A toric variety and a 2-parameter family of elliptic curves

Stefan Giesing
Supervisor: J. Stienstra

21st August 2008

Contents

Introduction

In the paper *The complex geometry of the spherical pendulum* [1] the phase space of the complexified spherical pendulum is described as a $\mathbb{C}^*$-bundle over a two-parameter family of elliptic curves. The original aim of the work that led to this paper, was to find some kind of $\mathbb{C}^*$-bundle over the toric family of curves given by equation (1), perhaps allowing a comparison to the bundle of elliptic curves from the aforementioned paper [1]. To this end we look at the family of curves defined by the equation

$$a_1 s^{-1} t^{-1} + a_2 t^{-1} + a_3 s^{-1} + a_4 + a_5 st = 0 \tag{1}$$

for $(s,t) \in \mathbb{C}^* \times \mathbb{C}^*$. The coefficients $a_1, \ldots, a_5$ are what we want to get a clearer picture about.

Our first try has been to realize these curves as a two-parameter family in $\mathbb{P}^1 \times \mathbb{P}^1$. Even after repeated blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$ we do not get a clear one-to-one relation to the parameters, so we don't pursue this further.

The equation above is related to a toric variety: the solution set is given by the intersection of a compactification of $\mathbb{C}^* \times \mathbb{C}^*$ in $\mathbb{P}^4$ with a hyperplane in $\mathbb{P}^4$. As an alternative perspective on the problem, we have subsequently tried to understand more about this toric variety. This has turned out to become the main subject of the paper.

There are different ways to construct a toric variety, but they generally rely on a lattice, which in our case is $\mathbb{Z}^2$, and a fan or polytope. To introduce the notation, we have given a short introduction on affine toric varieties, for the most part following [2]. When we treat $\mathbb{P}^1 \times \mathbb{P}^1$ as an example, we show that the series of blow-ups we conducted in the first section corresponds to modifying the fan of $\mathbb{P}^1 \times \mathbb{P}^1$ by refining some of its cones.

We then consider the fan in $\mathbb{R}^2$ which comes from the polygon pictured on the front page. The exponents of the Laurent monomials in equation (1) correspond to the points of the lattice $\mathbb{Z}^2$ that are also contained in this polygon; they define the action of $\mathbb{C}^* \times \mathbb{C}^*$ on $\mathbb{P}^4$ which leads to the compactification of the torus. We show that the toric variety constructed from the fan has quotient singularities, and that the resolution of these singularities is the same toric variety we arrived at by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$.

Another way to construct the same toric variety is as a quotient of (most of) $\mathbb{C}^4$ by an action of $\mathbb{C}^* \times \mathbb{C}^*$. This leads to the definition of monomials in $(z_1, z_2, z_3, z_4) \in \mathbb{C}^4$ that again correspond to the lattice points of the polygon on the front page, and can be used as homogeneous coordinates on $\mathbb{P}^4$. In these homogeneous coordinates, the toric variety we are interested in is given by the two equations

$$\begin{aligned} w_1 w_4 &= w_2 w_3 \\ w_1 w_5 &= w_4^2 \end{aligned}$$

In the last two sections we return to the family of curves from the beginning. We find a 3-dimensional variety in $\mathbb{P}^4$ as a model for this family, but unfortunately we haven't been able to find any evidence of a relation to the bundle of elliptic curves encountered in the paper [1] at the beginning of this introduction.

1 Blow-ups of $\mathbb{P}^1 \times \mathbb{P}^1$

Consider the following equation for $s, t \in \mathbb{C}^*$:

$$a_1 s^{-1} t^{-1} + a_2 t^{-1} + a_3 s^{-1} + a_4 + a_5 st = 0, \tag{2}$$

and substitute $s = \frac{u_1}{u_2}$, and $t = \frac{v_1}{v_2}$. This leads to a family of elliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$ parametrized by $a_1, a_2, \ldots, a_5 \in \mathbb{C}^*$ given by the equation

$$a_1 {u_2}^2 {v_2}^2 + a_2 u_1 u_2 {v_2}^2 + a_3 {u_2}^2 v_1 v_2 + a_4 u_1 u_2 v_1 v_2 + a_5 {u_1}^2 {v_1}^2 = 0$$

where $([u_1 : u_2], [v_1 : v_2])$ are homogeneous coordinates on $\mathbb{P}^1 \times \mathbb{P}^1$.

First we observe that the number of coëfficients may be reduced by rescaling the coordinates:

$$([u_1 : u_2], [v_1 : v_2]) \mapsto ([\mu u_1 : u_2], [\nu v_1 : v_2]) \quad \mu, \nu \in \mathbb{C}^*$$

and multiplying the equation with $\lambda \in \mathbb{C}^*$. This leads to:

$$\lambda a_1 {u_2}^2 {v_2}^2 + \lambda \mu a_2 u_1 u_2 {v_2}^2 + \lambda \nu a_3 {u_2}^2 v_1 v_2 + \lambda \mu \nu a_4 u_1 u_2 v_1 v_2 + \lambda \mu^2 \nu^2 a_5 {u_1}^2 {v_1}^2 = 0$$

set $\lambda = {a_1}^{-1}$, $\mu = a_1 {a_2}^{-1}$, $\nu = a_1 {a_3}^{-1}$, and this reduces to

$${u_2}^2 {v_2}^2 + u_1 u_2 {v_2}^2 + {u_2}^2 v_1 v_2 + b_1 u_1 u_2 v_1 v_2 + b_2 {u_1}^2 {v_1}^2 = 0 \tag{3}$$

with $b_1 = a_1 {a_2}^{-1} {a_3}^{-1} a_4$, $b_2 = {a_1}^3 {a_2}^{-2} {a_3}^{-2} a_5$.

The aim of this section is to try and find a smooth parametrization of the family of curves defined by equation (3).

Clearly, all curves K^{b_1, b_2} given by (3) pass through the points $p = ([0 : 1], [1 : 0])$ and $q = ([1 : 0], [0 : 1])$. We want to lift this degeneracy by blowing up these two points. First we construct the blow-up at p; define

$$\mathrm{Bl}_p = \{([u_1 : u_2], [v_1 : v_2], [w_1 : w_2]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 | u_1 w_1 = v_2 w_2\}$$

With the projection

$$\begin{array}{c} \mathrm{Bl}_p \\ \downarrow \pi \\ \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

onto the first two factors. Clearly at p, where $u_1, v_2 = 0$, the total inverse of this projection is $\pi^{-1}(p) = \{p\} \times \mathbb{P}^1$, which is called the *exceptional curve* over

p. Over any other point $([u_1 : u_2], [v_1 : v_2]) \in \mathbb{P}^1 \times \mathbb{P}^1$ the value of $[w_1 : w_2]$ is fixed by the equation $u_1 w_1 = v_2 w_2$, making the projection restricted to a suitable neighborhood of that point an isomorphism. In effect, we have replaced the point p with a copy of $\mathbb{P}^1$ leaving the rest of $\mathbb{P}^1 \times \mathbb{P}^1$ the same.

Next we also blow up the point q. Define

$$\begin{aligned}\mathrm{Bl}_{p,q} = \{([u_1 : u_2], [v_1 : v_2], [w_1 : w_2], [x_1 : x_2]) \in & \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 | \\ & u_1 w_1 = v_2 w_2, \quad u_2 x_2 = v_1 x_1\}\end{aligned}$$

with again the projection onto the first two factors

$$\begin{tikzcd} \mathrm{Bl}_{p,q} \arrow[d, "\pi"] \\ \mathbb{P}^1 \times \mathbb{P}^1 \end{tikzcd}$$

There are now two exceptional curves $\mathrm{E}_p = \pi^{-1}([0 : 1], [1 : 0])$ and $\mathrm{E}_q = \pi^{-1}([1 : 0], [0 : 1])$ over the points p and q respectively.

We now want to investigate what happens to the family K^{b_1,b_2} after blowing up the points p and q. This amounts to looking at the closure of the inverse image of K^{b_1,b_2} under π: $\widetilde{K}^{b_1,b_2} := \overline{\pi^{-1}(K^{b_1,b_2} - \{p\} - \{q\})}$, which is called the *strict transform* of K^{b_1,b_2}.

We pass to affine coordinates $(u, v) = (\frac{u_1}{u_2}, \frac{v_2}{v_1})$ in the neighborhood of the point p, and set $u_2 = v_1 = 1$. Equation (3) takes the form:

$$v^2 + uv^2 + v + b_1 uv + b_2 u^2 = 0 \tag{4}$$

The equations

$$\begin{aligned} u_1 w_1 &= v_2 w_2 \\ u_2 x_2 &= v_1 x_1 \end{aligned}$$

defining the set $\mathrm{Bl}_{p,q}$ reduce to

$$\begin{aligned} u w_1 &= v w_2 \\ x_1 &= x_2 \end{aligned}$$

So $\mathrm{Bl}_{p,q}$ reduces to $X = \{(u, v, [w_1 : w_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 | uw_1 = vw_2, \}$ And finally the projection $\mathrm{Bl}_{p,q} \xrightarrow{\pi} \mathbb{P}^1 \times \mathbb{P}^1$ becomes

$$\begin{tikzcd} X \arrow[d] \\ \mathbb{C}^2 \end{tikzcd}$$

This is $\mathbb{C}^2$ with the origin blown up, and the strict transform of the family ${K_p}^{b_1,b_2}$ given by (4) is the closure of the inverse image of ${K_p}^{b_1,b_2} - \{0\}$ under the projection (which is an isomorphism away from the origin.) Now suppose that $w_2 \neq 0$ so we can take w_2 equal to 1 and use $w = w_1$ as an affine coordinate for the exceptional curve. Then $v = uw$; substituting this into (4) gives

$$u(uw^2 + u^2w^2 + w + b_1uw + b_2u) = 0.$$

So we have on the one hand $u = 0$, $v = 0$ with w arbitrary, which gives the exceptional curve, while the equation

$$uw^2 + u^2w^2 + w + b_1uw + b_2u = 0 \tag{5}$$

together with $v = uw$ defines the strict transform $\widetilde{K_p}^{b_1,b_2} \subset X$ of ${K_p}^{b_1,b_2}$. Setting $u = 0$ implies $w = 0$ by equation (5), so for any b_1, b_2 the curve $\widetilde{K_p}^{b_1,b_2}$ meets the exceptional curve in $[w_1 : w_2] = [0 : 1]$. This means that all curves in the family $\widetilde{K}^{b_1,b_2}$ pass through the point $([0 : 1], [1 : 0], [0 : 1], [1 : 1])$ in $\mathrm{Bl}_{p,q}$. Note that the strict transform of the u-axis given by $v = 0$ is given by $uw = 0$, so the u-axis intersects the exceptional curve in the same point $([0 : 1], [1 : 0], [0 : 1], [1 : 1])$.

The same can be done at the point q: passing to affine coordinates $(u, v) = (u_2, v_1)$ reduces equation (3) to

$$u^2 + u + u^2v + b_1uv + b_2v^2 = 0 \tag{6}$$

which defines the family ${K_q}^{b_1,b_2} \subset \mathbb{C}^2$. Again we consider the strict transform of ${K_q}^{b_1,b_2}$ under the blow-up of $\mathbb{C}^2$ in the origin given by

$$\begin{array}{ccl} U & & = \{(u, v, [x_1 : x_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 | ux_2 = vx_1\} \,. \\ \big\downarrow & & \\ \mathbb{C}^2 & & \end{array}$$

Temporarily assuming $x_2 \neq 0$, and setting $x_2 = 1,\ x_1 = x$ we derive

$$\begin{aligned} &u = vx \\ &v(vx^2 + x + v^2x^2 + b_1vx + b_2v^2) = 0 \end{aligned}$$

Then if $vx^2 + x + v^2x^2 + b_1vx + b_2v^2 = 0$, $v = 0$ implies $x = 0$ so ${K_q}^{b_1,b_2}$ intersects the exceptional curve in $[x_1 : x_2] = [0 : 1]$ for all (b_1, b_2). Going back to homogeneous coordinates we see that all curves in $\widetilde{K}^{b_1,b_2}$ pass through $([1:0],[0:1],[1:1],[0:1])$ in $\mathrm{Bl}_{p,q}$. This is also the point where the exceptional curve meets the v-axis given by $u=0$.

So now we have the family of curves $\widetilde{K}^{b_1,b_2}$ in $\mathrm{Bl}_{p,q}$ with each curve going through the points $p' = ([0:1],[1:0],[0:1],[1:1])$ and $q' = ([1:0],[0:1],[1:1],[0:1])$. This is somewhat disappointing since we are looking for some restriction on the parameters (b_1, b_2). On the other hand it isn't very surprising either: not only does every curve given by (3) pass through p and q, they also all have the same tangent direction in these points. Since blowing up a point is essentially replacing that point with all tangent directions at that point, where each tangent direction is given by a point on the exceptional curve, it is to be expected that all curves in K^{b_1,b_2} intersect the exceptional curves over p and q in the same point. We will try again by blowing up p' and q' once more and calculate the intersection of the family K^{b_1,b_2} with the exceptional curves over p' and q'
To this end, define

$$\begin{aligned}\mathrm{Bl}_{p',q'} = \{([u_1:u_2],[v_1:v_2],&[w_1:w_2],[x_1:x_2],[y_1:y_2],[z_1:z_2]) \in \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 | \\ &u_1w_1 = v_2w_2,\ u_2x_2 = v_1x_1,\ u_1y_1 = w_1y_2,\ v_1z_1 = x_1z_2\}\end{aligned}$$

With the projection

$$\begin{array}{c}\mathrm{Bl}_{p',q'} \\ \downarrow{\scriptstyle \pi'} \\ \mathrm{Bl}_{p,q}\end{array}$$

induced by projection onto the first four factors of $\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1\times\mathbb{P}^1$.

We will first consider the point p'. Choose affine coordinates $(u,v,w) = (u_1, v_2, w_1)$ around p' so that we get:

$$\begin{array}{ccc} Y & = & \{(u,v,w,[y_1:y_2]) \in \mathbb{C}^3 \times \mathbb{P}^1 | uy_1 = wy_2,\ v = uw\} \\ \downarrow & & \\ X' & = & \{(u,v,w) \in \mathbb{C}^3 | v = uw\} \\ \downarrow & & \\ \mathbb{C}^2 & = & \{u, w \in \mathbb{C}^2\} \end{array}$$

The family K^{b_1,b_2} is given by

$$uw^2 + u^2w^2 + w + b_1uw + b_2u = 0 \tag{7}$$

in X'. To find the equations defining the strict transform in Y, we look at the affine part of Y where $y_2 \neq 0$, with coordinate $y = y_1$, and obtain

$$\begin{aligned} w &= uy \\ u(u^2y^2 &+ u^3y^2 + y + b_1uy + b_2) = 0 \end{aligned}$$

If $u^2y^2 + u^3y^2 + y + b_1uy + b_2$, then $u = 0$, gives $y = -b_2$. Going back to $\mathrm{Bl}_{p',q'}$, this means that K^{b_1,b_2} intersects the exceptional curve over p' in the point

$$([u_1 : u_2], [v_1 : v_2], [w_1 : w_2], [x_1 : x_2], [y_1 : y_2], [z_1 : z_2]) = \\ ([0 : 1], [1 : 0], [0 : 1], [1 : 1], [-b_2 : 1], [1 : 1])$$

Now for the point q'. Using affine coordinates $(u, v, x) = (u_2, v_1, x_1)$ we get

$$\begin{array}{ccc} V & = & \{(u, v, x, [z_1 : z_2]) \in \mathbb{C}^3 \times \mathbb{P}^1 | vz_1 = xz_2,\ u = vx\} \\ \downarrow & & \\ U' & = & \{(u, v, x) \in \mathbb{C}^3 | u = vx\} \\ \downarrow & & \\ \mathbb{C}^2 & = & \{(v, x) \in \mathbb{C}^2\} \end{array}$$

with $vx^2 + x + v^2x^2 + b_1vx + b_2v^2 = 0$ defining the affine family of curves corresponding to K^{b_1,b_2} in U'. Assuming $z_2 \neq 0$ for the moment, setting $z = z_1$ and substituting $x = vz$, we find the strict transform given by $v^2z^2 + z + v^3z^2 + b_1zv + b_2 = 0$, which clearly intersects the exceptional curve in $(0, 0, [-b_2 : 1]) \in V$.

This means that all members of K^{b_1,b_2} pass through $([1 : 0], [0 : 1], [1 : 1], [1 : 0], [1 : 1], [-b_2 : 1])$ in $\mathrm{Bl}_{p',q'}$.

The conclusion is that if we fix b_1, we get a unique elliptic curve through the points $([1 : 0], [0 : 1], [1 : 1], [1 : 0], [1 : 1], [-b_2 : 1])$ and $([0 : 1], [1 : 0], [0 : 1], [1 : 1], [-b_2 : 1], [1 : 1])$ for every b_2.

The aim was to obtain a smooth bundle of elliptic curves parametrized by $(b_1, b_2) \in \mathbb{C}^{*2}$. Trying to resolve the degeneracy in b_1 and b_2 rather directly by (repeatedly) blowing up the points p and q only works for the parameter b_2.

The point of departure at the beginning of this section was the equation

$$a_1 s^{-1}t^{-1} + a_2 t^{-1} + a_3 s^{-1} + a_4 + a_5 st = 0$$

for $s, t \in \mathbb{C}^* \times \mathbb{C}^*$. By passing to $\mathbb{P}^1 \times \mathbb{P}^1$ we have actually compactified $\mathbb{C}^* \times \mathbb{C}^*$ in $\mathbb{P}^1 \times \mathbb{P}^1$ by adding the four lines corresponding $u_1 = 0, u_2 = 0, v_1 = 0, v_2 = 0$ to it. By blowing up the points p, q, p', q' we have added an additional four projective lines to $\mathbb{P}^1 \times \mathbb{P}^1$, so without being too precise about it we can say we have added 8 lines to $\mathbb{C}^* \times \mathbb{C}^*$ resulting in the octagon in Figure 1 (where for example "E_p"denotes the appropriate affine part of the exceptional curve)

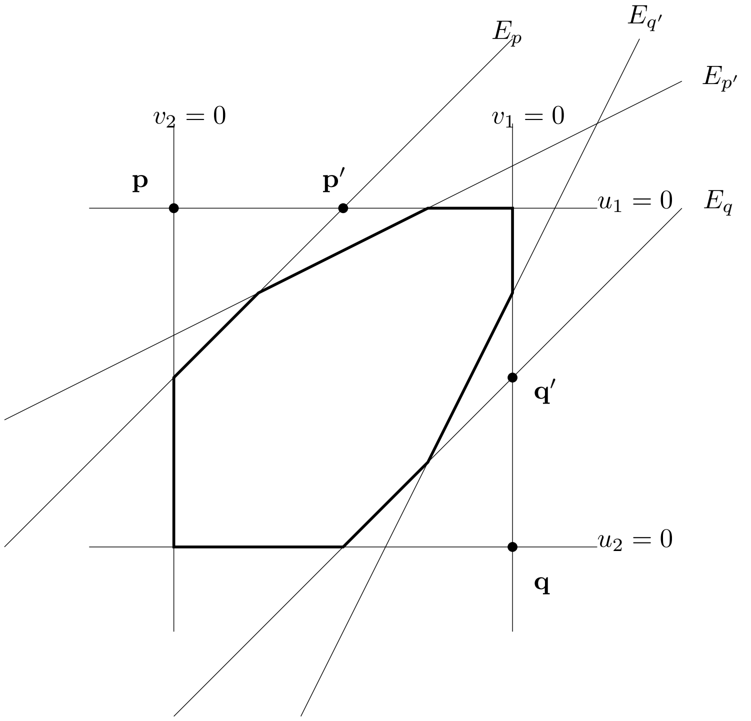


Figure 1: 8 lines added to $\mathbb{C}^* \times \mathbb{C}^*$

<u>Coordinates</u>

For later reference, we list the various equations giving the strict transform of equation (3) in affine coordinates after the appropriate substitutions.

coordinates	equation $K^{b_1,b_2}=0$
$(u,v)=(u_1,v_2)$	$v^2+uv^2+v+b_1uv+b_2u^2=0$
$(u,w)=(u_1,w_1)$	$uw^2+u^2w^2+w+b_1uw+b_2u=0$
$(u,y)=(u_1,y_1)$	$u^2y^2+u^3y^2+y+b_1uy+b_2=0$
$(u,v)=(u_2,v_1)$	$u^2+u+u^2v+b_1uv+b_2v^2=0$
$(v,x)=(v_1,x_1)$	$vx^2+x+v^2x^2+b_1vx+b_2v^2=0$
$(v,z)=(v_1,z_1)$	$v^2z^2+z+v^3z^2+b_1zv+b_2=0$

Table 1: affine coordinates for $\mathrm{Bl}_{p',q'}$

2 Toric varieties

2.1 Affine toric varieties

In this section we will show how to construct the toric variety corresponding to a fan in $\mathbb{R}^n$. To clarify the notation we will first sketch the general case. More details can be found in Chapter 1 of [2].

Let an action $\mathbb{C}^{*n} \times X \to X$ of the algebraic torus $\mathbb{C}^*$on a (normal) variety X be given. Then X is called a toric variety if it contains the torus $\mathbb{C}^{*n}$ as an open dense orbit (in the Zariski topology) such that the natural action of $\mathbb{C}^{*n}$ on itself is extended to the given action of $\mathbb{C}^{*n}$ on X.

A important example is projective space $\mathbb{P}^n$: the action $\mathbb{C}^{*n}$ on $\mathbb{P}^n$ is defined by

$$((\lambda_1, ..., \lambda_n), [x_0 : ... : x_n]) \longrightarrow [x_0 : \lambda_1 x_1 : ... : \lambda_n x_n],$$

so $\mathbb{P}^n$ contains $\mathbb{C}^{*n}$ as the orbit through $[1 : ... : 1]$. To see that the closure of this orbit is $\mathbb{P}^n$, note that it is the complement of the Zariski closed set defined by $x_0 x_1 \cdots x_n = 0$ in $\mathbb{P}^n$.

A useful and characteristic feature of (normal) toric varieties is that they can be constructed from combinatorial data encoded in so-called *fans*. But we start with the definition of the affine toric variety corresponding to a *strongly convex rational polyhedral cone* in $\mathbb{R}^n$, because a fan is a collection of such cones.

Definition 2.1. A *polyhedral cone* in $\mathbb{R}^n$ is a set

$$\sigma = \{a_1 v_1 + \cdots + a_n v_n \in \mathbb{R}^n | a_i \geq 0\}$$

generated by a finite number of $v_i \in \mathbb{R}^n$. Such a cone is *rational* if its generators are vectors in $\mathbb{Z}^n$. In addition, it is *simplicial* if the generators form a basis for $\mathbb{R}^n$, and *smooth* if they form a $\mathbb{Z}$-basis for $\mathbb{Z}^n$. Finally it is *strongly convex* if it contains no line through the origin.

Remark 2.1. Since we will only be dealing with strongly convex rational polyhedral cones, from now on, when we write *cone*, we mean *strongly convex rational polyhedral cone*.

If V is any vector space, and $V^\vee$ the dual vector space, we will denote $u(v) = \langle u, v \rangle$ for any two elements $u \in V^\vee$, $v \in V$.

If σ is a cone in $\mathbb{R}^n$ then its *dual* cone $\sigma^\vee \subset \mathbb{R}^{n\vee}$ is the set

$$\sigma^\vee = \{u \in \mathbb{R}^{n\vee} | \langle u, v \rangle \geq 0 \quad \forall v \in \sigma\}$$

If σ is a convex rational polyhedral cone, then $\sigma^\vee$ is as well, see [2] Section 1.2.

Now we can associate to σ a commutative semigroup S_σ:

$$S_\sigma = \sigma^\vee \cap \mathbb{Z}^{n\vee} = \{u \in \mathbb{Z}^{n\vee} |\, \langle u, v\rangle \geq 0 \quad \forall v \in \sigma\}$$

Lemma 2.1 (*Gordan's lemma*)**.** *Let $\sigma \subset \mathbb{R}^n$ be a convex rational polyhedral cone, and $\sigma^\vee$ its dual cone. Then S_σ as defined above is finitely generated.*

Proof. Let $u_1, \ldots, u_k \in \sigma^\vee \cap \mathbb{Z}^{n\vee}$ be generators for $\sigma^\vee$ as a cone in $\mathbb{R}^{n\vee}$. Define $K = \{\sum t_i u_i | 0 \leq t_i \leq 1\}$, then $K \cap \mathbb{Z}^n$ is finite since K is compact. It suffices to show that K generates S_σ. Let $u \in S_\sigma$ and write $u = \sum r_i u_i$ with $r_i \geq 0$, then $r_i = m_i + t_i$ with $0 \leq t_i \leq 1$ and m_i a nonnegative integer. So $u = \sum m_i u_i + \sum t_i u_i$ with $\sum t_i u_i \in K \cap \mathbb{Z}^{n\vee}$ and each $u_i \in K \cap \mathbb{Z}^{n\vee}$. $\square$

The semigroup algebra $\mathbb{C}[S_\sigma]$ is the complex vector space generated by elements χ^u, $u \in S_\sigma$ with multiplication given by

$$\chi^u \cdot \chi^v = \chi^{u+v}$$

The toric variety X_σ associated with the cone σ is the maximal spectrum of this algebra

$$X_\sigma = \mathrm{Spec}(\mathbb{C}[S_\sigma])$$

Example 2.1. Let $\{e_1, \ldots, e_n\}$ be a basis for $\mathbb{R}^n$, and $\{{e_1}^*, \ldots, {e_n}^*\}$ the dual basis. Let $\sigma \subset \mathbb{R}^n$ be the cone generated by $e_1, \ldots, e_k$, $k \leq n$. Then $u \in \sigma^\vee$ if and only if $\langle u, v\rangle \geq 0$ $\forall v \in \sigma$, which is equivalent to $u(e_i) \geq 0$ for all $0 \leq i \leq k$. This means that the semigroup $S_\sigma = \sigma^\vee \cap \mathbb{Z}^{n\vee}$ is

$$S_\sigma = \{u \in \mathbb{Z}^{n\vee} | u = a_1 {e_1}^* + \ldots + a_n {e_n}^*,\ a_1, \ldots, a_k \in \mathbb{Z}_{\geq 0},\ a_{k+1}, \ldots, a_n \in \mathbb{Z}\},$$

so S_σ is generated by ${e_1}^*, \ldots, {e_k}^*, \pm {e_{k+1}}^*, \ldots, \pm {e_n}^*$. If we denote $X_i := \chi^{{e_i}^*}$, then the complex algebra $\mathbb{C}[S_\sigma]$ is given by

$$\mathbb{C}[S_\sigma] = \mathbb{C}[X_1, \ldots, X_k, X_{k+1}, X_{k+1}^{-1}, \ldots, X_n, X_n^{-1}]$$

and the toric variety associated to σ is

$$X_\sigma = \underbrace{\mathbb{C} \times \ldots \times \mathbb{C}}_{k} \times \underbrace{\mathbb{C}^* \times \ldots \times \mathbb{C}^*}_{n-k} \tag{8}$$

In general, if a cone is generated by vectors which can be completed to a basis for $\mathbb{Z}^n$, the associated toric variety will be of the form (8), and in particular, it will be non-singular.

2.2 Toric varieties

General toric varieties are obtained by 'gluing together' affine toric varieties X_σ. We will need the definition of a fan, but first we need to specify what we mean by the *face* of a cone:

Definition 2.2. Let $\sigma \in \mathbb{R}^n$ be a cone. A *face* τ of σ is given by

$$\tau = \sigma \cap u^\perp = \{v \in \sigma | \langle u, v \rangle = 0\}$$

for some $u \in \sigma^\vee$, where

$$u^\perp = \{v \in \mathbb{R}^n | \langle u, v \rangle = 0\}$$

is a *supporting hyperplane* for τ if $u \in \sigma^\vee$. A cone is regarded as a face of itself.

Definition 2.3. A fan Σ is a set of cones (in the sense of Remark 2.1 above) which satisfy the following conditions:

(i) Any face of a cone in Σ is also a cone in Σ.

(ii) The intersection $\sigma \cap \sigma'$ of two cones $\sigma, \sigma' \in \Sigma$ is a face of σ as well as σ'.

If σ is a cone, and τ is a face of σ, then X_τ is a Zariski open subset of X_σ. (See [2] Section 1.3.) If τ is a common face of two cones σ and σ', then we glue the two affine varieties X_σ and $X_{\sigma'}$ together by identifying them on the open subset X_τ.

If Σ is a fan, the toric variety X_Σ is obtained as the disjoint union of all affine varieties X_σ corresponding to cones σ in Σ, which are glued together according to the rule sketched above.

Note that if $\sigma \subset \mathbb{R}^n$ is any cone, the origin is always a face of σ. It follows that the affine toric variety X_σ always contains $X_{\{0\}}$ as a Zariski open subset. Considering $\{0\}$ as a cone in $\mathbb{R}^n$, the corresponding semigroup $S_{\{0\}}$ equals all of $\mathbb{Z}^n$, so $\mathbb{C}[S_{\{0\}}] = \mathbb{C}[X_1, X_1^{-1}, \dots, X_n, X_n^{-1}]$ which means that $X_{\{0\}} = \mathbb{C}^{*n}$. This shows how any toric variety X_Σ corresponding to a fan Σ contains $\mathbb{C}^{*n}$ as an open dense orbit, in accordance with the general definition af a toric variety which was mentioned at the beginning of this section.

2.3 Gluing maps

From this point we will only be concerned with toric surfaces, i.e. toric varieties arising from fans in $\mathbb{R}^2$ and the lattice $\mathbb{Z}^2$.

Suppose that σ, σ' are two-dimensional cones in $\mathbb{R}^2$, we want to describe the gluing map $X_\sigma \to X_{\sigma'}$ more explicitly. Let $\{x, y\} \in \mathbb{Z}^2$ and $\{z, y\} \in \mathbb{Z}^2$ be the minimal vectors (i.e. not an integer multiple of any other vector) generating the edges of σ and σ' respectively, see Figure 2.

The vectors

$$\{u = \begin{pmatrix} x_2 \\ -x_1 \end{pmatrix}, v = \begin{pmatrix} -y_2 \\ y_1 \end{pmatrix}\}$$
$$\{s = \begin{pmatrix} -z_2 \\ z_1 \end{pmatrix}, t = -v\}$$

are the minimal vectors generating the edges of $\sigma^\vee$ and $\sigma'^\vee$.

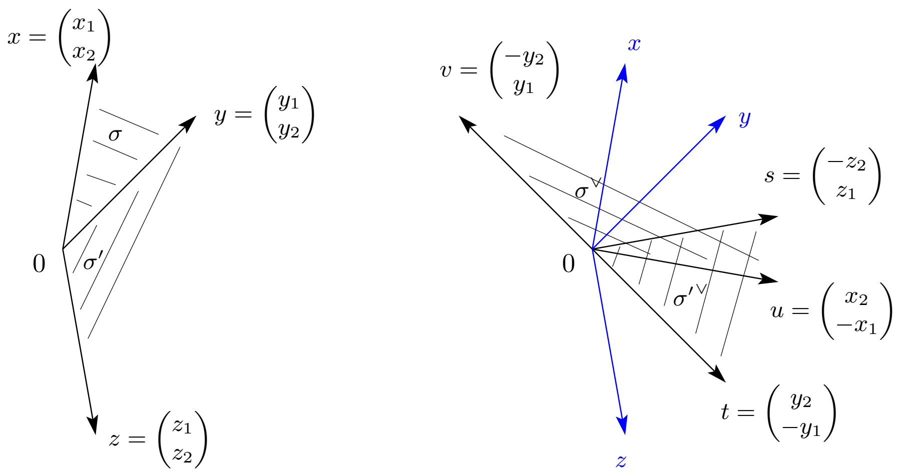


Figure 2: The cones σ, σ' and their dual cones

If the cones σ and σ' are smooth, then by definition the pairs of vectors (u, v) and (s, t) are both a $\mathbb{Z}$-basis for $\mathbb{Z}^2$, so they generate the semigroups $S_\sigma = \sigma^\vee \cap \mathbb{Z}^2$ and $S_{\sigma'} = \sigma'^\vee \cap \mathbb{Z}^2$. In that case $\mathbb{C}[S_\sigma] = \mathbb{C}[\chi^u, \chi^v] = \mathbb{C}[U, V]$, and $X_\sigma = \operatorname{Spec} \mathbb{C}[U, V] = \mathbb{C}^2$. In the same way $X_{\sigma'} = \mathbb{C}^2$.

The common face of σ and σ' is the halfline generated by y. The associated complex algebra is $\mathbb{C}[\chi^y, \chi^v, \chi^{-v}]$, and to glue X_σ and $X_{\sigma'}$ together we identify them on the common open subset $\operatorname{Spec}(\mathbb{C}[\chi^y, \chi^v, \chi^{-v}]) \cong \mathbb{C} \times \mathbb{C}^*$. To find the gluing map, we represent $(u, v) \in S_\sigma$ on the basis $(s, t) \in S_{\sigma'}$:

$$s = pu + qv$$
$$t = -v$$

with $p, q \in \mathbb{Z}$. Using 2×2 matrices and the notation $(u \;\; v) = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ these equations become

$$(s \;\; t) = (u \;\; v) \begin{pmatrix} p & 0 \\ q & -1 \end{pmatrix}, \tag{9}$$

and (u, v) in terms of (s, t) is given by

$$(u \;\; v) = (s \;\; t) \begin{pmatrix} \frac{1}{p} & 0 \\ \frac{q}{p} & -1 \end{pmatrix}$$

Note that (s, t) is a $\mathbb{Z}$-basis if and only if

$$\det \begin{pmatrix} p & 0 \\ q & -1 \end{pmatrix} = \pm 1$$

so $p = \pm 1$. The sign of p depends on the orientation of (u, v) and (s, t), which in this case is opposite, so $p = 1$.

Denote $(U, V) = (\chi^u, \chi^v)$ and $(S, T) = (\chi^s, \chi^t)$, then $(S, T) = (\chi^s, \chi^t) = (\chi^{u+qv}, \chi^{-v}) = (UV^q, V^{-1})$, and $(U, V) = (\chi^u, \chi^v) = (\chi^{s \pm qt}, \chi^{-t}) = (ST^{-q}, T^{-1})$. For $(x, y) \in \mathbb{C} \times \mathbb{C}^*$ we get the gluing isomorphism

$$(x, y) \longmapsto (xy^q, y^{-1}) \tag{10}$$

Now suppose that (s, t) is not a $\mathbb{Z}$-basis, then $p \neq 1$ and the above map becomes

$$(x, y) \longmapsto (\sqrt[p]{xy^q}, y^{-1})$$

which is not an isomorphism at $x = 0$.

The above illustrates how the combinatorial data from the fan and lattice are related to properties of the associated variety. In particular, if all the cones in the fan $\Sigma \subset \mathbb{R}^2$ are smooth, they are glued together with isomorphisms (10), and the result will be a smooth complex surface.

We will come back to this when we discuss the main example of a toric variety in this paper related to the family of curves from Section 1.

But first we want to show a familiar example of a toric variety, and show how the construction of blowing up points we used in section 1 arises in the context of fans and lattices.

2.4 Example: $\mathbb{P}^1 \times \mathbb{P}^1$

Let Σ be the fan pictured in Figure 3, leaving out the dashed line for now. The complex algebras corresponding to the cones σ_i are $\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[X, Y]$, $\mathbb{C}[S_{\sigma_2}] = \mathbb{C}[X, Y^{-1}]$, $\mathbb{C}[S_{\sigma_3}] = \mathbb{C}[X^{-1}, Y^{-1}]$, $\mathbb{C}[S_{\sigma_4}] = \mathbb{C}[X^{-1}, Y]$, where we used the notation $X := \chi^{e_1^*}$, $Y := \chi^{e_2^*}$. Clearly, Each X_{σ_i} is a copy of $\mathbb{C}^2$. The common faces of the adjacent σ_i are the rays from the origin generated by the standard basisvectors of $\mathbb{R}^2$.

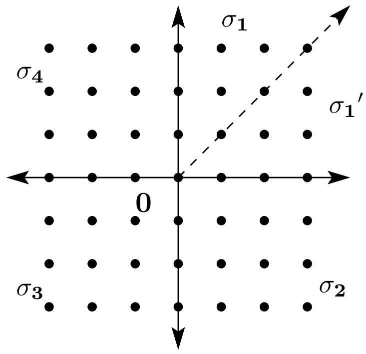


Figure 3: Fan corresponding to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in a point

Denote the affine variety corresponding to $\pm e_i$ by $X_{\pm e_i}$, then by example 2.1 $X_{e_1} = \mathbb{C} \times \mathbb{C}^*$, $X_{e_2} = \mathbb{C}^* \times \mathbb{C}$, $X_{-e_1} = \mathbb{C} \times \mathbb{C}^*$, $X_{-e_2} = \mathbb{C}^* \times \mathbb{C}$. Gluing together the four copies of $\mathbb{C}^2$ along common open subsets $X_{\pm e_i}$ results in the toric variety $\mathbb{P}^1 \times \mathbb{P}^1$. Take for example X_{σ_1} and X_{σ_2}, then

$$X_{\sigma_1} \hookleftarrow \mathbb{C} \times \mathbb{C}^* \hookrightarrow X_{\sigma_2}$$

with the gluing isomorphism $(x, y) \mapsto (x, \frac{1}{y})$. Set $X = \frac{u_1}{u_2}$, $Y = \frac{v_1}{v_2}$, then the X_{σ_i} are the four affine charts of $\mathbb{P}^1 \times \mathbb{P}^1$, with $([u_1 : u_2], [v_1 : v_2])$ homogeneous co-ordinates:

$$\begin{array}{ccc}
X_{\sigma_1} = \{(\frac{u_1}{u_2}, \frac{v_1}{v_2}) \in \mathbb{C}^2 | u_2 \neq 0,\ v_2 \neq 0\} & \overset{(x,\frac{1}{y})}{\longleftrightarrow} & X_{\sigma_2} = \{(\frac{u_1}{u_2}, \frac{v_2}{v_1}) \in \mathbb{C}^2 | u_2 \neq 0,\ v_1 \neq 0\} \\
\updownarrow (\frac{1}{x}, y) & & \updownarrow (\frac{1}{x}, y) \\
X_{\sigma_4} = \{(\frac{u_2}{u_1}, \frac{v_1}{v_2}) \in \mathbb{C}^2 | u_1 \neq 0,\ v_2 \neq 0\} & \overset{(x,\frac{1}{y})}{\longleftrightarrow} & X_{\sigma_3} = \{(\frac{u_2}{u_1}, \frac{v_2}{v_1}) \in \mathbb{C}^2 | u_1 \neq 0,\ v_1 \neq 0\}
\end{array}$$

Now we take the dashed line in Figure 3 into account. This leaves al cones with the exception of σ_1 unaffected. The cone σ_1 is divided into σ_1 and $\sigma_1{}'$ which results in two semigroups S_{σ_1} generated by e_1^* and $e_2^* - e_1^*$, and $S_{\sigma_1{}'}$

generated by $e_1^* - e_2^*$ and e_2^*.The corresponding complex algebras $\mathbb{C}[S_{\sigma_1}]$ and $\mathbb{C}[S_{\sigma_1'}]$ are then given by

$$\mathbb{C}[S_{\sigma_1}] = \mathbb{C}[X, X^{-1}Y], \quad \mathbb{C}[S_{\sigma_1'}] = \mathbb{C}[Y, XY^{-1}],$$

so X_{σ_1} and $X_{\sigma_1'}$ are both isomorphic to $\mathbb{C}^2$, and glued together along $X_{e_1^*+e_2^*} = \mathbb{C} \times \mathbb{C}^*$ via the map $(u, v) \mapsto (uv, v^{-1})$.

The blow-up of $\mathbb{C}^2$ at the origin can be realized as the set $B = \{(x, y, [w_1 : w_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 | xw_1 = yw_2\}$ with projection onto $\mathbb{C}^2$. The set B has an open cover by the two sets $U_1 = \{(x, y, [w_1 : w_2]) | w_2 \neq 0\}$ and $U_2 = \{(x, y, [w_1 : w_2]) | w_1 \neq 0\}$, with co-ordinates $(x, \frac{w_1}{w_2})$ on U_1, $(y, \frac{w_2}{w_1})$ on U_2, and gluing given by $(u, v) \mapsto (uv, v^{-1})$. Since $\frac{w_1}{w_2} = x^{-1}y$, and $\frac{w_2}{w_1} = xy^{-1}$, we have $X_{\sigma_1} = U_1$ and $X_{\sigma_1'} = U_2$ with the same gluing.

The result is $\{([u_1 : u_2], [v_1 : v_2], [w_1 : w_2]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 | u_1w_1 = v_1w_2\}$, which is $\mathbb{P}^1 \times \mathbb{P}^1$ blown up in the point $([0 : 1], [0 : 1])$.

What we have done is to refine the fan defining $\mathbb{P}^1 \times \mathbb{P}^1$ by subdividing one of its cones. In general, Σ' is a *refinement* of the fan Σ, if any cone of Σ is a union of cones in Σ'. In that case one gets a birational, proper map $X_{\Sigma'} \to X_\Sigma$ between the associated toric varieties, which in the case of this example is the projection

$$\begin{array}{c} \{([u_1 : u_2], [v_1 : v_2], [w_1 : w_2]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 | u_1w_1 = v_1w_2\} \\ \downarrow \\ \mathbb{P}^1 \times \mathbb{P}^1 \end{array}$$

In the case of a cone $\sigma \subset \mathbb{R}^2$ which is not smooth, which is to say that the vectors generating the edges of the cone do not form a basis of the lattice, refining the cone provides a way to resolve the singularity due to the non-smoothness of σ. This will be discussed further in the next section.

Recall that in Section 1, we constructed the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in the four points p, q, p', q'. Following the above example, we can now perform the same construction in terms of subdividing the fan of $\mathbb{P}^1 \times \mathbb{P}^1$: blowing up the points p, q, and then p', q' amounts to adding the edges:

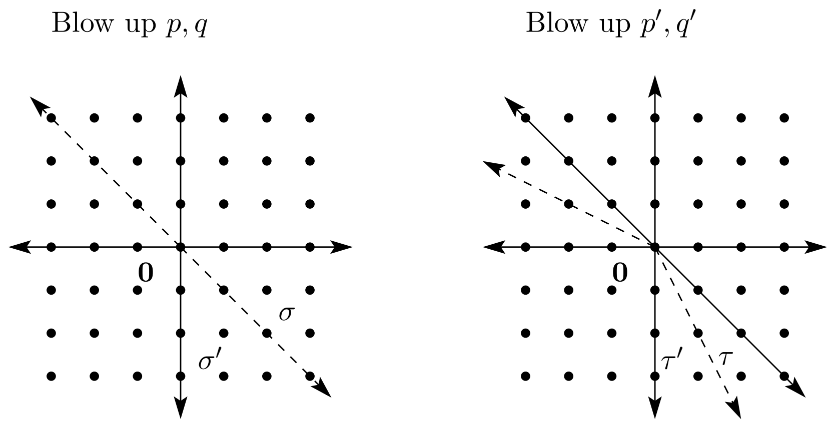


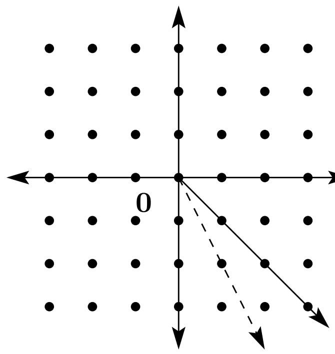
Figure 4: Blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ by refining the fan

To show that the toric variety associated to the refined fan is indeed the same as $\mathrm{Bl}_{p',q'}$ defined in Section 1, consider the cones σ, σ' and τ, τ' in Figure 4. We use the same homogeneous coordinates for $\mathbb{P}^1 \times \mathbb{P}^1$ as above. Clearly, by the calculations above, the subdivision σ, σ' corresponds to the blow-up of $p = ([0 : 1], [1 : 0])$. In Section 1 the next step was to blow up the point p' which is the point where the exceptional divisor intersects the (strict transform of the) u_1-axis given by $v_2 = 0$.

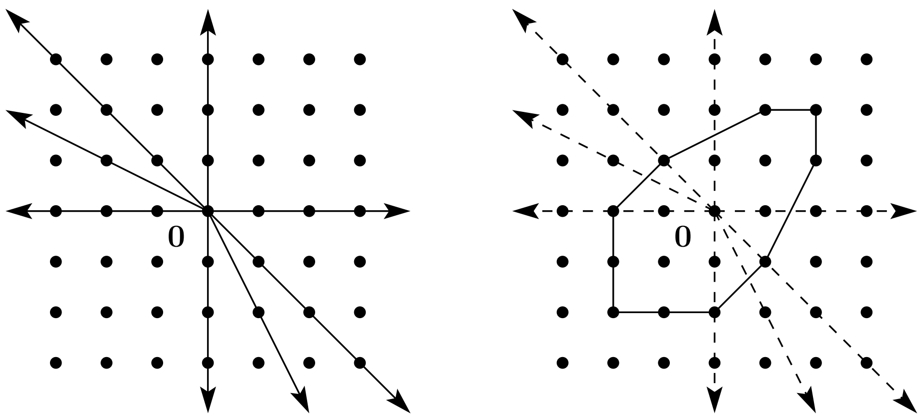
Above we had the blow-up of $\mathbb{C}^2$ in the origin given by $B = \{(x, y, [w_1 : w_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 | xw_1 = yw_2\}$. The strict transform of the x-axis is given by $xw_1 = 0$, and intersects the exceptional curve in $w_1 = 0$. Define $B' = \{(x, \frac{w_1}{w_2}, [z_1 : z_2]) \in \mathbb{C}^2 \times \mathbb{P}^1 | xz_1 = wz_2,\ w_2 \neq 0\}$, then B' is covered by the two open sets $U_1' = \{(x, \frac{w_1}{w_2}, [z_1 : z_2]) | z_2 \neq 0\}$ with coordinates $(x, \frac{z_1}{z_2}) = (x, \frac{w_1}{w_2}x^{-1}) = (x, x^{-2}y)$ and $U_2' = \{(x, \frac{w_1}{w_2}, [z_1 : z_2]) | z_1 \neq 0\}$ with coordinates $(\frac{w_1}{w_2}, \frac{z_2}{z_1}) = (x^{-1}y, x\frac{w_2}{w_1}) = (x^{-1}y, x^2y^{-1})$, and gluing $(u, v) \mapsto (uv, v^{-1})$.[1]

[1]Remark: in the example above we considered X_{σ_1} with coordinates (X, Y), while we are now considering X_{σ_2} with coordinates (X, Y^{-1}), so to get the formula's right substitute y^{-1} for y in the foregoing

The coordinate algebras of τ and τ' are given by $\mathbb{C}[X^{-1}Y^{-1}, X^2Y]$ and $\mathbb{C}[X, X^{-2}Y^{-1}]$ respectively, so X_τ and $X_{\tau'}$ are both isomorphic to $\mathbb{C}^2$, with gluing map $(u,v) \mapsto (uv, v^{-1})$. With the substitution $(X,Y) = (\frac{u_1}{u_2}, \frac{v_1}{v_2})$ we conclude that the fan



corresponds to $\{([u_1 : u_2],[v_1 : v_2],[w_1 : w_2],[z_1 : z_2]) \in \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 | u_1w_1 = v_2w_2, \;\; u_1z_1 = w_1z_2\}$. Similarly the other subdivisions in Figure 4 correspond to the blowing up of q and q', and the final conclusion is that the set $\mathrm{Bl}_{p',q'}$, defined in Section 1 in an attempt to obtain a clear description of the family of curves K^{b_1,b_2}, is actually the toric variety corresponding to the fan:



Note that this is the (inward) normal fan to a polygon carrying a striking but not at all coincidental resemblance to the one in Figure 1.

3 Toric surface

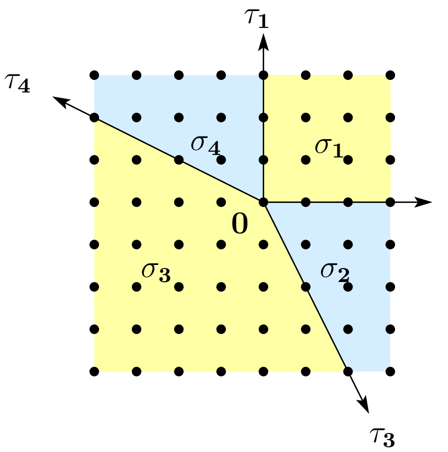


Figure 5: The fan $\Sigma \subset \mathbb{R}^2$ with the lattice $\mathbb{Z}^2$

We want to examine the toric variety X_Σ corresponding to the fan Σ pictured in Figure 5. Σ consists of four 2-dimensional cones $\sigma_1, \ldots, \sigma_4$, four 1-dimensional cones $\tau_1, \ldots, \tau_4$, and one zero-dimensional cone which is the origin. It is clear from the outset that X_Σ will have singularities, since the cones σ_2, σ_3 and σ_4 are simplicial (their generating vectors span $\mathbb{R}^2$) but not smooth (their generating vectors do not span $\mathbb{Z}^2$.)

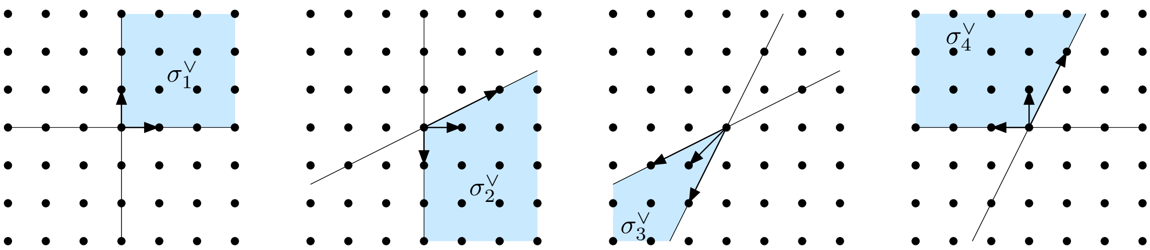


Figure 6: Dual cones of the σ_i

The dual cones $\sigma_i{}^\vee$ are pictured in Figure 6. The commutative semigroups

$S_i = {\sigma_i}^\vee \cap \mathbb{Z}^2$ are

$$\begin{aligned}
S_1 &= \mathbb{Z}_{\geq 0}(e_1^*) + \mathbb{Z}_{\geq 0}(e_2^*) \\
S_2 &= \mathbb{Z}_{\geq 0}(2e_1^* + e_2^*) + \mathbb{Z}_{\geq 0}(e_1^*) + \mathbb{Z}_{\geq 0}(-e_2^*) \\
S_3 &= \mathbb{Z}_{\geq 0}(-e_1^* - 2e_2^*) + \mathbb{Z}_{\geq 0}(-e_1^* - e_2^*) + \mathbb{Z}_{\geq 0}(-2e_1^* - e_2^*) \\
S_4 &= \mathbb{Z}_{\geq 0}(-e_1^*) + \mathbb{Z}_{\geq 0}(e_2^*) + \mathbb{Z}_{\geq 0}(e_1^* + 2e_2^*)
\end{aligned}$$

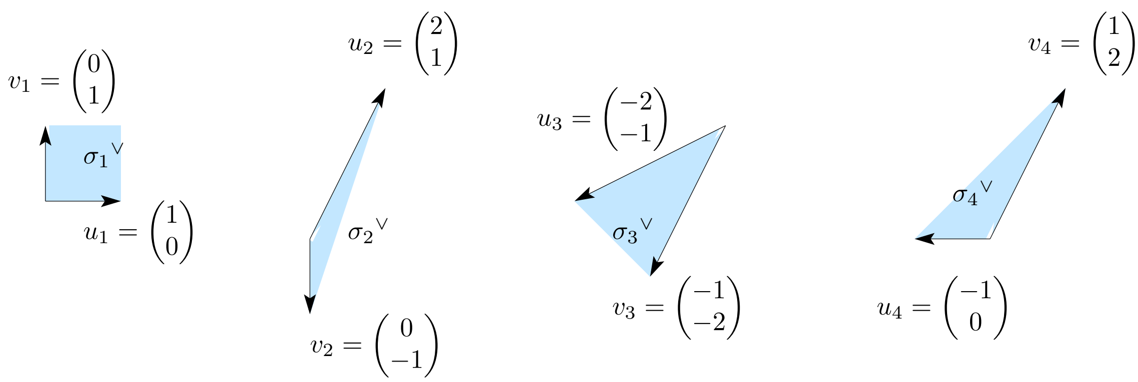
So the affine toric varieties $X_i = \operatorname{Spec} \mathbb{C}[S_i]$ are given by

$$\begin{aligned}
&X_1 = \operatorname{Spec} \mathbb{C}[X, Y] \cong \mathbb{C}^2, \\
&X_2 = \operatorname{Spec} \mathbb{C}[X^2Y, X, Y^{-1}] \cong \{(u, v, w) \in \mathbb{C}^3 | v^2 = uw\}, \\
&X_3 = \operatorname{Spec} \mathbb{C}[X^{-2}Y^{-1}, X^{-1}Y^{-1}, X^{-1}Y^{-2}] \cong \{(u, v, w) \in \mathbb{C}^3 | v^3 = uw\}, \\
&X_4 = \operatorname{Spec} \mathbb{C}[X^{-1}, Y, XY^2] \cong \{(u, v, w) \in \mathbb{C}^3 | v^2 = uw\},
\end{aligned}$$

which shows that X_2, X_3, X_4 are singular at the origin in $\mathbb{C}^3$.

If we try to glue the affine varieties X_i together along the sets $X_{\tau_1} \cong \mathbb{C}^* \times \mathbb{C}$, $X_{\tau_2} \cong \mathbb{C} \times \mathbb{C}^*$, $X_{\tau_3} \cong \mathbb{C}^* \times \mathbb{C}$, $X_{\tau_4} \cong \mathbb{C} \times \mathbb{C}^*$ in the same way as in the non-singular case, we do not get isomorphisms, as was also indicated in paragraph 2.3.

Denote by



the vectors generating the edges of the cones.

Write

$$\begin{aligned}
s &= au + bv \\
t &= cu + dv
\end{aligned}$$

as

$$\begin{pmatrix} s & t \end{pmatrix} = \begin{pmatrix} u & v \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

then

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix} = (u \;\; v)^{-1}(s \;\; t)$$

is the matrix with columns corresponding to the coordinates of the vectors $(s \;\; t)$ with respect to the basis $(u \;\; v)$. Each $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then gives the coordinate change

$$\mathbb{C}[\chi^u, \chi^v] \to \mathbb{C}[\chi^s, \chi^t]$$

by

$$(\chi^s, \chi^t) = (\chi^{au+bv}, \chi^{cu+dv}) = ((\chi^u)^a(\chi^v)^b, (\chi^u)^c(\chi^v)^d). \tag{11}$$

We calculate these matrices for the $(u_i \;\; v_i)$ corresponding to adjacent cones

$$(u_1 \;\; v_1)^{-1}(u_2 \;\; v_2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}$$

$$(u_2 \;\; v_2)^{-1}(u_3 \;\; v_3) = \begin{pmatrix} 2 & 0 \\ 1 & -1 \end{pmatrix}^{-1} \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & -\frac{1}{2} \\ 0 & \frac{3}{2} \end{pmatrix}$$

$$(u_3 \;\; v_3)^{-1}(u_4 \;\; v_4) = \begin{pmatrix} -2 & -1 \\ -1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} & 0 \\ -\frac{1}{3} & -1 \end{pmatrix}$$

$$(u_4 \;\; v_4)^{-1}(u_1 \;\; v_1) = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{pmatrix}$$

Denote $(U_i, V_i) := (\chi^{u_i}, \chi^{v_i})$, then using equation (11) we have:

$$(U_2, V_2) = (U_1^2 V_1, V_1^{-1})$$
$$(U_3, V_3) = (U_2^{-1}, U_2^{-\frac{1}{2}} V_2^{\frac{3}{2}})$$
$$(U_4, V_4) = (U_3^{\frac{2}{3}} V_3^{-\frac{1}{3}}, V_3^{-1})$$
$$(U_1, V_1) = (U_4^{-1}, U_4^{\frac{1}{2}} V_4^{\frac{1}{2}})$$

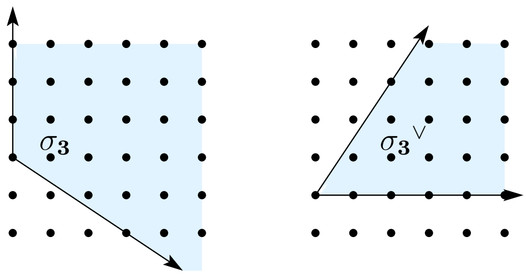
The inverse coordinate changes are found by taking the inverses of the matrices $(u_i \;\; v_i)^{-1}(u_{i+1} \;\; v_{i+1})$. The diagrams below show the maps between the common subsets $\mathbb{C} \times \mathbb{C}^*$ or $\mathbb{C}^* \times \mathbb{C}$ of the X_i:

$$\begin{tikzcd}
X_1 \arrow[r, dashed, "{(x^2y,y^{-1})}", "{y\neq 0}"'] & X_2 \arrow[d, dashed, "{x\neq 0}"', "{(x^{-1},\sqrt{\frac{y^3}{x}})}"] \\
X_4 \arrow[u, dashed, "{(x^{-1},\sqrt{xy})}", "{x\neq 0}"'] & X_3 \arrow[l, dashed, "{y\neq 0}"', "{(\sqrt[3]{\frac{x^2}{y}},y^{-1})}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
X_1 \arrow[d, dashed, "{(x^{-1},xy^2)}"', "{x\neq 0}"] & X_2 \arrow[l, dashed, "{(\sqrt{xy},y^{-1})}"', "{y\neq 0}"] \\
X_4 \arrow[r, dashed, "{y\neq 0}", "{(\sqrt{\frac{x^3}{y}},y^{-1})}"'] & X_3 \arrow[u, dashed, "{x\neq 0}", "{(x^{-1},\sqrt[3]{\frac{y^2}{x}})}"']
\end{tikzcd}$$

3.1 Singularities

To take a closer look at the singularities of the affine parts X_2, X_3, X_4 of X_Σ, first consider X_3 which corresponds to the cone σ_3 with edges generated by $(-2, 1)$ and $(1, -2)$. We choose a basis for the lattice $\mathbb{Z}^2$ so that σ_3 (and $\sigma_3{}^\vee$) take a more convenient form.

Let this basis be given by $b_1 = (0, -1)$, $b_2 = (1, -2)$, then the vectors generating the edges of σ_3 with respect to (b_1, b_2) are $(0, 1)$ and $(3, -2)$. The



semigroup $S_3 = \sigma_3{}^\vee \cap \mathbb{Z}^2$ is given by

$$S_3 = \mathbb{Z}_{\geq 0}(b_1^*) + \mathbb{Z}_{\geq 0}(b_1^* + b_2^*) + \mathbb{Z}_{\geq 0}(2b_1^* + 3b_2^*),$$

and, denoting $(X, Y) = (\chi^{b_1^*}, \chi^{b_2^*})$, $\mathbb{C}[S_3] = \mathbb{C}[X, XY, X^2Y^3]$, so that

$$X_3 = \operatorname{Spec}(\mathbb{C}[X, XY, X^2Y^3]) \cong \{(u, v, w) \in \mathbb{C}^3 | uw = v^3\}.$$

Loosely speaking, the reason X_3 has a singularity is that the vectors generating the edges of σ_3 do not generate the whole lattice $\mathbb{Z}^2$. To clarify this, suppose that $\mathbb{L} = \mathbb{Z}(0, 1) \oplus \mathbb{Z}(3, -2)$ is the lattice that these vectors do generate, and leave $\sigma_3 \subset \mathbb{L} \otimes \mathbb{R} = \mathbb{R}^2$ the same. The lattice $\mathbb{L}$ is generated by $3b_1$ and b_2, so the dual lattice $\mathbb{L}^\vee$ has generators $\frac{1}{3}b_1^*$ and b_2^*.

The semigroup $S_3' = \sigma_3{}^\vee \cap \mathbb{L}^\vee$ is given by $S_3' = \mathbb{Z}_{\geq 0}(\frac{1}{3}b_1^*) \oplus \mathbb{Z}_{\geq 0}(\frac{2}{3}b_1^* + b_2^*)$, and $\mathbb{C}[S_3'] = \mathbb{C}[U, V]$ with $U^3 = X$ and $V = U^2Y$, so $X_3' = \operatorname{Spec}(\mathbb{C}[S_3]) = \mathbb{C}^2$.

The inclusions $\mathbb{L} \subset \mathbb{Z}^2 \Rightarrow \mathbb{Z}^2 \subset \mathbb{L}^\vee \Rightarrow \mathbb{C}[S_3] \subset \mathbb{C}[S_3']$, induce a map $X_3' = \mathbb{C}^2 \to X_3$.

Proposition 3.1. *The map $X_3' = \mathbb{C}^2 \to X_3$ is the quotient map of the action of the group of roots of unity $G = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$ on $\mathbb{C}^2$ given by*

$$\eta(x, y) = (\eta x, \eta^2 y),$$

so that $X_3 = X_3'/G$, and X_3 has a cyclic quotient singularity.

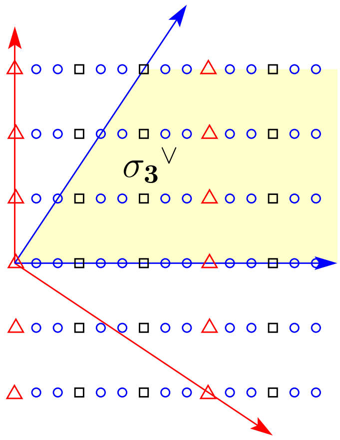


Figure 7: The lattices $\mathbb{Z}^2$ (black squares), $\mathbb{L}$ (red triangles), $\mathbb{L}^\vee$ (blue circles). (When lattice points overlap, the coarsest lattice is shown.)

Proof. Via the canonical duality pairing

$$\mathbb{L}^\vee/\mathbb{Z}^2 \times \mathbb{Z}^2/\mathbb{L} \to \mathbb{Q}/\mathbb{Z} \hookrightarrow \mathbb{C}^*$$

given by $\langle\ ,\ \rangle$ and $q \mapsto e^{2\pi i q}$ respectively, $\mathbb{Z}^2/\mathbb{L}$ acts on $\mathbb{C}[S_i']$:

$$v \cdot \chi^u = e^{2\pi i \langle u,v\rangle}\chi^u$$

for $u \in S_3' \subset \mathbb{L}^\vee$ and $v \in \mathbb{Z}^2$. This action is completely determined by

$$b_1 \cdot \chi^u = b_1 \cdot \chi^{p\frac{1}{3}b_1^*}\chi^{qb_2^*} = (e^{2\pi i/3})^p\chi^u$$

where $u = p\frac{1}{3}b_1^* + qb_2^*$ for some $p, q \in \mathbb{Z}_{\geq 0}$. So $G = \{1, e^{2\pi i/3}, e^{4\pi i/3}\} = \{1, \eta, \eta^2\}$ acts on $\mathbb{C}[S_3'] = \mathbb{C}[\chi^{\frac{1}{3}b_1^*}, \chi^{\frac{2}{3}b_1^*+b_2^*}] = \mathbb{C}[U, V]$ by $U \mapsto \eta U$, $V \mapsto \eta^2 V$.

The algebra $\mathbb{C}[S_3]$ is given by $\mathbb{C}[S_3] = \mathbb{C}[X, XY, X^2Y^3] = \mathbb{C}[U^3, UV, V^3]$ for $U^3 = X, V = U^2Y$. This is exactly the invariant subalgebra $\mathbb{C}[U, V]^G \subset \mathbb{C}[U, V]$ under the action of G.

The inclusion

$$\mathbb{C}[S_3] = \mathbb{C}[S_3']^G \subset \mathbb{C}[S_3']$$

induces

$$\mathbb{C}^2/G = X_3$$

□

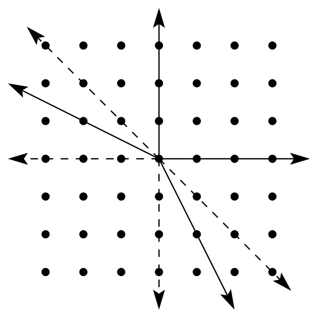
In the same way it can be shown that X_2 and X_4 are equal to the quotient $\mathbb{C}^2/\mu_2$ of $\mathbb{C}^2$ by the action $(x, y) \mapsto (e^{\pi i}x, e^{\pi i}y)$ of the finite group $\mu_2 = \{1, e^{\pi i}\}$.

3.2 Resolution of the singularities

In the previous section, the singularities due to the lack of smoothness of some cones of X_Σ were analysed by making a change to the lattice, while leaving the cone the same. In this section we do the opposite: the lattice $\mathbb{Z}^2$ remains fixed, while we will subdivide the non-smooth cones in such a way, that the vectors generating the edges of the new cones form a $\mathbb{Z}$-basis for $\mathbb{Z}^2$, i.e. such that all cones of the fan Σ are smooth.

Let $\Sigma \subset \mathbb{R}^2$ be a *complete fan*, which means that the union of all its cones is $\mathbb{R}^2$. Suppose that Σ' is a refinement of Σ, then for each cone $\sigma' \in \Sigma'$, there is a $\sigma \in \Sigma$ such that $\sigma' \subset \sigma$. This implies that for the corresponding semigroups $S_\sigma \subset S_{\sigma'}$, so $\mathbb{C}[S_\sigma] \subset \mathbb{C}[S_{\sigma'}]$. This determines a morphism $X_{\sigma'} \longrightarrow X_\sigma \subset X_\Sigma$. The morphisms $X_{\sigma'} \longrightarrow X_\Sigma$ patch together to a morphism $X_{\Sigma'} \longrightarrow X_\Sigma$, which is birational and proper under our assumptions. (For details see [2])

In the picture below, the cones which are not smooth are divided along the dashed lines, resulting in the refined fan Σ' which has only smooth cones, as is readily seen from the fact that the determinants of the matrices $(u_i \;\; v_i)$ with u_i, v_i the vectors generating the edge of some twodimensional cone in Σ' are all equal to ± 1.



This is exactly the same fan we showed to correspond to the (repeated) blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$ in Section 2.4, so what we constructed in Section 1 was actually a resolution of the singularities of X_Σ.

4 Toric surface embedded in $\mathbb{P}^4$

4.1 Toric variety X_Δ from the polytope Δ

In Section 2.1 a toric variety was defined as a variety which contains $\mathbb{C}^{*n}$ as an open dense orbit. This gives an easy way to generate examples of toric varieties: for any finite set of points $A = \{a^1, \dots, a^k \in \mathbb{Z}^n\}$ of the lattice $\mathbb{Z}^n$, there is a corresponding toric variety $X_A \subset \mathbb{P}^{k-1}$.

There is a bijective correspondence between elements of the lattice $\mathbb{Z}^n$ and *Laurent monomials*, which are group homomorphisms $\mathbb{C}^{*n} \longrightarrow \mathbb{C}^*$. For $k = (k_1, \dots k_n) \in \mathbb{Z}^n$ and $\phi = (\phi_1, \dots, \phi_n) \in \mathbb{C}^{*n}$ this correspondence is given by

$$k \longleftrightarrow (\phi \longmapsto \phi^k = \phi_1^{k_1}\phi_2^{k_2}\cdots\phi_n^{k_n} \in \mathbb{C}^*)$$

Let $\phi^{a^i} : \mathbb{C}^{*n} \to \mathbb{C}^*$ be the Laurent monomials corresponding to the elements of A, with $\phi^{a^i} = \phi_1^{a_1^i}\phi_2^{a_2^i}\cdots\phi_n^{a_n^i}$. Then $\mathbb{C}^{*n}$ acts on $\mathbb{P}^{k-1}$ by

$$(\phi, [x_1 : \dots : x_k]) \mapsto [\phi^{a^1}x_1 : \dots : \phi^{a^k}x_k]$$

and the toric variety X_A is defined as the closure of the orbit of this action through $[1 : 1 : \dots : 1]$.

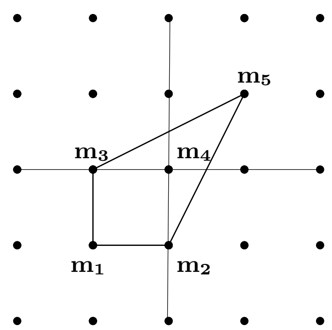


Figure 8: The polygon Δ

Figure 8 shows a polygon $\Delta \in \mathbb{R}^2$ and the lattice $\mathbb{Z}^2$. The *lattice points* $\Delta \cap \mathbb{Z}^2$ of Δ are $m_1 = (-1,-1)$, $m_2 = (0,-1)$, $m_3 = (-1,0)$, $m_4 = (0,0)$, $m_5 = (1,1)$. These lattice points of Δ determine the toric variety $X_\Delta \subset \mathbb{P}^4$ as the closure of the orbit through $[1 : 1 : 1 : 1 : 1]$ of the $\mathbb{C}^* \times \mathbb{C}^*$-action

$$((\lambda, \mu), [x_1 : x_2 : x_3 : x_4 : x_5]) \mapsto [\lambda^{-1}\mu^{-1}x_1 : \mu^{-1}x_2 : \lambda^{-1}x_3, x_4, \lambda\mu x_5] \quad (12)$$

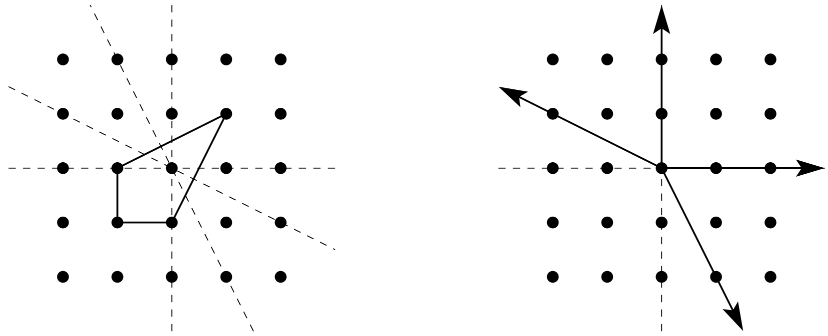


Figure 9: The polygon Δ and its normal fan Σ

4.2 Normal fan

By taking the outer normal vectors to Δ, we retrieve the fan used to define X_Σ in section 3, see Figure 9.

In section 2.1 each two-dimensional cone σ in the fan was used to obtain an affine variety X_σ, which was subsequently glued to the affine varieties corresponding to neighboring cones along their common edges.

In this section we use the same fan to construct the toric variety X_Σ in an alternative way. First we will obtain a quotient of (an open subset of) $\mathbb{C}^4$ by an action of $\mathbb{C}^* \times \mathbb{C}^*$ via the quotient of $\mathbb{Z}^4$ by a sublattice of rank two associated to the fan. Via the lattice points m_i of the polytope Δ we will obtain monomials x_i^m which all transform in the same way under this action, i.e. they are all multiplied by the same factor. This gives an embedding into $\mathbb{P}^4$.

Taking normal vectors of minimal length in $\mathbb{Z}^2$, we get four unique vectors in $\mathbb{Z}^2$, which we will denote by

$$n_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, n_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, n_3 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}, n_4 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Note that in terms of the n_i, Δ is given by the inequalities:

$$\langle x, n_i \rangle \geq -1,\ i = 1, \ldots, 4 \tag{13}$$

Define the map $\alpha : \mathbb{Z}^4 \mapsto \mathbb{Z}^2$ as follows:

$$\alpha : e_i \longmapsto n_i,\ i = 1, \ldots, 4$$

with $\{e_1, e_2, e_3, e_4\}$ the standard basis of $\mathbb{Z}^4$. Since α is surjective, we have the exact sequence

$$0 \longrightarrow \mathbb{L} \xrightarrow{\beta} \mathbb{Z}^4 \xrightarrow{\alpha} \mathbb{Z}^2 \longrightarrow 0$$

Where $\mathbb{L} := \mathbb{Z}(1,1,1,1) \oplus \mathbb{Z}(0,3,1,2)$. Identifying $\mathbb{L}$ with $\mathbb{Z}^2$, and choosing standard bases for $\mathbb{Z}^2$ and $\mathbb{Z}^4$, α and β can be represented as

$$\alpha = \begin{pmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix} \qquad \beta = \begin{pmatrix} 1 & 0 \\ 1 & 3 \\ 1 & 1 \\ 1 & 2 \end{pmatrix}$$

By taking duals we also get the exact sequence

$$0 \longrightarrow \mathbb{Z}^{2\vee} \xrightarrow{\alpha^T} \mathbb{Z}^{4\vee} \xrightarrow{\beta^T} \mathbb{Z}^{2\vee} \longrightarrow 0 \tag{14}$$

where for $k \in \mathbb{Z}^{2\vee}$,

$$\alpha^T(k) = k \circ \alpha : \mathbb{Z}^4 \to \mathbb{Z},$$

and for $l \in \mathbb{Z}^{4\vee}$,

$$\beta^T(l) = l \circ \beta : \mathbb{Z}^2 \to \mathbb{Z}.$$

The exact sequence (14) induces the following exact sequence of complex groups:

$$1 \longrightarrow \mathrm{Hom}(\mathbb{Z}^{2\vee}, \mathbb{C}^*) \xrightarrow{\hat{\beta^T}} \mathrm{Hom}(\mathbb{Z}^{4\vee}, \mathbb{C}^*) \xrightarrow{\hat{\alpha^T}} \mathrm{Hom}(\mathbb{Z}^{2\vee}, \mathbb{C}^*) \longrightarrow 1 \tag{15}$$

which we write as

$$1 \longrightarrow G \xrightarrow{\hat{\beta^T}} \mathbb{C}^{*4} \xrightarrow{\hat{\alpha^T}} H \longrightarrow 1 \tag{16}$$

where $\hat{\alpha^T}$ and $\hat{\beta^T}$ are just the appropriate compositions with α^T and β^T. We use the identifications $\mathbb{Z}^{n\vee} \cong \mathbb{Z}^n$, and

$$\mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^*) \cong \mathbb{C}^{*n} \tag{17}$$

given by: $\phi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^*) \sim (\phi_1, \ldots, \phi_n)$ where ϕ_i is equal to $\phi(e_i) \in \mathbb{C}^*$, with $e_i \in \mathbb{Z}^n$ the nth standard basis vector. Then for $\phi \in \mathrm{Hom}(\mathbb{Z}^n, \mathbb{C}^*)$, $(a_1, \ldots, a_n) \in \mathbb{Z}^n$, $\phi(a_1, \ldots, a_n) = {\phi_1}^{a_1} {\phi_2}^{a_2} \ldots {\phi_n}^{a_n}$.

We will verify the exactness of (16) explicitly, the calculation will be useful later. Let $\xi \in G \cong \mathbb{C}^* \times \mathbb{C}^*$, then $\hat{\beta^T}(\xi) = \xi \circ \beta^T : \mathbb{Z}^{4^\vee} \to \mathbb{C}^*$. β^T is given by

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 3 & 1 & 2 \end{pmatrix}$$

so for $(l_1, l_2, l_3, l_4) : \mathbb{Z}^4 \to \mathbb{Z}$

$$\xi \circ \beta^T(l_1, l_2, l_3, l_4) = \xi_1^{l_1+l_2+l_3+l_4} \xi_2^{3l_2+l_3+2l_4} \in \mathbb{C}^*$$

and the four components $(\hat{\beta^T}(\xi))_i = \xi \circ \beta^T(e_i)$ are

$$\begin{aligned} \xi \circ \beta^T(1,0,0,0) &= \xi_1 \\ \xi \circ \beta^T(0,1,0,0) &= \xi_1 {\xi_2}^3 \\ \xi \circ \beta^T(0,0,1,0) &= \xi_1 \xi_2 \\ \xi \circ \beta^T(0,0,0,1) &= \xi_1 {\xi_2}^2 \end{aligned} \tag{18}$$

Similarly, for any $\lambda : \mathbb{Z}^{4^\vee} \to \mathbb{C}^*$,

$$\hat{\alpha^T}(\lambda) = \lambda \circ \alpha^T : \mathbb{Z}^{2^\vee} \to \mathbb{C}^*$$

So that if $k = (k_1, k_2) : \mathbb{Z}^2 \to \mathbb{Z}$,

$$\lambda \circ \alpha^T(k) = \lambda_1^{k_1} \lambda_2^{k_2} \lambda_3^{k_2 - 2k_1} \lambda_4^{k_1 - 2k_2} \tag{19}$$

so in components

$$\hat{\alpha^T}(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (\lambda_1 \lambda_3^{-2} \lambda_4, \lambda_2 \lambda_3 \lambda_4^{-2})$$

Applying this formula to $\hat{\beta^T}(\xi_1, \xi_2) = (\xi_1, \xi_1 {\xi_2}^3, \xi_1 \xi_2, \xi_1 {\xi_2}^2)$ gives

$$\hat{\alpha^T} \hat{\beta^T}(\xi) = (\xi_1 (\xi_1 \xi_2)^{-2} \xi_1 {\xi_2}^2, (\xi_1 {\xi_2}^3)(\xi_1 \xi_2)(\xi_1 {\xi_2}^2)^{-2}) = (1,1)$$

We now have all the ingredients to define the toric variety associated to the normal fan of Δ via the action of G.

First consider the natural action of $\mathbb{C}^{*4}$ on $\mathbb{C}^4$ given by

$$\lambda \cdot x = (\lambda_1 x_1, \lambda_2 x_2, \lambda_3 x_3, \lambda_4 x_4), \quad \lambda \in \mathbb{C}^{*4},\ x \in \mathbb{C}^4$$

This restricts to the action

$$G \times \mathbb{C}^4 \to \mathbb{C}^4,$$

which from (18) is given by

$$(\xi_1,\xi_2)\cdot(x_1,x_2,x_3,x_4) = (\xi_1 x_1, \xi_1{\xi_2}^3 x_2, \xi_1\xi_2 x_3, \xi_1{\xi_2}^2 x_4) \tag{20}$$

To ensure this action has no singular orbits, we let G act on the complement of a closed subset of $\mathbb{C}^4$: define

$$Z = \{(x_1,x_2,x_3,x_4)\in\mathbb{C}^4 | x_1x_2 = 0, x_2x_3 = 0, x_3x_4 = 0, x_1x_4 = 0\}$$

Then the complement $\mathbb{C}^4\backslash Z$ has at least one of the monomials $x_ix_j \neq 0$ for $1 \leq i < j \leq 4$.

The toric variety associated to the normal fan Σ is defined as the quotient

$$X_\Sigma = (\mathbb{C}^4\backslash Z)/G$$

It follows from (16) that X_Σ contains the torus $H \cong \mathbb{C}^{*2}$ via the inclusion $\mathbb{C}^{*4} \subset \mathbb{C}^4\backslash Z$:

$$H \cong \mathbb{C}^{*4}/G \subset (\mathbb{C}^4\backslash Z)/G = X_\Sigma$$

In the next section we describe an embedding of this quotient into $\mathbb{P}^4$.

4.3 Embedding in $\mathbb{P}^4$

We turn back to the polygon Δ in Figure 8. The lattice points $m_i \in \Delta \cap \mathbb{Z}^2$ are given by

$$\begin{aligned} m_1 &= (-1,-1) & m_4 &= (0,0)\\ m_2 &= (0,-1) & m_5 &= (1,1)\\ m_3 &= (-1,0) \end{aligned}$$

and the minimal vectors normal to the faces of Δ by

$$\begin{aligned} n_1 &= (1,0) & n_3 &= (-2,1)\\ n_2 &= (0,1) & n_4 &= (1,-2) \end{aligned}$$

Recall that Δ is given by the inequalities (13): $\langle x, n_i\rangle \geq -1,\ i = 1,\ldots,4,$ for any $m_j \in \Delta\cap\mathbb{Z}^2$:

$$\langle m_j, n_i\rangle + 1 \geq 0.$$

We use this fact to define the following monomials in $z_1, \ldots, z_4 \in \mathbb{C}$:

$$z^{m_j} = \prod_{i=1}^{4} z_i^{\langle m_j, n_i \rangle + 1} \tag{21}$$

These monomials have the property that they are all multiplied by the same factor by the action of G defined in the previous section:

Proof. first observe that for $(\lambda_1, \ldots, \lambda_4) \in \mathbb{C}^{*4}$

$$\lambda \cdot z^{m_j} = \prod_{i=1}^{4} (\lambda_i z_i)^{\langle m_j, n_i \rangle + 1} = \lambda_1 \lambda_2 \lambda_3 \lambda_4 z^{m_j} \prod_{i=1}^{4} \lambda_i^{\langle m_j, n_i \rangle} \tag{22}$$

Now interpret $(\lambda_1, \ldots, \lambda_4)$ as $\lambda : \mathbb{Z}^4 \to \mathbb{C}^*$ via the correspondence (17), then from (19) we see that

$$\prod_{i=1}^{4} \lambda_i^{\langle m_j, n_i \rangle} = \hat{\alpha^T} \lambda(\mathbf{m}_j) = \lambda \circ \alpha^T(\mathbf{m}_j)$$

where $\mathbf{m}_j := \langle m_j, \_ \rangle \in \mathbb{Z}^{2\vee}$.[2]

To restrict the action to G, substitute $\lambda = \hat{\beta^T}(\xi)$ for some $\xi \in G$. Then (16) gives

$$\prod_{i=1}^{4} \lambda_i^{\langle m_j, n_i \rangle} = \hat{\alpha^T} \lambda(\mathbf{m}_j) = (\hat{\alpha^T} \hat{\beta^T} \xi)(\mathbf{m}_j) = 1$$

So $(\xi_1, \xi_2) \cdot z^{m_j} = \xi_1 \xi_1 \xi_2^3 \xi_1 \xi_2 \xi_1 \xi_2^2 z^{m_j} = \xi_1^4 \xi_2^6 z^{m_j}$ for any $1 \leq j \leq 5$. □

Define $w_j := z^{m_j}$, so that

$$\begin{aligned} w_1 &= z_3^2 z_4^2 \\ w_2 &= z_1 z_4^3 \\ w_3 &= z_2 z_3^3 \\ w_4 &= z_1 z_2 z_3 z_4 \\ w_5 &= z_1^2 z_2^2 \end{aligned} \tag{23}$$

and define a map

[2] Actually, the polygon Δ naturally sits in 'dual space', as can be seen if one views the defining inequalities of Δ as $x \in \Delta \subset \mathbb{R}^{2\vee} \Leftrightarrow x(n_i) = \langle x, n_i \rangle \geq -1$ for all outer normal vectors n_i.

$$\mathbb{C}^4 \backslash Z \longrightarrow \mathbb{P}^4$$

by

$$(z_1, z_2, z_3, z_4) \longmapsto [w_1 : w_2 : w_3 : w_4 : w_5]$$

where

$$Z = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 | z_1 z_2 = 0, z_2 z_3 = 0, z_3 z_4 = 0, z_1 z_4 = 0\}$$

Since $(\xi_1, \xi_2) \cdot w_j = \xi_1^4 \xi_2^6 w_j$ for any $(\xi_1, \xi_2) \in G$, $1 \leq j \leq 5$, this map factors over X_Σ, and we get

$$\rho : (\mathbb{C}^4 \backslash Z)/G = X_\Sigma \longrightarrow \mathbb{P}^4$$

Via the $\mathbb{C}^{*4}$-action on the monomials w_j, G there is an action of $H = \mathbb{C}^{*4}/G$ on $\mathbb{P}^4$. The action of $\mathbb{C}^{*4}$ on the monomials $w_j = z^{m_j}$ is given by (22):

$$\begin{aligned} \lambda \cdot w_j &= \lambda_1 \lambda_2 \lambda_3 \lambda_4 \prod_{i=1}^{4} \lambda_i^{\langle m_j, n_i \rangle} w_j \\ &= \lambda_1 \lambda_2 \lambda_3 \lambda_4 \hat{\alpha^T} \lambda(\mathbf{m}_j) w_j \\ &= \hat{\alpha^T} \lambda(\mathbf{m}_j) w_j \qquad (w_j \text{ homogeneous}). \end{aligned}$$

Define $\eta := \hat{\alpha^T} \lambda \in H \cong \mathbb{C}^* \times \mathbb{C}^*$, then

$$\eta = (\eta_1, \eta_2) = (\lambda_1 \lambda_3^{-2} \lambda_4, \lambda_2 \lambda_3 \lambda_4^{-2}) \tag{24}$$

and the action of H on w_j is given by $\eta \cdot w_j = \eta^{\mathbf{m}_j} w_j$, so

$$\eta \cdot [w_1 : w_2 : w_3 : w_4 : w_5] = [\eta_1^{-1} \eta_2^{-1} w_1 : \eta_2^{-1} w_2 : \eta_1^{-1} w_3 : w_4 : \eta_1 \eta_2 w_5] \tag{25}$$

This is the action (12) of the Laurent monomials corresponding to the lattice points of Δ on $\mathbb{P}^4$ we defined at the beginning of this section, so we recover the description of X_Σ as the closure of the orbit of this action through $[1 : 1 : 1 : 1 : 1]$ in $\mathbb{P}^4$.

4.4 Lines

We take another look at the polygon Δ in Figure 8. There are the following affine relations among the lattice points $m_j \in \Delta \cap \mathbb{Z}^2$:

$$\begin{aligned} m_1 + m_4 &= m_2 + m_3 \\ m_1 + m_5 &= 2m_4 \\ m_2 + m_3 + m_5 &= 3m_4 \end{aligned}$$

where the last one is redundant because it follows from the first two relations. For the w_j it follows that

$$\begin{aligned} w_1 w_4 &= w_2 w_3 \\ w_1 w_5 &= w_4^2 \end{aligned} \tag{26}$$

since $w_1 w_4 = z^{m_1} z^{m_4} = z^{m_1+m_4} = z^{m_2+m_3} = z^{m_2} z^{m_3} = w_2 w_3$, and similarly for the second equation.

These equations are invariant under the action of H, which follows from the definition of this action as: $\eta \cdot w_j = \eta^{m_j} w_j$.

So the image of ρ in $\mathbb{P}^4$ is given by the equations (26), and we obtain yet another description of X_Σ as the projective variety in $\mathbb{P}^4$ defined by these homogeneous equations.

From these equations we derive the following: If $w_4 = 0$, then $w_1 w_5 = 0$ and $w_2 w_3 = 0$. It follows that the four projective lines

$$\begin{aligned} L_1 &= [p : 0 : q : 0 : 0] \\ L_2 &= [p : q : 0 : 0 : 0] \\ L_3 &= [0 : p : 0 : 0 : q] \\ L_4 &= [0 : 0 : p : 0 : q] \end{aligned}$$

are in $X_\Sigma \subset \mathbb{P}^4$. If we draw a line through the m_j that corresponds to the w_j which are not equal to zero in the definition of L_i above, it is a line through the face of the polygon Δ, see Figure 10. The defining equations of $w_1, \dots, w_5$ (23) show that the set $\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \backslash Z | z_i = 0\}$ is mapped onto the line L_i.

Via the map ρ, these lines are added to the torus $H \cong \mathbb{C}^* \times \mathbb{C}^*$, compactifying it in $\mathbb{P}^4$:

$$H \cong \mathbb{C}^{*4}/G \subset \mathbb{C}^4/G \hookrightarrow \mathbb{P}^4$$

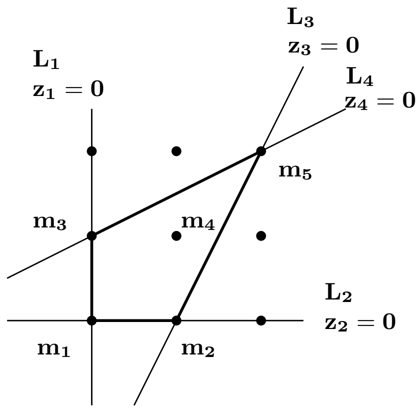


Figure 10: The lines $L_1, \ldots, L_4$

4.5 Comparison to the blow-up of $\mathbb{P}^1 \times \mathbb{P}^1$

In section 1 we looked at the family of curves given by

$$a_1 s^{-1}t^{-1} + a_2 t^{-1} + a_3 s^{-1} + a_4 + a_5 st = 0 \tag{27}$$

The set of solutions to this equation, relative to some given $a_1, \ldots, a_5 \in \mathbb{C}$ not all equal to zero, can be realized as follows.

Take a general hyperplane in $\mathbb{P}^4$, given by

$$a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 + a_5x_5 = 0.$$

Intersecting this hyperplane with the image of the map

$$(s,t) \longrightarrow [s^{-1}t^{-1} : t^{-1} : s^{-1} : 1 : st]$$

gives the set of solutions to (27) for a fixed $[a_1 : \ldots : a_5] \in \mathbb{P}^4$.

Remark 4.1. Note that the Laurent monomials from the above equation are exactly those defining the action (25) of H on $\mathbb{P}^4$, corresponding to the lattice points of the polygon Δ. So the set of solutions to (27) is given by the intersection of X_Δ with a hyperplane in $\mathbb{P}^4$.

After the substitution $s = \frac{u_1}{u_2}$, $t = \frac{v_1}{v_2}$ with $([u_1 : u_2], [v_1 : v_2]) \in \mathbb{P}^1 \times \mathbb{P}^1$, the above map becomes

$$([u_1 : u_2], [v_1 : v_2]) \longrightarrow [r_1 : r_2 : r_3 : r_4 : r_5]$$

with

$$
\begin{aligned}
r_1 &= u_2^2 v_2^2 \\
r_2 &= u_1 u_2 v_2^2 \\
r_3 &= u_2^2 v_1 v_2 \\
r_4 &= u_1 u_2 v_1 v_2 \\
r_5 &= u_1^2 v_1^2
\end{aligned}
\tag{28}
$$

This map is not defined in the points $p = ([0:1],[1:0])$, $q = ([1:0],[0:1])$. In section 1 we obtained the set

$$
\begin{aligned}
X = \{([u_1:u_2],[v_1:v_2],[w_1:w_2],&[x_1:x_2],[y_1:y_2],[z_1:z_2]) \in \mathbb{P}^1 \times \ldots \times \mathbb{P}^1 | \\
& u_1 w_1 = v_2 w_2,\ u_1 y_1 = w_1 y_2,\ u_2 x_2 = v_1 x_1,\ v_1 z_1 = x_1 z_2\}
\end{aligned}
$$

by blowing up the points p and q and the points of intersection of the exceptional curves over p and q and the lines $u_1 = 0$ and $v_1 = 0$ respectively.

We get a well-defined map by extending the above map to X via projection onto $\mathbb{P}^1 \times \mathbb{P}^1$:

$$
\begin{tikzcd}
X \arrow[r] \arrow[d] & \mathbb{P}^4 \\
\mathbb{P}^1 \times \mathbb{P}^1 \arrow[ur, dashed] &
\end{tikzcd}
$$

To get equation (3)

$$
{u_2}^2 {v_2}^2 + u_1 u_2 {v_2}^2 + {u_2}^2 v_1 v_2 + b_1 u_1 u_2 v_1 v_2 + b_2 {u_1}^2 {v_1}^2 = 0
$$

from

$$
a_1 r_1 + a_2 r_2 + \ldots + a_5 r_5 = 0
$$

we have to set $a_1 = 1$ and normalize the other coefficients a_i under the action of $\mathbb{C}^* \times \mathbb{C}^*$ on r_i given by

$$
([u_1 : u_2],[v_1 : v_2]) \mapsto ([\mu u_1 : u_2],[\nu v_1 : v_2])
$$

which we now recognize as the action of H on $\mathbb{P}^4$:

$$
\begin{aligned}
(\mu,\nu)\cdot r_1 &= r_1 & (\mu,\nu)\cdot r_2 &= \mu r_1 \\
(\mu,\nu)\cdot r_3 &= \nu r_3 & (\mu,\nu)\cdot r_4 &= \mu\nu r_4 \\
(\mu,\nu)\cdot r_5 &= \mu^2 \nu^2 r_5 & &
\end{aligned}
\tag{29}
$$

(Multiply everything with the factor $\mu^{-1}\nu^{-1}$ to get the same as in (25).)

Setting $a_1 = 1, \mu = a_3^{-1}, \nu = a_2^{-1}, b_1 = a_2^{-1}a_3^{-1}a_4, b_2 = a_2^{-2}a_2^{-2}a_5$ in $r_1 + \nu a_2 r_2 + \mu a_3 r_3 + \mu\nu a_4 r_4 + \mu^2\nu^2 a_5 r_5 = 0$ gives

$$r_1 + r_2 + r_3 + b_1 r_4 + b_2 r_5 = 0$$

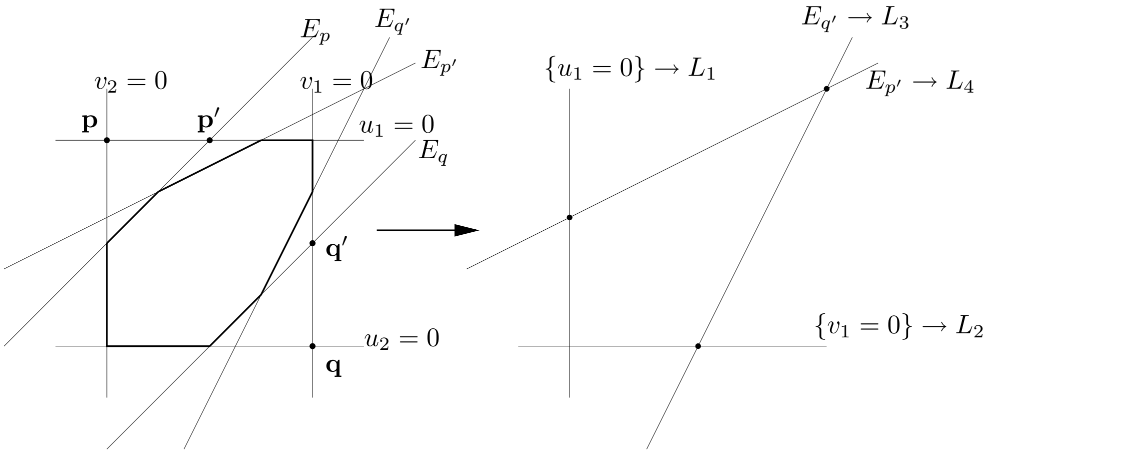
The r_i satisfy the same equations as the w_j of the previous section, since they define the same toric variety X_Δ (Remark 4.1.)

$$r_1 r_4 = r_2 r_3, \qquad r_1 r_5 = r_4^2,$$

so the lines $L_1, \dots, L_4 \subset \mathbb{P}^4$ are also contained in the image of $X \longrightarrow \mathbb{P}^4$. From definition (28) it is easy to see that the projective line $[0:1] \times \mathbb{P}^1$ is mapped to $L_1 = [p:0:q:0:0]$, $\mathbb{P}^1 \times [0:1]$ to $L_2 = [p:q:0:0:0]$, and $[1:0] \times \mathbb{P}^1$, $\mathbb{P}^1 \times [1:0]$ both to the point $[0:0:0:0:1]$.

Using the affine coordinates from Table 1, we calculate the image of the exceptional curves in X by expressing the monomials r_1, r_2, r_3, r_4, r_5 in these coordinates, and then putting $u, v = 0$:

coordinates		image exceptional curve
(u, w)	$[r_1 : \ldots : r_5] = [uw^2 : u^2w^2 : w : uw : u]$	$[0:0:1:0:0]$
(u, y)	$[r_1 : \ldots : r_5] = [u^2y^2 : u^3y^2 : y : uy : 1]$	$[0:0:y:0:1] = L_4$
(v, x)	$[r_1 : \ldots : r_5] = [vx^2 : x : v^2x^2 : vx : v^2]$	$[0:1:0:0:0]$
(v, z)	$[r_1 : \ldots : r_5] = [v^2z^2 : z : v^3z^2 : zv : 1]$	$[0:z:0:0:1] = L_3$



5 Conclusions

We have now seen a lot of different descriptions of the same toric variety X (we don't write X_Σ or X_Δ anymore, since they are really the same) constructed from either the polytope Δ, or its normal fan Σ (Figure 9.) The first one, treating each of the (maximal) cones seperately to construct from it an affine variety, and subsequently gluing these together, turns out to be useful to study the singularities arising from the non-smoothness of some cones, and moreover the resolution of these singularities.

Another perspective is provided by using the lattice points of a polytope to define an action of $\mathbb{C}^{*2}$ via Laurent monomials on the projective space $\mathbb{P}^4$, and then taking X to be the closure of the orbit through $[1 : 1 : \ldots : 1]$.

Yet another way to describe X is as the quotient of a suitable open subset of $\mathbb{C}^4$ by an action of $\mathbb{C}^* \times \mathbb{C}^*$ determined by a sublattice of rank two in $\mathbb{Z}^4$, which in turn is determined by the fan Σ. The monomials in $(z_1, \ldots, z_4) \in \mathbb{C}^4$ which transform equally under the action of $\mathbb{C}^* \times \mathbb{C}^*$ can be used as homogeneous coordinates for $\mathbb{P}^4$ to embed X into $\mathbb{P}^4$.

Finally, the last description of X is given as the projective variety in $\mathbb{P}^4$ given by the equations

$$\begin{aligned} w_1 w_4 &= w_2 w_3 \\ w_1 w_5 &= w_4^2 \qquad [w_1 : \ldots : w_5] \in \mathbb{P}^4 \end{aligned}$$

From the previous section we have that the intersection of $X \subset \mathbb{P}^4$ with a fixed hyperplane in $\mathbb{P}^4$ gives a curve which intersects the lines L_3 and L_4. To let the hyperplane defining the curve vary, we take the intersection of $X \subset \mathbb{P}^4$ with the incedence relation on $\mathbb{P}^4 \times \mathbb{P}^{4\vee}$ given by $I = \{(p, V) \in \mathbb{P}^4 \times \mathbb{P}^{4\vee} | p \in V\}$. This set $X \cap I$ then consists of pairs $([w_1 : \ldots : w_5], V_a)$ of a point $[w_1 : \ldots : w_5] \in \mathbb{P}^4$ satisfying

$$\begin{aligned} w_1 w_4 &= w_2 w_3 \\ w_1 w_5 &= w_4^2 \\ a_1 w_1 + a_2 w_2 &+ a_3 w_3 + a_4 w_4 + a_5 w_5 = 0 \end{aligned} \tag{30}$$

and the hyperplane $V_a = \{[x_1 : \ldots : x_5] \in \mathbb{P}^4 | a_1 x_1 + \ldots + a_5 x_5 = 0, [a_1 : \ldots : a_5 \in \mathbb{P}^4]\}$ through that point. We denote this hyperplane V_a with $[a_1 : \ldots : a_5]$.

The group $H \cong \mathbb{C}^* \times \mathbb{C}^*$ acts on $\mathbb{P}^4 \times \mathbb{P}^{4\vee}$ by the action (25) on $\mathbb{P}^4$, and the same action with opposite signs on $\mathbb{P}^{4\vee}$:

$$\begin{aligned} &\eta \cdot ([w_1 : \ldots : w_5], [a_1 : \ldots : a_5]) \\ &\quad = ([\eta_1^{-1}\eta_2^{-1} w_1 : \eta_2^{-1} w_2 : \eta_1^{-1} w_3 : w_4 : \eta_1 \eta_2 w_5], [\eta_1 \eta_2 w_1 : \eta_2 w_2 : \eta_1 w_3 : w_4 : \eta_1^{-1} \eta_2^{-1} w_5]) \end{aligned}$$

Since the equations (30) are invariant under this action of H, $X \cap I$ is left invariant by H. So we can restrict the projection $(\mathbb{P}^4 \times \mathbb{P}^{4^\vee})/H \longrightarrow \mathbb{P}^{4^\vee}$ to $X \cap I$:

$$
\begin{tikzcd}
X \cap I \arrow[r, hook] & \mathbb{P}^4 \times \mathbb{P}^{4^\vee}/H \arrow[d] \\
 & \mathbb{P}^{4^\vee}/H
\end{tikzcd}
$$

This models the family of curves given by (30) as a threedimensional variety in $\mathbb{P}^4 \times \mathbb{P}^{4^\vee}$, with parameter space $\mathbb{P}^4/H$.

References

[1] R. Cushman and F. Beukers. The complex geometry of the spherical pendulum. *Celestial Mechanics, Contemporary Mathematics 292*, 2002.

[2] W. Fulton. *Introduction to Toric Varieties*. Princeton University Press, 1993.