GEOMETRIC ABELIAN CLASS FIELD THEORY

By

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The undersigned hereby certify that they have read and recommend to the Faculty of Science for acceptance a thesis entitled "**Geometric Abelian Class Field Theory**" by **Péter Tóth** in partial fulfillment of the requirements for the degree of **Master of Science**.

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To Anett and Belián.

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Abstract

The purpose of this thesis has been twofold. First to give a detailed treatment of unramified geometric abelian class field theory concentrating on Deligne's geometric proof in order to remedy the unfortunate situation that the literature on this topic is very deficient, partial and sketchy written[1]. In the second place to give also a detailed treatment of ramified geometric abelian class field theory and more importantly to find a new geometric proof for the ramified theory by trying to adapt or to lean on Deligne's geometric argument in the unramified case.

What was achieved is the following: we begin with discussing and building up the unramified theory in details, which describes a remarkable connection between the Picard group and the abelianized étale fundamental group of a smooth, projective, geometrically irreducible curve over a finite field. We give the necessary background culminating in the fully presented geometric proof of Deligne. Then we turn to a detailed discussion of the tamely ramified theory, which transforms the classical situation to the open complement of a finite set of points of the curve, establishing a connection between a modified Picard group and the tame fundamental group of the curve with respect to this finite subset of points. In the end we finally present a geometric proof for the tamely ramified theory.

[1]Probably due to the fact that people working on this field are mostly concentrating on the higher dimensional theory, that is on the geometric Langlands program, instead of writing down relatively classical works in a fairly didactic way.

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Utrecht, The Netherlands
May 15, 2011

Péter Tóth

...δῆλον γάρ ὡς ὑμεῖς μὲν ταῦτα (τί ποτε βούλεσθε σημαίνειν ὁπόταν ᾽ὸν φθέγγεσθε) πάλαι γιγνώσκετε, ἡμεῖς δὲ πρὸ τοῦ μὲν ᾠόμεθα, νῦν δ᾽ ἠπορήκαμεν...

...for manifestly you have long been aware of what you mean when you use the expression 'being'. We, however, who used to think we understood it, have now become perplexed...

Plato, Sophist 244a

Introduction

Abelian class field theory aims to provide a description of the abelian extensions of a global field K in terms of arithmetic data attached to it. A natural way to encode this information is to consider all algebraic extensions of K at once resulting the big Galois group $Gal(\overline{K}/K)$ and take its abelianization $Gal(K^{ab}/K)$ describing the abelian extensions. The main goal is then to capture this Galois world relying on the arithmetic of the field K.

Mysteriously the suitable arithmetic data associated to the field K is its idèle group $\mathbb{I}_K$ which is a topological group defined by the restricted product

$$\mathbb{I}_K := \prod_v' K_v^\times$$

using the completions K_v at all primes (places) of the field K. On the Galois side one can associate to each prime v of K the Frobenius element $Frob_v \in Gal(K^{ab}/K)$ which enables us to define Artin's Reciprocity Map

$$\Phi_K \colon \mathbb{I}_K \longrightarrow Gal(K^{ab}/K)$$

given by

$$\Phi_K \colon (\dots, a_v, \dots) \mapsto \prod_v Frob_v^{ord_v(a_v)}.$$

For precise definitions and proofs of the statements what follows the reader is referred to the classical references, such as V.5. in [Mil08b] and Chapters VII., VIII. in [AT90].

Different base fields need different treatments and provide different end results. For number fields the high-light of the classical theory is Artin's Reciprocity Law, which states that the Reciprocity Map is surjective and factors through the idèle class group $\mathbf{C}_K := K^\times \backslash \mathbb{I}_K$ of K giving an isomorphism

$$\Phi_K \colon \mathbf{C}_K/\mathcal{O} \xrightarrow{\cong} Gal(K^{ab}/K)$$

where $\mathcal{O}$ is the connected component of $1 \in \mathbf{C}_K$.

For function fields the Reciprocity Map is no longer surjective but factors also through the the idèle class group and results an injective map with dense image

$$\Phi_K \colon K^\times \backslash \mathbb{I}_K / \prod_v \mathcal{O}_v^\times \hookrightarrow Gal(K^{ab}/K).$$

In this thesis we will concentrate on the function field case, i.e. geometric abelian class field theory, starting with the unramified theory where we take the function field $K = k(C)$ of a smooth, projective, geometrically irreducible curve C defined over a finite base field $k = \mathbb{F}_q$ and consider all abelian unramified extensions $Gal(K^{un}/K)^{ab}$. We will proceed with the theory in a geometric manner interpreting the Reciprocity Map in purely geometric terms, namely the adelic double quotient on the left hand side can be identified with the Picard group $Pic_C(k)$ of C and on the Galois side we have the ablianized étale fundamental group $\pi_1^{ab}(C)$. Then relying on brilliant ideas of Deligne in order to prove Artin's Reciprocity Law we will begin a long geometric journey going on a by-pass road through l-adic representations, l-adic local systems and Grothendieck's faisceaux-fonctions correspondence leading finally to a geometric proof of the unramified theory.

Then we turn our attention to the tamely ramified theory considering a finite set of points $S \subset C$ and its open complement $U := C \setminus S$ and we transplant the results of the unramified theory by establishing a Reciprocity Map

$$\Phi_{K,S} \colon Pic_{C,S}(k) \longrightarrow \pi_1^{t,ab}(U)$$

between the k-rational points of a modified Picard group and the abelianized tame étale fundamental group of C with respect to S. In this case we will be able to adapt Deligne's geometric argument leading to a geometric proof for the tamely ramified theory.

Geometric class field theory is now embedded as the one dimensional case in big theories, such as the geometric Langlands Program or higher dimensional class field theory. There are many references for the subject, but as far as I know none of them endeavors completeness and/or does particularly not dwell into the details concerning Deligne's geometric proof in this one dimensional case. A good overview is given in [Fre05] or in [Gai04]. Deligne's argument is contained in [Lau90], [Hei07] and [Hei04]. We give later specific references for all other geometric instruments needed for fulfilling Deligne's idea. In the higher dimensional class field theory direction we refer to [Sch05], which gives a detailed overview and [SS99] which deals with our subject in a different way. Based upon these articles, the concerned reader will find other useful references in these directions. I want to mention also [Sza09a] and last but not least the classical book of Serre [Ser88], which gives a self-contained geometric approach discussing also the ramified case, but with slightly different end results and particularly with different techniques.

Chapter 1

Unramified Geometric Abelian Class Field Theory

In this chapter we will present unramified geometric abelian class field theory which establishes a remarkable connection between the Picard group and the abelianized étale fundamental group of a smooth projective curve over a finite field.

We begin with stating the main theorem of the unramified theory in different forms without going into the details and trace out a way how we will prove it.

Then in the subsequent sections we will discuss and develop the background needed to fully understand and to be able to follow in details the theory. In particular we will define the basic concepts appearing in both sides of the correspondence, namely the Picard scheme and the étale fundamental group of a smooth, projective curve and we will also perform the necessary constructions leading to a proof of the main theorem, which finally will be provided in the next chapter.

1.1 The Main Theorem: Artin's Reciprocity Law

Let C be a smooth, projective, geometrically irreducible curve over a finite field $k = \mathbb{F}_q$. Let $K = k(C)$ be its function field and for every closed point $p \in |C|$ let $\widehat{\mathcal{O}_p}$ be the completion of the local ring at the point p and K_p its quotient field. Let $\mathbb{A}_K := \prod'_{p\in|C|} K_p$ be the ring of adèles and $\mathbb{I}_K$ the idèle group of C. Recall that every $p \in |C|$ defines an (additive) discrete valuation ord_p of the field K and for an idèle $(\dots, a_p, \dots)_{p\in|C|}$ it holds that for all but finitely many $p \in |C|$ $ord_p(a_p) = 0$. The main theorem of unramified geometric abelian class field theory is the following:

Theorem 1.1.1 (Artin's Reciprocity Law, adelic form)**.** *The Artin Reciprocity Map*

$$\begin{aligned} \Phi_K \colon \mathbb{I}_K / \textstyle\prod_{p\in|C|} \widehat{\mathcal{O}_p^\times} &\xrightarrow{\Phi_K} \pi_1^{ab}(C) \\ [(\dots, a_p, \dots)_{p\in|C|}] &\longmapsto \textstyle\prod_{p\in|C|} Frob_p^{ord_p(a_p)} \end{aligned}$$

factors through the quotient

$$\Phi_K \colon K^\times \backslash \mathbb{I}_K / \textstyle\prod_{p\in|C|} \widehat{\mathcal{O}_p^\times} \xrightarrow{\Phi_K} \pi_1^{ab}(C)$$

and fits into the commutative diagram

$$\begin{array}{ccccc} Ker(deg) & \hookrightarrow & K^\times \backslash \mathbb{I}_K / \prod_{p\in|C|} \widehat{\mathcal{O}_p^\times} & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow & & \downarrow \Phi_K & & \downarrow can \\ Ker(\varphi) & \hookrightarrow & \pi_1^{ab}(C) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}} \end{array}$$

such that there is an induced isomorphism on the kernels

$$Ker(deg) \xrightarrow{\cong} Ker(\varphi)$$

where $\varphi\colon \pi_1^{ab}(C) \longrightarrow \widehat{\mathbb{Z}}$ *is the map between the abelianized étale fundamental groups induced by the structure morphism* $C \longrightarrow Spec(k)$ *(section 1.4).*

We can characterize the adelic double quotient in terms of geometric data associated to the curve C in the following way

Proposition 1.1.2. *There is an isomorphism*

$$K^\times \backslash \mathbb{I}_K / \prod_{p\in|C|} \widehat{\mathcal{O}_p^\times} \cong Pic_C(k)$$

between the adelic double coset space and isomorphism classes of invertible sheaves on C.

Proof. This is a special case of the more general statement (cf. 2.1 in [Gai04]), which gives an adelic description of isomorphism classes of vector bundles on C.[1]

Given an invertible sheaf $\mathcal{F}$ on C, we choose a trivialization at the generic point ξ of C

$$f_\xi\colon \mathcal{F} \otimes_{\mathcal{O}_C} K \xrightarrow{\cong} K$$

as the local ring $\mathcal{O}_\xi$ is isomorphic to the function field K. We choose also a trivialization for every closed point $p \in |C|$

$$f_p\colon \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \xrightarrow{\cong} \mathcal{O}_p$$

as $\mathcal{F}_p$ is a free $\mathcal{O}_p$-module of rank 1. The natural morphism $\mathcal{O}_p \longrightarrow K$ induces the diagram

[1] There is a one-to-one correspondence between isomorphism classes of vector bundles of rank n on C and the double coset space $GL_n(K)\backslash GL_n(\mathbb{A}_K)/\prod_{p\in|C|} GL_n(\widehat{\mathcal{O}_p})$.

$$\begin{array}{ccc} \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \otimes_{\mathcal{O}_p} K & \xrightarrow{f_p \otimes id_K} & \mathcal{O}_p \otimes_{\mathcal{O}_p} K \\ \downarrow{\scriptstyle f_\xi} & & \downarrow{\scriptstyle \cong} \\ K & \xrightarrow[\cong]{g_p} & K\ . \end{array}$$

The isomorphism g_p is given by multiplication by an element $a_p \in K^\times$. Moreover for all but finitely many closed point $p \in |C|$ we have that $a_p \in \mathcal{O}_p^\times$ (cf. Lemma I.6.5 in [Har06]), hence the invertible sheaf $\mathcal{F}$ defines an element

$$(\ldots, a_p, \ldots)_{p\in|C|} \in {\prod}'_{p\in|C|} K^\times.$$

If we choose another trivialization at the generic point ξ

$$f'_\xi \colon \mathcal{F} \otimes_{\mathcal{O}_C} K \xrightarrow{\cong} K$$

then every a_p will be changed via left multiplication by an element in $K^\times$. Also if we choose another trivializations at each $p \in |C|$

$$f'_p \colon \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \xrightarrow{\cong} \mathcal{O}_p$$

then each a_p will be changed via right multiplication by an element in $\mathcal{O}_p^\times$. Hence $\mathcal{F}$ defines an element in $K^\times \backslash {\prod}'_{p\in|C|} K^\times / \prod_{p\in|C|} \mathcal{O}_p^\times$.

Claim 1.1.3. The natural map

$$K^\times/\mathcal{O}_p^\times \xrightarrow{\cong} K_p^\times/\widehat{\mathcal{O}_p^\times}$$

is an isomorphism.[2]

Proof. The natural inclusion $K \hookrightarrow K_p$ shows that the map is injective. For surjectivity it is enough to prove that

[2]In fact as $\mathcal{O}_p$ is a discrete valuation ring, both groups are isomorphic to $\mathbb{Z}$.

$$K_p^\times = K^\times \widehat{\mathcal{O}_p^\times}.$$

Let $a \in K_p^\times$ be given. By Theorem II.4.4 in [Neu07] we can write a uniquely as

$$a = u\pi^m(a_0 + a_1\pi + a_2\pi^2 + \dots)$$

where $u \in \mathcal{O}_p^\times$, $\pi \in \mathcal{O}_p$ is a primelement $ord_p(\pi) = 1$, $m \in \mathbb{Z}$, $a_i \in R \subset \mathcal{O}_p$, where R is a representative system for the residue field $\mathcal{O}_p/\mathfrak{m}_p$, $a_0 \neq 0$, hence

$$(a_0 + a_1\pi + a_2\pi^2 + \dots) \in \widehat{\mathcal{O}_p^\times}.$$

By Theorem II.3.8 in [Neu07] we have that $u\pi^m \in K^\times$ completing the proof of the claim. □

Now we use this isomorphism to get an element in $K^\times \backslash \mathbb{I}_K / \prod_{p \in |C|} \widehat{\mathcal{O}_p^\times}$ defined by $\mathcal{F}$, which depends only on the isomorphism class of $\mathcal{F}$ by construction.

On the other hand given an element $a = (\dots, a_p, \dots) \in \mathbb{I}_K$ we define a sheaf $\mathcal{F}_a$ on C by

$$\mathcal{F}_a(U) := \{x \in K |\ a_p^{-1}x \in \widehat{\mathcal{O}_p}\ \forall\ p \in U\}.$$

It follows from this local description that $\mathcal{F}_a$ defines a sheaf on C. Moreover changing the coset representative a from the right by an element in $\prod_{p \in |C|} \widehat{\mathcal{O}_p^\times}$ does not change anything in $\mathcal{F}_a$ and changing from the left by an element $b \in K^\times$ gives an isomorphism of sheaves $\mathcal{F}_a \xrightarrow{b\times} \mathcal{F}_{ba}$, so the only thing to prove is that $\mathcal{F}_a$ is locally free of rank 1. If $a_p \in \widehat{\mathcal{O}_p^\times}$ for all $p \in U$ then $\mathcal{F}_a(U) = \mathcal{O}_C(U)$ by construction, hence it is free on U. Otherwise using the above claim 1.1.3 we take an element $t \in K^\times$ such that $ta_p \in \widehat{\mathcal{O}_p^\times}$. Now define $U := \{p \in |C| :\ t \in \widehat{\mathcal{O}_p^\times}\}$ and use the isomorphism $\mathcal{F}_a \xrightarrow{t\times} \mathcal{F}_{ta}$ which gives that $\mathcal{F}_a$ is locally free of rank 1 on U.

These two constructions are inverses to each other which completes the proof. □

Now we can give the Artin Reciprocity Law in a more geometric form:

Theorem 1.1.4 (Artin's Reciprocity Law, geometric form)**.** *The Artin Reciprocity Map*

$$\Phi_K\colon Div(C) \longrightarrow \pi_1^{ab}(C)$$
$$p \longmapsto Frob_p$$

factors through rational equivalence

$$\Phi_K\colon Pic_C(k) \longrightarrow \pi_1^{ab}(C)$$

and fits into the commutative diagram

$$\begin{array}{ccccc} Pic^0_C(k) & \hookrightarrow & Pic_C(k) & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle \Phi_K} & & \downarrow{\scriptstyle can} \\ Ker(\varphi) & \hookrightarrow & \pi_1^{ab}(C) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}} \end{array} \tag{1.1.1}$$

such that there is an induced isomorphism on the kernels

$$Pic^0_C(k) \xrightarrow{\cong} Ker(\varphi)$$

where $\varphi\colon \pi_1^{ab}(C) \longrightarrow \widehat{\mathbb{Z}}$ *is the map between the abelianized étale fundamental groups induced by the structure morphism* $C \longrightarrow Spec(k)$ *(section 1.4).*

We can rephrase this theorem in a slightly weaker form - which is of more number theoretic nature - using the general theory of profinite groups. Namely assuming the above Theorem holds true we can pass to the profinite completions in diagram 2.2.1 and get an isomorphism of profinite groups

$$\widehat{Pic_C(k)} \xrightarrow{\cong} \widehat{\pi_1^{ab}(C)}$$

by Corollary 10.3 in [AM69] (as the isomorphism $Pic^0_C(k) \xrightarrow{\cong} Ker(\varphi)$ induces an isomorphism between the completions and $\widehat{\widehat{\mathbb{Z}}} = \widehat{\mathbb{Z}}$, hence there is an isomorphism between the completions in the middle).

Now applying Proposition 3.2.2 in [RZ00] we get directly

Theorem 1.1.5 (Artin's Reciprocity Law, weaker form)**.** *There is a one-to-one correspondence between*

normal subgroups of finite index of $K^\times \backslash \mathbb{I}_K / \prod_{p\in|C|} \widehat{\mathcal{O}_p^\times}$	$\overset{1:1}{\longleftrightarrow}$	*finite, abelian, unramified extensions* L/K

Remark 1.1.6. Here and later on as well we mean by one-to-one correspondence that we can associate bijectively objects of one set to the other in a constructive, natural way that arises from the underlying geometric structures, as the problem with this terminology is that in this case and later also, we are speaking about sets of the same cardinality, hence one could make a correspondence relying on this fact resulting that the statement is empty.

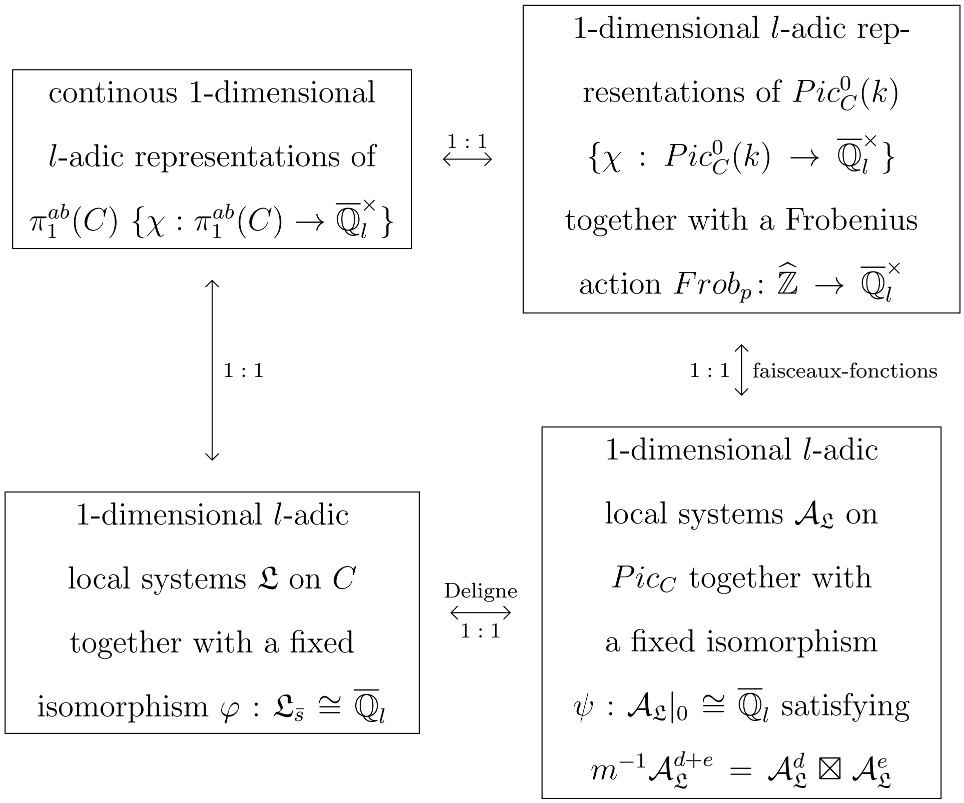
We have stated the Main Theorem of the unramified theory in different forms and from now on we will concentrate on its geometric formulation. Let us begin with tracing out the main steps toward a proof of Theorem 1.1.4.

The strategy of the proof is first to consider continuous, 1-dimensional l-adic representations of $\pi_1^{ab}(C)$ and 1-dimensional l-adic representations of $Pic^0_C(k)$, where l is a prime number different from $char(k)$. Assume we have a closed point $p \in C(k)$, then we can characterize these representations as follows

- the continuous 1-dimensional l-adic representations of $\pi_1^{ab}(C)$ (which are the same as the continuous 1-dimensional l-adic representations of the algebraic fundamental group $\pi_1(C, \bar{s})$) are in one-to-one correspondence with 1-dimensional l-adic local systems $\mathfrak{L}$ on C together with a rigidification, i.e. a fixed isomorphism $\varphi\colon \mathfrak{L}_{\bar{s}} \cong \overline{\mathbb{Q}}_l$, where $\bar{s}\colon Spec(\Omega) \longrightarrow C$ is a geometric point (cf. section 1.5);

- 1-dimensional l-adic representations of $Pic^0_C(k)$ together with a Frobenius action $Frob_p\colon \widehat{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}_l^\times$ are in one-to-one correspondence with 1-dimensional l-adic local systems $\mathcal{A}_{\mathfrak{L}}$ on Pic_C together with a rigidification, i.e. a fixed isomorphism $\psi : \mathcal{A}_{\mathfrak{L}}|_0 \cong \overline{\mathbb{Q}}_l$ satisfying $m^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e} \cong \mathcal{A}_{\mathfrak{L}}^d \boxtimes \mathcal{A}_{\mathfrak{L}}^e$, where $m\colon Pic^d_C \times Pic^e_C \longrightarrow Pic^{d+e}_C$ is the group operation on Pic_C and $0\colon Spec(k) \to Pic^0_C$ is the identity section (the class of the trivial bundle) (cf. section 1.6).

Using these correspondences we will present Deligne's geometric argument in section 2.2, which gives a one-to-one correspondence between rigidified 1-dimensional l-adic local systems on C and rigidified multiplicative 1-dimensional l-adic local systems on Pic_C. Having done this we will get the following diagram:



(1.1.2)

Finally in section 2.2 we will use the correspondences appearing in this diagram to prove Artin's Reciprocity Law.

1.2 Symmetric Powers of a Curve

In this section we will recall the construction of symmetric powers of a curve over a field k and investigate their relationship to effective divisors on the curve. The main references will be [Mum70] and [Mil08a].

First we will work in a fairly general setting and then descend to the case of curves. So let X be a quasi-projective scheme of finite presentation over a field k. For an integer $d \geq 1$ let us consider the d-fold product

$$X^d := X \times_{Spec(k)} \cdots \times_{Spec(k)} X.$$

There is an action of the symmetric group S_d on X^d by permuting the factors.

Definition 1.2.1. A morphism $g : X^d \longrightarrow Y$ of schemes over k is said to be *symmetric* if it is invariant under the S_d action on X^d.

Then the main theorem is the following:

Theorem 1.2.2. *Let X be a quasi-projective scheme of finite presentation over a field k and $d \geq 1$ a positive integer. Then there exists a scheme $X^{(d)}$ over k and a symmetric morphism $\pi : X^d \longrightarrow X^{(d)}$ called the* ***dth symmetric power*** *of the scheme X/k having the following properties:*

1. *the underlying topological space is the quotient $X^{(d)} := X^d/S_d$ of X^d by the action of S_d;*
2. *for an affine open subset $U \subset X$, $U^{(d)} \subset X^{(d)}$ is affine open and it holds that*

$$\mathcal{O}_{X^{(d)}}(U^{(d)}) = (\mathcal{O}_{X^d}(U^d))^{S_d}.$$

The pair $(X^{(d)}, \pi)$ has the following universal property: every symmetric k-morphism $g : X^d \longrightarrow Y$ factors uniquely through π and $(X^{(d)}, \pi)$ is uniquely determined up to a unique isomorphism by this universal property. Moreover the map π is finite, surjective and separated.

Proof. See II.7 in [Mum70]. □

In the case of curves we have additionally the very important property:

Proposition 1.2.3. *Let C be a nonsingular curve over a field k and $d \geq 1$ a positive integer. Then the dth symmetric power $C^{(d)}$ is also nonsingular.*

Proof. See Proposition 3.2. in [Mil08a] and Proposition 11.24 in [AM69]. □

Next we turn to the connection between effective divisors and symmetric powers. First we note the following

Proposition 1.2.4. *For a noetherian, integral, locally factorial separated scheme X the group $Div(X)$ of Weil divisors on X is isomorphic to the group of Cartier divisors $H^0(X, \mathcal{K}^*/\mathcal{O}_X^*)$ on X (principal Weil divisors corresponding to principal Cartier divisors).*

Proof. This is Proposition 6.11 in [Har06]. □

Now we define effective Cartier divisors in a relative situation.

Definition 1.2.5. Let $f : X \longrightarrow T$ be a morphism of schemes over k. A *relative effective Cartier divisor* on X/T is an effective Cartier divisor D on X, which is flat over T considered as a subscheme of X.

If the base scheme is affine $T = Spec(A)$, then a subscheme $D \subset X$ is a relative effective Cartier divisor, if there exists an open affine covering $X = \bigcup_{i \in I} Spec(A_i)$ such that for all $i \in I$

1. $D \bigcap Spec(A_i) = Spec(A_i/(h_i))$ where $h_i \in A_i$ is not a zero divisor
2. $A_i/(h_i)$ is a flat A-algebra.

We can characterize relative effective Cartier divisors D on X/T in the following way. Assume D is represented by $(U_i, g_i)_{i \in I}$ where $X = \bigcup_{i \in I} U_i$ is an open covering, and $g_i \in \Gamma(U_i, \mathcal{O}_X)$ such that for all i, j, $\frac{g_i}{g_j} \in \Gamma(U_i \times_X U_j, \mathcal{O}_X^*)$. Then the ideal sheaf $\mathcal{I}(D)$ of the relative effective Cartier divisor D is the invertible sheaf defined locally on U_i by g_i, denoted also by $\mathcal{O}(-D)$ suggesting that the invertible sheaf $\mathcal{O}(D)$ is defined locally on U_i by $\frac{1}{g_i}$. So we have the exact sequence of sheaves on X

$$0 \longrightarrow \mathcal{O}(-D) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_D \longrightarrow 0$$

where $\mathcal{O}_D$ is the structure sheaf of D considered as a closed subscheme of X. Tensoring by $\mathcal{O}(D)$ we get the exact sequence of sheaves X

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{s_D} \mathcal{O}(D) \longrightarrow \mathcal{O}(D)/s_D\mathcal{O}_X \longrightarrow 0$$

where s_D is the canonical non-zero global section of $\mathcal{O}(D)$ defined by the inclusion $\mathcal{O}_X \hookrightarrow \mathcal{O}(D)$, such that the subscheme defined by s_D is flat over T.

Proposition 1.2.6. *The map $D \mapsto (\mathcal{O}(D), s_D)$ establishes a one-to-one correspondence between relative effective Cartier divisors D on X/T and isomorphism classes of pairs $(\mathcal{G}, s)$, where $\mathcal{G}$ is a locally free $\mathcal{O}_X$-module of rank 1 and $s \in H^0(X, \mathcal{G}) \setminus \{0\}$ is a non-zero global section such that the subscheme defined by s is flat over T.*

Proof. See Remark 3.6. in [Mil08a]. □

Next we state some basic properties of relative effective Cartier divisors.

Proposition 1.2.7. *1. Let D_1 and D_2 be relative effective Cartier divisors for X/T. Then also the sum $D_1 + D_2$ is a relative effective Cartier divisor for X/T.*

2. Let us consider the cartesian base change diagram

$$\begin{array}{ccc} X' & \longrightarrow & X \\ \downarrow & & \downarrow \\ T' & \longrightarrow & T \end{array}$$

and let D be a relative effective Cartier divisor on (X/T). Then the pull-back D' of D is a relative effective Cartier divisor on (X'/T').

Proof. See Lemma 3.5 and Proposition 3.7 in [Mil08a]. □

Now let $f\colon X \longrightarrow T$ be a smooth, proper morphism of schemes with fibres of dimension 1. If D is a relative effective Cartier divisor on X/T, then for a point $t \in T$ $D_t := D \times_T \{t\}$ is an effective divisor on $X_t := X \times_T \{t\}$, and if T is connected then the degree of D_t is constant, and it is called the *degree* of the relative effective Cartier divisor D.

Proposition 1.2.8. *Let $f\colon X \longrightarrow T$ be a smooth, proper morphism of schemes with fibres of dimension 1. A closed subscheme $D \subset X$ is a relative effective Cartier divisor on X/T if and only if it is finite and flat over T. In particular for a section $s\colon T \longrightarrow X$, $s(T)$ is a relative effective Cartier divisor of degree 1 on X/T.*

Proof. Corollary 3.9. in [Mil08a]. □

Now we restrict ourselves to the case of smooth, projective curves over a field k and we define the following functor:

Definition 1.2.9. Let C be a smooth, projective curve over a field k, $d \geq 1$ a positive integer and define the functor

$$Div_C^d : Sch/k \longrightarrow Set$$

which to a k-scheme T associates the set $Div_C^d(T)$ of relative effective Cartier divisors of degree d on $(C \times_{Spec(k)} T)/T$.

By the above Proposition 1.2.7 this is well-defined concerning morphisms. For the proof that this functor is representable by $C^{(d)}$, we construct a canonical relative effective Cartier divisor on $(C \times_{Spec(k)} C^{(d)})/C^{(d)}$. For that let us consider the projection

$$pr : C \times_{Spec(k)} C^d \longrightarrow C^d$$

which has canonical sections for all $i = 1, \dots, d$

$$s_i : C^d \longrightarrow C \times_{Spec(k)} C^d$$

defined by

$$s_i((p_1, p_2, \dots, p_d)) := (p_i, (p_1, p_2, \dots, p_d)).$$

Then define the relative effective Cartier divisor $D = \sum_{i=1}^d D_i$ on $C \times_{Spec(k)} C^d / C^d$, where $D_i := s_i(C^d)$ is a relative effective Cartier divisor for all $i = 1, \dots, d$. The divisor D is stable as a subscheme under the action of S_d, and so is C^d, hence it defines a relative effective Cartier divisor D_{can} on $C \times_{Spec(k)} C^{(d)}/C^{(d)}$.

Now we can state the main theorem about the representability of the relative effective Cartier divisor functor:

Theorem 1.2.10. *Let C be a smooth, projective curve over a field k and $d \geq 1$ a positive integer. Then for any relative effective Cartier divisor D on $(C \times_{Spec(k)} T)/T$ there exists a unique morphism $\alpha : T \longrightarrow C^{(d)}$ such that*

$$D = (id_C \times \alpha)^{-1}(D_{can}),$$

that is the functor Div^d_C is representable by $C^{(d)}$.

Proof. This is Theorem 3.13 in [Mil08a]. $\square$

1.3 The Picard Scheme of a Curve

In this section let C be a smooth, projective curve over a field k. We will define Picard functors Pic_C^d of various degrees from the category of k-schemes to the category of abelian groups and investigate under which circumstances are these representable by a group scheme, denoted also by Pic_C^d over k, such that the k-rational points of this group scheme should be isomorphic to the group of invertible sheaves of degree d on C.

First assume for simplicity that $C(k) \neq \emptyset$ and let $p \in C(k)$ be a k-rational point.

Definition 1.3.1. For an integer $d \in \mathbb{Z}$ define the *Picard functor of degree* d of C as the functor from the category of schemes over k to the category of abelian groups

$$Pic_C^d : Sch/k \longrightarrow Ab$$

which to a k-scheme T associates the abelian group

$$Pic_C^d(T) := \{\mathcal{G} \in Pic(C \times_{Spec(k)} T) |\ deg(\mathcal{G}_t) = d \ \ \forall t \in T\}/pr_2^{-1}(Pic^d(T))$$

i.e. families of invertible sheaves of degree d on C parametrized by T modulo those coming from T.

We note that for any $d \in \mathbb{Z}$ and a scheme T over k we have an isomorphism

$$Pic_C^0(T) \xrightarrow{\cong} Pic_C^d(T)$$

given by

$$\mathcal{G} \mapsto \mathcal{G} \otimes pr_1^{-1}\mathcal{O}(dp).$$

Remark 1.3.2. Hence concerning representability of the Picard functors it is enough to prove that a Picard functor of some appropriate degree d is representable. Now the main theorem is the following

Theorem 1.3.3. *There exists an abelian variety* Jac *over* k *and a natural transformation of functors* $\alpha : Pic^0_C \longrightarrow h_{Jac}$ *(where* h_{Jac} *is the functor of points of* Jac*) such that* $\alpha(T) : Pic^0_C(T) \xrightarrow{\cong} h_{Jac}(T)$ *is an isomorphism of abelian groups if* $C(T) \neq \emptyset$*.*

Even if $C(k) = \emptyset$ by definition there exists a finite extension $k^{'}/k$ in a fixed algebraic closure $k \subset k^{'} \subset \bar{k}$ such that $C_{k'} := C \times_{Spec(k)} Spec(k^{'})$ has a $k^{'}$-rational point. Then the following proposition ensures that we can always assume that C has a k-rational point:

Proposition 1.3.4. *If for a finite separable extension* $k^{'}/k$ *Theorem 1.3.3 holds for* $C_{k'}$*, then it also holds for* C*.*

Proof. This is Chapter III.,Proposition 1.14 in [Mil08a]. □

Now we turn to the construction of this Jacobian variety taking into account the fact (1.3.2) that it is enough to show representability of a Picard functor Pic^d_C of an arbitrary degree d.

We choose a fixed degree d satisfying $d \geq 2g-1$, where g is the genus of the curve C. We define the following natural transformation of functors

Definition 1.3.5. The *Abel-Jacobi map*

$$\mathfrak{AJ}_d \colon Div^d_C \longrightarrow Pic^d_C$$

is defined for a scheme T over $Spec(k)$ and for a relative effective Cartier divisor D of degree d on $(C \times_{Spec(k)} T)/T$ by

$$\mathfrak{AJ}_d(T)(D) := [\mathcal{O}(D)]$$

where $[\mathcal{O}(D)]$ is the class of the invertible sheaf $\mathcal{O}(D)$ on $(C \times_{Spec(k)} T)/T$. Equivalently if D is represented by the pair $(\mathcal{G}, s)$ (1.2.6), then the Abel-Jacobi map is given by

$$\mathfrak{AJ}_d(T)((\mathcal{G}, s)) := [\mathcal{G}].$$

Note that by Theorem 1.2.10 the functor Div_C^d is representable by the dth symmetric power $C^{(d)}$. Now consider the following construction:

Construction 1.3.6. *Assume that the natural transformation of functors*

$$\mathfrak{AJ}_d : Div_C^d \longrightarrow Pic_C^d$$

has a section

$$s : Pic_C^d \longrightarrow Div_C^d$$

i.e. a natural transformation such that $\mathfrak{AJ}_d \circ s \cong id_{Pic_C^d}$*, then the functor* Pic_C^d *is representable by a closed subscheme of* $C^{(d)}$ *denoted by* Jac *or* Pic_C^d *later on.*

Proof. Given the section $s : Pic_C^d \longrightarrow Div_C^d$ we can define a natural transformation of functors

$$\lambda = s \circ \mathfrak{AJ}_d : Div_C^d \longrightarrow Div_C^d$$

which induces a morphism of schemes $\lambda : C^{(d)} \longrightarrow C^{(d)}$. Now consider the diagram

$$\begin{array}{ccc} Jac & \xrightarrow{\text{closed}} & C^{(d)} \\ \downarrow & & \downarrow {\scriptstyle id_{C^{(d)}} \times \lambda} \\ C^{(d)} & \xrightarrow{\Delta} & C^{(d)} \times_{Spec(k)} C^{(d)} \end{array}$$

where Jac is defined as the fibre product

$$Jac := C^{(d)} \times_{(C^{(d)} \times_{Spec(k)} C^{(d)})} C^{(d)}.$$

By definition we have for a k-scheme T

$$Jac(T) := \{(x, y) \in C^{(d)}(T) \times C^{(d)}(T) | (x, x) = (y, \lambda(y))\}$$

hence we have that

$$Jac(T) = \{x \in C^{(d)}(T) | \lambda(x) = x\}$$

which means that

$$Jac(T) = \{x \in C^{(d)}(T) | x = s(z),\ z \in Pic_C^d(T)\}$$

but as the section $s(T)$ is injective for every k-scheme T, so we finally get that

$$Jac(T) = Pic_C^d(T)$$

as we wanted. The statement concerning closedness follows from the separability of $C^{(d)}$, i.e. from the closedness of the diagonal Δ. □

Now the question is how to find such a section. Unfortunately there is no such section, because - as we will see later in Corollary 2.1.4 - for large enough d there is a $d-g$-dimensional family of effective divisors over every invertible sheaf of degree d and there is no canonical way to choose one such effective divisor. However one can define representable open subfunctors of Pic_C^d covering Pic_C^d using this section trick and hence one can construct Jac locally glueing together this open parts. For more details see III.4 in [Mil08a], pp.99-101.

1.4 The Étale Fundamental Group

In this section we will define the étale fundamental group of a scheme and present some of its most important properties according to the scope of our purposes. The main reference for this section is the wonderful book [Sza09b].

For the section let k be a (base) field with fixed separable and algebraic closures $k^s \subseteq \bar{k}$ and S a (base) scheme.

Definition 1.4.1. A finite dimensional k-algebra A is *étale* over k if it is isomorphic to a finite direct product of separable field extensions of k.

We can characterize étale k-algebras as follows:

Lemma 1.4.2. *For a finite dimensional k-algebra A the followings are equivalent:*

1. *A is étale*
2. *$A \otimes_k \bar{k}$ is isomorphic to a finite product of copies of $\bar{k}$*
3. *$A \otimes \bar{k}$ is reduced.*

Proof. Proposition 1.5.6 in [Sza09b]. □

Now we can define the notion étale for schemes:

Definition 1.4.3. A finite morphism $f : X \longrightarrow S$ of schemes is *locally free* if the direct image sheaf $f_*\mathcal{O}_X$ is a locally free $\mathcal{O}_S$-module of finite rank. If each fibre $X_s = Spec(k(s)) \times_S X$ for a point $s \in S$ is the spectrum of a finite étale $k(s)$-algebra, then we say f is a *finite étale* morphism. A *finite étale cover* is a surjective finite étale morphism.

Let us see some very important examples:

Example 1.4.4.

- Let $S = Spec(A)$ and $X = Spec(B)$ be affine schemes with $B = A[x]/(f)$ with a monic polynomial $f \in A[x]$ of degree d. Then B is a free A-module of finite rank generated by the images of $1, x, x^2, \ldots, x^{d-1}$ in B. Hence the morphism $Spec(B) \longrightarrow Spec(A)$ is finite and locally free. Let $s \in Spec(A)$ be a point, then the fibre is $X_s = Spec(k(s) \otimes_A B)$, which is isomorphic to $Spec(k(s)[x]/(\bar{f}))$, where $\bar{f}$ is the image of f in $k(s)[x]$. So we see that if $f \in A[x]$ is separable, then the $k(s)$-algebra $k(s)[x]/(\bar{f})$ is étale over $k(s)$ hence the morphism $Spec(B) \longrightarrow Spec(A)$ is finite étale.

- Let $\bar{s} : Spec(\Omega) \longrightarrow S$ be a geometric point of S, i.e. the image of $\bar{s}$ is a point $s \in S$ such that Ω is an algebraically closed extension of the residue field $k(s)$. The geometric fibre $X_{\bar{s}}$ of f over $\bar{s}$ is the spectrum of a finite étale algebra if and only if it is of the form $Spec(\Omega \times \cdots \times \Omega)$, i.e. a finite disjoint union of points defined over Ω.

Let us list some basic properties of finite étale morphisms:

- If $f : X \longrightarrow S$ and $g : Y \longrightarrow X$ are finite étale morphisms of schemes, then so is the composite $f \circ g : Y \longrightarrow S$.

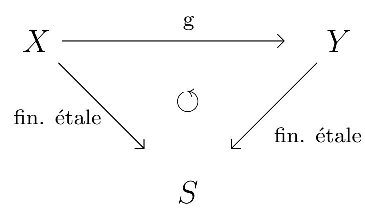
- If $f : X \longrightarrow S$ is a finite étale morphism and $g : Y \longrightarrow S$ is any morphism, then the base change $X \times_S Y \longrightarrow Y$ is a finite étale morphism.

Let us now turn to the construction of the étale fundamental group.

Let Fet/S be the category of finite étale covers of the scheme S with morphisms beeing the S-morphisms of schemes. Let $\bar{s} : Spec(\Omega) \longrightarrow S$ be a fixed geometric point and define a functor, called the fibre functor at the geometric point $\bar{s}$

$$F_{\bar{s}} : Fet/S \longrightarrow Set$$

which to a finite étale cover $f : X \longrightarrow S$ associates the underlying set of the geometric fibre over $\bar{s}$, i.e. $F_{\bar{s}}(X, f) := \{\text{the underlying set of } X_{\bar{s}} := X \times_S Spec(\Omega)\}$ and for a morphism



it associates the set-theoretic map $F_{\bar{s}}(X) \longrightarrow F_{\bar{s}}(Y)$ induced by the morphism of geometric fibres $X \times_S Spec(\Omega) \longrightarrow Y \times_S Spec(\Omega)$.

Definition 1.4.5. Let S be a scheme and $\bar{s}$ a geometric point of it. Then the *algebraic fundamental group* $\pi_1(S, \bar{s})$ of S at $\bar{s}$ is defined as the automorphism group of the fibre functor $F_{\bar{s}}$ on Fet/S.

Recall that for a functor $F : \mathcal{C}_1 \longrightarrow \mathcal{C}_2$ between categories, an automorphism is defined as a natural transformation of functors $\alpha : F \longrightarrow F$, which is an isomorphism (i.e. has a two-sided inverse). Then the set of automorphisms $Aut(F)$ has a natural structure of a group. Moreover, if $\mathcal{C}_2 = Set$, then for all object $C \in Ob(\mathcal{C}_1)$ there is a natural left action of $Aut(F)$ on $F(C)$. In particular, for every finite étale morphism $f : X \longrightarrow S$ there is a natural left action of $\pi_1(S, \bar{s})$ on the fibre $F_{\bar{s}}(X)$. The main theorem along these lines is the following:

Theorem 1.4.6 (Grothendieck)**.** *Let S be a connected scheme and $\bar{s}$ a geometric point of it. Then*

- *the group $\pi_1(S,\bar{s})$ is profinite and its action on $F_{\bar{s}}(X)$ is continous for every X in Fet/S.*
- *The fibre functor $F_{\bar{s}}$ induces an equivalence of the category Fet/S with the category of finite continous left $\pi_1(S,\bar{s})$-sets. Connected covers corresponds to sets with transitive action, and Galois covers to finite quotients of $\pi_1(S,\bar{s})$.*

Proof. Theorem 5.4.2 in [Sza09b]. □

This is a vast generalization of basic Galois Theory, at least in the reformulation of Grothendieck, namely:

Example 1.4.7. Let $S = Spec(k)$ for a field k. Then by definition a finite étale cover X of S is the spectrum of a finite étale k-algebra $X = Spec(L)$. For a geometric point $\bar{s}$ the fibre functor gives $F_{\bar{s}}(X) = Spec(L \otimes_k \Omega)$ (if X is connected), which is the finite set of k-algebra homomorphisms $L \longrightarrow \Omega$ (the image of such a morphism lies in the separable closure k^s of k in Ω). So we see that $F_{\bar{s}}(X) = Hom_k(L, k^s)$ and hence $\pi_1(S,\bar{s}) \cong Gal(k^s/k)$. As $Spec(k^s)$ is not a finite étale cover of $Spec(k)$, the functor is not representable by $Spec(k^s)$, but it is pro-representable. For more details see section 5.4 in [Sza09b].

We see that we need to fix a geometric point $\bar{s}$ for defining the algebraic fundamental group of S. But just as in topology, algebraic fundamental groups of a scheme S defined at different base points are non-canonically isomorphic:

Proposition 1.4.8. *For a connected scheme S and two different geometric points $\bar{s}, \bar{s}'$ of it, there exists an isomorphism of fibre functors $\alpha : F_{\bar{s}} \xrightarrow{\cong} F_{\bar{s}'}$, hence there exists a continous isomorphism of profinite groups $\pi_1(S,\bar{s}) \xrightarrow{\cong} \pi_1(S,\bar{s}')$.*

Proof. For the existence of an isomorphism of fibre functors see Proposition 5.5.1 in [Sza09b]. Such an isomorphism is called a *path* between the geometric points $\bar{s}$ and $\bar{s}^{'}$. Then an isomorphism $\alpha : F_{\bar{s}} \xrightarrow{\cong} F_{\bar{s}'}$ of the fibre functors induces an isomorphism of their automorphism group via $\varphi \mapsto \alpha^{-1} \circ \varphi \circ \alpha$. □

In particular

Corollary 1.4.9. *The isomorphism* $\pi_1(S, \bar{s}) \xrightarrow{\cong} \pi_1(S, \bar{s}^{'})$ *induced by a path depends on the path but it is unique up to an inner automorphism of* $\pi_1(S, \bar{s})$ *or* $\pi_1(S, \bar{s}^{'})$ *respectively. In particular the maximal abelian quotient of* $\pi_1(S, \bar{s})$ *is independent of the choice of a geometric point and hence denoted by* $\pi_1^{ab}(S)$*.*

Next we investigate functoriality. Let S and T be connected schemes with geometric points $\bar{s}\colon Spec(\Omega) \longrightarrow S$ and $\bar{t}\colon Spec(\Omega) \longrightarrow T$ respectively, together with a morphism $\varphi\colon S \longrightarrow T$ preserving base points, i.e. $\bar{t} = \varphi \circ \bar{s}$. Then we have a base change functor

$$B_{S,T}\colon Fet/T \longrightarrow Fet/S$$

by associating to each object $X \longrightarrow T$ in Fet/T the base change $S \times_T X$ in Fet/S and to a morphism $X \longrightarrow Y$ in Fet/T the induced morphism $S \times_T X \longrightarrow S \times_T Y$ in Fet/S. The base point preserving property gives an equality of functors

$$F_{\bar{t}} = F_{\bar{s}} \circ B_{S,T}$$

hence every automorphism of the fibre functor $F_{\bar{s}}$ induces an automorphism of the fibre functor $F_{\bar{t}}$ giving the induced map on the algebraic fundamental groups

$$\varphi_*\colon \pi_1(S, \bar{s}) \longrightarrow \pi_1(T, \bar{t})$$

which is a continuous homomorphism of profinite groups.

Proposition 1.4.10. *The induced map φ_* is surjective if and only if for every connected finite étale cover $X \longrightarrow T$ the base change $S \times_T X \longrightarrow S$ is connected as well.*

Proof. Proposition 5.5.4 in [Sza09b]. □

Now we state the very important homotopy exact sequence theorem:

Theorem 1.4.11. *Let $X \longrightarrow Spec(k)$ be a quasi-compact and geometrically integral scheme over a field k. Fix an algebraic closure $\bar{k}$ of k and let k^s/k be the corresponding separable closure. Let $\overline{X} := Spec(k^s) \times_{Spec(k)} X$ be the geometric fibre and let $\bar{x}$: $Spec(k^s) \longrightarrow \overline{X}$ be a geometric point of $\overline{X}$. Then the sequence of profinite groups*

$$1 \longrightarrow \pi_1(\overline{X}, \bar{x}) \longrightarrow \pi_1(X, \bar{x}) \longrightarrow Gal(k^s/k) \longrightarrow 1$$

induced by the maps $\overline{X} \longrightarrow X$ and $S \longrightarrow Spec(k)$ is exact.

Proof. Proposition 5.6.1 in [Sza09b]. □

We give an important example which we will use vigorously:

Example 1.4.12. Let X be a normal, connected scheme and take $\bar{x}$ as the generic point of X. Let $K(X)$ be the function field of X and $K(X)^S$ be the composite of all finite separable subextensions $L/K(X)$ of the separable closure $K(X)^s$ such that the normalization of X in L is étale over X. Then $K(X)^S/K(X)$ is Galois and $Gal(K(X)^S/K(X)) \cong \pi_1(X, \bar{x})$. In particular take $X = \mathbb{P}^n_k$ than we have that $\pi_1(\mathbb{P}^n_k, \bar{x}) \cong Gal(k^s/k)$, in particular for a separable closed field $k = k^s$ we have that $\pi_1(\mathbb{P}^n_{k^s}, \bar{x}) = 1$ is trivial.(see in [Sza09b] Proposition 5.4.9)

Finally we give a relative version of the above homotopy exact sequence:

Theorem 1.4.13. *Let $f : X \longrightarrow S$ be a proper, surjective morphism of finite presentation with geometrically connected fibres. Let $\bar{s} : Spec(\Omega) \longrightarrow S$ be a geometric point of S such that Ω is the algebraic closure of the residue field of the image of $\bar{s}$ in S, and let $\bar{x}$ be a geometric point of the fibre $X_{\bar{s}} := Spec(\Omega) \times_S X$. Then the sequence*

$$\pi_1(X_{\bar{s}}, \bar{x}) \longrightarrow \pi_1(X, \bar{x}) \longrightarrow \pi_1(S, \bar{s}) \longrightarrow 1$$

is exact.

Proof. This is Corollaire 6.11, Exposé IX. in [Gro61]. □

1.5 l-adic Local Systems and Representations

In this section we will define l-adic sheaves and study some of their basic properties, especially concentrating on l-adic local systems. Then we will investigate the relationship between l-adic local systems on a connected scheme X and continous l-adic representations of its étale fundamental group and give the correspondence appearing in the left vertical side of diagram 1.1.2. The main references for this section are Exposé VI. in [Gro66], [Shi05] and § 12. in [FK88].

Let X be a connected scheme and $X_{ét}$ its étale site, i.e. the underlying category is Et/X whose objects are the étale morphisms $U \longrightarrow X$ and whose arrows are the morphisms $\varphi\colon U \longrightarrow V$ over X together with the Grothendieck topology whose coverings are the surjective families of morphisms $(U_i \longrightarrow U)_{i\in I}$ in Et/X. We fix also a prime number l invertible on X.

Definition 1.5.1. An *étale sheaf* $\mathfrak{F}$ on X (or a sheaf on $X_{ét}$) is a contravariant functor

$$\mathfrak{F}\colon Et/X \longrightarrow Set \text{ or } (Ab, \dots)$$

such that

$$\mathfrak{F}(U) \to \textstyle\prod_{i\in I} \mathfrak{F}(U_i) \rightrightarrows \prod_{i,j\in I} \mathfrak{F}(U_i \times_U U_j)$$

is an equalizer diagram for every object $U \longrightarrow X$ in Et/X and for every étale covering $(U_i \longrightarrow U)_{i\in I}$.

An étale sheaf $\mathfrak{F}$ on X is called

1. *local constant* if there is a covering $(U_i \longrightarrow X)_{i\in I}$ such that the restrictions $\mathfrak{F}|_{U_i}$ are constant étale sheaves for all $i \in I$;

2. *constructible* if for every closed immersion $j\colon Z \longrightarrow X$ there exists an open subset $U \subset Z$ such that the étale sheaf $j^{-1}\mathfrak{F}|_U$ is locally constant having finite stalks.

In particular for a smooth, geometrically irreducible curve C over a field k an étale sheaf $\mathfrak{F}$ is constructible if it has finite stalks and there exists an open subset $U \subset C$ such that $\mathfrak{F}|_U$ is locally constant.

Also for a connected scheme X there is an equivalence between the category of finite étale coverings Fet/X and the category of locally constant étale sheaves with finite stalks (Proposition 6.16 in [Mil08c]).

Definition 1.5.2. A sheaf of $\mathbb{Z}_l$-modules $\mathfrak{F}$ on X is an inverse system

$$(\mathfrak{F}_n, f_{n+1}\colon \mathfrak{F}_{n+1} \longrightarrow \mathfrak{F}_n)_{n\in\mathbb{N}}$$

such that

1. for all $n \in \mathbb{N}$, $\mathfrak{F}_n$ is a constructible sheaf of $\mathbb{Z}/l^n\mathbb{Z}$-modules;

2. for all $n \in \mathbb{N}$ the map $f_{n+1}\colon \mathfrak{F}_{n+1} \longrightarrow \mathfrak{F}_n$ induces an isomorphism

$$\mathfrak{F}_{n+1} \otimes_{\mathbb{Z}/l^{n+1}\mathbb{Z}} \mathbb{Z}/l^n\mathbb{Z} \cong \mathfrak{F}_n.$$

Definition 1.5.3. A sheaf of $\mathbb{Z}_l$-modules $\mathfrak{F}$ on X is called *locally constant* if each $\mathfrak{F}_n$ is locally constant.

We can extend the previous definitions by passing to a finite field extension $K/\mathbb{Q}_l$, where denote by $\mathcal{O}_K$ and $\mathfrak{m}_K$ resp. the valuation ring of K and its maximal ideal. Then we make the exact same definitions

Definition 1.5.4. A sheaf of $\mathcal{O}_K$-modules $\mathfrak{F}$ on X is an inverse system

$$(\mathfrak{F}_n, f_{n+1} \colon \mathfrak{F}_{n+1} \longrightarrow \mathfrak{F}_n)_{n\in\mathbb{N}}$$

such that

1. for all $n \in \mathbb{N}$, $\mathfrak{F}_n$ is a constructible sheaf of $\mathcal{O}_K/\mathfrak{m}^n$-modules;

2. for all $n \in \mathbb{N}$ the map $f_{n+1} \colon \mathfrak{F}_{n+1} \longrightarrow \mathfrak{F}_n$ induces an isomorphism

$$\mathfrak{F}_{n+1} \otimes_{\mathcal{O}_K/\mathfrak{m}^{n+1}} \mathcal{O}_K/\mathfrak{m}^n \cong \mathfrak{F}_n.$$

Definition 1.5.5. A sheaf of $\mathcal{O}_K$-modules $\mathfrak{F}$ on X is called *locally constant* if each $\mathfrak{F}_n$ is locally constant.

The morphisms $Hom_{\mathcal{O}_K-sheaves}(\mathfrak{F}, \mathfrak{G})$ between sheaves of $\mathcal{O}_K$-modules are defined as the compatible systems of morphisms of sheaves of $\mathcal{O}_K/\mathfrak{m}^n$-modules for all $n \in \mathbb{N}$ giving an $\mathcal{O}_K$-module structure to $Hom_{\mathcal{O}_K-sheaves}(\mathfrak{F}, \mathfrak{G})$.

Then for each finite field extension $K/\mathbb{Q}_l$ we define the category of K-sheaves, whose objects are the sheaves of $\mathcal{O}_K$-modules and the morphisms are defined by

$$Hom_{K-sheaves}(\mathfrak{F}, \mathfrak{G}) := Hom_{\mathcal{O}_K-sheaves}(\mathfrak{F}, \mathfrak{G}) \otimes_{\mathcal{O}_K} K.$$

(We will sometimes use the notation $\mathfrak{F}^K$ for the image of a sheaf of $\mathcal{O}_K$-modules $\mathfrak{F}$ in the category of K-sheaves.)

If K runs through all the finite extensions of $\mathbb{Q}_l$, the categories of K-sheaves form a directed system, as for all inclusion $K \subset L$ an L-sheaf can be considered naturally as a K-sheaf as well, so we take the direct limit obtaining the category of $\overline{\mathbb{Q}}_l$-sheaves.

Definition 1.5.6. Let $\mathfrak{F}$ be a sheaf of $\mathcal{O}_K$-modules on X. Then the K-sheaf $\mathfrak{F}^K$ is said to be *locally constant*, if the sheaf of $\mathcal{O}_K$-modules $\mathfrak{F}$ is locally constant. A $\overline{\mathbb{Q}}_l$-sheaf $\mathfrak{F}$ on X is called *locally constant* if it is a direct limit of locally constant K-sheaves.

Finally we arrived to our first main object:

Definition 1.5.7. An *l-adic local system* $\mathfrak{L}$ on X is a locally constant $\overline{\mathbb{Q}}_l$-sheaf on X.

We can define stalks for l-adic local systems.

Definition 1.5.8. Let X be a connected scheme, $\mathfrak{L}$ a sheaf of $\mathcal{O}_K$-modules on X and $\bar{s}\colon Spec(\Omega) \longrightarrow X$ a geometric point. Then we define the stalk to be

$$\mathfrak{L}_{\bar{s}} := \varprojlim_{n \in \mathbb{N}} (\mathfrak{L}_n)_{\bar{s}}$$

where $(\mathfrak{L}_n)_{\bar{s}}$ are the étale stalks (II.2. in [Mil80]). For a K-sheaf $\mathfrak{L}^K$ we define the stalk to be

$$\mathfrak{L}^K_{\bar{s}} := \mathfrak{L}_{\bar{s}} \otimes_{\mathcal{O}_K} K.$$

To define the stalk of a $\overline{\mathbb{Q}}_l$-sheaf we just take the direct limit of the stalks of the K-sheaves defining the $\overline{\mathbb{Q}}_l$-sheaf.

For a locally constant $\overline{\mathbb{Q}}_l$-sheaf $\mathfrak{L}$ on a connected scheme X the stalk $\mathfrak{L}_{\bar{s}}$ is a finite dimensional $\overline{\mathbb{Q}}_l$-vector space whose dimension is called the *rank* of the l-adic local system $\mathfrak{L}$.

An étale sheaf $\mathfrak{F}$ on X is said to be *representable* if it is representable by an étale covering $U \longrightarrow X$ in Et/X

$$\mathfrak{F}(-) = Hom_X(-, U).$$

Constant étale sheaves are representable (Chapter 6 in [Mil08c]), hence locally constant (constructible) étale sheaves are also representable by an étale covering $U \longrightarrow X$ (3.18 Lemma in [FK88]). For a geometric point $\bar{s}\colon Spec(\Omega) \longrightarrow X$ the fibre $U_{\bar{s}} =$

$U \times_X Spec(\Omega)$ is canonically isomorphic to the stalk $\mathfrak{F}_{\bar{s}}$ and by definition has a natural continuous left action $\varrho\colon \pi_1(X,\bar{s}) \times \mathfrak{F}_{\bar{s}} \longrightarrow \mathfrak{F}_{\bar{s}}$. By Theorem 1.4.6 we get

Proposition 1.5.9. *The assignment $\mathfrak{F} \mapsto (\mathfrak{F}_{\bar{s}}, \varrho)$ establishes an equivalence between the category of locally constant (constructible) étale sheaves of abelian groups and the category of finite continuous $\pi_1(X,\bar{s})$-modules.*

Proof. A I.7 Proposition in [FK88]. □

By passing to the projective limits we get the main result we are seeking for

Theorem 1.5.10. *The assignment $\mathfrak{L} \mapsto (\mathfrak{L}_{\bar{s}}, \varrho)$ establishes an equivalence between the category of locally constant sheaves of $\mathbb{Z}_l$-modules and the category of finitely generated $\mathbb{Z}_l$-modules on which $\pi_1(X,\bar{s})$ acts continuously with respect to the l-adic topology.*

For a locally constant $\mathbb{Q}_l$-sheaf $\mathfrak{L}^{\mathbb{Q}_l}$ there is a continuous action of $\pi_1(X,\bar{s})$ on the stalk $\mathfrak{L}^{\mathbb{Q}_l}_{\bar{s}} = \mathfrak{L}_{\bar{s}} \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Then the functor $\mathfrak{L}^{\mathbb{Q}_l} \mapsto \mathfrak{L}^{\mathbb{Q}_l}_{\bar{s}}$ establishes an equivalence between the category of locally constant $\mathbb{Q}_l$-sheaves and the category of continuous representations of $\pi_1(X,\bar{s})$ on finite dimensional vector spaces over $\mathbb{Q}_l$.

Proof. A I.8 Proposition in [FK88] and Exposé VI., Proposition 1.2.5 in [Gro66]. □

The theorem remains true word by word if we pass to locally constant K-sheaves and hence also for l-adic local systems.

Specializing to our situation let C be a smooth projective, geometrically irreducible curve over a finite field k, l a prime number different from $char(k) = p$ and $\bar{s}\colon Spec(\Omega) \longrightarrow C$ a geometric point. Let $\mathfrak{L}$ be a 1-dimensional l-adic local system on C, hence the fibre $\mathfrak{L}_{\bar{s}}$ is a 1-dimensional vector space over $\overline{\mathbb{Q}}_l$. Let us fix an isomorphism $\varphi\colon \mathfrak{L}_{\bar{s}} \cong \overline{\mathbb{Q}}_l$. Then the action of $\pi_1(C, \bar{s})$ on $\mathfrak{L}_{\bar{s}}$ will be a continuous 1-dimensional l-adic representation $\chi : \pi_1(C, \bar{s}) \to \overline{\mathbb{Q}}_l^*$, which necessarily factors through the maximal abelian quotient $\pi_1^{ab}(C)$ (see Corollary 1.4.9 for notation).

Finally we get

Theorem 1.5.11. *There is a one-to-one correspondence between*

continuous 1*-dimensional* *l-adic representations of* $\pi_1^{ab}(C)$ $\{\chi : \pi_1^{ab}(C) \to \overline{\mathbb{Q}}_l^*\}$	$\overset{1:1}{\longleftrightarrow}$	1*-dimensional l-adic* *local systems $\mathfrak{L}$ on C* *together with a fixed* *isomorphism $\varphi : \mathfrak{L}_{\bar{s}} \cong \overline{\mathbb{Q}}_l$*

To finish this section we discuss the behaviour of l-adic local systems under basic operations, which we will need heavily in section 2.1.

Proposition 1.5.12. *Let $\mathfrak{L}$ and $\mathfrak{M}$ be l-adic local systems on a scheme X. Then the following hold true*

1. *if $f\colon Y \longrightarrow X$ is a morphism of schemes, then $f^{-1}\mathfrak{L}$ is also an l-adic local system on Y;*

2. *the direct sum $\mathfrak{L} \oplus \mathfrak{M}$ is also an l-adic local system on X;*

3. *the tensor product $\mathfrak{L} \otimes \mathfrak{M}$ is also an l-adic local system on X.*

Proof. (Sketch) Let l-adic local systems $\mathfrak{L}$ and $\mathfrak{M}$ be given as the direct limits $\varinjlim_K \mathfrak{L}_K$ and $\varinjlim_K \mathfrak{M}_K$ of locally constant K-sheaves. First we note that pull-back, direct sum and tensor product commute with direct limit, hence it is enough to prove the statements for locally constant K-sheaves. Such a locally constant K-sheaf is an inverse system of locally constant, constructible sheaves of $\mathcal{O}_K/\mathfrak{m}^n$-modules, so again we can restrict ourselves to prove the statements for ”ordinary” locally constant, constructible étale sheaves.

So let $\mathfrak{F}$ be a locally constant étale sheaf on X. Then there exists an étale covering $(U_i \longrightarrow X)_{i \in I}$ such that $\mathfrak{F}|_{U_i}$ is constant. As pull-backs of constant étale sheaves are obviously constant, we get that $f^{-1}\mathfrak{F}$ is constant on $Y \times_X U_i$, hence locally constant on Y. Let $\mathfrak{G}$ be another locally constant étale sheaf on X such that it is constant on the étale covering $(V_j \longrightarrow X)_{j \in J}$. As the direct sum and tensor product of constant sheaves are constant, the direct sum $\mathfrak{F} \oplus \mathfrak{G}$ and the tensor product $\mathfrak{F} \otimes \mathfrak{G}$ will be constant on the étale covering $(U_i \times_X V_j \longrightarrow X)_{i \in I, j \in J}$, hence locally constant on X.

Finally let $\mathfrak{F}$ and $\mathfrak{G}$ be constructible étale sheaves on X, i.e. for every closed immersion $j \colon Z \hookrightarrow X$ there exists open subsets $U \subset Z$ and $V \subset Z$ such that the restrictions $\mathfrak{F}|_U$ and $\mathfrak{G}|_V$ are locally constant on U and V resp. We can reduce to the case that X is irreducible. Then using the above arguments we see that on the open subset $U \times_Z V \subset Z$ the direct sum $\mathfrak{F} \oplus \mathfrak{G}$ and the tensor product $\mathfrak{F} \otimes \mathfrak{G}$ will be locally constant. As what pull-backs concern let $f \colon Y \longrightarrow X$ be a morphism and $j \colon T \hookrightarrow Y$ be a closed immersion. Take the closure $\overline{T}$ of the image of T in X, and an open subset $U \subset \overline{T}$ such that $\mathfrak{F}|_U$ is locally constant.

$$\begin{array}{ccc} T & \xrightarrow{f_T := f \circ j} & \overline{T} \supset U \\ {\scriptstyle j}\big\downarrow & & \big\downarrow \\ Y & \xrightarrow{\quad f \quad} & X \end{array}$$

Then we get that $f^{-1}\mathfrak{F}$ is locally constant on $f_T^{-1}U$. □

1.6 The Faisceaux-Fonctions Correspondence

In this section we will discuss Grothendieck's function-sheaf correspondence and make major steps toward establishing the connection appearing in the right vertical side of diagram 1.1.2, which will be accomplished in section 2.2. The main references are [Sommes trig.] in [Del77], [Gai04] and [Shi05].

First we need to define the Frobenius action on local systems.

Definition 1.6.1 (construction of the Frobenius action)**.** Let X be a connected scheme over a finite field $k = \mathbb{F}_q$ and $\mathfrak{L}$ be an l-adic local system of rank r on X. Let $x\colon Spec(\mathbb{F}_{q^n}) \longrightarrow X$ be a closed point and $\bar{x}\colon Spec(\overline{\mathbb{F}}_q) \longrightarrow X$ a geometric point lying above it. Then we have isomorphisms

$$\overline{\mathbb{Q}}_l^r \cong \mathfrak{L}_{\bar{x}} \cong \bar{x}^{-1}\mathfrak{L}$$

where the last space is a discrete $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_{q^n})$-module by Chapter II., Theorem 1.9 in [Mil80]. The automorphism in $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_{q^n})$ defined by

$$Frob^n\colon x \mapsto x^{q^n}$$

is called the n^{th} power of the *arithmetic Frobenius* and it defines an automorphism of the r-dimensional $\overline{\mathbb{Q}}_l$-vector space $\mathfrak{L}_{\bar{x}}$ denoted by $Frob_x$. This whole construction depends on the choice of a geometric point and hence only well-defined up to inner automorphism by Corollary 1.4.9, but the trace of the Frobenius is a well-defined element in $\overline{\mathbb{Q}}_l$ and denoted by $tr_{\mathfrak{L}}(Frob_x)$.

Let G be a connected, separated, commutative group scheme of finite type defined over a finite field $k = \mathbb{F}_q$. As usual let l be a prime number different from $char(k) = p$ and denote by $m\colon G \times G \longrightarrow G$, $i\colon G \longrightarrow G$ and $e\colon Spec(k) \longrightarrow G$ the multiplication map, the inverse map and the identity map respectively, where the image of the identity will be also denoted by $0 \in G$.

Definition 1.6.2. An *l-adic character sheaf* $\mathfrak{L}$ on G is a 1-dimensional l-adic local system on G together with a trivialization $\mathfrak{L}_0 \cong \overline{\mathbb{Q}}_l$ satisfying $m^{-1}\mathfrak{L} \cong \mathfrak{L} \boxtimes \mathfrak{L}$.

The main theorem of this section is the following:

Theorem 1.6.3. *Let G be a connected, separated, commutative group scheme of finite type defined over a finite field $k = \mathbb{F}_q$. Then there is a one-to-one correspondence between*

$$\boxed{\begin{array}{c} l\text{-adic character} \\ \text{sheaves } \mathfrak{L} \text{ on } G \end{array}} \xleftrightarrow{\ 1:1\ } \boxed{\begin{array}{c} \text{group homomorphisms} \\ \chi\colon G(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_l^{\times} \end{array}}$$

Proof. First let an l-adic character sheaf $\mathfrak{L}$ be given. For a point $x \in G(\mathbb{F}_q)$ we defined the Frobenius action $Frob_x$ on the fibre $\mathfrak{L}_{\bar{x}}$ which is a 1-dimensional vector space over $\overline{\mathbb{Q}}_l$ and we define

$$\chi(x) := tr_{\mathfrak{L}}(Frob_x).$$

We use the following properties:

1. $tr_{\mathfrak{L}\oplus\mathfrak{K}} = tr_{\mathfrak{L}} + tr_{\mathfrak{K}}$;

2. $tr_{\mathfrak{L}\otimes\mathfrak{K}} = tr_{\mathfrak{L}} tr_{\mathfrak{K}}$;

3. for a morphism $f\colon X \longrightarrow Y$ we get $tr_{f^{-1}(\mathfrak{L})} = f^{-1} tr_{\mathfrak{L}}$.

We have to prove that χ is a group homomorphism. From the character sheaf property we have that $\mathfrak{L}_{\overline{x+y}} \cong \mathfrak{L}_{\bar{x}} \otimes \mathfrak{L}_{\bar{y}}$, hence using the above properties we get that

$$tr_{\mathfrak{L}}(Frob_{x+y}) = tr_{\mathfrak{L}}(Frob_x) tr_{\mathfrak{L}}(Frob_y)$$

that is $\chi(x+y) = \chi(x)\chi(y)$.

($\Leftarrow$) First we define the absolute Frobenius.

Definition 1.6.4. Let G be a connected group scheme defined over a finite field $k = \mathbb{F}_q$. The *absolute Frobenius* is the scheme morphism over $Spec(\mathbb{F}_q)$

$$Fr_q \colon (G, \mathcal{O}_G) \longrightarrow (G, \mathcal{O}_G)$$

defined as the identity map on the underlying topological space and as $g \mapsto g^q$ on the structure sheaf of rings $\mathcal{O}_G$.

Now we recall the definition of the Lang isogeny (Chapter VI., §1. in [Ser88]).

Definition 1.6.5. Let G be a connected, commutative group scheme of finite type defined over a finite field $k = \mathbb{F}_q$. The *Lang isogeny* φ is the composite map

$$\begin{array}{c} \overset{\varphi}{\overbrace{\hphantom{G \xrightarrow{(Fr_q, i)} G \times G \xrightarrow{m} G}}} \\ G \xrightarrow[(Fr_q, i)]{} G \times G \xrightarrow{m} G \end{array}$$

that is given by $g \mapsto Fr_q(g) - g$ on the functor of points.

The Lang isogeny is a group homomorphism because G is commutative implying that m and i are both group homomorphisms and Fr_q is a group homomorphism. There is an exact sequence

$$0 \longrightarrow G(\mathbb{F}_q) \longrightarrow G \xrightarrow{\varphi} G \longrightarrow 0$$

which makes G into a finite, étale, Galois covering of itself via the Lang isogeny φ with Galois group of the group of translations $G(\mathbb{F}_q)$ (VI.1. Proposition 3. in [Ser88] and p.7. in [Shi05]).

Now let a group homomorphism $\chi\colon G(\mathbb{F}_q) \longrightarrow \overline{\mathbb{Q}}_l^\times$ be given. Then first to give a 1-dimensional l-adic local system $\mathfrak{L}$ on G is the same as giving a continuous, 1-dimensional l-adic representation $\pi_1^{ab}(G) \longrightarrow \overline{\mathbb{Q}}_l^\times$ (section 1.5). But as $\varphi\colon G \longrightarrow G$ is a finite, étale, Galois covering with Galois group $G(\mathbb{F}_q)$, there exists a natural surjection

$$\psi\colon \pi_1^{ab}(G) \longrightarrow\!\!\!\!\rightarrow G(\mathbb{F}_q)$$

by Example 1.4.12. The composite map $\chi \circ \psi\colon \pi_1^{ab}(G) \longrightarrow \overline{\mathbb{Q}}_l^\times$ gives us a 1-dimensional l-adic local system $\mathfrak{L}$ on G. Note that by construction $\varphi^{-1}\mathfrak{L}$ is a constant $\overline{\mathbb{Q}}_l$-sheaf, as to $\varphi^{-1}\mathfrak{L}$ corresponds the trivial l-adic representation by the above Lang isogeny exact sequence. We have to prove that $\mathfrak{L}$ satisfies the character sheaf property $m^{-1}\mathfrak{L} \cong \mathfrak{L} \boxtimes \mathfrak{L}$. For that we consider the following commutative diagram

$$\begin{array}{ccc} G \times G & \xrightarrow{\ m\ } & G \\ \big\downarrow{\scriptstyle (\varphi,\varphi)} & & \big\downarrow{\scriptstyle \varphi} \\ G \times G & \xrightarrow{\ m\ } & G \end{array}$$

The commutativity is ensured by the fact that the absolute Frobenius commutes with arbitrary morphisms (cf. 3.2.4 Lemma 2.22 in [Liu02]). As $\varphi^{-1}\mathfrak{L}$ is a constant $\overline{\mathbb{Q}}_l$-sheaf on G, then

$$m^{-1}\varphi^{-1}\mathfrak{L} = (\varphi,\varphi)^{-1}m^{-1}\mathfrak{L}$$

is also a constant $\overline{\mathbb{Q}}_l$-sheaf on $G \times G$. Similarly

$$\varphi^{-1}\mathfrak{L} \boxtimes \varphi^{-1}\mathfrak{L} = (\varphi,\varphi)^{-1}\mathfrak{L} \boxtimes \mathfrak{L}$$

is a constant $\overline{\mathbb{Q}}_l$-sheaf on $G \times G$. So to prove the character sheaf property we only need to show that the Galois group $G(\mathbb{F}_q) \times G(\mathbb{F}_q)$ of $G \times G \overset{(\varphi,\varphi)}{\longrightarrow} G \times G$ acts on the same way on the sheaves $(\varphi,\varphi)^{-1}m^{-1}\mathfrak{L}$ and $(\varphi,\varphi)^{-1}\mathfrak{L}\boxtimes\mathfrak{L}$. For that it is enough to check the action on the stalks at the geometric point $\overline{(0,0)}$ lying over $(0,0)$. These stalks are the same as the stalks of $m^{-1}\mathfrak{L}$ and $\mathfrak{L}\boxtimes\mathfrak{L}$ at $\overline{(0,0)}$. We have also that $m^{-1}\mathfrak{L}_{\overline{(0,0)}} \cong \mathfrak{L}_{\overline{0}}$. So let $(g,h) \in G(\mathbb{F}_q)\times G(\mathbb{F}_q)$ be given. The action of (g,h) is transformed via m to the action of $g+h \in G(\mathbb{F}_q)$ by the commutativity of the above diagram. But the action of $g+h$ is just multiplication by $\chi(g+h)$. Also by similar arguments the action of (g,h) on $\mathfrak{L}\boxtimes\mathfrak{L}$ is given by multiplication by $\chi(g)\chi(h)$. Hence (g,h) acts both on $m^{-1}\mathfrak{L}$ and $\mathfrak{L}\boxtimes\mathfrak{L}$ by multiplication by $\chi(g+h)=\chi(g)\chi(h)$, as wanted.

These two constructions are inverses to each other completing the proof of the Theorem. $\square$

We will use this result in section 2.2 applying to the Picard scheme Pic_C^0 of degree 0 finishing the proof of the connection appearing in the right vertical side of diagram 1.1.2.

Chapter 2

The Proof of the Main Theorem of the Unramified Theory

In this chapter we will begin with final preliminary works proving the correspondences appearing in diagram 1.1.2 and after that we will give Deligne's geometric proof of the Main Theorem 1.1.4.

2.1 Preliminary Constructions

Let C be a smooth, projective, geometrically irreducible curve of genus g over a finite field $k = \mathbb{F}_q$. Let us begin with building up the connection between 1-dimensional l-adic local systems on the curve C and 1-dimensional l-adic local systems on the Picard scheme Pic_C appearing in the bottom row of diagram 1.1.2.

So let $\mathfrak{L}$ be a 1-dimensional l-adic local system on C. We will construct a 1-dimensional l-adic local system $\mathfrak{L}^{(d)}$ on $C^{(d)}$ using the following construction. First we define the l-adic local system $\mathfrak{L}^{\boxtimes d}$ on C^d by

$$\mathfrak{L}^{\boxtimes d} := \bigotimes_{i=1}^{d} pr_i^{-1} \mathfrak{L}$$

using the natural projections $pr_i : C^d \to C$ for $i = 1, 2, \ldots, d$. This is an l-adic local system by Proposition 1.5.12. Next we want to analyze the stalks of this l-adic local system. To do this, we can restrict ourselves to the case of locally constant, constructible étale sheaves using the reduction argument in Proposition 1.5.12. Hence in the following we assume that $\mathfrak{L}$ is a locally constant, constructible étale sheaf of rank 1.[1]

Now by definition we have that the stalk at a geometric point $\underline{p} = (p_1, p_2, \ldots, p_d) \in C^d$ is

$$(\mathfrak{L}^{\boxtimes d})_{\underline{p}} = \bigotimes_{i=1}^{d} \mathfrak{L}_{p_i}.$$

Also we have that for a $\sigma \in S_d$ the stalk at $\sigma\underline{p} = (p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(d)})$ is

$$(\mathfrak{L}^{\boxtimes d})_{\sigma\underline{p}} = \bigotimes_{i=1}^{d} \mathfrak{L}_{p_{\sigma(i)}}.$$

But we have an isomorphism

$$\psi_{\underline{p},\sigma} : \bigotimes_{i=1}^{d} \mathfrak{L}_{p_i} \xrightarrow{\cong} \bigotimes_{i=1}^{d} \mathfrak{L}_{p_{\sigma(i)}}$$

by the commutativity of tensor product. This means that $\mathfrak{L}^{\boxtimes d}$ is an S_d-equivariant locally constant étale sheaf on C^d.

Now we need to descend to the quotient $C^{(d)}$ and investigate how sheaves behave:

Proposition 2.1.1. *Let $\mathfrak{L}$ be a locally constant étale sheaf of rank 1 on C and consider the symmetrization morphism $\pi : C^d \to C^{(d)}$ for an integer $d \geq 1$. Then the étale sheaf $\mathfrak{L}^{(d)} := (\pi_* \mathfrak{L}^{\boxtimes d})^{S_d}$ of S_d-invariants of the push-forward of $\mathfrak{L}^{\boxtimes d}$ is a locally constant étale sheaf of rank 1 on $C^{(d)}$.*

[1] Of course Proposition 1.5.12 holds true a fortiori in that case.

Proof. First we note that in general for a morphism $f : X \to Y$ of schemes and an étale sheaf $\mathfrak{F}$ on X there is a natural morphism $f^{-1}f_*\mathfrak{F} \to \mathfrak{F}$. If a finite group G acts on X and the sheaf $\mathfrak{F}$ is G-equivariant then there is a natural action of G on the push-forward $f_*\mathfrak{F}$. Hence the inclusion of sheaves $(f_*\mathfrak{F})^G \to f_*\mathfrak{F}$ on Y induces a morphism of sheaves $f^{-1}(f_*\mathfrak{F})^G \to f^{-1}f_*\mathfrak{F}$ on X such that the composition

$$f^{-1}(f_*\mathfrak{F})^G \longrightarrow f^{-1}f_*\mathfrak{F} \longrightarrow \mathfrak{F}$$

gives us a morphism of sheaves $f^{-1}(f_*\mathfrak{F})^G \to \mathfrak{F}$ on X. Now in the case of our proposition we have

Claim 2.1.2. The natural morphism of étale sheaves $\pi^{-1}(\pi_*\mathfrak{L}^{\boxtimes d})^{S_d} \to \mathfrak{L}^{\boxtimes d}$ on C^d is an isomorphism.

Proof. To prove the claim first we consider the stalks of the étale sheaves $\pi_*\mathfrak{L}^{\boxtimes d}$ and $(\pi_*\mathfrak{L}^{\boxtimes d})^{S_d}$ on $C^{(d)}$. As the symmetrization morphism is a finite, hence proper, surjective map, we can assume that a geometric point in $C^{(d)}$ is given by

$$\pi(\underline{p}) := \pi((p_1, p_2, \ldots, p_d)) \in C^{(d)}$$

where $\underline{p}$ is a geometric point of C^d. Then it follows that

$$\pi^{-1}\pi(\underline{p}) = S_d\,\underline{p} = \{(p_{\sigma(1)}, p_{\sigma(2)}, \ldots, p_{\sigma(d)} |\ \sigma \in S_d)\}$$

i.e. the orbit of $\underline{p}$ under the action of S_d. Now consider the following commutative diagram:

$$\begin{array}{ccc} \pi^{-1}\pi(\underline{p}) & \xrightarrow{j'} & C^d \\ \downarrow{\scriptstyle \pi'} & & \downarrow{\scriptstyle \pi} \\ \pi(\underline{p}) & \xrightarrow{j} & C^{(d)} \end{array}$$

where the j and j' are the inclusion maps and π' is the first projection of the fibre product. By the Proper Base Change Theorem (Theorem 17.10 in [Mil08c]) we have a canonical isomorphism

$$j^{-1}\pi_*\mathfrak{L}^{\boxtimes d} \xrightarrow{\cong} \pi'_*(j')^{-1}\mathfrak{L}^{\boxtimes d}.$$

Now we will use the following theorem:

Theorem 2.1.3. *Let $\pi : X \longrightarrow Y$ be a morphism of schemes.*

For any étale sheaf $\mathfrak{F}$ on Y and any geometric point $\bar{x} \in X$ we have

$$(\pi^{-1}\mathfrak{F})_{\bar{x}} \cong \mathfrak{F}_{\pi(\bar{x})}.$$

If $\pi : X \longrightarrow Y$ is a finite morphism and $\mathfrak{G}$ is an étale sheaf on X, then for any geometric point $\bar{y} \in Y$ we have

$$(\pi_*\mathfrak{G})_{\bar{y}} = \textstyle\prod_{\pi(\bar{x})=\bar{y}} \mathfrak{G}_{\bar{x}}.$$

Proof. The first statement (a) is proved in Ch.II.Theorem 3.2.(a) in [Mil80]. The second statement (b) is proved in Ch.II.Corollary 3.5.(c) in [Mil80]. □

By this theorem we have that

$$j^{-1}\pi_*\mathfrak{L}^{\boxtimes d} = (\pi_*\mathfrak{L}^{\boxtimes d})_{\pi(\underline{p})}$$

and also that

$$\pi'_*(j')^{-1}\mathfrak{L}^{\boxtimes d} = \textstyle\prod_{\underline{p}' \in S_d\underline{p}} (\mathfrak{L}^{\boxtimes d})_{\underline{p}'}.$$

If $\underline{p}' = (p_1', p_2', \dots, p_d') \in S_d\,\underline{p}$ then using the property of external tensor products above we have that

$$(\pi_*\mathfrak{L}^{\boxtimes d})_{\pi(\underline{p})} = \prod_{\underline{p}'\in S_d\underline{p}} \bigotimes_{i=1}^{d} \mathfrak{L}_{p_i'}.$$

A permutation $\sigma \in S_d$ acts on the stalk in the following way:

$$\sigma(\pi_*\mathfrak{L}^{\boxtimes d})_{\pi(\underline{p})} = \prod_{\underline{p}'\in S_d\underline{p}} \bigotimes_{i=1}^{d} \mathfrak{L}_{p'_{\sigma(i)}}$$

hence we get for the S_d-invariants

$$(\pi_*\mathfrak{L}^{\boxtimes d})_{\pi(\underline{p})}^{S_d} \cong \bigotimes_{i=1}^{d} \mathfrak{L}_{p_i}.$$

Using again the property of the pull-back on stalks (Theorem 2.1.3 (a)) we have for any $\underline{p}' \in \pi^{-1}\pi(\underline{p})$ that

$$\pi^{-1}((\pi_*\mathfrak{L}^{\boxtimes d})^{S_d})_{\underline{p}'} = (\pi_*\mathfrak{L}^{\boxtimes d})_{\pi(\underline{p})}^{S_d}$$

which is

$$\pi^{-1}((\pi_*\mathfrak{L}^{\boxtimes d})^{S_d})_{\underline{p}'} = \bigotimes_{i=1}^{d} \mathfrak{L}_{p_i} \cong \mathfrak{L}_{\underline{p}'}^{\boxtimes d}$$

hence these isomorphisms on the stalks induce an isomorphism between the étale sheaves on C^d

$$\pi^{-1}(\mathfrak{L}^{(d)}) \xrightarrow{\cong} \mathfrak{L}^{\boxtimes d}$$

completing the proof of the claim. □

It remained to prove that $\mathfrak{L}^{(d)}$ is a locally constant étale sheaf on $C^{(d)}$. We note that $\mathfrak{L}^{\boxtimes d}$ being locally constant is representable by an étale covering. Then as the push-forward and a subsheaf of a representable sheaf is representable, we get that at least $\mathfrak{L}^{(d)}$ is representable by a scheme Z on $C^{(d)}$. By Theorem 2.1.3 (a) we know that for any geometric point $\pi(\underline{p}) \in C^{(d)}$ the stalks are isomorphic $\pi^{-1}(\mathfrak{L}^{(d)})_{\underline{p}} \xrightarrow{\cong} \mathfrak{L}^{(d)}_{\pi(\underline{p})}$.

Hence there exists an étale neighborhood U of $\pi(\underline{p})$ and a section $s\colon U \longrightarrow Z$ which generates the stalk $\mathfrak{L}^{(d)}_{\pi(\underline{p})}$. The pull-back of this section must generate all stalks of $\pi^{-1}(\mathfrak{L}^{(d)})$ in an étale neighborhood of $\underline{p} \in C^d$, as $\pi^{-1}(\mathfrak{L}^{(d)})$ is a locally constant étale sheaf on C^d. As the stalks of the sheaf $\mathfrak{L}^{(d)}$ are 1-dimensional and the pull-back of the section generates the stalks $\pi^{-1}(\mathfrak{L}^{(d)})_{\underline{p}}$ in some étale neighborhood, it follows that the section must generate the stalks $\mathfrak{L}^{(d)}_{\pi(\underline{p})}$ in some étale neighborhood of $\pi(\underline{p})$ as well, i.e. the sheaf $\mathfrak{L}^{(d)}$ becomes constant on an étale covering. This completes the proof of the Proposition. □

Recall that in section 1.3 we have defined the Abel-Jacobi natural transformation of the functors Div^d_C and Pic^d_C for an integer $d \geq 1$ and indicated also that we can assume that C has a k-rational point, consequently the functors are representable inducing a morphism of schemes (using the same notation for the Picard scheme)

$$\mathfrak{AJ}_d\colon C^{(d)} \longrightarrow Pic^d_C$$

such that on the geometric points the fibre of the Abel-Jacobi map over the class of an invertible sheaf $\mathcal{G}$ of degree d on C is isomorphic to

$$\mathfrak{AJ}_d^{-1}([\mathcal{G}]) \cong \mathbb{P}H^0(C, \mathcal{G})$$

by Proposition 1.2.6.

It follows that $\mathbb{P}H^0(C, \mathcal{G}) \neq \emptyset$ if and only if $h^0(C, \mathcal{G}) := \dim_k H^0(C, \mathcal{G}) \geq 1$. By the Riemann-Roch Theorem we have that

$$h^0(C, \mathcal{G}) - h^0(C, \omega_C \otimes \mathcal{G}^{-1}) = d - g + 1$$

hence our condition is equivalent to

$$h^0(C, \omega_C \otimes \mathcal{G}^{-1}) + d - g \geq 0.$$

This is certainly the case if $d \geq g$. But we can say more:

Proposition 2.1.4. *If the degree d satisfies $d \geq 2g - 1$ whenever $g \geq 1$ (or $d \geq g$ whenever $g = 0$) then the Abel-Jacobi map $\mathfrak{AJ}_d : C^{(d)} \longrightarrow Pic_C^d$ is a proper, surjective morphism with geometric fibres isomorphic to the projective space $\mathbb{P}_{k^s}^{d-g}$.*

Proof. It follows from the above considerations that if $h^0(C, \omega_C \otimes \mathcal{G}^{-1}) = 0$, which is the case exactly if the degree of $\mathcal{G}$ satisfies $d \geq 2g - 1$, and the fibre is not empty $\mathfrak{AJ}_d^{-1}(\mathcal{G}) \neq \emptyset$, i.e. $d \geq g$, then putting together the two conditions we have that if $d \geq \max\{2g-1, g\}$ then the fibres $\mathfrak{AJ}_d^{-1}(\mathcal{G})$ will be all isomorphic to the projective space of dimension $d-g$ via the isomorphisms $\mathfrak{AJ}_d^{-1}(\mathcal{G}) \cong \mathbb{P}H^0(C, \mathcal{G})$. The properness follows from the fact the $C^{(d)}$ is projective. $\square$

2.2 Deligne's Geometric Proof

In this section we turn to the proof of the Main Theorem 1.1.4 using a geometric argument of Deligne. The main references are [Lau90], [Hei07] and [Hei04].

First we prove

Theorem 2.2.1. *(Deligne's Theorem): There is a one-to-one correspondence between*

<table><tr><td>1-dimensional l-adic
local systems $\mathfrak{L}$ on C
together with a fixed
isomorphism $\varphi : \mathfrak{L}_{\bar{s}} \cong \overline{\mathbb{Q}}_l$</td><td>$\overset{1:1}{\longleftrightarrow}$</td><td>1-dimensional l-adic
local systems $\mathcal{A}_{\mathfrak{L}}$ on
Pic_C together with
a fixed isomorphism
$\psi : \mathcal{A}_{\mathfrak{L}}|_0 \cong \overline{\mathbb{Q}}_l$ satisfying
$m^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e} = \mathcal{A}_{\mathfrak{L}}^d \boxtimes \mathcal{A}_{\mathfrak{L}}^e$</td></tr></table>

Proof. Let be given a 1-dimensional l-adic local system $\mathfrak{L}$ on C with a fixed isomorphism $\varphi : \mathfrak{L}_{\bar{s}} \cong \overline{\mathbb{Q}}_l$ where $\bar{s}$: $Spec(\Omega) \longrightarrow C$ is a geometric point. Then for an integer $d \geq 1$ we constructed a 1-dimensional l-adic local system $\mathfrak{L}^{(d)} := (\pi_* \mathfrak{L}^{\boxtimes d})^{S_d}$ on $C^{(d)}$ in the previous section 2.1. It follows from the correspondence between l-adic local systems and representations of the fundamental group (section 1.5) that there is associated to $\mathfrak{L}^{(d)}$ a unique continuous 1-dimensional l-adic representation $\rho_{\mathfrak{L}^{(d)}} : \pi_1^{ab}(C^{(d)}) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$. Now we assume that $d \geq 2g-1$ and apply the relative homotopy exact sequence theorem (1.4.13) to the Abel-Jacobi map $\mathfrak{AJ}_d : C^{(d)} \longrightarrow Pic_C^d$ taking into account Proposition 2.1.4 to get the exact sequence

$$\pi_1(\mathbb{P}_{k^s}^{d-g}, \bar{s}) \longrightarrow \pi_1(C^{(d)}, \bar{s}) \longrightarrow \pi_1(Pic_C^d, \bar{s}) \longrightarrow 1$$

where k^s is the separable closure of the base field k. We know from Example 1.4.12 that $\pi_1(\mathbb{P}_{k^s}^{d-g}, \bar{s}) = 1$ is trivial, hence we get an isomorphism

$$\pi_1(C^{(d)}, \bar{s}) \cong \pi_1(Pic_C^d, \bar{s})$$

which induces an isomorphism between the abelianized étale fundamental groups

$$\pi_1^{ab}(C^{(d)}) \cong \pi_1^{ab}(Pic_C^d).$$

This means that to the continuous 1-dimensional l-adic representation

$$\rho_{\mathfrak{L}^{(d)}} : \pi_1^{ab}(C^{(d)}) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$$

corresponds a unique continuous 1-dimensional l-adic representation

$$\rho_{\mathcal{A}_{\mathfrak{L}}^d} : \pi_1^{ab}(Pic_C^d) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$$

which again corresponds (section 1.5) to a unique 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}^d$ on Pic_C^d together with a fixed isomorphism $\psi\colon \mathcal{A}_{\mathfrak{L}}^d|_{d0} \cong \overline{\mathbb{Q}}_l$.

For the 1-dimensional l-adic local systems $\mathcal{A}_{\mathfrak{L}}^d$ we have the following

Proposition 2.2.2. *The* 1*-dimensional l-adic local systems $\mathcal{A}_{\mathfrak{L}}^d$ satisfy*

$$(+)^{-1}\mathcal{A}_{\mathfrak{L}}^{d+1} = \mathfrak{L} \boxtimes \mathcal{A}_{\mathfrak{L}}^d.$$

Proof. Let us consider the following commutative diagram

$$\begin{array}{ccc}
(p, D) & \longmapsto & (p + D) \\
C \times C^{(d)} & \xrightarrow[\cong]{\tilde{+}} & C^{(d+1)} \\
\Big\downarrow{\scriptstyle id \times \mathfrak{AJ}_{(d)}} & & \Big\downarrow{\scriptstyle \mathfrak{AJ}_{(d+1)}} \\
C \times Pic_C^d & \xrightarrow{+} & Pic_C^{d+1} \\
(p, \mathcal{G}) & \longmapsto & \mathcal{O}(p) \otimes \mathcal{G}\ .
\end{array}$$

Because of the correspondence between $\mathcal{A}_{\mathfrak{L}}^d$ and $\mathfrak{L}^{(d)}$ constructed above we just have to prove that $(\tilde{+})^{-1}\mathfrak{L}^{(d+1)} = \mathfrak{L} \boxtimes \mathfrak{L}^{(d)}$. For this we consider the diagram

$$\begin{array}{ccc} C \times C^d & \xrightarrow{\tilde{\tilde{+}}} & C^{d+1} \\ \downarrow{\scriptstyle id \times \pi} & & \downarrow{\scriptstyle \pi} \\ C \times C^{(d)} & \xrightarrow[\cong]{\tilde{+}} & C^{(d+1)} \end{array}.$$

Using the Claim 2.1.2 we have an isomorphism $\pi^{-1}\mathfrak{L}^{(d+1)} \cong \mathfrak{L}^{\boxtimes(d+1)}$. Now as $\mathfrak{L}^{\boxtimes d}$ is a local system we get that

$$(\tilde{\tilde{+}})^{-1}\mathfrak{L}^{\boxtimes(d+1)} = \mathfrak{L} \boxtimes \mathfrak{L}^{\boxtimes d}.$$

On the other hand we also have by the above Claim 2.1.2 that $(id \times \pi)^{-1}(\mathfrak{L} \boxtimes \mathfrak{L}^{(d)}) \cong \mathfrak{L} \boxtimes \mathfrak{L}^{\boxtimes d}$ and both are S_d-equivariant, hence we get that

$$(\tilde{+})^{-1}\mathfrak{L}^{(d+1)} = \mathfrak{L} \boxtimes \mathfrak{L}^{(d)}$$

completing the proof of the Proposition. □

Using this we can extend the construction of $\mathcal{A}^d_{\mathfrak{L}}$ for every $d \in \mathbb{Z}$, hence to all of Pic_C in the following way.

We assume that we have a k-rational point $p \in C(k)$. Then for an integer $d \leq 2g-2$ we choose a positive integer N such that $d + N \geq 2g - 1$ and we consider the isomorphism

$$Np \colon Pic^d_C \xrightarrow{\cong} Pic^{d+N}_C$$

induced by successively applying the map

$$+ \colon p \times Pic^d_C \longrightarrow Pic^{d+1}_C.$$

Now we can define $\mathcal{A}^d_{\mathfrak{L}}$ on Pic^d_C for any $d \in \mathbb{Z}$ using the property in Proposition 2.2.2 by

$$\mathcal{A}_{\mathfrak{L}}^{d} := (Np)^{-1}\mathcal{A}_{\mathfrak{L}}^{d+N} \otimes \mathfrak{L}_p^{\otimes -N}.$$

We have to prove the multiplicative property of $\mathcal{A}_{\mathfrak{L}}$, i.e. that for the multiplication map

$$m\colon Pic_C^d \times Pic_C^e \longrightarrow Pic_C^{d+e}$$

defined by

$$(\mathcal{G}, \mathcal{H}) \mapsto \mathcal{G} \otimes \mathcal{H}$$

we have that

$$m^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e} \cong \mathcal{A}_{\mathfrak{L}}^{d} \boxtimes \mathcal{A}_{\mathfrak{L}}^{e}.$$

For that we consider the following commutative diagram

$$\begin{array}{ccc} p \times Pic_C^d \times Pic_C^e & \xrightarrow[\cong]{+ \times id} & Pic_C^{d+1} \times Pic_C^e \cong Pic_C^0 \times Pic_C^0 \\ \Big\downarrow {\scriptstyle id \times m} & & \Big\downarrow {\scriptstyle pr_1 \otimes pr_2} \\ p \times Pic_C^{d+e} & \xrightarrow[\cong]{+} & Pic_C^{d+e+1} \cong Pic_C^0 \ . \end{array}$$

Then we start with $\mathcal{A}_{\mathfrak{L}}^{d+e+1}$ on Pic_C^{d+e+1} satisfying $+^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e+1} \cong \mathfrak{L}_p \otimes \mathcal{A}_{\mathfrak{L}}^{d+e}$. The pull-back of the latter on the left vertical side is

$$(id, m)^{-1}(\mathfrak{L}_p \otimes \mathcal{A}_{\mathfrak{L}}^{d+e}) = \mathfrak{L}_p \otimes m^{-1}(\mathcal{A}_{\mathfrak{L}}^{d+e}).$$

Under the isomorphism $Pic_C^{d+e+1} \cong Pic_C^0$ the local system $\mathcal{A}_{\mathfrak{L}}^{d+e+1}$ goes to $\mathcal{A}_{\mathfrak{L}}^{d+e+1} \otimes \mathfrak{L}_p^{\otimes -d-e-1}$ which is just $\mathcal{A}_{\mathfrak{L}}^{0}$ and it pulls back to $Pic_C^0 \times Pic_C^0$ along the right vertical side to

$$\mathcal{A}_{\mathfrak{L}}^{d+e+1} \otimes \mathfrak{L}_p^{\otimes -d-e-1} \boxtimes \mathcal{A}_{\mathfrak{L}}^{d+e+1} \otimes \mathfrak{L}_p^{\otimes -d-e-1}$$

which again goes back under the isomorphism $Pic_C^{d+1} \times Pic_C^e \cong Pic_C^0 \times Pic_C^0$ to

$$\mathcal{A}_{\mathfrak{L}}^{d+e+1} \otimes \mathfrak{L}_p^{\otimes -e} \boxtimes \mathcal{A}_{\mathfrak{L}}^{d+e+1} \otimes \mathfrak{L}_p^{\otimes -d-1}.$$

Using successively the property $+^{-1}\mathcal{A}_{\mathfrak{L}}^{d+1} \cong \mathfrak{L}_p \otimes \mathcal{A}_{\mathfrak{L}}^d$ we get

$$\mathfrak{L}_p \otimes \mathcal{A}_{\mathfrak{L}}^d \boxtimes \mathcal{A}_{\mathfrak{L}}^e$$

hence we get the isomorphism

$$m^{-1}(\mathcal{A}_{\mathfrak{L}}^{d+e}) \cong \mathcal{A}_{\mathfrak{L}}^d \boxtimes \mathcal{A}_{\mathfrak{L}}^e.$$

On the other way round let a rigidified multiplicative 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}$ on Pic_C be given. Then we consider the Abel-Jacobi map

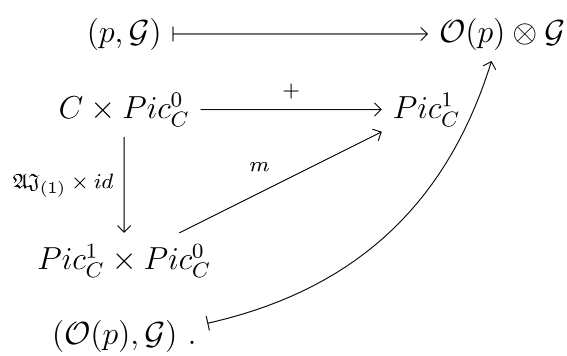
$$\mathfrak{AJ}_{(1)} \colon C \longrightarrow Pic_C^1$$

and by Proposition 1.5.12 the pull-back

$$\mathfrak{L} := \mathfrak{AJ}_{(1)}^{-1}\mathcal{A}_{\mathfrak{L}}^1$$

is a 1-dimensional l-adic local system on C together with a rigidification.

Finally we have to prove that these two constructions are inverses to each other. For both we consider the commutative diagram



Then given a rigidified, 1-dimensional l-adic local system $\mathfrak{L}$ on C, we can associate to it a rigidified, multiplicative 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}$ on Pic_C, such that we define $\mathfrak{L}' = \mathfrak{AJ}_{(1)}^{-1}\mathcal{A}_{\mathfrak{L}}^1$. Then using the above diagram

$$\begin{array}{ccc} \mathfrak{L}' \boxtimes \mathcal{A}_{\mathfrak{L}}^0 = \mathfrak{L} \boxtimes \mathcal{A}_{\mathfrak{L}}^0 & \xmapsto{\;+\;} & \mathcal{A}_{\mathfrak{L}}^1 \\ \Big\downarrow{\scriptstyle \mathfrak{AJ}_{(1)} \times id} & \nearrow{\scriptstyle m} & \\ \mathcal{A}_{\mathfrak{L}}^1 \boxtimes \mathcal{A}_{\mathfrak{L}}^0 & & \end{array}$$

we get $\mathfrak{L} = \mathfrak{L}'$.

Similarly starting with a rigidified, multiplicative 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}$ on Pic_C, we define the rigidified, 1-dimensional l-adic local system $\mathfrak{L} := \mathfrak{AJ}_{(1)}^{-1}\mathcal{A}_{\mathfrak{L}}$ on C. We can associate to it a rigidified, multiplicative 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}'$ on Pic_C. Then using the above diagram

$$\begin{array}{ccc} \mathfrak{L} \boxtimes \mathcal{A}_{\mathfrak{L}}^0 = \mathfrak{L} \boxtimes (\mathcal{A}_{\mathfrak{L}}')^0 & \xmapsto{\;+\;} & (\mathcal{A}_{\mathfrak{L}}')^1 = \mathcal{A}_{\mathfrak{L}}^1 \\ \Big\downarrow{\scriptstyle \mathfrak{AJ}_{(1)} \times id} & \nearrow{\scriptstyle m} & \\ \mathcal{A}_{\mathfrak{L}}^1 \boxtimes \mathcal{A}_{\mathfrak{L}}^0 & & \end{array}$$

we get $\mathcal{A}_{\mathfrak{L}}' = \mathcal{A}_{\mathfrak{L}}$ completing the proof of the Theorem. □

Next we want to accomplish the proof of the correspondence appearing in the right vertical side of diagram 1.1.2.

Theorem 2.2.3. *There is a one-to-one correspondence between*

1-*dimensional* l-*adic* *representations of* $Pic^0_C(k)$ $\{\chi : Pic^0_C(k) \to \overline{\mathbb{Q}}^*_l\}$ *together with a Frobenius action* $Frob_p\colon \widehat{\mathbb{Z}} \to \overline{\mathbb{Q}}_l^\times$	$\xleftrightarrow{1:1}$	1-*dimensional* l-*adic local systems* $\mathcal{A}_{\mathfrak{L}}$ *on* Pic_C *together with a fixed isomorphism* $\psi : \mathcal{A}_{\mathfrak{L}}\vert_0 \cong \overline{\mathbb{Q}}_l$ *satisfying* $m^{-1}\mathcal{A}^{d+e}_{\mathfrak{L}} \cong \mathcal{A}^d_{\mathfrak{L}} \boxtimes \mathcal{A}^e_{\mathfrak{L}}$

Proof. First let be given a 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}$ on Pic_C together with a fixed isomorphism $\psi : \mathcal{A}_{\mathfrak{L}}|_0 \cong \overline{\mathbb{Q}}_l$ satisfying $m^{-1}\mathcal{A}^{d+e}_{\mathfrak{L}} \cong \mathcal{A}^d_{\mathfrak{L}} \boxtimes \mathcal{A}^e_{\mathfrak{L}}$. Then the restriction $\mathcal{A}^0_{\mathfrak{L}}$ on Pic^0_C will be a 1-dimensional local system together with a fixed isomorphism $\mathcal{A}^0_{\mathfrak{L}}|_0 \cong \overline{\mathbb{Q}}_l$ satisfying the character sheaf property

$$m^{-1}\mathcal{A}^0_{\mathfrak{L}} \cong \mathcal{A}^0_{\mathfrak{L}} \boxtimes \mathcal{A}^0_{\mathfrak{L}}$$

hence by the faisceaux-fonctions correspondence (section 1.6) it will give us a 1-dimensional l-adic representation of $Pic^0_C(k)$. Also by Theorem 2.2.1 $\mathcal{A}_{\mathfrak{L}}$ defines a 1-dimensional l-adic local system $\mathfrak{L}$ on C, hence a sheaf $\mathfrak{L}_p$ on the point $p \in C(k)$, which is the same as a Frobenius action $Frob_p\colon \widehat{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}_l^\times$ by Definition 1.6.1.

Going on the other way round let be given a 1-dimensional l-adic representation

$$\chi : Pic^0_C(k) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

together with a Frobenius action $Frob_p\colon \widehat{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}_l^\times$. Then by Definition 1.6.1 the Frobenius action defines a sheaf $\mathfrak{L}_p$ on $p \in C(k)$. Also by section 1.6 χ defines a character sheaf $\mathcal{A}^0_\chi$ on Pic^0_C together with a fixed trivialization $\mathcal{A}^0_\chi|_0 \cong \overline{\mathbb{Q}}_l$. Similarly as we did in the proof of Deligne's Theorem we can extend this local system to all of Pic_C by

$$\mathcal{A}_\chi^d := (-dp)^{-1}\mathcal{A}_\chi^0 \otimes \mathfrak{L}_p^d$$

where $-dp$ is the morphism

$$-dp := p \times Pic_C^d \xrightarrow{\cong} Pic_C^0$$

defined by sending $\mathcal{G}$ to $\mathcal{G} \otimes \mathcal{O}(-dp)$. Using the commutative diagram in the proof of Theorem 2.2.1 and the argument presented there it follows from this definition that the 1-dimensional l-adic local system $\mathcal{A}_\chi$ on Pic_C will satisfy

$$m^{-1}\mathcal{A}_\chi^{e+d} \cong \mathcal{A}_\chi^d \boxtimes \mathcal{A}_\chi^e.$$

These two constructions are inverses to each other completing the proof of the Theorem. $\square$

With this theorem we finally finished the long journey leading to the proof of the one-to-one correspondences appearing in diagram 1.1.2.

Now we are able to prove Theorem 1.1.4.

Theorem 2.2.4 (Artin's Reciprocity Law, geometric form). *The Artin Reciprocity Map*

$$\begin{aligned} \Phi_K \colon Div(C) &\longrightarrow \pi_1^{ab}(C) \\ p &\longmapsto Frob_p \end{aligned}$$

factors through rational equivalence

$$\Phi_K \colon Pic_C(k) \longrightarrow \pi_1^{ab}(C)$$

fitting into the commutative diagram

$$\begin{array}{ccccc} Pic^0_C(k) & \hookrightarrow & Pic_C(k) & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle \Phi_K} & & \downarrow{\scriptstyle can} \\ Ker(\varphi) & \hookrightarrow & \pi_1^{ab}(C) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}} \end{array} \tag{2.2.1}$$

such that there is an induced isomorphism on the kernels

$$Pic^0_C(k) \xrightarrow{\cong} Ker(\varphi) \ .$$

Proof. First we prove that the diagram

$$\begin{array}{ccccc} Div^0(C) & \hookrightarrow & Div(C) & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle \Phi_K} & & \downarrow{\scriptstyle \text{can}} \\ Ker(\varphi) & \hookrightarrow & \pi_1^{ab}(C) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}} \end{array}$$

is commutative. For that we define the maps appearing in the diagram. Let us begin with the Frobenius element:

Definition 2.2.5. (construction) Let $p\colon Spec(\mathbb{F}_{q^n}) \longrightarrow C$ be a closed point, which has a degree defined as the degree $deg(p) = [k(p) : k]$ of the field extension $k(p)/k$, where $k(p)$ is the residue field of p. We choose a geometric point $\bar{p}\colon Spec(\overline{\mathbb{F}}_q) \longrightarrow C$ lying above p. By Example 1.4.7 we have that $\pi_1(Spec(\mathbb{F}_{q^n}), \bar{p}) \cong Gal(\overline{\mathbb{F}}_q/\mathbb{F}_{q^n})$ hence the induced map between the étale fundamental groups becomes

$$p_*\colon Gal(\overline{\mathbb{F}}_q/\mathbb{F}_{q^n}) \longrightarrow \pi_1(C, \bar{p}).$$

Now the Galois group $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_{q^n})$ is an open subgroup of the Galois group $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ which is a profinite group topologically generated by the Frobenius automorphism

$$Frob\colon \overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q$$

defined by $x \mapsto x^q$, hence the Galois group $Gal(\overline{\mathbb{F}}_q/\mathbb{F}_{q^n})$ is topologically generated by the $deg(p)^{\text{th}}$ power of the Frobenius automorphism (where now $deg(p) = n$)

$$Frob^{deg(p)} \colon \overline{\mathbb{F}}_q \longrightarrow \overline{\mathbb{F}}_q$$

defined by $x \mapsto x^{q^n}$. Its image under the map

$$p_* \colon Gal(\overline{\mathbb{F}}_q/\mathbb{F}_{q^n}) \longrightarrow \pi_1(C, \bar{p})$$

is only well-defined up to an inner automorphism, but by Corollary 1.4.9 its image under the composite map

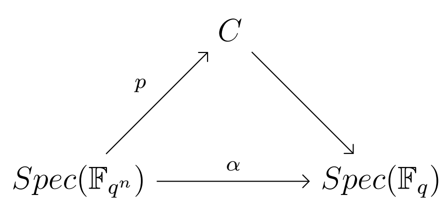
$$\pi_1(C, \bar{p}) \longrightarrow \pi_1^{ab}(C)$$

is a well-defined element, denoted by $Frob_p \in \pi_1^{ab}(C)$, which is called the Frobenius element associated to $p \in C(\mathbb{F}_{q^n})$.

Now as $Div(C)$ is freely generated by the closed points, the Artin Reciprocity Map

$$\Phi_K \colon Div(C) \longrightarrow \pi_1^{ab}(C)$$

sending $p \mapsto Frob_p$ is well-defined. Next we define the map $\varphi \colon \pi_1^{ab}(C) \longrightarrow \widehat{\mathbb{Z}}$ on the Frobenius elements. Recall that φ is the map between the abelianized fundamental groups induced by the structure morphism $C \longrightarrow Spec(k)$. A closed point $p \colon Spec(\mathbb{F}_{q^n}) \longrightarrow C$ gives a commutative triangle



where the map α is induced by the natural inclusion $\mathbb{F}_q \longrightarrow \mathbb{F}_{q^n}$, hence the map between the étale fundamental groups

$$\alpha_*\colon Gal(\overline{\mathbb{F}}_q/\mathbb{F}_{q^n}) \hookrightarrow Gal(\overline{\mathbb{F}}_q/\mathbb{F}_q)$$

is just the natural inclusion. Then by the functoriality property of the étale fundamental group we have that

$$\varphi(Frob_p) = Frob^{deg(p)}.$$

Moreover this map is surjective by Proposition 1.4.10. The other two maps are defined by

$$deg(\textstyle\sum_i d_i p_i) := \sum_i d_i deg(p_i) \text{ and } can(1) := Frob$$

where deg is surjective and can is injective.[2] Now the commutativity of the diagram follows from

$$\varphi(\Phi_K(p)) = Frob^{deg(p)} = can(deg(p))$$

and the fact that by the universal property of the kernel, the Reciprocity Map Φ_K restricted to the divisors $Div^0(C)$ of degree 0 factors through $Ker(\varphi)$.

It remained to prove that the Reciprocity Map factors through the principal divisors. For that let us take a continuous, 1-dimensional l-adic representation

$$\chi\colon \pi_1^{ab}(C) \longrightarrow \overline{\mathbb{Q}}_l^\times.$$

By the geometric constructions we carried out and appearing in diagram 1.1.2, we can associate to χ a 1-dimensional l-adic local system $\mathfrak{L}$ on C, such that for a closed point $p\colon Spec(\mathbb{F}_{q^n}) \longrightarrow C$, the Frobenius element $Frob_p$ acts on the fibre $\mathfrak{L}_{\bar{p}}$ by

[2] Hopefully it may not cause any confusion that deg denotes two different maps, as the notations are so natural in both cases, that the change of notation may have caused more confusion.

multiplication by $\chi(Frob_p) \in \overline{\mathbb{Q}}_l^{\times}$. By Deligne's Theorem we can associate to $\mathfrak{L}$ a multiplicative 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}$ on Pic_C and by the faisceaux-fonctions correspondence we can associate to $\mathcal{A}_{\mathfrak{L}}$ a 1-dimensional l-adic representation

$$\psi\colon Pic^0_C(k) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$$

together with a Frobenius action $Frob_x\colon \widehat{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}_l^{\times}$, where $x \in C(k)$. The pair $(\psi, Frob_x)$ induces a 1-dimensional l-adic representation[3] (using the same notation)

$$\psi\colon Pic^1_C(k) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$$

defined by

$$\psi([\mathcal{O}(p)]) := tr_{\mathcal{A}^1_{\mathfrak{L}}}(Frob_{[\mathcal{O}(p)]}),$$

i.e. for a closed point $p\colon Spec(\mathbb{F}_{q^n}) \longrightarrow C$ the Frobenius element $Frob_{[\mathcal{O}(p)]}$ acts on the fibre $\mathcal{A}^1_{\mathfrak{L}}|_{\overline{[\mathcal{O}(p)]}}$ by multiplication by $\psi([\mathcal{O}(p)]) \in \overline{\mathbb{Q}}_l^{\times}$. Now consider the Abel-Jacobi map

$$\mathfrak{AJ}_{(1)}\colon C \longrightarrow Pic^1_C$$

defined by $p \mapsto [\mathcal{O}(p)]$. By Deligne's Theorem 2.2.1 we have that

$$\mathfrak{AJ}_{(1)}^{-1}(\mathcal{A}^1_{\mathfrak{L}}) = \mathfrak{L}$$

hence for a closed point $p\colon Spec(\mathbb{F}_{q^n}) \longrightarrow C$ the Frobenius element $Frob_p$ acts on the fibre $\mathfrak{L}_{\bar{p}}$ in the same way as the Frobenius element $Frob_{[\mathcal{O}(p)]}$ acts on the fibre $\mathcal{A}^1_{\mathfrak{L}}|_{\overline{[\mathcal{O}(p)]}}$, that is

$$\chi(Frob_p) = \psi([\mathcal{O}(p)]),$$

[3]The pair $(\psi, Frob_x)$ induces 1-dimensional l-adic representations on all degree components $\psi\colon Pic^d_C(k) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$, hence on all of $Pic_C(k)$.

which means that we have the following commutative diagram

$$\begin{array}{ccc} Div(C) & \xrightarrow{\Phi_K} & \pi_1^{ab}(C) \\ \downarrow\!\downarrow & \overset{\Phi_K}{\nearrow}\ ? & \downarrow \chi \\ Pic_C(k) & \xrightarrow{\psi} & \overline{\mathbb{Q}}_l^{\times} \end{array} \tag{2.2.2}$$

where to be able to prove that Φ_K factorizes through $Pic_C(k)$ we need to show that rationally equivalent divisors $D_1 \sim D_2$, that is $[\mathcal{O}(D_1)] = [\mathcal{O}(D_2)]$ in $Pic_C(k)$, have the same image $\Phi_K(D_1) = \Phi_K(D_2)$ in $\pi_1^{ab}(C)$. This reduces to show that different closed points $p_1, p_2 \in |C|$ having the same image in $Pic_C(k)$, have the same image $Frob_{p_1} = Frob_{p_2}$ in $\pi_1^{ab}(C)$. By the commutativity of the diagram above, the closed points $p_1, p_2 \in |C|$ with this property must satisfy $\chi(Frob_{p_1}) = \chi(Frob_{p_2})$ for all continuous 1-dimensional l-adic representation $\chi\colon \pi_1^{ab}(C) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$. But if $Frob_{p_1} \neq Frob_{p_2}$ then we can construct a representation $\chi\colon \pi_1^{ab}(C) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ satisfying $\chi(Frob_{p_1}) \neq \chi(Frob_{p_2})$ by using Lang's Theorem 2.2.6 stated below, as the Frobenius elements $Frob_p$ associated to the closed points $p \in |C|$ generate topologically the group $\pi_1^{ab}(C)$, hence any representation $\chi\colon \pi_1^{ab}(C) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ is given by determining its values on the Frobenius elements. It follows that $Frob_{p_1} = Frob_{p_2}$ hence the Reciprocity Map factors through rational equivalence

$$\Phi_K\colon Pic_C(k) \longrightarrow \pi_1^{ab}(C).$$

Now we give

Theorem 2.2.6 (Lang's Theorem)**.** *The Frobenius elements $Frob_p$ associated to the closed points $p \in |C|$ generate a dense subgroup in the topological group $\pi_1^{ab}(C)$.*

Proof. The original proof uses the properties of the *zeta function* of schemes, but here we will give a more geometric proof again using Deligne's construction (diagram 1.1.2).

Assume that the image $H := \Phi_K(Div(C)) \subset \pi_1^{ab}(C)$ of the Reciprocity Map is not dense and denote by $\overline{H}$ its closure. Then there exists an open subset U containing H. By Theorem 1.4.6 it corresponds to a finite, abelian, étale cover $Y \longrightarrow C$ of some degree d, which by Proposition 1.5.9 corresponds to a locally constant étale sheaf $\mathfrak{F}$ on C and to a d-dimensional representation

$$\chi\colon \pi_1^{ab}(C) \longrightarrow GL_d(\mathbb{Q}_l).$$

Here we can pass to a $\overline{\mathbb{Q}}_l$-representation by making $\mathfrak{F}$ into an l-adic local system. By construction we have that for every closed point $p \in |C|$ the action of the Frobenius element $Frob_p$ on the fibre $\mathfrak{F}_{\bar{p}}$ is trivial, hence the representation $\chi|_H\colon H \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$ is trivial. On the other hand taking the decomposition into irreducible, hence 1-dimensional representations

$$\chi = \chi_1 \oplus \chi_2 \oplus \cdots \oplus \chi_d$$

each χ_i will correspond to a 1-dimensional l-adic representation

$$\psi_i\colon Pic_C^0(k) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$$

satisfying $\psi_i = \chi_i \circ \Phi_K$ which are the trivial representations by construction. Hence by diagram 4.2.3 we get that each χ_i and consequently the representation

$$\chi\colon \pi_1^{ab}(C) \longrightarrow GL_d(\overline{\mathbb{Q}}_l)$$

is trivial. It follows that $U = \pi_1^{ab}(C)$ and $\overline{H} = \pi_1^{ab}(C)$ completing the proof of Lang's Theorem. $\square$

To prove the remaining statement concerning the induced isomorphism on the kernels

$$Pic^0_C(k) \xrightarrow{\cong} Ker(\varphi)$$

we note that the factorized Reciprocity Map

$$\Phi_K \colon Pic_C(k) \longrightarrow \pi_1^{ab}(C)$$

is injective and has dense image by Lang's Theorem and diagram 4.2.3, so we get the following commutative diagram

$$\begin{array}{ccccc}
Pic^0_C(k) & \hookrightarrow & Pic_C(k) & \xrightarrow{deg} & \mathbb{Z} \\
\downarrow & & \downarrow \cong & & \downarrow \cong \\
Ker(\varphi) & \hookrightarrow & \Phi_K(Pic_C(k)) & \xrightarrow{\varphi} & \mathbb{Z}
\end{array}$$

which induces an isomorphism

$$Pic^0_C(k) \xrightarrow{\cong} Ker(\varphi)$$

completing the Proof of Artin's Reciprocity Law. □

Chapter 3

Tamely Ramified Geometric Abelian Class Field Theory

In this chapter we will present tamely ramified geometric abelian class field theory which establishes a connection between the Picard group $Pic_{C,S}$ of a smooth projective curve over a finite field, where S is a finite set of points of the curve and the abelianized tame étale fundamental group $\pi_1^{t,ab}(U)$ of the open complement.

We will begin with stating the main theorem of the tamely ramified theory in different forms without going into the details and trace out a way how we will prove it.

Then in the subsequent sections we will discuss and develop the background. In particular we will define the basic concepts appearing in both sides of the correspondence, namely the Picard group $Pic_{C,S}$ and the tame étale fundamental group of the open complement and we will also perform the necessary constructions leading to a geometric proof of the main theorem, which finally will be provided in the last chapter.

3.1 The Main Theorem

Let C be a smooth, projective, geometrically irreducible curve over a finite field $k = \mathbb{F}_q$, $S = \{p_1, p_2, \ldots, p_n\} \subset |C|$ a finite set of closed points and $U := C \setminus S$ the open complement of S. Let $K = k(C)$ be the function field and for every closed point $p \in |C|$ let $\widehat{\mathcal{O}_p}$ be the completion of the local ring at the point p and K_p its quotient field. For every $p \in S$ we consider the kernel

$$\widehat{\mathcal{O}_p}^1 := Ker(\widehat{\mathcal{O}_p}^\times \longrightarrow (\mathcal{O}_p/\mathfrak{m}_p)^\times)$$

where we have $\mathcal{O}_p/\mathfrak{m}_p \cong \widehat{\mathcal{O}_p}/\widehat{\mathfrak{m}}_p$[1]. Let $\mathbb{I}_K := \prod'_{p\in|C|} K_p^\times$ be the idèle group of C, $\mathbb{I}_{K,S} := (\prod_{p\notin S} K_p^\times \times \prod_{p\in S} \widehat{\mathcal{O}_p}^1) \bigcap \mathbb{I}_K$ be the subgroup of the idèles relative to S. Let $K_S := K^\times \bigcap \mathbb{I}_{K,S} \subset \mathbb{I}_K$ be the set of rational functions satisfying $ord_p(1-g) \geq 1$ for all $p \in S$. We consider also the normal subgroup

$$\prod_{p\in S} \widehat{\mathcal{O}_p}^1 \times \prod_{p\notin S} \widehat{\mathcal{O}_p^\times} \subseteq \mathbb{I}_K.$$

Furthermore let $\pi_1^{ab,t}(U)$ be the abelianized tame étale fundamental group of the open subscheme $U \subset C$. The main theorem of tamely ramified abelian class field theory is the following:

Theorem 3.1.1 (Tamely Ramified Reciprocity Law, adelic form)**.** *The tamely ramified Reciprocity Map*

$$\Phi_{K,S}\colon \mathbb{I}_{K,S}/\prod_{p\notin S} \widehat{\mathcal{O}_p^\times} \xrightarrow{\Phi_{K,S}} \pi_1^{t,ab}(U)$$
$$[(\ldots, a_p, \ldots)_{p\in U}] \longmapsto \prod_{p\in U} Frob_p^{ord_p(a_p)}$$

induces a map (also denoted by $\Phi_{K,S}$)

[1] cf. Theorem II.4.3 in [Neu07]

$$\Phi_{K,S} : K^{\times}\backslash \mathbb{I}_K / \prod_{p\in S} \widehat{\mathcal{O}_p}^{1} \times \prod_{p\notin S} \widehat{\mathcal{O}_p^{\times}} \longrightarrow \pi_1^{t,ab}(U)$$

fitting into the commutative diagram

$$\begin{array}{ccccc}
Ker(deg) & \hookrightarrow & K^{\times}\backslash \mathbb{I}_K / \prod_{p\in S} \widehat{\mathcal{O}_p}^{1} \times \prod_{p\notin S} \widehat{\mathcal{O}_p^{\times}} & \xrightarrow{deg} & \mathbb{Z} \\
\downarrow & & \downarrow \Phi_{K,S} & & \downarrow can \\
Ker(\varphi) & \hookrightarrow & \pi_1^{t,ab}(U) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}}
\end{array}$$

such that there is an induced isomorphism on the kernels

$$Ker(deg) \xrightarrow{\cong} Ker(\varphi)$$

where $\varphi\colon \pi_1^{t,ab}(U) \longrightarrow \widehat{\mathbb{Z}}$ *is the map given by composing the natural map* $\pi_1^{t,ab}(U) \longrightarrow \pi_1^{ab}(C)$ *with the induced map* $\pi_1^{ab}(C) \longrightarrow \widehat{\mathbb{Z}}$ *(section 3.3).*

In order to interpret this statement geometrically we need the following

Proposition 3.1.2. *The inclusion* $\mathbb{I}_{K,S} \hookrightarrow \mathbb{I}_K$ *induces an isomorphism*

$$K_S\backslash \mathbb{I}_{K,S} \cong K^{\times}\backslash \mathbb{I}_K$$

and hence an isomorphism

$$K_S\backslash \mathbb{I}_{K,S} / \prod_{p\notin S} \widehat{\mathcal{O}_p^{\times}} \cong K^{\times}\backslash \mathbb{I}_K / \prod_{p\in S} \widehat{\mathcal{O}_p}^{1} \times \prod_{p\notin S} \widehat{\mathcal{O}_p^{\times}}.$$

Proof. The first statement is Proposition 4.6 (b) in [Mil08b], the second statement then follows by dividing out with $\prod_{p\in S} \widehat{\mathcal{O}_p}^{1}$. □

We can characterize this adelic double quotient in terms of geometric data associated to the curve C in the following way

Proposition 3.1.3. *There is an isomorphism*

$$K^\times \backslash \mathbb{I}_K / \prod_{p\in S} \widehat{\mathcal{O}_p}^1 \times \prod_{p\notin S} \widehat{\mathcal{O}_p^\times} \cong Pic_{C,S}(k)$$

between the adelic double coset space and isomorphism classes of invertible sheaves on C together with fixed isomorphisms at each point $p \in S$

$$Pic_{C,S}(k) := \{([\mathcal{G}], \{\psi_p\}_{p\in S}) \mid [\mathcal{G}] \in Pic_C(k) \text{ and } \psi_p \colon \mathcal{G} \otimes_{\mathcal{O}_C} \mathcal{O}_p/\mathfrak{m}_p \cong \mathcal{O}_p/\mathfrak{m}_p\}.$$

Proof. Given an invertible sheaf $\mathcal{F}$ on C with fixed isomorphisms at each $p \in S$

$$\psi_p : \mathcal{G} \otimes_{\mathcal{O}_C} \mathcal{O}_p/\mathfrak{m}_p \cong \mathcal{O}_p/\mathfrak{m}_p,$$

we choose a trivialization at the generic point ξ of C

$$f_\xi \colon \mathcal{F} \otimes_{\mathcal{O}_C} K \xrightarrow{\cong} K,$$

also a trivialization for every closed point $p \in U$

$$f_p \colon \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \xrightarrow{\cong} \mathcal{O}_p,$$

and at each $p \in S$ a trivialization

$$f_{p\in S} \colon \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \xrightarrow{\cong} \mathcal{O}_p$$

such that $f_{p\in S} \equiv \psi_p \bmod \mathfrak{m}_p$. The natural morphism $\mathcal{O}_p \longrightarrow K$ gives the diagram for all $p \in C$

$$\begin{array}{ccc} \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \otimes_{\mathcal{O}_p} K & \xrightarrow{f_p \otimes id_K} & \mathcal{O}_p \otimes_{\mathcal{O}_p} K \\ \downarrow f_\xi & & \downarrow \cong \\ K & \xrightarrow[\cong]{g_p} & K \end{array} .$$

The isomorphism g_p is given by multiplication by an element $a_p \in K^\times$. Moreover for all but finitely many closed point $p \in |C|$ we have that[2] $a_p \in \mathcal{O}_p^\times$, hence the invertible sheaf $\mathcal{F}$ defines an element

[2]cf. Lemma I.6.5 in [Har06]

$$(\dots, a_p, \dots)_{p\in|C|} \in \prod'_{p\in|C|} K^\times.$$

If we choose another trivialization at the generic point ξ

$$f'_\xi \colon \mathcal{F} \otimes_{\mathcal{O}_C} K \xrightarrow{\cong} K$$

then each element a_p will be changed via left multiplication by an element in $K^\times$. If we choose another trivializations at each $p \in U$

$$f'_p \colon \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \xrightarrow{\cong} \mathcal{O}_p$$

then the element a_p will be changed via right multiplication by an element in $\mathcal{O}_p^\times$. Also if we choose another trivializations at each $p \in S$

$$f'_{p\in S} \colon \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \xrightarrow{\cong} \mathcal{O}_p$$

satisfying $f_{p\in S} \equiv \psi_p \bmod \mathfrak{m}_p$ then the element a_p will be changed via right multiplication by an element in $\widehat{\mathcal{O}_p}^1$. Hence $\mathcal{F}$ defines an element in

$$K^\times \backslash \prod'_{p\in|C|} K^\times / \prod_{p\in S} \widehat{\mathcal{O}_p}^1 \times \prod_{p\notin S} \mathcal{O}_p^\times.$$

Now we use Claim 1.1.3 to get an element in

$$K^\times \backslash \mathbb{I}_K / \prod_{p\in S} \widehat{\mathcal{O}_p}^1 \times \prod_{p\notin S} \widehat{\mathcal{O}_p^\times}$$

which depends only on the isomorphism class of $\mathcal{F}$ by construction.

On the other hand given an element $a = (\dots, a_p, \dots) \in \mathbb{I}_K$ we define a sheaf $\mathcal{F}_a$ on C by

$$\mathcal{F}_a(V) := \{x \in K |\ a_p^{-1} x \in \widehat{\mathcal{O}_p}\ \forall\ p \in V\}.$$

It follows from this local description that $\mathcal{F}_a$ defines a sheaf on C. Moreover changing the coset representative a from the right by an element in $\prod_{p\in S} \widehat{\mathcal{O}_p}^1 \times \prod_{p\notin S} \widehat{\mathcal{O}_p^\times}$ does not change anything in $\mathcal{F}_a$ and does not change $a_p \bmod \mathfrak{m}_p$ at $p \in S$. Changing from the left by an element $b \in K^\times$ gives an isomorphism of sheaves $\mathcal{F}_a \xrightarrow{b\times} \mathcal{F}_{ba}$, so we have to prove that $\mathcal{F}_a$ is locally free of rank 1. If $a_p \in \widehat{\mathcal{O}_p^\times}$ for all $p \in V$ then $\mathcal{F}_a(V) = \mathcal{O}_C(V)$ by construction, hence it is free on V. Otherwise using Claim 1.1.3 we take an element $t \in K^\times$ such that $ta_p \in \widehat{\mathcal{O}_p^\times}$. Now define $V := \{p \in |C| : \ t \in \widehat{\mathcal{O}_p^\times}\}$ and use the isomorphism $\mathcal{F}_a \xrightarrow{t\times} \mathcal{F}_{ta}$ which gives that $\mathcal{F}_a$ is locally free of rank 1 on V. It remains to prove the existence of fixed isomorphisms at each point $p \in S$. For that we use the approximation theorem[3], which enables us to find an element $a_S \in K^\times$ satisfying $a_S^{-1}a_p \equiv 1 \bmod \mathfrak{m}_p$ for all $p \in S$. Hence we can modify the elements a_p at each $p \in S$ such that the trivializations defined by a_p

$$f_{p\in S} \colon \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p \xrightarrow{\cong} \mathcal{O}_p$$

will give fixed isomorphisms

$$\psi_p \colon \mathcal{F} \otimes_{\mathcal{O}_C} \mathcal{O}_p/\mathfrak{m}_p \cong \mathcal{O}_p/\mathfrak{m}_p$$

well-defined modulo $\mathfrak{m}_p$ at each $p \in S$. These two constructions are inverses to each other which completes the proof. □

Now we can state the tamely ramified Reciprocity Law in a more geometric form:

Theorem 3.1.4 (Tamely Ramified Reciprocity Law, geometric form)**.** *The tamely ramified Reciprocity Map*

$$\begin{aligned} \Phi_{K,S} \colon Div(U) &\longrightarrow \pi_1^{t,ab}(U) \\ p &\longmapsto [Frob_p] \end{aligned}$$

induces a map

$$\Phi_{K,S} : Pic_{C,S}(k) \longrightarrow \pi_1^{t,ab}(U)$$

[3]cf. Theorem II.3.4 in [Neu07].

and fits into the commutative diagram

$$\begin{array}{ccccc} Pic^0_{C,S}(k) & \hookrightarrow & Pic_{C,S}(k) & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle \Phi_{K,S}} & & \downarrow{\scriptstyle can} \\ Ker(\varphi) & \hookrightarrow & \pi_1^{t,ab}(U) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}} \end{array} \tag{3.1.1}$$

such that there is an induced isomorphism on the kernels

$$Pic^0_{C,S}(k) \xrightarrow{\cong} Ker(\varphi)$$

where $\varphi\colon \pi_1^{t,ab}(U) \longrightarrow \widehat{\mathbb{Z}}$ *is the map given by composing the natural map* $\pi_1^{t,ab}(U) \longrightarrow \pi_1^{ab}(C)$ *with the induced map* $\pi_1^{ab}(C) \longrightarrow \widehat{\mathbb{Z}}$ *(section 3.3).*

As in the unramified theory, the strategy of the proof is first to consider continuous, 1-dimensional l-adic representations of $\pi_1^{t,ab}(U)$ and 1-dimensional l-adic representations of $Pic^0_{C,S}(k)$, where l is a prime number different from $char(k)$. Assume we have a closed point $p \in U(k) \subset C(k)$, then we can characterize these representations as follows

- the continuous 1-dimensional l-adic representations of $\pi_1^{t,ab}(U)$ (which are the same as the continuous 1-dimensional l-adic representations of the tame étale fundamental group $\pi_1^t(U,\bar{u})$) are in one-to-one correspondence with 1-dimensional l-adic local systems $\mathfrak{L}$ on U, which are tame at S together with a rigidification, i.e. a fixed isomorphism $\varphi\colon \mathfrak{L}_{\bar{u}} \cong \overline{\mathbb{Q}}_l$, where $\bar{u}\colon Spec(\Omega) \longrightarrow U$ is a geometric point (cf. section 4.1);

- the 1-dimensional l-adic representations of $Pic^0_{C,S}(k)$ together with a Frobenius action $Frob_p\colon \widehat{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}_l^\times$ are in one-to-one correspondence with 1-dimensional

l-adic local systems $\mathcal{A}_{\mathfrak{L}}$ on $Pic_{C,S}$ together with a rigidification, i.e. a fixed isomorphism $\psi : \mathcal{A}_{\mathfrak{L}}|_0 \cong \overline{\mathbb{Q}}_l$ satisfying

$$m^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e} \cong \mathcal{A}_{\mathfrak{L}}^{d} \boxtimes \mathcal{A}_{\mathfrak{L}}^{e}$$

where

$$m\colon Pic^{d}_{C,S} \times Pic^{e}_{C,S} \longrightarrow Pic^{d+e}_{C,S}$$

is the group operation on $Pic_{C,S}$ and $0\colon Spec(k) \longrightarrow Pic_{C,S}$ is the identity section (cf. section 4.1).

Using these correspondences we will give a geometric construction adapting Deligne's argument in section 4.2, which gives a one-to-one correspondence between rigidified 1-dimensional l-adic local systems on U which are tame at S and rigidified multiplicative 1-dimensional l-adic local systems on $Pic_{C,S}$. Having done this we will get the following diagram:

$$
\begin{array}{ccc}
\boxed{\begin{array}{c}\text{continuous 1-dimensional} \\ l\text{-adic representa-} \\ \text{tions of } \pi_1^{t,ab}(U) \\ \{\chi : \pi_1^{t,ab}(U) \to \overline{\mathbb{Q}}_l^\times\}\end{array}} & \overset{1:1}{\longleftrightarrow} & \boxed{\begin{array}{c}\text{1-dimensional } l\text{-adic rep-} \\ \text{resentations of } Pic^0_{C,S}(k) \\ \{\chi : Pic^0_{C,S}(k) \to \overline{\mathbb{Q}}_l^\times\} \\ \text{together with a Frobenius} \\ \text{action } Frob_p \colon \widehat{\mathbb{Z}} \to \overline{\mathbb{Q}}_l^\times\end{array}} \\
\updownarrow {\scriptstyle 1:1} & & \updownarrow {\scriptstyle 1:1} \\
\boxed{\begin{array}{c}\text{1-dimensional } l\text{-adic} \\ \text{local systems } \mathfrak{L} \text{ on } U \\ \text{which are tame at } S \\ \text{together with a fixed} \\ \text{isomorphism } \varphi : \mathfrak{L}_{\bar{u}} \cong \overline{\mathbb{Q}}_l\end{array}} & \overset{1:1}{\longleftrightarrow} & \boxed{\begin{array}{c}\text{1-dimensional } l\text{-adic} \\ \text{local systems } \mathcal{A}_{\mathfrak{L}} \text{ on} \\ Pic_{C,S} \text{ together with} \\ \text{a fixed isomorphism} \\ \psi : \mathcal{A}_{\mathfrak{L}}|_0 \cong \overline{\mathbb{Q}}_l \text{ satisfying} \\ m^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e} = \mathcal{A}_{\mathfrak{L}}^d \boxtimes \mathcal{A}_{\mathfrak{L}}^e\end{array}}
\end{array}
\tag{3.1.2}
$$

Finally in section 4.2 we will use the correspondences appearing in this diagram to prove geometrically the tamely ramified Reciprocity Law.

3.2 The Symmetric Power $U^{(d)}$ and the relative Picard Scheme $Pic_{C,S}$

In this section we will define the symmetric powers $U^{(d)}$ and the Picard scheme $Pic_{C,S}$ functorially and establish the tamely ramified Abel-Jacobi map between them.

Let C be a smooth, projective curve over a field k, $S = \{p_1, p_2, \ldots, p_n\} \subset |C|$ a finite set of closed points, $U := C \setminus S$ the open complement of S and $d \geq 1$ an integer. Recall that in section 1.2 we defined the functor

$$Div_C^d \colon Sch/k \longrightarrow Set$$

which to a k-scheme T associates the set $Div_C^d(T)$ of relative effective Cartier divisors of degree d on $(C \times_{Spec(k)} T)/T)$. By Proposition 1.2.6 this functor parametrizes pairs $(\mathcal{G}, s)$, where $\mathcal{G}$ is an invertible sheaf on $(C \times_{Spec(k)} T)/T)$ and $s \in H^0((C \times_{Spec(k)} T)/T), \mathcal{G}) \setminus \{0\}$ is a non-zero global section such that the subscheme defined by s is flat over T.

Definition 3.2.1. We define an open subfunctor of Div_C^d

$$Div_U^d \colon Sch/k \longrightarrow Set$$

which to a k-scheme T associates the set $Div_U^d(T)$ of relative effective Cartier divisors of degree d on $(C \times_{Spec(k)} T)/T$ such that if such a $D \in Div_U^d(T)$ is represented by a class $(\mathcal{G}, s)$, then $s|_{S \times T} \neq 0$.

Now the proof of Theorem 1.2.10 can be applied word by word for our situation and one can construct a canonical relative effective Cartier divisor D_{can} on $(C \times_{Spec(k)} U^{(d)})/U^{(d)})$, which implies the following

Theorem 3.2.2. *Let C be a smooth, projective curve over a field k, $S = \{p_1, p_2, \dots, p_n\} \subset |C|$ a finite set of closed points, $U := C \setminus S$ the open complement of S and $d \geq 1$ an integer. Then for any relative effective Cartier divisor D on $(C \times_{Spec(k)} T)/T$, that is represented by a class $(\mathcal{G}, s)$, where $s|_{S\times T} \neq 0$ there exists a unique morphism $\alpha : T \longrightarrow C^{(d)}$ factorizing through $\alpha : T \longrightarrow U^{(d)}$ such that*

$$D = (id_C \times \alpha)^{-1}(D_{can}),$$

that is the functor Div^d_U is representable by $U^{(d)}$.

Now let us turn to the Picard scheme $Pic_{C,S}$.

Definition 3.2.3. For an integer $d \in \mathbb{Z}$ we define the *Picard functor of degree d with respect to S* as the functor from the category of schemes over k to the category of abelian groups

$$Pic^d_{C,S} \colon Sch/k \longrightarrow Ab$$

which to a k-scheme T associates the abelian group

$$Pic^d_{C,S}(T) := (\{\mathcal{G}, \{\psi_p\}_{p\in S}\})/pr_2^{-1}(Pic^d(T))$$

where $\mathcal{G} \in Pic(C \times_{Spec(k)} T)$ such that $deg(\mathcal{G}_t) = d$ for all $t \in T$ together with fixed isomorphisms $\psi_p \colon \mathcal{G} \otimes_{\mathcal{O}_{p\times T}} \mathcal{O}_{p\times T}/\mathfrak{m}_{p\times T} \xrightarrow{\cong} \mathcal{O}_{p\times T}/\mathfrak{m}_{p\times T}$ at each $p \in S$.

We note that for a point $p \in S$ we have an isomorphism of functors

$$Pic^d_C \xrightarrow{\cong} Pic^d_{C,p}$$

given by sending $\mathcal{G} \in Pic^d_C(T)$ to $\mathcal{G} \otimes pr_2^{-1}(\mathcal{G}|_{p\times T})$, which has a canonical trivialization along $p \times T$, where $pr_2 \colon C \times_{Spec(k)} T \longrightarrow T$ is the second projection. On the other way round we use the forgetful functor sending $\{\mathcal{G}, \{\psi_p\}_{p\in S}\}$ to $\mathcal{G}$. So the fibres of the natural (forgetful) transformation

$$Pic^d_{C,S} \longrightarrow Pic^d_C$$

are the possible trivializations of bundles at the other $\sharp S - 1$ points in S. From that we can conclude that the functor $Pic^d_{C,S}$ is an extension of the (representable) functor Pic^d_C by the (representable) functor $\mathbb{G}_m^{\sharp S-1}$

$$\mathbb{G}_m^{\sharp S-1} \longrightarrow Pic^d_{C,S} \longrightarrow Pic^d_C,$$

thus itself representable.

Now we can define a natural transformation between the functors Div^d_U and $Pic^d_{C,S}$

Definition 3.2.4. The *tamely ramified Abel-Jacobi map*

$$\mathfrak{AJ}_{S,d}\colon Div^d_U \longrightarrow Pic^d_{C,S}$$

is defined for a scheme T over $Spec(k)$ and for a relative effective Cartier divisor D of degree d on $(C \times_{Spec(k)} T)/T$ represented by the pair $(\mathcal{G}, s)$, where $s|_{S\times T} \neq 0$ by

$$\mathfrak{AJ}_{S,d}(T)((\mathcal{G}, s)) = \{\mathcal{G}, \{s_p\}_{p\in S}\}$$

where the section s gives canonical isomorphisms $s_p\colon \mathcal{G} \otimes_{\mathcal{O}_{p\times T}} \mathcal{O}_{p\times T}/\mathfrak{m}_{p\times T} \xrightarrow{\cong} \mathcal{O}_{p\times T}/\mathfrak{m}_{p\times T}$ at each $p \in S$.

3.3 The Tame Fundamental Group

In this section we will define the tame fundamental group of a regular integral scheme that is separated of finite type over a (finite) field and present some of its most important properties according to the scope of our purposes. The main references for this section are [GM71], 5.7.15-16 in [Sza09b], A.I. in [FK88] and II.7. in [Neu07].

Let K be a field together with a discrete valuation v, denote $\mathcal{O}_v$, $k(v)$ and $v(K^\times)$ the valuation ring, the residue field and the value group of v respectively. Let L/K be a finite separable field extension, then by II.8.1 in [Neu07] there exist only finitely many non-equivalent extensions w/v of the discrete valuation v to the field L and these are also discrete valuations.

Definition 3.3.1. A finite separable field extension L/K is said to be *tamely ramified* with respect to the valuation v of K if for each extension w of v to L we have

1. $char(k(v)) \nmid e$ where $e := [w(L^\times) : v(K^\times)]$ is the ramification index of w/v and
2. the residue field extension $k(w)/k(v)$ is separable.

Tamely ramified extensions have the following properties[4]

Proposition 3.3.2. *1. For a tower of field extensions $M \supset L \supset K$ the extension M/K is tamely ramified if and only if L/K and M/L are tamely ramified with respect to any extension $z/w/v$;*

2. if L/K and M/K are tamely ramified with respect to v, then the compositum LM/K is also tamely ramified with respect to v;

[4] We assume that every field extension $L, M, K'/K$ and their composita are contained in a fixed algebraic closure $\bar{K}/K$.

3. *let L/K be tamely ramified with respect to v and K'/K be an arbitrary field extension together with a discrete valuation v' extending v, then the extension LK'/K' is tamely ramified with respect to v';*

4. *let L/K be tamely ramified with respect to v and let M/K be the smallest Galois extension of K containing L, then M/K is also tamely ramified with respect to v;*

5. *if L/K is tamely ramified with respect to v, then $L \otimes_K K_v/K_v$ is also tamely ramified, meaning that each summand L_w is tamely ramified over K_v (see Theorem II.8.3 in [Neu07]).*

Proof. cf. Lemma 2.1.3. and Corollaire 2.1.4. in [GM71]. □

Now we define the notion tamely ramified for schemes:

Definition 3.3.3. Let X be a normal, integral scheme, $U \subset X$ an open subscheme, such that the closed subset $S = X \setminus U$ is of at least codimension 1. Then a finite morphism $f: Y \longrightarrow X$ is said to be *tamely ramified with respect to S* if

1. $f|_{f^{-1}U}$ is a finite étale cover $f^{-1}U \subseteq Y \longrightarrow U$ and

2. Y is tamely ramified over $Spec(\mathcal{O}_{X,s})$ for each codimension 1 point $s \in S$, that is the fibre $Y_s := Y \times_X Spec(\mathcal{O}_{X,s})$ is a finite direct product $Y_s = \prod_{i=1}^{n} Spec(L_i)$ of spectra of finite separable field extensions of the quotient field K of $\mathcal{O}_{X,s}$, as the generic point of $Spec(\mathcal{O}_{X,s})$ is contained in U, and each extension L_i/K is tamely ramified with respect to the valuation given by $\mathcal{O}_{X,s}$.

Tamely ramified coverings have the following properties:

Proposition 3.3.4. *Let $f\colon Y \longrightarrow X$ be a morphism of schemes and $g\colon X^{'} \longrightarrow X$ a surjective, étale base change morphism. Then Y is tamely ramified over X with respect to S if and only if $Y \times_X X^{'}$ is tamely ramified over $X^{'}$ with respect to $g^{-1}(S)$.*

Proposition 3.3.5.

1. *Let Y/X and Z/X be schemes over X, $f\colon Y \longrightarrow Z$ and $g\colon Z \longrightarrow X$ be finite morphisms g being surjective. Then Y is tamely ramified over X with respect to S if and only if Z is tamely ramified over X with respect to S and Y is tamely ramified over Z with respect to $g^{-1}(S)$.*

2. *Let $f\colon Y \longrightarrow X$ be a tamely ramified cover with respect to S, where S is a divisor with normal crossings. Let $g\colon X^{'} \longrightarrow X$ be a morphism, where $X^{'}$ is a normal scheme such that $S^{'} := g^{-1}(S)$ is defined and is again a divisor with normal crossings. Then $f^{'}\colon X^{'} \times_X Y \longrightarrow X^{'}$ is a tamely ramified cover with respect to $S^{'}$.*

Proof. cf. Lemma 2.2.5, Lemma 2.2.7 and Lemma 2.3.6 in [GM71]. □

Assume now that X is a connected regular integral scheme that is separated of finite type over a (finite) field k and $S \subset X$ is a divisor with normal crossings[5]. Let $U := X \setminus S$ be the open complement.

Let $Fet^{t,S}/X$ be the category of tamely ramified coverings of X with respect to S. Let $\bar{u}\colon Spec(\Omega) \longrightarrow U$ be a fixed geometric point and define the tame fibre functor at the geometric point $\bar{u}$

$$F^{t,S}_{\bar{u}}\colon Fet^{t,S}/X \longrightarrow Set$$

[5]In particular the case we are most interested in when X is a smooth projective geometrically irreducible curve over a finite field and $S \subset X$ is a finite set of closed points.

which to a tamely ramified cover $f : Y \longrightarrow X$ with respect to S associates the underlying set of the fibre over $\bar{u}$, i.e. $F_{\bar{u}}^{t,S}(Y, f) := \{$the underlying set of $Y_{\bar{u}} := Y \times_X Spec(\Omega)\}$ and for a morphism

$$\begin{array}{ccc} Y & \xrightarrow{\quad g \quad} & Z \\ \text{tame w.r.t. S} \searrow & \circlearrowleft & \swarrow \text{tame w.r.t. S} \\ & X & \end{array}$$

it associates the set-theoretic map $F_{\bar{u}}^{t,S}(Y) \longrightarrow F_{\bar{u}}^{t,S}(Z)$ induced by the morphism of geometric fibres $Y \times_X Spec(\Omega) \longrightarrow Z \times_X Spec(\Omega)$.

Definition 3.3.6. Let X, S, U and $\bar{u}$ be as above. Then the *tame étale fundamental group* $\pi_1^t(X, S, \bar{u})$ of X with respect to S at $\bar{u}$ is defined as the automorphism group of the tame fibre functor $F_{\bar{u}}^{t,S}$ on $Fet^{t,S}/X$. If there is no confusion we will denote this group as $\pi_1^t(U, \bar{u})$ and call it the tame étale fundamental group of U at $\bar{u}$.

Theorem 3.3.7. *Let X be a connected regular integral scheme that is separated and of finite type over a (finite) field, $S \subset X$ a divisor with normal crossings, $U := X \setminus S$ its open complement and $\bar{u}$ a geometric point of $U \subset X$. Then*

- *the group $\pi_1^t(U, \bar{u})$ is profinite and its action on $F_{\bar{u}}^{t,S}(Y)$ is continuous for every Y in $Fet^{t,S}/X$.*

- *The fibre functor $F_{\bar{u}}^{t,S}$ induces an equivalence of the category $Fet^{t,S}/X$ with the category of finite continuous left $\pi_1^t(U, \bar{u})$-sets. Connected covers corresponds to sets with transitive action, and Galois covers to finite quotients of $\pi_1^t(U, \bar{u})$.*

Proof. cf. Corollaire 2.4.4. in [GM71]. □

Example 3.3.8. Let $X = \mathbb{P}^n_{k^s}$, $S = \mathbb{P}^{n-1}_{k^s}$ and $U = \mathbb{A}^n_{k^s}$, where k^s is a separable closed field. Then $\pi_1^t(\mathbb{A}^n_{k^s}, \bar{u}) = \{1\}$ is trivial. To prove this we use an induction argument.

For $n = 1$ it is enough to show that any finite map $f\colon Y \longrightarrow \mathbb{P}^1_{k^s}$, that is étale over $\mathbb{A}^1_{k^s} \subset \mathbb{P}^1_{k^s}$ and tamely ramified at infinity, is an isomorphism. Let ω be the differential dt on $\mathbb{P}^1_{k^s}$. It has a double pole at infinity and no other poles or zeroes. By the Hurwitz-Riemann formula we have that $2g(Y) - 2 = n(-2) + r_\infty - 1$, where n is the degree of the map f and r_∞ is the ramification index at infinity. Now we have the inequality $-2n + n - 1 \geq 2g(Y) - 2$, from which it follows that $n = 1$. Now assume that $f\colon Y \longrightarrow \mathbb{P}^{n+1}_{k^s}$ is a finite map, that is étale over $\mathbb{A}^{n+1}_{k^s} \subset \mathbb{P}^{n+1}_{k^s}$ and tamely ramified along $\mathbb{P}^n_{k^s}$. But then the restriction of f to any n-dimensional affine subspace will be tame along any $n-1$-dimensional projective subspace of the complement, hence trivial by the induction hypothesis. So it implies that f must be an isomorphism, completing the proof.

Let Fet/U, Fet/X and Fin/X be the categories of finite étale covers of U, finite étale covers of X and finite covers of X respectively. Then we have the following inclusions of categories as full subcategories

$$Fet/X \subseteq Fet^{t,S}/X \subseteq Fin/X$$

and also

$$Fet^{t,S}/X \subseteq Fet/U$$

which induce the following surjective maps

$$\pi_1^t(U, \bar{u}) \longrightarrow \pi_1(X, \bar{u}) \longrightarrow 1$$

and

$$\pi_1(U, \bar{u}) \longrightarrow \pi_1^t(U, \bar{u}) \longrightarrow 1 \ .$$

The tame étale fundamental group has the usual properties of the algebraic fundamental group:

Proposition 3.3.9. *1. Changing the base point $\bar{u}$ to $\bar{u}'$ the tame fundamental groups become isomorphic $\pi_1^t(U, \bar{u}) \xrightarrow{\cong} \pi_1^t(U, \bar{u}')$ determined up to an inner automorphism.*

2. Let $g \colon X' \longrightarrow X$ be a morphism, where X' is a normal scheme such that $S' := g^{-1}(S)$ is defined and is again a divisor with normal crossings and $\bar{u}'$ a geometric point of $U' \subset X'$, where $U' := X' \setminus S'$ is the open complement. Then there is an induced group homomorphism $\pi_1^t(U', \bar{u}') \longrightarrow \pi_1^t(U, \bar{u})$ determined up to an inner automorphism of $\pi_1^t(U, \bar{u})$.

Proof. cf. 2.4.5 in [GM71]. □

Chapter 4

The Proof of the Main Theorem of the Tamely Ramified Theory

In this chapter we begin with final preliminary works proving the correspondences appearing in diagram 3.1.2 and after that we give a geometric proof of the Main Theorem 3.1.4 adapting Deligne's geometric argument.

4.1 Preliminary Constructions

Let C be a smooth, projective, geometrically irreducible curve of genus g over a finite field $k = \mathbb{F}_q$, $S = \{p_1, p_2, \ldots, p_n\} \subset |C|$ a finite set of closed points and $U := C \setminus S$ the open complement of S.

Let us begin to prove the correspondence between representations of $\pi_1^{t,ab}(U)$ and local systems on U which are tame at S. First we define 1-dimensional l-adic local systems on U which are tame at S.

Definition 4.1.1. Let $\mathfrak{L}$ be a 1-dimensional l-adic local system on $U \subset C$ defined by a continuous 1-dimensional l-adic representation $\chi\colon \pi_1(U, \bar{u}) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$. Then $\mathfrak{L}$ is said to be tame at $S = X \setminus U$ if for every $s \in S$ the induced map

$$\pi_1(Spec(K_s),\bar{s}) \longrightarrow \pi_1(U,\bar{u}) \xrightarrow{\chi} \overline{\mathbb{Q}}_l^{\times}$$

where K_s is the stalk at the generic point of $Spec(\mathcal{O}_{X,s})$, induces a factorization

$$\begin{array}{ccc} \pi_1(Spec(K_s),\bar{s}) \longrightarrow \pi_1(U,\bar{u}) & \xrightarrow{\quad\chi\quad} & \overline{\mathbb{Q}}_l^{\times} \\ \searrow & \circlearrowleft \quad \exists & \nearrow \\ & \pi_1^t(Spec(K_s),\bar{s}) \ . & \end{array}$$

In other words the induced representation $\pi_1(Spec(K_s),\bar{s}) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ is trivial on the *inertia subgroup* $I_s \subset \pi_1(Spec(K_s),\bar{s})$, where the inertia subgroup is defined as the kernel of the natural surjective map $\pi_1(Spec(K_s),\bar{s}) \twoheadrightarrow \pi_1^t(Spec(K_s),\bar{s})$.

Now we can state

Theorem 4.1.2. *There is a one-to-one correspondence between*

continuous 1-dimensional l-adic representations of $\pi_1^{t,ab}(U)$ $\{\chi\colon \pi_1^{t,ab}(U) \to \overline{\mathbb{Q}}_l^{\times}\}$	$\overset{1:1}{\longleftrightarrow}$	1-dimensional l-adic local systems $\mathfrak{L}$ on U which are tame at S together with a fixed isomorphism $\varphi : \mathfrak{L}_{\bar{u}} \cong \overline{\mathbb{Q}}_l$

Proof. We consider the following diagram of categories and the natural functors between them

$$\begin{array}{ccc} Fet/U & \xrightarrow{\text{restriction}} & Fet/Spec(K_s) \\ \uparrow & & \uparrow \\ Fet^{t,S}/X & \xrightarrow{\text{restriction}} & Fet^t/Spec(K_s) \end{array}$$

which is a push out diagram, hence it induces a push out diagram

$$\begin{array}{ccc} \pi_1(U,\bar{u}) & \longleftarrow & \pi_1(Spec(K_s),\bar{s}) \\ \downarrow\!\!\downarrow & & \downarrow\!\!\downarrow \\ \pi_1^t(U,\bar{u}) & \longleftarrow & \pi_1^t(Spec(K_s),\bar{s}) \end{array} .$$

Now given a continuous 1-dimensional l-adic representation $\chi\colon \pi_1^t(U,\bar{u}) \longrightarrow \overline{\mathbb{Q}}_l^\times$, it induces a continuous 1-dimensional l-adic representation $\chi\colon \pi_1(U,\bar{u}) \longrightarrow \overline{\mathbb{Q}}_l^\times$ such that the restriction map

$$\pi_1(Spec(K_s),\bar{s}) \longrightarrow \pi_1(U,\bar{u}) \xrightarrow{\chi} \overline{\mathbb{Q}}_l^\times$$

induces a factorization at each $s \in S$

$$\begin{array}{ccccc} \pi_1(Spec(K_s),\bar{s}) & \longrightarrow & \pi_1(U,\bar{u}) & \xrightarrow{\chi} & \overline{\mathbb{Q}}_l^\times \\ & \searrow & \circlearrowleft & \overset{\exists}{\nearrow} & \\ & & \pi_1^t(Spec(K_s),\bar{s}) & & \end{array}$$

that is we have a rigidified 1-dimensional l-adic local system $\mathfrak{L}$ on U which is tame at S by section 1.5 and by the definition of tameness of a local system.

On the other hand if we are given a rigidified 1-dimensional l-adic local system $\mathfrak{L}$ on U which is tame at S, it induces a continuous 1-dimensional l-adic representation $\chi\colon \pi_1(U,\bar{u}) \longrightarrow \overline{\mathbb{Q}}_l^\times$ together with the above factorization, hence the above push out diagram induces a continuous 1-dimensional l-adic representation $\chi\colon \pi_1^t(U,\bar{u}) \longrightarrow \overline{\mathbb{Q}}_l^\times$. These constructions are inverses to each other. Now we can pass to the abelianized fundamental groups, as 1-dimensional representations naturally factor through them completing the proof of the Theorem. $\square$

Next we want to analyze the tamely ramified Abel-Jacobi map (cf. section 3.2) and determine the fibres.

Given an integer $d \geq 1$ and a geometric point $([\mathcal{G}], \{\psi_p\}_{p\in S}) \in Pic^d_{C,S}$ we have the following condition concerning the surjectivity of the tamely ramified Abel-Jacobi map

Condition 4.1.3. The fibre $\mathfrak{AJ}^{-1}_{S,d}([\mathcal{G}], \{\psi_p\}_{p\in S}) \neq \emptyset$ is not empty if there exists a non-zero global section $s \in H^0(C, \mathcal{G}) \setminus \{0\}$ generating the fibre $\mathcal{G} \otimes_{\mathcal{O}_C} \mathcal{O}_p$ at each $p \in S$ such that the induced map $s_p \colon \mathcal{G} \otimes_{\mathcal{O}_C} \mathcal{O}_p/\mathfrak{m}_p \longrightarrow \mathcal{O}_p/\mathfrak{m}_p$ coincides with the fixed trivialization $\psi_p \colon \mathcal{G} \otimes_{\mathcal{O}_C} \mathcal{O}_p/\mathfrak{m}_p \longrightarrow \mathcal{O}_p/\mathfrak{m}_p$ for all $p \in S$.

Now consider the exact sequence of sheaves on C

$$0 \longrightarrow \mathcal{O}(-S) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_S \longrightarrow 0$$

where $\mathcal{O}_C$ is the structure sheaf of C, $\mathcal{O}(-S)$ is the ideal sheaf of the closed subscheme $S \subset C$ and $\mathcal{O}_S$ is the structure sheaf of S. As $\mathcal{G}$ is locally free, tensoring with $\mathcal{G}$ we get the exact sequence

$$0 \longrightarrow \mathcal{G}(-S) \longrightarrow \mathcal{G} \longrightarrow \mathcal{G}/\mathcal{G}(-S) \longrightarrow 0$$

where $\mathcal{G}/\mathcal{G}(-S) = \bigoplus_{p\in S}(\mathcal{G} \otimes_{\mathcal{O}_C} \mathcal{O}_p/\mathfrak{m}_p)$. This exact sequence induces a long exact sequence in cohomology

$$0 \longrightarrow H^0(C, \mathcal{G}(-S)) \longrightarrow H^0(C, \mathcal{G}) \xrightarrow{ev} H^0(C, \mathcal{G}/\mathcal{G}(-S)) \xrightarrow{\delta} H^1(C, \mathcal{G}(-S)) \longrightarrow \cdots$$

Then Condition 4.1.3 is equivalent to

$$\{\psi_p\}_{p\in S} \in Ker(\delta)$$

In particular this is the case if the evaluation map in cohomology is surjective

$$H^0(C,\mathcal{G}) \xrightarrow{ev} H^0(C,\mathcal{G}/\mathcal{G}(-S)) \longrightarrow 0.$$

From the long exact sequence it follows that this is the case if

$$H^1(C,\mathcal{G}(-S)) = 0.$$

By Serre duality

$$H^1(C,\mathcal{G}(-S)) \cong H^0(\omega_C \otimes \mathcal{G}(-S)^{-1})^\vee,$$

hence $H^1(C,\mathcal{G}(-S))$ vanishes if $deg(\mathcal{G}(-S)) \geq 2g-1$. From the exact sequence we started with it follows that

$$deg(\mathcal{G}(-S)) = d - \sharp S$$

where $d = deg(\mathcal{G})$. Putting everything together we get

Proposition 4.1.4. *The tamely ramified Abel-Jacobi map $\mathfrak{AJ}_{S,d}$ is surjective if $d \geq 2g-1+\sharp S$.*

Now we can determine the fibres of the map $\mathfrak{AJ}_{S,d}$ for $d \geq 2g-1+\sharp S$.

Proposition 4.1.5. *Given a geometric point $(\mathcal{G},\{\psi_p\}_{p\in S}) \in Pic^d_{C,S}$ and an integer d satisfying $d \geq 2g-1+\sharp S$, we have the exact sequence*

$$0 \longrightarrow H^0(C,\mathcal{G}(-S)) \longrightarrow H^0(C,\mathcal{G}) \xrightarrow{ev} H^0(C,\mathcal{G}/\mathcal{G}(-S)) \xrightarrow{\delta} 0$$

and for the fibre we have the isomorphism

$$\chi : ev^{-1}(\{\psi_p\}_{p\in S}) \xrightarrow{\cong} \mathfrak{AJ}^{-1}_{S,d}((\mathcal{G},\{\psi_p\}_{p\in S}))$$

given by

$$s \mapsto div(s)$$

for an $s \in ev^{-1}(\{\psi_p\}_{p\in S}) \subset H^0(C, \mathcal{G})$.

Proof. Injectivity: given sections $s, s^{'} \in ev^{-1}(\{\psi_p\}_{p\in S})$ with $div(s) = div(s^{'})$, then by definition we have that

$$s - s^{'} \in Ker(ev) \cong H^0(C, \mathcal{G}(-S))$$

so $s^{'} = s - t$ for some $t \in H^0(C, \mathcal{G}(-S))$. But we also have that

$$t \in H^0(C, \mathcal{G} \otimes \mathcal{O}(-div(s)))$$

because of the equality $div(s) = div(s^{'})$. Now the degree

$$deg(\mathcal{G} \otimes \mathcal{O}(-div(s))) = 0$$

is zero, hence $H^0(C, \mathcal{G} \otimes \mathcal{O}(-div(s))) = k$, that is t must be a constant. But as $t \in H^0(C, \mathcal{G}(-S))$ as well, which means that t vanishes at the points $p \in S$, it follows that $t = 0 \in k$, hence $s = s^{'}$.

Surjectivity: Given an effective divisor $D \in \mathfrak{AJ}_{S,d}^{-1}((\mathcal{G}, \{\psi_p\}_{p\in S}))$, it defines a canonical section $s_D\colon \mathcal{O}_C \hookrightarrow \mathcal{O}(D)$ satisfying $0 \neq s_D|_p = \psi_p$ for all $p \in S$ and $\mathcal{O}(D) \cong \mathcal{G}$. This means that $s_D \in ev^{-1}(\{\psi_p\}_{p\in S}) \subset H^0(C, \mathcal{G})$ and $div(s) = D$, which completes the proof of the Proposition. □

The main result that we will need in the next section is

Theorem 4.1.6. *If the degree satisfies* $d \geq 2g - 1 + \sharp S$ *then the tamely ramified Abel-Jacobi map* $\mathfrak{AJ}_{S,d}\colon U^{(d)} \longrightarrow Pic_{C,S}$ *is a surjective morphism. For a geometric point* $([\mathcal{G}], \{\psi_p\}_{p\in S}) \in Pic^d_{C,S}$ *the fibre is isomorphic to the affine subspace*

$$\mathfrak{AJ}_{S,d}^{-1}((\mathcal{G},\{\psi_p\}_{p\in S}))\cong ev^{-1}(\{\psi_p\}_{p\in S})\subset H^0(C,\mathcal{G})$$

which is an $H^0(C,\mathcal{G}(-S))$*-torsor.*

4.2 A Geometric Proof

In this section we turn to the proof of the Main Theorem 3.1.4 using a geometric argument adapting Deligne's proof of the unramified theory.

First we want to prove the tame version of Deligne's Theorem which gives the connection between rigidified 1-dimensional l-adic local systems on U which are tame at S and rigidified multiplicative 1-dimensional l-adic local systems on the Picard scheme $Pic_{C,S}$ appearing in the bottom row of diagram 3.1.2.

Theorem 4.2.1. *There is a one-to-one correspondence between*

1-*dimensional* l-*adic local systems* $\mathfrak{L}$ *on* U *which are tame at* S *together with a fixed isomorphism* $\varphi : \mathfrak{L}_{\bar{u}} \cong \overline{\mathbb{Q}}_l$	$\overset{1:1}{\longleftrightarrow}$	1-*dimensional* l-*adic local systems* $\mathcal{A}_{\mathfrak{L}}$ *on* $Pic_{C,S}$ *together with a fixed isomorphism* $\psi : \mathcal{A}_{\mathfrak{L}}\vert_0 \cong \overline{\mathbb{Q}}_l$ *satisfying* $m^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e} = \mathcal{A}_{\mathfrak{L}}^{d} \boxtimes \mathcal{A}_{\mathfrak{L}}^{e}$

Proof. Given a 1-dimensional l-adic local system $\mathfrak{L}$ on U which is tame at S we can construct a 1-dimensional l-adic local system $\mathfrak{L}^{(d)}$ on $U^{(d)}$ in the same way as we did in section 2.1 by defining

$$\mathfrak{L}^{(d)} := ((Sym^d|_{U^d})_* \mathfrak{L}^{\boxtimes d})^{S_d}$$

where $Sym^d \colon C^d \longrightarrow C^{(d)}$ is the symmetrization morphism. Moreover this 1-dimensional l-adic local system $\mathfrak{L}^{(d)}$ will be tame at the codimension 1 part of the complement $C^{(d)} \setminus U^{(d)} \supset Z := \bigcup_{s \in S} s \times U^{(d-1)}$ by the following argument. Let η be the generic point of $s \times U^{(d-1)}$ and $\tilde{\eta}$ a generic point of $s \times U^d$ mapping to η under the symmetrization morphism. Assume that the 1-dimensional l-adic local system $\mathfrak{L}^{(d)}$ on $U^{(d)}$ is defined by the continuous 1-dimensional l-adic representation $\chi\colon \pi_1(U^{(d)}, \bar{u}) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$. Then we have to prove that the induced representation $\chi\colon \pi_1(K_\eta, \bar{\eta}) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ factors through $\pi_1^t(K_\eta, \bar{\eta})$. The morphism Sym^d is étale over the open subscheme $C^{(d)} \setminus \Delta \subset C^{(d)}$, where Δ is the big diagonal, that is where some of the points coincide, which means that Sym^d is étale over the generic point η. By Proposition 3.3.4 we know that tame étale coverings of $U^{(d)}$ form a full subcategory of the tame étale coverings of U^d, hence there is a surjective morphism between the inertia groups $I_{\tilde{\eta}} \twoheadrightarrow I_\eta$. Now assume that $\mathfrak{L}^{\boxtimes d}$ is tame at $\tilde{\eta}$. Then we claim that $\mathfrak{L}^{(d)}$ is tame at η. To prove this we recall that $(Sym^d)^{-1}\mathfrak{L}^{(d)} \xrightarrow{\cong} \mathfrak{L}^{\boxtimes d}$, so tameness of $\mathfrak{L}^{\boxtimes d}$ at $\tilde{\eta}$ means that we have the commutative diagram

$$\begin{array}{ccccc}
I_{\tilde{\eta}} & \twoheadrightarrow & I_\eta & & \\
\downarrow & & \downarrow & & \\
\pi_1(K_{\tilde{\eta}}, \bar{\tilde{\eta}}) & \longrightarrow & \pi_1(K_\eta, \bar{\eta}) & \xrightarrow{\chi} & \overline{\mathbb{Q}}_l^{\times} \\
\downarrow & & \downarrow & \nearrow & \\
\pi_1^t(K_{\tilde{\eta}}, \bar{\tilde{\eta}}) & \longrightarrow & \pi_1^t(K_\eta, \bar{\eta}) & & .
\end{array}$$

As by assumption the representation $\pi_1(K_{\tilde{\eta}}, \bar{\tilde{\eta}}) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ becomes trivial on the inertia subgroup $I_{\tilde{\eta}}$ and the map between the inertia subgroups $I_{\tilde{\eta}} \twoheadrightarrow I_\eta$ is surjective, it follows that the representation $\pi_1(K_\eta, \bar{\eta}) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ also becomes trivial on the inertia subgroup I_η, hence $\mathfrak{L}^{(d)}$ is tame at η, as wanted.

It remains to prove that $\mathfrak{L}^{\boxtimes d}$ is tame along $\bigcup_{s \in S} s \times U^{d-1}$. As this is a product, we have to prove that $\mathfrak{L}^{\boxtimes d-1}$ is tame along $\bigcup_{s \in S} s \times U^{d-2}$. It finally reduces to show that $\mathfrak{L}$ is tame along $S \subset C$, which is our starting assumption, completing the proof of the claim, that $\mathfrak{L}^{(d)}$ is tame along $Z := \bigcup_{s \in S} s \times U^{(d-1)}$.

Now consider the universal family of invertible sheaves $\mathcal{G}_{univ}$ on

$$C \times_{Spec(k)} Pic^d_{C,S} \xrightarrow{\pi} Pic^d_{C,S}.$$

By definition for each closed point $y := [\mathcal{G}, \{\psi_p\}_{p\in S}] \in Pic^d_{C,S}$ the restrictions $\mathcal{G}_{univ,y}$ of the universal invertible sheaf to the fibres C_y have degree d and if the degree satisfies $d \geq 2g - 1 + \sharp S$ then we have fibrewise exact sequences

$$0 \to H^0(C_y, \mathcal{G}_{univ,y}(-S)) \to H^0(C_y, \mathcal{G}_{univ,y}) \to H^0(C_y, \mathcal{G}_{univ,y}/\mathcal{G}_{univ,y}(-S)) \to 0.$$

Now we need the following theorem

Theorem 4.2.2 (Grauert, Grothendieck)**.** *Let $f\colon X \longrightarrow Y$ be a projective morphism of noetherian schemes with Y integral and $\mathcal{F}$ a coherent sheaf on X, flat over Y. Suppose that for some i the function*

$$h^i(y, \mathcal{F}) := dim_{k(y)} H^i(X_y, \mathcal{F}_y)$$

is constant on Y. Then $R^i f_*(\mathcal{F})$ is locally free on Y and for every y the natural map

$$R^i f_*(\mathcal{F}) \otimes k(y) \longrightarrow H^i(X_y, \mathcal{F}_y)$$

is an isomorphism.

Proof. cf. II.12.9 Corollary in [Har06]. □

By this theorem we get an exact sequence of locally free sheaves on $Pic^d_{C,S}$

$$0 \longrightarrow \pi_*\mathcal{G}_{univ}(-S) \longrightarrow \pi_*\mathcal{G}_{univ} \xrightarrow{ev} \pi_*\mathcal{G}_{univ}/\mathcal{G}_{univ}(-S) \longrightarrow 0. \tag{4.2.1}$$

As the restrictions $\mathcal{G}_{univ,y}$ have trivializations $\psi_{univ,p}\colon \mathcal{G}_{univ,y} \otimes_{\mathcal{O}_{C_y}} \mathcal{O}_{C_y}/\mathfrak{m}_{p,y} \xrightarrow{\cong} \mathcal{O}_{C_y}/\mathfrak{m}_{p,y}$ at each point $p \in S$, the sheaf $\pi_*\mathcal{G}_{univ}/\mathcal{G}_{univ}(-S)$ has a canonical global section s_{univ} given by these trivializations. Now we claim that $U^{(d)}$ is isomorphic to the closed subscheme $ev^{-1}s_{univ} \subset \pi_*\mathcal{G}_{univ}$ fitting into the commutative diagram

$$\begin{array}{ccc} ev^{-1}s_{univ} \subset \pi_*\mathcal{G}_{univ} & \xrightarrow{ev} & \pi_*\mathcal{G}_{univ}/\mathcal{G}_{univ}(-S) \\ \cong \uparrow & \searrow \qquad \swarrow \ \uparrow s_{univ} & \\ U^{(d)} & \xrightarrow{\mathfrak{AJ}_{S,d}} & Pic^d_{C,S} \ . \end{array}$$

To prove this we consider a closed point $y := [\mathcal{G}, \{\psi_p\}_{p\in S}] \in Pic^d_{C,S}$. For the fibre $\mathfrak{AJ}^{-1}_{S,d}(y)$ we have the isomorphism $\mathfrak{AJ}^{-1}_{S,d}(y) \cong ev^{-1}\{\psi_p\}_{p\in S}$. The image of y under the universal section is just $s_{univ}(y) = \{\psi_p\}_{p\in S}$, thus we get the isomorphism $\mathfrak{AJ}^{-1}_{S,d}(y) \cong ev^{-1}s_{univ}(y)$. This gives a morphism $U^{(d)} \longrightarrow ev^{-1}s_{univ}$. To construct the inverse map we note that $\pi_*\mathcal{G}_{univ}$ parametrizes tuples $(\mathcal{G}, s, \{\psi_p\}_{p\in S})$, where $\mathcal{G}$ is an invertible sheaf on C, $s \in H^0(C,\mathcal{G}) \setminus \{0\}$ is a non-zero global section which does not vanish at the points $p \in S$ and $\{\psi_p\}_{p\in S}$ are fixed isomorphisms $\psi_p \colon \mathcal{G} \otimes_{\mathcal{O}_C} \mathcal{O}_p/\mathfrak{m}_p \cong \mathcal{O}_p/\mathfrak{m}_p$. It follows from this description that $ev^{-1}s_{univ}$ parametrizes tuples $(\mathcal{G}, s, \{\psi_p\}_{p\in S})$ where $\psi_p(s_p) = 1 \in \mathcal{O}_p/\mathfrak{m}_p$ at each $p \in S$ and we can define a morphism $ev^{-1}s_{univ} \longrightarrow U^{(d)}$ by sending

$$(\mathcal{G}, s, \{\psi_p\}_{p\in S}, \psi_p(s_p) = 1) \mapsto div(s),$$

that is inverse to the morphism $U^{(d)} \longrightarrow ev^{-1}s_{univ}$. Moreover $ev^{-1}s_{univ} \subset \pi_*\mathcal{G}_{univ}$ is a closed subscheme being the inverse image of the closed subscheme $s_{univ}Pic^d_{C,S} \subset \pi_*\mathcal{G}_{univ}/\mathcal{G}_{univ}(-S)$, proving the claim.

Next we take a projective closure of $U^{(d)}$ defined by the following diagram

$$
\begin{array}{ccccc}
\mathbb{P}(\pi_*\mathcal{G}_{univ} \oplus \mathcal{O}) & \hookleftarrow & \pi_*\mathcal{G}_{univ} & & \\
\uparrow & & \uparrow & & \\
\overline{U^{(d)}} & \overset{j}{\hookleftarrow} & U^{(d)} & \xrightarrow{\mathfrak{AJ}_{S,d}} & Pic^d_{C,S}
\end{array}
$$

$$\overline{\mathfrak{AJ}_{S,d}} : \overline{U^{(d)}} \longrightarrow Pic^d_{C,S}$$

where the morphism $\overline{\mathfrak{AJ}_{S,d}}$ is defined by using the restriction of the natural projection $\mathbb{P}(\pi_*\mathcal{G}_{univ} \oplus \mathcal{O}) \longrightarrow \pi_*\mathcal{G}_{univ}$ followed by the bundle map $\pi_*\mathcal{G}_{univ} \longrightarrow Pic^d_{C,S}$.

As the sheaves in the sequence 4.2.2 are locally free, the closure of $U^{(d)}$ will look like locally as the closure of an affine subspace in the projective closure of affine space, hence it is isomorphic to a projective space of dimension one less than the dimension of the affine subspace. It follows that the codimension 1 part of the complement $Z \subseteq \overline{U^{(d)}} \setminus U^{(d)}$ is a smooth divisor.

Now I claim that the local system $\mathfrak{L}^{(d)}$ on $U^{(d)}$ can be extended to $\overline{U^{(d)}}$.

Claim 4.2.3. The push forward $\overline{\mathfrak{L}^{(d)}} := j_*\mathfrak{L}^{(d)}$ of the 1-dimensional l-adic local system $\mathfrak{L}^{(d)}$ defined on $U^{(d)}$ which is tame at $\bigcup_{s\in S} s \times U^{(d-1)}$ is a 1-dimensional l-adic local system on $\overline{U^{(d)}}$.

Proof. It suffices to prove this locally at the generic points of the smooth divisor $Z \subseteq \overline{U^{(d)}} \setminus U^{(d)}$, because any subset $V \subset \overline{U^{(d)}} \setminus U^{(d)}$ of codimension ≥ 2 can be disregarded by I.§5.(h) in [Mil80]. So let $\tilde{\eta}$ be such a generic point with image $\overline{\mathfrak{AJ}_{S,d}}(\tilde{\eta}) = \eta \in Pic^d_{C,S}$. Now the tamely ramified Abel-Jacobi map is Zariski locally trivial, as the exact sequence of locally free sheaves on $Pic^d_{C,S}$

$$0 \longrightarrow \pi_*\mathcal{G}_{univ}(-S) \longrightarrow \pi_*\mathcal{G}_{univ} \xrightarrow{ev} \pi_*\mathcal{G}_{univ}/\mathcal{G}_{univ}(-S) \longrightarrow 0 \qquad (4.2.2)$$

is Zariski locally a split exact sequence, so we get for the geometric fibre

$$\mathfrak{AJ}_{S,d}^{-1}(\bar{\eta}) \cong \bar{\eta} \times \mathbb{A}_{k^s}^n$$

where $n = d - g + 1 - \sharp S$. Now assume that $\mathfrak{L}^{(d)}$ is tame on $U^{(d)}$ along the divisor in the compactification $\overline{U^{(d)}}$. Then the restriction

$$\mathfrak{L}^{(d)}|_{\bar{\eta} \times \mathbb{A}_{k^s}^n}$$

is defined by a continuous 1-dimensional l-adic representation

$$\chi \colon \pi_1^{t,ab}(\mathbb{A}_{k^s}^n) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

which must be trivial, as $\pi_1^{t,ab}(\mathbb{A}_{k^s}^n) = 1$ is trivial by Example 3.3.8. It then follows that $\mathfrak{L}^{(d)}$ extends naturally to

$$\eta \times \overline{\mathbb{A}_{k^s}^n} = \eta \times \mathbb{P}_{k^s}^{n-1}$$

and hence defines a 1-dimensional l-adic local system $\overline{\mathfrak{L}^{(d)}}$ on $\overline{U^{(d)}}$, which is uniquely defined by the construction.

So it remains to prove that $\mathfrak{L}^{(d)}$ is tame on $U^{(d)}$ along the divisor in the compactification $\overline{U^{(d)}}$. For that we recall that the closed subscheme $U^{(d)} \subset \pi_* \mathcal{G}_{univ}$ can be described as isomorphism classes of tuples $(\mathcal{G}, s, \{\psi_p\}_{p \in S})$ where $\psi_p(s_p) = 1 \in \mathcal{O}_p/\mathfrak{m}_p$ at each $p \in S$. So the projective closure $\overline{U^{(d)}} \subset \mathbb{P}(\pi_* \mathcal{G}_{univ} \oplus \mathcal{O})$ is given as the isomorphism classes of $(\mathcal{G}, [s : t], \{\psi_p\}_{p \in S})$ where $\psi_p(s) = t$ for all $p \in S$ and $(s, t) \in H^0(C, \mathcal{G}) \oplus k$ (as U^d is given by setting $t = 1$). The section s can not be 0 at any point in $\overline{U^{(d)}}$, because the condition $\psi_p(s) = t$ would imply $t = 0$, which can not be the case. Thus we can define a map

$$\alpha \colon \overline{U^{(d)}} \longrightarrow C^{(d)}$$

by sending $(\mathcal{G}, [s : t], \{\psi_p\}_{p \in S}, (\psi_p(s) = t)_{p \in S}) \mapsto (\mathcal{G}, s))$. We can describe the boundary as follows

$$\partial\overline{U^{(d)}} := \overline{U^{(d)}} \setminus U^{(d)} = \{(\mathcal{G}, [s:t], \{\psi_p\}_{p\in S}) \mid t = 0, s_p = 0\} = \mathbb{P}H^0(C, \mathcal{G}(-S)).$$

Thus the image of the boundary is $\alpha(\partial\overline{U^{(d)}}) = S \times C^{(d-\sharp S)} \subset C^{(d)}$. Now consider the following fibre product diagram

$$\begin{array}{ccc} C^d \times_{C^{(d)}} \overline{U^{(d)}} & \xrightarrow{\ q\ } & C^d \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle Sym} \\ \overline{U^{(d)}} & \xrightarrow{\ \alpha\ } & C^{(d)} \end{array} .$$

Since the local system $\mathfrak{L}^{\boxtimes d}$ is tame on C^d along $\bigcup_{s\in S} s\times C^{d-1}$, by Proposition 3.3.4 the pull-back $q^{-1}\mathfrak{L}^{\boxtimes d}$ will be tame on $C^d \times_{C^{(d)}} \overline{U^{(d)}}$ along the pull-back $q^{-1}(\bigcup_{s\in S} s\times C^{d-1})$. Now as before at the generic point η of $\alpha(\partial\overline{U^{(d)}})$ the symmetrization morphism Sym is étale, thus the projection p is étale over a generic point $\tilde{\eta}$ of $\partial\overline{U^{(d)}}$, hence using the descent argument as before we can conclude that $\mathfrak{L}^{(d)}$ is tame on $\overline{U^{(d)}}$ along $\partial\overline{U^{(d)}}$, completing the proof of the claim. □

Now we can apply the relative homotopy exact sequence theorem (1.4.13) to the compactified tamely ramified Abel-Jacobi map $\overline{\mathfrak{AJ}_{S,d}}\colon \overline{U^{(d)}} \longrightarrow Pic^d_{C,S}$ taking into account Theorem 4.1.6 to get the exact sequence

$$\pi_1(\mathbb{P}^{d-g-\sharp S}_{k^s}, \bar{u}) \longrightarrow \pi_1(\overline{U^{(d)}}, \bar{u}) \longrightarrow \pi_1(Pic^d_{C,S}, \bar{u}) \longrightarrow 1$$

where k^s is the separable closure of the base field k. We know from Example 1.4.12 that $\pi_1(\mathbb{P}^{d-g-\sharp S}_{k^s}, \bar{u}) = 1$ is trivial, hence we get an isomorphism

$$\pi_1(\overline{U^{(d)}}, \bar{u}) \cong \pi_1(Pic^d_{C,S}, \bar{u})$$

which induces an isomorphism between the abelianized étale fundamental groups

$$\pi_1^{ab}(\overline{U^{(d)}}) \cong \pi_1^{ab}(Pic^d_{C,S}).$$

This means that to the continuous 1-dimensional l-adic representation

$$\chi_{\overline{\mathfrak{L}^{(d)}}}\colon \pi_1^{ab}(\overline{U^{(d)}}) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

corresponds a unique continuous 1-dimensional l-adic representation

$$\chi_{\mathcal{A}_{\mathfrak{L}}^d} : \pi_1^{ab}(Pic_{C,S}^d) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

which again corresponds (section 1.5) to a unique 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}^d$ on $Pic_{C,S}^d$ together with a fixed isomorphism $\psi\colon \mathcal{A}_{\mathfrak{L}}^d|_{d0} \cong \overline{\mathbb{Q}}_l$. To prove further that $\mathcal{A}_{\mathfrak{L}}^d$ extends to $Pic_{C,S}$ and satisfies the multiplicative property we refer to the proof of Deligne's Theorem 2.2.1.

On the other way round let a rigidified multiplicative 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}$ on $Pic_{C,S}$ be given. Then we consider the tamely ramified Abel-Jacobi map

$$\mathfrak{AJ}_{S,1}\colon U \longrightarrow Pic_{C,S}^1$$

and by Proposition 1.5.12 the pull-back

$$\mathfrak{L} := \mathfrak{AJ}_{S,1}^{-1}\mathcal{A}_{\mathfrak{L}}^1$$

is a 1-dimensional l-adic local system on U together with a rigidification. We have to prove that $\mathfrak{L}$ is tame along $S \subset C$. For that we note that $\mathcal{A}_{\mathfrak{L}}^0$ on $Pic_{C,S}^0$ is defined by a 1-dimensional l-adic representation $\chi\colon Pic_{C,S}^0(k) \longrightarrow \overline{\mathbb{Q}}_l^\times$ by Theorem 4.2.4. Now consider the multiplication by $q-1$ map

$$[q-1]\colon Pic_{C,S}^0(k) \longrightarrow Pic_{C,S}^0(k)$$

fitting into the diagram

$$\begin{array}{ccccc}
(k^{\times})^{\sharp S-1} & \hookrightarrow & Pic^0_{C,S}(k) & \twoheadrightarrow & Pic^0_C(k) \\
\downarrow [q-1] & & \downarrow [q-1] & & \downarrow [q-1] \\
(k^{\times})^{\sharp S-1} & \hookrightarrow & Pic^0_{C,S}(k) & \twoheadrightarrow & Pic^0_C(k)\ .
\end{array}$$

It follows from this that the pull-back $[q-1]^{-1}\mathcal{A}^0_{\mathfrak{L}}$ on $Pic^0_{C,S}$ will come from a local system on Pic^0_C as the induced representation $Pic^0_{C,S}(k) \stackrel{[q-1]}{\longrightarrow} Pic^0_{C,S}(k) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ is trivial on $(k^{\times})^{\sharp S-1}$. The map $[q-1]$ is certainly tame for any smooth compactification $\overline{Pic^0_{C,S}}$, so using the isomorphism $Pic^0_{C,S} \cong Pic^1_{C,S}$ and pulling back everything to U we get a finite cover $U' \longrightarrow U$ and a local system $\mathfrak{A}\mathfrak{J}^{-1}_{S,1}[q-1]^{-1}\mathcal{A}^1_{\mathfrak{L}}$ on U'. But the induced map $U' \longrightarrow Pic^1_{C,S} \longrightarrow Pic^1_C$ will extend to a smooth compactification C' of U' as the scheme Pic^1_C is compact. In particular the finite cover $C' \longrightarrow U$ is tame along $C' \setminus U'$. So we find that the pull-back $\mathfrak{A}\mathfrak{J}^{-1}_{S,1}[q-1]^{-1}\mathcal{A}^1_{\mathfrak{L}}$ is unramified over C' as it comes from Pic^1_C, thus the local system $\mathfrak{L} = \mathfrak{A}\mathfrak{J}^{-1}_{S,1}\mathcal{A}^1_{\mathfrak{L}}$ becomes unramified after pull-back to a tame covering, hence certainly tame on U with respect to $C \setminus U$, completing the proof of the Theorem. □

Next we want to discuss the tame version of the faisceaux-fonctions correspondence appearing in the right vertical side of diagram 3.1.2. Assume we have a closed point $p \in U(k) \subset C(k)$, then we have the following

Theorem 4.2.4. *There is a one-to-one correspondence between*

1-*dimensional* l-*adic representations of* $Pic^0_{C,S}(k)$ $\{\chi : Pic^0_{C,S}(k) \to \overline{\mathbb{Q}}_l^\times\}$ *together with a Frobenius action* $Frob_p \colon \widehat{\mathbb{Z}} \to \overline{\mathbb{Q}}_l^\times$	$\overset{1:1}{\longleftrightarrow}$	1-*dimensional* l-*adic local systems* $\mathcal{A}_{\mathfrak{L}}$ *on* $Pic_{C,S}$ *together with a fixed isomorphism* $\psi : \mathcal{A}_{\mathfrak{L}}\|_0 \cong \overline{\mathbb{Q}}_l$ *satisfying* $m^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e} = \mathcal{A}_{\mathfrak{L}}^d \boxtimes \mathcal{A}_{\mathfrak{L}}^e$

Proof. First let be given a 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}$ on $Pic_{C,S}$ together with a fixed isomorphism $\psi : \mathcal{A}_{\mathfrak{L}}|_0 \cong \overline{\mathbb{Q}}_l$ satisfying $m^{-1}\mathcal{A}_{\mathfrak{L}}^{d+e} \cong \mathcal{A}_{\mathfrak{L}}^d \boxtimes \mathcal{A}_{\mathfrak{L}}^e$. Then the restriction $\mathcal{A}_{\mathfrak{L}}^0$ on $Pic^0_{C,S}$ will be a 1-dimensional l-adic local system together with a fixed isomorphism $\mathcal{A}_{\mathfrak{L}}^0|_0 \cong \overline{\mathbb{Q}}_l$ satisfying the character sheaf property

$$m^{-1}\mathcal{A}_{\mathfrak{L}}^0 \cong \mathcal{A}_{\mathfrak{L}}^0 \boxtimes \mathcal{A}_{\mathfrak{L}}^0$$

hence by the faisceaux-fonctions correspondence (section 1.6) it will give us a 1-dimensional l-adic representation of $Pic^0_{C,S}(k)$. Also by Theorem 4.2.1 $\mathcal{A}_{\mathfrak{L}}$ defines a 1-dimensional l-adic local system $\mathfrak{L}$ on U which is tame at S, hence a sheaf $\mathfrak{L}_p$ on the point $p \in U(k)$, which is the same as a Frobenius action $Frob_p \colon \widehat{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}_l^\times$ by Definition 1.6.1.

Going on the other way round let be given a 1-dimensional l-adic representation

$$\chi : Pic^0_{C,S}(k) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

together with a Frobenius action $Frob_p \colon \widehat{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}_l^\times$. Then by Definition 1.6.1 the Frobenius action defines a sheaf $\mathfrak{L}_p$ on $p \in U(k)$. Also by section 1.6 χ defines a character sheaf $\mathcal{A}_\chi^0$ on $Pic^0_{C,S}$ together with a fixed trivialization $\mathcal{A}_\chi^0|_{\bar{u}} \cong \overline{\mathbb{Q}}_l$. We can extend this local system to all of $Pic_{C,S}$ by

$$\mathcal{A}_\chi^d := (-dp)^{-1}\mathcal{A}_\chi^0 \otimes \mathfrak{L}_p^d$$

where $-dp$ is the isomorphism

$$-dp := p \times Pic_{C,S}^d \xrightarrow{\cong} Pic_{C,S}^0$$

defined by sending $(\mathcal{G}, \{\psi_p\}_{p\in S})$ to $(\mathcal{G}\otimes\mathcal{O}(-dp), \{\psi_p s_p\}_{p\in S}))$, where s_p is the canonical trivialization at each $p \in S$ defined by the canonical section of $\mathcal{O}(-dp)$. Using now the argument in the proof of Theorem 2.2.3 it follows from this definition that the 1-dimensional l-adic local system $\mathcal{A}_\chi$ on $Pic_{C,S}$ will satisfy

$$m^{-1}\mathcal{A}_\chi^{e+d} \cong \mathcal{A}_\chi^d \boxtimes \mathcal{A}_\chi^e.$$

These two constructions are inverses to each other completing the proof of the Theorem. □

With this theorem we finally finished the proofs of the one-to-one correspondences appearing in diagram 3.1.2.

Now we are able to prove Theorem 3.1.4.

Theorem 4.2.5 (Tamely Ramified Reciprocity Law, geometric form)**.** *The tamely ramified Reciprocity Map*

$$\Phi_{K,S}\colon Div(U) \xrightarrow{\Phi_{K,S}} \pi_1^{t,ab}(U)$$
$$p \longmapsto [Frob_p]$$

induces a map

$$\Phi_{K,S} : Pic_{C,S}(k) \longrightarrow \pi_1^{t,ab}(U)$$

and fits into the commutative diagram

$$\begin{array}{ccccc} Pic^0_{C,S}(k) & \hookrightarrow & Pic_{C,S}(k) & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow & & \downarrow \Phi_{K,S} & & \downarrow can \\ Ker(\varphi) & \hookrightarrow & \pi_1^{t,ab}(U) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}} \end{array}$$

such that there is an induced isomorphism on the kernels

$$Pic^0_{C,S}(k) \xrightarrow{\cong} Ker(\varphi).$$

Proof. First we prove that the homomorphism

$$Div(U) \longrightarrow \pi_1^{t,ab}(U)$$

induces a homomorphism

$$\Phi_{K,S} \colon Pic_{C,S}(k) \longrightarrow \pi_1^{t,ab}(U).$$

For that let us take a continuous, 1-dimensional l-adic representation

$$\chi \colon \pi_1^{t,ab}(U) \longrightarrow \overline{\mathbb{Q}}_l^{\times}.$$

By the geometric constructions we carried out and appearing in diagram 3.1.2, we can associate to χ a 1-dimensional l-adic local system $\mathfrak{L}$ on U which is tame at S, such that for a closed point $p \colon Spec(\mathbb{F}_{q^n}) \longrightarrow U$, the Frobenius element $Frob_p$ acts on the fibre $\mathfrak{L}_{\bar{p}}$ by multiplication by $\chi([Frob_p]) \in \overline{\mathbb{Q}}_l^{\times}$. By Theorem 4.2.1 we can associate to $\mathfrak{L}$ a rigid multiplicative 1-dimensional l-adic local system $\mathcal{A}_{\mathfrak{L}}$ on $Pic_{C,S}$ and by the faisceaux-fonctions correspondence we can associate to $\mathcal{A}_{\mathfrak{L}}$ a 1-dimensional l-adic representation

$$\psi \colon Pic^0_{C,S}(k) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$$

together with a Frobenius action $Frob_x \colon \widehat{\mathbb{Z}} \longrightarrow \overline{\mathbb{Q}}_l^{\times}$, where $x \in U(k)$. The pair $(\psi, Frob_x)$ induces a 1-dimensional l-adic representation (using the same notation)

$$\psi\colon Pic^1_{C,S}(k) \longrightarrow \overline{\mathbb{Q}}_l^\times$$

defined by

$$\psi([\mathcal{O}(p)], \{s_p\}_{p\in S}) := tr_{\mathcal{A}^1_{\mathfrak{L}}}(Frob_{([\mathcal{O}(p)],\{s_p\}_{p\in S})}),$$

i.e. for a closed point $p\colon Spec(\mathbb{F}_{q^n}) \longrightarrow U$ the Frobenius element $Frob_{([\mathcal{O}(p)],\{s_p\}_{p\in S})}$ acts on the fibre $\mathcal{A}^1_{\mathfrak{L}}|_{\overline{([\mathcal{O}(p)],\{s_p\}_{p\in S})}}$ by multiplication by $\psi([\mathcal{O}(p)], \{s_p\}_{p\in S}) \in \overline{\mathbb{Q}}_l^\times$. Now consider the tamely ramified Abel-Jacobi map

$$\mathfrak{AJ}_{S,1}\colon U \longrightarrow Pic^1_{C,S}$$

defined by $p \mapsto ([\mathcal{O}(p)], \{s_p\}_{p\in S})$. By Theorem 4.2.1 we have that

$$\mathfrak{AJ}^{-1}_{S,1}(\mathcal{A}^1_{\mathfrak{L}}) = \mathfrak{L}$$

hence for a closed point $p\colon Spec(\mathbb{F}_{q^n}) \longrightarrow U$ the Frobenius element $Frob_p$ acts on the fibre $\mathfrak{L}_{\bar{p}}$ in the same way as the Frobenius element $Frob_{([\mathcal{O}(p)],\{s_p\}_{p\in S})}$ acts on the fibre $\mathcal{A}^1_{\mathfrak{L}}|_{\overline{([\mathcal{O}(p)],\{s_p\}_{p\in S})}}$, that is

$$\chi([Frob_p]) = \psi([\mathcal{O}(p)], \{s_p\}_{p\in S}),$$

which means that we have the following commutative diagram

$$\begin{array}{ccc} Div(U) & \xrightarrow{\Phi_{K,S}} & \pi_1^{t,ab}(U) \\ \downarrow & \overset{\Phi_{K,S}}{\underset{?}{\nearrow}} & \downarrow \chi \\ Pic_{C,S}(k) & \xrightarrow{\psi} & \overline{\mathbb{Q}}_l^\times \end{array} \tag{4.2.3}$$

where to be able to prove that $\Phi_{K,S}$ factorizes through $Pic_{C,S}(k)$ we need to show that if different closed points $p_1, p_2 \in |U|$ have the same image in $Pic_{C,S}(k)$ then they have the same image $[Frob_{p_1}] = [Frob_{p_2}]$ in $\pi_1^{t,ab}(U)$. By the commutativity of the diagram above, the closed points $p_1, p_2 \in |U|$ with this property must satisfy $\chi([Frob_{p_1}]) =$

$\chi([Frob_{p_2}])$ for all continuous 1-dimensional l-adic representation $\chi\colon \pi_1^{t,ab}(U) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$. But if $[Frob_{p_1}] \neq [Frob_{p_2}]$ then there exists an open normal subgroup $G \subset \pi_1^{t,ab}(U)$ of finite index which contains one of $[Frob_{p_i}]$, but not the other, as $\pi_1^{t,ab}(U)$ being a profinite group is Hausdorff. Hence in the finite abelian group $\pi_1^{t,ab}(U)/G$ it holds for the images that $\overline{[Frob_{p_1}]} \neq \overline{[Frob_{p_2}]}$. As $\pi_1^{t,ab}(U)/G$ is finite abelian, we can construct a representation $\chi\colon \pi_1^{t,ab}(U)/G \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ satisfying $\chi(\overline{[Frob_{p_1}]}) \neq \chi(\overline{[Frob_{p_2}]})$, which induces a representation $\chi\colon \pi_1^{t,ab}(U) \longrightarrow \overline{\mathbb{Q}}_l^{\times}$ satisfying $\chi([Frob_{p_1}]) \neq \chi([Frob_{p_2}])$. It follows that $[Frob_{p_1}] = [Frob_{p_2}]$ hence we have the tamely ramified Reciprocity Map

$$\Phi_{K,S}\colon Pic_{C,S}(k) \longrightarrow \pi_1^{t,ab}(U).$$

Now as we did in the proof of Theorem 1.1.4 we can prove in the same way that we have a commutative diagram

$$\begin{array}{ccccc}
Div^0(U) & \hookrightarrow & Div(U) & \xrightarrow{deg} & \mathbb{Z} \\
\downarrow & & \downarrow{\scriptstyle \Phi_{K,S}} & & \downarrow{\scriptstyle \text{can}} \\
Ker(\varphi) & \hookrightarrow & \pi_1^{ab}(U) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}}\ .
\end{array}$$

Moreover we have that the maps deg and φ extend to $Pic_{C,S}(k)$ and $\pi_1^{t,ab}(U)$, that is we have the commutative diagrams

$$\begin{array}{ccc}
Div(U) & \longrightarrow & Pic_{C,S}k \\
{\scriptstyle deg}\searrow & & \swarrow{\scriptstyle deg} \\
 & \mathbb{Z} &
\end{array}$$

and

$$\begin{array}{ccc} \pi_1^{ab}(U) & \twoheadrightarrow & \pi_1^{t,ab}(U) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\varphi} & \\ & \widehat{\mathbb{Z}} & \end{array}$$

hence we have the commutative diagram

$$\begin{array}{ccccc} Pic^0_{C,S}(k) & \hookrightarrow & Pic_{C,S}(k) & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle \Phi_{K,S}} & & \downarrow{\scriptstyle \text{can}} \\ Ker(\varphi) & \hookrightarrow & \pi_1^{t,ab}(U) & \xrightarrow{\varphi} & \widehat{\mathbb{Z}} \end{array}$$

including the kernels into the picture as well. Now the injectivity of the tamely ramified Reciprocity Map follows from the argument that we used to prove the factorization property of $\Phi_{K,S}$, thus as in the unramified case we have the following commutative diagram

$$\begin{array}{ccccc} Pic^0_{C,S}(k) & \hookrightarrow & Pic_{C,S}(k) & \xrightarrow{deg} & \mathbb{Z} \\ \downarrow & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\ Ker(\varphi) & \hookrightarrow & \Phi_{K,S}(Pic_{C,S}(k)) & \xrightarrow{\varphi} & \mathbb{Z} \end{array}$$

which induces an isomorphism

$$Pic^0_{C,S}(k) \xrightarrow{\cong} Ker(\varphi)$$

completing the Proof of the tamely ramified Reciprocity Law. $\square$

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