

Determining the Cheapest-to-Deliver Bonds for Bond Futures

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Abstract

In this research futures on bonds are studied and since this future has several bonds as its underlyings, the party with the short position may decide which bond it delivers at maturity of the future. It obviously wants to give the bond that is the Cheapest-To-Deliver (CTD). The purpose of this project is to develop a method to determine, which bond is the CTD at expiration of the future. To be able to compare the underlying bonds, with different maturities and coupon rates, conversion factors are used.

We would like to model the effects that changes in the term structure have on which bond is cheapest-to-deliver, because when interest rates change, another bond could become the CTD. We assume that the term structure of the interest rates is stochastic and look at the Ho-Lee model, that uses binomial lattices for the short rates. The volatility of the model is supposed to be constant between today and delivery, and between delivery and maturity of the bonds.

The following questions will be analysed:

- Is the Ho-Lee model a good model to price bonds and futures, i.e. how well does the model fit their prices?
- How many steps are needed in the binomial tree to get good results?
- At what difference in the term structure is there a change in which bond is the cheapest?
- Is it possible to predict beforehand which bond will be the CTD?
- How sensitive is the futures price for changes in the zero curve?
- How stable are the volatilities of the model and how sensitive is the futures price for changes in these parameters?

To answer these questions, the German Euro-Bunds are studied, which are the underlying bonds of the Euro-Bund Future.

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List of symbols

a_k	drift at time k in the discrete Ho-Lee model
A_T	value of an asset at time T
α	constant used in the bond pricing formula of the Ho-Lee model
b_k	volatility at time k in the discrete Ho-Lee model
β	constant used in the bond pricing formula of the Ho-Lee model
c_t	coupon payment at time t
C_0	cash price of a bond at time 0
$\text{Caplet}((k, s), t)$	value of the caplet with maturity t at node (k, s)
$d_{k,s}$	one-period discount rate at node (k, s) in the discrete Ho-Lee model
Δt	length of time interval
\mathbb{E}	expectation
$\mathbb{E}_{\mathbb{Q}}$	expectation under the probability measure \mathbb{Q}
\mathcal{F}	σ -algebra
\mathcal{F}_t	natural filtration containing all the information up to t
$F(0, T)$	price of a futures contract with maturity T , but fixed at time 0
$F((M, s), t_d)$	price of a futures contract at node (M, s) with maturity t_d
$f(t, T)$	instantaneous forward rate at time t for the maturity T
$f(t, T_1, T_2)$	continuously compounded forward rate at time t for maturity T_2 as seen from expiry T_1
$f((M, s), t_{n+1})$	continuously compounded forward rate at node (M, s) for maturity t_{n+1}
$H((M, s), t)$	discount factor or bond price at node (M, s) with maturity t
I_0	value of coupon payments of a bond at time 0
$J(\vec{c}, \vec{t}_c, t)$	price at time t of a bond with coupons \vec{c} at times \vec{t}_c
k	time variable
$K(\vec{c}^j, \vec{t}_c^j, (M, s))$	price at node (M, s) of a bond j with coupons \vec{c}^j at times \vec{t}_c^j
M	number of time steps in a tree
μ	drift
$\mathcal{N}(\mu, \sigma^2)$	normal distribution with mean μ and variance σ^2
$O(m, s, t_o)$	value of an option with maturity t_o , at node (m, s) of the tree
Ω	set of all possible outcomes
$(\Omega, \mathcal{F}, \mathbb{P})$	probability space
\mathbb{P}	physical measure or real-world measure
$P(t, T)$	discount factor or bond price at time t with maturity T
$P_0(k, s)$	elementary price or bond price at time 0 paying 1 at time k in state s
$P_f(t, T_1, T_2)$	forward zero-coupon bond price at time t for maturity T_2 as seen from expiry T_1
\mathbb{Q}	martingale measure or risk-neutral measure
$r(t)$	short rate or instantaneous spot rate at time t
$\hat{r}(t)$	discrete short rate at time t
\hat{R}	discrete time short rate process on the tree
s	state variable
S	strike level of an option
S_k	cap rate or strike at time k
σ	volatility
σ_1	volatility from $t = 0$ until delivery of the future, $t = t_d$
σ_2	volatility from $t = t_d$ until maturity of the bonds
σ_m	market cap volatility
t	time
t_n	reset date of caplet n
t_{n+1}	payoff date of caplet n

t_N^j	maturity of bond j , where $j = 1, 2$ or 3
T	maturity
W_t	Brownian motion process
X	random variable
y	yield
$z(t)$	continuously compounded zero or spot rate, at time zero for the maturity T
$z(t, T)$	continuously compounded zero or spot rate, at time t for the maturity T
$\tilde{z}(t, T)$	annually compounded zero or spot rate, at time t for the maturity T

1 Introduction

1.1 Saen Options

Since the change from the floor-based open outcry trading to screen trading in 2000, a lot has changed for market makers, such as Saen Options. Technology has become one of the most important facets of the trading. The software used by Saen Options, has to be faster than the software of its competitors, so when previously a second would count to do a trade, nowadays, every nanosecond counts.

To be able to be the fastest on every market, software is needed, that incorporates the latest changes in the field. Only half of the people that work at Saen Options are traders, and a big number of people works at either the IT, Development or Research department. At Research new products are investigated, problems that traders encounter in the markets are solved, and investigations are conducted to find the optimal trading. At the Development department new software and programs are designed, according to what is needed in the market.

It is a great opportunity to be able to write my thesis at Saen Options and to know that my research is useful for them. As described above, the whole business is driven by being the fastest, the smartest and the best on the trading markets, and it is a great experience to be part of such a challenging business.

1.2 Financial introduction

In this section the most important financial terms are explained.

A **futures contract** is a contract between two parties to buy or sell a commodity, at a certain future time at a delivery price, that is determined beforehand. The delivery date, or **final settlement date**, is also fixed in the agreement. Futures are standardized contracts that are traded on an exchange and can refer to many different types of commodities, like gold, silver, aluminium, wool, sugar or wheat, but also financial instruments, like stock indices, currencies or bonds, can be the underlying of the contract. The **quoted price** of a certain contract is the price at which traders can buy or sell the commodity and it is determined by the laws of supply and demand. The **settlement price** is the official price of the contract at the end of a trading day.

Forward contracts are similar contracts, but unlike the futures contracts, they are traded over-the counter instead of on an exchange, i.e., they are traded between two financial institutions. This makes it a much less secure contract, because if one of the companies does not obey the rules, e.g., if the buyer goes bankrupt and wants to back out of the deal, the other company has a problem.

To make sure that this does not happen when trading the futures contract on the exchange, a **broker** intervenes. This is a party that mediates between the buyer and the seller. An investor that wants to buy a futures contract, tells his broker to buy the contract on the exchange, which is the seller of the future, and the broker requires the investor to deposit funds in a **margin account**. The money that must be paid at the entering of the contract is the **initial margin**. When, at a later time point, the investor's losses are more than what the **maintenance margin** allows, the investor receives a **margin call** from the broker, that he should top up the margin account to the initial margin level before the next day. The broker checks if all of this happens and makes sure that in case the investor does not answer his margin calls, that he can end the

contract on time and is able to pay for the debts.

The party with the **short position** in the futures contract agrees to sell the underlying commodity for the price and date fixed in the contract. The party with the **long position** agrees to buy the commodity for that price on that date.

A **bond** is an interest rate derivative, which certifies a contract between the borrower (bond issuer) and the lender (bond holder). The issuer, usually a government, credit institution or company, is obliged to pay the bond's principal, also known as **notional**, to the bond holder on a fixed date, the **maturity date**. Such debt securities are very important, because in almost every financial transaction, one is exposed to interest rate risk and it is possible to control this risk using bonds. A **discount bond** or **zero-coupon bond** only provides the notional at maturity, while a **coupon bond** also pays a monthly, semiannually or annually coupon.

The **spot rate**, **zero-coupon interest rate** or simply **zero rate** $z(t, T)$, is defined as the interest rate at time t , that would be earned on a bond with maturity T , that provides no coupons. A **term structure model** describes the relationship between these interest rates and their maturities. It is usually illustrated in a zero-coupon curve or **zero curve** at some time point t , which is a plot of the function $T \rightarrow z(t, T)$, for $T > t$.

The **discount rate** is the rate with which you discount the future value of the bond. Since we assume that the bond is worth 1 at maturity T , the discount rate is actually the **value of the zero-coupon bond at time t for the maturity T** , $P(t, T)$. By denoting the annually compounded zero rate from time t until time T by $\tilde{z}(t, T)$, the discount rate is

$$P(t, T) = \frac{1}{(1 + \tilde{z}(t, T))^{(T-t)}},$$

but we will use continuously compounded zero rates $z(t, T)$ instead to compute the discount rate:

$$P(t, T) = e^{-(T-t)z(t, T)}. \quad (1)$$

The zero rate $z(t, T)$ applies to the period $[t, T]$. In most of the thesis, the zero rate $z(T)$, which is an abbreviation for $z(0, T)$, is used.

We first take a look at the **discrete time** and next we look at the continuous time. The **forward rate** is the interest rate for money to be borrowed between two dates in the future (T_1, T_2) , where $T_1 < T_2$, but under terms agreed upon at an earlier time point t . It is denoted by $f(t, T_1, T_2)$ at time t for the dates T_1, T_2 , and defined as

$$P_f(t, T_1, T_2) = e^{-(T_2-T_1)f(t, T_1, T_2)}, \quad (2)$$

where $P_f(t, T_1, T_2)$ is defined as the forward zero-coupon bond price at time t for maturity T_2 as seen from expiry T_1 and it equals

$$P_f(t, T_1, T_2) = \frac{P(t, T_2)}{P(t, T_1)}. \quad (3)$$

Borrowing an amount of money at time t until time T_1 at the known interest rate $z(t, T_1)$, and combining it from time T_1 to T_2 at the rate $f(t, T_1, T_2)$, known at time t , should give the same discount rates as when borrowing the amount of money at time t until T_2 against the interest rate $z(t, T_2)$:

$$P(t, T_1) \cdot P_f(t, T_1, T_2) = P(t, T_2) \quad (4)$$

or

$$e^{-(T_1-t)z(t,T_1)} \cdot e^{-(T_2-T_1)f(t,T_1,T_2)} = e^{-(T_2-t)z(t,T_2)}. \quad (5)$$

From Equation (5) one finds that:

$$f(t, T_1, T_2) = \frac{(T_2 - t)z(t, T_2) - (T_1 - t)z(t, T_1)}{T_2 - T_1}. \quad (6)$$

When looking at discrete time models, the **discrete short rate** is defined to be the one-period interest rate $\hat{r}(t)$ for the next period $[t, t + 1]$. It is actually the forward rate spanning a single time period,

$$\hat{r}(t) = f(0, t, t + 1).$$

We now take a look at the **continuous time**.

The **instantaneous forward rate** is the forward interest rate for an infinitesimally short period of time, and is defined as

$$f(t, T) := \lim_{\epsilon \downarrow 0} f(t, T, T + \epsilon), \text{ for all } t < T,$$

which equals

$$\begin{aligned} f(t, T) &\stackrel{(2)}{=} -\lim_{\epsilon \downarrow 0} \frac{\ln P_f(t, T, T + \epsilon)}{\epsilon} = -\lim_{\epsilon \downarrow 0} \frac{\ln P_f(t, T, T + \epsilon) - \ln P_f(t, T, T)}{\epsilon} \\ &= -\frac{\partial}{\partial T_2} \ln P_f(t, T, T_2)|_{T_2=T} \stackrel{(3)}{=} -\frac{\partial}{\partial T_2} \ln P(t, T_2)|_{T_2=T} = -\frac{\partial}{\partial T} \ln P(t, T), \end{aligned} \quad (7)$$

which follows from the equations that are mentioned on top of the equal signs.

The **instantaneous short rate** $r(t)$ is defined as the interest rate, for an infinitesimally short period of time after time t :

$$r(t) := \lim_{\epsilon \downarrow 0} z(t, t + \epsilon).$$

In Chapter 2 both the continuous time and the discrete time short rate models are studied. When the term ‘short rate’ is mentioned, the instantaneous short rate is meant, unless stated otherwise.

To indicate the difference between a zero-coupon bond and a **coupon-bearing bond**, we define $J(\vec{c}, \vec{t}_c, t)$ as the price of a bond at time t with coupons $\vec{c} = [c_{t_1}, c_{t_2}, \dots, c_{t_N}]$, at the coupon dates $\vec{t}_c = [t_1, t_2, \dots, t_N]$ for $t \leq t_1 < t_2 < \dots < t_N$, where the last coupon date is the maturity of the bond.

When the zero rates at time t until time t_i are $z(t, t_i)$, for $i = 1, \dots, N$, then at time t , the price of a coupon-bearing bond with the above coupons at the above dates, is:

$$J(\vec{c}, \vec{t}_c, t) = \sum_{i=1}^N c_{t_i} P(t, t_i) + P(t, t_N). \quad (8)$$

$P(t, t_i)$ is the price of a bond at time t that pays one at time t_i , so when a coupon of c_{t_i} is paid at time t_i , we have to discount with $P(t, t_i)$ to find the value of the coupon at time t . The total price of the coupon-bearing bond is the sum of the discounted coupon payments plus the discounted notional. We can rewrite this as:

$$J(\vec{c}, \vec{t}_c, t) = \sum_{i=1}^N c_{t_i} e^{-(t_i-t)z(t,t_i)} + e^{-(t_N-t)z(t,t_N)}. \quad (9)$$

A bond's **yield** y is defined as the interest rate at which the present value of the stream of cash flows, consisting of the coupon payments and the notional of one, is exactly equal to the current price of the bond, i.e.,

$$J(\vec{c}, \vec{t}_c, t) = \sum_{i=1}^N c_{t_i} e^{-(t_i-t)y} + e^{-(t_N-t)y}, \quad (10)$$

As one can see, every cash flow is discounted by the same yield.

A **future on a bond** is a contract that obliges the holder to buy or sell a bond at maturity. Often, this future consists of a basket of bonds. In this thesis, the **Euro-Bund future** or **FGBL contract**, will be studied. The market data for this future and its underlying bonds can be extracted from **Bloomberg**, which is a computer system that financial professionals use to view financial market data movements. It provides news, price quotes, and other information of the financial products.

Since the party with the short position may decide which bond to deliver, he chooses the **Cheapest-to-Deliver bond (CTD)**. The basket of bonds to choose from, consists of several bonds with different maturities and coupon payments. To be able to compare them, **conversion factors** are used. They represent the set of prices that would prevail in the cash market if all the bonds were trading at a yield equivalent to the contract's notional coupon. They are calculated by the exchanges according to their specific rules. The FGBL contract, that we look at, has a notional coupon of six percent, see Chapter 3. It is assumed that:

- the cash flows from the bonds are discounted at six percent,
- the notional of the bond to be delivered equals 1.

In Equation (10) the bond price for a given yield y can be seen. Since the contract's notional is six percent, the conversion factor of this contract can be found by filling in $y = 0.06$ in Equation (10):

$$\text{Conversion factor} = \sum_{i=1}^N c_{t_i} e^{-(t_i-t)0.06} + e^{-(t_N-t)0.06}.$$

When bonds have a yield of six percent, the conversion factors are equal to one. If the bonds have a yield larger than six percent, then the conversion factors are larger than one, but the shorter the maturity, the closer the factor comes to one. Similarly, when the yields are smaller than six percent, the conversion factors are smaller than one, but the longer the maturity, the closer the conversion factor comes to one, see [9].

When pricing a bond, it is necessary to look at what moments the coupons are paid. The bond is worth less on the days that the coupons are provided, because there will be one less future cash flow at that point. For the same reason, when approaching the next coupon payment date, the bond will be worth more. To give the bond holder a share of the next coupon payment that he has the right to, **accrued interest** should be added to the price of the bond. This new price is called the **cash price** or **dirty price**. The quoted price without the accrued interest is referred to as the **clean price**. The accrued interest can be calculated by multiplying the interest earned in the reference period by

$$\frac{\text{the number of days between today and the last coupon date}}{\text{the number of days in the reference period}}.$$

The reference period is the time period over which you receive the coupon. There are different ways to count the number of days of such a period, the most common are:

- actual/actual day count takes the exact number of days between the two dates and assumes the reference period is the exact number of days of the year (either 365 or 366 days in a year),
- 30/360 day count assumes there are 30 days in a month and 360 days in a year,
- actual/360 day count takes the exact number of days in a year, but assumes the reference period has 360 days.

We use the actual/actual day count, because this is the type of day-count used for the Euro-Bund future.

To determine which bond is the CTD, one needs to look at what cash flows there are. By selling the futures contract, the party with the short position receives:

$$(\text{Settlement price} \times \text{Conversion factor}) + \text{Accrued interest}.$$

By buying the bond, that he should deliver to the party with the long position, he pays:

$$\text{Quoted bond price} + \text{Accrued interest}.$$

The CTD is therefore the bond with the least value of

$$\text{Quoted bond price} - (\text{Settlement price} \times \text{Conversion factor}).$$

The corresponding price of the future fixed at time 0 with maturity T is:

$$F(0, T) = (C_0 - I_0)e^{z(T)T}, \quad (11)$$

where C_0 is the cash price of the bond at time 0, I_0 is the present value of the coupon payments during the life of the futures contract, T is the time until the maturity of the futures contract, and $z(T)$ is the risk-free zero rate from today to time T . Before showing why Equation (11) must hold, we introduce a new term: **arbitrage**. This is the possibility for investors to make money without taking a risk. Such an investor is called an arbitrageur. We want the economy to be arbitrage-free, because we do not want these self-financing strategies to lead to sure profit. If $F(0, T) > (C_0 - I_0)e^{z(T)T}$, an arbitrageur can make a profit by

- buying the bond; it costs him C_0 today, but he will receive coupon payments worth I_0 today. At maturity T his costs for buying the bond have become $(C_0 - I_0)e^{z(T)T}$.
- shorting a future contract on the bond, for which he receives $F(0, T)$ at maturity, which is more than what he paid for the bond.

If $F(0, T) < (C_0 - I_0)e^{z(T)T}$, an arbitrageur would be able to take advantage of the situation, by

- shorting the bond, for which he receives C_0 , but he has to pay the coupon payments, which are worth I_0 today. His gains from this are $(C_0 - I_0)e^{z(T)T}$ at maturity T .
- taking a long position in a future contract on the bond, for which he only pays $F(0, T)$, which is less than the profit that he has made from shorting the bond.

In both ways, the arbitrageur has made a riskless profit. Since we want the price of a future to be arbitrage-free, it cannot be larger than $(C_0 - I_0)e^{z(T)T}$, neither can it be smaller than this, so the futures price should be exactly as in Equation (11).

A **call option** is an agreement between two parties, which gives the holder the right, but not the obligation, to buy the underlying asset for a certain price at a certain time. This price is called the **strike** and the future time point is called the **maturity**. Regular types of assets are stocks, bonds or futures (on bonds). In Figure 1a one can see that a call only has a strictly positive payoff when the price of the underlying, A_T , rises above the strike level S , at maturity T :

$$\text{Payoff of a call option} = \max(A_T - S, 0).$$

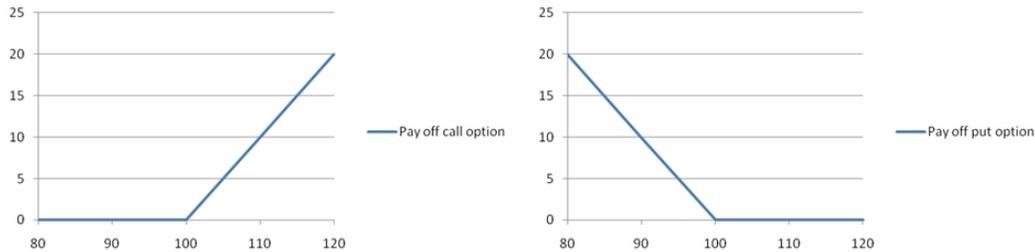


Figure 1:

- The payoff of a call option with strike $K = 100$,
- The payoff of a put option with strike $K = 100$

A **put option** is an agreement between two parties, which gives the holder the right, but not the obligation, to sell the underlying asset for a certain price at a certain time. In Figure 1b one can see that it provides a strictly positive payoff only if the underlying, A_T , is worth less than the strike price S , at maturity T :

$$\text{Payoff of a put option} = \max(S - A_T, 0).$$

A **swap** is an agreement between two companies to exchange one cash flow stream for another in the future. One interest rate is received, while at the same time the other one is paid. The swaps are netted, which means that only the difference in payments is made by the company that owes this difference. A notional principal is fixed at the entering of the contract. It is used to set the payments, but it will never be paid out.

The most common type of swap is a **plain vanilla interest rate swap**, for which a fixed rate cash flow is exchanged for a variable rate cash flow or vice versa. The fixed rate is chosen in such a way that the payoff of the swap would be zero. Because of this, and since the principal is never paid out, swaps have a very low **credit risk**. Potential losses from defaults on a swap are much less than the potential losses from defaults on a loan with the same principal, because for a loan, the lender has the risk that the borrower cannot pay the whole notional back, while for a swap it is only the difference in rates, taken over this principal, that one of the parties of the swap cannot gather.

An interest rate **cap** is an option that gives a payoff at the end of each period, when the interest rate is above a certain level, which we call the **cap rate** or strike S_n at time n . The interest rate is a floating rate that is reset periodically and it is taken over a principal amount. The caps that we will look at, have the Euribor rate as the floating rate. **Euribor** is short for Euro Interbank Offered Rate and the rates they offer are the average interest rates at which more than fifty European banks borrow funds from one another. The time between resets is

called the **tenor** and is usually three or six months.

Interest rate caps are invented to provide insurance against the floating rate. If the tenor is three months and today's Euribor rate is higher than today's cap rate, then in three months the cap will provide a payoff of the difference in rates times the notional amount. Vice versa, when today's Euribor rate is lower than today's cap rate, the payoff in three months will be zero.

A cap can be analyzed as a series of European call options or so-called **caplets**, which each have a payoff at time t_{n+1} :

$$\max(f(t, t_n, t_{n+1}) - S_n, 0),$$

where t_n is the reset date, t_{n+1} is the payoff date, $f(t, t_n, t_{n+1})$ is the forward rate, at time t , between times t_n and t_{n+1} , and S_n is the strike at time n . The total payoff of a cap with N caplets, at time t is:

$$\sum_{n=1}^N (t_{n+1} - t_n) P(t, t_{n+1}) \max(f(t, t_n, t_{n+1}) - S_n, 0), \quad (12)$$

where $t_{n+1} - t_n$ is the tenor and $P(t, t_{n+1})$ is the discount factor from t to t_{n+1} .

1.3 Mathematical introduction

Although most of the mathematical background that will be used, is explained in this section, the reader is assumed to have some knowledge in probability theory. More information on the subjects can be found in [10, 11, 13, 14].

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, (E, \mathcal{E}) be a measurable space and $[0, T]$ be a set. A **stochastic process** is defined as a collection $X = (X_t)_{t \in [0, T]}$ of measurable maps X_t from the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to (E, \mathcal{E}) . The **probability space** $(\Omega, \mathcal{F}, \mathbb{P})$ needs to satisfy a few properties. The collection of subsets \mathcal{F} , of the set Ω , should be a **σ -algebra**:

- $\emptyset \in \mathcal{F}$,
- if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$, and
- for any countable collection of $A_i \in \mathcal{F}$, we have $\bigcup_i A_i \in \mathcal{F}$.

This means that $\{\emptyset, \Omega\} \in \mathcal{F}$, and \mathcal{F} is closed under complements and countable unions. It should also hold that \mathbb{P} , the **probability measure**, is a function from \mathcal{F} to $[0, 1]$, such that

- $\mathbb{P}(\Omega) = 1$, and
- for any disjoint countable collection $\{A_i\}$ of elements of \mathcal{F} , one has $\mathbb{P}(\bigcup_i A_i) = \sum_i \mathbb{P}(A_i)$.

If the previous holds, then $(\Omega, \mathcal{F}, \mathbb{P})$ is indeed a probability space.

We say that a random variable X (from Ω to \mathbb{R}) is **measurable** with respect to \mathcal{F} if for all $x \in \mathbb{R}$, $\{\omega : X(\omega) \leq x\} \in \mathcal{F}$.

For a random variable $X \in \mathcal{L}^1(\Omega, \mathcal{F}, \mathbb{P})$, we define the **expectation** $\mathbb{E}(X)$ of X by

$$\mathbb{E}(X) := \int_{\Omega} X d\mathbb{P} = \int_{\Omega} X(\omega) \mathbb{P}(d\omega).$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and X a random variable with $\mathbb{E}(|X|) < \infty$. Let \mathcal{G} be a sub- σ -algebra of \mathcal{F} . Then there exists a random variable Y such that Y is \mathcal{G} -measurable, $\mathbb{E}(|Y|) < \infty$ and for every set $G \in \mathcal{G}$, we have

$$\int_G Y d\mathbb{P} = \int_G X d\mathbb{P}.$$

Y is called a version of the **conditional expectation** $\mathbb{E}(X|\mathcal{G})$ of X given \mathcal{G} , and we write $Y = \mathbb{E}(X|\mathcal{G})$, a.s.

If a collection $(\mathcal{F}_t)_{0 \leq t < \infty}$ of sub- σ -algebras has the property that $s \leq t$ implies $\mathcal{F}_s \subset \mathcal{F}_t$, then the collection is called a **filtration**. \mathcal{F}_t is the **natural filtration** $(\mathcal{F}_t)_{t \geq 0}$ and it contains all the information up to time t .

A real-valued stochastic process X , indexed by $t \in [0, T]$, is called a **martingale** w.r.t. the filtration \mathcal{F}_t , if the following conditions hold:

- (i) X_t is adapted for all $t \in [0, T]$, i.e., X_t is \mathcal{F}_t -measurable for all $t \in [0, T]$,
- (ii) X_t is integrable, $\mathbb{E}|X_t| < \infty$ for all $t \in [0, T]$,
- (iii) for discrete time: $\mathbb{E}(X_{s+1}|\mathcal{F}_s) = X_s$ a.s. for all $s \in [0, T]$,
for continuous time: $\mathbb{E}(X_t|\mathcal{F}_s) = X_s$ a.s. for all $s \leq t$ and $s, t \in [0, T]$.

By the third property we know that, given all information up to time s , the conditional expectation of observation X_{s+1} (resp. X_t), is equal to the observation at the earlier time s . In particular, $\mathbb{E}X_t = \mathbb{E}X_0$ for all $t \in [0, T]$.

A **Brownian motion** or **Wiener process** $W = (W_t)_{t \geq 0}$ is a continuous-time stochastic process that satisfies:

- W_t is adapted to \mathcal{F}_t ,
- $W_0 = 0$ a.s.,
- W has independent increments, i.e., $W_t - W_s$ is independent of $(W_u : u \leq s)$ for all $s \leq t$,
- W has stationary increments, i.e., $W_t - W_s$ has a $\mathcal{N}(0, t - s)$ -distribution for all $s \leq t$,
- the sample paths of W are almost surely continuous.

An **Itô process** is defined to be an adapted stochastic process which can be expressed as

$$X(t) = X(0) + \int_0^t \mu(s, X(s))ds + \int_0^t \sigma(s, X(s))dW_s. \quad (13)$$

It is usually written in differential form as

$$dX(t) = \mu(t, X(t))dt + \sigma(t, X(t))dW_t. \quad (14)$$

$X(t)$ consists of a **drift** term $\mu(t, X(t))dt$ and a **noise** term $\sigma(t, X(t))dW_t$.

If $\gamma \in C^{1,2}(\mathbb{R}^+ \times \mathbb{R})$ and $X(t)$ is a process that satisfies Equation (13), then the process $Y(t) = \gamma(t, X(t))$ can be written as:

$$Y(t) = Y(0) + \int_0^t \frac{\partial \gamma}{\partial x}(s, X(s))dX(s) + \int_0^t \frac{\partial \gamma}{\partial t}(s, X(s))ds + \frac{1}{2} \int_0^t \frac{\partial^2 \gamma}{\partial x^2}(s, X(s))\sigma^2(s, X(s))ds, \quad (15)$$

where $\frac{\partial}{\partial x}$ is the partial derivative with respect to the second variable W_s . This is called **Itô's formula** and it equals:

$$dY(t) = \frac{\partial \gamma}{\partial t}(t, X(t))dt + \frac{\partial \gamma}{\partial x}(t, X(t))dX(t) + \frac{1}{2} \frac{\partial^2 \gamma}{\partial x^2}(t, X(t))\sigma^2(t, X(t))dt. \quad (16)$$

The class $\mathcal{H}^2 = \mathcal{H}^2[0, T]$ consists of all measurable adapted functions ϕ that satisfy the integrability constraint:

$$\mathbb{E} \left(\int_0^t \phi^2(\omega, s) ds \right) < \infty,$$

which is a closed linear subspace of $\mathcal{L}(dP \times dt)$ (see [10]). If $\phi \in \mathcal{H}^2$, then for all $t \in [0, T]$:

$$\mathbb{E} \left(\int_0^t \phi(\omega, s) dW_s \right)^2 = \mathbb{E} \left(\int_0^t \phi^2(\omega, s) ds \right), \quad (17)$$

which is called the **Itô isometry**.

The **risk neutral measure** or martingale measure, denoted by \mathbb{Q} , is a probability measure that results, when all tradeables have the same expected rate of return, regardless of the ‘riskiness’, i.e., the variability in the price, of the tradeable. This expected rate of return is called the **risk-free rate**, so under \mathbb{Q} , $\mu(s, X(s)) \equiv r(s)$ for all tradeables’ price processes X .

In the physical or **real-world measure** \mathbb{P} this is the opposite case, more risky assets or assets with a higher price volatility, have a greater expected rate of return, than less risky assets. In [1] it can be seen how one can switch from the real-world measure to the risk-neutral measure by applying Girsanov’s theorem. The measure that will be used from now on is the risk-neutral measure.

The **fundamental theorem of arbitrage-free pricing** roughly states that there is no arbitrage if and only if there exists a unique risk neutral measure \mathbb{Q} , that is equivalent to the original probability measure \mathbb{P} .

For fixed T , the process $t \rightarrow P(t, T)_{0 \leq t \leq T}$ is a nonnegative, càdlàg (continue à droite, limite á gauche) semimartingale defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0 \leq t \leq T}, \mathbb{P})$, with $P(T, T) = 1$, because the bond is worth 1 at maturity. At time $t > T$, the bond is worthless, therefore $P(t, T)$ is only defined when $t \in [0, T]$. The adaptedness property must hold, because at time t the price of the bond must be known.

When the instantaneous short rate r_t is a stochastic process, the expectation under the risk-neutral measure \mathbb{Q} of the value of a bond equals the current arbitrage-free price once discounted by the short rate. The discounted value of a bond at time t , paying 1 at maturity T , is: $e^{-\int_t^T r(s) ds}$. The short rate being random, applying the conditional expectation operator under the risk-neutral measure \mathbb{Q} gives:

$$P(t, T) = \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r(s) ds} | \mathcal{F}_t \right), \quad (18)$$

for all $t < T$. The term $e^{-\int_t^T r(s) ds}$ can be interpreted as a random discount factor applied to the notional of 1. Equation (18) is called the **bond pricing equation**. If the short rate is deterministic, then for all $t < T$:

$$P(t, T) = e^{-\int_t^T r(s) ds}.$$

1.4 Outline of Thesis

In the introduction of Chapter 2 we give an overview of the short rate models that are most common and in Section 2.2 it is explained how the Ho-Lee model can be used to find the short rate in continuous and discrete time. The two methods are compared in Section 2.2.3 and in

the succeeding section a numerical test of the approximations of this comparison is made. In Section 2.3 it can be seen how spot rates can be computed from a series of coupon-bearing bonds, and how they can be interpolated. A number of interpolation methods is listed with their properties and it is explained why raw interpolation is used in this project. To conclude this chapter, an example is given of how to bootstrap and interpolate with real market data.

Chapter 3 starts with an introduction about the Euro-Bunds and the Euro-Bund futures. In the next section it is looked at how to determine the Cheapest-to-Deliver bond and the futures and bonds are priced. An example is given of how to calculate today's bond and futures prices and how to find the CTD, when the zero curve, the volatility and the bond prices at delivery are given. In Section 3.3 it is explained how to find all the variables necessary to calculate these prices.

In Chapter 4 real market data is used to fit the model. In Section 4.1 it is investigated how many steps are needed to get a good fit and what happens to the futures and bond prices when there is only one volatility used in the model. In Section 4.2 we take a look at what values the volatilities should have to get a nice fit and it can be seen which futures and bond prices are obtained with these optimized volatilities. In Section 4.3 it can be found which bond is the Cheapest-to-Deliver and what change in the short rate makes the CTD change from a certain bond to another. The influence of the bonds and the volatilities on the futures price is studied in Section 4.4 and in the last section of this chapter we look at the possibility to get a nice prediction of the futures price, when fixing the volatilities on a certain date.

The conclusion can be found in Chapter 5 and in the appendix, starting on page 63, all Matlab commands, used in the project, are listed.

2 Short rate models

Over the last decades people have invented and improved many short rate models. In this section the most popular models are discussed and it is explained how one of these models, the Ho-Lee model can be solved in continuous and discrete time.

All models that are studied are **one-factor models**, depending on a single Wiener process.

2.1 Introduction

Since bond prices can be characterized by Equation (18), we know that whenever we can characterize the distribution of $e^{-\int_t^T r(s)ds}$ in terms of a chosen model for r , conditional on the information available at time t , we are able to compute the bond prices. From the bond prices the zero rates are computable, so by knowing the characterization of the short rate, the whole zero curve can be constructed.

The short rate process r is assumed to satisfy the stochastic differential equation (14) under the risk-neutral measure \mathbb{Q} . By defining the short rate as an Itô stochastic differential equation, we are able to use continuous time instead of discrete time. The short rate that we look at in this section is the instantaneous short rate, because the rate applies to an infinitesimally short period of time. For more information on the short rate models, see [1].

When choosing a model, it is important to consider the following questions:

- What distribution does the future short rate have?
- Does the model imply positive rates, i.e., is $r(t) > 0$ a.s. for all t ?
- Are the bond prices, and therefore the zero rates and forward rates, explicitly computable from the model?
- Is the model suited for building **recombining trees**? These are binomial trees for which the branches come back together, as can be seen in Figure 2a. The opposite of recombining trees are **bushy trees**, of which an example is given in Figure 2b, but we will not use this type of tree, because the computation is too cumbersome.
- Does the model imply **mean reversion**? This is a phenomenon, where the expected values of interest rates are pulled back to some long-run average level over time. This means that when the interest rate is low, mean reversion tends to give a positive drift and when the interest rate is high, mean reversion tends to give a negative drift.

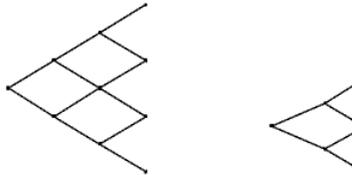


Figure 2: a. Recombining tree, b. Bushy tree

In this section these questions will be answered for each considered short rate model and in Table 1 on page 27 the most important properties are summarized.

The first short-rate models that were proposed were **time-homogeneous**, which means that the functions μ and σ in the stochastic differential equation for the short rates r do not depend on time:

$$dr(t) = \mu(r(t))dt + \sigma(r(t))dW_t.$$

The advantage of such models is that bond prices can be calculated analytically, but the term structure is **endogenous**, which means that the term structure of interest rates is an output rather than an input of the model, so the rates do not necessarily match the market data.

One of the first to model the short rate, was **Vasicek** [12] in 1977, who proposed that the short rate can be modeled as

$$dr(t) = a(\theta - r(t))dt + \sigma dW_t,$$

where θ, a and σ are positive constants. The short rate $r(u)$ conditional on \mathcal{F}_t is normally distributed with mean respectively variance:

$$\begin{aligned}\mathbb{E}(r(u)|\mathcal{F}_t) &= r(t)e^{-a(u-t)} + \theta(1 - e^{-a(u-t)}), \\ \text{Var}(r(u)|\mathcal{F}_t) &= \frac{\sigma^2}{2a}(1 - e^{-2a(u-t)}).\end{aligned}$$

The derivation can be found in the appendix, on page 63. For more information about the characteristics of this short-rate model, see [1]. A disadvantage is that for each time u , the short rate $r(u)$ can be negative with positive probability. The model has the following advantages: the distribution of the short rates is Gaussian, and the bond prices can be solved explicitly by computing the expectation (18). It does incorporate mean reversion, because the short rate tends to be pulled to level θ at rate a .

In 1978, **Dothan** [3] introduced the following short rate model:

$$dr(t) = ar(t)dt + \sigma r(t)dW_t,$$

where a is a real constant and σ is a positive constant. By integrating, one finds for $t \leq u$:

$$r(u) = r(t)e^{(a - \frac{1}{2}\sigma^2)(u-t) + \sigma(W_u - W_t)}.$$

Therefore $r(u)$, conditional on \mathcal{F}_t is lognormally distributed with mean respectively variance:

$$\begin{aligned}\mathbb{E}(r(u)|\mathcal{F}_t) &= r(t)e^{a(u-t)}, \\ \text{Var}(r(u)|\mathcal{F}_t) &= r^2(t)e^{2a(u-t)}(e^{\sigma^2(u-t)} - 1).\end{aligned}$$

The short rate $r(u)$ is always positive for each u , because of its lognormal distribution. The bond prices can be computed analytically, but the formulae are quite complex. For more information about the characteristics of this short-rate model and for the details of the derivation, see [1].

The **Cox-Ingersoll-Ross model** [2], developed in 1985, looks as follows:

$$dr(t) = a(\theta - r(t))dt + \sigma\sqrt{r(t)}dW_t,$$

and takes into account mean reversion. It also adds another quality, namely multiplying the stochastic term by \sqrt{r} , implying that the variance of the process increases as the rate r itself increases. For the positive parameters θ, a , and σ ranging in a reasonable region ($2a\theta > \sigma^2$),

the model implies positive interest rates and the instantaneous rate is characterized by a noncentral chi-squared distribution, with mean respectively variance:

$$\begin{aligned}\mathbb{E}(r(u)|\mathcal{F}_t) &= r(t)e^{-a(u-t)} + \theta(1 - e^{-a(u-t)}), \\ \text{Var}(r(u)|\mathcal{F}_t) &= r(t)\frac{\sigma^2}{a} \left(e^{-a(u-t)} - e^{-2a(u-t)} \right) + \theta\frac{\sigma^2}{2a} \left(1 - e^{-a(u-t)} \right)^2.\end{aligned}$$

For more information of the characteristics of this short-rate model and for the details of the derivation, see [1]. The model has been used often, because of its analytical tractability and the fact that the short rate is positive.

As already mentioned briefly, the time-homogeneous models have an important disadvantage, which is that today's term structure is not automatically fitted. It is possible to choose the parameters of the model in such a way that the model gives an approximation of the term structure, but it will not be a perfect fit. Therefore Ho and Lee [5] came up with an **exogenous term structure model** in 1986. The term structure is an input of the model, hence it perfectly fits the term structure. For these models, the drift does depend on t .

The **Ho-Lee model** is defined as

$$dr(t) = \theta(t)dt + \sigma dW_t, \tag{19}$$

where $\theta(t)$ should be chosen such that the resulting forward rate curve matches the current term structure. It is the average direction that the short rate moves at time t . We will now determine how $\theta(t)$ should be chosen.

By integrating (19) we obtain:

$$\begin{aligned}\int_t^u dr(s) &= \int_t^u \theta(s)ds + \sigma \int_t^u dW_s \\ r(u) &= r(t) + \int_t^u \theta(s)ds + \sigma(W_u - W_t).\end{aligned}$$

The short rate $r(u)$, conditional on \mathcal{F}_t , is normally distributed with mean respectively variance:

$$\begin{aligned}\mathbb{E}(r(u)|\mathcal{F}_t) &= r(t) + \int_t^u \theta(s)ds, \\ \text{Var}(r(u)|\mathcal{F}_t) &= \mathbb{E}(\sigma^2(W_u - W_t)^2|\mathcal{F}_t) \\ &= \sigma^2(u - t).\end{aligned}$$

The bond price at time t with maturity T equals:

$$\begin{aligned}P(t, T) &= \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r(u)du} | \mathcal{F}_t \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T [r(t) + \int_t^u \theta(s)ds + \sigma(W_u - W_t)]du} | \mathcal{F}_t \right).\end{aligned}$$

The integral $Z = -\sigma \int_t^T (W_u - W_t) du$ is normally distributed with mean zero and variance:

$$\begin{aligned}
\text{Var} \left(-\sigma \int_t^T (W_u - W_t) du \middle| \mathcal{F}_t \right) &= \sigma^2 \text{Var} \left(\int_0^{T-t} W_u du \right) \\
&= \sigma^2 \text{Var} \left((T-t)W_{T-t} - \int_0^{T-t} u dW_u \right) \\
&= \sigma^2 \mathbb{E} \left(\int_0^{T-t} (T-t-u) dW_u \right)^2 \\
&\stackrel{(17)}{=} \sigma^2 \int_0^{T-t} (T-t-u)^2 du \\
&= \frac{1}{3} \sigma^2 (T-t)^3.
\end{aligned}$$

The second equality follows from Itô's formula (16):

$$d(uW_u) = u dW_u + W_u du \Rightarrow \int W_u du = uW_u - \int u dW_u.$$

We have proved that the \mathcal{F}_t -conditional variance of the variable Z , which has a normal distribution on \mathcal{F}_t , equals $\frac{1}{3} \sigma^2 (T-t)^3$, therefore $\mathbb{E}_{\mathbb{Q}}(e^Z | \mathcal{F}_t) = e^{\frac{1}{6} \sigma^2 (T-t)^3}$, hence

$$P(t, T) = e^{-(T-t)r(t) - \int_t^T \int_t^u \theta(s) ds du + \frac{1}{6} \sigma^2 (T-t)^3}. \quad (20)$$

We want the bond prices to satisfy (7):

$$f(0, T) = -\frac{\partial}{\partial T} \ln P(0, T),$$

for a given function $f(0, T)$, therefore

$$\begin{aligned}
f(0, T) &= -\frac{\partial}{\partial T} \left(-Tr(0) - \int_0^T \int_0^u \theta(s) ds du + \frac{1}{6} \sigma^2 T^3 \right) \\
&= r(0) + \int_0^T \theta(s) ds - \frac{1}{2} \sigma^2 T^2,
\end{aligned}$$

from which we conclude that

$$\theta(t) = \frac{\partial}{\partial t} f(0, t) + \sigma^2 t.$$

The model has many advantages, namely the bond prices are explicitly computable from the model, it is very well suited for building recombining lattices (see Section 2.2.2) and it is an exogenous model. A disadvantage of the model is that $r(t)$ becomes negative with positive probability and that it does not incorporate mean reversion.

The well-known **Hull-White model** [7] is very similar to the Ho-Lee model except that a mean reversion term is added:

$$dr(t) = (\theta(t) - ar(t))dt + \sigma dW_t, \quad (21)$$

where a and σ are positive constants. It is often called the extended Vasicek model, because θ is no longer a constant, but a function of time, which is chosen to ensure that the model fits the

term structure. By integrating (21) we obtain:

$$\begin{aligned}\int_t^u dr(s) &= \int_t^u (\theta(s) - ar(s)) ds + \sigma \int_t^u dW_s \\ r(u) &= r(t)e^{-a(u-t)} + \int_t^u e^{-a(u-s)}\theta(s)ds + \sigma \int_t^u e^{-a(u-s)}dW_s,\end{aligned}$$

which is done in a similar way as for the Vasicek model, see appendix. The short rate $r(u)$, conditional on \mathcal{F}_t , is normally distributed with mean respectively variance:

$$\begin{aligned}\mathbb{E}(r(u)|\mathcal{F}_t) &= r(t)e^{-a(u-t)} + \int_t^u e^{-a(u-s)}\theta(s)ds, \\ \text{Var}(r(u)|\mathcal{F}_t) &= \frac{\sigma^2}{2a} \left(1 - e^{-2a(u-t)}\right).\end{aligned}$$

The bond price at time t with maturity T equals:

$$\begin{aligned}P(t, T) &= \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T r(u)du} | \mathcal{F}_t \right) \\ &= \mathbb{E}_{\mathbb{Q}} \left(e^{-\int_t^T [r(t)e^{-a(u-t)} + \int_t^u e^{-a(u-s)}\theta(s)ds + \int_t^u \sigma e^{-a(u-s)}dW_s] du} | \mathcal{F}_t \right) \\ &= e^{-\int_t^T r(t)e^{-a(u-t)} du - \int_t^T \int_t^u e^{-a(u-s)}\theta(s)ds du + \int_t^T \frac{\sigma^2}{2a}(1 - e^{-2au}) du}.\end{aligned}$$

We want the bond prices to satisfy (7):

$$f(0, T) = -\frac{\partial}{\partial T} \ln P(0, T),$$

therefore

$$\begin{aligned}f(0, T) &= -\frac{\partial}{\partial T} \left(-\int_0^T r(0)e^{-au} du - \int_0^T \int_0^u e^{-a(u-s)}\theta(s)ds du + \int_0^T \frac{\sigma^2}{2a}(1 - e^{-2au}) du \right) \\ &= r(0)e^{-aT} + e^{-aT} \int_0^T e^{as}\theta(s)ds - \frac{\sigma^2}{2a}(1 - e^{-2aT}),\end{aligned}$$

from which we conclude that

$$\int_0^T e^{as}\theta(s)ds = e^{aT} f(0, T) + e^{aT} \cdot \frac{\sigma^2}{2a}(1 - e^{-2aT}) - r(0),$$

hence

$$\begin{aligned}e^{aT}\theta(T) &= \frac{\partial}{\partial T} \left(e^{aT} f(0, T) + \frac{\sigma^2}{2a}(e^{aT} - e^{-aT}) - r(0) \right) \\ &= e^{aT} \cdot \frac{\partial}{\partial T} f(0, T) + ae^{aT} f(0, T) + \frac{\sigma^2}{2a}(e^{aT} - e^{-aT}),\end{aligned}$$

thus

$$\theta(t) = \frac{\partial}{\partial t} f(0, t) + af(0, t) + \frac{\sigma^2}{2a}(1 - e^{-2at}).$$

In Table 1 the properties of the short rate models are summarized. $(\log)\mathcal{N}$ stands for (log)normally distributed, $\text{NC}\chi^2$ for noncentral χ^2 distributed and AB for analytical bond price.

It is made clear of all the models what their advantages and disadvantages are. In this research we will work with the Ho-Lee model, because it is an exogenous term structure model, that can compute the bond prices, zero rates, and forward rates analytically and it is very suitable for building recombining trees, which can be seen in Section 2.2.2.

Table 1: Properties of the short-rate models

Model	Dynamics	$r > 0$	$r \sim$	AB
Vasicek	$dr(t) = a(\theta - r(t))dt + \sigma dW_t$	NO	\mathcal{N}	YES
Dothan	$dr(t) = ar(t)dt + \sigma r(t)dW_t$	YES	$\log\mathcal{N}$	YES
Cox-Ingersoll-Ross	$dr(t) = a(\theta - r(t))dt + \sigma\sqrt{r(t)}dW_t$	YES	$\text{NC}\chi^2$	YES
Ho-Lee	$dr(t) = \theta(t)dt + \sigma dW_t$	NO	\mathcal{N}	YES
Hull-White	$dr(t) = (\theta(t) - ar(t))dt + \sigma dW_t$	NO	\mathcal{N}	YES

2.2 Solving the short-rate models

In this section we show how the Ho-Lee model can be solved for continuous and discrete time, the two methods are compared, and an approximation of the comparison is given.

2.2.1 Continuous time Ho-Lee model

We have already seen in Equation (20) that the bond price at time t for maturity T is:

$$P(t, T) = e^{-(T-t)r(t) - \int_t^T \int_t^u \theta(s) ds du + \frac{1}{6}\sigma^2(T-t)^3}.$$

If the interest rates are constant, then $\theta(t) = 0$ and $\sigma = 0$, hence

$$P(t, T) = e^{-(T-t)r(t)}. \quad (22)$$

When $\theta(t)$ equals some constant α , we find

$$\begin{aligned} P(t, T) &= e^{-(T-t)r(t) - \int_t^T \int_t^u \alpha ds du + \frac{1}{6}\sigma^2(T-t)^3} \\ &= e^{-(T-t)r(t) - \frac{1}{2}\alpha(T-t)^2 + \frac{1}{6}\sigma^2(T-t)^3}. \end{aligned} \quad (23)$$

If $\theta(t)$ is a polynomial function βt^n for some constant β , we find

$$\begin{aligned} P(t, T) &= e^{-(T-t)r(t) - \int_t^T \int_t^u \beta s^n ds du + \frac{1}{6}\sigma^2(T-t)^3} \\ &= e^{-(T-t)r(t) - \frac{\beta}{n+1} \int_t^T u^{n+1} - t^{n+1} du + \frac{1}{6}\sigma^2(T-t)^3} \\ &= e^{-(T-t)r(t) - \frac{\beta}{n+1} [\frac{u^{n+2}}{n+2} - t^{n+1}u]_t^T + \frac{1}{6}\sigma^2(T-t)^3} \\ &= e^{-(T-t)r(t) - \frac{\beta}{n+1} (\frac{T^{n+2}}{n+2} - t^{n+1}T + \frac{n+1}{n+2}t^{n+2}) + \frac{1}{6}\sigma^2(T-t)^3}. \end{aligned} \quad (24)$$

It can be seen that $\theta(t)$ is not necessary to calculate the bond price. By filling in $\theta(s) = \frac{\partial}{\partial s} f(0, s) + \sigma^2 s$ in the integral $\int_t^T \int_t^u \theta(s) ds du$, one finds:

$$\begin{aligned} \int_t^T \int_t^u \theta(s) ds du &= \int_t^T \int_t^u \left(\frac{\partial}{\partial s} f(0, s) + \sigma^2 s \right) ds du \\ &= \int_t^T (f(0, u) - f(0, t) + \frac{1}{2}\sigma^2(u^2 - t^2)) du \\ &= -(T-t)f(0, t) - \frac{1}{2}\sigma^2(T-t)t^2 + \int_t^T \left(-\frac{\partial}{\partial u} (\ln P(0, u)) + \frac{1}{2}\sigma^2 u^2 \right) du \\ &= -(T-t)f(0, t) - \frac{1}{2}\sigma^2(T-t)t^2 - \ln P(0, T) + \ln P(0, t) + \frac{1}{6}\sigma^2(T^3 - t^3). \end{aligned}$$

The bond pricing formula (20) becomes:

$$\begin{aligned}
P(t, T) &= e^{-(T-t)r(t) - \int_t^T \int_t^u \theta(s) ds du + \frac{1}{6}\sigma^2(T-t)^3} \\
&= e^{-(T-t)r(t) + (T-t)f(0,t) + \frac{1}{2}\sigma^2(T-t)t^2 + \ln P(0,T) - \ln P(0,t) - \frac{1}{6}\sigma^2(T^3 - t^3) + \frac{1}{6}\sigma^2(T-t)^3} \\
&= \frac{P(0, T)}{P(0, t)} e^{-(T-t)(r(t) - f(0,t)) - \frac{1}{2}\sigma^2 t(T-t)^2}, \tag{25}
\end{aligned}$$

because

$$\begin{aligned}
\frac{1}{2}\sigma^2(T-t)t^2 - \frac{1}{6}\sigma^2(T^3 - t^3) + \frac{1}{6}\sigma^2(T-t)^3 &= \frac{1}{2}\sigma^2(T-t)t^2 - \frac{1}{6}\sigma^2(T^3 - t^3) \\
&\quad + \frac{1}{6}\sigma^2(T^3 - 3tT^2 + 3t^2T - t^3) \\
&= -\frac{1}{2}\sigma^2(tT^2 - 2t^2T + t^3) \\
&= -\frac{1}{2}\sigma^2 t(T-t)^2.
\end{aligned}$$

2.2.2 Discrete time Ho-Lee model

In this section we show how the continuous-time Ho-Lee model can be discretized, see [8]. This will be done by using a binomial lattice. We show that the discrete time version converges to the continuous time case for very small time intervals, in Section 2.2.3. In this section, the term ‘short rate’ is used to indicate the discrete short rate.

We set up a lattice with a time span between the nodes equal to the time period we want to use to represent the term structure. At each node, we assign a short rate, which is the one-period interest rate for the next period. We then assign probabilities to the various node transitions to create a fully probabilistic process for the short rate, where the probabilities are risk-neutral node transition probabilities of $\frac{1}{2}$.

The nodes in the lattice are indexed by (k, s) , where k is the time variable, $k = 0, \dots, T$, for maturity T , and s represents the state, $s = 0, \dots, k$, at time k , as can be seen in Figure 3. We assume that there are M steps in the tree, which means the time intervals have length $\Delta t = \frac{T}{M}$. The short rate at node (k, s) is given by $\hat{r}(k, s) \geq 0$.

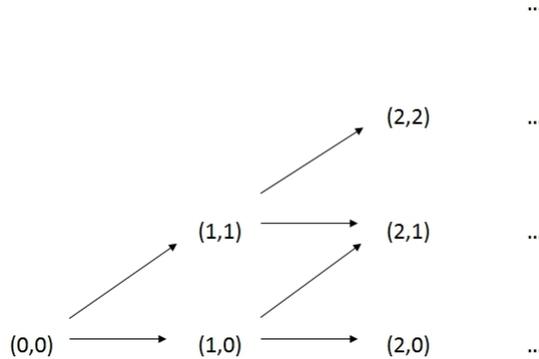


Figure 3: The nodes are indexed as (k, s) , where k refers to the time and s refers to the state

From the short rates, one can calculate the one-period discount rates at the nodes by:

$$d(k, s) = e^{-\Delta t \hat{r}(k, s)}.$$

We use a different notation for these discount rates than for the original discount rates, to distinguish between the one-period case and the usual case.

This lattice forms the basis for pricing bonds by using risk-neutral pricing. In [8] it is showed that it is much more computationally extensive to determine the term structure by working backwards through the lattice, than by applying a forward recursion, as we will do in this section. This is based on calculating at each node the **elementary price** $P_0(k, s)$, which is the price at time zero of a contract paying one unit at time k and state s only.

If we look at the node $(k + 1, s)$, for $s \neq 0$ and $s \neq k + 1$, we see that at the previous time k , there are two nodes leading to this state, namely nodes $(k, s - 1)$ and (k, s) , as can be seen in Figure 4.

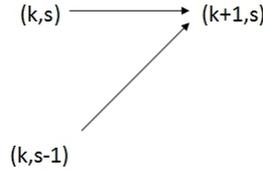


Figure 4: From nodes (k, s) and $(k, s - 1)$ to $(k + 1, s)$

When a bond pays 1 unit at node $(k + 1, s)$ and nothing elsewhere, then by going backwards, one finds that the bond would have value $\frac{1}{2}d(k, s - 1)$ at node $(k, s - 1)$ and value $\frac{1}{2}d(k, s)$ at node (k, s) .

By definition of the elementary prices, at time zero, these represent values

$$\frac{1}{2}d(k, s - 1)P_0(k, s - 1)$$

at time zero and

$$\frac{1}{2}d(k, s)P_0(k, s)$$

at time zero. Since $P_0(k + 1, s)$ is the expectation of the contract at time zero under the risk-neutral measure, we find

$$P_0(k + 1, s) = \frac{1}{2}d(k, s - 1)P_0(k, s - 1) + \frac{1}{2}d(k, s)P_0(k, s). \quad (26)$$

These are the elementary prices in the middle of the lattice ($0 < s < k + 1$). In this way, we can compute the elementary prices at every node, but at the top or bottom of the lattice the node only has one predecessor and therefore only depends on that node.

For the nodes at the bottom ($s = 0$) the elementary prices are:

$$P_0(k + 1, 0) = \frac{1}{2}d(k, 0)P_0(k, 0),$$

for the nodes at the top of the lattice ($s = k + 1$):

$$P_0(k + 1, k + 1) = \frac{1}{2}d(k, k)P_0(k, k).$$

By definition of the elementary prices, we know that the price of a bond that pays one unit at time $k = 0$ and state $s = 0$ is one, so the first elementary price $P_0(0,0)$ equals one. It is possible to determine all the prices at later time points by the forward recursion. These prices are strictly positive, because by moving step-by-step through the lattice, they are multiplied by $\frac{1}{2}$ and by the strictly positive discount factors and they are summed up eventually.

If the elementary prices would not be strictly positive, it would mean that at some node (k, s) the payout would be one unit, by definition of the elementary prices, although the price of this contract would be zero or negative, which is an arbitrage opportunity. Since this is not the case, we know that there is no arbitrage, hence the probabilities of $\frac{1}{2}$ are indeed risk-neutral. We could also have chosen to use for example probabilities $\frac{3}{4}$ and $\frac{1}{4}$, which would lead to no-arbitrage as well, but we decided to use the probabilities of $\frac{1}{2}$, because we want the short rate to have the same probability to go up as to go down.

Summing all the elementary prices at time k , which are the elements in column k of the lattice, gives the price of a zero-coupon bond with maturity k ,

$$P(0, k) = \sum_{s=0}^k P_0(k, s). \quad (27)$$

From these bond prices, the zero rates can easily be computed.

In the Ho-Lee model, the short rate at node (k, s) is represented as:

$$\hat{r}(k, s) = a(k) + b(k) \cdot s, \quad (28)$$

where $a(k)$ is a measure of aggregate drift from 0 to k and $b(k)$ is the volatility parameter. In Figure 5 it can be seen how the short rate tree is set up.

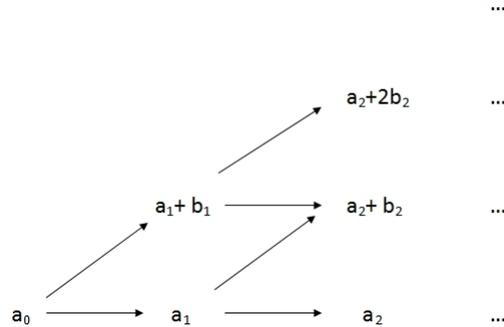


Figure 5: Ho-Lee short rate tree

From node (k, s) at time k , the short rate goes to node $(k + 1, s)$ with the risk-neutral probability $\frac{1}{2}$ and to node $(k + 1, s + 1)$ with probability $\frac{1}{2}$, as can be seen in Figure 6, with corresponding values

$$\hat{r}(k + 1, s) = a(k + 1) + b(k + 1) \cdot s,$$

respectively

$$\hat{r}(k + 1, s + 1) = a(k + 1) + b(k + 1) \cdot (s + 1).$$

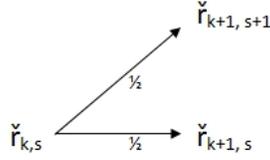


Figure 6: Short rate tree from node (k, s) to nodes $(k+1, s)$ and $(k+1, s+1)$

By matching the zero rates implied by the tree method, with the known zero rates from the market data, one can adjust the parameters $a(k)$ in such a way that the term structure is perfectly fit. To ensure that this is a good approximation to the continuous case, we compare the parameters $a(k)$, $b(k)$ with the parameters $\theta(t)$, σ in the continuous case in the following section.

2.2.3 Comparing the continuous and discrete time Ho-Lee models

Obviously we want the discrete time model for very small Δt to give results close to the continuous time model. To check this, the expectations and variances are compared in this section.

If we define by \hat{R} the discrete time short rate process on the tree, then, given the short rate $\hat{r}(k, s)$ as in Equation (28), this satisfies:

$$\hat{R}(k+1) - \hat{R}(k) = a(k+1) - a(k) + (b(k+1) - b(k))s,$$

with probability $\frac{1}{2}$, and

$$\hat{R}(k+1) - \hat{R}(k) = a(k+1) - a(k) + (b(k+1) - b(k))s + b(k+1),$$

with probability $\frac{1}{2}$. The conditional expectation and variation equal:

$$\mathbb{E}\left(\hat{R}(k+1) - \hat{R}(k) \mid \hat{R}(k) = \hat{r}(k, s)\right) = a(k+1) - a(k) + (b(k+1) - b(k))s + \frac{b(k+1)}{2}, \quad (29)$$

$$\text{Var}\left(\hat{R}(k+1) - \hat{R}(k) \mid \hat{R}(k) = \hat{r}(k, s)\right) = \frac{b(k+1)^2}{4}. \quad (30)$$

In the continuous time Ho-Lee model, one easily finds the conditional expectation and variance:

$$\mathbb{E}(r(t + \Delta t) - r(t) \mid \mathcal{F}_t) = \theta(t)\Delta t + o(\Delta t), \quad (31)$$

$$\text{Var}(r(t + \Delta t) - r(t) \mid \mathcal{F}_t) = \sigma^2\Delta t + o(\Delta t). \quad (32)$$

The time steps of the tree in the discrete time Ho-Lee model, are chosen to represent the term structure, which means that Δt is the same time interval as the difference between times k and $k+1$. This makes the conditional expectations and variances of the discrete and continuous time versions comparable. When the first and second moment of the discrete time version equal the first and second moment of the continuous time version, we know that the discrete model converges to the continuous model. This holds true when the conditional expectations of Equations (29) and (31) are equal:

$$a(k+1) - a(k) + (b(k+1) - b(k))s + \frac{b(k+1)}{2} = \theta(t)\Delta t, \quad (33)$$

and the standard deviations, the square roots of Equations (30) and (32), are equal:

$$\frac{b(k+1)}{2} = \sigma\sqrt{\Delta t}. \quad (34)$$

Since the Ho-Lee model assumes that the volatility per unit of time σ is constant, $\frac{b(k+1)}{2}$ is independent of k and equal to $\sigma\sqrt{\Delta t}$. Therefore $b(k+1) - b(k) = 0$, hence Equation (33) becomes

$$a(k+1) - a(k) = \theta(t)\Delta t - \sigma\sqrt{\Delta t}. \quad (35)$$

In case the tree satisfies Equation (35) for $\Delta t \rightarrow 0$, we know that the discrete-time version converges to the continuous-time Ho-Lee model. For this to happen, the parameters $a(k)$ should increase every time step by $\theta(t)\Delta t - \sigma\sqrt{\Delta t}$. A numerical test of this approximation can be found in the next section.

2.2.4 Numerical test of the approximations

For **constant interest rates**, $\theta(t)$ and σ equal zero, and the increase of a is zero, hence all a 's are equal.

When $\theta(t)$ **equals some constant** α , then $a(k+1) - a(k) = \alpha\Delta t - \sigma\sqrt{\Delta t}$, so the a 's grow with a constant value, because Δt and σ are constant. This means that a is a linear function. This is confirmed in Figure 7a, where the time t is on the horizontal axis and a is on the vertical axis.

For $\theta(t)$ **linear**, $\theta(t) = \alpha t$ and $a(k+1) - a(k) = \alpha t\Delta t - \sigma\sqrt{\Delta t}$, where Δt and σ are constant. Since a increases with αt per time step, we know a should be a quadratic function, as can be seen in Figure 7b, where the time t is on the horizontal axis and a on the vertical axis.

In all cases, when equating the bond prices from the discrete model (27) with the bond prices from the continuous model, either (22), (23), or (24), we indeed find that Equation (35) holds for Δt very small (in Figure 7a and b, $\Delta t = \frac{T}{M} = \frac{1}{100}$). Therefore we can say that this discrete time model gives a good approximation of the continuous time Ho-Lee model.

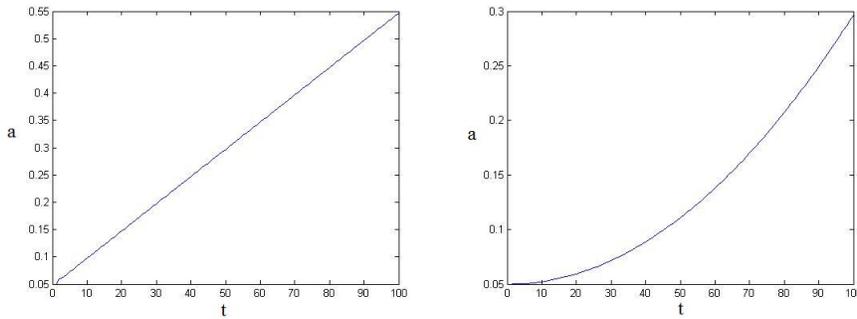


Figure 7:

- a. Graph of a , with $r = 0.05, \sigma = 0.0001, T = 1, M = 100$, and $\theta(t) = 0.5$
- b. Graph of a , with $r = 0.05, \sigma = 0.0001, T = 1, M = 100$, and $\theta(t) = 0.5t$

2.3 Bootstrap and interpolation of the zero rates

Bootstrapping is a method to calculate the zero curve from a series of coupon-bearing bonds. We have seen in Equation (9), that when the zero rates at time t for the maturities t_i are $z(t, t_i)$, for $i = 1, \dots, N$, then at time t , the price of a coupon-bearing bond with coupons \vec{c} at times \vec{t}_c , is:

$$J(\vec{c}, \vec{t}_c, t) = \sum_{i=1}^N c_{t_i} e^{-(t_i-t)z(t, t_i)} + e^{-(t_N-t)z(t, t_N)}. \quad (36)$$

The total price of the coupon-bearing bond is the sum of the discounted coupon payments plus the discounted notional. To give a reliable rate, the bonds, that are used to bootstrap, must be liquid, which means that they are traded often. If not, it is better to exclude them from the bootstrapping.

Rewriting Equation (36) gives:

$$z(t, t_N) = \frac{-1}{t_N - t} \ln \left(\frac{J(\vec{c}, \vec{t}_c, t) - \sum_{i=1}^{N-1} c_{t_i} e^{-(t_i-t)z(t, t_i)}}{1 + c_{t_N}} \right). \quad (37)$$

By knowing a bond price $J(\vec{c}, \vec{t}_c, t)$ and the zero-rates $z(t, t_i)$, for $i = 1, \dots, N-1$, it is possible to find the next zero-rate $z(t, t_N)$. For example, if the zero-rates up to time t_{N-1} are known and the zero-rate from time t until time t_N is required, this is easily done, by discounting all earlier cash flows with $z(t, t_1), z(t, t_2), \dots, z(t, t_{N-1})$, then the only unknown in Equation (36) is $z(t, t_N)$. This can be done for every maturity t_N .

In general, the earlier rates are not known exactly, because it is unlikely that there are always bonds available that expire exactly at the times t_i , for $i = 1, \dots, N-1$, as can be seen in Example 1. The information that is lacking can be completed by using a method called **interpolation**, see [4]. This is a technique that calculates the intermediate zero rates when the zero rates are only known for a few time points. Since the zero curve cannot be determined uniquely by the bootstrap, an interpolation scheme is necessary. Therefore the interpolation method is closely related to the bootstrap.

It could be the case that the rates are not even known after interpolation; for the smallest t_i , the rates might be available (eventually after interpolation), but the later ones might not be available. However, since Equation (37) is an iterative solution algorithm, it is possible to guess $z(t, T_N)$.

To see why this holds true, we give an example. Assume we have determined the zero curve at time t , up to ten years. If there is only a 15-years bond available for the succeeding years, we can do the following:

In Equation (37) it can be seen that, to calculate $z(t, 15)$, we also need the zero rates for 11, 12, 13 and 14 years. But bonds with these maturities are not available. Therefore we guess $z(t, 15)$, interpolate between ten and fifteen years to find $z(t, 11), z(t, 12), z(t, 13)$ and $z(t, 14)$ and check if the bond price, that follows from implementing these rates in Equation (36), is right. If not, then we try a bigger (or smaller) zero rate $z(t, 15)$, interpolate and check, if the bond price is fitted well. We do this iteratively, until the bond is fitted perfectly and reach the zero curve in this manner.

There are different ways to interpolate, but when choosing an interpolation method we should pay attention to the following:

- Are the forward rates positive? This is necessary to avoid arbitrage.

- Are the forward rates continuous? This is required, because pricing of interest sensitive instruments is sensitive to the stability of forward rates.
- How local is the interpolation method? When a change is made in the input, does it only have an effect locally or also for the rest of the rates?
- Are the forwards stable, i.e., if an input is changed with one basis point up or down, what is the change in the forward rate?

In Table 2, these questions are answered for a few interpolations methods. More on this can be found in [4].

Table 2: Properties of the interpolation methods

Interpolation type	Forwards positive?	Forward smoothness	Method local?	Forwards stable?
Linear on zero rates	no	not continuous	excellent	excellent
Linear on the log of zero rates	no	not continuous	excellent	excellent
Linear on discount factors	no	not continuous	excellent	excellent
Linear on the log of discount rates	yes	not continuous	excellent	excellent
Piecewise linear forward	no	continuous	poor	very poor
Quadratic	no	continuous	poor	very poor
Natural cubic	no	smooth	poor	good
Quartic	no	smooth	poor	very poor
Monotone convex	yes	continuous	good	good

As already stated before, the forward rates have to be positive to avoid arbitrage. The most ideal situation would be that forwards are also continuous, but these two properties are only fulfilled by the monotone convex interpolation, which is a method that is quite complex in use. Therefore we will use **raw interpolation** or **linear interpolation on the log of discount factors**, because despite the fact that the forward rates are not continuous, it is really easy to work with and it gives positive forward rates. The method corresponds to piecewise constant forward rates, which can be seen as follows.

For $t_i \leq t \leq t_{i+1}$, the continuously compounded risk-free rate for maturity t is:

$$z(t) = \frac{t - t_i}{t_{i+1} - t_i} \frac{t_{i+1}}{t} z(t_{i+1}) + \frac{t_{i+1} - t}{t_{i+1} - t_i} \frac{t_i}{t} z(t_i). \quad (38)$$

Taking the exponential gives

$$e^{z(t)} = e^{\frac{t-t_i}{t_{i+1}-t_i} \frac{t_{i+1}}{t} z(t_{i+1})} \cdot e^{\frac{t_{i+1}-t}{t_{i+1}-t_i} \frac{t_i}{t} z(t_i)}.$$

By taking the t -th power one finds:

$$\begin{aligned} \left(e^{-z(t)}\right)^t &= \left(e^{-\frac{t-t_i}{t_{i+1}-t_i} \frac{t_{i+1}}{t} z(t_{i+1})}\right)^t \cdot \left(e^{-\frac{t_{i+1}-t}{t_{i+1}-t_i} \frac{t_i}{t} z(t_i)}\right)^t \\ \Leftrightarrow e^{-tz(t)} &= e^{-\frac{t-t_i}{t_{i+1}-t_i} t_{i+1} z(t_{i+1})} \cdot e^{-\frac{t_{i+1}-t}{t_{i+1}-t_i} t_i z(t_i)} \\ &= \left(e^{-t_{i+1} z(t_{i+1})}\right)^{\frac{t-t_i}{t_{i+1}-t_i}} \cdot \left(e^{-t_i z(t_i)}\right)^{\frac{t_{i+1}-t}{t_{i+1}-t_i}}. \end{aligned}$$

Since the zero rates are continuously compounded, we can substitute $P(0, t) = e^{-tz(t)}$ and perceive:

$$P(0, t) = P(0, t_{i+1})^{\frac{t-t_i}{t_{i+1}-t_i}} P(0, t_i)^{\frac{t_{i+1}-t}{t_{i+1}-t_i}}, \quad (39)$$

which is equivalent to linear interpolation of the logarithm of the discount factors. Hence by linearly interpolating $\log(P)$, and then reverting it to P by taking the exponential, we can find all the intermediate discount factors. We then find the corresponding spot rates by $z(t) = -\frac{1}{t} \ln P(0, t)$. Since the instantaneous forward rates equal $f(0, t) = -\frac{\partial}{\partial t} \ln P(0, t)$ and we are interpolating linearly on the log of the discount rates, we find that the forward curves are piecewise constant, because the derivative is constant.

We give a general explanation of how to interpolate. Suppose that from the bond prices that are known, we have computed the zero rates (either at the given maturities or at other time points). The spot rates are given by

$$z(1), z(2), \dots, z(T),$$

where T is the maturity.

We want the new tree to have M steps, so the time steps in the interpolated tree are $\Delta t = \frac{T}{M}$. The new interpolated spot rates are

$$z(\Delta t), z(2\Delta t), \dots, z((M-1)\Delta t), z(M\Delta t).$$

By choice of Δt , the latter equals $z(T)$. For the case that $\frac{M}{T}$ is not an integer, we introduce the ceiling function, which also works if it is indeed an integer. For $t = \lceil \frac{M}{T} \rceil \Delta t$ until $t = T$ we will interpolate using raw interpolation (39), as it is explained above. For $t = \Delta t$ until $t = (\lceil \frac{M}{T} \rceil - 1)\Delta t$ we will extrapolate using

$$P(0, t) = P(0, 1)^{t\Delta t}. \quad (40)$$

In this case we find that $P(0, \Delta t), P(\Delta t, 2\Delta t), \dots$ are equal. This is exactly what we want. By extrapolating like this, we find that the spot rates between 0 and $t = (\lceil \frac{M}{T} \rceil - 1)\Delta t$ are equal to the original $z(1)$.

In Bloomberg, one can find the prices of coupon-bearing bonds, that can be used to bootstrap and interpolate. In Example 1, the zero curve is computed from three months, six months, one, two, and three year bonds.

Example 1: Finding the zero curve using bootstrapping and interpolation

Instead of the bonds with maturities equal to exactly 3 months, 6 months, 1 year, 2 years, and 3 years, we have the bonds that are listed in Table 3. Bund 1 and 2 only provide the notional at maturity and do not provide any coupons. We assume that the day count is act/365. Filling in Equation (36), we find that

$$P(0, 77 \text{ days}) = e^{-\frac{77}{365} z(77 \text{ days})} = 0.9940,$$

hence

$$z(77 \text{ days}) = -\frac{365}{77} \ln(0.9940) = 0.0285.$$

In the same way:

$$P(0, 170 \text{ days}) = e^{-\frac{170}{365}z(170 \text{ days})} = 0.9875,$$

hence

$$z(170 \text{ days}) = -\frac{365}{170} \ln(0.9875) = 0.0270.$$

For Bund 3 we have

$$P(0, 362 \text{ days}) = (1 + 0.04)e^{-\frac{362}{365}z(362 \text{ days})} = 1.0135,$$

hence

$$z(362 \text{ days}) = -\frac{365}{362} \ln\left(\frac{1.0135}{1 + 0.04}\right) = 0.0260.$$

For Bund 4 we have

$$\begin{aligned} P(0, 758 \text{ days}) &= 0.035e^{-\frac{393}{365}z(393 \text{ days})} + (1 + 0.035)e^{-\frac{758}{365}z(758 \text{ days})} \\ &= 1.0075, \end{aligned} \tag{41}$$

hence

$$z(758 \text{ days}) = -\frac{365}{758} \ln\left(\frac{1.0075 - 0.035e^{-\frac{393}{365}z(393 \text{ days})}}{1 + 0.035}\right).$$

To calculate $z(758 \text{ days})$, we need to know $z(393 \text{ days})$. However, until now, we have only obtained the zero rates for today until 77, 170, and 362 days, so $z(393 \text{ days})$ cannot be found by interpolating. There are two ways to solve this:

- extrapolate, or
- try a value for $z(758 \text{ days})$, interpolate to find $z(393 \text{ days})$ and check if this satisfies Equation (41). If not, try a bigger (or smaller) value for $z(758 \text{ days})$, until the bond price is fit to the equation.

The first method is not a nice way to solve this, because it usually does not get close to the actual rates. The second method is the most precise, but for larger maturities with unknown zero rates, it could take some time to solve. In this case, we only have to interpolate one zero rate, which we do by applying Equation (38) for $t=393 \text{ days}$, $t_i=362 \text{ days}$ and $t_{i+1}=758 \text{ days}$:

$$z(393 \text{ days}) = \frac{393 - 362}{758 - 362} \frac{758}{393} z(758 \text{ days}) + \frac{758 - 393}{758 - 362} \frac{362}{393} z(362 \text{ days}).$$

Table 3: Data of the bonds in the example

Bond	Price	Time to maturity	Time to first coupon	Coupon Rate
Bund 1	0.9940	77 days = 0.2110 years	-	0
Bund 2	0.9875	170 days = 0.4658 years	-	0
Bund 3	1.0135	362 days = 0.9918 years	362 days = 0.9918 years	4
Bund 4	1.0075	758 days = 2.0767 years	393 days = 1.0767 years	3.5
Bund 5	1.0178	1080 days = 2.9589 years	350 days = 0.9589 years	3.75

If we substitute this in Equation (41), then the only unknown is $z(758 \text{ days})$. Matlab can solve this equation easily and finds that $z(758 \text{ days}) = 0.0295$.

The price of Bund 5 is:

$$\begin{aligned} P(0, 1080 \text{ days}) &= 0.0375e^{-\frac{350}{365}z(350 \text{ days})} + 0.0375e^{-\frac{715}{365}z(715 \text{ days})} \\ &\quad + (1 + 0.0375)e^{-\frac{1080}{365}z(1080 \text{ days})}. \end{aligned} \tag{42}$$

$$= 1.0178.$$

We need $z(350 \text{ days})$ and $z(715 \text{ days})$ to compute $z(1080 \text{ days})$. We will find these rates easily by interpolating.

By applying Equation (38) for $t = 350 \text{ days}$, $t_i = 170 \text{ days}$, and $t_{i+1} = 362 \text{ days}$, we obtain:

$$\begin{aligned} z(350 \text{ days}) &= \frac{350 - 170}{362 - 170} \frac{362}{350} z(362 \text{ days}) + \frac{362 - 350}{362 - 170} \frac{170}{350} z(170 \text{ days}) \\ &= 0.0261. \end{aligned}$$

By applying Equation (38) for $t = 715 \text{ days}$, $t_i = 362 \text{ days}$, and $t_{i+1} = 758 \text{ days}$, we obtain:

$$\begin{aligned} z(715 \text{ days}) &= \frac{715 - 362}{758 - 362} \frac{758}{715} z(758 \text{ days}) + \frac{758 - 715}{758 - 362} \frac{362}{715} z(362 \text{ days}) \\ &= 0.0293. \end{aligned}$$

With these zero rates, it is possible to compute $z(1080 \text{ days})$, which equals 0.0313. In Figure 8 we can see what the zero curve looks like.

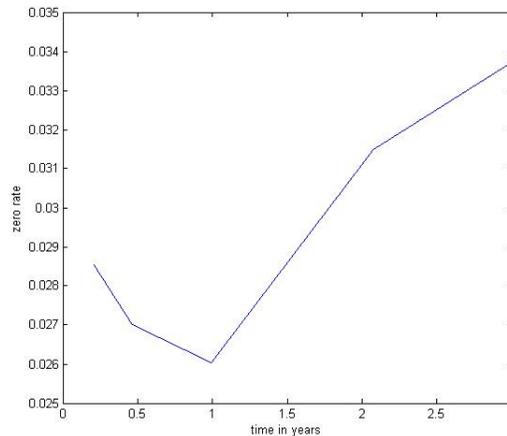


Figure 8: Zero curve

2.4 Conclusion

In this chapter a number of short rate models have been discussed, and their advantages and disadvantages were pointed out. We have chosen to work with the Ho-Lee model, because the bond prices are explicitly computable, which we showed in Equation (20), it is very well

suitable for building a recombining lattice, which we saw in Section 2.2.2 and it is an exogenous model. The continuous and discrete time Ho-Lee models were compared in Section 2.2.3 with a numerical approximation in Section 2.2.4. How to interpolate and bootstrap was explained in Section 2.3 and an example is given to show why and how we need to interpolate.

3 Future and bond pricing

In this chapter an introduction about the Euro-Bunds and the Euro-Bund futures is given. It is looked at how to price the bonds and futures and how to determine which bond is the cheapest at delivery of the future. An example is given to clarify how this can be done. Later on in the chapter we take a look at the variables necessary to calculate the bond and futures prices.

3.1 Introduction

The bonds that we look at, are the underlyings of the **FGBL contract** or **Euro-Bund futures contract**. This contract is traded on the **Eurex**, one of the world's largest derivatives exchanges and the leading clearing house in Europe (www.eurexchange.com). It is jointly operated by Deutsche Börse AG and SIX Swiss Exchange. Since it has market participants connected from 700 locations worldwide, the trading volume at Eurex exceeds 1,5 billion contracts a year.

The contract is unique in the sense that it only trades bonds with a maturity of 8.5 to 10.5 years. To get an idea of what other contracts are traded on the Eurex, in Table 4 the notional short-, medium- or long-term debt instruments, issued by the Federal Republic of Germany, are summarized. The currency is in euros.

Table 4: Data of the futures traded on the Eurex

Contract	Product ID	Remaining Term Years	Coupon rate
Euro-Schatz Futures	FGBS	1.75 to 2.25	6
Euro-Bobl Futures	FGBM	4.5 to 5.5	6
Euro-Bund Futures	FGBL	8.5 to 10.5	6
Euro-Buxl Futures	FGBX	24.0 to 35.0	4

The value of the FGBL contract or par value is 100.000 euro and the price quotation is in percent of the par value. The minimum price change should be one basis point (0.01 percent) or 10 euro. The delivery day is the tenth calendar day of the maturity month, if this day is an exchange day; otherwise, it is the exchange day immediately succeeding that day. For the December 2008 FGBL contract that we study this means that delivery takes place on December 10, 2008. The **last trading day** is two exchange days prior to the delivery day. On this day, the members with open short positions should notify Eurex which of the bonds they will deliver. This is on December 8, 2008, and this is the day on which the short position determines which bond is the Cheapest-to-Deliver.

The **final settlement price** is established by Eurex on the last trading day at 12:30 and it is based on the volume-weighted average price of all trades during the final minute of trading, provided that more than ten trades occurred during this minute. Otherwise the volume-weighted average price of the last ten trades of the day is taken as the final settlement price, provided that these are not older than thirty minutes. If such a price cannot be determined, or does not reasonably reflect the prevailing market conditions, Eurex will establish the final settlement price.

The December 2008 FGBL contract has three bonds as its underlyings, i.e., the bonds with ISIN codes DE0001135333, DE0001135341, and DE0001135358. They will be abbreviated by Bund 1, 2, resp. 3. Bund 1 has the shortest maturity and Bund 3 the longest, as can be seen

in Table 5, where also their coupon rates and conversion factors can be found. Note that the day-count convention is actual/actual, the coupon rates are annual and the settlement date is the day on which the bond is issued.

Table 5: Data of the bonds in the December 2008 FGBL contract

Bond	Settle	First Coupon	Maturity	Coupon Rate	Conversion factor
Bund 1	25-5-2007	4-7-2008	4-7-2017	4.25	0.885104
Bund 2	16-11-2007	4-1-2009	4-1-2018	4.00	0.863086
Bund 3	30-5-2008	4-7-2009	4-7-2018	4.25	0.874950

The option on the FGBL contract is called the **OGBL**, or option on the Euro-Bund futures. It expires six exchange days prior to the first calendar day of the futures expiration month. For the December 2008 FGBL contract the option expires on November 21, 2008. Unlike other options, the price of the OGBL option is settled daily, just like the future, with a margin account.

3.2 Cheapest-to-Deliver bond

The main goal of this thesis is to find out which of the three bonds, is the Cheapest-to-Deliver on the last trading day December 8, 2008. This is the day that the party with the short position on the future, has to notify which bond it will deliver. From Bloomberg, we have collected information of the dates from October 27, 2008 until December 8, 2008, and we would like to predict which bond will be the cheapest on the last trading day. We call the day that we look at, ‘today’, or $t = 0$, and the last trading day, December 8, 2008, is called ‘delivery’, or $t = t_d$, although it is actually delivered on December 10, 2008.

The bond that is cheapest-to-deliver on $t = t_d$, is the bond with the least value of

$$\text{Quoted bond price} - (\text{Settlement price} \times \text{Conversion factor}). \quad (43)$$

The quoted bond price is the clean bond price and the settlement price is the price of the future. The conversion factors of the bonds can be seen in Table 5.

On the website of the Eurexchange can be found that for FGBL contracts with delivery March, June, and September 2008 the short parties always delivered the cheapest bond. For the FGBL of December 2008, it states that in 98 % of the cases, the short position delivered the cheapest bond on the delivery day, which was Bund 1. The final settlement price of the future was 122.47 and the price of Bund 1 was 108.40 at that time point. The cost to deliver this bond would therefore be:

$$108.40 - 122.47 \times 0.885104 = 0.$$

This means that it would cost nothing to deliver Bund 1, and to deliver any of the other bonds would cost more. We can conclude that at delivery the futures price only depends on the bond for which Equation (43) is zero, so at delivery the price of the future is the minimum of

$$\text{Futures price} = \min_{i=1,2,3} \left(\frac{\text{Clean price of bond } i}{\text{Conversion factor of bond } i} \right).$$

We want to calculate the prices of the bonds at delivery of the future. One way to do this, is by making a Ho-Lee short rate lattice from ‘today’ until maturity of the bonds, calculate all

the bond and future prices at the nodes by going backwards through the lattice, and find out which one is the cheapest to deliver. It is an extensive work to go through all the steps of this enormous Ho-Lee tree, but luckily there is another less extensive way to do this.

Since we are only interested in the bond prices between today and delivery, we could also price the bonds more efficiently, by making a short rate tree from ‘today’ until delivery of the future, and calculate the bond prices at delivery analytically, by applying Equations (8) and (25). When a bond j , for $j = 1, 2, 3$, has a series of coupons $c_{t_i}^j$ that are paid at times t_i^j , $i = 1, \dots, N$, where t_N^j is the maturity of the bond, then the price of this bond at time t_d is:

$$J(\vec{c}^j, \vec{t}_c^j, t_d) = \sum_{i=1}^N c_{t_i}^j P(t_d, t_i^j) + P(t_d, t_N^j). \quad (44)$$

$P(t_d, t_i^j)$ is the price of a bond at time t_d that pays one at time t_i^j , so when a coupon of $c_{t_i}^j$ is paid at time t_i^j , we have to discount with $P(t_d, t_i^j)$ to find the value of the coupon at time t_d . The total price of the coupon-bearing bond is the sum of the discounted coupon payments plus the discounted notional.

We want to price the bonds at delivery $t = t_d$, at the end nodes of a binomial tree with M steps, hence we want to know the prices of the three bonds at the nodes (M, s) , where $s = 1, \dots, M$. We introduce two new notations, the first is the definition of $H((M, s), t)$ as the discount factor at node (M, s) with maturity t , hence it is the price at node (M, s) of a bond that pays one unit at time t . Next, we define $K(\vec{c}^j, \vec{t}_c^j, (M, s))$ as the price at node (M, s) of a bond j with coupons \vec{c}^j at times \vec{t}_c^j , and it can be calculated by using the above formula:

$$K(\vec{c}^j, \vec{t}_c^j, (M, s)) = \sum_{i=1}^N c_{t_i}^j H((M, s), t_i^j) + H((M, s), t_N^j). \quad (45)$$

The coupon payments $c_{t_i}^j$ are known for every payment date of every bond $j = 1, 2, 3$ and so are the dates on which the coupons are paid t_i^j . How this equation can be solved, will be shown in Section 3.3.

When all the bond prices $K(\vec{c}^j, \vec{t}_c^j, (M, s))$ are known at delivery, for the nodes (M, s) , $s = 1, \dots, M$, we can calculate the bond prices at the nodes prior to delivery, by doing the following. At some node $(M-1, s)$, then one time step later, we are either at node (M, s) , with probability $\frac{1}{2}$, or at node $(M, s+1)$, with probability $\frac{1}{2}$. The price of Bund j , at node $(M-1, s)$ is therefore:

$$K(\vec{c}^j, \vec{t}_c^j, (M-1, s)) = \frac{1}{2} d_{M-1, s} \left[K(\vec{c}^j, \vec{t}_c^j, (M, s)) + K(\vec{c}^j, \vec{t}_c^j, (M, s+1)) \right], \quad (46)$$

where $d_{M-1, s}$ is the discount factor at state s for the period $M-1$ to M . Note that if a bond would have a coupon payment between today and delivery of the future, then this cash flow should be added to the bond price at that time point, hence it should be added to the bond prices in the nodes (k, s) for k closest to the coupon date. The three bonds that we study, do not have any coupons in this time interval.

We have seen that the futures price at delivery can be found by taking the minimum of the bond prices divided by their conversion factors:

$$F((M, s), t_d) = \min_{j=1,2,3} \frac{K(\vec{c}^j, \vec{t}_c^j, (M, s))}{CF_j}. \quad (47)$$

This is what we do at the final nodes of the tree (M, s) , for $s = 1, \dots, M$ to find the price of the FGBL contract at delivery. One time step before, at $t = M - 1$, the future price equals

$$F((M - 1, s), t_d) = \frac{1}{2} [F((M, s), t_d) + F((M, s + 1), t_d)], \quad (48)$$

because from node $(M - 1, s)$, it goes to node (M, s) with probability $\frac{1}{2}$ and to node $(M, s + 1)$ with probability $\frac{1}{2}$. Since the future is settled daily, we do not have to encounter a discount factor. In other words, the futures price is the average of the two next prices using the risk-neutral probabilities without discounting. By going backwards through the lattice and computing the averages like this, we can find today's price of the future with delivery t_d at the initial node $(0, 0)$.

Example 2: Finding the cheapest-to-deliver and the futures price

In this example we will show how to use the above theory in practice. Assume that in exactly three months the futures contract will expire. The zero curve for today until one, two and three months is given by:

$$Z(t) = (0.0260, 0.0269, 0.0275),$$

for $t = (\frac{1}{12}, \frac{2}{12}, \frac{3}{12})$. Assume the volatility σ_1 equals 0.01. Since the zero rates and the volatility are known, we can build a short rate tree and its corresponding discount factor tree, see Figures 9a and b. The difference of 0.0058 in the short rates between the states follows from Equation (34):

$$\text{for all } k : \frac{b(k+1)}{2} = \sigma\sqrt{\Delta t} \Rightarrow b(1) = b(2) = 2 \cdot 0.01\sqrt{\frac{1}{12}} = 0.0058.$$

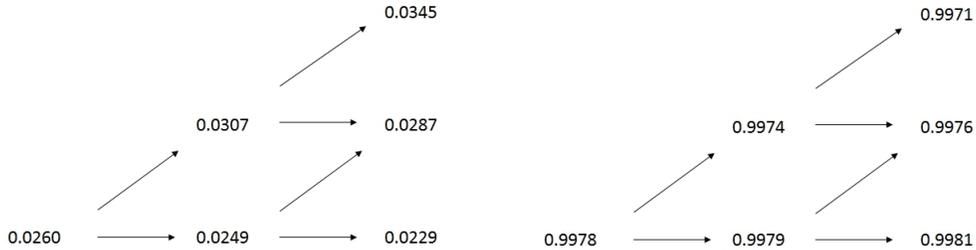


Figure 9:a. Short rate tree, b. Discount factor tree

At delivery, at the last nodes, the Bund prices are as follows:

Table 6: Bond prices at the nodes of the tree

	Bund 1	Bund 2	Bund 3
Node (3,1)	108.88	107.53	109.34
Node (3,2)	105.39	103.75	105.26
Node (3,3)	100.18	98.66	98.79

The conversion factors are as given in Table 5: 0.885104, 0.863086, and 0.874950 for Bund 1, 2, resp. 3. To find out which bond is the cheapest-to-deliver at the nodes $(3, 1)$, $(3, 2)$, and

(3, 3), we can compare the values of

$$\frac{K(\vec{c}^j, \vec{t}_c^j, (M, s))}{CF_j},$$

for the bonds $j = 1, 2$, and 3 , as can be seen in Table 7.

Table 7: Calculation of the cheapest bond and the corresponding futures price

	$\frac{K(\vec{c}^1, \vec{t}_c^1, (M, s))}{CF_1}$	$\frac{K(\vec{c}^2, \vec{t}_c^2, (M, s))}{CF_2}$	$\frac{K(\vec{c}^3, \vec{t}_c^3, (M, s))}{CF_3}$	CTD	$F((M, s), \frac{3}{12})$
Node (3,1)	123.01	124.59	124.79	1	123.01
Node (3,2)	119.07	120.21	120.30	1	119.07
Node (3,3)	113.18	114.31	112.91	3	112.91

The futures price is the minimum of these values. To calculate the futures price at the earlier nodes, we use Equation (48). The futures price at nodes (2, 1), resp. (2, 2) equal:

$$F((2, 1), \frac{3}{12}) = \frac{1}{2} (F((3, 1), \frac{3}{12}) + F((3, 2), \frac{3}{12})) = \frac{1}{2} (123.01 + 119.07) = 121.04,$$

resp.

$$F((2, 2), \frac{3}{12}) = \frac{1}{2} (F((3, 2), \frac{3}{12}) + F((3, 3), \frac{3}{12})) = \frac{1}{2} (119.07 + 112.91) = 115.99.$$

Today's price of the future can be found in the initial node (1, 1) of the tree:

$$F((1, 1), \frac{3}{12}) = \frac{1}{2} (F((2, 1), \frac{3}{12}) + F((2, 2), \frac{3}{12})) = \frac{1}{2} (121.04 + 115.99) = 118.52.$$

In Figure 10 the probabilities that either Bund 1, 2, or 3 will be the cheapest, are summarized.

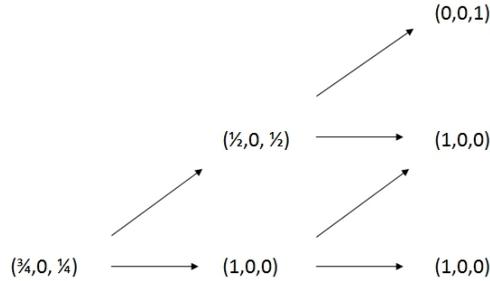


Figure 10: Tree with the probabilities that Bund 1, 2, or 3 will be the cheapest on delivery

We can also calculate the bond prices at the earlier nodes by applying Equation (46), as can be seen in Table 8. We can conclude that Bund 1 is the cheapest-to-deliver in the nodes (3, 1) and (3, 2) and Bund 3 is cheapest-to-deliver in node (3, 3). Today's price of the future is 118.52 and today's bond prices are 104.72, 103.18, and 104.42 for Bund 1, 2, resp. 3.

Table 8: Bond prices at the nodes of the tree

	Bund 1	Bund 2	Bund 3
Node (2,1)	106.91	105.42	107.08
Node (2,2)	102.52	100.95	101.76
Node (1,1)	104.72	103.18	104.42

3.3 Finding all the elements to compute the bond prices at delivery

In the previous section we have seen that the bond prices at delivery can be priced by Equation (45). To calculate the discount factors $H((M, s), t_i^j)$ of this equation, with which the cash flows have to be discounted, we can make use of Equation (25):

$$P(t_d, t_i^j) = \frac{P(0, t_i^j)}{P(0, t_d)} e^{-(t_i^j - t_d)(\hat{r}(t_d) - f(0, t_d)) - \frac{1}{2}\sigma_2^2 t_d (t_i^j - t_d)^2}, \quad (49)$$

hence at node (M, s) the discount factor $H((M, s), t_i^j)$ equals:

$$H((M, s), t_i^j) = \frac{P(0, t_i^j)}{P(0, t_d)} e^{-(t_i^j - t_d)(\hat{r}(M, s) - f(0, t_d)) - \frac{1}{2}\sigma_2^2 t_d (t_i^j - t_d)^2}. \quad (50)$$

At every state s of the tree, there is a different short rate and therefore a different bond price.

We will explain how the parameters that we need for Equation (50) can be found.

- the discount factors $P(0, t_i^j)$ and $P(0, t_d)$ are obtained by linearly interpolating the logarithm of the discount factors, which follow from the zero curve, see Section 3.3.1
- the short rates $\hat{r}(M, s)$ follow from the binomial tree from $t = 0$ until $t = t_d$, that we construct according to the Ho-Lee method, as can be seen in Section 2.2.2. We need the zero curve and the volatility for the period $[0, t_d]$, which we denote by σ_1 , to build this tree, see Sections 3.3.1, 3.3.2 and 3.3.3
- the forward rate of the equation is found by Equation (7):

$$f(0, t_d) = -\frac{\partial}{\partial t} \ln P(0, t)|_{t=t_d}.$$

Since we interpolate linearly over the logarithm of the discount factors, we know that its derivative must be constant on the time intervals.

- σ_2 is the volatility for $t = t_d$ until $t = t_N^j$, which we assume to be constant for all bonds j , see Section 3.3.4. The volatility σ_2 that we use in Equation (50) is another volatility than the volatility σ_1 , that we use in the tree.

The reason why we have chosen to use two different volatilities in the model is the following. We can fit the volatility σ_1 by pricing the call options on the FGBL contract, which is explained in detail in Section 3.3.3. This way to fit the volatility is very appropriate, because the options are based on government interest rates, just like the future and the bonds. Unfortunately we can only find the volatility for a very short time period, because the options expire on November 21, 2008.

Instead of applying the short term volatility for the larger time interval, we expect the model to

fit the bond and futures prices better, when using another volatility until maturity of the bonds. Since the call options are not available for this time interval, we make use of caps to fit these volatilities, see Section 3.3.4. This method is less appropriate, because it uses interbank interest rates instead of government interest rates. However, we suppose that this method still fits the bond and futures prices better, than applying the same short term volatility in the whole model.

3.3.1 Zero Curve

We start by creating a zero curve, from which the discount factors follow immediately. From Bloomberg we find the prices of a series of Euro Bunds with maturities of two until ten years that are listed in Table 9. These bonds are coupon-bearing bonds, not zero-coupon bonds, which means we first have to eliminate the coupons from the bond prices, by applying the bootstrapping method, as was showed in Section 2.3.

Since the first bond, that we use for bootstrapping, is a two years bond, we do not have information about the zero curve within these two years. We can instead find this information from the **Eonia swap**, or Euro Over-Night Index Average swap, which is a type of plain vanilla interest rate swap. It gives the zero rate from today until one week, two weeks, three weeks, one month, . . . , twelve months. Now that we have found the zero curve, which can be seen in Figure 11, we can calculate the discount factors by applying Equation (1).

Note that in Bloomberg, there is also information available about a three and six months bond and a one year bond, but since they are not liquid, which means that they are not traded much, it results in a zero curve with extreme fluctuations. The Eonia swap rates, on the contrary, are very liquid and therefore give a more reliable rate.

We are able to use the Eonia rates together with the zero curve from the government bonds, because in both cases, the credit risk is very low. For swaps this was already described in the introduction, and for government bonds it holds that they are supposed to be very safe and therefore they also have a low credit risk.

We want the zero curve to be as precise as possible, to price bonds 1, 2, and 3, so all three bonds have been used for bootstrapping, as can be seen in the table, Bund 1 is the 8.5 years bond, Bund 2 is the 9 years bond, and Bund 3 is the 10 years bond. In this way, we are sure that the bonds are priced well.

3.3.2 Short Rate Tree

We want to build a short rate tree with M steps, from ‘today’, $t = 0$, until December 8, 2008, $t = t_d$. To do so, according to the Ho-Lee model, we need to know the zero curve and the volatility σ_1 on $[0, t_d]$.

The zero curve has been determined by bootstrapping and the Eonia swap rates. To find the volatility of the tree, is much more complicated, because the Ho-Lee volatility cannot be observed in the market. The method that we use to find volatility σ_1 is first introduced shortly here and in the next section it is explained in detail.

Assume the volatility on the interval $[0, t_d]$ equals $\hat{\sigma}_1$, which is an estimator of the volatility σ_1 . We can build a short rate tree and if all the other elements of Equations (45) and (50) are known, we can find the bond prices $K(\vec{c}^j, \vec{t}_c^j, (M, s))$ at the nodes (M, s) of the tree and calculate the futures price $F((M, s), t_d)$ in the nodes. Subsequently, we can calculate the theoretical price of the OGBL option and compare it with the market option price. When the theoretical price is larger than the market option price, we try a smaller $\hat{\sigma}_1$ and vice versa, when it is smaller, we

Table 9: Data of the bonds used for bootstrapping the zero curve

Bond	Settle	Next Coupon	Maturity	Coupon Rate
Bund 2 year	9/10/2008	9/10/2009	9/10/2010	4
Bund 3 year	9/29/2006	10/14/2009	10/14/2011	3.5
Bund 4 year	9/28/2007	10/12/2009	10/12/2012	4.25
Bund 5 year	9/26/2008	10/11/2009	10/11/2013	4
Bund 6 year	11/26/2004	1/4/2009	1/4/2015	3.75
Bund 7 year	11/25/2005	1/4/2009	1/4/2016	3.5
Bund 8 year	11/17/2006	1/4/2009	1/4/2017	3.75
Bund 8.5 year	5/25/2007	7/4/2009	7/4/2017	4.25
Bund 9 year	11/16/2007	1/4/2009	1/4/2018	4
Bund 10 year	5/30/2008	7/4/2009	7/4/2018	4.25
Bund 15 year	1/4/1994	1/4/2009	1/4/2024	4.25

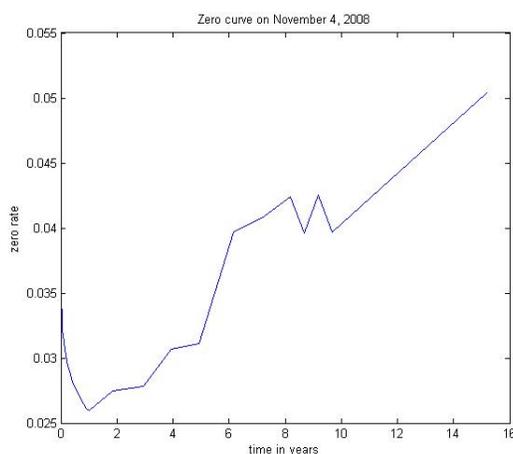


Figure 11: Zero curve on November 4, 2008

try a larger $\hat{\sigma}_1$. We do this iteratively until we find the volatility $\hat{\sigma}_1$ that fits the prices perfectly. Then we know that $\hat{\sigma}_1 = \sigma_1$.

3.3.3 Volatility σ_1

The price of an option depends on its underlying, in case of the OGBL the underlying is the FGBL contract. To compute today's value of the option, we use part of the short rate lattice that we constructed earlier with the estimator of σ_1 . The lattice starts today at $t = 0$ and ends at expiration of the option $t = t_o$. We assume that there are m time steps in this lattice, so at expiration of the option, we are at time m .

We calculate the futures prices at the nodes with Equation (47). At maturity of the call option, $t = t_o$, the payoff of the option is

$$O((m, s), t_o) = \max(0, F((m, s), t_d) - S),$$

where S is the strike of the option.

The price of the option at one time step before expiration, at $t = m - 1$, and at state s , can be calculated as follows:

$$O((m - 1, s), t_o) = \frac{1}{2} (O((m, s), t_o) + O((m, s + 1), t_o)),$$

where $O((m, s), t_o)$ and $O((m, s + 1), t_o)$ are the prices of the option at nodes (m, s) resp. $(m, s + 1)$. Since the option is settled daily, just like the futures price, we do not encounter a discount factor. Therefore, the price at delivery of the option, does not have to be discounted with a discount factor to find the price of the option today.

By going backwards through the lattice from $t = m - 1$ to $t = 0$, one can find today's value of the option at node $(0, 0)$ and compare this theoretical price with the price of the OGBL that we have found in Bloomberg and adjust σ_1 in case they are not equal. We change the volatility until the theoretical and market option price are equal, and then we have found the volatility σ_1 that fits the option price.

Note that the volatility that we look at in this section is actually the volatility on the time interval $[0, t_o]$, but since we assume the volatility to be constant between $t = 0$ and $t = t_d$, we use this volatility on the whole interval $[0, t_d]$.

There are also other possibilities to find the volatility σ_1 , that we have not used:

- calculating the theoretical option price according to Black's model, see [6], and comparing it with the market OGBL price. This is not possible, because the underlying of the option needs to be lognormally distributed. This is not the case, because the underlying future has three bonds as its underlyings and the minimum of the prices of three bonds, that are lognormally distributed, is not lognormal.
- using Black's model for bond options instead of bond futures options, because the bonds are lognormally distributed, but these options are not available, so this is not possible either.

3.3.4 Volatility σ_2

The objective is to find the volatility from $t = t_d$ until the maturity of the bonds. We want to use one volatility in the model, that is used for all three bonds and we have chosen to work with the volatility that corresponds to the longest maturity, that of Bund 3, of 9.5 years. The reason for this is that the differences between the 8, 9, and 10-years volatilities are very small, so we do not expect it to have a big influence which one of them we take for the three bonds.

Actually we should calculate the 'forward volatility', because the volatility that we obtain from the 10-years cap is from today until 10 years from today, but we want to find the volatility from delivery of the future until maturity of the bonds. Since we expect this difference to be very small, we decided to take today's 10-years cap.

Although the cap volatilities are not equal to the Ho-Lee volatilities, we are indeed able to find the Ho-Lee volatilities from the data that we have of the cap. In the introduction it was explained that a cap is a series of caplets or call options on interest rates. We can price these options on our short rate tree, as it is explained in detail at the end of this section. By fitting our theoretical Ho-Lee price of the cap, which is the sum of the caplets, to the price of the cap in the market, we can find the volatility of our model. We will first show how the market cap

price can be calculated from the cap strike and volatility and later on in this section it will be showed how the theoretical price of the cap is computed.

In Bloomberg we have found that the 10-years cap has the six months Euribor rate as its floating rate. This means that the tenor is six months, so every half a year the cap rate or strike is compared to the Euribor rate and when the Euribor rate is more than the strike, the difference will be paid out six months later. The 10-years cap has 19 reset dates: 0.5, 1, 1.5, ..., 9, 9.5 years and 19 payoff dates: 1, 1.5, 2, ..., 9.5, 10 years.

In the market, the price of a cap is calculated according to Black's model. It assumes that the underlyings of the option, the forward rates, are lognormal. When the cap strikes and volatilities are known, the cap can be priced easily. The Ho-Lee model does not assume that the forward rates (and short rates) are lognormally distributed, so when calculating the theoretical cap price later on in the section, Black's model will not be used.

Black's model states that at time t , the payoff of a caplet, with cap rate S_n and reset date t_n and payoff date t_{n+1} , is:

$$(t_{n+1} - t_n)P(t, t_{n+1}) [f(t, t_n, t_{n+1})\mathcal{N}(d_1(n)) - S_n\mathcal{N}(d_2(n))],$$

where

$$d_1(n) = \frac{\ln\left(\frac{f(t, t_n, t_{n+1})}{S_n}\right) + \frac{\sigma_m^2 t_n}{2}}{\sigma_m \sqrt{t_n}},$$

$$d_2(n) = \frac{\ln\left(\frac{f(t, t_n, t_{n+1})}{S_n}\right) - \frac{\sigma_m^2 t_n}{2}}{\sigma_m \sqrt{t_n}} = d_1(n) - \sigma_m \sqrt{t_n},$$

where $P(t, t_{n+1})$ is the discount factor for the period from t to t_{n+1} , $f(t, t_n, t_{n+1})$ is the forward rate at time t for the period between t_n and t_{n+1} , and \mathcal{N} is the normal distribution. The tenor is 6 months, hence $t_{n+1} - t_n = \frac{1}{2}$. The cap volatility σ_m and the cap rates S_n can be found in Bloomberg and the discount factors follow from the zero curve. The forward rates satisfy Equation (2), hence we can rewrite this as:

$$f(t, t_n, t_{n+1}) = -\frac{\log P_f(t, t_n, t_{n+1})}{t_{n+1} - t_n} = -\frac{\log\left(\frac{P(t, t_{n+1})}{P(t, t_n)}\right)}{t_{n+1} - t_n} = -\frac{\log P(t, t_{n+1}) - \log P(t, t_n)}{t_{n+1} - t_n}$$

$$= 2(\log P(t, t_n) - \log P(t, t_{n+1})). \quad (51)$$

The 10-years cap consists of 19 caplets, so we can sum up the caplet payoffs to find the value of the cap at time t :

$$\frac{1}{2} \sum_{n=1}^{19} P(t, t_{n+1}) [f(t, t_n, t_{n+1})\mathcal{N}(d_1(n)) - S_n\mathcal{N}(d_2(n))]. \quad (52)$$

Note that for all caplets, we have taken the same volatility σ_m . These volatilities are referred to as **flat volatilities** and all have the same maturity, namely the maturity of the cap. We could also have taken a different volatility for every caplet, which are called **spot volatilities**, that have the maturities of the caplets. Since the volatilities of the market caps, that we have obtained from Bloomberg, are flat volatilities, we also use flat volatilities for the pricing of the caps with the Ho-Lee model. The strikes or cap rates of the caps are at the money, which means

they equal the value of the underlying forward rate.

We will now take a look at how to price the caps on our Ho-Lee tree.

The 10-years cap is a series of 19 call options with 19 different maturities (the payoff dates) with the forward rates as its underlying. For every option, we can build a short-rate tree with some estimator of σ_2 , M steps, from $t = 0$ until the reset date of the caplet $t = t_n$. At time t_n we are in any of the nodes (M, s) , for $s = 1, \dots, M$, and we want to know the half-year forward rates at these nodes, that depend on the corresponding short rates.

To achieve this, we start by calculating the discount factor for the interval t_n to t_{n+1} , at node (M, s) , that we already defined as $H((M, s), t_{n+1})$. From the Ho-Lee bond pricing formula (25) follows:

$$\begin{aligned} H((M, s), t_{n+1}) &= \frac{P(0, t_{n+1})}{P(0, t_n)} e^{-\frac{1}{2}(\hat{r}(M, s) - f(0, t_n)) - \frac{1}{2} t_n \sigma_2^2 (t_{n+1} - t_n)^2} \\ &= \frac{P(0, t_{n+1})}{P(0, t_n)} e^{-\frac{1}{2}(\hat{r}(M, s) - f(0, t_n)) - \frac{1}{8} t_n \sigma_2^2}, \end{aligned}$$

where $f(0, t_n)$ is the instantaneous forward rate. Since we linearly interpolate over the logarithm of the discount factors, the forward rate is constant on the interval $[t_n, t_n + \epsilon]$ according to Equation (7). Therefore:

$$f(0, t_n) = -\frac{\partial}{\partial t} \log P(0, t)|_{t=t_n} = \frac{\log P(0, t_n) - \log P(0, t_n + \epsilon)}{\epsilon}.$$

We are now able to compute the discount factor $H((M, s), t_{n+1})$, from which we can calculate the forward rate $f((M, s), t_{n+1})$, which is defined as the forward rate at node (M, s) for maturity t_{n+1} . Just like in Equation (51), we find:

$$f((M, s), t_{n+1}) = -\frac{\log H((M, s), t_{n+1})}{t_{n+1} - t_n} = -2 \log H((M, s), t_{n+1}).$$

Since the payoff of the caplet is given at time t_{n+1} , but we want to know the payoff at the end nodes of the tree, at time t_n , we have to discount it with $H((M, s), t_{n+1})$ to find the payoff at node (M, s) :

$$\text{Caplet}((M, s), t_{n+1}) = \max(f((M, s), t_{n+1}) - S_n, 0) \cdot H((M, s), t_{n+1}).$$

By going backwards through the lattice, one can find that at node $(M - 1, s)$, the option is worth:

$$\text{Caplet}((M - 1, s), t_{n+1}) = \frac{1}{2} d(M - 1, s) (\text{Caplet}((M, s), t_{n+1}) + \text{Caplet}((M, s + 1), t_{n+1})),$$

which follows from the risk-neutral pricing and by stepwise discounting instead of discounting from t to t_{n-1} in one time, like in Equation (52). Continuing like this, one finds today's price of the option at the node $(0, 0)$.

The payoff at time zero of a cap with cap rates S_n , reset dates t_n , payoff dates t_{n+1} , for $n = 1, \dots, 19$, is:

$$\frac{1}{2} \sum_{n=1}^{19} \text{Caplet}((0, 0), t_{n+1}). \quad (53)$$

If the theoretical value of the cap in Equation (53) is lower than the market value in Equation (52), then we need to try a bigger estimator for σ_2 . If it is higher than the market price, we need to try a smaller estimator for σ_2 . We adjust the estimator of σ_2 in this manner, until the theoretical and the market price of the cap are equal. Then we have found the volatility σ_2 that fits the cap price.

4 Fitting with real market data

In this chapter we take a look at the results of the model using real market data. At time $t = t_d$ of the short rate tree, we calculate the bond prices and the futures prices, according to Equations (44), (50) and (47), and we fit the volatilities σ_1 and σ_2 as we described in Sections 3.3.3, resp. 3.3.4.

We take the data on a daily basis at 9.30 h, and in the first section it is studied how many steps are needed in the tree to give reliable results. There is also showed how well the futures and bond prices would be fitted when assuming that the volatilities are equal, $\sigma_1 = \sigma_2$. In the next section the volatilities are fitted at every time point and it is looked at how well the futures and bond prices are fitted. In Section 4.3 we take a look at which bond is the cheapest to deliver and at what change in the short rates the bond changes from being the cheapest to deliver. In Section 4.4 it is analyzed how sensitive the futures price is to changes in the bond prices or volatilities and in the last section we look at how well the futures price is fitted in future time points, when taking fixed volatilities of an earlier time point.

4.1 Increasing the number of steps in the tree

In this section we want to find out how many steps are needed to obtain reliable values for the bond and futures prices and the volatilities. We expect that the futures and bond prices converge to some value, because the more steps we take, the more the discrete time model converges.

When increasing the number of steps in the Ho-Lee tree, today's bond prices converge to some values between 0.027 and 0.033, which is 0.026 and 0.032 percent of the market price, see Figure 12a. Since the bond prices are used as an input of the model, they should be fitted 'perfectly'. The reason why they do not converge to zero is that we have taken two different volatilities in the tree, although the Ho-Lee model assumes that there is only one constant volatility.

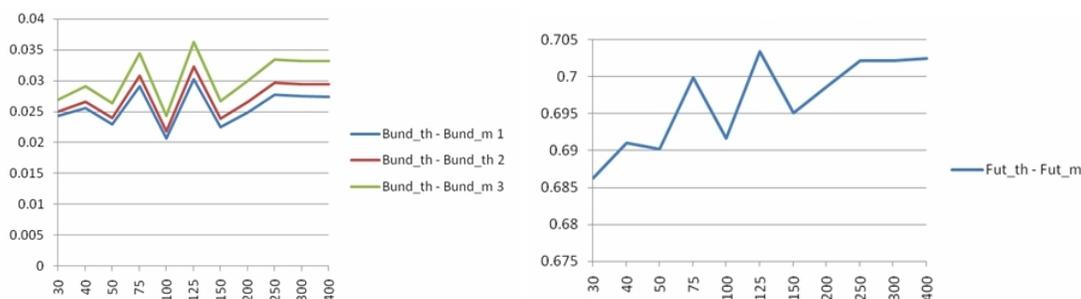


Figure 12:

- The difference between the Ho-Lee prices and the market prices of Bund 1, 2, and 3, when increasing the number of steps in the tree, on October 27, 2008
- The difference between the Ho-Lee futures price and the market futures price, when increasing the number of steps in the tree, on October 27, 2008

We take a closer look at Equation (50) to see how this error is caused. When the zero curve is fixed, the only elements that can influence the equation, are the volatility σ_2 and the short rate tree, constructed according to the fixed zero curve and the volatility σ_1 . Equation (50) is made according to the Ho-Lee model, that assumes that the volatility is constant. However, we

decided to take a different volatility on the time interval $[0, t_d]$ than on the time interval $[t_d, T^i]$, until maturity of the bonds. This is the reason why the theoretical prices of the bonds do not equal the market prices exactly.

At the end of this section we show what happens if we take $\sigma_1 = \sigma_2$, which means we take one volatility in the model.

In Figure 12b one can see that when increasing the number of steps from 30 to 400, the futures price converges to the value 0.7025. One of the reasons that it does not converge to zero, is that we have taken two different volatilities in the model, like we just described.

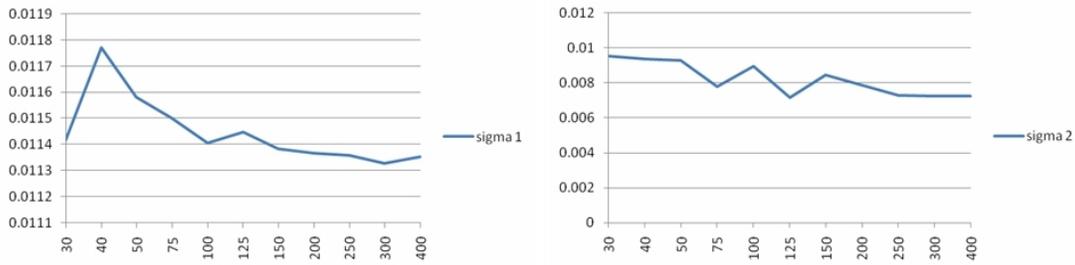


Figure 13:

- a. σ_1 fitted by comparing the Ho-Lee and market call options, when increasing the number of steps in the tree, on October 27, 2008
- b. σ_2 fitted by comparing the Ho-Lee and market caps, when increasing the number of steps in the tree, on October 27, 2008

The volatilities σ_1 and σ_2 converge to the values 0.01135, respectively 0.00725, when the number of steps increases, as can be seen in Figures 13a, resp. 13b. In all the figures in this section it can be seen that from 300 steps on, the values do not improve significantly. Therefore we will use a binomial tree of 300 steps throughout the rest of this chapter.

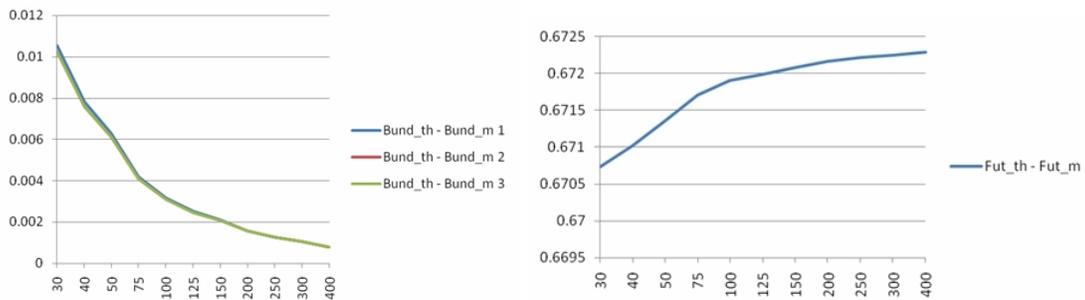


Figure 14:

- a. The difference between the Ho-Lee prices and the market prices of Bund 1, 2, and 3, when $\sigma_2 = \sigma_1$, on October 27, 2008
- b. The difference between the Ho-Lee futures price and the market futures price, when $\sigma_2 = \sigma_1$, on October 27, 2008

When taking volatility σ_2 equal to σ_1 , it means that we do not calculate σ_2 by fitting the market and theoretical cap prices, but we assume that the volatility from delivery of the future until maturity of the bonds equals the volatility from today until delivery of the future. This volatility σ_1 is found by fitting the OGBL market and theoretical price. In Figure 14a one can see that the differences between the Ho-Lee and market Bund prices get smaller when the number of steps increases. The error is much smaller than the error that we found, when assuming that there are two different volatilities σ_1 and σ_2 , and it is caused by the discretization of the continuous Ho-Lee model. When taking more steps, the discrete time model converges to the continuous time model and the errors becomes smaller.

4.2 Fitting the volatilities σ_1 and σ_2

In this section we take a look at the results found by fitting the parameters σ_1 and σ_2 . First we discuss the volatilities, then the futures prices and at the end we look at the bond prices.

In Figure 15 one can see that the prices of the call option with strike 118 have extreme fluctuations, so when fitting the volatility σ_1 with these prices, we find unreliable values. Since the option with strike 119 has a much smoother graph, we decided to fit the volatility σ_1 with this option. The values that we have found for σ_1 can be seen in Figure 16a.

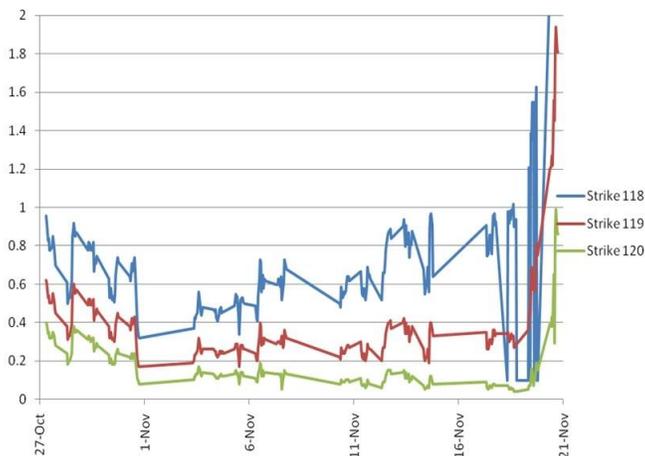


Figure 15: Values of the call options with strikes $S = 118, 119,$ and 120

The figure gives the volatility at the dates up to and including November 18, 2008. The volatility σ_1 has a very nice value, because the difference between the smallest and largest value is only 4.1 percent. On November 20, 2008, one day before expiration of the option, we cannot fit the volatility. The reason for this is that the closer we get to maturity, the closer the payoff of the option comes to

$$\max(\text{Futures price} - \text{Strike}, 0),$$

so the price of the option is not so sensitive to the volatility anymore. On November 20, 2008, the theoretical price of the call option can be calculated as:

$$\text{Price of call option} = \max(\text{Futures price} - \text{Strike}, 0) = \max(120.2 - 119.0) = 1.20,$$

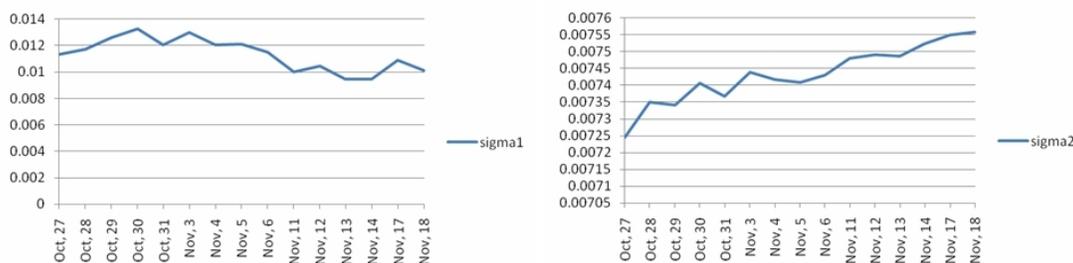


Figure 16:
a. σ_1 fitted by comparing the Ho-Lee and market call option with strike 119
b. σ_2 fitted by comparing the Ho-Lee and market caps

which is very close to the market call price of 1.21. This confirms what we just explained, the price of the call option does not depend on the volatility, but it converges to the value at delivery. Therefore we were not able to fit the volatility on November 20, 2008.

In Figure 16b it can be seen that the volatility σ_2 grows as the time evolves. The difference between the highest and lowest volatility is 4.3 percent.

We would like to check how well the model fits the theoretical futures price to the market futures price, when fitting the volatilities as described in Sections 3.3.3 and 3.3.4. In Figure 17a we can see the difference between these futures prices on the days between October 27, 2008 and November 18, 2008. We cannot look at the days after November 21, 2008, because the OGBL expires on this day, so we cannot calculate σ_1 on the succeeding days, because there is no market price of the call option available. We have just seen that one day before the expiration of the option, the OGBL is not sensitive to volatility and therefore the value that we found for σ_1 was not representative either. We exclude these dates in the rest of the analysis of this section and only look at the futures price until November 18, 2008.

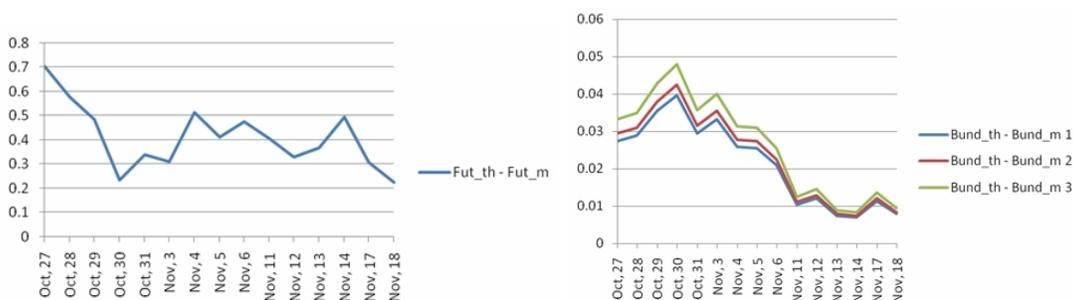


Figure 17:
a. The difference between the Ho-Lee futures price and the market futures price
b. The difference between the Ho-Lee prices and the market prices of Bund 1, 2, and 3

From Figure 17a we can deduce that the closer we get to delivery of the future, the better the model fits the futures price. This holds, because when we approach delivery, it becomes

clearer, which bond will be the cheapest at delivery and the futures price depends on this bond. Also, as the time to delivery of the future decreases, it becomes more unlikely that there will be enormous changes in the zero curve. The largest difference between the market and theoretical futures price is on October 27, 2008 and is 6.0 percent, but the differences of the other dates are much smaller.

We will now take a look at the differences between today's theoretical prices and today's market prices of Bund 1, 2, and 3, which can be seen in Figure 17b. Just like the futures price differences, it also holds for the bond price differences that they become smaller when the time evolves. The errors are very small, but still larger than zero.

4.3 Which bond is the cheapest to deliver

By calculating the bond and futures prices at the delivery nodes, we can find out at every node, which bond is the cheapest to deliver. In Table 10 it can be seen per day, what the lowest and the highest short rate is in the tree, and at which value of the short rate Bund 1 and 3 switch from being the cheapest to deliver. Bund 1 is the CTD between the lowest short rate and the value in the second column of the table and Bund 3 is CTD between the value in column 3 and the highest short rate. The short rates can be negative, as was already mentioned in Section 2, which can be seen very clearly in Figure 18a. The part between the blue line and the green line is where Bund 1 is CTD, and the part between the red line and the purple line is where Bund 3 is CTD.

Table 10: Overview of the short rate levels (in percent) at which the CTD changes from Bund 1 to 3

Date	Lowest short rate	Bund 1	Bund 3	Highest short rate
October 27, 2008	-3.60	5.10	5.14	9.67
October 28, 2008	-3.77	4.92	4.96	9.76
October 29, 2008	-4.20	4.79	4.84	10.18
October 30, 2008	-4.55	4.61	4.66	10.42
October 31, 2008	-3.85	4.65	4.70	9.60
November 3, 2008	-4.03	4.55	4.59	9.83
November 4, 2008	-3.47	4.71	4.75	9.21
November 5, 2008	-3.43	4.75	4.80	9.12
November 6, 2008	-3.14	4.73	4.77	8.62
November 11, 2008	-1.84	4.72	4.75	7.54
November 12, 2008	-2.04	4.57	4.60	7.59
November 13, 2008	-1.59	4.52	4.54	6.94
November 14, 2008	-1.52	4.37	4.40	6.83
November 17, 2008	-1.45	4.82	4.85	7.56
November 18, 2008	-1.17	4.71	4.74	6.97

We have seen which bond is the cheapest at the delivery of the future. In the nodes prior to the delivery nodes, we can calculate the probabilities that Bund 1, Bund 2, or Bund 3 will be the cheapest-to-deliver at delivery, just like we have done in Example 2. We look at the data of October 27, 2008 and the graph demonstrates the probabilities between this date and delivery, that the bonds become the CTD. Bund 2 is in neither of the end nodes of the tree the cheapest, so the probability that Bund 2 becomes cheapest is zero in the whole tree. In Figure 18b the blue area is the part where Bund 1 is the cheapest, the red area is where Bund 3 is the cheapest

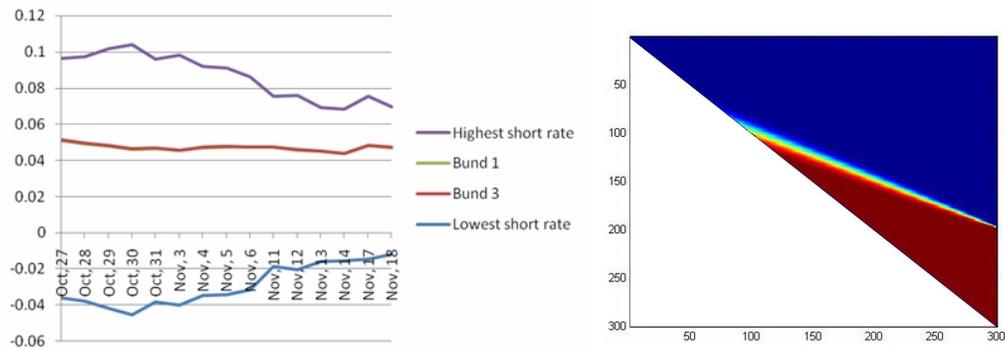


Figure 18:

- The short rate levels, between the green and blue line Bund 1 is cheapest and between the red and purple line Bund 3 is cheapest
- The probability tree, in the blue area Bund 1 is cheapest, in the red area Bund 3 is cheapest and in the mixed colors they both have a probability of becoming the cheapest at delivery

and the colors in between state that there is a probability that Bund 1 will become the cheapest at delivery as well as there is a probability that Bund 3 will become the cheapest at delivery.

4.4 Sensitivity of the futures price

In this section we take a look at how sensitive the futures price is towards the bond prices and the volatilities.

4.4.1 Influence of the bond prices on the futures price

First we show what happens to the theoretical futures price when one of the three bond prices increases or decreases by one. We assume that the bond prices are not correlated. We look at the ratio:

$$\frac{\text{original futures price} - \text{futures price after changing the bond price}}{\text{original bond price} - \text{bond price after changing}} = \frac{\Delta F_j}{\Delta \text{bond } j},$$

where $j = 1, 2, \text{ or } 3$. In Section 4.2 we have fitted the volatilities σ_1 and σ_2 , such that the theoretical and market call, resp. cap prices were equal. We fix these volatilities and change one of the three bond prices by one, which is approximately a one percent change. When the bond price is changed, this means that the zero curve, the short rate tree and therefore the futures price is different. How much influence it has on the futures price can be seen in Figures 19a, b and c.

In Section 3.2 we have seen that at delivery of the future, its price equals the minimum of the three bonds divided by their conversion factors. In case we change one of the bond prices it is possible that another bond becomes the cheapest to deliver. In Figure 19c it can be seen that even if Bund 3 would decrease with one, then its influence on the futures price is still very small.

If Bund 1 decreases with one, then the futures price decreases with approximately 1.13 at every time point, see Figure 19a. This follows from the fact that Bund 1 is always cheaper than



Figure 19:
a. The influence of Bund 1 on the futures price
b. The influence of Bund 2 on the futures price
c. The influence of Bund 3 on the futures price

the other bonds, so the futures price depends uniquely on this bond. This change in futures price is explainable, because when dividing the change in bond price by the conversion factor, we find:

$$\frac{1}{CF_1} = \frac{1}{0.885104} \approx 1.13.$$

If Bund 1 increases by one, we see that its influence of the futures price is smaller. This follows because Bund 1 is no longer the cheapest at all time points. For example, on November 3, 2008,

$$\frac{\text{price of Bund 1} + 1}{CF_1} = \frac{104.02}{0.885104} \approx 117.53,$$

and

$$\frac{\text{price of Bund 2}}{CF_2} = \frac{101.09}{0.863086} \approx 117.13,$$

so Bund 2 becomes the cheapest when Bund 1 increases by one.

The same holds when we decrease the price of Bund 2 by one, while keeping the other prices fixed, then Bund 2 also becomes cheaper than Bund 1 at some time points. Figure 20 is a scatter diagram with on the horizontal axis

$$\frac{\text{price of Bund 1}}{CF_1} - \frac{\text{price of Bund 2} - 1}{CF_2}, \tag{54}$$

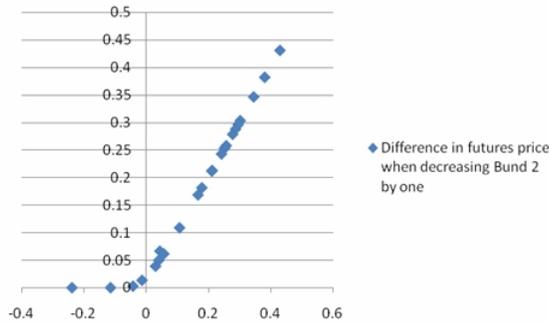


Figure 20: On the horizontal axis: difference as described in Equation (54) and on the vertical axis the difference in futures price when decreasing Bund 2 by one

and on the vertical axis the difference in futures price when decreasing Bund 2 by one. It can be seen on the left side of the graph that when the term in Equation (54) is negative, Bund 1 is the cheapest and Bund 2 barely effects the futures price. Otherwise, when it is positive, Bund 2 is the cheapest and has a big effect on the futures price.

4.4.2 Influence of the volatilities on the futures price

In a similar way, we look at the difference in futures price when one of the volatilities is changed. We want to know vega, which is the ratio:

$$\nu_i = \frac{\text{original futures price} - \text{futures price after changing the volatility}}{\text{original volatility} - \text{volatility after changing}} = \frac{\Delta F_i}{\Delta \text{volatility } i'}$$

for $i = 1$ or 2 . The value of ν_i that we obtain in this manner, is not very representative. It shows the difference in futures price, when the volatility is changed by one, but since the volatilities are around 0.01 or even smaller, the ratio ν_i does not make sense. Therefore we look at the one percent futures price change, so we divide look at vega divided by hundred.

We fix the bond prices and the volatility σ_2 , and change σ_1 with approximately one percent, or 0.0001. We construct a new short rate tree and find different futures prices. The influence of σ_1 on the futures price, can be found in Figure 21a. From November 20, 2008, on, it is not possible to determine the volatility σ_1 , so we use the volatility of November 18, 2008 for these days. We can deduce from the results that when everything is kept fixed, except for the volatility σ_1 , that has been increased, then the bond prices increase, and hence the futures price increases.

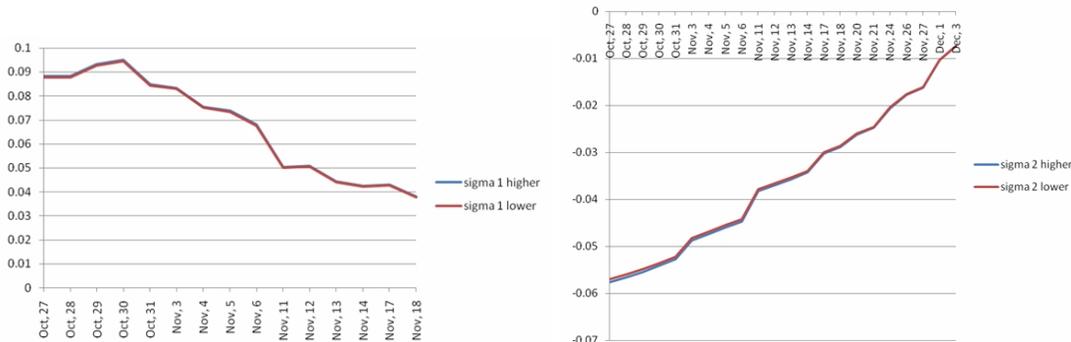


Figure 21:
a. The influence of σ_1 on the futures price
b. The influence of σ_2 on the futures price

We change volatility σ_2 with approximately one percent, or 0.00007, while fixing the bond prices and volatility σ_1 . We calculate the new bond and futures prices. The influence of σ_2 on the futures price, can be found in Figure 21b. In Equation (50), we can see that when everything is kept fixed, except for the volatility σ_2 , that has been increased, the bond prices decrease, and hence the futures prices decrease. Vice versa, when σ_2 decreases, the bond and futures prices increase.

One can see in the graphs of both volatilities that the lines of an up- or downwards change in the volatilities are very close to each other, which means that for small changes, the difference in futures price is linear with respect to the difference in volatilities.

To get a better idea of how much influence the volatilities have, we take a closer look at the volatility σ_1 .

In Figure 21 it can be seen that the largest difference in futures price is 0.095, which means that when changing σ_1 by 0.01, the futures price changes by 0.095. This is very little compared to the effect of Bund 1 on the futures price. For example, assume we want to decrease the futures price with 1.13 by changing the volatility σ_1 , just like we did when decreasing the price of Bund 1. Volatility σ_1 needs to decrease with $\frac{1.13}{0.095} \cdot 0.01 = 0.119$, which is ten times the value of the original volatility of 0.011 and which we cannot even subtract from the volatility, because it cannot be negative. Hence, the influence of volatility σ_1 is very small and the same holds for volatility σ_2 , especially when we compare the influences of the volatilities to those of the bond prices.

We would like to see how well the model fits the futures prices for future time points, when fixing the volatilities on November 18, 2008. We look at the futures price on later time points, between November 21, 2008 and December 5, 2008, and want to see how close they are to the market values. This is a check to see how well we can predict the futures price, at a certain time point.

We have just seen that the futures price does not depend much on the volatilities, but is above all determined by the zero curve, which is calculated at every time point again. Therefore, we find that the futures price is fitted well to the model, when we use the volatilities of November 18, 2008.

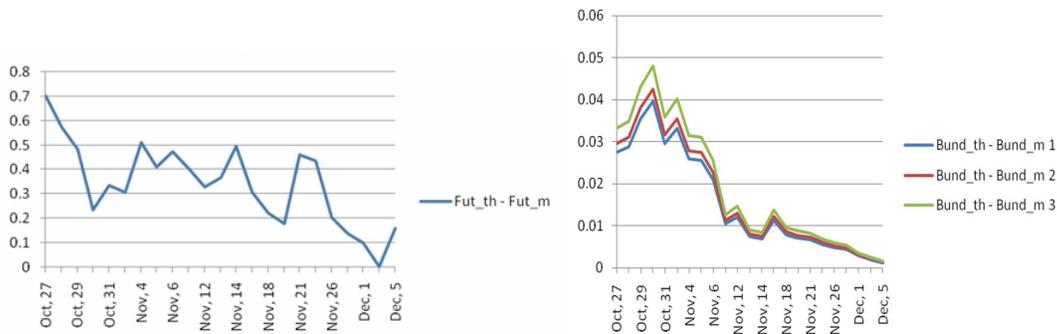


Figure 22:

- The difference between the Ho-Lee futures price and the market futures price, where until November 18, 2008 the volatilities are fitted at every date and from November 20, 2008, the volatilities of November 18, 2008 are taken
- The difference between the Ho-Lee prices and the market prices of Bund 1, 2, and 3, when σ_1 and σ_2 are fixed on November 18, 2008

When we look at the bond prices between November 20, 2008 and December 5, 2008, when fixing the volatilities, on November 18, 2008, we find that the theoretical bond prices get closer to the market bond prices as the time evolves. They are very well fit, which is for the same reason as we just explained, because the bond prices are determined mostly by the zero curve.

5 Conclusion

In this thesis we have studied the questions that were stated in the abstract. To answer these questions, we first examined a range of different short rate models in Chapter 2 and decided that the Ho-Lee model was the most appropriate model for our analysis. The reasons for this choice are that the Ho-Lee model is an exogenous term structure model, that can compute the bond prices analytically, which is done in Section 2.2, and that it is very suitable for building recombining trees, which we explained in detail in Section 2.2.2. The discrete and continuous time models were compared in the succeeding section and a numerical approximation of this comparison was given. In the same chapter a series of interpolation methods was listed with their properties and it was pointed out why we decided to use the raw interpolation method in our model. The chapter was concluded with an example of how to bootstrap and interpolate with real market data to show that the bootstrapping and interpolation method go hand in hand.

In Chapter 3 we took a closer look at the Euro-Bunds and their futures. It was described how the bond and futures prices are calculated and how we can determine the cheapest bond at delivery from the bond prices and their conversion factors. We gave an overview of all the elements that were necessary to calculate the bond prices at delivery and made a Ho-Lee short rate tree from ‘today’ until delivery of the future, on which we could price the bonds, futures, options and caps that we needed to find the volatilities and the bond and futures prices. We took two different volatilities in the model, one for ‘today’ until delivery of the future, and one from delivery until maturity of the bonds. We expected it to give a better fit than when using one single volatility σ_1 in the model, because this volatility could only be calculated for a very short time period.

In Chapter 4 real market data was used to fit the model and we were able to find the answers to the questions of the abstract.

How many steps are needed in the binomial tree to get good results?

We investigated how many steps were needed to get nicely converged bond and futures prices and discovered that from 300 steps on, the bond and futures prices and the volatilities σ_1 and σ_2 did not improve significantly anymore, hence we used a 300 steps-binomial tree in the rest of our calculations.

Is it possible to predict beforehand which bond will be the CTD?

Since the term structure is stochastic, we cannot give an exact prediction of which bond will be the CTD on December 8, 2008, but what we can do, is make a tree of the probabilities that Bund 1, 2 or 3 will become the CTD. In Figure 18b these probabilities were demonstrated of October 27, 2008. It gave an impression of the probabilities that Bund 1, resp. Bund 3 would be the Cheapest-to-Deliver on December 8, 2008. The probability that Bund 2 would become cheapest was zero. In the tree one can see that it is very likely that Bund 1 would become the cheapest at delivery, because only when the short rate would increase enormously, the CTD would change from Bund 1 to Bund 3.

At what difference in the term structure is there a change in which bond is the cheapest?

In Figures 18a and b it could be seen very clearly that in most cases Bund 1 was the cheapest. Together with Table 10 we can conclude that Bund 3 only becomes the cheapest, when the short rate is very high, between 4.37 and 5.14. For more on this topic, see the next question.

How sensitive is the futures price for changes in the zero curve?

The futures price is very sensitive to changes in the term structure. When Bund 1, that appeared

to be the cheapest at delivery, is decreased by one, the futures price decreases by approximately 1.13 at all time points. When Bund 1 is increased by one, the change in futures price was between 0.8 and 1.13, because at most dates, Bund 1 remained the CTD. Bund 2 also effects the futures price, but much less than Bund 1. If it decreases by one, the change in futures price is at most 0.43. Bund 3 barely effects the futures price, because the change in futures price is at most 0.058.

Note that the bond prices are usually highly correlated, so when looking at a change of one in the price of a bond, it is very unlikely that the other bonds stay fixed. Since Bund 1 has the most influence on the futures price, even with it is increased by one, we can conclude that there is only a very small probability that any of the other bonds would become the CTD. For example, when we mentioned that Bund 2 could become cheaper to deliver than Bund 1, if its price would decrease by one, this was based on the fact that the other bond prices were kept fixed. However, Bund 1 and Bund 3 will also decrease if Bund 2 decreases, so there is only a very small probability that Bund 2 would indeed become the cheapest.

How stable are the volatilities of the model and how sensitive is the futures price for changes in these parameters?

The volatilities that we fitted, appeared to be quite stable, because the difference between their smallest and biggest values was only about four percent of the total value. They did not influence the futures price much, because when changing σ_1 by 0.01, the change in futures price was between 0.01 and 0.095. Since volatilities cannot be negative and the value of the volatility σ_1 itself is only around 0.011, this range actually gives the maximal decrease that is possible in the futures price. Of course increasing the volatility can be done endlessly to increase the futures price, but to perceive a significant effect, the volatility should attain unreasonable high values. Changing σ_2 with 0.01, led to a futures price change between -0.0575 and -0.0073 , which is even less than for volatility σ_1 . Indeed, the maximal futures change that can be reached when decreasing this volatility does not attain these values, because the value of the volatility is around 0.007, so it cannot decrease by 0.01. We can conclude that the volatilities both have a very small effect on the futures price.

Because of this, we have taken a look at how far in the future we can still use the volatilities of a certain date. We have looked at the bond and futures prices between November 20, 2008 and December 5, 2008, when we use the volatilities that were fitted on November 18, 2008. Since the futures price is mainly determined by the bond prices and much less by the volatilities, we already expected the futures price to be priced well. In Section 4.4 this was confirmed. Since the model uses the bond prices as an input of the model, the bond prices were also fitted well at the future time points, when using these fixed volatilities.

Is the Ho-Lee model a good model to price bonds and futures, i.e. how well does the model fit their prices?

After the analysis in Chapter 4 we are able to state that the Ho-Lee model is a very satisfactory model to price bonds and futures. We fitted the volatilities, and found a futures price that was very close to the market price. In the worst case, which was on October 27, 2008, there was a difference of 0.7 between the market and the theoretical price, which was only 0.6 percent of the market price. At the later time points this difference was a lot smaller.

A reason for the (small) difference between the theoretical and the market futures prices, could be that the market uses a slightly different term structure of interest rates than we have used to price the future. For the first two years, we applied the Eonia swap rates, which are interbank interest rates, and since they have a very low credit risk, just like the government bonds, we decided that they would be a good replacement for the short term government rates. It could be that the gap between the market and theoretical futures price is caused by this difference between the two rates.

We checked what the differences between the market and theoretical bond and futures prices were, when using one volatility for the whole model, $\sigma_1 = \sigma_2$. We calculated this volatility by fitting the market and theoretical OGBL prices, just like we did for σ_1 and found a better fit for the futures and bond prices. As the steps in the tree increased, the bond price differences became much smaller than in the case we used two different volatilities. This means that we were wrong in our assumption that it would be better to use a different volatility for the long term instead of using the same volatility in the whole model. The short term volatility can indeed be used for the long term.

To conclude this thesis, here are some ideas for future investigation:

- taking one volatility for the whole model, as we just mentioned. This gave an improved fit for the futures and bond prices.
- looking at a time-varying volatility, that is fitted at many time points. We have looked at a volatility with two different values, but maybe when increasing this number, it gives a better fit.
- finding a better way to fit the long term volatility, until maturity of the bonds. This could improve the futures and bond prices. We have used the caps to do so, but as we mentioned earlier, they depend on interbank interest rates and the bonds and its future depend on government interest rates.
- taking the forward volatility from t_d to the maturity of the bonds, when optimizing volatility σ_2 . We have looked at the volatility from ‘today’ until the largest maturity of the three bonds, but it is more precise to take the volatility from time t_d on, and take some ‘average forward volatility’ for the three maturities, eventually by calculating the spot and forward volatilities between the maturities and computing the intermediate volatility.
- using an alternative source for short term rates for the first two years, such as repo rates. These are rates at which one prime bank offers funds in euro to another prime bank if in exchange the former receives from the latter Eurepo as collateral, see www.eurepo.org.

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6 Appendix

6.1 Derivation of the Vasicek model

$$dr(t) = a(\theta - r(t))dt + \sigma dW_t,$$

where θ , a and σ are positive constants. Applying Itô's formula with $Y(u) = \gamma(u, r(u)) = r(u)e^{au}$ gives:

$$\begin{aligned} dY(u) &= ar(u)e^{au} du + e^{au} dr(u) \\ &= ar(u)e^{au} du + e^{au} ((\theta - ar(u))du + \sigma dW_u) \\ &= a\theta e^{au} du + \sigma e^{au} dW_u \end{aligned}$$

Integrating this equation for $t \leq u$, leads to:

$$\begin{aligned} r(u)e^{au} &= r(t)e^{at} + \int_t^u a\theta e^{as} ds + \int_t^u \sigma e^{as} dW_s \\ &= r(t)e^{at} + \theta(e^{au} - e^{at}) + \sigma \int_t^u e^{as} dW_s \\ r(u) &= r(t)e^{-a(u-t)} + \theta(1 - e^{-a(u-t)}) + \sigma \int_t^u e^{-a(u-s)} dW_s, \end{aligned}$$

so $r(u)$ conditional on \mathcal{F}_t is normally distributed with mean respectively variance:

$$\begin{aligned} \mathbb{E}(r(u)|\mathcal{F}_t) &= r(t)e^{-a(u-t)} + \theta(1 - e^{-a(u-t)}), \\ \text{Var}(r(u)|\mathcal{F}_t) &= \mathbb{E} \left(\sigma \int_t^u e^{-a(u-s)} dW_s | \mathcal{F}_t \right)^2 \\ &\stackrel{(17)}{=} \sigma^2 \mathbb{E} \left(\int_t^u e^{-2a(u-s)} ds | \mathcal{F}_t \right) \\ &= \frac{\sigma^2}{2a} (1 - e^{-2a(u-t)}). \end{aligned}$$

6.2 Matlab codes

In this section, the following Matlab codes can be found:

1. bootstrapping per time, this file is the main file, which bootstraps the zero curve, adds the Eonia rates to the zero curve, builds the Ho-Lee short rate lattice and calculates the futures prices, bond prices, σ_1 and σ_2 , by making use of the following other files, see page 64.
2. vol1c calculates the σ_1 by comparing the market option price by the option price that follows from the Ho-Lee tree, see page 66.
3. vol2a and vol2b calculate the σ_2 by comparing the market cap price by the Ho-Lee cap price, see page 68.
4. InterpolatedSpot1 and InterpolatedSpot2 calculate the zero rates in the intermediate time steps, see page 70.
5. fita1a, fita1b, fita2a, fita2b calculate the Ho-Lee short rate trees, see page 71.

```

function [sigma1,fval1,sigma2,fval2] = bootstrapping_per_time(k)

M = 300;
[num1,txt1] = xlsread('BondData_Compact.xls','BondInformation');
[num2,txt2] = xlsread('BondData_Compact.xls','Aligned_HistoricalPrices');
[num3,txt3] = xlsread('BondData_Compact.xls','Aligned Eonia Swap');
[num4,txt4] = xlsread('BondData_Compact.xls','FGBL');

Start = datenum(txt4(k+1,1));
StartNum = floor(Start);
StartDate = datestr(StartNum);
BundDate = datenum(txt2(2:size(txt2,1),1));
for i=1:length(BundDate)
if Start < BundDate(i) && Start > BundDate(i+1)
    Bund_m = num2(i+1,8:10)
    end
end
Fut_m = num4(k,1)

EoniaNum = datenum(txt3(3:size(txt3,1),1));
for i = 1:length(EoniaNum)
if EoniaNum(i) == StartNum
    ZR_Eonia = num3(i+1,+)/100;
    end
end
t_Eonia = num3(1,+)/365;
ZR_Eonia;

BootDate = datenum(txt2(2:size(txt2,1),1));
for i = 1:length(BootDate)
    if Start < BootDate(i) && Start > BootDate(i+1)
        BPclean = num2(i+1,);
    elseif k==1
        BPclean = num2(1,);
    end
end

MaturityDates(1:(size(txt1,1)-1)) = txt1(2:size(txt1,1),11);
for i = 1:length(MaturityDates)
    Maturity(i) = (datenum(cell2mat(MaturityDates(i)))-datenum(StartDate))/365;
end
FirstCouponDate = txt1(2:size(txt1,1),10);
for j = 1:length(FirstCouponDate)
    FirstCoupon(j) = (datenum(cell2mat(FirstCouponDate(j)))-datenum(StartDate))/365;
end

c = num1(:,1);
freq = [2,3,4,5,7,8,9,9,10,10,16];
C = zeros(length(freq),freq(end));
for i = 1:length(freq)

```

```

        C(i,1:(freq(i)-1))=c(i);
        C(i,freq(i))=100+c(i);
    end
    CouponDates = zeros(length(freq),freq(end));
    for i=1:length(freq) % All coupons payment dates
        CouponDates(i,1:freq(i)) = FirstCoupon(i)+(0:(freq(i)-1));
    end
    BP=BPclean;

    t=[t_Eonia,Maturity];
    length_Eonia=length(t_Eonia);
    ZR=ZR_Eonia;

    for i=(length_Eonia+1):length(t)
        dd=zeros(1,freq(i-length_Eonia));
        g=@(x) bootstrapping2(i,freq,dd,BP,C,t,length_Eonia,CouponDates,[ZR(1:(i-1)) x]);
        ZR(i)=fzero(g,ZR(i-1));
    end
    logP=-t.*ZR;

    [num8,txt8]=xlsread('BondData_Compact.xls','Options');
    OptDate = datenum(txt8(2:size(txt8,1),1),1);
    for i=1:length(OptDate)
        if Start == OptDate(i)
            Call_m = num8(i,3);
        elseif Start < OptDate(i) && Start > OptDate(i+1)
            Call_m = num8(i+1,3);
        end
    end

    tto=(datenum('21-Nov-2008')-datenum(Start))/365;
    K=119;

    [CapValue_m,CapletValue_m,sigma2,fval2] = vol2a(Start,t,ZR,logP,M);
    l = @(xxx) vol1c(Start,t,ZR,logP,M,Fut_m,Call_m,tto,K,sigma2,xxx);
    [sigma1,fval1] = fzero(l,[0,0.05],optimset('Display','iter'));

    *****

    function f = bootstrapping2(i,freq,dd,BP,C,t,length_Eonia,CouponDates,ZZ)

    for k=i-length_Eonia
        for y=1:freq(k)
            dd(y) = interp1(t(1:i),-( t(1:i) ).* ZZ(1:i),CouponDates(k,y));
        end
    end

    f = BP(k) - sum( C(k,1:freq(k)).*exp(dd(1:freq(k))) );

```

```

function f = volic(Start,t,ZR,logP,M,Fut_m,Call_m,tto,K,sigma2,xxx)

StartDate = datestr(floor(Start));
ttd=(datenum('8-Dec-2008')-datenum(StartDate))/365;

if ttd<=7/365
t= [0,t];
    ZR = [ZR(1),ZR];
    logP = -t.*ZR;
    P_ttd = exp(interp1(t,logP,ttd));
    forw_ttd = (logP(1)-logP(2))/(t(2)-t(1));
    dt1 = ttd/M;
    NewSpot1(1:M) = ZR(1);
    [d1,SR1] = fit_a_1a(dt1,NewSpot1,M,xxx);
end

for i=1:length(t)-1
    if ttd>t(i) && ttd<=t(i+1)
        P_ttd = exp(interp1(t,logP,ttd));
        forw_ttd = (logP(i)-logP(i+1))/(t(i+1)-t(i));
        Spot1 = ZR(1:i+1);
        tt = t(1:i+1);
        [dt1,NewSpot1] = InterpolatedSpot1(ttd,tt,Spot1,M);
        [d1,SR1] = fit_a_1a(dt1,NewSpot1,M,xxx);
    end
end
SRlastcol = SR1(:,M);

[num5,txt5]=xlsread('BondData_Compact.xls','BundInformation');

Bund_FirstCouponDate = txt5(2:4,10);
Bund_FirstCoupon = (datenum(cell2mat(Bund_FirstCouponDate))-datenum(StartDate))/365;
Bund_c = num5(:,1);
Bund_freq = [9,10,10];
Bund_C = zeros(3,10);
P_total = zeros(M,M,3);
for bb=1:3
    Bund_C(bb,1:(Bund_freq(bb)-1)) = Bund_c(bb);
    Bund_C(bb,Bund_freq(bb)) = 100 + Bund_c(bb);
    Bund_CouponDates(bb,1:Bund_freq(bb))=Bund_FirstCoupon(bb)+(0:(Bund_freq(bb)-1));
    P_coupon(bb,1:Bund_freq(bb))=exp(interp1(t,logP,Bund_CouponDates(bb,1:Bund_freq(bb))));
    P_ttd_coupon = zeros(M,Bund_freq(bb),3);
    for s=1:M
        for j=1:Bund_freq(bb)
            P_ttd_coupon(s,j,bb)=P_coupon(bb,j)./P_ttd...
                .*exp(-(Bund_CouponDates(bb,j)-ttd)...
                    *(SR1(s,M)-forw_ttd)-1/2*(sigma2)^2*ttd...
                    *(Bund_CouponDates(bb,j)-ttd)^2);
        end
        P_total(s,M,bb)=sum(P_ttd_coupon(s,1:Bund_freq(bb),bb).*Bund_C(bb,1:Bund_freq(bb)));
    end
end

```

```

end

CF = [0.885104,0.863086,0.874950];
Fut = zeros(M,M);
CTD = zeros(M,1);
for s=1:M
    for i=1:3
        Fut_total(s,i) = (P_total(s,M,i)./CF(i));
    end
    if Fut_total(s,1) < Fut_total(s,2) && Fut_total(s,1) < Fut_total(s,3)
        CTD(s) = 1;
        Fut(s,M) = Fut_total(s,1);
    elseif Fut_total(s,2) < Fut_total(s,1) && Fut_total(s,2) < Fut_total(s,3)
        CTD(s) = 2;
        Fut(s,M) = Fut_total(s,2);
    elseif Fut_total(s,3) < Fut_total(s,1) && Fut_total(s,3) < Fut_total(s,2)
        CTD(s) = 3;
        Fut(s,M) = Fut_total(s,3);
    end
end
for j=M-1:-1:1
    for k=j:-1:1
        Fut(k,j) = 1/2 * (Fut(k,j+1) + Fut(k+1,j+1));
        for i=1:3
            P_total(k,j,i) = 1/2 * d1(k,j)*(P_total(k,j+1,i) + P_total(k+1,j+1,i));
        end
    end
end

Fut_th = Fut(1,1)
P_th = P_total(1,1,1:3)

ii = round(tto/dt1);
for kk = 1:ii
    if Fut(kk,ii) - K > 0
        Call(kk,ii) = Fut(kk,ii) - K;
    else
        Call(kk,ii) = 0;
    end
end
for j=ii-1:-1:1
    for k=j:-1:1
        Call(k,j) = 1/2 * (Call(k,j+1)+Call(k+1,j+1));
    end
end
Call_th = Call(1,1)

f = Call_th - Call_m

```

```

function [CapValue_m,sigma2,fval2] = vol2a(Start,t,ZR,logP,M)

[num7,txt7]=xlsread('CapData2.xls','Aligned vol-strike');

CapDate = datenum(txt7(2:size(txt7,1),1),1);
CapVolList = num7(:,2)/100;          % cont. comp. act/365
CapStrikeList = num7(:,4)/100;      % cont. comp. act/365

for i=1:length(CapDate)
    if Start < CapDate(i) && Start > CapDate(i+1)
        CapStrike = CapStrikeList(i+1);
        CapVol = CapVolList(i+1);
    else
        CapStrike = CapStrikeList(end);
        CapVol = CapVolList(end);
    end
end
NumCaplets = 19;
CapletResets = 1/2 * (1:NumCaplets);
CapletPayments = 1/2 + CapletResets;
Totaltimes = [1/2,CapletPayments];
logP_CapletPayments=interp1(t,logP,Totaltimes);
P_CapletPayments=exp(logP_CapletPayments)
for i = 1:NumCaplets
    F_caplet(i)=(logP_CapletPayments(i)-logP_CapletPayments(i+1))/(1/2)
    d1(i) = (log(F_caplet(i)/CapStrike)+(1/2)*(CapVol^2)*CapletResets(i)) ...
            ./ (CapVol*sqrt(CapletResets(i)))
    d2(i) = d1(i) - CapVol * sqrt(CapletResets(i))
    CapletValue_m(i) = 1/2 * P_CapletPayments(i+1) .* (F_caplet(i)*normcdf(d1(i)) ...
    - CapStrike .* normcdf(d2(i)))
end
CapValue_m = sum(CapletValue_m);

h = @(yy) vol2b(t,ZR,M,CapStrike,NumCaplets,Totaltimes,CapletPayments,logP,...
logP_CapletPayments,P_CapletPayments,CapValue_m,yy);
[sigma2,fval2] = fzero(h,[0,0.05],optimset('Display','iter'));

*****

function f = vol2b(t,ZR,M,CapStrike,NumCaplets,Totaltimes,CapletPayments,logP,...
logP_CapletPayments,P_CapletPayments,CapValue_m,yy)

[dt2,NewSpot2] = InterpolatedSpot2(t,ZR,M,CapletPayments);
[d2,SR2] = fit_a_2a(dt2,NewSpot2,M,yy);
SRlastcol = SR2(:,M);

eps = dt2/10;
logP_eps=interp1(t,logP,Totaltimes+eps);

for cc=1:NumCaplets
    i = round((cc*(1/2))/dt2);          % i = reset_step

```

```

reset_time = cc/2;
F_instan(cc)=(logP_CapletPayments(cc)-logP_eps(cc))/eps;
CV = zeros(i,i);
for j=1:i
    P_reset_payoff(j) = (P_CapletPayments(cc+1)/P_CapletPayments(cc))...
        *exp(-(1/2)*(SR2(j,i)-F_instan(cc))-(1/8)*(yy^2)*reset_time);
    forw_reset_payoff(j) = -2*log(P_reset_payoff(j));
    if forw_reset_payoff(j) - CapStrike > 0
        CV(j,i) = (forw_reset_payoff(j) - CapStrike)* P_reset_payoff(j);
    else
        CV(j,i) = 0;
    end
end
for j=(i-1):-1:1
    for k=j:-1:1
        CV(k,j) = 1/2 * d2(k,j) * (CV(k,j+1)+CV(k+1,j+1));
    end
end
CV;
CapletValue_hl(cc) = 1/2 * CV(1,1);
end

CapValue_hl = sum(CapletValue_hl);
f = CapValue_hl - CapValue_m

```

```

function [dt1,NewSpot1] = InterpolatedSpot1(ttd,tt,Spot1,M)

D = exp(-Spot1.*(tt));
logD_ttd = interp1(tt,log(D),ttd);
Spot_ttd = -(1/ttd)*(logD_ttd);

tttt = [tt(1:length(tt)-1),ttd];
Spot1 = [Spot1(1:length(tt)-1),Spot_ttd];

T=ttd;
dt1=T/M;

% The zero rates are known from 7 days on, we extrapolate for the rates
% < 7 days and interpolate in the usual way from > 7 days

D=exp(-Spot1.*tttt);
NewD=zeros(M,1);
for i=1:M
    if i*dt1 < tttt(1)
        NewD(i)=D(1)^(i*dt1/tttt(1));
    elseif i*dt1 >= tttt(1)
        NewD(i) = interp1(tttt,log(D),i*dt1);
        NewD(i) = exp(NewD(i));
    end
end

for i=1:M;
    NewSpot1(i)=-1/(i*dt1)*log(NewD(i));
end

*****

function [dt,NewSpot] = InterpolatedSpot2(t,Spot,M,CapletPayments)

dt=CapletPayments(end)/M;
D = exp(-Spot.*(t));
for i=1:M;
    if i*dt < t(1)
        NewSpot(i)=Spot(1);
    elseif i*dt >= t(1)
        NewD(i) = interp1(t,log(D),i*dt);
        NewSpot(i)=-1/(i*dt)*(NewD(i));
    end
end
end

```

```

function [d1,SR1] = fit_a_1a(dt1,NewSpot1,M,sigma1)

global EP1 d1

a1=NewSpot1(1);
b=zeros(2,1);
b=b+2*sigma1*sqrt(dt1);
[indxa,indxb]=meshgrid(1:2,1:2);
d1=exp(-dt1*(a1+b(indxb).*(indxb-1)));

EP1=zeros(M+1,M+1);
EP1(1,1)=1;
EP1(1,2)=1/2*d1(1,1)*EP1(1,1);
EP1(2,2)=1/2*d1(1,1)*EP1(1,1);

% M is the length of the NewSpot=the interpolated spot rates
for ii=2:M
    N=length(a1)+1; % The vector of parameters a is known up to time N-1,
                    % so it has length N-1.
    g=@(x) fit_a_1b(dt1,NewSpot1,[a1(1:(N-1)) x],sigma1);
    % we put a(N)=x and then let f=0.
    a1(ii) = fzero(g,a1(N-1));
end

SR1=(-1/dt1).*log(d1);

*****

function f = fit_a_1b(dt1,NewSpot1,aa,sigma1)

global EP1 d1

M=length(NewSpot1);
N=length(aa);

b=2*sigma1*sqrt(dt1);

[indxa,indxb]=meshgrid(1:N,1:N);
d1=exp(-dt1*(aa(indxa)+b*(indxb-1)));

EP1(1,N+1)=1/2*d1(1,N)*EP1(1,N);

EP1(N+1,N+1)=1/2*d1(N,N)*EP1(N,N);

for i=2:N;
EP1(i,N+1)=1/2*(d1(i,N)*EP1(i,N)+d1(i-1,N)*EP1(i-1,N));
end

zcbvolgensSpot=exp(-NewSpot1(N).*N*dt1); % zcb=zero-coupon-bond
zcbvolgensHoLee=sum(EP1(:,N+1));
f=zcbvolgensSpot-zcbvolgensHoLee;

```

```

*****

function [d2,SR2] = fit_a_2a(dt,NewSpot,M,yy)

global EP2 d2

a2=NewSpot(1);

b = zeros(2,1);
b = b+2*yy*sqrt(dt);
[indxa,indxb] = meshgrid(1:2,1:2);
d2 = exp(-dt*(a2+b(indxb).*(indxb-1)));

EP2 = zeros(M+1,M+1);
EP2(1,1) = 1;
EP2(1,2) = 1/2 * d2(1,1) * EP2(1,1);
EP2(2,2) = 1/2 * d2(1,1) * EP2(1,1);

% M is the length of the NewSpot=the interpolated spot rates
for ii = 2:M
    N = length(a2)+1; % The vector of parameters a is known up to time N-1,
                    % so it has length N-1.
    g = @(x) fit_a_2b(dt,NewSpot,[a2(1:(N-1)) x],yy);
    % we put a(N)=x and then let f=0.
    a2(ii) = fzero(g,a2(N-1));
end

SR2=(-1/dt)*log(d2);

*****

function f = fit_a_2b(dt,NewSpot,aa,yy)

global EP2 d2

M=length(NewSpot);
N=length(aa);

b=2*yy*sqrt(dt);

[indxa,indxb]=meshgrid(1:N,1:N);
d2=exp(-dt*(aa(indxa)+b*(indxb-1)));

EP2(1,N+1)=1/2*d2(1,N)*EP2(1,N);

EP2(N+1,N+1)=1/2*d2(N,N)*EP2(N,N);

for i=2:N;
EP2(i,N+1)=1/2*(d2(i,N)*EP2(i,N)+d2(i-1,N)*EP2(i-1,N));
end
end

```

```
zcbvolgensSpot=exp(-NewSpot(N).*N*dt); % zcb=zero-coupon-bond
zcbvolgensHoLee=sum(EP2(:,N+1));
f=zcbvolgensSpot-zcbvolgensHoLee;
```