Solitons

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Abstract

The history of the soliton is discussed and two methods of obtaining soliton solutions are presented. The bilinear method is applied to the Boussinesq, shallow water wave and nonlinear Schrödinger equation. A number of exact soliton and breather solutions are obtained and the dependence of the solutions on the parameters is investigated. Some rational solutions to these equations are obtained and it is shown that the breather solutions to the nonlinear Schrödinger equation can be written as an imbricate series of rational growing-and-decaying mode solutions. Exact 1- and 2-soliton solutions to the nonlinear Schrödinger, modified Korteweg-de Vries and Sine-Gordon equations are obtained from the inverse scattering transform. Besides the standard initial conditions also the use of other initial conditions for inverse scattering is investigated. In the appendix a discussion of the extended homoclinic test technique is given.

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Contents

Part I

Introduction

Chapter 1

General Introduction

In this thesis I present the results of my research into solitons and their behavior. I performed most of this research whilst staying at the University of Kent in Canterbury, under supervision of Professor Peter Clarkson. I will describe two separate methods of constricting solutions to solitons equations, i.e. the Hirota bilinear method and the inverse scattering transform. Using these I will construct multi solitons solutions as well as breather and rational solutions to a number of nonlinear partial differential equations. These equations are the Boussinesq equation, the shallow water wave equation and the nonlinear Schrödinger equation for the bilinear method and the focusing nonlinear Schrödinger equation, the modified Korteweg-de Vries equation and the Sine-Gordon equation for the inverse scattering transform. The outline of this thesis is as follows. It consists of three parts of which each consist of a number of chapters. The first part gives an introduction into the subject and consists of 3 chapters. The first one is a general introduction, which you are reading at the moment. The second chapter offers a brief history of the solitary wave and the solitons and will introduce the reader to their definitions. The third chapter will give a short introduction into both methods I will use later on. The second part consists of Chapters 4 up to and including 6 and gives the results obtained from the bilinear method. In Chapter 4 we discuss the Boussinesq equation and obtain soliton and breather solutions to it. The next chapter deals with the shallow water wave equation and besides the soliton and breather solutions we now also obtain some rational solutions. The final chapter of this part deals with the nonlinear Schrödinger equation and now, besides obtaining breather, soliton and rational solutions, we also show that the breather solutions can be written as imbricate series of rational solutions. The third and final part contains the results obtained from the inverse scattering transform and consists of Chapters 7 until 10. In Chapter 7 I derive the formalism needed and in Chapters 8, 9 and 10 I give soliton solutions for the focusing nonlinear Schrödinger equation, the modified Korteweg-de Vries equation and the Sine-Gordon equation respectively. Finally in the Appendix I will discuss 2 recent articles which claim to introduce a new method to obtain new solutions to the Korteweg-de Vries equation and the shallow water wave equation. However I will show that these solutions can also be obtained from the bilinear method in a way that has been known for at least two decades.

In this thesis I hope to offer an introduction into the vast field of soliton equations and to explain the two most used methods for solving them. Hopefully after reading this thesis the reader can understand these methods and appreciate their differences. Furthermore I tried to give an overview of the kind of solutions that exist and to illustrate their behavior. Obviously there are more solutions known then I could list here and probably there are also still more solutions to be found as can be seen from the new solutions obtained in Sections 5.6 and 6.2. As the calculations involved are very long and tedious I used the computer program MAPLE to do most of the calculations and make the plots. However other programs like MATHEMATICA could probably be used equally well.

Chapter 2

History

2.1 The history of the solitary wave

Here I will explain what the soliton is and how it was first discovered. I will start by outlining its discovery and will give the definitions along the way. The first observation of it was made by John Scott Russell in 1834 whilst riding along a canal in Scotland. 10 years later he described his observation as follows [25]:

"I was observing the motion of a boat which was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped - no so the mass of water in the channel which it had put in motion; it accumulates round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without change of form or diminution of speed. I followed it on horseback, and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles I lost it in the windings of the channel. Such, in the month of August 1834, was my first chance interview with that rare and beautiful phenomenon which I have called the **Wave of Translation**..."

Following this observation Russell performed numerous experiments in a laboratory to study this wave more closely. He managed to create solitary waves, which are long, shallow, water waves of permanent form, in a wave tank and this was his most important result. As he observed them, they must exist. Furthermore he found that their propagation speed c is related to the water depth h by the formula

$$c^2 = g(h + \eta), \tag{2.1}$$

where η is the amplitude of the wave and g the gravitational constant. However there was not yet a mathematical description for this phenomenon and work by Airy and Stokes contradicted Russell's observations and his results were questioned. However Boussinesq ([5], [6]) and Rayleigh[24] both independently obtained approximate descriptions of a solitary wave with the profile

$$\eta(x, t) = a\,\mathrm{sech}^2(\kappa(x - ct)), \tag{2.2}$$

with $\kappa^2 = \frac{3a}{4h^2(a+h)}$. Furthermore Boussinesq derived a 1-dimensional nonlinear partial differential equation, which is now known as the Boussinesq equation, and is given by

$$\eta_{tt} = \frac{c^2}{h}\left(h\eta_{xx} + \frac{3}{2}(\eta^2)_{xx} + \frac{h^3}{3}\eta_{xxxx}\right). \tag{2.3}$$

Here the subscript x or t denotes differentiation with respect to the variable x or t. He also showed that it has the solitary wave solution given by

$$\eta(x,t) = a\,\mathrm{sech}^2(\kappa(x \pm \beta t)), \tag{2.4}$$

where $\kappa^2 = \frac{3a}{4h^3)}$ and $\beta = c\sqrt{1+\frac{4}{3}h^2\kappa^2}$. In Section 4 I will study this equation in detail (although I will start by transforming it into a slightly easier form) and derive this solution as well as multiple soliton solutions. The next important development came in 1895 with the celebrated paper by Korteweg and de Vries [19] introducing the equation

$$\eta_\tau = \frac{3}{2}\sqrt{\frac{g}{h}}\left(\frac{\eta^2}{2} + \frac{2\alpha\eta}{3} + \frac{\sigma\eta_{\xi\xi}}{3}\right)_\xi, \tag{2.5}$$

where $\sigma = \frac{h^3}{3} - \frac{Th}{\rho g}$, η is the surface elevation, h is the equilibrium level, α is a small arbitrary constant related to the uniform motion of the liquid, g is the gravitational constant, T is the surface tension and ρ is the density. This equation governs long one dimensional, small amplitude, surface gravity waves and is now known as the Korteweg-de Vries equation (or KdV for short). The equation however had already been given by Boussinesq in [6]. The KdV equation can be transformed into an easier form by introducing

$$t = \frac{1}{2}\sqrt{gh\sigma}\tau, \qquad x = -\sigma^{-1/2}\xi, \qquad u = \frac{1}{2}\eta + \frac{1}{3}\alpha \tag{2.6}$$

and then we get

$$u_t + 6uu_x + u_{xxx} = 0. \tag{2.7}$$

This equation has a solitary wave solution (already found by Korteweg and de Vries) given by

$$u(x,t) = 2\kappa^2\mathrm{sech}^2(\kappa(x - 4\kappa^2 t - \delta_0)), \tag{2.8}$$

where κ and δ_0 are constants. It is now possible and instructive to give the exact definition of a solitary wave before proceeding.

Definition *A solitary wave solution of a PDE*

$$\triangle(x,t,u) = 0, \tag{2.9}$$

where t and x are temporal and spatial variables and u is the dependent variable, is a traveling wave solution of the form

$$u(x,t) = w(z), \tag{2.10}$$

with $z = x - \gamma t$ and γ a constant, whose transition is from one constant asymptotic state as $z \to -\infty$ to another constant asymptotic as $z \to +\infty$.

We see that the solution given in equation (2.8) satisfies this definition with both asymptotic states being equal to 0.

2.2 History of the soliton

After the paper by Korteweg and de Vries nothing happened for a long time in the field, until the second half of the twentieth century. The rekindle of the interest started with a paper by Fermi, Pasta and Ulam from 1955 [9] in which they numerically studied a one-dimensional inharmonic lattice of identical masses connected by nonlinear springs with the force law $F(\triangle) = -K(\triangle + \alpha\triangle^2)$, where $\triangle$ denotes the displacement from the equilibrium position. If we denote the displacement of the nth mass from its equilibrium position by y_n the following equations of motion can be derived

$$m\frac{\mathrm{d}^2 y_n}{\mathrm{d}t^2} = K(y_{n+1} - 2y_n + y_{n-1}) + K\alpha[(y_{n+1} - y_n)^2 - (y_n - y_{n-1})^2], \tag{2.11}$$

for $n = 1, 2, \ldots, N-1$ and the boundary and initial conditions

$$y_0 = y_N = 0, \qquad y_n(0) = \sin(n\pi/N) \qquad \text{and} \qquad \frac{\mathrm{d}y_n}{\mathrm{d}t}(0) = 0. \tag{2.12}$$

Fermi, Pasta and Ulam expected that any smooth initial state would eventually relax into an equilibrium. However to their surprise this did not happen, but the energy recurred. It flowed back and forth amongst some of the lower order modes before returning to the lowest mode within an accuracy of a couple percent. And this process then repeated itself.

The next step came a decade later when Zabusky and Kruskal considered the same model in the continuum limit [31]. They called the length between the springs h and introduced the new coordinates $t' = \sqrt{K/m}t$ and $x' = x/h$ with $x = nh$ and then expanded $y_{n\pm1}$ in a Taylor series. If we drop the primes we see that equation (2.11) reduces to

$$y_{tt} = y_{xx} + \epsilon y_x y_{xx} + \frac{h^2}{12} y_{xxxx} + O(\epsilon h^2, h^4), \tag{2.13}$$

where $\epsilon = 2\alpha h$. Next they sought an asymptotic solution of the form

$$y \sim \phi(\xi, \tau), \qquad \xi = x - t, \qquad \tau = \frac{1}{2}\epsilon t, \tag{2.14}$$

and inserting this in 2.13 they found

$$\phi_{\xi\tau} + \phi_\xi \phi_{\xi\xi} + \delta^2 \phi_{\xi\xi\xi\xi} + O(h^2, h^4\epsilon^{-1}) = 0, \tag{2.15}$$

where they introduced $\delta^2 = \frac{h^2}{12\epsilon}$. Now they assumed that the higher order terms in h are small so they can be neglected and set $u = \phi_\xi$ which then gave the KdV equation:

$$u_\tau + uu_\xi + \delta^2 u_{\xi\xi\xi} = 0. \tag{2.16}$$

They numerically studied this equation, starting with the initial condition $u(\xi, 0) = \cos(\pi\xi)$ with $0 \leq \xi \leq 2$ and $\delta = 0.022$. Because of the small value of δ the third term in equation (2.16) becomes small and can be neglected unless the curve is very steep. Therefore initially the first two terms dominate and a classical overtaking phenomenon occurs, i.e. the curve steepens where it has negative slope. Then however the third term comes into play and prevents the forming of an discontinuity. It causes oscillations with a small wavelength to develop along the entire wavefront. They grow in amplitude and finally obtain a constant size and a form very similar to solitary waves. Then every "solitary wave pulse" or soliton, as Zabusky and Kruskal call them from then on, starts to move with a speed linearly proportional to its amplitude. Therefore we see that the taller ones overtake the smaller ones and when this happens they interact nonlinearly. However they both emerge seemingly unaffected from the interaction. Zabusky and Kruskal noted that the interaction is not irreversible which is very remarkable given that the KdV equation is a nonlinear equation. As said before they named these kind of waves solitons, mainly due to their analogy with particles. Nowadays the official definition of a soliton is the following

Definition *A soliton is a solitary wave which asymptotically preserves its shape and velocity after a nonlinear interaction with other solitary waves or even an arbitrary localized disturbance.*

The observation of Zabusky and Kruskal obviously sparked the interest in the KdV equation and solitons and beckoned an analytical explanation. This came only two years later when Gardner, Green, Kruskal and Miura presented the Inverse Scattering Transform [10].

Chapter 3

Introduction of the methods used

3.1 Inverse Scattering Transform

I will give a short introduction here to the Inverse Scattering Transform (IST) for the KdV equation. I will not go into too many details, but these can be found in almost any textbook on the subject, for instance [18]. In Part III I will present the IST in detail for the nonlinear Schrödinger, modified KdV and Sine-Gordon equation. The genius idea behind the IST is the following: if ϕ satisfies both the time-independent Schrödinger equation given by

$$\phi_{xx} + u\phi = \lambda\phi \tag{3.1}$$

and the time dependence equation

$$\phi_t = (\gamma + u_x)\phi - (4\lambda + 2u)\phi_x \tag{3.2}$$

where, γ is an arbitrary constant. If we then assume that $\lambda_t = 0$, we find that if we require $\phi_{xxt} = \phi_{txx}$, u satisfies the KdV equation (2.7). Therefore we can use equations (3.1) and (3.2) to obtain solutions to the KdV equation. Note that u is often called the potential as it fulfils the role of potential in the time-independent Schrödinger equation. We start by assuming an initial condition $u_0 = u(x, 0)$. The first Part of IST is then to map the initial potential u_0 into the scattering data $S(\lambda, 0)$ using equation (3.1) and is called the direct problem. The second Part is the time dependence Part and here we use equation (3.1) to determine the time dependence of the scattering data $S(\lambda, t)$. The final part, called the inverse problem, consists of retrieving the potential $u(x, t)$ from $S(\lambda, t)$. The name Inverse Scattering Transform is derived from the similarity with the Fourier Transform method for solving ODE's, i.e. the direct problem is similar to the Fourier transform, the time-dependence to the dispersion relation and the inverse problem to the inverse Fourier transform. I will now give a short description of how all three parts of IST work. As said before, I will not go into details, nor will I give any proofs, but these can for instance be found in [18].

3.1.1 Direct problem

Assume that at time $t = 0$ the initial potential u_0 is given and that it decays rapidly enough for large x (for details on what is rapidly enough see [18]). It is then known that equation (3.1) has a finite number of discrete eigenvalues with $\lambda > 0$ and a continuous spectrum for $\lambda < 0$. We can compute the corresponding eigenfunctions and writing $\lambda = \kappa_n^2$ we find for the eigenfunctions corresponding to the discrete spectrum

$$\phi_n(x, 0) \sim c_n(0) \exp(-\kappa_n x) \qquad \text{for} \quad x \to -\infty, \tag{3.3}$$

with

$$\int_{-\infty}^{+\infty} \phi_n^2 \mathrm{d}x = 1. \tag{3.4}$$

And here we have found the first two parts of the scattering data, i.e. κ_n and c_n. The rest can be found from the continuous spectrum. If we write $\lambda = -k^2$ we find that the corresponding eigenfunctions behave as

$$\phi(x,0) \sim \tau(k,0)e^{-ikx} \qquad \text{for} \quad x \to -\infty, \tag{3.5}$$
$$\phi(x,0) \sim a(k,0)e^{-ikx} + b(k,0)e^{ikx} \qquad \text{for} \quad x \to +\infty. \tag{3.6}$$

This defines the other two parts of the scattering date, which we call the reflection coefficient $\tau(k,0) = 1/a(k,0)$ and the transmission coefficient $\rho(k,0) = b(k,0)/a(k,0)$. The scattering data is therefore given by

$$S(\lambda,0) = \left(\kappa_n, c_n(0)_{n=1}^N, \rho(k,0), \tau(k,0)\right). \tag{3.7}$$

3.1.2 Time evolution

From equation (3.2) it is easy to determine the time evolution of the scattering data. It can be shown that

$$\begin{aligned} \kappa_n(t) &= \kappa_n(0) && (3.8)\\ c_n(t) &= c_n(0)\exp(4\kappa_n^3 t) && (3.9)\\ \tau(k,t) &= \tau(k,0) && (3.10)\\ \rho(k,t) &= \rho(k,0)\exp(8ik^3 t). && (3.11) \end{aligned}$$

So from equation (3.7) we can now find the scattering data at time t.

3.1.3 Inverse problem

I will here give the solution to the inverse problem given by Gel'fand and Levitan [11]. Note that the problem also can be solved by considering it as a Riemann-Hilbert boundary-value-problem, but I will not give this view here. I will use this viewpoint however in Part III when I discuss the IST for the nonlinear Schrödinger equation. The first step in the inverse problem is to define the function

$$F(x,t) = \sum_{n=1}^{N} c_n^2(t)e^{-\kappa_n x} + \frac{1}{2\pi}\int_{-\infty}^{+\infty} \rho(k,t)e^{ikx}\mathrm{d}k \tag{3.12}$$

and then to solve the integral equation

$$K(x,y,t) + F(x+y,t) + \int_{x}^{+\infty} K(x,z,t)F(z+y,t)\mathrm{d}z = 0, \tag{3.13}$$

which is known as the Gel'fand-Levitan-Marchenko equation. If we have found $K(x,y,t)$ we can reconstruct the potential from the relation

$$u(x,t) = 2\frac{\partial}{\partial x}K(x,x,t). \tag{3.14}$$

In principle we can now solve the KdV equation with every initial condition u_0 which decays rapidly enough. However the inverse problem is far from trivial to solve and the direct problem can present some problems as well. If we start with an initial condition leading to pure soliton solutions the reflection coefficient $\rho(k,t)$ will vanish and therefore $F(x,t)$ simplifies significantly. In this case it is therefore possible to explicitly solve equation (3.13) and exact solutions for any n-soliton interactions can be found as noted by Gardner, Green, Kruskal and Miura [10]. However for other initial conditions where the reflection coefficient does not vanish this is not as easy and the exact solution can often not be obtained.

3.1.4 Lax's Generalization

Peter Lax put the Inverse Scattering Transform as described by Gardner, Green, Kruskal and Miura in a more general framework. If we define the operators $\mathcal{L}$ and $\mathcal{M}$ by

$$\mathcal{L} = \frac{\partial^2}{\partial x^2} + u(x,t), \tag{3.15}$$

$$\mathcal{M} = u_x + \gamma - (2u + 4\lambda)\frac{\partial}{\partial x}, \tag{3.16}$$

then we see that equations (3.1) and (3.2) are given by

$$\mathcal{L}\phi = \lambda\phi, \tag{3.17}$$

$$\phi_t = \mathcal{M}\phi. \tag{3.18}$$

We call $\mathcal{L}$ and $\mathcal{M}$ the Lax pair of the KdV equation. In general if we have two operators $\mathcal{L}$ and $\mathcal{M}$ satisfying equations (3.17) and (3.18), they are called a Lax pair and then it is an easy calculation to show that

$$\mathcal{L}_t + [\mathcal{L}, \mathcal{M}] = 0 \tag{3.19}$$

if and only if $\lambda_t = 0$ and where $[\mathcal{L}, \mathcal{M}]$ denotes the commutator $[\mathcal{L}, \mathcal{M}] = \mathcal{L}\mathcal{M} - \mathcal{M}\mathcal{L},$. This equation is known as the Lax equation and if the Lax pair is given by equation (3.15) and (3.16) it reduces to the KdV equation.

3.2 Hirota's bilinear method

A couple of years later Hirota developed a different method for obtaining soliton solutions [13]. This method is now known as the bilinear method or the direct method, to signify its difference with inverse scattering. I will introduce this method here to solve the KdV equation, before giving a more general formulation. It can be shown that the N-soliton solution to the KdV equation can be written in the form

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}(\ln[\det(\mathbf{I} + \mathbf{C})]), \tag{3.20}$$

where $\mathbf{I}$ is the $N \times N$ identity matrix and $\mathbf{C}$ is an $N \times N$ matrix. For the 1-soliton solution this is easily seen

$$u(x,t) = 2k^2 \text{sech}^2[k(x - 4k^2 t)] = 2\frac{\partial^2}{\partial x^2}\ln F(x,t), \tag{3.21}$$

where $F(x,t) = 1 + \exp[2k(x - 4k^2 t)]$ and for the 2-soliton solution this is for instance shown in [18]. Motivated by this Hirota proposed the transformation

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln F(x,t). \tag{3.22}$$

To keep the calculation easy I will do this transformation in 2 steps; first we insert $u = v_x$ in equation (2.7) and after integrating with respect to x we obtain the potential KdV equation

$$v_t + 3v_x^2 + v_{xxx} = 0. \tag{3.23}$$

If we now insert $v = 2\frac{\partial}{\partial x}\ln F(x,t) = 2\frac{F_x}{F}$, we find that F will have to satisfy

$$FF_{xt} - F_x F_t + FF_{xxxx} - 4F_x F_{xxx} + 3F_{xx}^2 = 0. \tag{3.24}$$

We see that we now have a second order equation in F, which is therefore called bilinear. Hirota now introduced the operator $D_z(F \bullet G)$ defined by

$$D_z(F \bullet G) = \left(\frac{d}{dz_1} - \frac{d}{dz_2}\right) F(z_1)G(z_2)|_{z_1=z,z_2=z} = F_zG - FG_z. \tag{3.25}$$

Now we can easily derive that for n a non-negative integer

$$D_z^n(F \bullet G) = \left(\frac{d}{dz_1} - \frac{d}{dz_2}\right)^n F(z_1)G(z_2)|_{z_1=z,z_2=z} = \sum_{k=0}^{n}(-1)^k \binom{n}{k} \frac{d^{n-k}F}{dz^{n-k}} \frac{d^kG}{dz^k}. \tag{3.26}$$

The crucial step is the introduction of the bilinear operator

$$D_t^m D_x^n(F \bullet G) = \left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2}\right)^m \left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right)^n F(x_1,t_1)G(x_2,t_2)|_{x_1=x,x_2=x,t_1=t,t_2=t}, \tag{3.27}$$

where m and n are non-negative integers. This operator has a large number of interesting properties of which I will now give only 3, which I will need later on.

(i). $D_t^m D_x^n(F \bullet 1) = \frac{\partial^{m+n}F}{\partial x^n \partial t^m}$,

(ii). $D_x^{2n+1}(F \bullet F) = 0$,

(iii). $D_t^m D_x^n(\exp(\eta_1) \bullet \exp(\eta_2)) = (\omega_2 - \omega_1)^m(\kappa_1 - \kappa_2)^n \exp(\eta_1 + \eta_2)$,

where $\eta_i = \kappa_i x - \omega_i t$, with κ_i and ω_i constants. I will not prove these properties here, but the proof is an easy and straightforward calculation. Now we see that using the bilinear operator we can rewrite equation (3.24) in the form

$$(D_xD_t + D_x^4)(F \bullet F) = 0. \tag{3.28}$$

This we call the bilinear form of the equation. The final step to obtain the soliton solutions is to make a correct ansatz on the form of F. We start by writing

$$F(x,t) = 1 + \sum_{n=1}^{\infty} \epsilon^n F_n(x,t) = 1 + \epsilon F_1(x,t) + \epsilon^2 F_2(x,t) + \ldots, \tag{3.29}$$

where ϵ is a formal expansion parameter which is not necessarily small. However, in order to find the N-soliton solution we will impose $f_n(x,t) = 0$ for $n > N$, so we see that $F(x,t)$ is defined by a finite sum and therefore has no problems with convergence. If we now insert (3.29) into equation (3.28)and split it up according to order in ϵ we get using property (i) derived above

$$\frac{\partial^2 F_1}{\partial x \partial t} + \frac{\partial^4 F_1}{\partial x^4} = 0, \tag{3.30}$$

$$\frac{\partial^2 F_2}{\partial x \partial t} + \frac{\partial^4 F_2}{\partial x^4} = -\frac{1}{2}(D_xD_t + D_x^4)(F_1 \bullet F_1), \tag{3.31}$$

$$\frac{\partial^2 F_3}{\partial x \partial t} + \frac{\partial^4 F_3}{\partial x^4} = -(D_xD_t + D_x^4)(F_1 \bullet F_2), \tag{3.32}$$

$$\frac{\partial^2 F_n}{\partial x \partial t} + \frac{\partial^4 F_n}{\partial x^4} = -\frac{1}{2}(D_xD_t + D_x^4)\left(\sum_{m=1}^{n-1} F_m \bullet F_{n-m}\right). \tag{3.33}$$

So we see that we have obtained a set of recursion equations which will allow us to find all the F_n once we have made a choice for F_1 which satisfies equation (3.30).
To find the 1-soliton solution we choose

$$F_1(x,t) = \exp(\eta_1(x,t)), \tag{3.34}$$

with

$$\eta_1(x,t) = \kappa_1 x - \omega_1 t + \eta_1^0, \tag{3.35}$$

where κ_1, ω_1 and η_1^0 are all constants. If we insert this in equation (3.30) and use property (iii) we see that this yields $\omega_1 = \kappa_1^3$ and therefore the right hand side (RHS) of equation (3.31) vanishes. So we can choose $F_2(x,t) = 0$ and from the other recursion equations we see that we can also fix $F_n(x,t) = 0$ for $n > 1$. So we get

$$F(x,t) = 1 + \exp(\eta_1(x,t)), \tag{3.36}$$

where the constant ϵ is absorbed into η_1^0. After some rewriting we then find that

$$u(x,t) = 2\frac{\partial^2}{\partial x^2} \ln F(x,t) = \frac{\kappa_1^2}{2} \operatorname{sech}^2 \left(\frac{\eta_1(x,t)}{2} \right) \tag{3.37}$$

and it can easily be verified by a computer algebra program as MAPLE that this is indeed a solution to the KdV equation given in equation (2.7).
In a similar way we can find the 2-soliton solution by making the ansatz

$$F_1(x,t) = \exp(\eta_1(x,t)) + \exp(\eta_2(x,t)), \tag{3.38}$$

where η_1 and ω_1 satisfy the same relations as before and η_2 and ω_2 are defined in a similar way. We therefore see that equation (3.30) is still satisfied, but now the RHS of equation (3.31) no longer vanishes. However, if we take

$$F_2(x,t) = A_{12} \exp(\eta_1(x,t) + \eta_2(x,t)), \tag{3.39}$$

where the constant A_{12} is given by

$$A_{12} = \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \right)^2, \tag{3.40}$$

we see that it is satisfied. Furthermore the RHS of equation (3.32) now vanishes so we can take $F_n = 0$ for $n > 2$. So for the 2-soliton solution we get

$$F = 1 + \exp(\eta_1(x,t)) + \exp(\eta_2(x,t)) + A_{12} \exp(\eta_1(x,t) + \eta_2(x,t)) \tag{3.41}$$

and it is an easy calculation to find the solution $u(x,t)$ from this. In principle we can find N-soliton solutions to the KdV equation for any N in this way; however the algebra will get more and more involved for larger N. According to Hietarinta [12] the N-soliton solution $F(x,t)$ is given by

$$F(x,t) = \sum_{\mu_i = 0,1} \exp \left(\sum_{1 \le i < j \le N} \phi(i,j) \mu_i \mu_j + \sum_{j=1}^{N} \mu_j \eta_j \right), \tag{3.42}$$

where η_j is as before, $\exp(\phi(i,j)) = A_{ij}$ and the first summation is over all possible combinations with all the $\mu_i \in \{0,1\}$. It is important to note that this way of writing the solutions was first introduced by Hirota and used for instance in [17].
So we have found the N-soliton solution to the KdV equation; however the same method is also applicable to other nonlinear evolution equations. As I will use this method to study the Boussinesq, shallow water wave and nonlinear Schrödinger equation in Part II, I will now give a more general description of the method for other equations. Assume that we have our equation in the bilinear form, i.e. in the form

$$B(D_x, D_t)(F \bullet F) = 0, \tag{3.43}$$

where F is related to the solution of the nonlinear evolution equation in a given way and $B(D_x, D_t)$ is a polynomial in D_x and D_t. We should note here that the process of bilinearization is far from trivial and that there are also more complicated bilinear forms, but I will not go into that here and in the next sections I will only derive the bilinear forms of equations when needed. For a more general discussion on the bilinearization of equations see [12]. Note that from property (ii), $B(D_x, D_t)$ only contains even terms. Similar to what we did for the KdV equation above we now expand $F(x,t)$ in ϵ

$$F = 1 + \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 + ... \tag{3.44}$$

and if we now insert this in equation (3.43) we see that by splitting it up into orders of ϵ we again get a set of recursion relations similar to the ones found in (3.30)-(3.33), now given by

$$B(\partial_x, \partial_t)F_1 = 0 \tag{3.45}$$

$$B(\partial_x, \partial_t)F_2 = -\frac{1}{2}B(D_x, D_t)(F_1 \bullet F_1) \tag{3.46}$$

$$B(\partial_x, \partial_t)F_3 = -B(D_x, D_t)(F_1 \bullet F_2) \tag{3.47}$$

$$B(\partial_x, \partial_t)F_n = -\frac{1}{2}B(D_x, D_t)\left(\sum_{m=1}^{n-1} F_m \bullet F_{n-m}\right). \tag{3.48}$$

The ansatz to obtain a 1-soliton solution is now the same as before, i.e. $F_1 = \exp(\eta_1)$; however the relation between ω_1 and κ_1 will now be different and depend on the exact form of B. The same thing holds for the 2-soliton solution but if we have found the correct form of A_{12} (which will obviously also depend on the exact form of B), we will see that we then can still find the higher order solutions by equation (3.42). So if we have a nonlinear evolution equation with a bilinear form as in equation (3.43), then we only need to find ω and A_{12} in order to find all N-soliton solutions.

Part II

Results from the bilinear method

Chapter 4

The Boussinesq equation

I will now start my research into soliton solutions of nonlinear evolution equations. In this Part I will present the results obtained from Hirota's bilinear method and in the next Part I will present those obtained from inverse scattering. In the first Chapter of this Part I will discuss the Boussinesq equation, in Chapter 5 the shallow water wave equation and in Chapter 6 the nonlinear Schrödinger equation. At the end of every chapter I will compare my results with those obtained in the literature.
The Boussinesq equation has been studied a lot and appears in many physical systems such as water waves. It was first derived by Boussinesq [6]. It has many different forms but here I will use the form

$$u_{tt} = \mu u_{xx} + 3p(u^2)_{xx} + pu_{xxxx} \tag{4.1}$$

where μ is a real number and $p = \pm 1$. This can be obtained from the one given in equation (2.3) by introducing the new variables $u = \frac{3\mu}{2hp}\eta$, $x' = \sqrt{\frac{3p}{h^2\mu}}x$ and $t' = \frac{\sqrt{3pc}}{h\mu}$ and then dropping the primes. We first need to obtain the bilinear form of this equation and to do this I will start with transforming the equation into the potential Boussinesq equation by substituting $u = v_{xx}$ and integrating the equation twice with respect to x. We then find

$$v_{tt} = \mu v_{xx} + 3p(v_{xx})^2 + pv_{xxxx} \tag{4.2}$$

and if we now substitute $v = 2\ln(F)$ it is an easy exercise to show that this is equivalent to the bilinear form

$$(D_t^2 - pD_x^4 - \mu D_x^2)(F \bullet F) = 0. \tag{4.3}$$

So we have now found the bilinear form with $B(D_x, B_t) = D_t^2 - pD_x^4 - \mu D_x^2$. If we insert this into the recursion relations (eq. (3.45)-(3.48)) we find

$$\left(\frac{\partial^2}{\partial t^2} - p\frac{\partial^4}{\partial x^4} - \mu\frac{\partial^2}{\partial x^2}\right)F_1 = 0 \tag{4.4}$$

$$\left(\frac{\partial^2}{\partial t^2} - p\frac{\partial^4}{\partial x^4} - \mu\frac{\partial^2}{\partial x^2}\right)F_2 = -\frac{1}{2}(D_t^2 - pD_x^4 - \mu D_x^2)(F_1 \bullet F_1) \tag{4.5}$$

$$\left(\frac{\partial^2}{\partial t^2} - p\frac{\partial^4}{\partial x^4} - \mu\frac{\partial^2}{\partial x^2}\right)F_3 = -(D_t^2 - pD_x^4 - \mu D_x^2)(F_1 \bullet F_2) \tag{4.6}$$

$$\left(\frac{\partial^2}{\partial t^2} - p\frac{\partial^4}{\partial x^4} - \mu\frac{\partial^2}{\partial x^2}\right)F_n = -\frac{1}{2}(D_t^2 - pD_x^4 - \mu D_x^2)\left(\sum_{m=1}^{n-1} F_m \bullet F_{n-m}\right). \tag{4.7}$$

To find the one-soliton solution we again make the ansatz $F_1 = e^{\eta_1}$ and $F_n = 0$ for $n > 1$ where η_1 is still given by

$$\eta_i = \kappa_i x - \omega_i t - \eta_i^0, \tag{4.8}$$

with κ_i, ω_i and η_i^0 constants. Similar to the example of the KdV equation we can find ω_i from the first recursion relation and this gives

$$\omega_i = \pm\sqrt{\mu\kappa_i^2 + p\kappa_i^4} \tag{4.9}$$

and we see that equations (4.5), (4.6) and (4.7) are then satisfied. For the 2-soliton solution we start again with $F_1 = e^{\eta_1} + e^{\eta_2}$ and find that $F_2 = A_{12}e^{\eta_1+\eta_2}$ where

$$A_{ij} = \frac{\omega_i\omega_j - \mu\kappa_i\kappa_j - 2p\kappa_i^3\kappa_j - 2p\kappa_i\kappa_j^3 + 3p\kappa_i^2\kappa_j^2}{\omega_i\omega_j - \mu\kappa_i\kappa_j - 2p\kappa_i^3\kappa_j - 2p\kappa_i\kappa_j^3 - 3p\kappa_i^2\kappa_j^2}. \tag{4.10}$$

from equation (4.5). We fix again $F_n = 0$ for $n > 2$ and then it is an easy calculation to verify that equations (4.6) and (4.7) are satisfied.

4.1 2-soliton solutions to the Boussinesq equation

So we have now found F and therefore the 2-soliton solution. Note that we need to take the κ_i in such a way that they are both real and that both the ω_i, given by equation 4.9, are real. Then in order to rewrite the solution in a decent form we first have to fix the sign of A_{12}. I will discuss three cases separately, A_{12} positive, negative and zero. First if we assume A_{12} to be positive, we find

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2} && (4.11)\\ &= 2e^{\frac{\eta_1+\eta_2}{2}}\left(\sqrt{A_{12}}\cosh\left(\frac{\eta_1+\eta_2+\ln(A_{12})}{2}\right) + \cosh\left(\frac{\eta_1-\eta_2}{2}\right)\right). && (4.12)\end{aligned}$$

Note that I incorporated the ϵ term in η_1^0 and that we can eliminate the $\ln(A_{12})$-term by a smart choice of the η_i^0; therefore I will leave it out from now on. If we use that

$$u = v_{xx} = 2\frac{\partial^2}{\partial x^2}\ln(F) = 2\frac{FF_{xx} - F_x^2}{F^2} \tag{4.13}$$

we find that our solution becomes (after using the well known addition formulas for cosh-functions)

$$u = \frac{2\sqrt{A_{12}}\left(\kappa_2^2\cosh(\eta_1) + \kappa_1^2\cosh(\eta_2)\right) + (\kappa_1-\kappa_2)^2 + A_{12}(\kappa_1+\kappa_2)^2}{2\left(\sqrt{A_{12}}\cosh\left(\frac{\eta_1+\eta_2}{2}\right) + \cosh\left(\frac{\eta_1-\eta_2}{2}\right)\right)^2} \tag{4.14}$$

We note that our solution has no poles and is therefore well behaved. If we now look at the case $A_{12} = 0$ we find that

$$F = 1 + 2e^{\frac{\eta_1+\eta_2}{2}}\cosh\left(\frac{\eta_1-\eta_2}{2}\right) \tag{4.15}$$

and therefore we get the solution

$$u = 2e^{\frac{\eta_1+\eta_2}{2}}\frac{(\kappa_1-\kappa_2)^2e^{\frac{\eta_1+\eta_2}{2}} + (\kappa_1^2+\kappa_2^2)\cosh(\frac{\eta_1-\eta_2}{2}) + (\kappa_1^2-\kappa_2^2)\sinh(\frac{\eta_1-\eta_2}{2})}{(1+2e^{\frac{\eta_1+\eta_2}{2}}\cosh(\frac{\eta_1-\eta_2}{2}))^2}. \tag{4.16}$$

We see that this solution also has no poles and is therefore again well behaved. If we now finally look at the situation for negative A_{12} we find

$$F = 1 + 2e^{\frac{\eta_1+\eta_2}{2}}\left(-\sqrt{-A_{12}}\sinh\left(\frac{\eta_1+\eta_2}{2}\right) + \cosh\left(\frac{\eta_1-\eta_2}{2}\right)\right). \tag{4.17}$$

However we see that F now has zeroes and as a result our solution will have poles and is not well behaved. As these solutions therefore are not physical I will not consider this case but instead require $A_{12} \geq 0$ from now on. Note that with a programme like MAPLE it is easy to verify that all three equations (4.14), (4.16) and (4.17) are indeed solutions to the Boussinesq equation.
So we now have to investigate the behavior of A_{12}, paying special attention to its sign. However we need to split this up into different cases again. We have two split it up into $p = +1$ and $p = -1$ but also need to distinguish between the different sign possibilities for ω in equation (4.9). I distinguish two different cases here, either both the ω's are positive or one is positive and the other negative. The case with both negative is mapped to the positive case under time reversal and will therefore not be discussed here. So we have four cases in total to discuss. I will start with $p = +1$ and both ω positive. In this case we see that A_{12} has no zeroes or poles so A_{12} cannot change sign. Furthermore we see that it is positive everywhere. This means we only have one kind of behavior and this is illustrated in Figure 4.1. In the left part we have two

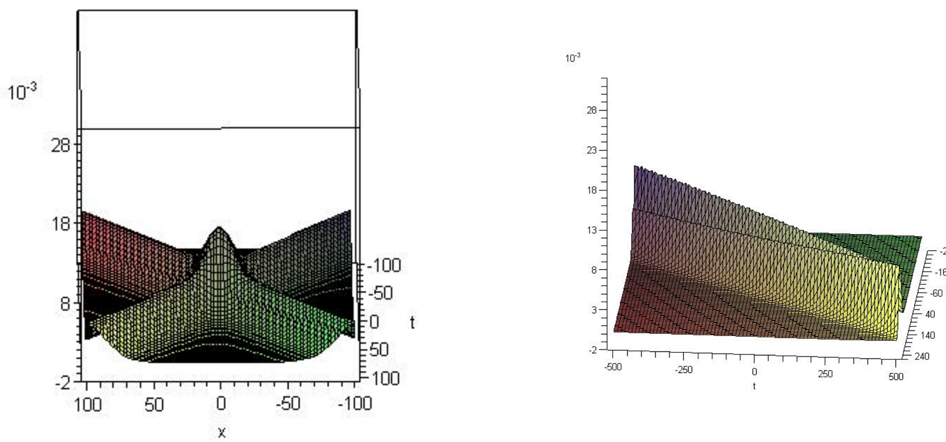

Figure 4.1: 2-soliton solution to equation (4.1) with the parameter values (a) $\mu = 1$, $\kappa_1 = -0.1$ and $\kappa_2 = 0.1$ (left) and (b) $\mu = 0.1$, $\kappa_1 = 0.15$ and $\kappa_2 = 0.2$ (right).

waves coming in from opposite directions and moving on unperturbed after a short and local interaction. In the right part we have both waves coming in from the same direction and we see that now the area of the interaction is larger, but again they emerge unperturbed. So we see that they are indeed soliton solutions.
If we now move on to the case with one positive and one negative ω, we see that not much changes. If we keep requiring that the ω's are real (in the next section I will discuss what happens when they become complex), we find that A_{12} seen as a function of μ, κ_1 and κ_2 has again no zeroes or poles and is again positive everywhere. Therefore the behavior is as we have seen in Figure 4.1, with the small change that one of the waves is traveling in the opposite direction. But as we had already the freedom to choose this by choosing the signs of the κ's we see that this introduces nothing new.
The situation for $p = -1$ is much more interesting as we will see that A_{12} now has zeroes. Again I will start with the case with 2 positive ω's. Note that in order to keep the ω's real we have to impose a constraint on our κ's in the form of $\kappa_i^2 \leq \mu$ for $i = 1, 2$. I will impose this constraint throughout the rest of this section. Furthermore I will fix $\mu = 0.1$ as the exact value of μ does not matter much now. We see that as long as it is positive only its size in comparison to the κ's matters. I have chosen this value instead of $\mu = 1$ because the pictures will become a bit smoother. Now we can examine the zeroes and poles of A_{12} and this is shown graphically in Figure 4.2. In this Figure the blue lines indicate the constraint $\kappa_i^2 \leq \mu$ and we therefore cannot pas through them. The red lines indicate the position of the zeroes of A_{12} and the green lines the position of the poles. The straight red line through the origin corresponds to the case $\kappa_1 = \kappa_2$ and this reduces to

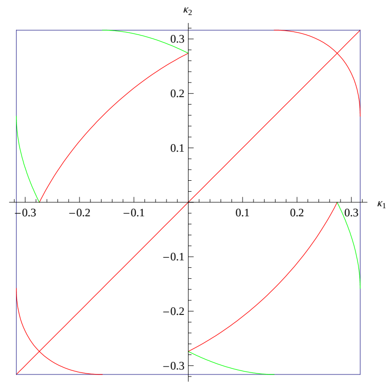

Figure 4.2: Zeroes (red) and poles (green) of A_{12} as given in equation (4.10) with $p = -1$ and $\mu = 0.1$. The blue lines give the constraint $\kappa_i^2 < \mu$ for $i = 1, 2$.

the 1-soliton solution. Note that as our solution is invariant under interchanging κ_1 and κ_2, A_{12} will not change sign when we pass through this line. It will however change sign when we pass through any of the other red or green lines as these correspond to zeroes and poles of order 1. Therefore we have the following situation. In the large area in the center of the picture we have $A_{12} > 0$ (with exception of the line $\kappa_1 = \kappa_2$, but that has already been discussed) and the solution is given by equation (4.14). In this area we therefore have the normal 2-soliton behavior as we saw in picture 4.1. However because we have changed the sign of p now the larger of the two waves travels slower than the smaller one, but as this is the only difference I will not illustrate this case again. In the four areas in the corners of the graph enclosed by the red, blue and green lines A_{12} will be negative and therefore our solution will have poles as we saw earlier. This solution is therefore not physical and I will not study this case. On the green lines A_{12} is infinite and therefore our solution will have infinities again. This case I will therefore also eliminate. On the red lines finally we have $A_{12} = 0$ and therefore our solutions are governed by equation (4.16). Here we observe two new kinds of behavior illustrated in Figure 4.3. In the left figure we see that we have one big stationary incoming wave

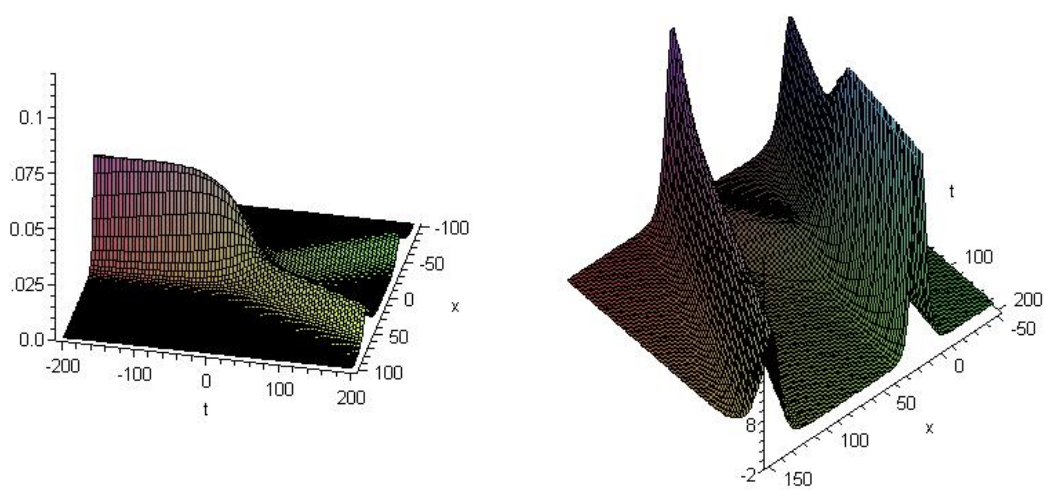

Figure 4.3: 2-soliton solution to equation (4.1) with the parameter values $p = -1$, $\mu = 0.1$, κ_2 on the top red curve of figure 4.2 and (a) $\kappa_1 = -0.15$ (left) and (b)$\kappa_1 = 0.22$ (right).

that splits up at $t = 0$ into two smaller waves traveling in opposite directions. Similarly the opposite can happen with two smaller waves merging into a larger wave (I did not illustrate this case). In the right part we see two waves coming in from the same side with the smaller faster one overtaking the bigger slower one. However the interaction now does not happen as we seen before in Figure 4.1 but now the bigger wave emits a smaller wave traveling backwards. This is then absorbed by the smaller wave whose size increases as a result (and thus slows down). Effectively this means that the smaller wave has now overtaken the bigger wave as before, but the interaction in between was different. We have now seen all possible behavior for the 2-soliton solutions. In the next subsection I will look at breather solutions that emerge when we allow the κ's and ω's to be complex. However before that I will look a little closer at one property of the 2-soliton solution.

Phase shift

Although it might seem that both solitons are totally unaffected by the interaction, this is not the case. It can be shown analytically that both solitons get a phase shift as a result of the interaction. To show this I first fix the values of κ_1, κ_2, μ and p to keep the calculations as clear as possible. I choose $\kappa_1 = 1$, $\kappa_2 = 2$, $\mu = 1$ and $p = 1$ but their exact value does not matter (as long as we keep A_{12} positive). Inserting this in equation (4.14) we find

$$u = \frac{2\sqrt{A_{12}}\left(4\cosh(x-\sqrt{2}t)+\cosh(2x-2\sqrt{5}t)\right)+1+9A_{12}}{2\left(\sqrt{A_{12}}\cosh\left(\frac{3x-(\sqrt{2}+2\sqrt{5})t}{2}\right)+\cosh\left(\frac{-x-(\sqrt{2}-2\sqrt{5})t}{2}\right)\right)^2}. \tag{4.18}$$

Now we introduce two new coordinate systems, the first given by

$$x_1 = x - \sqrt{2}t, \qquad t_1 = t \tag{4.19}$$

and the second one given by

$$x_2 = 2x - 2\sqrt{5}t, \qquad t_2 = t. \tag{4.20}$$

We see that the first coordinate system is equal to riding on top of the smaller, slower soliton and the second to riding on top of the bigger, faster soliton. If we insert the first new coordinates into equation (4.18) and take the limit for $t_1 \to -\infty$ we get

$$u \sim \frac{1}{1+\cosh(x_1+\ln(A)/2)} \tag{4.21}$$

and in the limit for $t_1 \to +\infty$

$$u \sim \frac{1}{1+\cosh(x_1-\ln(A)/2)}. \tag{4.22}$$

So we see that the smaller soliton picks up a phase shift, which means that the top is shifted forward $\ln(A)$ during the interaction. However as $\ln(A)$ is negative, (as can easily been seen from equation (4.10)), in effect the top of the smaller soliton is shifted backward. From exactly the same reasoning we see that the top of the bigger soliton is shifted backward by $\ln(A)$. This is also illustrated in Figure 4.4, where we see that the smaller soliton is shifted backwards and the bigger soliton is shifted forwards. Similar phase shifts also occur in the 3-soliton solutions to the Boussinesq equation and in soliton solutions to the SWW and NLS equations, but I will not explicitly show this, as this can be done in exactly the same way. Similarly this can also be shown for the KdV equation and a calculation can for instance be found in [18].

4.2 Breather solutions to the Boussinesq equation

In this part I will derive the stationary breather, growing-and-decaying mode and moving breather solutions to the Boussinesq equation. I will do this only for the case of $p = +1$, as in the other case as a result of the behavior of A_{12} I have not been able to find any such solutions. I will start by deriving the stationary breather solution.

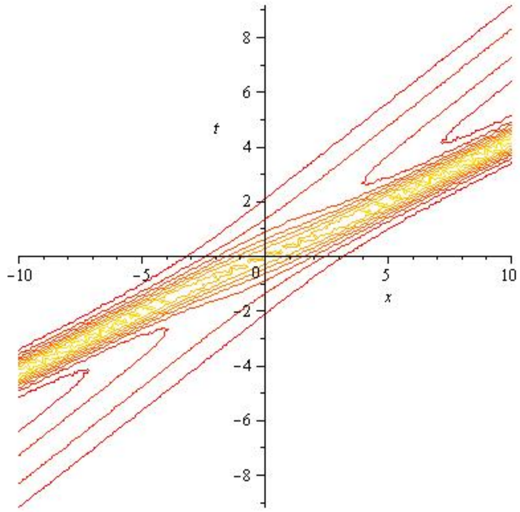

Figure 4.4: Contourplot of 2-soliton solution to equation (4.1) given by equation (4.14) with the parameter values $p = 1$, $\mu = 1$, $\kappa_1 = 1$ and $\kappa_2 = 2$.

4.2.1 Stationary breather solution

To find the stationary breather solution I will look at real values of the κ's and μ such that the ω's become complex. From equation (4.9) we see that this corresponds to $\mu < -\kappa_i^2$ for $i = 1, 2$. It will become clear later that in order for our final solution to be real, we will need $\kappa_2 = \kappa_1$ and $\omega_2 = \omega_1^*$, where $*$ indicates the normal complex conjugate. We can easily achieve this by choosing the sign of ω_2 to be opposite from the sign of ω_1 in equation (4.9). I therefore choose

$$\kappa_1 = \kappa_2 = \kappa \tag{4.23}$$
$$\omega_1 = -\omega_2 = \sqrt{\mu\kappa^2 + \kappa^4} = i\omega, \tag{4.24}$$

where I assumed that $\mu < -k^2$ such that ω is a real number. Furthermore, for now I will assume $\eta_i^0 = 0$ to keep the formulas as clear as possible. If we now insert this into equation (4.10), we see that it reduces to

$$A_{12} = \frac{\mu + k^2}{\mu + 4k^2}. \tag{4.25}$$

We see that A_{12} has a zero at $\mu = -k^2$ and a pole at $\mu = -4k^2$. However as we assumed that $\mu < -k^2$ the zero is not interesting, the pole however is as A_{12} changes sign there. For $\mu < -4k^2$ we have a positive A_{12} and for $\mu > -4k^2$ a negative. We will see that this makes an important difference. If we assume A_{12} to be positive we can then rewrite the solution we found in a nicer form. We start by rewriting F in the form

$$F = 1 + e^{\eta_1} + e^{\eta_2 + A_{12}} e^{\eta_1 + \eta_2} = 1 + e^{\kappa x - i\omega t} + e^{\kappa x + i\omega t} + A_{12} e^{2\kappa x} \tag{4.26}$$
$$= 2e^{\kappa x}\left(\sqrt{A_{12}}\cosh\left(\kappa x - \frac{\ln(A_{12})}{2}\right) + \cos(\omega t)\right) \tag{4.27}$$

and we see that if we insert this in equation (4.13) we get after some algebra

$$u = 2\kappa^2 \frac{\sqrt{A_{12}}\cos(\omega t)\cosh(\kappa x + \frac{1}{2}\ln(A_{12})) + A_{12}}{\cos(\omega t) + \sqrt{A_{12}}\cosh(\kappa x + \frac{1}{2}\ln(A_{12}))}. \tag{4.28}$$

We see that we can again simplify this formula if we take a different value for the η_i^0. For instance if we

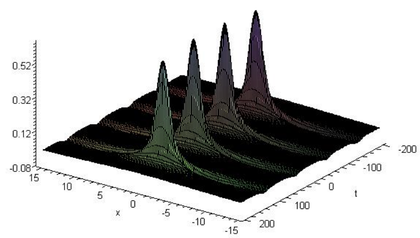

Figure 4.5: Stationary breather solution with parameter values $\kappa = -0.1$ and $\mu = -0.5$.

take $\eta_1^0 = \eta_2^0 = \ln(A)/2$, we can then simplify it to

$$u = 2\kappa^2 \frac{\sqrt{A_{12}}\cos(\omega t)\cosh(\kappa x) + A_{12}}{\cos(\omega t) + \sqrt{A_{12}}\cosh(\kappa x)}. \tag{4.29}$$

Note that since we assumed that $A_{12} > 0$ we have a real valued solution. Furthermore since this requires that $\mu < -4\kappa^2$ we see from equation (4.25) that $A_{12} > 1$ for all allowed values of μ. Therefore we see that u has no poles and should be well behaved everywhere. If we plot the behavior, which is done in Figure 4.5, we see that we have obtained the stationary breather solution. The solution is periodic in time and only nonzero in a finite part of space. This was to be expected as ω was chosen complex and therefore the time dependence is governed by trigonometric functions instead of hyperbolic functions.
Let us now investigate what happens when A_{12} is negative. Then the derivation is very similar and we find

$$u = 2\kappa^2 \frac{-\sqrt{-A_{12}}\cos(\omega t)\sinh(\kappa x) + A_{12}}{\cos(\omega t) - \sqrt{-A_{12}}\sinh(\kappa x)}, \tag{4.30}$$

where we see that the only differences are signs and the replacement of the cosh-function by the sinh-function. The latter however does present us with a problem as u will now have poles. Therefore our solution is not well behaved in this case and I will not consider it any further.

4.2.2 Growing-and-decaying mode solution

In this part I will derive the growing-and-decaying mode solution in a similar way as I found the stationary breather solution before. To do this I now choose $\kappa_1 = \kappa_2^* = i\kappa$ with κ a real number and $\eta_1^0 = \eta_2^0 = 0$. Note that we now get $\omega_1 = \omega_2 = \sqrt{-\mu\kappa^2 + \kappa^4}$ and I will require these numbers to be real, i.e. $\mu < \kappa^2$. If we insert all this in equation (4.10) we find

$$A_{12} = \frac{-\mu + 4\kappa^2}{-\mu + \kappa^2}. \tag{4.31}$$

Note that A_{12} is always positive in the range of μ we allowed earlier. Furthermore for $\kappa \neq 0$, A_{12} is larger than 1. We can rewrite our solution in a similar way as before and we find

$$u = -2\kappa^2 \frac{\sqrt{A_{12}}\cos(\kappa x)\cosh(\omega t - \frac{1}{2}\ln(A_{12})) + 1}{\cos(\kappa x) + \sqrt{A_{12}}\cosh(\omega t - \frac{1}{2}\ln(A_{12}))}. \tag{4.32}$$

Choosing $\eta_1^0 = \eta_2^0 = \ln(A_{12})/2$ we see that this simplifies to

$$u = -2\kappa^2 \frac{\sqrt{A_{12}}\cos(\kappa x)\cosh(\omega t) + 1}{\cos(\kappa x) + \sqrt{A_{12}}\cosh(\omega t)}. \tag{4.33}$$

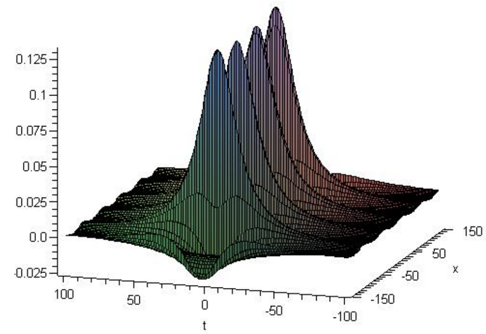

Figure 4.6: Growing-and-decaying mode solution with parameter values $\kappa = 0.1$ and $\mu = -0.1$.

We see that besides x and t changing places not much has changed. We also see this in Figure 4.6, where we see that the solution is now periodic in x and rapidly decaying in t. This solution is therefore known as the growing-and-decaying mode solution.

4.2.3 Moving breather solution

The idea behind the derivation of the moving breather solution is the same as for the stationary breather and growing-and-decaying mode solution. However, the algebra will become more difficult. For now I again take $\eta_1^0 = \eta_2^0 = 0$. We now allow the κ_i and ω_i to be general complex numbers. Motivated by the previous calculations I do require that $\kappa_2 = \kappa_1^*$. I therefore put

$$\kappa_1 = a + ib \qquad \text{and} \qquad \kappa_2 = a - ib, \tag{4.34}$$

where a and b are real numbers. We see that we now get

$$\omega_1 = \sqrt{c + id}, \tag{4.35}$$

with

$$c = \mu a^2 - \mu b^2 + a^4 + b^4 - 6a^2 b^2 \qquad \text{and} \qquad d = 2\mu ab + 4a^3 b - 4ab^3. \tag{4.36}$$

We see that we can now write $\omega_1 = \sqrt{R}\exp(i\phi/2)$ with $R = \sqrt{c^2 + d^2}$ and

$$\phi = \arctan(d/c) + \text{sign}(d)\frac{1 - \text{sign}(c)}{2}\pi \qquad \text{if} \qquad c \neq 0 \tag{4.37}$$

$$\phi = \text{sign}(d)\frac{\pi}{2} \qquad \text{if} \qquad c = 0. \tag{4.38}$$

Furthermore we see that $\omega_2 = \sqrt{c - id} = \sqrt{R}\exp(-i\phi/2)$ so that $\omega_2 = \omega_1^*$ as we had in the previous sections. If we insert all this into equation (4.10) we get after some calculations

$$A_{12} = \frac{-R + \mu(a^2 + b^2) + a^4 - 7b^4 - 6a^2 b^2}{-R + \mu(a^2 + b^2) + 7a^4 - b^4 + 6a^2 b^2}. \tag{4.39}$$

Note that A_{12} is again a real valued variable but that the sign of A_{12} is not easy to determine and can be both positive and negative depending on the choice of a, b and μ. For now I will assume that A_{12} is positive and deal with the other case later. Before I start rewriting the solution I introduce some new, real valued, variables y and z by $\omega_1 = y + iz$. We see that we therefore have $y = \sqrt{R}\cos(\phi/2)$ and $z = \sqrt{R}\sin(\phi/2)$. If $c > 0$ and $d > 0$ we can simplify this into

$$y = \sqrt{\frac{R+c}{2}} \tag{4.40}$$

$$z = \sqrt{\frac{R-c}{2}} \tag{4.41}$$

Furthermore I introduce $\eta_R = ax - yt$ and $\eta_I = bx - zt$, such that

$$\eta_1 = \eta_R + i\eta_I \qquad \text{and} \qquad \eta_2 = \eta_R - i\eta_I. \tag{4.42}$$

Using this we can rewrite F as follows

$$\begin{aligned} F &= 1 + \exp(\eta_1) + \exp(\eta_2) + A_{12}\exp(\eta_1 + \eta_2) \\ &= 1 + \exp(\eta_R)(\exp(i\eta_I) + \exp(-i\eta_I)) + A_{12}\exp(2\eta_R) \\ &= \exp(\eta_R)(2\cos(\eta_I) + \exp(-\eta_R) + A_{12}\exp(\eta_R)) \\ &= 2\exp(\eta_R)\left(\cos(\eta_I) + \sqrt{A_{12}}\cosh\left(\eta_R + \frac{1}{2}\ln(A_{12})\right)\right) \end{aligned} \tag{4.43}$$

and we see it is now an easy calculation to find F_x and F_{xx}. If we insert all this in equation (4.13) we get

$$u = 2\frac{\sqrt{A_{12}}\cos(\eta_I)\cosh\left(\eta_R + \frac{1}{2}\ln(A_{12})\right)(a^2 - b^2) + 2ab\sqrt{A_{12}}\sin(\eta_I)\sinh\left(\eta_R + \frac{1}{2}\ln(A_{12})\right) + A_{12}a^2 - b^2}{\left(\cos(\eta_I) + \sqrt{A_{12}}\cosh\left(\eta_R + \frac{1}{2}\ln(A_{12})\right)\right)^2} \tag{4.44}$$

which can be simplified to

$$u = 2\frac{\sqrt{A_{12}}\cos(\eta_I)\cosh(\eta_R)(a^2 - b^2) + 2ab\sqrt{A_{12}}\sin(\eta_I)\sinh(\eta_R) + A_{12}a^2 - b^2}{\left(\cos(\eta_I) + \sqrt{A_{12}}\cosh(\eta_R)\right)^2}, \tag{4.45}$$

if we choose $\eta_1^0 = \eta_2^0 = \ln(A_{12})/2$. We have now found the moving breather solution. If we compare this with the form given in paragraph 3 of [26], we see that it coincides almost and that the solutions are equivalent if we change formula 5.3 in [26] such that the LHS contains an extra factor 2. This is apparently an error in the paper as this factor is included in a later paper by the same author [27]. We note that the real parts of the η_i appear in hyperbolic functions whereas the complex parts appear in trigonometric functions. This is similar to what was observed in the previous two sections. We see that the behavior of our solution is again strongly dependent of the value of A_{12}. If $A_{12} < 1$ then the solution has poles and is not very well behaved (I do not illustrate this case here). However if $A_{12} > 1$ we see that we have no poles and a well-behaved solution, known as the moving breather solution and which is illustrated in Figure 4.7. We see that the behavior is now periodic along a line in the (x, t)-plane not corresponding to either of the axes, but besides that it still looks quite similar to what we have seen before.
Let us now take a look at what happens when A_{12} is negative. This means that equation (4.45) will change and the new form becomes

$$u = -2\frac{\sqrt{-A_{12}}\cos(\eta_I)\sinh(\eta_R)(a^2 - b^2) + 2ba\sqrt{-A}\sin(\eta_I)\cosh(\eta_R) - Aa^2 + b^2}{\left(\cos(\eta_I) - \sqrt{-A_{12}}\sinh(\eta_R)\right)^2}. \tag{4.46}$$

We see that as before we now have a sinh-function in the denominator and again this means that we will always have poles. Therefore the solution is not very well behaved and I do not illustrate it here. Because of the complicated form of equation (4.39) I have not investigated for which initial values we will exactly get a moving breather solution, but we do see from Figure 4.7 that they exist.

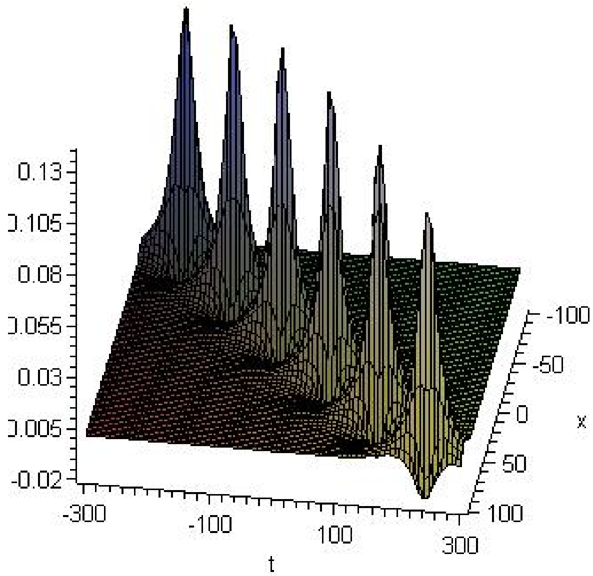

Figure 4.7: Moving breather solution with parameter values $a = 0.08$, $b = -0.1$ and $\mu = -0.1$.

4.3 3-soliton solution to the Boussinesq equation

Now we can use the same ideas to find the 3-soliton solution. To do this we make the ansatz

$$F_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}, \tag{4.47}$$

and if we insert this into equations (4.4)-(4.7) we find

$$\begin{aligned} F_2 &= A_{12}e^{\eta_1+\eta_2} + A_{23}e^{\eta_2+\eta_3} + A_{13}e^{\eta_1+\eta_3}, && (4.48)\\ F_3 &= B_{123}e^{\eta_1+\eta_2+\eta_3}, && (4.49)\\ F_n &= 0 \qquad \text{for } n > 3, && (4.50)\end{aligned}$$

where η_i and A_{ij} are as before and $B_{123} = A_{12}A_{23}A_{13}$. Before we can rewrite the solution we have to take a look at the signs of the A_{ij}'s again. As we have seen before that negative A_{ij} leads to infinities in the solutions I here only consider $A_{ij} \geq 0$. Now we have four different cases, corresponding to putting either 0,1,2 or 3 of the A_{ij} to zero and keeping the others positive. If non of the A_{ij}'s are zero we get

$$F = 2e^{\eta_+}\left(\sqrt{A_{12}A_{13}A_{23}}\cosh(\eta_+) + \sum_{i<j\neq k}\sqrt{A_{ij}}\cosh(\eta_{ijk})\right), \tag{4.51}$$

where $\eta_+ = (\eta_1 + \eta_2 + \eta_3)/2$, $\eta_{ijk} = (\eta_i + \eta_j - \eta_k)/2$ and the summation is over $i, j, k \in \{1, 2, 3\}$ such that $i < j$, $j \neq k$ and $i \neq k$ respectively. Now it is an easy exercise to find the formula for u, but I will not give this here as this becomes a very long and hideous equation. However we do see that F has no poles so our solution should be well behaved. If we now put $A_{12} = 0$ and keep the other two positive we get

$$F = 2e^{\frac{\eta_3}{2}}\left(\cosh\left(\frac{\eta_3}{2}\right) + e^{\frac{\eta_1+\eta_2}{2}}\left(\sqrt{A_{13}}\cosh\eta_{132} + \sqrt{A_{23}}\cosh\eta_{231}\right)\right). \tag{4.52}$$

We see that we again get a well behaved solution as F has no zeroes. Similarly if we put $A_{12} = A_{23} = 0$ we get

$$F = 1 + e^{\frac{\eta_1+\eta_3}{2}} \left(\cosh\left(\frac{\eta_1 - \eta_3}{2} \right) + e^{\eta_2} \sqrt{A_{13}} \cosh(\eta_{132}) \right), \tag{4.53}$$

and as expected this again leads to a solution without infinities. If we finally put all three A_{ij} to zero we get

$$F = 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3}. \tag{4.54}$$

Note that there are multiple ways to rewrite this in terms of hyperbolic functions, but as this does not make it any clearer I have not done that here. Again we see our solution is well behaved.
To illustrate these four solutions I will directly go to the $p = -1$ case. I do this because as in the $p = +1$ case all the A_{ij} have no zeroes, there is not much to illustrate. The only behavior we can have is with all positive A_{ij} and this will be similar to that for the $p = -1$ case. So for $p = -1$ let us start to look at the case of all positive A_{ij}. This case is shown in Figure 4.8. I choose here to let two of the solitons come from

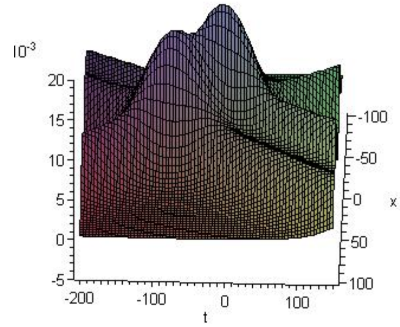

Figure 4.8: 3-soliton solution to the Boussinesq equation (4.1) with all $A_{ij} > 0$.

one side and the third from the opposite side to keep things clear, although we clearly also have solutions with 3 solitons coming from one side. We see that we have the expected three soliton behavior with a local interaction happening as expected. Next we look at the behavior with one A_{ij} set to zero. This is done in Figure 4.9. What we see here is that we have one large incoming wave that splits up in the origin and is similar to the one shown in Figure 4.3(a). Besides that we have another soliton which interacts normally with the rest. The situation becomes more interesting when we put two of the A_{ij} to zero as shown in Figure 4.10. We see that here we have two incoming large waves which both break up into 2 smaller waves at different times. Two of these smaller waves then merge into a new large wave so that we end up with three waves again. Finally we have the case with all three A_{ij} zero. However it turns out that the only way to achieve this is to put $\kappa_i = \kappa_j$ for a $i \neq j$ and therefore this reduces to the 2-soliton case and the behavior is similar to that shown in Figure 4.3(a).

4.4 Discussion

None of the results obtained for the Boussinesq equation in this section are new. The first results using the bilinear method for the Boussinesq equation were obtained by Hirota in 1973 [15], however he worked only with $\mu = p = 1$. He gave the N-soliton solution obtained in a similar way as I obtained the 1- and

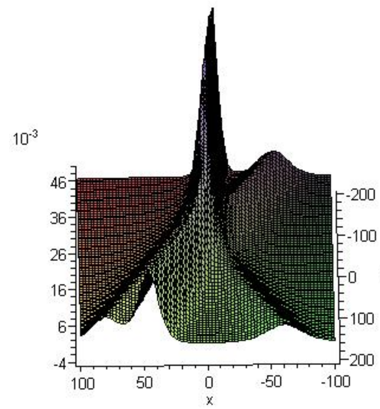

Figure 4.9: 3-soliton solution to the Boussinesq equation (4.1) with one $A_{ij} = 0$ and the other $A_{ij} > 0$.

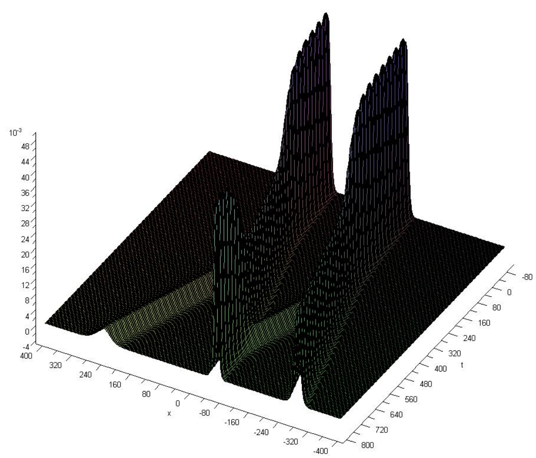

Figure 4.10: 3-soliton solution to the Boussinesq equation (4.1) with two $A_{ij} = 0$ and the other $A_{ij} > 0$.

2-soliton solution by writing F in the way given in (3.42). He also calculated the phase shifts and showed that the total phase shift is always equal to 0. Furthermore he also illustrated the 2-soliton solution. As he only looked at positive μ and p he did not investigate the position of the zeroes of A_{12} nor point out their importance. So far I have not found a discussion of the zeroes of A_{12} in connection with the 2-soliton solution in literature. The breather solutions have been found before and are given in [26]. The authors obtain them in a similar way as I did, by taking the κ_i and ω_i to be complex numbers. Furthermore they also discuss the conditions on μ and p in order to have well behaved breather solutions. They also look into the stability of the solutions which I have not done. Furthermore they also derive the interaction between a growing-and-decaying mode and a stationary breather, which can be derived from a 4-soliton solution. In another article [27] the authors also show that the growing-and-decaying mode, stationary breather and breather solution can all be constructed from a rational growing-and-decaying mode solution. However as I have not been able to find how they found this rational growing-and-decaying mode solution I have not reproduced their results here. However they do exactly the same for the NLS equation in [28] and I will reproduce their results in part 6.7 to illustrate their method.

Chapter 5

Shallow water wave equation

In this part I will present some solutions to the shallow water wave equation:

$$u_{xxxt} + 3u_x u_{xt} + 3u_t u_{xx} - u_{xx} - bu_{xt} = 0, \tag{5.1}$$

where b is either $+1$ or -1. I will use the Hirota method again to find the 1-, 2- and the 3-soliton solution. However we will see that this does not look like the solitons we saw for the Boussinesq equation earlier. I will start by deriving the solutions and then illustrate the 2-soliton solution. Next I will derive the breather solution in a similar way as we did for the Boussinesq equation. Then I will derive the 3-soliton solution and look at the 1-soliton breather interaction. I will finish with the derivation of a number of rational solutions obtained from the limits of the soliton solutions and of a new solution derived from a different ansatz inserted in the bilinear form.

First I will derive the 1- and 2-soliton solution to the shallow water wave equation given in equation (5.1). I will start with transforming this equation into the potential shallow water wave equation by substituting $u = v_x$ and integrating the equation with respect to x. We then find

$$v_{xxxt} + 3v_{xx}v_{xt} - v_{xx} - bv_{xt} = 0 \tag{5.2}$$

and if we now substitute $v = 2\ln(F)$ it is an easy exercise to show that this is equivalent to the Hirota bilinear form

$$(D_t D_x^3 - D_x^2 - bD_x D_t)(F \bullet F) = 0. \tag{5.3}$$

From now one we can do exactly the same as before for the Boussinesq equation. We start by making the ansatz

$$F = 1 + \epsilon F_1 + \epsilon^2 F_2 + \epsilon^3 F_3 + ... \tag{5.4}$$

and if we insert this into equation (5.3), we can split it up according to order in ϵ. We see that we then get a set of recursive equations:

$$\left(\frac{\partial^4}{\partial t \partial x^3} - \frac{\partial^2}{\partial x^2} - b\frac{\partial^2}{\partial t \partial x}\right)F_1 = 0 \tag{5.5}$$

$$\left(\frac{\partial^4}{\partial t \partial x^3} - \frac{\partial^2}{\partial x^2} - b\frac{\partial^2}{\partial t \partial x}\right)F_2 = -\frac{1}{2}(D_t D_x^3 - D_x^2 - bD_x D_t)(F_1 \bullet F_1) \tag{5.6}$$

$$\left(\frac{\partial^4}{\partial t \partial x^3} - \frac{\partial^2}{\partial x^2} - b\frac{\partial^2}{\partial t \partial x}\right)F_3 = -(D_t D_x^3 - D_x^2 - bD_x D_t)(F_1 \bullet F_2) \tag{5.7}$$

$$\left(\frac{\partial^4}{\partial t \partial x^3} - \frac{\partial^2}{\partial x^2} - b\frac{\partial^2}{\partial t \partial x}\right)F_n = -\frac{1}{2}(D_t D_x^3 - D_x^2 - bD_x D_t)\left(\sum_{m=1}^{n-1} F_m \bullet F_{n-m}\right). \tag{5.8}$$

To find the one-soliton solution we now make the ansatz $F_1 = e^{\eta_1}$ with $\eta_i = \kappa_i x - \omega_i t + \eta_i^0$ and $F_n = 0$ for $n > 1$. Inserting this in equation (5.5) we find

$$\omega_i = \frac{\kappa_i}{b - \kappa_i^2} \tag{5.9}$$

and we see that equations (5.6), (5.7) and (5.8) are then satisfied. To derive the 2-soliton equation we now make the ansatz $F_1 = e^{\eta_1} + e^{\eta_2}$ and find that $F_2 = A_{12}e^{\eta_1+\eta_2}$ and $F_n = 0$ for $n > 3$, where η_i and ω_i are as before and

$$A_{ij} = \frac{(\kappa_i - \kappa_j)^2(-\kappa_i^2 + \kappa_i\kappa_j - \kappa_j^2 + 3b)}{(\kappa_i + \kappa_j)^2(-\kappa_i^2 - \kappa_i\kappa_j - \kappa_j^2 + 3b)}. \tag{5.10}$$

To find the 2-soliton solutions we now need to take the κ's real and if we make them complex we will get a breather solution. I will start with the 2-soliton solution.

5.1 2-soliton solutions to the shallow water wave equation

As said before we now take the κ's to be real and then we see that F is given by

$$F = 1 + \exp(\eta_1) + \exp(\eta_2) + A_{12}\exp(\eta_1 + \eta_2), \tag{5.11}$$

where η_i and A_{12} are as before. In order to write this in a more intuitive form we will need to treat 3 cases separately, i.e. when A_{12} is positive, negative and zero, just as we did for the Boussinesq equation. We will start with A_{12} positive. If we now introduce $\eta_+ = (\eta_1 + \eta_2)/2$ and $\eta_- = (\eta_1 - \eta_2)/2$ we can rewrite 5.11 in the form

$$\begin{aligned} F &= \exp(\eta_+)\left[\exp(-\eta_+) + \exp(\eta_-) + \exp(-\eta_-) + A_{12}\exp(\eta_+)\right] && (5.12) \\ &= 2\exp(\eta_+)[\cosh(\eta_-) + \sqrt{A_{12}}\cosh(\eta_+ + \ln(A)/2)]. && (5.13) \end{aligned}$$

We see that we can take out the $\ln(A_{12})/2$ term by a smart choice of the η_i^0 and therefore I will leave it out from now on. As we have that

$$u = v_x = 2(\ln(F))_x = 2\frac{F_x}{F}, \tag{5.14}$$

we now get

$$u = \frac{(\kappa_1 + \kappa_2)[\cosh(\eta_-) + \sqrt{A_{12}}(\cosh(\eta_+) + \sinh(\eta_+))] + (\kappa_1 - \kappa_2)\sinh(\eta_-)}{\cosh(\eta_-) + \sqrt{A_{12}}\cosh(\eta_+)}. \tag{5.15}$$

Note that in this case u has no poles and is therefore well behaved everywhere. In a similar way we find for negative A_{12} that

$$u = \frac{(\kappa_1 + \kappa_2)[\cosh(\eta_-) + \sqrt{-A_{12}}(\cosh(\eta_+) + \sinh(\eta_+))] + (\kappa_1 - \kappa_2)\sinh(\eta_-)}{\cosh(\eta_-) - \sqrt{-A_{12}}\sinh(\eta_+)}. \tag{5.16}$$

Although the changes seem minor, we note that our solution now does have poles and is therefore not physical. I will therefore not study this case any further. If we finally look at the case of $A_{12} = 0$ we see that we get

$$u = \frac{\exp(\eta_+)((\kappa_1 + \kappa_2)\cosh(\eta_-) + (\kappa_1 - \kappa_2)\sinh(\eta_-))}{1 + \exp(\eta_+)(\cosh(\eta_-))}, \tag{5.17}$$

and we see that in this case we have no poles, so again a well behaved solution. We see that the behavior of the solution is strongly related to that of A_{12} and therefore I will study this behavior first.
If $b = +1$ we see that A_{12} has zeroes if $\kappa_2 = \kappa_1$ or if

$$\kappa_2 = \frac{1}{2}\left(\kappa_1 \pm \sqrt{-3\kappa_1^2 + 12}\right). \tag{5.18}$$

Note that the first zero is of order 2 whereas the second zero is of order 1. Furthermore we see that A_{12} has poles at $\kappa_2 = -\kappa_1$ and at

$$\kappa_2 = -\frac{1}{2}\left(\kappa_1 \pm \sqrt{-3\kappa_1^2 + 12}\right). \tag{5.19}$$

Again we note that the first pole is of order 2 and the second of order 1. We can represent this graphically as is done in Figure 5.1. In this figure the positions of the zeroes are shown in red and those of the poles are

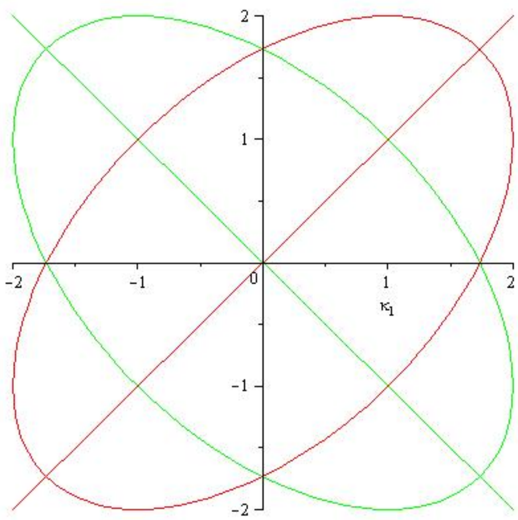

Figure 5.1: Zeroes (red) and poles (green) of A_{12} for $b = 1$

shown in green. Furthermore note that, in contrast to the case for the Boussinesq equation, we now do not have a upper limit on the values of the κ's. We see that the zeroes of order 2 form a straight line and that as the order is 2 the sign of A_{12} will not change when we pass through this line. I will not examine the behavior on this line as this reduces to a 1-soliton solution. Similarly we see that the poles of order 2 also form a straight line and the sign of A_{12} will again not change when we pass through this. As A_{12} (and therefore u) is infinite on this line, I will not investigate what happens on this line. We furthermore note that the zeroes and poles of order 1 both form an ellipse and the sign of A_{12} will change when we pass through it. Therefore we see that A_{12} is positive when we are either inside or outside both ellipses and that A_{12} is negative when we are inside one ellipse and outside the other. As negative A_{12} leads to infinities in our solution, this case will not be discussed further. Furthermore on the red ellipse $A_{12} = 0$ and therefore the solution is given by equation (5.17) and is finite. On the green ellipse however, A_{12} is infinite and therefore our solution has infinities as well, therefore I will not discuss this here. So we see that we have three possible choices for κ_1 and κ_2 remaining, we can choose them inside both ellipses, on the red ellipse or outside both ellipses. As the situation inside and outside both ellipses is similar I only illustrate the behavior outside both ellipses; this is done in Figure 5.2. We see that we have two solitons both coming in from the positive x-direction. The larger, slower moving one has size $2\kappa_2$ and is overtaken in the origin by the smaller, faster moving wave with size $2\kappa_1$. If we change the sign of one of the κ_i, then the wave will have a negative amplitude and

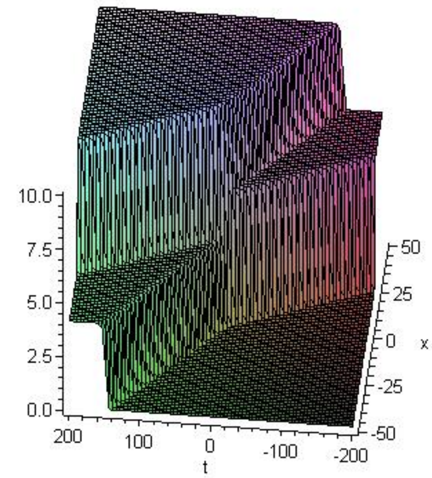

Figure 5.2: 2-soliton solution of the shallow water wave equation (5.1) with parameter values $\kappa_1 = 2$, $\kappa_2 = 3$ and $b = 1$.

travel in the opposite direction. This however is the only change in behavior that is possible for positive A_{12}. Note that these solitons are very different from what we have seen for the Boussinesq equation. If we move to $A_{12} = 0$ (so onto the ellipse), we see that the behavior does change as is shown in Figure 5.3. We see that we now have one big soliton coming in from the positive x-direction which splits up into two smaller waves when it reaches the origin. One of these continues to travel toward the negative x-direction but with a higher speed than before (so again we note that smaller waves travel faster) and the other travels in the opposite direction. Again not many variations on this behavior are possible, the main being that changing signs will again give waves with a negative amplitude and can lead to the opposite behavior of two waves coming in from opposite directions and merging into one bigger wave. Note that although the waves look different from what we saw for the Boussinesq equation, the behavior is similar.
Let us now finally look at the case $b = -1$. This case is less interesting as now the only zeroes of A_{12} are when $\kappa_2 = \kappa_1$ and the only poles are when $\kappa_2 = -\kappa_1$. Again these are both of order 2, so we see that outside those two straight lines A_{12} is always larger then zero and therefore the behavior is similar to that in Figure 5.2. Therefore I do not illustrate this case here again.

5.2 Breather solutions to the shallow water wave equation

Here the situation is different from what we have seen for the Boussinesq equation. We cannot find the stationary breather in the same way as we did there, as equation (5.9) now gives us no possibilities for complex ω's with real κ's. Neither can we find the growing-and-decaying mode solution by taking $\kappa_1 = \kappa_2^* = i\kappa$ as this now corresponds to a pole of A_{12}. However, we can find the moving breather solution. I will therefore take κ_1 and κ_2 to be general complex numbers again and to make sure our solution stays real I will again require $\kappa_2 = \kappa_1^*$. Therefore I will choose $\kappa_1 = c + id$ and $\kappa_2 = c - id$ with c, d real numbers and then we get

$$\omega_1 = \omega_2^* = \frac{c(b - c^2 - d^2) + id(b + c^2 + d^2)}{(b - c^2 + d^2)^2 + 4c^2d^2} \tag{5.20}$$

and

$$A_{12} = -\frac{d^2(-c^2 + 3d^2 + 3b)}{c^2(-3c^2 + d^2 + 3b)}. \tag{5.21}$$

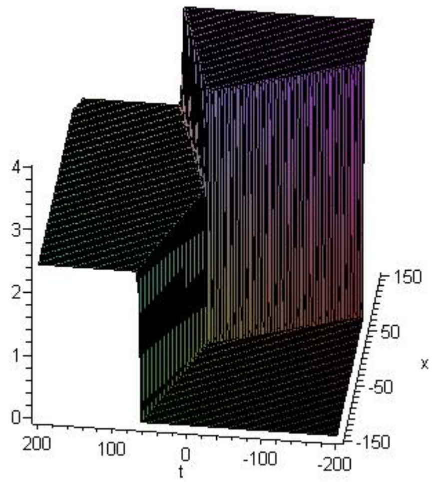

Figure 5.3: 2-soliton solution of the shallow water wave equation (5.1) with parameter values $\kappa_1 = 2$, $b = 1$ and κ_2 chosen to satisfy equation (5.18).

We can now introduce η_R and η_I such that $\eta_1 = \eta_R + i\eta_I$ and $\eta_2 = \eta_R - i\eta_I$. Note that this means $\eta_+ = \eta_R$ and $\eta_- = i\eta_I$. Again we need to split our analysis into the three different cases, A_{12} positive, negative and zero. For positive A_{12} we get in a similar fashion as before

$$u = 2\frac{c[\cos(\eta_I) + \sqrt{A_{12}}(\cosh(\eta_R) + \sinh(\eta_R))] - d\sin(\eta_I)}{\cos(\eta_I) + \sqrt{A_{12}}\cosh(\eta_R)}. \tag{5.22}$$

We first of all note that if we want a solution without poles we now must have $A_{12} > 1$. If A_{12} is negative the solution is given by

$$u = 2\frac{c[\cos(\eta_I) - \sqrt{-A_{12}}(\cosh(\eta_R) + \sinh(\eta_R))] - d\sin(\eta_I)}{\cos(\eta_I) - \sqrt{-A_{12}}\sinh(\eta_R)}. \tag{5.23}$$

and we see that in this case our solution always has poles. Finally if $A_{12} = 0$ we get the solution

$$u = \frac{\exp(\eta_R)(c\cos(\eta_I) - d\sin(\eta_I))}{1 + \exp(\eta_R)(\cos(\eta_I))}, \tag{5.24}$$

and we see that this solution also has poles. Therefore the only interesting case is $A_{12} > 1$ and we start by solving $A_{12} = 1$. We see that this gives

$$d = \pm\sqrt{c^2 - b}. \tag{5.25}$$

Furthermore we see that A_{12} has poles at

$$d = \pm\sqrt{3c^2 - 3b}, \tag{5.26}$$

which are of interest as the sign of A_{12} will change here. Again I will first look at the case $b = 1$. In that case the lines where $A_{12} = 1$ are given in Figure 5.4 in red and the poles are given in green. We see that we only have a small area of the complex plane in between the red and green hyperbola where $A_{12} > 1$ and where our solutions are therefore well behaved. The solution is plotted in Figure 5.5. We now only have one wave and around this wave we now get periodic behavior. This is to be expected as some of the hyperbolic functions

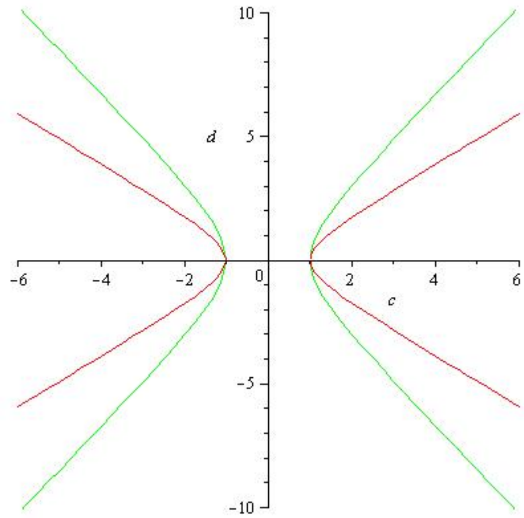

Figure 5.4: Solutions to $A_{12} = 1$ (red) and poles of A_{12} (green) for $b = 1$

were replaced by trigonometric functions. Again there is not much variation to this behavior possible for other values of c and d (as long as we keep $A_{12} > 1$). If we change the sign of c we get again a negative wave but no other changes can occur.
If we now look at the case $b = -1$ we need to again look at the behavior of A_{12} and this is given in Figure 5.6. The red lines are again solutions to $A_{12} = 1$ and the green lines are poles of A_{12} where it changes sign. We see that we again have only a small part of the complex plain in between the two hyperbola where $A_{12} > 1$ and our solution is therefore well behaved. The behavior of the solution is similar to that given in Figure 5.5 for the case $b = +1$ and I will therefore not illustrate it again.

5.3 3-soliton solutions to the shallow water wave equation

Now we can use the same techniques and ideas to find 3-soliton solutions to the shallow water wave equation (5.1). We make the ansatz

$$F_1 = e^{\eta_1} + e^{\eta_2} + e^{\eta_3}, \tag{5.27}$$

and from equations (5.5)-(5.8) we find

$$F_2 = A_{12}e^{\eta_1+\eta_2} + A_{23}e^{\eta_2+\eta_3} + A_{13}e^{\eta_1+\eta_3}, \tag{5.28}$$

$$F_3 = B_{123}e^{\eta_1+\eta_2+\eta_3}, \tag{5.29}$$

$$F_n = 0 \qquad \text{for } n > 3, \tag{5.30}$$

where η_i and A_{ij} are as before, and $B_{123} = A_{12}A_{23}A_{13}$. Now that we have found the 3-soliton solutions, we can investigate what they look like.
Again I will start with $b = +1$ and before we can rewrite our solution we must again first consider the signs of the A_{ij}. As negative A_{ij} will lead to singularities as we have seen before, I will only consider positive or zero values. If we choose all A_{ij} positive we get after some rewriting

$$u = \Bigg\{ \sqrt{A_{12}A_{23}A_{13}}(\kappa_1+\kappa_2+\kappa_3)(\cosh(\eta_+)+\sinh(\eta_+)) + \sum_{i<j\neq k} \Big[(\kappa_i+\kappa_j-\kappa_k)\sqrt{A_{ij}}\sinh(\eta_{ijk})\Big]$$

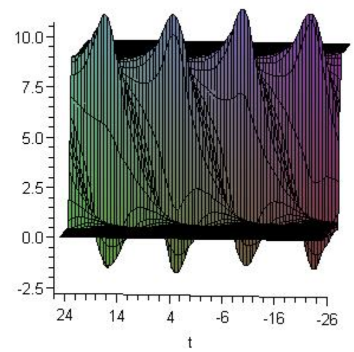

Figure 5.5: Breather solution of the shallow water wave equation (5.1) with parameter values $c = 2$, $d = 2$ and $b = +1$.

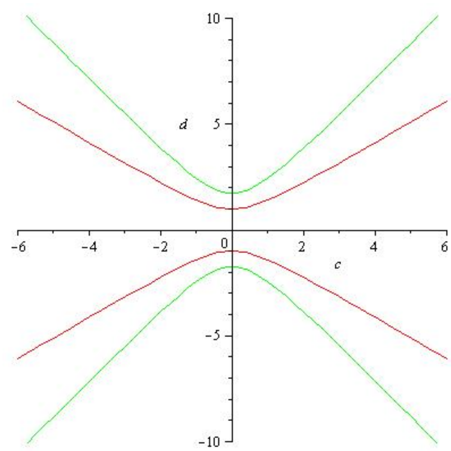

Figure 5.6: Solutions to $A_{12} = 1$ (red) and poles of A_{12} (green) for $b = -1$

$$
\begin{aligned}
&+ \sum_{i<j\neq k} \left[(\kappa_i + \kappa_j + \kappa_k) \sqrt{A_{ij}} \cosh(\eta_{ijk}) \right] \Bigg\} \qquad (5.31) \\
&/ \left(\sqrt{A_{12} A_{23} A13} \cosh(\eta_+) + \sum_{i<j\neq k} \left[(\kappa_i + \kappa_j + \kappa_k) \sqrt{A_{ij}} \cosh(\eta_{ijk}) \right] \right),
\end{aligned}
$$

where $\eta_+ = (\eta_1 + \eta_2 + \eta_3)/2$, $\eta_{ijk} = (\eta_i + \eta_j - \eta_k)/2$ and the summation is over $i, j, k \in \{1, 2, 3\}$ such that $i < j$, $j \neq k$ and $i \neq k$. Next I will investigate what happens if one of the A_{ij} is zero. So I choose $A_{12} = 0$ and the other two to be positive and then we get after some more rewriting

$$
\begin{aligned}
u &= \left\{ \kappa_3 \left[\cosh\left(\frac{\eta_3}{2}\right) + \sinh\left(\frac{\eta_3}{2}\right) \right] + \exp\left(\frac{\eta_1 + \eta_2}{2}\right) \left[(\kappa_1 + \kappa_2 + \kappa_3) \left(\sqrt{A_{23}} \cosh(\eta_{231}) + \sqrt{A_{13}} \cosh(\eta_{132}) \right) \right. \right. \\
&+ \left. \left. (\kappa_2 + \kappa_3 - \kappa_1) \sqrt{A_{23}} \sinh(\eta_{231}) + (\kappa_1 + \kappa_3 - \kappa_2) \sqrt{A_{13}} \sinh(\eta_{132}) \right] \right\} \qquad (5.32)
\end{aligned}
$$

$$/ \quad \left(\cosh\left(\frac{\eta_3}{2}\right) + \exp\left(\frac{\eta_1+\eta_2}{2}\right)\left[\sqrt{A_{23}}\cosh(\eta_{231}) + \sqrt{A_{13}}\cosh(\eta_{132})\right]\right).$$

If we now choose two of the A_{ij} to be zero (I take A_{12} and A_{23}) and the third one to be positive we get

$$\begin{aligned} u &= \left\{2\exp\left(\frac{\eta_1+\eta_3}{2}\right)\left[(\kappa_1+\kappa_3)\cosh\left(\frac{\eta_1-\eta_3}{2}\right) + (\kappa_1-\kappa_3)\sinh(\eta_1-\eta_3 2)\right.\right. \\ &+ \left.\left.\sqrt{A_{13}}\exp\left(\frac{\eta_2}{2}\right)((\kappa_1+\kappa_2+\kappa_3)\cosh(\eta_{132}) + (\kappa_1-\kappa_2+\kappa_3)\sinh(\eta_{132}))\right]\right\} \\ &/ \left(1+2\exp\left(\frac{\eta_1+\eta_3}{2}\right)\left((\kappa_1+\kappa_3)\cosh\left(\frac{\eta_1-\eta_3}{2}\right) + \sqrt{A_{13}}\exp\left(\frac{\eta_2}{2}\right)\cosh(\eta_{132})\right)\right). \end{aligned} \tag{5.33}$$

If we finally choose all three A_{ij} to be zero we get

$$u = \frac{\kappa_1 e^{\eta_1} + \kappa_2 e^{\eta_2} + \kappa_3 e^{\eta_3}}{1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3}}. \tag{5.34}$$

Note that this last solution can also be rewritten in terms of cosh-functions; however as this rewriting does not reduce the number of exponential functions necessary and does not simplify it I do not do this here. As we already know the places of the zeroes of the A_{ij} in the (κ_i, κ_j)-plane (see Figure 5.2) we can now examine the solutions derived above numerically. If we plot the solution given in equation (5.32), so for all A_{ij} nonzero and positive, we get the result as shown in Figure 5.7. We see that we have here a situation

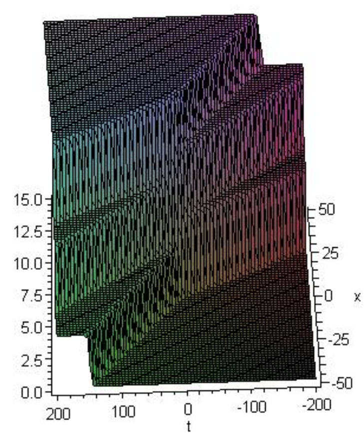

Figure 5.7: 3-soliton solution of the shallow water wave equation (5.1) with all $A_{ij} > 0$.

very similar to the 2-soliton case, only now we have 3 incoming solitons, with the bigger ones again traveling slower and they all interact in the origin. After the interaction they proceed unperturbed but the order has reversed as the smaller waves have overtaken the bigger ones. The situation becomes less trivial when we put one of the A_{ij} to zero. This situation (corresponding to equation (5.33)) is given in Figure 5.8. We see that we now have two incoming solitons, one from positive and one from negative x-direction. When they meet in the origin they break up into three waves, two traveling in the positive and one in the negative x-direction. Note that this can be seen just as adding an extra soliton to the 2-soliton solution with zero A_{12}. Therefore the next two cases will be more interesting as they do not have a 2-soliton analogue. The case for two A_{ij} equal to zero is depicted in Figure 5.9. We see that we have two solitons coming in from the positive x-direction, the larger one trailing the smaller one. The larger one then breaks up into two solitons traveling in opposite directions. The largest of those two continues in the negative x-direction and when it almost overtakes the smaller original soliton, this breaks up into two new waves again. The smaller of those travels in the opposite direction and is absorbed into the soliton that was trailing it. The situation changes

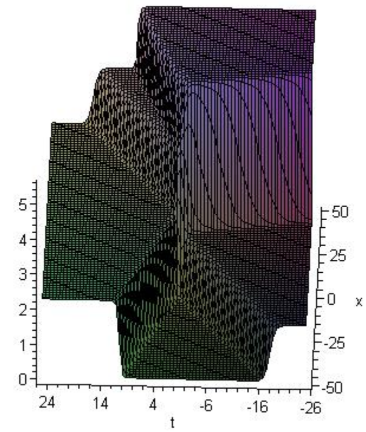

Figure 5.8: 3-soliton solution of the shallow water wave equation (5.1) with $A_{12} = 0$ and the other $A_{ij} > 0$.

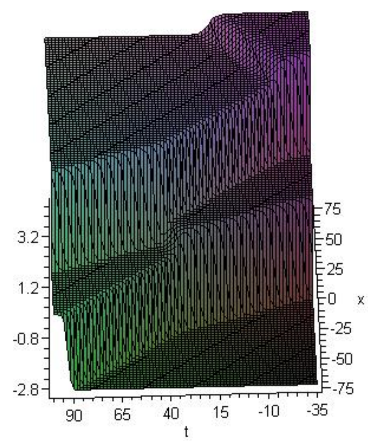

Figure 5.9: 3-soliton solution of the shallow water wave equation (5.1) with $A_{12} = A_{23} = 0$ and $A_{13} > 0$.

again when all three A_{ij} are zero and this is depicted in Figure 5.10. We see that we now have incoming solitons traveling in opposite directions which interact at $x = 0$. When they meet they form a larger wave which stays stationary for a short time before breaking up into two waves traveling in opposite directions again. We see that effectively there are only two solitons here. This is because in order to get all three A_{ij} zero, you need to have two κ_i equal. However as we did not see this kind of behavior in the 2-soliton case I still included it here. We note although that this behavior can be split up into two kinds of behavior that we did observe for the 2-soliton solution with $A_{12} = 0$, i.e. the merging of two incoming waves into a larger wave and the splitting up of a larger wave into two smaller ones. As a final note I want to stress the similarity in behavior with the 3-soliton solutions to the Boussinesq equation observed earlier. We see that although the solitons look different, their qualitative behavior is the same.
The case for $b = -1$ is again not very interesting. As all the A_{ij} are always larger then zero (outside the "trivial" lines $\kappa_i = \pm\kappa_j$), nothing special can happen here again and the behavior is similar to that seen in Figure 5.7.

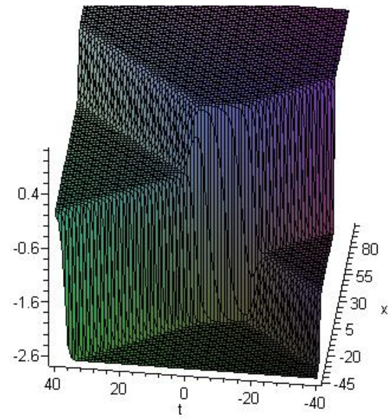

Figure 5.10: 3-soliton solution of the shallow water wave equation (5.1) with all $A_{ij} = 0$.

5.4 Breather soliton interaction

Next we can look at the interaction between a breather and a soliton. To do this I choose $\kappa_1 = \kappa_2^* = c + id$ with c and d real and also κ_3 to be real. We see that now A_{12} is real and given by equation (5.21), but A_{13} and A_{23} are complex numbers. However we do note that $A_{13}^* = A_{23}$ and therefore our final solution will be real as we will see shortly. If we now take η_R and η_I as before we get for F:

$$\begin{aligned} F &= 1 + e^{\eta_1} + e^{\eta_2 +} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{13}e^{\eta_1+\eta_3} + A_{23}e^{\eta_2+\eta_3} + A_{12}A_{13}A_{23}e^{\eta_1+\eta_2+\eta_3} \\ &= 1 + 2e^{\eta_R}\cos(\eta_I) + e^{\eta_3} + A_{12}e^{2\eta_R} + 2e^{\eta_R}(\mathrm{Re}(A_{13})\cos(\eta_I) - \mathrm{Im}(A_{13})\sin(\eta_I)) + A_{12}|A_{13}|^2 e^{2\eta_r+\eta_3} \\ &= 2e^{\eta_R+\frac{\eta_3}{2}}\Big(\sqrt{A_{12}}|A_{13}|\cosh(\eta_R + \frac{\eta_3}{2}) + 2\sqrt{\mathrm{Re}(A_{13})}\cos(\eta_I)\cosh(\frac{\eta_3}{2}) \qquad (5.35) \\ &+ \sqrt{A_{12}}\cosh(\eta_R - \frac{\eta_3}{2}) - \mathrm{Im}(A_{13})e^{\eta_r/2}\sin\eta_I\Big), \end{aligned}$$

where I assumed that $A_{12} > 0$ and $\mathrm{Re}(A_{13}) > 0$ in the last line (otherwise the cosh turns into a sinh). Note that this is indeed a real valued solution. It is now an easy exercise to find $u = F_x/F$; however I will not give the full form here as it is very large and not very instructive. Furthermore, we see that it is now not clear what demands we have to make on the values of the A_{ij} in order to have a well behaved solution. However a numerical investigation with MAPLE showed that if for $b = +1$ we choose c and d in the area between the hyperbola in Figure 5.4, it seems that we then get solutions without poles. This situation is given in Figure 5.11. We see that we have obtained an interaction between the breather obtained earlier and a soliton. The situation for $b = -1$ is similar.

5.5 Rational solutions

Here I will apply the method first developed by Ablowitz and Satsuma [3] to find rational solutions to the SWWE. We note from the discussion above that the 1-, 2- and 3-soliton solutions are given by

$$\begin{aligned} F^1 &= 1 + e^{\eta_1}, && (5.36) \\ F^2 &= 1 + e^{\eta_1} + e^{\eta_2} + A_{12}e^{\eta_1+\eta_2}, && (5.37) \\ F^3 &= 1 + e^{\eta_1} + e^{\eta_2} + e^{\eta_3} + A_{12}e^{\eta_1+\eta_2} + A_{23}e^{\eta_2+\eta_3} + A_{13}e^{\eta_1+\eta_3} + A_{12}A_{23}A_{13}e^{\eta_1+\eta_2+\eta_3}, && (5.38) \end{aligned}$$

where F^n denotes the F corresponding to the n-soliton solution, with η_i as before and A_{ij} as given in equation (5.10). Now I introduce $\alpha_i = e^{\eta_i^0}$ and we start by looking at the 1-soliton solution in the limit

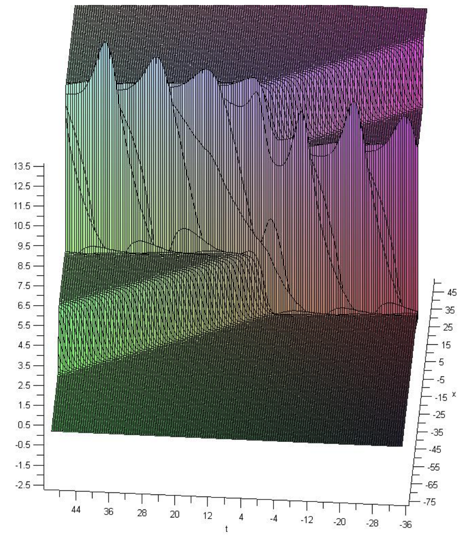

Figure 5.11: 3-soliton solution of the shallow water wave equation (5.1) with $c = 2$, $d = 2$ and $\kappa_3 = 1.5$.

$\kappa_1 \to 0$. We see from equation (5.36) that we get for $\kappa_1 \to 0$

$$F^1 = 1 + \alpha_1(1 + \kappa_1(x - bt)) + O(\kappa_1^2). \tag{5.39}$$

If we now choose $\alpha_1 = -1$ we see that the zeroth order in κ_1 cancels and we get for u_1

$$u_1 = 2(\ln F^1)_x = \frac{2}{x - bt}. \tag{5.40}$$

We can easily verify directly that this is indeed a solution to equation (5.1). So we see that we have found a rational solution to the shallow water wave equation, but unfortunately it does have a pole. We can do the same thing for the 2-soliton solution. For this I will replace the κ_i by $\epsilon\kappa_i$ and take the limit $\epsilon \to 0$. First of all we then see that for A_{12} we have

$$A_{12} = \left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)^2 \left(1 + \frac{2}{3}\epsilon^2 b\kappa_1\kappa_2 + O(\epsilon^3)\right) \tag{5.41}$$

and motivated by the example of the Boussinesq equation in [3] we now choose

$$\alpha_1 = \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} + \frac{1}{3}\epsilon^2 b\kappa_1\kappa_2 \qquad \text{and} \qquad \alpha_2 = -\frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} + \frac{1}{3}\epsilon^2 b\kappa_1\kappa_2. \tag{5.42}$$

If we insert all this in equation (5.37) and expand the exponents, we see that the zeroth, first and second order in ϵ now vanishes. We therefore find for F^2

$$F^2 = \epsilon^3 \frac{\kappa_1\kappa_2(\kappa_1 + \kappa_2)}{6}(-x^3 + 3x^2bt - 2bx - 3xt^2 - 10t + bt^3) \tag{5.43}$$

and this gives us the solution

$$u_2 = \frac{-6x^2 + 6xbt - 2b - 3t^2}{-x^3 + 3x^2bt - 2bx - 3xt^2 - 10t + bt^3}. \tag{5.44}$$

Again it is an easy check that this is indeed a solution to equation (5.1). Furthermore our solution has poles again. We can extend this method to the 3-soliton solution as well, although the algebra now gets more and more complicated as we need to go into even higher orders of ϵ. We start from the 3-soliton solution given in equation (5.38) and as before we replace the κ_i by $\epsilon\kappa_i$. Now we make the Taylor approximation of F^3 around the point $\epsilon = 0$ and we will need it up to sixth order in ϵ. Furthermore we will also need to approximate the A_{ij} up to sixth order in ϵ and this becomes

$$A_{ij} = \frac{(\kappa_i - \kappa_j)^2(27 + 18b\kappa_i\kappa_j\epsilon^2 + 6(\kappa_i^3\kappa_j + \kappa_i^2\kappa_j^2 + \kappa_i\kappa_j^3)\epsilon^4 + 2b(\kappa_i^5\kappa_j + 2\kappa_i^4\kappa_j^2 + 3\kappa_i^3\kappa_j^3 + 2\kappa_i^2\kappa_j + \kappa_i\kappa_j^5)\epsilon^6)}{27(\kappa_i + \kappa_j)^2}.$$

Next we need to choose the a_i and this can be done as follows:

$$\begin{aligned}
a_1 &= \left(\frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2} - \frac{b\epsilon^2}{3}\left(\kappa_1\kappa_2 - \frac{2\kappa_1\kappa_2\kappa_3}{\kappa_1 + \kappa_2 + \kappa_3}\right)\right)\left(\frac{\kappa_1 + \kappa_3}{\kappa_3 - \kappa_1} + \frac{b\epsilon^2}{3}\left(\kappa_1\kappa_3 - \frac{2\kappa_1\kappa_2\kappa_3}{\kappa_1 + \kappa_2 + \kappa_3}\right)\right) + c_1\epsilon^2 \\
a_2 &= \left(\frac{\kappa_1 + \kappa_2}{\kappa_2 - \kappa_1} - \frac{b\epsilon^2}{3}\left(\kappa_1\kappa_2 - \frac{2\kappa_1\kappa_2\kappa_3}{\kappa_1 + \kappa_2 + \kappa_3}\right)\right)\left(\frac{\kappa_2 + \kappa_3}{\kappa_3 - \kappa_2} - \frac{b\epsilon^2}{3}\left(\kappa_1\kappa_3 - \frac{2\kappa_1\kappa_2\kappa_3}{\kappa_1 + \kappa_2 + \kappa_3}\right)\right) + c_2\epsilon^2 \\
a_3 &= \left(\frac{\kappa_1 + \kappa_3}{\kappa_3 - \kappa_1} - \frac{b\epsilon^2}{3}\left(\kappa_1\kappa_3 - \frac{2\kappa_1\kappa_2\kappa_3}{\kappa_1 + \kappa_2 + \kappa_3}\right)\right)\left(\frac{\kappa_2 + \kappa_3}{\kappa_2 - \kappa_3} + \frac{b\epsilon^2}{3}\left(\kappa_1\kappa_3 - \frac{2\kappa_1\kappa_2\kappa_3}{\kappa_1 + \kappa_2 + \kappa_3}\right)\right) + c_3\epsilon^2,
\end{aligned}$$

where the c_i are constants depending on the κ_i but not on x, t or ϵ. We can find c_1, c_2 and c_3 respectively by requiring that the third, forth and fifth order terms in ϵ in the Taylor approximation of equation (5.38) cancel. This is easily done by a computer programme such as MAPLE (which I used) and as the results are very lengthy and not very intuitive, I will not give them here. If we insert this all into equation (5.38) we get

$$F^3 \sim C\left(3(x - bt)^6 + 30bx^4 + 60x^3t - 360bx^2t^2 + 420xt^3 - 150bt^4 - 60x^2 - 1680bxt - 420t^2 - 280b\right)\epsilon^6,$$

where C is a constant only depending on the κ_i and whose exact form is not important as we are only interested in $u_3 = (\ln F^3)_x$. So now we have found the third rational solution, but given the amount of algebra involved here, it is not very practical to proceed any further in this way. In [3] the authors give a recursion formula for the rational solutions obtained for the KdV-equation in this way. However it is very hard to find a similar relation for the SWWE. So far I have been unable to find one and wonder if one can be found in the way described in [3]. I will not go into the problems that arise any further here. I do want to make one final remark about the rational solutions obtained in this part and this is that we can also apply our limiting procedure to the breather solution given in equation (5.22). To do this we now take $c = \epsilon\gamma$ and $d = \epsilon\delta$ and we will be looking at the limit $\epsilon \to 0$. It is not hard to see that equation (5.21) now reduces to

$$A_{12} = -\frac{\delta^2}{\gamma^2} - \frac{2\delta^2(\gamma^2 + \delta^2)}{\gamma^2 b}\epsilon^2. \tag{5.45}$$

However if we now proceed in the same way as before we will get the same rational solution as we derived earlier in the 2-soliton case. So I will not go into this case any further and end my research into rational solutions to the SWWE here. In the next part I will derive one more solution to the SWWE by making a new ansatz and inserting this into the bilinear form.

5.6 New soliton solution

We have seen that we can find the one-soliton solution by making the ansatz $F_1 = \exp(\eta_1)$ with $\eta_i = \kappa_i x - \omega_i t + \eta_i^0$ and $F_n = 0$ for $n > 1$ and inserting this in equation (5.5). Furthermore we saw that this can then be extended to find all N-soliton solutions. However we should note that this is only one of the special solutions to equation (5.5). Here I will make a different ansatz, motivated by [30] I choose

$$F_1 = (\alpha x + \beta t + \gamma)e^{\kappa x - \omega t} \tag{5.46}$$

and if we insert this in equation (5.5) we find that

$$\omega = \frac{\kappa}{1-\kappa^2} \qquad \text{and} \qquad \beta = \frac{-\alpha(\kappa^2+1)}{(\kappa^2-1)^2}. \tag{5.47}$$

I will limit myself to the case $b = 1$ here, as this is the only case in which we get decent solutions. If we now insert this into equation (5.6) we get the equation

$$\left(\frac{\partial^4}{\partial t \partial x^3} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t \partial x}\right) F_2 = -\alpha(\beta+\alpha)e^{2(\kappa x - \omega t)} \tag{5.48}$$

and this is solved by

$$F_2 = \frac{-\alpha^2(\kappa^2-3)}{12\kappa^2(\kappa^2-1)} e^{2(\kappa x - \omega t)}. \tag{5.49}$$

It is now easily verified that equation (5.8) is satisfied for all n if we take $F_n = 0$ for $n > 2$. So we have found our F satisfying (5.3). Note however that now we cannot just leave the ϵ out, because we no longer have the ϵ_i^0-terms in which we included them earlier. In order to rewrite our solution it will be necessary to require that $A = \frac{-12\kappa^2(\kappa^2-1)}{\alpha^2(\kappa^2-3)}$ is positive (thus require that $1 < \kappa < \sqrt{3}$). The necessity of this will become clear later. If we now introduce $x_0 = -\frac{1}{\kappa}\ln\left(\frac{\epsilon}{\sqrt{A}}\right)$ we can rewrite F in the form

$$F = e^{\kappa(x-x_0)-\omega t}(2\cosh(\kappa(x-x_0)-\omega t) + \sqrt{A}(\alpha x + \beta t + \gamma)). \tag{5.50}$$

And from this we see that our solution becomes

$$u = 2\frac{F_x}{F} = 2\frac{2\kappa(\cosh(\kappa(x-x_0)-\omega t) + \sinh(\kappa(x-x_0)-\omega t)) + \sqrt{A}(a + \kappa(ax+bt+c))}{2\cosh(\kappa(x-x_0)-\omega t) + \sqrt{A}(ax+bt+c)}. \tag{5.51}$$

Note that it is easy to verify directly that this is indeed a solution. Furthermore we now see why I imposed $A > 0$. If not the cosh-term in F would have turned into a sinh, which would have created poles in u and furthermore ϵ would have become complex and therefore we would have got a complex solution. Now we see that u is finite as long as F is non-zero. However as F contains terms linear in both x and t we see that it will always have zeroes for some parts of the x, t-plane. Although if $a > 0$ we see that our solution is finite in the part of the plane where $x > 0$ and $t < 0$ and for large values of c in an even larger area. Similarly we see that for $a < 0$ u is finite for $x < 0$ and $t > 0$. These two situations are illustrated in Figure 5.12. We see that we have a 2-soliton solution here but unfortunately we cannot examine how the interaction goes as it has infinities as soon as we go through the origin.
We can find another solution if we put $A = 0$ (this corresponds to $\kappa = \sqrt{3}$). We then find

$$u = \frac{a + \kappa(ax+bt+c)}{e^{-\kappa x - \omega t} + ax + bt + c} \tag{5.52}$$

and this solution is illustrated in Figure 5.13 (note that it is well behaved in the same regions of (x, t)-space as before).

5.7 Discussion

The soliton solutions were first obtained for the SWWE as given in equation (5.1) by Hirota and Satsuma [17]. However they looked at a different form of the equation given by

$$v_t - v_{xxt} - 3vv_t + 3v_x \int_x^\infty v_t \mathrm{d}x + v_x = 0, \tag{5.53}$$

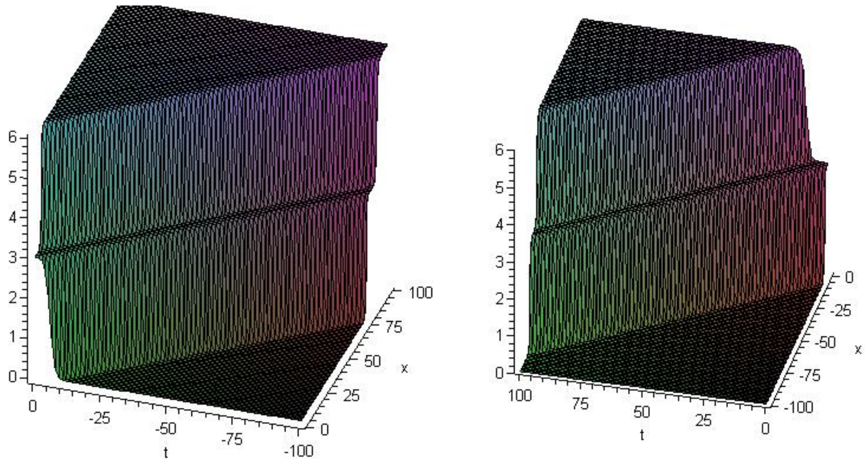

Figure 5.12: Special solution given in equation (5.51) of the shallow water wave equation (5.1) with parameter values $\kappa = 1.5$, $c = 0$, $x_0 = 0$ and $a = 1$ (left) and $a = -1$ (right).

which reduces to equation (5.1) under the substitution $v = u_x$. Therefore their solutions are given by the x-derivative of mine and look quite different. They look more similar to the solutions obtained for the Boussinesq equation, with the solution returning to 0 for $|x| \to \infty$. The method of bilinearization is exactly the same and they obtain the same 1-, 2- and 3-soliton solutions, but they do not obtain the breather solutions nor do they study the dependence of the behavior of the solutions on the sign of A_{12}. I have not found an article in the literature where the breather solutions are obtained from a similar method as I obtained them. I suspect that this has been done before as the method I used dates back to at least 1989, when it was introduced to obtain breather solutions of the Boussinesq equation by Tajiri and Murakami [26]. Similar solutions have been obtained from non-classical symmetry reductions by Clarkson and Mansfield [7]. Furthermore Li and Zhao claim to have obtained them from a new method, called the extend homoclinic test technique (EHTT), in a very recent article [22]. From this group a number of articles has appeared during the last 3 years in which they apply the EHTT to a number of equations. However in my opinion they have not obtained any results that can not be obtained easier and more completely by the method introduced by Tajiri and Murakami [26]. In appendix A I will show how the results obtained from the EHTT for both the KdV and the SWWE can be obtained more easily and completely from this technique. The results I obtained in sections 5.5 and 5.6 are new as far as I can tell. As said before the methods used are not new and the solutions are not very well behaved as they still have a number of infinities. Nevertheless the new solution obtained in Section 5.6 shows that we can still find new solutions using the bilinear method. With these results I have concluded my study of the SWWE. In the next section I will take a look at the nonlinear Schrödinger equation (NLS), which has a bilinear form consisting of two coupled equations. However we will see that a lot of ideas we saw for the Boussinesq and shallow water wave equation still apply.

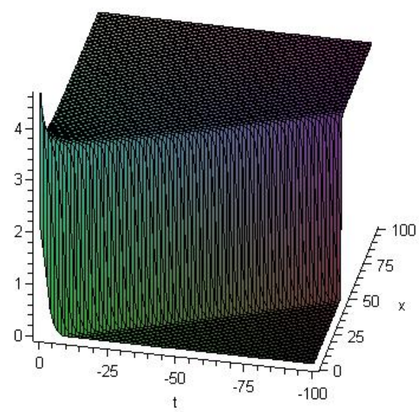

Figure 5.13: Special solution given in equation (5.52) of the shallow water wave equation (5.1) with parameter values $c = 0$, $x_0 = 0$ and $a = 1$ (left)

Chapter 6

The Nonlinear Schrödinger Equation

In this section I will present the results of my study of the nonlinear Schrödinger equation (NLS) given by

$$iu_t + u_{xx} + p|u|^2 u = 0, \tag{6.1}$$

where $p = \pm 1$. For $p = +1$ this is called the focusing NLS (FNLS) and for $p = -1$ the defocusing NLS (DNLS). I will study this using the Hirota bilinear form and I will bilinearize equation (6.1) in two different ways. I will start with the most obvious way, which will give us the soliton solutions for the FNLS. However as we will see, this does not work for the DNLS and therefore I will use a second bilinearization for this. I will also derive a special solution to the FNLS from the first bilinearization and from the second bilinearization breather and rational solutions to the FNLS will be found as well as rational solutions to the DNLS. I will conclude by showing that for the FNLS breather solutions can be written as imbricate series of rational growing-and-decaying mode solutions.

6.1 Soliton solutions to the FNLS

To find the soliton solutions to the FNLS, I will use the standard bilinear form of the NLS first derived by Hirota [16]. We start by substituting

$$u = \frac{G}{F} \tag{6.2}$$

in equation (6.1), where G and F are both functions of x and t, with the constraint that F is a real valued function. We see that it then splits up into two equations

$$(iD_t + D_x^2)(G \bullet F) = 0 \tag{6.3}$$

and

$$D_x^2(F \bullet F) - p|G|^2 = 0. \tag{6.4}$$

Now we make the following assumptions for F and G:

$$\begin{aligned} F &= 1 + \epsilon F_1 + \epsilon^2 F_2 + \dots, & (6.5)\\ G &= \epsilon G_1 + \epsilon^2 G_2 + \dots, & (6.6) \end{aligned}$$

where ϵ is a formal expansion parameter, not necessarily small. Inserting this into equations (6.3) and (6.4) and splitting up into order in ϵ, we get a set of recursion equations

$$(i\partial_t + \partial_x^2)G_1 = 0, \tag{6.7}$$

$$\begin{aligned}
2\partial_x^2 F_1 &= 0, && (6.8)
\end{aligned}$$

$$\begin{aligned}
(i\partial_t + \partial_x^2)G_2 &= -(iD_t + D_x^2)(G_1 \bullet F_1), && (6.9)\\
2\partial_x^2 F_2 &= -D_x^2(F_1 \bullet F_1) + p|G_1|^2, && (6.10)
\end{aligned}$$

$$\begin{aligned}
(i\partial_t + \partial_x^2)G_n &= -(iD_t + D_x^2)(\sum_{m=1}^{n-1} G_m \bullet F_{n-m}), && (6.11)\\
2\partial_x^2 F_n &= -D_x^2(\sum_{m=1}^{n-1} F_m \bullet F_{n-m}) + p\sum_{m=1}^{n-1} G_m G_{n-m}^*. && (6.12)
\end{aligned}$$

I will now first start by deriving the 1-soliton solution in the next subsection and after that I will derive the 2-soliton solution.

6.1.1 1-soliton solution

To derive the 1-soliton solution I will make the following ansatz

$$\begin{aligned}
F_1 &= 0, && (6.13)\\
G_1 &= e^{\eta_1}, && (6.14)
\end{aligned}$$

where $\eta_i = \kappa_i x - \omega_i t + \eta_i^0$ as always. Note that as we now do not require our final solution to be real (as our equation is not real itself), all the constants involved can be complex. We see that equation (6.8) is obviously satisfied and that from equation (6.7) we get

$$\omega_i = -i\kappa_i^2. \tag{6.15}$$

From equation (6.9) we see that $G_2 = 0$ and from (6.10) we find

$$F_2 = A_{11} e^{\eta_1 + \eta_1^*}, \tag{6.16}$$

where

$$A_{ij} = \frac{p}{2(\kappa_i + \kappa_j^*)^2}. \tag{6.17}$$

Note that F_2 is indeed real valued as required. Furthermore we see from equations (6.11) and (6.12) that $G_n = F_n = 0$ for $n \geq 3$. So we have now found the 1-soliton solution and we see from (6.2) that it is given by

$$u = \frac{e^{\eta_1}}{1 + A_{11} e^{\eta_1 + \eta_1^*}}, \tag{6.18}$$

where I now took the ϵ into η_1^0. In the case $p = +1$ (so the FNLS) we see that $A_{11} > 0$ and therefore we can rewrite 6.18 into

$$u = \frac{\sqrt{2}a \exp(i(bx + (a^2 - b^2)t))}{\cosh(a(x - 2bt) - \eta_1^0)}, \tag{6.19}$$

where I introduced $\kappa_i = a + ib$, with a and b real. We note that this is a well-behaved function without poles. However in the case of $p = -1$ (DNLS) we get $A_{11} < 0$ and therefore

$$u = -\frac{\sqrt{2}a \exp(i(bx + (a^2 - b^2)t))}{\sinh(a(x - 2bt) - \eta_1^0)}. \tag{6.20}$$

We see that in this case our solution is not well behaved and therefore it does not seem possible to derive the 1-soliton solution to the DNLS in this bilinearization. In the next section I will therefore look at another bilinearization which is more successful. Furthermore from equation (6.19) we note that a controls the height of the soliton and b its speed. So in contrast to the soliton solutions to the Boussinesq and shallow water wave equation we have seen so far, it is now possible to control both the size and speed of the solitons independent of each other.

6.1.2 2-soliton solution

To find the 2-soliton solution we now start with the ansatz

$$F_1 = 0, \tag{6.21}$$
$$G_1 = e^{\eta_1} + e^{\eta_2}, \tag{6.22}$$

with η_i and ω_i as before. We see that they satisfy equation (6.7) and (6.8) and from equations (6.9)-(6.12) we find

$$G_2 = 0, \tag{6.23}$$
$$F_2 = A_{11}e^{\eta_1+\eta_1^*} + A_{12}e^{\eta_1+\eta_2^*} + A_{21}e^{\eta_2+\eta_1^*} + A_{22}e^{\eta_2+\eta_2^*} \tag{6.24}$$
$$G_3 = B_{121}e^{\eta_1+\eta_2+\eta_1^*} + B_{122}e^{\eta_1+\eta_2+\eta_2^*} \tag{6.25}$$
$$F_3 = 0 \tag{6.26}$$
$$G_4 = 0 \tag{6.27}$$
$$F_4 = C_{1212}e^{\eta_1+\eta_2+\eta_1^*+\eta_2^*} \tag{6.28}$$

and $F_n = G_n = 0$ for $n \geq 5$, where A_{ij} is as in equation (6.17),

$$B_{ijk} = \frac{p(\kappa_i - \kappa_j)^2}{2(\kappa_i + \kappa_k^*)^2(\kappa_j + \kappa_k^*)^2} \tag{6.29}$$

and

$$C_{ijkl} = \frac{p^2(\kappa_i - \kappa_j)^2(\kappa_k^* - \kappa_l^*)^2}{4(\kappa_i + \kappa_k^*)^2(\kappa_i + \kappa_l^*)^2(\kappa_j + \kappa_k^*)^2(\kappa_j + \kappa_l^*)^2}. \tag{6.30}$$

We note that because of the symmetry in these constants both F_2 and F_4 are real valued functions as required. Furthermore we note that $C_{1212} > 0$ and that the sign of A_{11} and A_{22} is the same as the sign of p. We also see that $\text{Re}(A_{12})$=$\text{Re}(A_{21})$ and that this can change sign as function of the κ's for a fixed value of p. Similarly the signs of $\text{Re}(B_{121})$ and $\text{Re}(B_{122})$ can change without changing p. If we assume these last two to be positive and $p = +1$ we can write the solution as

$$u = \frac{\sqrt{B_{122}}e^{i\eta_{1,I}}\cosh(\eta_{2,R}) + \sqrt{B_{121}}e^{i\eta_{2,I}}\cosh(\eta_{1,R})}{\sqrt{C_{1212}}\cosh(\eta_{1,R}+\eta_{2,R}) + \sqrt{A_{11}A_{22}}\cosh(\eta_{1,R}-\eta_{2,R}) \pm \sqrt{A_{12}A_{21}}\cos(\eta_{1,I}-\eta_{2,I})}, \tag{6.31}$$

where the $\pm$ corresponds to the sign of $\text{Re}(A_{12})$, $\eta_{i,R} = \frac{\eta_i+\eta_i^*}{2}$ and $\eta_{i,I} = \frac{\eta_i-\eta_i^*}{2}$. We see that the solution is well behaved if $\sqrt{C_{1212}} + \sqrt{A_{11}A_{22}} > \sqrt{A_{12}A_{21}}$, which is always the case as can be seen from a simple calculation. Furthermore when the sign of $\text{Re}(B_{121})$ or $\text{Re}(B_{122})$ changes then the corresponding cosh gets replaced by -sinh. If $p = -1$ then the second cosh-term in the denominator picks up a minus sign and we see that our solution therefore is singular again. So again we see that we only find well behaved solutions to the FNLS in this way.

Next I will investigate what the solutions of equation (6.1) look like depending on different values of the parameters. I will do this for the cases given in the table below, where I introduced $\kappa_j = a_j + ib_j$, with a_j and b_j real.

case	a_1	a_2	b_1	b_2	$\text{Re}(A_{12})$	$\text{Re}(B_{122})$	$\text{Re}(B_{121})$
(i)	0.12	1	0.1	-2	< 0	> 0	> 0
(ii)	-0.1	-0.1	0.15	0.2	< 0	< 0	> 0
(iii)	0.12	0.1	0.1	0.2	> 0	< 0	< 0
(iv)	0.2	0.1	0.1	0.2	> 0	> 0	< 0
(v)	-0.2	0.1	0.1	0.2	$= 0$	< 0	> 0
(vi)	0.1	0.15	-0.1	-0.11	> 0	> 0	> 0
(vii)	0.1	0.15	-0.1	-0.1	> 0	> 0	> 0
(viii)	0.1	0.15	0	0	> 0	> 0	> 0

I will now illustrate these cases in Figure 6.1. In case (i) and (ii) we see the 2-soliton behavior we know from

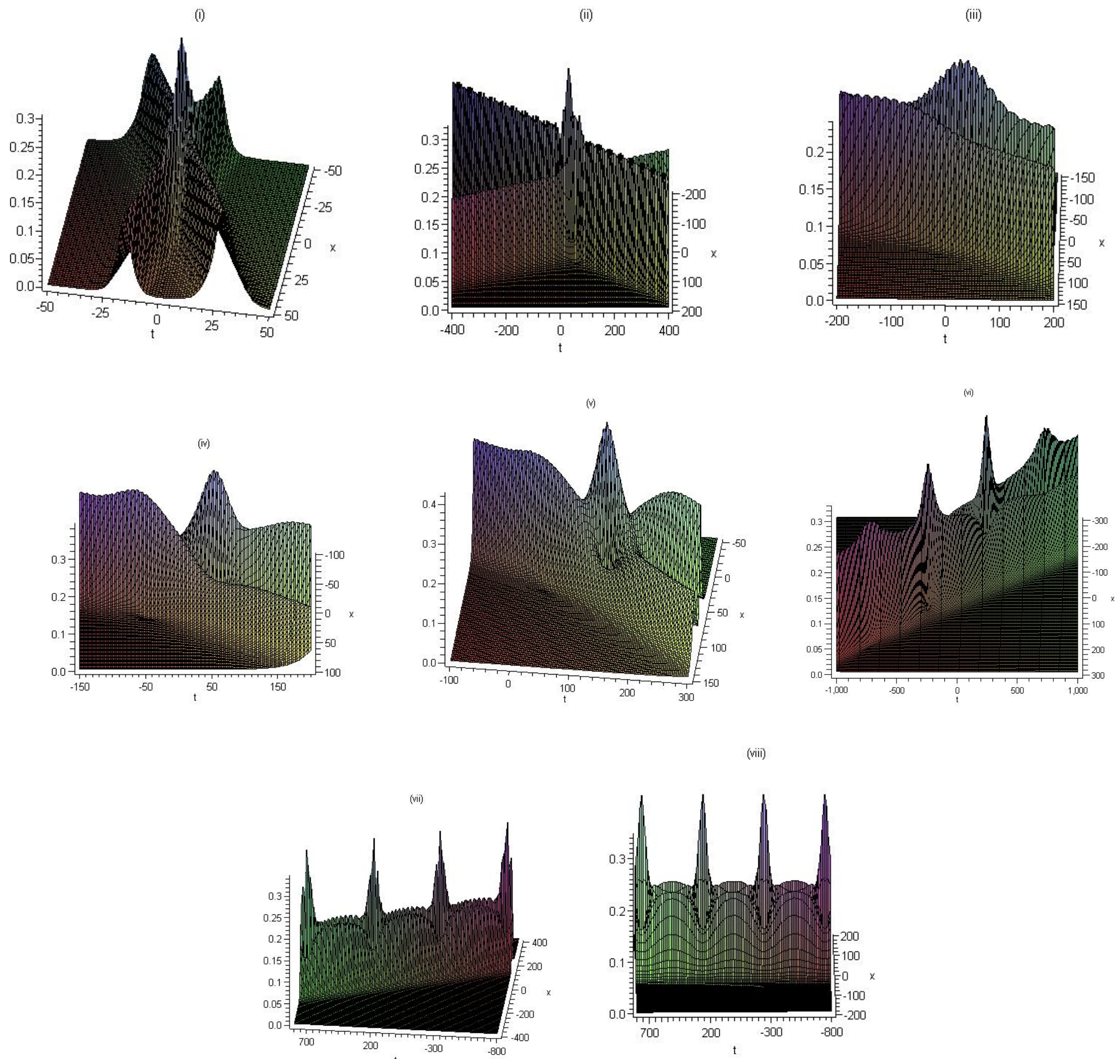

Figure 6.1: Absolute value of 2-soliton solutions to the FNLS for the parameter values shown in the table.

other equations, with only a limited interaction around the point where the two solitons collide. In case (iii) and (iv) the waves now come from the same side and we see that one overtakes the other. However we note that the two waves never truly meet but that they change into each other when they get close. In case (v) we see that the situation is again very similar to that in case (i) and (ii), but now the interaction takes place over a much longer time. In case (vi) this is even more so, now the interaction happens through multiple peaks. I think that this is because the two waves have almost equal speeds and therefore the interaction is smeared out. If we take two waves with similar speeds, as done in case (vii), we see that we get only one wave with periodic behavior, and this corresponds to a moving breather solution. We can see this from the solution given in equation (6.31). If $b_1 = b_2$ we see from equation (6.15) that $\omega_1 = 2a_1b_1 - i(a_1^2 - b_1^2)$ and $\omega_2 = 2a_2b_1 - i(a_2^2 - b_1^2)$. Therefore we get that $\eta_{1,R} = a_1(x - 2b_1t)$, $\eta_{2,R} = a_2(x - 2b_1t)$ and $\eta_{1,I} - \eta_{2,I} = i(a_1^2 - a_2^2)t$. We see now that if we introduce the new coordinates

$$x' = x - 2b_1t \tag{6.32}$$
$$t' = t \tag{6.33}$$

$\eta_{1,R}$ and $\eta_{2,R}$ depend only on x' and $\eta_{1,I} - \eta_{2,I}$ depends only on t'. Therefore we see that we get a stationary breather solution in the new coordinates as the t-dependence only appears in trigonometric functions, whereas the x-dependence appears only in hyperbolic functions. Furthermore if $b_1 = 0$ (case (viii)), we see that the two coordinate systems are the same and we therefore get the stationary breather in our normal coordinate system. A similar reduction with $a_1 = a_2 = 0$ would give us the growing-and-decaying mode solution, but this fails as then A_{11} and A_{22} become infinite as can be easy seen from equation (6.17).

6.2 Special solution to the FNLS

Before deriving the soliton solutions to the DNLS for which I will use a different bilinearization, I will first derive another 2-soliton solution to the FNLS using the bilinearization introduced in the previous section. In this part I follow the article by Jia-Ren et al. [30] and the method is similar to what I did for the shallow water wave equation in Section 5.6. We start by noticing that the solution given in equation (6.14) is not the only special solution to equation (6.7). Another solution is given by

$$G_1 = i(b(x - 2(a + ib)t - 1)) \exp(-bx + 2abt + i(ax - (a^2 - b^2)t)), \tag{6.34}$$

where a and b are real constants, as can be easily verified directly. If we furthermore take $F_1 = 0$ we find from the recursion formulae (6.9)-(6.12):

$$\begin{aligned}
G_2 &= 0, && (6.35)\\
F_2 &= \frac{p(4b^2(x - 2at)^2 + 16b^4t^2 + 2)}{32b^2} \exp(-2b(x - 2at)), && (6.36)\\
G_3 &= \frac{p(2b^2t - i(bx - 2abt + 1))}{32b^2} \exp(-3b(x - 2at) + iax - i(a^2 - b^2)t), && (6.37)\\
F_3 &= = 0, && (6.38)\\
G_4 &= = 0, && (6.39)\\
F_4 &= \frac{1}{1024b^4} \exp(-4b(x - 2at)), && (6.40)\\
G_n &= F_n = 0 \qquad \text{for } n \geq 5, && (6.41)
\end{aligned}$$

where I used $p^2 = 1$. Note that we now do not have the freedom of translation in x and t, normally represented by the η_i^0-terms. This has the consequence that we now cannot remove the ϵ from equations (6.5) and (6.6) by including them in the η_i^0 terms. Therefore I introduce $x_0 = \frac{1}{b} \ln\left(\frac{\epsilon}{4\sqrt{2}b}\right)$ and we see that if $p = +1$ we therefore find the solution

$$u = 2\sqrt{2}be^{i(ax-(a^2-b^2)t)} \frac{(2b^2t - i)\cosh(b(x - x_0 - 2at)) + ib(x - 2at)\sinh(b(x - x_0 - 2at))}{\cosh^2(b(x - x_0 - 2at)) + b^2((x - 2at)^2 + 4b^2t^2)}. \tag{6.42}$$

We see that this solution is well behaved. However if $p = -1$ then we get

$$u = 2\sqrt{2}be^{i(ax-(a^2-b^2)t)} \frac{(2b^2t - i)\sinh(b(x - x_0 - 2at)) + ib(x - 2at)\cosh(b(x - x_0 - 2at))}{\cosh^2(b(x - x_0 - 2at)) - b^2((x - 2at)^2 + 4b^2t^2) - 1}, \qquad (6.43)$$

and we see that our solution now has poles and is no longer well behaved. Therefore I will only study the $p = +1$ case further. In Figure 6.2 we see some plots for different values of x_0. I made all the plots for the same values of a and b as the behavior does not change much when we change them. The only difference is in the propagation speed of the waves. We see that for x_0 the interaction is symmetrical, but for increasing values of x_0 it becomes more and more asymmetrical. We see therefore clear difference with the 2-soliton behavior observed in the previous section. Jia-Ren et al. point out that this is a new solution and that this should prove that there are still more solutions to the NLS equation to be found. However it is not easy to change the ansatz made in (6.34) in order to obtain more solutions. If we take a function that is more complicated then a linear function in front of the exponent, it becomes very hard and seemingly impossible to obtain solutions to all the recursion equations (6.9)-(6.12). It is also unclear if this method can be used for different equations. I tried it for the Boussinesq equation, but there it failed to provide a real valued solution. We have seen that it does provide a new solution to the SWWE, but this solution is only well behaved in a part of the (x, t)-plane. Therefore the true value of this method is not yet proven.

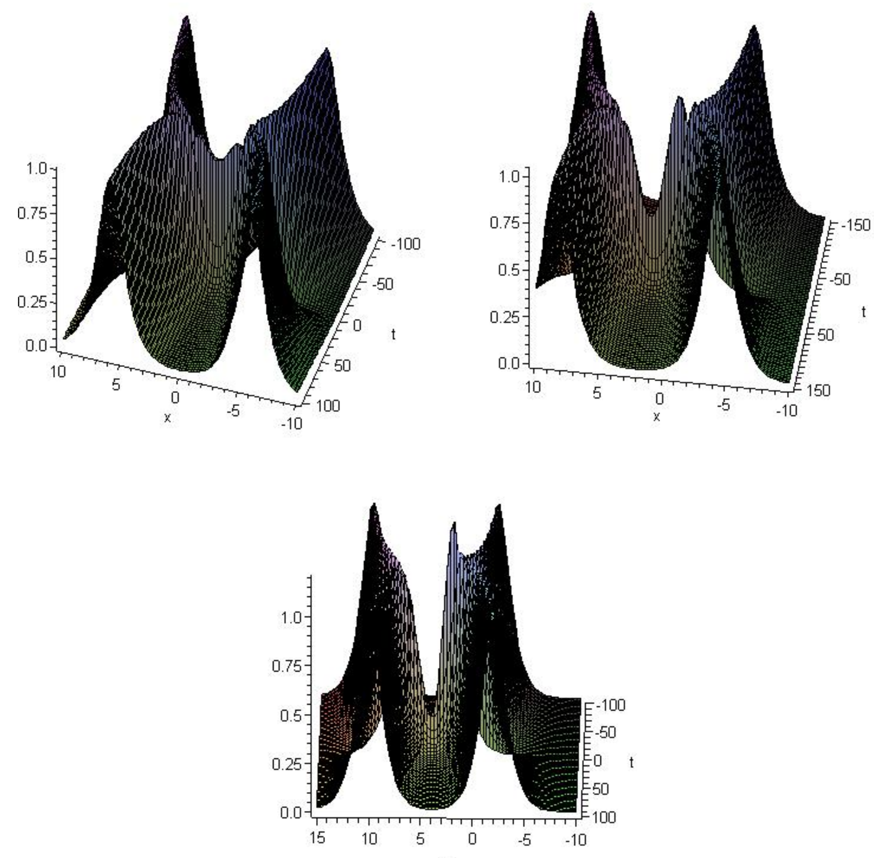

Figure 6.2: Absolute value of the solution given by equation (6.42) for parameter values $a = 0$, $b = 1$ and $x_0 = 0$ (top left), $x_0 = 2$ (top right) and $x_0 = 4$ (bottom).

6.3 Soliton solutions to the DNLS

As we saw above we cannot derive well behaved solutions to the DNLS with the substitution given in equation (6.2). However we can find soliton solutions but then we need to drop the requirement $u \to 0$ as $|x| \to \infty$ which was implicitly assumed by making this substitution. Now we will enforce the boundary condition $|u|^2 \to$ constant as $|x| \to \infty$. We can achieve this by making the substitution

$$u = \rho e^{i\theta} \frac{G}{F}, \tag{6.44}$$

where $\theta = Kx - \Omega t$, K and ρ are real constants, $\Omega = K^2 - p\rho^2$ and $\frac{G}{F} \to 1$ for $|x| \to \infty$. If we insert this in equation (6.1) we see that it again splits up into two bilinear equations given by

$$\begin{aligned} (iD_t + 2ikD_x + D^2)(G \bullet F) &= 0 & (6.45)\\ (D_x^2 + p\rho^2)(F \bullet F) - p\rho^2 |G|^2 &= 0. & (6.46) \end{aligned}$$

We now make the ansatz

$$\begin{aligned} F &= 1 + \epsilon F_1 + \epsilon^2 F_2 + \dots, & (6.47)\\ G &= 1 + \epsilon G_1 + \epsilon^2 G_2 + \dots, & (6.48) \end{aligned}$$

and then get the recursive equations

$$\begin{aligned} (i\partial_t + 2ik\partial_x + \partial^2)(G_1 - F_1) &= 0, & (6.49)\\ \partial_x^2 F_1 + p\rho^2 F_1 &= p\rho^2 (G_1 + G_1^*), & (6.50) \end{aligned}$$

$$\begin{aligned} (i\partial_t + 2ik\partial_x + \partial^2)(G_2 - F_2) &= -(iD_t + 2ikD_x + D^2)(G_1 \bullet F_1), & (6.51)\\ 2\partial_x^2 F_2 + p\rho^2 (2F_2 + F_1^2) &= -D_x^2 (F_1 \bullet F_1) + p\rho^2 (G_2 + G_2^* + |G_1|^2), & (6.52) \end{aligned}$$

$$\begin{aligned} (i\partial_t + 2ik\partial_x + \partial^2)(G_n - F_n) &= -(iD_t + 2ikD_x + D^2)(\sum_{m=1}^{n-1} G_m \bullet F_{n-m}), & (6.53)\\ 2\partial_x^2 F_n + 2p\rho^2 F_n &= -D_x^2 (\sum_{m=1}^{n-1} F_m \bullet F_{n-m}) + p \sum_{m=1}^{n-1} (-F_m F_{n-m} + G_m G_{n-m}^*). & (6.54) \end{aligned}$$

6.3.1 1-soliton solution

To find the 1-soliton solution we now take

$$\begin{aligned} F_1 &= e^{\eta_1}, & (6.55)\\ G_1 &= e^{\eta_1 + 2i\phi_1}, & (6.56) \end{aligned}$$

and $F_n = G_n = 0$ for $n \geq 2$ and where η_1 is as before, but now all constants involved (including ϕ_1 are real). If we insert this into equation (6.49) we find

$$\omega_1 = 2K\kappa_1 - \kappa_1^2 \cot(\phi_1) \tag{6.57}$$

and from equation (6.50) we find

$$\kappa_1^2 = -2p\rho^2 \sin^2(\phi_1). \tag{6.58}$$

We note here that only if $p = -1$ k_1 will be real. If $p = 1$ we see that because k_1 is complex also F_1 will become a complex valued function and this is in conflict with the assumption that F has to be real. Therefore I will now only consider the DNLS. We see that we can rewrite the solution using (6.44) into

$$u = \rho e^{i(\theta + \phi_1)} \left(\cos(\phi_1) + i \tanh\left(\frac{\eta_1}{2}\right) \sin(\phi_1) \right), \tag{6.59}$$

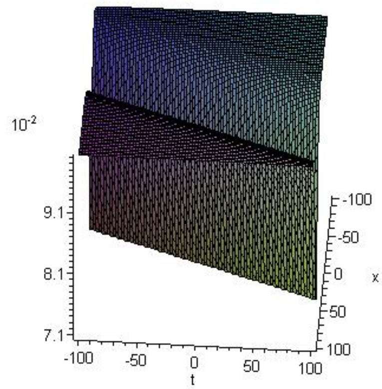

Figure 6.3: Absolute value of 1-soliton solution to the DNLS for the parameter values $\rho = 0.1$, $k = 2$ and $\phi_1 = \pi/4$.

and this solution is given in Figure 6.3. We see that we have a single wave but now it has a negative amplitude. Furthermore we note that as required for $|x| \to \infty$ indeed our solution goes to ρ.

6.3.2 2-soliton solution

We can now proceed to derive the 2-soliton solution in a similar way and we start with the ansatz

$$F_1 = e^{\eta_1} + e^{\eta_2}, \tag{6.60}$$
$$G_1 = e^{\eta_1+2i\phi_1} + e^{\eta_2+2i\phi_2}, \tag{6.61}$$

and from equations (6.49)-(6.54) we then find

$$F_2 = A_{12}e^{\eta_1+\eta_2}, \tag{6.62}$$
$$G_2 = e^{\eta_1+\eta_2+2i(\phi_1+\phi_2)}, \tag{6.63}$$

and $F_n = G_n = 0$ for $n \geq 3$, where η_i is as before, ω_i as in equation (6.57), κ_i as in equation (6.58) and

$$A_{12} = \left[\frac{\sin\left(\frac{1}{2}(\phi_1 - \phi_2)\right)}{\sin\left(\frac{1}{2}(\phi_1 + \phi_2)\right)}\right]^2. \tag{6.64}$$

In order to rewrite our solution in a nice form we will first need to make assumptions on the sign of A_{12} again. We note that $A_{12} \geq 0$ for all real values of ϕ_1 and ϕ_2. If $A_{12} > 0$ we get from equations (6.44), (6.47), (6.48) and (6.61)-(6.63) after a short calculation

$$\begin{aligned} u &= \frac{\rho e^{i(\theta+2\phi_+)}}{\sqrt{A_{12}}\cosh(\eta_+) + \cos(\eta_1)}\Big\{\sqrt{A_{12}}\cosh(\eta_+)\cos(2\phi_+) + \cosh(\eta_-)\cos(2\phi_-) \\ &+ i\sqrt{A_{12}}\sinh(\eta_+)\sin(2\phi_+) + i\sinh(\eta_-)\sin(2\phi_-)\Big\}, \end{aligned} \tag{6.65}$$

where $\eta_\pm = \frac{\eta_1 \pm \eta_2}{2}$ and $\phi_\pm = \frac{\phi_1 \pm \phi_2}{2}$. This case is illustrated in the Figure 6.4. We see that as expected we have two incoming waves (both with negative amplitude) and they interact only where they meet. Furthermore we see that the only way to have $A_{12} = 0$ is if $\phi_2 = \phi_1 + n2\pi$ with $n \in \mathbb{Z}$ and in this case the solution simply reduces to a 1-soliton solution. So I will not illustrate this case here.

We note now that in the derivations so far we have never used explicitly that $p = -1$ and therefore the calculations will also hold in the case $p = +1$. I will examine this case in the next section and we will see that this leads to breather solutions to the FNLS.

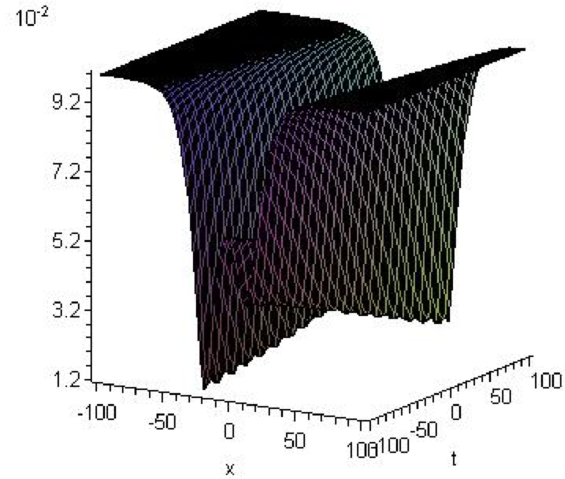

Figure 6.4: Absolute value of 2-soliton solution to the DNLS for the parameter values $\rho = 0.1$, $k = 0.1$, $\phi_1 = \pi/3$ and $\phi_2 = \pi/2$.

6.4 Breather solutions to the FNLS

As said before the 2-soliton solution obtained in the previous section also holds in the case $p = +1$ although we do see from equation (6.58) that both κ_1 and κ_2 now become pure imaginary for real ϕ. As this will also lead to complex ω_i (unless $K = 0$), it makes sense to allow the ϕ_i to be complex numbers, such that all parameters now are complex. However we noticed in the 1-soliton case that this was impossible because F would then no longer be a real valued function. This is not a problem now as we can assure that F is real valued by demanding $\eta_2 = \eta_1^*$, just as we did for the Boussinesq and the shallow water wave equation. We see from equations (6.57) and (6.58) that this comes down to requiring $\phi_2 = \phi_1^* \pm \pi$. If we introduce $\eta_1 = \eta_R + \eta_I$ and $\phi_1 = \phi_R + \phi_I$ we see that we can rewrite A_{12} in the form

$$A_{12} = \left[\frac{\cosh(\phi_I)}{\cos(\phi_R)}\right]^2. \tag{6.66}$$

We note that this is always positive. Therefore we can rewrite our solution in the form

$$\begin{aligned} u &= \frac{\rho e^{i(\theta+2\phi_R)}}{\sqrt{A_{12}}\cosh(\eta_R)+\cos(\eta_I)}\left\{\sqrt{A_{12}}\cosh(\eta_R)\cos(2\phi_R)+\cos(\eta_I)\cosh(2\phi_I)\right. \\ &+ \left. i\left(\sqrt{A12}\sinh(\eta_R)\sin(2\phi_R)-\sin(\eta_I)\sinh(2\phi_I)\right)\right\}. \end{aligned} \tag{6.67}$$

I will now illustrate this in a number of cases. First we see that for $K = 0$, $\phi_I = 0$ and $\phi_R \neq 0$ the x-dependence goes through trigonometric functions and the t-dependence through hyperbolic functions. Therefore we get a solution which is periodic in x and localized in time, which is known as the growing-and-decaying mode solution. This is illustrated in the left part of Figure 6.5. Similarly if $K = 0$, $\phi_R = 0$ and $\phi_I \neq 0$ we get a solution periodic in time and localized in space, which is known as the stationary breather solution and is illustrated in the right part of Figure 6.5. If $K \neq 0$, $\phi_I \neq 0$ and $\phi_R \neq 0$ then we get a moving breather solution, which is periodic around a line in the (x,t)-plane not equal to one of the axis. This solution is illustrated in Figure 6.6.

We see again that our derivation could equally well be done for the DNLS. However then we see from equation (6.58) that in order to have $\eta_2 = \eta_1^*$ we must require $\phi_2 = \phi_1^*$ and this means that equation (6.64) reduces to

$$A_{12} = -\left[\frac{\sinh(\phi_I)}{\sin(\phi_R)}\right]^2 \tag{6.68}$$

and we see that this is always negative and therefore F will have poles and our solution is not well-behaved. So we can not find breather solutions to the DNLS this way.

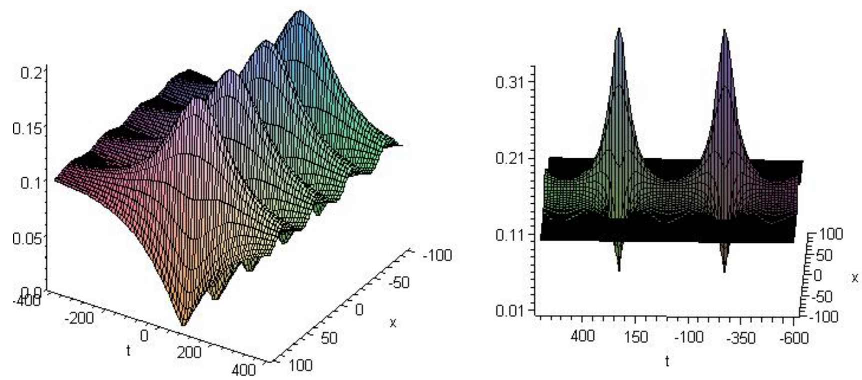

Figure 6.5: Absolute value of growing and decaying mode solution (left) and stationary breather solution (right) for the FNLS.

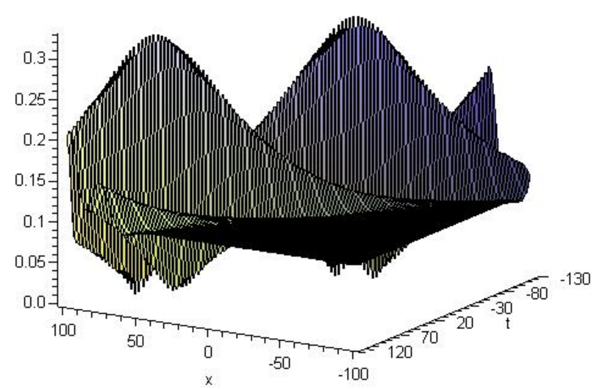

Figure 6.6: Absolute value of the moving breather solution for the FNLS.

6.5 Rational solution to the FNLS

Next I will use a limiting procedure similar to the one I used for the SWWE in Section 5.5 and which was first developed by Ablowitz and Satsuma [3] to derive the rational growing and decaying mode. To do this I now introduce $\phi_R = \epsilon\gamma$ and $\phi_I = \epsilon\delta$ where ϵ is a small expansion parameter. If we know introduce $\kappa_1 = \kappa_R + i\kappa_I$ and $\omega_1 = \omega_R + i\omega_I$ we find from equations (6.57) and (6.58) after some calculations

$$\begin{aligned}
\kappa_R &= -\sqrt{2p}\rho\delta\epsilon + O(\epsilon^3), && (6.69)\\
\kappa_I &= \sqrt{2p}\rho\gamma\epsilon + O(\epsilon^3), && (6.70)\\
\omega_R &= (2p\rho^2\gamma - 2K\rho\sqrt{2p}\delta)\epsilon + O(\epsilon^3), && (6.71)\\
\omega_I &= (2p\rho^2\delta + 2K\rho\sqrt{2p}\gamma)\epsilon + O(\epsilon^3), && (6.72)
\end{aligned}$$

and from equation (6.66)

$$A_{12} = (1 + \frac{\epsilon^2}{2}(\gamma^2 + \delta^2) + O(\epsilon^4))^2. \quad (6.73)$$

Furthermore we see that if we choose our η_R^0 and η_I^0 in such a way that the zeroth order terms in ϵ cancel we get for F and G

$$\begin{aligned}
F &= (\bar{\eta}_R^2 + \bar{\eta}_I^2 + \gamma^2 + \delta^2)\epsilon^2 + O(\epsilon^3), && (6.74)\\
G &= (\bar{\eta}_R^2 + \bar{\eta}_I^2 - 3(\gamma^2 + \delta^2) + 4i(\gamma\bar{\eta}_R + \delta\bar{\eta}_I))\epsilon^2 + O(\epsilon^3), && (6.75)
\end{aligned}$$

where $\eta_R - \eta_R^0 = \epsilon\bar{\eta}_R + O(\epsilon^2)$ and $\eta_I - \eta_I^0 = \epsilon\bar{\eta}_I + O(\epsilon^2)$. If we know insert all this in equation (6.44) and take the limit $\epsilon \to 0$ we get

$$u = \rho e^{i\theta}\left(1 - \frac{4 + 8ip\rho^2 t}{1 + 2p\rho^2(x - 2Kt)^2 + 4\rho^4 t^2}\right). \quad (6.76)$$

In Figure 6.7 we see that we have a solution here localized in space and time and which is known as the rational growing-and-decaying mode. We see again that this solution is only well behaved in the case $p = +1$ so, although it is a solution to the DNLS, it is not the growing-and-decaying mode solution to it. Note furthermore that it is not possible to derive rational solutions from the soliton solutions to the FNLS

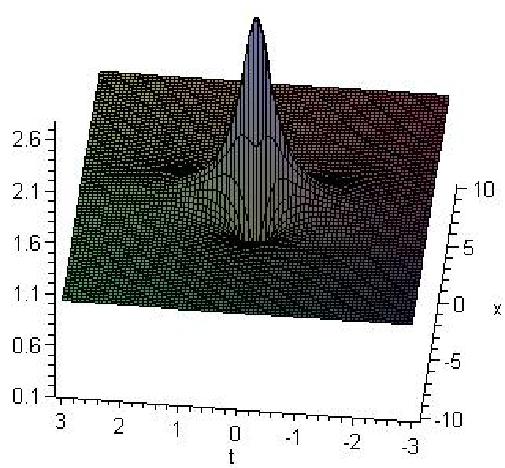

Figure 6.7: Absolute value of the rational growing and decaying mode solution for the FNLS.

obtained in Section 6.1 as it is not possible to cancel the zero-order terms in G as can easily be seen for the

1-soliton solution. Our results so far are equal to those published by Tajiri and Watanabe [28]. At this point they proceeded to show that the breather and growing-and-decaying mode solutions are imbricate series of this rational growing-and-decaying mode solution. I will reproduce their results later on. Before that I want to first move in another direction. In the next section I will apply the same approximation technique to the 1- and 2-soliton solution to DNLS obtained earlier and derive rational solutions to the DNLS.

6.6 Rational solutions to the DNLS

In this part I will use the same approximations as in the previous section, but now I will apply them to the 1- and 2-soliton solutions to the DNLS equation. I will start by approximating the 1-soliton solution to the DNLS. We know from Section 6.3.1 that the 1-soliton solution is given by

$$\begin{aligned} F &= 1 + e^{\eta_1}, & (6.77) \\ G &= 1 + e^{\eta_1 + 2i\phi_1}, & (6.78) \end{aligned}$$

where η_1 is as before. We now introduce $\phi_1 = \epsilon\gamma$ and from equations (6.57) and (6.58) we get

$$\begin{aligned} \kappa_1 &= \sqrt{-2p}\rho\epsilon\gamma + O(\epsilon^3) = \kappa\epsilon\gamma + O(\epsilon^3), & (6.79) \\ \omega_1 &= 2\sqrt{-2p}\rho K\epsilon\gamma - 2prho^2\epsilon\gamma + O(\epsilon^3) = \omega\epsilon\gamma + O(\epsilon^3), & (6.80) \end{aligned}$$

so we see that we can introduce $\epsilon\gamma\bar{\eta}_1 = \eta_1 - \eta_1^0$. If we furthermore introduce $a_1 = \exp(\eta_0^1)$ we can approximate F for $\epsilon \to 0$ by

$$F = 1 + a_1(1 + \gamma\bar{\eta}_1\epsilon) + O(\epsilon^2) \tag{6.81}$$

and we see that the zeroth order in ϵ cancels for $a_1 = -1$. Then G reduces to

$$G = 1 + a_1(1 + (\gamma\bar{\eta}_1 + 2i\gamma)\epsilon) + O(\epsilon^2) \tag{6.82}$$

and from equation (6.44) we find after taking the limit $\epsilon \to 0$

$$u = \rho e^{i\theta}(1 + \frac{2i}{\bar{\eta}}). \tag{6.83}$$

We see that as $\bar{\eta}$ has zeroes our solution has poles and is not well behaved. Furthermore we see from equations (6.79) and (6.80) that if $p = +1$, κ and ω become complex and therefore F becomes complex which we assumed it not to be. Therefore (6.83) is only a solution to the DNLS equation and not to the FNLS equation.

We can repeat the procedure for the 2-soliton solution to the DNLS equation. Now we introduce $\phi_1 = \epsilon\gamma$ and $\phi_2 = \epsilon\delta$ and we see from equations (6.58) and (6.57) that we get

$$\begin{aligned} \kappa_1 &= \sqrt{-2p}\rho(\epsilon\gamma - \frac{1}{6}\epsilon^3\gamma^3) + O(\epsilon^5) = \kappa(\epsilon\gamma - \frac{1}{6}\epsilon^3\gamma^3) + O(\epsilon^5), & (6.84) \\ \omega_1 &= 2\sqrt{-2p}\rho K(\epsilon\gamma - \frac{1}{6}\epsilon^3\gamma^3) - 2p\rho^2(\epsilon\gamma - \frac{2}{3}\epsilon^3\gamma^3) + O(\epsilon^5) = \omega_a\epsilon\gamma - \frac{1}{6}\omega_b\epsilon^3\gamma^3 + O(\epsilon^5), & (6.85) \\ \kappa_2 &= \sqrt{-2p}\rho(\epsilon\delta - \frac{1}{6}\epsilon^3\delta^3) + O(\epsilon^5) = \kappa(\epsilon\delta - \frac{1}{6}\epsilon^3\delta^3) + O(\epsilon^5), & (6.86) \\ \omega_2 &= 2\sqrt{-2p}\rho K(\epsilon\delta - \frac{1}{6}\epsilon^3\delta^3) - 2p\rho^2(\epsilon\delta - \frac{2}{3}\epsilon^3\delta^3) + O(\epsilon^5) = \omega_a\epsilon\delta - \frac{1}{6}\omega_b\epsilon^3\delta^3 + O(\epsilon^5). & (6.87) \end{aligned}$$

So we see that we can write $\eta_1 - \eta_1^0 = \eta_a\gamma\epsilon - \frac{1}{6}\eta_b\gamma^3\epsilon^3$ (and similar for η_2) where $\eta_a = \kappa x - \omega_a t$ and $\eta_b = \kappa x - \omega_b t$. Furthermore we see that A_{12} now reduces to

$$A_{12} = \left(\frac{\gamma - \delta}{\gamma + \delta}\right)^2 + \left(\frac{\gamma - \delta}{\gamma + \delta}\right)^2 \frac{\gamma\delta}{3}\epsilon^2 + O(\epsilon^4). \tag{6.88}$$

If we now approximate F and G around $\epsilon = 0$ again (due to length of expressions I will not give the full Taylor series approximation here) and choose $a_1 = \frac{\gamma+\delta}{\gamma-\delta} + \frac{\gamma\delta\epsilon^2}{6}$ and $a_2 = -\frac{\gamma+\delta}{\gamma-\delta} + \frac{\gamma\delta\epsilon^2}{6}$ we see that the zeroth, first and second order terms in ϵ cancel in both F and G and we see that they therefore reduce to

$$F = -\frac{\gamma\delta(\delta+\gamma)}{6}(\eta_a + \eta_a^3 + 2\eta_b)\epsilon^3 + O(\epsilon^4) \tag{6.89}$$

$$G = -\frac{\gamma\delta(\delta+\gamma)}{6}(-6i - 11\eta_a + 6i\eta_a^2 + \eta_a^3 + 2\eta_b)\epsilon^3 + O(\epsilon^4). \tag{6.90}$$

If we insert this in equation (6.44) and take the limit $\epsilon \to 0$ we get

$$u = \rho e^{i\theta}\frac{-6i - 11\eta_a + 6i\eta_a^2 + \eta_a^3 + 2\eta_b}{\eta_a + \eta_a^3 + 2\eta_b} \tag{6.91}$$

and we see that this solution again has poles. Furthermore following exact the same reasoning as before we see that this is only a solution to the DNLS equation and not the FNLS equation (which can also be easily verified directly).

6.7 Breather solutions to the FNLS as imbricate series of rational growing-and-decaying modes

In this part I will use the rational growing-and-decaying mode solution derived in Section 6.5 and show that breather solutions appear as imbricate series of them. I will follow the derivation outlined in [28] and which was first developed by Tajiri and Watanabe. I will start by deriving the growing-and-decaying mode solution in this way.

6.7.1 Growing-and-decaying mode solution

I will only look at the case with $K = 0$ in order to keep the algebra to a minimum. As the FNLS equation is invariant under the transformation

$$x' = x - 2Kt, \tag{6.92}$$

$$t' = t, \tag{6.93}$$

$$u(x', t') = \exp(-(Kx + K^2t))u(x,t), \tag{6.94}$$

we can always introduce this dependence later on. We note that with $K = 0$ we can rewrite the rational growing-and-decaying mode solution given by equation (6.76) in the form

$$u = \rho e^{ip\rho^2 t}\left(1 + \frac{1}{iq\rho^2 t + \frac{1}{2}\sqrt{1+2p\rho^2x^2}}\right)\left(1 + \frac{1}{iq\rho^2 t - \frac{1}{2}\sqrt{1+2p\rho^2x^2}}\right) \tag{6.95}$$

and therefore we guess the form of the growing-and-decaying mode solution to be

$$u = \rho e^{i(\sigma t+\phi)}\left(1 + b\sum_{n=-\infty}^{n=\infty}\frac{1}{i\alpha t + v(x) + n}\right)\left(1 + b\sum_{N=-\infty}^{N=\infty}\frac{1}{i\alpha t - v(x) + n}\right), \tag{6.96}$$

where $v(x)$ is a real function of x and α and σ are real constants to be determined. In a moment it will become clear why this is a good assumption. We now use the identity $\pi\cot(\pi z) = \sum_{n=-\infty}^{n=\infty}\frac{1}{z+n}$ which can be easily proven using Liouville's theorem. A proof of a similar identity which can readily be adapted to the current case can be found on page 379 of [20]. Using this identity we can rewrite equation (6.96) in the form

$$u = \rho e^{i(\sigma t+\phi)}(1 + b\pi\cot(\pi[v(x) + i\alpha t]))(1 - b\pi\cot(\pi[v(x) - i\alpha t])). \tag{6.97}$$

Now it is clear why this is a good ansatz, as we see that the solution given by this formula will be periodic in x and will decay exponentially in t, just as is required of the growing-and-decaying mode solution. Now we insert this in the NLS equation given in (6.1). We then split the cot-terms up in separate x and t parts. Our result, which due to its length I will not give here, can then be split up according to order in $\cosh(\pi\alpha t)$. The sixth order term (or equivalently also the fifth order term) then gives

$$\sigma = p\rho(1+\pi^2 b^2)^2. \tag{6.98}$$

We can now use this to derive from the first and third order terms (or equivalently the second and fourth order terms again) the following two equations

$$\begin{aligned} \left(\frac{dv(x)}{dx}\right)^2 &= \frac{p\rho^2 b^2}{2}(1-\pi^2 b^2\cot^2(2\pi v(x))), && (6.99)\\ \frac{d^2 v(x)}{dx^2} &= p\rho^2\pi^3 b^4\cot(2\pi v(x))\left(\frac{1+2\pi^2 b^2}{\pi^2 b^2} - \frac{\alpha}{p\rho^2\pi^2 b^3} + \cot^2(2\pi v(x))\right). && (6.100)\end{aligned}$$

If we now take the derivative of equation (6.99) and subtract this from equation (6.100) we find

$$\alpha = p\rho^2 b(1+\pi^2 b^2). \tag{6.101}$$

Furthermore we can integrate equation (6.99) to find

$$v(x) = \frac{1}{2\pi}\arccos\left(\frac{1}{\sqrt{1+\pi^2 b^2}}\cos(\sqrt{2\pi^2\alpha b}x + C)\right), \tag{6.102}$$

where C is an integration constant. If we now combine all this we find

$$\begin{aligned} u &= \rho(1+\pi^2 b^2)\exp(i[p\rho^2(1+\pi^2 b^2)^2 t + \phi])\\ &\times \left(1 - \frac{2\pi}{1+\pi^2 b^2}\frac{\pi b\cosh(2\pi\alpha t) + i\sinh(2\pi\alpha t)}{\cosh(2\pi\alpha t) - (1/\sqrt{1+\pi^2 b^2})\cos(\sqrt{2\pi^2\alpha b}x + C)}\right), \end{aligned} \tag{6.103}$$

where α is given by equation (6.101). We see that our solution is indeed periodic in x as we wanted it to be. As this solution is similar to the growing-and-decaying mode solution given in Section 6.4 and which is illustrated in the left part Figure 6.5 I will not illustrate it here again.

6.7.2 Stationary breather solution

The procedure for deriving the stationary breather solution is the same as for the growing-and-decaying mode solution, but now the t-dependence will come through the trigonometric functions and the x-dependence through the hyperbolic functions, thus getting a solution that is periodic in time. On the basis of (6.95) we guess the form

$$u = \rho e^{i(\sigma t+\phi)}\left(1 + ib\sum_{n=-\infty}^{n=\infty}\frac{1}{\alpha t + iv(x) + n}\right)\left(1 + ib\sum_{N=-\infty}^{N=\infty}\frac{1}{\alpha t - iv(x) + n}\right). \tag{6.104}$$

Making use of a similar identity as before (which can be proven in an similar way) we get

$$u = \rho e^{i(\sigma t+\phi)}(1 + b\pi\coth(\pi[v(x) - i\alpha t]))(1 - b\pi\cot(\pi[v(x) + i\alpha t])). \tag{6.105}$$

If we insert this in equation (6.1) we can derive in the same way as we derived equations (6.98), (6.99) and (6.100) above

$$\begin{aligned} \sigma &= p\rho(1-\pi^2 b^2)^2 && (6.106)\\ \left(\frac{dv(x)}{dx}\right)^2 &= \frac{p\rho^2 b^2}{2}(1-\pi^2 b^2\coth^2(2\pi v(x))), && (6.107)\\ \frac{d^2 v(x)}{dx^2} &= p\rho^2\pi^3 b^4\coth(2\pi v(x))\left(\frac{1-2\pi^2 b^2}{\pi^2 b^2} + \frac{\alpha}{p\rho^2\pi^2 b^3} + \coth^2(2\pi v(x))\right). && (6.108)\end{aligned}$$

Again we can derive α by subtracting the derivative of eq. (6.107) of eq. (6.108) and this gives

$$\alpha = -p\rho^2 b(1-\pi^2 b^2). \tag{6.109}$$

Integrating equation (6.107) now gives us

$$v(x) = \frac{1}{2\pi}\text{arccosh}\left(\frac{1}{\sqrt{1-\pi^2 b^2}}\cosh(\sqrt{-2\pi^2\alpha b}x + C)\right), \tag{6.110}$$

and this all together gives us the solution

$$\begin{aligned} u &= \rho(1-\pi^2 b^2)\exp(i[p\rho^2(1-\pi^2 b^2)^2 t + \phi]) \\ &\times \left(1 + \frac{2\pi}{1-\pi^2 b^2}\frac{\pi b\cos(2\pi\alpha t) - i\sin(2\pi\alpha t)}{\cos(2\pi\alpha t) - (1/\sqrt{1-\pi^2 b^2})\cosh(\sqrt{-2\pi^2\alpha b}x + C)}\right). \end{aligned} \tag{6.111}$$

We see that apart from some sign changes the main difference is that all hyperbolic functions in eq. (6.103) are now replaced by the corresponding trigonometric functions and vice versa. Therefore our solution is now indeed periodic in time as demanded and looks similar to the one illustrated in the right part of Figure 6.5.

6.7.3 Breather solution

In principle we could construct the breather solution in the same way as we did for the stationary breather and growing-and-decaying mode solution. However as Tajiri and Watanabe [28] point out, the algebra then gets very complicated and they therefore use a different approach, which I will follow here. This starts by noting that using equations (6.57), (6.58) and (6.61)-(6.63) the absolute value of equation (6.44) can be rewritten in the form

$$|u|^2 = \rho^2 + \frac{2}{p}\frac{\partial^2}{\partial x^2}\ln(F), \tag{6.112}$$

where $F = 1 + \exp(\eta_1) + \exp(\eta_2) + A_{12}\exp(\eta_1 + \eta_2)$. As this also holds for the breather solution and furthermore we see that $|u|^2$ does not change if we multiply F with $\exp(ax+b)$ with a and b constants we see that we can rewrite equation (6.67) in the form

$$|u|^2 = \rho^2 + \frac{2}{p}\frac{\partial^2}{\partial x^2}\ln\left(\sqrt{A_{12}}\cosh(\eta_R) - \cos(\eta_I + \pi)\right), \tag{6.113}$$

where η_R and η_I are as before and the reason for adding the π in the last term will become clear later on. Now we note that if $K \neq 0$ then the equivalent of equation (6.95) is given by

$$\begin{aligned} u &= \rho\exp[i(Kx - (K^2 - p\rho^2)t)]\left(1 + \frac{1}{ip\rho^2 t + \frac{1}{2}\sqrt{1 + 2p\rho^2(x-2Kt)^2}}\right) \\ &\times \left(1 - \frac{1}{ip\rho^2 t + \frac{1}{2}\sqrt{1 + 2p\rho^2(x-2Kt)^2}}\right), \end{aligned} \tag{6.114}$$

and from this we can derive

$$|u|^2 = \rho^2 - \frac{1}{p}\frac{\partial^2}{\partial x^2}\ln\left(\frac{1}{\left(\frac{1}{2}\sqrt{1+2p\rho^2(x-2Kt)^2} + ip\rho^2 t\right)^2}\frac{1}{\left(\frac{1}{2}\sqrt{1+2p\rho^2(x-2Kt)^2} - ip\rho^2 t\right)^2}\right). \tag{6.115}$$

Therefore we now assume the imbricate series of the breather solution to have the form

$$|u|^2 = \rho^2 - \frac{1}{p}\frac{\partial^2}{\partial x^2}\ln\left[\left(\sum_{n=-\infty}^{\infty}\frac{1}{(\phi(x,t) - i\psi(x,t) - n)^2}\frac{1}{(\phi(x,t) - i\psi(x,t) - n)^2}\right)\right], \tag{6.116}$$

where $\phi(x,t)$ an $\psi(x,t)$ are unknown functions yet to be determined. We note that this can be rewritten in the form

$$|u|^2 = \rho^2 + \frac{2}{q}\frac{\partial^2}{\partial x^2}\ln(\cosh(2\pi\psi) - \cos(2\pi\phi)). \tag{6.117}$$

If we now compare this with the form given in equation (6.113) we see that we either must have

$$\cosh(2\pi\psi) = \sqrt{A_{12}}\cosh(\eta_R), \tag{6.118}$$
$$\cos(2\pi\phi) = \cos(\eta_I + \pi), \tag{6.119}$$

or

$$\cosh(2\pi\psi) = \cosh(\eta_R), \tag{6.120}$$
$$\cos(2\pi\phi) = \frac{1}{\sqrt{A_{12}}}\cos(\eta_I + \pi). \tag{6.121}$$

We can solve equations (6.118) and (6.119) to give

$$\psi = \frac{1}{2\pi}\ln\left(\sqrt{A_{12}}\cosh(\eta_r) + \sqrt{A_{12}\cosh^2(\eta_R) - 1}\right), \tag{6.122}$$
$$\phi = \frac{1}{2\pi}(\eta_I + \pi). \tag{6.123}$$

Similarly we can solve equations (6.120) and (6.121) to find

$$\psi = \frac{1}{2\pi}\eta_R \tag{6.124}$$
$$\phi = \frac{1}{2\pi}\arccos\left(\frac{1}{\sqrt{A_{12}}}\cos(\eta_I + \pi)\right). \tag{6.125}$$

So we see that equation (6.117) gives the breather solution to the FNLS equation if ϕ and ψ are given by either equations (6.122) and (6.123) or by equations (6.124) and (6.125). Therefore we see that the breather solution can also be written as the imbricate series of rational growing-and-decaying mode solutions. As this solution is just a different form of the one illustrated in Figure 6.6 I will not illustrate it here again.

6.8 Discussion

Almost all of the solutions derived in this chapter have been known for a long time. The soliton solutions to both the FNLS and DNLS were first given by Hirota [16]. The discussion in Section 6.2 follows the article by Yan et al. [30] and the results obtained in sections 6.4, 6.5 and 6.7 were given by Tajiri and Watanabe [28]. Although I might have used slightly different notation at some places, all calculations are done in a similar way to what has been done in the literature mentioned above. Only the results in Section 6.6 I have not found in the literature. However the method is already well known and was first published by Ablowitz and Satsuma [3].

Here I end my study of soliton equations using the bilinear form. In the next part I will look at three equations (NLS, modified Korteweg-de Vries (mKdV) and the Sine-Gordon equation (SG)) using the inverse scattering method.

Part III

Results from the Inverse Scattering Transform

Chapter 7

Inverse Scattering Transform for the NLS

In this Part I will look at the inverse scattering transform (IST). As we saw in the introduction this method predates the bilinear method and led to the revived interest in soliton equations. I will study here the nonlinear Schrödinger equation (NLS) instead of the KdV equation witch is the standard example and which I discussed without going into detail in Section 3.1. I will start by developing the IST in a more general form in Chapter 7 before applying it to the NLS to derive the 1- and 2-soliton solutions in Chapter 8. In chapters 9 and 10 I will then show how we can obtain 1- and 2-soliton solutions to the modified Korteweg-de Vries (mKdV) and the Sine-Gordon (SG) equation from the same method. Actually as we will see that the scattering problem is very similar to that for the NLS, we can apply all theory derived in Chapter 7 with only a small number of changes.
Here I will present the inverse scattering transform for the nonlinear Schrödinger equation (NLS). I will follow the discussion in Chapter 2 of the book by Ablowitz, Prinari and Trubatch [2]. I will use the following form of the NLS

$$iq_t - q_{xx} - 2|q|^2 q = 0, \tag{7.1}$$

which is slightly different from the one I used when I studied it using the bilinear method in Chapter 6, but it is clear how the solutions of this equation can be mapped onto those derived earlier. The idea behind inverse scattering for the NLS equation is the same as what we saw in Section 3.1 for the KdV equation. We start with the Lax pair, first give by Zakharov and Shabat in [32] and which consists of the two linear equations

$$v_x = \begin{pmatrix} -ik & q \\ r & ik \end{pmatrix} v \tag{7.2}$$

and

$$v_t = \begin{pmatrix} 2ik^2 + iqr & -2kq - iq_x \\ -2kr + ir_x & -2ik^2 - iqr \end{pmatrix} v. \tag{7.3}$$

From a simple calculation we see that Lax equation (3.19) then gives

$$iq_t - q_{xx} - 2rq^2 = 0 \tag{7.4}$$

$$ir_t + r_{xx} + 2r^2 q = 0 \tag{7.5}$$

and this reduces to equation (7.1) under the reduction $r = -q^*$. I will do most of the inverse scattering for general r and only use this reduction to find the soliton solutions at the end. The function q is often called the potential, because in the inverse scattering transform for the KdV-equation it has the role of potential in the time-independent Schrödinger equation and I will use this terminology here. As we saw in the introduction the inverse scattering transform now consists of the following steps: first we will have to map this potential, given at $t = 0$ into the scattering data using equation (7.2). This is done in the next part. Then we have to figure out how to map the scattering data back into the potential, the so-called inverse scattering problem, and this is discussed in Section 7.2. Finally we have to use equation (7.3) to determine the time evolution of the scattering data and this is done in part 7.3. If we are then given an initial condition $q(x, 0)$ we can derive the solution to equation (7.1) as follows: we use the direct scattering to obtain the scattering data at $t = 0$. Using the time evolution we obtain the scattering data at the time t and this we map back into the potential $q(x, t)$ using the inverse scattering. Due to the similarity in approach to the method of the Fourier transform for solving ODE's, i.e. the direct scattering is similar to the Fourier transform, the time evolution to the dispersion relation and the inverse scattering to the inverse Fourier transform, this method has been named the Inverse Scattering Transform (IST).

7.1 Direct scattering problem

In order to make the IST work we need one important assumption on the potentials q and r and that is that they decay rapidly as $|x| \to \infty$. I will not go into the details of how rapid decay is necessary, but more details can be found in [2]. In general when q and r are in $L^1(\mathbf{R})$ it will be sufficient. We will refer to the solution v of equation (7.2) as an eigenfunction and we see that for large $|x|$ they satisfy

$$v_x = \begin{pmatrix} -ik & 0 \\ 0 & ik \end{pmatrix} v, \tag{7.6}$$

as we assumed that q and r are rapidly decaying. So we see that we can introduce the eigenfunctions defined by the boundary conditions

$$\phi(x, k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx}, \qquad \bar{\phi}(x, k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx} \qquad \text{as } x \to -\infty, \tag{7.7}$$

$$\psi(x, k) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{ikx}, \qquad \bar{\psi}(x, k) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-ikx} \qquad \text{as } x \to +\infty. \tag{7.8}$$

However it is easier to use functions with constant boundary conditions, so called Jost functions defined by

$$M(x, k) = e^{ikx}\phi(x, k), \qquad \bar{M}(x, k) = e^{-ikx}\bar{\phi}(x, k), \tag{7.9}$$

$$N(x, k) = e^{-ikx}\psi(x, k), \qquad \bar{N}(x, k) = e^{ikx}\bar{\psi}(x, k). \tag{7.10}$$

Note that we can rewrite equation (7.2) in the form

$$v_x = (ik\mathbf{J} + \mathbf{Q})v \tag{7.11}$$

where

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{Q} = \begin{pmatrix} 0 & q \\ r & 0 \end{pmatrix}. \tag{7.12}$$

If we now define $\mathbf{I}$ to be the identity matrix then M and $\bar{N}$ are solutions of the equation

$$\chi_x(x, k) = ik(\mathbf{J} + \mathbf{I})\chi(x, k) + (\mathbf{Q}\chi)(x, k) \tag{7.13}$$

whereas $\bar{M}$ and N satisfy

$$\bar{\chi}_x(x, k) = ik(\mathbf{J} - \mathbf{I})\bar{\chi}(x, k) + (\mathbf{Q}\bar{\chi})(x, k). \tag{7.14}$$

It is well known that the solutions to these equations can be represented using Green's functions. We can write them as

$$\chi(x,k) = w + \int_{-\infty}^{\infty} \mathbf{G}(x-\xi,k)(\mathbf{Q}\chi)(\xi,k)d\xi, \tag{7.15}$$

$$\bar{\chi}(x,k) = \bar{w} + \int_{-\infty}^{\infty} \bar{\mathbf{G}}(x-\xi,k)(\mathbf{Q}\bar{\chi})(\xi,k)d\xi, \tag{7.16}$$

where $w = (w_1, 0)^T$, $\bar{w} = (0, \bar{w}_2)^T$ and $\mathbf{G}(x,k)$ and $\bar{\mathbf{G}}(x,k)$ are the Green's functions that satisfy

$$(\mathbf{I}\partial_x - ik(\mathbf{J}+\mathbf{I}))G(x,k) = \delta(x)\mathbf{I}, \tag{7.17}$$
$$(\mathbf{I}\partial_x - ik(\mathbf{J}-\mathbf{I}))\bar{G}(x,k) = \delta(x)\mathbf{I}. \tag{7.18}$$

These equations can be solved by means of the Fourier transform and we find

$$\mathbf{G}(x,k) = \frac{1}{2\pi i}\int_C \begin{pmatrix} p^{-1} & 0 \\ 0 & (p-2k)^{-1} \end{pmatrix} e^{ipx}dp, \tag{7.19}$$

$$\bar{\mathbf{G}}(x,k) = \frac{1}{2\pi i}\int_{\bar{C}} \begin{pmatrix} (p+2k)^{-1} & 0 \\ 0 & p^{-1} \end{pmatrix} e^{ipx}dp, \tag{7.20}$$

where C and $\bar{C}$ are appropriate contours. We now define the contours $C_\pm$ as the one passing either below/above both the singularities. If we now take $\mathbf{G}_\pm$ to be defined by equation (7.19) with $C = C_\pm$ and similarly for $\bar{\mathbf{G}}_\pm$ we find

$$\mathbf{G}_\pm(x,k) = \pm\Theta(\pm x)\begin{pmatrix} 1 & 0 \\ 0 & e^{2ikx} \end{pmatrix}, \tag{7.21}$$

$$\bar{\mathbf{G}}_\pm(x,k) = \mp\Theta(\mp x)\begin{pmatrix} e^{-2ikx} & 0 \\ 0 & 1 \end{pmatrix}, \tag{7.22}$$

where $\Theta(x)$ denotes the Heaviside function. We note that it is clear that the $+$-functions are analytic in the upper half of the k-plane and the $-$-functions in the lower half. From this we can derive the following Volterra integral equations for the Jost functions

$$M(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{\infty} \mathbf{G}_+(x-\xi,k)((Q)M)(\xi,k)d\xi), \tag{7.23}$$

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{\infty} \bar{\mathbf{G}}_+(x-\xi,k)((Q)N)(\xi,k)d\xi), \tag{7.24}$$

$$\bar{M}(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \int_{-\infty}^{\infty} \bar{\mathbf{G}}_-(x-\xi,k)((Q)\bar{M})(\xi,k)d\xi), \tag{7.25}$$

$$\bar{N}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \int_{-\infty}^{\infty} \mathbf{G}_-(x-\xi,k)((Q)\bar{N})(\xi,k)d\xi). \tag{7.26}$$

It is shown in Lemma 2.1 and the subsequent proof in [2] that for q, r in $L^1(\mathbf{R})$, M and N as defined above are analytic functions of k for $\Im(k) > 0$ and continuous for $\Im(k) \geq 0$, whereas $\bar{M}$ and $\bar{N}$ are analytic for

$\Im(k<0)$ and continuous for $\Im(k)\leq 0$. Furthermore the Jost functions are uniquely determined by equations (7.23)-(7.26) in the space of continuous functions. I will not reproduce this proof here, but it is quite straight forward, depending mainly on a number of integral identities.
We can now go back to the eigenfunctions ϕ, $\bar{\phi}$, ψ and $\bar{\psi}$. We first note that for any u and v that are solutions to equation (7.2), we have

$$\frac{d}{dx}W(u,v)=0, \tag{7.27}$$

where we define the Wronskian W of u and v by

$$W(u,v)=u^{(1)}v^{(2)}-u^{(2)}v^{(1)}. \tag{7.28}$$

This can be shown by an elementary calculation. Using this and equations (7.7) and (7.8), we can very easily derive the Wronskian of the eigenfunctions to be

$$W(\phi,\bar{\phi}) = \lim_{x\to-\infty} W(\phi,\bar{\phi})=1, \tag{7.29}$$
$$W(\psi,\bar{\psi}) = \lim_{x\to+\infty} W(\psi,\bar{\psi})=-1. \tag{7.30}$$

So we see that ϕ and $\bar{\phi}$ are linearly independent and the same holds for ψ and $\bar{\psi}$. However we know that equation (7.2) has only two linearly independent eigenfunctions, so it follows that ϕ and $\bar{\phi}$ must be linear combinations of ψ and $\bar{\psi}$. Therefore we get

$$\phi(x,k) = b(k)\psi(x,k)+a(k)\bar{\psi}(x,k), \tag{7.31}$$
$$\bar{\phi}(x,k) = \bar{a}(k)\psi(x,k)+\bar{b}(k)\bar{\psi}(x,k), \tag{7.32}$$

where we note that this only holds when all four eigenfunctions exist. This is the case on the real k-axis and this therefore defines the scattering coefficients $a(k)$, $\bar{a}(k)$, $b(k)$ and $\bar{b}(k)$. If we insert this in equation (7.29) and use equation (7.30) we find

$$a(k)\bar{a}(k)-b(k)\bar{b}(k)=1. \tag{7.33}$$

We see that we can also express the scattering coefficients as Wronskians of the eigenfunctions as follows

$$a(k)=W(\phi,\psi) \qquad \bar{a}(k)=-W(\bar{\phi},\bar{\psi}) \tag{7.34}$$
$$b(k)=-W(\phi,\bar{\psi}) \qquad \bar{b}(k)=W(\bar{\phi},\psi). \tag{7.35}$$

Therefore we see that $a(k)$ is analytic in the upper half of the complex k-plane, whereas $\bar{a}(k)$ is analytic in the lower half. Furthermore $b(k)$ and $\bar{b}(k)$ can in general not be extended off the real k-axis. For later use I will rewrite equations (7.31) and (7.32) using the Jost functions with fixed boundary conditions in the form

$$\mu(x,k)=\bar{N}(x,k)+\rho(k)e^{2ikx}N(x,k) \tag{7.36}$$
$$\bar{\mu}(x,k)=N(x,k)+\bar{\rho}(k)e^{-2ikx}\bar{N}(x,k) \tag{7.37}$$

where I introduced

$$\mu(x,k)=\frac{M(x,k)}{a(k)}, \qquad \bar{\mu}(x,k)=\frac{\bar{M}(x,k)}{\bar{a}(k)} \tag{7.38}$$

and the reflection coefficients are given by

$$\rho(k)=\frac{b(k)}{a(k)}, \qquad \bar{\rho}(k)=\frac{\bar{b}(k)}{\bar{a}(k)}. \tag{7.39}$$

Next we can look at the proper eigenvalues of equation (7.2). By a proper eigenvalue I mean a complex value of k such that the corresponding eigenstate v is bounded and decays to zero for large x. I will show that proper eigenvalues in the upper half of the k-plane correspond to the zeroes of $a(k)$. Suppose that $k_j = \xi_j + i\eta_j$ is a zero of $a(k)$. If we then introduce $\phi_j(x) = \phi(x, k_j)$ and similar for $\psi_j(x)$ we find from equation (7.34) that $W(\phi_j, \psi_j) = 0$ and so they are linearly dependent. Thus we find that

$$\phi_j(x) = c_j \psi_j(x), \tag{7.40}$$

with c_j a complex constant defined by this relation. If we now look at the behavior for large x we see that

$$\phi_j(x) \sim \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{\eta_j x - i\xi_j x} \qquad \text{as } x \to -\infty \tag{7.41}$$

$$\phi_j(x) = c_j \psi_j(x) \sim \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-\eta_j x + i\xi_j x} \qquad \text{as } x \to +\infty. \tag{7.42}$$

So we see that k_j is a proper eigenvalue if and only if η_j is positive. Furthermore if $a(k) \neq 0$ then the solution will blow up in at least one direction. So we see that in the upper half plane the proper eigenvalues correspond to the zeroes of $a(k)$. The same things can be done in the lower half plane and we therefore find $\bar{c}_j$ by

$$\bar{\phi}_j(x) = \bar{c}_j \bar{\psi}_j(x) \tag{7.43}$$

and the proper eigenvalues $\bar{k}_j$ located in the lower half plane as the zeroes of $\bar{a}(k)$. Furthermore for later use I note that we can also find c_j and $\bar{c}_j$ from the relations

$$M_j(x) = c_j e^{2ik_j x} N_j x, \tag{7.44}$$

$$\bar{M}_j(x) = \bar{c}_j e^{-2i\bar{k}_j x} \bar{N}_j(x). \tag{7.45}$$

Now we have found all the scattering data consisting of the scattering coefficients $a(k)$, $\bar{a}(k)$, $b(k)$ and $\bar{b}(k)$, the proper eigenvalues k_j and $\bar{k}_j$ and the normalization constants c_j and $\bar{c}_j$. Before moving on to the inverse problem however I will first look at the reductions we get in the case $r = \mp q^*$, which will simplify the problem significantly. It is easy to see that if this symmetry between q and r holds that if $v(x, k) = (v^{(1)}(x, k), v^{(2)}(x, k))^T$ satisfies equation (7.2) then also $\hat{v}(x, k) = (v^{(2)}(x, k^*), \mp v^{(1)}(x, k^*))^H$ satisfies it. Therefore we find, if we take equations (7.7) and (7.8) into account, that

$$\bar{\phi}(x, k) = \begin{pmatrix} \mp\phi(2)(x, k*) \\ \phi(1)(x, k^*) \end{pmatrix}^*, \qquad \bar{\psi}(x, k) = \begin{pmatrix} \psi(2)(x, k^*) \\ \mp\psi(1)(x, k^*) \end{pmatrix}^*. \tag{7.46}$$

If we insert this in equations (7.34) and (7.35) we directly see that

$$\bar{a}(k) = a^*(k^*), \tag{7.47}$$

$$\bar{b}(k) = \mp b^*(k^*) \tag{7.48}$$

and from them we get that $\bar{\rho}(k) = \mp\rho^*(k^*)$. Furthermore we note from equation (7.47) that if k_j is a zero of $a(k)$ then k_j^* is a zero of $\bar{a}(k)$. If we now insert (7.46) into equation (7.43) we directly see that

$$\bar{c}_j = \mp c_j^*. \tag{7.49}$$

We have found all scattering data and also have seen how they behave under the reduction $r = \mp q^*$, so we can proceed with the inverse scattering problem.

7.2 Inverse scattering

The recovering of the potentials from the scattering data will consist of two steps. First I will recover the Jost functions N and $\bar{N}$ and from them we can then recover the potentials q and r. I will treat it as an Riemann-Hilbert problem and start by introducing the projection operators defined by

$$P^{\pm}(f)(k) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{f(\zeta)}{\zeta - (k \pm i0)} d\zeta. \tag{7.50}$$

If we now define $f_{\pm}(k)$ to be analytic in the upper/lower half of the k-plane and require it to go to zero for $|k| \to \infty$ for $\Im(k)\gtrless 0$, then it is an easy exercise to show

$$P^{\pm}(f_{\mp})(k) = 0 \tag{7.51}$$

and

$$P^{\pm}(f_{\pm})(k) = \pm f_{\pm}(k). \tag{7.52}$$

Now we can apply P^{-} to both sides of equation (7.36) and P^{+} to both sides of equation (7.37) and if we take into account all the analytic properties and asymptotic behavior derived earlier and use equations (7.44) and (7.45) we find after evaluating all projections

$$\bar{N}(x,k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x} N_j(x)}{k - k_j} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\zeta) e^{2ikx} N(x,\zeta)}{\zeta - (k - i0)} d\zeta, \tag{7.53}$$

$$N(x,k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{\bar{C}_j e^{-2ik_j x} \bar{N}_j(x)}{k - \bar{k}_j} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\bar{\rho}(\zeta) e^{-2ikx} \bar{N}(x,\zeta)}{\zeta - (k - i0)} d\zeta, \tag{7.54}$$

where J and $\bar{J}$ denote the number of zeroes in $a(k)$ and $\bar{a}(k)$ and I introduced

$$C_j = \frac{c_j}{a'(k_j)}, \qquad \bar{C}_j = \frac{\bar{c}_j}{\bar{a}'(\bar{k}_j)}. \tag{7.55}$$

We can close this system by evaluating the first equation at $\bar{k}_l$ and the second at k_l and thus obtain the following equations

$$\bar{N}_l(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x} N_j(x)}{\bar{k}_l - k_j} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho(\zeta) e^{2i\zeta x} N(x,\zeta)}{\zeta - \bar{k}_l} d\zeta, \tag{7.56}$$

$$N_l(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{\bar{C}_j e^{-2ik_j x} \bar{N}_j(x)}{k_l - \bar{k}_j} - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\bar{\rho}(\zeta) e^{-2i\zeta x} \bar{N}(x,\zeta)}{\zeta - k_l} d\zeta. \tag{7.57}$$

We have thus obtained 4 equations from which we can in principle recover N and $\bar{N}$. Furthermore if we compare the behavior for large k of N and $\bar{N}$ thus obtained with that obtained from equations (7.24) and (7.26) we find

$$r(x) = -2i \sum_{j=1}^{J} e^{2ik_j x} C_j N_j^{(2)}(x) + \frac{1}{\pi}\int_{\infty}^{+\infty} \rho(\zeta) e^{2i\zeta x} N^{(2)}(x,\zeta) d\zeta, \tag{7.58}$$

$$q(x) = 2i \sum_{j=1}^{\bar{J}} e^{-2i\bar{k}_j x} \bar{C}_j \bar{N}_j^{(1)}(x) + \frac{1}{\pi}\int_{\infty}^{+\infty} \bar{\rho}(\zeta) e^{-2i\zeta x} \bar{N}^{(1)}(x,\zeta) d\zeta. \tag{7.59}$$

Thus we can recover the potentials and the inverse problem is solved. In [2] the authors also show that the inverse problem can be solved using Gel'fand-Levitan-Marchenko equations, however as I will not use that anywhere I will not reproduce the result here. Now all that is left to do is look at the time dependence and that will be done in the next section.

7.3 Time evolution

To determine the time evolution of the scattering data we start by rewriting the operator from equation (7.3) in the form

$$\partial_t v = \begin{pmatrix} A & B \\ C & -A \end{pmatrix} v. \tag{7.60}$$

We note that B and C go to zero for large x and therefore we see that for large x we have

$$\partial_t v = \begin{pmatrix} A_\infty & 0 \\ 0 & -A_\infty \end{pmatrix} v \qquad \text{as } x \to \pm\infty. \tag{7.61}$$

Here I introduced

$$A_\infty = \lim_{|x|\to\infty} A = 2ik^2. \tag{7.62}$$

Clearly this system has solutions that are linear combinations of

$$v^+ = \begin{pmatrix} e^{A_\infty t} \\ 0 \end{pmatrix}, \qquad v^- = \begin{pmatrix} 0 \\ e^{-A_\infty t} \end{pmatrix}. \tag{7.63}$$

However we will have to match this to the fixed boundary conditions of the Jost functions we used earlier. We define the time-dependent functions

$$\Phi(x,t) = e^{A_\infty t}\phi(x,t), \qquad \bar{\Phi}(x,t) = e^{A_{-\infty} t}\bar{\phi}(x,t), \tag{7.64}$$

$$\Psi(x,t) = e^{-A_\infty t}\psi(x,t), \qquad \bar{\Psi}(x,t) = e^{A_\infty t}\bar{\psi}(x,t), \tag{7.65}$$

to be solutions of equation (7.60). From this it is easy to see that ϕ and $\bar{\phi}$ satisfy

$$\partial_t \phi = \begin{pmatrix} A - A_\infty & B \\ C & -A - A_\infty \end{pmatrix} \phi, \qquad \partial_t \bar{\phi} = \begin{pmatrix} A + A_\infty & B \\ C & -A + A_\infty \end{pmatrix} \bar{\phi} \tag{7.66}$$

and similarly ψ and $\bar{\psi}$ satisfy

$$\partial_t \psi = \begin{pmatrix} A + A_\infty & B \\ C & -A + A_\infty \end{pmatrix} \psi, \qquad \partial_t \bar{\psi} = \begin{pmatrix} A - A_\infty & B \\ C & -A - A_\infty \end{pmatrix} \bar{\psi} \tag{7.67}$$

If we now take the time derivative of equation (7.31) we see that we get the equation

$$\partial_t \phi = \partial_t b(k)\psi + b(k)\partial_t \psi + \partial_t a(k)\bar{\psi} + a(k)\partial_t \bar{\psi}. \tag{7.68}$$

Using the forms we found above for the time derivatives of ϕ and ψ and taking the limit for $x \to \infty$ we get the equation

$$\begin{pmatrix} 0 & 0 \\ 0 & -2A_\infty \end{pmatrix} \phi = \partial_t b(k)\psi + b(k) \begin{pmatrix} -2A_\infty & 0 \\ 0 & 0 \end{pmatrix} \psi + \partial_t a(k)\bar{\psi} + a(k) \begin{pmatrix} 0 & 0 \\ 0 & -2A_\infty \end{pmatrix} \bar{\psi}. \tag{7.69}$$

So it is clear that we get

$$\partial_t a(k) = 0, \tag{7.70}$$

$$\partial_t b(k) = -2A_\infty b(k) \tag{7.71}$$

and from doing exactly the same with equation (7.32) we find

$$\partial_t \bar{a}(k) = 0, \tag{7.72}$$

$$\partial_t \bar{b}(k) = 2A_\infty \bar{b}(k) \tag{7.73}$$

So we get

$$a(k,t) = a(k,0), \qquad \bar{a}(k,t) = \bar{a}(k,0), \tag{7.74}$$

$$b(k,t) = e^{-4ik^2 t} b(k,0), \qquad \bar{b}(k,t) = e^{4ik^2 t} \bar{b}(k,0). \tag{7.75}$$

As we see that $a(k)$ and $\bar{a}(k)$ are independent of time, so are the proper eigenvalues, as they correspond to their zeroes. Furthermore we find that

$$\rho(k,t) = \frac{b(k,t)}{a(k,t)} = e^{-4ik^2 t} \rho(k,0), \tag{7.76}$$

$$\bar{\rho}(k,t) = \frac{\bar{b}(k,t)}{\bar{a}(k,t)} = e^{4ik^2 t} \bar{\rho}(k,0). \tag{7.77}$$

If we take the time derivative of equation (7.40) and then take the limit for $x \to \infty$ we get in a similar way as we did above

$$c_j(t) = c_j(0) e^{-4ik_j^2 t}, \qquad \bar{c}_j(t) = \bar{c}_j(0) e^{4ik_j^2 t} \tag{7.78}$$

and therefore we get

$$C_j(t) = C_j(0) e^{-4ik_j^2 t}, \qquad \bar{C}_j(t) = \bar{C}_j(0) e^{4ik_j^2 t}. \tag{7.79}$$

We have now found how all the scattering data evolve in time and therefore we have everything we need to perform the inverse scattering transform. In the next chapter I will use this to find 1- and 2-soliton solutions to the NLS-equation and after that I will also use this to derive soliton solutions to the mKdV and SG equations.

Chapter 8

Soliton solutions to the NLS

Now we can use the IST to find the soliton solutions to the NLS. To make everything as clear as possible I will start by deriving the 1-soliton solution in Section 8.1 before showing how IST can be used to obtain the N-soliton solutions in Section 8.2. This I will then illustrate by deriving the 2-soliton solution explicitly in Section 8.3. Finally in part 8.4 I will show how we can find a different 2-soliton solution if we start with a different initial condition.

8.1 One soliton solution

Direct scattering

We have to start by choosing the initial potential to do the direct scattering with. It turns out that the initial condition

$$q = -r^* = 2\eta e^{-2i\xi x}\text{sech}(2\eta x), \tag{8.1}$$

will give us the 1-soliton solution. Now we need to insert this into equation (7.2). From the first equation we can find

$$v^{(2)} = \frac{v_x^{(1)} + ikv^{(1)}}{q} \tag{8.2}$$

and if we insert this into the second equation we get

$$q\left(v_{xx}^{(1)} + k^2 v^{(1)} - rqv^{(1)}\right) - q_x\left(v_x^{(1)} - ikv^{(1)}\right) = 0. \tag{8.3}$$

If we now use equation (8.1) we get

$$\begin{aligned}\cosh^2(2\eta x)v_x^{(1)}x + 2i\xi\cosh^2(2\eta x)v_x^{(1)} + 2\eta\cosh(2\eta x)\sinh(2\eta x)v_x^{(1)} - 2k\xi\cosh^2(2\eta x)v^{(1)}\\ +2ik\eta\cosh(2\eta x)\sinh(2\eta x)v^{(1)} + k^2\cosh^2(\eta x)v^{(1)} + 4\eta^2 v^{(1)} = 0.\end{aligned} \tag{8.4}$$

We can simplify this by taking $v^{(1)}(x) = \Phi(y)$ with $y = \tanh(2\eta x)$ and then we get

$$\frac{\mathrm{d}}{\mathrm{d}x} = 2\eta(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y}, \quad \frac{\mathrm{d}^2}{\mathrm{d}x^2} = 4\eta^2(1-y^2)\frac{\mathrm{d}}{\mathrm{d}y}\left((1-y^2)\frac{\mathrm{d}}{\mathrm{d}y}\right). \tag{8.5}$$

Therefore equation (8.4) reduces to

$$\begin{aligned}&(-4\eta^2y^4 + 8\eta^2y^2 - 4\eta^2)\Phi_{yy} + (4\eta^2y - 4\eta^2y^3 - 4i\eta\xi + 4i\eta\xi y^2)\Phi_y\\ &+(2k\xi - 2ik\eta y - k^2 - 4\eta^2 + 4\eta^2y^2)\Phi = 0.\end{aligned} \tag{8.6}$$

This equation can be solved analytically and MAPLE gives the solution

$$\phi = A\left(\frac{y-1}{y+1}\right)^{\frac{ik}{4\eta}}(-2i\eta y+2k-2\xi)+B\left(\frac{y+1}{y-1}\right)^{\frac{ik-2i\xi}{4\eta}}(\sqrt{y^2-1}). \tag{8.7}$$

If we now substitute x back in we get after some rewriting

$$v^{(1)} = 2A(-1)^{\frac{ik}{4\eta}}e^{-ikx}(-i\eta\tanh(2\eta x)+k-\xi)+iB(-1)^{\frac{ik}{4\eta}}e^{i(k-2\xi)x}\text{sech}(2\eta x). \tag{8.8}$$

We now demand that our solution goes as $v^{(1)}(x) \sim e^{-ikx}$ for $x \to -\infty$ as demanded in equation (7.7) and we see that this results in $B = 0$. Before we can determine the value of A we need to fix the sign of η. From now on I will assume η to be positive, but the calculation for negative η is exactly the same apart from some sign changes. We therefore get

$$A = \frac{1}{2(-1)^{\frac{ik}{4\eta}}(i\eta+k-\xi)} \tag{8.9}$$

and thus equation (8.8) reduces to

$$v^{(1)} = \frac{-i\eta\tanh(2\eta x)+k-\xi}{i\eta+k-\xi}e^{-ikx} \tag{8.10}$$

and from equation (8.2) we find

$$v^{(2)} = -\frac{i\eta e^{-i(k-2\xi)x}}{(i\eta+k-\xi)\cosh(2\eta x)}. \tag{8.11}$$

We note that $\lim_{x\to-\infty} v^{(2)} = 0$ and therefore we have now found the eigenfunction ϕ as defined in (7.7). If we now examine the behavior for $x \to \infty$ we can find $a(k)$ and $b(k)$ from equation (7.31) (note that these in principle only hold for $\Im(k) = 0$). We see that if we fix $\Im(k) = 0$ we get for $x \to \infty$

$$v^{(1)} \sim \frac{-i\eta+k-\xi}{i\eta+k-\xi}e^{-ikx} \tag{8.12}$$

$$v^{(2)} \sim 0. \tag{8.13}$$

Therefore we find that $b(k) = 0$ for $k \in \mathbf{R}$ and

$$a(k) = \frac{-i\eta+k-\xi}{i\eta+k-\xi}, \tag{8.14}$$

where we note that as we assumed η to be positive $a(k)$ can indeed be analytically continued in the upper half-plane and has only one zero and therefore one discrete eigenvalue k_1 given by

$$k_1 = \xi + i\eta. \tag{8.15}$$

Furthermore we also note that $\rho(k) = 0$ for $k \in \mathbf{R}$. To calculate c_1 we now look at the behavior of $v^{(1)}(x, k_1)$ and $v^{(2)}(x, k_1)$ as $x \to \infty$. Not surprisingly we find

$$v^{(1)}(x, k_1) = 0, \tag{8.16}$$

which was expected as $a(k_1) = 0$, and

$$v^{(2)}(x, k_1) = -e^{ikx}. \tag{8.17}$$

We therefore find from equation (7.40) that

$$c_1 = -1. \tag{8.18}$$

From equation (7.55) it now follows that

$$C_1 = -2i\eta \tag{8.19}$$

and with this result all the scattering data are found and the direct scattering problem is complete.

Time evolution

From equations (7.74)-(7.79) we can easily derive that

$$\begin{aligned} a(k,t) &= a(k,0) = \frac{-i\eta + k - \xi}{i\eta + k - \xi} && (8.20)\\ b(k,t) &= b(k,0)e^{-4ik^2t} = 0 && (8.21)\\ \rho(k,t) &= \rho(k,0)e^{-4ik^2t} = 0 && (8.22)\\ C_1(t) &= C_1(0)e^{-4ik^2t} = -2i\eta e^{-4ik^2t}. && (8.23)\end{aligned}$$

So the time evolution is known and we therefore now have all the information we need to do the inverse scattering.

Inverse scattering

As $\rho(k,t) = 0$ we see that the set of equations (7.53), (7.54), (7.56) and (7.57) reduce to

$$\bar{N}_l(x) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{J} \frac{C_j e^{2ik_j x} N_j(x)}{\bar{k}_l - k_j} \tag{8.24}$$

$$N_l(x) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \sum_{j=1}^{\bar{J}} \frac{\bar{C}_j e^{-2ik_j x} \bar{N}_j(x)}{k_l - \bar{k}_j}. \tag{8.25}$$

And if we use the scattering data above then this system can easily be solved to give N_1 and $\bar{N}_1$. After some algebra we then find from equation (7.59)

$$q(x,t) = -2i\eta \frac{C_1(t)^*}{|C_1(t)|} e^{-2i\xi x} \text{sech}(2\eta x - 2\delta), \tag{8.26}$$

with $2\delta = \ln(|C_1(t)|/(2\eta))$. If we insert the value of $C_1(t)$ found above we get

$$q(x,t) = 2\eta e^{-2i\xi x + 4i(\xi^2 - \eta^2)t} \text{sech}(2\eta x - 8\xi\eta t). \tag{8.27}$$

We have found the 1-soliton solution and we see that for $t = 0$ this indeed reduces to equation (8.1). Furthermore we note that the form is similar to the one found from the bilinear form.

8.2 Direct scattering for N-soliton solutions

To find the N-soliton solutions we use the same approach, we only modify the initial potential slightly:

$$q = -r^* = Qe^{-2i\xi x} \text{sech}(2\eta x), \tag{8.28}$$

with Q a constant to be determined. Again we can insert this in the system of differential equations (7.2) and substitute $y = \tanh(x)$. Now we find the solution $v^{(1)}$ by using MAPLE to be

$$\begin{aligned} v^{(1)} &= A\, {}_2F_1\left(\alpha, -\alpha; \gamma; \frac{1}{1+e^{-4\eta x}}\right) e^{-ikx} \\ &+ B\, {}_2F_1\left(\gamma+\alpha+1, \gamma-\alpha+1; \gamma+1; \frac{1}{1+e^{-4\eta x}}\right) \frac{(\text{sech}^2(2\eta x))^{\frac{-2i\xi x + ikx}{4\eta}}}{\sqrt{1+e^{-4\eta x}}}, \end{aligned} \tag{8.29}$$

where A and B are constants, ${}_2F_1$ denotes the hypergeometric function,

$$\alpha = \frac{Q}{2\eta} \qquad \text{and} \qquad \gamma = \frac{\eta + i\xi - ik}{2\eta}. \tag{8.30}$$

Note that I will modify the constants A and B from time to time without renaming them or explicitly mentioning this, as their exact value still has to be fixed anyway. We again want to fix the behavior of $v^{(1)}$ for $x \to -\infty$ and we see that if $\eta > 0$ then we get in this limit that

$$v^{(1)} \sim Ae^{-ikx} + Be^{2\eta x - 2i\xi x}, \tag{8.31}$$

where I used that ${}_2F_1(\alpha, \beta; \gamma; 0) = 1$. We see that we therefore get $A = 1$ and $B = 0$. So we find

$$v^{(1)} = {}_2F_1\left(\alpha, -\alpha; \gamma; \frac{1}{1+e^{-4\eta x}}\right) e^{-ikx} \tag{8.32}$$

and inserting this into equation (8.2) we get after some rewriting

$$v^{(2)} = \frac{Q\, {}_2F_1\left(\alpha+1, -\alpha+1; \gamma+1; \frac{1}{1+e^{-4\eta x}}\right) e^{2i\xi x - ikx}}{2(-\eta - i\xi + ik)\cosh(2\eta x)}. \tag{8.33}$$

We note that as required $v^{(2)} \to 0$ for $x \to -\infty$. In order to find $a(k)$ and $b(k)$ we now have to examine the behavior of v as $x \to \infty$. For this I need the following identity

$$\begin{aligned} {}_2F_1(\alpha, \beta; \gamma; z) &= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}\, {}_2F_1(\alpha, \beta; \alpha+\beta-\gamma+1; 1-z) \\ &+ (1-z)^{\gamma-\alpha-\beta} \frac{\Gamma(\gamma)\Gamma(-\gamma+\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\, {}_2F_1(\gamma-\alpha, \gamma-\beta; -\alpha-\beta+\gamma+1; 1-z). \end{aligned} \tag{8.34}$$

Using this we can rewrite equation (8.33) in the form

$$\begin{aligned} v^{(2)} &= \frac{Q\, e^{2i\xi x - ikx}}{2(-\eta - i\xi + ik)\cosh(2\eta x)} \left\{ \frac{\Gamma(\gamma+1)\Gamma(\gamma-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)}\, {}_2F_1\left(\alpha+1, -\alpha+1; 1-\gamma; \frac{1}{1+e^{4\eta x}}\right) \right. \\ &+ \left. \left(\frac{1}{1+e^{4\eta x}}\right)^{\gamma-1} \frac{\Gamma(\gamma+1)\Gamma(-\gamma+1)}{\Gamma(\alpha+1)\Gamma(-\alpha+1)}\, {}_2F_1(\gamma-\alpha, \gamma+\alpha; \gamma; \frac{1}{1+e^{4\eta x}}) \right\} \end{aligned} \tag{8.35}$$

and if we assume $\Im(k) = 0$ we see that for $x \to \infty$ we get

$$v^{(2)} \sim \frac{\Gamma(\gamma+1)\Gamma(-\gamma+1)}{\Gamma(\alpha+1)\Gamma(-\alpha+1)} \frac{Q}{2(-\eta - i\xi + ik)} e^{ikx}. \tag{8.36}$$

We therefore find that

$$b(k) = \frac{\Gamma(\gamma+1)\Gamma(-\gamma+1)}{\Gamma(\alpha+1)\Gamma(-\alpha+1)} \frac{Q}{2(-\eta - i\xi + ik)} \tag{8.37}$$

and so we have found the reflection coefficient. As soliton solutions move through the potential unchanged we now require $b(k) = 0$ for all k and we will therefore look at the poles of $\Gamma(\alpha+1)\Gamma(-\alpha+1)$. Note that this has a pole when $\alpha = n$ with $n \in \mathbf{Z}$ and $n \neq 0$. Therefore we find

$$Q = 2n\eta \tag{8.38}$$

and if we take n positive this will give us the n-soliton solution. Note that this indeed coincides with the Q used to derive the 1-soliton solution above. We can rewrite equation (8.32) in a similar fashion using identity (8.34) and for $x \to \infty$ we get

$$v^{(1)} \sim \frac{\Gamma^2(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)} e^{-ikx} \tag{8.39}$$

and therefore we find

$$a(k) = \frac{\Gamma^2(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)}. \tag{8.40}$$

Next we need to look at the zeroes of a in order to find the discrete spectrum. Using $\alpha = n$ (with $n \in \mathbf{Z}$), we see that this is for $\gamma = -m \pm n$ with $m \in \mathbf{N}$ and with the condition that $\gamma > 0$. This last condition comes along because it is easy to see that $\gamma \in \mathbf{Z}$ and therefore $\Gamma(\gamma)$ would have a pole for $\gamma \leq 0$. If we assume $n > 0$ then we see that we must have

$$\gamma = \frac{\eta + i\xi - ik}{2\eta} = n - m, \tag{8.41}$$

and therefore we note that m can have values between 0 and $n-1$. If we now take k_m to be the value of k that satisfies equation (8.41) we get that for $x \to \infty$

$$v^{(1)}(x, k_m) \sim 0 \tag{8.42}$$

and

$$v^{(2)} \sim -\frac{\sin(\pi(1+n))}{\sin(\pi(1-m+n))} e^{ikx} \tag{8.43}$$

so from equation (7.40) we see that

$$c_m = \begin{cases} 1 & \text{if } m = n+1+2l; \\ -1 & \text{if } m = n+2l, \end{cases} \tag{8.44}$$

where $l \in \mathbf{Z}$. So we now have found all the scattering data and the direct scattering problem is solved.

8.3 2-Soliton solution

In this section I will try to explicitly derive the 2-soliton solution using the scattering data derived above. I will start, as always, with the direct problem.

Direct problem

To find the 2-soliton solution I fix $n = 2$ and we see that we then get $Q = 4\eta$. Furthermore from equation (8.41) we find that the discrete spectrum is given by

$$k_0 = \xi + 3i\eta \qquad \text{and} \qquad k_1 = \xi + i\eta \tag{8.45}$$

and therefore we get

$$c_0 = -1 \qquad \text{and} \qquad c_1 = 1. \tag{8.46}$$

From equation (7.55) we find

$$C_0 = \frac{c_0}{a'(k_0)} = -12i\eta \tag{8.47}$$

$$C_1 = \frac{c_1}{a'(k_1)} = -4i\eta \tag{8.48}$$

and as we already knew that $\rho(k) = b(k) = 0$ we see that we have found all the scattering data.

Time evolution

From equations (7.74)-(7.79) we now see that

$$
\begin{aligned}
a(k,t) &= a(k,0) = \frac{\Gamma^2(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)} && (8.49)\\
b(k,t) &= b(k,0)e^{-4ik^2t} = 0 && (8.50)\\
\rho(k,t) &= \rho(k,0)e^{-4ik^2t} = 0 && (8.51)\\
C_0(t) &= C_0(0)e^{-4ik_0^2t} = -12i\eta e^{24\xi\eta t-4i\xi^2t+36i\eta^2t}. && (8.52)\\
C_1(t) &= C_1(0)e^{-4ik_1^2t} = -4i\eta e^{8\xi\eta t-4i\xi^2t+4i\eta^2t}. && (8.53)
\end{aligned}
$$

We have therefore now all the information we need to do the inverse scattering.

Inverse Scattering

We see that from the system of equations (7.53), (7.54), (7.56) and (7.57) we can deduce the following equations

$$
\begin{aligned}
\bar{N}_0^{(1)} &= 1+\frac{C_0e^{2ik_0x}N_0^{(1)}}{k_0-\bar{k}_0}+\frac{C_1e^{2ik_1x}N_1^{(1)}}{\bar{k}_0-k_1} && (8.54)\\
\bar{N}_1^{(1)} &= 1+\frac{C_0e^{2ik_0x}N_0^{(1)}}{k_1-\bar{k}_0}+\frac{C_1e^{2ik_1x}N_1^{(1)}}{\bar{k}_1-k_1} && (8.55)\\
N_0^{(1)} &= \frac{\bar{C}_0e^{-2i\bar{k}_0x}\bar{N}_0^{(1)}}{k_0-\bar{k}_0}+\frac{\bar{C}_1e^{-2i\bar{k}_1x}\bar{N}_1^{(1)}}{k_0-\bar{k}_1} && (8.56)\\
N_1^{(1)} &= \frac{\bar{C}_0e^{-2i\bar{k}_0x}\bar{N}_0^{(1)}}{k_1-\bar{k}_0}+\frac{\bar{C}_1e^{-2i\bar{k}_1x}\bar{N}_1^{(1)}}{k_1-\bar{k}_1}. && (8.57)
\end{aligned}
$$

If we now use the values of k_0 and k_1 we derived earlier then this system is easily solvable. Using equation (7.59) we therefore find

$$
q=\frac{32i\eta^2\left(9\bar{C}_1\bar{C}_0C_1e^{-2\eta x}+144\eta^2\bar{C}_0e^{2\eta x}+\bar{C}_1\bar{C}_0C_1e^{-6\eta x}+144\eta^2\bar{C}_1e^{6\eta x}\right)e^{(-2i\xi-8\eta)x}}{144\bar{C}_1C_0\eta^2e^{-8\eta x}+|C_0C_1|^2e^{-16\eta x}+576|C_1|^2\eta^2e^{-4\eta x}+2304\eta^4+144\eta^2C_1\bar{C}_0e^{-8\eta x}+64|C_0|^2\eta^2e^{-12\eta x}}
$$

and we see that this can be rewritten in the form

$$
q=\frac{8i\eta e^{-2i\xi x}\left(3|C_1|\bar{C}_0\cosh\left(2\eta x+\ln\left(\frac{4\eta}{|C_1|}\right)\right)+|C_0|\bar{C}_1\cosh\left(6\eta x+\ln\left(\frac{12\eta}{|C_0|}\right)\right)\right)}{\frac{3}{2}(\bar{C}_0C_1+C_0\bar{C}_1)+|C_0C_1|\cosh\left(8\eta x+\ln\left(\frac{48\eta^2}{|C_0C_1|}\right)\right)+4|C_0C_1|\cosh\left(4\eta x+\ln\left(\frac{3|C_1|}{|C_0|}\right)\right)}. \quad (8.58)
$$

If we now use equations (8.52) and (8.53) we find the solution

$$
q=\frac{8\eta\exp(-2i\xi x+4i(\xi^2-\eta^2)t)\left(3e^{-32i\eta^2t}\cosh(2\eta x-8\eta\xi t)+\cosh(6\eta x-24\xi\eta t)\right)}{3\cos(32\eta^2t)+\cosh(8\eta x-32\eta\xi t)+4\cosh(4\eta x-16\eta\xi t)}, \quad (8.59)
$$

where I also used that $\bar{C}_j=-C_j^*$ as can be seen from equations (7.47), (7.49) and (7.55). We see that this solution reduces to equation (8.28) for $t=0$ and can also verify directly that it satisfies the FNLS equation. The solution is shown in Figure 8.1. We see that we do not have the expected 2-soliton behavior. This is because the real parts of k_0 and k_1 are equal and therefore both solitons move with the same speed. They therefore interact constantly leading to breather like behavior. Note that similar behavior was observed in some solutions obtained from the bilinear method as seen in parts (vii) and (viii) of Figure 6.1.

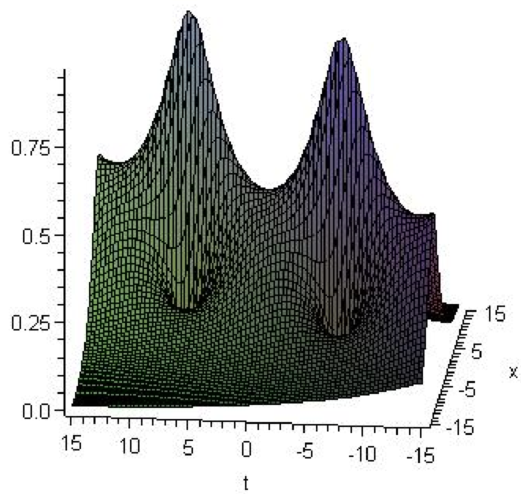

Figure 8.1: Absolute value of the solution given in equation (8.59) for $\xi = 0.1$ and $\eta = 0.12$.

8.4 2-Soliton solution with different initial condition

We have seen that we can derive the 2-soliton equation using Hirota's bilinear method and this is given by

$$u = \frac{\sqrt{B_{122}} e^{i\eta_{1,I}} \cosh(\eta_{2,R}) + \sqrt{B_{121}} e^{i\eta_{2,I}} \cosh(\eta_{1,R})}{\sqrt{C_{1212}} \cosh(\eta_{1,R} + \eta_{2,R}) + \sqrt{A_{11}A_{22}} \cosh(\eta_{1,R} - \eta_{2,R}) \pm \sqrt{A_{12}A_{21}} \cos(\eta_{1,I} - \eta_{2,I})}, \tag{8.60}$$

with

$$A_{ij} = \frac{p}{2(\kappa_i + \kappa_j^*)^2}, \tag{8.61}$$

$$B_{ijk} = \frac{p(\kappa_i - \kappa_j)^2}{2(\kappa_i + \kappa_k^*)^2(\kappa_j + \kappa_k^*)^2}, \tag{8.62}$$

$$C_{ijkl} = \frac{p^2(\kappa_i - \kappa_j)^2(\kappa_k^* - \kappa_l^*)^2}{4(\kappa_i + \kappa_k^*)^2(\kappa_i + \kappa_l^*)^2(\kappa_j + \kappa_k^*)^2(\kappa_j + \kappa_l^*)^2}. \tag{8.63}$$

Furthermore we saw above that if we use the initial condition $q = Q\eta e^{-2i\xi x}\text{sech}(2\eta x)$ we can find the following 2-soliton solution from inverse scattering

$$q = \frac{8\eta e^{-2i\xi x + 4i(\xi^2 - \eta^2)t} \left(3e^{-32i\eta^2 t} \cosh(2\eta x - 8\eta\xi t) + \cosh(6\eta x - 24\xi\eta t)\right)}{3\cos(32\eta^2 t) + \cosh(8\eta x - 32\eta\xi t) + 4\cosh(4\eta x - 16\eta\xi t)}. \tag{8.64}$$

We see that this corresponds to putting $\kappa_1 = 2\eta + 2i\xi$ and $\kappa_2 = 6\eta + 2i\xi$ in equation (8.60). I want to see if it is possible to derive solutions corresponding to different values of κ_1 and κ_2 using inverse scattering. First of all I want to try using a κ_2 with a different imaginary part then κ_1 as this would correspond to 2 solitons traveling at different speeds. However we see that if $\Im(\kappa_1) \neq \Im(\kappa_2)$ then $\eta_{1,I} - \eta_{2,I}$ will depend on x and not vanish for $t = 0$. Therefore the trigonometric term in equation (8.60) contains an x and this presents us with a problem in the direct problem. Now the substitution $x = \tanh(y)$ no longer works and therefore the direct problem seems unsolvable. For instance for $\kappa_1 = 2 + i$ and $\kappa_2 = 1 + 3i$ we get

$$u(,0) = \frac{\sqrt{2}\cosh(x)e^{ix}(1 - i + e^{2ix} + ie^{2ix})}{\cosh^2(x) + \cos^2(x)} \tag{8.65}$$

and we see that there is no substitution that makes this easy enough to make the direct problem solvable. However if $\Im(\kappa_1) = \Im(\kappa_2)$ I have been able to do inverse scattering for another value than found earlier. If we take $\kappa_1 = 1 + i$ and $\kappa_2 = 2 + i$ we get

$$u(,0) = \frac{3\cosh(x)e^{-ix}(\cosh(2x) + 2\cosh(x))}{2\cosh^3(x) + 3\cosh(x) + 4}. \tag{8.66}$$

Now we can do the same substitutions as before and solve the resulting equation using MAPLE. I will not give the eigenfunction here as it is very lengthy, but it is quite easy to obtain the scattering date from it. Indeed we find $b(k) = 0$ as expected and

$$a(k) = \frac{4k^2 - 4k - 6ik - 1 + 3i}{4k^2 - 4k + 6ik - 1 - 3i}. \tag{8.67}$$

So we find that

$$k_1 = \frac{1}{2} + \frac{i}{2} \qquad \text{and} \qquad k_2 = \frac{1}{2} + i \tag{8.68}$$

and note that this corresponds to $\kappa_1 = 2 + i$ and $\kappa_2 = 1 + 2i$ as we would expect. Next we find that $c_1 = 1$ and $c_2 = -1$ and therefore we get

$$C_1 = -3i \qquad \text{and} \qquad C_2 = -6i. \tag{8.69}$$

Now the direct scattering part is completed and we have obtained the scattering data. The time evolution and inverse scattering can be done in exactly the same way as before and so we find after some rewriting

$$q = \frac{6e^{-ix}\cosh(2x - 4t) + 12e^{-4ix}\cosh(x - 2t)}{\cosh(3x + 6t) + 9\cosh(x + 2t) + 8\cos(3t)}. \tag{8.70}$$

So we see that we can obtain solutions to the NLS equation using inverse scattering for different initial conditions, but only if the solitons are traveling at the same speed as otherwise the trigonometric terms will present us with a problem. We will see that this problem will arise more often.

8.5 Discussion

We have seen that we can also obtain 1- and 2-soliton solutions to the FNLS equation from inverse scattering. We only find solutions to the FNLS because in inverse scattering it is required that the solutions are rapidly decaying for large x. However, as we saw earlier, the soliton solutions to the DNLS do not decay for large x and can therefore not be obtained from inverse scattering in this way. As said before the IST for the NLS equation was first derived by Zakharov and Shabat [32] and they found the soliton solutions. Since them a lot of theory has been written about it. I mainly used the material in Chapter 2 of [2]. The authors approach is somewhat different from mine. They do not explicitly solve the direct scattering problem, but assume that $\rho = 0$ and then derive the 1-soliton solution from equations (7.53)-(7.57). They furthermore do not derive the 2-soliton solution which becomes very hard to do without explicitly solving the direct scattering problem.

Chapter 9

Soliton solutions to the mKdV

Here I will apply the inverse scattering transform to the modified Korteweg-de Vries Equation (mKdV) given by

$$q_t + 6pq^2 q_x + q_{xxx} = 0, \tag{9.1}$$

where $p = \pm 1$. This equation was derived in 1968 by Miura [23] and he showed that if q is a solution to the mKdV equation with $p = -1$ then

$$u = -q^2 - q_x \tag{9.2}$$

is a solution to the KdV equation (2.7). Furthermore we note that if q is a solution to equation (9.1) with $p = +1$ then iq is a solution corresponding to $p = -1$. I will do the scattering transform in the case that $p = -1$, but at the end I will transform the solution to the $p = +1$ case. I will do this as only in the $p = +1$ case the soliton solution is real valued and we obviously want a real valued solution to this equation as the equation itself is a real valued equation. Similar to the NLS equation the Lax pair is also given by two linear equations [29]:

$$v_x = \begin{pmatrix} -ik & q \\ q & ik \end{pmatrix} v \tag{9.3}$$

and

$$v_t = \begin{pmatrix} -4ik^3 - 2iq^2 k & 4qk^2 + 2iq_x k + 2q^3 - q_{xx} \\ 4qk^2 - 2iq_x k + 2q^3 - q_{xx} & 4ik^3 + 2iq^2 k \end{pmatrix} v. \tag{9.4}$$

We note that the scattering problem is the same as for the NLS only now we have $r = q$ instead of $r = -q^*$. Therefore it is not surprising that the direct and inverse problems stay almost the same. Furthermore as we will see that q is a pure imaginary function, we also have $r = -q^*$ and therefore all the symmetry reductions obtained for the NLS equation still hold. Especially we still have $\bar{C}_j = -C_j^*$, which we will need later on. I will again first do inverse scattering with the easiest possible potential, a sech-shaped potential, before showing that it can also be done in a more difficult case. Finally I will investigate if we can also obtain breather solutions from inverse scattering.

9.1 Direct scattering for N-soliton solutions

As said before I will start again with a sech-shaped potential. This time I will directly put in the constant Q and therefore our potential becomes

$$q = Q\text{sech}(\eta x). \tag{9.5}$$

We note the similarity with the initial condition for the NLS, although we see that the complex exponent is not present in this case. Again we can insert this in the system of differential equations (7.2) and substitute $y = \tanh(x)$. The resulting system can be solved by MAPLE and if we match it to the initial condition such that $v^{(1)} \to e^{-ikx}$ for $x \to -\infty$ we find

$$v^{(1)} = \ {}_2F_1\left(\alpha, -\alpha; \gamma; \frac{1}{1+e^{-2\eta x}}\right) e^{-ikx} \tag{9.6}$$

where ${}_2F_1$ denotes the hypergeometric function,

$$\alpha = \frac{iQ}{\eta} \qquad \text{and} \qquad \gamma = \frac{\eta - 2ik}{2\eta}, \tag{9.7}$$

where I assumed η to be positive in order to be able to determine the behavior in the limit of $x \to -\infty$. From this and the original scattering problem we find that

$$v^{(2)} = -Q\ {}_2F_1\left(1+\alpha, 1-\alpha; 1+\gamma; \frac{1}{1+e^{-2\eta x}}\right) \frac{e^{-ikx}}{(2ik-\eta)\cosh(\eta x)} \tag{9.8}$$

We note that as required $v^{(2)} \to 0$ for $x \to -\infty$. In order to find $a(k)$ and $b(k)$ we now have to examine the behavior of v as $x \to \infty$. I will need again the identity (8.34). We can rewrite equation (9.8) in the form

$$\begin{aligned} v^{(2)} &= \left(\frac{-Q\, e^{-ikx}}{(2ik-\eta)\cosh(\eta x)}\right)\{\frac{\Gamma(\gamma+1)\Gamma(\gamma-1)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)}\ {}_2F_1\left(1+\alpha, 1-\alpha; 1-\gamma; \frac{1}{1+e^{2\eta x}}\right) \\ &+ \left(\frac{1}{1+e^{2\eta x}}\right)^{\gamma-1} \frac{\Gamma(1+\gamma)\Gamma(1-\gamma)}{\Gamma(1+\alpha)\Gamma(1-\alpha)}\ {}_2F_1(\gamma-\alpha, \gamma+\alpha; \gamma; \frac{1}{1+e^{4\eta x}})\}. \end{aligned} \tag{9.9}$$

We see that in the limit $x \to \infty$ we get

$$v^{(2)} \sim \frac{\Gamma(\gamma+1)\Gamma(-\gamma+1)}{\Gamma(\alpha+1)\Gamma(-\alpha+1)} \frac{Q}{2ik-\eta} e^{ikx}. \tag{9.10}$$

We therefore find that

$$b(k) = \frac{\Gamma(\gamma+1)\Gamma(-\gamma+1)}{\Gamma(\alpha+1)\Gamma(-\alpha+1)} \frac{Q}{2ik-\eta} \tag{9.11}$$

and so we have found the reflection coefficient. Again we require $b(k) = 0$ for all k and we will therefore look at the poles of $\Gamma(\alpha+1)\Gamma(-\alpha+1)$. Note that this has a pole when $\alpha = n$ with $n \in \mathbf{Z}$ and $n \neq 0$. Therefore we find

$$Q = -2in\eta \tag{9.12}$$

and if we take n positive this will give us the n-soliton solution. Note that if η is real our initial condition is indeed purely imaginary as we required such that the symmetry reductions still hold. We can rewrite equation (9.6) in a similar fashion using identity (8.34) and for $x \to \infty$ we get

$$v^{(1)} \sim \frac{\Gamma^2(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)} e^{-ikx} \tag{9.13}$$

and therefore we find

$$a(k) = \frac{\Gamma^2(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)}. \tag{9.14}$$

Notice that the formulas for $a(k)$ and $b(k)$ have not changed in comparison to the NLS equation, only the definition of α and γ is different. Next we need to look at the zeroes of a in order to find the discrete spectrum. Using $\alpha = n$ (with $n \in \mathbf{Z}$) we see that this is again for $\gamma = -m \pm n$ with $m \in \mathbf{N}$ and also again with the condition that $\gamma > 0$. This last condition comes along for exactly the same reasons as before. If we assume $n > 0$ then we see that we must have

$$\gamma = \frac{\eta - 2ik}{2\eta} = n - m, \tag{9.15}$$

and therefore we note that m can have values between 0 and $n-1$. In order to find the values of the c_j we need to fix the exact values of m and n, as it is not as easy here to give a general formula for them. Therefore I will derive them when I give the 1- and 2-soliton solutions.

9.2 1-Soliton solution

Direct problem

We obtain the 1-soliton solution by setting $n = 1$ and therefore we get $m = 0$. So we see that $k_1 = \frac{i\eta}{2}$. Furthermore we now find that

$$v^{(2)}(x, k_1) = -\frac{ie^{\frac{\eta x}{2}}}{2\cosh(\eta x)} \tag{9.16}$$

and therefore we get

$$c_1 = \lim_{x \to +\infty} v^{(2)}(x, k_1) e^{-ik_1 x} = -i. \tag{9.17}$$

Using that $C_1 = \frac{c_1}{a'(k_1)}$ and equation (9.14) we find

$$C_1 = \eta. \tag{9.18}$$

So we have found all the scattering data and can proceed.

Time dependence

We see that the time dependence is governed by quite a different equation then we saw for the NLS equation. However in the limit for large x it is very similar, only now we have $A_\infty = -4ik^3$. Therefore we now get

$$\begin{aligned} a(k,t) &= a(k,0) = \frac{\Gamma^2(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)} && (9.19)\\ b(k,t) &= b(k,0)e^{8ik^2t} = 0 && (9.20)\\ \rho(k,t) &= \rho(k,0)e^{8ik^2t} = 0 && (9.21)\\ C_1(t) &= C_1(0)e^{8ik_1^2t} = \eta e^{\eta^3 t}. && (9.22) \end{aligned}$$

As the same symmetry reductions hold as in the case of the NLS equation we can still find the scattering data with a bar from equations (7.47)-(7.49). Therefore we have everything we need for the inverse scattering.

Inverse problem

The inverse problem is also very similar to that for the NLS equation, so I will not go into to much detail here. From equations (7.59), (8.24) and (8.25) we can find

$$q = -i\eta \operatorname{sech}(\eta x - \eta^3 t), \tag{9.23}$$

where I used that $\bar{C}_1 = -C_1$ as C_1 is real. It is easy to verify that the thus obtained q is a solution to equation (9.1) with $p = -1$. However we see that as expected the solution is purely imaginary, whereas the differential equation it solves is real. We can solve this by taking η a purely imaginary number, but then the sech changes into a sec and we will have poles in our solution. However we know that our solution corresponds to a solution to equation (9.1) with $p = +1$ and this solution is given by

$$q = \eta \text{sech}(\eta x - \eta^3 t). \tag{9.24}$$

So we have now obtained the 1-soliton solution to the mKdV-equation with $p = +1$. In the next part I will derive a 2-soliton solution.

9.3 2-Soliton solution

Direct problem

To find the 2-soliton solution we take $n = 2$ and we see that we can now have $m = 0$ or $m = 1$. Therefore we find from equation (9.15) that $k_1 = \frac{i\eta}{2}$ and $k_2 = \frac{3i\eta}{2}$. If we insert this into equation (9.10) we get

$$v^{(2)}(x, k_1) = \frac{-i(\cosh(\eta x) - 3\sinh(\eta x))e^{\frac{\eta x}{2}}}{4\cosh^2(\eta x)} \tag{9.25}$$

$$v^{(2)}(x, k_2) = \frac{i(\sinh(\eta x) - \cosh(\eta x))e^{\frac{3\eta x}{2}}}{4\cosh^2(\eta x)}. \tag{9.26}$$

From this we easily find

$$c_1 = \lim_{x \to +\infty} v^{(2)}(x, k_1)e^{-ik_1 x} = i, \tag{9.27}$$

$$c_2 = \lim_{x \to +\infty} v^{(2)}(x, k_2)e^{-ik_2 x} = -i. \tag{9.28}$$

Now we can find C_j in a similar fashion as before and we get

$$C_1 = 2\eta \qquad \text{and} \qquad C_2 = 6\eta. \tag{9.29}$$

So we have again found all scattering data and can proceed to the time-dependence part.

Time dependence

This part is exactly the same as before and we get

$$a(k, t) = a(k, 0) = \frac{\Gamma^2(\gamma)}{\Gamma(\gamma - \alpha)\Gamma(\gamma + \alpha)} \tag{9.30}$$

$$b(k, t) = b(k, 0)e^{8ik^2 t} = 0 \tag{9.31}$$

$$\rho(k, t) = \rho(k, 0)e^{8ik^2 t} = 0 \tag{9.32}$$

$$C_1(t) = C_1(0)e^{8ik_1^2 t} = 2\eta e^{\eta^3 t} \tag{9.33}$$

$$C_2(t) = C_2(0)e^{8ik_2^2 t} = 6\eta e^{27\eta^3 t} \tag{9.34}$$

Inverse problem

I will not go into details here again as this is already done when I discussed the NLS equation but from equations (7.59) and (8.54)-(8.57) we find after some algebra

$$q = \frac{-4i\eta\left(\cosh\left(3\eta x - 27\eta^3 t\right) + 3\cosh\left(\eta x - \eta^3 t\right)\right)}{4\cosh\left(2\eta x - 26\eta^3 t\right) + \cosh\left(4\eta x + 28\eta^3 t\right) + 3}. \tag{9.35}$$

Again our solution is purely imaginary but we can map it to a real solution of the mKdV equation with $p = +1$ given by

$$q = \frac{4\eta\left(\cosh\left(3\eta x - 27\eta^3 t\right) + 3\cosh\left(\eta x - \eta^3 t\right)\right)}{4\cosh\left(2\eta x - 26\eta^3 t\right) + \cosh\left(4\eta x + 28\eta^3 t\right) + 3}. \tag{9.36}$$

We see that we have thus obtained the 2-soliton solution and this is plotted in Figure 9.1. We see that we have the normal 2-soliton behavior here, which we have seen many times before. So far we have however only obtained solutions in which the faster soliton travels exactly 9 times faster than the smaller one. In the next part I will show that it is also possible to obtain 2-soliton solutions when the faster one travels exactly four times as fast, however the algebra then becomes more involved.

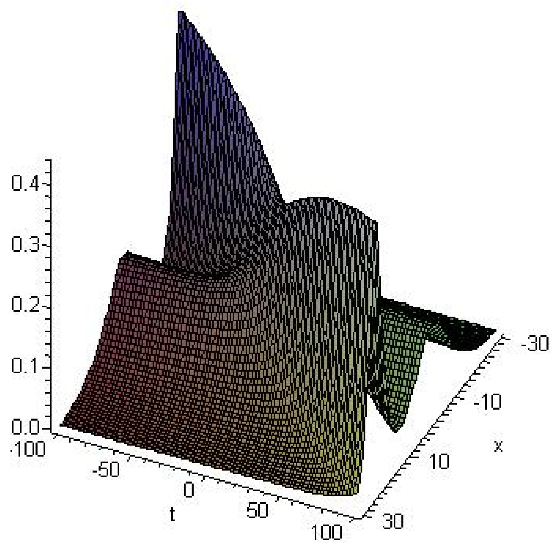

Figure 9.1: 2-soliton solution to the mKdV with $\eta = 0.15$

9.4 2-Soliton solution from different initial condition

Direct problem

To obtain the 2-soliton solution with relative speed equal to 4, I will need a different initial condition. As this one is so complicated that I cannot guess it, I derive it from the general 2-soliton solution to the mKdV, which for $p = +1$ is given by

$$q = \frac{2\sqrt{A}\left(\kappa_2\cosh\left(\kappa_1\eta x - \kappa_1^3 t\right) + \kappa_1\cosh\left(\kappa_2 x - \kappa_2^3 t\right)\right)}{1 - A + A\cosh\left((\kappa_1+\kappa_2)x - (\kappa_1^3+\kappa_2^3)t\right) + \cosh\left((\kappa_1-\kappa_2)x + (\kappa_1^3-\kappa_2^3)t\right)} \tag{9.37}$$

where

$$A = \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2}. \tag{9.38}$$

This solution can be obtained from the bilinear form in similar fashion as I did for other equations earlier and it is an easy calculation to verify that this is equal to the 2-soliton solution given in [15]. If we now choose $\kappa_1 = \eta$ and $\kappa_2 = 2\eta$ we see that the initial condition at $t = 0$ becomes (note that I moved to the $p = -1$ case here, so therefore the factor i)

$$q = \frac{3i\eta(2\cosh(\eta x) + 2\cosh^2(\eta x) - 1)}{4 + 2\cosh^3(\eta x) + 3\cosh(\eta x)}. \tag{9.39}$$

Now we can do the direct problem in exactly the same way as before and we can easily find $v^{(1)}$ and $v^{(2)}$. However as these have become quite lengthy expressions I will not give their full form here. From them we find that $b(k) = 0$ as expected and

$$a(k) = \frac{2k^2 - \eta^2 - 3ik\eta}{2k^2 - \eta^2 + 3ik\eta} \tag{9.40}$$

and therefore we find $k_1 = \frac{i\eta}{2}$ and $k_2 = i\eta$. In a similar way as before we obtain c_j from the limit of $v^{(2)}(x, k_j)$ for x to infinity. We find $c_1 = -i$ and $c_2 = i$ and this then gives

$$C_1 = -3\eta \qquad \text{and} \qquad C_2 = -6\eta. \tag{9.41}$$

So the scattering data is obtained.

Time dependence

The time dependence is done similar as before and we find

$$a(k,t) = a(k,0) = \frac{2k^2 - \eta^2 - 3ik\eta}{2k^2 - \eta^2 + 3ik\eta} \tag{9.42}$$

$$b(k,t) = b(k,0)e^{8ik^2t} = 0 \tag{9.43}$$

$$\rho(k,t) = \rho(k,0)e^{8ik^2t} = 0 \tag{9.44}$$

$$C_1(t) = C_1(0)e^{8ik_1^2t} = -3\eta e^{\eta^3 t} \tag{9.45}$$

$$C_2(t) = C_2(0)e^{8ik_2^2t} = -6\eta e^{8\eta^3 t} \tag{9.46}$$

Inverse problem

This is also done in exactly the same way as before and we find the solution of the mKdV with $p = -1$ to be

$$q = \frac{6i\eta\left(\cosh\left(2\eta x + 8\eta^3 t\right) + 2\cosh(\eta x + \eta^3 t)\right)}{8 + 9\cosh(\eta x + 7\eta^3 t) + \cosh(3\eta x + 9\eta^3 t)}. \tag{9.47}$$

Again we can map this to a real valued solution to the mKdV with $p = +1$, which becomes

$$q = \frac{6\eta\left(\cosh\left(2\eta x + 8\eta^3 t\right) + 2\cosh(\eta x + \eta^3 t)\right)}{8 + 9\cosh(\eta x + 7\eta^3 t) + \cosh(3\eta x + 9\eta^3 t)}. \tag{9.48}$$

So we see that for the mKdV we can obtain solutions traveling at different relative speeds to each other. I assume this is possible for quite a range of speeds, although I suspect that at some point the algebra might become to hard. However I did not investigate if and where this happens exactly.

9.5 Breather solutions

It is known that the mKdV equation also has breather solutions. These can be obtained from equation (9.37) by taking κ_1 and κ_2 to be complex and each others conjugate, similar as what we did for the Boussinesq and SWWE before. If we take $\kappa_1 = \mu + i\lambda$ and $\kappa_2 = \mu - i\lambda$ we get

$$q = \frac{4\lambda\mu(\mu\sinh(\eta_R)\cos(\eta_I) + \lambda\cosh(\eta_R)\sin(\eta_I))}{\lambda^2\cosh(2\eta_R) + \mu^2\cos(2\eta_I) + \lambda^2 + \mu^2}, \tag{9.49}$$

with

$$\eta_R = \mu x - (\mu^3 - 3\mu\lambda^2)t \qquad \text{and} \qquad \eta_I = \lambda x - (3\mu^2\lambda - \lambda^3)t. \tag{9.50}$$

Next I examined if a breather solution could be found from the inverse scattering transform. However we see that if $\mu \neq 0$ we again have a trigonometric term which does not vanish at $t = 0$ and this term presents us again with problems if we want to use the substitution $y = \tanh(x)$. It is known that the breather solution can also be written in the form

$$q = -2\frac{\partial}{\partial x}\arctan\left(\frac{\mu\cos(\eta_I)}{\lambda\cosh(\eta_R)}\right), \tag{9.51}$$

as can be verified by an elementary calculation. Therefore it might be a good idea to use the substitution

$$y = \arctan\left(\frac{\mu\cos(\lambda x)}{\lambda\cosh(\mu x)}\right) \tag{9.52}$$

instead of $y = \tanh(x)$. However this substitution has the problem that I am not able to invert it, i.e. write x as a function of y. This is however necessary in order to write $\frac{\partial}{\partial x}$ as an operator of y. Therefore this substitution does not work either. So far I have not found a substitution which enables me to do direct scattering with an initial condition that leads to breather behavior. I therefore doubt very much if it is at all possible.

9.6 Discussion

We have seen that we can obtain 1- and 2-soliton solutions for the mKdV equation from inverse scattering. However we have been unable to obtain breather solutions in this way. As mentioned before the inverse scattering transform for the mKdV was first carried out by Wadati in [29]. He already points out the similarity with the IST for the NLS equation and as a result presents his results in a very short note only listing the differences. He does not do the direct scattering part explicitly but assumes $b(k) = 0$ and derives the N-soliton solutions in a similar way as presented in the book by Ablowitz, Prinari and Trubatch [2] I mentioned before. Furthermore he does not discuss possible breather solutions, which strengthens my believe that it is not possible to obtain those from inverse scattering.

Chapter 10

Sine-Gordon equation

There is one more equation which has the same scattering problem. This is the Sine-Gordon (SG) equation, which is given by

$$u_{xt} = \sin(u). \tag{10.1}$$

It is known [1] to have the Lax pair consisting of

$$v_x = \begin{pmatrix} -ik & -\frac{1}{2}u_x \\ \frac{1}{2}u_x & ik \end{pmatrix} v \tag{10.2}$$

and

$$v_t = \frac{i}{4k} \begin{pmatrix} \cos(u) & \sin(u) \\ \sin(u) & \cos(u) \end{pmatrix} v. \tag{10.3}$$

If we take $q = -r = -\frac{1}{2}u_x$ we see that the scattering equation coincides with the one given in equation (7.2). Furthermore as we want u and therefore u_x to be real (as it is a solution to a real equation) we see that we still have $r = -q^*$ and therefore the symmetry reductions made in equations (7.47)-(7.49) still hold.

10.1 Direct scattering for N-solitons

We start with the same initial condition as before, the sech-shaped potential given by

$$q = Q\text{sech}(2kx) \tag{10.4}$$

and we see that the only difference with the calculations done for the mKdV equation above is that now we have $r = -q$ instead of $r = q$. Not surprisingly most results derived in Section 9.1 therefore still hold. This includes the eigenfunctions $v^{(1)}$ and $v^{(2)}$ as given in equation (9.6) and (9.8) as well as the reflection and transmission coefficients $a(k)$ and $b(k)$ as given in equations (9.14) and (9.11), but the α entering in this equations has changed slightly and is now given by

$$\alpha = \frac{Q}{\eta}, \tag{10.5}$$

whereas γ does not change. This then gives us that in order to have a reflectionless solution we need $Q = \eta n$ and we see that our q will indeed be real as we said earlier. As neither γ nor $a(k)$ has changed equation (9.15) is still correct and our proper eigenvalues are the same as for the mKdV. Now we can use this to derive the 1- and 2-soliton solutions.

10.2 1-soliton solution

Direct problem

Again we obtain the 1-soliton solution by setting $n = 1$ and therefore we get $m = 0$. So we see that $k_1 = \frac{i\eta}{2}$. Furthermore we now find that

$$v^{(2)}(x, k_1) = -\frac{e^{\frac{\eta x}{2}}}{2\cosh(\eta x)} \tag{10.6}$$

and therefore we get

$$c_1 = \lim_{x\to+\infty} v^{(2)}(x, k_1)e^{-ik_1 x} = -1. \tag{10.7}$$

Using that $C_1 = \frac{c_1}{a'(k_1)}$ and equation (9.14) we find

$$C_1 = -i\eta. \tag{10.8}$$

So we have found all the scattering data and can proceed.

Time dependence

We see that the time dependence is governed by quite a different equation then we saw in the setting of the mKdV equation. However in the limit for large x it is very similar, only now we have $A_\infty = \frac{i}{4k}$. Therefore we now get

$$\begin{aligned}
a(k,t) &= a(k,0) = \frac{\Gamma^2(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)} &(10.9)\\
b(k,t) &= b(k,0)e^{-\frac{it}{2k}} = 0 &(10.10)\\
\rho(k,t) &= \rho(k,0)e^{-\frac{it}{2k}} = 0 &(10.11)\\
C_1(t) &= C_1(0)e^{-\frac{it}{\eta}} = -i\eta e^{-\frac{t}{\eta}}. &(10.12)
\end{aligned}$$

As the same symmetry reductions hold as before we can still find the scattering data with a bar from equations (7.47)-(7.49). Therefore we have everything we need for the inverse scattering.

Inverse problem

The inverse problem is also very similar to what we did before, so I will not go into to much detail here. From equations (7.59), (8.24) and (8.25) we can find

$$q = \eta \operatorname{sech}(\eta x + \frac{t}{\eta}), \tag{10.13}$$

where I used that $\bar{C}_1 = -C_1^* = C_1$ as C_1 is purely imaginary. So we now have found q but we are not done yet. We still need to find u, which is given by

$$u(x) = -2\int_{-\infty}^{x} q(\xi)d\xi + A, \tag{10.14}$$

where A is an integration constant which we will need to determine by inserting the solution in the original equation (10.1). If we perform the integration and insert this into equation (10.1) we see that for $A = 0$ it is a solution so we get

$$u(x) = \pi + 2\arctan\left(\sinh\left(\eta x + \frac{t}{\eta}\right)\right) \tag{10.15}$$

and we have now found the 1-soliton solution.

10.3 2-soliton solution: kink-kink solution

Direct problem

To find the 2-soliton solution we take $n = 2$ and we see that we can now have $m = 0$ or $m = 1$. Therefore we find from equation (9.15) that $k_1 = \frac{i\eta}{2}$ and $k_2 = \frac{3i\eta}{2}$. If we insert this into equation (9.10) we get

$$v^{(2)}(x, k_1) = \frac{-(\cosh(\eta x) - 3\sinh(\eta x))e^{\frac{\eta x}{2}}}{4\cosh^2(\eta x)} \tag{10.16}$$

$$v^{(2)}(x, k_2) = \frac{(\sinh(\eta x) - \cosh(\eta x))e^{\frac{3\eta x}{2}}}{4\cosh^2(\eta x)}. \tag{10.17}$$

From this we easily find

$$c_1 = \lim_{x\to+\infty} v^{(2)}(x, k_1)e^{-ik_1 x} = -1, \tag{10.18}$$

$$c_2 = \lim_{x\to+\infty} v^{(2)}(x, k_2)e^{-ik_2 x} = 1. \tag{10.19}$$

Now we can find C_j in a similar fashion as before and we get

$$C_1 = 2i\eta \qquad \text{and} \qquad C_2 = 6i\eta. \tag{10.20}$$

So we have again found all scattering data and can proceed to the time-dependence part.

Time dependence

This part is exactly the same as before and we get

$$\begin{aligned} a(k,t) &= a(k,0) = \frac{\Gamma^2(\gamma)}{\Gamma(\gamma-\alpha)\Gamma(\gamma+\alpha)} && (10.21)\\ b(k,t) &= b(k,0)e^{-\frac{it}{2k}} = 0 && (10.22)\\ \rho(k,t) &= \rho(k,0)e^{-\frac{it}{2k}} = 0 && (10.23)\\ C_1(t) &= C_1(0)e^{-\frac{it}{2k_1}} = 2\eta e^{\frac{t}{\eta}} && (10.24)\\ C_2(t) &= C_2(0)e^{-\frac{it}{2k_2}} = 6\eta e^{\frac{t}{3\eta}} && (10.25) \end{aligned}$$

Inverse problem

I will not go into details here again as we already did this multiple times, but from equations (7.59) and (8.54)-(8.57) we find after some algebra

$$q = \frac{-4\eta\left(\cosh\left(3\eta x + \frac{t}{3\eta}\right) + 3\cosh\left(\eta x + \frac{t}{\eta}\right)\right)}{4\cosh\left(2\eta x - \frac{2t}{3\eta}\right) + \cosh\left(4\eta x + \frac{4t}{3\eta}\right) + 3}. \tag{10.26}$$

So we again found q but still need to find u using equation (10.14). It turns out that we can again take $A = 0$ but the solution becomes nicer for $A = 4\pi$. In this case we get

$$\begin{aligned} u &= 2i\left\{\ln\left(e^{4\left(\eta x+\frac{t}{\eta}\right)} - e^{\frac{8t}{3\eta}} + 2ie^{3\left(\eta x+\frac{t}{\eta}\right)} + 2ie^{\left(\eta x+\frac{11t}{3\eta}\right)}\right)\right.\\ &- \left.\ln\left(e^{4\left(\eta x+\frac{t}{\eta}\right)} - e^{\frac{8t}{3\eta}} - 2ie^{3\left(\eta x+\frac{t}{\eta}\right)} - 2ie^{\left(\eta x+\frac{11t}{3\eta}\right)}\right)\right\}. \end{aligned} \tag{10.27}$$

We note that this solution is real, although it might not be very obvious from first sight. The solution is illustrated in Figure 10.1 for $\eta = 1$. We see that it is a kink-kink solution and it looks similar to the solutions of the SWWE obtained earlier. In the next section I will show how we can derive a kink-antikink solution from inverse scattering. Note that if we change the sign of u we find an antikink-antikink solution, so I will not derive this separately.

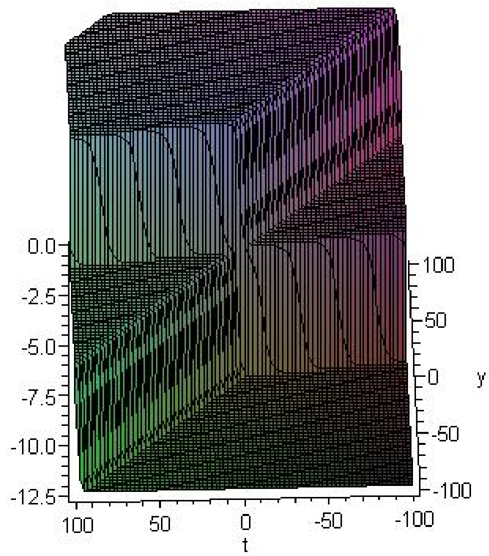

Figure 10.1: 2-soliton solution to the SG equation of the kink-kink type with $\eta = 1$

10.4 2-soliton solution: kink-antikink

It is possible to derive a general 2-soliton solution to the SG equation from a Bäcklund transformation [4] and this is given by

$$u = 4 \arctan\left[\left(\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2}\right)\frac{\exp(\eta_1) - \exp(\eta_2)}{1 + \exp(\eta_1 + \eta_2)}\right], \tag{10.28}$$

where $\eta_j = \kappa_j x + \frac{t}{\kappa_j}$. It is known that this gives a kink-kink interaction if $\kappa_1 > -\kappa_2 > 0$ and a kink-antikink interaction for $\kappa_1 > \kappa_2 > 0$. We see that the q corresponding to this solution is given by

$$q = \frac{-2(\kappa_1^2 - \kappa_2^2)(\kappa_1 \cosh(\eta_2) - \kappa_2 \cosh(\eta_1))}{(\kappa_1 - \kappa_2)^2 \cosh(\eta_1 + \eta_2) + (\kappa_1 + \kappa_2)^2 \cosh(\eta_1 - \eta_2) - 4\kappa_1\kappa_2}. \tag{10.29}$$

If we now take $\kappa_1 = 3\eta$ and $\kappa_2 = -\eta$ we see that this reduces to (10.26). In order to find a kink-antikink solution we now evaluate (10.29) with $\kappa_1 = 3\eta$ and $\kappa_2 = \eta$ at $t = 0$ and use this as the initial condition for inverse scattering. This gives us the initial condition

$$q = \frac{4\eta \cosh(\eta x)(2\cosh(\eta x)^2 - 3)}{4\cosh(\eta x)^4 - 3}. \tag{10.30}$$

I will not go into to many details of the direct scattering as the eigenfunctions are lengthy and not very instructive and all calculations are the same as before. We find $b(k) = 0$ and

$$a(k) = \frac{4k^2 - 3\eta^2 - 8ik\eta}{4k^2 - 3\eta^2 + 8ik\eta} \tag{10.31}$$

and therefore get

$$k_1 = \frac{i\eta}{2} \qquad \text{and} \qquad k_2 = \frac{3i\eta}{2}. \tag{10.32}$$

Finally we also find $C_1 = -2i\eta$ and $C_2 = 6i\eta$. So the scattering data is found. I will not go into any details of the time dependence or inverse scattering as this is all the same as before but we find after inverse scattering that

$$q = \frac{4\eta\left(\cosh\left(3\eta x + \frac{t}{3\eta}\right) - 3\cosh\left(\eta x + \frac{t}{\eta}\right)\right)}{4\cosh\left(2\eta x - \frac{2t}{3\eta}\right) + \cosh\left(4\eta x + \frac{4t}{3\eta}\right) - 3}. \tag{10.33}$$

If we insert this in equation (10.14) we find that

$$\begin{aligned} u &= 2i\left\{\ln\left(e^{4\left(\eta x+\frac{t}{\eta}\right)} - e^{\frac{8t}{3\eta}} - 2ie^{3\left(\eta x+\frac{t}{\eta}\right)} + 2ie^{\left(\eta x+\frac{11t}{3\eta}\right)}\right)\right. \\ &- \left.\ln\left(e^{4\left(\eta x+\frac{t}{\eta}\right)} - e^{\frac{8t}{3\eta}} + 2ie^{3\left(\eta x+\frac{t}{\eta}\right)} - 2ie^{\left(\eta x+\frac{11t}{3\eta}\right)}\right)\right\}. \end{aligned} \tag{10.34}$$

We note that it is not clear that this corresponds to the solution given in equation (10.28), but as their derivatives are equal, they can at most defer by a constant. This solution is illustrated in Figure (10.2) and we see that it is indeed a kink-antikink solution as required.

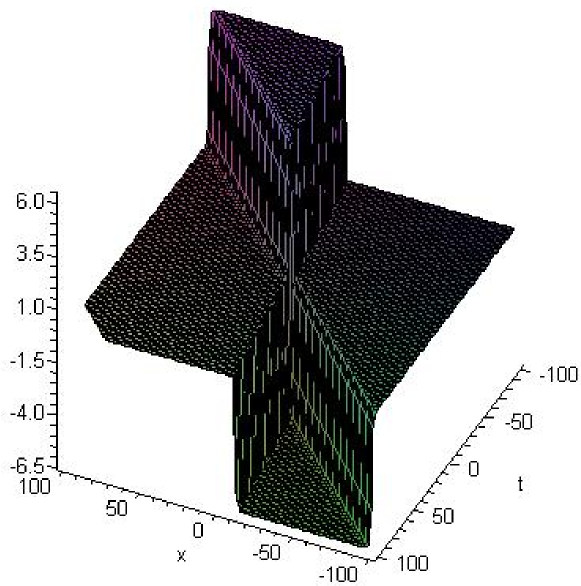

Figure 10.2: 2-soliton solution to the SG equation of the kink-antikink type with $\eta = 1$

10.5 Breather solution

So far we have shown that the kink-kink and kink-antikink solutions to the Sine-Gordon equation can be found from inverse scattering. However it is known that this equation also possesses a breather solution which can be found from equation (10.28) by setting $\kappa_1 = \mu + i\lambda$ and $\kappa_2 = \mu - i\lambda$ and after some rewriting we see that this becomes

$$u = 4\arctan\left(\frac{\mu\cos(\eta_I)}{\lambda\cosh(\eta_R)}\right), \tag{10.35}$$

with $\eta_R = \frac{\eta_1+\eta_2}{2}$ and $\eta_I = \frac{\eta_1-\eta_2}{2i}$. We note that the q corresponding to this solutions is the same as the one given in equation (9.51), only the η_R and η_I have a different time dependence. Therefore if we take this solution at $t = 0$ as an initial condition for inverse scattering it is no surprise that the direct scattering runs into the same problems as before. Again I have been unable to find a substitution which enables me to solve the scattering equation.

10.6 Discussion

We have seen that we can derive multiple 1- and 2-soliton solutions to the SG equation from IST. However we have again been unable to obtain breather solutions in this way. The IST for the SG equation was first developed by Ablowitz, Kaup, Newell and Segur and was presented in a fairly short note in [1]. Again they do not explicitly do the direct scattering part, but rather assume $\rho = 0$ and proceed directly with inverse scattering. In this way they do not only derive the N-soliton solutions but also find the breather solution. To do this they fix $k_2 = k_1^*$ and $k_1 k_2 = \frac{1}{4}$. This then enables them to solve equations (8.54)-(8.57) without a need for the exact value of C_1 and C_2 (however they do need their time-dependence). Their result coincides with the one given in equation (10.35) upon inserting $k_2 = k_1^*$ and $k_1 k_2 = \frac{1}{4}$. So we see that although it is again apparently impossible to do direct scattering for breather solutions, it is possible to obtain them from IST by skipping the direct problem and only solving the inverse problem.

Appendix A

Extended Homoclinic Test Technique

A.1 Introduction

In this short note I will present a number of comments on two recently published articles considering a "new" method for studying nonlinear evolution equations, called the extended homoclinic test. The two articles were both published in the magazine Chinese Physics Letters, the first one in January 2008 and the second one in January 2009. The first article [8] was written by Dai, Liu and Li and applies the method to the KdV equation. The second article [22] was written by Li and Zhao and applies the same method to the shallow water wave equation (SWWE). I will discuss their results and show that all results can be derived from the known 1- and 2-soliton solutions obtained from the Hirota bilinear form. Furthermore I will also point out a number of mathematical flaws in the articles. I will start with discussing their results for the KdV equation in Section A.2 and I will discuss the SWWE in Section A.3.

A.2 KdV equation

We have seen in Section 3.2 that the KdV equation

$$u_t + 6uu_x + u_{xxx} = 0 \tag{A.1}$$

can be put into the bilinear form[13]:

$$(D_x D_t + D_x^4) F \bullet F = 0, \tag{A.2}$$

where we made the substitution

$$u = 2(\ln F)_{xx}. \tag{A.3}$$

The 1-soliton solution is then given by $F = 1 + e^{\eta_1}$ with $\eta_1 = \kappa_1 x - \omega_1 t + \eta_1^0$ and

$$\omega_1 = \kappa_1^3. \tag{A.4}$$

Furthermore we saw that the two soliton solution is given by

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} \tag{A.5}$$

and with

$$A_{12} = \frac{(\kappa_1 - \kappa_2)^2}{(\kappa_1 + \kappa_2)^2}. \tag{A.6}$$

We note from equation (A.3) that u does not change if we multiply F by e^{ax+b}, where a and b do not depend on x but may depend on t. Therefore from now on we call F and F' conjugate if $F = e^{ax+b}F'$. It is also important to point out that we nowhere assumed the two κ_i to be real, so this also holds for complex κ_i.
Now Dai et al.[8] propose a method to find new solutions to the KdV equation, called the extended homoclinic test technique (EHTT), which consist of making the ansatz (their so-called test function)

$$f = e^{-\Omega(x+\alpha t)} + b_1 \cos(p(x-\alpha t)) + b_2 e^{\Omega(x+\alpha t)}, \tag{A.7}$$

with Ω, α, p, b_1 and b_2 constants still to be determined. They then fix $\Omega = p$ and insert this into equation (A.2) and find that

$$p = \pm\sqrt{\frac{\alpha}{2}}, \tag{A.8}$$

$$b_2 = -\left(\frac{b_1}{2}\right)^2. \tag{A.9}$$

They then insert this into equation (A.7) for 2 different cases corresponding to different signs of p. After introducing $\beta = \ln\left(\frac{b_1}{2}\right)$ and taking $\alpha = 2\kappa^2$ they find

$$f_1 = \frac{b_1}{2}\left[2\cos(\kappa(x-2\kappa^2 t)) - 2\sinh(\kappa(x+\kappa^2 t+\beta))\right], \tag{A.10}$$

$$f_2 = \frac{b_1}{2}\left[2\cos(\kappa(x-2\kappa^2 t)) + 2\sinh(\kappa(x+\kappa^2 t-\beta))\right]. \tag{A.11}$$

First of all we note that these are not two different solutions as Dai et al. claim. We know that the KdV equation is invariant under

$$x' \rightarrow -x, \tag{A.12}$$

$$t' \rightarrow -t, \tag{A.13}$$

and we see that under this transformation f_1 is mapped onto f_2 and vice versa. Furthermore we note that as f_1 has zeroes the solution u to the KdV equation will have poles, although the plots made by Dai et al. are made in such a way as to hide this fact. Now I will show that the same solution can be derived from the 2-soliton solution to the KdV given in equation (A.5). To do this we now take $\kappa_1 = a+ib$ and $\kappa_2 = a-ib$, with a and b real constants. Note that this is chosen in such a way that our F stays real. Note that this is not a new idea, but that a similar method for the Boussinesq equation was introduced by Tajiri and Watanabe[26] in 1989 already, however I have not yet seen it applied to the KdV equation before. This is not surprising as we already saw that the resulting solutions are not well behaved. Upon inserting this choice of κ_1 and κ_2 in equation (A.4) we get

$$\omega_1 = a^3 - 3ab^2 + i(-b^3+3a^2b) = \omega_R + i\omega_I, \tag{A.14}$$

$$\omega_2 = a^3 - 3ab^2 - i(-b^3+3a^2b) = \omega_R - i\omega_I, \tag{A.15}$$

$$\tag{A.16}$$

and from equation (A.6) we get

$$A_{12} = -\frac{b^2}{a^2}. \tag{A.17}$$

We note here that $A_{12} \leq 0$ for all a and b. If we know substitute $\eta_1 = \eta_R + i\eta_I$ and $\eta_2 = \eta_R - i\eta_I$, we see that we can then rewrite equation (A.5) in the form

$$\begin{aligned} F &= 1 + e^{\eta_R+i\eta_I} + e^{\eta_R+i\eta_I} - \frac{b^2}{a^2}e^{2\eta_R} \\ &= e^{\eta_R}\left(\cos(\eta_I) - \sqrt{\frac{b^2}{a^2}}\sinh(\eta_R+\delta)\right), \end{aligned} \tag{A.18}$$

where $\delta = \ln\left(\sqrt{\frac{b^2}{a^2}}\right)$. Upon setting $b = a$, we now get $\delta = 0$, $\eta_R = a(x - 2a^2 t) + \eta_R^0$ and $\eta_I = a(x + 2a^2 t) + \eta_I^0$. Therefore if we choose $\eta_R^0 = \beta$, $\eta_I^0 = 0$ and $a = \kappa$ we see that equation (A.18) reduces to a form conjugate to equation (A.10). So we see that we can find a bigger class of exact periodic solitary-wave solutions from the standard approach and which reduces to the one found by the extended homoclinic test technique in the limit $a = b$. Therefore this "new" technique does not seem to add anything in this case. In the next Section I will show similarly that also in the case of the SWWE the new technique does not add anything new.

A.3 Shallow Water Wave Equation

In this part I will look at the article by Li et al. [22] in which they apply the EHTT to the SWWE. They use the following form of the SWWE

$$u_t - u_{xxt} - 3uu_t + 3u_x \int_x^{\infty} u_t dx' + u_x = 0 \tag{A.19}$$

which was also studied by Hirota and Satsuma [17]. In Section 5.7 we saw how solutions to this equation can be mapped onto those of the SWWE given in equation (5.1). In this Section I will use the form given in A.19 as I will be discussing the article by Li et al. We see that we can put this into the bilinear form

$$D_x(D_x - D_x^2 D_t + D_x)F \bullet F = 0 \tag{A.20}$$

by making the transformation

$$u = 2(\ln F)_{xx}. \tag{A.21}$$

Note that we see that if F andF' are conjugate as defined in the previous section, they still give rise to the same solution u. We saw earlier that the one soliton solution is given by $F = 1 + e^{\eta_1}$ with $\eta_1 = \kappa_1 x - \omega_1 t + \eta_1^0$ and

$$\omega_1 = \frac{\kappa_1}{1 - \kappa_1^2}. \tag{A.22}$$

Furthermore the two soliton solution is given by

$$F = 1 + e^{\eta_1} + e^{\eta_2} + A_{12} e^{\eta_1 + \eta_2} \tag{A.23}$$

with

$$A_{12} = \frac{(\kappa_1 - \kappa_2)^2(-\kappa_1^2 + \kappa_1\kappa_2 - \kappa_2^2 + 3)}{(\kappa_1 + \kappa_2)^2(-\kappa_1^2 - \kappa_1\kappa_2 - \kappa_2^2 + 3)}. \tag{A.24}$$

We point out again that we nowhere assumed the two κ_i to be real, so this also holds for complex κ_i. Now Li et al. proceed by making the "new" ansatz

$$f = b_1 e^{-\Omega(x+\alpha t)} + b_2 \cos(p(x - \beta t)) + b_3 e^{\Omega(x+\alpha t)}, \tag{A.25}$$

where Ω, α, β, p, b_1, b_2 and b_3 are constants still to be determined. Upon substituting this into equation (A.20) and equating the different powers of $e^{-\Omega(x+\alpha t)}$ they derive the set of equations

$$\alpha\Omega^2(3p^2 + 1 - \Omega^2) + \beta p^2(p^2 + 1 - 3\Omega^2) + \beta p^2(p^2 + 1 - 3\Omega^2) = p^2 - \Omega^2, \tag{A.26}$$

$$(1 - \Omega^2 + 3p^2)\beta - (p^2 - 3\Omega^2 + 1)\alpha = 2, \tag{A.27}$$

$$b_3 = \frac{1}{4}\frac{b_2^2 p^2(4p^2\beta + \beta - 1)}{b_1\Omega^2(4\Omega^2\alpha - \alpha - 1)}. \tag{A.28}$$

Now Li et al. study 4 different cases depending on different choices of the constants. I will study the same cases and show that all their solutions can be derived from either the 1- or 2-soliton equation we already know.

A.3.1 Case 1

In the first case they choose $\Omega = ip$, $b_2 = b_1$ and $b_3 = b_1$. Then they derive from equations (A.26) and (A.27)

$$\alpha = -\frac{1}{4p^2+1} \qquad \text{and} \qquad \beta = \frac{1}{4p^2+1}. \tag{A.29}$$

However from an easy calculation we see that these values do not satisfy equation (A.28). But the strange thing is that if we insert these values in equation (A.25) it does satisfy (A.20). This is because there is a huge flaw in the derivation of equations (A.26)-(A.28), namely this does not hold if $\Omega = \pm ip$ because now the $\cos(p(x-\beta t))$ and the $e^{-\Omega(x+\alpha t)}$-terms are linearly dependent and we can therefore not split up into powers of $e^{-\Omega(x+\alpha t)}$. How they then found this choice of constants is unclear, but as it does give them a solution lets proceed. It is easy to see from equation (A.25) that we get

$$f = 3b_1 \cos\left(p\left(x - \frac{1}{1+4p^2}t\right)\right). \tag{A.30}$$

I will now show that we can find an F conjugate to this one from the 1-soliton solution. To do this we take $\kappa_1 = 2ip$ and see that we then get from equation (A.22)

$$\omega = \frac{2ip}{1+4p^2} \tag{A.31}$$

and therefore we get if we choose $\eta_1^0 = 0$

$$F = 1 + e^{\eta_1} = 2e^{\eta_1/2}\cos\left(p\left(x - \frac{1}{1+4p^2}t\right)\right). \tag{A.32}$$

We see that F is conjugate to f and therefore we see that the solution could have been equally well derived from the 1-soliton solution.

A.3.2 Case 2

Now they choose $\Omega = ip$, $\beta = 0$ and $b_3 = b_1$ and say to derive from equations (A.26)-(A.28)

$$\alpha = -\frac{2}{4p^2+1} \qquad \text{and} \qquad b_2 = \pm 2b_1 \tag{A.33}$$

Note that these values do satisfy all the equations, although they are not valid in this case. Inserting this into equation (A.25) gives

$$f = 2b_1\left[\cos\left(p\left(x - \frac{2}{4p^2+1}t\right)\right) \pm \cos(px)\right]. \tag{A.34}$$

I will now reproduce their solution from the 2-soliton solution. For this I fix $\kappa_1 = 2ip$ and $\kappa_2 = 0$. We note that according to equation (A.22) this would mean $\omega_1 = \frac{2ip}{1+4p^2}$ and $\omega_2 = 0$. However we note that $F = 1 + \exp(ct + d)$ with c and d constants is also a solution to equation (A.20) corresponding to $\kappa = 0$ and therefore we have the freedom to set $\omega_2 = \omega_1 = \frac{2ip}{1+4p^2} = i\omega$. We note that we get $A_{12} = 1$ and if we now set $e^{\eta_1^0} = e^{\eta_2^0} = \pm 1$ with the sign corresponding to the sign of b_2 we find

$$\begin{aligned} F &= 1 + e^{2ipx - i\omega t + \eta_1^0} + e^{-i\omega t + \eta_2^0} + e^{2ipx - 2i\omega t + \eta_1^0 + \eta_2^0} && \text{(A.35)} \\ &= e^{ipx - i\omega t}\left(e^{-ipx + i\omega t} \pm e^{ipx} \pm e^{-ipx} + e^{ipx - i\omega t}\right) && \text{(A.36)} \\ &= 2e^{ipx - i\omega t}\left(\cos(px) \pm \cos\left(p\left(x - \frac{2p}{1+4p^2}t\right)\right)\right). && \text{(A.37)} \end{aligned}$$

So we see that F is conjugate to f and therefore the solution can equally well be found from the 2-soliton solution.

A.3.3 Case 3

In this case they take $\Omega = p$ and $b_3 = b_1$ and find from equations (A.26)-(A.28)

$$\alpha = \frac{2p^2 - 1}{4p^2 + 1}, \qquad \beta = \frac{2p^2 + 1}{4p^2 + 1} \qquad \text{and} \qquad b_1 = \pm\frac{1}{2}\sqrt{\frac{2p^2 + 3}{2p^2 - 3}}b_2, \tag{A.38}$$

with $|p| > \sqrt{6}/2$. This then gives for f

$$f = 2b_1 \cosh\left(p\left(x + \alpha t\right)\right) + b_2 \cos\left(p\left(x - \beta t\right)\right). \tag{A.39}$$

We can reproduce this result from the 2-soliton solution again. We do this by taking $\kappa_1 = \kappa_2^* = a + ib$.We see that now equation (A.24) reduces to

$$A_{12} = -\frac{b^2(-a^2 + 3b^2 + 3)}{a^2(-3a^2 + b^2 + 3)}. \tag{A.40}$$

If we now assume that $A_{12} > 0$ and fix $e^{\eta_1^0} = e^{\eta_2^0} = \mp 1$ (where the sign is the opposite of the sign of b_1) we get from equation (A.23)

$$F = e^{ax - \omega_R t}\left(\cos(\eta_I) \pm \sqrt{A_{12}}\cosh(\eta_R)\right), \tag{A.41}$$

where ω_R and η_R denote the real parts of ω_1 and η_1 and η_I denotes the imaginary part of η_I. This gives us a class of breather solutions and if we now fix $a = b = p$ we see that equation (A.40) reduces to

$$A_{12} = \frac{2p^2 + 3}{2p^2 - 3} \tag{A.42}$$

and that this is indeed positive for $|p| > \sqrt{6}/2$. Furthermore we get

$$\omega_1 = \omega_2^* = \frac{p - 2p^3 + i(p + 2p^3)}{1 + 4p^4} = -p\alpha + ip\beta. \tag{A.43}$$

and therefore we find from equation (A.41)

$$F = e^{ax + \alpha t}\left(\pm\cos(p(x - \beta t)) + \sqrt{\frac{2p^2 + 3}{2p^2 - 3}}\cosh(p(x + \alpha t))\right), \tag{A.44}$$

and we see that we find again that F is conjugate to f and the solution Li et al. give is just a special case of the bigger class of breather solutions. Note that this is the same class of breather solutions I studied in Section 5.2.

A.3.4 Case 4

In the final case they take $\Omega = -p$ and $b_3 = -b_1$ and find from equations (A.26)-(A.28)

$$\alpha = \frac{2p^2 - 1}{4p^2 + 1}, \qquad \beta = \frac{2p^2 + 1}{4p^2 + 1} \qquad \text{and} \qquad b_1 = \mp\frac{1}{2}\sqrt{\frac{2p^2 + 3}{2p^2 - 3}}b_2, \tag{A.45}$$

where they now assume $|p| < \sqrt{6}/2$.This then gives for f

$$f = 2b_1 \sinh\left(p\left(x + \alpha t\right)\right) + b_2 \cos\left(p\left(x - \beta t\right)\right). \tag{A.46}$$

Again we can reproduce this result from the 2-soliton solution and I start again with taking $\kappa_1 = \kappa_2^* = a+ib$. We now note that if we assume $A_{12} < 0$ and fix $e^{\eta_1^0} = e^{\eta_2^0} = \mp 1$ (where the sign is the opposite of the sign of b_1) we now get from equation (A.25)

$$F = e^{ax-\omega_R t}\left(\cos(\eta_I) \mp \sqrt{-A_{12}}\sinh(\eta_R)\right), \tag{A.47}$$

with η_R, η_I and ω_R as before. Again this gives us a class periodic solutions. If we now choose $a = b = -p$ and we get

$$A_{12} = \frac{2p^2+3}{2p^2-3} \tag{A.48}$$

from which we note that $A_{12} < 0$ indeed for $|p| < \sqrt{6}/2$. Furthermore we find that

$$\omega_1 = \omega_2^* = -\frac{p - 2p^3 + i(p+2p^3)}{1+4p^4} = p\alpha - ip\beta. \tag{A.49}$$

Therefore equation (A.47) reduces to

$$F = e^{ax+\alpha t}\left(\cos(p(x-\beta t)) \mp \sqrt{\frac{2p^2+3}{-2p^2+3}}\cosh(p(x+\alpha t))\right), \tag{A.50}$$

and therefore we find again that F is conjugate to f and the solution Li et al. give is just a special case of the bigger class of periodic solutions.

A.4 Discussion

We have seen that the results obtained from the "new" extended homoclinic test technique can equally well be obtained by taking the parameters to be complex in the well known 1- and 2-soliton solutions as was introduced in [26]. Furthermore more general solutions can be obtained from the latter method. The general assumption of the EHTT comes down to looking for a solution of the form

$$f = e^{-\epsilon_1} + b_1\cos(\epsilon_2) + b_2 e^{\epsilon_1} = \sqrt{b2}\cosh(\epsilon_1) + b_1\cos(\epsilon_2), \tag{A.51}$$

where $\epsilon_i = \kappa_i x - \omega_i t$. However we know that the well known 2-soliton solution can be written in the form

$$F = e^{\eta_+}\left(\sqrt{A_{12}}\cosh(\eta_+) + \cosh(\eta_-)\right), \tag{A.52}$$

with $\eta_\pm$ and A_{12} as defined before. We see therefore if we can choose our constants such that $\eta_+ = \epsilon_1$, $\eta_- = i\epsilon_2$ and $A_{12} = b_2/b_1^2$ this directly reduces to the ansatz made in (A.51). We have seen that this is possible in all the cases studied above with the exception of case 1 for the SWWE, but here we could easily reproduce their ansatz from the 1-soliton solution. Therefore I fail to see the benefit of the EHTT and see no reason why it should be used. However I do note that from the group around Dai and Li more articles have been published using this technique on other equations, which I have not studied in detail and maybe in those cases it does give some new information. Although I do not expect this to happen as we saw that the ansatz of the EHTT reduces almost always to the 2-soliton solution and therefore can not give any new solutions.

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