

Asian options and stochastic volatility

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Abstract

In modern asset price models, stochastic volatility plays a crucial role in order to explain several stylized facts of returns. Recently, [3] introduced a class of stochastic volatility models (so called BNS SV model) based on superposition of Ornstein-Uhlenbeck processes driven by subordinators. The BNS SV model forms a flexible class, where one can easily explain heavy-tails and skewness in returns and the typical time-dependency structures seen in asset return data. In this thesis the effect of stochastic volatility on Asian options is studied. This is done by simulation studies of comparable models, one with and one without stochastic volatility.

Introduction

Lévy processes are popular models for stock price behavior since they allow to take into account jump risk and reproduce the implied volatility smile. Barndorff-Nielsen [3] introduces a class of stochastic volatility models (BNS SV model) based on superposition of Ornstein-Uhlenbeck processes driven by subordinators (Lévy processes with only positive jumps and no drift). The distribution of these subordinators will be chosen such that the log-returns of asset prices will be distributed approximately normal inverse Gaussian (NIG) in stationarity. This family of distributions has proven to fit the semi-heavy tails observed in financial time series of various kinds extremely well (see Rydberg [17], or Eberlein and Keller [9]).

In the comparison of the BNS SV model, we will use an alternative model NIG Lévy process model (LP model) which has NIG distributed log-returns of asset prices, with the same parameters as in the BNS SV case. Unlike the BNS SV model, this model doesn't have stochastic volatility and time-dependency of asset return data. Both models are described and provided with theoretical background. Moreover difference in pricing Asian option with the two different models will be studied.

Unlike the Black-Scholes model, closed option pricing formulae are not available in exponential Lévy models and one must use either deterministic numerical methods (see Carr [8] for the LP model and Benth [7] for the BNS SV model) or Monte Carlo methods. In this thesis we will restrict ourselves to Monte Carlo methods.

As described in Benth [6] the best way of simulating a NIG Lévy process is by a quasi-Monte Carlo method. We will use a simpler Monte-Carlo method, which needs bigger samplesize to reduce the error. Simulating from the BNS SV model involves simulating of an Inverse Gaussian Ornstein-Uhlenbeck (IG-OU) process. The oldest algorithm of simulating a IG-OU process is described in Barndorff [3]. This is a quiet bothersome algorithm, since it includes a numerical inversion of the Lévy measure of the Background driving Lévy process (BDLP). Therefore it has a large processing time, hence we will not deal with this algorithm.

The most popular algorithm is a series representation by Rosinski [16] . The special case of the IG-OU process is described in Barndorff [5]. Recently Zhang & Zhang [21] introduced an exact simulation method of an IG-OU process, using the rejection method. We will compare these last two algorithms.

Acknowledgments

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Chapter 1

Theory:

In this chapter the necessary theory for understanding the Barndorff-Nielsen Shephard stochastic volatility model will be given. First some basic notations and definitions are introduced and then some theory about Levy processes and Ornstein-Uhlenbeck processes is outlined.

1.1 Basic notations and definitions

We assume as given a filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ is a *filtration*. By a filtration we mean a family of σ -algebras $(\mathcal{F}_t)_{0 \leq t \leq \infty}$ that is increasing, i.e. $\mathcal{F}_s \subset \mathcal{F}_t$, if $s \leq t$. For convenience we will write \mathbb{F} for the filtration $(\mathcal{F}_t)_{0 \leq t \leq \infty}$.

Definition 1 (Usual condition) A filtered complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is said to satisfy the usual conditions if,

- (i) \mathcal{F}_0 contains all the P -null sets of \mathcal{F}_i
- (ii) $\mathcal{F}_t = \bigcap_{u \leq t} \mathcal{F}_u$, all t , $0 \leq t < \infty$; i.e. the filtration is right continuous.

A random variable X is a mapping from a sample space Ω into \mathbb{R} .

Two random variables X and Y are equal in distribution if they have the same distribution functions:

$$\mathbb{P}(X \leq x) = \mathbb{P}(Y \leq x) \quad \text{for all } x.$$

This property is notated as $X \stackrel{D}{=} Y$.

A stochastic process X on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a collection of \mathbb{R} -valued or \mathbb{R}^d -valued random variables $\{X(t)\}_{t \geq 0}$. The process X is said to be adapted if $X(t) \in \mathcal{F}_t$ (i.e. $X(t)$ is \mathcal{F}_t measurable) for each t .

A process X is said to be of finite variation on the interval $[a, b]$ if,

$$\sup_{P \in \mathcal{P}} \sum_{i=0}^{n_p-1} |X(t_{i+1}) - X(t_i)| < \infty$$

where the supremum runs over the set $\mathcal{P} = \{P = \{t_0, \dots, t_{n_p}\} | P \text{ is a partition of } [a, b]\}$ of all partitions of the given interval.

The function $t \rightarrow X(t, \omega)$ mapping $[0, \infty)$ into \mathbb{R} are called *sample paths* of the stochastic process X . A stochastic process X is said to be *càdlàg* if it a.s. has sample paths which are right continuous, with left limits and a stochastic process X is said to be *càglàd* if it a.s. has sample paths which are left continuous, with right limits.

With ϕ_X we will denote the characteristic function of the random variable X .

$$\phi_X(\zeta) = \mathbb{E} [e^{i\zeta x}]$$

The cumulant generating function will be denoted by κ_X ,

$$\kappa_X(\theta) = \mathbb{E} [e^{\theta x}] \quad \bar{\kappa}_X = \mathbb{E} [e^{-\theta x}]$$

Definition 2 (Martingale) A real-valued, adapted process X is called a martingale with respect to the filtration \mathbb{F} if,

- (i) $X(t) \in L^1(dP)$; i.e. $\mathbb{E}[|X(t)|] < \infty$.
- (ii) for $s \leq t$, $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ a.s.

1.2 Lévy processes

In literature Lévy processes were introduced as *processes with independent increments*. In the 20th century the name 'Lévy process' became popular in honor of the french mathematician Paul Lévy, who did extensive research on these processes.

Suppose we have a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying the usual conditions then,

Definition 3 (Lévy process) An adapted process $X = \{X(t)\}_{t \geq 0}$ with $X(0) = 0$ a.s. is a Lévy process if

- (i) X has increments independent of the past; that is, $X(t) - X(s)$ is independent of \mathcal{F}_s , $0 \leq s < t < \infty$.
- (ii) X has stationary increments; that is, $X(t) - X(s)$ has the same distribution as $X(t - s)$. $0 \leq s < t < \infty$.
- (iii) X_t is continuous in probability; that is $\lim_{t \rightarrow s} X(t) = X(s)$, where the limit is taken in probability.

The most common known examples of Lévy processes are the Poisson process and Brownian motion (also named Wiener process).

An advantage of Lévy processes is that they are *infinitely divisible*, which makes them flexible for modeling and simulating. Infinitely divisible means that for each positive integer n , $\phi(\zeta)$ is also the n -th power of a characteristic function. An other way of expressing this; for an infinitely divisible random variable X and every positive integer n , there exist n independent identically distributed random variables X_1, \dots, X_n whose sum is equal in distribution to X .

Theorem 1 Suppose $X = \{X(t)\}_{t \geq 0}$ is a Lévy process then it holds that,

$$\log \{ \phi_{X(t)}(\zeta) \} = t \log \{ \phi_{X(1)}(\zeta) \} \tag{1.1}$$

Proof:

$$\begin{aligned} \log \{ \phi_{X(t)}(\zeta) \} &= \log \left\{ \mathbb{E} \left[e^{i\zeta x(t)} \right] \right\} \\ &= \log \left\{ \mathbb{E} \left[e^{i\zeta \sum_{i=1}^t (x(i) - x(i-1))} \right] \right\} \\ &\stackrel{indep}{=} \log \left\{ \prod_{i=1}^t \mathbb{E} \left[e^{i\zeta (x(i) - x(i-1))} \right] \right\} \\ &= \sum_{i=1}^t \log \left\{ \mathbb{E} \left[e^{i\zeta (x(i) - x(i-1))} \right] \right\} \\ &\stackrel{stat}{=} \sum_{i=1}^t \log \left\{ \mathbb{E} \left[e^{i\zeta x(1)} \right] \right\} \\ &= t \log \{ \phi_{X(1)} \} \end{aligned}$$

□

The following theorem is a special case of the well known 'Lévy-Kintchine representation'. It is stated here without proof.

Theorem 2 (Lévy-Kintchine representation) *Let X be a infinitely divisible random variable, then*

$$\psi(\zeta) := \log \{\phi_X(\zeta)\} = -\frac{\eta}{2}\zeta^2 + i\xi\zeta + \int (e^{i\zeta x} - 1 - i\zeta\tau(x))U(dx), \quad (1.2)$$

where $\eta \in \mathbb{R}_+$; $\xi \in \mathbb{R}$; τ is a bounded Borel function which "behaves like x " near the origin and U is a measure with no atom at 0 satisfying

$$\int_{\mathbb{R}} \min\{1, x^2\}U(dx) < \infty \quad (1.3)$$

Moreover the representation of $\psi(\zeta)$ by ξ, η and U is unique. So conversely, if $\xi \in \mathbb{R}$, $\eta \in \mathbb{R}_+$ and U a measure satisfying (1.3) then there exists an infinitely divisible random variable with characteristic function given by (1.2).

Proof: See Sato [18] □

Note: If Y is a Lévy process that the above theorem also holds for $Y(1)$.

The function τ makes sure that the integrand of the integral on the right hand side is integrable with respect to U . Standard is choosing τ equal to $1_{|x| \leq 1}$. However there are many other ways of getting an integrable integrand. For instance bounded measurable functions τ satisfying,

$$\begin{cases} \tau(x) = 1 + o(|x|) & \text{as } |x| \rightarrow 0 \\ \tau(x) = O(1/|x|) & \text{as } |x| \rightarrow \infty. \end{cases}$$

lead to an integrand that is integrable. We will choose

$$\tau(x) = \begin{cases} x & \text{for } |x| \leq 1 \\ \frac{x}{|x|} & \text{for } |x| > 1. \end{cases}$$

(ξ, η, U) is referred to as the *characteristic triplet* of the Lévy process.

Theorem 3 *Suppose $X = \{X(t)\}_{t \geq 0}$ is a Lévy process with characteristic triplet (ξ, η, U) , then there exists a decomposition such that,*

$$X(t) \stackrel{\mathcal{D}}{=} \xi t + \sqrt{\eta}w(t) + X_0(t)$$

Where $w(t)$ and $X_0(t)$ are independent Lévy processes, $w(t)$ a brownian motion and $X_0(t)$ is such that $\log \{\phi_{X_0(1)}(\zeta)\} = \int (e^{i\zeta x} - 1 - i\zeta\tau(x))U(dx)$.

proof: By checking the conditions it is not hard to see that $\xi t + \sqrt{\eta}w(t) + X_0(t)$ is a Lévy process. Now by theorem 1,

$$\begin{aligned} \log \{\phi_{\xi t + \sqrt{\eta}w(t) + X_0(t)}\} &= t \log \{\phi_{\xi + \sqrt{\eta}w(1) + X_0(1)}\} \\ &\stackrel{\text{indep.}}{=} t \log \{\phi_{\xi} \cdot \phi_{\sqrt{\eta}w(1)} \cdot \phi_{X_0(1)}\} \\ &= t (\log(\phi_{\xi}) + \log(\phi_{\sqrt{\eta}w(1)}) + \log(\phi_{X_0(1)})). \end{aligned}$$

Since $w(t)$ is a brownian motion, $w(1)$ has a standard normal distribution, hence

$$\phi_{\sqrt{\eta}w(1)}(\zeta) = e^{-\frac{\eta}{2}\zeta^2}.$$

So we may conclude that,

$$\begin{aligned} \log \{\phi_{\xi t + \sqrt{\eta}w(t) + X_0(t)}(\zeta)\} &= t \left(i\xi\zeta - \frac{\eta}{2}\zeta^2 + \int (e^{i\zeta x} - 1 - i\zeta\tau(x))U(dx) \right) \\ &= t \log \phi_{X(1)}(\zeta) \stackrel{(1.1)}{=} \log \phi_{X(t)}(\zeta). \end{aligned}$$

Now the result follows by the uniqueness of the characteristic function. \square

Due to this decomposition we can divide a Lévy process into three different contributions; a drift part ξt , an infinite variation part $\sqrt{\eta}w(t)$ and a jump part $X_0(t)$. The term ξt gives a linear line which the random variable should follow. Since $w(t)$ has infinite variation η corresponds to an infinite variation part. However $\eta = 0$ does not imply that the process is of finite variation, since the measure U can still cause infinite activity. But if $\eta = 0$ and $\int_{-1}^1 |x|U(dx) < \infty$ then it follows from standard Lévy process theory that the process is of finite variation. In this case the triplets ξ and U are invariant of the choice of τ . Moreover we don't have problems with integrability of the integrand on the right hand side of (1.2) hence we can choose τ equal to the zero function. This implies we can get a representation of the form,

$$\psi(\zeta) := \log \{\phi_X(\zeta)\} = i\xi\zeta + \int (e^{i\zeta x} - 1)U(dx) \quad (1.4)$$

The Lévy measure $U(dx)$ dictates how the jumps occur. Jumps in the set A occur according to a Poisson process with intensity $\int_A U(dx)$.

Suppose X is a Lévy process with characteristic triplet (ξ, η, U) , then if $\eta = 0$, X is a *Lévy jump process*, if also $\xi = 0$, X is a *Lévy pure jump process* and if X only has positive increments and no drift then it is a *subordinator*.

1.3 Economical terminology

In this section we will explain some standard terminology used in economics.

1.3.1 Complete market

Whenever the number of different ways to obtain payoffs equals the number of probabilistic states, we can attain any payoff. In such a circumstance, financial economists say there is a complete market. According to the second fundamental theorem of financial economics, risk-neutral probabilities are unique if and only if the market is complete.

In practice the market is rarely complete. For instance brokers like to work with round numbers, while the probability distribution is continuous.

1.3.2 Arbitrage

The essence of arbitrage is that with no initial capital it should not be possible to make a profit without exposure to risk. Were it possible to do so, arbitrageurs would do so, in unlimited quantity. They would use the market as money-pump to extract arbitrarily large quantities of riskless profit. This would make it impossible for the market to be in equilibrium.

1.3.3 Volatility

An important feature which is missing in the Black-Sholes model and in the LP model is volatility modeling, or more generally, the environment is changing stochastically over time.

It has been observed that the estimated parameters of uncertainty (volatilities) change stochastically over time. One can see this by looking at historical volatility. This is a measure which reflects how volatile the asset has been in the recent past.

Out of practice there is evidence for volatility clusters. There seems to be a succession of periods with high return variance and with low variance. In practice this means that large price variations are more likely to be followed by large price variations. This observation motivate the introduction of a model where the volatility itself is stochastic.

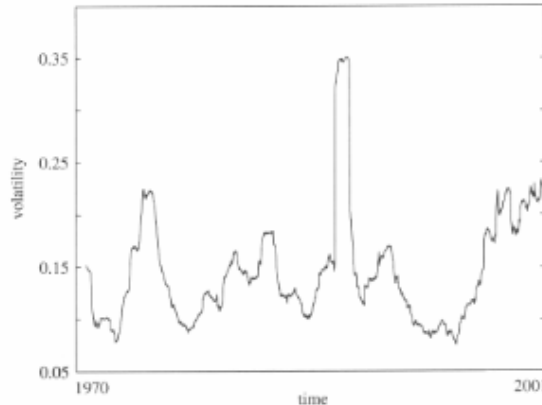


Figure 1.1: Historical volatility of the S&P 500 index

1.4 Ornstein-Uhlenbeck processes

An Ornstein-Uhlenbeck (OU) processes $x(t)$ is defined as the solution of a stochastic differential equation of the form

$$dx(t) = -\lambda x(t)dt + dz(t)$$

where $z(t)$ is a Lévy process. The rate λ is arbitrary. As z is used to drive the OU process, we shall call $z(t)$ a background driving Lévy process (BDLP). If the BDLP is a subordinator, then the process $x(t)$ is positive if its initial value is positive i.e. $x(0) > 0$ and $x(t)$ is bounded from below by the deterministic function $x(0)e^{-\lambda t}$.

We will use an unusual timing in $z(\lambda t)$ such that for an arbitrary $\lambda > 0$ the marginal distribution will be unchanged. This makes it possible to parameterize the volatility and the dynamic structure separately. So we consider OU processes of the form,

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dz(\lambda t), \quad \lambda > 0 \quad (1.5)$$

where $z(\lambda t)$ is a Lévy process and $\lambda > 0$ arbitrary.

Note: We will only consider Lévy processes such that the stochastic integral

$$f \bullet z := \int_{\mathbb{R}_+} f(t)dz(t)$$

is an ordinary stochastic integral.

1.4.1 Solution and existence

In this section we will describe a general form of solutions to a stochastic differential equation of the form (1.5).

Definition 4 (self-decomposable) A probability measure P on \mathbb{R} is self-decomposable or belongs to Lévy's class L , if for each $t \geq 0$, there exists a probability measure Q_t on \mathbb{R} such that,

$$\phi(\zeta) = \phi(e^{-t}\zeta)\phi_t(\zeta),$$

where ϕ and ϕ_t denote the characteristic functions of P and Q_t , respectively. A random variable x with law in L is also called self-decomposable.

From this definition it can be seen that every self-decomposable random variable is also infinitely divisible. A further important characterization of the class L as a subclass of the set of all infinitely divisible distributions in terms of the Lévy measure is the following equivalence.

- (1) P is self-decomposable.
- (2) The functions on \mathbb{R}_+ given by $U((-\infty, -e^s])$ and $U([e^s, \infty))$ are both convex.
- (3) $U(dx)$ is of the form $U(dx) = u(x)dx$ with $|x|u(x)$ increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

If u is differentiable, then (2) may be re-expressed as

$$u(x) + xu'(x) \leq 0 \quad \text{for } x \neq 0 \quad (1.6)$$

The equivalence of (1), (2) and (3) is due to Lévy. A proof may be found in Sato [18].

We can write self-decomposability in terms of random variables. Suppose σ^2 is self-decomposable then there exists an $Y(t)$ independent of σ^2 such that,

$$\sigma^2 \stackrel{\mathcal{D}}{=} e^{-t}\sigma^2 + Y(t) \quad (1.7)$$

Or similarly for each $\lambda > 0$

$$\sigma^2 \stackrel{\mathcal{D}}{=} e^{-\lambda t}\sigma^2 + Y(\lambda t) \quad (1.8)$$

Lemma 1 *If z is a stochastic process with càdlàg sample paths then*

$$Y(t) = \int_0^t e^{-s} dz(s) \quad (1.9)$$

is also a stochastic process with càdlàg sample paths. Moreover, if z has independent increments then so has Y .

Proof: Clearly the sample paths of Y are càdlàg. Out of ordinary integration theory we have that

$$\int_0^t de^{-s}z(s),$$

can be approximated arbitrarily close by Riemann-Stieltjes sums of the form,

$$\sum_{j=1}^n (e^{-s_j} - e^{-s_{j-1}}) z(s_{j-1})$$

where $0 = s_0 < \dots < s_n = t$ is a partition. Now by stochastic integration by parts we have that,

$$\begin{aligned} \int_0^t e^{-s} dz(s) &= e^{-t}z(t) - e^0z(0) - \int_0^t de^{-s}z(s) \\ &\approx \sum_{j=1}^n (e^{-s_j} - e^{-s_{j-1}}) z(s_{j-1}) \\ &= \sum_{j=1}^n e^{-s_j} (z(s_j) - z(s_{j-1})) \end{aligned} \quad (1.10)$$

Hence $Y(t)$ is measurable for all $t \geq 0$ and therefore it is a stochastic process. Moreover the last equation also shows that Y has independent increments whenever z has. \square

Lemma 1 can be extended to account for negative axis of the process $z(s)$ by taking an independent copy of z but modified such that it is càglàd. Call this process \bar{z} . Take for all $s > 0$, $z(-s) = \bar{z}(s)$. Then with a similar proof on $[-t, 0)$ we can conclude that,

$$Y(s) = \int_{-t}^0 e^{-s} dz(s)$$

is a stochastic process. Moreover, if z has independent increments then so has Y .

In Jurek and Mason [11] has even been proved that the integral representation (1.9) converges in distribution as $t \rightarrow \infty$ if and only if $\mathbb{E}[\max\{0, \log(|z(1)|)\}] < \infty$. More precise,

$$\int_0^\infty e^{-s} dz(s) \quad (1.11)$$

exists if and only if

$$\mathbb{E}[\max\{0, \log(|z(1)|)\}] < \infty. \quad (1.12)$$

From now on we assume that this condition is satisfied.

Theorem 4 σ^2 is a self-decomposable random variable if and only if there exists a càdlàg random variable z such that z has stationary independent increments, $z(0) = 0$ a.s., and

$$\int_0^\infty e^{-s} dz(s) \stackrel{\mathcal{D}}{=} \sigma^2,$$

in which case the random variable $Y(t)$ in (1.7) satisfies,

$$Y(t) \stackrel{\mathcal{D}}{=} \int_0^t e^{-(t-s)} dz(s) \quad \text{for each } t > 0 \quad (1.13)$$

Proof: Let $\sigma^2 \stackrel{\mathcal{D}}{=} \int_0^\infty e^{-s} dz(s)$, this is just (1.9) rewritten into a different form. For each $t > 0$ it holds

$$\begin{aligned} \sigma^2 &\stackrel{\mathcal{D}}{=} \int_t^\infty e^{-s} dz(s) + \int_0^t e^{-s} dz(s) \\ &\stackrel{\mathcal{D}}{=} e^{-t} \int_0^\infty e^{-s} dz((s+t)) + \int_0^t e^{-s} dz(s). \end{aligned}$$

The last two terms are independent since increments of z are independent. Moreover from stationarity and independence we have that

$$\int_0^\infty e^{-s} dz((s+t)) \stackrel{\mathcal{D}}{=} \int_0^\infty e^{-s} dz(s) \stackrel{\mathcal{D}}{=} \sigma^2$$

and

$$\int_0^t e^{-s} dz(s) = \int_0^t e^{-(t-s)} dz((t-s)) \stackrel{\mathcal{D}}{=} \int_0^t e^{-(t-s)} dz(s).$$

Thus, $\sigma^2 \stackrel{\mathcal{D}}{=} e^{-t}\sigma^2 + Y(t)$ with σ^2 and $Y(t)$ independent and $Y(t)$ as in (1.13). Hence σ^2 is self-decomposable and $Y(t)$ satisfies (1.13).

Conversely, assume σ^2 is self-decomposable then we claim that there exists a stochastic process $\{X(t)\}_{t \geq 0}$ with independent increments such that $X_0 = 0$ a.s. and, for $t, u \geq 0$

$$X(t+u) - X(t) \stackrel{\mathcal{D}}{=} e^{-t}\sigma^2, \quad (1.14)$$

so that in particular,

$$X(t) \stackrel{\mathcal{D}}{=} Y(t) \quad (1.15)$$

To proof this it is suffices to show that if (1.14) and (1.15) hold for two particular values, say t and u in \mathbb{R}_+ and $X(t)$ is independent of $X(t+u) - X(t)$, then (1.15) holds for $t+u$.

From (1.7) it follows that,

$$\begin{aligned} e^{-(t+u)}\sigma^2 + Y(t+u) &\stackrel{\mathcal{D}}{=} \sigma^2 \stackrel{\mathcal{D}}{=} e^{-t}(e^{-u}\sigma^2 + Y(u)) + Y(t) \\ &= e^{-t+u}\sigma^2 + e^{-t}Y(u) + Y(t) \end{aligned}$$

with each $Y(t+u)$, $Y(u)$ and $Y(t)$ independent of σ^2 . Now we have that,

$$Y(t+u) \stackrel{\mathcal{D}}{=} e^{-t}Y(u) + Y(t)$$

and hence,

$$X(t+u) = (X(t+u) - X(t)) + X(t) \stackrel{\mathcal{D}}{=} e^{-t}Y(u) + Y(t) \stackrel{\mathcal{D}}{=} Y(t+u)$$

Hence our claim is proved now. Let X be a càdlàg version of the process $\{X(t)\}_{t \geq 0}$ with independent increments. Now, set

$$z(t) := \int_0^t e^s dX(s) \quad \text{for } t > 0$$

From Lemma 1 we know that z is a stochastic process with càdlàg sample paths and independent increments. Moreover z has stationary increments. To see this, note that,

$$X(t+\cdot) - X(t) \stackrel{\mathcal{D}}{=} e^{-t}X(\cdot)$$

since both sides have independent increments and the same marginal distribution. Consequently, for fixed $t, u \geq 0$ we get

$$\begin{aligned} z(t+u) - z(t) &= \int_t^{t+u} e^s dX(s) = \int_0^u e^s e^t dX(s+t) \\ &\stackrel{\mathcal{D}}{=} \int_0^u e^s dX(s) = z(u) \end{aligned}$$

Finally by partial integration and the definition of z ,

$$\int_0^t e^{-s} dz(s) = \int_0^t e^{-s} e^s dX(s) = X(t)$$

Now by (1.15) and (1.7)

$$\int_0^t e^{-(t-s)} dz(s) \stackrel{\mathcal{D}}{=} \int_0^t e^{-s} dz(s) \stackrel{\mathcal{D}}{=} Y(t) \xrightarrow{\mathcal{D}} \sigma^2 \quad \text{as } t \rightarrow \infty$$

□

Theorem 5 *If σ^2 is self-decomposable, then there exists a stationary stochastic process $\{\sigma^2(t)\}_{t \geq 0}$ and a Lévy process $\{z(t)\}_{t \geq 0}$ such that $\sigma^2(t) \stackrel{\mathcal{D}}{=} \sigma^2$ and*

$$\sigma^2(t) = e^{-\lambda t} \sigma^2(0) + \int_0^t e^{-\lambda(t-s)} dz(\lambda s)$$

for all $\lambda > 0$ and $\int_0^t e^{-\lambda(t-s)} dz(\lambda s)$ is independent of $\sigma^2(0)$. Moreover $\sigma^2(t)$ is a solution to SDE (1.5)

Proof: Since σ^2 is self-decomposable there exists a Lévy process $\{z(t)\}_{t \geq 0}$ as in Theorem 4 such that,

$$\sigma^2 \stackrel{\mathcal{D}}{=} \int_0^\infty e^{-s} dz(s)$$

We can extend this Lévy process to the negative half line by taking an independent copy of the process z but modified such that it is càglàd. Call this process \bar{z} . Now take for all $t > 0$, $z(-t) = \bar{z}(t)$. Moreover we have that $\mathbb{E}[\max\{0, \log(|z(1)|)\}] < \infty$ by assumption (we will come back on this in the next paragraph). Now define,

$$\sigma^2(t) := e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} dz(\lambda s)$$

As described near (1.11) we know that $\sigma^2(t)$ exists. Furthermore $\sigma^2(t)$ is a stationary process, since for each $t \in \mathbb{R}$,

$$\begin{aligned}\sigma^2(t) &= e^{-\lambda t} \int_{-\infty}^t e^{\lambda s} dz(\lambda s) = e^{\lambda t} \int_{-\infty}^{\lambda t} e^s dz(s) \\ &= \int_{-\infty}^0 e^s dz(s + \lambda t) \stackrel{\mathcal{D}}{=} \int_{-\infty}^0 e^s dz(s) \\ &\stackrel{\mathcal{D}}{=} \int_0^{\infty} e^{-\lambda s} dz(\lambda s) \stackrel{\mathcal{D}}{=} \sigma^2\end{aligned}$$

From the above we may also conclude that $\sigma^2(t)$ has the same law as σ^2 . Moreover $\sigma^2(t)$ can be represented as,

$$\begin{aligned}\sigma^2(t) &= e^{-\lambda t} \int_{-\infty}^0 e^{\lambda s} dz(\lambda s) + e^{-\lambda t} \int_0^t e^{\lambda s} dz(\lambda s) \\ &= e^{-\lambda t} \sigma^2(0) + \int_0^t e^{-\lambda(t-s)} dz(\lambda s)\end{aligned}\tag{1.16}$$

and is therefore a solution to the stochastic differential equation,

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dz(\lambda t)$$

Also by the independent increments of z we have that $\int_0^t e^{-\lambda(t-s)} dz(\lambda s)$ is independent of $\sigma^2(0)$. \square

The integral of the right hand side of (1.16) is the most difficult to compute in a simulation. We will focus on this in chapter 5.

1.4.2 Relation characteristics

In this section we will derive relationships between the characteristic triplet of the OU-process with the characteristic triplet of the BDLP process.

Theorem 6 (key formula) *Suppose X is a Lévy process, then it holds that,*

$$\log \{\phi_{f \bullet z}(\zeta)\} = \int_{\mathbb{R}_+} \log \{\phi_{z(1)}(f(\tau)\zeta)\} d\tau\tag{1.17}$$

Proof:

$$\begin{aligned}\phi_{f \bullet z}(\zeta) &= \mathbb{E} \left[\exp \left(i\zeta \int_{\mathbb{R}_+} f dz \right) \right] = \mathbb{E} \left[\prod_{\tau \in \mathbb{R}_+} e^{i\zeta f(\tau) dz(\tau)} \right] \\ &= \prod_{\tau \in \mathbb{R}_+} \mathbb{E} \left[e^{i\zeta f(\tau) dz(\tau)} \right] = \prod_{\tau \in \mathbb{R}_+} e^{\log \{\phi_{dz(\tau)}(f(\tau)\zeta)\}} \\ &= \prod_{\tau \in \mathbb{R}_+} e^{\log \{\phi_{z(1)}(f(\tau)\zeta)\}} d\tau = \exp \left(\int_{\mathbb{R}_+} \log \{\phi_{z(1)}(f(\tau)\zeta)\} d\tau \right)\end{aligned}$$

\square

$\sigma^2(t)$ is equal in distribution to σ^2 . Hence it is sufficient to get a relation between the characteristic triplet of σ^2 with the characteristic triplet of $z(1)$. Suppose that σ^2 has characteristic triplet (ξ, η, U) , and suppose that $z(1)$'s characteristic triplet is given by (ξ_b, η_b, W) . Recall that σ^2 has an integral representation of the form $\int_0^\infty e^{-t} dz(t)$. Now by the key formula and the Lévy-Kintchin representation

we can conclude that both characteristic triplets are related according to,

$$\eta = \int_{\mathbb{R}_+} \eta_b (e^{-t})^2 dt \quad (1.18)$$

$$U(dx) = \int_{\mathbb{R}_+} W(e^t dx) dt \quad (1.19)$$

$$\xi = \int_{\mathbb{R}_+} e^{-t} (\xi_b + b_W(t)) dt \quad (1.20)$$

where $b_W(t)$ is given by,

$$b_W(t) = \int x \frac{x^2 - e^{-2t}}{(1+x^2)(1+e^{-2t}x^2)} W(dx)$$

This integral exists since z is a Lévy process and hence equation (1.3) holds for Lévy measure W .

If the Levy measures U and W are differentiable and u, w are their corresponding densities, then we can derive from equation (1.19),

$$u(x) = \int_1^\infty w(\tau x) d\tau$$

Now since σ^2 is infinitely divisible distributed we have that his corresponding Lévy measure satisfies (1.3) and hence,

$$\begin{aligned} \infty &> \int_{0+}^\infty \min\{1, x^2\} u(x) dx = \int_1^\infty \int_{0+}^\infty \min\{1, x^2\} w(\tau x) dx d\tau \\ &= \int_1^\infty \int_{0+}^\infty \min\{1, \tau^{-2} y^2\} \tau^{-1} w(y) dy d\tau \\ &= \int_{0+}^1 y^2 w(y) dy \int_1^\infty \tau^{-3} d\tau + \int_1^\infty w(y) \left(\int_1^y \tau^{-1} d\tau + y^2 \int_y^\infty \tau^{-3} d\tau \right) dy \\ &= \frac{1}{2} \int_{0+}^1 y^2 w(dy) + \int_1^\infty \log(y) w(y) dy + \frac{1}{2} \int_1^\infty w(y) dy \end{aligned}$$

Since in this final expression the first and third integrals are finite we can conclude

$$\mathbb{E}[\max\{0, \log(|z(1)|)\}] = \int_1^\infty \log(y) w(y) dy < \infty.$$

So assumption (1.12) was justified as long as σ^2 is infinitely divisible distributed. Moreover this gives us a way to define an OU-process on a infinitely divisible distribution. Suppose σ^2 is an infinitely divisible distributed random variable with characteristic triplet (ξ, η, U) then there exists a stationary OU-process $\sigma^2(t)$ with corresponding SDE (1.5) where z is a Lévy process with characteristic triplet (ξ_b, η_b, W) chosen according to equations (1.20), (1.18) and (1.19).

We can be more specific about the relationship between the characteristic function of the OU-process and his BDLP. With the key formula we can deduce a relationship between the characteristic functions of σ^2 and $z(1)$.

$$\begin{aligned} \log \{ \phi_{\sigma^2(t)}(\zeta) \} &= \log \{ \phi_{\sigma^2}(\zeta) \} = \log \{ \phi_{f \bullet z}(\zeta) \} \\ &= \int_{\mathbb{R}_+} \log \{ \phi_{z(1)}(f(t)\zeta) \} dt \\ &= \int_0^\infty \log \{ \phi_{z(1)}(e^{-t}\zeta) \} dt \\ &= \int_0^\zeta \log \{ \phi_{z(1)}(\xi) \} \xi^{-1} d\xi \end{aligned}$$

and hence,

$$\log \{\phi_{z(1)}(\zeta)\} = \zeta \frac{\partial \log \{\phi_{\sigma^2}(\zeta)\}}{\partial \zeta}$$

The relationship between their measures can be totally spelled out. Let U be the Lévy measure of σ^2 and W the Lévy measure of $z(1)$. Take the following tail mass function,

$$U(x) := U([x, \infty))$$

and similarly W . Now for self-decomposable law U , it holds by equation (1.19),

$$\begin{aligned} U(x) &= \int_0^\infty W(e^t[x, \infty)) dt = \int_1^\infty s^{-1} W([sx, \infty)) ds \\ &= \int_x^\infty s^{-1} W(s) ds \end{aligned}$$

The following important relation can be derived.

$$u(x) = x^{-1} W(x)$$

Hence if moreover the Levy measures U and W are differentiable and u, w are their corresponding densities, it holds that

$$w(x) = -u(x) - xu'(x). \tag{1.21}$$

We can rewrite this as,

$$\begin{aligned} W(x) &= \int_x^\infty w(y) dy = \int_{-\infty}^x -w(y) dy \\ &= \int_{-\infty}^x u(y) + yu'(y) dy \\ &= xu(x) \end{aligned} \tag{1.22}$$

Chapter 2

Examples of processes

2.1 Inverse Gaussian Lévy process

The inverse gaussian (IG) distributions belong to the family of generalized inverse gaussian (GIG) distributions. The name 'inverse gaussian' was first applied to a certain class of distributions by Tweedie in 1947. He noted an inverse relationship between the cumulant generating functions of these distributions and those of Gaussian distributions.

The inverse Gaussian distribution $IG(\delta, \gamma)$ is the distribution on \mathbb{R}_+ given in terms of its density,

$$f_{IG}(x; \delta, \gamma) = \frac{\delta}{\sqrt{2\pi}} \exp(\delta\gamma) x^{-\frac{3}{2}} \exp\left\{-\frac{1}{2}(\delta^2 x^{-1} + \gamma^2 x)\right\}, \quad x > 0$$

where the parameters δ and γ satisfy $\delta > 0$ and $\gamma \geq 0$.

It has characteristic function given by,

$$\phi_{IG}(u; \delta, \gamma) = \exp(-\delta(\sqrt{-2iu + \gamma^2} - \gamma)).$$

From this we can see that $\phi_{IG}(u; \delta, \gamma) = (\phi_{IG}(u; \delta/n, \gamma))^n$, thus the IG distribution is infinitely divisible. We define the IG process $X^{IG} = \{X^{IG}(t)\}_{t \geq 0}$, with parameters $\delta, \gamma > 0$, as the process which starts at zero and has independent and stationary increments such that,

$$\mathbb{E} \left[e^{iuX^{IG}(t)} \right] = \phi_{IG}(u; \delta, \gamma) = e^{at(\sqrt{-2iu + \gamma^2} - \gamma)}$$

Hence by definition the IG-process is a Lévy process.

Since the inverse Gaussian distribution is infinitely divisible, we can specify its Lévy triplet. The Lévy measure of the $IG(\delta, \gamma)$ law is given by,

$$U_{IG}(dx) = \frac{\delta}{\sqrt{2\pi}} x^{-3/2} e^{-1/2\gamma^2 x} \mathbf{1}_{\{x>0\}} dx$$

The IG process doesn't have an infinite variation part hence $\eta = 0$. Moreover the drift parameter is given by,

$$\xi = \frac{\delta}{\gamma} (2\Phi(\gamma) - 1)$$

where Φ is the cumulative distribution function of the standard normal distribution.

Properties: It holds that,

$$\begin{aligned} u(x) + xu'(x) &= \frac{\delta}{\sqrt{2\pi}} x^{-3/2} e^{-1/2\gamma^2 x} \mathbf{1}_{x>0} - \frac{3}{2} \frac{\delta}{\sqrt{2\pi}} x^{-3/2} e^{-1/2\gamma^2 x} \mathbf{1}_{x>0} \\ &= -\frac{1}{2} \frac{\delta}{\sqrt{2\pi}} x^{-3/2} e^{-1/2\gamma^2 x} \mathbf{1}_{x>0} \leq 0 \end{aligned}$$

Hence by (1.6) the IG distribution is self-decomposable.
Since $\eta = 0$ and,

$$\int_{-1}^1 |x|U(dx) = \int_0^1 \frac{\delta}{\sqrt{2\pi}} x^{-1/2} e^{-1/2\gamma^2 x} dx = \frac{\delta}{\gamma} (2\Phi(\gamma) - 1) < \infty$$

the IG distribution is of finite variation.

All positive and negative moments exist. If $X \sim \text{IG}(\delta, \gamma)$, then

$$\mathbb{E}[X^a] = \left(\frac{\gamma}{\delta}\right)^{2a+1} \mathbb{E}[X^{\alpha+1}], \quad a \in \mathbb{R}$$

Hence we have that,

	$\text{IG}(\delta, \gamma)$
mean	$\frac{\delta}{\gamma}$
variance	$\frac{\gamma}{\delta^3}$
skewness	$\frac{3}{\sqrt{\delta\gamma}}$
kurtosis	$3(1 + \frac{5}{\delta\gamma})$

2.2 Normal inverse Gaussian Lévy process

The normal inverse gaussian (NIG) distribution is a member of the family of generalized hyperbolic (GH) distributions. The generalized hyperbolic distributions were introduced by Barndorff-Nielsen in 1977 to model the log size distribution of aeolian sand deposits. Later, it was found that the distribution had applications in turbulence and in financial mathematics. In finance the NIG distribution is often used as a good approximation of the heavy tailed distribution of log-returns (see Barndorff-Nielsen [2], Eberlein and Keller [9]).

The NIG distribution has values on \mathbb{R} , it is defined by its density function,

$$f_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) = \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} - \beta\mu\right) \frac{K_1\left(\delta\alpha\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}} e^{\beta x}$$

where K_1 is the modified Bessel function of the third kind and index 1. Moreover the parameters are such that $\mu \in \mathbb{R}, \delta \in \mathbb{R}_+$ and $0 \leq \beta < |\alpha|$. By calculation one can conclude that the characteristic function is given by

$$\phi_{\text{NIG}}(u; \alpha, \beta, \delta, \mu) = \exp\left(\delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + iu)^2}\right) + \mu iu\right)$$

Again, we can clearly see that this is an infinitely divisible characteristic function. Hence we can define the NIG process $X^{\text{NIG}} = \{X^{\text{NIG}}(t)\}_{t \geq 0}$ with $X^{\text{NIG}}(0) = 0$ and stationary independent NIG distributed increments. i.e. $X^{\text{NIG}}(t)$ has a $\text{NIG}(\alpha, \beta, t\delta, \mu)$ law.

The Lévy triplet of the normal inverse Gaussian distribution is $(\xi, 0, U)$. Where Lévy measure U is given by,

$$U_{\text{NIG}}(dx) = \frac{\delta\alpha}{\pi} \frac{K_1(\alpha|x|)e^{\beta x}}{|x|} dx$$

and

$$\xi = \mu + \frac{2\delta\alpha}{\pi} \int_0^1 \sinh(\beta x) K_1(\alpha x) dx$$

Out of the characteristic function one can extract the cumulant generating function,

$$\kappa(u; \alpha, \beta, \delta, \mu) = \exp\left(\delta\left(\sqrt{\alpha^2 - \beta^2} - \sqrt{\alpha^2 - (\beta + u)^2}\right) + \mu u\right)$$

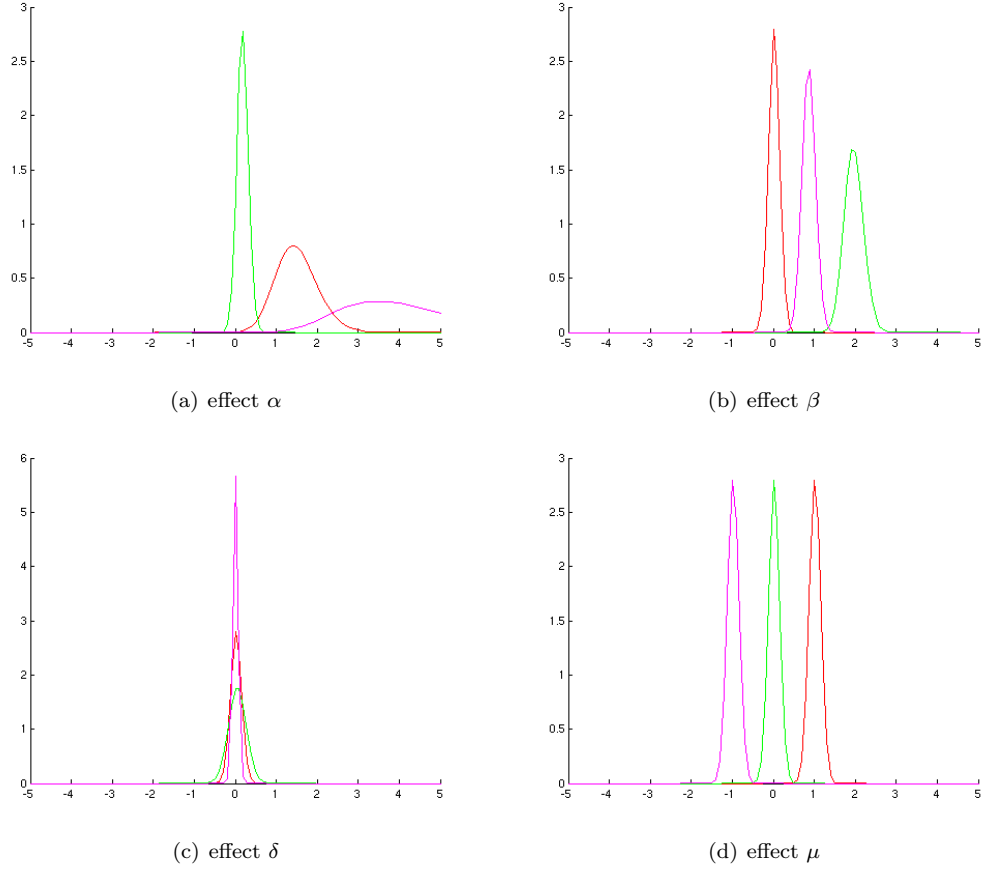


Figure 2.1: typical shapes of the NIG density function

Properties: In particular, it holds that, if x_1, \dots, x_m are independent NIG random variables with common parameter α and β but having individual location-scale parameter μ_i and δ_i for $i = 1, \dots, m$, then $x_+ = x_1 + \dots + x_m$ is again distributed according to a NIG law with parameters $(\alpha, \beta, \mu_+, \delta_+)$. If $X \sim \text{NIG}(\alpha, \beta, \delta, \mu)$ then $-X$ is distributed according to a $\text{NIG}(\alpha, -\beta, \delta, \mu)$ distribution. If $\beta = 0$ the distribution is symmetric. This can be seen from its cumulants, all odd moments are zero in this case.

NIG($\alpha, \beta, \delta, \mu$)	
mean	$\mu + \frac{\delta\beta}{\sqrt{\alpha^2 - \beta^2}}$
variance	$\frac{\alpha^2\delta}{(\alpha^2 - \beta^2)^{3/2}}$
skewness	$\frac{3\beta}{\alpha\sqrt{\delta}(\alpha^2 - \beta^2)^{1/4}}$
kurtosis	$3 \left(1 + \frac{\alpha^2 + 4\beta^2}{\delta\alpha^2\sqrt{\alpha^2 - \beta^2}} \right)$

The NIG distribution has semi-heavy tails, in particular,

$$f_{\text{NIG}}(x; \alpha, \beta, \delta, \mu) \sim |x|^{-3/2} e^{\alpha|x| + \beta x} \quad \text{as } x \rightarrow \infty$$

up to a multiplicative constant.

The NIG distribution can be written as a variance-mean mixture of a normal distribution with an $\text{IG}(\delta, \sqrt{\alpha^2 - \beta^2})$ distribution (see Barndorff-Nielsen [1]). More specifically, if we take $\sigma^2 \sim \text{IG}(\delta, \sqrt{\alpha^2 - \beta^2})$ independently distributed of $\epsilon \sim N(0, 1)$, then $x = \mu + \beta\sigma^2 + \sigma\epsilon$ is distributed $\text{NIG}(\alpha, \beta, \delta, \mu)$.

2.3 Inverse Gaussian Ornstein-Uhlenbeck process

Recall that IG distribution is self-decomposable and infinitely divisible. Hence out of Theorem 5 we can conclude that there exists a stationary stochastic process $\{\sigma^2(t)\}_{t \geq 0}$ with marginal law IG and a Lévy process $\{z(t)\}_{t \geq 0}$ such that,

$$\sigma^2(t) = e^{-\lambda t} \sigma^2(0) + \int_0^t e^{-\lambda(t-s)} dz(\lambda s)$$

for all $\lambda > 0$. Moreover $\sigma^2(t)$ satisfies differential equation (1.5) and is therefore an inverse Gaussian Ornstein-Uhlenbeck (IG-OU) process.

With the theory of Section 1.4.2 we can characterize the Lévy triplet (ξ_b, η_b, W) of the background driven Lévy process z .

Recall that the Lévy measure U_{IG} of the IG distribution is given by,

$$U_{\text{IG}}(dx) = \frac{\delta}{\sqrt{2\pi}} x^{-3/2} e^{-1/2\gamma^2 x} \mathbf{1}_{x>0} dx$$

Then the density w of $z(1)$ is as in equation (1.19). From (1.21) it can be derived that

$$w(x) = \frac{\delta}{\sqrt{2\pi}} \frac{1}{2} (x^{-1} + \gamma^2) x^{-1/2} e^{-1/2\gamma^2 x} \mathbf{1}_{x>0}. \quad (2.1)$$

The second parameter η_b can be calculated from (1.18),

$$0 = \eta = \int_0^\infty \eta_b e^{-2t} dt \quad \Rightarrow \quad \eta_b = 0. \quad (2.2)$$

Recall that the IG distribution is of finite variation hence by (1.4) it has a representation of the form,

$$\log \{\phi_{\text{IG}}(\zeta)\} = i\xi\zeta + \int (e^{i\zeta x} - 1) U(dx)$$

We can recalculate equation (1.20) for this special case. Recall that the IG distribution has a representation of the form $\int_0^\infty e^{-t} dz(t)$. Now by the key formula,

$$\begin{aligned} \log \{\phi_{\text{IG}}(\zeta)\} &= i\xi\zeta + \int (e^{i\zeta x} - 1) U(dx) = \int \left\{ i\xi_b \zeta e^{-t} + \int (e^{i\zeta e^{-t} x} - 1 - i\zeta e^{-t} \tau(x)) W(dx) \right\} dt \\ &= \int i\xi_b \zeta e^{-t} dt + \int \int (e^{i\zeta e^{-t} x} - 1) W(dx) dt - \int_0^\infty \int i\zeta e^{-t} \tau(x) W(dx) dt \\ &= \int i\xi_b \zeta e^{-t} dt + \int \int (e^{i\zeta y} - 1) W(e^t dy) dt - \int i\zeta \int_0^\infty e^{-t} dt \tau(x) W(dx) \\ &= \int i\xi_b \zeta e^{-t} dt + \int (e^{i\zeta y} - 1) U(dy) - \int i\zeta \tau(x) W(dx) \end{aligned}$$

hence,

$$\begin{aligned} \xi &= \int \xi_b e^{-t} dt - \int \tau(x) W(dx) \\ &= \int \xi_b e^{-t} dt + \frac{\delta}{\gamma} (2\Phi(\gamma) - 1) \\ &= \int \xi_b e^{-t} dt + \xi \end{aligned}$$

So $\xi_b = 0$. Thus from (2.1) and (2.2) we can conclude that $z(1)$ has characteristics $(0, 0, W)$, which implies it is a subordinator. Therefore it is a process which only consists of positive jumps. $\sigma^2(t)$ moves up entirely by jumps and then tails off exponentially with rate λ .

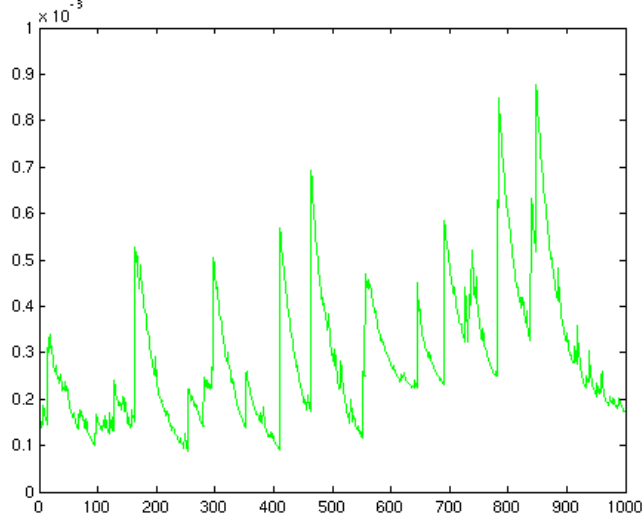


Figure 2.2: Sample path of an IG-OU process.

Since we have all the characteristics of the BDLP z , we can calculate its characteristic function,

$$\log \{\phi_z(u)\} = -iu \frac{\delta}{\gamma} \left(\frac{1}{\sqrt{1 + 2iu\gamma^{-2}}} \right)$$

There holds that,

$$\begin{aligned} \log \{\phi_z(u)\} &= \frac{\delta\gamma}{2} \left(\frac{1 - (2iu\gamma^{-2} + 1)}{\sqrt{1 + 2iu\gamma^{-2}}} \right) \\ &= \frac{\delta\gamma}{2} \left(\frac{1}{\sqrt{1 + 2iu\gamma^{-2}}} - \sqrt{2iu\gamma^{-2} + 1} \right) \\ &= \frac{\delta\gamma}{2} \left(\frac{1}{\sqrt{1 + 2iu\gamma^{-2}}} - 1 \right) - \frac{\delta}{2} \left(\sqrt{2iu + \gamma^2} - \gamma \right) \\ &:= \log \phi_{z^{(1)}}(u) + \log \phi_{z^{(2)}}(u). \end{aligned}$$

$\phi_{z^{(2)}}(u)$ corresponds to the characteristic function of an IG Lévy process with parameters $\delta/2$ and γ for $z^{(2)}(1)$. And $\phi_{z^{(1)}}(u)$ corresponds to the characteristic function of a compound Poisson process of the form,

$$z^{(1)}(t) = \gamma^{-2} \sum_{i=1}^{N_t} v_i^2$$

where N_t is a Poisson process with intensity $\frac{\delta\gamma}{2}$ and v_i are independent standard normal variables independent of N_t .

Hence if we have an $\text{IG}(\delta, \gamma)$ -OU process then the BDLP is a sum of two independent Lévy processes

$z(t) = z^{(1)}(t) + z^{(2)}(t)$, where $z^{(1)}$ is an inverse gaussian Lévy process with parameters $\delta/2$ and γ for $z^{(2)}(1)$, while $z^{(1)}(t)$ is compound Poisson process of the form,

$$z^{(1)}(t) = \gamma^{-2} \sum_{i=1}^{N_t} v_i^2$$

where N_t is a Poisson process with intensity $\frac{\delta\gamma}{2}$ and v_i 's are independent standard normal variables independent of N_t .

Chapter 3

The model:

To price derivative securities, it is crucial to have a good model of the probability distribution of the underlying product. The most famous continuous-time model is the celebrated Black-Scholes model, which uses the Normal distribution to fit the log-returns of the underlying asset. However the log-returns of most financial assets do not follow a Normal law. They are skewed and have actual kurtosis higher than that of the Normal distribution. Hence other more flexible distributions are needed. Moreover to model the behavior through time we need more flexible stochastic processes. Lévy processes have proven to be good candidates, since they preserve the property of having stationary and independent increments and they are able to represent skewness and excess kurtosis.

In this chapter we will describe the Lévy process models we use. We also give a method how to fit the model on historical data.

3.1 Lévy process model

As model without stochastic volatility we will use a fitted NIG distribution to the log-returns. i.e.

$$d \log S(t) = dX(t),$$

where S is an arbitrary stock-price and X is an NIG Lévy process. This model is flexible to model with, but a drawback is that the returns are assumed independently.

In literature this model has also been referred to as exponential Lévy process model, since the solution is of the form,

$$S(t) = S(0)e^{X(t)}$$

3.2 BNS SV model:

The Barndorf-Nielsen and Shephard stochastic volatility (BNS SV) model is an extension of the Black-Scholes Samuelson model. In contradiction to the Lévy process (LP) model and the Black-Scholes model the BNS SV models volatility (see Section 1.3.3). The volatility follows an OU process driven by a subordinator.

In the Black-Scholes Samuelson model the asset price process $\{S(t)\}_{t \geq 0}$ is given as a solution to the SDE,

$$d \log S(t) = \{\mu + \beta \sigma^2\} dt + \sigma dw(t) \tag{3.1}$$

where $w(t)$ is a standard Brownian motion. We want to take into account stochastic volatility. We do this by taking the parameter σ^2 stochastically. So we will look for a stochastic process $\{\sigma^2(t)\}_{t \geq 0}$ describing the nervousness of the market through time.

As described in Barndorf-Nielsen [3] an OU process or a superposition of OU processes is a good choice

for $\sigma^2(t)$. i.e.

$$\sigma^2(t) = \sum_{j=1}^n a_j \sigma_j^2(t)$$

with the weights a_j all positive, summing up to one and the processes $\{\sigma_j(t)\}_{t \geq 0}$ are OU processes satisfying,

$$d\sigma_j^2(t) = -\lambda_j \sigma_j^2(t) dt + dz_j(\lambda_j t)$$

where the processes z_j are independent subordinators i.e. Lévy processes with positive increments and no drift. Since the process z_j is a subordinator the process σ_j will jump up according to z_j and decay exponentially afterwards with a positive rate λ_j . We can rewrite equation (3.1) into,

$$d \log S(t) = \{\mu + \beta \sigma^2(t)\} dt + \sigma(t) dw(t) \quad (3.2)$$

we can reformulate this into,

$$\log S(t) = \mu t + \beta \sigma^{2*}(t) + \int \sqrt{\sigma^2(s)} dw(s)$$

where $\sigma^{2*}(t)$ is the integrated process, i.e.

$$\sigma^{2*}(t) = \int_0^t \sigma^2(u) du$$

Barndorff-Nielsen [4] showed that the tail behavior (or superposition) of an integrated IG process is approximately Inverse Gaussian. Moreover an integrated superposition of IG processes will show similar behavior as just one integrated IG process with the same mean. Hence by taking the $\sigma^2(t)$ an IG process one gets the same tail behavior as taking $\sigma^2(t)$ a superposition of IG processes. Moreover the log returns will be approximately NIG distributed, since the NIG is a mean-variance mixture (see Section 2.2).

3.3 Parameter estimation:

The stationary distribution σ_j is independent of λ_j (see Theorem 5). Hence its stationary distribution is determined by z_j . Therefore the distribution of the returns is independent of the λ_j 's. This allows us to calibrate the log return distribution to the data separately of the λ_j 's.

3.3.1 Maximum likelihood estimation

In order to get comparable models we will assume that both models have NIG distributed returns with the same parameters. The parameters will be estimated with a maximum likelihood estimation. The maximum-likelihood estimator $(\hat{\alpha}, \hat{\beta}, \hat{\delta}, \hat{\mu})$ is the parameter set that maximizes the likelihood function

$$L(\alpha, \beta, \delta, \mu) = \prod_{i=1}^n f_{NIG}(x_i; \alpha, \beta, \delta, \mu)$$

where x_i are points of our data-set. Since the NIG distribution is a mean-variance mixture (see section 2.2) we can choose the parameters of the IG resp. IG-OU process such that returns will be NIG resp. approximately NIG distributed.

3.3.2 Least square estimation

In Barndorff-Nielsen [3] the analytic auto-covariance function of the squared log returns is calculated and is found to be given by,

$$\text{cov}(y_n^2, y_{n+s}^2) = \omega^2 \sum a_i \lambda_i^{-2} (1 - e^{-\lambda_i \Delta t})^2 e^{-\lambda_i \Delta t (s-1)} \quad (3.3)$$

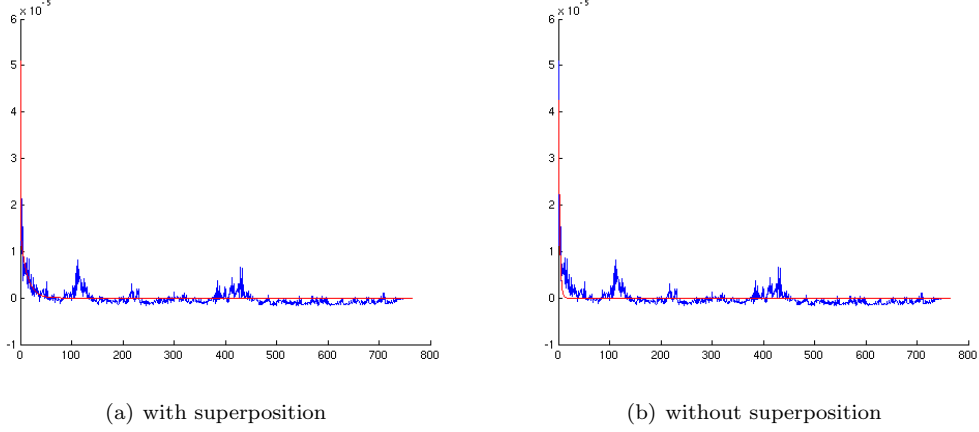


Figure 3.1: Least square fitting to the auto-covariance function

where ω^2 is equal to the variance of $\sigma^2(t)$ and a_i 's are the weights of the different OU processes. Looking at the empirical auto-covariance function of the squared log-returns of financial data it seems that this function fits the data really well (see Figure 3.1).

One could see that the empirical auto-covariance function of real financial data looks like a sum of exponentially decaying functions. We estimated the empirical auto-covariance function $\gamma(s)$ for a data-set x_1, \dots, x_n by,

$$\gamma(s) = \frac{1}{n-s} \sum_{i=1}^{n-s} \left(x_{i+s}^2 - \frac{1}{n-s} \sum_{j=s}^n x_j^2 \right) \left(x_i^2 - \frac{1}{n-s} \sum_{j=1}^{n-s} x_j^2 \right)$$

For the calibration we will use a non-linear least square comparison of the empirical auto-covariance function with the analytic auto-covariance function (3.3). Hence we will minimize,

$$\sum_s \{ \gamma(s) - cov(y_n^2, y_{n+s}^2) \}^2$$

For a non-linear least square comparison several algorithms are available. We used a standard function of Matlab which is based on the Gauss-Newton method.

Note that the least square comparison is only done on the λ_i 's and a_i 's since ω is already given from the maximum likelihood estimation.

Chapter 4

Pricing:

4.1 Equivalent martingale measures

Suppose we are on a filtered probability space $(\Omega, \mathcal{F}_t, P)$ with P denoting the 'physical' probability measure. Moreover we have a derivative product written on an asset with price process $S(t)$. To value the arbitrage-free price of the derivative product we need to know an equivalent martingale measure (EMM) or risk neutral measure Q , i.e. a probability measure equivalent to P under which the discounted price process $S(t)e^{-rt}$ evolves as a martingale.

Unfortunately, as in most realistic models, there is no unique equivalent martingale measure; the proposed Lévy models are incomplete. This means there exist an infinite number of EMM's for both models. Hence for each model we have to choose one EMM without restricting the possible price of our derivative product. Finally we give a method to price Asian options.

4.1.1 Lévy process model

For ease we are interested in equivalent martingale measures Q which preserve the structure of our model. In the case of an NIG distributed increments, the Esscher transform is a good candidate since it preserves the structure and is not restrictive on the range of viable prices.

Let $f_t(x)$ be the density of log price $\log S(t)$ under P . For some number $\theta \in \{\theta \in \mathbb{R} \mid \int_{-\infty}^{\infty} \exp(\theta y) f_t(y) dy < \infty\}$ we can define a new density,

$$f_t^{(\theta)}(x) = \frac{\exp(\theta x) f_t(x)}{\int_{-\infty}^{\infty} \exp(\theta y) f_t(y) dy}$$

Now we choose θ such that the discounted price process $\{e^{-rt} S_t\}_{t \geq 0}$ is a martingale, i.e.

$$S_0 = e^{-rt} \mathbb{E}^{(\theta)}[S_t]$$

where the expectation is taken with respect to the law with density $f_t^{(\theta)}$. We can rewrite this into a characteristic function representation. Hence in order to let the discounted price process be a martingale, we need to have,

$$e^r = \frac{\phi(-i(\theta + 1))}{\phi(-i\theta)}$$

In case of a NIG process this leads to,

$$\begin{aligned} e^r &= e^{\mu - \delta(\sqrt{\alpha^2 - (\beta + \theta + 1)^2} - \sqrt{\alpha^2 - (\beta + \theta)^2})} \\ &= e^{\mu - \delta(\sqrt{\alpha^2 - (\beta + \theta)^2 - 2(\beta + \theta) - 1} - \sqrt{\alpha^2 - (\beta + \theta)^2})} \end{aligned}$$

From which we can conclude that,

$$\frac{\mu - r}{\delta} = \sqrt{\alpha^2 - (\beta + \theta)^2 - 2(\beta + \theta) - 1} - \sqrt{\alpha^2 - (\beta + \theta)^2} \quad (4.1)$$

and hence,

$$\hat{\beta} = \beta + \theta = -\frac{1}{2} + \text{sgn}(\beta) \sqrt{\frac{(\mu - r)^2}{\delta^2 + (\mu - r)^2} \alpha^2 - \frac{(\mu - r)^2}{4\delta^2}}$$

Moreover the Esscher transform is structure preserving in case of an NIG distribution.

Lemma 2 *The Esscher transform of a NIG distributed random variable is again NIG distributed. In particular,*

$$f^{(\theta)}(x; \alpha, \beta, \delta, \mu) = f(x; \alpha, \beta + \theta, \delta, \mu)$$

Proof: From

$$\begin{aligned} f^{(\theta)}(x; \alpha, \beta, \delta, \mu) &= \frac{e^{\theta x} f(x; \alpha, \beta, \delta, \mu)}{\phi(-i\theta)} \\ &= \frac{e^{\theta x}}{\phi(-i\theta)} \cdot \frac{\alpha}{\pi} \exp\left(\delta\sqrt{\alpha^2 - \beta^2} - \beta\mu\right) \frac{K_1\left(\delta\alpha\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}} \exp(\beta x) \\ &= \frac{\alpha}{\pi} \frac{\exp\left(\theta\mu + \delta\sqrt{\alpha^2 - \beta^2} - \delta\sqrt{\alpha^2 - (\beta + \theta)^2}\right)}{\phi(-i\theta)} \exp\left(\delta\sqrt{\alpha^2 - (\beta + \theta)^2} - (\beta + \theta)(x - \mu)\right) \\ &\quad \cdot \frac{K_1\left(\delta\alpha\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}} \end{aligned}$$

it follows that we have to prove

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\alpha}{\pi} \cdot \exp\left(\theta x + \delta\sqrt{\alpha^2 - \beta^2} + \beta(x - \mu)\right) \frac{K_1\left(\delta\alpha\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}} dx \\ = \exp\left(\theta\mu + \delta\sqrt{\alpha^2 - \beta^2} - \delta\sqrt{\alpha^2 - (\beta + \theta)^2}\right) \end{aligned}$$

wich is equivalent to

$$\int_{-\infty}^{\infty} \frac{\alpha}{\pi} \cdot \exp\left(\delta\sqrt{\alpha^2 - (\beta + \theta)^2} + (\beta + \theta)(x - \mu)\right) \frac{K_1\left(\delta\alpha\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}\right)}{\sqrt{1 + \left(\frac{x-\mu}{\delta}\right)^2}} dx = 1$$

The integral on the left hand side integrates a $\text{NIG}(\alpha, \beta + \theta, \delta, \mu)$ density over its domain. Hence the above equation holds for $|\beta + \theta| < \alpha$. The last condition is satisfied since θ is chosen according to equation (4.1). \square

From the above lemma we may conclude that under an Esscher change of measure the distribution of the log price stays in the class NIG only with a different parameter for β . i.e. under EMM Q the distribution of $\log S(t)$ is $\text{NIG}(\alpha, \hat{\beta}, \delta, \mu)$.

4.1.2 BNS SV model

The structure of a general equivalent martingale measure for the BNS SV case and some relevant subsets are studied in Nicolato and Vernardos [14]. Of special interest is the structure preserving subset of martingale measures under which the log returns are again described by a BNS SV model, although possibly with different parameters and different stationary laws. Nicolato and Vernardos argue that it

suffices to consider only equivalent martingale measures of this subset. Moreover they show that the dynamics of the log price under such an equivalent martingale measure Q are given by,

$$\begin{aligned} d \log S(t) &= \left\{ r - \frac{1}{2} \sigma^2(t) \right\} dt + \sigma(t) dw(t) \\ d\sigma(t) &= -\lambda \sigma^2(t) dt + dz(\lambda t) \end{aligned}$$

where $\{w(t)\}_{t \geq 0}$ is a Brownian motion under Q independent of the BDLP. We will take the stationary distribution of σ^2 unchanged, i.e. $\sigma^2 \sim \text{IG}(\delta, \gamma)$. However it is possible to change γ to any value without the structure of the distribution being altered and such that the price process is a martingale. Economically our choice of martingale measure corresponds to the (questionable) idea that the jumps represent only non-systematic risk that is not reflected in derivatives prices [13]. However changing the variance process without any restrictions would be questionable as well.

An other possibility is to use Esscher transform in which the distribution of the variance process is changing as well. Hubalek and Sgarra [13] proved that these Esscher transform coincide the minimal martingale measure. Hence it is a good candidate. Unfortunately the Esscher transform is not in the set of structure preserving equivalent martingale measures. But, one can define the Esscher transform for every increment. If we work with a regular time partition $t_1 < t_2 < \dots < t_N = T$ with mesh Δ , we will get a set of θ_i 's satisfying,

$$e^{r\Delta} = \frac{\kappa^{t_i}(\theta_i + 1)}{\kappa^{t_i}(\theta_i)} \quad (4.2)$$

where κ^{t_i} is the conditional cumulant generating function given \mathcal{F}_{t_i} , i.e. $\kappa_X^{t_i}(u) = \mathbb{E}[e^{-uX^{(t_i+1)}} | \mathcal{F}_{t_i}]$. As shown in Nicolato and Vernardos [14], the cumulant function of the log returns is given by,

$$\kappa_{\log S}^{t_i}(u) = \exp \left(u(\log S(t_i) + \mu\Delta) + (u^2 + 2\beta u) \frac{\epsilon(t_i, t_{i+1})}{2} \sigma^2(t_i) + \int_{t_i}^{t_{i+1}} \lambda \kappa_z(f(s, u)) ds \right) \quad (4.3)$$

With $\epsilon(s, T) = \lambda^{-1}(1 - e^{-\lambda(T-s)})$ and $f(s, u) = \frac{1}{2}(u^2 + 2\beta u)\epsilon(s, T)$.

Putting this in equation 4.2 leads to,

$$r\Delta = \log S(t_i) + \mu\Delta + \frac{\epsilon(t_i, t_{i+1})}{2} \sigma^2(t_i)(\theta + 2\beta + 1) + \int_{t_i}^{t_{i+1}} \lambda \kappa_z(f(s, \theta + 1)) ds - \int_{t_i}^{t_{i+1}} \lambda \kappa_z(f(s, \theta)) ds$$

There are closed form formulas available for the integrals on the right hand side. Hence it is possible to calculate a θ_i for every time increment. However in most of the times the 'new' measure will not be of the same structure. In our case of dealing with an IG-OU process for the variance process, an Esscher transform would give a BDLP with non-stationary increments. Hence the BDLP would lose the property of being a Lévy process. This is too bothersome to work with. It would lead to a structure which is hard to simulate. Hence we stick to our EMM from Nicolato and Vernardos.

4.2 Asian option

Asian option is an option where the payoff is not determined by the underlying price at maturity but by the average underlying price over some pre-set period of time. Asian options were originated in Asian markets to prevent option traders from attempting to manipulate the price of the underlying on the exercise date. We will use Asian option because it is a path dependent option. Therefore it might be possible to see a difference in pricing between our two models.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $\{\mathcal{F}_t\}_{t \in [0, T]}$ satisfying the usual conditions, with $T < \infty$ being the time horizon. Let $X(t)$ be a Lévy process with càdlàg sample paths, and consider the following exponential model for the asset price dynamics,

$$S(t) = S(0)e^{X(t)} \quad (4.4)$$

Now the aim is to price arithmetic Asian call options written on $S(t)$. Consider such an option with exercise at time T and strike price K on the average over the time span up to T . The risk-neutral price is,

$$A(0) = e^{-rT} \mathbb{E}_Q \left[\max \left\{ \frac{1}{T} \int_0^T S(t) dt - K, 0 \right\} \right]$$

Where r is the risk-free interest rate. As described above we work with equivalent martingale measures for which the structure of the distribution of the log price is preserved. Hence under Q , $X(t)$ is still following the model. Only with different parameters, i.e. $X(t)$ is an $\text{NIG}(\alpha, \hat{\beta}, \delta, \mu)$ Lévy process in the case of Lévy process model and $X(t)$ has parameters $\mu = r$ and $\beta = -\frac{1}{2}$ in the BNS SV case. Moreover one can approximate the integral with a Riemann sum,

$$A(0) \approx e^{-rT} \mathbb{E} \left[\max \left\{ \frac{S(0)}{N} \sum_{i=1}^N e^{X(t_i)} \Delta - K, 0 \right\} \right] \quad (4.5)$$

For simplicity we will work with regular time partitions $t_1 < t_2 \cdots < t_N$ with mesh Δ . In the next chapter we will describe several methods to value the expectation in equation (4.5).

Chapter 5

Simulation:

There are two trends to simulate Inverse Gaussian random variates and processes. One is using exact simulation by using general rejection method and the other is a series expansion based on path rejection proposed by J.Rosinski. Earlier Rosinski had a technique using the inverse of the Lévy measure of the BDLP process. In the IG case this measure is not analytically invertible, hence this can only be done numerically. This is time-consuming calculation and is therefore bothersome to work with.

In the algorithms it is assumed that one can simulate from standard distributions. Moreover for simplicity we will work with a regular time partition $t_1 < t_2 < \dots < t_N = T$ with mesh Δ .

5.1 Inverse Gaussian Lévy process

5.1.1 Exact simulation

To simulate from an IG random number we can use the IG random number generator proposed by Michael, Schucany and Haas [12].

Algorithm 1 (*Generate of the $IG(\delta, \gamma)$ random variate; exact*)

-
- Generate a random variate Y with density χ_1^2 .
 - Set $y_1 = \frac{\delta}{\gamma} + \frac{Y}{2\gamma^2} - \frac{1}{2\gamma^2} \sqrt{4\delta\gamma Y + Y^2}$.
 - Generate a uniform $[0, 1]$ random variate U and if $U \leq \frac{\delta}{\delta + \gamma y_1}$, set $\sigma^2 = y_1$.
If $U > \frac{\delta}{\delta + \gamma y_1}$, set $\sigma^2 = \frac{\delta^2}{\gamma^2 y_1}$.
-

Then σ^2 is the desired random variate, namely $\sigma^2 \sim IG(\delta, \gamma)$.

This simulation method is based on the fact that if $Y \sim IG(\delta, \gamma)$ then $A = \frac{(\gamma Y - \delta)^2}{Y}$ is distributed χ_1^2 (see [12]). A has two roots.

$$y_1 = \frac{\delta}{\gamma} + \frac{Y}{2\gamma^2} - \frac{1}{2\gamma^2} \sqrt{4\delta\gamma Y + Y^2}$$

and

$$y_2 = \frac{\delta^2}{\gamma^2 y_1}$$

y_1 should be chosen with probability $p_1 = \frac{\delta}{\delta + \gamma y_1}$ and y_2 should be chosen with probability $1 - p_1$.

Next is to simulate a sample path of an IG Lévy process $\{\sigma^2(t)\}_{t \geq 0}$, where $\sigma^2(t)$ follows an $IG(\delta t, \gamma)$ law. Take as start value $\sigma^2(0) = 0$. Then for every time increment Δ , first generate independent $IG(\delta\Delta, \gamma)$ random number i , then set,

$$\sigma^2(t + \Delta) = \sigma^2(t) + i,$$

5.1.2 Series representation

Alternatively Rosinski's series representation (see [16]) can be used instead of the exact simulation. We can use the following algorithm to approximate a path of an $IG(\delta, \gamma)$ process $\{\sigma^2(t)\}_{(0 \leq t \leq T)}$.

Algorithm 2 (*Generate of the $IG(\delta, \gamma)$ Lévy process; series representation*)

-
- Take $n = \text{large}$ (to approximate an infinite sum).
 - Sample n arrival times $a_1 < a_2 < \dots < a_n$ from a Poisson process with intensity 1.
 - Sample n independent random variates $\{e_i\}_{i=1}^n$ according to an exponential distribution with mean $2/\gamma^2$.
 - Sample $\{\tilde{u}_i\}_{i=1}^n, \{u_i\}_{i=1}^n$ as i.i.d uniform $[0, 1]$ random variates.
 - Set $\sigma^2(t) = \sum_{i=1}^n \min \left\{ \frac{2}{\pi} \left(\frac{\delta T}{a_i} \right)^2, e_i \tilde{u}_i^2 \right\} 1_{(u_i T < t)}$.
-

Then σ^2 is the desired random process. This algorithm is based on the fact that $IG(\delta, \gamma)$ Lévy process can be approximated by,

$$\sigma^2(t) = \sum_{i=1}^{\infty} \min \left\{ \frac{2}{\pi} \left(\frac{\delta T}{a_i} \right)^2, e_i \tilde{u}_i^2 \right\} 1_{(u_i T < t)} \quad (5.1)$$

where $\{e_i\}$ is a sequence of independent exponential random numbers with mean $\frac{2}{\gamma^2}$, $\{\tilde{u}_i\}$ and $\{u_i\}$ are sequences of independent uniforms and $a_1 < a_2 < \dots < a_i < \dots$ are arrival times of a Poisson process with intensity parameter 1. The series is converging uniformly from below. How large n should be depends on the parameters δ and γ . The summation should generally run over 100 to 10000 terms.

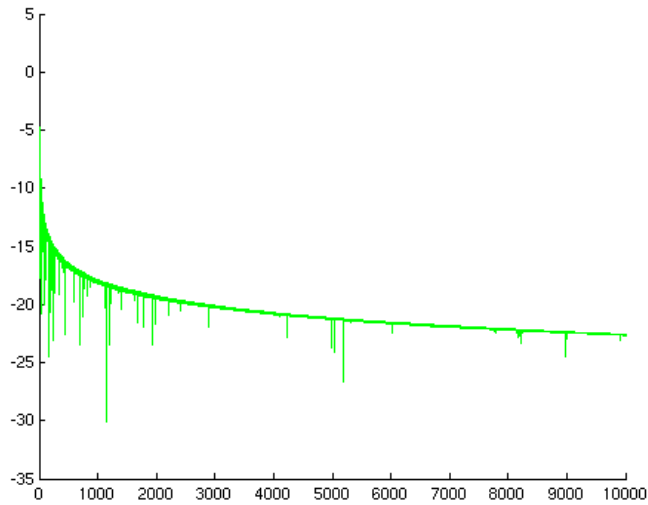


Figure 5.1: Graph of 100 simulations of the log of the individual terms $\min \left\{ \frac{2}{\pi} \left(\frac{\delta T}{a_i} \right)^2, e_i \tilde{u}_i^2 \right\}$ of the infinite series (5.1) for an $IG(0.5, 1)$ problem.

An example of how the infinite sum behaves is given in Figure 5.1 which depicts the logarithm of the individual terms, against the value of the index i . One can see that the series is dominated by the first few terms. In contradiction to the previous method the whole path is simulated directly. Moreover from the definition of a Lévy process we may conclude that if we take $T = 1$ then $\sigma^2(1)$ is an $IG(\delta, \gamma)$ random variable.

5.2 IG OU process

From Theorem 5 we know that a solution to a SDE of the Ornstein-Uhlenbeck type,

$$d\sigma^2(t) = -\lambda\sigma^2(t)dt + dz(\lambda t)$$

is given by,

$$\sigma^2(t) = e^{-\lambda t}\sigma^2(0) + \int_0^t e^{-\lambda(t-s)}dz(\lambda s) \quad (5.2)$$

Moreover, up to indistinguishability, this solution is unique (Sato [18]). Recall that for an IG(δ, γ, δ)-OU process, $\sigma^2(t)$ has stationary marginal law IG(δ, γ). So $\sigma^2(0)$ is IG(δ, γ) distributed. In the above section we already described a way to simulate IG(δ, γ) random variates. So the most difficult term to simulate in (5.2) is the integral $\int_0^t e^{-\lambda(t-s)}dz(\lambda s)$.

5.2.1 Exact simulation

Take,

$$\sigma^2(\Delta) = \int_0^\Delta e^{-\lambda(\Delta-s)}dz(\lambda s) \quad (5.3)$$

As shown in Zhang & Zhang [21], for fixed $\Delta > 0$ the random variable $\sigma^2(\Delta)$ can be represented to be the sum of an inverse Gaussian random variable and a compound Poisson random variable in distribution, i.e.,

$$\sigma^2(\Delta) \stackrel{\mathcal{D}}{=} W_0^\Delta + \sum_{i=1}^{N_\Delta} W_i^\Delta \quad (5.4)$$

where $W_0^\Delta \sim \text{IG}(\delta(1 - e^{-\frac{1}{2}\lambda\Delta}), \gamma)$, random variable N_Δ has a Poisson distribution of intensity $\delta(1 - e^{-\frac{1}{2}\lambda\Delta})\gamma$ and $W_1^\Delta, W_2^\Delta, \dots$ are independent random variables having a common specified density function,

$$f_{W^\Delta}(w) = \begin{cases} \frac{\gamma^{-1}}{\sqrt{2\pi i}} w^{-3/2} (e^{\frac{1}{2}\lambda\Delta} - 1)^{-1} \left(e^{-\frac{1}{2}\gamma^2 w} - e^{-\frac{1}{2}\gamma^2 w e^{\lambda\Delta}} \right) & \text{for } w > 0, \\ 0 & \text{otherwise} \end{cases}$$

Moreover for any $w > 0$ the density function $f_{W^\Delta}(w)$ satisfies,

$$f_{W^\Delta}(w) \leq \frac{1}{2} \left(1 + e^{\frac{1}{2}\lambda\Delta} \right) \frac{(\frac{1}{2}\gamma^2)^{1/2}}{\Gamma(\frac{1}{2})} w^{-\frac{1}{2}} e^{-\frac{1}{2}\gamma^2 w}$$

Hence we can use the rejection method on a $\Gamma(\frac{1}{2}, \frac{1}{2}\gamma^2)$ distribution to simulate random variables with density function $f_{W^\Delta}(w)$.

Algorithm 3 (*Generation of the random variate from the density $f_{w^\Delta}(w)$*)

-
- Generate a $\Gamma(\frac{1}{2}, \frac{1}{2}\gamma^2)$ random variate Y .
 - Generate a uniform $[0, 1]$ random variate U .
 - If $U \leq \frac{f_{w^\Delta}(w)}{\frac{1}{2}(1+e^{\frac{1}{2}\lambda\Delta})g(Y)}$, set $w_i^\Delta = Y$, where $g(Y) = \frac{(\frac{1}{2}\gamma^2)^{1/2}}{\Gamma(\frac{1}{2})} Y^{-\frac{1}{2}} e^{-\frac{1}{2}\gamma^2 Y}$.
-
- Otherwise return to the first step.
-

Since σ^2 is a stationary process we can conclude with equation 5.2, 5.3 and 5.4 that for all $t > 0$ we have the following equality in distribution,

$$\sigma^2(t + \Delta) \stackrel{\mathcal{D}}{=} e^{-\lambda\Delta}\sigma^2(t) + w_0^\Delta + \sum_{i=1}^{N_\Delta} w_i^\Delta$$

which can be translated in the following algorithm,

Algorithm 4 (*Generation of the random variate $\sigma^2(t_0 + \Delta)$ given the value $\sigma^2(t_0)$.*)

-
- Generate a $IG(\delta(1 - e^{-\frac{1}{2}\lambda\Delta}), \gamma)$ random variate W_0^Δ .
 - Generate a random variate N^Δ from the Poisson distribution with intensity $\delta(1 - e^{-\frac{1}{2}\lambda\Delta})\gamma$.
 - Generate $W_1^\Delta, W_2^\Delta, \dots, W_{N^\Delta}^\Delta$ from the density $f_{W^\Delta}(w)$ as independent identically distributed random variate.
 - Set $\sigma^2(t_0 + \Delta) = e^{-\lambda\Delta}\sigma^2(t_0) + \sum_{i=0}^{N^\Delta} W_i^\Delta$.
-

Moreover σ^2 is a stationary process hence the initial value $\sigma^2(0)$ can be generated from the density $IG(\delta, \gamma)$. We are simulating using an exact method so we will use Algorithm 1 to simulate $\sigma^2(0)$ from an $IG(\delta, \gamma)$ distribution.

5.2.2 Series representation

For simulating according to a series representation we make use of the fact that the BDLP can be decomposed into an IG Lévy process and a compound Poisson process as notated in Section 2.3. Using the series representation described in Section 5.1.2 we can conclude that $\int_0^t e^{-\lambda(t-s)} dz(\lambda s)$ can be approximated by,

$$e^{-\lambda t} \sum_{i=1}^{N_{\lambda t}} \frac{v_i^2}{\gamma^2} e^{d_i} + e^{-\lambda t} \sum_{i=1}^{\infty} e^{\lambda t \bar{u}_i} \min \left\{ \frac{1}{2\pi} \left(\frac{\delta \lambda T}{a_i} \right)^2, e_i \tilde{u}_i^2 \right\} 1_{(u_i T < t)}$$

where $\{e_i\}$ is a sequence of independent exponential random numbers with mean $\frac{2}{\gamma^2}$, $a_1 < a_2 < \dots < a_i < \dots$ are arrival times of a Poisson process with intensity parameter 1, $\{N_t\}_{t \geq 0}$ is a Poisson process with intensity parameter $\frac{\delta \gamma}{2}$ and interarrival times $d_1 < d_2 < \dots$, v_i independent standard normal random variables and $\{\bar{u}_i\}, \{\tilde{u}_i\}, \{u_i\}$ are sequences of independent uniform random numbers. By using equation (5.2) we can conclude that an IG-OU(δ, γ, λ) process $\sigma^2(t)$ can be approximated by,

$$\sigma^2(t) = e^{-\lambda t} \sigma^2(0) + e^{-\lambda t} \sum_{i=1}^{N_{\lambda t}} \frac{v_i^2}{\gamma^2} e^{d_i} + e^{-\lambda t} \sum_{i=1}^{\infty} e^{\lambda t \bar{u}_i} \min \left\{ \frac{1}{2\pi} \left(\frac{\delta \lambda T}{a_i} \right)^2, e_i \tilde{u}_i^2 \right\} 1_{(u_i T < t)}$$

As in the exact simulation case σ^2 is a stationary process, hence the initial value $\sigma^2(0)$ can be generated from the density $IG(\delta, \gamma)$. Since we are working with series representations we will use Algorithm 2 with $T = 1$ and $t = 1$ to simulate $\sigma^2(0)$ from an $IG(\delta, \gamma)$ distribution. As in the IG Lévy process case, this series is dominated by the first few terms and generally converges quickly.

5.3 Model

In this section we will describe a method to simulate a sample path from the logarithmic price process $\log S(t)$ for the models described in Chapter 3. For notational simplicity we will denote the process $\log S(t)$ by X .

5.3.1 Lévy process model

In the Lévy process model $X(t)$ is simply a NIG Lévy process. Hence we will describe a method to simulate a NIG Lévy process.

The most convenient way to simulate a NIG Lévy process is by a quasi-Monte Carlo Algorithm. We will use a Monte-Carlo algorithm based on exact simulation. The NIG distribution is a mean-variance mixture, i.e. if we take $\sigma^2 \sim IG(\delta, \sqrt{\alpha^2 - \beta^2})$ independently distributed of $\epsilon \sim N(0, 1)$, then $x = \mu + \beta\sigma^2 + \sigma\epsilon$ is distributed $NIG(\alpha, \beta, \delta, \mu)$.

Hence we can simulate a NIG random variate by,

Algorithm 5 (*Generate of the $NIG(\alpha, \beta, \delta, \mu)$ random variate.*)

-
- Sample σ^2 from $IG(\delta, \sqrt{\alpha^2 - \beta^2})$.
 - Sample Z from $N(0, 1)$.
 - $X = \mu + \beta\sigma^2 + \sigma Z$.
-

where σ^2 is taken according to Algorithm 1. Next is to simulate a sample path from a NIG Lévy process. This can be done by simply summing the increments.

Algorithm 6 (*Generate a sample path of a NIG($\alpha, \beta, \delta, \mu$) Lévy process*)

- Take $X(0) = 0$.
 - For each increment Δ sample $x \sim \text{NIG}(\alpha, \beta, \delta\Delta, \mu)$ set,
 $X(t + \Delta) = X(t) + x$.
-

5.3.2 BNS SV model

By discretising time we may conclude that equation (3.2) can be rewritten into,

$$C(t) := \log S(t + \Delta) - \log S(t) = \mu \cdot \Delta + \beta\sigma^2(t) \cdot \Delta + \sqrt{\sigma^2(t)\Delta} \cdot \epsilon$$

where ϵ is a standard normal distributed random variable. This is based on the fact that for a brownian motion w , $w(t + \Delta) - w(t)$ is equal in distribution to the random variate $\epsilon\sqrt{\Delta}$. We can use the following Algorithm to generate log-returns on discrete points.

Algorithm 7 (*Generate a sample path according to the BNS SV model.*)

- Generate a sample path $\sigma^2(t_i)$ from IG-OU($\delta, \sqrt{\alpha^2 - \beta^2}, \lambda$) process, for $i = 0, \dots, n$.
 - Sample $\{\epsilon_i\}_{i=0}^n$ as a sequence i.i.d standard normal variables.
 - Set $X(t_i) = \mu \cdot \Delta + \beta\sigma^2(t_i) \cdot \Delta + \sigma(t_i) \cdot \epsilon_i\sqrt{\Delta}$, for $i = 0, \dots, n$.
-

As described in Section 5.2 there are two methods to sample the IG-OU random variate. We will consider both. Note that in the case σ^2 is generated by an exact simulation method the steps of the algorithm can be performed simultaneously.

Superposition

In case of superposition in which $\sigma^2(t)$ is given by, $\sum_{j=1}^m a_j \sigma_j^2(t)$. one can replace the first step of Algorithm 7 by,

- Generate sample paths $\sigma_j^2(t_i)$ from IG-OU($\delta, \sqrt{\alpha^2 - \beta^2}, \lambda_j$) process, for $i = 0, \dots, n$ and $j = 1, \dots, m$.
 - Set $\sigma^2(t_i) = \sum_{j=1}^m a_j \sigma_j^2(t_i)$ for $i = 0, \dots, n$.
-

Chapter 6

Comparison simulation methods

6.1 Processes

In Figure 6.1 we made quantile-quantile plots of the increments of the BNS SV model versus NIG random variates. One can see that exact simulation method and the series representation produce series from the same distribution. Moreover the increments of the BNS SV model are approximately NIG distributed. However this is not always the case. The parameters have a large influence how the returns look in

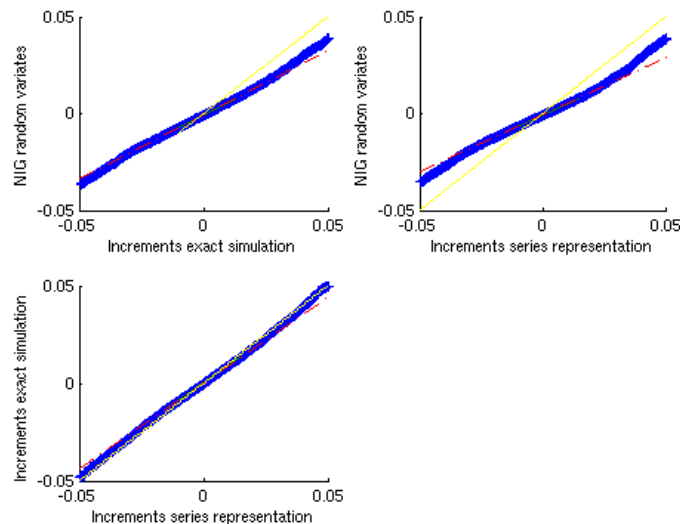


Figure 6.1: Quantile-quantile plot of 10000 simulated points with parameters given by, $\alpha = 64.0317$, $\beta = 4.3861$, $\delta = 0.0157$, $\lambda_1 = 0.0299$, $\lambda_2 = 0.3252$, $a_1 = 0.9122$ and $a_2 = 0.0878$.

comparison to NIG random variates, especially in the case of series representation. In the case of series representation the size of δ has a large influence on the distribution of the returns. The sampling with series representation is based on the fact that an IG Lévy process can be approximated with, (see (5.1))

$$\sigma^2(t) = \sum_{i=1}^{\infty} \min \left\{ \frac{2}{\pi} \left(\frac{\delta T}{a_i} \right)^2, e_i \tilde{u}_i^2 \right\} 1_{(u_i T < t)}.$$

From the definition of a Lévy process we know that the increments should be distributed Inverse Gaussian. But if we plot the empirical cumulative distribution function of the increments from a sample of 1000 points, and compare it with the theoretic cumulative distribution function of the IG distribution, then there is a large difference. This difference increases with the value of δ .

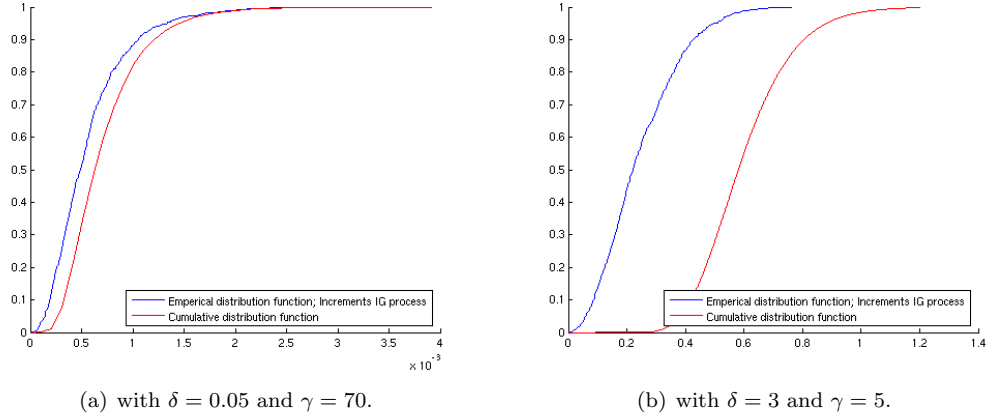


Figure 6.2: Empirical cumulative distribution function of a 1000 points sample generated with series representation compared with the theoretic cumulative IG-distribution function.

In case of financial assets the value of δ is normally between 0.01 and 0.03. In this case the empirical distribution function and the theoretic distribution function coincide. However this still doesn't make the simulation method reliable.

The above analyses is done with $n = 10000$ to approximate the infinite sum in the series representation. As mentioned before it is hard to decide how large n should be. It depends on the parameter values of δ and γ . However the size of n makes a huge difference in processing time using series representation.

Sample size	n	Series representation	Exact simulation
100	100	0.0140	0.0107
100	1000	0.0539	0.0117
100	10000	0.8800	0.0112
1000	100	0.1305	0.0906
1000	1000	0.6065	0.1090
1000	10000	8.4996	0.0918
10000	100	10.2784	3.0246
10000	1000	15.7861	2.7024
10000	10000	70.7537	2.7129

All times are in seconds measured on a not too new 1.3 Ghz i-book G4. The values are taken as the mean of 100 times simulating an IG-OU process. One can see that in all cases the exact simulation is faster.

6.2 Pricing Asian options

We consider the problem of pricing Asian options written on an asset dynamics given by an exponential NIG-Lévyprocess resp. BNS SV model. We will handle a representative case of pricing with calibrated parameters on log-returns of the AEX-index. In the Appendix a similar case of a pricing with calibrated parameters on log-returns of ING, a major dutch bank, is handled. For ease we assume that the stock price today is $S(0) = 100$ and that the risk-free interest rate is $r = 3.75\%$ yearly.

After calibration on a set of daily return data of the AEX-index the parameters are given by,

$$\alpha = 94.1797 \quad \beta = -16.0141 \quad \delta = 0.0086 \quad \mu = 0.0017 \quad \gamma = \sqrt{\alpha^2 - \beta^2} = 92.8082$$

So under the risk neutral measure Q we have,

$$\alpha = 94.1797 \quad \beta = -17.3709 \quad \delta = 0.0086 \quad \mu = 0.0017$$

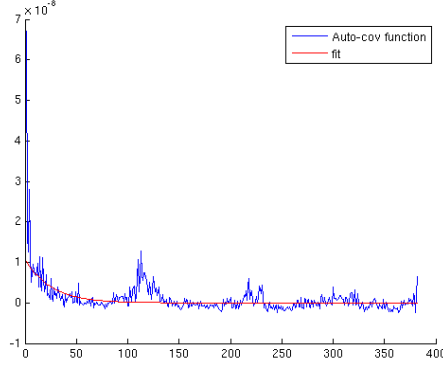


Figure 6.3: Least square fitting with 1 superposition of AEX daily return data.

in the exponential Lévy process case and,

$$\beta = -\frac{1}{2} \quad \delta = 0.0086 \quad \mu = r = \log(1.0375^{1/365}) \quad \gamma = 92.8082$$

in the BNS SV case. Moreover by least square fitting with 1 superposition λ is given by, (see Figure 6.3),

$$\lambda = 0.0397$$

With the above parameters we priced Asian options with a common strike $K = 100$ and exercise horizons of four, eight, or twelve weeks.

T	Lévy Pr.		Series repr.		Exact	
	Mean	Variance	Mean	Variance	Mean	Variance
20	1.0486	2.6390	0.9819	3.3204	0.9913	3.3100
20	1.0620	2.6348	0.9931	3.3584	0.9826	3.3368
20	1.0631	2.6981	0.9885	3.3814	0.9893	3.4301
40	1.5100	5.1793	1.4021	6.4351	1.3968	6.4039
40	1.5187	5.2249	1.3801	6.3452	1.4269	6.6618
40	1.4898	5.1594	1.3892	6.4103	1.4006	6.4840
60	1.8653	7.8627	1.7400	9.7761	1.7582	9.8833
60	1.8608	7.7685	1.7449	9.7804	1.7534	9.7123
60	1.8723	7.8704	1.7446	9.7466	1.7608	9.7476

The mean and variance of the option price are taken over 100,000 times simulating and the variance is only taken over the non-zero values of the option price. Moreover to approximate the infinite sum in the series representation, n is taken 1000.

Remarkable is that the variance of the option prices is higher in the BNS SV case than in the NIG Lévy process case. In the Table is also visible that the NIG Lévy process model is overpricing in comparison to the BNS SV model. Moreover the two simulation methods of the BNS SV model behave similarly. A quantile-quantile plot shows the last property clearly (see Figure 6.4). A Kolmogorov - Smirnov test confirms that the simulated price processes from the two simulation methods come from the same continuous distribution at a 5% significance level. The same test rejects the hypothesis that the price process of the NIG Lévy process model and the price process of the BNS SV model come from the same continuous distribution.

On the interval $[0, 6]$ the NIG Lévy price is slightly higher than the BNS SV price. Moreover the interval $[0, 6]$ consists of approximately 95% of the simulated points, hence the difference in prices on

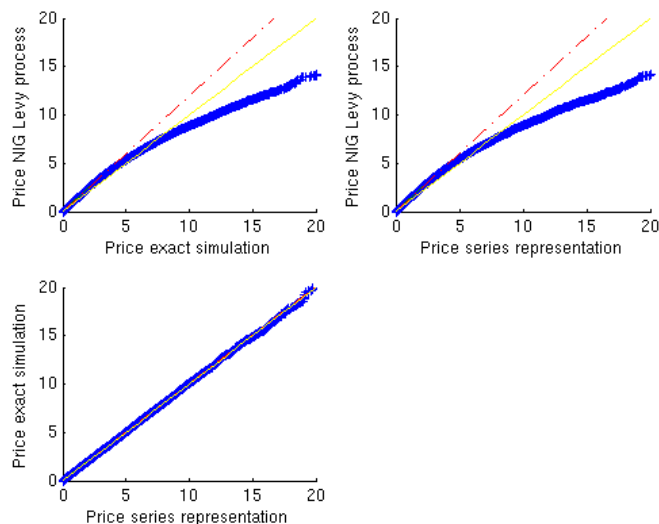


Figure 6.4: QQ-plot of 100,000 points of AEX prices with exercise horizon 40 days and strike price $K = 100$.

this interval leads to a slight higher mean in the NIG Lévy process model. In the other 5% of the cases the BNS SV model prices higher and sometimes even extensively higher.

When alternating with the strike price K , the relation between the option prices of the models is changing (see Figure 6.5, 6.6). The corresponding mean and variance of the option prices for 100000 times simulating are given by,

K	Lévy Pr.		Series repr.		Exact	
	Mean	Variance	Mean	Variance	Mean	Variance
80	20.1196	12.6174	20.1255	12.5494	20.1329	12.8147
100	1.5090	5.1640	1.3912	6.3118	1.4088	6.3986
110	0.0077	1.7775	0.0302	8.4686	0.0285	7.7898

In case of a strike $K = 80$ the mean option prices are approximately the same, but the distribution of the price processes from the different models seem totally not to coincide (see Figure 6.5). However approximately 90% of the simulated prices lie in the vertical part between 15 and 26. Hence the difference in pricing of the two models is caused by the outliers. The BNS SV model exaggerates the price of the extremes in comparison to the NIG Lévy process model i.e. BNS SV model will price low priced outliers even lower and high priced outliers higher compared to the NIG Lévy process model.

At least 95% of the of the simulated prices is zero in the case of a strike $K = 110$. So in the QQ-plot (Figure 6.6) it is again visible that the BNS SV model exaggerates the extreme values of the pricing process compared to the NIG Lévy process model.

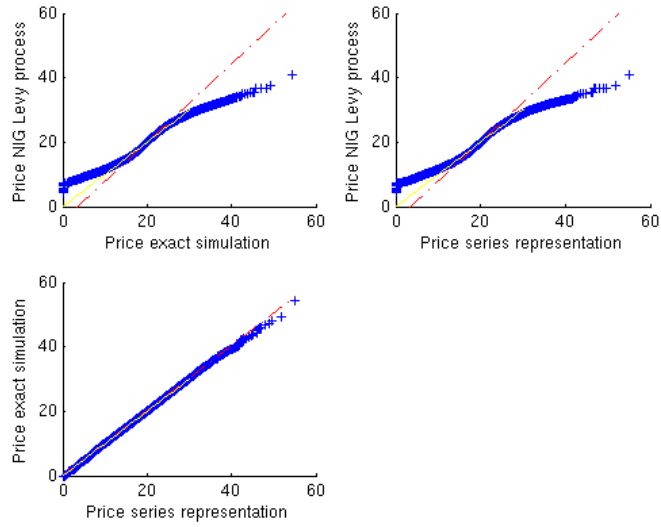


Figure 6.5: QQ-plot of 100,000 generated AEX prices with exercise horizon 40 days and strike price $K = 80$.

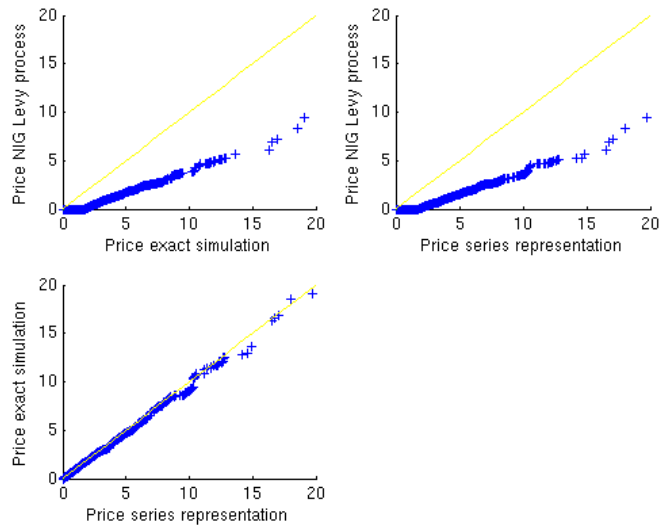


Figure 6.6: QQ-plot of 100,000 generated AEX prices with exercise horizon 40 days and strike price $K = 110$.

Chapter 7

Conclusion:

The Barndorff-Nielsen and Shephard stochastic volatility model is a complex but tractable model. It preserve nice properties as having skewed and heavy-tailed distribution of the log-returns. Moreover it models stochastic volatility.

Under an Esscher change of measure the distribution of the log price in the NIG Lévy process model stays in the class NIG only with a different parameter for β .

The structure preserving martingale measure of the BNS SV model coincide with the (questionable) idea that the jumps represent only non-systematic risk that is not reflected in derivatives prices. Unfortunately simple martingale transforms as the Esscher transform lead to non-tractable structures from which it is hard to simulate.

Although the Algorithm of an IG-OU process based on series representation of J.Rosinski is popular, it is questionable whether it is reliable. The algorithm is based on a method to simulate an inverse Gaussian Lévy process from which the increments not always follow an IG law. The exact simulation of Zhang & Zhang is not only faster but also more accurate.

The log-returns of the BNS SV model are approximately NIG distributed. How they approximate an NIG distribution depends on the parameters.

The price processes of the two simulation methods behave similarly for small δ . In pricing the NIG Lévy process model slightly overprices in comparison to the BNS SV model. The BNS SV price process exaggerates the extremes. Meaning that the BNS SV model will price low priced outliers even lower and high priced outliers higher compared to the NIG Lévy process model.

The variance of the BNS SV price process is higher then the variance of the NIG Lévy price process.

7.1 What I still can do...

Maybe something about Economical meaning????

Chapter 8

Appendix:

In this Appendix a similar case as in Section 6.2 will be handled.

We consider the case of pricing Asian options with calibrated parameters on a set of daily returns from ING, a major dutch bank.

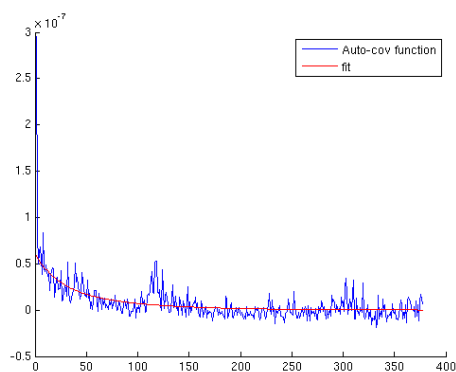


Figure 8.1: Least square fitting with 2 superposition's of ING daily return data.

The calibrated values on a set of daily return data of ING are given by,

$$\alpha = 64.8133 \quad \beta = -3.6096 \quad \delta = 0.0156 \quad \mu = 0.0014 \quad \gamma = \sqrt{\alpha^2 - \beta^2} = 64.7127$$

Hence in the risk neutral world (under Q) we have,

$$\alpha = 64.8133 \quad \beta = -6.0049 \quad \delta = 0.0156 \quad \mu = 0.0014$$

in the exponential Lévy process case and,

$$\beta = -\frac{1}{2} \quad \delta = 0.0156 \quad \mu = r = \log(1.0375^{1/365}) \quad \gamma = 64.7127$$

in the BNS SV case. Moreover by least square fitting with 2 superposition's the other parameters are (see Figure 8.1),

$$\lambda_1 = 0.0174 \quad \lambda_2 = 0.0381 \quad a_1 = 0.5812 \quad a_2 = 0.4188$$

With the above parameters we priced Asian options with a common strike $K = 100$ and exercise horizons of four, eight, or twelve weeks.

T	Lévy Pr.		Series repr.		Exact	
	Mean	Variance	Mean	Variance	Mean	Variance
20	1.7076	7.4142	1.6424	8.3324	1.6568	8.2188
20	1.7070	7.4617	1.6453	8.2154	1.6314	8.1157
20	1.7092	7.3450	1.6491	8.1685	1.6431	8.2075
40	2.4234	14.5726	2.3185	16.2304	2.3377	16.4316
40	2.4101	14.5189	2.3076	16.3691	2.3264	16.4073
40	2.3929	14.7033	2.3506	16.7763	2.3455	16.3278
60	2.9872	22.3489	2.8567	24.7078	2.8731	25.1728
60	2.9942	22.3526	2.8332	24.4307	2.8826	24.6833
60	2.9362	21.5672	2.8672	24.9073	2.8438	24.5923

The mean and variance of the option price are taken over 100000 times simulating. Moreover the variance is only taken over the non-zero values of the option price. The value n , to approximate the infinite sum in the series representation, is taken 1000.

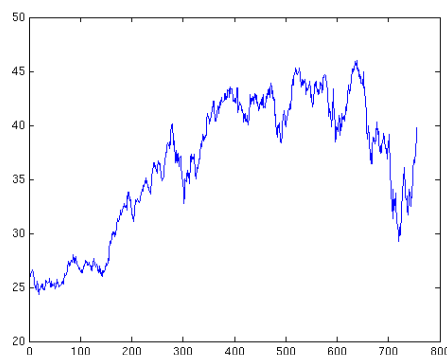


Figure 8.2: Plot of daily return data of ING from Apr.2005 - Apr.2008

In the Table is visible that the variance in the option price of the BNS SV model is higher than in the Lévy process case.

As in the AEX-index case the NIG Lévy process model is overpricing in comparison to the BNS SV model. Moreover the two simulation methods of the BNS SV model behave similarly. A QQ-plot (Figure 8.3) makes the last result visible.

The interval $[0, 10]$ consists of more than 95% of the simulated outcomes. On this interval the NIG Lévy price is slightly higher than the BNS SV price, hence the difference in prices on this interval leads to a slight higher mean of the price process in the NIG Lévy process case. In the top 5% of the cases the BNS SV model prices higher and sometimes even a extensively higher.

When alternating with the strike price K , the relation between the option prices of the models is changing (see Figure 8.4, 8.5). The corresponding mean and variance of the option prices for 100000 times simulating are given by,

K	Lévy Pr.		Series repr.		Exact	
	Mean	Variance	Mean	Variance	Mean	Variance
80	20.1232	32.9166	20.1162	32.2809	20.1429	32.8836
100	2.3945	14.5559	2.3177	16.4932	2.3145	16.1470
110	0.1493	7.4167	0.2047	24.2876	0.1990	22.3488

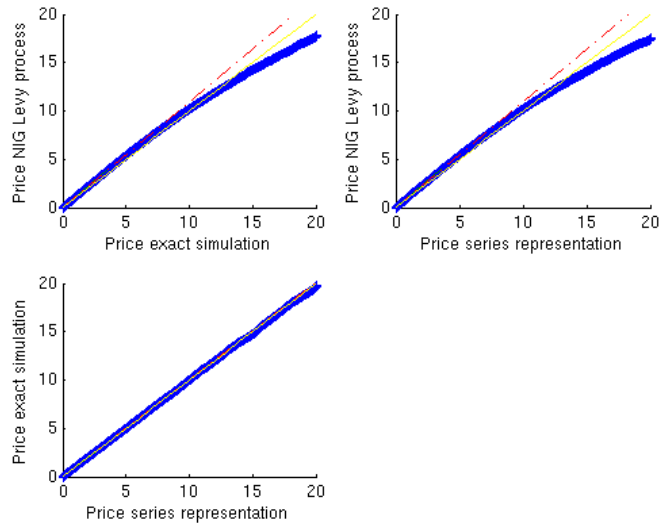


Figure 8.3: QQ-plot of 100000 generated prices.

In case of a strike $K = 80$ the mean of the price processes of the two models is approximately the same. But the price processes of the different models do not coincide (see Figure 8.4). However they coincide in 90% of the cases since 90% of the generated points lie in the interval $[11, 30]$ where the QQ-plot is vertically. In the other 10% of the cases the BNS SV model exaggerates the price of the extremes in comparison to the NIG Lévy process model i.e. BNS SV model will price low priced outliers even lower and high priced outliers higher compared to the NIG Lévy process model.

At least 95% of the of the simulated prices is zero in the case of a strike $K = 110$. So in the QQ-plot (Figure 8.5) it is again visible that the BNS SV model exaggerates the extreme values of the pricing process compared to the NIG Lévy process model.

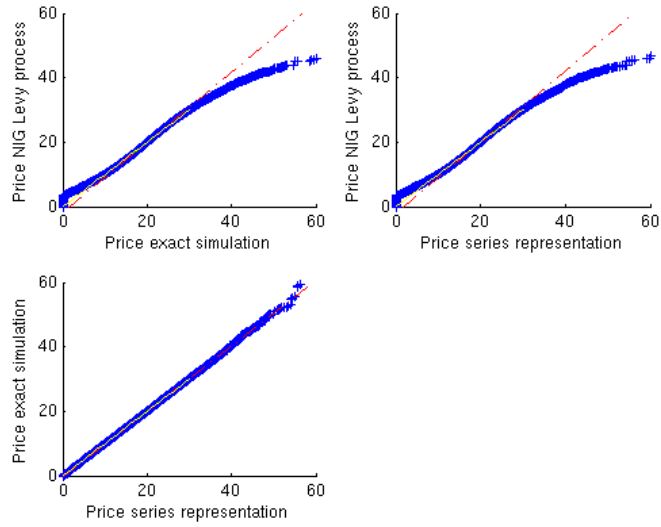


Figure 8.4: QQ-plot of 100,000 generated ING prices with exercise horizon 40 days and strike price $K = 80$.

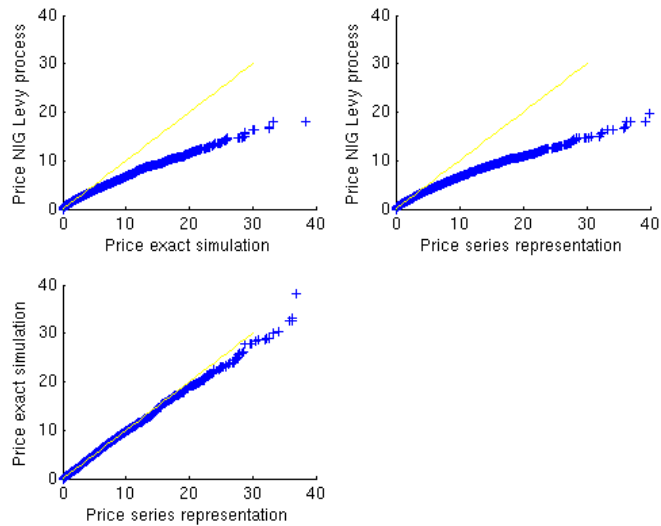


Figure 8.5: QQ-plot of 100,000 generated ING prices with exercise horizon 40 days and strike price $K = 110$.

Bibliography

- [1] O.E. Barndorff-Nielsen and N.Shephard(1977), *Exponentially decreasing distributions for the logarithm of particle size*, Proc. Roy. Soc. London A 353, 401-419.
- [2] O.E. Barndorff-Nielsen and N.Shephard (2001), *Lévy processes for financial econometrics*, Birkhäuser, Boston.
- [3] O.E. Barndorff-Nielsen and N.Shephard (2001), *Non-gaussian OU-based models and some of their uses in financial economics*, Journal of the Royal Statistical Society. Series B, vol 63, No. 2.
- [4] O.E. Barndorff-Nielsen and N.Shephard (2001), *Integrated OU processes and non-Gaussian OU-based stochastic volatility models*, Scandinavian Journal of statistics 2003
- [5] O.E. Barndorff-Nielsen and N.Shephard (2002), *Normal modified stable processes*, Theory Probab. Math. Statist. 65, 1-20.
- [6] F.E. Benth, M. Groth, and P.C. Kettler (2005), *A quasi-monte carlo algorithm for the normal inverse gaussian distribution and valuation of financial derivatives*, dept. of math Univ. of Oslo, Pure mathematics no. 14.
- [7] F.E.Benth and M.Groth (2005), *The minimal entropy martingale measure and numerical option pricing for hte Barndorff-Nielsen-Shephard stochastic volatility model*, E-print no. 26, September 2005, Department of Mathematics, University of Oslo.
- [8] P. Carr and D. Madan (1998), *Option valuation using the fast Fourier transform*, Journal of Computational Finance 2 , pp. 6173.
- [9] E. Eberlein and U. Keller (1995), *Hyperbolic distributions in finance*, Bernoulli, 1:281299.
- [10] J.Jacod, and A.N. Shiryaev (2003), *Limit theorems for stochastic processes*, Springer-Verlag, Berlin-Heidelberg.
- [11] J.Jurek and J.Mason(1993), *Operator-limit distributions in probability theory*, Wiley series in probability and mathematical statistics, New York.
- [12] J.R. Michael, W.R.Schucany, and R.W. Haas (1976), *Generating random variates using transformations with multiple roots*, The American Statistician, Vol. 30, No. 2, pp. 88-90.
- [13] Friedrich Hubalek and Carlo Sgarra(2007), *On the Esscher transforms and other equivalent martingale measures for Barndorff-Nielsen and Shephard stochastic volatility models with jumps*, Research Report number 501 in the Stochastics Series at Department of Mathematical Sciences, University of Aarhus, Denmark.
- [14] E.Nicolato, and E. Venardos (2003), *Option pricing in stochastic volatility models of the OU type*, Mathematical finance, Vol. 13, No. 4, 445-466.
- [15] P.Protter (1990), *Stochastic integration and differential equations*, Springer-Verlag, Berlin-Heidelberg.
- [16] J. Rosinski (2007), *Tempering stable processes*, Stochastic Processes and their Applications 117, 677707.

- [17] T.H.Rydberg (1997), *The normal inverse gaussian Lévy process: simulation and approximation*, Stochastic Models, 13:4, 887-910.
- [18] K.Sato (1999), *Lévy processes and infinitely divisible distributions*, Cambridge University Press, Cambridge
- [19] W.Schoutens (2003), *Lévy processes in finance, pricing derivatives*, Wiley series in probability and statistics. West Sussex, England.
- [20] D.Williams (2005), *Probability with martingales*, Cambridge university press, Cambridge.
- [21] S. Zhang, and X. Zhang (2007), *Exact simulation of IG-OU processes*, Methodol Comput appl probab, Springer Science + business media, LLC 2007.