Utrecht University

Mathematical Sciences

Master Thesis

Optimal design for auctioning CO_2 Emission Allowances under the European Emissions Trading System

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Executive Summary

In order to meet the obligations under the Kyoto Protocol, the European Union decided in 2003 to introduce the first cap-and-trade system for greenhouse gas emissions in the world.[2] In 2005 the European Emissions Trading System (EU ETS) was launched. Whereas until now the majority of the European Emission Allowances (EUA's) was handed out for free to the installations in the system, starting from 2013, a large amount of EUA's in the ETS will be auctioned. This raises the question how this new primary auction market will affect the secondary market. For a smooth functioning of the ETS, it is important to minimize price distortions. Price distortions on the secondary market will increase the volatility of the secondary market price. The uncertainty about the price in the market will increase, which makes investments in the ETS more expensive. Furthermore, from Member States revenue perspectives, it is important to assure the clearing price paid in the auctions is close to the secondary market price.

Since the ETS is still very young and the large-scale auctioning of approximately 1 billion EUA's per year was never done before, it is very hard to estimate how the market will react to these auctions. Not in the least because the trading system is very complex. Answering questions about the price impact of the auctions on the secondary market is only possible within a framework in which the properties of both primary and secondary market are combined and interaction between the two can be modeled. As far as we know, this thesis is the first attempt to construct such an integrated model. The model we constructed builds on two branches of mathematics, namely auction theory and market impact models. Also from a mathematical point of view, it is a first attempt to build a bridge between these two theories. It will turn out to be a very powerful instrument in addressing a wide range of policy questions. Our framework makes it possible to address the following subjects:

- Determining under which conditions auctions are less distorting to the secondary market than regular sell market orders.
- Determining whether uniform price auctions allocate efficiently.
- Determining how auction revenue can be maximized while secondary market distortion caused by the auctions is minimized.
- Determining the effects of specific auction properties on secondary market distortion.
- Comparing the differences in secondary market distortion between different auction frequencies.
- Determining the optimal division of volumes over all auctions.

First, we apply auction theory and market impact models to the European Emissions Trading System. After that, we combine these theories to be able to address the questions concerning interactions between primary and secondary markets.

In auction theory, bidding strategies and equilibrium outcomes are studied. An equilibrium is a situation in which no bidder can improve his position by individually changing his bidding strategy. The auction mechanism which will probably be used in the ETS is a uniform price sealed bid

auction. This means there will be only one price paid by the winning bidders, the clearing price. Under specific conditions, bidders in a uniform price auction may adopt a strategy of demand reduction. This means they bid lower than their true values for the EUA's. When all bidders adopt this strategy, the clearing price in the auction is below the secondary market price.
In general, a uniform price auction does not allocate efficiently. Efficient allocation means that the bidders who value the items the most, will win the items. However, because bidders may adopt a strategy of demand reduction, efficient allocation is not always achieved. Also reselling after the auction does not automatically lead to efficient allocation, because the auction does not reveal complete information about bidders' valuations.
To reduce the risk of a low-price equilibrium, it is crucial to attract a sufficient number of bidders to participate in the auction. When the number of bidders is high enough, bidders will compete over the price. The higher the number of bidders, the lower equilibrium underpricing will be. A small tick size, which allows bidders to make very small changes to their bid prices, can make equilibrium underpricing arbitrarily low. By making the tick size small, bidders are encouraged to compete over the price. So instead of using 0,01 euro as a tick size, it could be considered to use 0,001 euro.

Market impact models address the question how to optimally sell a large volume of shares in an uncompetitive market. An optimal selling strategy maximizes the revenue to the seller and minimizes market distortion. Market impact models study markets which are operated through Limit Order Books. In these electronic books, sell and bid orders are collected. Whenever a sell and bid order meet, i.e. have the same price, a trade is executed. That way prices are established. Dynamic Limit Order Book models study supply/demand dynamics at a micro-level.
Suppose a large trader wants to sell a very high volume compared to the volumes normally traded in the order book and he wants to execute this order within a short period of time. The trader will temporarily distort the Limit Order Book, because his large trade 'eats' a lot of bid orders from the book. Potentially many of these bid orders are lower than the best bid price. Consequently, the large trader has to make 'costs' for selling a large volume very quickly by accepting bids below the best bid price. After the large trade, the new best bid price in the Limit Order Book will be lower than before. The market is distorted. So market distortion and costs to the large seller are equivalent in these models.
By splitting up the trade in smaller pieces, costs to the seller and market distortion can be minimized. Market impact models study the question how small these pieces should be and how the volumes should be divided over these trades. The answers to these questions depend highly on specific properties of the market. In general, when the market shows quick recovery from trades, it is optimal to trade very small equal volumes at a high rate.

There are two main reasons why auctions will have an impact on the secondary market price. Firstly, because the auctions are a specific type of sell orders in a non-competitive market. Because of the auctions, bid orders may be (temporarily) detracted from the secondary market. This may lead to distortions of the secondary market price, which can be studied using market impact models. Secondly, because the auction mechanism itself may give rise to irregularities. The price paid in the auction is not always equal to the market price. When the clearing price differs from the market price, it is likely that the market price will be (temporarily) affected after the auction.
To increase revenues and minimize market distortion it is crucial to make the auctions as attractive as possible both for bidders active on the secondary market and for other bidders. Market distortion will be minimized when the overlap between the primary and secondary market is maximal, while extra bidders are attracted to the auction as well. Furthermore, choosing a sufficiently small tick size can make the risk of equilibrium underpricing low. If this is combined with frequent auctions, the market impact is minimized.
These conditions are crucial in answering the question whether auctions are less distorting than regular selling. Smart and smooth auction design is necessary to minimize market distortion. Otherwise, the outcome of the auctions may cause high price distortions in the secondary market, and the Member States could do better by selling the EUA's directly on the secondary market.

The approach and the resulting model in this thesis might be useful for other purposes as well. Applications one could think of are the following:

- Determining criteria for choosing an auction platform while aiming at minimal distortion of the secondary market.
- Monitoring functioning (and possible manipulations or distortions) of these markets in general and the impact of auctions in particular.
- Determining optimal selling strategies for *selling* large amounts of EUA's directly on the market: (how) should volumes be split up and divided over time?
- Determining optimal selling strategies for auctioning other assets, e.g. government bonds.

Contents

Preface

If it were not for me not even being able to hang a bookshelf on my own wall, I think I would have become a builder. As a builder, I would be specialized in the building of bridges. How exciting would that be! First there is a river and two river banks. Then on both sides of the river people would like to be able to meet and see each others worlds. And that's where I would come in. Equipped with a drill, nails and other bridge building gear, I would make a construction plan and start building. I would really be able to do something for these people. When the sides of the river meet, new ideas will emerge, new friendships would be made and of course traffic jams on the already existing bridge 10 km north will be diminished.
Unfortunately, my life is not that romantic. Some people say mathematics is the mother of all sciences. I think they might be right, and I love mathematics because it's pure and beautiful. However, as I found out while I was studying mathematics, the real world is not always pure and beautiful. But that is why I think it is far more interesting. As it turned out, I got a bit bored by everything being perfect in mathematics, while in reality everything is just one big happy mess.
This is why I think probability theory is one of the most exciting topics in mathematics. It is fascinating that people are trying to construct a completely pure model for describing structures in a random world. In fact, probability theory tries to build a bridge between mathematics and an entire real world of applications.
In fact, in this thesis I tried to compensate for not being able to handle hammers and nails, by building imaginary bridges. There were a lot of them. A bridge between government policies and scientific research, a bridge between the university and the ministry, a bridge between mathematics and the European Union Emissions Trading System, a bridge between auction theory and market impact models.
Of course, I would never have been able to build these bridges without my gear. I would like to thank Utrecht University and especially the Mathematics department for creating a really challenging environment. I would like to thank Peter Spreij and Remco van der Molen for giving me the trust to start this somewhat exotic thesis project, for providing critical insights and support me through the process (and for all the good coffee). Also I would like to thank the Ministry of Finance and all the people who work at the Foreign Financial Relations directorate. Above all, for making me feel very welcome in an extremely interesting, challenging and enthusiastic environment. I also really appreciated the critical thinking, no matter how technical this subject may be, the comments from the economists' point of view from which I learned a lot and all the other inspiring conversations on topics not closely related to my thesis but therefore not less interesting.
Finally I would like to thank Anne-Theo Seinen to be so welcoming to invite me to come to Brussels to discuss views on the auctioning regulation proposed by the European Commission. This is how my thesis actually started to 'live'.
I hope someway somehow somebody will ever use something I wrote down. So that I will be a successful bridge builder after all.

Chapter 1

The European Emissions Trading System

1.1 Introduction

In order to meet the obligations under the Kyoto Protocol, the European Union decided in 2003 to introduce the first cap-and-trade system for greenhouse gas emissions in the world [2]. In 2005 the system was launched. The first period until 2007 was marked as a trial period, before the actual obligations under the Kyoto protocol would start. The main objective of this first period was to develop infrastructure and knowledge necessary for an effective trading system in the second period, from 2008 until 2012. The main achievements of this first period were the birth of a single market price for CO_2 emission allowances, a functioning market and an infrastructure of market institutions, registries, monitoring, reporting and verification [13, p. iii]. Companies quickly adapted their policies to the consequences of CO_2 emissions trading. In the first period, all emission allowances were distributed over the industries and companies by the Member States. No company had to pay in advance for their rights. In the period 2008-2012, the current period, Member States are experimenting with auctioning relatively small amounts of CO_2 emission rights. Several auction mechanisms are used and amounts, frequencies and types of allowances sold differ from auction to auction. These trials aim to build up knowledge on and experience in auctioning emission allowances, because in the third period, starting from 2013, there will be large-scale auctioning of these allowances.

1.2 The European Emissions Trading System

In the European Emissions Trading System (ETS), an absolute cap on carbon emissions has been placed on a collection of emitting facilities in the European Union. A total number of emission rights, equal to the cap, has been distributed among these facilities. The facilities must measure and report their CO_2 emissions and during annual compliance periods, they surrender an allowance for every ton of CO_2 they emit during that year.

1.3 Cap-setting

Initially, the cap was set by the individual Member States. Each Member State issued a number of European Union Allowances (EUA's) , a number which was subject to approval by the European Commission. In the Kyoto Protocol, an economy-wide cap on all greenhouse gas emissions was imposed in the Burden Sharing Agreement [13]. However, in the EU ETS only a part of the polluters in the economy is included, namely the power sector, some industrial sectors[1] and

[1]These sectors are iron and steel, cement and lime, refineries, pulp and paper, ceramics, glass, bricks and tile.

combustion facilities. These sectors are responsible for about half of the CO_2 emissions in the European Union. They cover about 40% of all Greenhouse Gas Emissions in the European Union. Large emittors which are not included are the transportation and building sectors. From 2012 on the European Union aims to implement Emission Allowances for the aviation industry as well.

1.4 The Linking Directive

An important aspect of reducing CO_2 emissions is that the global effect of the reduction is independent of the actual site where the actual reduction takes place. Following this basic assumption, companies in the ETS which are short on EUA's basically have three options regarding emission reductions [2]:

1. Buy emission allowances and surrender these - not reducing carbon emissions but paying for them
2. Invest in measures that will reduce the CO_2 emissions of their own company - reduce the need for EUA's
3. Submit credits for emission reductions accomplished outside of the European Union.[2] In the future, the Commission wants to link the European ETS with other emission trading systems in the world. In 2008, the EU ETS was already linked to the Norwegian cap-and-trade system [13, p. 5].

1.5 Transitions between trading periods

Within a trading period in the EU ETS, participants can borrow or bank their allowances. Every year in February the allowances are issued. In April the allowances for the preceding year must be surrendered. Consequently, companies can compensate for a possible shortage of emission allowances in any given year within a trading period by allowances which were issued for the year to come. On the other hand, companies can bank emission rights they didn't need in any given year within a period to cover emissions in the year(s) to come. However, no banking or borrowing was allowed over the first (2005-2007) and second (2008-2012) period.

1.6 The price of emission allowances

The price of EUA's for the first period showed large fluctuations. These were caused by a number of facts. Firstly, because there was initial uncertainty about the actual total amount of emissions, there was uncertainty about the demand for EUA's. The actual emissions data showed a smaller amount than expected. This caused demand reduction, which was reflected in the price for EUA's. Of course, this effect was largest in the beginning of the trading period, because from then on, the expectations were adjusted. However, during the second trading period, due to the international financial crisis, the number of allowances issued was again larger than the actual need, which caused the same phenomenon to occur again. The size of these jumps in prices depend on the frequency with which emission data are collected (from which demand can be determined) and the length of the trading period. When the trading period is short, a reduced demand leads to a quick drop in the price. When the frequency of the release of emissions data is high, the subsequent jump sizes are smaller.
A second cause of fluctuations in the prices for EUA's are external influences, such as temperature,

[2] Only specific emission reduction measures under the Kyoto Protocol can be exchanged, namely Certified Emission Reductions (CERs) under the Clean Development Mechanism (CDM) and Emission Reduction Units (ERUs) under the Joint Implementation Programme (JI). There is a maximum on the amount of exchangeable units, to certify that the major part of reduction takes place within the European Union. However, this maximum differs from country to country and from industry to industry [13, p. 4].

and gas and oil prices and economic growth. When the winter is cold, there is an increased demand for energy and thus a rise in prices for CO_2. High gas and oil prices make emission reduction more expensive.[3] Thirdly, there was initially an asymmetry in trading activities between parties in a long position (having allowances for sale) and in a short position (willing to buy allowances). The parties in short of allowances, namely the electric power generating sector, were much more active in the market than the non-power companies and all companies in Eastern Europe- which had more allowances than they actually needed. This effect was reduced during the first trading period itself, because other parties became more active and liquidity rose. In Europe there is now a number of the trading platforms,[4] from which the European Climate Exchange (ECX) is the largest, accounting for about 80 percent of trading volume. The fourth cause of fluctuations in the price was caused by the fact that no banking or borrowing of allowances was allowed over the first and second period, which made the price of allowances for the first period fall to zero by the end of 2007. Currently there's a discussion about the transition from the second to the third period. Banking of allowances will be allowed [13]. The debate is about the question whether early auctions for the period from 2013 on should take place from 2011, making it possible for companies to better hedge their risks.

1.7 From a de-centralized to a centralized approach

Currently, the ETS is changing its system from a de-centralized to a centralized approach. Main features of the centralized approach in the third trading period include the increased use of benchmarks and harmonization of allocation procedures. Best practices from every industry and historical emissions data will be used to determine EU-wide benchmarks, which will prescribe the future amount of emission allowances per industry, aiming at a significant reduction in carbon emissions and stimulating the use of energy efficient technologies [2, article 10a, 1]. Furthermore, there will be harmonization of the allocation to similar facilities in the Member States (instead of every member state determining their own Allocation Plans). Therefore, on June 30 2010 the Commission will publish the absolute amount of emission allowances for the third period for the whole European Union (EU), which will be based on the total of the National Allocation Plans for the second period, in which the disjoint national amounts of emission allowances are specified. The European Union will distribute the allowances per industry. From 2013 on, this absolute amount will be diminished yearly by a constant linear factor of 1.74 percent [2]. At the time of writing, there is a discussion between several stakeholders[5] and the European Commission about the centralization of the auctions of EUA's from the third period onwards, contrary to the independent auctions currently organized by several Member States.

1.8 Case for auctioning allowances

Auctioning allowances is expected to have two main positive effects:

1. In an efficient auction mechanism, the allowances will be awarded to the participant who values them the most. This means auctioning the allowances will have a beneficial effect on the efficiency of the market.

2. In liberal electricity markets, as in some European countries, companies are able to make *windfall profits*[6] on EUA's and the consumer prices for electricity will rise (as is the case in the

[3] The *switching price* is a term generally used in this context, meaning the price level at which it would be profitable to switch from a cheap coal-fired to a gas-fired installation in the power sector.

[4] OTC, BlueNext, ECX, NordPool, EEX

[5] Member States, industrial organizations, companies, trading platforms, non-governmental organizations and other parties

[6] A company makes a windfall profit on CO_2 allowances when it gets the allowances for free, but increases consumer prices based on the market value of these allowances, as if it had to pay for them. This causes an increase in income whereas no actual extra costs have been made (yet).

Netherlands). In countries where the electricity market is regulated, the issuance of EUA's has had a negligible effect on consumers prices, because companies are not able to make windfall profit. Auctioning allowances will force every company to pay for their allowances, which will make windfall profits disappear. In countries with a liberalized electricity market, this will probably not have any additional effect on consumer prices. However in countries with a regulated market, consumers prices are expected to rise. Consequently, auctioning of allowances will ensure that carbon prices are uniformly reflected in consumer prices in Europe.

From 2013 on, all emission allowances which are not allocated for free will be auctioned. On 31 December 2010, the amount of emission rights which will be auctioned will be published by the Commission [2, article 10,1].

1.9 Designing auctions in the third ETS period

According to the European Commission, the mode of auctioning is "important for emitters and more generally for participants in the secondary market. In fact, an effective and efficient auctioning system is crucial for the smooth functioning of the ETS itself [15, p. 2]." We investigate auctioning emission allowances from a technical perspective.
First, we identify the goals of these auctions and the criteria which are relevant for the technical aspects of the auctions. We investigated the current status quo from expert reports and historical auction data and knowledge (from experimental auctioning of emission allowances within the European Union and the auctioning of Dutch State Treasuries).
We use these data to focus on which questions are still relevant to be answered from a technical and a practical perspective.
There are two main **goals** in auction design for carbon allowances:

1. Allocation efficiency[7]

2. Minimize expected costs for all participants.[8]

There is a list of **criteria** which the design must satisfy. These criteria are derived from a number of reports.

1. Minimal disturbance to the secondary market. The auction should not result in clearing prices which are systematically different from the secondary market.

2. There should be enough liquidity in the markets, which means there should be enough buyers in the auction and enough sellers in the secondary market, to make it always possible to convert allowances into money and vice versa.

3. The auctions should have a minimal impact on the volatility of the price of EUAs on the secondary market.

4. Auction design should protect against collusion (cooperate to keep prices low) or manipulation of the market. The possibilities for hoarding (driving up the prices in the auction) and collusion should be minimized.

5. The design schedule should be familiar to the industry. When a particular auction format is already widely used, it might be preferred over another format which would ask for greater effort from participants to learn how the model works.

[7] Allocative efficiency means the participant in the auction who values the object the most obtains the object

[8] In the letter to the Parliament from the Dutch Ministries of Economic Affaris and Housing, Spatial Planning and the Environment *maximizing revenue* was explicitly stated as an auction objective [5]. However this is not generally mentioned as a key objective of the carbon auctions, because large expected revenues can have adverse effects, such as very few participants and carbon leakage. See for more explanation the list of criteria in the remainder of this section.

6. The design should align well with wholesale energy and capacity markets. Because carbon allowances are needed for the production of electricity, it might be necessary for the power sector to hedge their risks in advance because of insecurities in the energy markets [14].
7. The design should be implementable within the legal framework of the European Union.
8. There should be uniform entrance requirements for all participants.
9. The auction process should be transparent, harmonized and non-discriminating.
10. There should be a predictable process regarding frequency of auctioning and amounts being sold.[9]

There have been several studies about the technical design aspects of EUA's. These studies differ in the way they were conducted and the perspective they use. In Appendix A, a brief overview of these studies and their results can be found.

[9] Criteria 7-10 are formulated by the European Commission [2, art. 10.4].

Chapter 2

Key Questions and Structure

Starting from 2013, approximately fifty percent of the total yearly amount of European Emission Allowances (EUA's) will be auctioned. These auctions should be designed in such a way that they minimally distort the functioning of the secondary market. An indication of the volume to be auctioned compared to the total trading volume in the ETS market can be found in Figure 2.1.[1]

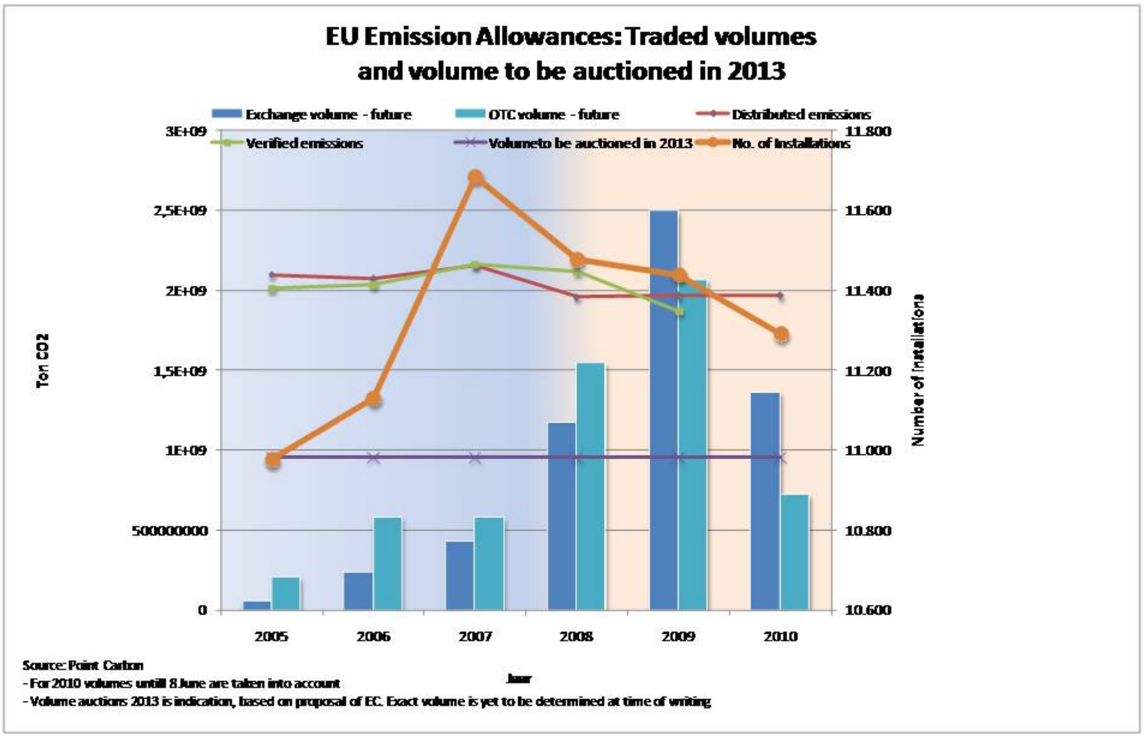


Figure 2.1: Traded volumes in the past and volumes to be auctioned in the future

From Figure 2.1 it is clear that the volumes for emissions traded are growing quickly. Furthermore, the volumes traded on the exchange are growing more quickly than the over the counter volumes,

[1]This picture should be interpreted as an indication of the volume to be auctioned proportional to the volumes traded in the market. However, actual volumes in the market in 2013 might be very different from the ones shown in the picture. Also, the volumes traded in the spot market are not visible in the figure. The futures market is much larger than the spot market. In the first five months of 2010, volumes in the future market were more than five times higher than in the spot markets. However, spot market volumes are growing.

with 2009 being the first year when exchange volumes exceeded over the counter volumes.[2] In a perfectly efficient market, the price of a stock which is traded will change only due to the arrival of new information. In the case of carbon emission allowances, historic price behavior has shown this is not the case. After the German administration performed the first of weekly spot auctions of 300,000 allowances in the beginning of January in 2010, Bloomberg reported prices fell 2,2 per cent 'due to an increase in supply' [11]. However, these auctions were announced well in advance. In a perfectly efficient market, this extra supply would already have been included in the price.
Furthermore, in uniform price auctions a bidding strategy of strategic bidding may be adopted. Consequently, with positive probability, the clearing price will not be equal to the market price. This theory is affirmed by some recent uniform price auctions performed by the United Kingdom and Germany, in which clearing prices were approximately 1% below market prices [3].
We can also assume the market is not *competitive*. A competitive market is one where any trader can buy or sell unlimited quantities of the relevant security without changing the security's price. In uncompetitive markets, prices are affected by traders and market structure. In the case of the carbon market, this is a realistic assumption, as the number of big traders is limited. There are currently 10,000 companies who need emission allowances within the European Union, not all of them are active in the market.

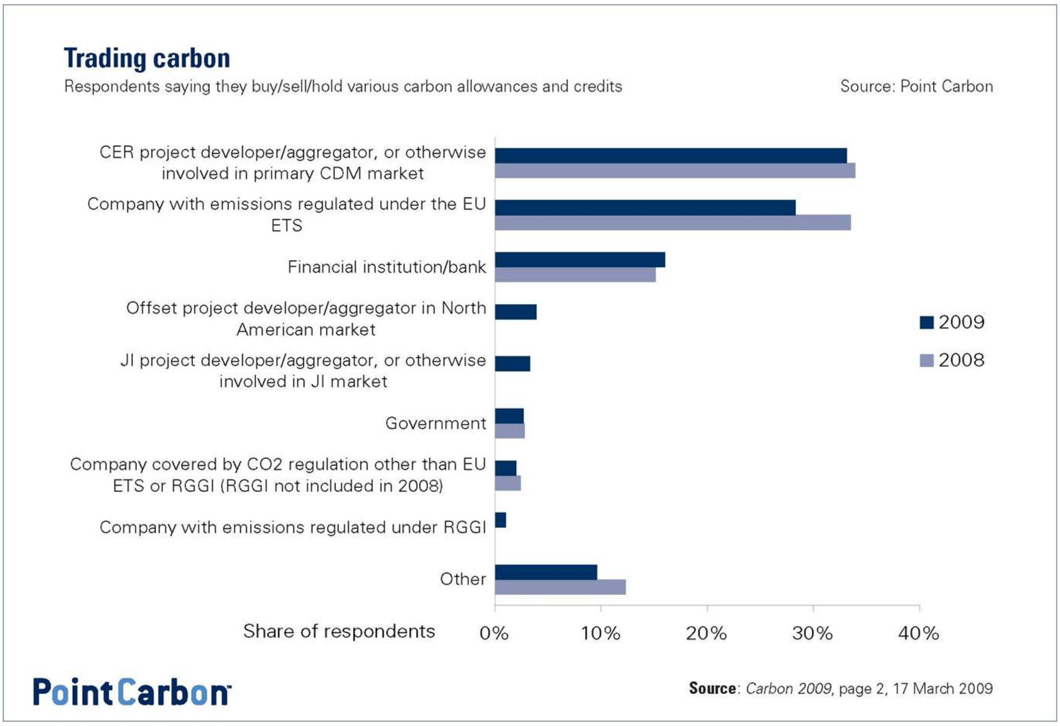


Figure 2.2: Market participants [27]

[2]Over the counter trading means trading which is not performed at an exchange, but directly between a seller and a buyer.

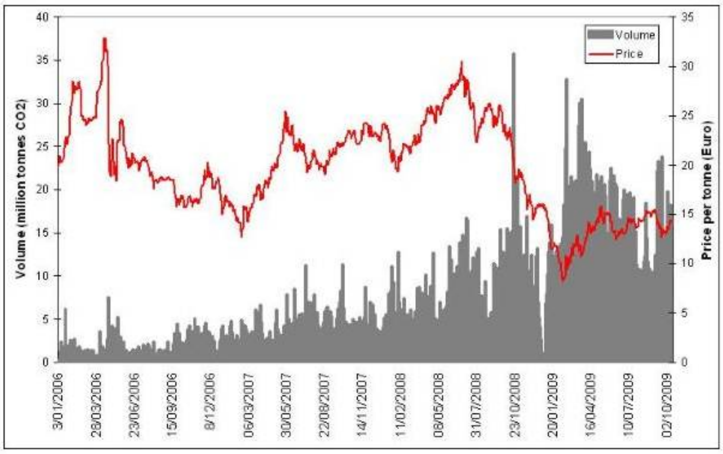


Figure 2.3: Volumes and prices traded in the ETS [4]

Changes in the price of EUA's Using our elementary observations, we can deduce the following causes of changes in the carbon price.

- **Arrival of new information**
  - *Change in demand.* Demand is of a stochastic nature. Main components which affect demand are fuel prices (carbon prices tend to move in the same direction as prices for natural gas [6]), weather (inverse relationship: when the winter is cold, the demand for energy increases, as does the demand for EUA's, because power generators will need more rights to cover their increased emissions), economic growth (monotone relationship: the higher the economic growth, the higher the energy consumption and the higher the demand for EUA's). Another possible driver for changes in demand are policy decisions. These can be system-related, for example about the question whether borrowing or banking over trading periods is permitted.[3] Policy decisions about carbon reduction goals also affect demand: the higher the future reduction goals which apply to the next trading period, the higher the demand in the current period (as banking is allowed).[4]
  - *Change of supply.* In the case of EUA's, the supply is fixed by a decision. The number of allowances for the whole trading period is determined before the trading period starts.
- **Inefficiencies**
  - *Market not competitive.* When a market is not competitive, there is not always enough liquidity in the market. Large market orders can have an impact on the price, both temporary and permanent. Large market orders can be placed by governments (in the form of auctions) or by any other player in the market.
  - *Demand reduction or overbidding in uniform price auctions.* When a large market order by a government is placed by using uniform price auctions, the price paid in the auction is not always equal to the market price. The clearing price differs from the market price, which will be likely to have an impact on the market price after the auction in an uncompetitive market.
  - *Incomplete information.* There are two main types of incomplete information in the EUA market which cause inefficiencies and consequently price fluctuations as information becomes known to more participants. The first type of incomplete information is

[3]This is explained in Section 1.6.

[4]After the failure of governments to achieve a binding and ambitious agreement setting high emission reduction targets in Copenhagen, prices for EUA's dropped.

ignorance of a large number of players, due to the fact that the ETS is a new phenomenon. The second main type of incomplete information is structural: it is caused by the fact that it is a market which is invented by governments. This gives rise to a type of 'system uncertainty', which is driven by politicians and bureaucrats rather than market players. Their decisions to change rules in the mechanism may highly affect valuations of players in the market, but of course future decisions are not known.

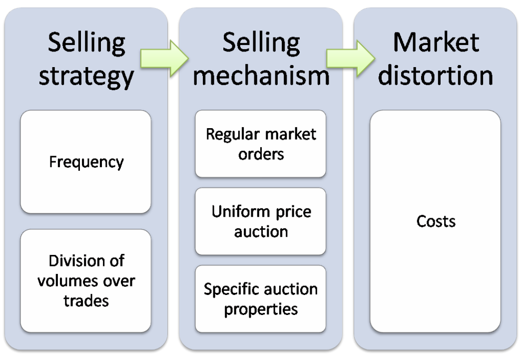
2.1 Key Questions

In this context, it is not unthinkable that auctions of EUA's will indeed have an impact on the secondary market price. Firstly, because the auctions are in fact large trading blocks of sell orders in a non-competitive market. Secondly, because the auction mechanism itself may give rise to irregularities. The nature of this disturbance can be divided into temporary and permanent components and will depend on several variables.
We investigate the following key questions:

- Do auctions have any distorting effect on the secondary market?
- If yes, is this distortion temporary or permanent?
- How can these distortions be minimized?

To find an answer to these questions, we construct a mathematical model. This model is based on a combination of auction theory and market impact models.
This approach and the resulting model can be useful for other purposes as well. Applications one could think of are the following:

- Determining criteria for choosing an auction platform while aiming at minimal distortion of the secondary market.
- Monitoring functioning (and possible manipulations or distortions) of these markets in general and the impact of auctions in particular.
- Determining the effects of specific auction properties on market distortion.
- Determining optimal selling strategies for *selling* large amounts of EUA's directly on the market (for example for CCS projects): (how) should volumes be split up and divided over time?
- Determining optimal selling strategies for auctioning other assets, e.g. government bonds.



Our objective in choosing a selling strategy and mechanism will be to minimize distortion of the secondary market. We will measure the distortion of the secondary market as *costs* to the seller.

2.2 Structure of this thesis: building bridges

This thesis is composed like a construction plan for building a bridge. First, we study the two sides of the river. We examine their properties and walk around on both sides. Sometimes, we will feel swamps underneath our feet. At times, we are confronted with lakes, which, however beautiful they may be, are particularly difficult to build bridges on. However, we will find some solid ground on both sides of the river to start building. The materials we use to construct our bridge are common sense and - of course - mathematical theory. The east-side of the river is called Auction Theory, the west-side of the river is called Market Impact Models. The river is called Large Auctions. The country is called a Market. The inhabitants are divided into two groups, called Buyers and Sellers. The sellers have a chairman, called the Big Seller. The bridge is built with a single objective: providing a framework to the Big Seller in the Market for designing auctions such that they minimize distortion of the market.

First, we describe our journey on the west-side of the river called Market Impact Models. This side of the river provides useful insights for Big Sellers. We consider the general question how to optimally sell a large amount of shares by splitting the volume in smaller pieces and trading sequentially in Chapter 3. By an optimal selling strategy we mean a strategy which minimizes distortion of the secondary market. We will show that this is closely related to a revenue maximizing strategy. We will find a closed-form solution for optimal selling strategies in terms of division of volumes over trades. We will see that it is not always optimal to divide volumes equally over all trades. Sometimes it is less distorting to trade high volumes in the beginning and at the end of the trading period.
However, market impact models are designed for studying *market orders*, and not specifically for auctions. To be able to build the bridge, we also study the east-side of the river, called Auction Theory. First, we take a tour to the hinterlands of this side of the river, to develop a rigorous mathematical framework in Chapter 4, which allows us to prove some strong results in Auction Theory. However, we will also show that these results generally do not hold when we want to apply them to auctions of European Union Emission Allowances. The conditions under which the strong theorems of auction theory hold, are not satisfied for the specific kind of auctions which will be used. The results in Auction Theory are beautiful lakes, but we cannot build bridges on them. However, in Chapter 5 we will study the specific auction mechanism which is probably going to be applied in the ETS, namely the uniform price sealed bid auction [1]. We will derive some results on the outcomes of these auctions, using the language we learned while walking around in Auction Theory. It turns out that the clearing price will be between bounds, which are influenced by the design of the auction. So in the end, we do find some solid ground on which we can build our bridge.
In the last part of this thesis, Chapter 6, we will put it all to work. We start on the west-side of the river in Market Impact Models. We first apply market impact models to find an optimal selling strategy for selling a large amount of EUA's through regular sell market orders. We will use this to graphically show the interdependence of the parameters in the model. We can derive conclusions about the importance of decisions about the frequency of sales and the influence of parameters which affect the optimal selling strategy. The framework from the west-side and direction of the bridge are ready.
In the last part of this thesis the really exciting work is done. We cross the river. We build the bridge between Auction Theory and Market Impact Models. We will first start from the Market Impact Models side of the river: we take the outcome of the auctions as exogenously given and then calculate market impact in the Limit Order Book. In our model this market impact depends on the relative dependence between the Limit Order Book and the auction. After having developed this model, we can cross the bridge and use the perspective from the Auction Theory side of the river. We will be able to study the implications of details in the auction design on bounds on the market impact of the auctions.
Finally, the bridge will be constructed and the Big Seller will have a framework, which was our objective to build the bridge. The Seller has a framework for examining how auctions should be

designed such that they minimize distortion of the secondary market. We will also be able to conclude under which conditions auctions are less distorting to the secondary market than regular sell market orders. Maybe surprisingly, this is not always the case. At the same time, we will be able to determine expected revenue to the seller and we will be able to see how to maximize this revenue.

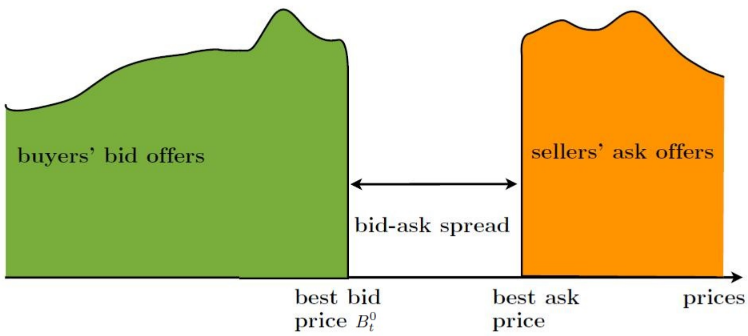
Chapter 3

Market Impact Models

Market impact can be described as the feedback effect on the quoted price of a stock caused by a trade. The basic observation is that liquidity costs of a large trade can be reduced significantly by breaking the trade into smaller pieces, thereby minimizing distortion of the price.
Market Impact Models are the subject of some recent studies among financial mathematicians. They provide interesting mathematics and applications to the real world. Their main application lies in providing trading strategies for large stock traders. We can think of a trader who wants to sell a large block of shares which comprises twenty percent of the daily traded volume of shares. An order of this size creates a significant impact on the asset price. In order to reduce the price impact, the question is how to 'allocate an optimal proportion of the entire order to each individual placement such that the overall price impact is minimized [8].'
We use a *Limit Order Book* approach to study any temporary and permanent effects of the trade. A *limit order* is an order to trade a certain amount of a security at a given price. More specifically, it is an order to buy a security at no more, or sell at no less, than a specific price. In a market operated through a Limit Order Book (LOB), traders (electronically) post their demand or supply in the form of limit orders to the system. As soon as a buy and a sell order 'meet' i.e., have the same price, a trade is executed. The collection of all limit orders posted can be viewed as the total demand and supply in the market [22]. A limit order is not the same as a *market order*. A limit order gives the trader certainty about the price, whereas there may be no actual execution, because his order is not met. Instead a market order is an order to buy or sell before a certain point in time at current market price. Market orders are therefore used when a trader prefers certainty of execution over the price of the execution. A Limit Order Book can be pictured as follows.[1]

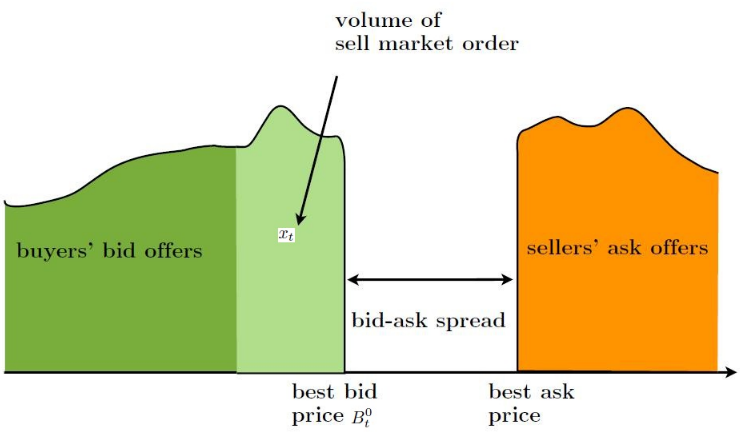
[1] These pictures are taken from [24].

Limit order book before market order



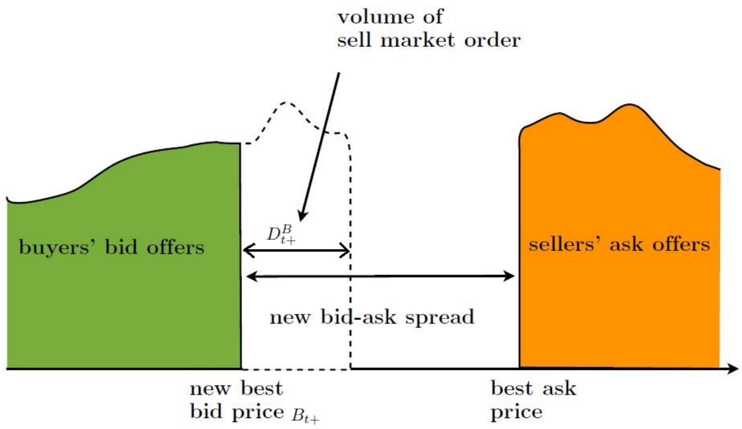
The dynamics of the Limit Order Book reflect supply and demand in the market. However, large market orders have a temporary distorting effect on the shape of the Limit Order Book. This can best be explained using some pictures. A sell market order of a large trader 'consumes' a block of shares located to the left of the best bid offer.

Limit order book before market order



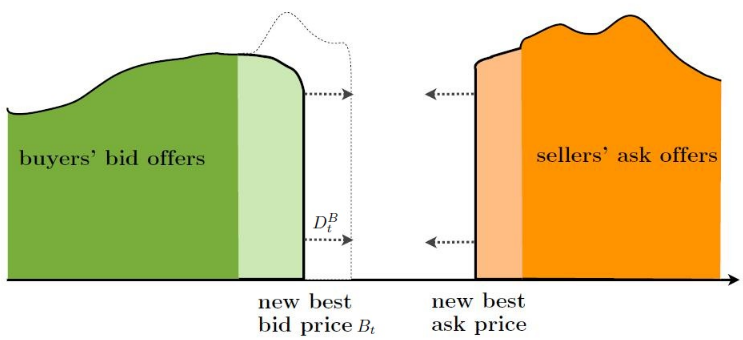
Consequently, the bid-ask spread increases as the new best bid price is lower than before the large market order.

Limit order book after market order



After the trade, resilience of the LOB takes place: new bids will be placed and a new best bid price will be formed.

Resilience of the limit order book after market order



The price impact depends on the size of the spread, the resilience, the order size, the shape of the order book and the trade rate. The impact can be studied as having a temporary and a permanent component.

A distinction can be made between market impact models on a microscopic level, a mesoscopic level and a macroscopic level. Market impact models on a microscopic level study order book dynamics. The optimal trading strategies are based on minimizing market distortion while maximizing revenue, with an emphasis on single trades. When the market is not completely perfect, a seller placing a market order has to accept bids lower than the best bid price. The *costs* of a sell market order can be calculated by evaluating the prices of the shares which have to be sold at a price below the best bid price. The higher these trading costs, the lower the revenue the seller receives from selling his shares. The *costs* which are related to a specific market order, can at the same time be taken as a measure of *market distortion*. The larger the costs to the seller, the greater the 'shock' in the Limit Order Book. High costs mean a large volume at a price lower than the best bid price is consumed from the Limit Order Book. The higher the costs, the greater the market distortion. This means that when a seller of a large volume is looking for an optimal selling strategy such that he maximizes his revenue, this means he wants to minimize his *costs*, which means his optimal strategy will be such that it minimizes market distortion.

In market impact models on a mesoscopic level, the effects of risk-averse behaviour are taken into account in the model. The costs of trading, which are reduced by trading not all stocks at once, but at a slower rate, are balanced against volatility risks incurred by holding the portfolio. In these models the variance of the stock price will also be affected by the trading rate itself. Macroscopic models emphasize the interaction between the large seller and the competitors. Optimal equilibrium strategies are derived and it is studied whether other large sellers can abuse the fact that a large trader has announced the selling of a large amount of stocks. In this thesis, we will use the microscopic approach. This provides us with a theoretical framework which we can connect to auction theory. However, it would also be interesting and possible to use the other approaches in further research. In this section, we will find a deterministic optimal selling strategy in terms of maximizing revenue and minimizing market distortion, for a large seller.

3.1 Microscopic market impact

3.1.1 Market Impact - a microscopic approach

In order to describe the supply/demand dynamics, we consider a Limit Order Book market and construct a dynamic model of the Limit Order Book. We will use a model which allows for a general shape function and exponential resilience, which is also observed in experimental studies such as [23]. Resilience can be described as the speed at which the Limit Order Book is rebuild after being hit by a trade [22, p. 2].

Dynamics of the LOB

In this section, we will construct a model for studying the dynamics of the LOB. We build on the models described in [22], [8] and [7]. We assume that a large trader is aiming to sell an amount $Z_0 > 0$ of shares within a given time period $[0, T], T < \infty$. So we will focus on sell orders, which is an arbitrary choice from a theoretical point of view. However, from a practical point of view this lies closer to our ultimate model for auctioning CO_2 allowances. Consequently, we focus on the bid side of the Limit Order Book. The highest price at which shares are requested is called the *best bid price*. First, we suppose that our large trader is inactive. This does not mean there is no trade: the order book will still show some dynamics. We assume that the unaffected best bid price B_t^0 is a martingale on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We assume $B_0^0 = B^0$. On purpose, we choose a very general description for B^0, so that we can allow for any specific model. For example, a geometric Brownian motion could be used, $B_t^0 = B^0 \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W_t\}$ where W_t is a standard Wiener Process. We use a shape function f to describe the number of requested shares under B_t^0 at a price $B_t^0 - x$. This number is given by $\int f(x)dx$ for $f : \mathbb{R} \to (-\infty, 0)$ a continuous function. Note that when f is constant, the Limit Order Book is block-shaped. This means at any price below the best bid price, an equal number of shares is requested in the order book.

Now the large trader comes in. Assume the large trader is placing a sell market order of $x_0 > 0$ shares at time $t = 0$. This means the large seller consumes all orders below B^0 until D_{0+}^B, where D_{0+}^B is given by

$$\int_0^{D_{0+}^B} f(x)dx = x_0.$$

The new best bid price is given by $B_{0+} := B_0^0 - D_{0+}^B$. We denote by B_t the actual bid price at time t, so after the price impact of previous sell orders is taken into account. The quantity $D_t^B := B_t^0 - B_t$ denotes the *extra spread* in the bid price distribution. After time $t = 0$ we can continue selling small trades. Another sell order of size $x_t > 0$ will now consume all shares demanded at prices between B_t and $B_{t+} := B_t - D_{t+}^B + D_t^B = B_t^0 - D_{t+}^B$, where D_{t+}^B satisfies

$$x_t = \int_{D_t^B}^{D_{t+}^B} f(x)dx. \tag{3.1}$$

D_t^B captures the market impact process caused by the large trades of the seller. When the shape function f is constant, the price impact $D_{t+} - D_t$ will be a linear function of the volume of the trade. However, when f is not constant, the price impact is a nonlinear function of trading volumes.
Speaking of volumes, we introduce a process E_t^B, which describes the number of shares 'eaten up' by the seller at time t:

$$E_t^B = \int_0^{D_t^B} f(x)dx. \tag{3.2}$$

Of course, after the Order Book is hit by the large trade and the best bid price has decreased, immediately new limit orders will come in. Some of these orders will be above the new best bid price. Generally, it is assumed that the Order Book recovers to it's original state after being hit by a trade. This is called the resilience of the Order Book. Some empirical studies (like [23] and [29]) have shown that the extra spread decays exponentially:

$$D_{t+s}^B = e^{-\rho s} D_t^B, \tag{3.3}$$

where $\rho \in \mathbb{R}^+$ denotes the *resilience speed*. Sometimes, we will also call this the *recovery* speed. It is also possible to develop a model in which the volume is assumed to recover exponentially:

$$E_{t+s}^B = e^{-\rho' s} E_t^B,$$

for some $\rho' > 0$, which leads roughly to the same results, depending on the shape of the order book [8]. We will use the model describing exponential resilience of the extra spread. Empirical research has shown that in markets with frequently traded stocks, this speed is such that the half-time of the decay is a few minutes [8, p. 6]. Similarly, we can model buy orders, which we can assume to be negative $x_t < 0$ and we will use the notation D_t^A to describe the ask price process in this case. We will not explicitly restate all the formulas here, since we will not use them, but they can be found in for example [8, p. 4-6]. See also [23], [9].

Minimize Market Impact

When the large seller consumes a big part of the order book under the best bid price, the (temporary) increase of the bid-ask spread will be very high and the size of this impact can be expressed in terms of the 'costs' for the large seller. When he sells all shares at once, he is expected to receive a lower price. So he might consider using a sequence of trades to sell his shares, in order to maximize his revenue and simultanously minimize market impact. We will assume the trader wants to sell his order before a certain time T and he wants to split his trade up into a maximum of $N + 1$ pieces.
When placing a market order of size $x_t \geq 0$ at time t , the seller sells $\int f(x)dx$ shares at price $B_t^0 - x$, where x is between D_t^B and D_{t+}^B. Consequently, the total *revenues* π_t of the sell market orders as a function of the trading size $x_t \geq 0$ can be expressed as follows:

$$\pi_t(x_t) := \int_{D_t^B}^{D_{t+}^B} (B_t^0 - x) f(x)dx = B_t^0 x_t - \int_{D_t^B}^{D_{t+}^B} x f(x)dx. \tag{3.4}$$

assuming that a trader wants to sell an amount $Z_0 > 0$ of shares before time T while using a maximum of $N + 1$ trades. To be flexible in our final trading schedule, we allow for non-equidistant trading intervals, so trading can occur at times $0 \leq t_1 \leq t_2 \ldots \leq t_N \leq T$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.
We define an *admissible strategy* $\xi = (\xi_0, \xi_1, \ldots, \xi_N)$ as a sequence of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ such that

1. $\sum_{n=0}^{N} \xi_n = Z_0$,
2. $\xi_n \in \mathcal{F}_{t_n}$,

3. ξ_n is bounded,

where ξ_n corresponds to a market order placed at time t_n. This means ξ_n can also be positive, meaning that we allow for buy orders in the strategy as well, however there is some upper bound on these buy orders. Now the *total revenue* $\mathcal{C}(\xi)$ of an admissible strategy is defined as the expected value of the sum of the revenues incurred by the individual market orders:

$$\mathcal{C}(\xi) = \mathbb{E}[\sum_{n=0}^{N} \pi_{t_n}(\xi_n)]. \tag{3.5}$$

3.1.2 Block-shaped LOB

To make life a little easier, for now we will assume the LOB is block-shaped, so $f(x) = q$, $q \in \mathbb{R}^+$. This means that market orders have the size $x_t = q(B_t - B_{t+})$. Consequently, D^B_{t+} is determined by the condition

$$q(D^B_{t+} - D^B_t) = x_t.$$

Furthermore, we assume *a priori* the possibility of an additional permanent price impact. The intuition behind a permanent price impact, is that a large trade may provide new information about supply and demand in the market and thereby have a permanent effect on the price of the share. This may be useful in studying optimal selling strategies for any type of shares. However, we will argue in a later stadium that in the case for auctioning EU emission allowances, the permanent price impact can be assumed to be zero. We also use a slightly more general formulation for the temporary price impact, allowing for the resilience speed to be dependent of the time. We use from now on ρ_t as a strictly positive measurable function on $[0, T]$. So in a more general formulation instead of (3.3) we use

$$D^B_{t+} = D^B_t e^{-\int_{t_n}^{t_n+} \rho_s ds} = (B_{t_n} - B_{t_n+})e^{-\int_{t_n}^{t_n+} \rho_s ds} = \frac{x_t}{q} e^{-\int_{t_n}^{t_n+} \rho_s ds},$$

for the temporary price impact of a sell market order. Now also taking an optional permanent price impact into account, we assume the price impact of a sell market order x_t placed at time t to be

$$\gamma x_t + \kappa e^{-\int_t^{t+u} \rho_s ds} x_t,$$

where $\gamma < 1/q$ quantifies the permanent impact as a proportion of the size of the trade and $\kappa := 1/q - \gamma$ that 'weights' the temporary price impact. Note that when $\gamma = 0$ we have a model with only temporary price impact. When $\gamma \to 1/q$ the permanent impact completely overshadows the temporary impact, which vanishes. The total extra spread caused by sell market orders x_{t_n} placed at times t_n is just the sum of the individual orders,

$$D^B_t = \gamma \sum_{t_n \le t} x_{t_n} + \kappa \sum_{t_n \le t} e^{-\int_{t_n}^{t} \rho_s ds} x_{t_n}. \tag{3.6}$$

In the block-shaped LOB, when a trader places a sell market order of size x_t, he consumes $\int q dx$ shares at price $B^0_t - x$, where $x \in [D^B_t, D^B_{t+}]$. So using formula (3.4) for π and $f(x) = q$, we get

$$\pi_t(x_t) = \int_{D^B_t}^{D^B_{t+}} (B^0_t - x) q dx = B^0_t x_t - \int_{D^B_t}^{D^B_{t+}} x q dx = B^0_t x_t - \frac{q}{2}((D^B_{t+})^2 - (D^B_t)^2), \tag{3.7}$$

for sell market orders and

$$\pi_t(x_t) = \int_{D^A_t}^{D^A_{t+}} (A^0_t + x) q dx = A^0_t x_t + \int_{D^A_t}^{D^A_{t+}} x q dx = A^0_t x_t + \frac{q}{2}((D^A_{t+})^2 - (D^A_t)^2).,\tag{3.8}$$

for buy market orders. We will now formulate a general result for optimal portfolio liquidation for sell market orders in a block-shaped order book and we introduce the possibility of having constraints on the strategy (for example no trading on specific dates or extra volumes on certain dates). We will use a quite elegant method and proof, which is introduced by [8].

Reduction to a deterministic problem

From now on we will use the shorthand notation

$$a_n := e^{-\int_{t_{n-1}}^{t_n} \rho_s ds}.$$

Problem 3.1.1. *Our aim is to maximize total revenues of the execution of the market order:*

$$\max_{\xi} C(\xi) = \max_{\xi} \mathbb{E}[\sum_{n=0}^{N} \pi_{t_n}(\xi_n)].$$

We will reduce the optimization problem to a deterministic problem, which is a lot easier to study and gives us a closed-form solution. Denote by $\mathbf{1}$ the (N+1)-dimensional column vector $(1, 1, \dots, 1)^T$. Observe that an admissible strategy can be written as an inner product with $\mathbf{1}$, such that $\langle \xi, \mathbf{1} \rangle = Z_0$. So the class of all admissible strategies Ξ is defined as

$$\Xi := \{x \in \mathbb{R}^{N+1} | \langle x, \mathbf{1} \rangle = Z_0\}.$$

Now we can model constraints on admissible strategies by allowing for strategies ξ such that $\xi : \Omega \to \Xi_0$, where Ξ_0 is a closed convex subset of Ξ. We allow for $\Xi_0 \neq \Xi$ in order to be able to set constraints on the set of possible strategies in a later stadium, for example when we want to exclude trades at specific dates. We define a the matrix M

$$M := \begin{bmatrix} 1 & a_1 & a_1a_2 & \dots & \dots & a_1a_2\dots a_N \\ a_1 & 1 & a_2 & a_2a_3 & \dots & a_2a_3\dots a_N \\ a_1a_2 & a_2 & 1 & a_3 & \dots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ a_2\dots a_N & & & a_{N-1} & 1 & a_N \\ a_1a_2\dots a_N & \dots & \dots & a_{N-1}a_N & a_N & 1 \end{bmatrix} \tag{3.9}$$

and now we are ready to prove the following lemma in which we would like to prove that maximizing

$$C(\xi) = \mathbb{E}[\sum_{n=0}^{N} \pi_{t_n}(\xi_n)]$$

with respect to ξ taking values in Ξ_0 is the same as minimizing

$$C(x) = \frac{1}{2}\langle x, Mx \rangle$$

with respect to $x \in \Xi_0$.

Lemma 3.1.2. *Let Ξ_0 be a closed convex subset of Ξ., Then*

(i) The matrix M is positive definite and so the quadratic form

$$C(x) = 1/2\langle x, Mx \rangle, x \in \Xi_0$$

admits a unique minimizer $x^* = (x_0^*, \dots, x_N^*)$ in Ξ_0.

(ii) If $x_n^* \geq 0$ for $n = 0, \dots, N$ then $\xi_n^* = x_n^*$ is the unique optimal strategy within the class of strategies that take values in Ξ_0 [7].

Proof. To prove this, we consider a simplified problem, where we reduce the initial bid-ask spread to a single value. The distinction between best bid price and best ask price disappears: we will study a new pair of processes D and E which react to either buy or sell orders satisfying the following conditions:

$$D_0 = 0,$$

$$\text{for } k = 0, \dots, N-1 : D_{t_{k+1}} = e^{-\int_{t_k}^{t_{k+1}} \rho_s ds} D_{t_k}.$$

Note that $D = D^B$ when ξ consists of sell orders only. In general we have

$$D_t^B \le D_t \le D_t^A.$$

So we define the simplified revenue of ξ_n at time t_n:

$$\bar{\pi}_{t_n}(\xi_n) := B_{t_n}^0 \xi_n - \int_{D_{t_n}}^{D_{t_n+}} x f(x) dx. \tag{3.10}$$

Recall that for the original revenue function we had (3.4):

$$\pi_{t_n}(\xi_n) = B_{t_n}^0 \xi_n - \int_{D_{t_n}^B}^{D_{t_n+}^B} x f(x) dx.$$

Now we need one little lemma, which is taken from [7].

Lemma 3.1.3. *For any admissible strategy ξ,*

$$\bar{\pi}_{t_n}(\xi_n) \ge \pi_{t_n}(\xi_n), \tag{3.11}$$

with equality if $\xi_k \ge 0$ for all $k \le n$.

Proof. First observe that we can use $D_{t_n+} - D_{t_n} = \xi_n/q$ and $f(x) = q$ to write the simplified revenue function (3.10) as

$$\pi_{t_n}(\xi_n) = B_{t_n}^0 \xi_n - \frac{q}{2}(D_{t_n+}^2 - D_{t_n}^2) = \frac{q}{2}((B_{t_n}^0 - D_{t_n+})^2 - (B_{t_n}^0 - D_{t_n})^2).$$

First consider a sell order, $\xi_n \ge 0$. Then we use (3.6) and the fact that $D_{t_n+}^B = D_{t_n}^B \ge 0$ and $D_{t_n}^A \le 0$ to find

$$\begin{aligned}
\bar{\pi}_{t_n}(\xi_n) &= \frac{q}{2}((B_{t_n}^0 - D_{t_n+})^2 - (B_{t_n}^0 - D_{t_n})^2) \\
&= \frac{q}{2}((B_{t_n}^0 - D_{t_n+}^A - D_{t_n+}^B)^2 - (B_{t_n}^0 - D_{t_n}^A - D_{t_n}^B)^2) \\
&\ge \frac{q}{2}((B_{t_n}^0 - D_{t_n+}^B)^2 - (B_{t_n}^0 - D_{t_n}^B)^2) = \pi(\xi_n).
\end{aligned}$$

On the other hand, if we have a buy order $\xi_n \le 0$, we use that $D_{t_n+}^B = D_{t_n}^B$, $D_{t_n}^B \ge 0$ and $B_{t_n}^0 \le A_{t_n}$ to find

$$\begin{aligned}
\bar{\pi}_{t_n}(\xi_n) &= \frac{q}{2}((B_{t_n}^0 - D_{t_n+})^2 - (B_{t_n}^0 - D_{t_n})^2) \\
&= \frac{q}{2}((B_{t_n}^0 - D_{t_n+}^A - D_{t_n}^B)^2 - (B_{t_n}^0 - D_{t_n}^A - D_{t_n+}^B)^2) \\
&\ge \frac{q}{2}((B_{t_n}^0 - D_{t_n+}^A)^2 - (B_{t_n}^0 - D_{t_n}^A)^2) \\
&\ge \frac{q}{2}((A_{t_n}^0 - D_{t_n+}^A)^2 - (A_{t_n}^0 - D_{t_n}^A)^2) = \pi(\xi_n).
\end{aligned}$$

So $\bar{\pi}_{t_n}(\xi_n) \ge \pi_{t_n}(\xi_n)$. Finally, we observe that when $\xi_n \ge 0$ for all n, $D_{t_n+} - D_{t_n} = D_{t_n+}^B - D_{t_n}^B$ for all t_n, so $\pi_{t_n} = \bar{\pi}_{t_n}$. □

When the optimal strategy consists of sell orders only, $\xi_k \ge 0$ for all $k \le n$, with Lemma 3.1.2 we have $\bar{\pi}_i(\xi_i) = \pi_i(\xi_i)$ for all i. This means $\mathcal{C}(\xi) = \bar{\mathcal{C}}(\xi)$ where $\bar{\mathcal{C}}$ represents the *simplified revenue functional*

$$\bar{\mathcal{C}}(\xi) := \mathbb{E}[\sum_{n=0}^{N} \bar{\pi}_{t_n}(\xi_n)]. \tag{3.12}$$

With our function $f(x) = q$ and using (3.10) we find as a simplified revenue function

$$\bar{\pi}_t(\xi_n) = B_t^0 \xi_n - q/2((D_{t_{n+1}})^2 - (D_{t_n})^2). \tag{3.13}$$

Now we will show that the simplified revenue functional $\bar{\mathcal{C}}$ has a unique maximizer, which is equal to ξ^* in Lemma 3.1.2. Furthermore, we will show that this strategy consists of sell orders only. So $\mathcal{C}$ coincides with $\bar{\mathcal{C}}$ and thus ξ^* must be the unique maximizer of $\mathcal{C}$ we are looking for.
We define Z_t as the amount which still has to be sold at time t:

$$Z_t := Z_0 - \sum_{t_k \le t} \xi_k \text{ for } t \le T \text{ and } Z_{t_{N+1}} := 0. \tag{3.14}$$

Using (3.13) we find the accumulated simplified revenue of an admissible strategy ξ:

$$\sum_{n=0}^{N} \bar{\pi}_{t_n}(\xi_n) = \sum_{n=0}^{N} B_{t_n}^0 \xi_n - q/2 \sum_{n=0}^{N} (D_{t_n+}^2 - D_{t_n}^2). \tag{3.15}$$

Furthermore, using (3.14), we obtain

$$\sum_{n=0}^{N} B_{t_n}^0 \xi_n = -\sum_{n=0}^{N} B_{t_n}^0 (X_{t_{n+1}} - X_{t_n}) = Z_0 B_0 + \sum_{n=1}^{N} X_{t_n} (B_{t_n}^0 - B_{t_{n-1}}^0).$$

Now because ξ is admissible, X_{t_n} is a bounded predictable process ($X_{t_n} \in \mathcal{F}_{n-1}$). Furthermore, since B_t^0 is a martingale, $\mathbb{E}(B_t - B_s) = 0$ for any $s < t$, so

$$\mathbb{E}(\sum_{n=0}^{N} B_{t_n}^0 \xi_n) = B_0 Z_0. \tag{3.16}$$

Furthermore, once we know the realizations $\xi_0(\omega), \xi_1(\omega), \ldots, \xi_N(\omega)$, the extra spread D as defined in (3.1) is a deterministic process. So there exists a deterministic function $\bar{C} : \mathbb{R}^{n+1} \to \mathbb{R}$ such that

$$q/2 \sum_{n=0}^{N} (D_{t_n}^2 - D_{t_n}^2) = \bar{C}(\xi_0, \ldots, \xi_N). \tag{3.17}$$

Using the observations (3.12), (3.13), (3.15), (3.16) and (3.17), we see that

$$\bar{\mathcal{C}}(\xi) = B_0 Z_0 - \mathbb{E}(\bar{C}(\xi_0, \ldots, \xi_N)).$$

We take a closer look at the deterministic function which describes the extra spread. $D_{t_n+} = D_{t_n} + x_n/q$ for any deterministic strategy $(x_0, \ldots, x_N) \in \Xi_0$. Using (3.6) we find for the simplified process

$$D_t = \gamma \sum_{t_n \le t} x_{t_n} + \kappa \sum_{t_n \le t} e^{-\int_{t_n}^{t} \rho_s ds} x_{t_n}.$$

So

$$\begin{aligned}
\bar{C}(x_0, \ldots x_N) &= \frac{q}{2} \sum_{n=0}^{N} (D_{t_{n+1}}^2 - D_{t_n}^2) = \frac{q}{2} \sum_{n=0}^{N} (D_{t_n} + x_n/q)^2 - D_{t_n}^2) \\
&= \sum_{n=0}^{N} x_n D_{t_n} + \frac{1}{2q} \sum_{n=0}^{N} x_n^2 \\
&= \gamma \sum_{n=0}^{N} \sum_{k=0}^{n-1} x_n x_k + \kappa \sum_{n=0}^{N} \sum_{k=0}^{n-1} x_n d^{-\int_{t_k}^{t_n} \rho_s ds} x_k + \frac{\gamma + \kappa}{2} \sum_{n=0}^{N} x_n^2 \\
&= \frac{\kappa}{2} \sum_{n=0}^{N} x_n^2 + \kappa \sum_{n=0}^{N} \sum_{k=0}^{n-1} x_n d^{-\int_{t_k}^{t_n} \rho_s ds} x_k + \frac{\gamma}{2} \Big(\sum_{k=0}^{N} x_k \Big)^2 \\
&= \frac{\kappa}{2} \sum_{n=0}^{N} x_n^2 + \kappa \sum_{n=0}^{N} \sum_{k=0}^{n-1} x_n d^{-\int_{t_k}^{t_n} \rho_s ds} x_k + \frac{\gamma}{2} Z_0^2,
\end{aligned}$$

which means we can reduce Problem 3.1.1 to minimizing the function

$$C(x_0,\dots,x_N) := \frac{1}{2}\sum_{n=0}^{N} x_n^2 + \sum_{n=0}^{N}\sum_{k=0}^{n-1} x_n d^{-\int_{t_k}^{t_n}\rho_s ds} x_k \tag{3.18}$$

over Ξ_0. Now we are almost done. We introduce a matrix M with entries

$$M_{ij} = e^{-|\int_{t_i}^{t_j}\rho_s ds|} \text{ for } i,j \in \{0,\dots,N\}.$$

Observe that M can be represented as in (3.9). Now

$$C(x) = \frac{1}{2}\langle x, Mx\rangle \text{ for } x = (x_0,\dots,x_N) \in \mathbb{R}^{N+1}.$$

To conclude the proof, observe that matrix M is symmetric and C is a quadratic form. It only remains to show that M is positive definite. We will show this in the proof of the following theorem. When we know M is positive definite, there must be a unique minimizer x^* on Ξ_0. Now if all $x_i^* \in x^*$ are positive, we must have by Lemma 3.1.3 and the above reasoning that this optimal (deterministic) strategy for the simplified revenue functional equals the optimal strategy for the revenue functional. □

From now on, we will call C the *market distortion* of the trade. No matter whether we consider sell or buy orders, finding an optimal strategy always boils down to minimizing market distortion. From this we also observe that the problem is independent of γ and κ (at least when we assume $\kappa > 0$, which we did). So the optimum does not depend on the actual size of the permanent or temporary impacts. Given this lemma, we can reduce our optimization problem to finding an $x \in \Xi_0$ that minimizes $C(x) = \frac{1}{2}\langle x, Mx\rangle$. We use linear constraints $u^1,\dots,u^k, v^1,\dots v^l \in \mathbb{R}^{N+1}$ on the strategies, which gives us the constraints set

$$\Xi_0 = \{x \in \Xi | \langle u^i, x\rangle = 0, i = 1,\dots,k, \langle v^j, x\rangle \geq 0, j = 1,\dots,l\}. \tag{3.19}$$

where k and l in $\mathbb{N}$ denote the number of constraints of the two types. Suppose for example we want to sell at least half of our units in the first four trades: $x_0 + x_1 + \ldots + x_4 \geq Z_0/2$. This means $\langle v, x\rangle \geq 0$ for $v = (1,1,1,1,0,0,0,\dots,0) - \frac{1}{2}\mathbf{1}$, where $\mathbf{1}$ denotes the $N+1$-dimensional vector consisting of 1's only. The following theorem and it's proof are entirely based on [7].

Theorem 3.1.4. *Let x^* be the unique minimizer of $C(x) = \frac{1}{2}\langle x, Mx\rangle$ on the set Ξ_0 as in* (3.19) *and let J denote the set of indices for all active constraints on x. In other words, $J = \{j \in \{1,\dots,l\} | \langle v^j, x^*\rangle = 0\}$. We assume that the set of vectors $\{\mathbf{1}\} \cup \{u^1,\dots,u^k\} \cup \{v^j, j \in J\}$ is linearly independent. Then*

$$x^* = \lambda_0 M^{-1}\mathbf{1} + \sum_{i=1}^{k}\lambda_i M^{-1}u^i + \sum_{j\in J}\mu_j M^{-1}v^j, \tag{3.20}$$

for multipliers $\lambda_0, \lambda_1, \dots, \lambda_k \in \mathbb{R}$ and $\mu_j \geq 0$, $j \in J$. These multipliers are uniquely determined by the following system of equations.

$$Z_0 = \lambda_0\langle \mathbf{1}, M^{-1}\mathbf{1}\rangle + \sum_{i=1}^{k}\lambda_i\langle \mathbf{1}, M^{-1}u^i\rangle + \sum_{j\in J}\mu_j\langle \mathbf{1}, M^{-1}v^j\rangle \tag{3.21}$$

$$0 = \lambda_0\langle u^p, M^{-1}\mathbf{1}\rangle + \sum_{i=1}^{k}\lambda_i\langle u^p, M^{-1}u^i\rangle + \sum_{j\in J}\mu_j\langle u^p, M^{-1}v^j\rangle, \tag{3.22}$$

$$0 = \lambda_0\langle v^q, M^{-1}\mathbf{1}\rangle + \sum_{i=1}^{k}\lambda_i\langle v^q, M^{-1}u^i\rangle + \sum_{j\in J}\mu_j\langle v^q, M^{-1}v^j\rangle, \tag{3.23}$$

where $p \in \{1,\ldots,k\}$ and $q \in J$. The inverse of M is given by the matrix

$$M^{-1} = \begin{bmatrix} \frac{1}{1-a_1^2} & \frac{-a_1}{1-a_1^2} & 0 & \cdots & 0 \\ \frac{-a_1}{1-a_1^2} & \frac{1}{1-a_1^2}+\frac{a_2^2}{1-a_2^2} & \frac{-a_2}{1-a_2^2} & 0 & \cdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \frac{-a_{N-1}}{1-a_{N-1}^2} & \frac{1}{1-a_{N-1}^2}+\frac{a_N^2}{1-a_N^2} & \frac{-a_N}{1-a_N^2} \\ 0 & \cdots & 0 & \frac{-a_N}{1-a_N^2} & \frac{1}{1-a_N^2} \end{bmatrix}. \tag{3.24}$$

Proof. Let $e_0,\ldots,e_N$ denote the canonical basis of $\mathbb{R}^{N+1}$ and let us recursively define a set of vectors $(y_k)_{k\le N} \in \mathbb{R}^{N+1}$ as follows:

$$\begin{aligned} y_0 &= e_0 \\ y_n &= y_{n-1}a_n + e_n\sqrt{1-a_n^2}. \end{aligned}$$

Now observe that $\langle y_i, y_i\rangle = a_i^2 + 1 - a_i^2 = 1 = M_{ii}$. Furthermore $\langle y_i, y_{i+1}\rangle = \langle y_i, y_i\rangle a_{i+1}$ and repeating this reasoning we find $\langle y_i, y_j\rangle = \langle y_i, y_i\rangle a_{i+1}\ldots a_j = M_{ij}$ for all $j > i$. Furthermore, observe that

$$y_n = \sum_{j=0}^{n}\Big(\prod_{i=j+1}^{n} a_i\Big)\sqrt{1-a_j^2}e_j,$$

where we set $a_0 := 0$ and $\prod_{i\in\emptyset}(\ldots) = 1$. Now define the matrix Y as the matrix which has as columns the vectors y_i. This matrix has value zero anywhere below the diagonal, and is called the Cholesky factor of M ($Y^\top Y = M$).

$$Y := \begin{bmatrix} 1 & a_1 & a_1a_2 & \cdots & a_1a_2\ldots a_N \\ 0 & \sqrt{1-a_1^2} & \sqrt{1-a_1^2}a_2 & & \cdots \\ 0 & 0 & \sqrt{1-a_2^2} & \cdots & \vdots \\ \vdots & & & & \vdots \\ 0\ldots & 0 & & & 0 \end{bmatrix}. \tag{3.25}$$

We denote by $Y^\top$ the transpose of Y and observe that $Y^\top Y = M$. Since $t_n - t_{n-1} > 0$ and $\rho_t > 0$, $0 < a_n < 1$ $\forall n = 1,\ldots,N$. So all diagonal components of Y are strictly positive, which means Y is invertible. Consequently, we may write

$$C(x_0,\ldots,x_N) = \frac{1}{2}x^T Mx = \frac{1}{2}||Yx||^2 > 0 \ \ \forall x \neq 0.$$

But this means that M is positive definite ($z^\top Mz > 0 \ \forall z \neq 0$), which was needed in the proof of Lemma 3.1.2. Note that the gradient of C at x is given by Mx. Now we can apply the Kuhn-Tucker theorem[2] which states that there exist multipliers $\lambda_0, \lambda_1, \ldots, \lambda_k \in \mathbb{R}$ and $\mu_j \geq 0, j \in J$ such that

$$Mx^* = \lambda_0\mathbf{1} + \sum_{i=1}^{k}\lambda_i u^i + \sum_{j\in J}\mu_i v^j.$$

Now if we multiply both sides with M^{-1}, we find (3.20). To see that these multipliers are uniquely determined by (3.21), (3.22) and (3.23), we only have to use the assumption of linear independence of the constraints and orthogonality of u and v^j with x^* and the fact that $\langle x^*, \mathbf{1}\rangle = Z_0$.
It only remains to prove the formula for M^{-1} is correct. To show this, we use Y. Observe that

[2]The Kuhn-Tucker theorem is a generalization of the Lagrange multiplier method. A clear explanation of the theorem and a proof can be found in for example [16].

$M^{-1} = (Y^\top Y)^{-1} = Y^{-1}(Y^\top)^{-1}$. So all we have to do is find the inverse of $Y^\top$. Using $Y^\top x = z$, we get

$$\begin{aligned}
x_0 &= z_0 \\
x_0 a_1 + \sqrt{1-a_1^2}x_1 &= z_1 \\
x_0 a_1 a_2 + \sqrt{1-a_1^2}a_2 x_1 + \sqrt{1-a_2^2}x_2 &= z_2 \\
\vdots \quad &\quad \vdots \\
(a_1 \dots a_N)x_0 + \dots + \sqrt{1-a_N^2}x_N &= z_N.
\end{aligned}$$

We can take these expressions together and rewrite the set of equations as follows:

$$\begin{aligned}
x_0 &= z_0 \\
x_0 a_1 + \sqrt{1-a_1^2}x_1 &= z_1 \\
a_2 z_1 + \sqrt{a-a_2^2}x_2 &= z_2 \\
\vdots \quad &\quad \vdots \\
a_N x_{N-1} + \sqrt{1-a_N^2}x_N &= z_N.
\end{aligned}$$

So we find

$$(Y^\top)^{-1} = \begin{bmatrix}
1 & 0 & \dots & \dots & 0 \\
\frac{-a_1}{\sqrt{1-a_1^2}} & \frac{1}{\sqrt{1-a_1^2}} & 0 & & 0 \\
0 & 0 & \ddots & \dots & \vdots \\
\vdots & & & & \vdots \\
0\dots & 0 & & \frac{-a_N}{\sqrt{a-a_N^2}} & \frac{1}{\sqrt{1-a_n^2}}
\end{bmatrix}.$$

Now it is easy to see that $M^{-1} = Y^{-1}(Y^\top)^{-1}$ is given by (3.24). □

Of course we can also remove the constraints $u^1, \dots, u^k, v^1, \dots v^l$ from this theorem, which gives us a very nice result for the optimal strategy, see [7].

Corollary 3.1.5. *There exists a unique (unconstrained) optimal strategy $\xi^* = (\xi_0^*, \dots, \xi_N^*)$ in the class of all admissible strategies. In this case*

$$\lambda_0 = \frac{Z_0}{\frac{2}{a+a_1} + \sum_{n=2}^N \frac{1-a_n}{1+a_n}}, \tag{3.26}$$

with the first market order

$$\xi_0^* = \lambda_0\Big(\frac{1}{a+a_n} - \frac{a_{n+1}}{1+a_{n+1}}\Big), \quad n = 1, \dots, N-1,$$

and last market order

$$\xi_N^* = \frac{\lambda_0}{1+a_N}.$$

In particular, the optimal selling strategy is deterministic. Moreover $\xi_0^* > 0$ for all n, so the optimal selling strategy consists of sell orders only.

Proof. As in the general case, we can use Lemma 3.1.2 and reduce the problem to the deterministic case. Because we have no constraints, we can set here $\Xi_0 = \Xi$. Consequently, from (3.20) we see that in this case

$$x^* = \lambda_0 M^{-1}\mathbf{1}. \tag{3.27}$$

We use (3.21) in a reduced form (we can skip all the equations with the constraints) for finding λ_0:

$$Z_0 = \lambda_0 \langle \mathbf{1}, M^{-1}\mathbf{1} \rangle, \tag{3.28}$$

and $\langle \mathbf{1}, M^{-1}\mathbf{1} \rangle$ is equal to

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{j=1}^{N} M_{ij}^{-1} &= \frac{2-a_1}{1-a_1^2} + \sum_{n=2}^{N} \frac{1-2a_n+a_n^2}{1-a_n^2} \\
&= \frac{2}{1+a_1} + \sum_{n=2}^{N} \frac{(1-a_n)(1-a_n)}{(1-a_n)(1+a_n)} = \frac{2}{1+a_1} + \sum_{n=2}^{N} \frac{1-a_n}{1+a_n}.
\end{aligned} \tag{3.29}$$

So combining (3.28) and (3.29), we find (3.26). Furthermore, we see from (3.27) that

$$x_i^* = \lambda_0 \sum_{j=1}^{N} M_{ij}^{-1},$$

which yields the desired result. □

There is an even more elegant result, namely when trading times are equidistant, so $t_n - t_{n-1} = \tau$ for $\tau = T/N$ for all n. If we also assume that ρ is a constant, every a_n is the same, so we can write $a_n := a$. Also the proof of the following corollary is derived from [7]. Alternative (direct) proofs can be found in [22] and [8].

Corollary 3.1.6. *When trading intervals are equidistant and equally divided over the trading period, $t_n - t_{n-1} = \tau$ where $\tau = T/N$ for all n and ρ is a constant, the optimal strategy is given by*

$$\xi_0^* = \frac{Z_0}{2+(N-1)(1-a)} = \xi_N^*,$$

and

$$\xi_n^* = \xi_0^*(1-a).$$

Proof. Just take $a_i = a$ in the proof of Corollary 3.1.5. □

Remark 3.1.7. Observe that in a block-shaped LOB, the previous results from a model assuming exponential recovery of the bid-ask spread $D_{t+s} = e^{-\rho s} D_t$ also hold for a model assuming exponential recovery of the volume $E_{t+s} = e^{\rho s} E_t$. This is easy to see:

$$\begin{aligned}
E_{t+s} &= e^{-\rho s}(D_t \cdot q) = q e^{-\rho s} D_t \\
&\Rightarrow \frac{E_{t+s}}{q} = D_{t+s} = e^{-\rho s} D_t.
\end{aligned}$$

We will use this observation in Chapter 6.

3.1.3 Interpretation and remarks on these results

Remark 3.1.8. When the resilience $\rho\tau$ converges to zero, using the above results, it is optimal to trade according to $\xi_0^* = \frac{1}{2} Z_0 = \xi_N^*$ and $\xi_n^* = 0$ for all intermediate $n \in \{1, \ldots, N-1\}$.
When the resilience $\rho\tau$ converges to infinity, using the above results, it is optimal to trade according to $\xi_n^* = \frac{Z_0}{N+1}$ for all n, so all trades are equally sized.
When the resilience $\rho\tau$ is anywhere in $(0, \infty)$, the first and last trades in the optimal strategy will be bigger than the intermediate trades. This is called the *bath-tub* effect.

The first question we can ask ourselves, is how this bath-tub effect can be understood. In figures 3.1, 3.2 we show what happens when intra-trading times are shorter than the time it takes for the market to completely recover from the previous trade.

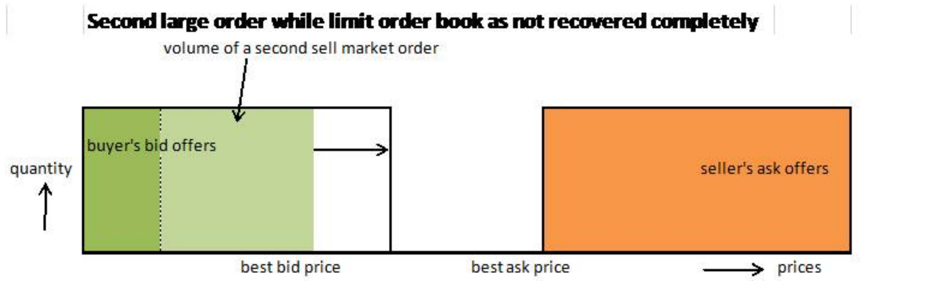


Figure 3.1: Second market order

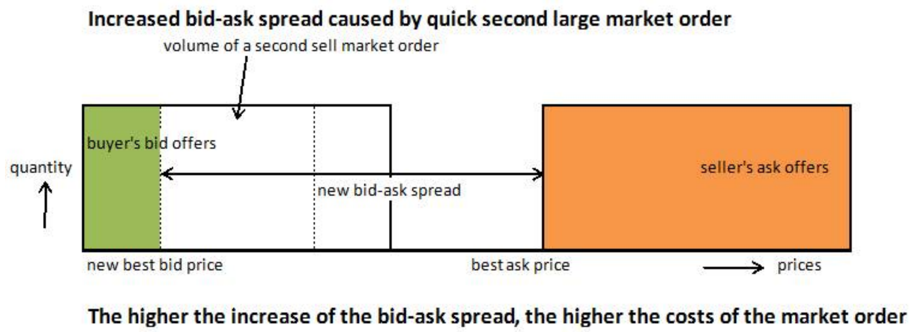


Figure 3.2: Increase bid-ask spread

As can be seen in the figure, the second trade will cost even more than the first one, because the prices accepted are even more below the best bid price. When the seller continues to adopt this strategy, every consecutive trade will be performed at lower prices and thus at greater costs and greater market distortion. In Figures 3.3 and 3.4, we show that a seller can lower this trading costs, by selling exactly the amount of shares by which the order book has recovered from the previous trade. By doing so, market distortion will not be increased by consecutive trades. Because the seller eventually will have to sell his total volume, selling volumes in the beginning and at the end are higher than the intermediate volumes. Because the recovery of the Limit Order Book is exponential, this is less distorting.

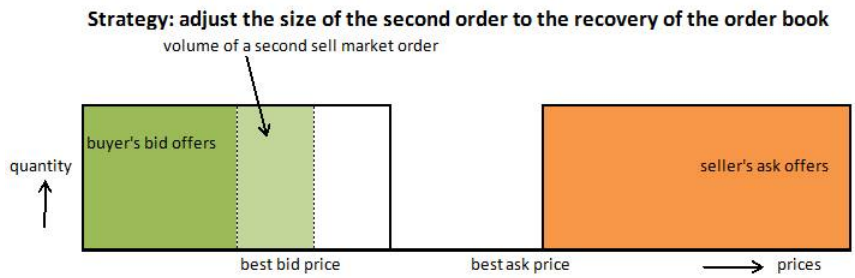


Figure 3.3: Smaller second order

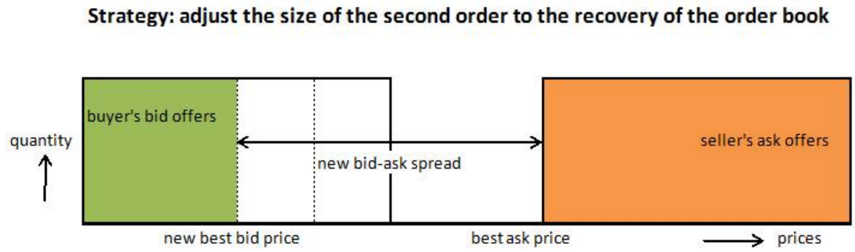


Figure 3.4: Smaller increase bid-ask spread

The bath-tub effect means that when time intervals between consecutive trades are small enough (so T is relatively small with respect to N) and the resilience speed ρ is relatively low, the volumes of the first and last trades in the optimal strategy are higher. In the above theorems, the optimal strategies will depend on the resilience rate ρ, which depends on the elasticity of the market. In determining the optimal strategy, it is thus crucial to make a realistic assumption on the resilience rate of the order book. This assumption can be made for any specific market by conducting empirical research, as is for example done in [9].
In Chapter 6, we will show for a specific example what Corollary 3.1.6 means for finding an optimal selling strategy. When empirical research can or has not yet been performed, large sellers should be aware of the risk they are taking of incurring great market distortion when the trades are frequent, while the market recovery is slow. However, as we will see in Chapter 6, large sellers are facing a trade-off between selling in small volumes to minimize market distortion, while at the same time making sure the market has enough time to recover between two trades, so the frequency cannot be too high. When recovery is quick enough, it is optimal to sell as frequently as possible in small portions. In that case market distortion is minimized, while at the same time the revenue to the seller is maximized.

Chapter 4

Auction Theory

4.1 Introduction

In his article *Counterspeculation, auctions and competitive sealed tenders* [28] William Vickrey starts by pointing out that

> In his *Economics of Control*, A. P. Lerner threw out an interesting suggestion that where markets are imperfectly competitive, a state agency, through "counterspeculation", might be able to create the conditions whereby the marginal conditions for efficient resource allocation could be maintained. [28, p. 8]

In this article Vickrey developed a mechanism for efficient allocation in auction design, thereby laying the foundations of modern auction theory. During the nineties, research on auctions experienced an incredible growth, triggered by large-scale spectrum auctions in Europe and the United States. In 1996 Vickrey was awarded a Nobel Prize in economics for his work in auction theory.

In general, in an auction the seller specifies a set of rules and the buyers will compete over the price at which the object is eventually sold. A famous example of an auction format is the Dutch flower auction, when the seller calls a descending price and the first buyer to accept the price wins as many flowers as he/she wishes. In principle, a seller can design any set of rules he likes. The main question underlying these choices is: how should the items which are for sale be allocated and which prices should be paid?
An often mentioned argument among economists to use auctions as a selling mechanism is *efficient allocation*. This notion is not always explained in the same way. In this thesis, as in [28] and [20], by speaking of efficient allocation we mean that the firm or individual who values the items the most will actually obtain the items. In the case of CO_2 emission allowances, it would be desirable to make the firms which are the worst polluters pay the most for their allowances. The idea behind using auctions for CO_2 emissions is that polluters will need emission rights, so they will attach high value to these rights.

4.1.1 The value of a theory

Traditional auction theory relies heavily on some standard assumptions from game theory, which are of course not always true descriptions of the real world. One of the central assumptions we will use is that buyers and sellers behave rationally, but (maybe not so surprising) in reality they do not always behave rationally. Consequently, some standard concepts in game theory underlying auction theory, such as optimal strategic behavior and equilibrium strategies which will be treated in this section, should be handled with care. As is the case with any model, also this auction model should be treated as an instrument, not as the truth.
Having said this as a disclaimer, we can happily start developing our (theoretical) model. By introducing our model we are trying to create a structure for the universe of auctions and designs,

which will hopefully be useful in a next stage, when we would like to investigate certain auctions and certain designs.

4.1.2 Basic auction types

There are four basic auction types, which can be divided into open and closed formats.

1. **Open Dutch descending price auction.** In a Dutch auction, the auctioneer calls a descending price which all potential buyers can hear and the first buyer to accept the price wins the item for the accepted price. This format is typically used in Dutch flower auctions.

2. **Sealed-bid first price auction.** In a sealed-bid first price auction, all bidders submit there bids to the seller, but none of the bidders can view any bids except his own. The item is sold to the highest bidder at the price he bid. This format is equivalent to the Dutch auction.[1] This can easily be seen from a heuristic point of view, because in a Dutch auction, none of the bidders can observe any of the private values of the other bidders, because as soon as information about these values is provided by one of the bidders, it can only be by placing a bid, but then the auction stops. Furthermore, the item will be sold to the highest bidder.

3. **Open English ascending price auction.** In an English auction, the auctioneer calls an ascending price, as long as there are at least two interested bidders. Bidders drop out whenever they want, they can let the auctioneer know when they do so, thereby also informing all other bidders (since it is an open format). When there is only one interested bidder left, the auction stops and the item is sold to the remaining bidder for the price at which the before-last bidder left the auction. This type of auctions is widely used for selling art, for example at Sotheby's.

4. **Sealed-bid second price auction.** In a sealed-bid second price auction, all bidders submit there bids to the seller, but none of the bidders can view any bids except his own. The item is sold to the highest bidder at a price equal to the second highest bid. This format is weakly equivalent to the English auction format[2], because also in an English auction, the item is sold at a price equal to the bid of the second-highest bidder, because when the price the auctioneer calls reaches that point the auction stops. It remains unknown what would have been the highest bid. However, these two are not strongly equivalent, because in the open format bidders receive information about each other's behavior. This is especially relevant when the values of the bidders are interdependent, because dropping out of a bidder might then affect behavior of other bidders.

4.2 Auction mechanisms

We will now start by setting up the basic framework and introducing some definitions. Next, we will prove some basic results in traditional auction theory. With the machinery we then develop, we will study a specific type of auctions, namely multiple unit auctions. These will be useful in a later stadium when we will try to apply auction theory to the auctioning of CO_2 emission allowances.

Auctions as games of incomplete information

Auction theory is a branch of game theory. Auction theory models the decisions economic agents are making as players of a game with incomplete information. The game takes place in an *environment.* We set an *environment* $(\mathcal{N}, \Omega, \mathcal{X})$ to be consisting of

[1]Two games are strategically equivalent, if they have the same normal form except for duplicate strategies. This means that for every strategy in one game, there exists a strategy in the other game which yields the same outcome [18, p. 4].

[2]Weakly equivalent is used here as an informal concept, meaning that the two auction formats are not strategically equivalent, but the optimal strategies in the two are the same only when values are private [18, p. 5].

- A set $\mathcal{N}$ of bidders
- A collection Ω of possible *outcomes* of the auction game
- For every bidder $i \in \mathcal{N}$ a set of *signals* $\mathcal{X}_i$. We denote by $\mathcal{X} = \prod_j \mathcal{X}_j$ the collection of all bidder's signals. We will specify below what we mean by outcomes and signals. Instead of signals, we will sometimes use the word *types*, which will mean exactly the same.

In games of incomplete information, a player determines his strategy on the basis of *expected payoff*, conditional on the information which is available to him. We assume all his information comes in the form of signals. In a game of incomplete information, a player may not have access to the full set of information which is available about the game (for example, other players' signals). Formally, an auction game consists of:

1. A set $\mathcal{N}$ of bidders
2. For every bidder $i \in \mathcal{N}$ a set $\mathcal{B}_i \neq \emptyset$ of bids, $\mathcal{B}_i \subset \mathbb{R}$
3. For every bidder $i \in \mathcal{N}$ a set of signals $\mathcal{X}_i$
4. For every bidder $i \in \mathcal{N}$ a payoff function $u_i : \prod_j \mathcal{B}_j \times_j \mathcal{X}_j \to \mathbb{R}$
5. A probability distribution F over the set of signals $\prod_j \mathcal{X}_j$

Definition 4.2.1. A (pure) *strategy* β_i is a monotone function $\beta_i : \mathcal{X}_i \to \mathcal{B}_i$ mapping signals into bids.[3]

When an auction starts, the signals $X = (X_1, \ldots, X_N)$ are drawn from the distribution F and player i knows the realization of his own signal $X_i = x_i$. Now every player chooses a bid $\beta_i(x_i) = b_i$. Finally, based on the realized signals $x = (x_1, \ldots, x_N)$ of all bidders and their bids $b = (b_1, \ldots, b_N)$, payoffs are realized.
Mechanism design studies the subset of rules in a universe which can be controlled by a designer, namely the *auction* - or *selling mechanism.* A selling mechanism $(\mathcal{B}, \omega)$ consists of $\mathcal{B} = \prod \mathcal{B}_j \subset \mathbb{R}^N$, a set of possible strategy profiles for each buyer and a function $\omega : \mathcal{B} \to \Omega$, which maps strategy profiles into outcomes. The *outcome* of an auction is a pair (δ, p) with δ a decision vector and p a payments vector.

- The decision vector $\delta \in \Delta$, where Δ is the set of all possible probability distributions over the set of buyers $\mathcal{N}$. When $\delta_i = 1$, bidder i wins the item and when $\delta_j = 0$, bidder j does not. Note that $\sum_{i=1}^{N} \delta_i = 1$ iff the number of bidders is N and the number of items is 1. In most of the models we will study, $\delta_i \in \{0, 1\}$, except when bids are tied (two bidders bid the same values) and ties are resolved randomly.
- The payment vector p is a vector with elements p_i representing the payment i has to make to the seller.

In auction design, a mechanisms designer must specify these two rules to determine how he's going to decide

1. **Who** gets the object: formalized by an allocation rule μ
2. **How much** the winner has to pay for the object: formalized by a payment rule π

The function ω, mapping strategy profiles into outcomes, can be decomposed as a pair (π, μ) of two separate mappings π and μ, where $\pi : \mathcal{B} \to \mathbb{R}^N$ and $\mu : \mathcal{B} \to \Delta$ such that $\mu_i(b) = \delta_i$ and $\pi_i(b) = p_i$.

[3]Strategies are not always pure. When strategies are not pure, we call them *mixed*, which means that the strategy consists of a probability distribution over the set of possible pure strategies.

Example 4.2.2. Suppose there is a single item to be sold by a single seller and there are four bidders. Suppose the seller is going to sell the item to the highest bidder. Then the allocation rule in any four of the standard auction formats mentioned in Section 4.1.2 can be formulated as follows:

$$\mu_i(b) = \mathbf{1}_{\{b_i > \max_{j \neq i} b_j\}} = \delta_i \tag{4.1}$$

Example 4.2.3. In a first price auction for a single unit, the highest bidder has to pay the price he bid:

$$\pi_i(b) = b_i \mathbf{1}_{\{b_i > \max_{j \neq i} b_j\}} = p_i \tag{4.2}$$

Example 4.2.4. In a second-price auction for a single unit, the highest bidder gets the object but he pays a price equal to the second highest bid, so the pricing rule becomes:

$$\pi_i(b) = \max_{\{j \neq i\}} b_j \mathbf{1}_{b_i > \max_{j \neq i} b_j\}} = p_i \tag{4.3}$$

We introduce a (common) notation: for any vector $x = (x_1, x_2, \dots, x_N)$ we denote by $x_{-i} \in \mathbb{R}^{N-1}$ the vector $((x_j)_{j \leq N, i \neq j})$ the vector of all elements in x except x_i.

Signals

The signals represent the information available to a bidder about the value of the item which is being sold. This information can be used to determine a bidding strategy. Information can be asymmetric among bidders. A signal tells the bidder something about what it is worth obtaining the item. Signals can be correlated or even the same among bidders. When there is asymmetric information among bidders, signals are not the same. We now formalize this idea.
The signals X_i are in principle uncertain, so we assume that a signal X_i is a random variable on a probability space $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$, which is distributed according to some probability density f_i. Note that these random variables and their distributions are specified per buyer and thus specific buyers may have different distribution functions, so for all i:

$$F_i(y) = \int_0^y f_i(x)dx.$$

We assume that $X_i \geq 0$ *a.s.* From now on we will assume the buyer's signals are independently distributed. Furthermore, we assume $\mathbb{E}(|X_i|) < \infty$. We use the following proposition from measure theory [26].

Proposition 4.2.5. *Two random variables X_1, X_2 on $(\Omega, \mathcal{F}, \mathbb{P})$ are independent iff the joint distribution $\mathbb{P}^{(X_1,X_2)}$ on $\mathcal{B}^2$ is the product measure $\mathbb{P}^{X_1} \times \mathbb{P}^{X_2}$. This in turn happens iff $F(x_1, x_2) = F_{X_1}(x_1)F_{X_2}(x_2)$, for all $x_1, x_2 \in \mathbb{R}$. Assume further that (X_1, X_2) has a joint probability density function f. Let f_1 and f_2 be the (marginal) probability density functions of X_1 and X_2 respectively. Then X_1 and X_2 are independent iff $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ for all (x_1, x_2) except in a set of Lebesgue measure zero.*

Of course we can extend this definition to our random vector X taking values in $\mathbb{R}^N$ by repeating the argument. This means that when we are assuming independence the joint distribution function F is given by

$$F(x_1, \dots, x_N) = F_{X_1}(x_1) \dots F_{X_N}(x_n).$$

Values

The *value* a bidder attaches to an item is the maximum amount he is willing to pay for the item(s), given the outcome of the auction. We assume that the value of the object to bidder i can be expressed as a function of all bidder's signals and the outcome of the allocation rule:

$$V_i = v_i(\delta, X_1, X_2, \dots, X_N) = v_i(\delta, X)$$

where v_i is i's valuation which depends on the outcome δ and we assume that the v_i are non-decreasing in X_j, $j \neq i$ and twice continuously differentiable. Furthermore, the v_i are *strictly* increasing in X_i. Now we thus assume that V_i is completely determined by the signals. More in general, it is also possible to study models in which the V_i are random variables, conditional on the given signals:

$$v_i(\delta, x_1, x_2, \ldots, x_N) = \mathbb{E}(V_i | \delta, X_1 = x_1, X_2 = x_2, \ldots, X_N = x_N).$$

In any case, we suppose $v_i(\delta, 0, 0, \ldots, 0) = 0$ and $\mathbb{E}(V_i) < \infty$.

Definition 4.2.6. A value is assumed to be *private*, when it is assumed to be independent of other bidder's types. More specifically, a value is private when

$$v_i(\delta, X_1, X_2, \ldots, X_N) = X_i \delta_i. \tag{4.4}$$

Another special case of interdependent values which will be useful for us is when there is a *common values* principle. This is the case when bidders in principle attach the same value to the items, but this value is uncertain.

Definition 4.2.7. We speak of a *pure common value* when the V_i are the same and we write

$$V = v(\delta, X_1, X_2, \ldots, X_N). \tag{4.5}$$

Utility

In general, a player's utility function $u_i : \mathcal{B} \times \mathcal{X} \to \mathbb{R}$ is formulated as a concave, increasing and continuous function of strategies and values. We wirte

$$u_i(b_1, \ldots, b_N, X).$$

Given a certain selling mechanism $(\mathcal{B}, \omega)$, bidding strategies and values determine bids, outcomes are then determined using the allocation and payment rules μ and π. In auction theory it is assumed in this perspective that when calculating utility, strategies matter only insofar as they determine outcomes. So from now on, utility functions will be formulated as functions of *outcomes* and type profiles, so $u_i : \Omega \times \mathcal{X} \to \mathbb{R}$.

Dominant strategies

In auction theory, bidders behave according to some strategy. Given their type, they adopt a certain strategy which makes them decide to place a certain bid.

Definition 4.2.8. A strategy β_i is *weakly dominating* β_i' if $\forall x \in \mathcal{X}$ and for all b_{-i}

$$u_i(\beta_i(x_i), b_{-i}, x) \geq u_i(\beta_i'(x_i), b_{-i}, x)$$

with strict inequality for some x and b_{-i}.

Definition 4.2.9. The strategy β_i^* is *dominant* if it (weakly) dominates every other strategy β_i'.

This means a bidder can never increase his payoff by changing his strategy to anything other than β_i^*.

Definition 4.2.10. If there exists a dominant strategy for every player i, then we call β^* the *dominant strategy equilibrium*.

In a dominant strategy equilibrium, no player has any incentive to change his or her bid, because this will never result in a higher payoff.

Bayesian-Nash equilibria

When bidders behave rationally, we assume they aim to maximize their expected payoff and will choose a strategy to accomplish this goal.

Definition 4.2.11. A (pure-strategy) *Bayesian-Nash equilibrium* of an auction is a vector of strategies β^* such that for all i and voor all $x_i \in \mathcal{X}_i$ and for all $b_i \in \mathcal{B}_i$

$$\mathbb{E}[u_i(\beta^*(X), X)|X_i = x_i] \geq \mathbb{E}[u_i(b_i, \beta^*_{-i}(X_{-i}), X)|X_i = x_i] \tag{4.6}$$

where $\beta^*(x) = (\beta_i^*(x_i))_{i \in \mathcal{N}}$ denotes the vector of bids of all players and $\beta^*_{-i}(x_{-i}) = (\beta_j^*(x_j))_{j \in \mathcal{N}, j \neq i}$.

We say that for all i, β^* is optimal at the *interim* stage, namely when values are known and players determine their bid strategy. So no player has anything to gain by changing his strategy at this point.
Now suppose β_i is some strategy, not equal to β_i^*. For all i and for all realizations x_i (4.6) holds for $b_i = \beta_i(x_i)$. When we take expectations in (4.6), by one of the fundamental properties of conditional expectations ($\mathbb{E}(\mathbb{E}(X|Y)) = \mathbb{E}(X)$), we see that then for all i and for all β_i

$$\mathbb{E}[u_i(\beta^*(X), X)] \geq \mathbb{E}[u_i(\beta_i(X_i), \beta^*_{-i}(X_{-i}), X)], \tag{4.7}$$

which we call an *ex ante* Bayesian-Nash equilibrium. The terminology is obvious: β_i is optimal for all i before the actual signals are known. Note that interim optimality implies ex ante optimality, by the above argumentation. Ex ante optimality implies interim optimality *almost surely* (there might exist a set of measure zero for which the implication does not hold).
Another concept we will use is that of *ex post* equilibrium, which refers to a Bayesian-Nash equilibrium with the additional requirement that not only a bidder's own type, but also all other types are known to the bidder. (This is often interpreted as a 'no-regrets' situation - even if a bidder knows all types, it is still optimal to bid according to β_i.)
An *ex post* equilibrium is a Bayesian-Nash equilibrium β^* with the property that for all i, for all $x \in \mathcal{X}$ and all b_i:

$$u_i(\beta^*(x), x) \geq u_i(b_i, \beta^*_{-i}(x_{-i}), x).$$

Definition 4.2.12. A bidder i is risk-neutral iff his Von-Neumann Morgenstern utility function is given by

$$u_i(\omega, X) = v_i(\delta, X) - p_i, \tag{4.8}$$

where v_i is the function determining the value the bidder attaches to the outcome of the auction and p_i is the payment he has to make.

When a bidder is risk-neutral, he only cares about his expected revenue and not about any risks involved. Note that in the private values case (4.4), (4.8) reduces to $u_i(\omega, X) = x_i\delta_i - p_i$.

4.2.1 Revenue and efficiency

In principle, when a seller chooses an auction mechanism, it would be intuitive to select the mechanism that maximizes his *expected revenue*.

Definition 4.2.13. We define the *revenue* R to the seller to be

$$R = \sum_{i=1}^{N} p_i.$$

From this, we can derive *ex ante* expected revenue for the seller to be $\mathbb{E}(R) = \mathbb{E}\sum_{i=1}^{N} \pi_i(\beta_i(X_i))$.
From a bidder's perspective, the best mechanism would be the most *efficient* one. An efficient mechanism we will define as a mechanism in which the participant with the highest valuation of the item attains the item.

Definition 4.2.14. An *efficient* mechanism maximizes

$$\mathbb{E}[\sum_{i=1}^{N} v_i(\delta, X)]. \tag{4.9}$$

The expression in (4.9) is sometimes called *social welfare*. These two principles do not necessarily result in the same design. However, when governments are trying to sell items via auctions to private institutions, as is the case for auctioning EUA's, it might be in the interest of the government to create the best mechanism for the buyers, thus aiming at efficiency instead of revenue maximization, because the best buyer's situation will ultimately probably also be the best situation for the government. We will come back to this issue in Section 4.4.4

4.3 Equilibrium strategies and revenue maximization in basic second- and first-price auctions

As stated in Section 4.2.1, in auction theory we are interested in efficient allocation and expected revenues, which we can examine by studying equilibrium strategies. We start by studying some fundamental results in elementary auction theory, which rely heavily on some ideal (and unrealistic) assumptions. Sometimes, these assumptions will turn out to be harmful - when some assumptions are removed, our results change significantly. We will gradually relax our assumptions and check the results, while trying to make a model that describes auction mechanisms for CO_2 emissions as close as possible. As noted before, every open auction format is (weakly) equivalent to a sealed-bid format, so it suffices to study sealed-bid formats. Results can then easily be transferred to the open formats, if one likes.

Assumptions 4.3.1. We assume the following:

1. Bidders have private - not interdependent- values.
2. Bidders are risk-neutral: they only aim to maximize their expected profit.
3. All components of the model are known to all participants, except the types of other bidders.
4. Bidders are not subject to any budget or liquidity constraints.
5. Bidder's signals are independent and identically distributed: The distribution function $F_i = F$ is the same for all bidders (and is known to all bidders): the bidders behave *symmetrically.*

4.3.1 Second-price auctions

We begin by studying equilibrium strategy and expected revenue in second-price auctions. At the *ex ante* stage, utility can be expressed as follows:

$$u_i(\omega, X) = u_i(\omega(\beta(X)), X) = u_i(\mu(\beta(X)), \pi(\beta(X)), X).$$

At the interim stage, bidder i knows his own type, but other bidder's types are unknown to bidder i, so the payoff function can be expressed as follows:

$$\begin{aligned} u_i(\mu_i(\beta(X)), \pi_i(\beta(X)), X|X_i = x_i) &= u_i((\delta, p), X|X_i = x_i) \\ &= v_i(\delta, X|X_i = x_i) - p_i \\ &\overset{Definition\ 4.2.12}{=} x_i\delta_i - p_i \\ &= \Big(x_i - \max_{j\neq i} \beta_j(X_j)\Big)\mathbf{1}_{\{\beta_i(x_i) > \max_{j\neq i}\beta_j(X_j)\}}. \end{aligned}$$

Proposition 4.3.2. *In a second price sealed bid auction, a weakly dominant strategy is to bid according to* $\beta(x) = x$[4]

Proof. Suppose $p_1 = \max_{j \neq 1} b_j$. Now if $x_1 > p_1$, bidder 1 wins and if $x_1 < p_1$, bidder 1 looses. Now suppose all bidders except bidder 1 bid according to $\beta(x) = x$. If bidder 1 places a bid $z_1 > x_1 > p_1$ or $x_1 > z_1 > p_1$ he still wins and has exactly the same payoff as when he bids x_1, namely $x_1 - p_1$. If he bids $z_1 < p_1 < x_1$ he looses (zero payoff) instead of winning (positive payoff). If he bids $x_1 < p_1 < z_1$ the reverse happens: he wins the item but has a negative payoff, whereas by bidding x_1 the payoff would have been 0. If he bids $z_1 < x_1 < p_1$ or $x_1 < z_1 < p_1$ he still looses and has zero payoff so nothing changes [18, p. 15]. □

Now we're interested in what a bidder is expected to pay. Denote by $X_{-i}^{(1)}$ the maximum of X_{-i}. Now let G denote the distribution function of $X_{-i}^{(1)}$. Then because the X_i are independent, we have that $G(x) = F(x)^{N-1}$. We can now calculate the expected payment $m_i(x_i)$ by a bidder. We assume all bidders act according to their weakly dominant strategy $\beta(x) = x$.

$$\begin{aligned} m_i(x_i) &= \int_{\mathbb{R}^+} \mu(\beta(x_i, x_{-i})) f(x_{-i}) dx_{-i} \\ &= \int_{\mathbb{R}^+} \max_{\{j \neq i\}} x_j \mathbf{1}_{\{x_j \leq x_i\}} f(x_{-i}) dx_{-i} \\ &= \int_0^{x_i} \Big(\int_0^{x_i} \max_{\{j \neq i\}} x_j f(x_{-i}) dx_{-i} \Big) f(x_{-i}) dx_{-i} \\ &= \int_0^{x_i} \mathbb{E}(X_{-i}^{(1)} | X_{-i}^{(1)} < x_i) f(x_{-i}) dx_{-i} \\ &= G(x_i) \mathbb{E}(X_{-i}^{(1)} | X_{-i}^{(1)} < x_i), \end{aligned} \tag{4.10}$$

where we used the assumption that $X_i \geq 0$ a.s.

4.3.2 First-price auctions

For the payoff function for a bidder in a first price auction we can use the same reasoning as for the second price auction with these assumptions. The payoff function at the interim stage is thus given by

$$u_i(\omega, x_i, X_{-i}) = (x_i - \beta_i(x_i)) \mathbf{1}_{\{\beta_i(x_i) > \max_{j \neq i} \beta_j(X_j)\}}.$$

To determine the optimal strategy for a bidder in a basic first-price auction, the maximization problem for an individual bidder is to maximize his expected utility. The only variable in a utility function the bidder can change is his bid b. In equilibrium, a bidder's bid will be changed if he acts according to an 'untrue' type. So the problem can be formulated as follows:

$$\max_z \mathbb{E}(u_i(\omega(\beta(z)), x, X_{-i})) = \max_z \mathbb{E}(x_i - \beta_i(z)) \mathbf{1}_{\{\beta_i(z) > \max_{j \neq i} \beta_j(X_j)\}}.$$

From now on, we will denote by $\Pi(\beta(z), x)$ the expected utility of a bidder with true type x who bids as if his type were z. So $\Pi(\beta(z), x) = \mathbb{E}(u_i(\omega(\beta(z)), X))$. But this reduces to

$$\max_z G(z)(x_i - \beta(z)), \tag{4.11}$$

using that $\prod_{j=1}^N \mathbb{P}(\beta_j(X_j) \leq \beta_i(z)) = \prod_{j=1}^N \mathbb{P}(X_j \leq z)$ because bidder's distributions are iid and the β_i are strictly monotone for all i, so in equilibrium their strategies are expected to be the same at the interim stage.
In a first-price auction, when a bidder follows the same strategy as is optimal in a second-price auction, namely bid his true value, his expected payoff is 0. Bidding an amount higher than his

[4] Note that it is thus optimal for each bidder to bid exactly the value he attains to the item. No one has anything to win by lying about his true values.

true value can only result in 0 or negative payoff, so it will be optimal to report a value $\beta_i(x_i) \leq x_i$. But which value? From now on, we use superscript I to indicate we talk about first-price auctions, whereas II will refer to strategies in second price auctions. The following proposition and proof are derived from [18, p 17-18].

Proposition 4.3.3. *Symmetric equilibrium strategies in a first-price auction are given by*

$$\beta^I(x) = \mathbb{E}(X_{-1}^{(1)} | X_{-1}^{(1)} < x). \tag{4.12}$$

Proof. Suppose all bidders follow strategy β^I in (4.12), except bidder 1. Suppose he bids $b > \beta(y)$, when $X_{-1}^{(1)} : \Omega \to [0, y]$. This is not optimal, because then he would surely win and he can always reduce his bid with $b - \beta(y) > \epsilon > 0$, such that $b - \epsilon > \beta(y)$. So we suppose that $b \leq \beta(y)$, and we denote by $z = \beta^{-1}(b)$ the value of bidder 1 such that b is an equilibrium bid. For the sake of clarity, we omit indices and denote $\mathbb{E}(u_i(\omega, x_1, X_{-1}))$ as $\mathbb{E}(u(\omega, x))$. The expected payoff Π of bidder 1 is then:

$$\begin{aligned}
\Pi(b, x) = \mathbb{E}(u(\omega(\beta(z))), x) &= G(z)[x - \beta(z)] \\
&= G(z)x - G(z)\mathbb{E}(X_{-1}^{(1)} | X_{-1}^{(1)} < z) \\
&= G(z)x - G(z)\frac{\mathbb{E}(X_{-1}^{(1)} \mathbf{1}_{\{X_{-1}^{(1)} < z)\}}}{\mathbb{E}(\mathbf{1}_{\{X_{-1}^{(1)} < z\}})} \\
&= G(z)x - G(z)[\frac{1}{G(z)} \int_0^z yg(y)dy] \\
&= G(z)x - \int_0^z yg(y)dy \\
&= G(z)x - G(z)z + \int_0^z G(y)dy.
\end{aligned}$$

Now by taking $z = x$ in the above equation, we see that that $\Pi(\beta(x), x) = \int_0^x G(y)dy$. Assume $x \leq z$. Then

$$\Pi(\beta(x), x) - \Pi(\beta(z), x) = G(z)(z - x) - \int_x^z G(y)dy.$$

Also if $x \geq z$:

$$\Pi(\beta(x), x) - \Pi(\beta(z), x) = -G(z)(x - z) + \int_z^x G(y)dy = G(z)(z - x) - \int_x^z G(y)dy.$$

Because $G(y)$ is non-decreasing, (it is a probability distribution function), we conclude that $\Pi(\beta(x), x) - \Pi(\beta(z), x) \geq 0 \ \forall z, x$. So it is indeed optimal to follow the strategy $\beta(x)$ as in (4.12). $\square$

4.4 The revenue equivalence principle in basic auctions

In our simple model, a quite remarkable, nice and often mentioned result, is called the *revenue equivalence* theorem. In this section we will prove the revenue equivalence principle for symmetric bidders. For notational simplicity, we introduce the function $m_i = \pi \circ \beta : \mathcal{X} \to \mathbb{R}^N$ for all i as a composition of β and π on $\mathcal{X}$. So m_i gives us the payment a bidder has to make, given a certain payment function and a strategy he chooses, as a function of his signal.
We defined the expected revenue $\mathbb{E}(R)$ to the seller to be the sum of the expected payments of the bidders,

$$\mathbb{E}(R) = \sum_{i=1}^{N} \mathbb{E}m_i(X_i).$$

The next proposition and its proof are based on [18, p. 30-13].

Proposition 4.4.1. *Suppose that values are independently, identically distributed among a group of risk neutral bidders. Assume that the expected payment by a bidder with private value* 0 *equals* 0*. Then any symmetric equilibrium of a standard auction yields the same expected revenue to the seller.*

Proof. Consider any auction format A, an equilibrium strategy β and a payment function $m^A(x)$ such that $m^A(0) = 0$. Suppose all bidders, except bidder 1, adopt strategy β. Bidder 1 has true value x, but he places a bid $\beta(z)$. Bidder 1 wins when $\beta(z) > \beta(X_{-1}^{(1)})$ or equivalently when $z > X_{-1}^{(1)}$. His expected payoff is then

$$\Pi^A(z,x) = G(z)x - m^A(z). \tag{4.13}$$

This holds because $\mathbb{E}(u_i(\omega(\beta(z))), X) = \mathbb{E}(v_i(\delta(\omega(z)), X) - p_i) = \mathbb{P}(X_{-i} \leq z)x - m_i(X)$ (because we assume private values and by definition of m_i). Note that his expected payment m^A does not depend on his true value x. To optimize his expected payoff, we set the derivative equal to zero:

$$\frac{\partial}{\partial z}\Pi^A(z,x) = g(z)x - \frac{d}{dz}m^A(z) = 0.$$

In equilibrium it is optimal to follow the strategy using $z = x$, such that

$$\frac{d}{dy}m^A(y) = g(y)y.$$

So

$$\begin{aligned}
m^A(x) &= m^A(o) + \int_0^x yg(y)dy \\
&= \int_0^x yg(y)dy \\
&= \mathbb{E}(X_{-1}^{(1)}\mathbf{1}_{\{X_{-1}^{(1)}<x\}}) \\
&= \mathbb{E}(X_{-1}^{(1)}|X_{-1}^{(1)} < x)\mathbb{E}(\mathbf{1}_{\{X_{-1}^{(1)}<x\}}) \\
&= \mathbb{E}(X_{-1}^{(1)}|X_{-1}^{(1)} < x)G(x). \qquad (4.14)
\end{aligned}$$

Now note that the righthandside (4.14) does not depend on A, so we see that the expected revenue in this auction does not depend on the specific auction type. □

4.4.1 Revenue equivalence under less ideal assumptions

Of course, it is interesting to study whether the revenue equivalence principle (Proposition 4.4.1) holds, when the assumptions are relaxed. In the following subsections we will study some of these relaxations. An overview of assumptions which affect revenue equivalence in single unit auctions can be found in the following table.[5]

4.4.2 Unknown number of bidders

Until now, we assumed the number of bidders known to each participant in advance. However, the number of bidders can be unknown to the bidders, as is often the case in sealed-bid formats. What happens if we relax the assumption of a known number of bidders to the revenue principle? Call $\mathcal{N} = \{1, 2, \ldots, N\}$ the list of *potential bidders* and $\mathcal{N}^A \subseteq \mathcal{N}$ the list of *actual bidders* . Consider a bidder $i \in \mathcal{N}^A$ and denote by p_n the probability that i thinks he's facing n competitive bidders, so that the total number of bidders equals $n+1$. We assume that this probability is the same for

[5] For a description about the results on budget constraints and interdependent values, we refer to [18]. The effects of the other assumptions on revenue equivalence will be described in the next subsections.

Table 4.1: Assumptions which affect revenue equivalence in single unit auctions

Assumption	Revenue equivalence?
Unknown number of bidders	✓
Risk averse bidders	✗
Budget constraints	✗
Asymmetric bidders	✗
Dependent values	✗

all i, and does not depend on their true values. Since bidders do not know how many competitors they're facing, their strategy cannot depend on this number. Consider bidder 1 who bids $b = \beta(z)$ when his true signal is x. Suppose he faces n competitors. The probability that this happens is p_n and he will win when his bid is higher than the other ones: $\beta(z) > \beta(X_{-1}^{(1)})$ or equivalently $z > X_{-1}^{(1)}$. The conditional probability that this happens we denote by $G^{(n)}(z) = F(z)^n$ We then have for the overall probability of player i to win if he bids $\beta(z)$:

$$G(z) = \sum_{n=0}^{N-1} p_n G^{(n)}(z),$$

His expected payoff is then

$$\Pi^A(z, x) = G(z)x - m^A(z),$$

so the revenue equivalence still holds, because this is independent of the auction format.

4.4.3 Risk-averse bidders

We assumed bidders behave risk-neutral, so their optimal strategy was only determined by their expected payoff Π. As is widely presumed, in the real world bidders will behave risk-averse (if you ask somebody to choose between directly getting 500 euros or flipping a coin and winning 1000 euro for heads and nothing for tails, expected revenue is the same, but most people - risk averse bidders - will prefer the first option.). To study the behaviour of risk-averse bidders, we use concave, strictly increasing and continuous utility functions $u_i : \Omega \times \mathcal{X} \to \mathbb{R}$ such that $u(0, 0) = 0$. For our model, it means that in this case, it no longer holds that $u_i(\omega, X) = v_i(\delta, X) - p_i$. Instead, $u_i(\omega, X) = u_i(v_i(\delta, X) - p_i)$. We assume u is differentiable on its domain. The following theorem is derived from [18, p. 38-40], however we modified it's proof to make it mathematically more rigorous.

Proposition 4.4.2. *When bidders are risk averse, expected revenue in a first price auction is higher than expected revenue in a second price auction.*

Proof. We begin by considering a second-price auction. When a bidder does not report his true value, due to the reasoning in the proof of Theorem 4.3.2 and the fact that u is a utility function, he will still be able to improve his position, so the dominant strategy is to bid his true value.
Now we consider the first price auction. Instead of only maximizing expected revenue, we seek to maximize expected utility. The maximization problem to find a dominant equilibrium strategy in a first price auction becomes:

$$\begin{aligned}
\max_z \mathbb{E} u_i(\omega(\beta(z)), x_i) &= \max_z \mathbb{E}(u_i(x_i - \gamma_i(z)) \mathbf{1}_{\{\gamma_i(z) > \max_{j \neq i} \gamma_j(X_j)\}}) \\
&= \max_z \prod_{j=1}^{N} [\mathbb{P}(\gamma_j(X_j) \leq \gamma_i(z))] u_i(x_i - \gamma_i(z))] \\
&= \max_z G(z) u_i(x_i - \gamma_i(z))],
\end{aligned}$$

where $\gamma_i : \mathcal{X}_i \to \mathbb{R}^+$ denotes the equilibrium bidding strategy for a risk-averse bidder.
In the derivation that follows, we omit the indices. We take the derivative and set it equal to zero to obtain:

$$g(z)u(x-\gamma(z)) - G(z)\gamma'(z)u'(x-\gamma(x)) = 0. \tag{4.15}$$

Now note that a risk-neutral bidder by Definition 4.2.12 has utility function $u(x) = x$, so when we plug this in we find

$$\beta'(x) = (x-\beta(x))\frac{g(x)}{G(x)}, \tag{4.16}$$

where β is the equilibrium strategy for risk-neutral bidders.
Now, because the function u is strictly concave and $u(0) = 0$, we have

$$\frac{u(y)-u(0)}{y-0} > u'(y) \Rightarrow \frac{u(y)}{u'(y)} > y.$$

We can use this in (4.15) to obtain

$$\gamma'(x) > (x-\gamma(x))\frac{g(x)}{G(x)}. \tag{4.17}$$

We see from (4.15) and (4.16) that

$$\beta(x) > \gamma(x) \Rightarrow \gamma'(x) > \beta'(x). \tag{4.18}$$

But we know that $\beta(0) = \gamma(0) = 0$, so the righthandside of (4.18) cannot hold and thus by contraposition $\beta(x) \leq \gamma(x)$.
This means that for a risk averse bidder in a First-Price auction it is optimal to place higher bids than a risk-neutral bidder. This means also expected payment increases, as this is directly affected by the bids. But that means that expected revenue increases. However, we already saw in the beginning of this proof that equilibrium bidding in a second-price auction does not change when bidders are risk-averse. Consequently, revenue equivalence no longer holds. □

4.4.4 Resale and efficiency

A relevant question in auction design when a government is the seller, is the question what to set as goal: *revenue maximization* or *efficiency*? Revenue maximization would risk higher costs for the participants, which could have a rebouncing effect on the economy. However, an often mentioned argument against setting efficiency as a goal is that resale would lead to efficient allocation after the auction. It is tempting to think that setting efficiency as a goal in auction design when resale is possible, is completely irrelevant. We give a counterexample to prove that this is not the case in general.
We consider two auction formats with *asymmetric* bidders. In this case, the second-price auction allocates efficiently, whereas the first-price auction with positive probability does not [18, p. 54]. Expected revenue is not necessarily the same in the two auction formats [18, p. 52,53]. We consider the first price auction and assume efficient allocation can be accomplished after resale - we will then argue by contradiction.
Assume $X_1 \sim F_1$ and $X_2 \sim F_2$ have different distribution functions and $\mathbb{E}(X_1) \neq \mathbb{E}(X_2)$. Assume that the bidding strategies β_1 and β_2 are invertible and denote $\phi_1 := \beta_1^{-1}$ and $\phi_2 := \beta_2^{-1}$. Assume $\beta_1(0) = 0 = \beta_2(0)$ and there exists a $\bar{b}$ such that $\beta_1(y) = \bar{b} = \beta_2(y)$ where y is the maximum value in the range of $\max\{X_1, X_2\}$. Suppose bidder 2 bids according to the equilibrium strategy and bidder 1 bids as if his true value were z_1. Then

$$\begin{aligned}
F_2(\phi_2(\beta_1(z_1))) &= \mathbb{P}(X_2 \leq \phi_2(\beta_1(z_1))) \\
&= \mathbb{P}(X_2 \leq \beta_2^{-1}(\beta_1(z_1))) \\
&= \mathbb{P}(\beta_2(X_2) \leq \beta_1(z_1)) = \mathbb{P}(\text{bidder 1 wins}).
\end{aligned}$$

Now if $\beta_1(X_1) > \beta_2(X_2)$ and $X_2 > z_1$, bidder 1 wins and resells the object to bidder 2 at a price $X_2 = \max\{z_1, X_2\}$. When $\phi_2(\beta_1(z_1)) < X_2 < z_1$ (this is just the reverse but in different notation which will help us) bidder 1 looses, but buys the object from bidder 2 at a price $z_1 = \max\{X_2, z_1\}$. This means that the total expected payment of bidder 1 will be

$$m_1(z_1)^I = F_2(\phi_2(\beta_1(z_1)))\beta_1(z_1) - \int_{z_1}^{\phi_2(\beta_1(z_1))} \max\{z_1, x_2\} f_2(x_2)dx_2.$$

Because we assume efficiency, we can apply the revenue equivalence theorem, Proposition 4.4.1, which tells us that for all x_1

$$m_1(x_1)^I = m_1(x_1)^{II}, \tag{4.19}$$

where $m_1(x_1)^{II} = \int_0^{x_1} x_2 f_2(x_2)dx_2$ is the expected payment for bidder 1 in a second price auction. Now we set $x_1 = y$, and see that $\phi_2(\beta_1(y)) = \phi_2(\bar{b}) = \beta_2^{-1}(\bar{b}) = y \Rightarrow \beta_1(y) = \bar{b}$. Plugging this into (4.19) will give us $\mathbb{E}(X_2) = \bar{b} = \mathbb{E}(X_1)$. But this is a contradiction, because we assumed $\mathbb{E}(X_1) \neq \mathbb{E}(X_2)$.

This contradiction shows us that invertibility of β can not be assumed in general. This assumption must be overthrown, ϕ does not always exist. In effect, resale must take place under incomplete information, which is in the literature reffered to as *pooling*: for at least one of the bidders i, there exists an interval $[x_i', x_i'']$ such that $\forall\ x_i \in [x_i', x_i'']:\ \ \beta_i(x_i) = b_i$. This will with positive probability lead to inefficiency.

4.5 Mechanism Design: a general approach

By using a more general approach, we can prove some very powerful results in Auction Theory. However, these results exist under ideal assumptions. The more complex the model and the less ideal the assumptions we make, the less powerful results remain as tools in studying auction formats.[6]

4.5.1 The Revelation Principle

To make auctions easier to study, we use a very powerful principle, which tells us we can in fact replicate every auction mechanism in which bidders do not report their true values, by a mechanism in which they do. We call the set of mechanisms for which the bids are exactly equal to the values $\mathcal{B}_i = \mathcal{X}_i$ for all i *direct mechanisms*. Formally, such a mechanism (Q, M) where $Q = (Q_1, \dots, Q_N)$ and $M = (M_1, \dots, M_N)$ consists of two functions $Q : \mathcal{X} \to \Delta$ and $M : \mathcal{X} \to \mathbb{R}^N$ where $Q_i(X)$ is the probability that i gets the object and $M_i(X)$ is the expected payment by i. We call $(Q(X), M(X))$ the *outcome* of the mechanism.

Theorem 4.5.1 (The Revelation Principle)**.** *Given any mechanism and an equilibrium for this mechanism, there exists a direct mechanism, in which every bidder reports his own value, such that the outcome of the direct mechanism equals the outcome of the original mechanism.*

Proof. We just define Q and M as compositions of β and π and μ, respectively:

$$Q(x) = \pi(\beta(x)) \text{ and } M(x) = \mu(\beta(x)) \tag{4.20}$$

In our original model, the outcome (δ, p) was determined by the pair of mappings μ and π, applied to $\beta(x)$. So the outcome of the original mechanism is equal to the outcome of the direct mechanism in this case. In both models, bidders bid according to their true values. □

[6]A description of most of the results in Table 4.5 is given in this section. The results which are not described, can be found in [18] and some of them in [19].

Table 4.2: Results in Auction Theory

Single Unit Independent values	**Single Unit Dependent values**
Revelation Principle Theorem 4.5.1	Revelation Principle
Revenue Equivalence Theorem Theorem 4.27	Revenue Ranking
First Price Auctions, Second Price Auctions, Vickrey Auctions all have efficient equilibria	Only under the condition that values are symmetric and the single crossing condition is satisfied, efficient equilibria exist.
Optimal mechanism design is possible but not efficient	Optimal mechanism: seller can extract all surplus from bidders
The Vickrey Clark Groves mechanism is optimal among all efficient mechanisms	Under the single crossing condition, truth-telling is an efficient ex post equilibrium of the generalized VCG mechanism
Multiple Unit Independent values	**Multiple Unit Interdependent values**
Revelation Principle	
Revenue Equivalence Theorem	
Only the VA has an efficient equilibrium. FPA and SPA are inefficient	
Under the single crossing condition, the VCG mechanism is efficient	Under informationally separable and additive signals and the single crossing condition, the VCG mechanism is efficient
Sequential sales: price path is a martingale	Sequential sales: price path is no martingale but shows a drift (up or down?)

We compute

$$q_i(z_i) = \int_{\mathcal{X}_{-i}} Q_i(z_i, x_{-i}) f_{-i}(x_{-i}) dx_i, \tag{4.21}$$

the probability that i gets the item if he bids z_i and

$$m_i(z_i) = \int_{\mathcal{X}_{-i}} M_i(z_i, x_{-i}) f_{-i}(x_{-i}) dx_i,$$

the expected payment for i when he bids z_i.

Definition 4.5.2. A direct mechanism (Q, M) is called *incentive compatible* (IC) if

$$q_i(x_i)x_i - m_i(x_i) \geq q_i(z_i)x_i - m_i(z_i) \ \forall z_i. \tag{4.22}$$

We call

$$U_i(x_i) := q_i(x_i)x_i - m_i(x_i), \tag{4.23}$$

the *equilibrium payoff function.*

Incentive compatibility means a bidder can never gain anything by not bidding according to his true value. Now incentive-compatibility implies that

$$U_i(x_i) = \max_{z_i \in \mathcal{X}_i} \{q_i(z_i)x_i - m_i(z_i)\}.$$

Note that all these expected payoff functions $q_i(z_i)x_i - m_i(z_i)$ are *affine* functions[7] of x_i. This implies U_i must be a convex function, because it is the maximum of a family of affine functions. Furthermore, we can use the equality

$$\begin{aligned} q_i(x_i)z_i - m_i(x_i) &= q_i(x_i)x_i - m_i(x_i) + q_i(x_i)(z_i - x_i) \\ &= U_i(x_i) + q_i(x_i)(z_i - x_i), \end{aligned} \tag{4.24}$$

[7]A function is called affine if it consists of a linear transformation followed by a translation.

and incentive compatibility (4.22) to derive that

$$U_i(z_i) \geq q_i(x_i)z_i - m_i(x_i). \tag{4.25}$$

Thus by combining (4.24) and (4.25) we get the following inequality:

$$\frac{U_i(z_i) - U_i(x_i)}{z_i - x_i} \geq q_i(x_i).$$

Following the same argumentation we can show that

$$U_i(x_i) + q_i(z_i)(z_i - x_i) \geq U_i(z_i).$$

So by taking limits $|z_i - x_i| \to 0$ we conclude that

$$U_i'(x_i) = q_i(x_i),$$

Because U_i is convex, it is absolutely continuous. The number of points at which U_i is not differentiable is at most countable. (Also q_i is nondecreasing.) From the fundamental theorem of calculus we can now conclude

$$U_i(x_i) = U_i(0) + \int_0^{x_i} q_i(t_i)dt_i. \tag{4.26}$$

Now we only need one line to prove the

Theorem 4.5.3 (Revenue Equivalence Theorem in single unit auctions with private values)**.** *When (Q, M) is an incentive compatible mechanism, then for all i and x_i*

$$m_i(x_i) = m_i(0) + q_i(x_i)x_i - \int_0^{x_i} q_i(t_i)dt_i. \tag{4.27}$$

So the expected payment in any auction format depends on the allocation alone (except for a constant).

Proof. From (4.23) we see that $U_i(0) = -m_i(0)$. We plug this into (4.26) and find (4.27). □

The above theorem and reasoning are based on [18, p. 62-66]. Note that this theorem only tells us something about revenue equivalence *when allocation is the same*. In many cases, when we'd like to compare different standard auction formats under realistic assumptions, this condition does not hold. It is important to note that when we observe that an auction format A allocates efficiently and B does not, the allocations must differ and consequently Theorem 4.5.3 does not hold. Furthermore, observe that this theorem is a generalization of Proposition 4.4.1.

4.6 Optimal Auction Design

In this section, we study an optimal mechanism for the seller, which means that we aim to maximize expected revenue.

Problem 4.6.1. *Maximize*

$$\mathbb{E}(R) = \sum_{i \in \mathcal{N}} \mathbb{E}[m_i(X_i)]. \tag{4.28}$$

We will construct an optimal mechanism. This construction is mainly based on the article by Myerson [20]. We assume individual rationality, which is defined as follows:

Definition 4.6.2. A bidder i is called *individually rational* (IR) iff

$$U_i(0) \geq 0.$$

This notion can be interpreted as the assumption that bidders never participate in the auction when they are expecting to loose money.
Consider a direct auction mechanism (Q, M). We study the sum in (4.28) component-wise. Let $y_i = \sup(range(X_i)) \leq \infty$. Then

$$\begin{aligned}
\mathbb{E}(m_i(X_i)) &= \int_0^{y_i} m_i(x_i) f_i(x_i) dx_i \\
&\overset{\text{rev.eq.}}{=} m_i(0) + \int_0^{y_i} q_i(x_i) f_i(x_i) dx_i - \int_0^{y_i} \int_0^{x_i} q_i(t_i) f_i(x_i) dt_i dx_i \\
&\overset{\text{Fubini}}{=} m_i(0) + \int_0^{y_i} q_i(x_i) f_i(x_i) dx_i - \int_0^{y_i} \int_{t_i}^{y_i} q_i(t_i) f_i(x_i) dx_i dt_i \\
&= m_i(0) + \int_0^{y_i} q_i(x_i) f_i(x_i) dx_i - \int_0^{y_i} (1 - F_i(t_i)) q_i(t_i) dt_i \\
&= m_i(0) + \int_0^{y_i} (x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}) q_i(x_i) f_i(x_i) dx_i \\
&\overset{(4.21)}{=} m_i(0) + \int_{\mathcal{X}} (x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}) Q_i(x) f(x) dx. \qquad (4.29)
\end{aligned}$$

So Problem 4.6.1 is finding a mechanism that maximizes

$$\sum_{i \in \mathcal{N}} m_i(0) + \int_{\mathcal{X}} (x_i - \frac{1 - F_i(x_i)}{f_i(x_i)}) Q_i(x) f(x) dx.$$

We assume that bidders are incentive compatible (Definition 4.5.2) and individually rational (Definition 4.6.2), which form the constraints to our optimization problem. We assume $(f > 0)$. Now we define the *virtual valuation function* ψ as

$$\psi_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)},$$

We say this problem is *regular* when ψ is increasing in x. Note that $\mathbb{E}(\psi_i(X_i)) = 0$. This is because $\mathbb{E}\frac{1-F(X)}{f(X)} = \int \frac{1-F(x)}{f(x)} f(x) dx = \int (1 - F(x)) dx = \mathbb{E}X$.

Definition 4.6.3. The *hazard rate function* λ_i is given by

$$\lambda_i := \frac{f_i}{1 - F_i}.$$

So, when $\lambda_i(.)$ is increasing, the problem is regular. From now on, we will assume this is the case.

Lemma 4.6.4. *Let the optimization problem (Problem 4.6.1) be regular. The optimal mechanism, maximizing expected revenue* (4.28) *to the seller, under the conditions that IR (Definition 4.6.2) and IC (Definition 4.5.2) hold, is given by*

$$Q_i(x) > 0 \Leftrightarrow \psi_i(x_i) = \max_{j \in \mathcal{N}} \psi_j(x_j) \geq 0, \qquad (4.30)$$

$$M_i(x) = Q_i(x) x_i - \int_0^{x_i} Q_i(z_i, x_i) dz_i. \qquad (4.31)$$

Proof. First we check whether this mechanism satisfies our conditions of individual rationality (IR) and incentive compatibility (IC).

1. IC. Note that $z_i \leq x_i \Leftrightarrow \psi_i(z_i) \leq \psi_i(x_i)$ by regularity of ψ_i. Thus the allocation rule Q in (4.30) satisfies $Q_i(z_i, x_{-i}) \leq Q_i(x_i, x_{-i})$.

2. IR. From the previous and (4.21), we see that q_i is non-decreasing. Furthermore, $M_i(0, x_{-i}) = 0 \ \forall x_{-i} \Rightarrow m_i(0) = 0$. So combining this with (4.26), we conclude that IR holds.

This mechanism separately maximizes the terms over all $Q(x) \in \Delta$ by (4.29), so it is indeed optimal [18, p. 67-71]. □

The maximized expected revenue is given by

$$\mathbb{E}[\max_i\{\psi_1(x_1), \psi_2(X_2), \dots \psi_i(X_i), \dots \psi_N(X_N)\}].$$

Theorem 4.6.5. *When the optimization problem (Problem 4.6.1) is regular, an optimal mechanism is given by*

$$Q_i(x) = \mathbf{1}_{\{z_i > y_i(x_{-i})\}}$$

and

$$M_i(x) = y_i(x_{-i})\mathbf{1}_{\{Q_i(x)=1\}},$$

where

$$y_i(x_{-i}) = \inf\{z_i : \psi_i(z_i) \geq 0 \textit{ and } \forall j \neq i, \ \ \psi_i(z_i) \geq \psi_j(x_j)\}.$$

Proof. This proof is based on [18, p. 67-71]. We can just take the mechanism we constructed in Lemma 4.6.4 as a start and observe that

$$y_i(x_{-i}) = \inf\{z_i : \psi_i(z_i) \geq 0 \text{ and } \forall j \neq i, \ \ \psi_i(z_i) \geq \psi_j(x_j)\}$$

is the smallest value for bidder i such that he can win the auction. Returning to (4.30) we see that the allocation rule becomes

$$Q_i(z_i, x_i) = \mathbf{1}_{\{z_i > y_i(x_{-i})\}},$$

and

$$\int_0^{x_i} Q_i(z_i, x_{-i}) dz_i = (x_i - y_i(x_{-i}))\mathbf{1}_{\{x_i > Y_i(x_{-i})\}}.$$

This gives us as a payment rule

$$M_i(x) = y_i(x_{-i})\mathbf{1}_{\{Q_i(x)=1\}}.$$

□

A special case arises when the bidders are symmetric. In this case, $f_i = f \ \forall i$, so $\psi_i = \psi$ for all i, so

$$y_i(x_{-i}) = \max\{\psi^{-1}(0), \max_{j \neq i} x_j\},$$

but this is the same as in a second-price auction with reserve price $\psi^{-1}(0)$.[8] Note that this also implies that in the symmetric case, setting a reserve price for a second-price auction is optimal in terms of maximizing expected revenue.

[8]When a seller promises to sell the object to the highest buyer but the seller himself attains positive value $x_0 > 0$ to the object, the seller takes a risk of negative payoff (and thus a loss), when the highest bid $\beta^{(1)}(x) < x_0$. To prevent this, the seller can set a *reserve price*, which is the minimum price at which the item will be sold. Reserve prices can be secret (as is often the case in art auctions) or public.

4.6.1 Efficient mechanisms

The optimal mechanism as constructed in the previous section, is not always efficient. If the seller's valuation of the item is 0 and all buyer's valuations are nonnegative, there is a possibility (namely when the virtual valuations are lower than the reserve price), that the item is not allocated to one of the buyers and consequently the seller keeps it, while his valuation is minimum among all participants.
Of course, we are interested in efficient mechanism design as well. In this section we will introduce the Vickrey-Clark-Groves mechanism, which is not only efficient, but also optimal *among all efficient mechanisms*. We will specify what we mean by this.
Suppose the value of a participant in the auction $X_i \in \mathcal{X}_i$ satisfies $X_i(\omega) \in [\alpha_i, y_i] \in \mathbb{R}$ where $\alpha_i < 0$, so we allow for the possibility of negative values (costs). Now an allocation rule $Q^* : \mathcal{X} \to \Delta$ is *efficient* if it maximizes *social welfare*:

$$\forall x \in \mathcal{X} : Q^*(x) \in \arg\max_{Q \in \Delta} \sum_{j \in \mathcal{N}} Q_j(x) x_j, \tag{4.32}$$

and (trivially) when the allocation rule is efficient, the mechanism is efficient. We define the *maximized social welfare* to be

$$W(x) = \sum_{j \in \mathcal{N}} Q_j^*(x) x_j. \tag{4.33}$$

Now introduce the payment rule $M^V : \mathcal{X} \to \mathbb{R}^N$ given by

$$M_i^V(x) = W(\alpha_i, x_{-i}) - W_{-i}(x), \tag{4.34}$$

where the first term can be interpreted as the social welfare at its lowest possible value and the second term as the welfare of all other participants. A mechanism which is optimal among all efficient mechanisms satisfies (4.32) and maximizes

$$\sum_{i \in \mathcal{N}} \mathbb{E}[m_i(X_i)] = \mathbb{E}(R).$$

Definition 4.6.6. VCG-mechanism. The mechanism $(Q_j^*(x), M^V)$ where $Q_j^*(x)$, M^V are as in (4.32) and (4.34) is called the Vickrey-Clark-Groves mechanism.

Remark 4.6.7. In the case of auctions as a selling mechanism, the VCG mechanism is the same as a second-price auction.To see this, suppose, as is the case in auctions, that $\alpha_i = 0$. Then

$$M_i^V(x) = W_{-i}(0, \boldsymbol{x_i}) - W_{-i}(x).$$

This is positive iff $x_i > \max_{i \neq j} x_j$ (because in that case $\sum_{j \neq i} Q_j^*(x) x_j = 0$ and $\sum_{j \neq i} Q_j^*(0, x_{-i}) x_j \geq 0$). And if this condition is satisfied, $M_i^V = \sum_{j \neq i} Q_j^*(0, x_{-i}) x_j \geq 0 = \max_{j \neq i} x_j$. So the size of the payment equals the second highest value, which means in the context of auctions the VCG mechanism is the same as a second-price auction.

The following proposition is based on [18, p. 76,77].

Proposition 4.6.8. *The VCG-mechanism is IR, IC and optimal among all efficient mechanisms.*

Proof. First we show Incentive Compatibility, Definition 4.5.2. Consider a bidder i who bids as if his true value were z_i, while is actual value is x_i. We use (4.33) to see that his payoff will be

$$Q_i^*(z_i, x_{-i}) x_i - M_i(z_i, x_{-i}) = \sum_{j \in \mathcal{N}} Q_j^*(z_i, x_{-i}) x_j - W(\alpha_i, x_{-i}). \tag{4.35}$$

By definition of Q^*, the first term on both sides is maximized when $z_i = x_i$, so it can never be optimal to report $z_i \neq x_i$. So we have incentive compatibility.

Next, we show individual rationality, Definition 4.6.2. From Section 4.5.1, we know that the equilibrium payoff function U_i is convex, because the mechanism is IC. Also, from (4.35) we see that for this mechanism ex ante payoff for bidder i is

$$U_i(x_i) = Q_i^*(x)x_i - M_i^V(x) = W(x) - W(\alpha_i, x_{-i}),$$

so ex ante we have

$$U_i(x_i) = \mathbb{E}(W(x_i, X_{-i}) - W(\alpha_i, X_{-i})).$$

We immediately see that $U_i(\alpha_i) = 0$ and for all $x_i \geq \alpha_i$ it is increasing in x_i, so we have IR.
It remains to show that the VCG mechanism is optimal among all efficient mechanisms. Suppose $(\tilde{Q}, \tilde{M})$ is some other efficient mechanism that maximizes social welfare. By revenue equivalence Theorem 4.27 the sum of the expected payments of this mechanism must be equal to the sum of the expected payments of the VCG mechanism. But that means that individual payoffs can differ by at most a constant from the payoffs in the VCG mechanism. Denote by $\tilde{U}$ the payoff of the new mechanism. Then $c_i = \tilde{U}_i(x_i) - U_i(x_i)$. Because the payoff functions are convex and nondecreasing and $U_i(\alpha_i) = 0$, if we require $(\tilde{Q}, \tilde{M})$ to be IR as well, this constant must be positive. This means the expected payoffs in the new mechanism are higher than for the VCG mechanism. However, the allocation is the same for both formats and because values are also equal, the payments made in $(\tilde{Q}, \tilde{M})$ can only be lower than in the VCG mechanism, which means the VCG mechanism is optimal among all efficient mechanisms. □

4.7 Sealed-bid auctions for selling multiple identical units

We will study three basic models for selling multiple identical units in a standard auction format but with different pricing rules. We use the framework and results we derived in this chapter.
Just like before, the selling mechanism $(\mathcal{B}, \pi, \mu)$ consists of $\mathcal{B} \subset \mathbb{R}^N$, a set of possible bids for each buyer, and two rules for allocation and payment, π and μ.

4.7.1 Bidders and values

In each of these models, a bidder i can submit K bids $b_j^i \in \mathbb{R}$ for obtaining additional objects, satisfying $b_1^i \geq b_2^i \geq \ldots \geq b_K^i$. Hence, we assume that the bidder is willing to pay most for the first object. To obtain a second object, he is willing to pay no more than for the first object. This assumption is widely used in economics and is usually referred to as *marginally declining values* A value is called marginally declining when the value of an additional unit decreases with the number of units already obtained [18, p. 166]. We denote the *bid-vector* by $b^i = (b_1^i, \ldots, b_K^i) \in \mathbb{R}^k$. Note that $\sum_{j=1}^l b_j^i$ equals the amount bidder i is willing to pay for l units. From the bid vector of bidder i we can derive the demand function $d^i : \Pi \to \mathbb{N}$ (where Π is the set of all possible prices of one item) of bidder i, by taking

$$d^i(p) := \max\{j : p \leq b_j^i\},$$

where p is the price of one unit. Note that a bidder's demand is a non-increasing function of the price.

4.7.2 Allocation rule

The total number of bids in a multiple unit auction equals $N \times K$. In the models we study, all the bids are collected and arranged in decreasing order. We can denote this vector of collected and ordered bids by $\tilde{b} = (b^{(1)}, b^{(2)}, b^{(3)}, \ldots, b^{(N \times K)})$. The first object is awarded to the bidder who issued the highest bid, the second object to the bidder with the second highest bid and so on. Consequently, the allocation rule μ can be formulated as follows:

$$\mu_i(b) = \sum_{j=1}^{K} \mathbf{1}_{\{b_j^i \geq b^{(K)}\}}. \tag{4.36}$$

Auctions in which the objects are awarded to the highest bidders are called *standard auctions* [18, p. 168]. In fact, this rule is just an extension of the allocation rule (4.1) for a standard auction for single units.

4.7.3 Pricing rule

Now there are three main formats which we will study, which are all standard auctions, but have different pricing rules: the *discriminatory* auction, the *uniform-price* auction and the *Vickrey* auction.

The discriminatory auction The discriminatory format is sometimes referred to as the "pay as you bid" format and this name gives an explicit hint on how the format works. In this format, the amount p^i each winning bidder pays, equals the sum of his winning bids:

$$p_i = \pi_i(\mathbf{b}) = \sum_{j=1}^{K} b_j^i \mathbf{1}_{\{b_j^i \geq b^{(K)}\}}. \tag{4.37}$$

Example 4.7.1. Suppose for example there are five identical items to be sold and there are two bidders. Bidder 1 bid as follows:

$$b^1 = (15, 12, 7, 4, 0),$$

and bidder two bid:

$$b^2 = (13, 8, 2, 1, 1).$$

The bids are collected and ordered in the following vector:

$$\tilde{b} = (15, 13, 12, 8, 7, 4, 2, 1, 1, 0).$$

Note that $b^{(5)} = 7$ Now the allocation rule μ (4.36) tells us:

$$\mu^1((15, 12, 7, 4, 0, 13, 8, 2, 1, 1)) = \sum_{j=1}^{K} \mathbf{1}_{b_j^1 \geq b^{(K)}} = \sum_{j=1}^{5} \mathbf{1}_{b_j^1 \geq b^{(5)}} = 3,$$

and

$$\mu^2((15, 12, 7, 4, 0, 13, 8, 2, 1, 1)) = \sum_{j=1}^{K} \mathbf{1}_{b_j^2 \geq b^{(K)}} = \sum_{j=1}^{5} \mathbf{1}_{b_j^1 \geq b^{(5)}} = 2.$$

The pricing rule π (4.37) tells us:

$$\pi^1(\mathbf{b}) = \sum_{j=1}^{5} b_j^1 \mathbf{1}_{b_j^1 \geq b^{(5)}} = 15 + 12 + 7 = 34,$$

$$\pi^2(\mathbf{b}) = \sum_{j=1}^{5} b_j^2 \mathbf{1}_{b_j^2 \geq b^{(5)}} = 13 + 8 = 21.$$

So in this example, bidder 1 wins 3 items for a total price of 34 and bidder 2 wins 2 items for a total price of 21.

Uniform price auction In a uniform price auction, as the name suggests, everyone who wins units pays the same price per unit. The price paid per unit is called the *clearing price* and the clearing price can be set equal to any value between the highest losing bid and the lowest winning bid. Let's assume that the clearing price equals the value of the highest losing bid.
We apply the same allocation rule as for the discriminatory auction (4.36). Note that because

$b^i_j \geq b^i_{j+1}$ $\forall j$, the highest losing bid of bidder i equals $b^i_{\mu^i(b)+1}$. Consequently the overall highest losing bid, (the clearing price p_c,) equals

$$p_c = \max_i b^i_{\mu^i(b)+1}. \tag{4.38}$$

So each bidder pays

$$\pi^i(b) = \sum_{j=1}^{K} p_c \mathbf{1}_{\{b^i_j \geq b^{(K)}\}}.$$

Example 4.7.2. Assume we have the same bids and the same items sold as in example Example 4.7.1. In a uniform auction, we use the same allocation rule, so bidder 1 wins 3 items and bidder 2 wins 2 items. The difference lies in the price they pay. The clearing price

$$p_c = \max_i b^i_{\pi^i(b)+1} = \max\{b^1_{(3+1)}, b^2_{(2+1)}\} = \max\{4, 2\} = 4,$$

and the pricing rule tells us the amounts the bidders have to pay:

$$\pi^1(b) = \sum_{j=1}^{5} 4 \cdot \mathbf{1}_{\{b^1_j \geq b^{(5)}\}} = 4 + 4 + 4 = 12,$$

$$\pi^2(b) = \sum_{j=1}^{5} 4 \cdot \mathbf{1}_{\{b^2_j \geq b^{(5)}\}} = 4 + 4 = 8.$$

Note that in this example, the revenue for the seller is $12 + 8 = 20$, whereas in Example 4.7.1, the revenue for the seller was $34 + 21 = 55$. However, now we cannot take the shortcut and conclude that expected revenue in a uniform price auction is lower than in a discriminatory price auction, because in general bidders will not adopt the same strategies in both auctions.

Vickrey Auctions In a Vickrey auction, we use the idea of the Vickrey-Clark-Groves mechanism, as explained in Section 4.6.1. We denote by c^{-i} the vector of competing and (decreasingly) ordered bids, so $c^{-i}_1 = b^{(1)}_{-i}$ is the highest competing bid facing bidder i. For bidder i to win one item, he must defeat the lowest competing bid c^{-i}_K. If he wins two items, he has to defeat the second-lowest competing bid for the second unit, and so on. In a Vickrey auction, a bidder who wins $\mu^i(b)$ units, pays

$$\pi^i(b) = \sum_{j=1}^{\mu^i(b)} c^{-i}_{K-\mu^i(b)+j}. \tag{4.39}$$

When a single unit is sold, the Vickrey auction reduces to the second-price auction, just like the uniform price auction. However, when more units are sold, the Vickrey auction differs from the uniform price auction. Note that in the Vickrey auction, the payment of a bidder depends on *other* bids, not on his own. This is a very important difference with respect to the uniform price auction, where it might be the case that a bid on an item which is not the first one, determines the clearing price, which has a direct effect on the payment of a winning bidder. Consequently, as we will see later in this section, there exist equilibria in which it is optimal for each bidder to shade their bids on every item except the first one, to make the clearing price decrease. This phenomenon is called *demand reduction.*

4.7.4 Efficient equilibria in multiple unit auctions with private values

We consider the three standard auction formats. Denote by y the maximum value the type (signal) X can take, so we assume $X(\omega) \leq y$. Take a set of types

$$\mathcal{X} = \{x \in [0, y]^K : \forall k \;\; x_k \geq x_{k+1}\},$$

and equilibrium bidding strategy for bidder i $\beta^i : \mathcal{X} \to \mathbb{R}^k_+$ such that for all items k it holds that $\beta^i_k(x^i) \geq \beta^i_{k+1}(x^i)$. Consider an auction equilibrium with bidding strategies $(\beta^1, \beta^2, \ldots, \beta^V)$ and types $(x^1, x^2, \ldots, x^N)$. Recall that in the private values case (Definition 4.4) $v_i(\delta, X|X_i = x_i) = x_i\delta_i$.

Suppose K units are awarded to the K highest *bids* b. Efficiency now requires that these K units are also allocated to the K highest *values* x^i_k. This means that if we want an equilibrium to be efficient, the ranking of the bids must correspond to the ranking of the values, so

$$x^i_k > x^j_l \Leftrightarrow \beta^i_k(x^i) > \beta^j_l(x^j). \tag{4.40}$$

To see when this is the case, we define the concepts of *separable* and *symmetric* bidding strategies.

Definition 4.7.3. A bidding strategy is called *separable* when the bid for the k-th object only depends on the k-th marginal value, i.e.

$$\beta_k(x) = \beta_k(x_k).$$

Definition 4.7.4. A bidding strategy is called *symmetric* when it is the same across items and bidders, i.e.

$$\beta^i_k(.) = \beta^j_l, \ \ \forall i, j, k, l.$$

The following proposition gives us the conditions under which a multiple unit auction allocates efficiently.

Proposition 4.7.5. *An equilibrium of a standard multiple unit auction is efficient iff bidding strategies are separable and symmetric across bidders and objects. In this case, there exists an increasing function β, such that for all i, k:*

$$\beta^i_k(x) = \beta(x^i_k).$$

Proof. First we prove the conditions are necessary. Suppose the bidding strategies are not separable. Suppose $x^i_k > x^j_{k'}$ and $\beta^i_k(x^i) > \beta^j_{k'}(x^j)$ where k' is the $k+1$ highest bid, just below the kth. Now if the bid strategy of bidder i depends on value x^i_l, $k \neq l$ say, then there exists a change from x^i_l to $x^i_l - \epsilon$, $\epsilon > 0$, such that

$$\beta^i_k(x^i_{(l-\epsilon)}, x^i_{-l}) < \beta^j_{k'}(x^j) \text{ but still } x^i_k > x^j_{k'}, \tag{4.41}$$

which means the allocation would be inefficient. So bidding strategies must be separable.

Now suppose the bidding strategies are not symmetric. Then it would be possible that $x^i_k > x^j_l$ but $\beta^i_k(x^i_k) < \beta^j_l(x^j_l)$. So the strategies must be symmetric.

Now we prove sufficieny. Suppose the bidding strategies are symmetric and separable. Then the values are mapped into the bids using a single bidding function β. But this immediately implies efficiency by (4.40). □

Corollary 4.7.6. *In standard multiple unit auctions, the Vickrey auction allocates efficiently.*

Proof. Following [18, p. 181-182], we will show that in Vickrey auctions, the optimal bidding strategy is just to report values, so β is just the identity mapping and thus satisfies the condition in Proposition 4.7.5.

Consider a bidder who bids $b^i = x^i$. He is awarded $k^i := \mu^i(b)$ units. Equation (4.39) tells us what he has to pay. For him to win the $k \leq k^i$th item, he must have $x^i_k \geq c^i_{K-k^i+k} = p^i_k$.

Now suppose a bidder who bids $b^i \neq x^i$, while he is awarded the same number of units k^i. In this case his payment (4.39) will not change, so he cannot improve his payoff.

Suppose he bids $b^i \neq x^i$, while he is awarded $l^i < k^i$ items. Now he receives fewer items. So he misses the extra positive payoff by winning those items.

On the other hand, suppose he bids $b^i \neq x^i$, while he is awarded $m^i > k^i$ items. Now for the first k^i units his payoff remains unchanged. However, for the additional units, he pays $p^i_m = c^{-i}_{K-k^i+m} \geq x^i_m$.

So he can never increase his payoff by choosing another strategy than bidding his true value x. □

4.7.5 Revenue Equivalence in Multi-unit Auctions

As we will show in Proposition 5.2.5, in the multi-unit case, the uniform price auction and the Vickrey auction do not allocate the same way. Because in a uniform price auction demand reduction is an equilibrium strategy, allocation is not necessarily efficient, while the Vickrey auction does allocate efficiently (Corollary 4.7.6). Also the discriminatory auction is in general inefficient [18, p. 192-195]. So if we have a revenue equivalence theorem, in the multi-unit case it does not mean that the standard auctions have the same expected revenues, because the condition under which the theorem holds (equal allocation between auction formats), does not hold. The theorem will not apply. For completeness we do state it here.

Theorem 4.7.7. *In any two multi-unit auctions which allocate the same way, the equilibrium payoff functions and individual payments of any bidder (and consequently the revenue to the seller) differ by at most an additive constant.*

Proof. We just follow the same reasoning as in the proof of the revenue equivalence in single unit auctions. Suppose all bidders behave according to an equilibrium strategy, just like in Section 4.7.4. Let $q^i_1(z^i)$ denote the probability that bidder i will win the first unit when he bids $\beta^i(z^i)$. We denote by $m^i(z^i)$ expected payoff in equilibrium. If his true value vector is x^i, a bidder's expected payoff can be written as

$$q^i(z^i)x^i - m^i(z^i).$$

Because in equilibrium it is optimal to bid $\beta(x)$, it holds that for all z^i

$$q^i(x^i)x^i - m^i(x^i) \geq q^i(z^i)x^i - m^i(z^i). \tag{4.42}$$

Now we define the maximized payoff function U^i,

$$U^i := \max_{z^i}\{q^i(z^i)x^i - m^i(z^i)\},$$

and like in Section 4.13 we deduce that also here U^i is convex. By (4.42) we know that

$$U^i(z^i) \geq U^i(x^i) + q^i(x^i) \cdot (z^i - x^i).$$

But now we have a little problem, because if we'd like to follow the same reasoning as in Section 4.13 to find an explicit form for U_i, we are confronted with a multi-dimensional instead of a one-dimensional problem. In [18], a trick is proposed to circumvent this problem. We define $V^i(t) = U^i(tx^i)$ such that $V^i(0) = U^i(0)$ and $V^i(1) = U^i(x^i)$. Since $U^i : \mathcal{X} \to \mathbb{R}$ is convex and absolutely continuous, also $V^i : [0,1] \to \mathbb{R}$ is convex and absolutely continuous. So V^i is differentiable almost everywhere on the interior of its domain. This means we can write

$$V^i(1) = V^i(0) + \int_0^1 v^i(t)dt, \tag{4.43}$$

for some L^1 function v^i. Now suppose V^i is differentiable at $t \in [0,1]$. Then for all $\Delta > 0$

$$\begin{aligned} V^i(t+\Delta) - V^i(t) &= U^i((t+\Delta)x^i) - U^i(tx^i) \\ &\geq q^i(tx^i)\Delta(x^i). \end{aligned} \tag{4.44}$$

Hence

$$\lim_{\Delta \downarrow 0} \frac{V^i(t+\Delta) - V^i(t)}{\Delta} \geq q^i(tx^i)x^i,$$

and for $\Delta < 0$

$$\lim_{\Delta \uparrow 0} \frac{V^i(t+\Delta) - V^i(t)}{\Delta} \leq q^i(tx^i)x^i.$$

But this means that

$$\frac{dV^i(t)}{dt} = q^i(tx^i)x^i,$$

for all $t \in [0,1]$ such that $V^i(t)$ is differentiable. Now if we combine this with (4.43), we get

$$U^i(x^i) = U^i(0) + \int_0^1 q^i(tx^i)x^i dt.$$

So U^i is determined by q^i only, except for a constant, which means expected revenue depends on the allocation rule alone, just like in single unit auctions. $\square$

Chapter 5

Auctions under the European Emissions Trading System

In the European Union Emissions Trading System, there is some agreement on the basics of the auction format which should be used. For auctioning CO_2 allowances, we derive the following input for a model.

- The potential bidders can either be primary participants[1] or any qualified bidder.[2]
- The goods will be multiple EUA's, which are interchangeable (it doesn't matter whether the bidder obtains allowance A or allowance B). The number of goods to be auctioned at once is one of the questions which has to be answered. Several approaches are possible, varying from auctioning all allowances at once to continuous auctions in which one EUA is sold at a time. Only the total number of allowances over the whole period is assumed to be determined and fixed.
- We assume that in every auction there is a certain predetermined maximum to the amount of allowances which will be sold.
- The auction format will be a uniform price sealed bid auction.
- Bids are submitted as price-quantity pairs.
- There might be a maximum bid size.
- There will be a minimum lot size, which means all bids have to be a multiple of a certain value (for example 500 EUA's).
- There will be a minimum tick size (which means prices must be increments of a certain value (for example 0,01 euro).)
- Tied bids have to be resolved some way. This can for example be done at random or proportional or giving advantage to a certain group of bidders[3].
- There might be a reservation price, which means there is a minimum price at which the items are sold. Any bids below the reservation price will be rejected.

[1] A primary participant is a selected institution which can perform bids on behalf of any potential buyer. Primary participants can be banks or financial institutions which have gained experience in performing financial transactions and auctions when large risks are involved.

[2] The European Commission can decide when a potential bidder is 'qualified'. This would mainly involve legal and financial demands to prevent from the risk of default or manipulation in the auction.

[3] For example, *compliance buyers*, companies under the ETS, can be given preference over other bidders, like banks.

To study these auctions, first, we construct a simple model for this type of auctions and gradually add complexity. We are interested in determining equilibrium strategies and outcome of the auction.

5.1 Selling mechanism and environment

Denote by K the quantity for sale.

Assumptions 5.1.1. We make the following assumptions for the simple set-up.

- We assume there are N risk-neutral bidders (Definition 4.2.12).
- $0 < K < \infty$.
- The maximum bid size is equal to $\bar{K}$.
- The reservation price is zero.
- All bidders are symmetric (Assumptions 4.3.1 - 5). The only signals they receive are the historical prices of EUA's on the secondary market.
- We assume there is no trading on the secondary market during the auction.
- Moreover, we assume the items sold have a pure common value to all bidders (Definition 4.2.7). This common value is equal to the secondary market price at the end of the auction. In our market impact model, we assumed the price of the allowances $S_i(t)$ to behave like a martingale on $(\Omega, \mathcal{F}, \mathbb{P})$, where $\mathbb{P}$ is the risk-neutral measure. Denote by t the time the auction starts (when bidders submit their values) and by $t+1$ the time the auction ends (when the outcome is announced). We then derive for the pure common value:

  $$v_i(\delta, x_1, \dots, x_N) = \delta \mathbb{E}(S_i(t+1)|\mathcal{F}_t) = \delta S_i(t) := \nu.$$

  So for every participant in the auction, the value of one item equals ν.

5.1.1 Demand functions

The possible prices of the allowances which are paid in the auction are denoted by $p \in [0, \infty)$. For simplicity and without doing any harm to our results, we assume the price cannot be negative and cannot take the value infinity either. Instead of placing discrete-valued bids, we assume bidders can submit continuous, non-increasing and left-differentiable *demand functions* $d^i : \mathcal{P} \to [0, \bar{K}]$ where $d^i(p) = q$ denotes the quantity q demanded at a price p. We call $\mathcal{D}_i$ the set of possible demand functions for bidder i and denote by $\prod_{i=1}^{N} \mathcal{D}_i = \mathcal{D}$. Assume that $\mathcal{D}_i \neq \varnothing$ for all i. Note that demand functions may be discontinuous, for we only require left-differentiability. We define the *aggregate demand function* $d(p) : [0, \infty) \to [0, N\bar{K}]$ to be

$$d(p) := \sum_{i=1}^{N} d^i(p).$$

We set the clearing price p_c to be

$$p_c = \sup\{p | d(p) \geq \tilde{K}\},$$

which is the lowest winning bid.

5.1.2 Allocation rule and tied bids

The *outcome* of a multi-unit auction is a sequence $\{(\delta_j, p_j)\}_{j=1}^{K}$ where (δ^i_j, p^i_j) denotes the allocation and payment for the jth item for the ith bidder. The collection of all possible outcomes is denoted as Ω. In this environment, we extend our definition of allocation and payment rule to the multi-unit case. The function $\omega : \mathcal{D} \to \Omega$, mapping strategy profiles into outcomes, can be decomposed as the pair (μ, π), where $\mu : \mathcal{D} \to \Delta$, such that $\mu^i(d^i) = \sum_{j=1}^{K}\{\delta^i_j\} := q^i$ denotes the amount bidder i receives when he submits a demand schedule d^i. We also define the payment function $\pi : \mathcal{D} \to \mathbb{R}^{N\bar{K}}$ such that $p^i := \pi^i(d^i)$ denotes the payment for bidder i when submitting a demand schedule d^i.
Now we specify the exact allocation and payment rules in our selling mechanism for selling EUA's. If we define the residual supply at the clearing price as $R(p_c) := \tilde{K} - d(p_c)$, we have by definition of p_c that $R(p_c) \leq 0$. There can be two situations when all bids are submitted. In the first situation, aggregate demand is exactly equal to the supply (the auction size) at the clearing price. This means $R(p_c) = 0$. The other situation is when aggregate demand *exceeds* the auction size at the clearing price. In this case $R(p_c) < 0$ and we speak of *tied bids* at the clearing price.[4] We denote by $d^i(p+) = \lim_{p' \downarrow p} d^i(p') = \sup\{d^i(p+\epsilon) : \epsilon > 0\}$ the quantity demanded by bidder i at his next bid just above p, where we use that d^i is non-increasing. By $R(p+) := \lim_{p' \downarrow p} R(p')$ we denote the residual supply at a price just above p. We define the payment and allocation as follows, assuming that demand at the clearing price is divided proportionally over the bidders.

$$q^i = \mu^i(d^i) = d^i(p_c)\mathbf{1}_{\{R(p_c)=0\}} + (d^i(p_c+) + R(p_c+)\frac{d^i(p_c) - d^i(p_c+)}{d(p_c) - d(p_c+)})\mathbf{1}_{\{R(p_c)<0\}}, \tag{5.1}$$

where the last term refers to the proportional division at the clearing price. As a payment rule we have

$$p^i = \pi^i(d^i) = \mu^i(d^i)p_c. \tag{5.2}$$

Recall that the payoff function $u_i(\omega, S)$ for risk-neutral bidders is given by $u^i(\omega, S) = v^i(\delta, S) - p^i$ for single unit auctions (Definition 4.2.12). In the EUA multi-unit case, when a bidder wins, the value of one item is ν, and the payment for a unit is p_c. When the bidder does not win, value and payment are 0. So we can simply extend the one-item case to the multiple unit-case, by multiplying with the number of items won. We have a value $v^i_j(\delta, S) = \nu\mathbf{1}_{\{\delta^i_j=1\}}$ for the ith bidder and jth unit. The price for a single jth unit is $p^i_j = p_c\mathbf{1}_{\{\delta^i_j=1\}}$ for a bidder i. To calculate an individual bidder's total payoff, we just add the single payoffs per item. So for the payoff function of an individual bidder we have

$$u^i(\omega, S) = \sum_{j=1}^{K}(\nu - p_c)\mathbf{1}_{\{\delta^i_j=1\}} = (\nu - p_c)\mu^i(d^i). \tag{5.3}$$

5.2 Equilibrium strategies

Assumptions 5.2.1. We make some additional assumptions:

- There is no maximum bid size: $\bar{K} = \infty$.
- Bidders $2, \ldots, N$ submit demand schedules $d(p)$ which are all the same, non-increasing and left-differentiable.
- $(N-1)d(\nu) < K$.

Observe that $d(0) \geq \frac{K}{N}$. The reverse would be inconsistent with equilibrium, because then the clearing price would be zero, which means every bidder would be able to increase his payoff by changing his bid size to $\bar{K}$, thereby receiving more free shares. We will now derive a formula to describe a class of equilibrium strategies for the auctions described in our model.

[4] Tied bids are equal bids at the clearing price.

Definition 5.2.2. When an equilibrium in a uniform price auction results in a clearing price $p_c < \nu$ lower than the secondary market price at the end of the auction, we speak of a *low-price equilibrium.*

Note that when $\bar{K} > \frac{K_{max}}{N-1}$ there exists an equilibrium without underpricing, namely $d(\nu) = \bar{K}$. Denote by d^1 bidder 1's demand schedule such that $d^1(p)$ denotes his demand at price p. Using (5.3) we can formulate the payoff maximizing optimization problem for bidder 1 as follows.

Problem 5.2.3. *Optimize*

$$\max_{d^1}(\nu - p_c)\mu^1(d^1). \tag{5.4}$$

Lemma 5.2.4. *Suppose Assumptions 5.1.1 and Assumptions 5.2.1 hold. Suppose bidders* $2, \dots, N$ *submit demand schedules* $d(p)$ *which are non-increasing and differentiable, with* $(N-1)d(\nu) < K$ *and* $d(0) \leq K/N$. *Then it is optimal for bidder 1 to submit a demand schedule* $d^1(p)$ *such that aggregate demand equals supply in equilibrium.*

Proof. First, assume that bidder 1 bids a demand schedule such that $p_c > \nu$. Then, because $K-(N-1)d(\nu) > 0$, he will raise the clearing price above ν, and he will receive a positive quantity, which gives him negative payoff. So bidder 1 will submit a demand schedule such that there is no excess demand at p_c. He will bid such that $(N-1)d(\nu) + d^1(p_c) \leq (N-1)d(\nu) + d^1(\nu) \leq K$.
Now assume bidder 1 bids such that his demand is lower than residual supply at the clearing price: $(N-1)d(\nu) + d^1(p_c) < K$. We will show this is not optimal. For then it will always be optimal for him to raise his demand such that he receives extra units of residual supply $R(p_c)$ and thereby increase his payoff, since we already see that he assures that $p_c \leq \nu$. So he will bid such that $K = (N-1)d(\nu) + d^1(p_c)$. □

The above lemma tells us that in equilibrium, there will be no excess demand and no excess supply. Consequently, as a constraint for our optimization problem (5.4) we have

$$K = d^1(p_c) + (N-1)d(p_c). \tag{5.5}$$

Given the other bidders' demand functions, we can think of p_c as a function of d^1 and rewrite the optimization problem (5.4) as follows:

$$\begin{aligned} &\max_{d^1}(\nu - p_c(d^1))d^1(p_c), \\ \text{such that} \quad &d^1(p_c) + (N-1)d(p_c) = K, \end{aligned}$$

where we additionally assume that d^1 is invertible. But now we see that optimization over d^1 is the same as optimization over p_c, because by d^1 bidder 1 uniquely determines the clearing price. So choosing a demand schedule means choosing a clearing price.

$$\max_{d^1}(\nu - p_c(d^1))(K - (N-1)d(p_c)) = \max_{p_c}(\nu - p_c)(K - (N-1)d(p_c)).$$

When we take a closer look at the meaning of this formula, we see that the payoff of bidder 1 is actually determined by the payoff per unit at a certain clearing price, times the residual supply at that clearing price. As a bidder lowers the clearing price p_c, his profit per unit increases, but the residual supply at that price decreases, so the share he will receive decreases as well. Consequently, bidder 1 has a certain degree of market power, because he can affect the clearing price by choosing this to be lower than ν, the actual value of the item sold [17, p. 855].
To find an optimal solution, we take the derivative of d with respect to p, denoted by d' and set this equal to zero:

$$(N-1)(\nu - p_c)d'(p_c) + K - (N-1)d(p_c) = 0. \tag{5.6}$$

In a symmetric equilibrium, $d^1 = d$, so the constraint (5.5) becomes

$$d(p_c) + (N-1)d(p_c) = Nd(p_c) = K.$$

Now we are ready to prove the following proposition [17, p. 856]:

Proposition 5.2.5. *Suppose Assumptions 5.1.1 and Assumptions 5.2.1 hold. Then there is a class of symmetric linear equilibria given by*

$$d(p) = \frac{K}{N} + \frac{\gamma K}{(1-\gamma)N(N-1)} - \frac{pK}{(1-\gamma)\nu N(N-1)},$$

in which case the clearing price is $\gamma\nu$ with $\gamma \in (0,1)$.

Proof. We prove the first assertion. We claim there exists a linear demand function $d(p) = a + bp$ which describes a class of equilibria. To verify this claim, observe that we can insert $d'(p) = b$ into (5.6) to find

$$\begin{aligned} (N-1)(\nu - p_c)b + \frac{K}{N} &= 0 \\ \Rightarrow \quad b &= -\frac{K}{N(N-1)(\nu - p_c)}. \end{aligned}$$

Now set $\gamma = \frac{p_c}{\nu}$. Then

$$b = -\frac{K}{N(N-1)\nu(1-\gamma)}.$$

Because our demand function $d(p)$ is non-increasing, $b \leq 0$, so $\gamma < 1$. Now we have to pick a such that $K = Nd(p)$, so

$$\begin{aligned} K &= N(a + bp) \\ &= N\Big(a - \frac{\gamma\nu K}{(1-\gamma)\nu N(N-1)}\Big), \end{aligned}$$

which gives us

$$a = \frac{K}{N} + \frac{\gamma K}{(1-\gamma)N(N-1)},$$

and we found the linear function that solves our equation. □

The interpretation of this result is obvious: there is a class of linear equilibrium demand schedules, which result in clearing prices $p_c \in (0, \nu)$. This means equilibrium underpricing can occur under the assumption bidders submit differentiable demand functions. In Section 5.2.1, we will investigate whether equilibrium underpricing still occurs when bidders submit discrete demand schedules, which may not be differentiable on their entire domains.

5.2.1 Discrete demand schedules

In the case of auctions for EUA's, bidders submit bids as price-quantity-pairs. In fact, they might even be restricted to placing bids with a certain tick size (a minimal amount with which they can raise their price, for example 0.01 euro) and a certain lot size (for example every bid must be a multiple of 500 EUA's). We will look at the effects of tick and lot sizes after we have studied equilibrium strategies for discrete demand schedules in general.
Suppose bidders are restricted to placing a finite number of bids. We denote a bid b^i of bidder i as

$$b^i = \{(p^i_k, q^i_k)\}_{k=1}^{T^i},$$

where $T^i < \infty$ is the number of bids for bidder i. Using this formulation, we can write the demand function of this bidder as

$$d^i(p) = \sum_{k=1}^{T^i} q^i_k \mathbf{1}_{\{p^i_k \geq p\}}.$$

This function is a non-increasing step-function in p. Now we will show that the discreteness property implies that the equilibrium clearing price equals ν [17, p. 858].

Theorem 5.2.6. *Suppose Assumptions 5.1.1 hold. Suppose there are N bidders and $T^i < \infty$ for all $i \in \{1, \ldots, N\}$. Suppose furthermore that $\bar{K} > \frac{K}{N-1}$, so removing one bidder from the auction still leaves enough potential demand to cover the auction size. Then the unique equilibrium clearing price is ν.*

Proof. Suppose there exists a low price equilibrium $\underline{p} < \nu$. Suppose the number of bids T^i is not restricted by the auctioneer, so it can be anything as long as it is finite. First, we prove that the bids at $\underline{p}$ will be rationed because aggregate demand exceeds the auction size at $\underline{p}$, which means each bidder will receive less than he demanded at $\underline{p}$.
To show this, we only have to prove that the situation in which demand is not rationed at the clearing price is inconsistent with equilibrium. So suppose this is the case: $R(\underline{p}) = K - d(\underline{p}) = 0$. Because $d(p)$ is a step-function, this can, by definition of p_c, only happen when there is a flat in the demand curve at $\underline{p}$: $d(\underline{p}) - d(\underline{p} - \epsilon) = 0$ for some $\epsilon > 0$. But that means, because aggregate demand is the sum of the individual demands and demand functions are not increasing, that there exists a set of constants $\{c_j^*\}_{j=1}^N$ such that $d^i(p) = c_j^*$ for $p \in [\underline{p} - \epsilon, \underline{p}]$. If the original bid size was smaller than the maximum bid size $\bar{K}$, a bidder can increase his payoff by increasing his bid size at $\underline{p}$ with $(\underline{p}, \bar{K} - c_j^*)$. This means he will bid $\bar{K}$ at $\underline{p}$, and this holds for every bidder.
Now since by assumption $(N-1)\bar{K} > K$, and more than 2 bidders will bid $\bar{K}$ at $\underline{p}$, the bids at $\underline{p}$ will be cut off (bidders will not receive the amount they requested).
Second, we will prove that the existence of such a clearing price $\underline{p}$ is inconsistent with equilibrium because it is always optimal for an individual bidder to slightly raise the price. In the original situation, we can use (5.3), (5.1) and (5.2) to find the payoff of bidder i equals

$$\begin{aligned} \Pi^i = u^i(\omega, S) &= (\nu - p_c)q^i = (\nu - p_c)(d^i(p_c+) + R(p_c+)\frac{d^i(p_c) - d^i(p_c+)}{d(p_c) - d(p_c+)}) \\ &= (\nu - p_c)(d^i(p_c+) + \alpha^i R(p_c+)), \end{aligned}$$

where $\alpha^i = \frac{d^i(p_c) - d^i(p_c+)}{d(p_c) - d(p_c+)}$ for the proportion of the shares allocated to bidder i at the cut-off. From the above reasoning, we know that in equilibrium $\alpha^i > 0$.
We consider an agent who has a bid at $\underline{p}$. Assume that for this bidder any bid at a price higher than $\underline{p}$ has size smaller than $\bar{K}$, so $c_j^* < \bar{K}$. Now pick an $\epsilon > 0$. Suppose the bidder moves the *price* of his original bid for $\bar{K} - c_i^*$ from $\underline{p}$ to $\underline{p} + \epsilon$. We will first show that this has a positive effect on his allocation and then we investigate the effect on his payoff. We denote by p^{old} and p^{new} the clearing prices before and after the bidder changed his bid. We consider two cases:

- $p^{old} = \underline{p}$ and $p^{new} = \underline{p} + \epsilon$. Originally, we have $d(\underline{p}) = N\bar{K} > K$ (as observed in the first part of the proof). At a slightly higher price $\underline{p} + \epsilon$ we have $d(\underline{p} + \epsilon) = \sum_j c_j^* < K$. Now the allocation q_i^{new} in the new situation is given by

  $$q_i^{new} = d^i(p_c+) + R(p_c+),$$

  because the clearing price was changed, so bidder i consumes the total residual supply. Of course, he has to pay the new clearing price for every item he won in this new situation. To see whether he can improve his payoff by use of this strategy, we take a look at the following equivalent inequalities:

  $$\begin{aligned} \Pi_i^{old} &\leq \Pi_i^{new} \\ (\nu - \underline{p})q_i^{old} &\leq (\nu - \underline{p} - \epsilon)q_i^{new} \\ (\nu - \underline{p})q_i^{old} &\leq (\nu - \underline{p})q_i^{new} - \epsilon q_i^{new}, \end{aligned}$$

  and because $\nu - p_c > 0$ and $q_i^{new} > q_i^{old}$ and ϵ can be chosen arbitrarily small, we see that the bidder always has an opportunity to improve his payoff by bidding slightly higher for additional units.

- $p^{old} = p^{new} = \underline{p}$. Because bidder i increased his bid, he now receives a higher amount $\bar{K}$ at the same, so this definitely affects his payoff positively because $(\nu - \underline{p})q_i^{old} < (\nu - \underline{p})q_i^{new}$.

The above reasoning tells us that if every bidder bids his maximum bid size $\bar{K}$ at ν, then no single bidder can improve his allocation by bidding lower, because he will not be able to lower the clearing price. By bidding lower he might only decrease his allocation. The only clearing price which is consistent with equilibrium is ν, since $(N-1)\bar{K} > K$. □

The effect described in the previous theorem is caused by the discreteness property of the bids, which causes bidders in a situation with many competitors to bid higher to prevent them from being rationed at the clearing price. By bidding at a slightly higher price, bidders could always increase their payoff. However, this is not always the case in auctions. For in reality, ϵ can not be chosen arbitrarily small. In fact, bidders will be constrained by a certain tick size, which means the shift in price is bounded from below by the tick size. Furthermore, the increases in bid size are constrained by multiples of the minimum lot size. This means we have to adjust the above theorem subject to these reality constraints. We will investigate this in the next section.

5.2.2 Tick size and quantity multiple

We now suppose there is a *tick size* of $h > 0$ and a *quantity multiple* (also sometimes referred to as *lot size*) of $w > 0$.

Assumptions 5.2.7. We assume:

- No supply uncertainty (so the amount to be auctioned is fixed and known in advance),
- $\frac{K}{w} \in \mathbb{N}$, $\frac{\nu}{h} \in \mathbb{N}$, $\frac{\bar{K}}{w} \in \mathbb{N}$, $\frac{\bar{K}}{w} \geq 2$
- $\bar{K} > \frac{K}{N-1}$,

where the last assumption assures that removing one bidder leaves enough potential demand to cover the auction. The second assumption is very weak and can easily be made without loss of generality. It means the auction size is a multiple of the minimum bid size, the clearing price is a multiple of the tick size and the quantity multiple is at least twice times smaller than the maximum bid size. The following theorem tells us that tick size and quantity multiple can determine bounds for underpricing equilibria [17, p. 860].

Theorem 5.2.8. *Let Assumptions 5.2.7 hold. Then there exists a number $t^* \geq 1$ such that $p_c = \nu - t_0 h$ is an equilibrium clearing price for $t_0 \in \{0, 1, \dots, \nu/h\}$ if and only if $t_0 \leq t^*$. The number t^* is bounded by*

$$\max\{1, \frac{K}{(N-1)w}\} \leq t^* \leq \frac{K}{(N-1)w} + 1.$$

Proof. Note that this theorem tells us that t^* determines a bound for equilibrium underpricing $t_0 h$. First, observe that there exists an equilibrium such that the clearing price is ν, because when every bidder bids the maximum bid size at ν, a single bidder can never improve his payoff by bidding at a lower price, for he cannot change the clearing price, because supply is already covered by his competitors.
Second, observe that there also exists an equilibrium such that $\nu - h$ is a clearing price. For if every bidder bids the maximum at this price, an individual bidder can only try to increase his payoff by increasing his bid at the clearing price $p_c = \nu - h$, but if he wins extra items by choosing this strategy, his payoff will be 0 for those items, which is the same as when he would not win the item at all by not increasing his bid. Consequently, no one can improve his situation by changing his strategy. So $t^* \geq 1$. Now suppose $p_c = \nu - t_0 h$, $t_0 \in \{2, \dots, \nu/h\}$. Then $t_0 \leq \nu/h$, because we still have the standing assumption that the reservation price equals zero.
Claim: p_c is supported as an equilibrium price[5] if and only if

[5]If we say p_c is supported as an equilibrium price, we mean there exists an equilibrium such that the clearing price equals p_c.

(i) $d^i(p_c) = \bar{K}$

(ii) $d(p_c + h) = K - w$.

We will first show that this claim is true.
(i) If $d^i(p_c) < \bar{K}$ for all i, there will be excess demand at the clearing price ($R(p_c) < 0$). Then the payoff $\Pi^i = (\nu - p_c)(d^i(p_c+) + R(p_c+)\frac{d^i(p_c)-d^i(p_c+)}{d(p_c)-d(p_c+)}) < (\nu - p_c)(d^i(p_c+) + R(p_c+)\frac{K-d_i(p_c+)}{d(p_c)-d(p_c+)})$. Of course this implication also holds in the other direction.
(ii) If $d(p_c+h) < K-w$, any bidder can increase his payoff by bidding an additional w at p_c+h. This will not affect the clearing price, but he will increase his allocation and thus his payoff will be higher.
It can never happen that $d(p_c + h) > K - w$, because this is inconsistent with the definition of the clearing price and our divisibility assumptions. So only $d(p_c + h) = K - w$ is consistent with equilibrium. Intuitively, this can be explained as the situation in which no bidder can increase his demand at a higher price without increasing the clearing price.
The smallest incentive to deviate arises when

(iii) $d(\nu) = \bar{K} - w$,

for the payoff with a clearing price ν will be zero. So p_c is supported in equilibrium, is equivalent to stating that p_c is supported in equilibrium when (iii) holds. We assume (iii) holds.
Because we assume the equilibrium clearing price is p_c, we have for all i that the bid size at p_c+h is smaller than $\bar{K}$ (for if not, this would have been the clearing price). Now we suppose only bidder i increases his bid size at $p_c + h$ such that he bids $\bar{K}$ at $p_c + h$. From (ii) we know that the aggregate demand $d(p_c + h) = K - w$. Because bidder i increases his bid size with at least w at $p_c + h$, the new aggregate demand d^{new} at the clearing price will be $d^{new}(p_c) \geq K$, so the new clearing price p_c^{new} takes the value $p_c^{new} = p_c + h$. We get the following chain of inequalities, which follow from each other.

$$\begin{aligned}
\Pi_i^{old} &\leq \Pi_i^{new} \\
(\nu - p_c)q_i^{old} &\leq (\nu - p_c - h)q_i^{new} \\
\frac{(\nu - (\nu - t_0 h))q_i^{old}}{h} &\leq \frac{(\nu - (\nu - t_0 h) - h)q_i^{new}}{h} \\
\frac{t_0 h q_i^{old}}{h} &\leq \frac{t_0 h + h)q_i^{new}}{h} \\
t_0 q_i^{old} &\leq (t_0 + 1)q_i^{new} \\
t_0 &\leq \frac{q_i^{new}}{q_i^{old} - q_i^{new}}.
\end{aligned}$$

So $p_c = \nu - t_0 h$ can only be supported in equilibrium if

$$t_0 \leq \frac{q_i^{new}}{q_i^{new} - q_i}.$$

We express the allocation q_i^{old} and q_i^{new} before and after the shift of the ith bidder in the parameters of our model.

$$q_i^{new} = d_i^{old}(p_c + h) + w,$$

and

$$\begin{aligned}
q_i^{old} &= d_i^{old}(p_c + h) + R(p_c+)\frac{d^i(p_c) - d^i(p_c+)}{d(p_c) - d(p_c+)} \\
&\overset{(ii)}{=} d_i^{old}(p_c + h) + w\frac{d^i(p_c) - d^i(p_c+)}{d(p_c) - d(p_c+)} \\
&\overset{(i)}{=} d_i^{old}(p_c + h) + w\frac{\bar{K} - d^i(p_c + h)}{N\bar{K} - (K - w)}.
\end{aligned}$$

So
$$\frac{q_i^{new}}{q_i^{new}-q_i}=\frac{d_i^{old}(p_c+h)+w}{d_i^{old}(p_c+h)+w-d_i^{old}(p_c+h)-\frac{\bar{K}-d^i(p_c+h)}{N\bar{K}-(K-w)}w},$$
which is increasing in $d^i(p_c+h)$. So for all i we have
$$t_0\leq\frac{d^i(p_c+h)+w}{w-\frac{\bar{K}-d^i(p_c+h)}{N\bar{K}-(K-w)}w}.$$
So as a uniform bound over all i we have
$$t_0\leq\min_i\frac{d^i(p_c+h)+w}{w-\frac{\bar{K}-d^i(p_c+h)}{N\bar{K}-(K-w)}w}.$$
Hence an equilibrium can be supported if there exists a collection of demand schedules $\{d^i\}_{i\in\{1,\ldots,N}$ such that
$$t_0\leq\max_d\min_i\frac{d^i(p_c+h)+w)}{w-\frac{\bar{K}-d^i(p_c+h)}{N\bar{K}-(K-w)}w}. \tag{5.7}$$
Now observe that in the collection of all possible demand schedules for all bidders, the minimal demand d_i of an individual bidder i is largest when the demand functions are as similar as possible, so when every bidder bids the same. In that case we have
$$y^*:=\max_d(\min_i(d_i(p_0+h))=\left\lfloor\frac{K-w}{Nw}\right\rfloor w=\left\lfloor\frac{d(p_c+h)}{Nw}\right\rfloor w\leq\frac{K-w}{Nw}\cdot w=\frac{K-w}{N}.$$
So from this we can arrive an upper bound for y^*, namely $\frac{K-w}{N}$. Furthermore we see that $y^*\geq\frac{K-Nw}{N}$. If we plug this into equation (5.7), we can easily get rid of a bunch of similar expressions and we find that
$$t_0\leq\frac{K}{(N-1)w}+1,$$
and
$$t_0\geq\frac{K}{(N-1)w}.$$
And because we already found 1 as a lower bound for t_0, this establishes our proof. □

Note that when we choose $h=0$ in the above theorem, we find the result of Theorem 5.2.6. This result is very useful in practice, because it has some very powerful implications on the occurrence of equilibrium underpricing. By adjusting the specific properties of the uniform price auction, the equilibrium difference between the clearing price and the market price can be made arbitrarily small.

5.3 Main results and intuition behind these results

We close this section with our main results and an intuition behind these results.

- In general, in a uniform price sealed bid auction with a common value and continuous demand schedules, any price between the common value and zero can be supported as an equilibrium clearing price. So equilibrium underpricing can be arbitrarily high in theory.
- However when demand schedules are discrete, the potential demand from the individual bidders is higher than the volume in the auction and the tick size is infinitely small, the equilibrium clearing price is equal to the common value.

- When demand schedules are discrete but the tick size is not infinitely small, there may be equilibrium underpricing. However, this equilibrium underpricing will be between specific bounds. These bounds depend on the quantity multiple w, the tick size h and the ratio between the number of bidders in the auction and the volume to be sold.
- The higher the quantity multiple w, the smaller equilibrium difference between clearing price and market price.
- The lower the tick size h, the smaller equilibrium difference between clearing price and market price.
- The higher the ratio between number of bidders and volume in the auction, the lower demand reduction.
- Equilibrium underpricing can be made arbitrarily small by choosing a sufficiently small tick size h.[6]

The intuition behind these results can easily be understood by thinking of extreme cases.
Suppose the number of bidders is one, then this bidder can bid as low as he likes to win the item (assuming there is no reserve price). The more bidders, the more competition to win the items. When there are very few bidders and very many items sold, the bidders can easily adopt a strategy of demand reduction by bidding at a low price, while still being certain of winning their items. When there are very many bidders and very few items sold, this strategy is more risky because bidders might not win anything.
Suppose the quantity multiple is as high as possible, which means it is equal to the auction size. In that case, the auction is equivalent to a single unit auction, because every bidder can either win or loose the items for sale, which can together be seen as a single 'unit'. Consequently, we can apply Proposition 4.3.2. It is a dominant strategy for every bidder to bid according to his true value. So choosing a large quantity multiple encourages bidders to bid close to their true values.
Suppose the tick size is very high, for example 10 euros. Then any bidder can choose between bidding either nothing or 10 or 20 euros or more. When the value of an EUA on the secondary market is for example 13 euros, no bidder will bid 20, so everybody will bid either 0 or 10 euros. This results in a very low clearing price. The smaller the tick size, the more the bidders will be encouraged to compete and bid closer to their true values.

[6]Of course in practice, there is a lower bound on the size of the ticks, for practical reasons.

Chapter 6

Putting the models to work

In Chapter 3, we found a closed-form solution for determining optimal selling strategies in terms of frequencies and volumes for selling a large amount of shares in a market which is not completely competitive. We would now like to use these results to determine an optimal selling strategy for auctioning European Union emission allowances. However, auctioning is not the same as regular selling. This means we need to adjust our market impact models for auctions instead of regular sell market orders. We will consider auctions as a specific mechanism to sell items. The model we derive in this section can be used to study the market impact of auctions compared to regular selling by placing market orders. We derive conditions for auction mechanisms such that distortion of the secondary market is minimized.

6.1 Approach

In the proposal of the European Commission on auctioning [1], the uniform price sealed bid auction is put forward as a mechanism. In Chapter 5, we derived some results on these auctions when bidders have a common value for the items which are sold. We saw that when demand schedules are continuous, any outcome between the common value and zero is consistent with equilibrium (Proposition 5.2.5). However, when the auction designer makes the choice for a sufficiently large lot size and sufficiently small tick size, Theorem 5.2.8 tells us equilibrium underpricing can be subjected to certain boundaries. We will use these boundaries to derive an upper bound for the difference between the clearing price in the uniform price auction and the secondary market price. We can calculate the *costs* of the auction to the seller, by calculating the number of bids accepted at a price lower than the secondary market price and the prices of these bids.[1] We use the lower bound on the clearing price to calculate a lower bound on the expected revenue to the seller and return to our market impact models. Because once again we are considering a seller aiming at minimizing market distortion. By minimizing the costs of auctions, the seller minimizes market distortion and at the same time maximizes his revenues, just as we saw in Chapter 3.
We will start by applying market impact models to selling large amounts of EUA's. First, we assume no auctions are used, but the items are sold by placing regular sell market orders, just as in Chapter 3.
Then we will 'cross the bridge': we will bring auction theory and market impact models together by studying clearing prices in equilibrium and the effects of these clearing prices on the secondary market. We will also study the impact of the *volume* of the auction on the Limit Order Book.

[1]We will assume all bidders behave rationally and symmetric. So no bidder will have any incentive to bid higher than the secondary market price. Nobody fears having to pay a higher price in the secondary market after the auction (for example because large traders use their market power to raise the prices).

6.2 Application of market impact models to selling emission rights

This section describes in which way our model might be applied to the third ETS period. However, there is a number of uncertainties, which makes it very hard to conduct empirical research on the correctness of our model. The main uncertainties are the following:

- **System uncertainty** Because the European Commission is at the time of writing designing proposals on auctioning, the auctioned amount, the exact regulation, benchmark categories, etc. the rules governing the system which we want to study are unknown.
- **Market uncertainties** There is a number of factors which are market specific and crucial to our outcomes, such as the shape of the Limit Order Book, the resilience speed and the types and number of players in the market. It is very hard to estimate these factors, because the ETS in the third period will differ on crucial points from any other market in present or past. Any estimate derived from bond markets, stock markets or the current ETS might be uninformative. We simply cannot check the justification of using those results.

The question arises: what *can* we say then? Well, we can describe the *interaction* between the several (currently) unknown regulations and market properties. To see how this works, we use the following example. In our example, we give some values to parameters which are a priori uncertain. When we fix some parameters, we can study the relative behavior of other parameters in the model. We can use this to determine the consequences of specific decisions for market distortion, given the values of other parameters. We can also study the sensitivity of the final market distortion caused by a strategy to variations in specific parameter values. This will give us information on the relative importance of specific decisions and parameter values. We use the notation from Section 3.

Example 6.2.1. Optimally selling 1 billion EUA's per year.

Objective: We want to determine an optimal selling strategy in terms of frequency of trades and division of volumes over trades, thereby minimizing distortion of the Limit Order Book. More formally, we are looking for a selling strategy that solves Problem 3.1.1. We measure the amount of distortion of the order book as costs to the seller.

System input:

- We want to sell an amount of 1 billion EUA's per year.
- We want to use daily, twice a week, weekly or monthly auctions.
- The auctions are announced in advance, before the trading period started.

Limit Order Book

Assumptions 6.2.2. We assume:

1. The Limit Order Book is block-shaped,
$$f(x) = q.$$
2. Intra-trading times are equal over all trades $\tau = \frac{T}{N}$.
3. The resilience rate is a constant, $\rho > 0$.
4. The resilience of the volume of the Limit Order Book is exponential, which is equivalent to exponential resilience of the bid-ask spread.[2]

[2]This only holds because the Limit Order Book is block-shaped. See Remark 3.1.7

5. The permanent price impact of the auctions is zero. The auctions are announced in advance, so no new information is provided to the market about supply and demand. Therefore, the price will not change permanently.[3]

Under these assumptions we can apply Corollary 3.1.6, which tells us the optimal selling strategy ξ^* is given by

$$\xi_0^* = \frac{Z_0}{2+(N-1)(1-a)} = \xi_N^*,$$

and

$$\xi_n^* = \xi_0^*(1-a), \quad 1 \leq n \leq N-1.$$

Using this and (3.4),

$$\pi_t(x_t) = \int_{D_t^B}^{D_{t+}^B} (B_t^0 - x) q dx = B_t^0 x_t - \int_{D_t^B}^{D_{t+}^B} x q dx = B_t^0 x_t - \frac{q}{2}((D_{t+}^B)^2 - (D_t^B)^2),$$

and

$$\mathcal{C}(\xi) = \mathbb{E}[\sum_{n=0}^{N} \pi_{t_n}(\xi_n)],$$

we can calculate the revenues of this strategy. We can use (3.18) to calculate costs of the market distortion (loss in revenue). We refer to Chapter 3 for the notation used in these formulas.

6.2.1 Results and observations from the application

In our example, we use the following input variables: $Z_0 = 1000000000$, half time of exponential resilience of the bid-ask spread $h \in [0.01, 0.5]$, $q = 50000$, $N \in \{10, 50, 100, 365\}$, $T = 365$. From the graph in Figure 6.1 we can quickly see what this means for finding an optimal division of volumes over trades.[4]

[3] In reality, auctions may provide new information about *demand* to the market. When the clearing price is higher than the secondary market price, for example, this could mean that actual demand is higher than expected and this might cause the price on the secondary market to rise. Therefore, it would be interesting for further research to study this model when a permanent price impact is taken into account.

[4] R source code can be found in the Appendix B.

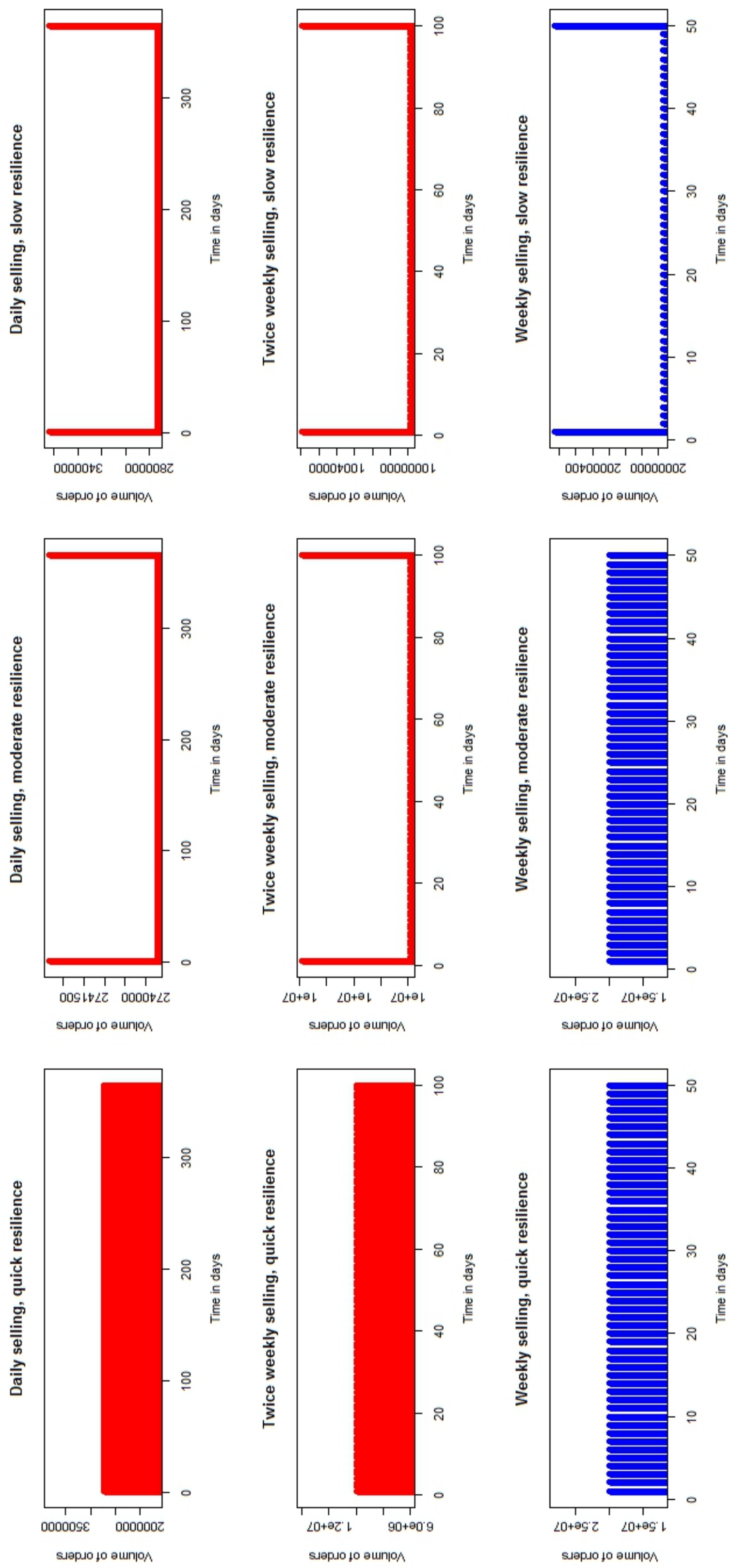


Figure 6.1: Optimal division of volumes over trades. Input: $Z_0 = 1000000000$, $h \in \{0.02, 0.1, 0.5\}$ (resp. quick, moderate, slow), $q = 50000$, $N \in \{365, 100, 50\}$ (resp. daily, twice weekly, weekly), $T = 365$.

An interpretation of these results is that the resilience speed is crucial in determining optimal selling strategies. First we consider the division of volumes over trades. The quicker the resilience, the quicker the absorption of the market order. Consequently, trades can be spread equally over time. However when resilience is slow, trading too fast requires trading smaller volumes, for every large quick trade will only increase the bid-ask spread further. Large trades in the beginning and the end are required to sell the total volume. This is the bath-tub effect we already described in Chapter 3. Note that the values at the vertical axes are not equal over all figures.
Also when we consider the trading frequency, resilience is crucial. We see that when market recovery is quick, daily trading is far less expensive than weekly trading. This can also be seen in the table in Figure 6.2.

Cost of selling strategies (€)				
Volume Yearly	1.000.000.000	EUA's		
Thickness Order Book	50.000			
Selling frequency	recovery			
	quick (h=0.002)	moderate (h=0.02)	slow (h=0.2)	very slow (h=0.5)
daily	27 bn	27 bn	336 bn	2511 bn
twice a week	100 bn	100 bn	100 bn	159 bn
weekly	200 bn	200 bn	200 bn	200 bn
monthly	1000 bn	1000 bn	1000 bn	1000 bn

Figure 6.2: The costs of optimal selling strategies

In this table, only optimal selling strategies are used and cumulative costs over the whole period of one year are calculated. We see that when recovery is quick, it is optimal to trade daily. However, when recovery is slow, trading twice a week is less distorting. Note that this table should be used to examine the *relative* results. No *absolute* values should be focused on, because this is just a purely imaginary example to illustrate the relation between the parameters in the model.
To examine the sensitivity of the trading strategies to the resilience speed more closely, we can calculate costs of the optimal selling strategies relative to the selling frequencies. In Figure 6.3, we assumed $q = 50.000$.

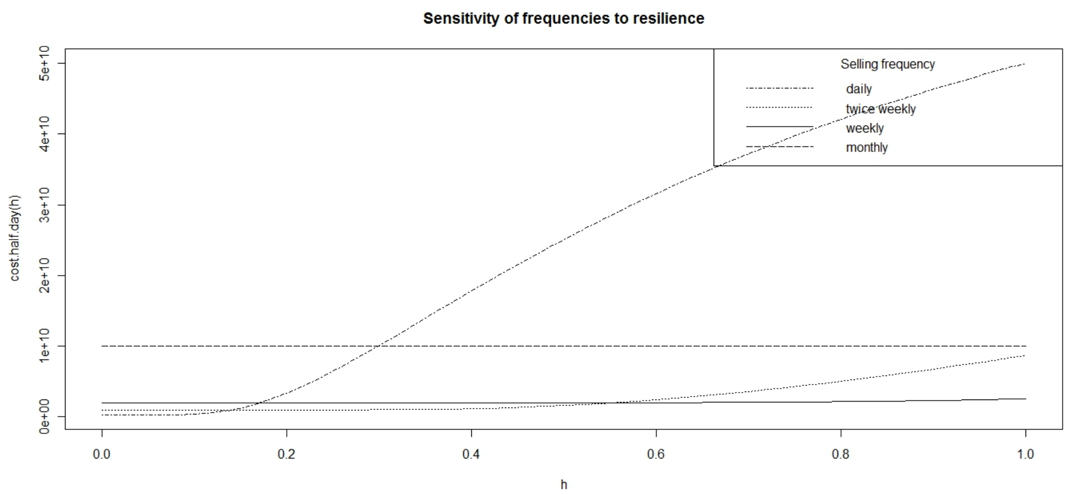


Figure 6.3: Sensitivity of optimal trading strategies to the resilience speed

In Figure 6.3, we clearly see that it is optimal to trade daily, as long as the resilience speed is high enough. However, when the half time of the recovery is a few hours, daily trading can become very distorting to the market. In stock markets with frequently traded stocks however, empirical studies show that half time of the recovery is a few minutes rather than a few hours [9]. In Figure

6.4, we see the reverse study: given a certain resilience speed, which selling frequency is optimal? When resilience is quick, the higher the frequency the better. However, when resilience is slow, optimal trading frequency becomes a trade-off. When the frequency is very low, we are trading huge volumes at a time. The costs of this are very high, no matter how quick the resilience is. Cutting the trades into smaller pieces is less distorting. However, when we cut the trades in pieces which are too small, we have to trade frequently and distort the market every time we place a new trade. The market does not get enough time to recover, which makes our trades very expensive because the best bid price decreases with every new trade. We can clearly see the effect of this trade-off between frequencies and volumes when the resilience speed is very slow.

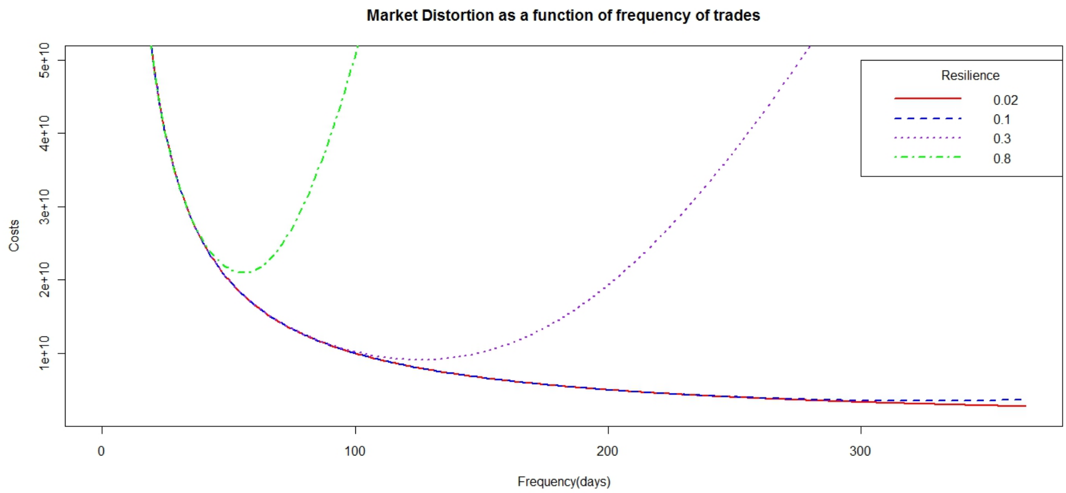


Figure 6.4: Costs of selling strategies depend highly on the resilience speed

From this, we can directly derive some recommendations with respect to optimal selling strategies.

- When recovery is quick, market distortion decreases with frequency of trades.
- When recovery is quick, it is optimal to have daily trades.
- When recovery is quick, market distortion when trades are twice a week is four times larger than when trades are daily.
- When recovery is slow, it is optimal to trade twice a week.
- When recovery is slow, daily trading is three times more distorting then trading twice a week.
- When recovery is slow and intratrading times are short, it is optimal to trade higher volumes in the beginning and at the end of the trading period: the bath-tub effect.

6.3 Approach to market impact of auctions in the symmetric situation

Often large sell orders are not placed as regular market orders, but by using auctions. It is reasonable to suppose that also large auctions will have some impact on the secondary market price. In this section, we construct an integrated model to study this impact. We assume the auction as described in Section 5 is used, namely a uniform price sealed bid auction. This is consistent with the proposal of the European Commission on auctioning EUA's in the third period

of the ETS [1].
Price impact of auctions can be decomposed and studied in several ways. We can split up the analysis in two situations. The first is the rational and symmetric situation. The second is the asymmetric and possibly irrational situation. In the first situation, our mathematical gear will be very useful and we can make some quantitative statements. We will focus on this situation in this thesis. The second situation we leave for further research. More information on asymmetric bidders, market impact models from a macroscopic point of view and the risk of equilibrium overpricing can be found in for example [21], [25] and [10].

6.4 Symmetric behaviour

We merge our results from auction theory in the symmetric case and microscopic market impact models. In reality, on the ETS market, there are several trading platforms, so several Limit Order Books. However, prices in these order books will be the same (for if the price would differ between platforms, there would be arbitrage opportunities). For simplicity, we will assume there exists only one Limit Order Book in which all limit orders are posted. Auctioning differs in several ways from regular market orders. In the previous section, we studied regular selling of EUA's in a Limit Order Book. So no new bidders were added to the Limit Order Book. Now, we will study the situation when there is a different platform in which orders are placed. This is the auction bid book. An auction bid book can be regarded as a specific type of Limit Order Book. Different rules are applied in the auction bid book. Because this analysis will lead to complexities, we begin by studying the auction bid book as if it were another Limit Order Book. We study different scenario's. In the clean scenario, we assume the total volume from the original Limit Order Book is copied to the new auction bid book and everything else stays the same. In the second scenario, the mixed scenario, we study what happens when only a fraction of bids from the Limit Order Book is transferred to the auction. Furthermore, other bids can be added to the auction bid book. In this case, the revenue of the sell order will be influenced and will depend on the fraction of bids which are transferred and the number of new bids attracted to the 'auction'. However, it is very important to note that these scenario's should not be interpreted as actual auctions, because in a real auction, bidders will behave strategically.
In the last scenario, the strategic mixed scenario, we will study strategic behaviour. This is the scenario with the closest resemblance to the real auctions. We will use our results from auction theory to predict bidding behaviour and outcome of the auction. We will consider the impact of the auction on the Limit Order Book caused by the volume in the auction and the clearing price. The three scenario's have an increasing degree of complexity. In every scenario, we have the following:

Market environment

1. A collection $\mathcal{N}$ of potential buyers.
2. A collection $\mathcal{S}$ of potential sellers in the LOB.
3. A number of items to be sold in the auctions in one year, denoted by Z_0.
4. A number of auctions per year, denoted by N.

We assume Assumptions 6.2.2 hold for the Limit Order Book.

The Auction

1. The auction format is uniform-price sealed bid.
2. The auction properties satisfy Assumptions 5.1.1. (Which also means we assume that the items have a pure common value.)

6.4.1 Clean Scenario

In the Clean Scenario, we make the following assumptions:

1. Every bid from the Limit Order Book is transferred to the auction at the time the auction starts.
2. There are no other bids added to the auction bid book.
3. Bidders do not change their strategies when they take their bids from the Limit Order Book to the auction bid book.

In this situation, the auction bid book is exactly equal to the Limit Order Book. This means the volume in the auction is equal to the volume detracted from the Limit Order Book by the auction. The auctioned volume is equal to the volume of a large sell market order of the size of the volume in the auction. The clearing price is equal to the new best bid price after the auction and the bid-ask spread temporarily increases.

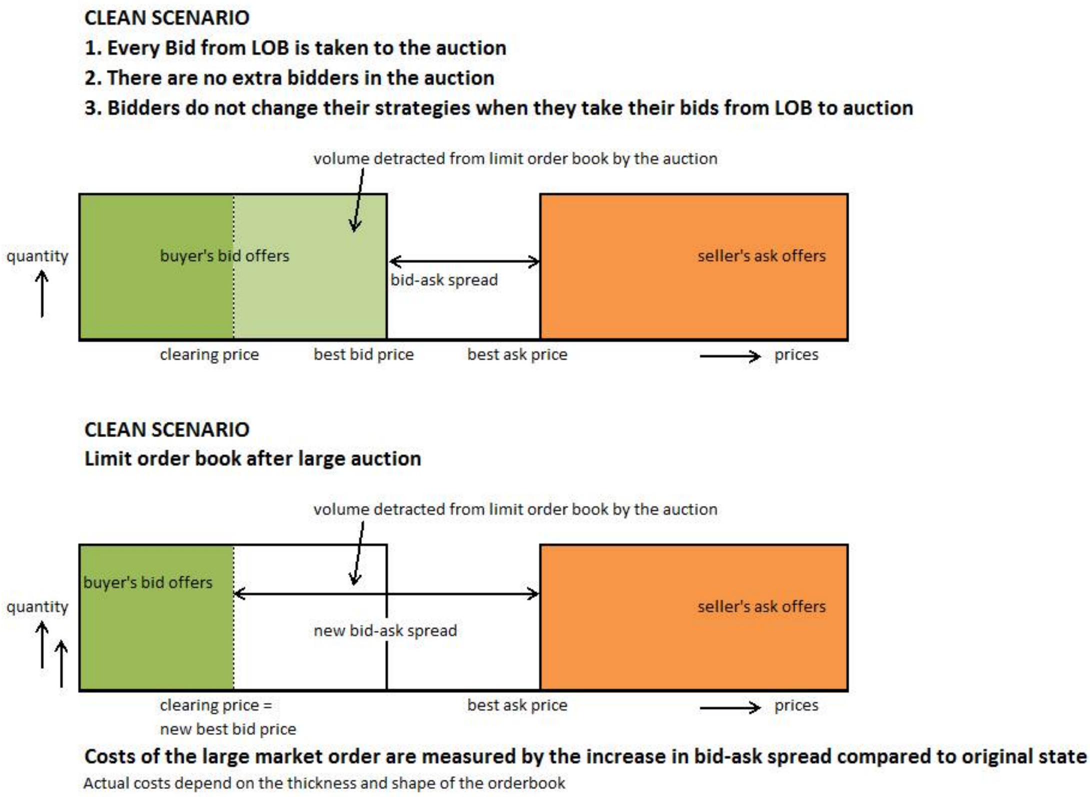


Figure 6.5: Market impact in the clean scenario

From Figure 6.5, we immediately see that the market impact of the auction is similar to the market impact of a large sell market order. The *costs* to the seller are equal to the bids accepted lower than the best bid price times the differences of their individual values and the best bid price. This is at the same time a measure for market distortion by the auction. A selling strategy which minimizes market distortion is what we are looking for. Consequently, we can apply Corollary 3.1.6 for determining optimal selling strategies. After that, we use (3.4) and (3.5), where we can plug-in the optimal strategy ξ^* from Corollary 3.1.6 to calculate market impact costs. Note that in this scenario any standard auction format can be used, the pricing rule is irrelevant. The results for this scenario with respect to market distortion are described in Example 6.2.1.

Consequences for the clearing price and revenue to the seller

Although we did not state revenue maximization as an explicit objective, the clean approach with market impact models provides us with an explicit solution to the outcome of the auction. We consider the first auction in the sequence of auctions over one year.
Assume the auction size is $x_0 > 0$ shares at time $t = 0$. This means the seller consumes all orders below the best bid price B_0 until $B_{0+} = B_0 - D^B_{0+}$, where D^B_{0+} is determined by

$$\int_0^{D^B_{0+}} f(x)dx = x_0. \tag{6.1}$$

If we write

$$F(x) = \int_0^x f(y)dy,$$

we can rewrite (6.1) as

$$F(D^B_{0+}) = x_0 \Leftrightarrow F^{-1}(x_0) = D^B_{0+}.$$

By assumption $f(x) = q$ for all x, so $F(x) = xq$, which means $D^B_{0+} = \frac{x_0}{q}$.
For sell orders, the price on the secondary market equals the best bid price, B^0_t. From now on, for notational simplicity we will denote the best-bid price at time t by B_t. Now we have for the clearing price $C_t = B_t - D^B_{t+}$. In Section 4.2.1 we defined revenue to the seller as

$$R = \sum_{i=1}^{N} p_i,$$

which is the sum of the individual payments of each bidder. In the multi-unit case, from (5.2) and (5.1) we conclude that the revenue to the seller equals

$$\mathbb{E}(R) = \mathbb{E}(x_0 C_0) = \mathbb{E}(x_0(B_0 - D^B_{0+})) = x_0(B_0 - D^B_{0+}),$$

for the first auction. (Where the last equation holds because D_t is deterministic and B_t is a martingale, so $B_0 = \mathbb{E}B_t$.)
We assume auctions take place at times t_n, where $0 \leq t_0 < t_1 < \ldots < t_n$. We assume the total number of auctions per year equals $1 \leq N+1 < \infty, N \in \mathbb{N}$. Now to determine the clearing price of the second auction, we observe that in general

$$F(D_{t_n+}) = x_n + F(D_{t_n}).$$

Furthermore

$$D^B_{t+s} = e^{-\rho s} D^B_t,$$

and $D_0 = 0$. We also have from Corollary 3.1.6 that for the optimal strategy, $x_n = x_0(1-a) = x_0 - e^{-\rho\tau} x_0$. By induction, it is now easy to see that

$$D_{t_n+} = F^{-1}(x_0),$$

and

$$D_{t_{n+1}} = e^{-\rho\tau} F^{-1}(x_0),$$

for all $n \in \{1, \ldots, N-1\}$. This means that for auction clearing prices we have

$$C_{t_n} = B_{t_n} - D_{t_n+} = B_{t_n} - \frac{x_0}{q},$$

for $n \in \{1, \ldots, N-1.\}$ At the terminal time, $t_N = T$, the extra spread is given by

$$\begin{aligned} D_{t_N+} &= F^{-1}(E_{t_N+}) = F^{-1}(E_{t_N} + x_N) \\ &= F^{-1}(ax_0 + Z_0 - (N-1)x_0(1-a) - x_0) = F^{-1}(Z_0 - N(x_0 - ax_0)) \\ &= \frac{Z_0 - N(x_0 - ax_0)}{q}, \end{aligned}$$

where we use Corollary 3.1.6. The last expression gives us an explicit solution to find the clearing price of the last auction,

$$C_{t_N} = B^0_{t_N} - D_{t_N+} = B^0_{t_N} - \frac{Z_0 - N(x_0 - ax_0)}{q}.$$

The expected revenue to the seller of the series of auctions can be calculated by

$$\mathbb{E}(R) = \mathbb{E}(\sum_{n=0}^{N}(B_{t_n} - D_{t_n+})x_n) = \sum_{n=0}^{N}(B_0 - D_{t_n+})x_n, \tag{6.2}$$

where we use that B_t is a martingale. The 'loss' $L = R - \sum_{n=0}^{N} B_{t_n} x_n$ of selling at a lower rate than the best bid price can be calculated by

$$\mathbb{E}(L) = L = \sum_{n=0}^{N} -D_{t_n+}x_n.$$

6.4.2 Mixed scenario

It is reasonable to assume that not every bid order from the Limit Order Book is transferred to the auction. Probably only a part of the orders is actually detracted from the order book by the auction. We may also assume that not every bidder who bids in the auction also had a bid in the Limit Order Book before the auction. Therefore, we study the mixed scenario, which is based on the following assumptions:

1. Not every bid from the Limit Order Book is transferred to the auction at the time the auction starts. Some bids stick to the LOB.
2. There are other bids added to the auction bid book.
3. Bidders do not change their strategies when they take their bids from Limit Order Book to auction bid book.

In Figure 6.6, we can easily see what these assumptions mean in an auction.

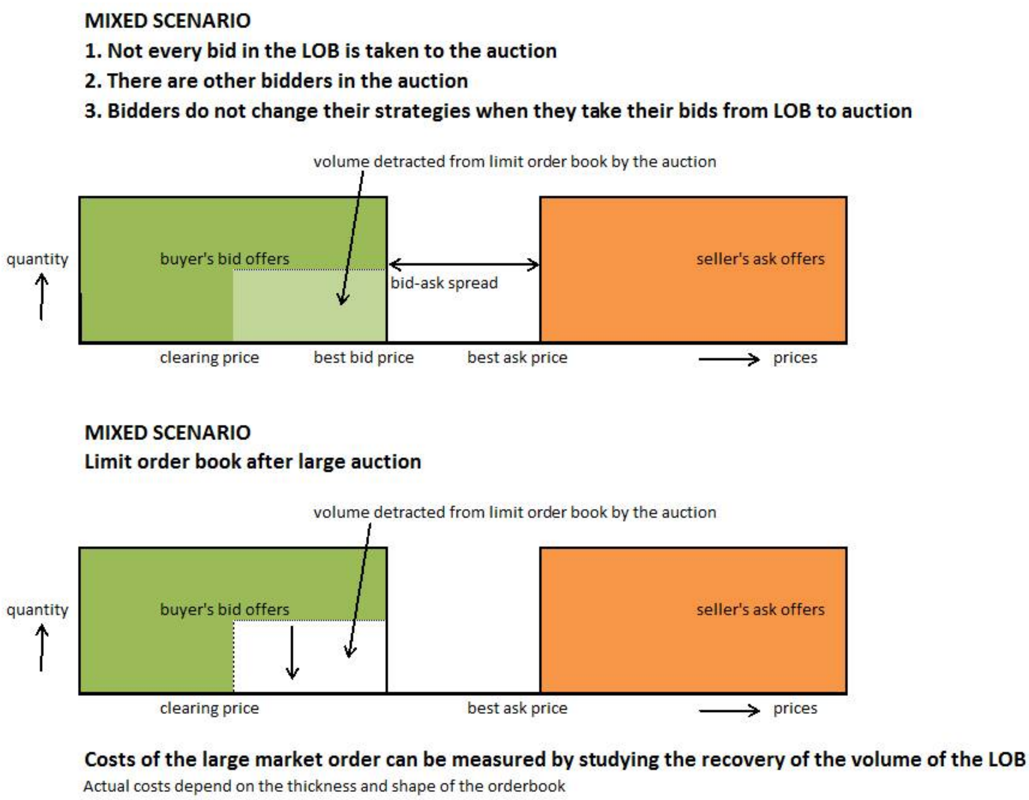


Figure 6.6: Market impact in the mixed scenario

We see from the figure that the smaller the proportion of bids transferred to the auction, the lower the clearing price. Suppose the fraction of bids transferred from the LOB to the auction is $b \in [0, 1]$. In the mixed scenario, the bid-ask spread does not necessarily increase as such, but the market impact is measured by the temporarily decrease in the volume of the Limit Order Book. However we saw in Remark 3.1.7 that this is equivalent to a distortion of the bid-ask spread, for we suppose the auction bid book is block-shaped as well. Just like in the LOB, we can formulate the shape function for the new orders in the auction bid book as $g(x) = r$, $r \geq 0$. The higher the number of new bids in the auction relative to the bids from the limit order book, the lower the volume of bids removed from the limit order book by the auction. We denote by $\frac{g(x)}{f(x)} = s$, $s \in [0, \infty)$ the ratio between auction and limit orders. This provides us with a new shape function. From the figure we see that the higher the number of new bids added to the auction bid book, the higher the clearing price. So for the first order, the auction takes $\int bf(x)dx + g(x)dx = \int (bf(x) + g(x))dx = \int (bf(x) + sf(x))dx = \int (b+s)f(x)dx$ shares at a price $B_0 + x$ where x is between B_0 and $B_0 - D^B_{0+}$:

$$\int_0^{D^B_{0+}} f(x)(b + s)dx = x_0. \tag{6.3}$$

We can rewrite (6.1) as

$$F(D^B_{0+}) = \frac{x_0}{b + s} \Rightarrow F^{-1}(\frac{x_0}{b + s}) = D^B_{0+}.$$

By assumption $f(x) = q$ for all x, so $F(x) = xq$, which means

$$D^B_{0+} = \frac{x_0}{(b + s)q}. \tag{6.4}$$

So the clearing price in the first auction equals $B_0 - \frac{x_0}{(b+s)q}$. We use the mixed scenario and adjust for our proportion s to find

$$D_{t_n+} = F^{-1}(\frac{x_0}{b+s}),$$

for all $n \in \{1, \ldots, N_1\}$. This means that for auction clearing prices we have

$$C_{t_n} = B_{t_n} - D_{t_n+} = B_{t_n} - \frac{x_0}{(b+s)q},$$

for $n \in \{1, \ldots, N-1.\}$ At the terminal time, $t_N = T$, the extra spread is given by

$$\begin{aligned} D_{t_N+} &= F^{-1}(E_{t_N+}) = F^{-1}(E_{t_N} + x_N) \\ &= F^{-1}(ax_0 + Z_0 - (N-1)x_0(1-a) - x_0) = F^{-1}(Z_0 - N(x_0 - ax_0)) \\ &= \frac{Z_0 - N(x_0 - ax_0)}{(b+s)q}, \end{aligned} \tag{6.5}$$

In Figure 6.7, we see what this means for the clearing price in these auctions.

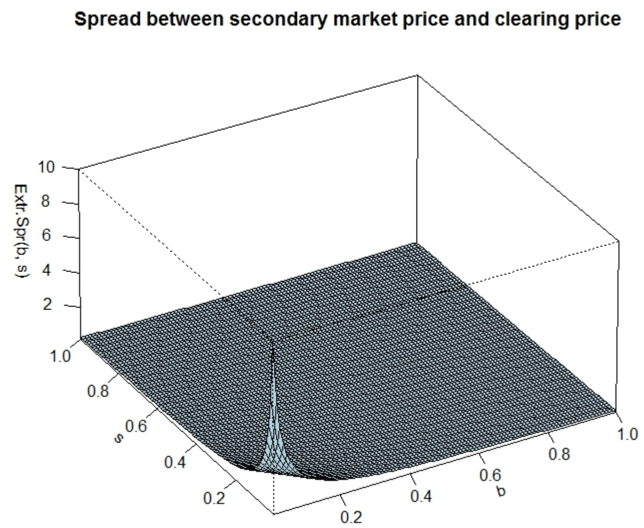


Figure 6.7: Spread between secondary market price and clearing price as a function of b and s

When b and s are both close to zero, the spread is highest. This means that when there are very few bids transferred from the LOB to this auction and there are almost no new orders added to the auction, the difference between the clearing price and the secondary market price is high. This is consistent with our intuition that a lower number of bidders leads to a lower clearing price. When more bids are added to the auction which were not in the LOB, s increases and the spread will be lower. When all bids from the LOB are taken into the auction, b increases and the spread will also be lower. The expected revenue to the seller of the series of auctions can again be calculated by (6.2). The lower the spread between secondary market price and clearing price, the higher the revenue to the seller. Note that the mixed scenario reduces to the clean scenario for $b = 1$ and $s = 0$.

6.4.3 Interpretation and remarks on results in the mixed scenario

- In the mixed scenario, the market impact is caused by the volume of the auction. A large volume sold in the auction may (temporarily) detract orders from the Limit Order Book. We introduced two parameters into our model to influence the measure of this impact. The first is the fraction of bids which is removed from the LOB and transferred to the auction. The second is the ratio between 'new bids' and bids from the Limit Order Book in the auction bid book. When the fraction of bids transferred from the LOB to the auction is low and there are no new bids added to the auction bid book, the risk for a high spread is highest. When the fraction of bids transferred from the LOB to the auction is high and there are many new bids coming in as well, the auction clearing price is likely to be equal to the secondary market price.

- In terms of market impact, however, while the clearing price may be low, if the fraction of bids transferred from the LOB to the auction is low and the number of extra bidders in the auction is high, the market impact caused by the *volume* of the auction will be low. So a low clearing price does not necessarily mean the market impact caused by the volume in the auction is high.

- The values of b and s depend on the market itself. When the threshold to transfer a bid from the LOB into the auction is very low, every participant has an incentive to move his bid from the LOB to the auction. This is because in the auction a bidder will only pay the price he bids or any price lower than his own bid (because the clearing price equals the lowest winning bid). Otherwise, he wins nothing in the auction but he can take his bid back to the Limit Order Book.

- This means that the more the auctions are integrated into the exchanges (which are operated through Limit Order Books), the higher b and the lower the spread in this auction. In practice, it is possible that the auctions are performed via a separate exchange platform. Bidders may be asked to subscribe separately to this auction platform and there might be additional costs for participating in the auctions. These thresholds will make the value of b lower and thus increase the probability of a high spread.
At the same time, when extra bidders are encouraged to participate in the auction, for example because it is very cheap and easy to participate in the auction instead of trading on the exchange, s increases and this also decreases the spread.

6.5 Strategic mixed scenario

In the previous scenario's, we assumed no bidder changes his bid when he transfers his bid from the order book to the auction. However, in general this assumption is not justified, for we know from Section 5 that an auction is a mechanisms which affects incentives. As we saw, bidders may adopt a strategy of so-called demand reduction. Adoption of this strategy by a single bidder means that he will not bid the same in the auction as in the Limit Order Book. More specifically, he will bid *lower* in the auction. When all bidders adopt this strategy, the clearing price might be significantly lower than the secondary market price. As we saw in Section 5, the chances this strategy is adopted in equilibrium depends on specific details of the auction design. To merge auction theory and market impact models, first we study the dynamics of the Limit Order Book when a certain clearing price is *given* (ex post). Given a certain clearing price, we ask ourselves what the market impact will be. Then, using auction theory, we will be able to make statements about the probability a certain clearing price (and with this clearing price a certain market impact) occurs, given a specific set of parameters.
The strategic mixed scenario is based on the following assumptions.

1. Not every bid from the Limit Order Book is transferred to the auction at the time the auction starts. Some bids remain in the LOB.

2. There are other bids added to the auction bid book.

3. Bidders may change their strategies when they take their bids from Limit Order Book to the auction bid book. However, all bidders are assumed to behave rationally and their positions

and strategies are symmetric, so no bidder will bid higher than the expected price on the secondary market right after the auction.

From Figure 6.8, we can easily see what this means in an auction.

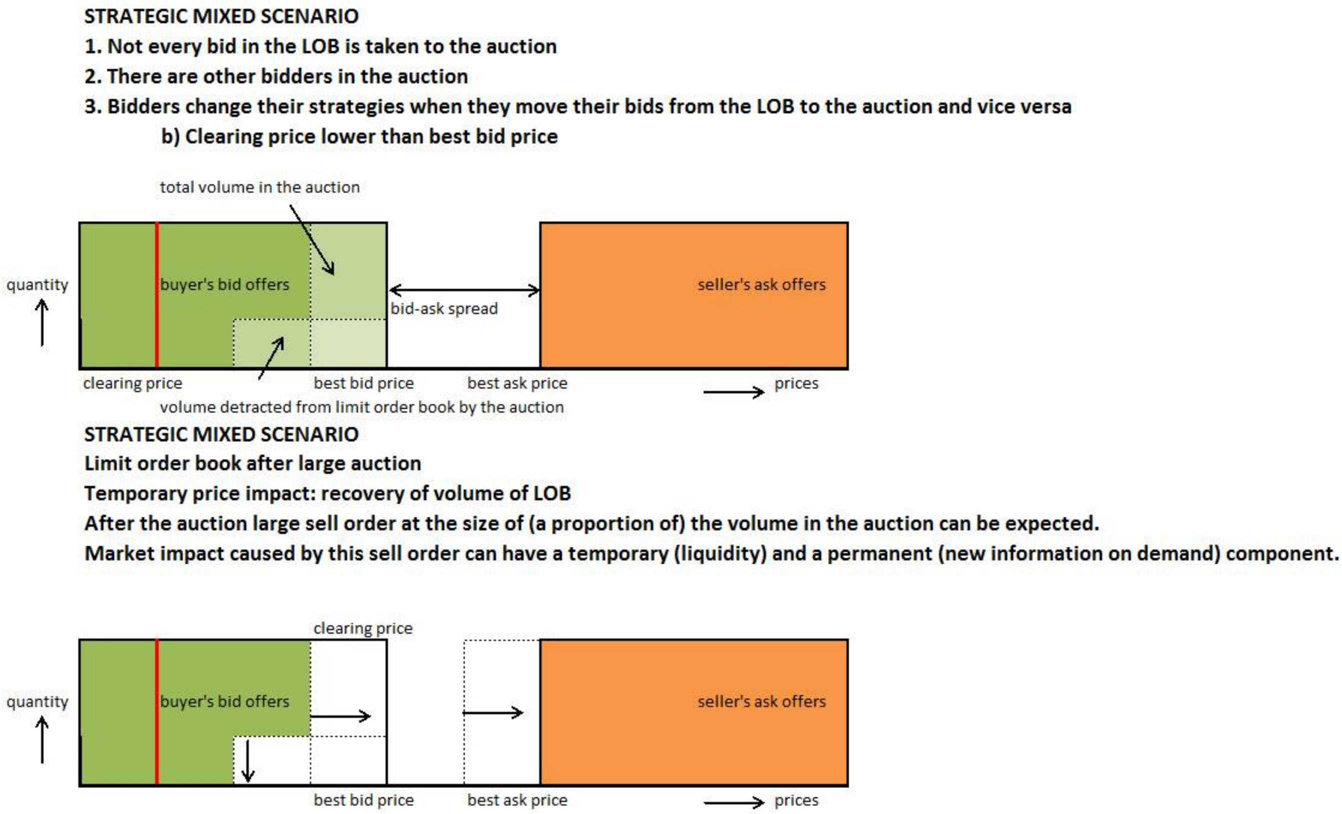


Figure 6.8: Market impact in the strategic scenario

In the strategic mixed scenario, the price impact of an auction has two components. The first is caused by the large volume of the auction, which may detract a certain volume of bid orders from the Limit Order Book. The second is caused by the outcome of the auction and a clearing price which is possibly lower than the price on the secondary market. The first component can be studied as in the mixed scenario.

6.5.1 Market impact of the outcome of the auction

The rationale behind the second component is the following. Consider a market with two assets, one risky with price S and one risk-less with price S^0. Suppose the interest rate is 0, so the value of the risk-less asset $S_t^0 = S^0$ for all times t. Denote by (Y^0, Y^1) the vector of quantities, which is we will call a *portfolio*. The value of a portfolio at time $t = 0$ is $V_0 = Y^0 S^0 + Y^1 S^1$. Consider a buyer with an initial position $V_0 = 0$. Suppose in the auction, the buyer acquires one risky share at price $C_0 < S_{0+}$. The value of his portfolio will change to

$$V_{0+} = S_{0+} - C_0.$$

After acquiring this share, the holder of the share can either keep it or sell it on the secondary market. If he decides to keep it, the value of his portfolio will develop as a martingale:

$$V_t = S_t - C_0,$$

where C_0 is a constant. However, if he sells it, the value of his portfolio will be fixed:

$$V_t = S_{0+} - C_0.$$

Depending on the preferences of the player, he will decide to keep or resell. In the market for EUA's, there are players who aim to make a cash profit by trading. These players may be willing to resell their acquired EUA's immediately after the auction. However, there are also compliance buyers, who have to surrender an amount of EUA's at some point in time. These players might be less willing to resell their acquired EUA's after the auction, for they fear having to buy EUA's at higher prices in the future.

Assumption on reselling after the auction: We assume in the strategic scenario that *if* the clearing price is lower than the price paid on the secondary market, a *proportion* of the players in the auction will take the chance to make a cash profit by selling their EUA's immediately after the auction on the secondary market. Furthermore, we take over all assumptions and notation from the mixed scenario without strategic behavior.

We are interested in the price impact of these strategies on the LOB. This impact is different from the impact caused by the volume of the auction. Therefore, we use a different notation. With the superscript $\mathcal{B}$ we denote the dynamics in the bid-ask spread of the order book after the auction when a resale of the total volume in the auction takes place. After the auction, a volume x_0 of shares is taken from the LOB at a price $B_0 - x$ where x is between B_0 and $B_0 - D^{\mathcal{B}}_{0+}$ where $D^{\mathcal{B}}_{0+}$ denotes the increase in bid-ask spread caused by the outcome of the auction. For the shape of the LOB we have from the mixed scenario that it equals $bf(x)$ for $x \in [B_0^0 - D^B_{0+}, B_0^0]$, where b is the fraction of bids transferred from the LOB to the auction and it equals $f(x)$ for $x \leq B_0 - D^B_{0+}$. So for the price impact after the auction we have the following:

$$\int_0^{D^{\mathcal{B}}_{0+}} f(x)(1-b)dx + (\int_{D^B_{0+}}^{D^{\mathcal{B}}_{0+}} bf(x)dx)\mathbf{1}_{s>1-2b} = x_0. \tag{6.6}$$

This follows from Figure 6.8. Notice that $s > 1-2b \Leftrightarrow D^{\mathcal{B}}_{0+} > D^B_{0+}$, which can be seen as follows. For the left-hand side of the above equation to hold, the volume in the auction must be larger than $(1-b)qD_{0+}$, so

$$\begin{aligned} (b+s)qD_{0+} &> (1-b)qD_{0+} \\ s &> 1-2b. \end{aligned}$$

We would like to rewrite (6.6). First, suppose $s \leq 1-2b$. Then

$$F(D^{\mathcal{B}}_{0+}) = \frac{x_0}{1-b} \Rightarrow F^{-1}(\frac{x_0}{1-b}) = D^{\mathcal{B}}_{0+}, \tag{6.7}$$

where we assume F is strictly increasing. Second, suppose $s > 1-2b$. Then

$$\begin{aligned} \int_0^{D^{\mathcal{B}}_{0+}} (f(x)(1-b) + f(x)b)dx - \int_0^{D^B_{0+}} f(x)bdx &= x_0 \\ \Rightarrow \int_0^{D^{\mathcal{B}}_{0+}} f(x)dx = x_0 + \int_0^{D^B_{0+}} bf(x)dx &\overset{(6.4)}{=} x_0 + \frac{b}{b+s}x_0 \\ \Rightarrow F(D^{\mathcal{B}}_{0+}) &= x_0(1+(\frac{b}{b+s})). \end{aligned} \tag{6.8}$$

By assumption $f(x) = q$ for all x, so $F(x) = xq$, which means from (6.7) and (6.8),

$$D^{\mathcal{B}}_{0+} = \frac{x_0}{(1+\frac{b}{b+s})q}\mathbf{1}_{s>1-2b} + \frac{x_0}{(1-b)q}\mathbf{1}_{s\leq 1-2b}. \tag{6.9}$$

However, for costs of our market order, using the reasoning above, we know that the price impact caused by this effect only holds when $D_{t_n+} > B^0_{t_n} - C_{t_n}$. Orders are consumed from the order book only when they lie above the clearing price. Denote the extra spread above the clearing price at time t by $D_{t+}^{C_t} := C_t - B_t^0$, and set $D_{t+}^{\omega} = \min\{D_{t+}^{\mathcal{B}}, D_{t+}^{C_t}\}$ for the price impact of the outcome of the auction. We use the superscript ω to denote the impact of the outcome of the auction, caused by a resale, but only when it is still profitable to resell in the Limit Order Book (so when the best bid price is higher than the clearing price). In this approach, the clearing prices C_{t_n} are exogenously given, so like in (6.2), the revenue to the seller just equals

$$R = \sum_{n=0}^{N} C_{t_n}.$$

To calculate the temporary costs of the outcome ω of one auction with a low-price equilibrium we use

$$\begin{aligned} D_{t_{n+1}}^{\omega} &= e^{-\rho\tau} D_{t_n}^{\omega} \\ &= e^{-\rho\tau} \min\{\frac{x_0}{q} I, C_{t_n} - B_t^0\} \\ &= \frac{1}{q} e^{-\rho\tau} \min\{x_0 I, q(B_t^0 - C_{t_n})\}, \end{aligned}$$

where $I := \frac{b+s}{2b+s}\mathbf{1}_{\{s>1-2b\}} + \frac{1}{1-b}\mathbf{1}_{\{s\leq 1-2b\}}$. Now also taking into account the permanent price impact, we derive the impact at time $t+\tau$ of the outcome of an auction of size x_t executed at time t to be

$$\gamma x_t + \kappa e^{-\rho\tau} \min\{x_0 I, q(B_t^0 - C_{t_n})\}, \tag{6.10}$$

where $\kappa = \frac{1}{q} - \gamma$. The total extra spread equals the sum of the individual orders

$$D_t^{\omega} = \gamma \sum_{t_n \leq t} x_{t_n} + \kappa \sum_{t_n \leq t} e^{-\rho\tau} \min\{x_0 I, q(C_{t_n} - B_t^0)\}.$$

We still use superscript ω to denote price impact of the reselling after the auction, also when we calculate revenues of the outcome of the auction. We will denote by π^{ω} the revenue impact of the *outcome* of the auction. So using formula (3.4) for π, we find

$$\begin{aligned} \pi_t(x_t)^{\omega} &= \int_{D_t^{\omega}}^{D_{t+}^{\omega}} (B_t^0 - x) q dx \\ &= B_t^0 x_t - \int_{D_t^{\omega}}^{D_{t+}^{\omega}} xq dx \\ &= B_t^0 x_t - \frac{q}{2}((D_{t+}^{\omega})^2 - (D_t^{\omega})^2), \end{aligned}$$

and for the revenues of the outcome of the auction $\mathcal{C}^{\omega}$ we find

$$\mathcal{C}^{\omega}(\xi) = \mathbb{E}[\sum_{n=0}^{N} \pi_{t_n}^{\omega}(\xi_n)].$$

6.5.2 The clearing price

In Section 6.5 we assumed the clearing price was given. However, using the results from Section 5, we are able to say something about the value of the clearing price. Suppose demand schedules are discrete. More specifically, suppose there are M bidders and $T^i < \infty$ for all $i \in \{1, \dots, N\}$. Suppose furthermore that $\bar{K} > \frac{x_t}{M-1}$ for all t. In that case Theorem 5.2.6 holds, so the clearing price equals ν almost surely. But that means $C_{t+} = \mathbb{E}(B_{t+}|\mathcal{F}_t) = B_t$. So there will be no expected market distortion caused by the outcome of the auction. However, note that this *only* holds when the maximum auction size satisfies $\bar{K} > \frac{x_t}{M-1}$. This means market impact is

- non-increasing in maximum bid size in a single auction
- non-increasing in number of bidders in a single auction
- non-decreasing in the volume of a single auction

We now suppose there is a *tick size* of $h > 0$ and a *quantity multiple* (also sometimes referred to as *lot size*) of $w > 0$. We assume Assumptions 5.2.7 and we can apply Theorem 5.2.8. To see what this means, we take a look at a small example.

Example 6.5.1. Suppose we are auctioning 4 million EUA's using the uniform price sealed bid auction where ties are resolved proportionally. Suppose the number of bidders in the auction $M = 20$, the quantity multiple $w = 500$ and the tick size $h = 0.01$. In this case we see

$$\max\{1, \frac{4.000.000}{10.000} \leq t^* \leq \frac{4.000.000}{10.000} + 1 \Rightarrow 400 \leq 401 \Rightarrow t^* \approx 400.$$

So in this case, equilibrium underpricing $t_0 h \leq t^* h = 4$. So the clearing price will not be more than 4 euro's lower than the expected secondary market price.

In general, when x_0 is large compared to $(M-1)w$, the bound t^* converges to $t^* \to \frac{x_0}{(M-1)w}$. That means for large x_0, $B_t^0 - C_{t_n} = t_0 h \leq t^* h = \frac{x_0 h}{(M-1)w}$.
We will use this fact in our market impact model. In determining the optimal selling strategy and costs of this strategy, we would like to know the deviation of the clearing price from the secondary market price, $t_0 h$, in advance. However, we do not know this in general, we only have a certain bound on this deviation. If we plug this into (6.10), we find for the price impact of a single auction at time $t + \tau$ that it is bounded from above by

$$\gamma x_t + \kappa e^{-\rho\tau} \min\{x_0 I, q\frac{x_0 h}{(M-1)w}\} = \gamma x_t + \kappa e^{-\rho\tau} x_0 \min\{I, q\frac{h}{(M-1)w}\}.$$

The optimal selling strategy is independent of the constants γ and κ and we can take the constant $\min\{I, q\frac{h}{(M-1)w}\}$ into κ. So using $\tilde{\kappa} = \kappa \min\{I, q\frac{h}{(M-1)w}\}$, the solution to the optimal selling strategy in terms of ξ^* can simply be found by applying Corollary 3.1.6. However, for determining the actual costs of this strategy we cannot be certain, we can only give a bound. The total extra spread is bounded as follows:

$$D_t^\omega \leq \gamma \sum_{t_n \leq t} x_{t_n} + \kappa \sum_{t_n \leq t} e^{-\rho\tau} \min\{x_0 I, x_0 q\frac{h}{(M-1)w}\}.$$

The upper bound $\bar{D}_t$ of this equals

$$D_t^\omega \leq \bar{D}_t^{\ \omega} = \gamma \sum_{t_n \leq t} x_{t_n} + \kappa \sum_{t_n \leq t} e^{-\rho\tau} x_0 I,$$

if $I < \frac{qh}{(M-1)w}$ and

$$D_t^\omega \leq \bar{D}_t^{\ \omega} = \gamma \sum_{t_n \leq t} x_{t_n} + \kappa \sum_{t_n \leq t} e^{-\rho\tau} x_0 q(\frac{h}{(M-1)w}),$$

if $I \geq \frac{qh}{(M-1)w}$.
We will denote by $\bar{\pi}^\omega$ the lower bound on the revenue of reselling after the auction. So using (3.4) we find

$$\bar{\pi}_t(x_t)^\omega = B_t^0 x_t - \frac{q}{2}((\bar{D}_{t+}^\omega)^2 - (\bar{D}_t^\omega)^2) \tag{6.11}$$

and for the lower bound of the revenues of the reselling after the auction $\bar{\mathfrak{C}}^\omega$, we find

$$\bar{\mathfrak{C}}^\omega(\xi) = \mathbb{E}[\sum_{n=0}^{N} \bar{\pi}_{t_n}^\omega(\xi_n)].$$

6.5.3 Combining volume impact and impact of the outcome of the auction

As in the mixed scenario, also in the strategic scenario we allow for price impact of the *volume* of the auction, which was in fact already calculated in the mixed scenario. Adding these two gives us the total price impact of the auction. We find that the lower bound of the *total revenues* (for the seller in the auction and the traders selling right after the auction) of one auction $\bar{\pi}^{auc}$ can be calculated by combining (3.4) and (6.11) to get

$$\bar{\pi}^{auc}(x_t) = B_t^0 x_t - \frac{q}{2}((D_{t+}^B)^2 - (D_t^B)^2) - \frac{q}{2}((\bar{D}_{t+}^{\omega})^2 - (\bar{D}_t^{\omega})^2),$$

where $D_{t_{n+1}} = e^{-\rho\tau} D_{t_n}$ and $D_{t_n+} = F^{-1}(\frac{x_0}{b+s})$ for $n \in \{1, \dots N-1\}$, and (6.4) for $n = 0$ and (6.5) for $n = N$ hold. For the lower bound on the total revenues $\bar{\mathcal{C}}^{auc}$ we find

$$\bar{\mathcal{C}}^{auc}(\xi) = \mathbb{E}[\sum_{n=0}^{N} \bar{\pi}_{t_n}^{auc}(\xi_n).$$

6.5.4 Results of building the bridge

We can use these results on lower bounds of total revenues from volume and outcome of the auction, to calculate an upper bound on total market distortion *costs* of the auctions.

We calculated market distortion caused by the volume of the auction, which may detract volume from the LOB, and a possible distortion by the outcome of the auction, which may encourage traders to resell their items directly after the auction.

For the volume impact, we saw that a high value of b and a high value of s decreases equilibrium underpricing and thus increases revenue to the seller. On the other hand, a low value of b decreases market impact of the volume in the auction.

For the impact of the outcome of the auction, the smaller the underpricing, the lower the market impact. This would mean that b should be as high as possible to decrease market impact.

Because these two observations are contradictory, it is very interesting to use our bridge to connect both views and see what happens. When we use the model from the strategic mixed scenario for upper bounds on market distortion and we make the calculations in R, we get the graph in Figure 6.9.

In this figure, we see that either a high value for b or a high value for s helps to minimize market distortion. When one of these two values is low, the market distortion can be very high. When both values are high, the distortion is minimized.

A very interesting application lies in comparing regular selling with auctioning. This can easily be done when we use the observation that market distortion in the clean scenario is equivalent to market distortion of regular selling. We plotted the results of this comparison in Figure 6.9. In this figure, we see that *only* when the auctions are aligned well with the secondary market (when b and s are high), auctions are certainly less distorting than selling using market orders. When additionally the number of bidders in the auction is very high, the distortion of auctions can be diminished even further. In this figure, note that the bars for auctions refer to *upper bounds* to distortion. So they can be interpreted as worst case scenarios for auctioning.

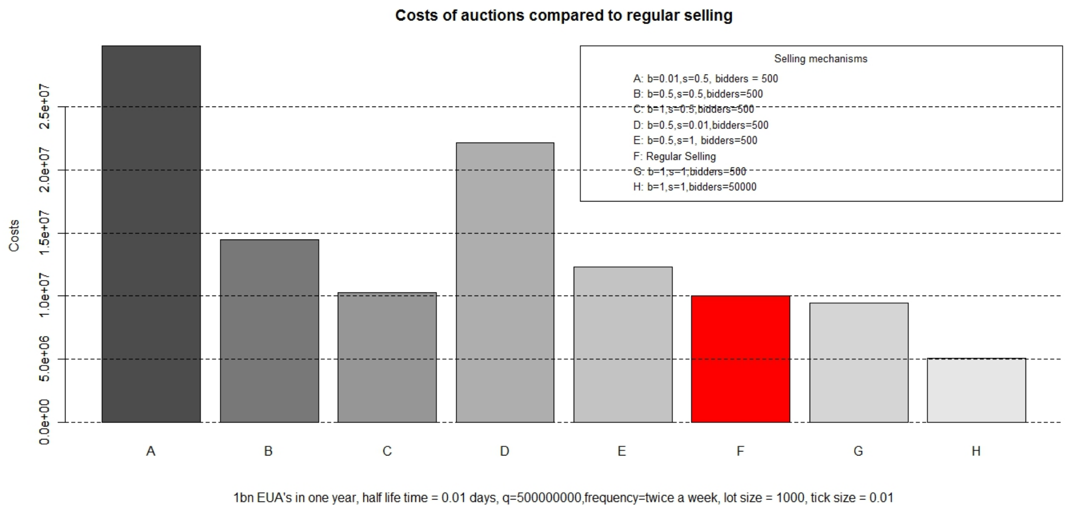


Figure 6.9: Only when the auction design fits well with the secondary market and the number of bidders is sufficiently large, auctions are certainly less distorting than regular sales

This result can also be understood from an intuitive point of view. The more integration between the secondary market and the auctions, the smaller the distortion of the secondary market by the auctions. Furthermore, when we choose a sufficiently small tick size and attract enough bidders to the auction to cover the auction size, this will result in a clearing price very close to the secondary market price. Thereby the price impact of the outcome of the auction can be minimized.
However, when the design of the auction is such that there remains a possibility for a low price equilibrium, the possible distortion of auctions can be even higher than a regular sell market order. We can model this in R and do some sensitivity analyses on the importance of choosing an appropriate tick size and quantity multiple. As can be seen in Figures 6.10 and 6.11, this depends highly on the thickness of the Limit Order Book and the number of bidders in the auction. Only when the order book is not very thick and the number of bidders is very high, choosing a high quantity multiple and low tick size has a positive effect on minimizing market distortion. When the thickness of the Order Book is high and the number of bidders large, market impact by auctions decreased significantly. In the figures, the impact of the volume and the outcome of the auction are plotted separately. The total impact is just the sum of the two. It should be noted that these are *upper bounds* on the impact.

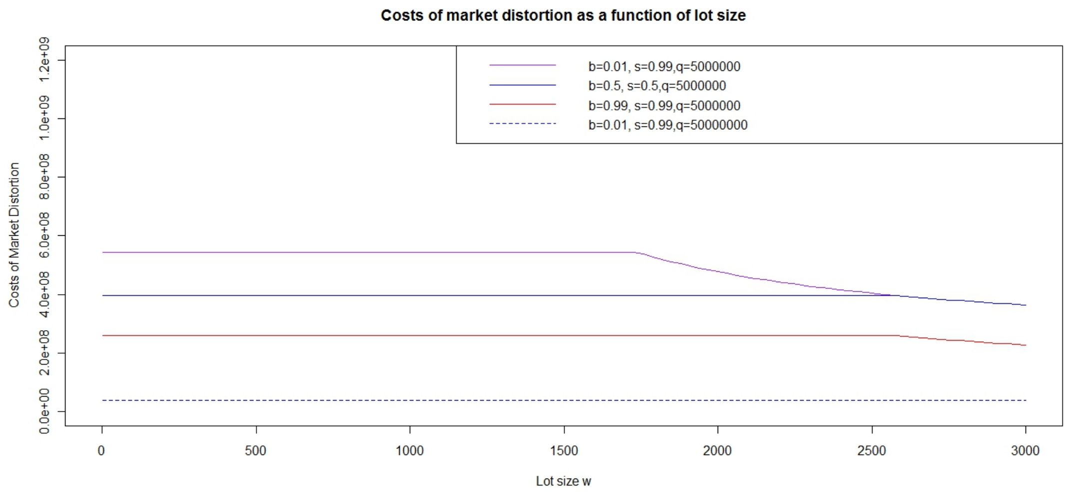


Figure 6.10: Choosing a lot size

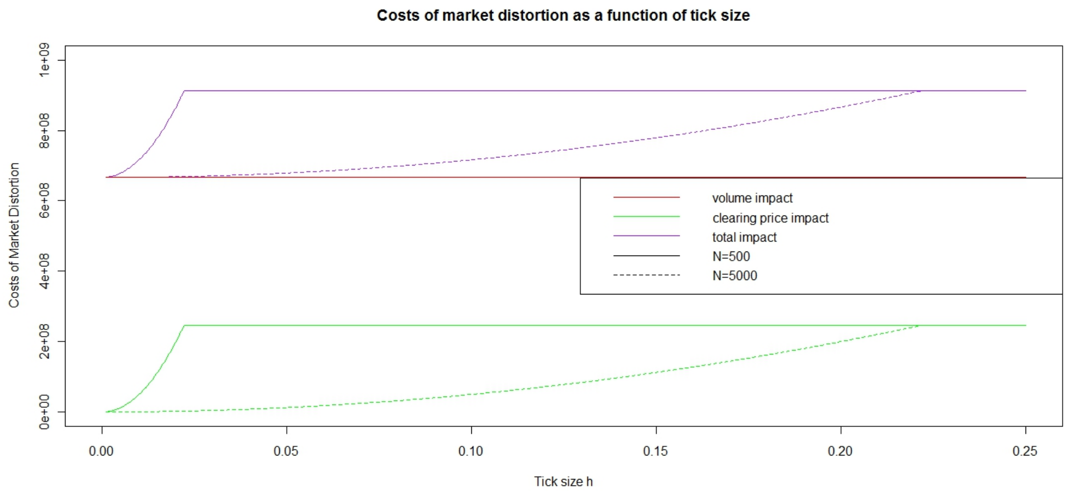


Figure 6.11: Choosing a tick size

6.5.5 Interpretation and remarks on these results

- In the strategic mixed scenario, we allowed for a second type of market impact caused by the *outcome* of the auction. We assumed that if the clearing price is lower than the price paid on the secondary market, a proportion of the players in the auction will take the chance to make a cash profit by selling their EUA's immediately after the auction on the secondary market. In fact, this means that after the auction, a large sell volume may 'eat' the bid orders from the Limit Order Book. This impact is affected by the clearing price in the auction. A low clearing price means that the market impact caused by the outcome of the auction might be high.

- The choice for a tick size and quantity multiple have an effect on the bound for equilibrium underpricing. However, they can only diminish market impact when the number of bidders is sufficiently high and/or the thickness of the order book is sufficiently low. See Figures 6.10

and 6.11. So while tick size and quantity multiple can be used to move the clearing price closer to the best bid price in the Limit Order Book, this only has an effect when threshold values are exceeded. The threshold values depend on the number of bidders in the auction, the auction size and the thickness of the Limit Order Book.

- When the number of bidders in the auction is low and the thickness of the order book is high, using a tick size and quantity multiple to diminish market impact of the outcome of the auction only makes sense in the extreme case (which means a very low tick size and very high quantity multiple must be taken).
- However, choosing a very high quantity multiple might deter small bidders from the auction, so this may have an adverse effect on the number of bidders in the auction.
- When the clearing price is low, the revenue to the seller is low as well. The revenue to the seller is thus increasing as b, the fraction of bids removed from the order book to the auction, gets higher.
- The market distortion of auctions is increasing when b approaches zero, as we see in Figure 6.9. So when the number of bidders who take their bids from the Limit Order Book to the auction is low, the market distortion will be higher. However, this effect can be diminished by attracting extra bidders to the auction which were not active in the Limit Order Book. When the number of other bidders is high relative to the number of bidders from the Limit Order Book, market distortion will be low. Also revenue to the seller becomes higher when this happens.
- To increase revenues and minimize market distortion it is thus crucial to make the auctions as attractive as possible, for bidders active on the secondary market and other bidders.
- Market distortion will be minimized when the overlap between Limit Order Book and auction bid book is maximal, while extra bids are added to the auction bid book as well. Furthermore, choosing a sufficiently small tick size can make the risk of equilibrium underpricing low. If this is combined with frequent auctions, the market impact is minimized.
- Even when the number of bidders is low, when resilience is quick enough it is still optimal in terms of minimizing market distortion and revenue maximization to have daily trades.
- Market impact of large auction is highly dependent on the thickness of the limit order book. The higher the thickness, the lower the impact.

Chapter 7

Conclusions and recommendations

We split up our conclusions in two parts: mathematical conclusions and policy recommendations.

7.1 Mathematical conclusions

Optimal selling strategy in market impact models

- It is possible to derive a closed-form solution for finding an optimal selling strategy. We derived a solution to the optimization problem which allows for restrictions on the strategy and irregular intra-trading times, Theorem 3.1.4. We assumed exponential resilience of the Limit Order Book.
- When we assume intra-trading times are equidistant and the Limit Order Book is block-shaped, we can derive a very simple formula for calculating the optimal selling strategy in terms of division of volume over trades, Corollary 3.1.6.
- The characteristics of the optimal selling strategy in terms of division of volumes over trades depend highly on the resilience rate ρ of the Limit Order Book. This rate depends on the volatility of the price and traded volumes in the market. In determining an optimal selling strategy, it is crucial to determine the resilience rate by empirical research.
- When recovery is slow, it can be an optimal selling strategy to trade higher volumes in the beginning and at the end of a trading period. We call this the bath-tub effect.
- When recovery of the Limit Order Book is quick (exponential half time of a few minutes), market distortion decreases with frequency of trades.

Auction design

- In auction design, an optimal mechanism (maximizing revenue to the seller), is not the same as an efficient mechanism. The Vickrey-Clark-Groves mechanism allocates efficiently and is optimal among all efficient mechanisms. However, this mechanism is generally not applicable in the real world when multiple units are auctioned.
- The uniform price auction for selling multiple identical units is not always efficient when bidders are asymmetric, because bidders may adopt a strategy of demand reduction. Reselling after the auction does not necessarily lead to efficient allocation after the auction either, because players in the market do not have access to complete information (because some bidders may not bid according to their true values).
- The revenue equivalence theorem can be extended to the multi-unit case. However, the theorem holds under the condition that allocation is the same among auction formats. This is generally not the case for discriminatory auctions and uniform price auctions. That means

expected revenue in a uniform price auction is not necessarily equal to expected revenue in a discriminatory price auction.

- In a uniform price auction for EUA's with a common value of the item in the auction ν, bidders behaving risk-neutrally, no reservation price and all bidders behaving symmetrically,equilibrium underpricing will be zero under conditions. These conditions are that the tick size must be infinitely small, bidders are only allowed to place a finite number of bids and removing one bidder still leaves enough potential demand to cover the auction ($\bar{K} > \frac{x_0}{N-1}$) (where x_0 is the auction size, $\bar{K}$ the maximum bid size and N the number of bidders). So when the common value of an EUA in the auction is the expected price of an EUA on the secondary market when the auction ends, under these conditions the auction clearing price will be exactly equal to the expected value for the secondary market price when the auction ends. Market distortion will be minimal.
- When there is a tick size greater than zero, and the auction size is large, equilibrium underpricing will be bounded by $\frac{x_0 h}{(N-1)w}$, where h is the tick size and w the lot size (quantity multiple).

Putting it to work

- We constructed an integrated model, based on auction theory and market impact models, which allows us to find an optimal selling strategy, when a uniform price sealed bid auction is used as a selling mechanism, such that market distortion is minimized.
- In this model, we allowed for two types of market impact in the Limit Order Book caused by large auctions. The first impact is caused by the volume of the auction. A large volume sold in the auction may (temporarily) detract orders from the Limit Order Book. We introduced two parameters into our model to influence the measure of this impact. The first is the fraction of bids which is removed from the LOB and transferred to the auction. The second is the ratio between 'new bids' and bids from the Limit Order Book in the auction bid book.
- In terms of market impact, while the clearing price may be low, if the fraction of bids transferred from the LOB to the auction is low and the number of extra bidders in the auction is high, the market impact caused by the *volume* of the auction will be low. So a low clearing price does not necessarily mean the market impact caused by the volume in the auction is high.
- In our model we allow for a second type of market impact caused by the *outcome* of the auction. We assume that if the clearing price is lower than the price paid on the secondary market, a proportion of the players in the auction will take the chance to make a cash profit by selling their EUA's immediately after the auction on the secondary market. In fact, this means that after the auction, a large sell volume may 'eat' the bid orders from the Limit Order Book. This impact is affected by the clearing price in the auction. A low clearing price means that the market impact caused by the outcome of the auction might be high.
- The choice for a tick size and quantity multiple have an effect on the bound for equilibrium underpricing. However, they can only diminish market impact when the number of bidders is sufficiently high and/or the thickness of the order book is sufficiently low. So while tick size and quantity multiple can be used to move the clearing price closer to the best bid price in the Limit Order Book, this only has an effect when threshold values are exceeded. The threshold values depend on the number of bidders in the auction, the auction size and the thickness of the Limit Order Book.
- When the clearing price is low, the revenue to the seller is low as well.
- Market distortion will be minimized when the overlap between Limit Order Book and auction bid book is maximal, while extra bids are added to the auction bid book as well. Furthermore,

choosing a sufficiently small tick size can make the risk of equilibrium underpricing low. If this is combined with frequent auctions, the market impact is minimized.

- To increase revenues and minimize market distortion it is thus crucial to make the auctions as attractive as possible, for bidders active on the secondary market and other bidders.

7.2 Policy Recommendations

From the mathematical conclusions, we can derive policy recommendations. To make sure the auctions minimally distort the secondary market price, it is crucial to attract enough bidders to participate in the auction and to make sure that the auction mechanism is an integrated part of the existing infrastructure for the secondary market. Furthermore, details in the auction design can be crucial and the optimal design and calendar will depend highly on the specific properties of the market in the third trading period. Therefore, flexibility in adjusting the auction calendar and specific properties of auction design such as tick size, quantity multiple and maximum bid size is desirable. We derive the following list of recommendations.

The objectives of auctioning

1. *Revenue maximization should not be stated as an explicit objective in auctions for EUA's.* This will make bidders reluctant to participate in the auctions, because they will fear having to pay an auction clearing price higher than the secondary market price.

2. *Policy makers should be careful in stating auctions are used to ensure items are allocated efficiently.* In general, a uniform price auction does not allocate efficiently. Efficient allocation means that the bidders who value the items the most, will win the items. However, because bidders may shade their true values in the auction, efficient allocation is not certain.

Auction design

1. *Instead of using 0,01 euro as a tick size, 0,001 euro or an even smaller should be used.* A small tick size can make equilibrium underpricing arbitrarily low. By making the tick size small, bidders are encouraged to compete over the price.

2. *The quantity multiple should be made as small as possible to attract small bidders to participate in the auction as well.* In theory, a large quantity multiple can be used to encourage bidders to bid more aggressively. However, in the case of auctioning EUA's, in our model only with a very large quantity multiple this effect is significant. But using this method is risky, because it might scare small bidders away from the auction and thereby decrease the number of bidders.

3. *It is crucial to attract bidders to participate in the auction.* When there are enough bidders to cover the auction size, they will compete over the price. The higher the number of bidders, the smaller the difference between the clearing price and the secondary market price (equilibrium underpricing) will be.

Auction platform

It is crucial to ensure the auction platform is aligned well with the existing infrastructure on the secondary market. This, together with a smart design, assures auctions will be less distorting to the secondary market than regular sell orders would be.

1. *There should not be any additional administrative burdens for participating in the auctions for players already active on a trading platform on the secondary market.* This makes it very easy for traders on the secondary market to also trade in the auction. This will reduce the distortion of the secondary market by the auctions.

2. *The fees to participate in the auction should be equal to or lower than fees on the secondary market.*

3. *It should be easy for small players in the market to participate in the auction as well, for example by allowing for trading via primary participants.*

4. *The infrastructure for auctions should be integrated as much as possible with the secondary market infrastructure.* This can be done by making sure the auctions are carried out by the existing trading platforms on the secondary market.

5. *It should be possible to make adjustments in the requirements and infrastructure after 2013 if necessary.* Small decisions in the design and organization of the auctions can have very large implications in terms of distortion and functioning of the market. It would be wise to make it possible to learn from the functioning of the auction platform after it started.

Auction calendar

1. *There should be a possibility for flexibility in the auction calendar by making it possible to adjust the auction calendar on a year-by-year basis.* The optimal calendar highly depends on the functioning of the market itself, which is hard to estimate beforehand. When the market is functioning very well and liquidity and recovery speed are high, it is probably best to have daily auctions. This way the volumes in the auctions can be made very small and market distortion through the auctions will be minimized. However, when the market is not competitive enough, daily auctions may be very distorting because the market does not have enough time to recover from previous auctions and because the number of bidders in the auction may be too low.

2. *As a start, auction frequency should be twice a week and market distortion by the auctions should be evaluated during the first year.* In general, the smaller the volume in the auction, the smaller the possible distortion of the secondary market by the auction. By auctioning twice a week, the volumes in the auctions are not too big while there is still plenty of time for the market to recover from a single auction before the next one starts.

Selling large amounts of shares

1. An optimal selling strategy in terms of division of volumes over these pieces can be calculated using market impact models. When the recovery of the Limit Order Book is exponential (which is generally the case) and the Limit Order Book is block-shaped, it is optimal to sell large volumes in the beginning and at the end of the trading period. Intermediate trades are small, so that they minimize additional price impact of trading. This is called the bath-tub effect. By choosing a smart selling strategy, trading costs for a large volume can be diminished significantly, depending on the thickness and the recovery rate of the order book.

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Appendix A

Previous technical studies

In October 2007, a research group from the University of Virginia, Resources for the Future and the California Institute of Technology performed an experimental analysis to the auctioning of 100% of greenhouse gas allowances under the 10 northeastern american states that comprise the Regional Greenhouse Gas Initiative [14].
In July 2009, an auctioning working group set up by the French Government and chaired by Mr Jean-Michel Carpin, Inspecteur général des finances in which all stakeholders were represented has tried to tackle the theoretical and technical issues in auction design [12].
During the summer of 2009, the European Commission itself held a consultation among Member States, emitters, carbon commodity traders, financial institutions and market operators on the technical aspects of an effective and efficient auction design [15].

The main theoretical and technical questions and answers can be divided into three sections, regaring auction calendar, auction design and organization of the auctions. We will briefly explore the main conclusions of the three studies on these issues.

1. **What and when to auction?**

   EU Consultation
   There is a debate going on about the question whether or not to start *early auctions* before the third trading period actually starts, to give firms the opportunity to hedge their risks. Companies are in general in favor of early auctions, whereas Member States are against, because they fear an oversupply of allowances which would depress the market price [15, p. 5].
   There is also widespread difference in opinion about whether there should be auctions of futures or only of spots. About the *frequency* of the auctions answers from the respondents differ widely. A high frequency of auctions would minimize potential disturbances on the secondary market price, whereas a lot of small auctions increases costs for all participants and involves the risk of very few participants, which can have a negative impact on the clearing price. Opinions on the frequency of the auctions differ from continuous auctions to daily, weekly, monthly or quarterly auctions. Most respondents were in favor of a weekly schedule. In any case, the calendar should be communicated well in advance.
   In the EU consultation, most of the respondents suggested a minimum bid size[1] of 500t.

   RGGI experimental study
   The optimal frequency of the auctions would be quarterly. Furthermore, there should also be auctioning of futures and there should be separate auctions for each maturity date.
   The minimum bid size should be 1000t.

   Report France

[1]A minimum bid size is set to reduce administrative costs and bidding costs. However, a maximum bid size which is relatively high will reduce participation of small emitters.

If the auctions are going to be implemented EU-wide, according to the French they should be held weekly. The minimum bid size should be 1000t.

2. **Auction Design**

   EU Consultation
   Mainly for the sake of simplicity, the main part of the respondents was in favour of a single-round sealed bid auction with uniform pricing. A reserve price[2] would not be necessary according to the majority of the respondents, because a robust design would be enough to assure that the clearing price would not deviate too much from the secondary market price. Most stakeholders are against a maximum bid size[3]. They advocate that a high frequency of the auctions will minimize the risk of a possible squeeze.[4] Tied bids should be resolved at random.

   RGGI experimental study
   Also the RGGI advocates a uniform price, single round, sealed bid auction, in which tied bids are resolved randomly. In the design of the RGGI, a maximum bid size of 33% of the amount of allowances sold is proposed to protect the market against a possible squeeze. Also a reserve price is set, because the possibility of collusion and weak competition among asymmetric bidders can possibly be very high [14, p. 56].

   Report France The report of the French research group advocates the same mechanism design as the other two reports. They do advocate the possibility of setting a reserve price on the basis of the secondary market price and a possibility to introduce a maximum bid size only when necessary.

3. **How will the auction(s) be implemented and who auctions?**

   EU Consultation
   The main part of the respondents was in favor of a harmonized registration and collateral process,[5] which is consistent with a central approach to the European auctions. Some Member States are against this approach because they fear that this is very difficult to implement safely. The risks are very high and it's complicated to harmonize all these procedures. They favor a hybrid model, in which Member States can auction the allowances, corresponding to a uniform European framework and calendar. There was little support for the primary participants model, in which only appointed institutions such as banks can perform bids. Auctions should be open to all qualified bidders. Moreover, a large majority of the stakeholders in the consultation was in favor of the use of an existing exchange platform for organizing the auctions. Furthermore, there should be uniform rules about information disclosure.

   RGGI experimental study and Report France
   Mainly the same conclusions are derived in these studies as in the EU consultation. The reason for this is understandable: the more transparent, easy and uniform the auction design, the greater the number of expected participants in the auction and the higher the expected efficiency and the expected return to the seller.

[2] A reserve price is an auction price below which the seller chooses to retain ownership of the item rather than sell it. A reserve price can protect the seller against a possible loss when the clearing price is below initial costs for the seller [14, p. 55].

[3] A maximum bid size is the maximum number of items the participants in the auction can bid for. A maximum bid size can be introduced to protect the market against a squeeze.

[4] A squeeze can occur in multiple unit auctions on a large scale, when a lot of players take a forward position on the units which are sold in the auction. During the auction they have two options: either buy the items on the auction or on the secondary market. The risk of waiting to buy after the auction until they have to buy on the secondary market, lies in the possibility that a few number of powerful participants buy so much, that they can highly influent the secondary market price. They can raise the price when the short players have to buy: this way they cause a short squeeze. This effect is also called the *loser's nightmare*.

[5] Collateral is a pledge of the bidder of specific property to the auctioneer, which protects the auctioneer against the risk of default of the bidder.

Appendix B

R source codes

Figure 6.1

```
#CALCULATE STRATEGY VECTOR XI
A<-50
strat.fn<-function(Z_0,A,T,half){
N<-A-1
tau<-T/N
rho<--log(1/2)/half
a<-exp(-rho*tau)
xi_0<-Z_0/((N-1)*(1-a)+2)
xi_n<-xi_0*(1-a)
xi<-c(xi_0,rep(xi_n,N-1),xi_0)
xi
}

# AND PLOT THIS STRATEGY
x<-seq(1,A, by=1)
plot(x,strat.fn(Z_0,A,T,half))
# IN EEN OVERZICHTJE
par(mfrow=c(3,3))
plot(strat.fn(Z_0,365,T,0.01),type="h",lwd=6,col="red",main="Daily selling, quick
    resilience", xlab="Time in days", ylab="Volume of orders")
plot(strat.fn(Z_0,365,T,0.1),type="h",lwd=6,col="red", main="Daily selling,
    moderate resilience", xlab="Time in days", ylab="Volume of orders")
plot(strat.fn(Z_0,365,T,0.5),type="h",lwd=6,col="red", main="Daily selling, slow
    resilience", xlab="Time in days", ylab="Volume of orders")

plot(strat.fn(Z_0,100,T,0.01),type="h",lwd=6,col="red",main="Twice weekly selling,
    quick resilience", xlab="Time in days", ylab="Volume of orders")
plot(strat.fn(Z_0,100,T,0.1),type="h",lwd=6,col="red", main="Twice weekly selling,
    moderate resilience", xlab="Time in days", ylab="Volume of orders")
plot(strat.fn(Z_0,100,T,0.5),type="h",lwd=6,col="red", main="Twice weekly selling,
    slow resilience", xlab="Time in days", ylab="Volume of orders")

plot(strat.fn(Z_0,50,T,0.01),type="h",lwd=6,col="blue", ,main="Weekly selling,
    quick resilience", xlab="Time in days", ylab="Volume of orders")
plot(strat.fn(Z_0,50,T,0.1),type="h",lwd=6,col="blue", main="Weekly selling,
    moderate resilience", xlab="Time in days", ylab="Volume of orders")
plot(strat.fn(Z_0,50,T,0.5),type="h",lwd=6,col="blue", main="Weekly selling, slow
    resilience", xlab="Time in days", ylab="Volume of orders")
```

Figure 6.3

```
# SENSITIVITY ANALYSIS: Allow for A=365, A=100, A=50 and A=10.
# We suppose only temporary market impact.
# First we determine the optimal trading strategy. With this strategy we can
    calculate costs.
```

```
# We use a simple model, assuming constant resilience, equidistant intratrading
    times and a block-shaped LOB.
# xi_0 is the size of the first and last order:
xi_0<-Z_0/((N-1)*(1-a)+2)
# All intermediate orders are xi_n
xi_n<-xi_0*(1-a)
# Input variables are:
# Z_0 = total volume traded in one period
# A= number of trades in one period.
# T= length of trading period
N=A-1
# a is a measure for the dynamics of the LOB, it describes the exponential recovery
    of volumes/spread (which is equivalent because we assume block-shape of LOB).
a<-exp(-rho*tau)
# Input variables are:
tau<-T/N
# rho = resilience speed factor
# To make this more intuitive, we can also ask for a half-life time as input
    variable and calculate a from this, instead of rho:
rho<--log(1/2)/half
# In this case the input variable is half
# half = half-life time for resilience (time it takes f0or the order book to
    recover half of the volume of the Order Book.)
# Now calculate the quadratic form, we use the matrix formula. First we set the
    matrix:
y<-numeric(N+1)
y<-rep(a,(N+1)*(N+1))
d<-diag(N+1)
z<-matrix(y,N+1,N+1)
M<-z+d-d*a
# Now we construct the vector with the optimal trading strategies:
xi<-c(xi_0,rep(xi_n,N-1),xi_0)
# And we calculate the quadratic form:
C<-0.5*xi%*%M%*%xi
# Now to calculate the costs we only have to divide by q.
# q = depth of order book
Ctemp<-C/q

# SET INPUT VARIABLES
par(mfrow=c(1,5))
Z_0<-1000000000
A<-50
T<-365
half<-0.02
q<-50000

# Now we define a function which describes this:
cost.fn<-function(Z_0,A,T,half,q){
N<-A-1
tau<-T/N
rho<--log(1/2)/half
a<-exp(-rho*tau)
xi_0<-Z_0/((N-1)*(1-a)+2)
xi_n<-xi_0*(1-a)

y<-numeric(N+1)
y<-rep(a,(N+1)*(N+1))
d<-diag(N+1)
z<-matrix(y,N+1,N+1)
M<-z+d-d*a
xi<-c(xi_0,rep(xi_n,N-1),xi_0)
C<-0.5*xi%*%M%*%xi
Ctemp<-C/q
Ctemp[1,1]
}
```

```
par(mfrow=c(1,1))
h<-seq(0,1, by=0.02)
c.f<-numeric(length(h))

cost.half.day<-numeric(length(h))
cost.half.day<-function(h){
for(i in 1:length(h)){
c.f[i]<- cost.fn(1000000000,365,365,h[i],5000000)}
c.f}

cost.half.twice<-numeric(length(h))
cost.half.twice<-function(h){
for(i in 1:length(h)){
c.f[i]<- cost.fn(1000000000,100,365,h[i],5000000)}
c.f}
plot(h,cost.half.twice(h), main="twice a week")

cost.half.month<-numeric(length(h))
cost.half.month<-function(h){
for(i in 1:length(h)){
c.f[i]<- cost.fn(1000000000,10,365,h[i],5000000)}
c.f}
plot(h,cost.half.month(h), main="monthly")


cost.half.week<-numeric(length(h))
cost.half.week<-function(h){
for(i in 1:length(h)){
c.f[i]<- cost.fn(1000000000,50,365,h[i],5000000)}
c.f}
plot(h,cost.half.day(h), lty=0, main="Sensitivity of frequencies to resilience")
lines(h, cost.half.week(h),lty=1)
lines(h, cost.half.twice(h),lty=3)
lines(h, cost.half.month(h),lty=5)
lines(h,cost.half.day(h),lty=0.5)
```

Figure 6.4

```
par(mfrow=c(1,1))
xrange<-range(0,365)
yrange<-range(2000000000,50000000000)
plot(xrange,yrange,type="n",xlab="Frequency(days)", ylab="Costs",main="Market
    Distortion as a function of frequency of trades")

A<-seq(2,T,by=1)
c.f<-numeric(T-1)
cost.freq<-numeric(T-1)
cost.freq<-function(A){
for(i in 1:364){
c.f[i]<- cost.fn(1000000000,A[i],365,0.001,500000)}
c.f}
lines(A,cost.freq(A), lty=1, lwd=2, col="red")

A<-seq(2,T,by=1)
c.f<-numeric(T-1)
cost.freq<-numeric(T-1)
cost.freq<-function(A){
for(i in 1:364){
c.f[i]<- cost.fn(1000000000,A[i],365,0.1,500000)}
c.f}
lines(A,cost.freq(A), lty=2, lwd=2, col="blue")


A<-seq(2,T,by=1)
c.f<-numeric(T-1)
cost.freq<-numeric(T-1)
cost.freq<-function(A){
```

```
for(i in 1:364){
c.f[i]<-cost.fn(1000000000,A[i],365,0.3,500000)}
c.f}
lines(A,cost.freq(A),lty=3, lwd=2, col="purple")

A<-seq(2,T,by=1)
c.f<-numeric(T-1)
cost.freq<-numeric(T-1)
cost.freq<-function(A){
for(i in 1:364){
c.f[i]<-cost.fn(1000000000,A[i],365,0.8,500000)}
c.f}
lines(A,cost.freq(A),lty=4, lwd=2, col="green")
legend(300,yrange[2],c("0.02","0.1","0.3", "0.8"), cex=0.8, col=c("red","blue", "
    purple", "green"), lty=1:4,lwd=2, title="Resilience")
```

Figure 6.7

```
# AUCTION CLEARING PRICES

b<-seq(0.01,1,length=100)
s<-seq(0.01,1,length=100)
u<-matrix(numeric(10000),100)
Extr.Spr<-matrix(numeric(10000),100)

q<-100000000
A<-50
Extr.Spr<-function(b,s){
for(i in 1:100){
for(j in 1:100){
u[i,j]<-xi_0/((b[i]+s[j])*q)
}
}
u
}

par(mfrow=c(1,1))

persp(b,s,Extr.Spr(b,s),col="lightblue",
theta=-30, phi=30,
r=50,
d=0.1,
expand=0.5,
ltheta=90, lphi=180,
shade=0.01,
ticktype="detailed",
nticks=5,
main="Equilibrium underpricing in the mixed scenario"
)
```

Figure 6.9

```
# WE USE COST.FN AND COMPUTE THE COSTS OF AUCTIONING COMPARED TO REGULAR SELLING
Z_0<-1000000000
M<-seq(1,50000, length=100)
half<-0.01
q<-500000000
T<-365
A<-100
w<-1000
b<-0.9
s<-0.6
h<-0.01

# A FORMULA TO CALCULATE THE COSTS OF AUCTIONS
c.sb.auc <-numeric(100)
cost.sb.auc<-numeric(100)
cost.sb.auc<-function(b,s,M,A){
```

```
for(i in 1:100){
c.sb.auc[i]<-cost.fn(Z_0*multi(b,s,q,h,M[i],w),A,T,half,q)+cost.fn(Z_0,A,T,half,q*(
    b+s))}
c.sb.auc}

# AND A FORMULA TO CALCULATE THE COSTS OF THE MIXED SCENARIO. WHEN WE CHOOSE B=1
    AND S=0 UNDER THE MIXED SCENARIO, THE AUCTION REDUCES TO THE REGULAR SELLING
    SITUATION.
c.sb.mixed <-numeric(100)
cost.sb.mixed<-numeric(100)
cost.sb.mixed<-function(b,s,M,A){
for(i in 1:100){
c.sb.mixed[i]<-cost.fn(Z_0,A,T,half,q*(b+s))}
c.sb.mixed}

#WE USE A VECTOR WITH ALL THE VALUES AND TAKE IT ALL TOGETHER IN A BARPLOT.
y<-c(cost.sb.auc(0.01,0.5,M,100)[2],cost.sb.auc(0.5,0.5,M,100)[2],cost.sb.auc
    (1,0.5,M,100)[2],cost.sb.auc(0.5,0.01,M,100)[2],cost.sb.auc(0.5,1,M,100)[2],
    cost.sb.mixed(1,0,M,100)[2],cost.sb.auc(1,1,M,100)[2],cost.sb.auc(1,1,M,100)
    [100])
l<-c(1,2,3,4,5,6,7,8)
jaja<-as.table(y,l)
col2=c(gray.colors(7)[1],gray.colors(7)[2],gray.colors(7)[3],gray.colors(7)[4],
    gray.colors(7)[5],"red",gray.colors(7)[6],gray.colors(7)[7])
barplot(jaja,main="Costs of auctions compared to regular selling", sub="1bn EUA's
    in one year, half life time = 0.01 days, q=500000000,frequency=twice a week,
    lot size = 1000, tick size = 0.01",col=col2, ylab="Costs")
axis(1,labels=c(1,2,3,4,5,6,7,8), col.axis="black", padj=1)
cost.sb.mixed(1,0,M,100)
legend("topright", cex=0.6, c("A: b=0.01,s=0.5, bidders = 500", "B: b=0.5,s=0.5,
    bidders=500", "C: b=1,s=0.5,bidders=500", "D: b=0.5,s=0.01,bidders=500", "E: b
    =0.5,s=1, bidders=500", "F: Regular Selling","G: b=1,s=1,bidders=500","H: b=1,s
    =1,bidders=50000"), title= "Selling mechanisms")
lines(cost.sb.mixed(1,0,M,100)[2], col="black")
```

Figure 6.11

```
# WE USED THE FOLLOWING PRESETTINGS
Z_0<-1000000000
M<-30
half<-0.01
q<-50000000
T<-365
A<-50
w<-2000
b<-0.9
s<-0.6

# AND DEFINE FUNCTIONS FOR CALCULATING THE COSTS OF VOLUME, OUTCOME AND TOTAL
    AUCTIONS.
h<-seq(0.001,0.25,length=1000)
cost.sb<-function(b,s,h,M){
for(i in 1:1000){
c.sb[i]<-cost.fn(Z_0*multi(b,s,q,h[i],M,w),A,T,half,q)}
c.sb}

c.sb.auc <-numeric(1000)
cost.sb.auc<-numeric(1000)
cost.sb.auc<-function(b,s,h,M){
for(i in 1:1000){
c.sb.auc[i]<-cost.fn(Z_0*multi(b,s,q,h[i],M,w),A,T,half,q)+cost.fn(Z_0,A,T,half,q*(
    b+s))}
c.sb.auc}

c.sb.mixed <-numeric(1000)
cost.sb.mixed<-numeric(1000)
cost.sb.mixed<-function(b,s,h,M){
for(i in 1:1000){
```

```
c.sb.mixed[i]<-cost.fn(Z_0,A,T,half,q*(b+s))}
c.sb.mixed}

# PLOT: COSTS OF MARKET DISTORTION AS A FUCTION OF TICK SIZE
par(mfrow=c(1,1))
xrange<-range(0,0.25)
yrange<-range(0,1000000000)
plot(xrange,yrange,type="n",ylab="Costs of Market Distortion", xlab="Tick size h",
    main="Costs of market distortion as a function of tick size")
lines(h,cost.sb.auc(0.1,0.2,h,500), col="purple")
lines(h, cost.sb.mixed(0.1,0.2,h,500), col="red")
lines(h,cost.sb(0.1,0.2,h,500), col= "green")

lines(h,cost.sb.auc(0.1,0.2,h,5000), col="purple", lty=2)
lines(h, cost.sb.mixed(0.1,0.2,h,5000), col="red", lty=2)
lines(h,cost.sb(0.1,0.2,h,5000), col= "green", lty=2)
legend("topright", cex=0.6, c("volume impact","clearing price impact","total impact
    ","N=500","N=5000"), col=c("red", "green", "purple", "black", "black"), lty=c
    (1,1,1,1,2))

par(mfrow=c(1,1))
xrange<-range(0,0.25)
yrange<-range(0,1000000000)
plot(xrange,yrange,type="n",ylab="Sensitivity of costs to h", xlab="Tick size h",
    main="Costs of market distortion as a function of tick size")
```

Figure 6.10

```
# WE DO THE SAME FOR LOT SIZE
Z_0<-1000000000
M<-30
half<-0.01
q<-5000000000
T<-365
A<-365
h<-0.01


w<-seq(1,3000,length=100)
cost.sb<-function(b,s,w,q){
for(i in 1:100){
c.sb[i]<-cost.fn(Z_0*multi(b,s,q,h,M,w[i]),A,T,half,q)}
c.sb}

c.sb.auc <-numeric(100)
cost.sb.auc<-numeric(100)
cost.sb.auc<-function(b,s,w,q){
for(i in 1:100){
c.sb.auc[i]<-cost.fn(Z_0*multi(b,s,q,h,M,w[i]),A,T,half,q)+cost.fn(Z_0,A,T,half,q*(
    b+s))}
c.sb.auc}

c.sb.mixed <-numeric(100)
cost.sb.mixed<-numeric(100)
cost.sb.mixed<-function(b,s,w,q){
for(i in 1:100){
c.sb.mixed[i]<-cost.fn(Z_0,A,T,half,q*(b+s))}
c.sb.mixed}

# PLOTS: COSTS OF MARKET DISTORTION AS A FUCTION OF LOT SIZE
par(mfrow=c(1,1))
xrange<-range(0,3000)
yrange<-range(0,1200000000)
plot(xrange,yrange,type="n",ylab="Costs of Market Distortion", xlab="Lot size w",
    main="Costs of market distortion as a function of lot size")
lines(w,cost.sb.auc(0.01,0.99,w,5000000), col="purple")
lines(w,cost.sb.auc(0.5,0.5,w,5000000), col="blue")
lines(w,cost.sb.auc(0.999,0.9999,w,5000000), col="red")
```

```
lines(w,cost.sb.auc(0.5,0.5,w,50000000), col="blue", lty=2)
legend("topright", cex=0.6, c("b=0.01, s=0.99,q=5000000","b=0.5, s=0.5,q=5000000","
    b=0.99, s=0.99,q=5000000","b=0.01, s=0.99,q=50000000"), col=c("purple", "blue",
     "red", "blue"), lty=c(1,1,1,2))
```