomorphic data and we should keep track of these isomorphisms. In turn, these isomorphisms themselves should not be unique, but only coherent up to higher isomorphisms. The conclusion is that we also want to extend our TQFT's to assign data to diffeomorphisms, isotopies between these, etc., i.e. extend *upwards.*

This poses a technical problem, which is solved by treating the n-dimensional cobordisms not as a category, but as an (∞, n)-category. A heuristic definition of an (∞, n)-category is that it consists of sets of objects and n-morphisms for all $n \in \mathbb{N}$, such that for all $k > n$ the k-morphisms are invertible. Of course, what is left vague about this definition are the compatibility conditions between these morphisms. There exist several models of (∞, n)-categories to write these down precisely. In the case $n = 1$, quasicategories have proven to be a great tool for proving statement about higher categories. Because in our case we are dealing with cobordisms, which are inherently topological, a topological model for (∞, n)-categories suits best. This is given by the model of n-fold complete Segal spaces.

Definition 7.13. A Segal space is a simplicial space $\mathcal{X}$ such that for all $n, m \geq 0$ the following diagram is a homotopy pullback square:

$$\begin{tikzcd} \mathcal{X}_{n+m} \arrow[r] \arrow[d] & \mathcal{X}_n \arrow[d] \\ \mathcal{X}_m \arrow[r] & \mathcal{X}_0 \end{tikzcd}$$

where the maps are the natural ones coming from the simplicial structure. A Segal space is said to be complete when the degeneracy map $\mathcal{X}_0 \to Z \subset \mathcal{X}_1$ is a weak homotopy equivalence, where Z is the subspace of $\mathcal{X}_1$ of elements which are invertible in the π_0 of following homotopy pullbacks $\{x\} \times^h_{\mathcal{X}_0} \mathcal{X}_1 \times^h_{\mathcal{X}_0} \{y\}$ for some $x, y \in X_0$.

Definition 7.14. A $(\infty, 1)$-category is a complete Segal space.

Heuristically, we can interpret the conditions of a complete Segal space in the language of categories as follows: the space $\mathcal{X}_k$ consists of the k-morphisms, the first condition tells us that composition is possible and unique up to homotopy and the second condition tells us that if a morphism is invertible it comes from a path in the space of objects.

To get a (∞, n)-category, we simple iterate this construction in categories of simplicial spaces: previously we have defined the case $n = 1$. In the category of k-fold simplicial spaces, a map is a weak equivalence if and only it is a levelwise one. A similar definition exists for homotopy pullbacks. A k-fold simplicial space is essentially constant if it weakly equivalent to a k-fold simplicial space which is constant in the first direction.

Definition 7.15. Let $n \geq 2$. A n-fold Segal space is a n-fold simplicial space $\mathcal{Y}$ such that the following conditions hold. Each $\mathcal{Y}_k$ is a $(n-1)$-dimensional Segal space and $\mathcal{Y}_0$ is essentially constant. Furthermore, for each $n, m \geq 0$ the following is a homotopy pullback diagram of $(n-1)$-fold simplicial spaces:

$$\begin{tikzcd} \mathcal{Y}_{n+m} \arrow[r] \arrow[d] & \mathcal{Y}_n \arrow[d] \\ \mathcal{Y}_n \arrow[r] & \mathcal{Y}_0 \end{tikzcd}$$

$\mathcal{Y}$ is said to be complete if each $\mathcal{Y}_k$ is, and if the simplicial space $\mathcal{Y}_{k,0,\ldots,0}$ is.

Definition 7.16. A (∞, n)-category is a n-fold complete Segal space.

The next question is how to describe the correct cobordism (∞, n)-category Bord_n. The idea is that a Bord_n should be a (∞, n)-category because we can cut the cobordism in n different orthogonal directions. The number of cuts in each direction is the simplicial degree in that direction.

Definition 7.17. For V a vector space $(\mathsf{PreBord}_n^V)_{k_1,\ldots,k_n}$ consists of the collections $(M, (t_0^1 \leq \ldots \leq t_{k_1}^1), \ldots, (t_0^n \leq \ldots \leq t_{k_n}^n))$ where M is a n-dimensional submanifold of $V \times \mathbb{R}^n$ and $n+1$ sequences of real number. These should satisfy the following conditions:

(1) the projection $M \hookrightarrow V \times \mathbb{R}^n \to \mathbb{R}^n$ is proper,
(2) the projection $M \hookrightarrow V \times \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^{\{i+1,\ldots,n\}}$ is submersive at each $x \in M$ such that its image in $\mathbb{R}^{\{i\}}$ lies at any of the points $\{t_{i_0}, \ldots, t_{i_n}\}$,

(3) for every subset S of $\{0, \ldots, n\}$ the projection $M \hookrightarrow V \times \mathbb{R}^n \to \mathbb{R}^n \to \mathbb{R}^S$ does not have the point $(t_{j_{s_1}}, \ldots, t_{j_{s_m}})$ as critical value for any choice $0 \leq j_{s_1} \leq k_{s_1}$.

We let $(\mathsf{PreBord}_n)_{k_1,\ldots,k_n} = \lim_{V \in \mathsf{Vect}} (\mathsf{PreBord}_n^V)_{k_1,\ldots,k_n}$.

This definition is given in [**Lur09**, section 2.2], inspired by [**GMTW10**]. There one also finds a sketch of the following result.

Lemma 7.18. *There is a completion* Bord_n *which is a n-fold complete Segal space and hence by definition a (∞, n)-category.*

The cobordism hypothesis is concerned with classifying fully extended TQFT's, which we can define precisely now.

Definition 7.19. A n-dimensional (fully extended) TQFT with values in a (∞, n)-category C is a symmetric monoidal (∞, n)-functor $F : \mathsf{Bord}_n \to \mathsf{C}$.

To write down the cobordism hypothesis, we need to discuss symmetric monoidal (∞, n)-categories and dualizable objects in these categories. We have seen a glimpse of this in our discussion for ordinary 2-dimensional TQFT's: a Frobenius algebra is an algebra satisfying some strong dualizability conditions. For example, these conditions force it to by finite-dimensional.

Because the general definition of a fully dualizable object is quite involved (see [**Lur09**, section 2.3]), we will just give the definition for a low n. Specifically, we will give the definition for $n = 2$ from [**Lur09**, section 4.2]. The general definition is similar to this one.

Definition 7.20. Let C be a symmetric monoidal $(\infty, 2)$-category. Then an object $X \in \mathsf{C}$ is called fully dualizable if the following two conditions hold:

(1) there exists an object $X^\vee$ dual to X: this means that there exist 1-morphisms
$$ev_X : X \otimes X^\vee \to 1$$
$$coev_X : 1 \to X^\vee \otimes X$$
such that the following equations hold: $(id_X \circ coev_X) \circ (ev_X \otimes id_X) = id_X$ and $(coev_X \otimes id_{X^\vee}) \circ (id_{X^\vee} \otimes ev_X) = id_{X^\vee}$.

(2) The map $ev_X : X \otimes X^\vee \to 1$ has a left and right adjoint: this means that there exist morphisms $ev_X^L : 1 \to X \otimes X^\vee$ and $ev_X^R : 1 \to X \otimes X^\vee$ and for each of these two 2-morphisms $\eta^R, \epsilon^R, \eta^L, \epsilon^L$ acting similarly to the unit and counit of an adjunction.

Heuristically, the first condition say that X has a dual. More general, it has a left and right dual, but the fact that the category is not only monoidal but symmetric monoidal implies that these coincide up to isomorphism. The second condition makes more sense in this light: the evaluation also has a left and right dual. It is a consequence that then the coevaluation has a left and right dual as well.

Theorem 7.21 (Hopkins-Lurie)**.** *Let* C *be a symmetric monoidal category,* C^{dual} *be its fully dualizable part and* $(\mathsf{C}^{dual})^\sim$ *by the $(\infty, 0)$-category obtained by throwing away all non-invertible morphisms. Then the last space has a canonical $SO(n)$-action and there is a canonical equivalence of (∞, n)-categories*
$$\mathrm{Fun}^\otimes(\mathsf{Bord}_n, \mathsf{C}) \to ((\mathsf{C}^{dual})^\sim)^{hSO(n)}$$

In particular, a n-dimensional TQFT considered as a symmetric monoidal (∞, n)-functor F : $\mathsf{Bord}_n \to \mathsf{C}$ *can by identified uniquely up to homotopy by the $SO(n)$-invariant fully dualizable object $F(*)$.*

Remark 7.22. The $SO(n)$-action acts on the framings in the framed case: we want to take invariants because we want to work with oriented cobordisms. The easiest case is that of a framed cobordisms, in which case no group actions or homotopy fixed points appear.

There are many interesting corollaries that follow from this statement. For example, Bord_n is itself the free symmetric monoidal (∞, n)-category generated by on a single fully dualizable object. This is the analogue of the theorem which says that the ordinary 2-dimensional cobordism category is the free symmetric monoidal category generated by a commutative Frobenius object. This makes Bord_n the universal example of a n-dimensional fully extended TQFT, which we will use later on to see what kind of TQFT operations we should get.

However, for string topology we are of course not interested in these full TQFT's: we have the positive boundary condition to worry about. The solution to this problem is given in the next subsection.

2.2.4. *The 2-dimensional non-compact cobordism hypothesis.* Now we discuss a special case of the cobordism hypothesis that is particularly relevant to string topology. We fix modifications of the previous results coming from the positive boundary condition and discuss a special type of target categories: associative algebras in an $(\infty, 1)$-category.

We first need to deal with the positive boundary condition. This is done by replacing the $(\infty, 2)$-category Bord_2 with a suitable $(\infty, 2)$-category of 'non-compact' cobordisms:

Definition 7.23. For V a vector space the space $(\mathrm{PreBord}_2^{V,nc})_{k_1,k_2}$ consists of the collections $(M, (t_0^1 \leq \ldots \leq t_{k_1}^1), (t_0^2 \leq \ldots \leq t_{k_2}^2))$ with M a 2-dimensional submanifold and $t_i^1, \ldots, t_i^{k_i}$ two sequences of real numbers.

These should satisfy the conditions of definition 7.17 together with the additional condition that if a connected component of M has non-empty intersection with the fiber of the projection on the first component $M \hookrightarrow V \times \mathbb{R}^2 \to \mathbb{R}^2 \to \mathbb{R}^1$ over t_i^1, then it has non-empty intersection with the fiber over t_j^1 for $j > i$.

We set Bord_2^{nc} to be the completion of $\lim_{V \in \mathrm{Vect}}(\mathrm{PreBord}_n^{V,nc})$.

The additional condition gives the positive boundary condition. As before Bord_2^{nc} is a 2-fold complete Segal space and hence can be seen as a $(\infty, 2)$-category. It is of course symmetric monoidal as well. There is an alternative way to define this non-compact version, which makes a lot of sense in the context of our proofs of the structure of ordinary TQFT's using Morse theory. Recall that Morse theory naturally gives handle decomposition of your manifold.

Remark 7.24. The (∞, n)-category Bord_n has a filtration

$$\mathcal{F}_{-1} \hookrightarrow \mathcal{F}_0 \hookrightarrow \mathcal{F}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{F}_n$$

given by taking $\mathcal{F}_k$ to be cobordisms with a handle decomposition with only handles of index $\leq k$. This filtration is used in one of the inductive steps of the proof of the cobordism hypothesis.

In the case $n = 2$, to impose the positive boundary condition we want make sure we don't attach a 0-handle, because it closes off a circle. So, we note that there is an alternative filtration

$$\mathcal{G}_{-1} \hookrightarrow \mathcal{G}_0 \hookrightarrow \mathcal{G}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{G}_n$$

by taking in $\mathcal{G}_k$ only cobordisms with a handle decomposition with only handles of index $\geq n - k$. So, one can also take Bord_2^{nc} to be the subcategory $\mathcal{G}_1$ of Bord_2.

We now define fully extended TQFT with positive boundary. The following definition is a natural extension of the definitions of the last subsection.

Definition 7.25. A 2-dimensional (fully extended) TQFT with positive boundary condition and values in a $(\infty, 2)$-category C is a symmetric monoidal $(\infty, 2)$-functor $F : \mathsf{Bord}_2 \to \mathsf{C}$.

There is a corresponding version of the cobordism hypothesis for this type for TQFT's. To write it down, we need to identify the correct analogue of a fully dualizable object. The positive boundary condition in fact means that one can drop some of the dualizability conditions. Intuitively, one might think that objects no longer need to be dualizable, but is the wrong intuition: the objects corresponds to the 0-dimensional cobordisms, the evaluation and coevaluation to the 1-dimensional cobordisms and finally adjoints of the evaluation to the 2-dimensional cobordisms. Because we are essentially only restricting the 2-dimensional cobordisms, we turn out to lose some of the conditions on the adjoints.

Definition 7.26. Let C be a symmetric monoidal $(\infty, 2)$-category. Then an object $X \in \mathsf{C}$ is positively dualizable (the opposite of what Lurie calls an Calabi-Yau object) if it satisfies the following two conditions:

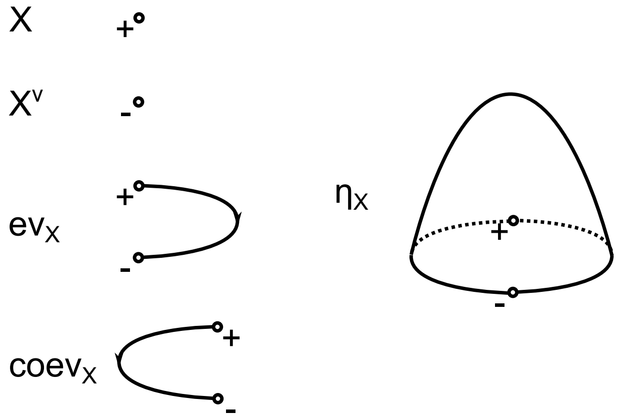


FIGURE 7.28. Part of the structure of the generating positively dualizable object in Bord_2^{nc}.

(1) X has a dual $X^\vee$.
(2) There is a $SO(2)$-equivariant morphism $1 \to ev_X \circ coev_X$ acting as a unit for the adjunction between ev_X and $coev_X$.

Here the $SO(2)$-equivariance of course turns up because we are dealing with oriented cobordisms. Now we can state the 2-dimensional non-compact cobordism hypothesis. There is a general version, which gives the full structure of all positively dualizable objects at the same time in terms of an equivalence of $(\infty, 2)$-categories. However, Because we only want to look at a single TQFT at a time we will just state a simple version [**Lur09**, theorem 4.2.11].

Theorem 7.27. *Let C be a symmetric monoidal $(\infty, 2)$-category. Then there is an equivalence between symmetric monoidal functors $F : \mathsf{Bord}_2^{nc} \to \mathsf{C}$ and positively dualizable objects in C, given by mapping F to $F(*)$.*

Again we have that Bord_2^{nc} is itself the free symmetric monoidal $(\infty, 2)$-category on a single positively dualizable object. In this example, it is easy to see how to interpret a positively dualizable object X and its structure maps geometrically: see figure 7.28.

Remark 7.29. There is an opposite notion of negatively dualizable object. This is called a Calabi-Yau object in [**Lur09**, section 4.2]. The notions of positively and negatively dualizable objects are opposite, in the sense that by reversing the directions of the 2-morphisms and higher dimensions, one is transformed in the other. Hence results about one are easy to translate to results about the other.

For string topology as developed in this thesis, positively dualizability makes more sense. For string topology developed from rational homotopy theory, negatively dualizability makes more sense.

2.2.5. *String topology as a positively dualizable object.* Our next goal is to explain how one can fit string topology into the previous framework. We start by looking at a way to create $(\infty, n+k)$-categories out of (∞, n)-categories and use this to find candidates for $(\infty, 2)$-categories in which one can realize string topology.

Then we look at specific choices for positively dualizable objects in these categories. This last part is speculative and may be incorrect.

We start by discussing E_n-algebras, as in [**Lur09**, section 4.1]. The idea is easy: given a symmetric monoidal $(\infty, 1)$-category C, a E_n algebras in C heuristically is an object in C possessing n commuting associative algebra structures. This is made precise in the following definition.

Definition 7.30. Let C be a symmetric monoidal $(\infty, 1)$-category. Then we define $\mathrm{alg}^1(\mathsf{C})$ to be the $(\infty, 1)$-category of associative algebra objects in C. Inductively, we can then define $\mathrm{alg}^n(\mathsf{C})$ to be $\mathrm{alg}^1(\mathrm{alg}^{n-1}(\mathsf{C}))$. We call $\mathrm{alg}^n(\mathsf{C})$ the $(\infty, 1)$-category of E_n-algebras in C.

An E_n-algebra in the $(\infty, 1)$-category of (∞, k)-categories is a E_n-monoidal (∞, k)-category.

The trick is that under certain technical assumptions of sifted colimits – summarized by the adjective "good" – an E_n-monoidal $(\infty,1)$-category can be used to produce a $(\infty,n+1)$-category or alternively a (∞,n)-category with duals (i.e. every object is fully dualizable).

Proposition 7.31. *If* C *is a good* E_n*-monoidal* $(\infty,1)$*-category then for each two* E_n*-algebras* A,B *in* C*, there is a* E_{n-1}*-monoidal* $(\infty,1)$*-category* $\mathrm{Bimod}_{A,B}$ *of* A,B*-bimodules in* C*.*

One can inductively define a $(\infty,n+1)$ *category* $\mathrm{Alg}_n(\mathsf{C})$ *with objects the* E_n*-algebras in* C *and morphism* (∞,n)*-category* $\mathrm{Alg}_{(n-1)}(\mathrm{Bimod}_{A,B}(\mathsf{C}))$*.*

The choice of name for these algebras is actually quite natural: the E_n-algebras can be identified with algebras over the operad $\mathcal{D}_n$ of little n-disks, see [**Lur09**, claim 4.1.16].

So, in particular we are interested in the case $n=2$. In this case, the construction gives interesting results when we apply to it produce E_1-algebras in a E_2-monoidal $(\infty,1)$-category. Any symmetric monoidal category is E_2-monoidal by taking both assocative algebra structure to be equal (the symmetry makes them commute). Thus, we look at categories of the form $\mathrm{Alg}_1(\mathsf{C})$. A criteria for an object of this category to be positively dualizable can be given in terms of a criteria on the underlying object in C. The result is the following:

Proposition 7.32. *A object* $A \in \mathrm{Alg}_1(\mathsf{C})$ *is positively dualizable if* A *is an associative algebra object in* C *with a* $SO(2)$*-equivariant cotrace*

$$\mathrm{cotr} : 1 \to \int_{S^1} A$$

where $\int_{S^1} A$ *is the Hochschild homology* $A \otimes_{A\otimes A^{\mathrm{op}}} A$*. This map should be such that the composition*

$$1 \to \int_{S^1} A \to \int_{S^0} A \to A \otimes A$$

identifies A *with its dual* $A^\vee$*.*

Let M be an even-dimensional manifold. Lurie proposes to use as C the chain complexes over the periodic ring $R = \mathbb{Q}[t,t^{-1}]$ with $|t| = 2$. Then $C_*(\Omega M;R)$ is easily seen to be an associative algebra object in C by concatenation of loops. Indeed, the Hochschild homology of $C_*(\Omega M;R)$ is known to be quasi-isomorphic to $C_*(LM;R)$. The fundamental class gives the cotrace. The reason for the assumption that M is even-dimensional and the use of R is that otherwise we need to deal with orientations and degree shifts.

So, how can we do that? Lurie suggests using 2-gerbes, but it is hard to makes this precise. Maybe it is easier to use a more geometric category than chain complexes. In particular, one is inclined to take $\mathsf{C} = \mathsf{Spectra}$ for a suitable model of spectra, e.g. symmetric or orthogonal spectra, because this would probably simplify the task of comparing it to Godin's construction of the string operations.

Although it seems like one can obtain the open part of string topology by taking the ring spectrum $M^{-TM} \wedge H\mathbb{Z}$ with cotrace given by the fundamental class, this doesn't seem to assign the correct value to the circle. Maybe we need to use a parametrized spectra over some base which allows one to keep track of the twistings.

Conjecture 7.33. *There is a suitable category of (possibly parametrized) ring spectrum in which there is a positively dualizable object such that one recovers string topology when one takes the homology of corresponding fully extended field theory.*

APPENDIX A

Spectra

In this appendix we sketch some of the background of the theory of spectra that is used in this thesis. In particular, the string operations are originally constructed as maps of (parametrized) spectra. We start with the ordinary case and then discuss the theory of families of spectra, which are known as parametrized spectra.

1. Spectra

1.1. Basic definitions. Spectra are a model for cohomology theories and are indispensable when working with generalized (co)homology theories other than singular (co)homology. But even in the case of singular (co)homology, they provide a nice framework in which to do homotopy theoretical manipulations.

One of the problems of working with spectra, however, is the multiplicity of available models. Much can be done in the classical theory of spectra, but then one has the large problem that the homotopical structure is not compatible with the internal smash product, which has the additional problem that it is difficult to define. To remedy this, people have come up with other models of spectra, like S-modules, symmetric spectra and orthogonal spectra. However, we will mostly need results from the classical and only occasionally remark that a certain manipulation can be justified by using a more advanced model of spectra.

Our main references for the classical theory of spectra are [**Ada74**] and [**Rud08**]. The main reference for symmetric spectra, our prefered advanced model of spectra, are [**HSS99**] and [**Sch07**]. Alternatively, one can use orthogonal spectra [**MS06**]. These have the advantage of being more convenient for the theory of parametrized spectra.

Convention A.1. Occasionally, our notation will be slightly ambiguous, as in the case the tensor $\wedge$ and the smash product $\wedge$. Usually it is clear from the context what is meant, and if this is not the case, it will be because the ambiguity doesn't matter up to homotopy. This way we keep the notation clean.

To start with our definition of spectra, we first recall the definition of the reduced suspension and wedge sum operations on pointed spaces.

Definition A.2. Let X and Y be pointed spaces then the *wedge sum* $X \vee Y$ is given by $X \sqcup Y/(*_X \sqcup *_Y)$. The *reduced suspension* $\Sigma X = S^1 \wedge X$ of a pointed space X is the pointed space $S^1 \times X/(S^1 \vee X)$.

Definition A.3. A *spectrum* E is a sequence $\{E_n\}_{n\in\mathbb{Z}}$ of pointed spaces together with embeddings $\sigma_n : \Sigma E_n \to E_{n+1}$ called structure maps. A morphism of spectra $f : E \to F$ is a sequence $f_n : E_n \to F_n$ compatible with the structure maps σ_n. This defines a category $\mathsf{Spectra}$.

Because we are concerned with (co)homology, we are interested in the homotopy theory of spectra and we need a notion of weak equivalence. To define this, we simply copy the definition of a weak equivalence of topological spaces. To do this, we first need to define homotopy groups of spectra.

Definition A.4. Let E be a spectrum, then we define the *homotopy groups of spectra* $\pi_k(E)$ to be $\lim_{n\to\infty} \pi_{k+n}(E_n)$ of usual homotopy groups of spaces over the system given by:

$$[S^{k+n}, E_n] \longrightarrow [\Sigma S^{k+n}, \Sigma E_n] \longrightarrow [S^{k+n+1}, E_{n+1}]$$

$$[f] \longmapsto [\Sigma f] \longmapsto [\sigma^n \circ \Sigma f]$$

Note that we can use this to define a functor π_* from $\mathsf{Spectra}$ to $\mathsf{GrAbgrps}$. It suffices to note that a morphism of spectra f induces maps $\pi_k(f)$ on homotopy groups and π_* is then simply obtained by taking a direct sum of all π_k. Note that in contrast with normal topological spaces, the homotopy groups can also be non-zero for negative integers.

Definition A.5. A map $f : E \to F$ is a *weak equivalence* if the induced maps $\pi_*(f) : \pi_*(E) \to \pi_*(F)$ is an isomorphism in all degrees. We say that two spaces are weakly equivalent if there is a zigzag of weak equivalences between them.

Definition A.6. The *homotopy category of spectra* $\mathsf{HSpectra}$ is the same with object spectra and as morphism the weak equivalence classes of morphisms.

For spectra E, F we usually denote the sets of morphisms from E to F in $\mathsf{HSpectra}$ by $[E, F]$.

Remark A.7. Recall that Whitehead's theorem says that a map between CW-complexes that induces an isomorphism on all homotopy groups, called a weak equivalence, is in fact a homotopy equivalence. There is a notion of homotopy equivalence for spectra such that a version of Whitehead's theorem holds. Using the standard theorems on CW approximation, up to homotopy we can assume that all E_n are CW complexes and the structure maps σ_n are cellular. Such a spectrum is called a CW spectrum. In the case of a map between two CW spectra, we can also assume that f_n are cellular.

A subspectrum E' of a CW-spectrum E is a sequence of cellular subcomplexes E'_n compatible with the structure maps σ_n, i.e. $\sigma_n(\Sigma E'_n) \subset E'_{n+1}$. Hence the sequence $\{E'_n\}_{n\in\mathbb{Z}}$ with restricted σ_n is itself a spectrum. E' is called cofinal if for each cell e of E there is an integer n such that $\Sigma^n e$ is a cell of E'. Then two maps f_0, f_1 from E to F are said to be homotopic if there exists a cofinal subspectrum E' of E and a map of spectra $f : E' \wedge I_+$ such that restriction of f to $E' \wedge \{i, *\}$ for $i = 0, 1$ is f_i restricted to E'. A homotopy equivalence is a morphism of spectra admitting a inverse up to homotopy.

In this situation the Whitehead theorem holds. A morphism between two CW spectra is a homotopy equivalence if and only if it is a weak equivalence.

Remark A.8. In symmetric spectra one should be careful because this is **not** the correct notion of a weak equivalence if one wants to get the same homotopy types as in the classical theory of spectra. This naive notion of weak equivalence does coincide with the correct one for fibrant objects.

1.2. Examples of spectra. We now give three basic methods of constructing spectra: suspension spectra, Eilenberg-Mac Lane spectra and the Brown representability theorem.

The easiest way to construct a map $\Sigma E_n \to E_{n+1}$ is to simply define E_{n+1} to be ΣE_n. In this way, one can start at any E_k and work upwards to get candidates for structure maps. If we start at $k = 0$ and thus set $E_k = *$ for negative k and E_0 arbitrary, we obtain the notion of a suspension spectrum.

Definition A.9. Let X be a pointed space. Then the *suspension spectrum* $\Sigma^\infty X$ is given by

$$\Sigma^\infty X_n = \begin{cases} \Sigma^n X & \text{if } n > 0 \\ X & \text{if } n = 0 \\ * & \text{if } n < 0 \end{cases}$$

Convention A.10. If there is no chance for confusion, we denote $\Sigma^\infty X$ simply by X. Later we will see that for the purposes of (co)homology there is no essential difference between a space and its corresponding spectrum.

Note that any map $f : X \to Y$ induces a morphism of spectra on the suspension spectra and this assignment is functorial. Hence we get a functor Σ^∞ from spaces to spectra. It is easy to see from the definitions that this functor preserves weak equivalences. Furthermore, note that the homotopy groups of spectra $\pi_*(\Sigma^\infty X)$ are the stable homotopy groups of X, usually denoted by $\pi_*^S(X)$.

Remark A.11. In general, one has be to careful identifying a space with its suspension spectrum. The reason for this is that although Σ^∞ preserves weak equivalence, it doesn't detect them. In other words, the stable homotopy type is less able to distinguish spaces than the ordinary homotopy type.

An example for a map of spaces that is not a weak equivalence, but whose image under Σ^∞ is one, can be given as follows: let X be any connected acyclic CW complex with non-trivial fundamental group (necessarily perfect). Then $X \to *$ is not a weak homotopy equivalence by looking at π_1, but $\Sigma X \to \Sigma *$ is. This implies that $\Sigma^\infty(f)$ induces a weak equivalence of spectra.

The most important of the suspension spectra is the one corresponding to S^0. It occurs enough to give it its own symbol: $\mathbb{S}$. One of the reason for its importance is the following lemma, which we will later state in a more general form:

Lemma A.12. *There is a natural isomorphism between* $\pi_0(E)$ *and* $[\mathbb{S}, E]$.

This concludes our discussion of suspension spectra. For the construction of Eilenberg-Mac Lane spectra, we remember that for any abelian group there exist CW-complexes $K(A, n)$ for $n \geq 0$ such that $H^n(-; A)$ is naturally isomorphic to $[-, K(A, n)]$. Now remember that $H^n \circ \Sigma$ is naturally isomorphic to H^{n+1}. From this one can deduce that there is a map $\Sigma K(A, n) \to K(A, n+1)$ which is a homotopy equivalence. Using a standard mapping cone construction, we can therefore find a new sequence of Eilenberg-Mac Lane spaces such that the following definition gives a spectrum:

Definition A.13. Let A be an abelian group. Then the *Eilenberg-Mac Lane spectrum* HA is given by

$$(HA)_n = \begin{cases} K(A, n) & \text{if } n \geq 0 \\ * & \text{if } n < 0 \end{cases}$$

Remark A.14. The construction HA can be made explicit using proposition B.37. Remember that for any group G, the functorial classifying space BG exists. However, if we apply this to an abelian group A then BA will again be a abelian group object in Top_+ and we can iterate the construction of B in the context of categories enriched in topological spaces to obtain a sequence of spaces $B^n A$. It turns out that are these are models for $K(A, n)$. If one looks at the details of the construction, one sees that $B^{n+1}A$ is constructed from $B^n A$ by taking a quotient of $\coprod_{k \geq 0} \Delta^k \times B^n A \times \ldots \times B^n A$. The term $\Delta^1 \times B^n A$ is attached to $*$ at the boundaries of Δ^1, and this gives a map $\Sigma B^n A \hookrightarrow B^{n+1} A$.

So for cohomology with coefficients in an abelian group A we have a representing spectrum. Natural questions may arise about the uniqueness of this spectrum and the uniqueness and existence of spectra representing other generalized cohomology theories. To formulate the answer we need to define a special class of spectra. First we note that a spectrum E leads to a cohomology theory E_* using $E_*(X) = [X, E_n]$ only if the adjoint $E_n \to \Omega E_{n+1}$ of σ_n is a weak equivalence. The class of spectrum with this property are called the Ω-spectra and in fact every spectrum is weakly equivalent to an Ω-spectrum. Then there is a theorem known as Brown representability, which tells us that Ω-spectra are essentially the same as cohomology theories.

Theorem A.15. *For each generalized cohomology theory* E^* *there is an* Ω*-spectrum* E *representing it and this is unique up to weak equivalence.*

Many explicit examples of spectra representing well-known generalized cohomology theories exist: Fredholm operators commuting with Clifford actions for K-theory and Thom spectra over classifying spaces for cobordism theories are well-known examples.

1.3. Suspension, desuspension, tensoring and cotensoring. In this section we discuss several easy to define operations on spectra and discuss there relations. The first thing we note is that the we can always shift the index of the spectrum. This gives the first and most general definition of suspension and desuspension.

Definition A.16. Let $k \in \mathbb{Z}$, then we define the functor $\Sigma^k : \mathsf{Spectra} \to \mathsf{Spectra}$ by $(\Sigma^k E)_n = E_{n+k}$. The structure maps σ_n are shifted accordingly.

The functor Σ^0 is the identity. If $k > 0$, we call Σ^k a suspension functor and $\Sigma := \Sigma^1$ is called the *suspension functor*. If $k < 0$, we call Σ^k a desuspension functor and Σ^{-1} is called the *desuspension functor.*

It is clear that Σ^k and Σ^{-k} are mutually inverse. As an example note that $\Sigma^k\mathbb{S}$ look as follows: in degree n, where $n \geq k$, it is S^{n+k}. In particular, directly using the definition of the homotopy groups of spectra, we can prove the following:

Lemma A.17. *There is a natural isomorphism $\pi_k(E) \cong [\Sigma^k\mathbb{S}, E]$ for all $k \in \mathbb{Z}$.*

However, this definition of the suspension functors may seem a bit forced. However, we will see that up to weak equivalence Σ is equal to functor on spectra that is defined topologically in a natural way. The desuspension functor Σ^{-1} will be a inverse to this functor up to weak equivalence. The existence of this inverse is one of the great advantages of the category of spectra and the reason for calling the study of spectra stable homotopy theory.

To prove the existence of topological version of the suspension functor, we first note that $\mathsf{Spectra}$ is tensored and cotensored over Top_*. We start by defining the tensor:

Definition A.18. There is a functor $\wedge : \mathsf{Top}_* \times \mathsf{Spectra} \to \mathsf{Spectra}$ given by $(X \wedge E)_n = X \wedge E_n$, where the latter is the smash product of topological spaces. It is called the *tensoring operation.* The new structure maps σ_n are given by the smash product of the identity map of X and the original σ_n.

Proposition A.19. *The tensoring operation $\wedge$ has the following properties:*

- **Preserves equivalences:** *For all pointed spaces X, Y and spectra E, F, if $f : X \to Y$ is a weak equivalence or $g : E \to F$ is a weak equivalence, then the induced maps $f \wedge id : X \wedge E \to Y \wedge E$ and $id \wedge g : X \wedge E \to X \wedge F$ are weak equivalences.*
- **Associativity:** *For all pointed spaces X, Y and each spectrum E there is a natural weak equivalence $(X \wedge Y) \wedge E \simeq X \wedge (Y \wedge E)$.*
- **Unit:** *The one point space $*$ is a unit for $\wedge$: for each spectrum E, there are natural weak equivalences $* \wedge E \simeq E \simeq E \wedge *$.*

A direct consequence of the definition is that $X \wedge \Sigma^\infty Y$ is naturally isomorphic to $\Sigma^\infty(X \wedge Y)$ as $(X \wedge \Sigma^\infty Y)_n$ is nothing but $X \wedge (S^1)^{\wedge n} \wedge Y$, and we can simply move the smash of circles to the left to obtain $(S^1)^{\wedge n} \wedge X \wedge Y = (\Sigma^\infty(X \wedge Y))_n$.

The smash with the circle plays an important role in the definition of a spectrum. This means that it is useful to define a functor implementing this operation on the level of spectra and the tensoring operation has given us a method to produce this implementation: $S^1 \wedge -$ is the topological version of the suspension functor. Its most important property is the following:

Proposition A.20. *Let E be a spectrum, then ΣE is naturally weakly equivalence to $S^1 \wedge E$.*

This is not strange, as $S^1 \wedge E$ and ΣE are equal in degree greater than zero. This is particularly fruitful when applied to $E = \mathbb{S}$. It is clear that $S^1 \wedge \mathbb{S} = \Sigma^\infty S^1$. If one now recalls that we wrote $\pi_0(E)$ as $[\mathbb{S}, E]$, and thinks of $\mathbb{S}$ as S^0 then the following will come as no suprise: using the tensor we can similarly define the higher homotopy groups of spectra in terms of a topological operation on spectra.

Lemma A.21. *There is a natural isomorphism $\pi_k(E) \cong [(S^1)^{\wedge k} \wedge \mathbb{S}, E]$, where we wedge with S^1 k times, for every $k \in \mathbb{Z}_{\geq 0}$.*

For completeness, we also explain the cotensoring on $\mathsf{Spectra}$. As expected, this is done using the pointed mapping spaces, which are adjoint to smash product of pointed topological spaces.

Definition A.22. There is a functor Map : $\mathsf{Top}_* \times \mathsf{Spectra} \to \mathsf{Spectra}$ called the *cotensoring operation.* It is contravariant in the first entry and covariant in the second entry and is given by $\mathrm{Map}(X, E)_n = \mathrm{Map}(X, E_n)$, where the latter is the mapping space of topological spaces in the compact open topology. The new structure maps σ_n are induced by postcomposition with the original ones.

Proposition A.23. *The cotensoring operation* Map *has the following properties:*

Preserves equivalences: *For all pointed spaces X, Y and spectra E, F, if $f : X \to Y$ is a weak equivalence or $g : E \to F$ is a weak equivalence, then the induced maps* $\mathrm{Map}(f, id) : \mathrm{Map}(X, E) \to \mathrm{Map}(Y, E)$ *and* $\mathrm{Map}(id, g) : \mathrm{Map}(X, E) \to \mathrm{Map}(X, F)$ *are weak equivalences.*

Associativity: *For all pointed spaces X, Y and each spectrum E there is a natural weak equivalence* $\mathrm{Map}(X, \mathrm{Map}(Y, E)) \simeq \mathrm{Map}(X \wedge Y, E)$.

Unit: *The one point space $*$ is a unit for $\wedge$: for each spectrum E there are natural weak equivalences* $\mathrm{Map}(*, E) \simeq E$.

1.4. The smash product. The most useful feature of the category of spectra is the existence of a symmetric monoidal structure up to homotopy, with product known as the smash product. This gives a symmetric monoidal structure on the homotopy category. The exact incarnation of this product depends on the model of spectra and in the classical theory is not easy to give a definition. The interested reader may look at [**Ada74**, chapter 5]. We will only claim that it should be thought of as the generalization of the smash product of pointed spaces.

In any case, for our purposes it suffices to note the most important properties of the smash product and to draw a few elementary conclusions from these.

Theorem A.24. *There exists a product $\wedge$ on* $\mathsf{Spectra}$ *with unit $\mathbb{S}$. This is called the smash product. More precisely this is a functor* $\wedge : \mathsf{Spectra} \times \mathsf{Spectra} \to \mathsf{Spectra}$ *with the following properties:*

Preserves equivalences: *For all spectra E, F, G and each weak equivalence $f : E \to F$ the induced map $E \wedge G \to F \wedge G$ is a weak equivalence.*

Associativity: *For all spectra E, F, G there is a natural equivalence $(E \wedge F) \wedge G \simeq E \wedge (F \wedge G)$.*

Commutativity: *For all spectra E, F there is a natural equivalence $E \wedge F \simeq F \wedge E$ whose square is weakly equivalent to the identity.*

Unit: *For each spectrum E there are natural equivalences $\mathbb{S} \wedge E \simeq E \simeq E \wedge \mathbb{S}$.*

Compatibility with suspension: *For all spectra E, F there are natural equivalences $E\ \Sigma E \wedge F \simeq \Sigma(E \wedge F)$.*

Tensor can be written as smash: *For a pointed spaces X and a spectrum E, the cotensoring $X \wedge E$ is naturally equivalent to $\Sigma^\infty X \wedge E$, where $\Sigma^\infty X$ is the suspension spectrum and $\wedge$ the smash product.*

Remark A.25. In the correct model of spectra, like symmetric spectra, this smash product will make the category of spectra into a symmetric monoidal model category, which means that the smash product is compatible with the model category structure. This property was actually the reason why we called symmetric spectra or orthogonal spectra correct models of spectra. For more details, see [**HSS99**] or [**MS06**].

An easy corollary of this is that $\Sigma^\infty X \wedge \Sigma^\infty Y$ is weakly equivalent to $\Sigma^\infty(X \wedge Y)$ by first taking out X by seeing it as a tensor and then using that $X \wedge \Sigma^\infty Y$ is naturally isomorphic to $\Sigma^\infty(X \wedge Y)$.

1.5. Homology and cohomology theories. Earlier in this section we gave the example of spectra arising from generalized cohomology theories. We want to write essentially all of the basic constructions in cohomology and homology theories in terms of spectra. Of course, a natural question is then whether we can also interpret generalized homology theories in terms of spectra. This is in fact the case under some mild conditions:

Theorem A.26. *Let E be an Ω-spectrum, then the functor $E_* : \mathsf{Top}_+ \to \mathsf{GrAbgrps}$ given on a pointed space X in degree k by*

$$E_k(X) = \lim_{n\to\infty} \pi_{n+k}(X \wedge E_n)$$

and given on a morphism f by the map induced by $f \wedge id_{E_n}$, is a generalized homology theory.

Dually, the functor $E^* : \mathsf{Top}_+ \to \mathsf{GrAbgrps}$ given on a pointed space X in degree k by

$$E^k(X) = [X, E_k]$$

and on a morphism f given by the map induced by postcomposition by f, is a generalized cohomology theory.

In fact this construction can be extended to the entire category of spectra, by precomposing with a functor $\Omega^\infty\Sigma^\infty = \lim_k \Omega^k\Sigma^k$ which makes every spectrum into a weakly equivalent Ω-spectrum. This can be seen as a fibrant replacement in a model category structure on spectra.

Using the fact that the tensor $X \wedge -$ can be written in terms of the smash product $\Sigma^\infty X \wedge -$, we see that $E_*(X)$ can also be defined as $\pi_k(\Sigma^\infty X \wedge E) = [\Sigma^k\mathbb{S}, \Sigma^\infty X \wedge E]$. Note that the latter describes homology purely as internal to homotopy category of spectra. Similar, $E^*(X)$ can be written as $[\Sigma^\infty X, E]$.

Using this description, we can find out how the suspension and desuspension functors act on the (co)homology theories corresponding to a spectrum.

Proposition A.27. *We have that $(\Sigma^n E)^*(X) = E^{*+n}(X)$ and $(\Sigma^n E)_*(X) = E_{*-n}(X)$.*

Proof. The first statement is easy: $(\Sigma^n E)^k(X) = [X, (\Sigma^n E)_k] = [X, E_{n+k}] = E^{k+n}(X)$. For the second statement, note that for a term in the limit defining $E_k(X)$ we have that $[\Sigma^k\mathbb{S}, \Sigma^\infty X \wedge \Sigma^n E] = [\Sigma^{k-n}, \Sigma^\infty X \wedge E]$, which is a term in the limit defining $E_{k-n}(X)$. □

Because we are mainly interested in singular (co)homology, one may ask whether the generalized homology theory corresponding to the Eilenberg-Mac Lane spectrum HA which represented cohomology with coefficients in an abelian group A, is in fact homology with coefficients in A. This is true because the homology theory associated to HA will satisfy the dimension axiom and hence must be equal to singlar homology. We record this result in the next lemma.

Lemma A.28. *Let HA be the Eilenberg-Mac Lane spectrum for an abelian group A. Then the associated generalized homology theory $(HA)_*$ is naturally isomorphic to singular homology with coefficients in A.*

Note that in this way we can naturally extend homology and cohomology to spectra: for a spectrum E we have that $H_*(E; A) = \pi_*(E \wedge HA)$ and $H^*(E; A) = [\Sigma^* E, HA]$.

If we have a pointed space X, then the suspension spectrum still has all the information about the homology. This can be seen by considering the following string of equations:

$$H_k(X; A) = \lim_{n\to\infty} \pi_{n+k}(X \wedge (HA)_k) = \pi_k(\Sigma^\infty X \wedge (HA)_k)$$

The reader should compare this to our earlier remark A.11, where we saw that up to homotopy we can lose information about the noncommutative part of the fundamental group.

An important statement is that when we are dealing with CW spectra, the Whitehead and homology Whitehead theorems hold under certain conditions. One might think that this would require some notion of a simply connected spectrum, however the previous remarks show that if we can just suspend, which is invertible in the category of spectra, we lose the non-commutative part. Hence we do not need to worry about simply connectedness. Alternatively, one can think of this that weak equivalences of spectra only care about the stable homotopy theory, which is weaker than ordinary homotopy type, and hence under milder conditions a Whitehead theorem holds.

Let's define some notions used in the Whitehead theorem for spectra. We say that a spectrum E is bounded below if there exist a $N \in \mathbb{Z}$ such that $\pi_k(E) = 0$ for $k < N$. One should this condition as saying that the spectrum looks like a space, in the sense that its Postnikov tower is bounded below.

Theorem A.29. *Let E, F be two spectrum which are bounded below and let $f : E \to F$ be a morphism of spectra such that the induced map in homology is an isomorphism in all degrees. Then f is a weak equivalence. Furthermore, if E, F are in addition CW spectra, then f is a homotopy equivalence.*

The next thing we want to describe in terms of spectra are the cup and cap products. These only exist for spectra satisfying some additional properties. In the case of cup products, one way to see this is to note that we only have cup products in singular cohomology if our coefficient group is a ring. The correct notion is that of a ring spectrum.

Definition A.30. A *ring spectrum* is a spectrum E together with an weakly associative product $\mu : E \wedge E$ and weak unit $\eta : \mathbb{S} \to E$. This means that the following diagrams should commute up to coherent weak equivalence:

$$\begin{tikzcd} E \wedge E \wedge E \arrow[r, "\mu\wedge id"] \arrow[d, "id\wedge\mu"'] & E \wedge E \arrow[d, "\mu"] \\ E \wedge E \arrow[r, "\mu"'] & E \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \wedge E \arrow[d, "\eta\wedge id"'] & E \arrow[l, "\simeq"'] \arrow[r, "\simeq"] \arrow[d, equal] & E \wedge \mathbb{S} \arrow[d, "\eta\wedge id"] \\ E \wedge E \arrow[r, "\mu"'] & E & E \wedge E \arrow[l, "\mu"] \end{tikzcd}$$

A ring spectrum is called commutative if, in addition, the product is commutative up to weak equivalence. This means that the following diagram should commute up weak equivalence:

$$\begin{tikzcd} E \wedge E \arrow[d, "\tau"] \arrow[dr, "\mu"] & \\ E \wedge E \arrow[r, "\mu"'] & E \end{tikzcd}$$

where τ is the twist mapping a point (e, e') to (e', e).

A morphism $\phi : E \to F$ of (commutative) ring spectra is a morphism of spectra intertwining the product and unit up to weak equivalence. This means that the following two diagrams should commute up to weak equivalence:

$$\begin{tikzcd} E \wedge E \arrow[r, "\phi\wedge\phi"] \arrow[d, "\mu_E"'] & F \wedge F \arrow[d, "\mu_F"] \\ E \arrow[r, "\phi"'] & F \end{tikzcd} \qquad \begin{tikzcd} \mathbb{S} \arrow[r, "\eta_E"] \arrow[dr, "\eta_F"'] & E \arrow[d, "\phi"] \\ & F \end{tikzcd}$$

The product of a ring spectrum E induces a product in the cohomology as follows. Let X be a pointed space and $\Delta : X \to X \wedge X$ be the diagonal map. Then we get a map $\cup : E^*(X) \otimes E^*(X) \to E^*(X)$ as follows:

$$[X, E_n] \times [X, E_m] \longrightarrow [X \wedge X, E_n \wedge E_m] \xrightarrow{\mu_*} [X \wedge X, E_{n+m}] \xrightarrow{\Delta^*} [X, E_{n+m}]$$

It is easy to see that if the ring spectrum is commutative then the induced product is graded commutative.

Because we are interested in singular cohomology, we want to compare this to the more elementary definitions of the cup product. First note that if R is a commutative ring, then HR is a commutative ring spectrum. To see this, note that because the classifying space construction is functorial and preserves products it sends algebra objects in Grps to algebra objects in Top. Using this technique, we obtain a spectrum HR which is an algebra object in $\mathsf{Spectra}$ and hence in particular a commutative ring spectrum. The consequence of this is the first part of the next lemma.

Lemma A.31. *Let R be a commutative ring, then HR is a commutative ring spectrum and the induced product on cohomology with coefficients in R is the cup product.*

Next we want to describe the cap product for spectra. The idea is that we can produce maps $\cap : E^d(X) \otimes E_*(X) \to E_{*-d}(X)$ for all $d \in \mathbb{Z}$ as follows: it suffices to produce the maps on each term $[S^{n+k}, X \wedge E_n]$ in the colimit defining the E-homology. On such a term we define it by

$$[X, E_d] \times [S^{n+k}, X \wedge E_n] \overset{id\times(\Delta\wedge id)_*}{\longrightarrow} [X, E_d] \times [S^{n+k}, X \wedge X \wedge E_n] \longrightarrow [S^{n+k}, X \wedge E_d \wedge E_n]$$
$$\overset{(id\wedge\mu)_*}{\longrightarrow} [S^{n+k}, X \wedge E_{d+n}] \longrightarrow [S^{d+n+k-d}, X \wedge E_{d+n}]$$

Of course, this definition has been chosen such that this construction indeed gives the usual cap product.

Lemma A.32. *The operation $\cap$ induced by HR on homology with coefficients in R is the cap product.*

2. Parametrized spectra

Finally, we make a few remarks about parametrized spectra. Our motivation for this is that ordinary spectra are insufficient to model homology with local coefficients, which we will need to describe the Thom isomorphism for vector bundles which are not orientable. Because every vector bundle is locally trivializable, one can apply the Thom isomorphism locally and try to glue all of these together. In essence, a parametrized spectrum is the result of this glueing procedure: it should be thought of as a bundle of spectra over a base space.

2.1. Definitions and basic constructions. As in the case of spectra, we give a naive definition. This is certainly not good enough for the applications we have in mind, so in reality one must use a good model of parametrized spectra based in turn on a good model of spectra. One such a framework is given in the book by May and Sigurdsson [**MS06**] and is based on the theory of orthogonal spectra. To give the naive description, we first discuss parametrized spaces and ex-spaces, which are the parametrized analogues of spaces and pointed spaces.

Definition A.33. The category of parametrized spaces over base B is the slice category Top/B. Thus a parametrized space is a pair of a space X and a map $p : X \to B$. Morphisms are commutative triangles.

The category of ex-spaces over B has as objects triple consisting of a space X with two maps $s : B \to X$ and $p : X \to B$ such that their composition $p \circ s : B \to B$ is the identity. Morphisms are commutative diagrams.

The first map of an ex-space should be though of as giving a basepoint for each fiber of the second map. It is possible to define a fiberwise suspension functor Σ_B for ex-spaces over B and this is enough to give a naive definition of a parametrized spectrum.

Definition A.34. A parametrized spectrum $\mathbb{E}$ over B is a sequence $\{\mathbb{E}_n\}_{n\in\mathbb{Z}}$ of ex-spaces over B with fiberwise embeddings $\sigma_n : \Sigma_B\mathbb{E}_n \to \mathbb{E}_{n+1}$. A morphism of parametrized spectra is a sequence of maps of ex-spaces compatible with structure maps σ_n.

Now one can repeat all the definitions of the theory of spectra in this fiberwise context to get fiberwise smash products, fiberwise mapping spectra, parametrized (commutative) ring specta, etc. On the other hand, one can also introduce many constructions from bundle theory to this context: pullbacks, associated bundles, etc. As an example we show how to define the generalized homology associated to a parametrized spectrum.

Definition A.35. Let $\mathbb{E}$ be a parametrized spectra over B, then the $\mathbb{E}$-homology $\mathbb{E}_*(B)$ of B is given by

$$\mathbb{E}_k(B) = \lim_{n\to\infty} [S^{n+k} \times B, \mathbb{E}_n]_B$$

where $S^{n+k} \times B$ is the trivial ex-space with fiber S^{n+k} and $[-,-]_B$ denotes the set of fiberwise homotopy classes of maps of ex-spaces over B.

The reader may wonder how one produces parametrized spectra. There is a general method which uses the theory of twists of generalized (co)homology theories. This theory roughly says that for each commutative ring spectrum E there is a topological group $\mathrm{gl}_1(E)$ of automorphisms of E as a commutative ring spectrum. With this in mind, we now describe several methods to produce parametrized spectra with an eye towards applications in this thesis.

Example A.36. Trivial parametrized spectra: There is a simple method to turn a spectrum E into a parametrized spectrum $\mathbb{E}$ over B: simply set $\mathbb{E}_n = E_n \times B$. This is an ex-space with p the projection on the second component and s the map sending a point in B to the basepoint of E_n in the corresponding fiber.

Alternatively, E can be seen as a parametrized spectrum over the single point $*$ and then $\mathbb{E}$ is the pullback of E along the terminal map $B \to *$.

Group quotients: Let G be a topological group. Suppose that we have a commutative ring spectrum E with automorphisms $\mathrm{gl}_1(E)$, a space $\tilde{B}$ with G-action and a homomorphism $G \to \mathrm{gl}_1(E)$, then we can construct a parametrized spectrum over $B := \tilde{B}/G$.

One starts with the trivial spectrum $\mathbb{E}$ over $\tilde{B}$ and defines $\mathbb{E}/G$ over $\tilde{B}/G$ by identifying the fibers of $\mathbb{E}$ over a orbit of G with each other using the elements of $\mathrm{gl}_1(E)$. The results indeed is a parametrized spectrum over B.

Associated bundles: The group quotient construction can be used to construct new parametrized spectra in a way reminiscent of the associated bundle construction. Suppose that $\mathbb{E}$ is a parametrized spectrum over B with fiber a commutative ring spectrum E and let F be another commutative ring spectrum. Then the automorphism spaces $\mathrm{gl}_1(E)$ and $\mathrm{gl}_1(F)$ are well-defined. Suppose that we have a topological group G with homomorphisms $G \to \mathrm{gl}_1(E)$ and $G \to \mathrm{gl}_1(F)$.

Then one can form the associated spectrum $\mathbb{E} \times_G F$ by taking the new base space $B \times G$ with fibered spectrum $\pi_1^* \mathbb{E} \wedge_{B \times G} F$ and taking the quotient by G as before.

Pulling back the universal parametrized spectrum: Like a principal G-bundle can be constructed by pulling back the universal G-bundle EG over BG, a parametrized spectra with fiber E in particular can arise from a map to $B\mathrm{gl}_1(E)$ by pulling a universal parametrized spectrum with fiber E.

2.2. Homology with local coefficients. We will use these techniques to discuss the properties of homology with local coefficients.

2.2.1. *Homology with local coefficients.* We start by looking at grading-preserving local systems. In this case homology with local coefficients can be obtained by using singular chains which don't have constant coefficients, but coefficients depending on the image of the barycenter. We will give the classical definition as in [**Hat02**, section 3.H]. For that we first need define the notion of a bundle of abelian groups.

Definition A.37. A bundle of abelian groups is a covering space $\mathbb{A}$ of B with projection map $p : \mathbb{A} \to B$ satisfying the following two properties:

(1) Each fiber has the structure of an abelian group.
(2) There is an abelian group A such that for each point of B there is an open neighborhood U and a homeomorphism of $p^{-1}(U)$ with $A \times U$ which is given by an isomorphism of groups in each fiber.

Remark A.38. A bundle of abelian groups with fiber A should not be confused with an A-principal bundle. They are in fact equivalent to $\mathrm{Aut}(A)$-principal bundles, as we will show later. This difference is similar to that between a $\mathbb{R}^n$-principal bundle and a n-dimensional vector bundle, which is equivalent to a $\mathrm{GL}(n)$-principal bundle.

The idea is to define homology with local coefficients in a bundle of groups $\mathbb{A}$ as follows: It is given by the homology of the chain complex of singular simplices of B with a choice of lift to $\mathbb{A}$. This chocie of lift is equivalent to a choice of point in the fiber above the barycenter because simplices are contractible, but it is easier to describe the boundary maps using lifts. The boundary of a singular simplex is given by its ordinary boundary together with the restriction of the lift to this boundary. We denote this homology by $H_*(B, \mathbb{A})$.

There are alternative constructions using parametrized spectra. The most obvious one generalizes the construction of the Eilenberg-Mac Lane spectrum HA by iterating the classifying space functor. Let A be an abelian group and suppose that $\mathbb{A}$ is a bundle of groups over a base space B with fiber A. Then applying the iterated classifying space fiberwise, one obtains a parametrized spectrum $H\mathbb{A}$.

Lemma A.39. *The homology with local coefficients $H_*(B;\mathbb{A})$ is naturally isomorphic to $H\mathbb{A}_*(B)$.*

It turns out that $H\mathbb{A}$ can also be obtained by the group quotient construction as described earlier. To see this we note that $\mathrm{gl}_1(HA)$ is weakly equivalent to $\mathrm{Aut}(A)$ and a bundle groups with fiber A is the same as a $\mathrm{Aut}(A)$-principal bundle. Let $\mathbb{A}$ be the total space of this $\mathrm{Aut}(A)$-principal bundle, then $H\mathbb{A}$ is obtained by taking the quotient by $\mathrm{Aut}(A)$ of the trivial parametrized spectrum HA over $\mathbb{A}$.

2.2.2. *Grading-preserving local systems.* These remarks allows us to relate local coefficients to twists. Although we have referred to them, let's give a naive definition of a twist.

Definition A.40. A grading-preserving *twist* of a commutative ring spectrum E is $\mathrm{gl}_1(E)$-principal bundle.

Using the associated bundle construction for parametrized spectrum, a non-trivial twist can be used to create a non-trivial parametrized spectrum. The twists of homology figure quite prominently in this thesis, so we give them a special name.

Definition A.41. A grading-preserving *local system* is a grading-preserving twist of an Eilenberg-Mac Lane spectrum.

We just describe the most important case of local system: those by $\mathbb{Z}_2$. These are give all the grading-preserving local systems for homology with coefficients in $\mathbb{Z}$ or equivalently $\mathbb{Q}$, but can twist all other coefficient groups as well. The idea is that the group $\mathbb{Z}_2$ should be thought of as multiplication by ± 1 and in this guise is homotopy equivalent to $\mathrm{gl}_1(H\mathbb{Z})$. Because multiplication by -1 is an automorphism of every abelian group A, we have a map $\mathbb{Z}_2 \to \mathrm{gl}_1(HA)$ for all abelian groups A and hence can twist all coefficient systems with these twists.

These local systems can be modelled in various ways, alluded to earlier.

Lemma A.42. *The following four pieces of data are equivalent up to homotopy:*

(1) A grading-preserving local system for $H\mathbb{Z}$.
(2) A $\mathbb{Z}_2$-principal bundle $\mathcal{L}$.
(3) A bundle of abelian groups $\mathcal{L}'$ with fiber $\mathbb{Z}$.
(4) A 1-dimensional real vector bundle $\mathbb{L}$.

Proof. **(1) $\Leftrightarrow$ (2):** A grading-preserving local system of $H\mathbb{Z}$ is a $\mathrm{gl}_1(H\mathbb{Z})$-principal bundle. Because $\mathrm{gl}_1(H\mathbb{Z})$ is homotopy equivalent to $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$, this is the same as a $\mathbb{Z}_2$-principal bundle.

(2) $\Leftrightarrow$ (3): This is the associated bundle construction in one direction and the automorphism bundle construction in the other.

(2) $\Leftrightarrow$ (4): Both are classified up to homotopy by $\mathbb{C}P^\infty$.

□

We will concretely describe the homology with local coefficients corresponding to corresponding to a piece of data as above in several different ways. This gives the promised alternatives to applying the classifying space functor fiberwise.

(1) In the most alternative is the associated bundle construction specialized to give a bundle of groups from a A grading-preserving local system in the guise of $\mathbb{Z}_2$-principal bundle. Here one uses that any abelian group A is a $\mathbb{Z}_2$-module under multiplication with ± 1. The principal bundle can be seen as a double cover and one then defines the $A_{\mathcal{L}}$ to be $A \times_{\mathbb{Z}_2} \mathcal{L}$. Now apply the iterated classifying space functor to obtain a parametrized spectrum $H\mathbb{A}_{\mathcal{L}}$.

(2) Alternatively, one can first form the trivial parametrized spectrum $H\mathbb{A}$ over $A \times \mathbb{Z}_2$ and use the group quotient construction.

This generalizes to give a way to construct $H\mathbb{A}_{\mathcal{L}\otimes\mathcal{L}'}$ out of $H\mathbb{A}_{\mathcal{L}}$ and a local system $\mathcal{L}'$. Indeed, the result of this is weakly equivalent to applying B fiberwise to the local system obtained by twisting A by $\mathcal{L} \otimes \mathcal{L}'$.

One can externally multiply local systems: the fiberwise smash gives us a fiberwise weak equivalence $H\mathbb{A} \wedge_B H\mathbb{A}' \to H(\mathbb{A} \times \mathbb{A}')$. If $H\mathbb{A}' = H\mathbb{Z}_{\mathcal{L}'}$ is the homology with local coefficients obtained by applying the previous construction to local systems with fiber $\mathbb{Z}$, then $H(\mathbb{A} \times \mathbb{Z}_{\mathcal{L}'})$ is naturally isomorphic to $H\mathbb{A}_{\mathcal{L}}$ as before. Hence we can also define $H\mathbb{A}_{\mathcal{L}}$ as $H\mathbb{A} \wedge_B H\mathbb{Z}_{\mathcal{L}'}$.

Suppose that one has a map of parametrized spectra $\tilde{f} : H\mathbb{A} \to H\mathbb{A}_{\mathcal{L}}$ covering a map of base spaces $f : B \to B'$. We claim that one can transfer the twist to the other side. The idea is that the associated bundle construction is natural. Given a local systems $\mathcal{L}'$ on B' we get a map

$$\tilde{f} \otimes \mathcal{L}' : H\mathbb{A}_{f^*\mathcal{L}'} \to H\mathbb{A}_{\mathcal{L}\otimes\mathcal{L}'}$$

If we set $\mathcal{L}' = \mathcal{L}^{-1}$, then we thus have found a way to move local systems from the codomain to the domain. Similarly, we can move them from the domain to the codomain.

2.2.3. *General local systems.* Finally, we explain the general twists for homology with local coefficients. The grading-preserving local systems were represented by $B\mathrm{gl}_1(E)$, but more general ones might modify the grading. For homology this is simply given by shifting the grading for each connected component, which can be represented by the space $\mathbb{Z}$. A full local system for homology with coefficients in an abelian group A over B can therefore be seen a map $B \to B\mathrm{gl}_1(HA) \times \mathbb{Z}$.

There is an obvious way to represent these twists is by graded 1-dimensional vector bundles which extends the method of representing grading-preserving twists by 1-dimensional vector bundles.

Definition A.43. A *graded line bundle* over B is a 1-dimensional real vector bundle together with a locally constant map $B \to \mathbb{Z}$.

Note that this is a like a vector bundle, but modelled on the category of graded lines of definition 5.47. Finally, we describe homology with local coefficients coming from a graded line.

Definition A.44. Let A be an abelian group and $\mathcal{L} = (\lambda, d)$ be a graded line bundle over B. Then the parametrized spectrum $HA_{\mathcal{L}}$ is by definition the suspension Σ^d of the parametrized spectrum HA_λ, where λ is considered as a grading-preserving local system.

This last definition is the one that appears naturally in the context of the Thom isomorphism for an unorientable vector bundle μ. The vector bundle will be the determinant line bundle $\det(\mu)$ and the degree shift will be its dimension.

APPENDIX B

Categorical constructions

In this second appendix we bring together the background of various definitions and constructions used in this thesis. They all fit together under the vague banner of categorical constructions, but include such diverse topics as category theory, homotopy theory and the theory of props and operads.

1. Categories, symmetric monoidal categories and topological categories

We will assume that the reader has a rudimentary knowledge of category theory. Therefore, we will only treat three specific refinements of the notion of category: symmetric monoidal categories, enriched categories and internal categories. In particular, we will look at the topological cases, because these appear naturally in this thesis. Our main sources for this are [**ML98**] and [**Kel05**]. To do this, we will quickly repeat two equivalent sets of axioms for categories.

1.1. Two definitions of a category. We start with the definition as given in most elementary texts about category theory.

Definition B.1. A (small) category is a set $\mathrm{Ob}(\mathsf{C})$ and for each pair $(X, Y) \in \mathrm{Ob}(\mathsf{C})$ a set $\mathrm{Hom}(X, Y)$. There should be a composition map $\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(X, Y) \to \mathrm{Hom}(X, Y)$ and an identity morphism $id_X \in \mathrm{Hom}(X, X)$. This data should satisfy the following axioms:

Associativity: The composition map is associative.
Identity: For any morphism f we have that $id \circ f = f$ and $f \circ id = f$.

This definition turns out to be well-suited for the enriched case. In particular, it is exactly the definition of a category enriched in Set. There is an alternative definition which is well-suited for categories internal to a category.

Definition B.2. A (small) category is a pair of sets $\mathrm{Ob}(\mathsf{C})$, $\mathrm{Hom}(\mathsf{C})$ together with maps $1 : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Hom}(\mathsf{C})$, $s, t : \mathrm{Hom}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{C})$ and $\circ : \mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \to \mathrm{Hom}(\mathsf{C})$, where $\mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C})$ is the subset of $\mathrm{Hom}(\mathsf{C}) \times \mathrm{Hom}(\mathsf{C})$ of those pairs (f, g) such that $t(f) = s(g)$.

This data should satisfy the following conditions.

Correct source and target of composition: The following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{C}) \arrow[d, "s"'] & \mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\circ"] & \mathrm{Hom}(\mathsf{C}) \arrow[d, "t"] \\
\mathrm{Ob}(\mathsf{C}) & \mathrm{Hom}(\mathsf{C}) \arrow[l, "s"] \arrow[r, "t"'] & \mathrm{Ob}(\mathsf{C})
\end{tikzcd}$$

Correct source and target of the identity: The following diagram commutes

$$\begin{tikzcd}
 & \mathrm{Ob}(\mathsf{C}) \arrow[dl, "1"'] \arrow[d, equal] \arrow[dr, "1"] & \\
\mathrm{Hom}(\mathsf{C}) \arrow[r, "s"'] & \mathrm{Ob}(\mathsf{C}) & \mathrm{Hom}(\mathsf{C}) \arrow[l, "t"]
\end{tikzcd}$$

Associativity: The following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \arrow[r, "\circ \times id"] \arrow[d, "id \times \circ"'] & \mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \arrow[d, "\circ"] \\
\mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \arrow[r, "\circ"'] & \mathrm{Hom}(\mathsf{C})
\end{tikzcd}$$

Identity: The following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Ob}(\mathsf{C}) \arrow[r, "id \times 1"] \arrow[dr, "\pi_1"'] & \mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \arrow[d, "\circ"] & \mathrm{Ob}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \arrow[l, "1 \times id"'] \arrow[dl, "\pi_2"] \\
& \mathrm{Hom}(\mathsf{C}) &
\end{tikzcd}$$

As said before, this definition will be seen to naturally generalize to an internal category. In particular, it is exactly the definition of a category internal to Set.

Lemma B.3. *These two definitions are equivalent.*

PROOF. If C is a category in the sense of the first definition, then setting the set of morphisms to be $\mathrm{Hom}(\mathsf{C}) = \coprod_{X,Y} \mathrm{Hom}(X,Y)$, $s(f) = X$ and $t(f) = Y$ for $f \in \mathrm{Hom}(X,Y)$, $\circ$ the coproduct of the original composition map and $1(X) = id_X$, defines a category in the sense of the second definition.

Similarly, if C is a category in the sense of the second definition, then setting $\mathrm{Hom}(X,Y) = s^{-1}(\{X\}) \cap t^{-1}(\{Y\})$, $\circ$ the restriction of the original $\circ$ and $id_X = 1(X)$, defines a category in the sense of the first definition. $\square$

Remark B.4. The reader may wonder about the word "small" in these two definitions. The reason for this is that they exclude examples like the category of sets, because these don't have sets of objects or morphisms, but classes. This can be solved by either keeping track of the cardinalities involved or working with a hierarchy of Grothendieck universes.

1.2. Symmetric monoidal categories. We will now give the definition of a symmetric monoidal category, which we will use when discussing the cobordism categories and enriched category theory.

Definition B.5. A monoidal category is a category C together with a functor $\otimes : \mathsf{C} \times \mathsf{C} \to \mathsf{C}$ and $1 : * \to \mathsf{C}$, where $*$ the discrete category with a single object. There should be three natural isomorphisms $\alpha : \otimes \circ (id \times \otimes) \dot{\longrightarrow} \otimes \circ (\otimes \times id)$ and $\eta_L : id \times 1 \dot{\longrightarrow} \pi_1$ and $\eta_R : 1 \times id \dot{\longrightarrow} \pi_2$. These are called the associator and the left and right units respectively. These functors and natural transformations should satisfy the following axioms:

Pentagon axiom: The following diagram commutes:

$$\begin{tikzcd}
((W \otimes X) \otimes Y) \otimes Z \arrow[r, "\alpha"] \arrow[d, "\alpha \otimes id"] & (W \otimes X) \otimes (Y \otimes Z) \arrow[r, "\alpha"] & W \otimes (X \otimes (Y \otimes Z)) \\
(W \otimes (X \otimes Y)) \otimes Z \arrow[rr, "\alpha"'] & & W \otimes ((X \otimes Y) \otimes Z) \arrow[u, "id \otimes \alpha"']
\end{tikzcd}$$

Identity: The following diagram commutes:

$$\begin{tikzcd}
X \otimes (1 \otimes Y) \arrow[rr, "\alpha"] \arrow[dr, "id \otimes \eta_L"'] & & (X \otimes 1) \otimes Y \arrow[dl, "\eta_R \otimes 1"] \\
& X \otimes Y &
\end{tikzcd}$$

If the associator α and the units η_R and η_L are always identity morphisms, then the symmetric monoidal category is called strict.

An important remark about monoidal categories is that we have Mac Lane's coherence theorem, which says that if we have two valid terms consisting of objects, units, tensors and brackets, then all ways of getting from the first term to the second by applying associators and units gives the same morphism.

Definition B.6. Let T denote the twist functor of $\mathsf{C} \times \mathsf{C}$, i.e. the functor given on objects by $T(X,Y) = (Y,X)$. A symmetric monoidal category is a monoidal category with an additional natural isomorphism $c : \otimes \circ T \stackrel{\cdot}{\longrightarrow} \otimes$ called the symmetry. This symmetry should satisfy the following axioms.

Idempotence: The following diagram commutes:

$$\begin{tikzcd} X \otimes Y \arrow[r, "c"] \arrow[dr, "id"'] & Y \otimes X \arrow[d, "c"] \\ & X \otimes Y \end{tikzcd}$$

Compatibility with associativity: The following diagram commutes:

$$\begin{tikzcd} (X \otimes Y) \otimes Z \arrow[r, "a"] \arrow[d, "c\otimes id"'] & X \otimes (Y \otimes Z) \arrow[r, "c"] & (Y \otimes Z) \otimes X \arrow[d, "a"] \\ (Y \otimes X) \otimes Z \arrow[r, "a"'] & Y \otimes (X \otimes Z) \arrow[r, "id\otimes c"'] & Y \otimes (Z \otimes X) \end{tikzcd}$$

Compatibility with units: The following diagram commutes:

$$\begin{tikzcd} 1 \otimes X \arrow[rr, "c"] \arrow[dr, "\eta_L"'] & & X \otimes 1 \arrow[dl, "\eta_R"] \\ & X & \end{tikzcd}$$

Mac Lane's coherence theorem extends to this context as well: if we have two valid terms consisting of objects, units, tensors and brackets, then all way of getting from the first term to the second by applying associators, symmetries and the unit maps gives the same morphism. In practice this means that we don't need to worry about the exact way in which we use symmetry and associativity.

Remark B.7. Some of these axioms are redundant. Because the compatibility with the units defines η_R in terms of η_L, we can remove it from the axioms.

Example B.8. We list some symmetric monoidal categories.

- The category Set of sets. This has as tensor product the usual products of sets. A generalisation of this is the category of simplicial sets SSet, where the tensor product is given by the degree-wise product of simplicial sets.
- The category Top of topological spaces with tensor products the space with underlying set the product set and the product topology. There are variations on this: the category CW of CW-complexes, the category of CW_h of the spaces with the homotopy type of CW-complexes, the category $\mathsf{CGWHTop}$ of compactly generated weakly Hausdorff spaces. In some of these cases we need to modify the product by applying the Kelleyfication functor. The last two variations are useful, because in this case the product admits a right adjoint. In particular in 1.2 of chapter 2 we decided to use $\mathsf{CGWHTop}$.

  A generalization of Top is the category of spectra $\mathsf{Spectra}$. There are several different models, which are morally equivalent. It is always true that homotopy category of spectra $\mathsf{HSpectra}$ is symmetric monoidal, but for a symmetric monoidal smash product on spectra one needs to pick a nice model of spectra. See appendix A for more information about these.

- The category Vect_k of vector spaces over a field k. The tensor product is the ordinary tensor product of vector spaces. This is a special case of the category Mod_R of R-modules for a commutative ring R. Another special case is $R = \mathbb{Z}$, in which cases we obtain the category of abelian groups Abgrps.

  We can also take into account a grading on the abelian groups. This is useful when dealing with homology or cohomology groups. The tensor product then becomes $(V \otimes W)_n = \bigoplus_{k+l=n} V_k \otimes W_l$. The resulting category is denoted $\mathsf{GrAbgrps}$.

  Alternatively, we can look at formal differences of vector spaces to get the category VirtV_k of virtual vector spaces over a field k. These play a role in modelling virtual vector bundles, just like vector spaces model vector bundles.
- The two-dimensional cobordism category Bord^{π_0} has a objects the non-negative integers and morphisms the isomorphism classes of oriented two-dimensional cobordisms. This is a symmetric monoidal category, with product on objects being addition and on morphisms disjoint union of cobordisms. This is the object underlying the theory of topological quantum field theories, as in section 3.2 of chapter 6.

1.3. Enriched categories. In category theory, we have learned that it is often useful to abstract algebraic structures such that we can define them in other categories than the category of sets. The idea is that any object whose axioms can be defined as a collection as sets and functions satisfying certain commutative diagrams can be abstracted to any category by defining it to be a collection of objects and morphisms which satisfying the same commutative diagrams in that category.

If we are working in a symmetric monoidal category, we can also abstract structures in which the commutative diagrams contain products. These are replaced by the tensor product.

For example, the notion of an algebra can be given in terms of axioms involving a product map and unit map which satisfy associativity, commutativity and identity properties. These axioms can be written in terms of diagrams and by demanding that similar commutative diagrams exist, we get the notion of an algebra object. For example, a monoid in Top is nothing but a strict H-space.

We can therefore apply this procedure to the first form of the axioms of a category, thereby getting a notion of a category enriched in a monoidal category. These are categories where the morphisms have been replaced by object of a symmetric monoidal category.

Definition B.9. Let $(\mathsf{C}, \otimes)$ be a monoidal category. A category D enriched in C is a set of objects $\mathrm{Ob}(\mathsf{D})$ and for each pair $(X, Y) \in \mathrm{Ob}(\mathsf{D})$ an object $\mathrm{Hom}(X, Y) \in \mathsf{C}$. There should be composition morphisms $\circ : \mathrm{Hom}(X, Y) \otimes \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z)$ and identity morphisms $id_X : 1 \to \mathrm{Hom}(X, X)$. This data should satisfy the following axioms.

Associativity.: The following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(W,X) \otimes (\mathrm{Hom}(X,Y) \otimes \mathrm{Hom}(Y,Z)) \arrow[r, "id\otimes\circ"] \arrow[d, "a"'] & \mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Z) \arrow[dr, "\circ"] & \\
 & & \mathrm{Hom}(W,Z) \\
(\mathrm{Hom}(W,X) \otimes \mathrm{Hom}(X,Y)) \otimes \mathrm{Hom}(Y,Z) \arrow[r, "\circ\otimes id"'] & \mathrm{Hom}(W,Y) \otimes \mathrm{Hom}(Y,Z) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

Identity.: The following diagrams commute:

$$\begin{tikzcd}
1 \otimes \mathrm{Hom}(X,Y) \arrow[r, "id_X\otimes id"] \arrow[dr, "\eta_L"'] & \mathrm{Hom}(X,X) \otimes \mathrm{Hom}(X,Y) \arrow[d, "\circ"] \\
 & \mathrm{Hom}(X,Y)
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(X,Y)\otimes 1 \arrow[r, "id\otimes id_Y"] \arrow[dr, "\eta_R"'] & \mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(Y,Y) \arrow[d, "\circ"] \\
& \mathrm{Hom}(X,Y)
\end{tikzcd}$$

We said before, an ordinary category is a category enriched in Set. There exists natural extensions of the notions of functors and natural transformations to the enriched context. For completeness' sake we spell them out.

Definition B.10. Let $(\mathsf{C},\otimes)$ be a monoidal category and D,E be categories enriched in C. An enriched functor $F:\mathsf{D}\to\mathsf{E}$ is a map of sets $f:\mathrm{Ob}(\mathsf{D})\to\mathrm{Ob}(\mathsf{D})$ together with a set of morphisms $F:\mathrm{Hom}(X,Y)\to\mathrm{Hom}(f(X),f(Y))$. These morphisms should satisfy the following axioms:

Compatibility with composition: The following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(X,Y)\otimes\mathrm{Hom}(Y,Z) \arrow[r, "F"] \arrow[d, "\circ"'] & \mathrm{Hom}(f(X),f(Y))\otimes\mathrm{Hom}(f(Y),f(Z)) \arrow[d, "\circ"] \\
\mathrm{Hom}(X,Z) \arrow[r, "F"'] & \mathrm{Hom}(f(X),f(Z))
\end{tikzcd}$$

Compatibility with identities: The following diagram commutes:

$$\begin{tikzcd}
& 1 \arrow[dl, "id_X"'] \arrow[dr, "id_{f(X)}"] & \\
\mathrm{Hom}(X,X) \arrow[rr, "F"'] & & \mathrm{Hom}(f(X),f(X))
\end{tikzcd}$$

Definition B.11. Let $(\mathsf{C},\otimes)$ be a monoidal category, D,E be categories enriched in C and $F,G:\mathsf{D}\to\mathsf{E}$ enriched functors. A enriched natural transformation $\beta:F\dot{\longrightarrow}G$ is a set of morphisms $\beta_X:1\to\mathrm{Hom}(f(X),g(X))$ such that the following diagram commutes:

$$\begin{tikzcd}
1\otimes\mathrm{Hom}(X,Y) \arrow[r, "\beta_X\otimes G"] & \mathrm{Hom}(f(X),g(X))\otimes\mathrm{Hom}(g(X),g(Y)) \arrow[dr, "\circ"] & \\
\mathrm{Hom}(X,Y) \arrow[u, "\eta_L^{-1}"] \arrow[d, "\eta_R^{-1}"'] & & \mathrm{Hom}(f(X),g(Y)) \\
\mathrm{Hom}(X,Y)\otimes 1 \arrow[r, "F\otimes\beta_Y"'] & \mathrm{Hom}(f(X),f(Y))\otimes\mathrm{Hom}(f(Y),g(Y)) \arrow[ur, "\circ"'] &
\end{tikzcd}$$

More complicated constructions like limits, colimits, Kan extensions, etc. are not always straightforward and we will only introduce them where they are required. The interested reader can consult [**Kel05**].

1.4. Internal categories. However, we can go one step further and try to get the objects to be an object of category instead of a set of object. To do this we use the second formulation of the axioms of a category.

First we remark that in the process of transferring definitions from the world of sets to category theory, sometimes the operation we need to transfer is to be defined on a certain subobject. If this suboject is given by equations, then we can write this subobject in terms of category theory, by replacing its definition with a certain limit, for example a pullback. To make this work, one needs to assume that the category one is working in is sufficiently complete. This will always be the case in those situations we will be working with.

A topological category is something stronger. We also put a topology on the objects.

Definition B.12. Let C be a category with pullbacks. A category D internal to C is an object $\mathrm{Ob}(\mathsf{D})$ of D together with an object $\mathrm{Hom}(\mathsf{D})$ of D, together with maps $1:\mathrm{Ob}(\mathsf{D})\to\mathrm{Hom}(\mathsf{D})$,

$s, t : \mathrm{Hom}(\mathsf{D}) \to \mathrm{Ob}(\mathsf{D})$ and $\circ : \mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \to \mathrm{Hom}(\mathsf{D})$, where the pullback is taken over the diagram

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[r, "\pi_2"] \arrow[d, "\pi_1"'] & \mathrm{Hom}(\mathsf{D}) \arrow[d, "s"] \\
\mathrm{Hom}(\mathsf{D}) \arrow[r, "t"'] & \mathrm{Ob}(\mathsf{D})
\end{tikzcd}$$

This data should satisfy the following conditions.

Correct source and target of composition: The following diagram commutes:

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{D}) \arrow[d, "s"'] & \mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] \arrow[d, "\circ"] & \mathrm{Hom}(\mathsf{D}) \arrow[d, "t"] \\
\mathrm{Ob}(\mathsf{D}) & \mathrm{Hom}(\mathsf{D}) \arrow[l, "s"] \arrow[r, "t"'] & \mathrm{Ob}(\mathsf{D})
\end{tikzcd}$$

Correct source and target of the identity: The following diagram commutes

$$\begin{tikzcd}
& \mathrm{Ob}(\mathsf{D}) \arrow[dl, "1"'] \arrow[d, equal] \arrow[dr, "1"] & \\
\mathrm{Hom}(\mathsf{D}) \arrow[r, "s"'] & \mathrm{Ob}(\mathsf{D}) & \mathrm{Hom}(\mathsf{D}) \arrow[l, "t"]
\end{tikzcd}$$

Associativity: The following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[r, "\circ \times id"] \arrow[d, "id \times \circ"'] & \mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[d, "\circ"] \\
\mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[r, "\circ"'] & \mathrm{Hom}(\mathsf{D})
\end{tikzcd}$$

Identity: The following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Ob}(\mathsf{D}) \arrow[r, "id \times 1"] \arrow[dr, "\pi_1"'] & \mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[d, "\circ"] & \mathrm{Ob}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[l, "1 \times id"'] \arrow[dl, "\pi_2"] \\
& \mathrm{Hom}(\mathsf{D}) &
\end{tikzcd}$$

We want to the generalize the notions of functor and natural transformation to this setting, looking at the set-valued case and enriched case for advice. For the rest of this section, we implicitly fix a category with pullbacks to work in.

Definition B.13. A internal functor $F : \mathsf{C} \to \mathsf{D}$ between internal categories is a pair of morphisms $\mathrm{Ob}(F) : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Ob}(\mathsf{D})$ and $\mathrm{Hom}(F) : \mathrm{Hom}(\mathsf{C}) \to \mathrm{Hom}(\mathsf{D})$, such that these morphisms commute with the morphisms s, t, $\circ$ and 1. More concretely, this means the data should satisfy the following conditions:

Correct source and target: The following diagram commutes

$$\begin{tikzcd}
\mathrm{Ob}(\mathsf{C}) \arrow[d, "\mathrm{Ob}(F)"'] & \mathrm{Hom}(\mathsf{C}) \arrow[l, "s"'] \arrow[r, "t"] \arrow[d, "\mathrm{Hom}(F)"] & \mathrm{Ob}(\mathsf{C}) \arrow[d, "\mathrm{Ob}(F)"] \\
\mathrm{Ob}(\mathsf{C}) & \mathrm{Hom}(\mathsf{D}) \arrow[l, "s"'] \arrow[r, "t"] & \mathrm{Ob}(\mathsf{C})
\end{tikzcd}$$

Compatible with composition: The following diagram commutes

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(C)} \mathrm{Hom}(\mathsf{C}) \arrow[r, "\mathrm{Hom}(F)\times\mathrm{Hom}(F)"] \arrow[d, "\circ"'] & \mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(D)} \mathrm{Hom}(\mathsf{D}) \arrow[d, "\circ"] \\
\mathrm{Hom}(\mathsf{D}) \arrow[r, "\mathrm{Hom}(F)"'] & \mathrm{Hom}(\mathsf{D})
\end{tikzcd}$$

Compatible with identities: The following diagram commutes

$$\begin{tikzcd}
\mathrm{Ob}(\mathsf{C}) \arrow[r, "\mathrm{Ob}(F)"] \arrow[d, "1"'] & \mathrm{Ob}(\mathsf{D}) \arrow[d, "1"] \\
\mathrm{Hom}(\mathsf{C}) \arrow[r, "\mathrm{Hom}(F)"'] & \mathrm{Hom}(\mathsf{D})
\end{tikzcd}$$

Definition B.14. A internal natural transformation η between internal functors $F, G : \mathsf{C} \to \mathsf{D}$ is a morphism $\eta : \mathrm{Ob}(\mathsf{C}) \to \mathrm{Hom}(\mathsf{D})$ such that the following diagrams commute

$$\begin{tikzcd}
 & \mathrm{Ob}(\mathsf{C}) \arrow[dl, "\mathrm{Ob}(F)"'] \arrow[d, "\eta"] \arrow[dr, "\mathrm{Ob}(G)"] & \\
\mathrm{Ob}(\mathsf{D}) & \mathrm{Hom}(\mathsf{D}) \arrow[l, "s"] \arrow[r, "t"'] & \mathrm{Ob}(\mathsf{D})
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{Hom}(\mathsf{C}) \arrow[r, "(\eta\circ s)\times\mathrm{Hom}(\mathsf{G})"] \arrow[d, "\mathrm{Hom}(\mathsf{F})\times(\eta\circ t)"'] & \mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[d, "\circ"] \\
\mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D}) \arrow[r, "\circ"'] & \mathrm{Hom}(\mathsf{D})
\end{tikzcd}$$

1.5. Topological categories. We will be interested in the previous notions of enriched and internal cateogires for the case $\mathsf{C} = \mathsf{Top}$. This is a very nice kind of symmetric monoidal category, because the categorical product coincides with the tensor product.

Definition B.15. A category enriched in topological spaces is a category enriched in Top, as above. A topological category is a category internal to Top.

Remark that a topological category with discrete object space is a category enriched in topological spaces. If the morphisms spaces are discrete as well, we simply get back an ordinary category.

1.5.1. *Constructing topological categories using spaces with action.* In this section we describe a method to produce new topological categories using so called spaces with an action of a category. These should be thought of as functors from a topological category to Top.

Definition B.16. Let C be a topological category. A space with action A is consists of a space $A(\mathrm{Ob}(\mathsf{C}))$ together with a surjective continuous map $p : A(\mathrm{Ob}(\mathsf{C})) \to \mathrm{Ob}(\mathsf{C})$ and a continuous map $a : A(\mathrm{Ob}(\mathsf{C})) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \to A(\mathrm{Ob}(\mathsf{C}))$, where the former space is obtained as the following pullback:

$$\begin{tikzcd}
A(\mathrm{Ob}(\mathsf{C})) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \arrow[r] \arrow[d] & \mathrm{Hom}(\mathsf{C}) \arrow[d, "s"] \\
A(\mathrm{Ob}(\mathsf{C})) \arrow[r, "p"'] & \mathrm{Ob}(\mathsf{C})
\end{tikzcd}$$

These should satisfy the following axioms:

Compatibility with source and target: For any point $(x, f) \in A(\mathrm{Ob}(\mathsf{C})) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C})$ we have that $t(f) = p(a(x, f))$.

Compatiblity with identity: For any point of the form (x, id), we have that $a(x, id) = x$.

Associativity of a: The two maps $A(\mathrm{Ob}(\mathsf{C})) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C}) \to X(\mathrm{Ob}(\mathsf{C}))$ obtained by either applying a twice, or by first composing the morphisms and then applying a, are equal.

Our remark that these spaces with action should be thought of as functor into Top can be made precise in the case that C is an ordinary category.

Lemma B.17. *Let C be an ordinary category seen as a discrete topological category, then the notion of an ordinary functor $F : \mathsf{C} \to \mathsf{Top}$ and a space with action A are equivalent.*

Proof. Suppose that we are given an ordinary functor, then this determines a space with action A by setting $A(\mathrm{Ob}(\mathsf{C})) = \coprod_{X \in \mathrm{Ob}(\mathsf{C})} F(X)$, giving p value X on the component $F(X)$ and determining a using the fact that $F(\mathrm{Ob}(\mathsf{C})) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C})$ is the space $\coprod_{X \in \mathrm{Ob}(\mathsf{C})} \left(\coprod_{f : X \to Y} F(X) \right)$ and setting on a component $F(X)_f$ by $F(f)$.

For the converse, we set $F(X)$ to be equal to $p^{-1}(X)$ for each object $X \in \mathrm{Ob}(\mathsf{C})$. Similarly a determines $F(f)$ by restricting a to the part of $F(\mathrm{Ob}(\mathsf{C})) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C})$ belonging to $p^{-1}(X) \times \{f\}$. ☐

An important construction is the topological version of the Grothendieck construction. We use it chapter 5 to interweave the data of the functors Split and Umk into the category $\nabla\mathsf{Fn}$.

Definition B.18. Let C be a topological category and A be a space with action. Then we define $\mathsf{C} \int A$ to be following topological category.

It has the object space $A(\mathrm{Ob}(\mathsf{C}))$ and the morphism space $A(\mathrm{Ob}(\mathsf{C})) \times_{\mathrm{Ob}(\mathsf{C})} \mathrm{Hom}(\mathsf{C})$. The source map is given on (x, f) by x and the target map by $a(x, f)$. The composition map is given by $(x', f') \circ (x, f) = (x', f' \circ f)$ and the identity map by $1(x) = (x, 1(p(x)))$.

2. Geometric realisation, homotopy colimits and homotopy limits

In this section we discuss an collection of related concepts that play a role in this thesis. All are concerned with making topological spaces out of other objects and determining the homotopy type of this space.

2.1. Geometric realisation of simplicial sets and simplicial spaces. Remember that Δ is the category with objects the finite ordered sets $[n] = \{0, \ldots, n\}$ for $n \in \mathbb{N} \sqcup \{0\}$ and morphisms the order-preservering maps. A simplicial set is a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$. Alternatively, we can characterize a simplicial set by a sequence of sets and morphisms satisfying certain commutative diagrams. The category of simplicial sets is denoted by SSet.

The description of a simplicial set as a functor of Δ^{op} into Set means that we can define a simplicial object in any category C as a functor $\Delta^{\mathrm{op}} \to \mathsf{C}$. A cosimplicial object is a functor $\Delta \to \mathsf{C}$.

Let X be a simplicial set. We will construct the geometric realisation of X as a space with one topological n-simplex for each n-simplex of X, which is attached to the $(n-1)$=simplices and $(n+1)$-simplices corresponding to degeneracy and boundary maps respectively.

Definition B.19. There is a cosimplicial object Δ^t in Top. This is given by $\Delta^n_t = \{(t_0, \ldots, t_n) \in [0,1]^{n+1} | \sum_i t_i = 1\}$.

Definition B.20. The geometric realisation of a simplicial set X is the topological space given by

$$\left(\coprod_{n \in \mathbb{N} \sqcup \{0\}} \coprod_{x \in X_n} \Delta^n_t \times \{x\} \right) / \sim$$

where the equivalence relation $\sim$ identifies the following points: $(\delta t, x) \sim (t, \partial x)$ and $(\sigma t, x) \sim (t, sx)$.

Proposition B.21. *The geometric realisation preserves products if one of the simplicial sets has countable many elements.*

This theorem can be made to hold for any simplicial set if one works in the category of compactly generated Hausdorff spaces or if one applies the Kelley functor.

We now want to generalize this to simplicial spaces. We start by defining these. Recall that we defined a simplicial set to be a functor $\Delta^{\mathrm{op}} \to \mathsf{Set}$. To define a simplicial space, we simply replace Set by Top. This gives a category STop of simplicial spaces.

Definition B.22. The geometric realisation is of a simplicial space $\mathcal{X}$ is the topological space given by

$$\left(\coprod_{n \in \mathbb{N} \sqcup \{0\}} \Delta^n \times \mathcal{X}_n \right) / \sim$$

where the equivalence relation $\sim$ identifies the following points: $(\delta t, x) \sim (t, \partial x)$ and $(\sigma t, x) \sim (t, sx)$ and the quotient is given the quotient topology.

Remark that a simplicial set can be seen as a simplicial space which is discrete in each degree. Then the definition of the geometric realisation of a simplicial set given previously coincides with this one. One can also extend the definition of geometric realisation to other cocomplete categories with a nice cosimpicial object or categories tensored over Top. In particular, we will sometimes be interested in simplicial spectra and their geometric realisation.

2.2. The nerve and geometric realisation of categories. We will discuss a method to create a space out of a category. This will be useful as a way to build complicated spaces from a combinatorial structure as encoded by a category.

2.2.1. *The never and geometric realisation of a category.* The nerve $\mathcal{N}\mathsf{C}$ of a category C is a simplicial set containing all the information about the category. It is given by considering the cosimplicial object in Cat given by $n \mapsto [n]$, the linear category with object $\{0, \dots, n\}$ and morphisms from $i+1$ to i. Then $\mathcal{N}_k\mathsf{C} = \mathrm{Fun}([k], \mathsf{C})$. More concretely, the k-simplices of $\mathcal{N}\mathsf{C}$ are composable strings of k morphisms:

$$X_n \xrightarrow{f_n} \dots \xrightarrow{f_1} X_0$$

We will now discuss some properties of the nerve construction.

Proposition B.23. *The nerve extends to a functor $\mathcal{N} : \mathsf{Cat} \to \mathsf{SSet}$. It preserves products, i.e. $\mathcal{N}(\mathsf{C} \times \mathsf{D})$ isomorphic to $\mathcal{N}(\mathsf{C}) \times \mathcal{N}(\mathsf{D})$.*

The geometric realisation gives a way to construct a space from a simplicial set, so it in particular applies to the nerve of a category.

Definition B.24. The geometric realisation $|\mathsf{C}|$ of a category C is the geometric realisation of the simplicial set $\mathcal{N}_*\mathsf{C}$. Since the construction of a nerve from a category and the geometric realisation of a simplicial set are both functorial, we obtain a functor $|-| : \mathsf{Cat} \to \mathsf{Top}$.

Note that because both the nerve $\mathcal{N}$ and the geometric realisation $|-|$ of simplicial sets preserve products, the geometric realisation of categories will preserve products as well.

2.2.2. *Natural transformations and homotopy equivalences.* We want to understand when the geometric realisations of two categories are homotopy equivalent. The most important proposition in this respect is the following.

Proposition B.25. *A natural transformation η between two functors $F, G : \mathsf{C} \to \mathsf{D}$ induces a homotopy $|\eta|$ between $|F|$ and $|G|$.*

Proof. We note that two functors with a natural transformation between then is the same as a functor $\mathsf{C} \times [1] \to \mathsf{D}$, where $[1]$ is the category consisting of objects $\{0, 1\}$ with a single morphism $0 \to 1$ in addition to the identities. Note that are functors $d^i : \mathsf{C} \hookrightarrow \mathsf{C} \times [1]$ for $i = 0, 1$ given by the $X \mapsto X \times \{i\}$ on objects and by $f \mapsto f \times \{i\}$ on morphisms. This way we obtain a commutative diagram in Cat:

$$\begin{tikzcd} \mathsf{C} \arrow[d, "d^0"'] \arrow[dr, "F"] & \\ \mathsf{C} \times [1] \arrow[r, "\eta"] & \mathsf{D} \\ \mathsf{C} \arrow[u, "d^1"] \arrow[ur, "G"'] & \end{tikzcd}$$

If we now apply geometric realisation everywhere, we obtain a commutative diagram:

$$\begin{tikzcd} |\mathsf{C}| \arrow[d, "i_0"'] \arrow[dr, "|F|"] & \\ |\mathsf{C}| \times I \arrow[r, "|\eta|"] & |\mathsf{D}| \\ |\mathsf{C}| \arrow[u, "i_1"] \arrow[ur, "|G|"'] & \end{tikzcd}$$

where we have used that $|[1]| = I$ and the $[1]$ has countable many simplices, from which it follows that $|\mathsf{C} \times [1]| \cong |\mathsf{C}| \times |[1]|$. This displays $|\eta|$ as a homotopy between $|F|$ and $|G|$. □

The proposition has many corollaries, most of them nothing more than looking at particular classes of functors and natural transformations and replacing each natural transformation by a homotopy upon geometric realisation.

Equivalent categories: Remember that an equivalence of categories is a pair of functors that are inverse up to a natural transformation.

Corollary B.26. *Equivalent categories have homotopy equivalent geometric realisations. In particular, a category has the same geometric realisation as its skeleton.*

Retraction on subcategory: A direct application is the following condition from a homotopy equivalence in the special case of a category which retracts onto a subcategory.

Corollary B.27. *Let D be a subcategory of C. Then $|\mathsf{C}|$ is homotopy equivalent to $|\mathsf{D}|$ if there exists a functor $R : \mathsf{C} \to \mathsf{C}$ which coincides with the identity functor on D and admits a natural transformation $\eta : R \dot{\longrightarrow} id_{\mathsf{C}}$ or $\eta' : id_{\mathsf{C}} \dot{\longrightarrow} R$. In this case we say that C retracts onto D.*

Proof. It suffices to note that R can be seen as a functor $\mathsf{C} \to \mathsf{D}$ and together with the inclusion $I : \mathsf{C} \to \mathsf{D}$ forms an equivalence of categories. □

Adjoint functors: Adjoint functors can be defined using the unit and counit natural transformations, leading to the following corollary.

Corollary B.28. *Let $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{C}$ be such that $F \dashv G$ are adjoint functors, then $|F|$ and $|G|$ are inverse up to homotopy and induce a homotopy equivalence between $|\mathsf{C}|$ and $|\mathsf{D}|$.*

Proof. One of the equivalent characterizations of adjoint functors, is that there exist natural transformations $\epsilon : FG \dot{\longrightarrow} id$ and $\eta : id \dot{\longrightarrow} GF$, known as counit and unit respectively. Applying the previous, this proves the result. □

Note that this result can be generalized quite easily. There is no difference between $FG \dot{\longrightarrow} id$ and $id \dot{\longrightarrow} FG$ after applying homotopy. Similarly $id \dot{\longrightarrow} GF$ and $GF \dot{\longrightarrow} id$ both give the same result.

Initial or terminal object: One example where the remark after the last corollary is useful is the following lemma.

Corollary B.29. *The geometric realisation of a category with a terminal object is contractible. Dually, the geometric realisation of a category with an initial object is contractible.*

Proof. Let X be the terminal object. In this case it suffices to show that there is a natural transformation from identity to the the functor C_X. Such a natural transformation $c : id \dot{\longrightarrow} C_X$ is given by setting c_Y the unique map from Y to X. □

Opposite category: A geometric realisation does not remember the direction of an arrow and therefore should not be able to distinguish between a category and its oppostie. This can be proven using the earlier corollary on adjoint functors:

Corollary B.30. *The geometric realisation of a category is homotopy equivalent to the geometric realisation of its opposite category.*

Proof. Let $\mathrm{ArTw}(\mathsf{C})$ be the twisted arrow category. An object is an arrow $f : X \to Y$ and morphism from f to g is a commutative diagram:

$$\begin{tikzcd} X \arrow[r, "f"] & Y \arrow[d] \\ Z \arrow[u] \arrow[r, "g"'] & W \end{tikzcd}$$

Projection on the source is a functor $\mathrm{ArTw}(\mathsf{C}) \to \mathsf{C}^{\mathrm{op}}$ and projection on the target is a functor $\mathrm{ArTw}(\mathsf{C}) \to \mathsf{C}$. Both of these functors admit adjoints, hence we obtain homotopy equivalences

$$|\mathsf{C}^{\mathrm{op}}| \simeq |\mathrm{ArTw}(\mathsf{C})| \simeq |\mathsf{C}|$$

□

Category of (cos)simplices: Let C be a category. Then we define $\nabla\mathsf{C}$ to be the category with objects the k-simplices of the nerve $N\mathsf{C}$ for $k \geq 0$ and morphism the maps induced by the boundary maps. Note that $\nabla\mathsf{C}$ is isomorphic to the category of simplices of $N\mathsf{C}$ ordered by coinclusion, i.e. there is a map $\sigma \to \sigma'$ if $\sigma' \subset \sigma$. We call $\nabla\mathsf{C}$ the category of simplices.

Corollary B.31. *The geometric realisation of $\nabla\mathsf{C}$ is homotopy equivalent to the geometric realisation of C.*

Proof. Let $P : \nabla\mathsf{C} \to \mathsf{C}$ be the functor which maps a sequence $(X_1 \to \ldots \to X_n)$ to X_1 and a morphism $(X_1 \to \ldots \to X_n) \to (X_{i_1} \to \ldots \to X_{i_n})$ to the composition $X_1 \to \ldots \to X_{i_0}$.

The comma category $X\backslash P$ of this functor is the category which has as objects the sequences $(X_1 \to \ldots \to X_n)$ together with a morphism $X \to X_1$, and morphisms as before which are additionally compatible with the morphisms from X. This category is isomorphic to the category of sequences $(X \to X_1 \to \ldots \to X_n)$ with morphisms which never remove X. This has a terminal object X, hence is contractible

Using alternative formulation of Quillen A, here given in the lemma following theorem B.38, we conclude that $|P| : |\nabla\mathsf{C}| \to |\mathsf{C}|$ is a homotopy equivalence. □

2.2.3. *Converging functor system.* Often if one wants to link the geometric realisation of two functors by a homotopy, one tries to construct a zig-zag functors and natural transformations between the two functors. Sometimes it is not possible to give to control the length of the zigzag for every object and morphism at same time, but we do know that the length of each zigzag is 'finite' for every individual object and morphism. This leads us to the definition of an converging functor system.

Definition B.32. A *converging functor system* is a sequence of functors $(F_n)_{n\in\mathbb{N}}$ from C to D and natural transformation $(\eta_n)_{n\in\mathbb{N}}$ where either $\eta_n : F_n \dot{\longrightarrow} F_{n+1}$ or $\eta_n : F_{n+1} \dot{\longrightarrow} F_n$, e.g.:

$$F_1 \dot{\longleftarrow} F_2 \dot{\longrightarrow} F_3 \dot{\longrightarrow} \ldots$$

The functors F_n and natural transformation η_n should satisfy the following properties:

(1) for each object X in C there exists a $N \in \mathbb{N}$ such that for $n > N$ the value $F_n(X)$ is constant and the component of η_n on X is the identity map for $n > N$,
(2) for each morphisms f there exists a $N \in \mathbb{N}$ such that the value $F_n(f)$ is constant.

The two conditions make precise the idea that length of the zig-zag is finite for each individual object and morphism. The next lemma explains what a converging functor system is supposed to converge to.

Lemma B.33. *The functor $F : \mathsf{C} \to \mathsf{D}$ given on an object X by $F_n(X)$ and on morphisms by $F_n(f)$ for n sufficiently large is well-defined.*

We say that such a converging functor system goes from F_1 to F.

Proof. The definition of an converging functor system implies that F is independent of the choices of n for each object. It clearly sends identities to identities, since F_n does this. Therefore it suffices to check associativity. To prove this, for a pair of arrows $f : X \to Y$, $g : Y \to Z$ pick n large enough such that $F_n(f)$ and $F_n(g)$ have stabilized. Then $F(g) \circ F(f) = F_n(g) \circ F_n(f) = F_n(g \circ f) = F(g \circ f)$. □

We will now prove that a converging functor system going from F_1 to F induces a homotopy between $|F_1|$ and $|F|$.

Proposition B.34. *A converging functor system induces a homotopy between $|F_1| : |\mathsf{C}| \to |\mathsf{D}|$ and $|F| : |\mathsf{C}| \to |\mathsf{D}|$.*

Proof. Let $h_n : |F_n| \times I \to |F_{n+1}|$ be the homotopy obtained from η_n, possibly reversing the direction if η_{n+1} is a natural transformation from F_{n+1} to F_n.

Then we set the homotopy h to be equal to h_1 on the interval $t = [0, \frac{1}{2}]$, h_2 on the interval $t = [\frac{1}{2}, \frac{1}{4}]$, etc. Since the η stabilize for each object and morphism and the geometric realisation has the weak topology, h will be a continuous map. Clearly at $t = 0$ it is $|F_1|$ and one can check that at $t = 1$ it is $|F|$. □

2.2.4. *Group actions.* Another way to calculate the homotopy type of the geometric realisation of a category is to look at group actions on it or on related categories.

An action of a group G on a category C is a map ρ from G to functors $\mathsf{C} \to \mathsf{C}$ such that $\rho(gh) = \rho(g) \circ \rho(h)$ and $\rho(e) = id_{\mathsf{C}}$. An action is called free it is free on objects. Necessarily it is then free on morphisms as well, because the source map partitions the morphism set and if an action is free on objects, it permutes these partitions.

This makes the nerve $\mathcal{N}(\mathsf{C})$ into a simplicial G-set, i.e. a simplicial object in the category of G-sets or equivalently a simplicial set with G-action compatible with the simplicial maps.

Proposition B.35. *Let C be a category and $G \to \mathrm{Fun}(\mathsf{C}, \mathsf{C})$ be a free action of a group G on C. If $|\mathsf{C}|$ is contractible and there is a surjective functor $\pi : \mathsf{C} \to \mathsf{D}$ such that π identifies two objects or two morphisms if and only if they are in the same G-orbit, then $|\mathsf{D}|$ is a $K(G, 1)$.*

Proof. The nerve $\mathcal{N}(\mathsf{C})$ is a simplicial G-set, hence $|\mathsf{C}|$ is a G-space. If the action of G on C is free, this action on $|\mathsf{C}|$ is free. Hence $|\mathsf{C}|/G$ is a $K(G, 1)$.

Hence it suffices to show that $|\mathsf{D}|$ can be identified with $|\mathsf{C}|/G$. Since quotients commute, it suffices to show that $\mathcal{N}(\mathsf{C})/G$ can be identified with $\mathcal{N}(\mathsf{D})$ using π by looking at the concrete construction of the geometric realisation. But this is clear from the demands on π. □

Remark B.36. This result is phrased in a way that is well-suited for applications. It is in fact possible to describe D uniquely up to isomorphism. It is the category C/G with objects $\mathrm{Ob}(\mathsf{C})/G$ and morphisms $\mathrm{Mor}(\mathsf{C})/G$. This only is a category if the G-action is free. Otherwise one might get problems with composition. This can be seen if one looks at the definition of the composition.

The composition is given as follows: let $[f_1]$ and $[f_2]$ be two morphisms, then their composition is given by choosing a representative of f_1 and the unique (this uses freeness) representative of f_2 which makes the composition of representatives in C possible.

If the action of G is not free, then it is still possible to describe a category D such that $|\mathsf{D}|$ is weakly homotopy equivalent to the homotopy orbit space $|\mathsf{C}|_{hG}$. It is given by Thomason's homotopy colimit construction for the diagram $G \to \mathsf{Cat}$ which describes the category with G-action.

A corollary is the following theorem, showing that indeed $|G|$ is a BG and hence the terminology of classifying space for the geometric realisation of a category has some merit to it.

Proposition B.37. *There is a functor $B : \mathsf{Grp} \to \mathsf{Top}$ such that BG is a $K(G, 1)$ and B preserves products.*

PROOF. The space BG is obtained by applying geometric realisation to a group G seen as a category with one object and only invertible morphisms. It suffices to show that this is a $K(G, 1)$. To do this, we define the category EG with objects indexed by G and the sets of morphisms from g to h just the single arrow (g, h) and composition $(g, h) \circ (h, k) = (g, k)$. Then $|EG|$ is contractible, because it has a terminal object e; every object h has a unique arrow (g, e) to it.

Furthmore, EG admits a G-action given by mapping g to the functor F_g which sends an objects h to the object gh and a arrow (h, k) to the arrow (gh, gk). This action is free.

The functor $\pi : EG \to G$ which maps an object g to the single object and a morphism (h, k) to the morphism kh^{-1} identifies two object or two morphisms if and only if they are in the same G-orbit. We conclude that $|G|$ is a $K(G, 1)$. □

2.3. Quillen's theorem A and B. Quillen's theorem A and B are terribly useful theorems to help calculate the homotopy type of the geometric realisation of categories. The original proofs can be found in [**Qui73**].

The context is as follows: we have a functor $F : \mathsf{C} \to \mathsf{D}$. Quillen's Theorem A tells us when the geometric realisation of this functor is a homotopy equivalence. In essence, it tells us that this is the case if the geometric realisation of each fiber is contractible. The fiber over an object Y of C in this case is the geometric realisation of the the comma category F/Y. This has as objects a pair (X, f) of an object X of C and a map $f : F(X) \to Y$ and as morphisms $\phi : (X, f) \to (X', f')$ the morphisms $w : X \to X'$ such that $f \circ F(w) = f'$.

Theorem B.38 (Quillen's theorem A)**.** *Suppose that $|F/Y|$ is (weakly) contractible for all objects Y of D, then $|F| : |\mathsf{C}| \to |\mathsf{D}|$ is a (weak) homotopy equivalence.*

There is an alternative formulation of this theorem, pointed out by Quillen directly after the statement of theorem A. It uses the comma category $Y\backslash F$. This has as objects a pair (X, f) of an object X of C and a map $f : Y \to F(X)$ and as morphisms $\phi : (X, f) \to (X', f')$ the morphisms $w : X \to X'$ such that $F(w) \circ f = f'$.

Lemma B.39. *In Quillen's theorem A, one can replace the condition that each $|F/Y|$ is contractible with the condition that each $|Y\backslash F|$ is contractible.*

PROOF. Consider $F^{\mathrm{op}} : \mathsf{C}^{\mathrm{op}} \to \mathsf{D}^{\mathrm{op}}$, then the comma category F^{op}/Y has as objects a pair (X, f) of an object X of C and a morphism $f : Y \to F(X)$ and as morphisms $\phi : (X, f) \to (X', f')$ the morphisms $w : X' \to X$ such that $F(w) \circ f' = f$. This category is clearly is isomorphic to $Y\backslash F$. Hence $|F^{\mathrm{op}}|$ is a homotopy equivalence if and only if each $|Y\backslash F|$ is contractible. But we know from the geometric realisation of a category is homotopy equivalent to that of its opposite, which proves the lemma. □

Quillen's theorem B is more advanced, explaining what can be done if the fibers are not contractible.

Theorem B.40 (Quillen's theorem B)**.** *Suppose that $|F/Y|$ is (weakly) homotopy equivalent to a space X for all objects Y of D and each morphism $f : Y \to Y'$ induces a homotopy equivalence. Then there is a homotopy fibration $X \longrightarrow |\mathsf{C}| \xrightarrow{|f|} |\mathsf{D}|$.*

2.4. The nerve and geometric realisation of topological categories. We will now describe a notion of nerve and geometric realisation of topological categories, which extends the usual definition of geometric realisation of a normal category when we see the latter as a topological category with discrete object and morphism spaces.

We start by noting that we can define the nerve $\mathcal{N}\mathsf{C}$ of a topological category D to be the following space in degree k:

$$\mathcal{N}_k\mathsf{C} = \underbrace{\mathrm{Hom}(\mathsf{D}) \times_{\mathrm{Ob}(\mathsf{D})} \ldots \times_{\mathrm{Ob}(\mathsf{D})} \mathrm{Hom}(\mathsf{D})}_{k+1}$$

As before, this construction satisfies the following properties:

Proposition B.41. *The nerve extends to a functor $\mathcal{N} : \mathsf{TopCat} \to \mathsf{STop}$, which preserves products.*

Definition B.42. The geometric realisation of topological category C is given the geometric realisation of the simplicial space $\mathcal{N}\mathsf{C}$. This gives a functor $|-| : \mathsf{TopCat} \to \mathsf{Top}$.

One can prove a lot of the same theorems about the geometric realisation of normal categories as well in the case of topological categories, with analogous proofs. However, there are some things to be careful about. In particular, one is interested in the situation where a map $\mathcal{X} \to \mathcal{Y}$ of simplicial spaces is a weak homotopy equivalence in each degree. This is the generalization of an isomorphism of categories and hence should give a weak equivalence of geometric realisations. It turns out that this is true only under certain conditions.

Definition B.43. A simplicial space $\mathcal{X}$ is said to be good if each degeneracy map $s_i : \mathcal{X}_{n-1} \to \mathcal{X}_n$ is a closed cofibration.

A closed cofibration is a Hurewicz cofibration which is also closed. It is known that a product of closed cofibration is a closed cofibration, which we will use at some point.

Proposition B.44. *If $\mathcal{X}$ and $\mathcal{Y}$ are good simplicial spaces and $f : \mathcal{X} \to \mathcal{Y}$ is a map of simplicial spaces which is a levelwise weak equivalence, then $|f| : |\mathcal{X}| \to |\mathcal{Y}|$ is a weak equivalence.*

2.5. The fat geometric realisation. The previous result only works for good simplicial spaces. Clearly this is a weakness and it turns out that one can trade in a good property of the geometric realisation to solve this problem.

The idea is not to quotient out the degeneracies in the definition of the geometric realisation

Definition B.45. The fat geometric realisation $||-|| : \mathsf{sSet} \to \mathsf{Top}$ is a functor given on an object X by

$$\left(\coprod_{n \in \mathbb{N} \sqcup \{0\}} \coprod_{x \in X_n} \Delta^n_t \times \{x\} \right) / \sim_{\mathrm{fat}}$$

where the equivalence relation $\sim_{\mathrm{fat}}$ identifies the following points: $(\delta t, x) \sim (t, \partial x)$.

The fat geometric realisation is homotopy equivalent to the ordinary geometric realisation, but no longer preserves products. It doesn't really help when working with simplicial sets, but is good for simplicial spaces.

Definition B.46. The fat geometric realisation $||-|| : \mathsf{sTop} \to \mathsf{Top}$ of simplicial spaces is a functor given on an object $\mathcal{X}$ by

$$\left(\coprod_{n \in \mathbb{N} \sqcup \{0\}} \Delta^n_t \times \mathcal{X}_n \right) / \sim_{\mathrm{fat}}$$

where the equivalence relation $\sim_{\mathrm{fat}}$ identifies the following points: $(\delta t, x) \sim (t, \partial x)$.

Now we can see the advantage of using the fat geometric realisation: we no longer need conditions like being a good simplicial space to get a weak equivalence of fat geometric realisations from a levelwise weak equivalence.

Proposition B.47. *If $\mathcal{X}$ and $\mathcal{Y}$ are simplicial spaces and $f : \mathcal{X} \to \mathcal{Y}$ is a map of simplicial spaces which is a levelwise weak equivalence, then $||f|| : ||\mathcal{X}|| \to ||\mathcal{Y}||$ is a weak equivalence.*

Of course, this condition of goodness must appear somewhere and in fact appears when trying to compare the fat and ordinary geometric realisation.

Proposition B.48. *If $\mathcal{X}$ is a good simplicial space the quotient map $||\mathcal{X}|| \to |\mathcal{X}|$ is a weak equivalence.*

2.6. Bisimplicial sets and homotopy colimits. In this section we will discuss homotopy colimits and in particular the model that we will use in this thesis.

2.6.1. *Bisimplicial sets and homotopy colimits of simplicial sets.* A bisimplicial set is a simplicial object in SSet or equivalently (using an internal hom adjunction) a functor $X_{**} : \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}} \to \mathsf{Set}$. These form a category SSet_2 From this second description it is easy to see that there is a diagonal map $\Delta^{\mathrm{op}} \to \Delta^{\mathrm{op}} \times \Delta^{\mathrm{op}}$ which induces functor $\mathrm{diag} : \mathsf{SSet}_2 \to \mathsf{SSet}$.

There is a categorical generalisation of the bar construction, usually used to construct nice resolutions in homological algebra. In this construction we will see a bisimplicial set as a simplicial object in simplicial sets.

Definition B.49. Let J be a small category and $F : \mathsf{J}^{\mathrm{op}} \to \mathsf{SSet}$ and $G : \mathsf{J} \to \mathsf{SSet}$ be functors. Then $B_{**}(F, \mathsf{J}, G)$ is the bisimplicial set given in degree n by the simplicial set

$$B_{n*}(F, \mathsf{J}, G) = \coprod_{j:[n]\to\mathsf{J}} F(j(n)) \times G(j(0))$$

where the coproduct runs over all functors $j : [n] \to \mathsf{J}$.

Definition B.50. The homotopy colimit $\mathrm{hocolim}_{\mathsf{J}} F$ is by definition the diagonal simplicial set $\mathrm{diag} B_{**}(C_*, \mathsf{J}, F)$, where C_* is the unique functor on J with codomain the terminal category $*$.

We state some theorems on the diagonal functor in relation to homotopy colimits.

Theorem B.51. *Let X, Y be bisimplicial sets $f : X \to Y$ be a map of bisimplicial sets such that f_n is weak equivalence of simplicial sets between X_n and Y_n for all $n \in \mathbb{Z}_{\geq 0}$. Then* $\mathrm{diag}(f) : \mathrm{diag}X \to \mathrm{diag}Y$ *is a weak homotopy equivalence.*

We prove that the homotopy colimit indeed satisfies one of the properties that one want the homotopy colimit to have.

Proposition B.52. *Let $\eta : F \to G$ be a natural transformation between functors $F, G : \mathsf{J} \to \mathsf{SSet}$. Then there is an induced map* $\mathrm{hocolim}\,\eta : \mathrm{hocolim}\,F \to \mathrm{hocolim}\,G$*. If η is such that $\eta_i : F(i) \to G(i)$ induces a weak homotopy equivalence in geometric realisation for all objects i of J, then* $\mathrm{hocolim}\,\eta$ *is a weak homotopy equivalence.*

PROOF. There is an induced map on bar constructions $B_* * (*, (J), \eta) : B_{**}(*, \mathsf{J}, F) \to B_{**}(*, \mathsf{J}, G)$ given by applying $id \times \eta_{j(0)}$ on each component.

The assumption that $\eta_i : F(i) \to G(i)$ induces a weak equivalence in geometric realisation implies that the restriction $B_{n*}(*, \mathsf{J}, F) \to B_{n*}(*, \mathsf{J}, G)$ induces a weak homotopy equivalence. This means that the previous theorem applies. ☐

2.6.2. *Homotopy colimits of geometric realisations of categories.* We will often be concerned with a set of categories that we want to glue in some way. This can achieved through an alternative construction of homotopy colimits in categories, known as Thomason's theorem [**Tho79**]. First we recall the Grothendieck construction.

Definition B.53. Let $F : \mathsf{J} \to \mathsf{Cat}$ be a functor. The Grothendieck construction $\mathsf{J} \int F$ is the category with object pairs (i, x) with i an object of J and x an object of $F(i)$. The morphisms $(i, x) \to (j, y)$ are pairs (f, φ) with $f : i \to j$ and $\varphi : F(f)(x) \to y$.

Theorem B.54. *Let $F : \mathsf{J} \to \mathsf{Cat}$ be a functor. The $|\mathsf{J} \int F|$ is weakly homotopy equivalent to* $\mathrm{hocolim}_{i\in\mathsf{J}} |F(i)|$.

This can be proven using a general theorem of Hirschhorn about homotopy right cofinal functors [**Hir03**, Theorem 19.6.7(a)] and some properties of homotopy left Kan extensions. The former is very useful and we therefore state it here.

Theorem B.55. *Suppose that we have functors $F : \mathsf{C} \to \mathsf{D}$ and $G : \mathsf{D} \to \mathsf{SSet}$ such that the comma category F/D for all objects $D \in \mathsf{D}$ is (weakly) contractible. Then there is a weak equivalence*

$$\operatorname*{hocolim}_{\mathsf{C}} G \circ F \simeq \operatorname*{hocolim}_{\mathsf{D}} G$$

This theorem is turn in a general model category, like Top or $\mathsf{Spectra}$.

2.6.3. *Homotopy colimits of spaces.* Finally, we end with a concrete model for the homotopy colimit of a diagram of spaces. Let $F : \mathsf{J} \to \mathsf{Top}$ be a diagram of spaces. Consider the associated functor $\nabla(\mathrm{Sing}_*(F)) : \mathsf{J} \to \mathsf{Cat}$, which sends an object $j \in J$ to the category of simplices of simplicial set $\mathrm{Sing}_*(F(j))$.

Now Thomason's theorem implies that the homotopy colimit $\mathrm{hocolim}_{\mathsf{J}} |\nabla(\mathrm{Sing}_*(F))|$ is given by:

$$\begin{aligned}\mathrm{hocolim}_{\mathsf{J}} |\nabla(\mathrm{Sing}_*(F))| &= |\mathsf{J} \int \nabla(\mathrm{Sing}_*(F))| \\ &= \left(\coprod_{n \in \mathbb{N} \sqcup \{0\}} \coprod_{j_n \to \ldots \to j_0 \in \mathcal{N}_n \mathsf{J}} \Delta^n \times |\nabla(\mathrm{Sing}_*(F))|(j_0) \right) / \sim\end{aligned}$$

where $\sim$ is the obvious equivalence relation used in the geometric realisation. The latter can be seen as the geometric realisation of the good simplicial space $\tilde{\mathcal{F}}$ given in degree k by

$$\tilde{\mathcal{F}}_k = \coprod_{j_k \to \ldots \to j_0 \in \mathcal{N}_n J} |\nabla(\mathrm{Sing}_*(F))|(j_0)$$

However, often it more convenient to use the fat geometric realisation. To do this, we note that $\tilde{\mathcal{F}}$ is good so that we have a homotopy equivalence between its ordinary and fat geometric realisation. Then if we form the simplicial space $\mathcal{F}$ given in degree k by

$$\mathcal{F}_k = \coprod_{j_k \to \ldots \to j_0 \in \mathcal{N}_n J} F(j_0)$$

there is a levelwise weak equivalence between $\tilde{\mathcal{F}}$ and $\mathcal{F}$. This uses the fact that every space is weakly equivalent to the geometric realisation of the simplex category of its singular simplicial set. We conclude that

$$\mathrm{hocolim}_{\mathsf{J}} |\nabla(\mathrm{Sing}_*(F))| \simeq |\tilde{\mathcal{F}}| \simeq ||\tilde{\mathcal{F}}|| \simeq ||\mathcal{F}||$$

We take the last to be the definition of the homotopy colimit of a diagram of spaces. This clearly agrees with theorem B.54 and is one of the many weakly equivalent definitions of a homotopy colimits.

Definition B.56. The homotopy colimits of $F : \mathsf{J} \to \mathsf{Top}$ is the fat geometric realisation of $\mathcal{F}$:

$$\mathrm{hocolim}_{\mathsf{J}} F = ||\mathcal{F}|| = \left(\coprod_{n \in \mathcal{N} \sqcup \{0\}} \coprod_{j_n \to \ldots \to j_0 \in \mathcal{N}_n \mathsf{J}} \Delta^n \times F(j_0) \right) / \sim_{\mathrm{fat}}$$

Remark B.57. This can be proven by noting that for the usual definitions of homotopy colimits of diagrams of spaces in the literature, the cofibrancy conditions is not needed for an objectwise weak equivalence to induce a weak equivalence of homotopy colimits. Hence the objectwise weak equivalences $|\nabla(\mathrm{Sing}_*(F))| \to F$ induce a weak equivalence

$$\mathrm{hocolim}_{\mathsf{J}} |\nabla(\mathrm{Sing}_*(F))| \simeq \mathrm{hocolim}_{\mathsf{J}} F$$

2.7. Homotopy limits. Finally, we give the definition of a homotopy limit of a diagram of spaces. Because this only appears once, we will not go into details much.

Definition B.58. Let $F : \mathsf{J} \to \mathsf{Top}$ be a diagram of spaces, then the homotopy limit $\mathrm{holim}_{\mathsf{J}} F$ is defined to be the subspace of

$$\prod_{n \in \mathbb{N} \sqcup \{0\}} \prod_{j_n \to \ldots \to j_0 \in \mathcal{N}_n \mathsf{J}} F(j_0)^{\Delta^n}$$

of sequences of maps f_τ for $\tau = (j_n \to \ldots \to j_0)$ such that the following diagrams commute: For all $1 \leq i \leq n$ we have the diagram

$$\begin{tikzcd} \Delta^{n-1} \arrow[r, "f_{d^i\tau}"] \arrow[d, "\delta_i"'] & F(j_0) \arrow[d, equal] \\ \Delta^n \arrow[r, "f_\tau"'] & F(j_0) \end{tikzcd}$$

and for $i = 0$ we have the diagram:

$$\begin{tikzcd}
\Delta^{n-1} \arrow[r, "f_{d^0\tau}"] \arrow[d, "\delta_i"'] & F(j_1) \arrow[d] \\
\Delta^n \arrow[r, "f_\tau"'] & F(j_0)
\end{tikzcd}$$

3. Props and operads

In this section of the appendix we describe how to encode algebraic structures using categorical techniques. In particular we will define props and operads and look at some of their properties.

3.1. (Partial) props. In this section we will describe a very general notion of objects encoding algebraic structures. These are modelled on the endomorphisms of an object in a symmetric monoidal category $(\mathsf{C}, \otimes)$. For any object X, set $\mathrm{End}_X(n,m) = \mathrm{Hom}(X^{\otimes n}, X^{\otimes m})$. There is a composition map

$$\mathrm{End}_X(n,m) \otimes \mathrm{End}_X(m,k) \to End_X(n,k)$$

and the tensor product induces a map

$$\mathrm{End}_X(n_1,m_1) \otimes \ldots\ldots \mathrm{End}_X(n_k,m_k) \to \mathrm{End}_X(n_1+\ldots+n_k, m_1+\ldots+m_k)$$

Our next generalisation uses larger set of objects $\{X, \ldots, Z\}$: in this case we set

$$\mathrm{End}_{X,\ldots,Z}((n_X,\ldots,n_Z),(m_X,\ldots,m_Z)) = \mathrm{Hom}(X^{\otimes n_X} \otimes \ldots \otimes Z^{\otimes n_Z}, X^{\otimes m_X} \otimes \ldots \otimes Z^{\otimes m_Z})$$

and we want to axiomatize the algebraic structure of these endomorphism objects. We begin by giving the definitions and some examples. After that we treat a general construction of new props from old ones. and use this to describe the homology prop associated to a topological prop.

3.1.1. *Definitions and examples of a props.* Recall that we want to axiomatize the algebraic structure in the endomorphisms

$$\mathrm{End}_{X,\ldots,Z}((n_X,\ldots,n_Z),(m_X,\ldots,m_Z)) = \mathrm{Hom}(X^{\otimes n_X} \otimes \ldots \otimes Z^{\otimes n_Z}, X^{\otimes m_X} \otimes \ldots \otimes Z^{\otimes m_Z})$$

between product of several products in a symmetric monoidal category. We start by giving a way to keep track of the products of objects using so-called colourings.

Definition B.59. Let $\mathcal{B}$ be a set of colours. A colouring is an element of the subset of $\mathbb{Z}_{\geq 0}^{\mathcal{B}}$ of functions such that only finitely many integers in the image are non-zero. We can add colourings pointwise, that is, $(f+g)(\beta) = f(\beta) + g(\beta)$.

We can consider the colourings in $\mathbb{Z}_{\geq 0}^{\mathcal{B}}$ as a symmetric monoidal category, discrete and with tensor product given by pointwise addition. We denote this category by $\mathsf{C}_1^{\mathcal{B}}$.

Definition B.60. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A $\mathcal{B}$-coloured sequence with values in C is a symmetric monoidal functor $\mathsf{C}_1^{\mathcal{B}} \to \mathsf{C}$.

Analogously, we can look at pairs of colourings. Again these can be added pointwise. We can consider the colourings in $\mathbb{Z}_{\geq 0}^{\mathcal{B}} \times \mathbb{Z}_{\geq 0}^{\mathcal{B}}$ as a symmetric monoidal category, discrete and with tensor product given by pointwise addition. We denote this category by $\mathsf{C}_2^{\mathcal{B}}$.

Definition B.61. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A $\mathcal{B}$-bicoloured sequence with values in C is a symmetric monoidal functor $\mathsf{C}_2^{\mathcal{B}} \to \mathsf{C}$.

A bicoloured sequence P may seem like something with not much structure, because there are no morphisms in $\mathsf{C}_2^{\mathcal{B}}$, but there is in fact quite a lot going on. To see this, we note that the data is equivalent to picking a set of objects $P(f,g)$ with maps coming by tensor product and symmetric group actions coming from the symmetry. This gives us the following equivalent but more explicit definition of a $\mathcal{B}$-bicoloured sequence.

Definition B.62. A $\mathcal{B}$-bicoloured sequence P with values C is a set of objects $P(f,g)$ of C indexed $f, g \in \mathrm{Ob}(\mathsf{C}_2^{\mathcal{B}})$ together with two sets of maps:

- A left action λ of $\prod_{\beta\in\mathcal{B}}\Sigma_{f(\beta)}$ and a right action ρ of $\prod_{\beta\in\mathcal{B}}\Sigma_{g(\beta)}$ on $P(f,g)$.
- Addition maps $\gamma : P(f_1, f_1') \otimes \ldots \otimes P(f_n, f_n') \to P(f_1 + \ldots + f_n, f_1' + \ldots + f_n')$ for each set $\{(f_1, f_1'), \ldots, (f_n, f_n')\}$ in $\mathrm{Ob}(\mathsf{C}_2^{\mathcal{B}})$.

These maps satisfy the following conditions:

Compatibility λ and ρ: Note by definition of an action, we have that first multiplying in the group and then applying λ or ρ is the same as applying λ or ρ twice. However, these are also compatible in the sense the following diagram commutes

$$\begin{tikzcd}
\prod_{\beta\in\mathcal{B}}\Sigma_{f(\beta)} \times P(f,g) \times \prod_{\beta\in\mathcal{B}}\Sigma_{g(\beta)} \arrow[r, "id\times\rho"] \arrow[d, "\lambda\times id"'] & \prod_{\beta\in\mathcal{B}}\Sigma_{f(\beta)} \times P(f,g) \arrow[d, "\lambda"] \\
P(f,g) \times \prod_{\beta\in\mathcal{B}}\Sigma_{g(\beta)} \arrow[r, "\rho"'] & P(f,g)
\end{tikzcd}$$

Associativity of γ: Because addition is associative, the same should hold for the tensor product. That is, for each set $\{(f_{1,1}, f_{1,1}'), \ldots, (f_{1,k_1}, f_{1,k}'), (f_{2,1}, f_{2,1}'), \ldots, (f_{n,1}, f_{n,k}')\}$ the following diagram commutes.

$$\begin{tikzcd}
\bigotimes_{i=1}^n \bigotimes_{j=1}^k P(f_{i,j}, f_{i,j}') \arrow[r, "\gamma"] \arrow[d, "\gamma"'] & \bigotimes_{i=1}^n P(\sum_{j=1}^k f_{i,j}, \sum_{j=1}^k f_{i,j}') \arrow[d, "\gamma"] \\
\bigotimes_{j=1}^k P(\sum_{i=1}^n f_{i,j}, \sum_{i=1}^n f_{i,j}') \arrow[r, "\gamma"'] & P(\sum_{i=1}^n \sum_{j=1}^k f_{i,j}, \sum_{i,j} f_{i,j}')
\end{tikzcd}$$

Equivariance of γ: The addition maps γ should be compatible with the groups actions. That means that the following diagram should commute for each $\{(f_1, f_1'), \ldots, (f_n, f_n')\}$.

$$\begin{tikzcd}
\bigotimes_{i=1}^n \left(\prod_{\beta\in\mathcal{B}}\Sigma_{f_i(\beta)} \times P(f_i, f_i') \times \prod_{\beta\in\mathcal{B}}\Sigma_{f_i'(\beta)}\right) \arrow[r, "\lambda\otimes\rho"] \arrow[d, "\beta\times\gamma\times\beta"'] & \bigotimes_{i=1}^n P(f_i, f_i') \arrow[d, "\gamma"] \\
\prod_{\beta\in\mathcal{B}}\Sigma_{\sum_{i=1}^n f_i(\beta)} \times P(\sum_{i=1}^n f_i, \sum_{i=1}^n f_i') \times \prod_{\beta\in\mathcal{B}}\Sigma_{\sum_{i=1}^n f_i'(\beta)} \arrow[r, "\lambda\otimes\rho"'] & P(\sum_{i=1}^n f_i, \sum_{i=1}^n f_i')
\end{tikzcd}$$

where $\beta : \prod \Sigma_{k_i} \to \Sigma_{\sum k_i}$ is the disjoint union map, i.e. puts a number of permutations next to each other.

We can now give the abstract definition of a prop, which is essentially a $\mathcal{B}$-bicoloured sequence with additional strucutre. However, we start with a more abstract definition.

Definition B.63. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A $\mathcal{B}$-coloured prop P in C is a strict symmetric monoidal category P enriched in C with objects the objects of $\mathsf{C}_1^{\mathcal{B}}$ such that the tensor product of P is induced by the tensor product of $\mathsf{C}_1^{\mathcal{B}}$.

We will unwind this definition to make it more tractable. On the objects, the fact that P is symmetric monoidal implies that we have an action of $\prod_{b\in\mathcal{B}}\Sigma_{f(b)}$ on the object f. This induces corresponding actions on the morphism objects, giving left and right actions of symmetric groups.

But there is of course much more structure on the morphism objects $P(f,g) \in \mathsf{C}$. The fact that the tensor product of P comes from addition in $\mathbb{Z}_{\geq 0}^{\mathcal{B}}$ implies that we have tensor product maps $\chi : P(f_1, f_1') \otimes \ldots \otimes P(f_n, f_n') \to P(f_1 + \ldots, f_n, f_1' + \ldots + f_n')$ for each pair of sets $\{f_1, \ldots, f_n\}$ and $\{f_1', \ldots, f_n'\}$ in $\mathbb{Z}_{\geq 0}^{\mathcal{B}}$. This should be associative, commutative and respect units.

In addition, we have composition maps between the morphism objects and these must be compatible with the tensor product. This gives us the following description.

Definition B.64. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category. A $\mathcal{B}$-coloured prop in C is a $\mathcal{B}$-bicoloured sequence P in C together with three sets of maps:

- A left action of $\prod_{\beta\in\mathcal{B}}\Sigma_{f(\beta)}$ and a right action of $\prod_{\beta\in\mathcal{B}}\Sigma_{g(\beta)}$ on $P(f,g)$.
- Composition maps $\chi : P(f,g) \otimes P(g,h) \to P(f,h)$ for all $f, g, h \in \mathbb{Z}_{\geq 0}^{\mathcal{B}}$.

- Unit maps $\eta_b : 1 \to P(1_b, 1_b)$ where 1_b is the function which is 1 on $b \in \mathcal{B}$ and zero on all other elements of $\mathcal{B}$.

These operations are subject to the following conditions:

Associativity: There are three associativity axioms: associativity for γ, associativity for χ and compatibility of γ and χ. The first already appears in the axioms of a $\mathcal{B}$-bicoloured sequence and therefore will not be repeated here. The associativity of χ is expressed by the demand that the following diagram commutes:

$$\begin{tikzcd} P(f,g)\otimes P(g,h)\otimes P(h,i) \arrow[r, "\chi\otimes id"] \arrow[d, "id\otimes\chi"'] & P(f,h)\otimes P(h,i) \arrow[d, "\chi"] \\ P(f,g)\otimes P(g,i) \arrow[r, "\chi"'] & P(f,i) \end{tikzcd}$$

The compatibility of γ and χ is expressed by the commutativity of the following diagram:

$$\begin{tikzcd} \bigotimes_{i=1}^n P(f_i,f_i')\otimes P(f_i',f_i'') \arrow[r, "\chi^{\otimes n}"] \arrow[d, "\gamma\otimes\gamma"'] & \bigotimes_{i=1}^n P(f_i,f_i'') \arrow[d, "\gamma"] \\ P(\sum_{i=1}^n f_i, \sum_{i=1}^n f_i')\otimes P(\sum_{i=1}^n f_i', \sum_{i=1}^n f_i'') \arrow[r, "\chi"'] & P(\sum_{i=1}^n f_i, \sum_{i=1}^n f_i'') \end{tikzcd}$$

Equivariance: Both the maps γ and χ should be equivariant with respect to the actions of the symmetric groups. Because the equivariance of γ is already included in the demand that P is a bicoloured sequence, we just need to write down the diagrams expressing equivariance for χ . There are two: the first one expressing "external equivariance" and the second one expressing "internal equivariance.

$$\begin{tikzcd} \prod_{\beta\in\mathcal{B}}\Sigma_{f(\beta)}\times P(f,g)\otimes P(g,h)\times\prod_{\beta\in\mathcal{B}}\Sigma_{h(\beta)} \arrow[r, "\lambda\otimes\rho"] \arrow[d, "id\times\chi\times id"'] & P(f,g)\otimes P(g,h) \arrow[d, "\chi"] \\ \prod_{\beta\in\mathcal{B}}\Sigma_{f(\beta)}\times P(f,h)\times\prod_{\beta\in\mathcal{B}}\Sigma_{h(\beta)} \arrow[r, "\lambda\times\rho"'] & P(f,h) \end{tikzcd}$$

$$\begin{tikzcd} P(f,g)\times\prod_{\beta\in\mathcal{B}}\Sigma_{g(\beta)}\times P(g,h) \arrow[r, "\rho\times id"] \arrow[d, "id\times\lambda"'] & P(f,g)\otimes P(g,f) \arrow[d, "\chi"] \\ P(f,g)\otimes P(g,h) \arrow[r, "\chi"'] & P(f,h) \end{tikzcd}$$

Identity: The maps η_b should acts as left and right identities for χ, in the sense that the following diagram commutes:

$$\begin{tikzcd} 1^{\otimes\sum f(b)}\otimes P(f,g) \arrow[d, "\prod\eta_b\otimes id"'] & P(f,g) \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[dd, equal] & P(f,g)\otimes 1^{\otimes\sum g(b)} \arrow[d, "id\otimes\prod\eta_b"] \\ \bigotimes_{b\in\mathcal{B}} P(1_b,1_b)^{f(b)}\otimes P(f,g) \arrow[d, "\gamma\otimes id"'] & & P(f,g)\otimes\bigotimes_{b\in\mathcal{B}} P(1_b,1_b)^{g(b)} \arrow[d, "id\otimes\gamma"] \\ P(f,f)\otimes P(f,g) \arrow[r, "\chi"'] & P(f,g) & P(f,g)\otimes P(g,g) \arrow[l, "\chi"] \end{tikzcd}$$

Example B.65. The collection $\mathrm{End}_X(n,m)$ is a prop of sets with a single colour. Similarly, the collection $\mathrm{End}_{X,\ldots,Z}((n_X,\ldots,n_Z),(m_X,\ldots,m_Z))$ is a prop of sets with set of colours $\{X,\ldots,Z\}$.

Every prop has an underlying category enriched in C, given by using the colourings as objects and the object $P(f,g)$ as hom-objects. The composition is then simply the map χ.

Sometimes we aren't as lucky as to be able to define a full prop, for example because we don't want to allow morphisms mapping to some objects. The result is a partial prop.

Definition B.66. A partial prop is a prop in the sense of definition B.64 such that the objects $P(f,g)$ and the composition are defined for only a subset of the pairs $f,g \in \mathbb{Z}^{\mathcal{B}}_{\geq 0}$ and then possibly only on a subobject of $P(f,g) \otimes P(g,h)$.

This may seem a bit of a unwieldy definition, but we will now recall two important examples of props that appear in this thesis, making clear the need for the definition of a partial prop.

Example B.67. We freely use the definitions from section 2.1. The isomorphism classes of 2-dimensional closed cobordisms form a prop Bord^{π_0} in Set on a single colour denoting the number of boundary circles. The open-closed cobordisms similarly are a prop in Set on two colours denoting the circle and the interval, and the $\mathcal{B}$-labelled cobordisms are a prop on colours the union of a single colour denoting the circle and pairs of lements in $\mathcal{B} \times \mathcal{B}$ denoting intervals with two boundary labels.

Of course intuitively we have an idea of when two cobordisms are "nearby", in the sense of the moduli space obtained from the Teichmüller space of conformal structure. This leads to a prop in Top given by moduli spaces of cobordisms with composition induced by glueing. This is made precise most easily using our nice cobordism prop $\mathsf{Bord}_{\mathcal{B}}$ consisting of the classifying spaces of the mapping class groups of $\mathcal{B}$-labelled cobordisms.

If we want to restrict to the positive boundary cobordisms, as is necessary in string topology, we need to restrict to the partial prop $\mathsf{Bord}^+_{\mathcal{B}}$ in $\mathsf{Bord}_{\mathcal{B}}$ where we don't allow connected component with empty outgoing boundary.

3.1.2. *Constructing props.* There is a simple way to construct a new prop from an old one. This requires a symmetric monoidal functor $F : \mathsf{C} \to \mathsf{D}$. Recall that a symmetric monoidal functor has natural maps $F(X) \otimes F(Y) \to F(X \otimes Y)$. It is clear that any such symmetric monoidal functor will send a bicoloured sequence to a bicoloured sequence, simply by composition of functors. It also sends props to props.

Lemma B.68. *If P is a prop in C, then $F(P)$ can be given the structure of a prop in D.*

Proof. It suffices to describe the composition maps for $F(P)$. But these are simply described by

$$F(P(f,g)) \otimes F(P(g,h)) \longrightarrow F(P(f,g) \otimes P(g,h)) \xrightarrow{F(\phi)} F(P(f,h))$$

$\square$

We will now discuss an important example of this construction, which will be used throughout this thesis.

Example B.69. Consider the case $\mathsf{C} = \mathsf{Top}$, $\mathsf{D} = \mathsf{GrAbgrps}$ and F is any type of homology with coefficients in a principal ideal domain R, denoted by $H_*(-;R)$. Here Top is a symmetric monoidal category with the cartesian product. One can also replace Top with $\mathsf{Spectra}$, which has a symmetric monoidal smash product in good models. Anyway, Künneth theorem tells us that there is a natural short exact sequence

$$0 \longrightarrow \bigoplus_{i+j=k} H_i(X;R) \otimes_R H_j(Y;R) \longrightarrow H_k(X \times Y;R) \longrightarrow \bigoplus_{i+j=k-1} \mathrm{Tor}_1^R(H_i(X;R);H_j(Y;R)) \longrightarrow 0$$

and hence the first map of this short exact sequence gives us the maps $H_*(X;R) \otimes H_*(Y;R) \to H_*(X \times Y;R)$ needed to make $H_*(-;R)$ into a symmetric monoidal functor. We conclude that if $\mathcal{P}$ is a prop in topological spaces, like the nice cobordism prop from section 2.1, then $H_*(\mathcal{P};R)$ is a prop in graded abelian groups.

Note that if R is a field k, then the maps $H_*(X;k) \otimes H_*(Y;k) \to H_*(X \times Y;k)$ are in fact isomorphisms. This will be important in the theory of algebras over a prop.

One can also ask about the situation for other generalized homology theories. In general one can prove the existence of a Künneth spectral sequence which has the tensor product of the generalized homology on the E^2-page and converges to the generalized homology of the product. One can use this to prove that for example complex K-theory and complex cobordism are give symmetric monoidal functors $\mathsf{Top} \to \mathsf{GrAbgrps}$.

3.1.3. *Algebra's over props.* We stated before that props should be though of as encoding generalized algebraic structures. How do they do this? We will now describe this by defining algebras over props. We first give the abstract definition and then describe a more down-to-earth version of it.

Definition B.70. Let C be a symmetric monoidal category and P a (partial) prop in C. A algebra over P is a C-enriched symmetric monoidal functor $A : P \to \mathsf{C}$.

If one unpacks all the definitions, this boils down to the following.

Definition B.71. Let P be a prop in C on a set of colours $\mathcal{B}$. An algebra A over P consists of objects A_β for $\beta \in \mathcal{B}$ together with operations:

$$\eta : P(f,g) \otimes \bigotimes_{\beta \in \mathcal{B}} (A_\beta)^{\otimes f(\beta)} \to \bigotimes_{\beta \in \mathcal{B}} (A_\beta)^{\otimes f(\beta)}$$

which are compatible with the addition maps, composition maps, group actions and identities.

Example B.72. The objects $X, \ldots, Z$ form an algebra over the endomorphism prop $\mathrm{End}_{X,\ldots,Z}$.

Note that applying a symmetric monoidal functor to an algebra A over a prop P doesn't return an algebra without further conditions. The problem that there is no guarantee of a map $F((A_\beta)^{\otimes f(\beta)}) \to F(A_\beta)^{\otimes f(\beta)}$. This does occur if we impose the conditions that the natural maps $F(X) \otimes F(Y) \to F(X \otimes Y)$ are isomorphisms. An example of this is the functor associating to a space its homology with coefficients in a field k. Hence an algebra over a topological prop gives a homology algebra over the homology prop as long as we work with field coefficients.

3.2. Operads. An operad is a restriction of the structure of a prop. In fact, any prop contains an operad. We will discuss the case that occurs most often: an operad on a single colour. The general case with multiple colours is slightly closer to a prop and therefore might seem more natural to discuss instead, but will not be used in this thesis.

Recall that the axioms of a prop were derived by looking at the structure of the endomorphisms

$$\mathrm{End}_{X,\ldots,Z}((n_X,\ldots,n_Z),(m_X,\ldots,m_Z)) = \mathrm{Hom}(X^{\otimes n_X} \otimes \ldots \otimes Z^{\otimes n_Z}, X^{\otimes m_X} \otimes \ldots \otimes Z^{\otimes m_Z})$$

of several objects. A single colour corresponds to a single object, so in that case we are in fact modelling our prop on $\mathrm{End}_X(n,m) = \mathrm{Hom}(X^{\otimes n}, Y^{\otimes m})$. These endomorphisms have addition operations, coming from the coproduct, and composition operations. The idea of an operad is just to look at the structure of

$$\mathrm{End}_X(n) = \mathrm{Hom}(X^{\otimes n}, X)$$

and just look at the composition operations, since then the addition operations no longer make sense. The result of axiomatizing this is the following set of definitions.

Definition B.73. Let Sym_1 be the category with objects $n \in \mathbb{N}$ and morphisms $\mathrm{Hom}(n,n) = \Sigma_n$ the symmetric group and $\mathrm{Hom}(n,m) = \emptyset$ for $n \neq m$. Let C be any category, then a symmetric sequence $\mathcal{O}$ in C is a functor $\mathcal{O} : \mathrm{Sym}_1 \to \mathsf{C}$.

In other words, a symmetric sequence consists of a sequence of object labelled by an integer together with an action of a symmetric group. This is the analog of a $\mathcal{B}$-bicoloured sequence for operad, which for convenience already includes symmetric group actions that appear only later in the definition of a prop.

Definition B.74. Let $(\mathsf{C}, \otimes)$ be a symmetric monoidal category, then an operad $\mathcal{O}$ is a symmetric sequence $\mathcal{O}$ together with a set of composition maps $\chi : \mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \ldots \otimes \mathcal{O}(k_n) \to \mathcal{O}(\sum_{i=1}^n k_i)$ for integers $n, k_1, \ldots, k_n$, and a unit map $u : 1 \to \mathcal{O}(1)$ satisfying the following axioms:

Associativity: The composition maps should be associative, in the sense that for all choices of integers the following diagram commutes:

$$\begin{tikzcd}
\mathcal{O}(n)\otimes\mathcal{O}(\vec{k})\otimes\mathcal{O}(\vec{l}_1)\otimes\ldots\otimes\mathcal{O}(\vec{l}_n) \arrow[r, "id\otimes\chi^{\otimes n}"] \arrow[d, "\chi\otimes id^{\otimes n}"'] & \mathcal{O}(n)\otimes\mathcal{O}(\sum_{j=1}^{k_1} l_{1j})\otimes\ldots\otimes\mathcal{O}(\sum_{j=1}^{k_n} l_{1j}) \arrow[d, "\chi"] \\
\mathcal{O}(\sum_{i=1}^{n} k_i)\otimes\mathcal{O}(\vec{l}_1)\otimes\ldots\otimes\mathcal{O}(\vec{l}_n) \arrow[r, "\chi"'] & \mathcal{O}(\sum_{i=1}^{n}\sum_{j=1}^{k_i} l_{ij})
\end{tikzcd}$$

where for a sequence $\vec{k}=(k_1,\ldots,k_n)$ of integers $\mathcal{O}(\vec{k})$ is by definition $\mathcal{O}(k_1)\otimes\ldots\otimes\mathcal{O}(k_n)$.

Unit: The unit map $u: 1\to\mathcal{O}(1)$ should behave as a left- and right-sided unit, in the sense that the following two diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(n) \arrow[r, "u\otimes id"] \arrow[dr, equal] & \mathcal{O}(1)\otimes\mathcal{O}(n) \arrow[d, "\chi"] \\
 & \mathcal{O}(n)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{O}(n) \arrow[r, "id\otimes u^{\otimes n}"] \arrow[dr, equal] & \mathcal{O}(n)\otimes\mathcal{O}(1)\otimes\ldots\otimes\mathcal{O}(1) \arrow[d, "\chi"] \\
 & \mathcal{O}(n)
\end{tikzcd}$$

Equivariance: The composition maps χ should be equivariant in the sense that the following two diagrams commute:

$$\begin{tikzcd}
\mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\mathcal{O}(k_n) \arrow[r, "\sigma\otimes p(\sigma)"] \arrow[d, "\chi"'] & \mathcal{O}(n)\otimes\mathcal{O}(k_{\sigma(1)})\otimes\ldots\otimes\mathcal{O}(k_{\sigma(n)}) \arrow[d, "\chi"] \\
\mathcal{O}(\sum_{i=1}^{n} k_i) \arrow[r, "\alpha(\sigma)"'] & \mathcal{O}(\sum_{i=1}^{n} k_i)
\end{tikzcd}$$

where $p(\sigma)$ denotes the permutation σ as acting by the symmetry of the symmetric monoidal category and $\alpha(\sigma):\Sigma_n\to\Sigma_{\sum_{i=1}^n k_i}$ is the strand splitting map which sends σ to the appropriate block permutation of set $(1,\ldots,\sum_{i=1}^n k_i)$.

$$\begin{tikzcd}
\mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\mathcal{O}(k_n) \arrow[r, "id\otimes\tau_1\otimes\ldots\otimes\tau_n"] \arrow[d, "\chi"'] & \mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\mathcal{O}(k_n) \arrow[d, "\chi"] \\
\mathcal{O}(\sum_{i=1}^{n} k_i) \arrow[r, "\beta(\tau_1,\ldots,\tau_n)"'] & \mathcal{O}(\sum_{i=1}^{n} k_i)
\end{tikzcd}$$

where $\beta:\prod_{i=1}^n\Sigma_{k_i}\to\Sigma_{\sum_{i=1}^n k_i}$ is the disjoint union map which puts the permutations next to each other.

Almost by definition, the following is true:

Lemma B.75. *Let P be a prop with set of colours $\mathcal{B}$. Then if one sets $\mathcal{O}(n)$ to be $\mathcal{P}(n\cdot 1_b, 1_b)$ where 1_b is the function having its only non-zero value 1 at $b\in\mathcal{B}$ with symmetric action from the left, the symmetric sequence $\mathcal{O}(n)$ can be given the structure of an operad by setting composition maps $\chi_{\mathcal{O}}$ to be*

$$\chi_{\mathcal{O}}:\mathcal{O}(n)\otimes\mathcal{O}(k_1)\otimes\ldots\otimes\mathcal{O}(k_n)\to\mathcal{O}(\sum_{i=1}^{n} k_i)$$
$$(x,x_1,\ldots,x_n)\mapsto\chi(x,\gamma(x_1,\ldots,x_n))$$

Remark B.76. Note that it suffices to start with a partial prop which is defined for the sets $\mathcal{P}(n\cdot 1_b,1_b)$ that we use. In particular, we can construct operads from the positive boundary cobordism prop of section 2.1.

Hence we can create at least $\#\mathcal{B}$ differents operads from a prop P. Furthermore, it is clear that we can still apply a symmetric monoidal functor F to $\mathcal{O}$ to get a new operad $F(\mathcal{O})$. If $\mathcal{O}$ is obtained from P in the manner described above then $F(\mathcal{O})$ is the same as the operad obtained from $F(P)$ in the manner describe above. Finally, algebras over operads can be defined. Because they feature prominently in some sections of this thesis, we give this definition.

Definition B.77. An algebra A over an operad $\mathcal{O}$ in C is an object A together with operations:

$$\eta : \mathcal{O}(n) \otimes A^{\otimes n} \to A$$

which are compatible with composition, the group actions and the units, in the following sense:

Compatibility with composition: The following diagram should commute:

$$\begin{tikzcd}
\mathcal{O}(n) \otimes \mathcal{O}(k_1) \otimes \ldots \otimes \mathcal{O}(k_n) \otimes A^{\otimes \sum_{i=1}^n k_i} \arrow[r, "id\otimes\eta^{\otimes n}"] \arrow[d, "\chi\otimes id"'] & \mathcal{O}(n) \otimes A^{\otimes n} \arrow[d, "\eta"] \\
\mathcal{O}(\sum_{i=1}^n k_i) \otimes A^{\otimes \sum_{i=1}^n k_i} \arrow[r, "\eta"'] & A
\end{tikzcd}$$

Equivariance: The following diagram should commute:

$$\begin{tikzcd}
\mathcal{O}(n) \otimes A^{\otimes n} \arrow[r, "\sigma\otimes id^{\otimes n}"] \arrow[d, "id\otimes p(\sigma)"'] & \mathcal{O}(n) \otimes A^{\otimes n} \arrow[d, "\eta"] \\
\mathcal{O}(n) \otimes A^{\otimes n} \arrow[r, "\eta"'] & A
\end{tikzcd}$$

where $p(\sigma)$ is σ acting by the symmetry of the symmetric monoidal category.

Units: The following diagram should commute:

$$\begin{tikzcd}
A \arrow[r, "u\otimes id"] \arrow[dr, equal] & \mathcal{O}(1) \otimes A \arrow[d, "\eta"] \\
& A
\end{tikzcd}$$

Again we can try to apply a symmetric monoidal functor to an algebra over an operad. Because in contrast to the case of an algebra over a prop we do not need to split the codomain, any symmetric monoidal functor F sends an algebra A over an operad $\mathcal{O}$ to an algebra $F(A)$ over an operad $F(\mathcal{O})$.

Example B.78. In section 3.2 of chapter 2 we see that commutative Frobenius algebras are algebras over the operad obtained by restricting prop Bord^{π_0} of two-dimensional cobordisms. In that chapter can also find how Poisson algebras and BV-algebras arise as algebras over an operad.

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CONSTRUCTING HIGHER STRING TOPOLOGY OPERATIONS USING RADIAL SLIT CONFIGURATIONS

A.P.M. KUPERS

Abstract. In this note we describe a new method to construct higher string topology operations and show that these endow $H_*(LM)$ with the structure of an HCFT. More generally the same techniques endow $E_*(LM)$ with the structure of a ECFT, where E is a commutative ring spectrum. Our construction is based on radial slit configurations and is elementary, except for some applications of parametrized spectra.

Contents

Introduction

String topology is the study of the algebraic structure on the homology $H_*(LM)$ of the free loop space of a compact oriented manifold induced by the merging and splitting of strings. More generally one can also look at the algebraic structure on $E_*(LM)$ for E_* a generalized homology theory. The worldsheets of strings merging or splittings are 2-dimensional compact oriented cobordisms with incoming and outgoing boundary circles.

Originally people defined such string operations for each isomorphism class of cobordism: Chas and Sullivan [CS99] constructed these on the chain level using transversal intersections, inspired by the Goldman bracket on free homotopy classes of curves on a surface. Later, Cohen and Jones [CJ02] found a homotopy-theoretic construction of these operations using the Pontryagin-Thom collapse map. In the end, all give the shifted homology $H_{*+d}(LM)$ the structure of BV-algebra [CV06].

The next step in this direction was to replace the isomorphism classes of cobordisms with moduli spaces of these cobordisms. This was done in an important article by Godin [God07], who defined string operations indexed by the homology groups of the moduli spaces of cobordisms by generalizing the technique of Cohen and Jones. This endows $H_*(LM)$ with the structure of a d-dimensional HCFT, as defined in [God07]. If one just considers the homology of the moduli spaces of genus zero cobordisms with many incoming but one outgoing boundary components, one recovers the BV-algebra structure.

The work of Godin contains many interesting ideas, but requires advanced techniques and is a bit intimidating on a first read. In our opinion, this is a consequence of the choice of model for the moduli space: the geometric realisation of a category of ribbon graphs. In this note we use a different model which significantly simplifies the construction. This model is Bödigheimer's radial slit

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configuration model [Böd06]. It has the additional advantage of having a natural compactification. It seems probable that one can extend the string topology operations to this compactification, see [Poi10] or section 6.3.

In the first two sections we describe those parts of Bödigheimer's model that will be used in our construction of the string operations. In the following two sections we construct the operations and show that they are compatible with the prop structure of the moduli space. After that, we show that our construction indeed recovers the degree zero string operations and make it plausible that they are equal to Godin's. Finally, we take a look at possible applications and extensions of this construction.

1. Radial slit configurations

We begin by describing the model for the moduli space of Riemann surfaces with incoming and outgoing boundary that we will use. It is a variation on Bödigheimer's model, as found in [Böd06].

Remark 1.1. Although we will define everything we need for our construction, a reader familiar with Bödigheimer's model may find it helpful to know that our model differs from that in [Böd06] in the following aspects:

- we have non-permutable boundary components,
- fixed inner radius 1 of the annuli,
- constant outer radius $R > 1$ of the each of the annuli,
- marked parametrisation point on each outgoing boundary component,
- no connectedness requirements,
- and additional parameters controlling partial glueings.

1.1. Radial slit preconfigurations.

We start by describing spaces of radial slit preconfigurations, which will be the basis of the model of the moduli space of Riemann surfaces with boundary that we will use. We fix integers $h \geq 0$, $m \geq 1$ and $n \geq 1$. One should think of h as minus the Euler characteristic, m as the number of incoming boundary circles and n as the number of outgoing boundary circles.

We first define our particular model for the annulus.

Definition 1.2. For $R > 1$, we define $\mathbb{A}_R$ to be the annulus $\{z \in \mathbb{C} | 1 \leq |z| \leq R\}$. The inner boundary $\partial_{in}\mathbb{A}_R$ is the subspace $\{z \in \mathbb{C} | |z| = 1\}$ and the boundary $\partial_{out}\mathbb{A}_R$ is the subspace $\{z \in \mathbb{C} | |z| = R\}$.

Using this, we can define the space of radial slit preconfigurations. In this definition we will use the notion of a radial segment. For any $z_0 \in \mathbb{A}_R$ the radial segment through z_0 is given by $\{z \in \mathbb{C} | \arg z_0 = \arg z \text{ and } |\zeta_0| \leq |z| \leq R\} \subset \mathbb{A}_R$.

Definition 1.3. The space of possibly degenerate preconfigurations $\mathrm{PRad}_h^{deg}(m,n)$ is given by a subspace of $(\coprod_{i=1}^m \mathbb{C})^{2h} \times S_{2h} \times S_{2h} \times \{0,1\}^{2h} \times (1,\infty) \times (\coprod_{i=1}^m \mathbb{C})^n \times [0,\infty)^{2h}$ of elements $L = (\zeta, \lambda, \omega, \Theta, R, \vec{P}, \vec{t})$. The ζ_i are called the slits, λ the slit pairing, ω the successor permutation, Θ the ordering set of exceptional indices, R the outer radius, the P_i the parametrisation points and the t_i the partial glueing parameters. We will use the subspace of such data subject to the following conditions:

- If a slit ζ_i lies in $\mathbb{C}_j$, then it lies in the annulus $\mathbb{A}_R \subset \mathbb{C}_j$.
- If a parametrisation point P_i lies in $\mathbb{C}_j$, then it lies in the outer boundary $\partial_{out}\mathbb{A}_R \subset \mathbb{C}_j$.
- The slit pairing λ consists of h cycles of length 2.
- The successor permutation ω consists of a disjoint union of m cycles.
- The cycles of ω consist exactly of those indices of slits ζ_i lying on one of the annuli and the permutation action of ω on these ζ_i preserves their weakly cyclic ordering coming from the argument in counterclockwise direction.
- The so-called boundary component permutation $\lambda \circ \omega$ consists of n cycles.

- We demand that P_i lies in the subset O_i of $\coprod_{i=1}^{n}(\partial_{out}\mathbb{A}_R)$ which we will now define. The n cycles of $\lambda \circ \omega$ partition the outer boundaries of the annuli into n parts O_i for $1 \leq i \leq n$, overlapping only in single points. To be precise, O_i is the union of the parts in the outer boundary between the intersection of outer boundary of the annulus with the radial segments through ζ_j and $\zeta_{\omega(j)}$ respectively, for all j in a cycle of ω. In terms of a notion to be defined later, in definition 1.9, O_i is the preimage of the i'th outgoing boundary component of the surface with boundary $F(L)$.
- The set Θ is a subset of the exceptional indices such that for each exceptional pair $\{i, j\}$ the subset Θ contains either i or j. Here the exceptional indices are by definition those indices i of slits ζ_i which lie on the same annulus with just a single other ζ_j and have the same argument as that ζ_j.
- Each t_i is zero unless ζ_i and $\zeta_{\omega(i)}$ have the same argument. In that case we demand $0 \leq t_i \leq \min(|\zeta_i|, |\zeta_{\omega(i)}|) - 1$.

We now explain the idea behind this definition; the precise definitions are given in section 1.2. The idea of the radial slit preconfigurations is that we have a number of annuli in which we have a number of slits, given by the ζ_k. We cut the annuli along the radial segments $\{z \in \mathbb{C} | \arg \zeta_k = \arg z \text{ and } |\zeta_k| \leq |z| \leq R\}$ from the outer boundary through the slit ζ_k. Along these cuts we glue the annuli together into a surface. There are several types of preconfigurations we sometimes need to worry about: if two or more slits lie on the same annulus and have the same argument. To be able to refer to these easily we introduce the following nomenclature.

Definition 1.4. A radial slit preconfiguration is called special if there exist ζ_i and ζ_j lying on the same annulus such that $\arg \zeta_i = \arg \zeta_j$. A radial slit preconfiguration is called exceptional if it is special and has non-empty set Θ of exceptional indices. If a radial slit preconfiguration is not special, it is called generic.

To decide which cuts to glue we use the pairing λ. In the generic case we have a canonical cylic ordering of the slits on an annulus using the argument. However, this is not the case if we are dealing with a special configuration, i.e. we have slits ζ_i which lie on the same radial segment. Then the surface obtained from glueing is not unambiguously defined unless we introduce some additional data: this is the ordering ω which keeps track of the cyclic ordering of the ζ_i on the same radial segments. However, even this cyclic ordering is not unambiguous in the case that there are only two slits ζ_i and ζ_j on a boundary component, which happen to lie on the same radial segment. The set Θ fixes this by keeping track of whether ζ_i touches the ζ_j from the left or right.

There is a canonical parametrisation of the incoming boundary given by the parametrisation of the inner boundaries of the annuli as subset of $\mathbb{C}$. However, to get a canonical parametrisation on the outgoing boundary, which corresponds to the decomposition O_i of the outer boundaries of the annuli given by the boundary component permutation $\lambda \circ \omega$, we require the parametrisation points P_i. Finally, there is a contractible space of possible t_i for each preconfiguration with slits lying on the same radial segment. These parameters control to which extent certain edges of graphs that we will obtain from a radial slit preconfiguration in section 2.4 are "zipped".

Note that we haven't demanded that there must be a slit on each annulus or that all annuli are connected by slits. The latter amounts to allowing disconnected surfaces and the former to including the cylinder as a possible connected component. This is a slight extension of Bödigheimer's model which will have the advantage of making the prop structure more natural.

1.2. **Non-degenerate preconfigurations and their corresponding surfaces.** We will now describe how the data of a preconfiguration allows one to construct a surface which is possibly degenerate, in the sense that it can be obtained from a non-degenerate surface by letting embedded arcs or cylinders shrink to points. In proposition 1.10 we will give conditions under which this surface is not degenerate.

We start by defining a space $\tilde{F}(L)$ of associated radial sectors and af that we will define the equivalence relation $\approx_L$ on $\tilde{F}(L)$ which explains how these radial sectors should be glued. The result will be a surface $F(L)$.

Definition 1.5. Let r denote the number of annuli containing no slits. The radial sector space $\tilde{F}(L)$ is a disjoint union of $2h+r$ spaces F_k called radial sectors. Let R denote the outer radius of the radial slit preconfiguration L. There are four cases, see figure 1.

Ordinary sectors: If $k \neq \omega(k)$ and $\arg \zeta_k \neq \arg \zeta_{\omega(k)}$, then we set,

$$F_k = \{z \in \mathbb{A}_R | \arg \zeta_k \leq \arg z \leq \arg \zeta_{\omega(k)}\}$$

Thin sectors: If $k \neq \omega(k)$, but $\arg \zeta_k = \arg \zeta_{\omega(k)}$ and $k \in \Theta$, then we set

$$F_k = \{z \in \mathbb{A}_R | \arg z = \arg \zeta_k\}$$

Note that in this case F_k is nothing but a line segment, which is called a thin sector.

Full sectors: If $k \neq \omega(k)$, but $\arg \zeta_k = \arg \zeta_{\omega(k)}$ and $\omega(k) \in \Theta$, then we set F_k to be the annulus $\mathbb{A}_R$, cut open along the segment $\arg z = \arg \zeta_k$, which is doubled.

Also, if $k = \omega(k)$, then we set F_k to be the annulus $\mathbb{A}_R$, cut open along the segment $\arg z = \arg \zeta_k$, which is doubled. These radial sectors are called full sectors. Note that full sectors only appear over exceptional components, i.e. components with points having exceptional indices.

Entire sectors: If an annulus is the j'th one in the ordering of the annuli which does not contain a slit, then we set $F_{2h+j} = \mathbb{A}_R$.

It is intuitively clear that these radial sectors depend continuously on the radial slit preconfigurations. As long as there are no full sectors, this can easily be made precise by seeing the sectors as subspaces of a trivial $\coprod_{i=1}^{2h+r} \mathbb{A}_R$-bundle over PRad^{deg}. However, for full sectors we need to be more careful, because in those a single radial segment of the cut annulus is doubled. The next lemma shows with what one has to replace the trivial bundle $\coprod_{i=1}^{2h+r} \mathbb{A}_R$ to describe the radial sectors as a bundle over PRad^{deg}.

Lemma 1.6. *There exists a space $\tilde{\mathbb{A}}$ over PRad^{deg} such that $\tilde{F}(L)$ is canonically a subspace of the fiber over L.*

Proof. Let $\mathrm{PRad}^2 \to \mathrm{PRad}^{deg}$ be the 2^{2h}-fold cover corresponding to the map $z \mapsto z^2$ on the slits. Thus a quotient by a free action of $(\mathbb{Z}_2)^{2h}$ of this cover gives us back our original space. Let $\mathbb{A}_2 \to \mathbb{A}_R$ be the double cover of the annulus corresponding to the map $z \mapsto z^2$. The group $\mathbb{Z}_2$ acts on this by permuting the two sheets. We let

$$\tilde{\mathbb{A}}_2 = \left((\coprod_{i=1}^{2h} \mathbb{A}_2) \sqcup (\coprod_{i=1}^{r} \mathbb{A}_R) \right) \times_{(\mathbb{Z}_2)^{2h}} \mathrm{PRad}^2$$

where $(\mathbb{Z}_2)^{2h}$ to left just acts on $(\coprod_{i=1}^{2h} \mathbb{A}_2)$ and acts on each of the components $\mathbb{A}_2$ by an individual $\mathbb{Z}_2$ of deck transformations. Thus we see that $F(L)$ has 2^{2h} possible lifts to $\tilde{\mathbb{A}}^2$. Pick one lift for a point in each connected component of PRad^2 and extend this lift uniquely to the rest of the connected component. This gives a subspace $\tilde{F} \subset \tilde{\mathbb{A}}_2$ over PRad^2. Now we can take the quotient by $(\mathbb{Z}_2)^{2h}$ of the base and identify the two copies of $F_k(L)$ in fiber by the natural homeomorphism given by the deck transformation. The result is the space $\tilde{\mathbb{A}}$ over PRad^{deg}. □

There are canonical subspaces of the radial sectors, along which we would like to glue.

Definition 1.7. Let $F_k \subset \tilde{F}(L)$ be an ordinary or thin radial sector, considered as a subspace of $\mathbb{A}_R$. Then we define the subsets $\alpha_k^{\pm}(L)$, $\beta_k^{\pm}(L)$ and $\gamma_k^{\pm}(L)$ as follows

$$\begin{aligned}
\alpha_k^+(L) &= \{z \in F_k | \arg z = \arg \zeta_{\omega(k)} \text{ and } |z| \leq |\zeta_{\omega(k)}|\} \\
\alpha_k^-(L) &= \{z \in F_k | \arg z = \arg \zeta_k \text{ and } |z| \leq |\zeta_k|\} \\
\beta_k^+(L) &= \{z \in F_k | \arg z = \arg \zeta_{\omega(k)} \text{ and } |z| \geq |\zeta_{\omega(k)}|\} \\
\beta_k^-(L) &= \{z \in F_k | \arg z = \arg \zeta_k \text{ and } |z| \geq |\zeta_k|\} \\
\gamma_k^+ &= \{z \in F_k || z| = R\} \\
\gamma_k^- &= \{z \in F_k || z| = 1\}
\end{aligned}$$

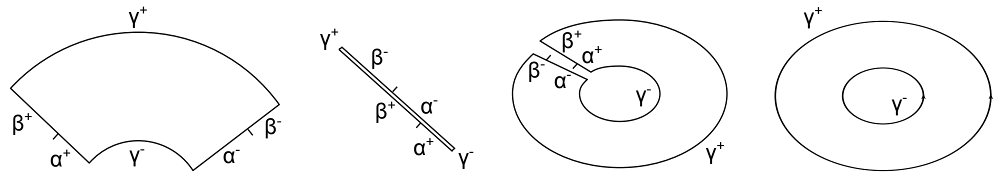

FIGURE 1. An ordinary, thin, full and entire sector with corresponding $\alpha^{\pm}$, $\beta^{\pm}$ and $\gamma^{\pm}$.

If F_k was full, then we note that the $\alpha_k^{\pm}(L)$, $\beta_k^{\pm}(L)$ and $\gamma_k^{\pm}(L)$ need a slightly different definition. Denote by S_k^+ the copy of the radial segment $\{z \in \mathbb{A}_{R_j} | \arg z = \arg z_k\}$ that bounds F_k in counterclockwise direction, the S_k^- the one that bounds F_k in clockwise direction. Then we define:

$$\begin{aligned}
\alpha_k^+(L) &= \{z \in S_k^+ \,|\, |z| \leq |\zeta_{\omega(k)}|\} \\
\alpha_k^-(L) &= \{z \in S_k^- \,|\, |z| \leq |\zeta_k|\} \\
\beta_k^+(L) &= \{z \in S_k^+ \,|\, |z| \geq |\zeta_{\omega(k)}|\} \\
\beta_k^-(L) &= \{z \in S_k^- \,|\, |z| \geq |\zeta_k|\} \\
\gamma_k^+ &= \{z \in F_k ||z| = R\} \\
\gamma_k^- &= \{z \in F_k ||z| = 1\}
\end{aligned}$$

Finally, if F_k was entire, no glueing should happen. See figure 1 for a picture of these subspaces.

Next, we define the equivalence relation $\approx_L$, which glues the radial sectors along $\alpha^{\pm}(L)$ and $\beta^{\pm}(L)$. We also define a subrelation $\sim_L$ which only partially glues the radial sectors.

Definition 1.8. The equivalence relation $\approx_L$ makes the following identifications:

- $z \in \alpha_k^+(L)$ is identified with $z \in \alpha^-_{\omega(k)}$,
- $z \in \alpha_k^-(L)$ is identified with $z \in \alpha^+_{\omega^{-1}(k)}$,
- $z \in \beta_k^+(L)$ is identified with $z \in \beta^-_{\lambda(\omega(k))}$, and
- $z \in \beta_k^-(L)$ is identified with $z \in \beta^+_{\omega^{-1}(\lambda^{-1}(k))}$.

There is a subrelation $\sim_L$ which just makes the following identifications:

- $z \in \alpha_k^+(L)$ is identified with $z \in \alpha^-_{\omega(k)}$ if $|z| \leq t_k + 1$,
- $z \in \alpha_k^-(L)$ is identified with $z \in \alpha^+_{\omega^{-1}(k)}$ if $|z| \leq t_{\omega^{-1}(k)} + 1$.

We can now define the surface corresponding to a radial slit preconfiguration. See [Böd06, section 4.6] for a proof that these surfaces are in many cases smooth and come with a canonical conformal structure.

Definition 1.9. We define the surface corresponding to a radial slit preconfiguration to be

$$F(L) = \tilde{F}(L)/\approx_L$$

In figure 2 we give an example of a radial slit configuration whose corresponding surface has genus 1 and a single incoming and outgoing boundary component. Further examples can be found in figure 12. We recommend that the reader tries to draw some examples as well.

In section 4.5 of [Böd06] the following criterion is proven.

Proposition 1.10. *The surface $F(L)$ is degenerate if and only if L satisfies at least one of the following two conditions.*

- *Some ζ_i lies in $\partial_{in}\mathbb{A}_R$ or $\partial_{out}\mathbb{A}_R$.*
- *There is a pair (i,j) such that ζ_i and ζ_j lie on the same annulus $\mathbb{A}_R$, $i \in \Theta$ and $\zeta_i = \zeta_j$ such that $|\zeta_k| \geq |\zeta_i| = |\zeta_j|$ for all $k \in I$ between i and j in the induced cyclic ordering of the indices of ζ on the annulus $\mathbb{A}_R$.*

If L doesn't satisfy either of these conditions we call it non-degenerate. If it is non-degenerate or just satisfies the first condition we call it semi-degenerate. If it is not non-degenerate or semi-degenerate, we call it degenerate.

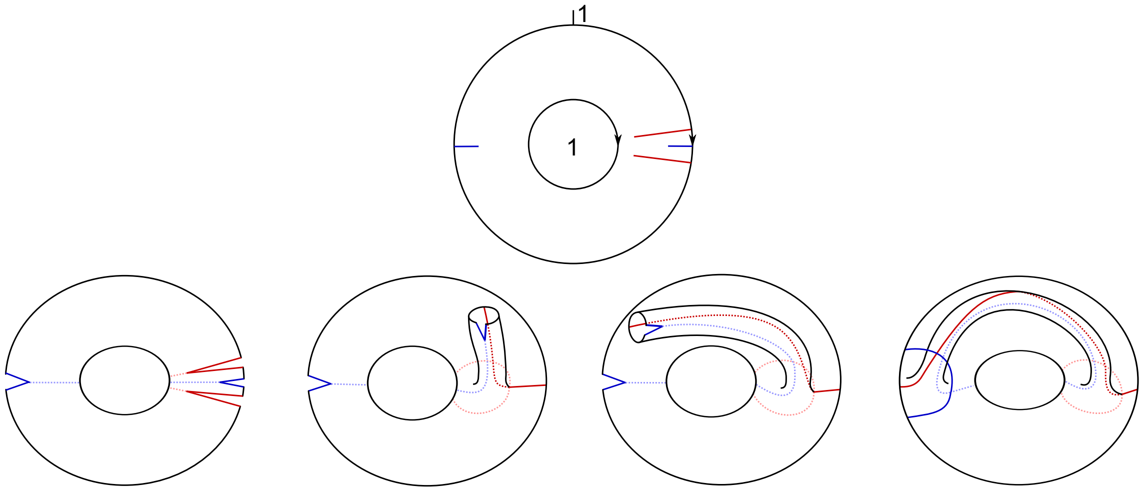

FIGURE 2. Four steps in the process of obtaining the surface corresponding to a radial slit configuration. For the convenience of visualization these steps do not correspond exactly to the steps of the glueing process defined in this section.

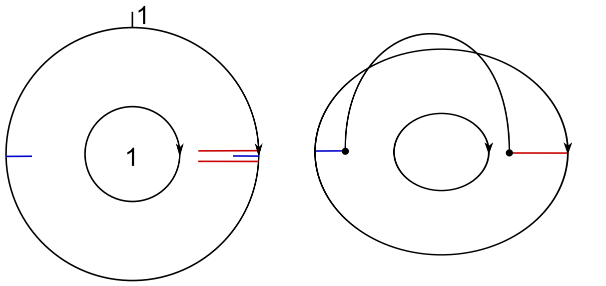

FIGURE 3. An example of a degenerate radial slit configuration with its associated surface.

In other words, we claim that there is a small problem when a slit lies on the boundary, but that there is a big problem if two slits coincide or a slit is squeezed in between two larger slits. By looking at the corresponding surfaces, e.g. figures 3 and 14, one sees that this is indeed the case. These types of degenerations are sometimes called harmonic degenerations.

Definition 1.11. The space of radial slit preconfigurations PRad is the subspace of PRad^{deg} consisting of those preconfigurations that are non-degenerate. Similarly, the space of semi-degenerate radial slit preconfigurations PRad^s is the subspace of PRad^{deg} consisting of those preconfigurations that are semi-degenerate.

Note that we have a sequence of inclusions

$$\mathrm{PRad} \hookrightarrow \mathrm{PRad}^s \hookrightarrow \mathrm{PRad}^{deg}$$

1.3. **Radial slit configurations and labelled radial slit configurations.** In fact, using the previous construction one almost gets a bijection between surfaces with conformal structure and non-degenerate preconfigurations. To make this a bijection, at least up to a small homotopy, we need to identify a small number of points. The only reasons that we don't get an actual bijection is that we introduced the partial glueing parameters $\vec{t}$ and required that all outer radii of the annuli are equal.

Anyway, if one tries to find preconfigurations which lead to the same suface with conformal structure, one is naturally led to the definition of the radial slit configurations. These will be the result of taking the quotient by the following two equivalence relations. To define these, we need look at a bit closer at the special configurations. For a special configuration, there is a tree

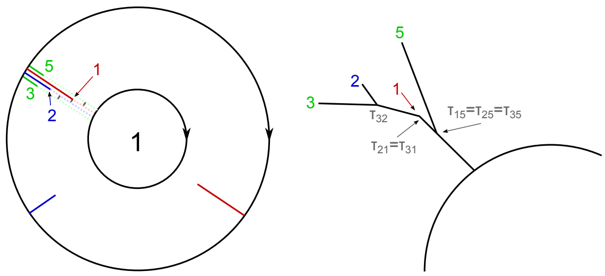

FIGURE 4. An example of the tree that appears for a part of a special configuration. The partial glueing parameters are denoted by grey lines. We have decided to leave out the parametrisation points to make the figure less crowded.

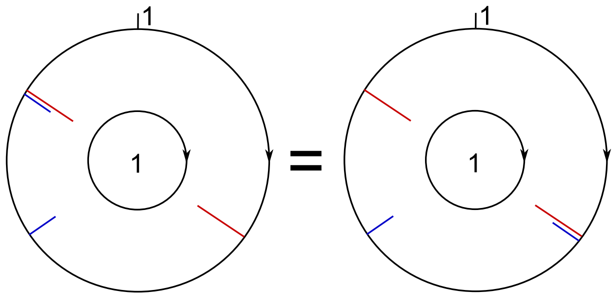

FIGURE 5. The blue slit jumps over the red one and hence these two configurations are identified by $\equiv_1$. Note that we implicitly assume that the partial glueing parameter is maximal.

for each point on the incoming boundary for which there are multiple slits on the radial segment through this point. This is given drawing an edge from this point to the slits on the radial segment and identifying points using the partial glueing parameters as in $\sim$. The modulus gives a height function and we can define points τ_{ij} be the points on that tree with maximal modulus where the edges corresponding to ζ_i and ζ_j are attached. See figure 4 for an example.

Definition 1.12. On PRad we have two equivalence relations $\equiv_1$ and $\equiv_2$. The relation $\equiv_1$ identifies two preconfigurations in two situations.

- **Jumps of slits:** Suppose that two or more slits lie on the same annulus and have the same argument, but different modulus. In other words, one lies on the radial segment through the other. Then under certain conditions the slits with larger modulus can jump across one with smaller modulus in counterclockwise direction. To be precise, a slit next to ζ_i can jump if the partial glueing parameter t_i corresponding to the jump slit is equal to $|\zeta_{\omega(i)}| - 1$. In this case all slits ζ_j such that τ_{ij} lies above ζ_i in the tree must jump as well. See figure 5 for a simple case and figure 6 for a more complicated one.
- **Jump of parametrisation points:** If a parametrisation point lies on the radial segment through a slit, we can move it to the other slit in the pairing. We say $L \equiv_1 L'$ if they differ as in figure 7.

The equivalence relation $\equiv_2$ identifies two preconfigurations if $\equiv_1$ identifies them or if the following additional situation occurs:

- **Renumbering:** The idea of this identification is that labels on the slits are superfluous for the surface obtained by taking the quotient by $\approx$. Thus, we let $\equiv_2$ take the quotient by the symmetric group S_{2h}, where the group S_{2h} acts L as follows: $\sigma \in S_{2h}$ acts on the ζ by permutation, sends λ to $\sigma \circ \lambda \circ \sigma^{-1}$, sends ω to $\sigma \circ \omega \circ \sigma^{-1}$, acts trivially on R and acts on $\vec{P}$, $\vec{t}$ and Θ by the obvious induced permutations.

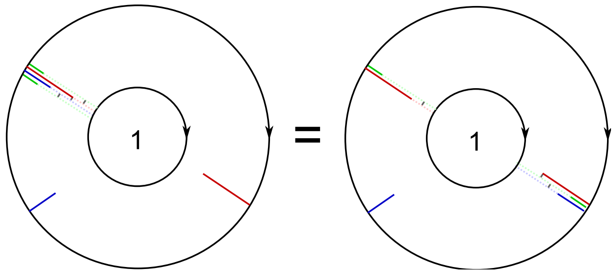

FIGURE 6. A more complicated slit jump. Note that because the partial glueing parameter of the bottom green slit is above the point where the blue slit attaches to the red one, it must jump as well. Also note that the other green one is not allowed to jump, because its partial glueing parameter is too small.

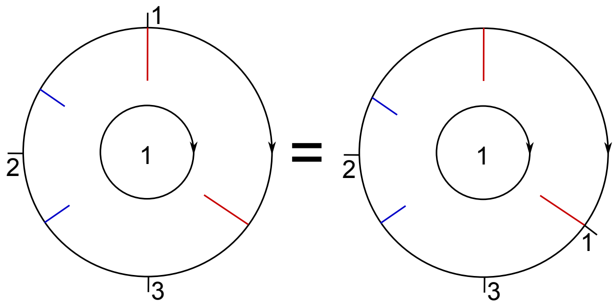

FIGURE 7. The parametrisation point of the first boundary component jumps across one of the red slits.

Of course it is possible to give explicit formules for the new data L' in terms of the original L when a slit or parametrisation point jumps. These can be found in [Böd06, section 7]. Furthermore, note that these equivalence relations make sense for PRad^s and PRad^{deg} as well.

Definition 1.13. We define a labelled radial slit configuration space LRad and a radial slit configuration space Rad as follows

$$\mathrm{LRad} = \mathrm{PRad}/\equiv_1$$
$$\mathrm{Rad} = \mathrm{Rad}/\equiv_2$$

Note that we have projections $p_{\equiv_1} : \mathrm{PRad} \to \mathrm{LRad}$, $p_{\equiv_2} : \mathrm{PRad} \to \mathrm{Rad}$ and $p : \mathrm{LRad} \to \mathrm{Rad}$, the latter of which is given by the quotient by a free S_{2h}-action. Again, one can make similar definitions where one allows semi-degenerate or even degenerate configurations. We will get back to this in the final section of this article.

We make some remarks about connectivity and cylinders. To do this, we introduce the connectivity graph G, which has vertices $\{1, \ldots, m\}$ and a edge between two vertices i, j if a slit on the i'th annulus is paired with a slit on the j'th annulus by λ.

Lemma 1.14. *The space* $\mathrm{Rad}_h(m, n)$ *splits into connected components according to different partitions of* $\{1, \ldots, m\}$*, corresponding to the components of* G*, and assignments of subsets of* $\{1, \ldots, 2h\}$ *to each element of such a partition. To be precise, the component of* $\mathrm{Rad}_h(m, n)$ *where* G *has* p *components splits into a disjoint union of products of* p *spaces* $\mathrm{Rad}_{h_j}(m_j, n_j)$ *such that* $\sum h_j = h$*,* $\sum m_j = m$ *and* $\sum n_j = n$*.*

This lemma implies that for the next proof it suffices to look only at configurations with connected G. Note that a point $L \in \mathrm{Rad}$ contains data to endow the surface with boundary $F(L)$ with the structure of a cobordism with parametrized incoming and outgoing boundary. Furthermore, all the cobordisms obtained this way from points in a single connected component of Rad are isomorphic as cobordisms with smooth structure. We denote a representative cobordism with Σ and recall that the mapping class group Γ_Σ of Σ is the group of connected components of the space of orientation-preserving self-homeomorphisms of Σ which preserves the boundary pointwise. The group operation is induced by the composition of these self-homeomorphisms. A consequence of Bödigheimer's results is the following theorem. Recall that Rad is short for $\mathrm{Rad}_h(m,n)$.

Theorem 1.15. *The space* Rad *has the homotopy type of a disjoint union of classifying spaces*

$$\mathrm{Rad} \simeq \coprod_{[\Sigma]} B\Gamma_\Sigma$$

where Γ_Σ is the mapping class group of the cobordism Σ and the coproduct is over all cobordisms with r incoming boundary circles, n outgoing boundary circle and a genus of $-2h$.

Proof. Because of lemma 1.14, without loss of generality we can restrict our attentions to configurations with connected connectivity graph G. A special case is that of an annulus with slits. In that case we have $\mathrm{Rad} \cong S^1$, which is a $B\mathbb{Z}$, and indeed $\mathbb{Z}$ is the mapping class of the cylinder as cobordism.

For the other cases we can use theorem 1.1 of [Böd06] and use that our space Rad is a $(S^1)^n$-bundle over a space which is homotopy equivalent to Bödigheimer's configuration space. The long exact sequence of a homotopy groups then implies that all higher homotopy groups of Rad vanish and we have a central extension

$$1 \to \mathbb{Z}^n \to \pi_1(\mathrm{Rad}) \to \Gamma'_\Sigma \to 1$$

where Γ'_Σ is the mapping class group of Σ of homeomorphisms which do not fix the outgoing boundary. Recall that Γ_Σ is obtained from Γ'_Σ by adjoining Dehn twists around the outgoing boundary components. The simple curves and regular neighborhoods supporting these n different Dehn twists can be chosen disjoint from each other and the other generators of Γ'_Σ (which is well known to be generated by Dehn twists), hence they are in central. $\square$

1.4. **Prop structure, HCFT's and ECFT's.** We will see that Rad forms a (partial) prop by glueing and taking disjoint union of the radial slit configurations. The glueing is well-defined on the non-degenerate radial slit configurations and the degenerate ones, but not on the semi-degenerate ones, because the result could become degenerate when a configuration with a slit on the outgoing boundary is glued into one with a slit on the incoming boundary. But note that it is well-defined when we restrict to semi-degenerate configurations without slits lying on the incoming (or outgoing) boundary. The disjoint union is always well-defined.

These operations will be associative on the nose but only in the case of glueing only have identity maps up to homotopy. This is not such a problem since we are really interested in these operations up to homotopy anyway, because we want apply generalized homology theories to these spaces and these are by definition homotopy-invariant.

Definition 1.16. We define a glueing operation

$$P\# : \mathrm{PRad}_h(m,n) \times \mathrm{PRad}_{h'}(n,k) \to \mathrm{PRad}_{h+h'}(m,k)$$

It is given on a pair $L = (\vec{\zeta}, \lambda, \omega, \theta, R, \vec{P}, \vec{t})$, $L' = (\vec{\zeta}', \lambda', \omega', \theta', R', \vec{P}', \vec{t}')$ by the following procedure.

- Take m annuli $\mathbb{A}_{R+R'}$ and note that we can write it as the union of the following two subspaces; an annulus $\mathbb{A}_R$ and non-standard annulus $\mathbb{A}_R^{R'} = \{z \in \mathbb{A}_{R+R'} | R \leq |z| \leq R'\}$.
- We insert L in $\mathbb{A}_R$.
- Recall that L divides the outer boundary of $(\coprod_{i=1}^m \mathbb{A}_R)$ naturally into n subsets O_i and on each of these lies a point P_i: for this see definition 1.3. The subsets O_i have angular length $|O_i|$.

  The subsets O_i can be subdivided by the radial segments through the slits and the P_i into a number of arcs $a_{i,1}, \ldots, a_{i,l}$ (most of which are equal to a segment γ^+ of the outer boundary, except at most one such γ^+ which is the one containing P_i in its interior). Here we number the arcs such that $a_{i,1}$ begins at P_i and continuous along the outer boundary

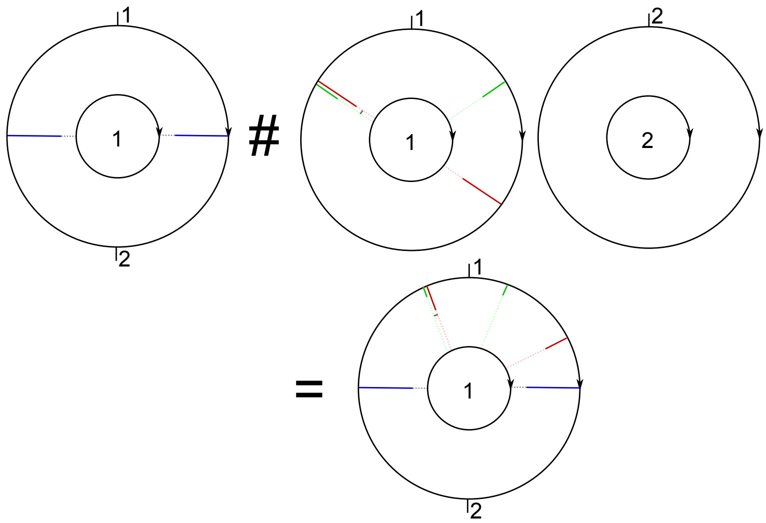

Figure 8. An example of the glueing operation. Note that the cylinder behaves as an identity up to homotopy.

of the annulus in counterclockwise direction, followed by $a_{i,2}$, etc. We denote the angular length of the segment $a_{i,j}$ by $|a_{i,j}|$.

- For L', divide the i'th annulus $\mathbb{A}_{R'}$ starting at radial segment above 1 in counterclockwise in radial sectors $F_{i,j}$ of relative angular lengths $\frac{|a_{i,j}|}{|O_i|}$. Some of the slits ζ_i will fall into the interior of these sectors, but some will lie on the boundaries of two sectors. In the latter case, we say that ζ_i lies in the radial sector of which it lies in the right boundary. This arbitrary choice will not matter once we take the quotient by $\equiv_1$.
  We can do the same for the P'_i.
- Linearly scale in the radial sectors $F_{i,j}$ in the radial direction to get them to lie in $\mathbb{A}_R^{R'}$ and insert them into $\mathbb{A}_{R+R'}$ after a rotation such that the intersection $F_{i,j} \cap \partial_{out}\mathbb{A}$ coincides with $a_{i,j}$.
- We know where the $\vec{\zeta}''$ and $\vec{P}''$ of $L\#L'$ should lie. The $\vec{P}''$ just keep their labelling as coming from $\vec{P}'$. The $\vec{\zeta}''$ are labelled such that those from $\vec{\zeta}$ appear first, followed by those from $\vec{\zeta}'$. This describes the $\vec{\zeta}''$ and $\vec{P}''$.
- The new pairing λ'' is simply the one induced by λ and λ''.
- The new successor permutation ω'' and exceptional index set Θ'' are implicit in our description of the $\vec{\zeta}''$.
- The new partial glueing parameters $\vec{t}''$ are given by the three sets: the first is the old set $\vec{t}$. the second the modified old set $\vec{t}' + R - 1$ and there is a new third set $\vec{t}''_n$ of partial glueing parameters between the rightmost slit to fall into a boundary of radial sector and the slit ζ_i of that boundary: its value will be $|\zeta_i| - 1$.

See figure 8 for a simple example of the glueing operation. Note that this construction is compatible with the equivalence relations $\equiv_1$ and $\equiv_2$, thus induces maps

$$L\# : \mathrm{LRad}_h(m,n) \times \mathrm{LRad}_{h'}(n,k) \to \mathrm{LRad}_{h+h'}(m,k)$$
$$\# : \mathrm{Rad}_h(m,n) \times \mathrm{Rad}_{h'}(n,k) \to \mathrm{Rad}_{h+h'}(m,k)$$

Next we get to the operation of disjoint union. This will be significantly easier.

Definition 1.17. We define a disjoint union operation

$$P\sqcup : \mathrm{PRad}_h(m,n) \times \mathrm{PRad}_{h'}(m',n') \to \mathrm{PRad}_{h+h'}(m+m',n+n')$$

It is given a pair $L = (\vec{\zeta}, \lambda, \omega, \theta, R, \vec{P}, \vec{t})$, $L' = (\vec{\zeta}', \lambda', \omega', \theta', R', \vec{P}', \vec{t}')$ by taking $m+m'$ annuli and then putting the data of L on the first m and the data of L' on the last m'.

It is also clear that disjoint union of configurations gives maps

$$L\sqcup : \mathrm{LRad}_h(m,n) \times \mathrm{LRad}_{h'}(m',n') \to \mathrm{LRad}_{h+h'}(m+m',n+n')$$
$$\sqcup : \mathrm{Rad}_h(m,n) \times \mathrm{Rad}_{h'}(m',n') \to \mathrm{Rad}_{h+h'}(m+m',n+n')$$

These operations are compatible in the following sense.

Proposition 1.18. *This makes* PRad*,* LRad *and* Rad *into prop's with identities up to homotopy.*

Proof. See [Böd06, section 9] for most of the proof. Our only modifications are dropping the connectedness assumptions and adding the cylinders. □

There is a general construction that takes operads or more generally prop's in Top and create new operads or prop's by applying a generalized homology theory. The result is an operad or prop in category of graded modules over the value of that generalized homology theory on a point, modulo Künneth. Such modules are also called E_*-modules. We will now apply this construction to our prop of radial slit configurations.

Let E be a commutative ring spectrum. For example, one can pick E to be the Eilenberg-Mac Lane spectrum HR or one of the K-theory spectra KU or KO. Then there are corresponding generalized homology and cohomology theories E_* and E^*. For more details see e.g. the classical book by Adams [Ada74] or the more recent book by May and Sigurdsson [MS06].

Suppose that all the finite products of spaces $\mathrm{Rad}_h(m,n)$ satisfy the Künneth theorem for E-homology. Then $E_*(\mathrm{Rad}_h(m,n))$ is a prop in the category of E_*-modules.

Definition 1.19. A 0-dimensional ECFT (with non-empty boundary) is an algebra over the prop $E_*(\mathrm{Rad})$ in the category of E_*-modules. If $E = HR$, we call it a HCFT (with non-empty boundary).

Here HCFT stands for homological conformal field theory and ECFT, not very originally, stands for E-homological conformal field theory. We will describe d-dimensional ECFT's and HCFT's later.

The reason we included the terminology "with non-empty boundary" is that as a consequence of our model we must have that every connected component of a cobordism has non-empty incoming and outgoing boundary. In the literature a HCFT, or more generally a ECFT, is usually not required to satisfy this condition. For string topology, the most natural condition is that the outgoing boundary of each connected component is non-empty. This is known as the positive boundary condition. In section 6.1 we will sketch how to extend Bödigheimer's model to include this case.

To connect to Godin's definition of a HCFT using the maps $B\Gamma_\Sigma \times B\Gamma'_\Gamma \to B\Gamma_{\Sigma\#\Sigma'}$ induced by glueing of cobordisms along the boundaries, we note the following proposition. This proposition should intuitively be clear. From it we conclude that these two notions of a HCFT coincide on their common domain of definition.

Proposition 1.20. *By taking π_1 the prop structure on* Rad *induces on on mapping class groups a prop structure. This is the same prop structure as the one induced by glueing cobordisms along boundaries or taking their disjoint union.*

Proof. See section 9.1 of [Böd06]. □

1.5. **The canonical local system $\mathcal{L}$.** We will now construct a local system over Rad which will keep track of the signs for the string topology operations. It will also be necessary to make the composition of the operations behave correctly.

We prefer to think of E-homology twisted by a local system as given by a non-trivial parametrized spectrum which is locally isomorphic to $U \times E$ for an open U up to fiberwise homotopy. For more information about parametrized spectra and good choices for them, see [MS06]. In the end our construction does not and should not depend on the intricacies of the model one chooses. We denote the trivial parametrized spectrum with fiber E by $\mathbb{E}$.

We will now describe a method to construct some examples of parametrized spectrum modelling E-homology with local coefficients.

(1) One can construct a local system from a double cover L. First note that there is a homomorphism from the group $\{\pm 1\}$ to the group of deck transformations, given by sending 1 to the identity and -1 to the map which switches the sheets.

To see this, recall that the E-homology of X with local coefficients is given by some parametrized spectrum over X. L determines a parametrized spectrum as follows. The multiplication by ± 1 gives us a map $\{\pm 1\} \to gl_1 E$. Since E was assumed to a commutative ring spectrum, $gl_1 E$ is well-defined. We then define a parametrized spectrum $\mathbb{E}_L$ over X by taking the quotient using the diagonal action of $\{\pm 1\}$.

$$\mathbb{E}_L = L \times_{\{\pm 1\}} E$$

(2) If X is obtained as a quotient by free action of a group G on a space $\tilde{X}$, one can construct a parametrized spectrum $\mathbb{E}_l$ over X from a trivial parametrized spectrum on $\tilde{X}$, given a homomorphism $l : G \to gl_1 E$. It is given as follows:

$$\mathbb{E}_l = (\tilde{X} \times \mathbb{E})/G$$

where G acts on $\tilde{X}$ as given and on G by l.

We will now describe a canonical local system $\mathcal{L}$ in two ways. Firstly, we define it to as follows. Let $\tilde{X} = \mathrm{LRad}$ and S_{2h} act on E via $\{\pm 1\}$ by the sign representation. Then we set $\mathbb{E}_{\mathcal{L}}$ to be the parametrized spectrum $(\mathrm{LRad} \times \mathbb{E})/S_{2h}$ over $\mathrm{Rad} = \mathrm{LRad}/S_{2h}$. This will be the way the local system appears in our construction.

Alternatively, consider the vector bundle $\mathrm{LRad} \times \Lambda^{2h}\mathbb{R}^{2h}$ and take the quotient by S_{2h}, which acts on $\Lambda^{2h}\mathbb{R}^{2h}$ by the map induced by the permutation of the coordinates on $\mathbb{R}^{2h}$. This gives us a line bundle over Rad, which inherits a metric since S_{2h} acts by isometries. Taking the two elements of norm 1 in this line bundle, we get a double cover L. By the earlier construction we obtain a parametrized spectrum $\mathbb{E}_L$, which is clearly isomorphic to $\mathbb{E}_{\mathcal{L}}$. We denote either one of these parametrized spectrum by $\mathbb{E}_{\mathcal{L}}$.

There is a tensor product of parametrized spectra of this type, because each fiber is a module over the commutative ring spectrum E. This is given by the fiberwise smash product over E. In the previous framework, there is an easy way to describe the product $\mathbb{E}_{\mathcal{L}}^{\otimes d}$ for $d \geq 1$. It is given by iterating the tensor product of double covers $L \otimes L' = L \times_{\{\pm 1\}} L'$ and taking the associated parametrized spectrum. Note that in our case $L^{\otimes 2}$ is canonically trivial, because there is a canonical global section. Hence we have that

$$\mathbb{E}_{\mathcal{L}}^{\otimes d} \cong \begin{cases} 1 & \text{if } d \text{ is even} \\ \mathbb{E}_{\mathcal{L}} & \text{if } d \text{ is odd} \end{cases}$$

1.6. **Compatibility of the local system with the prop structure.** We claim that $\mathbb{E}_{\mathcal{L}}$ is compatible with the prop structure on Rad. This means that it is in a natural way compatible with the $\#$ and $\sqcup$ maps.

Set $\mathrm{Rad} = \mathrm{Rad}_h(m,n)$, $\mathrm{Rad}' = \mathrm{Rad}_{h'}(n,k)$ and $\mathrm{Rad}'' = \mathrm{Rad}_{h+h'}(m,k)$ for convenience. Let $\pi_1 : \mathrm{Rad} \times \mathrm{Rad}' \to \mathrm{Rad}$ and $\pi_2 : \mathrm{Rad} \times \mathrm{Rad}' \to \mathrm{Rad}'$ be the projections. Then we must show that the following map induced by $\#$ by wedging over E in the fiber of the parametrized spectra is in fact well-defined.

$$\#_{\mathcal{L}} : \pi_1^*(\mathbb{E}_{\mathcal{L}}) \wedge_E \pi_2^*(\mathbb{E}_{\mathcal{L}'}) \to \mathbb{E}_{\mathcal{L}''}$$

Note that the first term is obtained by taking the quotient by $S_{2h} \times S_{2h'}$ of $\mathrm{Rad} \times \mathrm{Rad}' \times E \wedge_E E$, which is equal to the trivial parametrized spectrum over the product $\mathrm{Rad} \times \mathrm{Rad}'$ with fiber E. The map $\#_{\mathcal{L}}$ fits in the following diagram:

$$\begin{tikzcd} \mathrm{LRad} \times \mathrm{LRad}' \times E \arrow[r, "\# \times id_E"] \arrow[d, "q_{S_{2h} \times S_{2h'}}"'] & \mathrm{LRad}'' \times E \arrow[d, "q_{S_{2(h+h')}}"] \\ \pi_1^*(\mathbb{E}_{\mathcal{L}}) \wedge_E \pi_2^*(\mathbb{E}_{\mathcal{L}'}) \arrow[r, "\#_{\mathcal{L}}"'] & \mathbb{E}_{\mathcal{L}''} \end{tikzcd}$$

where $q_{S_{2h} \times S_{2h'}}$ and $q_{S_{2(h+h')}}$ are the quotient map coming from the group actions.

The commutativity of this then boils down to the claim that the standard map $S_{2h} \times S_{2h'} \to S_{2(h+h')}$, which puts the first permutation before the second, is a homomorphism. This is clear. The argument for $\sqcup$ is similar, but easier.

This means that we have a prop structure in parametrized spectra on the $\mathbb{E}_{\mathcal{L}}$ covering the prop structure on Rad in Top. In fact, from the definition we see that this prop structure in fact descends from the trivial prop structure of the trivial parametrized spectrum $\mathbb{E}$ covering the prop structure on LRad.

We can now define d-dimensional ECFT's as algebras over the corresponding twisted E-homology prop with shifted grading. We first fix some notation: the twisted E-homology group $E_*(\mathrm{Rad}; \mathcal{L})$ is the set of fiberwise homotopy classes of maps of parametrized spectra over Rad from suspensions of trivial sphere spectrum over Rad to $\mathbb{E}_{\mathcal{L}}$. In other words, it is given by fiberwise homotopy classes of sections of the spaces of the parametrized spectrum over Rad.

Definition 1.21. A d-dimensional ECFT is graded E_*-module which is an algebra over the shifted E-homology groups $E_{*+hd}(\mathrm{Rad}; \mathcal{L}^{\otimes d})$.

2. Canonical spaces over PRad, LRad and Rad

In this section we define several spaces lying over the space $\mathrm{Rad}_h(m,n)$ of non-degenerate radial slit configurations with $2h$ slits, m incoming boundary circles and n outgoing ones, or the spaces $\mathrm{LRad}_h(m,n)$ and $\mathrm{PRad}_h(m,n)$ of labelled non-degenerate radial slit configurations and non-degenerate radial slit preconfigurations respectively. For the remainder of this section, we fix h, m, n. For convenience, we denote the space $\mathrm{Rad}_h(m,n)$ by Rad, $\mathrm{LRad}_h(m,n)$ by LRad and $\mathrm{PRad}_h(m,n)$ by PRad.

The definitions of this section naturally extend to the semi-degenerate configurations, but for more remarks about this see the section 6.3.

2.1. Parametrised point-set topology.

We always work in the category Top of compactly generated weakly Hausdorff spaces. This means that the topologies are functorially replaced if necessary.

Definition 2.1. Let B be an object of Top, then the category Top/B over spaces over B is the slice category which has as objects pairs (X, f) of a space X and a map $f : X \to B$ and as morphisms the commutative diagrams

$$\begin{tikzcd} X \arrow[rr, "g"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & B & \end{tikzcd}$$

where g is a continuous map.

Many notions of topology have a simple fiberwise counterpart: a fiberwise homeomorphism is an isomorphism in Top/B, a fiberwise homotopy is a map $I \times X \to X'$ which commutes with the projections to B and a fiberwise homotopy equivalence is a map admitting an inverse in Top/B up to fiberwise homotopies.

The subspace $f^{-1}(\{b\}) \subset X$ is known as the fiber of X above b. We will use several constructions of parametrized topology, the first of which is the pullback functor generalizing the fiber.

Definition 2.2. Let $h : A \to B$ be a continuous map, then there is a functor $h^* : \mathsf{Top}/B \to \mathsf{Top}/A$ given by $h^*(X) = \{(a, x) \in A \times X | h(a) = f(x)\}$ with map to A given by projection on the first component. On morphisms this functor is given by $h^*(g) : (a, x) \mapsto (a, g(x))$.

Note that if $b : * \to B$ is the unique map which sends the singleton to a point b in B, then the fiber of X over b is exactly the pullback along this map. A point $*$ is terminal in Top and hence a topological space is the same as space over $*$. If $t_B : B \to *$ is the unique map to the terminal object and X any space, then $t_B^* X$ is $B \times X$ with the projection tp B as map to B.

Lemma 2.3. *The pullback functor h^* preserves fiberwise homeomorphisms and fiberwise homotopy equivalences.*

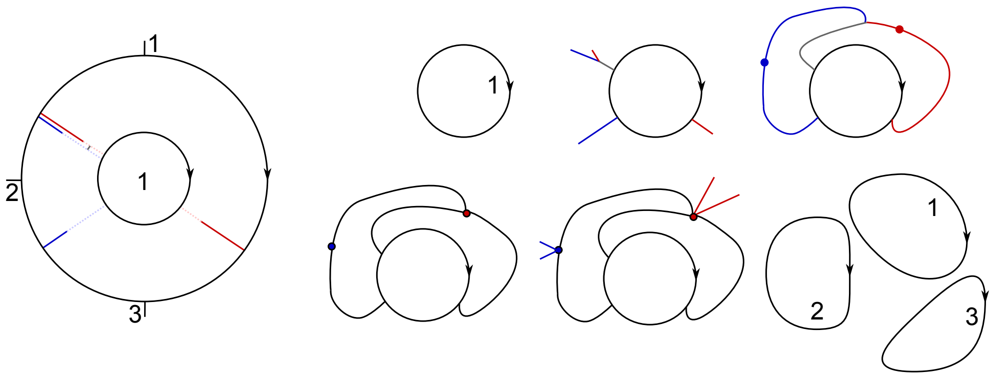

FIGURE 9. A radial slit preconfiguration with its associated graph spaces. The little grey line denotes the partial glueing parameter. On the top from left to right $P\Gamma_{in}$, $P\tilde{\Gamma}$, $P\Gamma$, on the bottom $P\Gamma_\alpha$, $P\Gamma_{\alpha,\beta}$ and $P\Gamma_{out}$.

Proof. Both claims are obvious as the candidates for the inverse and the fiberwise homotopy equivalence simply pull back as well. □

2.2. **The radial segment space.** The space $P\tilde{\Sigma}$ is a subspace of the space $\tilde{\mathbb{A}}$ over PRad and will inherit the subspace topology. Recall that for each radial slit configuration L, there is a space $\tilde{F}(L) \subset \tilde{\mathbb{A}}$ of radial sectors. We define

$$P\tilde{\Sigma} = \{(L, x)|x \in \tilde{F}(L)\} \subset \tilde{\mathbb{A}}$$

There is a canonical projection PRad. Note that when we take the fiber over $L \in$ PRad, we get exactly the space $\tilde{F}(L)$.

2.3. **The surface space.** Recall that there is an equivalence relation $\approx_L$ on $\tilde{F}(L)$ such that each $\tilde{F}(L)/\approx$ is a surface with Euler characteristic $-h$, m incoming boundary components and n outgoing boundary components, or a harmonic degeneration thereof. Combining all these equivalence relations gives an equivalence relation $\approx$ on $P\tilde{\Sigma}$ and we define

$$P\Sigma = P\tilde{\Sigma}/\approx$$

Note that the fiber over L is exactly the non-degenerate surface $F(L)$. There is also a natural isomorphism between the two surfaces over points in PRad which are identified by $\equiv_1$ when going to LRad, so $P\Sigma$ can be canonically made into a space $L\Sigma$ over LRad, with the correct fiber over each point. Similarly, because the surface doesn't depend on the labels, we have natural isomorphisms between surfaces over points that are identified by $\equiv_2$. Hence we get a space Σ over Rad. This space corresponds to the canonical surface bundle over the moduli space.

2.4. **The graph spaces.** In this section we define three radial subspaces of $P\tilde{\Sigma}$. Recall that each radial segment $\tilde{L}$ has canonical subspaces $\alpha^\pm(L), \beta^\pm(L), \gamma^\pm(L)$. The following six spaces over PRad are obtained by glueing parts of these subspaces

$$\begin{aligned}
P\Gamma_{in} &= \{(L, x)|x \in \gamma^-(L)\}/\approx \\
P\tilde{\Gamma} &= \{(L, x)|x \in \alpha^+(L) \sqcup \alpha^-(L) \cup \gamma^-(L)\}/\sim \\
P\Gamma &= \{(L, x)|x \in \alpha^+(L) \sqcup \alpha^-(L) \cup \gamma^-(L)\}/\sim_V \\
P\Gamma_\alpha &= \{(L, x)|x \in \alpha^+(L) \sqcup \alpha^-(L) \cup \gamma^-(L)\}/\approx \\
P\Gamma_{\alpha,\beta} &= \{(L, x)|x \in \alpha^+(L) \sqcup \alpha^-(L) \sqcup \beta^+(L) \sqcup \beta^-(L) \sqcup \gamma^-(L)\}/\approx \\
P\Gamma_{out} &= \{(L, x)|x \in \gamma^+(L)\}/\approx
\end{aligned}$$

where $\sim_V$ is the equivalence relation which identifies two points if they are identified by $\sim$ or if they are identified by $\approx$ and lie in both an α and a β. The use of the equivalence relation $\approx$ shows that $P\Gamma_{in}$, $P\Gamma_\alpha$, $P\Gamma_{\alpha,\beta}$ and $P\Gamma_{out}$ are naturally subspaces of $P\Sigma$. See figure 9 for an example.

Note that we have maps

$$P\Gamma_{in} \hookrightarrow P\tilde{\Gamma} \to P\Gamma \to P\Gamma_\alpha \hookrightarrow P\Gamma_{\alpha,\beta}$$

where each map except the collapse $P\tilde{\Gamma} \to P\Gamma$ is a homotopy equivalence. There is also a map $P\Gamma_{out} \hookrightarrow P\Sigma$, mapping $P\Gamma_{out}$ homeomorphically on the outgoing boundary.

Furthermore, for all these spaces except $P\tilde{\Gamma}$, there is a natural isomorphism between graphs which lie over points which are identified by $\equiv_1$ or $\equiv_2$, so these spaces descend to spaces $L\Gamma_{in}$, $L\Gamma$, $L\Gamma_\alpha$, $L\Gamma_{\alpha,\beta}$ and $L\Gamma_{out}$ over LRad and spaces Γ_{in}, Γ, Γ_α, $\Gamma_{\alpha,\beta}$ and Γ_{out} over Rad. This is *not* the case for $P\tilde{\Gamma}$.

2.5. **Incoming and outgoing circle bundles.** There are two simple torus bundles over PRad: $P(\mathbb{S}^1)^m$ and $P(\mathbb{S}^1)^n$ are just the trivial bundles, i.e. they are given by PRad $\times (S^1)^m$ and PRad $\times (S^1)^n$ respectively with the natural projection maps. These constructions also work over LRad or Rad, producing torus bundles $L(\mathbb{S}^1)^m$, $L(\mathbb{S}^1)^n$ and $(\mathbb{S}^1)^m$and $(\mathbb{S}^1)^n$.

However, $P(\mathbb{S}^1)^m$ admits an isomorphism with $P\Gamma_{in}$ as follows. We define a map by sending the j'th component S^1 over a point L in a counterclockwise angle preserving fashion onto the j'th circle of $F(L)$ in $P\Gamma_{in}$ obtained by glueing the circle segments γ^-. The map from $P(\mathbb{S}^1)^n$ to $P\Gamma_{out}$ is slightly more complicated. We map the j'th component S^1 in a counterclockwise angle preserving fashion onto the j'th outgoing boundary component, in such a way that 1 is mapped to the parametrisation point.

It is clear that both of these isomorphisms commute with the projection to PRad and hence we get that $P(\mathbb{S}^1)^m \cong P\Gamma_{in}$ and $P(\mathbb{S}^1)^n \cong P\Gamma_{out}$ in Top/PRad. Similar statements holds over LRad and Rad.

2.6. **The retraction map.** Next we define a fiberwise homotopy equivalence $Pr : P\Sigma \to P\Gamma_{\alpha,\beta}$ over PRad. We do this by defining a map $Pr : P\Sigma \to P\Gamma_{\alpha,\beta}$ in Top/Rad with the property that $Pr \circ Pi = id$, where $Pi : P\Gamma_{\alpha,\beta} \to P\Sigma$ is the inclusion. After that we show that it is compatible with the equivalence relations and hence descends to LRad and Rad.

Let F_k be a radial segment above L which is not thin or entire. We define the length $|\partial F_k| = |\beta_k^+| + |\alpha_k^+| + |\gamma_k^-| + |\beta_k^-| + |\alpha_k^-|$ as the sum of lengths of paths in $\mathbb{C}$. Then we define length ratios

$$l_k^1 = \frac{|\beta_k^+|}{|\partial F_k|}, \qquad l_k^2 = \frac{|\alpha_k^+| + |\gamma_k^-| + |\beta_k^-|}{|\partial F_k|} \qquad \text{and} \qquad l_k^3 = \frac{|\beta_k^-|}{|\partial F_k|}$$

We draw a line in polar coordinates between β_k^+ and the first part of γ_k^+ in clockwise fashion of relative length $|\beta_k^+|/|\gamma_k^+|$ and the same for the other two parts. We then define r on this segment by linearly retracting in polar coordinates the radial segment along the lines to part of its boundary given by $\alpha^+(L) \cup \alpha^-(L) \cup \beta^+(L) \cup \beta^-(L) \cup \gamma^-(L)$. See figure 10 and note that this is the identity on this boundary.

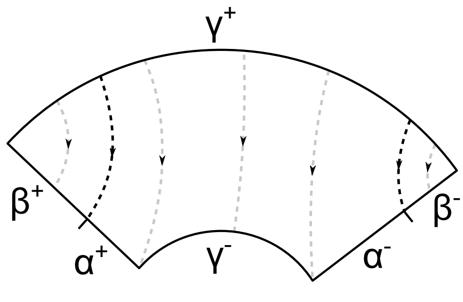

FIGURE 10. The retraction which is homotopy inverse to the inclusion of the part of boundary consisting of $\alpha^+(L) \cup \alpha^-(L) \cup \beta^+(L) \cup \beta^-(L) \cup \gamma^-(L)$.

If a radial segment is thin, then we define r to be the identity. If it is entire, then r is the linear retraction of the annulus onto its incoming boundary. Together, these definitions give a continuous map $P\Sigma \to P\Gamma_{\alpha,\beta}$ which has the desired property.

Since the glueing $\approx$ happens along the boundary and this homotopy is the identity there, this fiberwise homotopy equivalence between the radial sectors and their boundary factors to a fiberwise homotopy equivalence between the surface and a subspace of graphs. Furthermore, it is equal for points identified by $\equiv_1$, so descends to a fiberwise homotopy equivalence of $L\tilde{\Sigma}$ and $L\Gamma_{\alpha,\beta}$ over LRad. It is also S_{2h}-equivariant, hence descends to a fiberwise homotopy equivalence of Σ with $\Gamma_{\alpha,\beta}$ over Rad.

2.7. **Overview.** We have now defined the following commutative diagram of spaces, the top line over Rad, the middle line over LRad and the bottom lines over PRad. The vertical arrows cover the quotient maps LRad $\to$ Rad and PRad $\to$ LRad respectively.

(1)
$$\begin{tikzcd}
(\mathbb{S}^1)^m \arrow[r, "\cong"] & \Gamma_{in} \arrow[rr] & & \Gamma \arrow[r, "\simeq"] & \Gamma_{\alpha,\beta} \arrow[r, "\simeq"] & \Sigma & \Gamma_{out} \arrow[l] & (\mathbb{S}^1)^n \arrow[l, "\cong"'] \\
L(\mathbb{S}^1)^m \arrow[u] \arrow[r, "\cong"] & L\Gamma_{in} \arrow[u] \arrow[rr] & & L\Gamma \arrow[u] \arrow[r, "\simeq"] & L\Gamma_{\alpha,\beta} \arrow[u] \arrow[r, "\simeq"] & L\Sigma \arrow[u] & L\Gamma_{out} \arrow[u] \arrow[l] & L(\mathbb{S}^1)^n \arrow[u] \arrow[l, "\cong"'] \\
P(\mathbb{S}^1)^m \arrow[u] \arrow[r, "\cong"] & P\Gamma_{in} \arrow[u] \arrow[r, "\simeq"] & P\tilde{\Gamma} \arrow[r] & P\Gamma \arrow[u] \arrow[r, "\simeq"] & P\Gamma_{\alpha,\beta} \arrow[u] \arrow[r, "\simeq"] & P\Sigma \arrow[u] & P\Gamma_{out} \arrow[u] \arrow[l] & P(\mathbb{S}^1)^n \arrow[u] \arrow[l, "\cong"']
\end{tikzcd}$$

2.8. **Fiberwise mapping spaces.** To construct the mapping spaces we use the techniques of May-Sigurdsson. To define the fiberwise mapping spaces, one wants to construct a space of partially defined maps and then restrict to the subspace consisting of those maps that have a fiber as domain.

Definition 2.4. The partial map classifier $\tilde{X}$ of a space X is the following space: as a set it is the union of X with a disjoint point ω and it has the topology with closed sets $\tilde{X}$ itself and all closed sets of X.

The space $\tilde{X}$ has the property that a continuous map into $\tilde{X}$ is the same as a continuous map mapping a closed subset into X and the rest to ω. In other words, continuous maps into $\tilde{X}$ are continuous maps that map a closed subset to X. Note that fibers, being the inverse images of points, are closed.

Definition 2.5. There is a bifunctor $\mathrm{Map}_B(-,-) : (\mathrm{Top}/B)^{op} \times \mathrm{Top}/B \to \mathrm{Top}/B$ known as the fiberwise mapping space. For objects X and Y of Top/B, it is defined as the following pullback:

$$\begin{tikzcd}
\mathrm{Map}_B(X,Y) \arrow[r] \arrow[d] & \mathrm{Map}(X,\tilde{Y}) \arrow[d] \\
B \arrow[r] & \mathrm{Map}(X,\tilde{B})
\end{tikzcd}$$

where the bottom map sends a point b in B to the constant partially defined function $X_b \to b$ and a right map is induced by the map $Y \to B$. We use compact open topology for the spaces in the right column.

We will legitimize the name of this functor by proving the following lemma.

Lemma 2.6. *Let X, Y be spaces over B. Then $\mathrm{Map}_B(X,Y)$ is a space over B with the following properties.*

- *The fibers are the mapping spaces between the fibers: given $b : * \to B$ then we have $b^*\mathrm{Map}_B(X,Y) \cong \mathrm{Map}(b^*X, b^*Y)$ naturally.*
- *Maps between product spaces are a product space: if X and Y are spaces, then $\mathrm{Map}_B(t_B^*X, t_B^*Y) \cong t_B^*\mathrm{Map}(X,Y)$ naturally. Here $\mathrm{Map}(X,Y)$ has the compact open topology.*
- *Fiberwise mapping spaces preserve fiberwise homotopy equivalences.*

Proof. Note that spaces over B can be made into ex-spaces over B by taking the disjoint union with B. Hence we can use results of [MS06, section 2.2] to prove the first two claims. The last claim is a consequence of naturality. $\square$

In the next section we will apply this construction to the spaces over PRad, LRad and Rad considered earlier in this section.

3. Constructing the string operations

In this section we construct the higher string operations. Though the results we obtain are not new, this construction is substantially easier and more geometric than earlier constructions. The main reason for this is we only need to attach pairs of points at a time. Recall that E is a commutative ring spectrum, but we are in particular interested in HR for $R = \mathbb{Z}, \mathbb{Q}, \mathbb{Z}_p$. Fix a compact manifold M of dimension d which is E-oriented, then the higher string operations are a set of maps

$$\xi : E_*(\mathrm{Rad}_h(m,n); \mathcal{L}^{\otimes d}) \otimes E_*(LM)^{\otimes m} \longrightarrow E_{*-hd}(LM)^{\otimes n}$$

which satisfy certain compability relations giving $E_*(LM)$ the structure of a ECFT of dimension d.

3.1. **Overview of construction.** In this section, we shorten $\mathrm{Rad}_h(m,n)$ to Rad and similarly use abbreviations LRad and PRad. Similarly mapping spaces $\mathrm{Map}_B(X, t_B^* M)$ for a space X over B are denoted M^X. For $Map_B((\mathbb{S}^1)^m, M)$ we use the notation $\mathbb{L}M^m$. Using the fiberwise mapping space construction of the previous section we obtain the following diagram:

$$\begin{tikzcd}
(\mathbb{L}M)^m \arrow[r, "\cong"] & M^{\Gamma_{in}} \arrow[d, "\phi"'] & & M^{\Gamma} \arrow[ll, "\psi"'] \arrow[r, "\simeq"] \arrow[d, "\varphi"] & M^{\Gamma_{\alpha,\beta}} \arrow[r, "\simeq"] & M^{\Sigma} \arrow[r] & M^{\Gamma_{out}} \arrow[r, "\cong"] & (\mathbb{L}M)^n \\
& M^{L\Gamma_{in}} \arrow[d, "L\varphi"'] & & M^{L\Gamma_{\alpha}} \arrow[ll, "L\psi"] \arrow[d, "L\phi"] & & & & \\
& M^{P\Gamma_{in}} \arrow[r, "\simeq"'] & M^{P\tilde{\Gamma}} & M^{P\Gamma} \arrow[l, "P\psi"] & & & &
\end{tikzcd}$$

where the upper row are spaces over Rad, the second row consists of spaces over LRad and the third row of spaces over PRad. All the maps are induced by those in diagram 1. In the next section we want to apply E-homology fiberwise to this diagram to obtain a map in E-homology from the left to the right. Clearly, everything is fine, except the map ψ. We will bypass it using $P\psi$, but though this is a slightly nicer map for our purposes than ψ itself, it is still problematic. The trick is to notice it is given by fiberwise pullback along a fibration of an embedding of finite dimensional manifolds, hence we can apply the Thom collapse construction fiberwise to get a fiberwise umkehr map. It will turn out that we can descend the construction to Rad to obtain an umkehr map for ψ. The only problem will be that the orientations are changed by the S_{2h}-action which $\equiv_2$ quotients out. This can be resolved by including the local system $\mathcal{L}$.

3.2. **The umkehr map I: fiberwise tubular neighborhood.** To construct the umkehr maps fiberwise, we note the following: the quotient map $q : P\tilde{\Gamma} \to P\Gamma$ always takes $2h$ points which are identified to h points. This means that the map $M^{P\Gamma} \to M^{P\tilde{\Gamma}}$ is an embedding which is fiberwise given by a pullback over a finite-dimensional embedding. Hence we have a chance of applying the Thom collapse construction. To produce an umkehr map for $P\psi$, we define two additional spaces over PRad.

$$\tilde{V} = \{(L,x) | x \in (\alpha^+(L) \cap \beta^+(L)) \cup (\alpha^-(L) \cap \beta^-(L))\} \subset P\tilde{\Gamma}_\alpha$$
$$V = (\tilde{V}/\approx) \subset P\Gamma_\alpha$$

These spaces are chosen for the following reason: $\tilde{V}$ consists exactly of the points in $P\tilde{\Gamma}$ which are identified by $\sim_V$ and V is the result of this identification. We call points of these spaces "vertices" and the other parts of $P\tilde{\Gamma}$ and $P\Gamma$ which are not part of the incoming boundary the "edges".

Note $\tilde{V}$ is a trivial $2h$-fold covering of PRad, while V is h-fold one. That they are trivial follows from the fact that in PRad we have labelled the slits and hence the vertices. The quotient map gives a map $q_V : \tilde{V} \to V$ which induces a fiberwise diagonal map $\Delta_V^h : M^V \to M^{\tilde{V}}$ between the fiberwise mapping spaces into M. These spaces are isomorphic to the trivial bundles $t^*_{\mathrm{PRad}}(M^h)$ and $t^*_{\mathrm{PRad}}(M^{2h})$ over PRad and the map Δ_V under this isomorphism is pullback of the map $\iota^h : M^h \to M^{2h}$ which is the h-fold product of the diagonal map $\iota : M \hookrightarrow M^2$. Here our order for the product is that we order the pairs with respect to lowest index and within each pair we order in the usual manner.

Therefore, we have a canonical fiberwise normal bundle $P\nu$ over M^V, given by direct sum of copies of the pullback $t^*_{M^V}(\nu_\iota)$. The orientation on ν_ι is given by requiring that $TM^2 \cong TM \oplus \nu_\iota$ is an oriented isomorphism and defining the orientation of ν_{ι^h}, by definition a direct sum of ν_ι's, to be the orientation obtained from the natural ordering on the labels. We make this choice to make our lives easier when we quotient out the S_{2h}-action.

Next we fix a tubular neighborhood for ν_ι and therefore a tubular neighborhood f for $P\nu$. The tubular neighborhood is assumed to be symmetric, i.e. $f(-v)$ can be obtained from $f(v)$ be switching the components pairwise. We also fix a propagating flow $\mathcal{P}$ for ν_ι and a connection ∇ on ν_ι, which also are assumed to be symmetric and pull back to $P\nu$. All these choices come in contractible families [God07, section 3.1] and our construction will depend continuously on them, so these choices will not matter.

Note that there are evaluation maps $\mathrm{ev}_V : M^{P\Gamma} \to M^V$ and $\mathrm{ev}_{\tilde{V}} : M^{P\tilde{\Gamma}} \to M^{\tilde{V}}$. We can compose with the isomorphisms of V and $\tilde{V}$ with the trivial bundles M^h and M^{2h} over PRad to get a pullback diagram in spaces over PRad:

$$\begin{tikzcd}
M^{P\Gamma} \arrow[r] \arrow[d] & M^{P\tilde{\Gamma}} \arrow[d] \\
M^V \arrow[r, "\Delta_V^h"'] \arrow[d, "\cong"] & M^{\tilde{V}} \arrow[d, "\cong"'] \\
t^*_{\mathrm{PRad}}(M^h) \arrow[r, "t^*_{\mathrm{PRad}}(\iota^h)"'] & t^*_{\mathrm{PRad}}(M^{2h})
\end{tikzcd}$$

Suppose that we can construct a tubular neighborhood upstairs from a tubular neighborhood downstairs, a propogating flow and a connection, then we obtain a fiberwise tubular neighborhood $P\tilde{f} : P\tilde{\nu} \to M^{P\tilde{\Gamma}}$, where the $P\tilde{\nu}$ is the fiberwise vector bundle over $M^{P\Gamma}$ obtained by pulling back $P\nu$. To be precise $P\tilde{\nu}$ is given by the fiberwise pullback $\mathrm{ev}_V^* P\nu$. It turns out that such constructions exist and in the next subsection we take a closer look at a particular construction to see that is in fact not that hard. As a consequence we get a umkehr map between parametrized spectra

$$(P\tilde{f})^! : \Sigma^\infty_{\mathrm{PRad}}(M^{P\tilde{\Gamma}})_+ \to \mathrm{Thom}(P\tilde{\nu})$$

where $\Sigma^\infty_{\mathrm{PRad}}(-)_+$ is the fiberwise suspension functor over PRad and $\mathrm{Thom}(P\tilde{\nu})$ is obtained by applying the Thom spectrum construction fiberwise, hence obtaining a parametrized spectrum over PRad.

3.3. **The umkehr map II: explicitly constructing the tubular neighborhood.** In this section, we will show how to explicitly construct the tubular neigborhood upstairs. We do this for completeness, but also to make it easier to check the ECFT conditions.

The idea is that the pullback diagram is only really concerned with what happens near the vertices which are glued, as long as we are lucky enough to deal with configurations which aren't special. In the latter case, we need to allow interactions of the vertices to make sure that when a slit jumps, the map is the same. Essentially, the partial glueing parameters control the amount of this influence. If a partial glueing parameter is zero, it is like the corresponding vertices are still disjoint, while if it is maximal, it is like the collapse of overlying vertex happens after that of the underlying vertex.

On a more technical level, the idea is that we can use the tubular neighborhood f to see, given a normal vector, how to change value of our maps from $P\tilde{\Gamma}$ into M at the points of V and use a

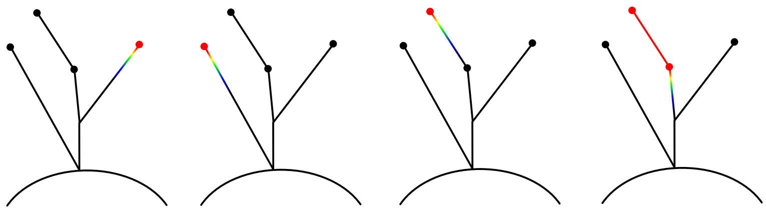

FIGURE 11. The flow control functions for four vertices on a tree. Here red denotes the value 1 and black the value 0. Note that a flow control vertex can never influence a vertex which lies below it due to our choice of ϵ.

flow to extend this continuously to the entire domain of the maps. To define this rigorously, we start by recalling the definition of a propagating flow.

Definition 3.1. Let ν be a vector bundle with projection π to the base. A propagating flow for ν is a continuous map $\mathcal{P} : \nu \to \chi_{vc}(\nu)$, where the latter denotes the space of compactly supported vertical smooth vector fields on ν, with the property that for all $v \in \nu$ and for each point p on the line segment between $\pi(v)$ in the 0-section and v in the fiber $\pi^{-1}(\pi(v))$ we have that $\mathcal{P}(v)(p) = v$.

We will now construct the tubular neighborhood. We note that there are canonical metrics $d_{\tilde{\Gamma}}$ on $P\tilde{\Gamma}$ and d_Γ on $P\Gamma$, which depend continuously on $L \in \mathrm{PRad}$. These metrics are given by glueing the metrics on the radial segments: the distance between points is the infimum over the lengths of all possible piecewise linear paths. The length of such a path is determined cutting it at the vertices and at the points where they cross the boundaries of the radial sectors, and then using the metric of the radial sectors to determine the length of the linear pieces.

We define a continuous function $\epsilon : \mathrm{PRad} \to \mathbb{R}_{>0}$. It is given by

$$\epsilon(L) = \inf\big(\{d_\Gamma(v,w) | v, w \in V \cap F(L)\} \cup \{d_\Gamma(v,a) | v \in V \cap F(L) \text{ and } a \in \gamma^-(L)\}\big)$$

where the first set gives us a radius for disjoint neighborhoods around the vertices and the second set guarantees that these neighborhoods are disjoint from the inner boundary (which will be useful when we glue the operations). Note that for non-degenerate radial slit preconfigurations the points of V are disjoint in $P\Gamma$ and disjoint from the incoming boundary. Hence $\epsilon(L)$ is non-zero for all non-degenerate L.

Then we define $P\tilde{f} : P\tilde{\nu} \to M^{P\tilde{\Gamma}}$ using the original tubular neighborhood f and propagating flow $\mathcal{P}$. Using the fiberwise tubular neighborhood, we can extend the vector fields to M^{2h} to obtain a continuous map $\mathcal{Z} : \nu \to \chi_c(M^{2h})$, where χ_c denotes the compactly supported vector fields. This induces a map $Z_i : \nu \times \mathbb{R} \to \mathrm{Diff}(M)$ by flowing along the i'th component $\mathcal{Z}_i(v)$ for a time $t \in \mathbb{R}$, which is defined for all times t because our vector fields are compactly supported.

We will now define the flow control functions. The following definition works for generic preconfigurations, but to treat special preconfigurations we need to make a few remarks. We have noted earlier that in the case of a special preconfiguration we have for each point on the incoming boundary for which there are multiple slits in the corresponding radial segment, a tree in $P\tilde{\Gamma}$ with natural height function. Let τ_{ij} be the point on that tree with maximal modulus where α_i^+ is attached to α_j^-. This was also discussion above definition 1.12.

We first pick a strictly decreasing smooth function $\eta : [0,\infty) \to [0,1]$ which has value 1 in 0 and is 0 on $[1,\infty)$. The space of these is contractible and our construction depends continuously on it, so this choice doesn't affect the result of our construction. We define the flow control functions $\eta_i : P\tilde{\Gamma} \to [0,1]$ as follows

$$\eta_i(y) = \begin{cases} \eta(3\frac{d_{\tilde{\Gamma}}(\zeta_i, \tau_{ji})}{\epsilon(L)}) & \text{if } y \text{ lies above } \tau_{ji} \text{ on } \alpha_i^+ \\ \eta(3\frac{d_{\tilde{\Gamma}}(\zeta_i, y)}{\epsilon(L)}) & \text{otherwise} \end{cases}$$

The level sets of the flow control functions on a tree in $P\tilde{\Gamma}$ of a special preconfigurations are shown in figure 11.

A point in $P\tilde{\nu}$ over L is given by a pair (g, v) of a function $g : (P\Gamma)_L \to M$, where $(P\Gamma)_L$ denotes the fiber over L, and a vector $v \in \nu_{\mathrm{ev}_V(g)}$, the fiber over the evaluation of g at the vertices. Order the indices as $i_1, \ldots, i_{2h}$ such that $|\zeta_{i_1}| \leq \ldots \leq |\zeta_{i_{2h}}|$. Let $y \in (P\tilde{\Gamma})_L$, then we set

$$P\tilde{f}(g,v)(y) = Z_{i_{2h}}(v, \eta_{i_{2h}}(y))[Z_{i_{2h-1}}(v, \eta_{i_{2h-1}}(y))[\ldots Z_{i_1}(v, \eta_{i_1}(y))[g(q(y))]\ldots]]$$

The reader may worry about the choice of ordering. However, because of our choice of ϵ, the supports of the flows are disjoint unless we have a special configuration. Hence in that case they commute and the order doesn't matter. In a special preconfiguration, only the flow of slits in a single radial segment might mix, but these are unambiguously ordered by modulus, because we look at non-degenerate preconfigurations.

Lemma 3.2. *The map $P\tilde{f}$ is continuous, injective and open. Therefore it is a homeomorphism onto its image. Furthermore is the identity map on zero section of $P\tilde{\nu}$.*

Proof. The continuity is a direct consequence of the fact that the maps Z_j depend continuously on their entries, the η_j are continuous, precomposition by q is continuous and the induced action of $\mathrm{Diff}(M)$ on the functions from the fibers of $P\tilde{\Gamma}_\alpha$ to M is continuous.

The injectivity and openness are a consequence of the fact that we can reverse the flows to get a continuous inverse to $\tilde{f}$ defined on its image. On the zero section we enter $v = 0$ in all the Z_i, which implies that we will flow along a zero vector fields and hence obtain the identity. ☐

As a consequence we see that $P\tilde{f}$ is a tubular neighborhood. It is an extension of the original tubular neighborhood for generic configurations, as shown by the following lemma. To write it down, recall that $P\tilde{f}$ gives a map $P\tilde{f}|_V : P\tilde{\nu} \to M^{\tilde{V}}$ by restriction to the vertices.

Lemma 3.3. *If L is generic then $P\tilde{f}|_V$ is given by the tubular neighborhood f.*

Proof. Because L is generic each flow is supported on a different edge of $P\tilde{\Gamma}$. This is a consequence of our choice of ϵ, which made the neighborhoods on which the flows change the value of the map to M disjoint. ☐

We will next apply the Thom collapse construction and investigate how the resulting map of spectra behaves when we look at two points that are identified by $\equiv_1$ and $\equiv_2$.

3.4. **The umkehr map III: descending to** LRad **and the fiberwise Thom isomorphism.** From the previous sections we know that we have a map

$$\Sigma^\infty_{\mathrm{PRad}}(M^{P\tilde{\Gamma}})_+ \to \mathrm{Thom}(P\tilde{\nu})$$

We can precompose this with the natural map $\Sigma^\infty_{\mathrm{PRad}}(M^{P\Gamma_{in}})_+ \to \Sigma^\infty_{\mathrm{PRad}}(M^{P\tilde{\Gamma}})_+$ to obtain a map of spectra

$$(P\tilde{f})^! : \Sigma^\infty_{\mathrm{PRad}}(M^{P\Gamma_{in}})_+ \to \mathrm{Thom}(P\tilde{\nu})$$

This map is determined by the 0-th level map $T : M^{P\Gamma_{in}} \to Thom(P\tilde{\nu})$ where the latter is the fiberwise Thom space of $\tilde{\nu}$. Suppose that $L \equiv_1 L'$, then we have a natural homeomorphism $\phi : (P\Gamma_{in})_{L'} \to (P\Gamma_{in})_L$. Suppose we have a map $g : (P\Gamma_{in})_L \to M$, then we can interpret g a either as a map $g_L : (P\Gamma_{in})_L \to M$ or as a map $g_{L'} = g_L \circ \phi : (P\Gamma_{in})_{L'} \to M$. We claim that $T(g_L) = T(g_{L'})$. This is exactly what is required to show that $(P\tilde{f})^!$ descends to a map of spectra

$$(L\tilde{f})^! : \Sigma^\infty_{\mathrm{LRad}}(M^{L\Gamma_{in}})_+ \to \mathrm{Thom}(L\tilde{\nu})$$

Because it is clear that the jumps of parametrisation points do not matter for this map, we can assume without loss of generality that L is obtained from L' by letting ζ_1, which is paired with ζ_2, jump over ζ_3 to ζ_4. We will therefore just treat this case.

We start by looking at the points mapped to infinity in the Thom space. To show that $T(g_L)$ is the point at infinity if and only if $T(g_{L'})$ is the point at infinity, it suffices to prove the following lemma.

Lemma 3.4. *The evaluation $\mathrm{ev}_{\tilde{V}}(g_L)$ lies in the image of the $P\tilde{f}|_V$ if and only if $\mathrm{ev}_{\tilde{V}}(g_{L'})$ lies in the image of the $P\tilde{f}|_V$.*

Proof. Suppose that the evaluation of g_L at the vertices is obtained by flowing a point in $(M^V)_L$ in the direction v. Then we note that the flow for the vertices of $(M^V)_{L'}$ differs only in the following way: where for L the vertex ζ_1 was flowed with $Z_3(v,1)$ and $Z_4(v,0)$, for L' it is flowed with $Z_3(v,0)$ and $Z_4(v,1)$. Thus by symmetry of the tubular neighborhood one obtains exactly the same result when v is replaced by $-v$. So $\mathrm{ev}_{\tilde{V}}(g_{L'})$ lies in the image of the $P\tilde{f}|_V$ as well. $\square$

One can determine the value of $P\tilde{f}_{in}$ on an arbitrary $g : (P\Gamma_{in})_L \to M$ which is not send to infinity as follows: by evaluation on the vertices and inverting $P\tilde{f}|_V$, which is possible by the injectivity, one determines the v that was used to flow a map $(P\Gamma)_L \to M$ in order to get g. The resulting vector is then used to reverse the flow to get a function $(P\tilde{\Gamma})_L \to M$.

Lemma 3.5. *If g_L is not mapped to infinity, then $(P\tilde{f}|_V)^{-1}$ applied to $\mathrm{ev}_{\tilde{V}}(g_L)$ is v if and only if $(P\tilde{f}|_V)^{-1}$ applied to $\mathrm{ev}_{\tilde{V}}(g_{L'})$ is v.*

Proof. For all pairs $(i, \lambda(i))$ except those that jumped across $(3,4)$ the components of ν are the same in both cases, since these can be determined from the corresponding components $\mathrm{ev}_{\tilde{V}}(g_L)$, which are the same. We can these flows, in particular including Z_3 and Z_4, to return to the value at ζ_1 to a state where it was not flowed. Our construction gives the same values for both L and L', which implies the statement. $\square$

Lemma 3.6. *If g_L is not mapped to infinity, then $T(g_L) = T(g_{L'})$.*

Proof. Similar reasoning as before shows that after returning to a state where the values at points in the tree which lie above ζ_3 or ζ_4, including ζ_1, were not flowed, the functions are the same in the case of L and L'. This implies the statement. $\square$

As noted before, the last two lemma's imply the following important corollary.

Corollary 3.7. *$(P\tilde{f}_{in})^!$ descends to a map of spectra $(L\tilde{f}_{in})^! : \Sigma^\infty_{\mathrm{LRad}}(M^{L\Gamma_{in}})_+ \to \mathrm{Thom}(L\tilde{\nu})$.*

3.5. **The umkehr map IV: descending to** Rad. We will now apply the Thom isomorphism. To do this, we must smash with E fiberwise. Because M is E-oriented, we can fix a Thom class on the spectrum level for the image of $M \hookrightarrow M^2$, hence for the image of f. This gives a unique map of spectra $\mathrm{Thom}(\nu) \wedge E \to \Sigma^\infty_+(M^h) \wedge \Sigma^{hd}E$ implementing the Thom isomorphism. Since $L\tilde{\nu}$ is obtained as a pullback, this pulls back to a map representing the Thom isomorphism for for $L\tilde{f}$. We get obtain a unique map of spectra

$$\mathrm{Thom}(L\tilde{\nu}) \wedge_{\mathrm{LRad}} \mathbb{E} \to \Sigma^\infty_{\mathrm{LRad}}(M^{L\Gamma})_+ \wedge_{\mathrm{LRad}} \Sigma^{hd}\mathbb{E}$$

Note that because ν is trivial, $L\tilde{\nu}$ is trivial over LRad as well. This will not be the case when we descend to Rad. Precomposing with the Thom collapse induced by the tubular neighborhood, we get a map of spectra

$$\Psi' : \Sigma^\infty_{\mathrm{LRad}}(M^{L\Gamma_{in}})_+ \wedge_{\mathrm{LRad}} \mathbb{E} \to \Sigma^\infty_{\mathrm{LRad}}(M^{L\Gamma})_+ \wedge_{\mathrm{LRad}} \Sigma^{hd}\mathbb{E}$$

Next we want to try and descend to Rad. Let's start with a point of $M^{\tilde{\Gamma}_{in}}$. When we pull this point back to LRad using ϕ we obtain several maps in $M^{L\tilde{\Gamma}_{in}}$ which are mapped to each other by action of S_{2h} on LRad. If we apply the construction of the previous subsection, because of the S_{2h}-invariance of our constructions, we do land in a parametrized spectrum over the image of M^Γ under φ in $M^{L\Gamma}$.

But the map $\Psi' \circ I$, where $I : \Sigma^\infty_{\mathrm{Rad}}(M^{\Gamma_{in}})_+ \to \Sigma^\infty_{\mathrm{LRad}}(M^{L\Gamma_{in}})_+$ is the natural map, is *not* S_{2h}-invariant. It only is up to a sign, due to our choice in orientation. We conclude that if we take the quotient by S_{2h} and note that our construction has the same result for points which are identified by $\equiv_2$, we obtain us a map of parametrized spectra

$$\Sigma^\infty_{\mathrm{Rad}}(M^{\Gamma_{in}})_+ \wedge_{\mathrm{Rad}} \mathbb{E} \to \left(\Sigma^\infty_{\mathrm{Rad}}(M^{L\Gamma})_+ \wedge_{\mathrm{Rad}} \Sigma^{hd}\mathbb{E}\right)/S_{2h}$$

An inspection of the signs occuring tells us that S_{2h} acts of E via $\{\pm 1\}$ by d'th power of the sign representation, where d is the dimension of ν, i.e. the dimension of M. Hence, we can twist both sides with $\mathcal{L}^{\otimes d}$ to obtain a map of spectra:

$$\Psi : \Sigma^\infty_{\mathrm{Rad}}(M^\Gamma_{in})_+ \wedge_{\mathrm{Rad}} \mathbb{E}^{\otimes d}_{\mathcal{L}} \to \Sigma^\infty_{\mathrm{Rad}}(M^\Gamma)_+ \wedge_{\mathrm{Rad}} \Sigma^{hd}\mathbb{E}$$

3.6. The higher string operations. Finally, diagram 1 allows us to compose and precompose with maps to obtain a map of parametrized spectra

$$\Phi : \Sigma^\infty_{\mathrm{Rad}}(\mathbb{L}M^m)_+ \wedge_{\mathrm{Rad}} \mathbb{E}_{\mathcal{L}}^{\otimes d} \to \Sigma^\infty_{\mathrm{Rad}}(\mathbb{L}M^n)_+ \wedge_{\mathrm{Rad}} \Sigma^{hd}\mathbb{E}$$

Because the space $\mathbb{L}M^m = \mathrm{Map}_{\mathrm{Rad}}((\mathbb{S}^1)^m, M)$ is trivial over Rad and the twist inducing $\mathbb{E}_{\mathcal{L}}$ is obtained by a pull back from Rad, we have that the fiberwise suspension $\Sigma^\infty_{\mathrm{Rad}}(\mathbb{L}M^n)_+ \wedge_{\mathrm{Rad}} \mathbb{E}_{\mathcal{L}}^{\otimes d}$ is naturally isomorphic to a wedge of an ordinary spectrum and a parametrized one: $\Sigma^\infty(LM^m)_+ \wedge \mathbb{E}_{\mathcal{L}}^{\otimes d}$, using the definition of $\mathcal{L}$. Taking global sections, i.e. composing with the projection $\mathbb{L}M^n \cong \mathrm{Rad} \times LM^n \to LM^n$, we obtain a map of spectra

$$\Xi : \Sigma^\infty(LM^m)_+ \wedge \mathbb{E}_{\mathcal{L}}^{\otimes d} \to \Sigma^\infty(LM^n)_+ \wedge \Sigma^{hd}E$$

Passing to homotopy classes of sections, we thus obtain degree-lowering map in E-homology

$$\xi : E_*(\mathrm{Rad}; \mathcal{L}^{\otimes d}) \otimes E_*(\mathrm{LM})^{\otimes m} \longrightarrow E_{*-hd}(\mathrm{LM})^{\otimes n}$$

where we have assumed that the E-homologies appearing are of such a type that the Künneth theorem applies. For example if $E = HR$, then requiring that $H_*(LM; R)$ is free and finitely generated in each degree suffices. This condition is always satisfied if R is a field, e.g. $\mathbb{Q}$, or if M is a compact Lie group G and R any commutative ring by Bott's theorem [Bot58]. Note that if d is even, then we have seen $\mathbb{E}_{\mathcal{L}}^{\otimes d}$ is in fact trivial and thus we can forget about it.

4. ECFT structure

Considering the fact that in the operations ξ the local system $\mathcal{L}^{\otimes d}$ and a dimension shift of $-hd$ appear, they are naturally candidates for the structure maps of a d-dimensional ECFT. We will show that this is in fact the case.

We must show that our Ξ are compatible with the $\#$ and $\sqcup$. For the former, set $\mathrm{Rad} = \mathrm{Rad}_h(m,n)$, $\mathrm{Rad}' = \mathrm{Rad}_{h'}(n,k)$ and $\mathrm{Rad}'' = \mathrm{Rad}_{h+h'}(m,k)$ for convenience and let $\pi_1 : \mathrm{Rad} \times \mathrm{Rad}' \to \mathrm{Rad}$ and $\pi_2 : \mathrm{Rad} \times \mathrm{Rad}' \to \mathrm{Rad}'$ be the projections. It suffices to show that the following diagram commute up to homotopy:

$$\begin{tikzcd}
\Sigma^\infty(LM^m)_+ \wedge (\pi_1^*(\mathbb{E}_{\mathcal{L}}^{\otimes d}) \wedge_{\mathbb{E}} \pi_2^*(\mathbb{E}_{\mathcal{L}'}^{\otimes d})) \arrow[r, "\Xi \wedge id"] \arrow[d, "id \wedge (\#_{\mathcal{L}})^{\otimes d}"'] & \Sigma^\infty(LM^n)_+ \wedge \Sigma^{hd}\mathbb{E}_{\mathcal{L}'}^{\otimes d} \arrow[d, "\Xi'"] \\
\Sigma^\infty(LM^m)_+ \wedge \mathbb{E}_{\mathcal{L}''}^{\otimes d} \arrow[r, "\Xi''"'] & \Sigma^\infty(LM^k)_+ \wedge \Sigma^{(h+h')d}E
\end{tikzcd}$$

Looking at the details of the previous section, one sees that this holds if the following diagram commutes up to homotopy:

$$\begin{tikzcd}
\Sigma^\infty_{\mathrm{Rad}''}(M^{\Gamma''_{in}})_+ \wedge_{\mathrm{Rad}''} (\pi_1^*(\mathbb{E}_{\mathcal{L}}^{\otimes d}) \wedge_{\mathbb{E}} \pi_2^*(\mathbb{E}_{\mathcal{L}'}^{\otimes d})) \arrow[r, "\bar{\Phi} \wedge id"] \arrow[d, "id \wedge \#_{\mathcal{L}}^{\otimes d}"'] & \Sigma^\infty_{\mathrm{Rad}''}(M^{\Gamma'_p})_+ \wedge_{\mathrm{Rad}''} \Sigma^{hd}\pi_2^*(\mathbb{E}_{\mathcal{L}'}^{\otimes d}) \arrow[d, "\bar{\Phi}'"] \\
\Sigma^\infty_{\mathrm{Rad}''}(M^{\Gamma''_{in}})_+ \wedge_{\mathrm{Rad}''} \mathbb{E}_{\mathcal{L}''}^{\otimes d} \arrow[r, "\Phi''"'] & \Sigma^\infty_{\mathrm{Rad}''}(M^{\Gamma''_{in}})_+ \wedge_{\mathrm{Rad}''} \Sigma^{(h+h')d}\mathbb{E}
\end{tikzcd}$$

where Γ'_p is the subgraph of Γ''_α containing only the α's of L. Then $\bar{\Phi}$ is the map induced by Φ by only glueing the α's of L. The map $\bar{\Phi}'$ is induced by Φ' by noting that Φ' is the identity near the inner boundary of the annuli and hence after rescaling we apply Φ' to the outer annuli $\mathbb{A}_R^{R'}$ and extend by the identity.

We have already seen that the glueing is compatible with our local system. Thus to show the commutativity up to homotopy of this previous diagram, we can show the commutativity up to homotopy of the corresponding diagram over LRad. The advance of this is that we don't have to worry about the local systems anymore.

$$\begin{tikzcd}
\Sigma^\infty_{\mathrm{LRad}''}(M^{L\Gamma''_{in}})_+ \wedge_{\mathrm{LRad}''} \mathbb{E} \arrow[r, "\bar{\Psi}\wedge id"] \arrow[dr, "\Psi''"'] & \Sigma^\infty_{\mathrm{LRad}''}(M^{L\Gamma'_p})_+ \wedge_{\mathrm{LRad}''} \Sigma^{hd}\mathbb{E} \arrow[d, "\bar{\Psi}'"] \\
 & \Sigma^\infty_{\mathrm{LRad}''}(M^{L\Gamma''_{in}})_+ \wedge_{\mathrm{LRad}''} \Sigma^{(h+h')d}\mathbb{E}
\end{tikzcd}$$

We now note that we can first do the Thom collapse construction for the slits coming from the first configuration, thus we can add a commutative triangle up to homotopy at the bottom. The result is that it suffices to show that the left-bottom and top-right compositions of the following diagram are equal up to homotopy.

$$\begin{tikzcd}
\Sigma^\infty_{\mathrm{LRad}''}(M^{L\Gamma''_{in}})_+ \wedge_{\mathrm{LRad}''} \mathbb{E} \arrow[r, "\bar{\Psi}_1\wedge id"] \arrow[d, "\bar{\Psi}_2\wedge id"'] \arrow[dr, "\Psi''"] & \Sigma^\infty_{\mathrm{LRad}''}(M^{L\Gamma'_p})_+ \wedge_{\mathrm{LRad}''} \Sigma^{hd}\mathbb{E} \arrow[d, "\bar{\Psi}'_1"] \\
\Sigma^\infty_{\mathrm{LRad}''}(M^{L\Gamma'_p})_+ \wedge_{\mathrm{LRad}''} \Sigma^{hd} \arrow[r, "\bar{\Psi}'_2"'] & \Sigma^\infty_{\mathrm{LRad}''}(M^{L\Gamma''_{in}})_+ \wedge_{\mathrm{LRad}''} \Sigma^{(h+h')d}\mathbb{E}
\end{tikzcd}$$

where we have renamed the maps that were originally called $\bar{\Psi}$ and $\bar{\Psi}'$ to $\bar{\Psi}_1$ and $\bar{\Psi}'_1$ respectively and $\bar{\Psi}_2$ and $\bar{\Psi}'_2$ are the result of doing the Thom collapse for the slits of respectively L and L' in the glued configuration. Now note that both maps out of $\Sigma^\infty(M^{L\Gamma''_{in}})_+ \wedge_{\mathrm{LRad}''} \mathbb{E}$ differ only in the number ϵ used. The homotopy $t \mapsto \epsilon(t) = \frac{1}{2}t\epsilon(L) + \frac{1}{2}(2-t)\epsilon(L'')$ for $t \in [0,1]$ does the trick. Thus it suffices to show that $\bar{\Psi}'_1$ and $\bar{\Psi}'_2$ are the same up to homotopy.

There are two ways in which they differ: the first difference is due to the induced retraction of the outer boundary of the inner annulus onto the graphs Γ'_p and the second is due to the differences between the maps controlling the flows. The first doesn't matter up to homotopy because we can change the retraction for the slits by a homotopy such that for all generic configurations, the retraction maps them to $\partial_{in}\mathbb{A} \subset \Gamma'_p$, and for all special configurations, the points to which the slits are mapped on the edges of Γ'_p depends on the partial glueing parameters. The second doesn't matter up to homotopy by changing ϵ in a smooth way.

A similar but much simpler argument works for the $\sqcup$ maps, because one only needs to homotope the ϵ. We conclude that indeed the ξ give $E_*(LM)$ the structure of a ECFT.

5. The classical string topology operations

In this section we explain how one can easily recover the classical definitions of the string product, string coproduct and BV-operator from this construction, when working with $E = HR$, i.e. ordinary homology. We will use the description found in [CV06]. To do compare to those descriptions, we will use the following proposition.

Proposition 5.1. *Let $a \in H_k(\mathrm{Rad}_h(m,n))$ be represented by submanifold A of dimension k such $\mathcal{L}$ is trivial over A. Then $\xi(a,-) : H_*(LM;R)^{\otimes m} \to H_{*-hd+k}(LM;R)^{\otimes n}$ is given by taking the restriction of the map Ξ to subspace A of the base $\mathrm{Rad}_h(m,n)$, taking fiberwise homotopy classes of sections and inserting the fundamental class $[A]$ into the resulting map $\xi|_A$.*

Proof. Let a map $f_a : S^{l+k} \to (\Sigma^\infty(\mathrm{Rad})_+ \wedge HR)_l$ represent the class a. The hypotheses imply that this map factors as

$$S^{l+k} \xrightarrow{f_{[A]}} (\Sigma^\infty(A)_+ \wedge HR)_l \xrightarrow{i} (\Sigma^\infty(\mathrm{Rad})_+ \wedge HR)_l$$

for sufficiently large l, where the first map represents the fundamental class $[A]$ and the second map is induced by the inclusion.

Let $g_b : S^{p+r} \to (\Sigma^\infty(LM^m)_+ \wedge HR)_p$ represent an element b of $H_r(LM^m;R)$. Because $\mathcal{L}$ is trivial over A, then $\xi(a,b)$ can be given by

$$\begin{aligned} S^{l+k+p+r} \cong S^{l+k} \wedge S^{p+r} &\xrightarrow{f_a \wedge g_b} (\Sigma^\infty(\mathrm{Rad})_+ \wedge HR)_l \wedge (\Sigma^\infty(LM^m)_+ \wedge HR)_p \\ &\longrightarrow (\Sigma^\infty(\mathrm{Rad} \times LM^m)_+ \wedge HR)_{l+p} \xrightarrow{\Xi} (\Sigma^\infty(LM^n)_+ \wedge \Sigma^{hd}HR)_{l+p} \end{aligned}$$

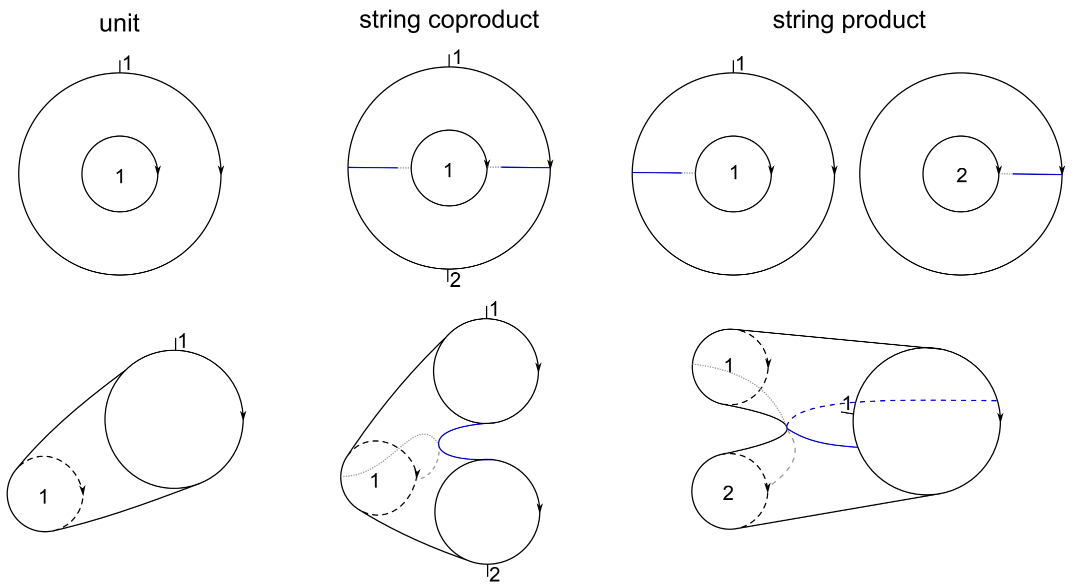

FIGURE 12. Radial slit configurations giving the unit, string coproduct and string product. The corresponding surfaces are pictured on the second line and are indeed equal the cylinder, reverse pair of pants and pair of pants, as expected.

On the other hand, if we write down $\xi|_A([A], b)$, we get the following:

$$S^{l+k+p+r} \cong S^{l+k} \wedge S^{p+r} \xrightarrow{f_{[A]} \wedge g_b} (\Sigma^\infty(A)_+ \wedge HR)_l \wedge (\Sigma^\infty(LM^m)_+ \wedge HR)_p$$
$$\longrightarrow (\Sigma^\infty(A \times LM^m)_+ \wedge HR)_{l+p} \xrightarrow{\Xi|_A} (\Sigma^\infty(LM^m)_+ \wedge \Sigma^{hd} HR)_{l+p}$$

The factorisation of f_a and the fact that its image is contained in the subspace A now show that both expressions are equal. □

5.1. **The string coproduct and product.** Since the string product and string coproduct should be given by the generators $H_0(\mathrm{Rad}_0(1, 2))$ and $H_0(\mathrm{Rad}_0(2, 1))$ respectively, it suffices to look at the operations we get when we restrict to a single point in these moduli spaces. For completeness, we have drawn in figure 12 two such points and another point giving the unit in $\mathrm{Rad}_0(1, 1)$.

Lemma 5.2. *Let $a_{1,1}$ be the generator $H_0(\mathrm{Rad}_0(1, 1); R)$. Then the operation*

$$I := \xi(a_{1,1}, -) : H_*(LM; R) \to H_*(LM; R)$$

is equal to identity map.

Proof. Using proposition 5.1, this is easy. When restricting to the point displayed in figure 12, note that in fact ψ is an isomorphism, as nothing is glued. This means that we can bypass the lower lines of the diagram 1 and write the operation as homology applied to the composition

$$LM \xrightarrow{\cong} M^{\Gamma_{in}} \longrightarrow M^\Gamma \xrightarrow{\simeq} M^\Sigma \longrightarrow M^{\Gamma_{out}} \xrightarrow{\cong} LM$$

where Γ_{in}, Γ, Σ, Γ_{out} now denote the fibers above our point in $\mathrm{Rad}_0(1, 1)$ and hence are simply graphs or surfaces. Now note that the map $M^{\Gamma_{in}} \to M^\Gamma$ is in fact the identity, as there are no slits and the map $M^\Sigma \to M^{\Gamma_{out}}$ is a homotopy equivalence, given by retraction of the cylinder onto its outgoing boundary, together with a rotation. This means that the entire composition is a homotopy equivalence and hence applying homology gives us the identity map. □

Lemma 5.3. *Let $a_{1,2}$ be the generator of $H_0(\mathrm{Rad}_0(1, 2); R)$. Then the operation*

$$C := \xi(a_{1,2}, -) : H_*(LM; R) \to H_{*-d}(LM \times LM; R)$$

is equal to the string coproduct.

Proof. Again we can just pick a single point representing $a_{1,2}$ and looking at the map of spectra in the fiber over that point. We can bypass the lower line and just work over Rad. Then we see that C is obtained from the following diagram of spaces:

$$LM \xrightarrow{\cong} M^{\Gamma_{in}} \xrightarrow{\simeq} M^{\tilde{\Gamma}} \xleftarrow[\phi]{} M^{\Gamma} \xrightarrow{\simeq} M^{\Sigma} \xrightarrow{\simeq} M^{\Gamma_{out}} \xrightarrow{\cong} LM^2$$

Here we use that $\tilde{\Gamma}$ is well-defined over a non-special configuration. By contracting the $\alpha^{\pm}$ and $\beta^{\pm}$ to a point everywhere, which is a homotopy equivalence, we see that the following diagram is commutative:

$$\begin{tikzcd}
LM \arrow[r, "\cong"] \arrow[d, equal] & M^{\Gamma_{in}} \arrow[r, "\simeq"] & M^{\tilde{\Gamma}_\alpha} & M^{\Gamma_\alpha} \arrow[l, "\psi"'] \arrow[r, "\simeq"] & M^{\Sigma} \arrow[r] & M^{\Gamma_{out}} \arrow[r, "\cong"] & LM^2 \arrow[d, equal] \\
LM \arrow[rr, equal] & & LM & M^E \arrow[l, "\chi"] \arrow[rrr, "\iota"'] & & & LM^2
\end{tikzcd}$$

where E is the figure eight space $S^1 \vee S^1$, χ is induced by the pinch map $S^1 \to S^1 \vee S^1$ and ι is induced by the projection map $S^1 \sqcup S^1 \to S^1 \vee S^1$, which traverses both circles once in positive direction, starting with the one labelled first. Thus, applying the umkehr map construction to ψ in the top row and composing with all the homeomorphisms and homotopy equivalences should give the same map as applying the umkehr map to χ and composing with ι. But the latter is exactly the classical definition of the string coproduct. □

Lemma 5.4. *Let $a_{2,1}$ be the generator of $H_0(\mathrm{Rad}_0(2,1); R)$. Then the operation*

$$P := \xi(a_{2,1}, -) : H_*(LM; R)^{\otimes 2} \to H_{*-d}(LM; R)$$

coincides with the string product.

Proof. The proof is the same as in the previous lemma, except now our commuting diagram is

$$\begin{tikzcd}
LM^2 \arrow[r, "\cong"] \arrow[d, equal] & M^{\Gamma_{in}} \arrow[r, "\simeq"] & M^{\tilde{\Gamma}_\alpha} & M^{\Gamma_\alpha} \arrow[l, "\psi"'] \arrow[r, "\simeq"] & M^{\Sigma} \arrow[r] & M^{\Gamma_{out}} \arrow[r, "\cong"] & LM \arrow[d, equal] \\
LM^2 \arrow[rr, equal] & & LM^2 & M^E \arrow[l, "\iota'"] \arrow[rrr, "\chi'"'] & & & LM
\end{tikzcd}$$

where ι' is induced by the projection map $S^1 \sqcup S^1 \to S^1 \wedge S^1$ and χ' is induced by the pinch map $S^1 \to S^1 \vee S^1$. Because applying the umkehr map construction to the lower line is exactly the classical definition of the string product, we are done. □

5.2. **The BV-operator.** It is simple to see that $\mathrm{Rad}_0(1,1)$ is homotopy equivalent to a circle, as the only parameters are the outer radius R, which lies in a contractible space of choices, and the parametrisation point P, the choice of which amounts essentially the choice of angle. We conclude that $\mathrm{Rad}_0(1,1)$ has homology concentrated in degree 0 and 1, where it is free. This is independent of the power of the local system $\mathcal{L}$, which is trivial over $\mathrm{Rad}_0(1,1)$ because $h = 0$. We have just seen that the generator of H_0 gives the identity. We claim that the generator of $H_1(\mathrm{Rad}_0(1,1); R)$ is the BV-operator.

Lemma 5.5. *Let $b_{1,1}$ be the generator of $H_1(\mathrm{Rad}_0(1,1); R)$, represented by the loop displayed in figure 13. Then the operation*

$$\Delta := \xi(b_{1,1}, -) : H_*(\mathrm{LM}; R) \to H_{*+1}(\mathrm{LM}; R)$$

coincides with the BV-operator.

Proof. Because there are no slits and hence no umkehr maps, the map in homology is simply the one obtained by inserting $[S^1]$ in the map induced in homology by the following diagram:

$$S^1 \times LM \xrightarrow{id \times \cong} S^1 \times M^{\Gamma_{in}} \xrightarrow{id \times \simeq} S^1 \times M^{\Sigma} \xrightarrow{\rho} M^{\Gamma_{out}} \xrightarrow{\cong} LM$$

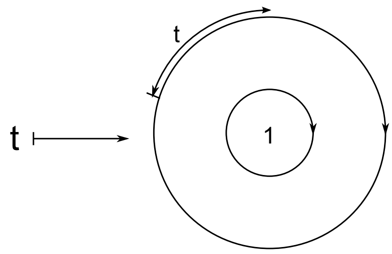

FIGURE 13. A loop in $\mathrm{Rad}_0(1,1)$ representing a generator of its first homology group. The induced string operation is the BV-operator.

where the map ρ sends a pair (θ, g) of $\theta \in S^1$ and a map $g : \mathbb{A} \to M$ to a map $S^1 \to M$ given by $\varphi \mapsto (g|_{\partial_{out}\mathbb{A}})(\varphi + \theta)$. Applying homology and inserting $[S^1]$, we see that the operation is given by

$$a \mapsto \rho_*([S^1] \times a)$$

which is exactly the classical definition of the BV-operator $\square$

5.3. **Comparing to Godin's higher string operations.** Godin's construction of higher genus operations [God07] precedes our construction. In this section we will indicate why our construction should give the same operations. We will not give a full proof, because a full proof will involve the details of Godin's construction and will therefore be too long for this note. Note that previous subsections already imply that our construction and Godin's construction are equal on all explicitly known operations, because both are HCFT's and assign the same operations to the homology classes corresponding to classical operations.

Godin's construction is very roughly as follows: (1) construct a string topology operation for each n-simplex of composable sequences of n morphisms of ribbon graphs, (2) in such way that one can glue all these operations together over the geometric realisation of the category of ribbon graphs. Note that each fiber of the graph space Γ is a ribbon graph using the cyclic structure from its embedding in the surface Σ. This category corresponds to the subcategory of ribbon graphs consisting of string diagrams [Poi10], which should be a deformation retract.

It is not hard to check that our construction for each of these single graphs coincides up to homotopy with Godin's, because we use the same Thom collapse construction.

The angular and radial movement of slits is modelled in the ribbon category by collapsing edges and a slit jump is modelled by the geometric realisation of a pair of arrows $\Gamma/e \leftarrow \Gamma \to \Gamma/e'$, collapsing $e = \alpha_i$ or $e' = \alpha_{\lambda(i)}$. Our construction for this path differs from Godin's in the following sense:in our construction, in the middle all edges and vertices are moved by the flow of the slit ζ_i, while in Godin's only a small part of the connecting edges is moved. But these two constructions are homotopy equivalent by smoothly tuning down the flow when one gets further away from the slit.

One can subdivide Rad into small subsimplices where only two slits move past each other or jump and restrict the parametrized spectra to these subsimplices. This way we should get a space with decomposition into simplices, which can be subdivided to be homeomorphic to a subdivision of geometric realisation of a deformation retract of the category of ribbon graphs, and over each simplex the construction is homotopy equivalent to Godin's. Hence both constructions should give the same maps in homology.

6. Possible applications and extensions

6.1. **Including units in the prop's.** Godin's construction in fact shows that $H_*(LM; R)$ has the structure of a HCFT with positive boundary. This means that the incoming boundary is allowed to be empty. Because Bödigheimer doesn't treat this situation in [Böd06], we decided not to include this possibilility in this note. However, it is clear that an appropriate modification of the radial slit configuration space where sometimes the annuli are replaced by should support similar operations as we constructed for annuli. These modified spaces should allow one to construct the units as well. We think that the following model should work: allow any inner boundary radius in $[0, 1]$ and add a parametrisation point on the inner boundary – which is just a point is the inner boundary radius is 0 – and then take the quotient by the S^1-action given by rotating the annulus and all the data on it.

6.2. **New string topology operations?** One of the great advantages of Bödigheimer's model of the moduli space of Riemann surfaces with boundary is that it is well-suited for calculations. For example, in [ABE08] a computer program is given that calculates the low-dimensional homology groups of the moduli space. The result are tantalizing, for example exhibiting non-zero classes in the unstable range, some of which are torsion.

By the previous construction or the one by Godin [God07], there are string operations corresponding to these. Although those operations coming from classes in the image of the stabilizing map vanish using results by Tamanoi [Tam08], the unstable ones stand a chance of being non-zero. Some of these of course arise from compositions of the classical string operations. However, the torsion classes cannot arise this way, unless there is a new relation between the previous operations which we do not know about.

The hope is that someone might be able to find specific submanifolds representing these unstable classes. Then our model of the higher string operations makes it particularly easy to calculate what this operation is in terms of elementary constructions in homology. In particular, combining this with a particular example of a space for which one knows the classical string operations already, e.g. Lie groups, spheres or projective spaces, would allow one to show that this new operation is non-zero and not a composition of the classical string operations. Alternatively, one can derive new relations between the classical operations.

6.3. **Extending to the compactification?** The space Rad has a natural compactification. It is simply given by extending the equivalence relations $\equiv_1$ and $\equiv_2$ from PRad to the larger space PRad^{deg} of possible degenerate preconfigurations. There is no problem doing this, as a quick inspection of the definition of $\equiv$ shows. It might be possible to extend our construction of the string operations to this compactification or some subspace thereof.

Clearly, we can extend to the semi-degenerate configurations Rad^s as the only difference that we now allow that length of $\alpha^\pm$ or $\beta^\pm$ to be zero. This shouldn't matter for our construction. However, it is also less interesting, as the homotopy type of the space doesn't change when one allows these configurations.

More interesting is whether we can extend to degenerate configurations as well. Naively, it seems more likely that we can only extend the operations to some subspace of the degenerate configurations, because when one naively tries to extend our earlier definitions to some of the degenerate configurations, one obtains an incorrect dimension shift. In particular, consider the configuration of figure 14. The shift here seems to be 0, while it is a degeneration of the string coproduct, which has a shift of degree $-hd$. Possibly the naive operation for the degenerate configuration is not correct. Alternatively, maybe one should be working in a more advanced context than parametrized spectra, where these jumps are allowed, e.g. twisted parametrized homotopy theory.

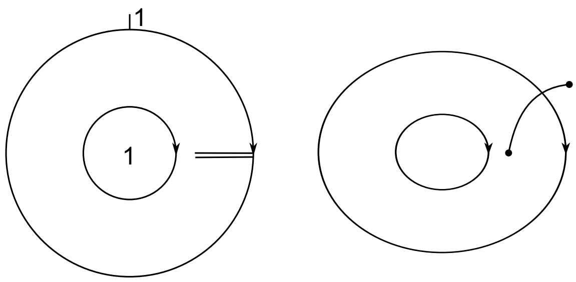

FIGURE 14. A degenerate radial slit configuration and the corresponding degenerate surface.

However, there is a lot of evidence indicating that one should be able to extend the operations to degenerate configurations in some way. First of all, Kate Poirier in her thesis [Poi10] describes a method to extend the operations to a closely related compactification on a chain level. In some cases, this method has been proven to work, but in other situations an anomaly arises. It seems likely that there exists a general method to deal with this anomaly.

Secondly, if we look at the compactification at genus zero, the homotopy type changes in such a way that the almost complete degeneracy of the string coproduct follows from the homology of the compactification. Previously, this was proven by explicit calculation of the string coproduct. To be precise, note that while the compactification $\mathrm{Rad}_0^{deg}(k,1)$ is homotopy equivalent to $\mathrm{Rad}_0(k,1)$, the compactification $\mathrm{Rad}_0^{deg}(1,k)$ is *not* homotopy equivalent to $\mathrm{Rad}_0(1,k)$. In particular $\mathrm{Rad}_0(1,2)$ is a trivial $(S^1)^2$-bundle over $(S^1) \times (0,1)$, but $\mathrm{Rad}_0^{deg}(1,2)$ is the quotient space of the trivial $(S^1)^2$-bundle over $(S^1) \times [0,1]$ where the first circle is collapsed over $S^1 \times \{0\}$ and the second over $S^1 \times \{1\}$. But this is what one gets when one glues two handlebodies of genus 1 along their boundary, showing that $\mathrm{Rad}_0^{deg}(1,2)$ is $S^3 \times S^1$. Thus, the homology classes of the two S^1's in fiber are homologous to zero in the compactification: but these correspond to the BV-operator applied to one of the output components of the string coproduct, which can be shown to be zero using Tamanoi's computation of the string coproduct [Tam07]. Thus the compactification here tells us relations that are already known to hold. We don't know of any relation incompatible with known results that would by implied by an extension of the string operations to the full compactification.

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THE STRING TOPOLOGY STRUCTURE OF THE LIE GROUPS $SU(n)$, $U(n)$, $Sp(n)$, G_2 AND F_4

A.P.M. KUPERS

Abstract. In this article we calculate the string topology structure of the Lie groups $SU(n)$, $U(n)$, $Sp(n)$, G_2 and F_4, by collecting known results and then applying Hepworth's theorem. We also describe a different technique to calculate the homology suspension. Futhermore, we use Tamanoi's results to elucidate more of the full HCFT structure of string topology for Lie groups in general.

Contents

Notation 0.1. In a graded ring, generators will be denoted with a subscript given by their degree. For example, the element a_k has degree k.

The intersection products and string products in homology give graded rings where the product has a shift. There are three ways to deal with this. Firstly one simply can keep the signs and accept that the commutativity relation has an additional sign. The second way is to let the product absorb the grading shift. Finally, one can regrade the rings. We think the final option is most natural, as then the unit will be in degree 0 and it simplifies the notation of many theorems (e.g.

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in spectral sequences for the string product or for the string topology structure of a product of manifolds). Such shifted homology is denoted by $\mathbb{H}$ instead of H for clarity.

Remark 0.2. At some points, we need the Künneth map $H_*(X;R) \otimes H_*(Y;R) \to H_*(X \times Y;R)$ or the universal coefficient map $H^*(X;R) \to Hom(H_*(X);R)$ to be invertible. If so, we implicitly assume this. For this to be the case, certain Tor and Ext groups have to vanish, which holds for example if R is a field or all relevant homology and cohomology groups are free R-modules. In all concrete cases in this article, this is in fact the case.

1. Introduction

1.1. **String topology in general.** We start with a quick introduction to string topology, beginning with an abstract description and becoming more concrete later on.

In one of its most general forms, for a oriented closed manifold M of dimension d, string topology is the structure of a d-dimensional positive boundary HCFT on $H_*(LM)$ [God07]. This means it is an algebra over a partial prop, given by the homology of moduli space of open-closed cobordisms with positive boundary under composition of cobordisms. An open-closed cobordism is a cobordism between one-dimensional manifolds with boundary. The cobordism consequently is a Riemann surface with boundary, divided into three parts: incoming boundary, outgoing boundary and free boundary. The free boundary is a cobordism between the zero-dimensional manifolds which make up the boundary of the incoming and outgoing boundary. See figure 1 for an example. An open-closed bordism has positive boundary if the boundary of every connected component is not completely included in the incoming boundary. This means that there must always be a non-empty free or outgoing part of the boundary of each connected component.

The moduli space of open-closed cobordisms splits into components depending on the diffeomorphism type of the open-closed bordisms. Each such diffeomorphism class has a fixed number of closed incoming, open incoming, closed outgoing and open outgoing boundaries, say n, m, r, s respectively. That $H_*(LM)$ is a d-dimensional positive boundary HCFT means that for each connected component $\mathcal{M}^{oc}(S)$ of the moduli space, we have operations:

$$H_*(\mathcal{M}(S); det^{\otimes d}) \otimes H_*(LM)^{\otimes n} \otimes H_*(M)^{\otimes m} \to H_*(LM)^{\otimes r} \otimes H_*(M)^{\otimes s}$$

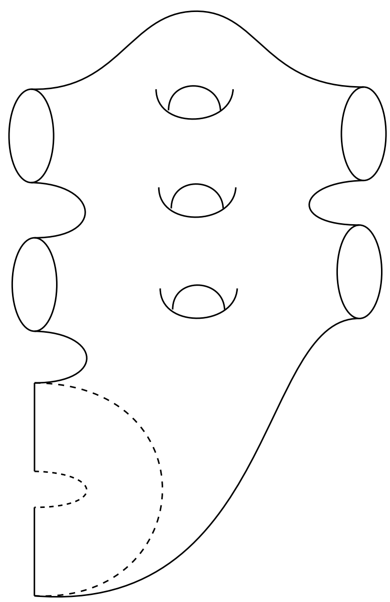

Figure 1. An open-closed cobordisms with incoming boundary two circles and two intervals and outgoing boundary consisting of two circles. Our convention is to draw the incoming boundary on the left and the outgoing boudary on the right. The free boundary is dotted.

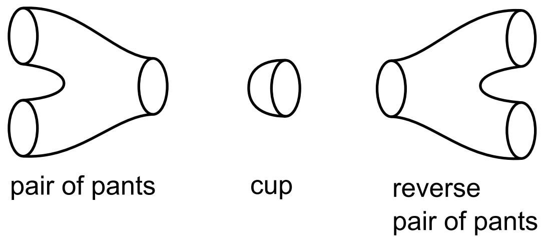

FIGURE 2. The cobordisms inducing the operations of a TQFT with positive boundary.

where det is local system. In many cases, in particular for all basic operations given below, it is locally trivial and hence gives a grading shift. In that case, increases the grading with $\dim(H_0(S, \partial_{in} S; \mathbb{Q})) - \dim(H_1(S, \partial_{in} S; \mathbb{Q}))$ for a open-closed cobordism S. These operations are compatible with glueing of open-closed cobordisms.

Alternatively, one can replace the moduli spaces by classifying spaces of the open-closed mapping class group $Mod^{oc}(S)$. Sometimes we will refer to elements in H_1 of the moduli space by the corresponding elements of the mapping class group. For technical reasons, these operations are constructed using ribbon graph models of surfaces. We will see examples of graphs modelling certain cobordisms later on.

1.2. **String topology for Lie groups.** Since compact Lie groups as pararellizable, they are always oriented closed manifolds and therefore they have an associated string topology structure. We will describe the basic operations below. In particular, the entire structure coming from H_0 of the moduli space is known.

1.2.1. *The TQFT structure from degree zero operations.* We first consider the operations coming from H_0 of the components of moduli space corresponding to closed cobordisms, i.e. those cobordisms with open or free boundary components. As we will see, these operations are fully known. There are three basic operations coming from closed cobordisms. We got their descriptions from [God07, section 1.1].[1] The cobordisms inducing these operations have been drawn in figure 2.

String product: The string product is the operation associated to the generator of H_0 of the component corresponding to the pair of paints with two incoming boundary components and one outgoing. By calculating the relative Euler characteristic, we see product has a degree shift of $-d$. The associativity and graded commutativity follow from the relevant statements for open-closed cobordisms, but can also easily be proven directly from the construction. Unwinding the construction of the HCFT operations, we see the product comes from the following diagram:

$$\begin{tikzcd} LM \times LM \arrow[d, "ev \times ev"] & Map(8, M) \arrow[l, "i"'] \arrow[r, "\gamma"] \arrow[d, "ev"] & LM \arrow[d, "ev"'] \\ M \times M & M \arrow[l, "\Delta"] \arrow[r, "id"'] & M \end{tikzcd}$$

by $\gamma_* i^! : H_*(LM) \otimes H_*(LM) \to H_{*-d}(8, M) \to H_{*-d}(LM)$. Note that the diagram implies that $(ev)_*$ intertwines the string product on $H_*(LM)$ with the intersection product on $H_*(M)$.

String coproduct: The string coproduct is the operation associated to the generator of H_0 of the component corresponding to the reverse pair of paints, i.e. the one with one incoming boundary component and two outgoing. It is coassociative and graded cocommutative. The construction uses a diagram similar to the one used to construct the string product.

[1]There is an issue with signs in string topology, depending on for example the choice of orientation of certain normal bundles. As Godin doesn't specify these choices, we have decided to use Tamanoi's signs in our description of the operations.

Tamanoi [Tam07] has shown that there is a Frobenius compatibility relation between the string product and coproduct, but more importantly, he has shown that the string coproduct applied to any element is a multiple of the Euler characteristic $\chi(M)$ of M.

Unit: This comes from H_0 of the disk with outgoing boundary, also known as the cup. It is simply the unit of the string product. Note there is no counit due to the positive boundary condition.

In particular, this operations gives the full structure of string topology operations coming from H_0 of the moduli space of closed cobordisms with positive boundary. The result is a TQFT with positive boundary. This can be proven using the same lemma about the decomposition of a closed cobordism into standard pieces which is often used to prove that a TQFT is equivalent to a commutative Frobenius algebra.

For Lie groups things simplify. Tamanoi's calculation of the string coproduct [Tam07] completely takes care of the string coproduct for Lie groups. In particular, he shows that the string coproduct of any element is a multiple of $\chi(G)$. However, $\chi(G) = 0$, which can be shown for example using the following cute proof:

Theorem 1.1. *For any Lie group, the Euler characteristic is zero.*

Proof. A Lie group is parallelizable, using either left or right translation. Hence TG admits a non-zero section V. This is a vector field without zeroes. The Poincaré-Hopf theorem gives us the Euler characteristic in terms of the indices of the zeroes of a vector field, as long as these are isolated. Since the sum is empty for V, we conclude that $\chi(G) = 0$. $\square$

Corollary 1.2. *For any Lie group, the string coproduct is identically zero. As a consequence, all operations coming from H_0 of higher genus operations vanish.*

Only two basic operators remain and these can be significantly simplified. To do this, we use the following decomposition of LG: LG is homeomorphic to $\Omega G \times G$, by the map which sends the loop γ to the pair $(\gamma \cdot \gamma(0)^{-1}, \gamma(0))$ of a based loop and an element of the group. The string topology product on $H_*(LM)$ should be seen as a mix between the intersection product of M and the Pontryagin product of ΩM. In the case of a Lie group, this heuristic can be made precise using nothing more that elementary algebraic topology, some knowledge of the behaviour of Thom classes and Bott's theorem [Bot58]:

Theorem 1.3. *Let G be a compact Lie group. Then $H_*(\Omega G;\mathbb{Z})$ is free abelian in each degree and concentrated in even degrees.*

The result uses several structures on $H_*(\Omega G)$ and $H_*(G)$, which we list below:

(1) $H_*(\Omega G)$ is a graded ring using the Pontryagin product. If $\mu : \Omega G \times \Omega G \to \Omega G$ is the concatenation of loops, then this product is $\mu_* : H_*(\Omega G) \otimes H_*(\Omega G) \to H_*(\Omega G)$.
(2) $H_*(G)$ is a graded ring using the intersection product. If $D : G \to G \times G$ is the diagonal map, then this product is $D^! : H_*(G) \otimes H_*(G) \to H_{*-\dim(G)}(G)$. Note that $[G_2]$ is the unit. We therefore shift the grading, setting $\mathbb{H}_*(G) = H_{*+\dim(G)}(G)$.

The result is part of Hepworth's theorem [Hep10], which allows us to completely determine the structure of a TQFT with positive boundary on $\mathbb{H}_*(LG)$ in terms of known invariants.

Theorem 1.4. *Let G be a Lie group. Then as graded rings*

$$\mathbb{H}_*(LG) \cong H_*(\Omega G) \otimes \mathbb{H}_*(G)$$

The unit is $1 \otimes [G]$, where 1 corresponds to the connected component of the identity and $[G]$ is the fundamental class of G.

1.2.2. *Degree zero operations from open-closed cobordisms.* There are more operations if we look at H_0 of the moduli space of all open-closed cobordisms. In fact [Tam08b, theorem B] gives a list of all possibly non-vanishing operations coming from H_0 of open-closed cobordisms. We describe the ones obtained in addition to those coming from closed cobordisms. These are drawn in figure 3.

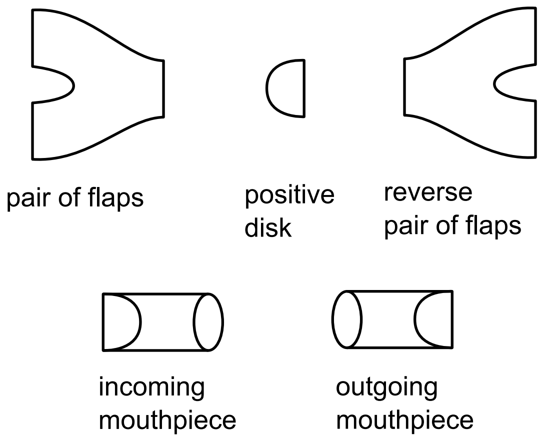

FIGURE 3. Open-closed cobordisms inducing additional operations.

Intersection product: This comes from the generator of H_0 of the connected component corresponding to the pair of flaps with two incoming and one going open boundary components. This is the well-known intersection product.

Unit: This comes from the positive disk and gives the unit for the intersection product, which is the fundamental class $[M]$.

Diagonal coproduct: This comes from the generator of H_0 of the connected component corresponding to the pair of flaps with one incoming and two going open boundary components. It is the diagonal coproduct dual to the cohomology cup product.

Module action: This comes from the generator of H_0 of the connected component corresponding to the "mouthpiece" with one incoming interval and an outgoing circle. It corresponds to the map $H_*(M) \to H_*(LM)$ sending a to $\chi(M)i_*(a)$, where $i : M \to LM$ is the inclusion of constant loops. It gives the structure of $H_*(M)$-module to $H_*(LM)$.

Evaluation map: This comes from the generator of H_0 of the connected component corresponding to the "mouthpiece" with one incoming circle and an outgoing interval. It corresponds to the map $(ev)_* : H_*(LM) \to H_*(M)$ induced by the evaluation of a loop at $1 \in S^1$.

Closed window operation: This comes from the generator of H_0 of the cylinder with one incoming and one outgoing boundary component plus an additional one free closed boundary component, the window. Hence it can be obtained as the composition of the module action with the evaluation map. Tamanoi has shown it is a multiple of $\chi(M)$.

Corollary 1.5. *The closed window operation and module action vanish for all Lie groups.*

In other words, for Lie groups, Tamanoi's type I and III operations vanish as well. As before, one can prove the H_0 of the full moduli space gives us operations corresponding to the structure of a open-closed TQFT with positive boundary. The previous results show how to completely determine the structure of an open-closed TQFT with positive boundary on $\mathbb{H}_*(LG)$ in terms of known invariants.

1.2.3. *Higher degree and higher genus operations.* For completeness, we summarize the results of [Tam08a] as well. These concern the vanishing of higher genus operations. Recall that the moduli space is homotopy equivalent to a disjoint union of classifying spaces of mapping class groups, see e.g. [God07].

Theorem 1.6. *All operations lying in the imaging of a stabilizing map of the mapping class group vanish. In particular, the operations coming from the stable range of the mapping class group vanish.*

Of course not all operations are in the image of a stabilizing map. One of those operations is particularily interesting:

BV operator: The BV-operator is the operation coming from generator of H_1 of the connected component corresponding to the cylinder with one incoming and one outgoing boundary. In terms of mapping class groups, it comes from a Dehn-twist around a curve homotopic to the boundary. Unwinding the construction, it comes from the map $\rho : S^1 \times LM \to LM$ given by $\rho(e^{i\theta}, \gamma)(s) = \gamma(s + \theta)$. Then $\Delta(a) = \rho_*([S^1] \times a)$.

Among its most important properties, it squares to zero and is a second order derivation with respect to the product. The latter follows essentially from the lantern relation in the mapping class group of the complement of three disjoint disks in a larger one.

Essentially, the BV-operator is all there is at degree 1 level: H_1 of the mapping class groups is generated by Dehn twists along simple curves. Cutting the cobordism along into three pieces: (i) a strip containing the curves and some cylinders (ii) a part before it and (iii) a part after it. The fact that the operations are compatible with composition of cobordisms then implies that we can write the degree 1 operations into a composition of degree zero operations and the BV-operator.

Also, we know the structure of the higher degree genus zero operations completely if there is one outgoing closed boundary component and no open or free boundary components: this is an operad generated by the product, the BV-operator subject to in particular the BV relation. This can be proven by showing that the genus zero operations can be modeled by the framed little disks operad, which can done by showing that the mapping class groups for genus zero cobordisms with one outgoing closed boundary component and no open or free boundary components, are pure ribbon braid groups. One can then apply Wahl's recognition principle [Wah01]. The calculation of the homology of the framed little disks operad was done by Wahl and Salvatore [WS03], using results by Cohen for the homology of the little disks operad. In particular, we conclude that the string topology operations give $\mathbb{H}_*(LG)$ the structure of a BV-algebra.

For Lie groups, the BV-operator admits a nice description with respect to the isomorphism $\mathbb{H}_*(LG) \cong H_*(\Omega G) \otimes \mathbb{H}_*(G)$ described previously. For this we need to additional pieces of information.

(1) $H_*(\Omega G)$ is a graded coalgebra using the diagonal coproduct. If $D : \Omega G \to \Omega G \times \Omega G$ is the diagonal, then this coproduct is given by $D_* : H_*(\Omega G) \to H_*(\Omega G \otimes \Omega G)$. Note that theorem 1.3 guarantees we can invert the Künneth map. We denote $D_*(a)$ by $\sum a_{(1)} \otimes a_{(2)}$, Sweedler's notation.
(2) There exist a map $\sigma : H_*(\Omega G) \to H_{*+1}(G)$, known as the homology suspension. It is given by $\sigma(a) = ev_*([S^1] \times a)$, where $ev : S^1 \times \Omega G \to G$ maps (θ, γ) to $\gamma(\theta)$, the loop evaluated at point θ.

Now we can give Hepworth's full theorem [Hep10, theorem 1 and proposition 3]:

Theorem 1.7. *Let G be a Lie group. Then as graded rings*

$$\mathbb{H}_*(LG) \cong H_*(\Omega G) \otimes \mathbb{H}_*(G)$$

The BV-operator is the given by

$$\Delta(a \otimes b) = \sum a_{(1)} \otimes \sigma(a_{(2)})b$$

Alternatively, if $H_*(G; R)$ admits the structure of a Hopf algebra with the Pontryagin product (e.g. if all homology groups are free, or if R is a field) the BV-operator can be written as $\sum \partial_i \otimes \delta_i$ over additive generators p_i of the odd-degree part of the module of primitive elements. Here $\partial_i : H_*(\Omega G; R) \to H_{*-|p_i|+1}(\Omega G; R)$ is given by $a \mapsto \langle p_i, \sigma(a_{(1)}) \otimes a_{(2)}$ and $\delta_i : H_*(G; R) \to H_{*+|p_i|}(G; R)$ is given by $x \mapsto p_i x$, the Pontryagin product of p_i and x.

We apply this theorem to $SU(n)$, $Sp(n)$, G_2 and F_4. As a corollary, we also obtain the string topology structure for $U(n)$. These results are new for $Sp(n)$, G_2, F_4, except in a small number of low dimensional cases of $Sp(n)$, where the groups coincide with the other spaces whose string topology structure has been calculated before. The groups $SU(n)$ and $U(n)$ were considered by Tamanoi in [Tam06], but there a different method was used to compute the string topology structure.

We think these results are interesting for several reasons.

(1) It gives examples of string topology structures and can therefore serve as a starting point for other calculations of string topology structures, for example using the spectral sequences described by Meier [Mei10].
(2) It serves as a starting point for calculations of the equivariant homology of LG. Tom Goodwillie pointed out on Mathoverflow that the E^3-page of the Serre spectral sequence in homology associated to the fibration $LG \to LG \times_{S^1} ES^1 \to BS^1$ can be described as follows: $E^3_{i,j} = H^{\Delta}_j(LG; R)$ for $i = 2n$ with $n \geq 1$, where H^{Δ}_* is defined to be $\ker\Delta/\mathrm{im}\Delta$, and $E^3_{0,j} = H_j(LG; R)/\mathrm{im}\Delta$. Here Δ is the BV-operator.
(3) The string topology structure can serve to distinguish between spaces, for example between $S^3 \times S^5$ and $SU(3)$.
(4) The vanishing or non-vanishing of the product or BV-operator might describe some obstruction. An example of such a statement can be found in the Abbaspour's article on string topology for 3-manifolds [Abb03], which links the string product to "algebraic hyperbolicity" of the manifold.

2. Calculating the string topology structure of Lie groups

2.1. **Two variations of a recipe.** The technique used by Hepworth in [Hep10] to calculate the string topology structure for $SO(n)$ will be the blueprint of our calculations. It consists of the following steps, where G denotes our Lie group:

(1) Compute $H^{pon}_*(G)$, the homology ring with Pontryagin product as a Hopf algebra and H^*, cohomology ring, as a Hopf algebra. If not all groups are free over the integers, we need to switch to field coefficients. Usually these results can be found in the literature, but in some cases one needs some more information about maps into homogeneous spaces. In these cases a nice cell decomposition of your Lie group is of help.
(2) Compute $\mathbb{H}^{int}_*(G)$, the homology intersection ring, from the cohomology ring using Poincaré duality and a grading shift of degree $\dim(G)$.
(3) Compute $H_*(\Omega G)$ as a Hopf algebra. Usually these can be found in the literature.
(4) Compute the string product using steps 2 and 3 and theorem 1.7.
(5) Compute the homology suspension for G. This can be done in the following two ways.
    (5a) One can calculate the homology suspension for a homogeneous space associated to G, e.g. a Stiefel manifold, or a subgroup and using naturality of the homology suspension to compute the homology suspension for G.
    (5b) One can calculate the homology suspension directly using the Serre spectral sequence for the multiplicative fibration $\Omega G \to PG \to G$.
(6) Use the description of the BV-operator as a sum of tensor products of derivation in 1.7 together with steps 1, 2 and 5 to calculate the BV-operator.

The results of such calculations for S^1, $SU(2)$ and $SO(n)$ can be found in [Hep10]. For those cases, step (5a) is the easiest option. However, in some cases, like F_4, there might not be a tractable homogeneous space available. In these cases, it is easier to use (5b). Furthermore, this technique is very powerful when one workes over a field of characteristic 0, because then proposition 2.11 terribly simplifies the trouble of determining the differentials in the spectral sequence.

In the next two sections we will describe the prerequisites for both techniques.

2.2. **The homology suspension and its naturality.** It is easy to describe the naturality of the homology suspension.

Proposition 2.1. *Let $f : X \to Y$ be a continuous map. Then the following diagram commutes:*

$$\begin{tikzcd} H_*(\Omega X) \arrow[r, "(\Omega f)_*"] \arrow[d, "\sigma"'] & H_*(\Omega Y) \arrow[d, "\sigma"] \\ H_{*+1}(X) \arrow[r, "f_*"'] & H_{*+1}(Y) \end{tikzcd}$$

Proof. The proof becomes trivial after one recalls that the homology suspension is defined as $\sigma(a) = (ev)_*([S^1] \otimes a)$, where $ev : (\theta, \gamma) \mapsto \gamma(\theta)$. □

2.3. **The homology suspension using spectral sequences.** In this subsection we take a look at the relation between the homology suspension and the Serre spectral sequence and exploit this to describe the properties of the homology suspension.

2.3.1. *The homology suspension as differential.* We start by describing the homology suspension as a differential in the Serre spectral sequence and from this derive a criterium for the homology suspension to be an isomorphism. The following proposition was proven by Serre, but a more modern source is McCleary [McC00, proposition 6.10].

Proposition 2.2. *Suppose that X is path-connected and ΩX is connected. Then the homology suspension makes the following diagram commute:*

$$\begin{tikzcd} E^q_{q,0} \arrow[r, "d^q", "\cong"'] \arrow[d, "i"] & E^q_{0,q-1} \\ H_q(X) & H_{q-1}(\Omega X) \arrow[l, "\sigma"] \arrow[u, "\pi"'] \end{tikzcd}$$

where $E^q_{i,j}$ are the entries of the (homological) Serre spectral sequence for the fibration $\Omega X \to PX \to X$, $i : E^q_{q,0} \to H_q(X)$ is the boundary inclusion, which is injective, and $\pi : H_{q-1}(\Omega X) \to E^q_{0,q}$ is the boundary projection, which is surjective.

This implies that the kernel of σ is the kernel of π and the image of σ is the image of i.

This means that in essence, computing the homology suspension is nothing but calculating the differentials $d^q : E^q_{q,0} \to E^q_{0,q-1}$.

Corollary 2.3. *If X is n-connected for $n \geq 2$, then $\sigma : H_{q-1}(\Omega X) \to H_q(X)$ is an isomorphism for $2 \leq q \leq 2n + 1$.*

Proof. If Y is n-connected, the entries $E^2_{i,j}$ such that $0 \leq i \leq n$ or $0 \leq j \leq n-1$ have zero entries, except of course $E^2_{0,0}$. In particular, this implies that the only non-zero differential can mapping out of $E^q_{q,0}$ for $0 \leq q \leq 2n+1$ could be d^q, and similarly the only non-zero differential can mapping into $E^q_{0,q-1}$ for $0 \leq q \leq 2n$ could be d^q. This means that in the range $2 \leq q \leq 2n$ both i and π are isomorphisms. For $q = 0, 1$, we don't have differentials d^q. The condition $n \geq 2$ guarantees that X is path-connected and ΩX is connected. □

Next we show how to use the product of the Lie groups to obtain additional structure on the Serre spectral sequence. This structure can then be used together with the previous reuslts to calculate the homology suspension directly for Lie groups.

2.3.2. *Spectral sequences of Hopf algebras.* We start by describing the correct notion to describe the structure on the Serre spectral sequence coming from certain nice fibrations: spectral sequences of Hopf algebras.

Let R be a ring. Each page of a spectral sequence is a R-module with two simultaneous gradings and a differential. An example of such a object is the tensor product of two chain complexes over R with total differential. In general, we have the following definition:

Definition 2.4. A differential bigraded module of bidegree (s,t) is a collection $E_{p,q}$ of R-modules for $p, q \in \mathbb{Z}$ and a differential d of bidegree (s,t), which is a collection of maps $E_{p,q} \to E_{p+s,q+t}$ which squares to zero.

A map of bigraded modules $f : E \to E'$ is a collection of maps $E_{p,q} \to E'_{p,q}$ commuting with the differential. In particular, the bidegrees of the differentials of E and E' must coincide.

There is a tensor product of differential bigraded modules, which is given by

$$(E \otimes E')_{p,q} = \bigoplus_{\substack{r+s=p \\ m+n=q}} E_{r,m} \otimes E'_{s,n}$$

with differential given by $d(a \otimes b) = da \otimes b + (-1)^{|a|} a \otimes db$, where $|a|$ is the total degree $r+m$ of $a \in E_{r,m}$. This turns the category of differential bigraded modules of bidegree (s,t) in a monoidal category with unit object 1 given by the differential bigraded module with R in bidegree $(0,0)$ and 0 elsewhere, and zero differential. This means that we can make sense of the notion of algebra, coalgebra, unit, counit and Hopf algebra in this category.

Definition 2.5. A differential bigraded module E is an algebra if there is a map $\phi : E \otimes E \to E$ of differential bigraded modules making the following diagram commute.

$$\begin{tikzcd}
E \otimes E \otimes E \arrow[r, "\phi \otimes id"] \arrow[d, "id \otimes \phi"'] & E \otimes E \arrow[d, "\phi"] \\
E \otimes E \arrow[r, "\phi"'] & E
\end{tikzcd}$$

A map $\eta : 1 \to E$ is a unit for an algebra if the following two diagrams commute.

$$\begin{tikzcd}
E \otimes 1 \arrow[r, "id \otimes \eta"] \arrow[dr, "="'] & E \otimes E \arrow[d, "\phi"] \\
& E
\end{tikzcd}
\qquad
\begin{tikzcd}
1 \otimes E \arrow[r, "\eta \otimes id"] \arrow[dr, "="'] & E \otimes E \arrow[d, "\phi"] \\
& E
\end{tikzcd}$$

A differential bigraded module E is an coalgebra if there is a map $\Delta : E \to E \otimes E$ of differential bigraded modules making the following diagram commute.

$$\begin{tikzcd}
E \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & E \otimes E \arrow[d, "\Delta \otimes id"] \\
E \otimes E \arrow[r, "id \otimes \Delta"'] & E \otimes E \otimes E
\end{tikzcd}$$

A map $\epsilon : E \to 1$ is a counit for a coalgebra if the following two diagrams commute.

$$\begin{tikzcd}
E \arrow[r, "\Delta"] \arrow[dr, "="'] & E \otimes E \arrow[d, "id \otimes \epsilon"] \\
& E \otimes 1
\end{tikzcd}
\qquad
\begin{tikzcd}
E \arrow[r, "\Delta"] \arrow[dr, "="'] & E \otimes E \arrow[d, "\epsilon \otimes id"] \\
& 1 \otimes E
\end{tikzcd}$$

A differential bigraded module E is a Hopf algebra is $\epsilon \circ \eta$ is the identity of 1 and the following diagram commutes.

$$\begin{tikzcd}
E \otimes E \arrow[r, "\phi"] \arrow[d, "\Delta \otimes \Delta"'] & E \arrow[r, "\Delta"] & E \otimes E \\
E \otimes E \otimes E \otimes E \arrow[rr, "1 \otimes \tau \otimes 1"'] & & E \otimes E \otimes E \otimes E \arrow[u, "\phi \otimes \phi"']
\end{tikzcd}$$

A spectral sequences of algebras is a spectral sequence such that each page forms a differential bigraded algebra of bidegree $(-r, r-1)$ and the product $\phi_{r+1} : E^{r+1} \otimes E^{r+1} \to E^{r+1}$ is given by the composition

$$E^{r+1} \otimes E^{r+1} \xrightarrow{\cong} H(E^r) \otimes H(E^r) \longrightarrow H(E^r \otimes H^r) \xrightarrow{H(\phi_r)} H(E^r) \xrightarrow{\cong} E^{r+1}$$

If the algebras have units, we require that the unit of E^r maps to the unit of E^{r+1} in homology. Note that $d(\eta(1)) = 0$, so this is well-defined.

If we work over a field k, a spectral sequences of coalgebras is a spectral sequence such that each page forms a differential bigraded coalgebra of bidegree $(-r, r-1)$ over k and the coproduct $\Delta_{r+1} : E^{r+1} \to E^{r+1} \otimes E^{r+1}$ is given by the composition

$$E^{r+1} \xrightarrow{\cong} H(E^r) \xrightarrow{H(\Delta_r)} H(E^r \otimes H^r) \longrightarrow H(E^r) \otimes H(E^r) \xrightarrow{\cong} E^{r+1} \otimes E^{r+1}$$

Note that to have a well-defined map $H(E^r \otimes H^r) \to H(E^r) \otimes H(E^r)$ we need to work over a field (as to guarantee vanishing Tor-terms in the Künneth theorem). If the coalgebra has a counit, we require that the counit of E^{r+1} is the restriction of the counuit of E^r, which is well-defined since $\epsilon(d(a)) = 0$.

Finally, we can define a spectral sequences of Hopf algebras by having ϕ_r and Δ_r as above, such that for each page E_r we have that diagram ensuring that ϕ_r and Δ_r are compatible in the sense of a Hopf algebra.

The reason we can interested in spectral sequences of Hopf algebras is that their differentials allow a very easy description in case one knows that primitives and indecomposables elements on each page. Recall that an element a of a Hopf algebra is called primitive if $\Delta(a) = 1 \otimes a + a \otimes 1$ and it is called indecomposable if $a = \phi(a_1, a_2)$ implies that either a_1 or a_2 lies in the image of the unit map. A general result is the following.

Proposition 2.6. *If $E^r_{p,q}$ is a spectral sequence of Hopf algebras, then for each $r \geq 2$, in the least degree such that d^r is non-zero it is defined on a indecomposable element, which it maps to a primitive element.*

Proof. Consider $d^r(x)$. If it is decomposable, note that $d^r(ab) = d^r(a)b + (-1)^{|a|} a d^r(b)$ with the degrees of a, b of lower degree, hence d^r must be zero. To see that $d^r(x)$ is primitive, note that $\Delta(d^r(x)) = d^r(\Delta(x)) = d^r(x) \otimes 1 + 1 \otimes d^r(x) + \sum d^r(x_{(1)}) \otimes x_{(2)} + (-1)^{|x_{(1)}|} x_{(1)} \otimes d^r(x_{(2)})$ with $x_{(1)}, x_{(2)}$ of lower degree. Hence d^r vanishes on those. □

In the special case discussed later, we will be able to exploit this proposition in combination with knowledge of the primitive and indecomposable elements to learn a lot about the differentials.

2.3.3. *Multiplicative fibrations.* Next we describe a type of fibration that can induce under mild conditions a spectral sequence of Hopf algebras in the Serre spectral sequences. These will of course be related Lie groups. Remember that a Lie groups has an associative product and identity elements and hence in particular is a an associative H-space with unit. This is the structure we want our fibrations to preserve.

Our general context is the following: let E, B be associative H-spaces with multiplications μ, ν and units e, b respectively.

Definition 2.7. We call a map $f : E \to B$ multiplicative if $f \circ \mu = \nu \circ (f \times f)$ and $f(e) = b$.

In fact, one can show that for our applications it suffices f is multiplicative up to homotopy, since one can always replace E and B with homotopy equivalent H-spaces and f with a homotopy equivalent map which is multiplicative. We now note that the fiber above the unit is a H-space itself.

Lemma 2.8. *The fiber $F = f^{-1}(b)$ is an associative H-space with unit e.*

Proof. To prove this, it suffices to note that $\mu|_{F \times F}$ maps into F. But this follows from the fact that if $\gamma, \gamma' \in F$, then $f \circ \mu(\gamma, \gamma') = \nu(f\gamma, f\gamma') = \nu(b, b) = b$. □

Furthermore, for the fiber above the identity we always have a simple system of coefficients.

Lemma 2.9. *The action of $\pi_1(B)$ on $H_*(F)$ is trivial.*

Proof. This is lemma 5.1 of [Bro63]. □

It is well-known that the homology of a associative H-space acquires the structure of a Hopf algebra if we use field coefficients. We will denote a generic field with k. Using naturality, we can then give the Serre spectral sequence the structure of a spectral sequence of Hopf algebras, under some assumptions.

Theorem 2.10 (Browder)**.** *Let E, B be associative H-spaces and $f : E \to B$ a multiplicative fibration. Assume that B is path-connected, E connected and $H_*(F)$ and $H_*(B)$ are of finite type over k. Then the Serre spectral sequence for $f : E \to B$ in homology with field coefficients is a spectral sequence of Hopf algebras whose E^2 term can be identified with $H_*(B; k) \otimes H_*(F; k)$,* *both of which are Hopf algebras using the multiplication for the product and the diagonal for the coproduct.*

Proof. See [Bro63] or the more readily available [FHT01]. □

Here a tensor product of Hopf algebras is given by the tensor product of the underlying vector spaces with product, resp. coproduct, the tensor product of the two product, resp. coproduct, maps. Since we know the structure of Hopf algebra on the E^2-page and we would like to apply proposition 2.6, it makes sense to take a closer look at the primitive elements and indecomposables in that situation.

Proposition 2.11. *Let E, F be graded Hopf algebras concentrated in non-negative degree and let $E \otimes F$ be bigraded Hopf algebra obtained as the tensor product of Hopf algebras.*

An element $e \otimes f$ of $E \otimes F$ is indecomposable if and only if either e is indecomposable and f is a multiple of 1, or e is a multiple of 1 and f is indecomposable.

If $R = k$, a field of characteristic zero, then an element $e \otimes f$ of $E \otimes F$ is primitive if and only if either e is primitive and f is a multiple of 1 or e is a multiple of 1 and f is primitive.

Proof. The first claim is trivial by definition of the product of a tensor product.

For the second claim, suppose that we work over a field of characteristic zero, $e \otimes f$ is primitive and both e and f are not multiples of 1. Recall that the coproduct for a tensor product is given by taking the twisted tensor product of the coproducts. Hence we must have

$$\begin{aligned}\Delta(e \otimes f) &= (1 \otimes \tau \otimes 1)(\Delta(e) \otimes \Delta(f)) \\ &= (1 \otimes \tau \otimes 1)((1 \otimes e + e \otimes 1 + \sum e_{(1)} \otimes e_{(2)}) \otimes (1 \otimes f + f \otimes 1 + \sum f_{(1)} \otimes f_{(2)})) \\ &= (e \otimes f) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (e \otimes f) + \dots\end{aligned}$$

where the additional term in bidegree $((k_1, |f|), (k_2, 0))$ with $k_1 + l_1 = |e|$ is given by $(e_{(k_1)} \otimes f) \otimes (e_{(k_2)} \otimes 1)$. On the other hand, using that $e \otimes f$ is primitive we also obtain $\Delta(e \otimes f) = (e \otimes f) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (e \otimes f)$. Since we work in characteristic zero this is only possible is $e_{(k_1)} \otimes e_{(k_2)} = 0$ and this implies that e is primitive or a multiple of 1. By symmetry we conclude that f must be primitive or a multiple of 1 as well. Therefore we only need to rule the case that both are primitive. If this was the case, then in bidegree $((0, |f|), (|e|, 0))$ we find the element $(1 \otimes f) \otimes (e \times 1)$ and obtain a contradiction with $\Delta(e \otimes f) = (e \otimes f) \otimes (1 \otimes 1) + (1 \otimes 1) \otimes (e \otimes f)$. □

The second claim of the previous proposition does not hold over fields of other characteristic. An example occurs in the calculation of the homology suspension of G_2 over $\mathbb{Z}_2$.

2.3.4. *Fibrations related to loops in a H-space.* In particular, if G is a Lie group or more generally an associative H-space, the results of last section apply to the fibrations $\Omega G \to LG \to G$ and $\Omega G \to PG \to G$. Note that PG and LG have pointwise multiplication as H-space maps. This may seem to induce a different H-space structure on ΩG than that of loop concatenation, namely that of pointwise multiplication. However, note the following lemma:

Lemma 2.12. *Loop concatenation and pointwise multiplication induce the same product map in homology. Furthermore, this map is commutative.*

Proof. It suffices to show that concatenation $\mu_c : \Omega G \times \Omega G \to \Omega G$ and pointwise multiplication $\mu_p : \Omega G \times \Omega G \to \Omega G$ induce the same map in homology. There are maps from ΩG to ΩG, $h_{[0,1/2]}$ and $h_{[1/2,1]}$, given by precomposing with the map $t \mapsto \max(2t, 1)$ and $t \mapsto \min(0, 2t-1)$ respectively. These are homotopic to the identity by precomposition with a family of piecewise linear maps $[0,1] \to [0,1]$. Consider the following commutative diagram:

$$\begin{tikzcd} \Omega G \times \Omega G \arrow[rr, "h_{[0,1/2]} \times h_{[0,1/2]}"] \arrow[dr, "\mu_c"'] & & \Omega G \times \Omega G \arrow[dl, "\mu_p"] \\ & \Omega G & \end{tikzcd}$$

This implies that $(\mu_c)_* = (\mu_p)_* \circ (h_{[0,1/2]} \times h_{[0,1/2]})_*$. But the last map is the identity using homotopy invariance of homology, hence $(\mu_c)_* = (\mu_p)_*$.

Also note that we have the interchange law $\mu_p \circ (\mu_c \times \mu_c) = \mu_c \circ (\mu_p \times \mu_p \circ (1 \times \tau \times 1))$ as maps $(\Omega G)^{\times 4} \to \Omega G$ and that both μ_p and μ_c have the same unit, the constant loop at the identity. From this it follows that they are commutative. □

In any case, we can derive the following corollary from the results of the previous subsection:

Corollary 2.13. *Let G be a path-connected Lie group. Then the Serre spectral sequence for the fibration $\Omega G \to PG \to G$ in homology with field coefficients is a spectral sequence of Hopf algebras whose E^2 term can be identified with $H_*(G;k) \otimes H_*(\Omega G;k)$, both of which are Hopf algebras using the multiplication (concatenation or pointwise) for the product and the diagonal for the coproduct.*

Proof. The only thing to check is that both $H_*(G;k)$ and $H_*(\Omega G;k)$ are of finite type. But this is a consequence of theorem 1.3 and the fact that for any compact manifold, the homology is of finite type over each field. □

2.3.5. *Interaction of homology suspension with Pontryagin product.* Finally, we can describe how the homology suspension interacts with the Pontryagin product. This allows us to reduce the calculation of the homology suspension by calculating its values on generators.

Proposition 2.14. *Let $\epsilon : H_*(\Omega X) \to \mathbb{Z}$ be the augmentation. Then $\sigma(ab) = \sigma(a)\epsilon(b) + \epsilon(a)\sigma(b)$, where we use the Pontryagin product on $H_*(\Omega X)$. In particular, if σ vanishes on all elements which can be written as the Pontraygin product of two lower degree classes.*

3. The string topology structure of $SU(n)$ and $U(n)$

In this section we prove the following theorem:

Theorem 3.1. *Let $n \geq 2$. As a graded ring $\mathbb{H}_*(LSU(n))$ is isomorphic to*

$$\mathbb{Z}[\sigma_2, \sigma_4, \ldots, \sigma_{2n-2}] \otimes \Lambda_{\mathbb{Z}}[b_{-3}, \ldots, b_{-(2n-1)}]$$

Under this isomorphism, the BV-operator Δ is given as a sum $\sum_{i=1}^{n-2} \partial_{2i+1} \otimes \delta_{2i+1}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: for $1 \leq j \leq i \leq n-2$ the operator $\partial_{2j+1} \otimes \delta_{2j+1}$ sends $\sigma_{2i} \otimes b_{-(2j+1)}$ to $\sigma_{2(i-j)} \otimes 1$.

Proof. Combine propositions 3.6 and 3.11. □

Some remarks:

- Note that $SU(1) = pt$, so we don't lose anything by requiring $n \geq 2$. This assumption will be implicit from now on.
- Since this theorem holds over $\mathbb{Z}$, it holds over any ring of coefficients. However, this generality means that we can't use the more powerful techniques available for working over fields. We will note when these techniques simplify the calculations.
- This result was already obtained by Tamanoi in [Tam06], where he calculated the string topology structure for complex Stiefel manifolds, since $V^{\mathbb{C}}_{n,n-1} \cong SU(n)$. Conversely, our result for $SU(n)$ also calculates the string topology structure for this class of complex Stiefel manifolds.
- A closer look at the induced maps i_* and $(\Omega i)_*$ for the inclusion $i : SU(k) \to SU(n)$ for $2 \leq k < n$, shows us that $(Li)_*$ induces a morphism of BV-algebras.

Using the isomorphisms $SU(2) \cong Spin(3) \cong Sp(1) \cong S^3$ and $SU(4) \cong Spin(6)$ of Lie groups, we also obtain the string topology structure of $Spin(3)$, $Spin(6)$ and $Sp(1)$. Futhermore, since $SU(2)$ is homeomorphic to S^3, we obtain the string topology structure of S^3. In fact, this result coincides with the one obtained by Hepworth [Hep10, proposition 9] and Menichi [Men09]. It is worthwhile to write out the BV-operator here, to allow easy comparison:

Corollary 3.2. *As a graded ring $\mathbb{H}_*(LS^3)$ is isomorphic to $\mathbb{Z}[\sigma_2] \otimes \Lambda_{\mathbb{Z}}[b_{-3}]$. The operator acts as follows: $\Delta(\sigma_2^i \otimes b_{-3}) = i\sigma_2^{i-1} \otimes 1$ and $\Delta(\sigma_2^i \otimes 1) = 0$*

Proof. This is a direct consequence of theorem 3.1 for $n = 2$ and the fact that Δ is a sum of tensor product of derivations. Alternatively, this follows from theorem 4.1 for $n = 1$. $\square$

3.1. **Yokota's cell structure of $SU(n)$.** For $n \geq 1$, let $\Sigma\mathbb{C}P^n$ denote the reduced suspension of $\mathbb{C}P^n$ using the interval $I = [\pi/2, -\pi/2]$, i.e. $\Sigma\mathbb{C}P^n = \mathbb{C}P^n \times I/\sim$ where $\sim$ is an equivalence relation collapsing the sets $\mathbb{C}P^n \times \{\pi/2\}$, $\mathbb{C}P^n \times \{-\pi/2\}$ and $I \times \{[1, 0, \ldots, 0]\}$. Because $\mathbb{C}P^n$ has a single cell in dimensions $0, 2, 4, \ldots, 2n-2$, we conclude that $\Sigma\mathbb{C}P^n$ has a single cell in dimensions $0, 3, 5, \ldots, 2n-1$. Let $\phi : D^{2n-1} \to \Sigma\mathbb{C}P^n$ denote the characteristic map of the top cell.

We choose to represent elements of $\mathbb{C}P^n$ by vectors in $\mathbb{C}^{n+1}$ with norm 1 modulo $U(1)$. We can give a map $r : I \times \mathbb{C}P^{n-1} \to SU(n)$ as follows:

$$r((\theta, [\vec{v}])) = \left(Id_n - 2\exp(-i\theta)\cos(\theta)([\vec{v}])^\dagger[\vec{v}]\right) \begin{pmatrix} -\exp(2i\theta) & 0 \\ 0 & Id_{n-1} \end{pmatrix}$$

Because the $\dagger$ contains a complex conjugation, this is independent of the choice of vector v representating the element of $\mathbb{C}P^n$. Note that if $\theta = \pi/2$, $\theta = -\pi/2$ or $[v_1, \ldots, v_n] = [1, 0, \ldots, 0]$, this map gives the identity and it is injective everywhere else and hence it factors as a map $\rho : \Sigma\mathbb{C}P^{n-1} \to SU(n)$. Using the inclusion $SU(m) \to SU(n)$ by adding an $(m-n) \times (m-n)$ identity matrix at the bottom, we obtain a map $\rho_m : \Sigma\mathbb{C}P^m \to SU(n)$ for $1 \leq m \leq n-1$.

We can use the group multiplication to extend this map to sequences I of integers between 1 and $n-1$. $\rho_I : \Sigma\mathbb{C}P^{i_1} \times \ldots \times \Sigma\mathbb{C}P^{i_k} \to SU(n)$ is given by

$$\rho_I((\theta_1, [\vec{v}_1]), \ldots, (\theta_k, [\vec{v}_k])) = \rho_{i_1}((\theta_1, [\vec{v}_1])) \cdots \rho_{i_k}(\theta_k, [\vec{v}_k])$$

We can precompose this map with the map $\phi_I : D^{2i_1-1} \times \ldots \times D^{2i_k-1} \to \Sigma\mathbb{C}P^{i_1} \times \ldots \times \Sigma\mathbb{C}P^{i_k}$ of top cells and an orientation-preserving homeomorphism between $D^{2i_1-1} \times \ldots \times D^{2i_k-1}$ and $D^{\sum(2i_j-1)}$ to get a map $\psi_I : D^{\sum(2i_j-1)} \to SU(n)$. We call a sequence I admissable if $n-1 \geq i_1 > i_2 > \ldots > i_k \geq 1$. Yokota proved the following result, which can be found in [Yok55] and [Yok56b]:

Theorem 3.3. *The maps ψ_I for admissible I are the characteristic maps of a cell decomposition of $SU(n)$. The primitive cells are those with $I = \{i\}$ and the interior of D^{2i-1} is mapped homeomorphically into $SU(i)$: the image is called E^i. The cell decomposition is compatible with the group multiplication in the sense that E^iE^j is homeomorphic to E^jE^i by a homeomorphism of degree -1 if $i \neq j$, and $E^iE^i \subset E^iE^{i-1}$. The inclusion of $SU(k) \to SU(n)$ for $k < n$ is cellular and sends the j'th primitive cell of $SU(k)$ homeomorphically to the j'th primitive cell of $SU(n)$.*

Proof. All of this can be found literally in [Yok56b], except the statements about the multiplication. These are consequences of the calculations of section 7 of [Yok56b]. $\square$

Corollary 3.4.

- *The homology ring with Pontryagin product $H_*^{pon}(SU(n))$ is given by $\Lambda_{\mathbb{Z}}[a_3, a_5, \ldots, a_{2n-1}]$. Because all homology groups are free, the diagonal induces a compatible coproduct, and for the Hopf algebra structure obtained this way the a_i are primitive. Futhermore, under the inclusion $SU(k) \to SU(n)$ for $k < n$, the generators of $H_*^{pon}(SU(k))$ are send to the corresponding generators of $H_*^{pon}(SU(n))$.*
- *The cohomology ring with cup product $H^*(SU(n))$ is given by $\Lambda_{\mathbb{Z}}[\alpha_3, \alpha_5, \ldots, \alpha_{2n-1}]$ with α_i dual to a_i. Because all cohomology groups are free, the diagonal induces a compatible coproduct, and for the Hopf algebra structure obtained this way the α_i are primitive.*
- *The intersection ring $\mathbb{H}_*^{int}(SU(n))$ is given by $\Lambda_{\mathbb{Z}}[b_{-3}, b_{-5}, \ldots, b_{-(2n-1)}]$, where we have shifted the graing by $-n^2 + 1 = \dim SU(n)$. The unit is the fundamental class $[SU(n)]$.*

Proof. First we get the homology with Pontryagin product. As consequence $\Sigma\mathbb{C}P^{n-1} \times \ldots \times \Sigma\mathbb{C}P^1$ maps by cellular map into $SU(n)$. One can therefore compute the boundary in the product of suspension of $\mathbb{C}P^n$, but there it must obviously vanish. Therefore are no non-zero differentials and the homology is easy to compute. Because of the compatibility of the group multiplication with the cell structure, we immediately obtain the Pontryagin product structure.

Next, we obtain the cohomology ring. Since the duals to cells have odd degree ≥ 3 or degree 0, again cohomology is easy to compute. Additively, it is free on generators α_i dual to a_i. The product structure comes from our knowledge of the product structure of $H^*(\mathbb{C}P^{n-1} \times \ldots \times \mathbb{C}P^1)$, into which $H^*(SU(n))$ embeds.

To show that the a_i and α_i are primitive, we note that the diagonal coproduct is dual to the cup product and the group coproduct is dual to the Pontryagin product. We know that a_i is dual to α_i and since only $1 \cdot a_i$ and $a_i \cdot 1$ have a_i as a result, α_i is primitive. Similarly, a_i is primitive.

The intersection product can be found by using the Poincaré duality to transfer the cup product to homology. If $[SU(n)] = a_3 \cdot a_5 \cdots a_{2n-1}$ is the fundamental class, then $b_{-(2i-1)} = (-1)^j a_3 \cdot a_5 \cdots \hat{a}_{2i-1} \cdots a_{2n-1}$. □

3.2. **The string product for $SU(n)$.** Using a auxilary space called the generating variety, Bott [Bot58, proposition 8.1] has calculated the homology of $\Omega SU(n)$.

Theorem 3.5. *The homology of $\Omega SU(n)$ is the Hopf algebra given by $\mathbb{Z}[\sigma_2, \sigma_4, \ldots, \sigma_{2n-2}]$ as Pontryagin product and coproduct given by:*

$$D_*(\sigma_{2i}) = \sum_{j+k=i} \sigma_{2j} \otimes \sigma_{2k}$$

where we denote 1 by σ_0.

Combining this information with corollary 3.4 gives us enough information to calculate the string product.

Proposition 3.6. *$H_*(LSU(n)) = \mathbb{Z}[\sigma_2, \sigma_4, \ldots, \sigma_{2n-2}] \otimes \Lambda_{\mathbb{Z}}[b_{-3}, b_{-5} \ldots, b_{-(2n-1)}]$ as a graded ring.*

Proof. Simply apply theorem 1.7. □

3.3. **The BV-operator for $SU(n)$.** We will now calculate the BV-operator using technique (5a), i.e. comparing it with the homology suspension of a homogeneous space. In this case the homogeneous spaces are complex Stiefel varieties.

3.3.1. *The topology of complex Stiefel varieties.* For later use, we do some calculations of topological invariants of a certain class of homogeneous space known as complex Stiefel varieties. The complex Stiefel varieties are given by $V^{\mathbb{C}}_{n,m} = SU(n)/SU(n-m)$ for $1 \leq m \leq n-1$. Since $SU(n-m)$ is a subcomplex of $SU(n)$ and the cellular structure is compatible with the group multiplication, we obtain the homology and cohomology of the Stiefel varieties as a corollary as well:

Corollary 3.7. *For $1 \leq m \leq n-1$, $H^*(V^{\mathbb{C}}_{n,m}) = \Lambda_{\mathbb{Z}}[\epsilon_{2(n-m+1)-1}, \ldots, \epsilon_{2n-1}]$ as a ring and $H_*(V^{\mathbb{C}}_{n,m}) = \Lambda_{\mathbb{Z}}[e_{2(n-m+1)-1}, \ldots, e_{2n-1}]$ additively, where ϵ_i is dual to e_i. Furthermore, for the projection map $p : SU(n) \to SU(n)/SU(n-m) = V^{\mathbb{C}}_{n,m}$, the induced map is homology is given by $p_*(a_{2i-1}) = e_{2i-1}$ if $n-m+1 \leq i \leq n$ and 0 otherwise, and the induced map in cohomology is given by $p^*(\epsilon_{2i-1}) = \alpha_{2i-1}$.*

Alternatively, one can recover this result by applying the Serre spectral sequence to the fibration $S^{2(n-m)-1} \to V^{\mathbb{C}}_{n,m+1} \to V^{\mathbb{C}}_{n,m}$.

Lemma 3.8. *Let $n \geq 2$ and $1 \leq m \leq n-1$, then $V^{\mathbb{C}}_{n,m}$ is $2(n-m)$-connected.*

Proof. Consider the fibration $SU(n-1) \to SU(n) \to S^{2n-1}$. From the long exact sequence of homotopy groups we conclude that map $\pi_i(SU(n-1)) \to \pi_i(SU(n))$ induced by the inclusion is an isomorphism for $0 \leq i \leq 2n-3$ and surjective for $i = 2n-1$. Using downwards induction on $k \geq 1$ we conclude that $\pi_i(SU(n-k)) \to \pi_i(SU(n))$ is an isomorphism for $0 \leq i \leq 2(n-k)-1$ and surjective for $2(n-k)+1$. In particular this holds for $k = m$.

Using the fibration $SU(n-m) \to SU(n) \to V^{\mathbb{C}}_{n,m}$ and the associated long exact sequence of homotopy groups, we see that $V^{\mathbb{C}}_{n,n-m}$ is $2(n-m)$-connected. □

Alternatively, this follows from the Hurewicz theorem once we know that $V^{\mathbb{C}}_{n,m}$ is simply connected, because the reduced homology of $V^{\mathbb{C}}_{n,m}$ has its first non-zero group in degree $2(n-m)+1$. The special class $V^{\mathbb{C}}_{n,1} \cong S^{2n-1}$ of Stiefel varieties admits particularily nice calculations. Note that its homology is concentrated in degree 0, $2n-1$.

Lemma 3.9. *Additively, $H_*(\Omega V^{\mathbb{C}}_{n,1}) = \mathbb{Z}[\tau_{2n-2}]$.*

Proof. This follows directly from the Serre spectral sequence for the path-loop fibration $\Omega V^{\mathbb{C}}_{n,1} \to PV^{\mathbb{C}}_{n,1} \to V^{\mathbb{C}}_{n-1}$, using the previous lemma to determine that $V^{\mathbb{C}}_{n,1}$ is simply connected. □

3.3.2. *The homology suspension for $SU(n)$ and the BV-operator.* We can now use these results to calculate the homology suspension for $SU(n)$.

Lemma 3.10. *$\sigma(\sigma_{2i})$ is $\pm a_{2i+1}$ for $1 \leq i \leq n-1$.*

Proof. We start by showing this for σ_{2n-2}, the generator of highest degree in the homology of $\Omega SU(n)$. Consider the Serre spectral sequence for the fibration $\Omega SU(n-1) \to \Omega SU(n) \to \Omega V^{\mathbb{C}}_{n,1}$. Since $\Omega V^{\mathbb{C}}_{n,1}$ is simply connected if $n \geq 2$, we do not need to bother with local coefficients.

We claim all non-zero entries on the E^2-page are in purely even degree. This is a consequence of Bott's theorem for $SU(n-1)$ and $SU(n)$. For $V^{\mathbb{C}}_{n,1}$, this is lemma 3.9. Since all non-zero entries on the E^2-page are in purely even degree and there are only two columns with non-zero entries, the spectral sequence reduces to a set of short exact sequences. In particular, in degree $2n-2$, we have:

$$0 \to H_{2n-2}(\Omega SU(n-1)) \to H_{2n-2}(\Omega SU(n)) \to H_{2n-2}(\Omega V^{\mathbb{C}}_{n,1}) \to 0$$

The second map is onto and coincides with the map $H_{2n-2}(\Omega SU(n)) \to H_{2n-2}(\Omega V^{\mathbb{C}}_{n,1})$ induced by Ωp. It is the upper row of the following commutative diagram:

$$\begin{tikzcd}
H_{2n-2}(\Omega SU(n)) \arrow[r, "(\Omega p)_*"] \arrow[d, "\sigma"'] & H_{2n-2}(\Omega V^{\mathbb{C}}_{n,1}) \arrow[d, "\sigma"] \\
H_{2n-1}(SU(n)) \arrow[r, "p_*"'] & H_{2n-1}(V^{\mathbb{C}}_{n,1})
\end{tikzcd}$$

Using the $2(n-1)$-connectedness of $V^{\mathbb{C}}_{n,1}$ (lemma 3.8) and the fact that $4n-3 \geq 2n-1 \geq 2$ if $n \geq 2$, corollary 2.3 tells us that $\sigma : H_{2n-2}(\Omega V^{\mathbb{C}}_{n,1}) \to H_{2n-1}(V^{\mathbb{C}}_{n,1})$ is an isomorphism and we conclude that it sends τ_{2n-2} to $\pm e_{2n-1}$. Furthermore p_* sends a_{2n-1} to e_{2n-1}. All this implies that $\sigma : H_{2n-2}(\Omega SU(n)) \to H_{2n-1}(SU(n))$ has as image the free group generated by a_{2n-1}. Note that $\sigma : H_{2n-2}(\Omega SU(n)) \to H_{2n-1}(SU(n))$ can only have a non-zero value on σ_{2n-2}, as other classes of degree $2n-2$ decompose as products of lower degree elements and proposition 2.14 tells us the homology suspension vanishes on these classes. This is only possible if $\sigma(\sigma_{2n-2}) = \pm a_{2n-2}$.

For the general case, consider the following commutative diagram, for $1 \leq m < n$ and $i : SU(m) \to SU(n)$ the inclusion:

$$\begin{tikzcd}
H_*(\Omega SU(m)) \arrow[r, "(\Omega i)_*"] \arrow[d, "\sigma"'] & H_*(\Omega SU(n)) \arrow[d, "\sigma"] \\
H_{*+1}(SU(m)) \arrow[r, "i_*"'] & H_{*+1}(SU(n))
\end{tikzcd}$$

Since $i_*(\sigma(\sigma_{2m-2})) = \pm a_{2m-1}$ and $\sigma : H_{2m-2}(\Omega SU(n)) \to H_{2m-1}(SU(n))$ can have non-zero value only on σ_{2m-2}, we conclude that $\sigma(\sigma_{2m-2}) = \pm a_{2m-1}$. This concludes the proof, as all σ_{2i} are obtained this way. □

If we had worked over a field, say $\mathbb{Q}$, then one could have used that $a_3, \ldots, a_{2n-1}$ are the only indecomposable elements in $H_*(SU(n)) \otimes H_*(\Omega SU(n))$ able to support a non-zero differential in the Serre spectral sequence for $\Omega G \to PG \to G$ and $\sigma_2, \ldots, \sigma_{2n-2}$ are the only primitive elements able to receive one. This determines the differentials completely and therefore also the homology suspension. Note that these results coincide with the previous lemma after tensoring with the rationals.

By possibly changing the generators a_{2i+1} and hence the $b_{-(2i+1)}$, we can assume that $\sigma(\sigma_{2i+2}) = a_{2i+1}$. This allows us to calculate the BV-operator.

Proposition 3.11. *The BV-operator Δ is given as a sum $\sum_{i=1}^{n-2} \partial_{2i+1} \otimes \delta_{2i+1}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: for $1 \leq j \leq i \leq n-2$ the operator $\partial_{2j+1} \otimes \delta_{2j+1}$ sends $\sigma_{2i} \otimes b_{-(2j+1)}$ to $\sigma_{2(i-j)} \otimes 1$.*

Proof. We use the expression of Δ as a sum over the primitive elements $a_3, \ldots, a_{2n-1}$ of $H_*(SU(n))$ of the tensor product of derivations. The derivation $\partial_{2i+1} : H_*(\Omega SU(n)) \to H_{*-(2i+1)-1}(\Omega SU(n))$ is given by $\partial_{2i+1}(\sigma_{2j}) = \sum_{k+l=j} \langle \alpha^{2i+1}, \sigma(\sigma_{2k}) \rangle \sigma_{2l}$. Since α^{2i+1} is non-vanishing only on a_{2i+1}, we conclude that:

$$\partial_{2i+1}(\sigma_{2j}) = \begin{cases} 0 & \text{if } i > j \\ \sigma_{2(j-i)} & \text{if } i \leq j \end{cases}$$

The derivation $\delta_{2i+1} : H_*(SU(n)) \to H_{*+2i+1}(SU(n))$ is given by the Pontryagin multiplication $x \mapsto a_{2i+1}x$. From the description of the b's in terms of the a's, we conclude that:

$$\delta_{2i+1}(b_{-(2j+1)}) = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$$

The result now follows from combining these two calculations. □

3.4. **Products and the string topology structure of $U(n)$.** It is a well-known fact that $U(n)$ is homeomorphic to $S^1 \times SU(n)$. The map is given as follows:

$$A \mapsto \left(\det(A), \begin{pmatrix} \det(A)^{-1} & \\ & Id_{n-1} \end{pmatrix} A \right)$$

This is not an isomorphism of Lie groups, as $U(n)$ is the semidirect product of S^1 with $SU(n)$. However, it is known that $\mathbb{H}_*(L(M \times N)) = \mathbb{H}_*(LM) \otimes \mathbb{H}_*(LN)$ as a graded ring and the BV-operator Δ is given by $\Delta(x \otimes y) = \Delta_M(x) \otimes y + (-1)^{|x|} x \otimes \Delta_N(y)$. In essence, this is a consequence of the fact that cross product of Thom classes is a Thom class. We can then use the homeomorphism $U(n) \cong S^1 \times SU(n)$ to calculate the string topology structure of $U(n)$.

We need the string topology structure of S^1 as input, but fortunately Menichi [Men09] and Hepworth [Hep10] independently calculated the string topology structure of S^1.

Theorem 3.12. *As a graded ring $\mathbb{H}_*(LS^1)$ is isomorphic to:*

$$\mathbb{Z}[\sigma_0, \sigma_0^{-1}] \otimes \Lambda_{\mathbb{Z}}[b_{-1}]$$

The BV-operator can be written as a tensor product of derivations $\partial_1 \otimes \delta_1$. This operator vanish on all generators, except $\sigma_0 \otimes a_{-1}$, which it sends to $1 \otimes 1$.

Corollary 3.13. *As a graded ring $\mathbb{H}_*(LU(n))$ is isomorphic to*

$$\mathbb{Z}[\sigma_0, \sigma_0^{-1}, \sigma_2, \ldots, \sigma_{2n-2}] \otimes \Lambda_{\mathbb{Z}}[b_{-1}, b_{-3}, \ldots, b_{2n-1}]$$

Under this isomorphism, the BV-operator Δ is given as a sum $\sum_{i=0}^{n-2} \partial_{2i+1} \otimes \delta_{2i+1}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: for $1 \leq j \leq i \leq n-2$ the operator $\partial_{2j+1} \otimes \delta_{2j+1}$ sends $\sigma_{2i} \otimes b_{-(2j+1)}$ to $\sigma_{2(i-j)} \otimes 1$. $\partial_1 \otimes \delta_1$ sends $x_0 \otimes b_{-1}$ to $1 \otimes 1$.

Consequently, we obtain the string topology structure for a second class of complex Stiefel manifolds, as $V_{n,n}^{\mathbb{C}} \cong U(n)$.

4. The string topology structure of $Sp(n)$

This calculation is essentially the same as that for $SU(n)$. The only difference is that the coproduct in the homology of $\Omega Sp(n)$ is significantly more complex. However, this poses no additional problems, since we never need to know the details of the complex terms.

Theorem 4.1. *As a graded ring $\mathbb{H}_*(LSp(n))$ is isomorphic to*

$$\mathbb{Z}[\sigma_2, \sigma_6, \ldots, \sigma_{4n-2}] \otimes \Lambda_{\mathbb{Z}}[b_{-3}, \ldots, b_{-(4n-1)}]$$

Under this isomorphism, the BV-operator Δ is given as a sum $\sum_{i=1}^{n} \partial_{4i-1} \otimes \delta_{4i-1}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following *cases: for $1 \leq j \leq i \leq n$ the operator $\partial_{4j-1} \otimes \delta_{4j-1}$ sends $\sigma_{4i-2} \otimes b_{-(4j-1)}$ to $\sigma_{4(i-j)} \otimes 1$. Here σ_{4i-j} is inductively defined by $\sigma_0 = 1$ and $\sigma_{2i} = \sum_{j+k=i,\, j,k>0} (-1)^{j-1} \sigma_{2j}\sigma_{2k}$.*

Proof. Combine propositions 4.5 and 4.9. □

This also gives the string topology structure for several other Lie groups, using isomorphisms in low dimensions. For example $SU(2) \cong Spin(3) \cong Sp(1) \cong S^3$, $Spin(4) \cong Sp(1) \times Sp(1)$ and $Spin(5) \cong Sp(2)$. Furthermore, we also obtain the string topology structure for a certain class of quaternionic Stiefel manifolds, as $V^{\mathbb{H}}_{n,n} \cong Sp(n)$.

4.1. **Yokota's cell structure of** $Sp(n)$**.** Yokota gave a cell structure of $Sp(n)$ similar to that of $SU(n)$ in [Yok56a]. The reader should be warned that the cell structure given in [Yok55] is incorrect, in fact [Yok56a] is a correction to that article.

A cell structure on $Sp(n)$ is given by first producing primitive cells $\phi_i : D^{4i-1} \to Sp(n)$ for $1 \leq i \leq n$. By group multiplication these maps extend to sequences I of integers i_j with $1 \leq i_j \leq n$: $\phi_I : D^{4i_1-1} \times \ldots \times D^{4i_k-1} \to Sp(n)$ is given by

$$\phi_I(x_1, \ldots, x_k) = \phi_{i_1}(x_1) \cdots \phi_{i_k}(x_k)$$

Precomposing with an orientation-preserving homeomorphism $D^{\sum(4i_j-1)} \to D^{4i_1-1} \times \ldots \times D^{4i_k-1}$ gives us a map $\psi_I : D^{\sum(4i_j-1)} \to Sp(n)$. We call a sequence admissible if $n \geq i_1 > i_2 > \ldots > i_k \geq 1$.

Theorem 4.2. *The maps ψ_I for admissible I are the characteristic maps of a cell decomposition of $Sp(n)$. The primitive cells are those with $I = i$, the image is called E^i. The cell decomposition is compatible with the group multiplication in the sense that E^iE^j is homeomorphic to E^jE^i by a homeomorphism of degree -1 if $i \neq j$ and $E^iE^i \subset E^iE^{i-1}$. The inclusion $Sp(k) \to Sp(n)$ for $k < n$ is cellular and sends the j'th primitive cell of $Sp(k)$ homeomorphically to the j'th primitive cell of $Sp(n)$.*

Proof. All these statements can be found literally in [Yok56a], but for full arguments look at the completely analogous [Yok56b]. This reasoning in section 7 again works here to prove the statements about the multiplication. □

As for $SU(n)$, the homology and cohomology are additively freely generated and easy to compute.

Corollary 4.3.
- *The homology ring with Pontryagin product $H^{pon}_*(Sp(n))$ is given by the ring $\Lambda_{\mathbb{Z}}[a_3, a_7, \ldots, a_{4n-1}]$. Because all homology groups are free, this is a Hopf algebra, of which the a_i are the primitive elements. Furthermore, under the inclusion $SU(k) \to SU(n)$ for $k < n$, the generators of $H^{pon}_*(SU(k))$ are send to the corresponding generators of $H^{pon}_*(SU(n))$.*
- *The cohomology ring with cup product $H^*(Sp(n))$ is given by $\Lambda_{\mathbb{Z}}[\alpha_3, \alpha_7, \ldots, \alpha_{4n-1}]$ with α_i dual to a_i. Because all homology groups are free, this is a Hopf algebra, of which the α_i are the primitive elements.*
- *Finally, homology with intersection product $\mathbb{H}^{int}_*(Sp(n))$ is given by the exterior algebra $\Lambda_{\mathbb{Z}}[b_{-3}, b_{-7}, \ldots, b_{-(4n-1)}]$ where we have shifted the degree by $n(2n+1) = \dim(Sp(n))$.*

4.2. **The string product for** $Sp(n)$**.** Using Bott periodicity to first determine the coproduct of ΩSp, where $Sp = \lim Sp(n)$, Kono and Kozima [KK78] determined the homology of $\Omega Sp(n)$. The result is the following theorem:

Theorem 4.4. *The homology of $\Omega Sp(n)$ is the Hopf algebra given by $\mathbb{Z}[\sigma_2, \sigma_6, \ldots, \sigma_{4n-2}]$ and coproduct given by:*

$$D_*(\sigma_{2i}) = \sigma_{2i} \otimes 1 + 2\left(\sum_{j+k=i,j,k>0} \sigma_{2j} \otimes \sigma_{2k}\right) + 1 \otimes \sigma_{2i}$$

where we denote 1 by σ_0 and σ_{2i} for $1 \leq i \leq 2n-1$ even is defined inductively by $\sigma_{2i} = \sum_{j+k=i,\, j,k>0} (-1)^{j-1} \sigma_{2j}\sigma_{2k}$.

Later on, it will useful to refer to the σ_{2i} with i odd as σ_{4j-2} and σ_{2i} with i even as σ_{4j}.

Proposition 4.5. *$\mathbb{H}_*(LSp(n)) = \mathbb{Z}[\sigma_2, \sigma_6, \ldots, \sigma_{4n-2}] \otimes \Lambda_{\mathbb{Z}}[b_{-3}, b_{-7}, \ldots, b_{-(4n-1)}]$ as a graded ring.*

4.3. **The BV-operator for $Sp(n)$.** We will now determine the BV-operator by calculating the homology suspension. This is done by using naturality and a calculation of the homology suspension for quaternionic Stiefel varieties

4.3.1. *The topology of quaternionic Stiefel varieties.* As before, we use the appropriate Stiefel varieties as a crutch for the homology suspension for G_2. For sympletic groups these are the quaternionic Stiefel varieties $V^{\mathbb{H}}_{n,m} = Sp(n)/Sp(n-m)$. Using the cell structure, we derive the following:

Corollary 4.6. *For $1 \leq m \leq n-1$, $H^*(V^{\mathbb{H}}_{n,m}) = \Lambda_{\mathbb{Z}}[\epsilon_{4(n-m+1)-1}, \ldots, \epsilon_{4n-1}]$ as a ring and $H_*(V^{\mathbb{H}}_{n,m}) = \lambda_{\mathbb{Z}}[e_{4(n-m+1)-1}, \ldots, e_{4n-1}]$ where ϵ_i is dual to e_i. The map induced by the projection $Sp(n) \to V^{\mathbb{H}}_{n,m}$ sends a_{4i-1} to e_{4i-1} for $n-m+1 \leq i \leq n$ and to zero otherwise.*

Furthermore $V^{\mathbb{H}}_{n,m}$ is $(4(n-m)+2)$-connected.

This calculation can also by done using the Serre spectral sequence in homology for the fibration $S^{4(n-m)-1} \to V^{\mathbb{H}}_{n,m+1} \to V^{\mathbb{H}}_{n,m}$ and the fact that $V^{\mathbb{H}}_{n,1} \cong S^{4n-1}$ or its corollary, the Leray-Hirsch theorem. The second statement can also be derived using the Hurewicz theorem. Again the $V^{\mathbb{H}}_{n,1}$ are nice in particular, being homeomorphic to S^{4n-1}. Using the Serre spectral sequence one easily derives:

Lemma 4.7. *Additively, $H_*(\Omega V^{\mathbb{H}}_{n,1}) = \mathbb{Z}[\tau_{4n-2}]$.*

4.3.2. *The homology suspension for $Sp(n)$ and the BV-operator.* We use the naturality of the homology suspension and our knowledge of the homology of the quaternionic Stiefel varieties to calculate the homology suspension for $Sp(n)$.

Lemma 4.8. *For i odd, $\sigma(\sigma_{2i}) = \pm a_{2i+1}$.*

Proof. We start with σ_{4n-2}. We use the Serre spectral sequence associated to the fibration $\Omega Sp(n-1) \to \Omega Sp(n) \to \Omega V^{\mathbb{H}}_{n,1}$, where the latter map is denoted by Ωp. All entries are in even degree, and there are only two non-zero columns in degrees 0, $4n-2$. Hence the induced map $(\Omega p)_* : H_{4n-2}(\Omega Sp(n)) \to H_{4n-2}(\Omega V^{\mathbb{H}}_{n,1})$ is onto, which forms the row in the following commutative diagram:

$$\begin{tikzcd} H_{4n-2}(\Omega Sp(n)) \arrow[r, "(\Omega p)_*"] \arrow[d, "\sigma"'] & H_{4n-2}(\Omega V^{\mathbb{H}}_{n,1}) \arrow[d, "\sigma"] \\ H_{4n-1}(Sp(n)) \arrow[r, "p_*"'] & H_{4n-1}(V^{\mathbb{H}}_{n,1}) \end{tikzcd}$$

Since $V^{\mathbb{H}}_{n,1}$ is $(4n-2)$-connected and $8n-4 \geq 4n-2$ for $n \geq 1$, proposition 2.2 implies the homology suspension is an isomorphism in degree $4n-2$. This means that e_{4n-1} is in the image of the composition $\sigma \circ (\Omega p)_*$. Since p_* maps only a_{4n-1} to e_{4n-1}, this implies a_{4n-1} is in the image of σ. Since σ vanishes on all products of lower order terms, we conclude $\sigma(\sigma_{4n-2}) = \pm a_{4n-1}$.

For the σ_{4k-2}, we use the map $\Omega i : \Omega Sp(k) \to \Omega Sp(n)$ and naturality. In degree $4k-2$ we get the commutative diagram:

$$\begin{tikzcd} H_{4k-2}(\Omega Sp(k)) \arrow[r, "(\Omega i)_*"] \arrow[d, "\sigma"'] & H_{4k-2}(\Omega Sp(n)) \arrow[d, "\sigma"] \\ H_{4k-1}(Sp(k)) \arrow[r, "i_*"'] & H_{4k-1}(Sp(n)) \end{tikzcd}$$

Since $i_*(\sigma(\sigma_{4k-2})) = \pm a_{4k-2}$ and σ can be non-vanishing only on σ_{4k-2}. We see that $\sigma(\sigma_{4k-2}) = a_{4k-2}$ for the homology suspension of $Sp(n)$. This concludes the proof. $\square$

Proposition 4.9. *The BV-operator Δ is given as a sum $\sum_{i=1}^n \partial_{4i-1} \otimes \delta_{4i-1}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: for $1 \le j \le i \le n$ the operator $\partial_{4j-1} \otimes \delta_{4j-1}$ sends $\sigma_{4i-2} \otimes b_{-(4j-1)}$ to $\sigma_{4(i-j)} \otimes 1$.*

Proof. We use the expression of Δ as a sum over the primitive elements $a_3, a_7, \ldots, a_{4n-1}$ of $H_*(Sp(n))$ of the tensor product of derivations.

To calculate the derivations $\partial_{4i-1} : H_*(\Omega Sp(n)) \to H_{*-(4i-1)-1}(\Omega Sp(n))$, we note that $\sigma(\sigma_{4i})$ is zero, since theorem 4.4 tells us σ_{4i} is a sum of products of lower order terms. Therefore, the only terms of a coproduct $D_*(\sigma_{4i-2})$ which we need to take into account are those such that only σ_{4j-2} terms appear on the left side. The pairing with α_{4i-1} is non-vanishing only on a_{4i-1}, and therefore we conclude that:

$$\partial_{4i-1}(\sigma_{4j-2}) = \begin{cases} \sigma_{4(i-j)} & \text{if } j \le i \\ 0 & \text{if } j > i \end{cases}$$

where $i - j$ is even and hence $\sigma_{4(i-j)}$ is a sum of lower order terms as described in theorem 4.4.

The derivation $\delta_{4i-1} : H_*(Sp(n)) \to H_{*+4i-1}(Sp(n))$ is given by the Pontryagin multiplication $x \mapsto a_{4i-1}x$. From the description of the b's in terms of the a's, we conclude that:

$$\delta_{4i-1}(b_{-(4j-1)}) = \begin{cases} 0 & \text{if } i \ne j \\ 1 & \text{if } i = j \end{cases}$$

The result now follows from combining these two calculations. □

5. The string topology structure of G_2

In this section, we calculate the string topology structure for the exceptional Lie group G_2. The torsion that appears makes the calculations a bit more intricate than those for $SU(n)$ or $Sp(n)$. In particular, the BV-operator requires a close look at the Serre spectral sequence associated to the path-loop fibration for the real Stiefel manifold $V_{7,2}^{\mathbb{R}}$.

The result for G_2 is the following:

Theorem 5.1. *As a graded ring $\mathbb{H}_*(LG_2; \mathbb{Q})$ is isomorphic to*

$$\mathbb{Q}[\sigma_2, \sigma_{10}] \otimes \Lambda_{\mathbb{Q}}[l_{-3}, l_{-11}]$$

Under this isomorphism, the BV-operator Δ is given as a sum $\partial_3 \otimes \delta_3 + \partial_{11} \otimes \delta_{11}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: $\partial_3 \otimes \delta_3$ sends $\sigma_2 \otimes l_{-3}$ to $1 \otimes 1$ and $\sigma_{10} \otimes l_{-3}$ to $3\sigma_2^4 \otimes 1$. $\partial_{11} \otimes \delta_{11}$ sends $\sigma_{10} \otimes l_{-11}$ to $1 \otimes 1$.

As a graded ring $\mathbb{H}_*(LG_2; \mathbb{Z}_2)$ is isomorphic to

$$\Lambda_{\mathbb{Z}_2}[\sigma_2] \otimes \mathbb{Z}_2[\sigma_4, \sigma_{10}] \otimes \mathbb{Z}_2[m_{-3}]/(m_{-3}^4) \otimes \Lambda_{\mathbb{Z}_2}[m_{-5}]$$

Under this isomorphism, the BV-operator Δ is given as a sum $\partial_3 \otimes \delta_3 + \partial_5 \otimes \delta_5$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: $\partial_3 \otimes \delta_3$ sends $\sigma_2 \otimes m_{-3}$ to $1 \otimes 1$, $\sigma_4 \otimes m_{-3}$ to $\sigma_2 \otimes 1$ abd $\sigma_{10} \otimes m_{-3}$ to $\sigma_4^8 \otimes 1$. $\partial_5 \otimes \delta_5$ sends $\sigma_4 \otimes m_{-5}$ to $1 \otimes 1$.

5.1. Mimura's cell structure of G_2.

Mimura has given a cellular decomposition of G_2 in terms of Yokota's cellular decomposition of $SU(3)$ and an additional 6-dimensional cell [MN02]. He provides maps $\phi_i : D^i \to G_2$ for $i = 3, 5, 6$. We can use group multiplication to extend these maps to sequences I of integers in $\{3, 5, 6\}$. $\phi_I : D^{i_1} \times \ldots \times D^{i_k} \to G_2$ is given by

$$\phi_I(x_1, \ldots, x_k) = \phi_{i_1}(x_1) \cdots \phi_{i_k}(x_k)$$

We can precompose with an orientation-preserving homeomorphism $D^{\sum i_j} \to D^{i_1} \times \ldots \times D^{i_k}$ to get a map $\psi_I : D^{\sum i_j} \to G_2$. We call a sequence I admissible if $6 \ge i_1 > i_2 > \ldots > i_k \ge 3$.

Theorem 5.2. *The maps ψ_I for admissible I are the characteristic maps of a cell decomposition of G_2. The primitive cells are those with $I = \{i\}$: the image is called E^i. The attaching map of E^6 is of degree 2.*

The fibration $SU(3) \to G_2 \to S^6$ is cellular. The primitive cells E^3, E^5 of $SU(3)$ are sent to the primitive cells E^3, E^5 of G_2 respectively and the primitive cell E^6 is sent to the top cell of S^6.

The fibration $SU(2) \to G_2 \to V^{\mathbb{R}}_{7,2}$ is cellular. The primitive cell E^3 of $SU(2)$ is sent to the primitive cell E^2 of G_2 and the primitive cells E^5, E^6 are sent to the primitive cells of $V^{\mathbb{R}}_{7,2}$ of degree $5, 6$ respectively.

Proof. The first statement is theorem 4.2 of [MN02] and the statement about the attaching maps follows from the calculation of the image of the boundary of E^6 in the proof of theorem 4.2.

The second and third statements are consequences of lemma 3.1 and lemma 4.2 of [MN02]. □

We conclude that G_2 has eight cells in dimensions $0, 3, 5, 6, 8, 9, 11, 14$ respectively. Because it is clear that homology and cohomology will acquire 2-torsion but no p-torsion for higher p, we split our calculations into a calculation over $\mathbb{Q}$ and a calculation over $\mathbb{Z}_2$.

Corollary 5.3.

- *The homology ring with Pontryagin product $H^{pon}_*(G_2;\mathbb{Q})$ is given by the ring $\Lambda_{\mathbb{Q}}[h_3, h_{11}]$. The Hopf algebra structure with coproduct induced by the diagonal is that of a exterior algebra and both h_3 and h_{11} are primitive.*
- *The homology ring with Pontryagin product $H^{pon}_*(G_2;\mathbb{Z}_2)$ is given by $\Lambda_{\mathbb{Z}_2}[k_3, k_5, k_6]$. The Hopf algebra structure with coproduct induced by the diagonal is a tensor product of a truncated polynomial algebra and an exterior algebra and both k_3 and k_5 are primitive.*
- *The cohomology ring with cup product $H^*(G_2;\mathbb{Q})$ is given by $\Lambda_{\mathbb{Q}}[\upsilon_3, \upsilon_{11}]$, with υ_3 dual to h_3 and υ_{11} dual to h_{11}. The Hopf algebra structure with coproduct induced by the multiplication is that of a exterior algebra and both υ_3 and υ_{11} are primitive.*
- *The cohomology ring with cup product $H^*(G_2;\mathbb{Z}_2)$ is given by $\mathbb{Z}_2[\kappa_3]/(\kappa_3^4) \otimes \Lambda_{\mathbb{Z}_2}[\kappa_5]$ with κ_3 dual to k_3 and κ_5 dual to k_5. The Hopf algebra structure with coproduct induced by the diagonal is a tensor product of a truncated polynomial algebra and an exterior algebra and an exterior algebra and both κ_3 and κ_5 are primitive.*
- *The homology ring with intersection product $\mathbb{H}^{int}_*(G_2;\mathbb{Q})$ is given by $\Lambda_{\mathbb{Q}}[l_{-3}, l_{-11}]$, where we have shifted with a degree $14 = \dim(G_2)$.*
- *The homology ring with intersection product $\mathbb{H}^{Int}_*(G_2;\mathbb{Z}_2)$ is given by $\mathbb{Z}_2[m_{-3}]/(m_{-3}^4) \otimes \Lambda_{\mathbb{Z}_2}[m_{-5}]$, where again we have shifted with a degree $14 = \dim(G_2)$*

Proof. The full Hopf algebra structure in rational cohomology is given by Mimura [Mim95, equation (2.3)]. For homology, we use that exterior algebras are self-dual. The statements about primitive elements follow from this as well.

For $\mathbb{Z}_2$, we again quote Mimura [Mim95, equation (2.6)], giving the full Hopf algebra structure for cohomology and the primitive elements. Using duality, we obtain the Hopf algebra structure for homology with the Pontraygin product, noting that exterior algebras over $\mathbb{Z}_2$ are self-dual, but the dual of $\mathbb{Z}_2[x]/(x^4)$ is $\Lambda_{\mathbb{Z}_2}[x] \otimes \Lambda_{\mathbb{Z}_2}[y]$ where y is the dual of x^2. Since x is a generator, its dual will be primitive, however the coproduct applied to y will give $y \otimes 1 + x \otimes x + 1 \otimes y$, so it is not primitive.

The intersection product is easily determined using Poincaré duality: as all groups are of dimension 1, the Poincaré duality map is easy to compute. □

5.2. **The string product for G_2.** Bott has calculated the Hopf algebra structure on $H_*(\Omega G_2)$ in [Bot58]. However, there is a misprint in the coproduct of w, which we call σ_{10} here. The 2 should be replace by a 3 [Wat78].

Theorem 5.4. *The homology of ΩG_2 is the Hopf algebra which as algebra under the Pontryagin product is given by $\mathbb{Z}[\sigma_2, \sigma_4, \sigma_{10}]/(2\sigma_4 - \sigma_2^2)$ and coproduct given by:*

$$D_*(\sigma_2) = \sigma_2 \otimes 1 + 1 \otimes \sigma_2$$
$$D_*(\sigma_4) = \sigma_4 \otimes 1 + \sigma_2 \otimes \sigma_2 + 1 \otimes \sigma_4$$
$$D_*(\sigma_{10}) = \sigma_{10} \otimes 1 + 3\sigma_4^2 \otimes \sigma_2 + 6\sigma_2\sigma_4 \otimes \sigma_4 + 6\sigma_4 \otimes \sigma_2\sigma_4 + 3\sigma_2 \otimes \sigma_4^2 + 1 \otimes \sigma_{10}$$

As all groups are free, changing coefficients to $\mathbb{Q}$ or $\mathbb{Z}_2$ is as simple as tensoring.

Corollary 5.5. *$H_*(\Omega G_2;\mathbb{Q})$ is $\mathbb{Q}[\sigma_2,\sigma_{10}]$ as an algebra with coproduct given by:*

$$D_*(\sigma_2)=\sigma_2\otimes 1+1\otimes\sigma_2$$
$$D_*(\sigma_{10})=\sigma_{10}\otimes 1+3\sigma_2^4\otimes\sigma_2+6\sigma_2^3\otimes\sigma_2^2+6\sigma_2^2\otimes\sigma_2^3+3\sigma_2\otimes\sigma_2^4+1\otimes\sigma_{10}$$

$H_*(\Omega G_2;\mathbb{Z}_2)$ is $\Lambda_{\mathbb{Z}_2}[\sigma_2]\otimes\mathbb{Z}_2[\sigma_4,\sigma_{10}]$ as an algebra, with coproduct given by:

$$D_*(\sigma_2)=\sigma_2\otimes 1+1\otimes\sigma_2$$
$$D_*(\sigma_4)=\sigma_4\otimes 1+\sigma_2\otimes\sigma_2+1\otimes\sigma_4$$
$$D_*(\sigma_{10})=\sigma_{10}\otimes 1+\sigma_4^2\otimes\sigma_2+\sigma_2\otimes\sigma_4^2+1\otimes\sigma_{10}$$

Proof. The case $\mathbb{Z}_2$ is directly clear by tensoring with $\mathbb{Z}_2$. For $\mathbb{Q}$, we tensor and rescale σ_4: $\sigma_4'=\frac{1}{2}\sigma_4$, and σ_{10}: $\sigma_{10}'=\frac{1}{4}\sigma_{10}$. Then the formulas are correct. □

Proposition 5.6. *$H_*(LG_2;\mathbb{Q})=\mathbb{Q}[\sigma_2,\sigma_{10}]\otimes\Lambda_{\mathbb{Q}}^{-14}[l_3,l_{11}]$ as a graded ring. $H_*(LG_2;\mathbb{Z}_2)=\Lambda_{\mathbb{Z}_2}[\sigma_2]\otimes\mathbb{Z}_2[\sigma_4,\sigma_{10}]\otimes\mathbb{Z}_2^{-14}[m_{11}]/(m_{11}^4)\otimes\Lambda_{\mathbb{Z}_2}^{-14}[m_9]$ as a graded ring.*

5.3. The BV-operator for G_2.

5.3.1. *The topology of the real Stiefel manifold $V^{\mathbb{R}}_{7,2}$.*

Lemma 5.7. *$V^{\mathbb{R}}_{7,2}\cong SO(7)/SO(5)$ is 4-connected and has homology given over $\mathbb{Q}$ by $\mathbb{Q}1\oplus\mathbb{Q}n_{11}$ and over $\mathbb{Z}_2$ by $\mathbb{Z}_21\oplus\mathbb{Z}_2o_5\oplus\mathbb{Z}_2o_6\oplus\mathbb{Z}_2o_{11}$.*

For the fibratrion $SU(3)\to G_2\to S^6$, the first map (denoted i) sends a_3 to k_3, a_5 to k_5 in homology with $\mathbb{Z}_2$-coefficients. For the fibration $SU(2)\to G_2\to V^{\mathbb{R}}_{7,2}$, the first map (denoted i) sends a_3 to k_3 in homology with rational coefficients. The second map (denoted p) sends k_{11} to n_{11} in homology with rational coefficients and k_{11} to o_{11} in homology with $\mathbb{Z}_2$-coefficients.

Proof. The first statement can be found in [Hat02, chapter 3.D], where a cell decomposition of $V^{\mathbb{R}}_{n,m}$ is derived from one of $SO(n)$. The statement about the fibrations follows from theorem 5.2. Alternatively, these result can be found in [McC00, section 5.2]. □

Lemma 5.8. *Additively, $H_*(\Omega V^{\mathbb{R}}_{7,2};\mathbb{Q})=\mathbb{Q}[\nu_{10}]$. The maps $i: E^{11}_{0,10}\to H_{10}(\Omega V^{\mathbb{R}}_{7,2};\mathbb{Q})$ and $\pi: H_{11}(V^{\mathbb{R}}_{7,2};\mathbb{Q})\to E^{11}_{11,0}$ in the Serre spectral sequence for the homology with rational coefficients associated to the fibration $\Omega V^{\mathbb{R}}_{7,2}\to PV^{\mathbb{R}}_{7,2}\to V^{\mathbb{R}}_{7,2}$ are isomorphisms.*

$H_i(\Omega V^{\mathbb{R}}_{7,2};\mathbb{Z}_2)$ is given by $\mathbb{Z}_2$ for $i=0,4,5,8,9,11,12$ and 0 for $i=1,2,3,6,7,10$. The map $i: E^{11}_{0,10}\to H_{10}(\Omega V^{\mathbb{R}}_{7,2};\mathbb{Z}_2)$ in the Serre spectral sequence for the homology with $\mathbb{Z}_2$-coefficients associated to the fibration $\Omega V^{\mathbb{R}}_{7,2}\to PV^{\mathbb{R}}_{7,2}\to V^{\mathbb{R}}_{7,2}$ is zero.

Proof. For the homology of the loop space, we consider the Serre spectral sequence for the fibration $\Omega V^{\mathbb{R}}_{7,2}\to PV^{\mathbb{R}}_{7,2}\to V^{\mathbb{R}}_{7,2}$, which is easy to construct since $V^{\mathbb{R}}_{7,2}$ is simply connected. Since $PV^{\mathbb{R}}_{7,2}$ is contractible, we must have that all differentials $d^q: E^q_{q,0}\to E^q_{0,q-1}$ are zero, with $q=11$ the only exception. Indeed, for $q=11$, we see that $d^{11}: E^{11}_{11,i}\to E^{11}_{0,i+10}$ must be an isomorphism if we use rational coefficients. This implies the statements about i, π.

We consider the Serre spectral sequence for the homology with $\mathbb{Z}_2$-coefficients associated to the fibration $\Omega V^{\mathbb{R}}_{7,2}\to PV^{\mathbb{R}}_{7,2}\to V^{\mathbb{R}}_{7,2}$. The lower corner of the E^2-page is displayed in figure 4, again with some non-zero differentials displayed. Since d^5 or d^6 already kills the $\mathbb{Z}_2$ that was originally located at $E^2_{0,10}$, the statement about i follows. □

5.3.2. *The homology suspension for G_2 and the BV-operator.*

Lemma 5.9. *Over $\mathbb{Q}$, $\sigma(\sigma_2)$ is a non-zero multiple of h_3 and $\sigma(\sigma_{10})$ is h_{11}.*

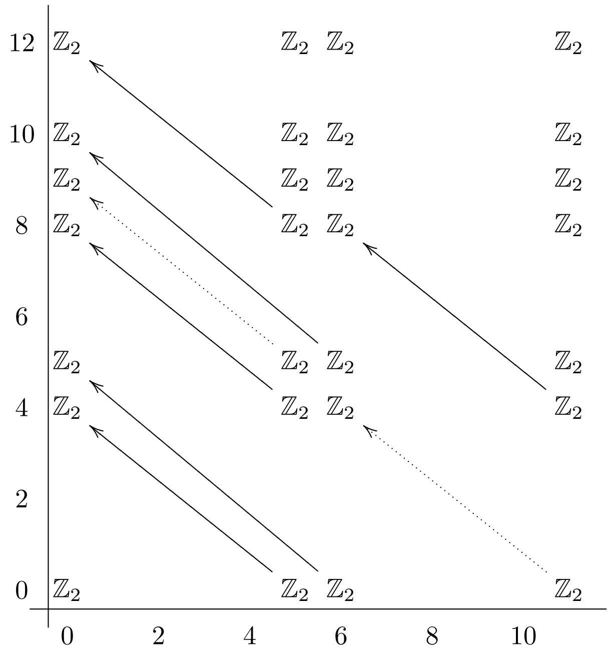

FIGURE 4. The lower corner of E^2-page of the Serre spectral sequence for the homology with $\mathbb{Z}_2$-coefficients associated to the fibration $\Omega V^{\mathbb{R}}_{7,2} \to PV^{\mathbb{R}}_{7,2} \to V^{\mathbb{R}}_{7,2}$ with some of the non-zero differentials. The dotted differentials need to be determined by using that the differentials in the homological spectral sequence are dual to the ones in the cohomological one, which is multiplicative.

Proof. Consider the map $H_2(\Omega SU(2);\mathbb{Q}) \to H_2(\Omega G_2;\mathbb{Q})$ is induced by Ωi. This gives the following commutative diagram, where the lower map is an isomorphism, since both groups are 1-dimensional $\mathbb{Q}$-vectorspaces and the properties of the map p described in theorem 5.2 imply that $p_*(a_3) = h_3$:

$$\begin{tikzcd} H_2(\Omega SU(2);\mathbb{Q}) \arrow[r, "(\Omega i)_*"] \arrow[d, "\sigma"'] & H_2(\Omega G_2;\mathbb{Q}) \arrow[d, "\sigma"] \\ H_3(SU(2);\mathbb{Q}) \arrow[r, "p_*"'] & H_3(G_2;\mathbb{Q}) \end{tikzcd}$$

From our calculations of the homology suspension for $SU(2)$, we already know that the map $\sigma : H_2(\Omega SU(2);\mathbb{Q}) \to H_3(SU(n);\mathbb{Q})$ is an isomorphism, hence the composite $\sigma \circ (\Omega i)_*$ must be an isomorphism. Since all spaces in the diagram are 1-dimensional, we conclude that $\sigma : H_2(\Omega G_2;\mathbb{Q}) \to H_3(G_2;\mathbb{Q})$ is an isomorphism and hence σ_2 is sent to a non-zero multiple of h_3.

For the homology suspension of σ_{10} we need the full strength of proposition 2.2. From this proposition and lemma 5.8 it follows that the homology suspension for $V^{\mathbb{R}}_{7,2}$ is an isomorphism in degree 11 if we use rational coefficients. We now use this to calculate $\sigma(\sigma_{10})$.

Consider the map $H_{10}(\Omega G_2;\mathbb{Q}) \to H_{10}(\Omega V^{\mathbb{R}}_{7,2};\mathbb{Q})$ induced by Ωp. This forms part of a commutative diagram where the lower map is an isomorphism:

$$\begin{tikzcd} H_{10}(\Omega G_2;\mathbb{Q}) \arrow[r, "(\Omega i)_*"] \arrow[d, "\sigma"'] & H_{10}(\Omega V^{\mathbb{R}}_{7,2};\mathbb{Q}) \arrow[d, "\sigma"] \\ H_{11}(G_2;\mathbb{Q}) \arrow[r, "p_*"'] & H_{11}(V^{\mathbb{R}}_{7,2};\mathbb{Q}) \end{tikzcd}$$

We just saw that $\sigma : H_{10}(\Omega V^{\mathbb{R}}_{7,2};\mathbb{Q}) \to H_{11}(V^{\mathbb{R}}_{7,2};\mathbb{Q})$ is an isomorphism, hence sends ν_{10} to a non-zero multiple of n_{11}. Since p_* maps h_{11} to n_{11}, we see that h_{11} is in the image of $\sigma : H_{10}(\Omega G_2;\mathbb{Q}) \to H_{11}(G_2;\mathbb{Q})$. This is only possible if $\sigma(\sigma_{10})$ is a non-zero multiple of h_{11}, because σ must vanish on σ_2^5, since it is a product of lower degree terms. □

In fact, using the Serre spectral sequences for homology with rational coefficients associated to the fibrations $\Omega SU(2) \to \Omega G_2 \to \Omega V^{\mathbb{R}}_{7,2}$ one can establish independently that $(\Omega p)_*$ is an isomorphism in degree 3. Note that we can change the generators h_3 and h_{11} to make sure $\sigma(\sigma_2) = h_3$ and $\sigma(\sigma_{10}) = h_{10}$.

Lemma 5.10. *Over $\mathbb{Z}_2$, $\sigma(\sigma_2) = k_3$, $\sigma(\sigma_4) = k_5$ and $\sigma(\sigma_{10}) = 0$.*

Proof. For σ_2 and σ_4 we consider the map $\Omega i : \Omega SU(3) \to \Omega G_2$. This induces a commutative diagram:

$$\begin{tikzcd} H_2(\Omega SU(3);\mathbb{Z}_2) \arrow[r, "(\Omega o)_*"] \arrow[d, "\sigma"'] & H_2(\Omega G_2;\mathbb{Z}_2) \arrow[d, "\sigma"] \\ H_3(SU(3);\mathbb{Z}_2) \arrow[r, "i_*"'] & H_3(G_2;\mathbb{Z}_2) \end{tikzcd}$$

The lower map is an isomorphism between 1-dimensional vectorspaces, as a consequence of the second statement of theorem 5.2. We also know that $\sigma : H_2(\Omega SU(3);\mathbb{Z}_2) \to H_3(SU(3);\mathbb{Z}_2)$ is an isomorphims. Since $H_2(\Omega G_2;\mathbb{Z}_2)$ is also 1-dimensional, we conclude that $(\Omega p)_*$ and $\sigma : H_2(\Omega G_2;\mathbb{Z}_2) \to H_3(G_2;\mathbb{Z}_2)$ are isomorphisms. The latter implies that $\sigma(\sigma_2) = k_3$.

The second statement follows from similar reasoning. We get the same commutative diagram in degree 4. The only difference now is that $H_4(\Omega SU(3))$ is 2-dimensional. However, σ is known to vanish on σ^2, but to send σ_4 to a_5, which is sent to k_5 by i_*. Hence $\sigma : H_4(\Omega G_2;\mathbb{Z}_2) \to H_5(G_2,\mathbb{Z}_2)$ must have a 1-dimensional image, which implies it is an isomorphism since $H_4(\Omega G_2;\mathbb{Z}_2)$ is 1-dimensional. Hence $\sigma(\sigma_4) = k_5$.

For the statement about σ_{10}, we consider the map $\Omega p : \Omega G_2 \to \Omega V^{\mathbb{R}}_{7,2}$. It induces the following commutative diagram:

$$\begin{tikzcd} H_{10}(\Omega G_2;\mathbb{Z}_2) \arrow[r, "(\Omega p)_*"] \arrow[d, "\sigma"'] & H_{10}(\Omega V^{\mathbb{R}}_{7,2};\mathbb{Z}_2) \arrow[d, "\sigma"] \\ H_{11}(G_2;\mathbb{Z}_2) \arrow[r, "p_*"'] & H_{11}(V^{\mathbb{R}}_{7,2};\mathbb{Z}_2) \end{tikzcd}$$

The lower map p_* is an isomorphism between 1-dimensional vectorspaces, hence $\sigma(\sigma_{10})$ is zero if and only if the composite $\sigma \circ (\Omega p)_*$ is. But we know the homology suspension of $V^{\mathbb{R}}_{7,2}$ vanishes in degree 10, combining proposition 2.2 with lemma 5.8. □

Alternatively, one can derive the last two results by directly calculating all differentials for the Serre spectral sequence associated to the fibration $\Omega G_2 \to PG_2 \to G_2$ for homology with rational of $\mathbb{Z}_2$-coefficients. The results of this calculation coincide with the earlier one, but are slightly more involved.

Proposition 5.11. *Over $\mathbb{Q}$, the BV-operator Δ is given as a sum $\partial_3 \otimes \delta_3 + \partial_{11} \otimes \delta_{11}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: $\partial_3 \otimes \delta_3$ sends $\sigma_2 \otimes l_{-3}$ to $1 \otimes 1$ and $\sigma_{10} \otimes l_{-3}$ to $3\sigma_2^4 \otimes 1$. $\partial_{11} \otimes \delta_{11}$ sends $\sigma_{10} \otimes l_{-11}$ to $1 \otimes 1$.*

Proof. We use the expression of Δ as a sum over the primitive elements h_3 and h_{11} of $H_*(G_2)$ of the tensor product of derivations. The derivation $\partial_3 : H_*(\Omega G_2) \to H_{*-3}(\Omega G_2)$ is given by $\partial_3(\sigma_2) = \langle \nu_3, \sigma(\sigma_2)\rangle 1$ and $\partial_3(\sigma_{10}) = \langle \nu_3, \sigma(\sigma_{\sigma_{10}})\rangle 1 + 3\langle \nu_3, \sigma(\sigma_2)\rangle \sigma_2^4$, where we have already used that σ vanishes on the unit and on product. Since ν_3 is the dual of h_3, we conclude that:

$$\partial_3(\sigma_k) = \begin{cases} 1 & \text{if } k = 2 \\ 2\sigma_2^4 & \text{if } k = 10 \end{cases}$$

The derivation $\partial_{11} : H_*(\Omega G_2) \to H_{*-10}(\Omega G_2)$ is given by $\partial_{11}(\sigma_2) = \langle \nu_{11}, \sigma(\sigma_2)\rangle 1$ and $\partial_{11}(\sigma_{10}) = \langle \nu_{11}, \sigma(\sigma_{\sigma_{10}})\rangle 1 + 2\langle \nu_{11}, \sigma(\sigma_2)\rangle \sigma_2^4$, where we have already used that σ vanishes on the unit and on product. Since ν_{11} is the dual of h_{11}, we conclude that:

$$\partial_{11}(\sigma_k) = \begin{cases} 0 & \text{if } k = 2 \\ 1 & \text{if } k = 10 \end{cases}$$

The derivation $\delta_3 : H_*(SU(n)) \to H_{*+3}(SU(n))$ is given by the Pontryagin multiplication $x \mapsto h_3 x$. From the description of the l's in terms of the h's, we conclude δ_3 sends l_{-11} to zero and l_{-3} to 1. Similarly, the derivation $\delta_{11} : H_*(SU(n)) \to H_{*+11}(SU(n))$ is given by the Pontryagin multiplication $x \mapsto h_{11}x$. From the description of the l's in terms of the h's, we conclude δ_3 sends l_{-11} to 1 and l_{-3} to 0.

The proposition now follows from combining these two calculations. □

Proposition 5.12. *Over $\mathbb{Z}_2$, the BV-operator Δ is given as a sum $\partial_3 \otimes \delta_3 + \partial_5 \otimes \delta_5$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: $\partial_3 \otimes \delta_3$ sends $\sigma_2 \otimes m_{-3}$ to $1 \otimes 1$, $\sigma_4 \otimes m_{-3}$ to $\sigma_2 \otimes 1$ abd $\sigma_{10} \otimes m_{-3}$ to $\sigma_4^8 \otimes 1$. $\partial_5 \otimes \delta_5$ sends $\sigma_4 \otimes m_{-5}$ to $1 \otimes 1$.*

Proof. For $\mathbb{Z}_2$, we use exactly the same technique. We need to describe ∂_3, ∂_5, δ_3 and δ_5. The derivation $\partial_3 : H_*(\Omega G_2) \to H_{*-3}(\Omega G_2)$ is given by $\partial_3(\sigma_2) = \langle \kappa_3, \sigma(\sigma_2)\rangle 1$, $\partial_3(\sigma_4) = \langle \kappa_3, \sigma(\sigma_{\sigma_4})\rangle 1 + \langle \kappa_3, \sigma(\sigma_2)\rangle \sigma_2$ and $\partial_3(\sigma_{10}) = \langle \kappa_3, \sigma(\sigma_{10})\rangle 1 + \langle \kappa_3, \sigma(\sigma_2)\rangle \sigma_4^2$, where we have already used that σ vanishes on the unit and on product. Since κ_3 is the dual of k_3, we conclude using lemma 5.10 that:

$$\partial_5(\sigma_k) = \begin{cases} 1 & \text{if } k = 2 \\ \sigma_2 & \text{if } k = 4 \\ \sigma_4^2 & \text{if } k = 10 \end{cases}$$

The derivation $\partial_5 : H_*(\Omega G_2) \to H_{*-5}(\Omega G_2)$ is given by $\partial_5(\sigma_2) = \langle \kappa_5, \sigma(\sigma_2)\rangle 1$, $\partial_3(\sigma_4) = \langle \kappa_5, \sigma(\sigma_{\sigma_4})\rangle 1 + \langle \kappa_3, \sigma(\sigma_2)\rangle \sigma_2$ and $\partial_3(\sigma_{10}) = \langle \kappa_5, \sigma(\sigma_{10})\rangle 1 + \langle \kappa_5, \sigma(\sigma_2)\rangle \sigma_4^2$, where we have already used that σ vanishes on the unit and on product. Since κ_5 is the dual of k_5, we conclude that:

$$\partial_3(\sigma_k) = \begin{cases} 0 & \text{if } k = 2 \\ 1 & \text{if } k = 4 \\ 0 & \text{if } k = 10 \end{cases}$$

The derivation $\delta_3 : H_*(SU(n)) \to H_{*+3}(SU(n))$ is given by the Pontryagin multiplication $x \mapsto k_3 x$. From the description of the m's in terms of the k's, we conclude δ_3 sends m_{-5} to zero and m_{-3} to $[G_2]$. Note that using the derivation property, it is defined on all other elements. Similarly, the derivation $\delta_5 : H_*(SU(n)) \to H_{*+11}(SU(n))$ is given by the Pontryagin multiplication $x \mapsto k_5 x$. We conclude δ_3 sends m_{-5} to 1 and m_{-3} to 0.

The proposition now follows from combining these two calculations. □

6. The string topology structure for F_4

Finally, we will calculate the string topology structure of F_4 with rational coefficients. In this case we will use the spectral sequence directly to calculate the homology suspension, as it is more involved to use homogeneous spaces.

We would like to note that for the exceptional Lie group E_6, there is sufficient literature to do the calculations of the string topology structure. The relevant articles are [Mim95] and [Nak03]. For E_7 and E_8, a calculation of the coproduct of the homology of their loop spaces seems to be missing in the literature. The homology and cohomology of F_4 have 2-torsion and 3-torsion. Therefore one should split the calculation calculation into rational coefficients, $\mathbb{Z}_2$ and $\mathbb{Z}_3$. However, because the results are quite involved, we just do the case of rational coefficients to illustrate the technique. We obtain the following result:

Theorem 6.1. *As a graded ring $\mathbb{H}_*(LF_4;\mathbb{Q})$ is isomorphic to*

$$\mathbb{Q}[\tau_2,\tau_{10},\tau_{14},\tau_{22}]\otimes\Lambda_{\mathbb{Q}}[b_{-3},b_{-11},b_{-15},b_{-23}]$$

Under this isomorphism, The BV-operator Δ is given as a sum $\partial_3\otimes\delta_3+\partial_{11}\otimes\delta_{11}+\partial_{15}\otimes\delta_{15}+\partial_{23}\otimes\delta_{23}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: $\partial_3\otimes\delta_3$ sends $\tau_2\otimes b_{-3}$ to 1, $\tau_{10}\otimes b_{-3}$ to $\frac{1}{12}\tau_2^4\otimes 1$, $\tau_{14}\otimes b_{-3}$ to $(\tau_{10}\tau_2-\frac{1}{72}\tau_2^6)\otimes 1$ and $\tau_{22}\otimes b_{-3}$ to $\frac{1}{3}(-\tau_{14}\tau_2^3+\frac{1}{2}\tau_{10}\tau_2^5-\frac{1}{144}\tau_2^{10})\otimes 1$; $\partial_5\otimes\delta_3$ sends $\tau_{10}\otimes b_{-11}$ to 1 and $\tau_{14}\otimes b_{-11}$ to $\frac{1}{2}\tau_2\otimes 1$; $\partial_{15}\otimes\delta_{15}$ sends $\tau_{14}\otimes b_{-15}$ to 1 and $\tau_{22}\otimes b_{-15}$ to $-\frac{1}{12}\tau_2^4\otimes 1$; finally, $\partial_{23}\otimes\delta_{23}$ sends $\tau_{22}\otimes b_{-23}$ to 1.

6.1. **The string product for F_4.** In the literature, one can find the (co)homology of F_4 and its loop space. For notational ease, we define $\tilde{D}_*(a)$ by $\tilde{D}_*(a)+\tau\tilde{D}_*(a)=D_*(a)-(a\otimes 1+1\otimes a)$ for a coproduct D_* of a Hopf algebra., where τ flips the components of the tensor product if they are not equal and is zero otherwise.

Theorem 6.2. *$H_*^{pon}(F_4;\mathbb{Q})=\Lambda_{\mathbb{Q}}[a_3,a_{11},a_{15},a_{23}]$ where all a_i are primitive. Dually, the cohomology ring is given by $H^*(F_4;\mathbb{Q})=\Lambda_{\mathbb{Q}}[\alpha_3,\alpha_{11},\alpha_{15},\alpha_{23}]$ with α_i dual to a_i and all α_i primitive.*

The intersection ring $\mathbb{H}_*^{int}(F_4;\mathbb{Q})=\Lambda_{\mathbb{Q}}[b_{-3},b_{-11},b_{-15},b_{-23}]$, where we shifted by $\dim F_4=54$.

The Hopf algebra $H_*(\Omega F_4;\mathbb{Q})$ is given by $\mathbb{Q}[\tau_2,\tau_{10},\tau_{14},\tau_{22}]$ with coproduct given by:

$$\begin{aligned}
\tilde{D}_*(\tau_2)&=0\\
\tilde{D}_*(\tau_{10})&=\frac{1}{12}\tau_2^4\otimes\tau_2+\frac{1}{12}\tau_2^3\otimes\tau_2^2\\
\tilde{D}_*(\tau_{14})&=(\tau_{10}\tau_2-\frac{1}{72}\tau_2^6)\otimes\tau_2+\frac{1}{2}\tau_{10}\otimes\tau_2^2+\frac{1}{72}\tau_2^4\otimes\tau_2^3\\
\tilde{D}_*(\tau_{22})&=\frac{1}{3}(-\tau_{14}\tau_2^3+\frac{1}{2}\tau_{10}\tau_2^5-\frac{1}{144}\tau_2^{10})\otimes\tau_2+\frac{1}{2}(-\tau_{14}\tau_2^2+\frac{1}{2}\tau_{10}\tau_2^4-\frac{1}{216}\tau_2^9)\otimes\tau_2^2\\
&\quad+\frac{1}{3}(-\tau_{14}\tau_2+\frac{1}{2}\tau_{10}\tau_2^3+\frac{1}{144}\tau_2^8)\otimes\tau_2^3+\frac{1}{12}(-\tau_{14}+\frac{1}{2}\tau_{10}\tau_2^2+\frac{8}{72}\tau_2^7)\otimes\tau_2^4\\
&\quad+\frac{1}{72}\tau_2^6\otimes\tau_2^5
\end{aligned}$$

In particular, there is a single primitive element in degree $2,10,14,22$, although these are not equal to the generators except for 2.

Proof. For the homology and cohomology of F_4, we cite Mimura [Mim95, section 2] and use duality of Hopf algebras and Poincaré duality.

For the homology of ΩF_4, consider Watanabe's result [Wat78]. Note that over $\mathbb{Q}$, $\sigma_2=\frac{1}{2}\sigma_1^2$ and $\sigma_3=\frac{1}{6}\sigma_1^3$. We set $\tau_{2i}=\sigma_i$ after these reductions. Then $\sigma_4=\sigma_2^2-\sigma_1\sigma_3=\frac{1}{12}\tau_2^4$ and $\sigma_6=\sigma_2^3-\sigma_3^2=\frac{1}{72}\tau_2^6$. □

As a consequence of these results, we obtain the string product.

Proposition 6.3. *As a graded ring $\mathbb{H}_*(LF_4;\mathbb{Q})$ is isomorphic to*

$$\mathbb{Q}[\tau_2,\tau_{10},\tau_{14},\tau_{22}]\otimes\Lambda_{\mathbb{Q}}[b_{-3},b_{-11},b_{-15},b_{-23}]$$

6.2. **The BV-operator for F_4.** We compute the homology suspension by directly looking at the spectral sequence.

Proposition 6.4. *The homology suspension of τ_2 is a non-zero multiple of a_3, the homology suspension of τ_{10} is a non-zero multiple of a_{11}, the homology suspension of τ_{14} is a non-zero multiple of a_{15} and the homology suspension of τ_{22} is a non-zero multiple of a_{23}.*

Proof. We apply the Serre spectral sequence for homology with rational coefficients to the fibration $\Omega F_4\to PF_4\to F_4$. Using the product structure one can determine the differentials completely.

For dimensional reasons, d^3 is non-zero on a_3, which is mapped to τ_2. In $E^4_{0,10}$, τ_2^5 was killed by $a_3\tau_2^4$. Hence the only way to get τ_{10} to vanish is if d^{11} sends a_{11} to τ_{10}. In $E^{12}_{0,14}$, τ_2^7 was killed by $a_3\tau_2^6$ and $\tau_{10}\tau_2^2$ was killed by $a_{10}\tau_2^2$. Hence d^{15} sends a_{15} to τ_{14}. Finally, in $E^{16}_{0,22}$, all elements τ_2^{11},

$\tau_{10}\tau_2^6$, $\tau_{10}^2\tau_2$ and $\tau_{14}\tau_2^4$ were killed already. The only element for which this is not already clear is $\tau_{10}^2\tau_2$. It is killed by $a_3\tau_{10}^2$.

Alternatively, one can use proposition 2.6 and 2.11 to see that the only indecomposables which could support a non-zero differential are $a_3, a_{11}, a_{15}, a_{23}$ and the only primitive elements which could receive a non-zero differential are the primitive elements in degree 2, 10, 14 and 22. Since we are dealing with a spectral sequences of Hopf algebras, this determines the differentials completely, giving the same result as using multiplicativity.

The description of the homology suspension in terms of the projection $H_q(F) \to E^q_{0,q-1}$, the differential d^q and the inclusion $E^q_{q,0} \to H_q(B)$ now proves the result. □

Without loss of generality, we can assume that the non-zero rational coefficients appearing in the last statement are 1. This is done by changing the a_i accordingly.

Theorem 6.5. *The BV-operator Δ is given as a sum $\partial_3 \otimes \delta_3 + \partial_{11} \otimes \delta_{11} + \partial_{15} \otimes \delta_{15} + \partial_{23} \otimes \delta_{23}$ of tensor products of derivations. These tensor products vanish on all generators, except in the following cases: $\partial_3 \otimes \delta_3$ sends $\tau_2 \otimes b_{-3}$ to 1, $\tau_{10} \otimes b_{-3}$ to $\frac{1}{12}\tau_2^4 \otimes 1$, $\tau_{14} \otimes b_{-3}$ to $(\tau_{10}\tau_2 - \frac{1}{72}\tau_2^6) \otimes 1$ and $\tau_{22} \otimes b_{-3}$ to $\frac{1}{3}(-\tau_{14}\tau_2^3 + \frac{1}{2}\tau_{10}\tau_2^5 - \frac{1}{144}\tau_2^{10}) \otimes 1$; $\partial_5 \otimes \delta_3$ sends $\tau_{10} \otimes b_{-11}$ to 1 and $\tau_{14} \otimes b_{-11}$ to $\frac{1}{2}\tau_2 \otimes 1$; $\partial_{15} \otimes \delta_{15}$ sends $\tau_{14} \otimes b_{-15}$ to 1 and $\tau_{22} \otimes b_{-15}$ to $-\frac{1}{12}\tau_2^4 \otimes 1$; finally, $\partial_{23} \otimes \delta_{23}$ sends $\tau_{22} \otimes b_{-23}$ to 1.*

Proof. For the BV-operator, we calculate each derivation seperately. The derivations ∂_i are given by $a \mapsto \sum a_{(1)}\langle \alpha_i, \sigma(a_{(2)})\rangle$. Using our calculation of the homology suspension and the fact that it vanishes on decomposable elements, we obtain the following results:

$$\partial_3(\tau_i) = \begin{cases} 1 & \text{if } i = 2 \\ \frac{1}{12}\tau_2^4 & \text{if } i = 10 \\ \tau_{10}\tau_2 - \frac{1}{72}\tau_2^6 & \text{if } i = 14 \\ \frac{1}{3}(-\tau_{14}\tau_2^3 + \frac{1}{2}\tau_{10}\tau_2^5 - \frac{1}{144}\tau_2^{10}) & \text{if } i = 22 \end{cases}$$

$$\partial_{11}(\tau_i) = \begin{cases} 0 & \text{if } i = 2 \\ 1 & \text{if } i = 10 \\ \frac{1}{2}\tau_2^2 & \text{If } i = 14 \\ 0 & \text{if } i = 22 \end{cases}$$

$$\partial_{15}(\tau_i) = \begin{cases} 0 & \text{if } i = 2, 10 \\ 1 & \text{if } i = 14 \\ -\frac{1}{12}\tau_2^4 & \text{if } i = 22 \end{cases}$$

$$\partial_{23}(\tau_i) = \begin{cases} 0 & \text{if } i = 2, 10, 14 \\ 1 & \text{if } i = 22 \end{cases}$$

The derivations δ_i are given by multiplication with a_i. This means that δ_i is non-zero only on b_{-i}, which is mapped to 1. □

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AN ELEMENTARY PROOF OF THE STRING TOPOLOGY STRUCTURE OF COMPACT ORIENTED SURFACES

A.P.M. KUPERS

Abstract. We give a new proof of the string topology structure of a compact-oriented surface of genus $g \geq 2$, using elementary algebraic topology. This reproves the result of Vaintrob.

Introduction

Let Σ_g be a compact oriented surface of genus g, then Σ_g is an Eilenberg-Mac Lane space $K(\pi_g, 1)$ for π_g the surface group $\pi_1(\Sigma_g; p)$, where $p \in \Sigma_g$ is any basepoint. This group has a presentation given by:

$$\pi_g = \langle A_1, \ldots, A_g, B_1, \ldots, B_g \rangle / ([A_1, B_1] \cdots [B_g, A_g])$$

We will calculate the string topology structure on $H_*(L\Sigma_g; \mathbb{Z})$ for $g \geq 2$, i.e. we will give the string product, BV-operator and string coproduct [CS99] [CV06]. These comprise all explicitly known non-zero string operations. In doing this we reprove the results of Vaintrob [Vai07]. The cases $g = 0$ and $g = 1$ are a consequence of Menichi's work on the string topology of spheres [Men09] and, in the second case, of the fact that the string topology structure of a product is the product of the string topology structures. From now we always assume $g \geq 2$.

1. A property of π_g

We will need the following fact: the centralizer of $h \in \pi_g$ is infinite cyclic unless $h = e$, in which case it is obviously the entire group π_g.

Proposition 1.1. *If $h \neq e \in \pi_g$, then the centralizer of h is infinite cyclic.*

Proof. We will use the theory of Riemann surfaces and in particular of $PSL(2, \mathbb{R})$ as isometries of the upper half-plane as model for hyperbolic space. We also use that all discrete subgroups of $\mathbb{R}$ are isomorphic to $\mathbb{Z}$.

A particular universal cover of Σ_g is the upper-half plane $\mathbb{H}$ and π_g can be made to act – of course still freely and properly discontinuous – on $\mathbb{H}$ by isometries. This means that we have an inclusion $\pi_g \hookrightarrow PSL(2, \mathbb{R})$. Because the fundamental domain has non-zero volume, the image has to be discrete. The non-identity elements of $PSL(2, \mathbb{R})$ can be classified into three types, depending on their fixed points when acting on the compactification $\bar{\mathbb{H}}$.

1. (1) Elliptic elements have a fixed point in $\mathbb{H}$, so can't be in the image of π_g.
2. (2) Parabolic elements have a single fixed point on the boundary and are all conjugate to $z \mapsto z + 1$. Thus, if h is sent to a parabolic element, without loss of generality we can assume it is sent to $z \mapsto z + 1$. Another element in $PSL(2, \mathbb{R})$ only commutes with this if it is of the form $z \mapsto z + t$. This means that the image of the centralizer of h must lie in $\{z \mapsto z + t | t \in \mathbb{R}\} \cong \mathbb{R}$ and be discrete, hence be infinite cyclic.
3. (3) Hyperbolic elements have two distinct fixed points on the boundary and are all conjugate to $z \mapsto \lambda z$ for some $\lambda \in \mathbb{R}_{>0}$. An element of $PSL(2, \mathbb{R})$ only commutes with this if it is of the form $z \mapsto \rho z$ for some $\rho \in \mathbb{R}_{>0}$. Hence, if h is sent to a hyperbolic element without loss of generality it is sent to $z \mapsto \lambda z$. Then the centralizer of h is sent to a discrete subgroup of $\{z \mapsto \rho z | \rho \in \mathbb{R}_{>0}\} \cong \mathbb{R}$ and again we conclude that it is infinite cyclic.

□

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Remark 1.2. In fact, because we required $g \geq 2$, the element h always has to be mapped to an hyperbolic element.

2. The homotopy and integral homology groups of $L\Sigma_g$

We will show that that the fact that Σ_g is a $K(\pi_g, 1)$ implies that $L\Sigma_g$ is a disjoint union of Eilenberg-Mac Lane spaces, which makes its homology easy to compute. First note that $\pi_0(L\Sigma_g)$ consists of the conjugacy classes in π_g, which in canonical bijection with the set L of isotopy classes of closed curves on Σ_g. This means we get a connected component $L_{[h]}\Sigma_g$ for each conjugacy class or equivalently for each isotopy class of closed curve.

The component $L_{[e]}\Sigma_g$ is canonically homotopy equivalent to Σ itself using the map $c : \Sigma_g \to L_{[e]}\Sigma_g$ assigning to a point a constant loop. For the other components, we use that the long exact sequence of homotopy groups for the fibration $\Omega\Sigma_g \to L\Sigma_g \to \Sigma_g$ over a point h which is a based representative of $[h]$ is as follows

$$\begin{aligned} \ldots \to \pi_2(\Omega\Sigma_g, h) \to \pi_2(L_{[h]}\Sigma_g, h) \to \pi_2(\Sigma_g, p) \to \pi_1(\Omega\Sigma_g, h) \to \pi_1(L_{[h]}\Sigma_g, h) \to \pi_1(\Sigma_g, p) \to \\ \to \pi_0(\Omega\Sigma_g) \to \pi_0(L_{[h]}\Sigma_g) \to \pi_0(\Sigma_g) \end{aligned}$$

This in particular tells us that the $L_{[h]}\Sigma_g$'s are $K(G, 1)$'s. But for which groups G? To find out, we use the following lemma:

Lemma 2.1. *$\pi_1(L_{[h]}\Sigma_g, h)$ is given by the centralizer in $\pi_1(\Sigma_g, p)$ of the representative h of the conjugacy class $[h]$.*

Proof. Although this fits into the more general context of Whitehead products on homotopy groups, we give an elementary proof. An element of this group is a homotopy class of maps $S^1 \to L\Sigma_g$ which sends 1 to the based loop h. This is the same as a homotopy class of maps $S^1 \times S^1 \to \Sigma_g$ which sends $S^1 \times 1$ to h. Consider the other restriction $f : 1 \times S^1 \to \Sigma_g$ and hence the map $(h, f) : S^1 \vee S^1 \to \Sigma_g$. The fact that this factors over $S^1 \times S^1$ is equivalent to the fact that $hfh^{-1}f^{-1}$ is homotopically trivial, which is in turn equivalent to f being a centralizer of h. Conversely, any element of the centralizer gives such a map. ☐

Using proposition 1.1 we obtain that $\pi_1(L_{[e]}\Sigma_g, e) = \pi_g$ and $\pi_1(L_{[h]}\Sigma_g, h) \cong \mathbb{Z}$ for $[h] \neq [e]$. Hence $L_{[h]\Sigma}$ is homotopy equivalent to a circle for $[h] \neq [e]$. This allows for a complete calculation of the homology of $L\Sigma_g$.

Theorem 2.2. *The integral homology of $L\Sigma_g$ is given by*

$$H_*(L\Sigma_g; \mathbb{Z}) = \bigoplus_{[h] \in \mathrm{Conj}(\pi_g)} H_*(L_{[h]}\Sigma_g; \mathbb{Z}) \cong \begin{cases} \mathbb{Z}[L] & \text{if } * = 0 \\ H_1(\Sigma_g; \mathbb{Z}) \oplus \mathbb{Z}[L]/(\mathbb{Z} \cdot e) & \text{if } * = 1 \\ \mathbb{Z} & \text{if } * = 2 \\ 0 & \text{otherwise} \end{cases}$$

Here we have to be careful: in H_1, the summand corresponding to a conjugacy class $[h]$ is *not* always generated by the class that one would expect.

Let k_h be a loop generating the centralizer of h. Note that we can arrange that $h = k_h^l$ for some $l \in \mathbb{N}$, since h lies in its own centralizer. The example of A_1^2 shows that we can have $l \neq 1$. It is easy to see that l is independent of the choice of representative h. We call $l = l([h])$ the level of the conjugacy class.[1] Using the proof of the lemma 2.1 we now see that a generator of $H_1(L_{[h]}\Sigma_g; \mathbb{Z})$ is given by the homology class of the cycle

$$\tilde{h} := \theta \mapsto \left(t \mapsto h\left(t + \frac{\theta}{l}\right)\right)$$

[1] It is essentially a winding number: l is the largest number such that there exists a representative $S^1 \to \Sigma_g$ of h which factors as $S^1 \overset{z \mapsto z^l}{\to} S^1 \to \Sigma_g$. If $l = 1$ then the corresponding isotopy class of closed curve is called primitive, exactly because it can not be written as a power of a smaller closed curve.

3. The BV-operator

Rotation of loops gives an action of S^1 on the free loop space:

$$\begin{aligned} \rho : S^1 \times L\Sigma_g &\to L\Sigma_g \\ (\theta, \gamma) &\mapsto (t \mapsto \gamma(t+\theta)) \end{aligned}$$

The BV-operator is given by $\Delta(a) = \rho_*([S^1] \times a)$ and we can calculate it explicitly. Note that ρ respects connected components, so it suffices to calculate the BV-operator for each connected component separately. We start with $[h] = [e]$. Then the action ρ is homotopy trivial, so Δ vanishes on the homology of $L_{[e]}\Sigma_g$.

Now let's do $[h] \neq [e]$. Only in degree one could the BV-operator be non-zero. If we pick a representative h as our generator for $H_0(L_{[h]}\Sigma_g; \mathbb{Z})$ we see directly that $\Delta(h) = l([h])\tilde{h}$, where l is the level of the conjugacy class of h as before.

4. The string product and coproduct

We will use the actual definition of the string product as being induced by the following diagram

$$\begin{tikzcd} L\Sigma_g \times L\Sigma_g \arrow[d] & \mathrm{Map}(8, \Sigma_g) \arrow[l, "i"'] \arrow[r, "j"] \arrow[d] & L\Sigma_g \\ \Sigma_g \times \Sigma_g & \Sigma_g \arrow[l, "\nabla"] & \end{tikzcd}$$

as $H_*(L\Sigma_g; \mathbb{Z}) \otimes H_*(L\Sigma_g; \mathbb{Z}) \ni a \otimes b \mapsto j_* i^!(a \otimes b) \in H_{*-d}(L\Sigma_g; \mathbb{Z})$. Because our surfaces are even-dimensional we do not need to worry about signs.

We know that the string product drops the degree by -2 and has a unit in $H_2(L\Sigma_g; \mathbb{Z})$ given by the image $c_*([\Sigma_g])$ of fundamental class of Σ_g under the induced map $c_* : H_*(\Sigma_g; \mathbb{Z}) \to H_*(L_{[e]}\Sigma_g; \mathbb{Z})$. This unit is therefore exactly the generator of $H_2(L\Sigma_g; \mathbb{Z})$. Hence it suffices to calculate the string product on $H_1(L\Sigma_g; \mathbb{Z}) \otimes H_1(L\Sigma_g; \mathbb{Z})$, which is mapped to $H_0(L\Sigma_g; \mathbb{Z})$.

The connected components of $L\Sigma_g \times L\Sigma_g$ are given by pairs of connected components $L_{[h_1]}\Sigma_g \times L_{[h_2]}\Sigma_g$. If $[h_1] = [e] = [h_2]$ then the vertical arrows become homotopy equivalences and the string product reduces to the ordinary intersection product $\langle -, - \rangle$ on the homology of Σ_g: $A_i \cdot A_j = B_i \cdot B_j = 0$ and $A_i \cdot B_j = \delta_{ij}$.

With two lemma's we calculate the other two cases. The first case is $[h_1] = [e]$ and $[h_2] = [h] \neq [e]$.

Lemma 4.1. *We have that* $A_i \cdot \tilde{h} = \frac{1}{l([h])} \langle A_i, h \rangle h$, *where* $\langle -, - \rangle$ *is the ordinary intersection product. A similar formula holds for* $B_i \cdot \tilde{h}$.

Proof. In this case the diagram reduces to

$$\begin{tikzcd} \Sigma_g \times L_{[h]}\Sigma_g \arrow[d, "id \times \mathrm{ev}"'] & L_{[h]}\Sigma_g \arrow[l, "\iota = \mathrm{ev} \times id"'] \arrow[r, "id"] \arrow[d, "\mathrm{ev}"] & L_{[h]}\Sigma_g \\ \Sigma_g \times \Sigma_g & \Sigma_g \arrow[l, "\nabla"] & \end{tikzcd}$$

The class $i_!(A_i \otimes \tilde{h})$ must be a multiple of the generator h of $H_0(L_{[h]}\Sigma_g; \mathbb{Z})$. It is given by $(id \times \mathrm{ev})^*(u) \cap (A_i \otimes \tilde{h})$, where u is Thom class for ∇. To determine which multiple we can compose with ev_*. We get the following formula:

$$\mathrm{ev}_*((id \times \mathrm{ev})^*(u) \cap (A_i \otimes \tilde{h})) = u \cap (A_i \otimes \mathrm{ev}_*(\tilde{h})) = u \cap (A_i \otimes k_h)$$

where k_h was the generator of the centralizer of h. This is the intersection product $\langle A_i, k_h \rangle$, which in turn is equal to $\frac{1}{l([h])} \langle A_i, h \rangle$ because by definition $h = k_h^{l([h])}$. This proves the formula. □

The second case is $[h_1], [h_2] \neq [e]$. The result uses the Goldman bracket [Gol86].

Lemma 4.2. *We have that $\tilde{h}_1 \cdot \tilde{h}_2$ is equal to a multiple of the Goldman bracket:*

$$\tilde{h}_1 \cdot \tilde{h}_2 = \frac{1}{l([h_1])l([h_2])}[h_1, h_2]$$

Proof. It is well-known that the Goldman bracket $[h_1, h_2]$ of h_1 and h_2 is given by $\Delta(h_1) \cdot \Delta(h_2)$ [CS99, example 7.1]. We just have to note that $\Delta(h_1) = l([h_1])\tilde{h}_1$, $\Delta(h_2) = l([h_2])\tilde{h}_2$ and that the string product and the Goldman bracket are bilinear. Because all homology groups are free abelian, we do not need to worry about torsion. □

Finally, we get to the string coproduct. The string coproduct is almost trivial: it vanishes except on $H_2(L_{[e]}\Sigma_g; \mathbb{Z})$ and is given on the unit as follows:

$$H_2(L_{[e]}\Sigma_g; \mathbb{Z}) \ni c_*([\Sigma_g]) \mapsto \chi(\Sigma_g)e \otimes e = (2-2g)e \otimes e \in H_0(L_{[e]}\Sigma_g; \mathbb{Z}) \otimes H_0(L_{[e]}\Sigma_g; \mathbb{Z})$$

This is a consequence of the calculations by Tamanoi [Tam07].

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