Seiberg-Witten theory and equivariant localization of instantons

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Introduction

History

In 1983, Simon Donaldson in his seminal work on invariants of four dimensional manifolds [Don83], proved that there the solutions of certain partial differential equations on these spaces are not invariant under general homeomorphisms, but are invariant under diffeomorphisms. These invariants were computed via the moduli space of $SU(2)$ connections satisfying the anti-self-dual equations, called instantons. The moduli spaces of instantons have some undesirable properties, for example, they are not compact, and the invariants were in general very hard to compute.

In 1988, Edward Witten in [Wit88] observed that these invariants could also be computed using a certain kind of quantum field theory, $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, which he called topological quantum field theory. Together with Nathan Seiberg, he argued in [SW94] this topological quantum field theory could also be understood by using the dual theory to the Yang-Mills quantum field theory, which is a much simpler theory of abelian monopoles. Their proof of the equivalence used certain aspects of quantum field theory which still have no sound mathematical basis, making the equivalence a conjecture. However the new $U(1)$ invariants were mathematically sound, and much simpler to compute than the Donaldson invariants, leading to a wealth of new information on four-manifolds, and establishing many conjectures within weeks of the publication of the Seiberg-Witten theory.

Following the work of Seiberg and Witten, many aspects of the new theory were investigated, leading to a relation between Seiberg-Witten theory and certain integrable systems. A programme [FL98][FL01] was set up to lay a direct, rigorous link between the Seiberg-Witten invariants and Donaldson invariants by investigating so called “non-abelian monopoles” in a $PU(2)$ theory, which in various limits lead to monopoles and instantons. This programme made some progress, but the technical difficulties were even harder than those of Donaldson theory, and a definitive proof is still lacking.

However in 2003, Nekrasov in [Nek03] proposed another way to directly calculate the defining function of the $SU(2)$ quantum field theory on $\mathbb{R}^4$ without using the conjectural duality. His proposal was to modify the original theory by imposing an extra symmetry, which lead to equivariant instantons, calculating the equivariant volume using generalizations of the Duistermaat-Heckman formula in equivariant cohomology, and then in the end taking the limit to the non-equivariant limit. This programme was taken up and completed in 2003 and 2004 by three different groups in different ways, by Nekrasov-Okounkov [NO06], Nakajima-Yoshioka [NY05a][NY05b], and Braverman-Etingof [Bra04][BE06].

Outline of thesis

In this thesis we will first introduce some concepts of differential geometry, symplectic geometry and equivariant cohomology. Especially important in the rest of this thesis are connections and curvature on principal fiber bundles, the symplectic cut, and the Berline-Vergne formula and its generalizations. After this, we will describe the Seiberg-Witten invariants and derive some of their properties.

Finally, we state the famous ADHM construction of instantons, which is vital for the Nekrasov programme, and follow the approach taken by Nekrasov and Okounkov, elaborated by Martens, to compute the full prepotential governing the $SU(2)$ quantum field theory. This gives a proof of the equivalence of the Seiberg-Witten and Donaldson invariants, at least on $\mathbb{R}^4$.

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Contents

Chapter 1

Preliminaries

In this chapter we will develop some of the mathematical background needed for the results in chapters 2 and 3. First some background on Lie groups and Lie algebras is given. Then the tools of symplectic and Kähler geometry are introduced. The theory of connections is built upon the basis of Lie groups, and finally we develop equivariant cohomology, which binds these subjects together.

In this thesis, we will use the notation $\Omega^k(M)$ for a differential k-form on a manifold M, $d : \Omega^k(M) \to \Omega^{k+1}(M)$ for the exterior differentiation on forms, $\iota_X : \Omega^k(M) \to \Omega^{k-1}(M)$ for the interior product with a vector field X. Manifolds will be finite dimensional, except when stated otherwise, vector spaces will be either over $\mathbb{R}$ or $\mathbb{C}$. The identity of a group will be denoted by e, or in the case of linear actions on a vector space, by I.

1.1 Lie groups and algebras

In this section we will clarify our notation on Lie groups and Lie algebras, and define some more general concepts, such as super Lie groups and super Lie algebras. We will give (almost) no proofs or motivations. See any standard text on Lie groups for these.

1.1.1 Lie groups

A *Lie group* is a smooth manifold G which acts by multiplication as a group on itself: $G \times G \to G$. We can also consider *conjugation*, that is for $h \in G$ the map $C_h : G \to G$ given by $g \mapsto hgh^{-1}$ Some examples are:

- $\mathbb{R}^n$ acting on itself by translation, this is abelian.
- The circle $U(1)$ acting on itself by rotations. This is also abelian.
- The torus group T^n, consisting of n copies of the circle. This is an abelian group.
- The group of orientation preserving rotations in $\mathbb{R}^n$, $SO(n)$. This not abelian for $n > 2$.
- The group of $n \times n$ unitary matrices, $U(n)$, acting on itself by matrix multiplication. If $n > 1$, this group is not abelian.

A *torus* in a Lie group G is an abelian subgroup of G. A *maximal torus* of G, T_G, is a subgroup which is maximal with respect to dimension among such groups. A Lie group need not have a torus, but if it has, all maximal tori are conjugate to each other. The *rank* of a Lie group is defined to be the dimension of T_G. Because all T_G are conjugate, this is well-defined. Some examples:

- A maximal torus of $SO(2n)$ is given by the set of all simultaneous rotations in n orthogonal 2-planes. Thus the rank is n.
- A maximal torus of $SO(2n + 1)$ is the same, so it also has rank n.
- For $SU(n)$ the maximal torus is given by the intersection of n copies of $U(1)$ with $SU(n)$, so the rank is $n - 1$.

Given a maximal torus T_G, we define

Definition 1.1.1. *Let $N(T_G)$ be the normalizer of T_G in G, that is the following subgroup of G : $\{x \in G | xT_G = T_G x\}$. Then the* Weyl group $W(T_G)$ *of T_G in G is $N(T_G)/T_G$.*

It can be proven that the Weyl group of a compact Lie group is finite.

1.1.2 Lie algebras

A *Lie algebra* is a vector space $\mathfrak{g}$ over a field k, in this thesis $\mathbb{R}$ or $\mathbb{C}$, with a binary operation $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ on it, which has the following properties:

- Anti-commuting: $[a, b] = -[b, a]$

- Bilinear: $[a+b,c] = [a,c] + [b,c]$
- Jacobi identity: $[a,[b,c]] + [c,[a,b]] + [b,[a,c]] = 0$

for all $a, b, c \in \mathfrak{g}$. The operation $[,]$ is called a *Lie bracket*.

Any Lie group G defines a corresponding Lie algebra $\mathfrak{g}$. The Lie algebra is identified with the tangent space at the identity in the Lie group. There is a map from $\mathfrak{g}$ to G, the *exponential map*, or exp given by

$$\exp X = \gamma(1)$$

with $\gamma : \mathbb{R} \to G$ the one-parameter subgroup of G given by the condition that $\gamma'(0) = X$. This is a bijection between a neighborhood of 0 in $\mathfrak{g}$ and a neighborhood of e in G. The conjugation map C_h fixes the origin, so we can define $\mathrm{Ad} : G \to \mathrm{Gl}(T_eG)$ by $\mathrm{Ad}_x = T_eC_x$. This called the *adjoint representation* of G. It can be shown that Ad is a Lie group homomorphism. We can also view Ad_x as an automorphism of $\mathfrak{g}$ via the identification of $\mathfrak{g}$ with T_eG. The map Ad maps e to the identity I in $\mathrm{Gl}(T_eG)$ and $T_I(\mathrm{Gl}(T_eG)) = \mathrm{End}(T_eG)$ and we define the linear map $\mathrm{ad} : T_eG \to \mathrm{End}(T_eG)$ as $\mathrm{ad} = d(\mathrm{Ad})_e$.

A Lie algebra with a zero Lie bracket is called abelian. In every Lie algebra one can construct an abelian subalgebra $\mathfrak{h}$, called the *Cartan algebra* with the following defining properties:

1. $\mathfrak{h}$ is a nilpotent Lie algebra, that is $[\mathfrak{h}, \mathfrak{h}] = 0$
2. $\mathfrak{h}$ is equal to its own normalizer.

The second property implies that $\mathfrak{h}$ is a maximal abelian subalgebra. We can then define for any $\alpha \in \mathfrak{h}^*$:

$$\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h,x] = \alpha(h)x, h \in \mathfrak{h}\}$$

This linear subspace of $\mathfrak{g}$ has the following properties

1. Let $\alpha, \beta \in \mathfrak{h}^*$. Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$
2. The Cartan subalgebra $\mathfrak{h}$ is $\mathfrak{g}_0$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, we call α a *root* of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and α is called the weight of $\mathfrak{g}_\alpha$

Lemma 1.1.2. *Denote by R the set of all roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$. Then*

$$\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in R} \mathfrak{g}_\alpha$$

It is easy to see that if $\alpha \in R$, then $-\alpha \in R$. We define R^+, the set of *positive roots* to be the subset of R such that for every root $\alpha \in R$, exactly one of the root $\alpha, -\alpha$ is contained in R^+ and for all $\alpha, \beta \in R^+$ such that $\alpha + \beta$ is a root, $\alpha + \beta \in R^+$. An element of R^+ is called *simple* if it cannot be written as the sum of two elements of R^+. For each choice of positive roots R^+, there exists a unique set of simple roots, such that R^+ is precisely the set of roots which can be written as positive combinations of the set of simple roots.

1.2 Symplectic and Kähler geometry

1.2.1 Symplectic geometry

In this section we will very briefly recall symplectic geometry and related concepts.

Definition 1.2.1. *A* symplectic form ω *is a two-form which is:*

1. *closed,* $d\omega = 0$
2. *non-degenerate, if* $\omega(X, Y) = 0$ *for all* Y *then* $X = 0$

A symplectic manifold is a manifold equipped with a symplectic form. As a consequence of the second property of the symplectic form, it follows that a symplectic manifold must be oriented and even-dimensional.

A simple example of a symplectic manifold is the cotangent bundle of a manifold M, which has a natural symplectic form via the tautological one-form. Set $\pi : T^*M \to M$ to be the projection operator, $T_{x,\alpha}\pi : T(T^*M) \to T_xM$ to be the induced map of tangent spaces. Then, since $\alpha_x \in T_x^*M$ is a linear map $T_xM \to \mathbb{R}$, we can compose with α_x again, and get the map:

$$\tau_\alpha = \alpha_x \circ T_{\alpha,x}\pi$$

a map from $T_\alpha(T_x^*M) \to \mathbb{R}$. This map varies smoothly with α_x and x, so this defines a global one-form, τ. The exterior derivative $\sigma = d\tau$ of this one-form is certainly closed, and it is also non-degenerate which can be shown using local coordinates, hence σ is a symplectic form.

Much of this thesis is concerned with the way in which Lie groups act on manifolds, and the interplay of that action with the symplectic structure.

Definition 1.2.2. *Let* v *be a vector field on* M*. The* Lie derivative $\mathcal{L}_v$ *is*

$$\mathcal{L}_X\alpha = \frac{d}{dt} exp(-tX)\bigg|_{t=0} \alpha$$

for $\alpha \in \Omega^k(M)$*.*

The Lie derivative can also be computed by the Cartan homotopy formula: $\mathcal{L}_v = d(\iota_v) + \iota_v d$. If ω is a symplectic form $d\omega = 0$, so $\mathcal{L}_v = d(\iota_v\omega)$. Now if a Lie group G acts on the manifold M, we can look at when this action preserves the symplectic form.

Definition 1.2.3. *An action of a Lie group* G *on a symplectic manifold* (M, ω) *is called a* Hamiltonian action *if there exists a map*

$$\mu : M \to \mathfrak{g}^*$$

called the moment map*, such that the following holds: For each* $X \in \mathfrak{g}$ *define*

- *the function* $\mu^X : M \to \mathbb{R}, \mu^X(p) = \langle \mu(p), X \rangle$
- *the vector field* $X^\#$ *on* M *generated by the one-parameter subgroup* $\{e^{tX} | t \in \mathbb{R}\}$*, that is* $(X^\# f)(x) = \frac{d}{dt} f(e^{-tX}x)|_{t=0}$

Then $d\mu^X = \iota_{X^\#}\omega$*, and so* $d\iota_{X^\#}\omega = 0$

Because $\mathcal{L}_v = d(\iota_v\omega)$, we see then that $\mathcal{L}_{X^\#}\omega = 0$

The moment map defined above is key to many interesting results, one of which is the *symplectic quotient*, also know as Marsden-Weinstein or symplectic reduction.

Definition 1.2.4. *Let* (M, ω) *be a symplectic manifold of dimension* $2n$*, and* G *a Lie group. A* symplectic reduction *of* M *at a point* $p \in \mathfrak{g}^*$ *with respect to a Hamiltonian action* G*, denoted by* $M_p = M/\!\!/_p G$ *is*

$$\mu^{-1}(p)/G$$

If p is a regular value of μ, the set $\mu^{-1}(p)$ is a manifold (possibly with some mild singularities), and if G acts freely on $\mu^{-1}(p)$ the symplectic reduction is a smooth manifold.

Theorem 1.2.5 (Marsden-Weinstein). *There exists a unique symplectic form* ν_p *on* M_p *such that*

$$\pi_p^*\nu_p = i_p^*\omega$$

where $pi_p : \mu^{-1}(p) \to M_p$ *is the projection map with* $\mu^{-1}(p)$ *considered as a* T*-fiber bundle over* M_p *and* $i_p : \mu^{-1}(p) \hookrightarrow M$ *the inclusion map.*

The proof of this theorem simplifies when using equivariant cohomology, so we will postpone the proof to section 1.4.

1.2.2 Kähler geometry

A Riemannian metric on a complex manifold M is called *Hermitian* if it is invariant under the complex structure, that is

$$\langle Ix, Iy\rangle = \langle x, y\rangle \ \forall x, y \in T_pM$$

This defines a two-form on M by

$$\omega(x, y) = \langle Ix, y\rangle$$

Definition 1.2.6. *A* Kähler manifold *is a complex manifold where the two-form* ω *is closed.*

Since the metric is non-degenerate, we see that ω is non-degenerate, hence the Kähler form is a symplectic form.

For this thesis the notion of *hyperkähler* structure is also important.

Definition 1.2.7. *A* hyperkähler *structure consists of 3 compatible complex structures* I, J, K *such that* $I^2 = J^2 = K^2 = IJK = -1$ *and three corresponding closed non-degenerate two forms* $\omega_I(x, y) = \langle Ix, y\rangle$*,* $\omega_J(x, y) = \langle Jx, y\rangle$ *and* $\omega_K(x, y) = \langle Kx, y\rangle$*.*

Examples of 4-dimensional hyperkähler manifolds are the 4-dimensional torus, flat Euclidean space $\mathbb{R}^4$, $K3$ surfaces and the non-compact moduli spaces of instantons. These will be defined later.

The symplectic quotient construction has generalizations to the Kähler and hyperkähler case.

Definition 1.2.8. *Let (M, ω) be a Kähler manifold of dimension $2n$, and G a Lie group, which acts Hamiltonian with respect to ω with moment map μ. The* Kähler quotient *of M at a point $p \in \mathfrak{g}^*$, denoted by $M_p = M/\!\!/_p G$ is*

$$\mu^{-1}(p)/G$$

If $(M, \omega_1, \omega_2, \omega_3)$ is a hyperkähler manifold of dimension $2n$, and G a Lie group which acts Hamiltonian with respect to ω_1, ω_2 and ω_3 with moment maps μ_1, μ_2 and μ_3. The hyperkähler quotient *of M at a point $p \in \mathfrak{g}^*$, denoted by $M_p = M/\!\!/\!\!/_p G$ is*

$$\left(\mu_1^{-1}(p) \cap \mu_2^{-1}(p) \cap \mu_3^{-1}(p)\right)/G$$

It can be proven that a manifold constructed by a Kähler or a hyperkähler quotient are themselves Kähler or hyperkähler respectively.

Given a symplectic form on a manifold, one can construct something slightly weaker than a complex manifold. An *almost complex structure* is a complex structure J_x on each fiber of the tangent bundle T_xM that varies smoothly with $x \in M$. This structure is *integrable* if for each $x_0 \in M$ there is a neighborhood $U \subset M$ of x_0 such that $x \mapsto J_x$ is constant for all $x \in U$. Every symplectic form induces an almost complex structure on the manifold. It is integrable if and only if the manifold is Kähler, with Kähler form the symplectic form.

On a complex manifold M, we have an almost complex structure J on the tangent space TM. This is a vector bundle endomorphism $TM \to TM$ such that $J^2 = -1$. We can thus decompose TM into the eigenspaces of J, $TM^{(1,0)}$ with eigenvalue $+i$ and $TM^{(0,1)}$ with eigenvalue $-i$. The map J induces a map J^* on T^*M via the metric, and we can define $T^*M^{(1,0)}$ and $T^*M^{(0,1)}$ similarly. We can make forms of arbitrary (p, q) by the wedge product:

$$T^*M^{(p,q)} = \bigwedge^p T^*M^{(1,0)} \otimes \bigwedge^q T^*M^{(0,1)}$$

A basis for $T^*M^{(1,0)}$ is denoted by dz_i and a basis for $T^*M^{(0,1)}$ by $d\bar{z}_i$. The space of complex k-forms decomposes into forms of type (p, q):

$$\Omega^k(M) \otimes \mathbb{C} = \sum_{p+q=k} \Omega^{p,q}(M)$$

The de Rham operator acting on this decomposition decomposes then into two components:

$$d = \partial + \bar{\partial}, \partial : \Omega^{p,q}(M) \to \Omega^{p+1,q}(M), \bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)$$

With this one can define the *Dolbeault complex*, a generalization of the de Rham complex for complex manifolds:

$$H^{p,q} = \frac{\mathrm{Ker}(\bar{\partial} : \Omega^{p,q}(M) \to \Omega^{p,q+1})}{\mathrm{Im}(\bar{\partial} : \Omega^{p,q-1}(M) \to \Omega^{p,q})} \text{ for q} > 0$$

The Kähler form ω is a form of type $(1, 1)$. Using the Hermitian inner product we define formal adjoints of these operators, ∂^*, $\bar{\partial}^*$. We have the following duality:

Theorem 1.2.9 (Serre duality). *Let M be a compact manifold of complex dimension n. Then there is a linear isomorphism*

$$\sigma : H^{p,q}(M) \to H^{n-p,n-q}(M)$$

For every Kähler manifold there exists local coordinates in which the metric can be written as the second derivative of a function K, $g = \partial\bar{\partial}K$. The function K is called the *Kähler potential*.

In the case of Seiberg-Witten theory, a slightly more special class of manifolds is important.

Definition 1.2.10. *Let M be a Kähler manifold with Kähler form ω and complex structure I. A* special Kähler *structure on M is a real flat torsionfree connection ∇ which satisfies*

$$d_\nabla \omega = 0 d_\nabla I = 0$$

This definition is equivalent to saying that there is a *prepotential*, a holomorphic function $\mathscr{F}$ which determines the Kähler potential in *special coordinates* z_i by

$$K = \frac{1}{2}\mathrm{Im}\left(\frac{\partial \mathscr{F}}{\partial z_i}\bar{z}_i\right) \tag{1.2.1}$$

A reference for all kinds of properties for these special Kähler manifolds is [Fre99].

1.2.3 Hodge theory

If we have an orientable compact Riemannian manifold M with metric g and dimension n, we can, by choosing an orientation, define an operator on differential forms on that manifold, the Hodge star $\star$.

The inner product g_p on T_pM induces an inner product on $\bigwedge^k T_p^*M$, so we can define an inner product on Ω^k by integrating the pointwise inner product. We have that for any $\beta \in \bigwedge^k$ the non-degenerate linear operation $\alpha \mapsto |g(\alpha,\beta)|\mu_M$, with μ_M the oriented volume element in $\bigwedge^n(M)$ and $\alpha \in \bigwedge^k$, is given by the following operation

$$\alpha \wedge \star\beta$$

with $\star\beta \in \bigwedge^{n-k}(M)$ uniquely determined. Thus $\star$ is a linear isomorphism between $\bigwedge^k$ and $\bigwedge^{n-k}$. In this notation, we can thus write

$$(\alpha,\beta) = \int_M g(\alpha,\beta)\mu_M = \int_M \alpha \wedge \star\beta$$

Lemma 1.2.11. *The square of the Hodge star $\star^2$ on a k-form and n-dimensional manifold M is given by $(-1)^{k(n-k)}$.*

Proof. Let α be a k-form, which can be written as $e_{i_1} \wedge \ldots \wedge e_{i_k}$ with $I = \{i_m\}_{m=1}^k \subset \{1,\ldots n\}$. By definition, $\beta = \star\alpha$ is a $n-k$ form such that $\alpha \wedge \beta$ is positively oriented. Thus $\beta = e_{j_1} \wedge \ldots \wedge e_{j_{n-k}}$ with $J = \{j_n\} = \{1,\ldots n\}\backslash I$. By permuting, we get immediately that $\star\beta = (-1)^{k(n-k)}\alpha$. □

With this star operator, we can define an adjoint to the differential operator, along with an elliptic operator from $\Omega^k(M)$ to $\Omega^k(M)$.

Definition 1.2.12. *The* codifferential $\delta : \wedge^k \to \wedge^{k-1}$ *is given by*

$$\delta\alpha = (-1)^{n(k+1)+1} \star d \star \alpha$$

*The*Hodge Laplacian $\Delta : \wedge^k \to \wedge^k$ *is the linear operator given by $d\delta + \delta d$.*

Proposition 1.2.13. *The codifferential and the Hodge laplacian have the following properties:*

a. The operator δ is the formal adjoint of d, that is $\langle d\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle$.

b. The Hodge Laplacian Δ is formally self-adjoint.

c. $\Delta\alpha = 0 \Leftrightarrow d\alpha = 0$ and $\delta\alpha = 0$

Proof. a. We have, for α a $k-1$ form and β a k-form:

$$\begin{aligned}\int_M d\alpha \wedge \star\beta = & \int_M d\left(\alpha \wedge \star\beta\right) - (-1)^{k-1} \int_M \alpha \wedge d(\star\beta) \\ =&(-1)^{k+(n-k+1)(n-(n-k+1))} \int_M \alpha \wedge \star\star d(\star\beta) \\ =&(-1)^{k+(n-k+1)(k-1)-(n(k+1)+1)} \int_M \alpha \wedge \star\delta\beta \\ =&(-1)^{k+nk-k^2+k-n+k-1-nk-n-1} \int_M \alpha \wedge \star\delta\beta \\ =&(-1)^{3k-k^2-2n} \int_M \alpha \wedge \star\delta\beta\end{aligned}$$

and since $3k-k^2-2n$ is even in all cases, δ is the adjoint of d.

b. This follows directly from the fact that d is adjoint to δ and vice-versa.

c. We have that $(\Delta\alpha, \alpha) = (d\delta + \delta d)\alpha, \alpha) = (d\delta\alpha, \alpha) + (\delta d\alpha, \alpha) = (\delta\alpha, \delta\alpha) + (d\alpha, d\alpha)$, both terms must be ≥ 0, hence it is only zero when $\delta\alpha = 0$ and $d\alpha = 0$.

□

We define the harmonic k-forms to be

$$\mathcal{H}^k(M) = \{\omega \in \Omega^k(M) : \Delta\omega = 0\}$$

We see that due to proposition 1.2.13c there is a mapping from the harmonic k-forms to the k-th de Rham cohomology class. In fact we even have the following theorem, due to Hodge :

Theorem 1.2.14 (Hodge theorem). *Every de Rham cohomology class on a compact oriented Riemannian manifold M has a unique harmonic representative, and so*

$$H^p(M;\mathbb{R}) \simeq \mathcal{H}^p(M).$$

Also, $\mathcal{H}^p(M)$ is finite dimensional and $\Omega^p(M)$ has the following decomposition into orthogonal subspaces:

$$\Omega^p(M) = \mathcal{H}^p(M) \oplus d\left(\Omega^{p-1}(M)\right) \oplus \delta\left(\Omega^{p+1}(M)\right)$$

For a proof see for example [HM89].

This follows partly from the fact that Δ is an elliptic operator and elliptic operators have finite dimensional kernels. We see from this theorem that the de Rham cohomology groups are finite dimensional. For Kähler manifolds, the Hodge theorem can be generalized as follows:

Theorem 1.2.15. *Let M be a compact Kähler manifold. Define $\mathcal{H}^{(p,q)}(M) = \{\omega \in \Omega^{(p,q)}(M) : \Delta\omega = 0\}$. Then there is an isomorphism between $\mathcal{H}^{(p,q)}(M)$ and $H^{(p,q)}(M)$*

We define the Betti numbers b_p as follows: $b_p = \dim(H^p(M;\mathbb{R})$. In the case interesting for Seiberg-Witten theory, compact oriented 4 manifolds, we have that $b_0 = 1$ since we only consider connected manifolds. We now see that the only interesting Betti numbers are b_1 and b_2, since the others follow by Poincaré duality. When we work in 4 dimensions, we can split b_2 into two different numbers, $b_2 = b_2^+ + b_2^-$, with b_2^- the dimension of the eigenspace of $\star$ with eigenvalue -1 and b_2^+ the dimension of the eigenspace of $\star$ with eigenvalue $+1$, since $\star^2 = 1$ on two-forms. We divide $\wedge^2 V$ up into the self-dual and anti-self-dual forms:

$$\wedge^2 V = \wedge_+^2 V \oplus \wedge_-^2 V,$$

where

$$\wedge_+^2 V = \{\omega \in \wedge^2 V : \star\omega = \omega\} \quad \wedge_-^2 V = \{\omega \in \wedge^2 V : \star\omega = -\omega\}$$

These spaces are both 3-dimensional subspaces of the six-dimensional space $\wedge^2 V$. If we have a smooth two-form on M, we can divide it into the self-dual and anti-self-dual part as follows:

$$\omega_+ = \frac{1}{2}(\omega + \star\omega) \in \Omega_+^2(M), \tag{1.2.2a}$$

$$\omega_- = \frac{1}{2}(\omega - \star\omega) \in \Omega_-^2(M), \tag{1.2.2b}$$

Since $\star$ interchanges the kernel of d and δ and the intersection of those kernels are precisely the harmonic forms, the self-dual and anti-self-dual parts of a harmonic form are harmonic themselves and we we have the decomposition:

$$\mathcal{H}^2(M) = \mathcal{H}_+^2(M) \oplus \mathcal{H}_-^2(M).$$

and we define

$$b_2^+ = \dim\mathcal{H}_+^2(M), \quad b_2^- = \dim\mathcal{H}_-^2(M)$$

Very important in Seiberg-Witten theory is the *fundamental elliptic complex* given by

$$0 \to \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^+} \Omega_+^2(M) \to 0 \tag{1.2.3}$$

where d^+ is given by restricting the image of d to the self-dual forms. We use Hodge theory to calculate the cohomology groups of this complex.

If $\omega \in \Omega_+(M)$ is orthogonal to the image of d^+, then $\delta\omega = 0$, and self-duality gives $d\omega = 0$ hence $\omega \in \mathcal{H}_+^2(M)$. If $\omega \in \Omega^1(M)$ lies in the kernel of d^+ and in the cokernel of d, then $\delta\omega = 0$ and $d \star \omega = 0$, so

$$(d + \delta)(\omega + \star\omega) = d\omega + \delta \star \omega = d\omega + \star d\omega = 2d^+\omega = 0$$

and we have that $\omega \in \mathcal{H}^1(M)$. So the cohomology groups of the complex 1.2.3 are

$$\mathcal{H}^0(M), \quad \mathcal{H}^1(M), \quad \mathcal{H}_+^2(M) \tag{1.2.4}$$

1.3 Connections and characteristic classes

1.3.1 Connections in differential geometry

A connection on a vector bundle E over a manifold M is a map

$$d_A : \Gamma(E) \to \Gamma(T^*M \otimes E)$$

with $\Gamma(E)$ a suitably chosen space of sections over E, satisfying the following axiom:

$$d_A(f\sigma + \tau) = (df) \otimes \sigma + f d_A\sigma + d_A\tau \tag{1.3.1}$$

with $\sigma, \tau \in \Gamma(E)$, $f : M \to \mathbb{R}$ (or $\mathbb{C}$) a function. If we contract d_A with a vector field X we get the following map:

$$\nabla : \Gamma(TM \otimes E) \to \Gamma(E)$$

satisfying

$$\nabla_X(f\sigma + \tau) = (Xf)\sigma + f\nabla_X\sigma + \nabla_X\tau$$

and

$$\nabla_{fX+Y}\sigma = f\nabla_X\sigma + \nabla_Y\sigma$$

with f, σ and τ as above and $X, Y \in \Gamma(TM)$. This can be taken as an alternative definition of a connection.

If we want to calculate with connections, it is usually convenient to go to local representatives. All vector bundles have a local trivialization of the form $U_\alpha \times \mathbb{R}^n$ (or $\mathbb{C}^n$), where the U_α cover M, together with smooth invertible transition functions $g_{\alpha\beta}$. A section $\sigma \in \Gamma(E)$ has a local representative σ_α, which is an n-tuple of functions on U_α, and so $d_A\sigma$ also has a local representative $(d_A\sigma)_\alpha$, which is an n-tuple of one-forms on U_α. View $e_1, \ldots e_n$, a basis of $\mathbb{R}^n$, as constant sections over U_α. Then any element of $\Gamma(T^*M \otimes E)$ can be written as

$$\begin{pmatrix} \omega^1 \\ \vdots \\ \omega^n \end{pmatrix} = \sum_{i=1}^n e_i\omega^i$$

with the ω_i one-forms, so we can write $d_A e_j = \sum_{i=1}^n e_i A^i_j$. Now we can apply equation 1.3.1 to write:

$$d_A\left(\sum_{i=1}^n e_i\sigma^i\right) = \sum_{i=1}^n e_i d\sigma^i + \sum_{i,j=1}^n e_i A^i_j \sigma^j$$

This can be written in short, suppressing indices as

$$d_A\sigma = (d + A)\sigma \tag{1.3.2}$$

in the trivialization. Now since the connection is well-defined on the overlap of U_α and U_β, we must have

$$d\sigma_\alpha + A_\alpha\sigma_\alpha = g_{\alpha\beta}(d\sigma_\beta + A_\beta\sigma_\beta)$$

on this overlap. Since sections transform as $\sigma_\beta = g_{\alpha\beta}^{-1}\sigma_\alpha$, we have

$$\begin{aligned} d\sigma_\alpha + A_\alpha\sigma_\alpha &= g_{\alpha\beta}\left(d(g_{\alpha\beta}^{-1}\sigma_\alpha) + A_\beta g_{\alpha\beta}^{-1}\sigma_\alpha\right) \\ &= d\sigma_\alpha + \left(g_{\alpha\beta}dg_{\alpha\beta}^{-1} + g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1}\right)\sigma_\alpha \end{aligned}$$

and we can conclude that

$$A_\alpha = g_{\alpha\beta}dg_{\alpha\beta}^{-1} + g_{\alpha\beta}A_\beta g_{\alpha\beta}^{-1} \tag{1.3.3}$$

If E is a fiber bundle with structure group G, then a *G-connection* is a connection whose local representatives ω_α take values in the Lie algebra of G, for example an $O(m)$ connection on a m-dimensional vector bundle is a connection which is of the form $d + A_\alpha$ with A_α a skew-symmetric matrix, and a $U(m)$ connection on a complex m-vector bundle is represented by skew-Hermitian matrices.

1.3.2 Parallel transport along curves

If we have a smooth map f from manifolds N to M, a vector bundle (E, π) on M, we can pull this bundle back to N, as follows:

$$f^*E = \{(p, v) \in N \times E : f(p) = \pi(v)\}, \quad \pi^*((p, v)) = p$$

We can then pull back sections in $\Gamma(E)$ as follows:

$$f^*\sigma : N \to f^*E \text{ by } f^*\sigma(p) = (p, \sigma \circ f(p)).$$

Theorem 1.3.1. *If d_A is a connection on the vector bundle (E, π) over M and $f : N \to M$ a smooth map, there is a unique connection d_{f^*A} on the pullback bundle f^*E over N such that the following diagram commutes:*

$$\begin{array}{ccc} \Gamma(E) & \xrightarrow{d_A} & \Gamma(T^*M \otimes E) \\ \downarrow{\scriptstyle f^*} & & \downarrow{\scriptstyle f^*} \\ \Gamma(F^*E) & \xrightarrow[d_{f^*A}]{} & \Gamma(T^*N \otimes f^*E) \end{array}$$

Proof. For existence, we note that d_{f^*A} defined locally by f^*A_α with A_α the local representatives of d_A defines a connection, since an easy calculation shows that this has the transformation properties of a connection form. The uniqueness follows from the diagram, we see that the local representative of d_{f^*A} on $f^{-1}(U_\alpha)$ must be given by f^*A_α. □

An application of the pullback construction is in the case of curves on manifolds. The existence of this pullback implies that we can transport vectors in a vector bundle along the curve. This is called *parallel transport* along the curve. It works by considering a differentiable curve $\gamma : [0, 1] \to M$. Any vector bundle over $[0, 1]$ is trivial, because of the following lemma:

Lemma 1.3.2. *Given a vector bundle $E \to I$ with I an interval, there exists a trivialization $\mathbb{R}^m \times I$ of E.*

Proof. Let $\{U_i\}$ be a trivialization of E over I such that U_i is connected. Choose a basis $\vec{s}_i$ for each E_i. Then on overlaps, there is a function $f_{ij} \in Gl(m, \mathbb{R})$ such that $\vec{s}_i|_{U_i \cap U_j} = f_{ij}\vec{s}_j|_{U_i \cap U_j}$. Now look at the oriented volume of the parallelepiped spanned by the $\vec{s}_i$, $\mathrm{vol}(\vec{s}_j)$. We see that if this volume never vanishes, we get a global basis of the vector bundle, hence it is trivial. Now on the overlaps $U_i \cap U_j$ we see that $\mathrm{vol}(\vec{s}_i)|_{U_i \cap U_j} = \det(f_{ij})\mathrm{vol}(\vec{s}_j)|_{U_i \cap U_j}$. Now since on the interval, if U_i and U_j are connected, $U_i \cap U_j$ is connected, $\epsilon_{ij} = \mathrm{sign}(\det(f_{ij}))$ is constant. Then we can define $\vec{s'}_j$ to be a suitable permutation of $\vec{s}_j$ such that $\mathrm{vol}(\vec{s}_j) = \epsilon_{ij}\mathrm{vol}(\vec{s'}_j)$, and we see that we have constructed a compatible series of sections that form a consistent basis of each E_i, hence we have constructed a global basis for E. □

Now it is easy to see that this global trivialization implies that there are globally flat sections, so if we look at the induced connection $\gamma^*\nabla$ on the interval, we can transport any vector $v \in E$ along these sections. This is called parallel transport.

1.3.3 Curvature of connection

We can extend the concept of connection to work on all differential forms by requiring that it is a derivation, that is, the Leibniz rule holds:

$$d_A(\alpha \otimes \sigma) = d\alpha \otimes \sigma + (-1)^p \alpha \wedge d_A\sigma \quad \text{for } \alpha \in \Omega^p(M), \sigma \in \Gamma(E)$$

In contrast to the normal exterior derivative d, d_A is not usually closed, $d_A \circ d_A \neq 0$ generically. However it is linear, by the following calculation:

$$\begin{aligned} d_A \circ d_A(f\sigma + \tau) &= d_A(df \otimes \sigma + f d_A\sigma + d_A\tau) \\ &= d(df)\sigma - df \otimes d_A\sigma + df \otimes d_A\sigma + f(d_A \circ d_A\sigma) + d_A \circ d_A\tau \\ &= f(d_A \circ d_A\sigma) + d_A \circ d_A\tau \end{aligned}$$

with $f \in C^\infty(M)$ and $\sigma, \tau \in \Omega^*(M)$ This means that $d_A \circ d_A$ is a tensor field called the *curvature of the connection* We can look at the local representatives of this curvature, and since it is linear over functions, we can see that in a local basis $\{e_i\}$ we have

$$d_A \circ d_A(\sigma) = d_A \circ d_A\left(\sum_{i=1}^m e_i\sigma^i\right) = \sum_{i,j=1}^m e_i F_{A}{}^i_j \sigma^j = F_A\sigma \tag{1.3.4}$$

with F_A a matrix of two forms. Now applying equation 1.3.2, we can see that for a trivialization $\{U_\alpha\}$, this matrix can be written as

$$F_{A\alpha} = dA_\alpha + A_\alpha \wedge A_\alpha \tag{1.3.5}$$

On $U_\alpha \cap U_\beta$ we have that

$$F_{A\alpha}\sigma_\alpha = g_{\alpha\beta}F_{A\beta}\sigma_\beta = g_{\alpha\beta}F_{A\beta}{g_{\alpha\beta}}^{-1}\sigma_\alpha \tag{1.3.6}$$

We see from equation 1.3.5 and using the fact that the local representatives of a $U(m)$ connection are skew-Hermitian, that the curvature of a $U(m)$ connection is skew symmetric.

One important identity in G-connections is the first of the *Bianchi identities*:

$$dF_{A\alpha} = F_{A\alpha} \wedge A_\alpha - A_\alpha \wedge F_{A\alpha} = [F_{A\alpha}, A_\alpha] \tag{1.3.7}$$

which can be proven by differentiating equation 1.3.2.

1.3.4 Characteristic classes

Characteristic classes in general are cohomology classes associated to vector bundles that are natural with respect to the pull-back bundles. They measure the "non-triviality" of the vector bundle. For different kind of vector bundles, different characteristic classes can be constructed. First we will focus on $U(m)$-bundles, and construct the Chern class.

Let M be a manifold, and $E \to M$ a complex vector bundle with rank m. It then has a structure group $U(m)$, meaning that the transition functions lie within $U(m)$. We equip E with a Hermitian metric. As seen before, $F_{A\alpha}$ is then skew-Hermitian, so the forms $\left(\frac{i}{2\pi}F_{A\alpha}\right)^k$ are Hermitian, hence Trace $\left[\left(\frac{i}{2\pi}F_{A\alpha}\right)^k\right]$ is real-valued for all $k \in \mathbb{N}$.

Definition 1.3.3. *The form* $\tau_k \in \Omega^2(M)$ *is called a* characteristic form

The transformation 1.3.6 shows that this form is left unchanged when going to different patches by the invariance of the trace, so the characteristic form is well-defined.

Lemma 1.3.4. *The characteristic form is closed,* $d\tau_k(A) = 0$

Proof. The $\frac{i}{2\pi}$ factor is inconsequential. We get

$$\begin{aligned}
d\tau_k(A) &= d\left[\text{Trace}\left({F_{A}}_\alpha^k\right)\right] = \text{Trace}\left[d\left({F_{A}}_\alpha^k\right)\right] \\
&= \text{Trace}\left\{\sum_{i=1}^{k} {F_{A}}_\alpha^{i-1}(dF_{A\alpha}){F_{A}}_\alpha^{k-i}\right\} \\
&= \text{Trace}\left\{\sum_{i=1}^{k} {F_{A}}_\alpha^{i-1}[F_{A\alpha}, A_\alpha]{F_{A}}_\alpha^{k-i}\right\} \\
&= \text{Trace}\left\{\sum_{i=1}^{k} {F_{A}}_\alpha^{i}A_\alpha{F_{A}}_\alpha^{k-i} - {F_{A}}_\alpha^{i-1}A_\alpha{F_{A}}_\alpha^{k-i+1}\right\} \\
&= \text{Trace}\left\{{F_{A}}_\alpha^{k}A_\alpha - A_\alpha{F_{A}}_\alpha^{k}\right\} = 0
\end{aligned}$$

where we use the Bianchi identity 1.3.7. Now the last line can be can be seen to be 0 by using that the trace is invariant under a cyclic permutation, hence and so the two terms cancel, hence $d\tau_k(A) = 0$ □

From this it follows that $\tau_k(A)$ represents a de Rham cohomology class $[\tau_k(A)] \in H^{2k}(M;\mathbb{R})$. The following two theorems prove that this class is a characteristic class.

Theorem 1.3.5. *The characteristic form $\tau_k(A)$ is natural with respect to the pull-back map:*

$$\tau_k(f^*A) = f^*\tau_k(A)$$

Proof. The local representative of the pull-back connection d_{F^*A} is F^*A_α so the curvature of the pull-back connection is

$$F_{A_\alpha}^* = f^*dA_\alpha + f^*A_\alpha \wedge f^*A_\alpha = f^*F_{A\alpha}$$

so

$$\text{Trace}\left[(F_{A_\alpha}^*)^k\right] = \text{Trace}\left[f^*\left(F_{A_\alpha}^k\right)\right] = f^*\text{Trace}\left[F_{A_\alpha}^k\right]$$

And by the definition of the characteristic form this gives the right identity. □

Theorem 1.3.6. *The de Rham cohomology class of $\tau_k(A)$, $[\tau_k(A)]$ is independent of the choice of unitary connection and Hermitian metric.*

Proof. We can connect any two connections A and B by a line in the space of connections, by setting $d_{C_t} = d + (1-t)A + tB$. Define $C_t = (1-t)A + tB$. We will show that $\frac{d}{dt}\tau_k(C_t) = d\theta$ for some function θ, hence that the class of the characteristic function is independent of the choice of connection. Note that we only have to establish this for some fixed $t \in [0,1]$ since $\frac{d^2}{dt^2}C_t = 0$ by construction. We have that the curvature of C_t, F_{C_t}, is $F_{C_t} = d\dot{C}_t + C_t \wedge C_t$, so $\frac{d}{dt}F_{C_t} = d\dot{C}_t + \dot{C}_t \wedge C_t + C_t \wedge \dot{C}_t$. Since

$$\frac{d}{dt}\text{Trace}(F_{C_t}^k) = \sum_{i=1}^{k} \text{Trace}\left(F_{C_t}^{i-1} \wedge \dot{F}_{C_t} \wedge F_{C_t}^{k-i}\right)$$

So by using the invariance of the trace we see

$$\frac{d}{dt}\text{Trace}(F_{C_t}^k) = -1$$

The same construction can be used for the case of the metric independence, by using that $(1-t)\langle,\rangle_0 + t\langle,\rangle_1$ is a Hermitian metric if $\langle,\rangle_0$ and $\langle,\rangle_1$ are. □

Since the characteristic class is not dependent on the connection or the metric chosen, we write $\tau_k(E) = [\tau_k(A)]$.

Definition 1.3.7. *The* Chern classes *of a vector bundle E are defined as the cohomology classes $c_k(E)$ given by representatives*

$$det\left(\frac{itF_A}{2\pi} + I\right) = \sum_k c_k(V)t^k$$

This is a finite sequence, since for $2k > \dim(M)$, $F_A^k = 0$. An easy calculation shows that

$$c_1(E) = \tau_1(E) \quad c_2(E) = \frac{1}{2}\left(\tau_1(E)^2 - \tau_2(E)\right)$$

For four dimensional manifolds, these are the only non-vanishing Chern classes.

We can simply calculate the first Chern class of a tensor product of two line bundles by a special case of the *Whitney product rule*:

Proposition 1.3.8. *If L_1 and L_2 are complex line bundles over M then*

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2)$$

Proof. Suppose that A_1 is a unitary connection on L_1 and A_2 a unitary connection on L_2. Then the curvature of the induced on $L_1 \otimes L_2$, $d_{A_1 \otimes A_2}$ is:

$$d_{A_1 \otimes A_2} \sigma_1 \otimes \sigma_2 = d_{A_1}(\sigma_1) \otimes \sigma_2 + \sigma_1 \otimes d_{A_2}(\sigma_2)$$

We can then calculate the curvature, $d_{A_1 \otimes A_2} \circ d_{A_1 \otimes A_2}$:

$$\begin{aligned} d_{A_1 \otimes A_2} \circ d_{A_1 \otimes A_2}(\sigma_1 \otimes \sigma_2) &= d_{A_1 \otimes A_2}\left(d_{A_1}(\sigma_1) \otimes \sigma_2 + \sigma_1 \otimes d_{A_2}(\sigma_2)\right) \\ &= d^2_{A_1}(\sigma_1) \otimes \sigma_2 + \sigma_1 \otimes d^2_{A_2}(\sigma_2) \end{aligned}$$

and we see that $F_{A_1 \otimes A_2} = F_{A_1} + F_{A_2}$, and so $c_1(L_1 \otimes L_2) = [\tau_1(L_1 \otimes L_2)] = [\frac{i}{2\pi} F_{A_1 \otimes A_2}] = c_1(L_1) + c_1(L_2)$ □

With the Chern classes we can also define for real vector bundles the *Pontrjagin classes*:

Definition 1.3.9. *Let E be a real vector bundle over M. The k-the* Pontrjagin class *is defined as:*

$$p_k(E) = (-1)^k c_{2k}(E \otimes \mathbb{C}) \in H^{4k}(M, \mathbb{Z})$$

Characteristic classes, though defined on vector bundles, can also be used to investigate properties of the underlying manifold, for example by studying the class of the tangent space.

1.3.5 Connections and projections

For a good understanding of the ADHM construction of instantons in section 3.1.2, we will review a way to construct connections on non-trivial vector bundles from flat connections on higher rank trivial vector bundles. We will follow [DK90].

Let M be a smooth manifold and K and L finite dimensional complex vector spaces. Let $R : M \to \mathrm{Hom}(K, L)$ be a smooth map. If R_x is surjective for all x, the kernels form a vector sub-bundle E over M of the trivial bundle $K \times M$, with $E_x = \ker(R_x)$. We know that $K \times M$ has a flat product connection d_0. Now suppose we have a smooth bundle projection $K \times M \to E$ which is left-inverse to the natural inclusion $i : E \hookrightarrow K \times M$. We then get an induced connection A on E given by the covariant derivative

$$d_A = \pi d_0 i$$

If K has an Hermitian metric we can define such a π as the orthogonal projection onto E, and this makes A into a unitary connection. Now we turn to the slightly more special case of holomorphic vector bundles.

Definition 1.3.10. *Let K_0, K_1 and K_2 be finite-dimensional complex vector spaces, M a complex vector manifold and assume we have holomorphic vector* *bundle maps:*

$$K_0 \times M \xrightarrow{\alpha} K_1 \times M \xrightarrow{\beta} K_2 \times M$$

with $\beta\alpha = 0$. *This sequence is called a* monad

This means there is a family of chain complexes parametrized by M:

$$K_0 \xrightarrow{\alpha_x} K_1 \xrightarrow{\beta_x} K_2$$

varying holomorphically with x.

Lemma 1.3.11. *The family of cohomology spaces defined by*

$$E_x = \frac{Ker(\beta_x)}{Im(\alpha_x)}$$

is a holomorphic vector bundle E *over* M*.*

Proof. It is clear that E_x has the structure of a vector bundle. The holomorphic structure is constructed by lifting to $K_1 \times M$.

We define a local section s of E to be holomorphic if it has a lift to a holomorphic section s' of $\mathrm{Ker}(\beta)K_1 \times M$.

Let $x_0 \in M$ and $k_1 \in \mathrm{Ker}(\beta_x)$. Choose a right inverse $P : K_2 \to K_1$ for β_{x_0}. Now we want to construct a holomorphic section of $\mathrm{Ker}(\beta)$ of the form $k_1 + j(x)$ with $j(x_0) = 0$. Define $\eta_x = \beta_x - \beta_{x_0}$. Then the condition that $k_1 + j(x)$ lies in $\mathrm{Ker}(\beta)$ is satisfied if $(1 + P\eta_x)j(x) = -P\eta_x(k_1)$. This can be done, since if x is close to x_0, $P\eta_x$ is small and $(1 + P\eta_x)$ can be inverted to find a unique solution for $j(x)$. Even more, $j(x)$ varies holomorphically with x, since η_x does. With this construction, we find a set of holomorphic local sections of E which form a basis of the fibers near x_0, so E is a holomorphic vector bundle. □

1.4 Equivariant cohomology

1.4.1 Introduction

Equivariant cohomology is a cohomology theory on a manifold where a group action works. This group action can give us much information about the underlying manifold. There are 2 definitions of equivariant cohomology, which were proven to be equivalent in the case of compact manifolds with compact group actions by Cartan. The first uses classifying spaces and is called the *Borel model*.

Definition 1.4.1. *Let G be a compact Lie group and assume that it acts on a contractible space E freely. This space is called a* classifying space *of G. There is then a principal G-bundle $E \to B = E/G$. The equivariant cohomology of M is defined to be the normal cohomology of $(M \times E)/G$:*

$$H_G^*(M) = H^*((M \times E)/G)$$

Since every compact Lie group has a faithful linear representation, it can be embedded into a $U(n)$ for n big enough. If H is a subgroup of G, then if G acts freely on a space E, then certainly H acts freely. Thus it suffices to prove the existence of a space E above for all $U(n)$. Let $\mathcal{E}_k$ be the space of orthonormal n-frames in $\mathbb{C}^{k+1}$ with $k \geq n$. Then $U(n)$ acts freely on $\mathcal{E}_k$. We can take the limit $k \to \infty$, and this gives one example of a space E. For details see chapter 1 of [GS99].

All classifying spaces for a group G can be shown to be homotopy equivalent, so the cohomology theory is well-defined and does not depend on the choice of E.

The action of a group G on M induces an action on differential forms on M. This action can be differentiated, leading to an action of $\mathfrak{g}$ on differential forms.

Definition 1.4.2. *An* equivariant form *is a map $\alpha : \mathfrak{g} \to \Omega^*(M)$ which is G-equivariant, that is, the following diagram commutes $\forall g \in G$*

$$\begin{array}{ccc} \mathfrak{g} & \xrightarrow{\alpha} & \Omega^*(M) \\ \downarrow{\scriptstyle Ad(g)} & & \downarrow{\scriptstyle g} \\ \mathfrak{g} & \xrightarrow[\alpha]{} & \Omega^*(M) \end{array}$$

or $g\alpha(X) = \alpha(Ad(g)(X))$ for all $X \in \mathfrak{g}$ and $g \in G$.

The wedge product of two equivariant forms is also equivariant, so we have the algebra $\Omega_G^*(M)$ of equivariant forms on M.

Definition 1.4.3. *The* twisted de Rham differential*, $d_{eq} : \Omega_G^*(M) \to \Omega_G^*(M)$ is defined as the map*

$$d_{eq}(\alpha)(X) = d(\alpha(X)) - \iota_{X^\#}(\alpha(X))$$

with d the normal de Rham differential, and $\iota_{X^\#}$ as defined in definition 1.2.3. So $d_{eq}\alpha$ is a map from $\mathfrak{g}$ to $\Omega^*(M)$.

It is easy to check that $(d_{eq})^2 = 0$ on equivariant forms, and $d_{eq}\alpha \in \Omega_G^*(M)$, so we can define equivariant cohomology in the *Cartan model*:

Definition 1.4.4. *The equivariant cohomology of M under the action of a group G is the cohomology of $(\Omega_G^*(M), d_{eq})$.*

These two definitions of equivariant cohomology are actually the same in many common cases [Car51]:

Theorem 1.4.5. *If G is a compact Lie group acting on a smooth compact manifold M then he twisted de Rham complex Ω_G^* computes the equivariant cohomology of M:*

$$H_G^*(M, \mathbb{C}) = H^*(\Omega_G^*(M), d_{eq})$$

Many common objects on a manifold can be made into equivariant objects. For example, if (M, ω, G, μ) is a symplectic manifold with Hamiltonian action G with moment map μ, then $\tilde{\omega} = \omega + \mu^X$ with μ^X the moment map defined in 1.2.3 is the *equivariant symplectic form*, a map from $\mathfrak{g}$ to $\Omega^*(M)$, for we see that

$$d\mu(X) = \iota_{X^\#}\omega$$

by definition of the moment map and $d\omega = 0$ so $d_{eq}\tilde{\omega} = 0$.

Now we can easily prove theorem 1.2.5:

Proof. We consider the equivariant symplectic form $\tilde{\omega} = \omega + \mu^X$. The restriction on the level set p of μ^X is equivariant closed: $(d - i_{X^\#})i_p^*\tilde{\omega} = 0 \; \forall X \in Lie(T)$ by definition of the moment map. We also have $i_p^*\mu^X = p$, so $i_p^*\tilde{\omega} = \omega + p$. Since ω is a symplectic form, we have $d\omega = 0$, hence we get (using that $dp = \iota_X p = 0$), $0 = d_{\mathrm{eq}} i_p^*\tilde{\omega} = \iota_a(i_p^*\omega) \otimes x^a$ with x^a a dual basis to the basis a of $\mathfrak{g}$. This means that $\iota_a(i_p^*\omega) = 0 \; \forall a$, and since $\iota_p^*\omega$ is G-invariant, $i_p^*\omega$ is in the image of $\pi^* : \Omega(M/G) \to \Omega(M)$. This map clearly is injective, so there is a unique two-form ν_p such that $\pi^*\nu_p = i_p^*\omega$.

We still need to show that this two-form is symplectic, i.e. closed and non-degenerate. It is certainly closed, since π^* is injective and $\pi^* d\nu_p = d\pi^*\nu_p = DI_p^*\omega = 0$. Set $d = \dim(G)$ and $2d = \dim(M)$. To prove that ν_p is non-degenerate, it suffices to show that $(\nu_p)^{d-n}$ vanishes nowhere. This is sufficient, since $\dim(\mu^{-1}(p)/G) = 2d - n - n = 2d - 2n$.

Since ν_p^{n-d} is nowhere vanishing if and only if $\pi^*(\nu_p)^{n-d}$ vanishes nowhere on $\mu^{-1}(p)$ and $\pi^*\nu_p = i_p^*\omega$, we need to show that ω^{n-d} vanishes nowhere on $\mu^{-1}(p)$. Now, using some basic combinatorics and definition 1.2.3 we see that

$$\iota_{\xi_1}\iota_{\xi_2} \dots i_{\xi_n}(\omega)^d = \frac{d!}{n!}\omega^{d-n} \wedge d\mu^1 \wedge \cdots \wedge d\mu^n$$

on $\mu^{-1}(p)$, where the ξ_i form a basis of $\mathfrak{g}$. Since the ξ_i are independent and ω is non-degenerate, we see that the right hand side is not identically zero, hence ν_p is symplectic. □

An important theorem in equivariant cohomology is the *splitting principle* which asserts that the equivariant cohomology of a manifold with respect to a group G can be computed by just looking at the equivariant cohomology with respect to a maximal torus T inside G:

Theorem 1.4.6 (Abstract splitting principle). *Let G be a connected compact Lie group, T a maximal torus of G and W the Weyl group. Then there is a natural isomorphism*

$$H_G^*(M) \simeq H_T^*(M)^W$$

with $H_T^*(M)^W$ the W invariant classes of the T-equivariant cohomology of M.

For a proof see for example [GS99].

This splitting principle has some important consequences, for example the splitting principle in topology

Corollary 1.4.7. *For every vector bundle $E \to M$ there exists a manifold N and a fibration $\pi : N \to M$ such that*

1. *The induced map $\pi^* H^*(M) \to N$ is injective*
2. *The pull-back bundle $\pi^* E$ splits into a direct sum of line bundles $\oplus L_i$*

1.4.2 Equivariant characteristic classes

The theory of characteristic classes also can be viewed within equivariant cohomology, instead of normal cohomology. Of particular interest in this thesis is the equivariant Euler class. First we will define the notions of equivariant connections and curvature forms.

Let $E \to M$ be a G-equivariant vector bundle, that is, G acts on E by vector bundle maps $E \to E$. The tangent bundle is just one example of a G-equivariant vector bundle, and the definition for G-equivariant differential forms from subsection 1.4.4 just carries over to the case of differential forms with values in E. We denote the action of X on this space by $X_E^\#$. We can build G-invariant connections d_A by averaging a normal connection over the Haar measure on G. We can then define the equivariant connection

Definition 1.4.8. *The* equivariant connection d_{Aeq} *corresponding to a G-invariant connection d_A is the operator on $\mathbb{C}[\mathfrak{g}] \otimes \Omega^*(M, E)$ defined by*

$$(d_{Aeq}\alpha)(X) = (d_A - \iota_X)\alpha(X)$$

where $X \in \mathfrak{g}$ and $\iota_X = \iota_{X_E^\#}$.

The definition of equivariant curvature cannot be just d_{Aeq}^2 as in the non-equivariant case as a simple example shows. Let $E = M \times \mathbb{C}^n$ be the trivial vector bundle of rank n and take $d_A = d$ be the standard de Rham operator. Then by definition $d_{Aeq} = d_{eq}$ and $(d_{Aeq})^2 = d_{eq}^2 = -(d\iota_X + \iota_X d) = -X_E^\#$, which is not 0 outside the equivariant forms. This motivates the definition:

Definition 1.4.9. *The* equivariant curvature F_{Aeq} *of an equivariant connection d_{Aeq} is defined as*

$$F_{Aeq}(X) = d_{Aeq}(X)^2 + X_E^\#$$

with $X \in \mathfrak{g}$

Then, almost by the same reasoning as above we have that $\mathrm{Trace}(F_{Aeq}^n)$ is equivariantly closed, and the equivariant cohomology class is independent of the choice of G-invariant connection on E. This leads to equivariant characteristic classes.

For the equivariant Euler class, we recall that any $A \in \mathfrak{so}(V)$ defines an anti-symmetric bilinear form on V by $\langle Av_1, v_2 \rangle$. Now identifying V with V^* by a metric we get an element $\alpha \in \bigwedge^2 V$, $\alpha = \sum_{i<j} \langle Ae_i, e_j \rangle e_i \wedge e_j$. We then define the Pfaffian Pf of a $A \in \mathfrak{so}(V)$ as

$$\mathrm{Pf}(A) = \mathrm{Pf}(\alpha) = \int (\exp \sum_{i<j} \langle Ae_i, e_j \rangle e_i \wedge e_j)$$

where the integral is the *Berezin integral*:

Definition 1.4.10. *Let V be a $\mathbb{R}$-vector space of dimension n, with oriented volume element $\mu \in \bigwedge^n V$. The* Berezin integral *is the linear function $\int$: $\bigwedge^* V \to \mathbb{R}$ defined as follows:*

- *If $\beta \in \bigwedge^k V$ with $k < n$ then $\int \beta = 0$.*
- *If $\beta = \mu$ then $\int \beta = 1$.*
- *If $\beta \in \bigwedge^n$ then β can be written as $ba \cdot \mu$ and we define $\int \beta = b$.*

Since α is a 2-form, this implies that $\mathrm{Pf}(A) = 0$ if V is not a even-dimensional vector space. If d_{Aeq} is a metric compatible connection, the curvature of the connection takes values in $\mathfrak{so}(V)$ so we can compute its Pfaffian.

Definition 1.4.11. *Let G be a Lie group acting on a manifold M, let A be an equivariant metric-compatible connection with equivariant curvature F_{Aeq} and $X \in \mathfrak{g}$. The* equivariant Euler class *$e_{eq} \in H^*_G(M)$ or e_G is*

$$e_{eq}(d_A)(X) = Pf(-F_{Aeq}(X))$$

1.4.3 Thom class

Given an n-dimensional compact oriented submanifold of an m-dimensional oriented manifold M, we can construct a characteristic class associated to this submanifold. Since every manifold can be viewed as a submanifold of a vector bundle by identification with the zero section, this gives a characteristic class of a vector bundle.

Integration over N gives a linear function on the de Rham cohomology group $H^n(M)$. By Poincaré duality we also have a non-degenerate pairing

$$H^n(M) \times H_c^{m-n}(M) \to \mathbb{C}, \quad (a,b) = \int_M a \wedge b$$

with $H_c^{m-n}(M)$ the compactly supported cohomology groups. This gives a unique cohomology class $\tau(N) \in H_c^{m-n}(M)$ such that

$$\int_N \alpha = \int_M \alpha \wedge \tau(N)$$

This class τ is called the *Thom class* of N. Any closed form τ_N representing $\tau(N)$ is called a Thom form.

Unfortunately, Poincaré duality does not hold in equivariant cohomology, which can be seen by using the upcoming Berline-Vergne localization theorem 1.4.15. We have a natural pairing

$$(,):\Omega_G^*(M)\times\Omega_G^*(M)\to\mathbb{C}[\mathfrak{g}]\quad(\alpha,\beta)=\int_M\alpha\wedge\beta$$

However, if the action of G is free on M, we see from equation 1.4.4 that the integral is always zero. However, the Thom class can be generalized to equivariant cohomology.

First we will say something about integration of equivariant forms. Let $\alpha\in\Omega_G^*(M)$, with M a n-dimensional manifold. Then we write

$$\alpha=\alpha_{[0]}+\alpha_{[1]}+\ldots+\alpha_{[n]}$$

with $\alpha_{[i]}$ of degree i in $\Omega_G^*(M)$. Then integration

$$\int:\Omega_G^*(M)\to\mathbb{C}[\mathfrak{g}]^G:\quad\left(\int\alpha\right)(X)\mapsto\int_M\left(\alpha_{[n]}(X)\right)$$

If α is equivariant exact, $\alpha=d_{eq}\beta$ we see that $\alpha_{[n]}(X)=d\beta_{[n-1]}(X)$ since i_X cannot map to an n-form, so Stokes' theorem holds for equivariant de Rham operators and we see that integration descends to equivariant de Rham classes

$$\int:H_G^*(M)\to\mathbb{C}[\mathfrak{g}]^G$$

Theorem 1.4.12 (Mathai-Quillen [MQ86]). *Let $\pi:V\to N$ be a G-equivariant vector bundle of rank k, with V and N oriented. Denote by $(H_c^*)_G$ the equivariant cohomology of compactly supported equivariant differential forms. Then the following holds*

1. *The map $\pi_*:(H_c^l)_G(V)\to H_G^{l-k}(N)$ is an isomorphism for all l*
2. *There exists a from $\tau_G\in\Omega_G^k(V)$ with compact support which is equivariantly closed so that $\pi_*\tau_G=1$*
3. *Let $e_G(V)$ be the equivariant Euler class associated to V, denote by $i_0:N\to V$ the embedding of N in V as the zero section, then $i_0^*\tau_G(V)=e_G(V)$*

The form τ_G is called the equivariant Thom form.

1.4.4 A first localization theorem

In this section we will prove some of the localization theorems which form the basis of the equivariant localization of instantons later in section 3.2.3. The basic idea behind localization is the exact stationary phase approximation. Suppose we want to calculate the integral of the function $e^{itf(x)}$ over a manifold. If t is large, we can take the exact stationary phase approximation, which says that the dominant contributions to the integral come from the critical points. These can then be calculated using a quadratic approximation to $f(x)$ near each critical point and calculating the resulting Gaussian function. Duistermaat and

Heckman proved in [DH82] that in case this approximation is exact if interpreted in equivariant cohomology. We will consider equivariant localization in this section in the most basic cases, and in section 3.2.3 in cases where we have an extra structure on the manifold we are integrating over. The later equivariant localizations are based on the basic localization theorems proven here.

Let G be a compact Lie group acting on a compact manifold M. Let $X \in \mathfrak{g}$. We define a corresponding vector field on M as follows. If α is an equivariant differential form, that is, $g\alpha(X) = \alpha(\mathrm{Ad}(g)(X))$ then if $Y \in \mathfrak{g}$, we set $g = \exp tY$, differentiate with respect to t and then set $t = 0$ to get an action of $\mathfrak{g}$ on forms, giving a vector field $X^\#$ associated to each element X of $\mathfrak{g}$, see also definition 1.2.3

$$(X^\# f)(x) = \frac{d}{dt} f(\exp(-tX) \cdot x)|_{t=0}$$

This is also a Lie algebra map by the relation $[X^\#, Y^\#] = [X, Y]^\#$. From now on, when $X \in \mathfrak{g}$ we will denote by abuse of notation $\iota_{X^\#} = \iota_X$.

Now set $M_0(X)$ to be the zeroes of the vector field $X^\#$.

Lemma 1.4.13. *If α is a smooth map from $\mathfrak{g}$ to $\Omega^*(M)$ and is equivariantly closed, then $\alpha(X)_{[dim(M)]}$ is exact on $M \backslash M_0(X)$.*

Proof. Set $\theta_X \in \Omega^1(M)$ to be a one-form such that $X^\# \theta_X = 0$ and $\iota_X \theta_X \neq 0$ on $M \backslash M_0$. Such a θ_X exists, for example $\theta_X(v) = \langle X^\#, v \rangle$. It is easy to see that $X^\# \theta_X = 0$ and $\iota_X \theta_X = \|X^\#\|^2 \neq 0$. Then

$$(d - \iota_X)^2 \theta_X = -d\iota_X \theta_X - \iota_X d\theta_X = -X^\# \theta_X = 0 \tag{1.4.1}$$

We can then invert $(d - \iota_X)\theta_X$ on $M \backslash M_0$ using a geometric series:

$$\frac{1}{(d - \iota_X)\theta_X} = -(\iota_X \theta_X)^{-1} - (\iota_X \theta_X)^{-2} d\theta_X - (\iota_X \theta_X)^{-3} (d\theta_X)^2 - \dots \tag{1.4.2}$$

Note that this geometric series is finite, because the $(d\theta_X)^n$ term vanishes if $2n > \dim(M)$. So

$$(d - \iota_X)\theta_X \wedge \frac{1}{(d - \iota_X)\theta_X} = 1$$

Applying $(d - \iota_X)$ again on the both sides, and using equation 1.4.1, we get

$$(d - \iota_X)\theta_X \wedge (d - \iota_X) \frac{1}{(d - \iota_X)\theta_X} = 0$$

and since $(d - \iota_X)\theta_X$ is invertible hence non-zero, we see that

$$(d - \iota_X) \frac{1}{(d - \iota_X)\theta_X} = 0$$

Since $\alpha(X)$ is equivariantly closed on $M \backslash M_0$ we can write on this set:

$$\alpha(X) = (d - \iota_X) \left(\frac{\theta_X \wedge \alpha(X)}{(d - \iota_X)\theta_X} \right)$$

If we only look at the n-form part with $n = \dim(M)$ we see immediately that

$$\alpha(X)_{[\dim(M)]} = d\left(\frac{\theta_X \wedge \alpha(X)}{(d - \iota_X)\theta_X}\right)_{[\dim(M)-1]} \tag{1.4.3}$$

□

We can look at the action of $X^\#$ on the tangent space of the fixed points of $X^\#$. We will first specialize to the case where the set of fixed points is finite. The Lie action of $X^\#$ on vector fields F, $F \mapsto [L_{X^\#}, F]$ induces a map on T_pM for $p \in M_0(X)$, $L(X,p)$

Lemma 1.4.14. *The transformation $L(X,p)$ on T_pM is invertible for each $p \in M_0$.*

Proof. Suppose $v \in T_pM$ $v \neq 0$ is in the kernel of $L(X,p)$. Then we can pick a G-invariant metric since G is compact, and look at the geodesic defined by $\exp_p(tv)$ for $t \in (-\epsilon, \epsilon) \subset \mathbb{R}$. This geodesic is then in the kernel of $\exp_p(sX)$ for $s \in (-\epsilon', \epsilon')$ hence p is not an isolated point and we have a contradiction. □

If we now pick a torus $T \subset G$ such that $X \in Lie(T)$, then T fixes $p \in M_0(X)$ and thus we have a representation of T on T_pM. Also, since T is compact, we have a Riemannian metric which is T invariant such that the metric on T_pM is preserved by T, called the *isotropy representation.* Hence $L(X,p) \in \mathfrak{so}(T_pM)$. Since by the above lemma $L(X,p)$ is invertible, it can then be written in block diagonal form, with blocks of the form

$$\begin{pmatrix} 0 & -c_j \\ c_j & 0 \end{pmatrix}$$

with $c_j \in \mathbb{R}$. Note also that M and thus T_pM must be even dimensional in this case. Define $n = \dim(M)$, then $\det^{1/2}(L(X,p)) = \prod_{i=1}^{n/2} c_i$.

Theorem 1.4.15 (Berline-Vergne localization [BV82]). *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, acting on a compact orientable manifold M with dimension n even. Let $\alpha : \mathfrak{g} \to \Omega^*(M)$ be a smooth equivariantly closed form and $X \in \mathfrak{g}$ such that $X^\#$ has a finite set of zeroes. Then*

$$\int_M \alpha(X) = (-2\pi)^{n/2} \sum_{p \in M_0} \frac{\alpha(X)_{[0]}(p)}{\det^{1/2}(L(X,p))} \tag{1.4.4}$$

If we take $\alpha(X)$ to be the equivariant symplectic volume form, that is $\exp(\tilde{\omega}) = \sum_{n=0}^{\infty} \frac{\tilde{\omega}^n}{n!}$, we get immediately the Duistermaat-Heckman formula from this theorem.

Proof. Without loss of generality we replace G by the torus $T \subset G$ such that $X \in Lie(T)$. Now we use the the exponential map at p: $\exp_p : T_pM \to M$ to construct local coordinates in an open subset U_p around p such that

$$X^\# = c_1\left(x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\right) + \ldots + c_{n/2}\left(x_n \frac{\partial}{\partial x_{n-1}} - x_{n-1} \frac{\partial}{\partial x_n}\right)$$

with c_i as in the discussion above. Now set θ_X^p to be the one-form given in U_p as

$$\theta_X^p = c_1^{-1}\,(x_2 dx_1 - x_1 dx_2) + \ldots + c_{n/2}^{-1}\,(x_n dx_n{-}1 - x_{n-1}dx_n)$$

This θ_X^p has the properties that $X^{\#}\theta_X^p = 0$ and $\iota_X\theta_X^p = \theta_X^p(X^{\#}) = \|x\|^2$ so $i_X\theta_X^p \neq 0$ at $U_p\backslash\{p\}$. Now we can use a G-invariant partition of unity to extend this θ_X^p to a one form θ_X on all of M such that $X^{\#}\theta_X = 0$ and $\iota_X\theta_X \neq 0$ on $M\backslash M_0$.

Around each fixed point p we cut out an open ball of radius ϵ, B_ϵ^p and we see with the help of formula 1.4.3:

$$\int_M \alpha(X) = \lim_{\epsilon\to 0}\int_{M\backslash\cup_p B_\epsilon^p}\alpha(X) = \lim_{\epsilon\to 0}\int_{M\backslash\cup_p B_\epsilon^p} d\left(\frac{\theta_X\wedge\alpha(X)}{(d-\iota_X)\theta_X}\right) \qquad (1.4.5)$$

Now we use Stokes' theorem to write this as an integral over the surfaces of the B_ϵ^p, the spheres S_ϵ^p with a minus sign added because the balls now lie in the exterior:

$$\lim_{\epsilon\to 0} - \int_{\cup_p S_\epsilon^p}\left(\frac{\theta_X\wedge\alpha(X)}{(d-\iota_X)\theta_X}\right) = -\lim_{\epsilon\to 0}\sum_p\int_{S_\epsilon^p}\frac{\theta_X^p\wedge\alpha(X)}{(d-\iota_X)\theta_X^p}$$

if the S_ϵ^p lie within the U_p

Now we use formula 1.4.2 to expand the denominator:

$$\frac{1}{(d-\iota_X)\theta_X^p} = -(\iota_X\theta_X^p) - (\iota_X\theta_X^p)(d\theta_X^p) - \ldots - (\iota_X\theta_X^p)^{-n/2-1}(d\theta_X^p)^{n/2}$$

On the sphere S_ϵ^p, $\iota_X\theta_X^p = \epsilon$ by construction and the rightmost term in the expansion vanishes, because together with the θ_X^p one-form in the numerator it is a $n+1$-form on the n-dimensional manifold M, so the series reduces to

$$-\epsilon^{-1} - \epsilon^{-2}d\theta_X^p - \ldots - \epsilon^{-n/2}(d\theta_X^p)^{n/2-1}$$

so we can write, remembering that we are integrating over an $(n-1)$-dimensional manifold:

$$\frac{\theta_X^p\wedge\alpha(X)}{(d-\iota_x)\theta_X^p} = -\epsilon^{-n/2}\alpha(X)_{[0]}\theta_X^p\wedge(d\theta_X^p)^{n/2-1} - \sum_{j=0}^{n/2-2}\epsilon^{-(j+1)}\theta_X^p\wedge\alpha_{[n-2j-2]}\wedge(d\theta_X^p)^j$$

Now the factor in the summations will vanish when we actually integrate over S_ϵ^p and take the limit $\epsilon\to 0$, so we write

$$\begin{aligned}\int_M \alpha(X) &= \lim_{\epsilon\to 0}\sum_p\int_{S_\epsilon^p}\epsilon^{-n/2}\alpha(X)_{[0]}\theta_X^p\wedge(d\theta_X^p)^{n/2-1}\\ &= \sum_p \alpha(X)_{[0]}(p)\lim_{\epsilon\to 0}\int_{S_\epsilon^p}\epsilon^{-n/2}\theta_X^p\wedge(d\theta_X^p)^{n/2-1}\\ &= \sum_p \alpha(X)_{[0]}(p)\lim_{\epsilon\to 0}\int_{B_\epsilon^p}\epsilon^{-n/2}(d\theta_X^p)^{n/2}\end{aligned}$$

Now since $d\theta_X^p = -2c_1^{-1}dx_1\wedge dx_2 - \ldots - 2c_{n/2}^{-1}dx_{n-1}\wedge dx_n$ we see that the n-form part of $(d\theta_X^p)^{n/2}$ is $(-2)^{n/2}(n/2)!\left(\prod_{i=1}^{n/2} c_i\right)^{-1} dx_1 \wedge \ldots \wedge dx_n$. The volume of a n-dimensional ball if n is even is $\frac{\pi^{n/2}}{(n/2)!}$ and so we get the asked formula:

$$\int_M \alpha(X) = (-2\pi)^{n/2} \sum_{p \in M_0(X)} \frac{\alpha(X)_{[0]}(p)}{\prod_{i=1}^{n/2} c_i}$$

□

As an example, we can compute the area of a sphere. Let $G = S^1$ the circle, act on $S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$ by rotation around the z-axis. The fixed points are $(0, 0, 1)$ and $(0, 0, -1)$. The sphere has a natural symplectic form $\iota_N dx \wedge dy \wedge dz$ with N the outward pointing normal. This is clearly invariant under the action of S^1. The vector field $L_{X^\#}$ are proportional to $x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$. A moment map $\mu : S^2 \to Lie(S^1) \simeq \mathbb{R}$ of this action is given by $\mu(t) = tz$, and set $\tilde{\omega}(t) = \mu(t) + \omega$. Now $c_1 = \mp t$, so we get

$$\int_{S^2} \mu(t) + \omega = \sum_{p=(0,0,\pm 1)} \frac{\tilde{\omega}_{[0]}(t)(p)}{\prod_{i=1}^{n/2} c_i} = \sum_{p=(0,0,\pm 1)} \frac{\mu(t)(p)}{\mp t} = \frac{t}{-t} + \frac{-t}{t} = -2$$

which together with the (-2π) factor give exactly the area of the sphere, which could also be directly computed by using the symplectic volume form (the moment map is of degree 0 so does not contribute to the integral in that case).

This theorem can also be proven in the case where the fixed points are not isolated, replacing the sum over the fixed points by an integral over each connected component of the set of fixed points. If we study the proof of theorem 1.4.15 carefully, we see that the only point in which we make use of the isolation of the fixed points is in lemma 1.4.14. This fact can be amended as follows.

Lemma 1.4.16. *Fix a $X \in \mathfrak{g}$ and let $M_0(X)$ be the set of zeroes of the vector field $X^\#$. Then connected components F_i of $M_0(X)$ are smooth submanifolds of M. The dimensions of different connected components do not have to be the same. The normal bundles $\mathcal{N}_i$ of F_i are orientable vector bundles with even-dimensional fibers.*

Proof. Choose a G-invariant Riemannian metric on M. Let $p \in M_0(X)$. The exponential map $T_pM \to M$ is a diffeomorphism near from a neighborhood of $0 \in T_pM$ to a neighborhood of $p \in M$. A vector $v \in T_pM$ is fixed by the induced transformation $L(X, p)$ if and only if the image $\exp(v)$ is contained within the connected component of $M_0(X)$ of p. From this we can construct adapted coordinates on F_i by taking a basis of the subspace of vectors fixed by $L(X, p)$, hence F_i is a submanifold.

The normal bundle $\mathcal{N}$ is defined fiberwise as the orthogonal complement of $T_pM_0(X)$ inside T_pM. We see that the linear transformation $L(X, p)$ induced by $X^\#$ is 0 on $T_pM_0(X)$ and invertible anti-symmetric on $\mathcal{N}_p$. Thus the fibers $\mathcal{N}_p$ must be even-dimensional. We can choose an orientation on $\mathcal{N}$ by requiring $\det\left(L(X, p)|_{\mathcal{N}_p}\right) > 0$ □

Now using the orientation of $\mathcal{N}$ and of M, we can set an orientation on $M_0(X)$ as follows. A basis of $T_pM_0(X)$ is positively oriented if and only if that basis forms a positively oriented basis of T_pM together with a positively oriented basis of $\mathcal{N}_p$.

Now consider the centralizer of X in $\mathfrak{g}$: $\mathfrak{g}_0(X) = \{Y \in \mathfrak{g} | [X, Y] = 0\}$ and let $G_0(X)$ be the connected component of the identity such that $\mathfrak{g}_0(X)$ is its Lie algebra. Clearly, since $\cdot^\#$ is a Lie algebra homomorphism, $G_0(X)$ preserves $M_0(X)$ and also acts on the normal bundle $\mathcal{N}$. Now choose a $G_0(X)$ invariant Riemannian metric and corresponding connection $d_A{}^{\mathcal{N}}$ on the vector bundle $\mathcal{N}$. We get an equivariant curvature form

$$F_{A_{eq}}^{\mathcal{N}}(Y) = F^{\mathcal{N}} + \mu^{\mathcal{N}}(Y), \mu^{\mathcal{N}}(Y) = Y^\#_{\mathcal{N}} - d_{A}{}^{\mathcal{N}}_Y, Y \in \mathfrak{g}_0(X)$$

Since $X^\#$ vanishes on $M_0(X)$, we have $\mu^{\mathcal{N}}(X) = X^\#_{\mathcal{N}}$ and $\mu^{\mathcal{N}}(X)|_p = L(X,p)|_{\mathcal{N}_p}$ which is invertible on $\mathcal{N}_p$. Since $\mathcal{N}$ has the orientation as above, we can calculate the $G_0(X)$-equivariant Euler class of $\mathcal{N}$:

$$\chi_{eq}(\mathcal{N})(Y) = \mathrm{Pf}(-F_{A_{eq}}^{\mathcal{N}}(Y))$$

If Y is sufficiently close to X, we have the 0-th order term of this equivariant Euler class is non-zero, hence the equivariant Euler class for Y sufficiently close to X is invertible. With some extra, technical observations, we can then prove:

Theorem 1.4.17 (Berline-Vergne localization with non-isolated fixed points). *Let G be a compact Lie group with Lie algebra $\mathfrak{g}$, acting on a compact orientable manifold M with dimension n even. Let $\alpha : \mathfrak{g} \to \Omega^*(M)$ be a smooth equivariantly closed form and $X \in \mathfrak{g}$. Then for Y in the centralizer of X and sufficiently close to X*

$$\int_M \alpha(Y) = (2\pi)^{Rank\mathcal{N}/2} \sum_i \int_{F_i} \frac{\alpha(Y)}{e_{eq}(\mathcal{N})(Y)} \tag{1.4.6}$$

with e_{eq} the equivariant Euler class of 1.4.11.

Note that the above reduces to theorem 1.4.15 in the case where the F_i are zero-dimensional, since then the integral over F_i just gives the degree 0 part in p of α, and the equivariant Euler class reduces to its degree 0 part as well.

1.4.5 Symplectic cut

Many interesting manifolds, especially in when we are considering instantons are not compact. This gives problems for many techniques, including equivariant localization. A method for dealing with this problems is the *symplectic cut*, described first by Lerman in [Ler95]. It is a generalization of the blowup operation in symplectic geometry, and makes use of the symplectic reduction of Marsden and Weinstein.

We have the standard symplectic form on $\mathbb{C}$: $idw \wedge d\bar{w}$. We use this, together with the symplectic reduction as described above, to get the symplectic cut:

Definition 1.4.18. *Let (M, ω) be a symplectic manifold, with a Hamiltonian circle action and a moment map $\mu : M \to \mathbb{R}$ and assume ϵ is a regular value of μ. Then the* symplectic cut *of M is*

$$\overline{M}_{\mu \leq \epsilon} = \{(m, z) \in M \times \mathbb{C} : \mu(m) + |z|^2 = \epsilon\}/S^1 = (M \times \mathbb{C})/\!\!/_\epsilon S^1$$

or equivalently, defining $\tilde{\mu} : M \times \mathbb{C} \to \mathbb{R}, \tilde{\mu}(m, z) = \mu(m) + |z|^2$*, as:*

$$\overline{M}_{\mu \le \epsilon} = \tilde{\mu}^{-1}(\epsilon)/S^1$$

We see that $\overline{M}_{\mu \le \epsilon}$ *has dimension* $2n$*, is compact if* $\tilde{\mu}$ *is proper and has a natural symplectic form by theorem 1.2.5.*

Of course, we can also consider the space

$$\overline{M}_{\mu \ge \epsilon} = \{(m, z) \in M \times \mathbb{C} : \mu(m) - |z|^2 = \epsilon\}/S^1$$

As an example, we calculate the symplectic cut in the simplest case, the complex plane $\mathbb{C}$ with circle action e^{it}. The moment map is $|z|^2$, so $\tilde{\mu}^{-1}(\epsilon) = S^3(\epsilon)$, the sphere in 4 real dimensions with radius ϵ. Now we need to divide out the circle actions on $\mathbb{C} \times \mathbb{C}$. We see that we can write $\tilde{\mu}^{-1}(\epsilon)$ as the disjoint union of two spaces, $\{(m, 0)|m \in \mu^{-1}(\epsilon)\}$ and $\{(m, w)|\mu(m) < \epsilon, w = e^{it}\sqrt{\epsilon - \mu(m)}\})$. The first is the boundary of the open disk with radius ϵ, $\mathbb{D}(\epsilon)$ and the second is the open disk $\mathbb{D}(\epsilon)$ with on each point a circle. We thus see that the action of S^1 on these spaces give the closed disk with radius ϵ with the boundary collapsed to a point.

This splitting is a general feature of the symplectic cut. We also see that if we let $\epsilon \to \infty$, we recover the original manifold as one of the components. If there is another group K acting on M whose action commutes with the S^1 action on M, we see that because this action descends to $\mu^{-1}(\epsilon)$ the group K acts on $\overline{M}_{\mu \le \epsilon}$.

Of course, this can easily be generalized to higher dimensional Lie groups.

Definition 1.4.19. *Take a maximal torus* T *of a Lie group* G *that acts Hamiltonian with respect to a symplectic form* ω *on a manifold* M*, and let* $\mu : M \to \mathfrak{t}^*$ *be its moment map. Let* $\beta = \{\beta_1, \ldots, \beta_k\}$ *be a set of weights that are linearly independent. We allow* k *to be less than the dimension of* T*. Let* T *act with weight* β *on* $\mathbb{C}_\beta \equiv \mathbb{C}^k$*. Of course,* T *is Hamiltonian with respect to the standard symplectic form* $\omega_\beta = \sum_i dz_i \wedge d\bar{z}_i$*, and the moment map is* $\psi(z) = \sum \beta_i |z_i|^2$*. The image of* $\mathbb{C}_\beta$ *under the moment map is the cone of dimension* k $\Sigma = \{\sum s_i\beta_i | s_i \ge 0\}$*. The combined action of* G *on* $M \times \mathbb{C}^k$ *is Hamiltonian with respect to the symplectic form* (ω, ω_β) *with moment map* $\phi(x, z) = \mu(x) + \psi(z)$*. If we have a set of indices* I *between* 1 *and* k *then we denote the open face of* Σ *given by* $\{\sum_{i\in I} s_i\beta_i | s_i > 0\}$ *by* Σ^I*. The subtorus of* T *that acts perpendicular to* Σ^I *is denoted by* T^I*.*

The symplectic cut with respect to a cone M_Σ *of* M *is then the symplectic reduction*

$$M_\Sigma = (M \times \mathbb{C}_\beta) /\!\!/_0 T$$

If μ is a proper map, then ϕ is also, so we have that M_Σ is compact. Furthermore, in [LMTW98] it is shown that the action of T descends to a Hamiltonian action T_Σ on M_Σ and that the image of the moment map μ_Σ is $\mu(M) \cap \Sigma$. Note that the action of T on M_Σ is not effective unless $\dim\Sigma = \dim T$, since the subtorus T^Σ of T which acts perpendicular to Σ acts trivially on M_Σ.

Now if H is a Lie group with an action on M which commutes with G, the action descends to $M /\!\!/_\Sigma G$. The fixed points of the action of H are related to the If we set M^H to be the set of fixed points of M under the action of H, we see that if M^H is compact, we recover M^H if the cone Σ is large enough, that is $\mu(M^H) \subset \Sigma$, and $k = \dim T$. There are also new fixed points in M_Σ.

For example, we can look at the symplectic cut of $\mathbb{C}^2$ by a circle S^1 acting on $\mathbb{C}^2$ as $s(z_1, z_2) = (sz_1, s^{-1}z_2)$, and look at the fixed points of a circle T acting as $t(z_1, z_2) = (tz_1, tz_2)$. Then for any $\epsilon > 0$, the symplectic cut $M_\epsilon = \{(z_1, z_2) \in \mathbb{C}^2 | |z_1|^2 + |z_2|^2 < \epsilon\} \sqcup \mathbb{CP}^1$. We see that T has now 3 fixed points: the original $(0,0)$, and two new fixed points in $\mathbb{CP}^1$, $[0:1]$ and $[1:0]$ in projective coordinates.

This technique can be used to easily prove the following generalization of the standard equivariant localization theorems.

Theorem 1.4.20 (Prato-Wu[PW94]). *Let (M, ω) be a symplectic manifold (possibly non-compact) of dimension $2n$ on which there is a Hamiltonian action of a torus T. If the fixed point set M^T is compact and there exists a $t_0 \in \mathfrak{t}$ such that the component of the moment map $\mu_{t_0} = \langle \mu, t_0 \rangle$ is proper and bounded on one side then the Duistermaat-Heckman formula*

$$\int_M e^{\langle \mu, t \rangle} \frac{\omega^n}{n!} = (-2\pi)^{n/2} \sum_{F \subset M^T} \int_F \frac{e^{\omega + \langle \mu, t \rangle}}{e_T(\nu_F)} \tag{1.4.7}$$

still holds true if we interpret the left hand side as the integral of $e^{\langle \mu, t \rangle}$ with respect to the measure induced by $\frac{\omega^n}{n!}$ and we restrict t to an open cone $C \subset \mathfrak{t}_\mathbb{C}$ such that the left hand side converges.

The Duistermaat-Heckman formula in the compact case can easily be proven by using the Berline-Vergne localization 1.4.15 if we notice that $\int_M e^\mu \frac{\omega^n}{n!} = \int_M e^{\tilde{\omega}}$, with $\tilde{\omega}$ the equivariant symplectic form. We also make use of the following result:

Theorem 1.4.21 (Duistermaat-Heckman [DH82]). *The cohomology class of the symplectic form ω_λ on $M /\!\!/_\lambda S^1$ behaves linearly in λ:*

$$[\omega_\lambda] = [\omega_0] + \langle \lambda, c \rangle \tag{1.4.8}$$

where $c \in H^*(M_0) \otimes \mathfrak{t}$ is the first Chern class of the circle bundle $\mu^{-1}(0) \to M_0$

Now we can prove theorem 1.4.20

Proof. We can choose a circle within the torus action such that the moment map of the circle is proper and bounded above, and we can take the symplectic cut with respect to this circle action:

$$M_\lambda = \mu^{-1}((-\infty, \lambda) \sqcup M /\!\!/_\lambda S^1$$

As in the above discussion, the fixed points of the torus action on this (compact) space are the old ones, together with new ones introduced by the cut:

$$M_\lambda^T = (\mu^{-1}((-\infty, \lambda)))^T \sqcup (M /\!\!/_\lambda S^1)^T$$

Since $\mu^{-1}((-\infty, \lambda))$ is open and dense in M_λ, inherits the symplectic structure of M and is equal to M in the limit $\lambda \to \infty$ we can recover the original integral over M as the limit $\lambda \to \infty$ of integrals over the symplectic cut M_λ.

If we take λ large enough, since M^T is compact, we have that $\mu^{-1}((-\infty, \lambda))$ contains all of M^T hence the original fixed points are restored. Since M_λ is compact, we can apply Duistermaat-Heckman, and if λ is large enough, the only λ dependence is in the new fixed points:

$$\frac{1}{(-2\pi)^{n/2}}\int_M e^{\omega+\mu} = \sum_{F\subset M^T}\int_F \frac{e^{\omega+\mu}}{e_T(\nu_F)} + \lim_{\lambda\to\infty}\sum_{F'\subset(M/\!\!/_\lambda S^1)^T}\int_{F'}\frac{e^{\omega+\mu}}{e_T(\nu'_F)}$$

The cohomology of the symplectic form on the new fixed point depends linearly on λ by theorem 1.4.21 and since we only run over regular values of μ, the topology of $\mu^{-1}(\lambda)$ does not change, so $e_T(\nu_F)$ is constant. Since we assume that t is in the open cone such that integral on the left-hand side of 1.4.7 converges, we must have $\langle\mu, t\rangle = \langle\lambda, t\rangle$ on $\mu^{-1}(\lambda)$ goes to $-\infty$ when $\lambda \to \infty$. This means the contributions of the new fixed point vanish:

$$\lim_{\lambda\to\infty}\sum_{F'\subset(M/\!\!/_\lambda U(1))^T}\int_{F'}\frac{e^{\omega+\langle\mu,t\rangle}}{e_T(\nu_{F'})} = 0$$

□

For example, if we calculate the equivariant integral on $\mathbb{C}$ with respect to a circle action acting with weight $a \in \mathbb{R}_+$, we get the Gaussian integral:

$$\int_{\mathbb{C}} e^{\omega+\mu_T(t)} = \int_C e^{-a||z||^2 t}\frac{i}{2}dz\wedge d\bar{z}$$

We already know that the value of the integral is $\frac{2\pi}{at}$.

This action has one fixed point, $z = 0$, and the weight of the isotropy representation on this point is $-at$, plugging this in in 1.4.7 we get

$$\int_M e^{\omega+\mu_T(t)} = (-2\pi)\sum_{p=0}\frac{e^0}{-at} = \frac{2\pi}{at}$$

We also see that the cone we have to choose t in, must lie within $(0, \infty) \times i\mathbb{R}$, otherwise the integral above does not converge, and the “new” fixed point in $\mu^{-1}(\lambda)/U(1)$ would give a contribution scaling with $e^{-\lambda t}$ which does not vanish when $\lambda \to \infty$.

Chapter 2

Seiberg-Witten theory

In this chapter we will derive the Seiberg-Witten invariants, which allows us to establish in some cases when two homeomorphic 4-manifolds are not diffeomorphic. The existence of manifolds which are homeomorphic but not diffeomorphic to each other was first established by Milnor in 1956 with his construction of the exotic 7-spheres. Donaldson constructed invariants of four-manifolds which were invariant under diffeomorphisms but not under general homeomorphisms, thus giving a powerful tool for checking whether two homeomorphic manifolds were diffeomorphic. In practice however, the Donaldson invariants were very hard to compute. In 1994 Witten, based on his work on supersymmetric Yang-Mills theory with Seiberg, proved that there exist much simpler invariants, which share the property that they are invariant under diffeomorphisms but not under homeomorphisms. They are much simpler to compute, and in this chapter we will establish their existence and some of their properties, along with some examples on how they can be used to classify certain manifolds.

We will first recall the notions of spin geometry, together with the Atiyah-Singer index theorem which allows us to calculate the dimensions of the moduli space of the Seiberg-Witten equations. We then state some results on Sobolev spaces and elliptic operators, tools which are essential to study partial differential equations. After this setup, we will introduce the Seiberg-Witten equations and study their properties. The compactness of the moduli space is proven, which makes the Seiberg-Witten invariants much easier to handle than the Donaldson invariants. This moduli space is then established to be a compact smooth finite dimensional manifold, where the dimension can be calculated by the Atiyah-Singer index theorem. Finally, the invariants are defined and some of their applications to the differential topology of 4-manifolds are mentioned.

2.1 Spin geometry

We recall some basic facts of Dirac operators and spin structures on manifolds. See for details [HM89]. Spin(n) is a double cover of $SO(n)$, the group of orientation-preserving rotations in n dimensions. In 4 dimensions, Spin(4) can be written as $SU(2) \times SU(2)$ and it is simply connected for all $n > 2$. A *spin structure* is a double covering of an $SO(n)$ principal bundle such that the covering is a Spin(n) principal fiber bundle. Such a covering exists if the second Stiefel-Whitney class $w_2(TM)$ vanishes. A spin^c structure is a double covering of $SO(n) \times U(1)$, the complexified rotations. It exists if the second Stiefel-Whitney class is the mod 2 reduction of an integral class. For 4-dimensional manifolds this is always the case.

In case M has a spin structure, we can write this as a $SU(2)\times SU(2)$ principal fiber bundle. We call one of these $SU(2)_+$ and the other $SU(2)_-$. With those fiber bundles we can associate vector bundles, in this case of rank 2 and we call these W_+ and W_-. We have

$$TM \otimes \mathbb{C} \simeq \mathrm{Hom}(W_+, W_-).$$

Since the transition functions for a complex line bundle live in $U(1)$ hence are commutative, they cancel each other out and we can write $TM \otimes \mathbb{C} \simeq \mathrm{Hom}(W_+ \otimes L, W_- \otimes L)$. Now associated to a spin^c structure we have two $U(2)$ bundles, and thus two complex vector bundles of rank two, which we call $W_+ \otimes L$ and $W_- \otimes L$. The bundles W_-, W_+ and L only exist if the manifold has a spin structure, but their products do exist for all 4-manifolds. Sections of $W_+ \otimes L$ and $W_- \otimes L$ are called *spinor fields*, respectively of positive or negative chirality. The bundle $L \otimes L = L^2$ also exists in all cases, and comes from the determinant map on the linear representation of spin^c on the vector bundle $W_+ \otimes L \otimes W_- \otimes L$.

The double covering of $SO(4)$ has a natural representation inside the *Clifford algebra* $C\ell_{4,0}$, which has as a basis $1, e_i$, where e_i is a basis for the 4 dimensional vector space $\mathbb{R}^4$, e_ie_j for $i < j$, $e_ie_je_k$ for $i < j < k$ and $e_1e_2e_3e_4$. There is a product on this algebra, generated by the relation $e_ie_j + e_je_i = -2\delta_{ij}$ with δ_{ij} the Kronecker delta. The spinors sit inside this Clifford algebra, and we have a natural identification of the exterior algebra $\sum_{k=0}^{4} \wedge^k V$ with the Clifford algebra as vector spaces. The product of the Clifford algebra in terms of the interior and wedge product of the exterior algebra is then

$$e_i \cdot \alpha = e_i \wedge \alpha - \imath(e_i)\alpha \tag{2.1.1}$$

for $\alpha \in \bigwedge^* V$. Now we can see that $\wedge^2_+ V$, see equation 1.2.2a for a definition, is exactly the space of trace free Hermitian endomorphisms of W_+, but this precisely the Lie algebra of $SU(2)_+$. Thus we can define a quadratic map $\sigma\colon W_+ \to \wedge^2_+ V$ to be the map

$$\sigma(\psi) = -\frac{i}{2} \sum_{i<j} \langle \psi, e_i \cdot e_j \cdot \psi \rangle e_i \cdot e_j \tag{2.1.2}$$

for $\psi \in W_+$. This can be extended to a map from $W_+ \otimes L \to \wedge^2_+ TM$. This has the following interpretation: a spinor is the “square root” of a self-dual two-form, together with a choice of phase. We also see that

$$|\sigma(\psi)|^2 = \frac{1}{2}|\psi|^2 \tag{2.1.3}$$

Definition 2.1.1. *The* Dirac operator $D_A : \Gamma(W \otimes L) \to \Gamma(W \otimes L)$ *on a Riemannian manifold with spin^c structure is the first order differential operator defined in local coordinates by:*

$$D_A(\psi) = \sum_{i=1}^{4} e_i d_A \psi(e_i)$$

with d_A a connection with group spin$^c(4)$, that is, a lift of the Levi-Civita connection on the $SO(4)$ bundle, together with a choice of a unitary connection A on the determinant line bundle L^2.

The Dirac operator is formally self-adjoint for smooth sections, that is

$$\int_M \langle D_A(\psi), \eta \rangle = \int_M \langle \psi, D_A(\eta) \rangle$$

and is essentially self-adjoint for L^2 sections.

We have the *Weitzenböck formula*:

$$D_A^2 \psi = \Delta^A \psi + \frac{s}{4}\psi - \sum_{i<j} F_A(e_i, e_j)(ie_i \cdot e_j \cdot \psi) \tag{2.1.4}$$

where Δ^A is the vector bundle Laplacian $\nabla^* \nabla$ with ∇ the induced covariant derivative on the spinor bundle by the Levi-Civita connection, ∇^* the formal adjoint of ∇, s is the scalar curvature and F_A the curvature of the connection on L.

Now the final piece of spin geometry we need is the *Atiyah-Singer index theorem*. This theorem gives the index of an elliptic operator in terms of topological data. The index of an operator is defined to be

$$\text{ind}(\Phi) = \dim(\ker(\Phi)) - \dim(\text{coker}(\Phi))$$

We will only state the theorem for the case of a Dirac operator with coefficients in the vector bundles $W_\pm \otimes L$. First we note that since the full Dirac operator D_A is formally self-adjoint, so it has index 0. We can, however split the Dirac operator in two pieces:

$$D_A^+ : \Gamma(W_+ \otimes L) \to \Gamma(W_- \otimes L), D_A^- : \Gamma(W_- \otimes L) \to \Gamma(W_+ \otimes L),$$

which have non-trivial index.

Theorem 2.1.2 (Atiyah-Singer index theorem). *If D_A is a Dirac operator with coefficients in a line bundle L on a compact oriented four-manifold M, then*

$$ind(D_A^+) = -\frac{1}{8}\tau(M) + \frac{1}{2}\int_M c_1(L)^2 \tag{2.1.5}$$

with $\tau(M) = b_2^+ - b_2^-$ the signature of M and $c_1(L)$ the first Chern class of L.

2.2 The Seiberg-Witten invariants

2.2.1 Sobolev spaces

In this section we define Sobolev spaces and summarize some results about them. See for proofs for example [Aub98] and [Rud86]. We want to consider the space of all $U(1)$ connections $\mathcal{A}$, the space of all $U(1)$ gauge transformation $\mathcal{G}$ and related spaces as manifolds modelled on infinite dimensional Hilbert or Banach spaces. Unfortunately, the spaces of C^∞ functions are not complete, hence not Banach, so we need a suitable completion.

Consider the norm $\|f\|_q = \left(\int |f(x)|^q|\right)^{1/q}$ on the space of continuous functions on $\mathbb{R}^n$ with standard measure. For compactly supported functions f this is a norm, and we call the completion of the space of continuous functions with respect to this norm the space L^q. The space L^∞ is defined to be the completion with respect to the norm $\|f\|_\infty = \sup_x |f(x)|$

Now we wish to control in some way not only the norm of the functions, but also the norm of derivatives of the functions. This leads to the notion of *Sobolev spaces*.

Definition 2.2.1. *Set $k \geq 0$ integer, then the* Sobolev space L^p_k *is the completion of the space of smooth compact supported functions under the norm:*

$$\|f\|_{L^p_k} = \left(\sum_{i=0}^{k} \|\nabla^i f\|^p_{L^p}\right)^{1/p}$$

We say a function f is locally in L^p_k if each point of an open subset U is contained in a neighborhood over which the L^p_k norm is finite.

We define L^p_k sections of a vector bundle V over a compact manifold M by choosing local coordinates and say a section is in L^p_k if it is represented by locally L^p_k functions in the trivializations.

Now we have a few theorems about these Sobolev spaces which are useful in this thesis. In these theorems, M is a compact Riemannian manifold without boundary.

Theorem 2.2.2. *The space $L^p_k(M)$ does not depend on the metric.*

Theorem 2.2.3. *If $r \geq k$ then $C^r(M)$ is dense in $L^p_k(M)$.*

Theorem 2.2.4 (Sobolev embedding theorem). *Let $n = dim\ M$. There is a natural bounded inclusion map from L^p_k into C^r if $k - \frac{n}{p} > r$*

Theorem 2.2.5 (Rellich's theorem). *Assume $k > m$. Then the inclusion $L^p_k(M) \hookrightarrow L^q_m(M)$ is bounded and compact if $k - \frac{n}{p} > m - \frac{n}{q}$ and bounded if $k - \frac{n}{p} = m - \frac{n}{q}$. In particular, there is a compact inclusion of L^p_{k+1} into L^p_k.*

Theorems 2.2.4 and 2.2.5 together can be combined to work out in which space the product $f \times g$ of two function $f, g \in L^p_k$ lies.

We have a compact embedding $L^p_k \to L^{2p}$ if $k > \frac{n}{2p}$. This gives by multiplication a bounded bilinear map $L^p_k \times L^p_k \to L^p$. Now if $k > \frac{n}{p}$ then there is a bounded inclusion map of L^p_k into C^0. We see that by expanding differentiation through the Leibniz rule, we see that the product fg must lie within L^p_k:

Theorem 2.2.6 (Sobolev Multiplication). *If $k > \frac{n}{p}$ there is a bounded multiplication map*

$$L_k^p \times L_k^p \to L_k^p$$

2.2.2 Some theorems on elliptic operators

Let $D : \Gamma(E) \to \Gamma(E)$ be a differential operator that sends smooth sections to smooth sections of a vector bundle E over M. The *principal symbol* of D is a map which assigns to each point $x \in M$ and each cotangent vector $\xi \in T_x^*M$ a linear map $\sigma_\xi(D) : E_x \to E_x$ as follows. If we can write D in local coordinates as

$$D = \sum_{|i| \leq m} A_i(x) \frac{\partial^{|i|}}{\partial x^i} \text{ and } \xi = \sum \xi_k dx_k$$

with m the order of D then

$$\sigma_x i(D) = i^m \sum_{|i|=m} A_i(x) \xi^i$$

It calculates the coefficients of the highest-order part of the operator. An operator is called *elliptic* if it's principal symbol is an isomorphism for all $\xi \neq 0$. Examples include the Dirac operator (by definition 2.1.1), the vector bundle Laplacian (can be seen using the Weitzenböck formula 2.1.4).

We need some fundamental results on elliptic operators. For proofs see for example [HM89]. The first result we need relates properties of elliptic operators on smooth functions to properties of those same operators on Sobolev spaces:

Theorem 2.2.7. *Let $P : \Gamma(E) \to \Gamma(F)$ be an elliptic operator of order m over a compact manifold M. Then the following is true:*

1. *For any open set $U \subset M$ and any $u \in L_s^2(E)$,*

$$Pu|_U \in C^\infty \Rightarrow u|_U \in C^\infty$$

2. *For each s, P extends to a Fredholm map $P : L_s^2(E) \to L_{s-m}^2(F)$ whose index is independent of s.*

3. *For each s there is a constant C_s such that*

$$\|u\| \leq C_s \left(\|u\|_{s-m} + \|Pu\|_{s-m} \right)$$

for all $u \in L_s^2$. If P has trivial kernel, the $\|u\|_{s-m}$ term can be left out.

The following theorem can be used to prove the Hodge theorem, but has other uses as well:

Theorem 2.2.8. *Let $P : \Gamma(E) \to \Gamma(E)$ be an elliptic self-adjoint differential operator over a compact Riemannian manifold. Then there is an L^2-orthogonal direct sum decomposition:*

$$\Gamma(E) = kerP \oplus imP$$

Finally the Dirac operator has a unique continuation property [Aro56]:

Theorem 2.2.9. *Let M be a connected manifold. If $D_A^+\psi = 0$ on M and ψ vanishes identically on an open subset of M, then $\psi = 0$.*

2.2.3 Seiberg-Witten equations

We now turn to the Seiberg-Witten equations and the moduli space of their solutions. If we take a 4 dimensional oriented Riemannian manifold with spin^c structure $(M, \langle,\rangle)$ with positive spinor bundle $W_+ \otimes L$, the Seiberg-Witten equations are:

$$D_A^+\psi = 0 \quad F_A^+ = \sigma(\psi) \tag{2.2.1}$$

where d_{2A} is a unitary connection on the determinant bundle L^2, $F_A^+ = \frac{1}{2}F_{2A}^+$ is the self-dual part of the curvature of this connection, and ψ a smooth section of $W_+ \otimes L$. Solutions to these equations are called *monopoles* because their physical interpretation is that of magnetic monopoles. The Seiberg-Witten equations are non-linear first order differential equations, but their non-linearity is mild compared to the Donaldson equations $F_A^+ = 0$, with A a $SU(2)$ connection. In fact the equations above have a compact solution space, which is a huge improvement over the Donaldson equations.

If we look at the total space of connections and sections, we can set some a priori bounds on the norms of solutions to the Seiberg-Witten equations. Set

$$\mathcal{A} = \{(A, \psi) : A \text{ is a } U(1) \text{ connection on } L,\ \psi \in \Gamma(W_+ \otimes L)\} \tag{2.2.2}$$

to be the total space of smooth sections and unitary connections. We can then define a functional on this space:

$$S(A, \psi) = \int_M \left[|D_A\psi|^2 + |F_A^+ - \sigma(\psi)|^2\right] dV \tag{2.2.3}$$

and we see immediately that solutions to the Seiberg-Witten equations 2.2.1 are global minima of this functional. This sets some a-priori bounds on these solutions.

Theorem 2.2.10. *Solutions to the Seiberg-Witten equations are bounded as follows:*

$$\int_M |F_A^+|^2 dv \leq \int_M \frac{s^2}{32} dV \tag{2.2.4a}$$

$$|\psi|^2(p) \leq -\frac{s}{2}(p) \tag{2.2.4b}$$

where the last inequality holds at the maximum of $|\psi|^2$ if ψ is not identically 0.

Proof. We have due to the Weitzenböck formula 2.1.4 and the self-adjointness of D_A that

$$\begin{aligned} 0 = S(A,\psi) &= \int_M |D_A^+\psi|^2 + |F_A^+|^2 - 2\langle F_A^+, \sigma(\psi)\rangle + |\sigma(\psi)|^2 \\ &= \int_M |D_A^+\psi|^2 + |F_A^+|^2 + |\sigma(\psi)|^2 + \sum_{i<j} F_A^+(e_i,e_j)\langle \psi, i\cdot e_i \cdot e_j \cdot \psi\rangle \\ &= \int_M |\nabla_A \psi|^2 + |F_A^+|^2 + |\sigma(\psi)|^2 + \frac{s}{4}|\psi|^2 \end{aligned}$$

We then use that $|\sigma(\psi)|^2 = \frac{1}{2}|\psi|^4$ to conclude the following:

1. We have a bound on $|F_A^+|^2$:

$$\int_M |F_A^+|^2 \leq \int_M \left(-\frac{s}{4}|\psi|^2 - \frac{1}{2}|\psi|^4\right)$$

   The left hand side is an integral over a positive function, thus must be ≥ 0, so we can complete the square to conclude $\int_M |F_A^+|^2 dv \leq \int_M \frac{s^2}{32} dV$.

2. Since $|\nabla_A\psi|^2 + |F_A^+|^2 + \frac{1}{2}|\psi|^4 + \frac{s}{4}|\psi|^2 = 0$ for solutions to the Seiberg-Witten equations, we have that $\frac{1}{2}|\psi|^4 + \frac{s}{4}|\psi|^2 \leq 0$ at all $x \in M$, so at the maximum of $|\psi|$ we have $-\frac{s}{2}(p) \geq |\psi|^2(p)$ if $\psi(p) \neq 0$.

□

From the preceding we can also conclude that if M has positive scalar curvature, the only solution to the Seiberg-Witten equations is $\psi = 0$. Before we investigating the moduli space of solutions to the Seiberg-Witten equations we first derive some properties of the total space $\mathcal{A}$. First choose a basepoint $A_0 \in \mathcal{A}$, then $\mathcal{A}$ can be written as:

$$\mathcal{A} = \{(d_{A_0} - ia, \psi) : a \in \Omega^1(M), \psi \in \Gamma(W_+ \otimes L)\}.$$

We use the *Fréchet derivative* to calculate derivatives of functions on this infinite dimensional space. A function f is Fréchet differentiable at $a \in \mathcal{A}$ if there is a bounded linear operator A_a such that

$$\lim_{h\to 0} \frac{\|f(a+h) - f(a) - A_a(h)\|_W}{\|h\|_{\mathcal{A}}} = 0$$

We have a gauge group $\mathcal{G} = \mathrm{Map}(M, S^1)$ which acts on $\mathcal{A}$ as

$$(g, (d_{A_0} - ia, \psi)) \to (d_{A_0} - ia + gd(g^{-1}), g\psi)$$

Lemma 2.2.11. *The gauge group leaves the set of Seiberg-Witten equations invariant*

Proof. We have, due to the chain rule:

$$D^+_{A+gdg^{-1}}(g\psi) = gD_A^+\psi + dg\cdot\psi + gd(g^{-1})g\cdot\psi = 0 + dg\cdot\psi - dg\cdot\psi = 0$$

For the second Seiberg-Witten equation we have:

$$F^+_{A+gdg^{-1}} - \sigma(\psi) = F_A^+ dgdg^{-1} - |g|^2\sigma(\psi) = F_A^+ - \sigma(\psi)$$

□

Also of interest is the group of *based gauge transformations* or the stabilizers of a point x_0, Stab_{x_0}:

$$\mathcal{G}_{x_0} = \{g \in \mathcal{G} : g(x_0) = 1\}$$

We see that $\mathcal{G} = S^1 \times \mathcal{G}_{x_0}$ with S^1 the space of constant transformations. This group acts freely on $\mathcal{A}$, so we can write $\widetilde{\mathcal{B}} = \mathcal{A}/\mathcal{G}_{x_0}$.

If the manifold is simply connected, the map $g \in \mathcal{G}$ has a global logarithm u such that $g = e^{iu(x)}$, with $u(x_0) = 0$ for based gauge transformations. In that case $\mathcal{G}$ acts on $\mathcal{A}$ as

$$(e^{iu}, (d_{A_0} - ia, \psi) \to (d_{A_0} - i(a + du), e^{iu}\psi) \tag{2.2.5}$$

Using this, we can describe $\widetilde{\mathcal{B}}$ as follows [Mor96]:

Lemma 2.2.12. *Let A be any connection. Then there is a unique gauge transformation g in the identity component, such that $\delta(gA - A_0) = 0$*

Proof. We can write $A = A_0 + \alpha_0$ for some α_0. We see that $\delta\alpha_0$ is L^2-orthogonal to the constant functions. On the orthogonal complement $\mathcal{I}$ of the constant functions, we can invert the Laplacian, since if $g \in \mathcal{I}$, g is either constant 0 or non-constant, hence if $g \neq 0$, $dg \neq 0$, hence by proposition c we have that $\Delta g \neq 0$.

Now define $\theta = -\Delta^{-1}(\delta\alpha_0)$. And set $g_0 = \exp(\theta)$, clearly a gauge transformation in the identity component of $\mathcal{G}$. Then $\alpha_1 = \alpha_0 + dg_0$ is such that $g_0 A = A_0 + \alpha_1$ by construction and

$$\delta\alpha_1 = \delta\alpha_0 - \delta d \Delta^{-1}(\delta\alpha_0) = 0$$

Uniqueness: If we have two representatives of $(d_{A_0} - ia, \psi)$, say $(d_{A_0} - ia_1, \psi_1)$ and $(d_{A_0} - ia_2, \psi_2)$ with $\delta a_1 = \delta a_2 = 0$ such that there is a gauge transformation e^{iu} such that $e^{iu}\psi_1 = \psi_2$ and $a_1 + du = a_2$. But then

$$\langle a_1 - a_2, a_1 - a_2\rangle = \langle du, a_1 - a_2\rangle = \langle u, \delta(a_1 - a_2\rangle = 0$$

so $a_1 = a_2$. □

We can thus, after choosing a base point, fix a point in the gauge orbit of A, gA such that $\delta(A - gdg^{-1} - A_0) = 0$. This gauge choice is called *Coulomb gauge*. This is not a unique choice though, we could pick a harmonic transformation in component not containing the identity and apply it, the resulting connection will still be in Coulomb gauge, but the action on ψ will certainly not give the same.

Now we look at the space $\mathcal{B} = \mathcal{A}/\mathcal{G}$. This quotient is more intricate, as the group of constant gauge transformations S^1 does not act freely on the whole of $\mathcal{A}$, but has fixed points at points $\psi = 0$, so $\mathcal{B}$ has singularities there. We call these points *reducible.*In order to work around these reducible points, we add a perturbation in the form of a self-dual two form ϕ:

$$D_A\psi = 0 \tag{2.2.6a}$$

$$F_A^+ = \sigma(\psi) + \phi \tag{2.2.6b}$$

Now, $\psi = 0$ can only be a solution of the perturbed Seiberg-Witten equations if ϕ is the self-dual part of the curvature of a connection.

Lemma 2.2.13. *Let L be a complex line bundle over a compact oriented Riemannian manifold M. Then an element $\phi \in \Omega^2_+(M)$ will be the self-dual part of a unitary connection on L if and only if ϕ lies in an affine subspace of $\Omega^2_+(M)$ of codimension b_2^+.*

For a proof, see [Tau83].

2.2.4 Structure of the moduli space

We have now enough materials to prove the compactness of the Seiberg-Witten moduli space.

Theorem 2.2.14 (Compactness of the moduli space). *Let M be a connected Riemannian 4-manifold, then the moduli space $\mathcal{M}$ of solutions to the Seiberg-Witten equations is compact.*

Proof. Choosing a base connection A_0, we can write all unitary connections on L as $d_{A_0} - ia$, for some $a \in \Omega^1(M)$. Imposing the Coulomb gauge, we can write the Seiberg-Witten equations as:

$$\begin{aligned} D_{A_0}\psi &= ia \cdot \psi \\ (da)^+ &= \sigma(\psi) - F^+_{A_0} \\ \delta\left(ia + gdg^{-1}\right) &= 0 \end{aligned}$$

We have that ψ is bounded in C^0 due to theorem 2.2.10, and is thus bounded in every L^p. The kernel of $d^+ \oplus \delta : \Omega^1(M) \to \Omega^2_+(M) \oplus \Omega^0(M)$ is precisely the set of harmonic forms, hence we can write, due to theorem 2.2.8: $a = (h, \beta)$ with h harmonic and β non-harmonic. We see that $\|a\|_{p,k+1} \leq \|h\|_{p,k+1} + \|\beta\|_{p,k+1}$. First we take a look at the non-harmonic part.

Because of theorem 2.2.7, we see that

$$\begin{aligned} \|\beta\|_{p,k+1} &\leq C_{p,k}\|(\delta + d^+)\beta\|_{p,k} \\ &= C_{p,k}\|0 + \sigma(\psi) - F^+_{A_0}\|_{p,k} \\ &= C_{p,k}\|F^+_A - F^+_{A_0}\|_{p,k} \\ &\leq C_{p,k}\left(\|F^+_A\|_{p,k} + \| - F^+_{A_0}\|_{p,k}\right) \\ &= C_{p,k}\left(\|F^+_A\|_{p,k} + C_{p,k}K_1\right) \end{aligned}$$

Now we need to bound the harmonic part h of a. We need a small lemma[Mor96]:

Lemma 2.2.15. *For every real harmonic one-form h whose periods all lie within $2\pi\mathbb{Z}$ there is a harmonic function $\phi : M \to S^1$ such that $h = d\phi$*

Proof. Choose a basepoint x_0 and integrate h along paths to define a C^∞-function $\tilde{\phi} : \tilde{M} \to \mathbb{R}$ on the universal cover $\tilde{M}$ of M. Now since the periods of h are in $2\pi\mathbb{Z}$, we see that $\tilde{\phi}$ descends to a map $\phi : M \to \mathbb{R}/(2\pi\mathbb{Z}) \simeq S^1$ and by construction $d\phi = h$, so ϕ is harmonic. □

Now the space of harmonic one-forms divided out by harmonic one-forms with periods all within $2\pi\mathbb{Z}$ is a compact torus, so there is a constant K_2 such that any harmonic one-form h can be written as $h = h_1 - h_2 = h_1 - d\phi$ with $\|h_1\|_{p,k+1} \leq K_2$ and h_2 with periods in $2\pi\mathbb{Z}$. Now apply this map ϕ to our connection A on L^2. We see that $(\det\phi)^*A = A_0 + ia + d\phi$ since ϕ is in the component of the identity so we see that $(\det\phi)^*A = A_0 + h_1 + \beta$. Applying Coulomb gauge to $(\det\phi)^*A$ we see that

$$\|a\|_{p,k+1} \leq \|h_1\|_{p,k+1} + \|\beta\|_{p,k+1} \leq K_2 + K_1 + C_{p,k}\|F_A^+\|_{p,k} = C_{p,k}\|F_A^+\|_{p,k} + K$$

To prove compactness, we need to show that every sequence $(d_{A_0} - a_i, \psi_i)$ of solutions to the Seiberg-Witten equations has a convergent subsequence. Now by the above bounds we have a_i is bounded in every L^1_p for all p. If $p > 4$ we can use the Sobolev Embedding Theorem 2.2.4 to see that it is bounded in C^0, and so $a_i\psi_i$ is bounded in L^p for all p. Then, by theorem 2.2.7 applied to $D_{A_0}\psi = ia \cdot \psi$ we see that this is bounded in L^p_1 for all p. Then, if $p > 4$ we can use the Sobolev multiplication theorem 2.2.6 to show that if a_i and ψ_i are bounded in L^p_k, then $a_i\psi_i\sigma(\psi_i)$ are bounded in L^p_k, hence by the above a_i, ψ_i are bounded in L^p_{k+1}. Repeating this process, we get that a_i, ψ_i are bounded in L^p_k for all k. We apply Rellich's theorem 2.2.5 to produce a subsequence which converges in L^p_k for all k, then use the Sobolev Embedding Theorem again to show this converges in C^l for all l, hence the moduli space is compact. □

This proof of the compactness also goes through in the case of the perturbed moduli space defined by 2.2.6a and 2.2.6b.

So now we have that the moduli space is compact, we want to show it is a smooth manifold. For this we need an infinite dimensional analogue of the implicit function theorem, and a theorem that guarantees the existence of regular values for this function. This can be done using *Fredholm maps*, an obvious generalization of Fredholm maps in normal, linear functional analysis.

Recall that a map F between Banach spaces E_1 and E_2 is *Fredholm* if:

1. the kernel of F is finite dimensional,
2. the cokernel of F is finite dimensional, meaning that the range of F has finite codimension, and
3. the range of F is closed.

The *index* of a Fredholm map is then

$$\dim(\ker(F) - \dim(\mathrm{coker}(F))$$

Now if M_1 and M_2 are Banach manifolds, we define a nonlinear smooth map F to be a Fredholm map of index k if for every $p \in M_1$, the map $dF(p) : T_pM_1 \to T_{F(p)}M_2$ is a Fredholm map of index k. Obviously, every smooth map between finite-dimensional manifolds M and N is a Fredholm map of index $\dim(M) - \dim(N)$. For the implicit function theorem, we are interested when the map $dF(p)$ has an empty cokernel. We say that $q \in M_2$ is *regular*, if for every $p \in F^{-1}(q)$ the map $dF(p)$ is surjective.

We then have an infinite dimensional version of the implicit function theorem:

Theorem 2.2.16. *Suppose $F : M_1 \to M_2$ is a smooth Fredholm map between separable Banach manifolds of index k. If $b \in M_2$ is regular, then $F^{-1}(b)$ is a smooth finite dimensional manifold with dimension equal to k.[Lan72]*

Also, for Fredholm maps, we have the Sard-Smale theorem (see for example [DK90]):

Theorem 2.2.17. *If $F : M_1 \to M_2$ is a C^k Fredholm map between separable Banach manifolds and $k > max(0, Ind(F))$, then the set of regular values of F is residual in M_2.*

The term *residual* means that it is the countable intersection of open dense sets. Due to the Baire Category theorem [Sch00] this means that it is dense. Hence the set of regular values of F is dense. The condition that the manifolds be separable is in the case at hand satisfied, since the L^p_k spaces are separable.

Now we can state the main theorem of this section

Theorem 2.2.18 (Transversality theorem). *Let M be a compact connected smooth four-manifold with a spin^c structure, with $b_2^+ > 0$. Then for a generic choice of self-dual two form ϕ, $\tilde{\mathcal{M}}_\phi$ is an oriented smooth manifold, with dimension*

$$dim\left(\tilde{\mathcal{M}}_\phi\right) = 2(complex\ index\ of\ D_A^+) - b_2^+ + b_1 \tag{2.2.7}$$

Proof. The proof goes in two steps: first we prove the surjectivity of our operator at hand, and second we prove that the manifold is oriented.

Surjectivity Define

$$F : \tilde{\mathcal{B}} \times \Omega^2_+(M) \to \Gamma(W_- \otimes L) \times \Omega^2_+(M)$$

as

$$F(A, \psi, \phi) = (D_A^+\psi, F_A^+ - \sigma(\psi) - \phi).$$

We can then calculate the differential of F at a point (A, ψ, ϕ):

$$dF(A, \psi, \phi)(a, \psi', \phi') = (D_A^+\psi' - ia \cdot \psi, (da)^+ - 2\sigma(\psi, \psi') - \phi'$$

First of all, by fixing a, ψ', we see that by varying ϕ', the entirety of $\Omega^2_+(M)$ is mapped upon. Now it suffices to show that for $\phi' = 0$ the first component is surjective.

We first consider the case where ψ is not identically 0. Then the linear map $a \mapsto a \cdot \psi$ is injective, because $a \cdot a \cdot \psi = -|a|^2\psi$, and because T^*M and $W^- \otimes L$ have the same dimension, it is an isomorphism. Then, because ψ is not identically 0, there is an open set U in M for which ψ does not vanish, hence the map $a \mapsto a \cdot \psi$ is an isomorphism for sections supported in this set. Now suppose that $\sigma \in L^2(W^- \otimes L)$ is orthogonal to the image of $dF(A, \psi)$, then $(-ia \cdot \psi, \sigma) = 0$ for all a supported in U, hence σ is 0 on U. But also $0 = (D_A^+\psi, \sigma) = (\psi, D_A^-\sigma)$, hence by the Unique Continuation Theorem 2.2.9 we have that $\sigma = 0$, hence we have surjectivity.

Now consider the case where ψ is identically 0. In this case, for the pair (A, ψ) to be a solution to the perturbed Seiberg-Witten equations, we must have $\phi = F_A^+$. But this space, according to lemma 2.2.13 has codimension $b_2^+ > 0$ hence the complement is open and dense. Call this complement $\mathcal{U} = \{\phi \in \Omega_+^2 : \phi \neq F_A^+$ for any connection $A\}$. Then we have that the following is a submanifold of $\tilde{\mathcal{B}} \times \mathcal{U}$:

$$\mathcal{N} = \{(A, \psi, \phi) \in \tilde{\mathcal{B}} \times \mathcal{U} : F(A, \psi, \phi) = 0\}$$

The tangents space at a point (A, ψ, ϕ) is given by

$$\{(a, \psi', \phi') : (D_A^+\psi' - ia \cdot \psi, \delta a, (da)^+ - 2\sigma(\psi, \psi') = (0, 0, \phi')\}$$

Define

$$L(a, \psi') = (D_A^+\psi' - ia \cdot \psi, \delta a, (da)^+ - 2\sigma(\psi, \psi'))$$

which is an elliptic operator Now the projection $\pi : \mathcal{N} \to \Omega_+^2(M)$ given by $\pi(A, \psi, \phi) = \phi$ is a Fredholm map, because the kernel of $d\pi$ is precisely the kernel of L. On the other hand

$$\text{Im}(d\pi) = \{\phi' \in \Omega_+^2 : (0, 0, \phi') = L(a, \psi') \text{ for a } a, \psi'\}$$

which is exactly $\text{Im}L \cap (0 \oplus 0 \oplus \Omega_+^2$, which means, due to the ellipticity of L that the image of $d\pi$ is of finite codimension, and is closed.

Now we use the unique continuation theorem to show that the cokernel of $d\pi$ has the same dimension as the cokernel of L. More precisely, we show that

$$\Big(\Gamma(W_- \otimes L) \oplus \tilde{\Omega}^0(M) \oplus \{0\}\Big) \cap (\text{Im}(L))^\perp \tag{2.2.8}$$

with $\tilde{\Omega}^0(M)$ the space of functions over M which integrate to 0.

Now if $(\sigma, u, 0) \in \Big(\Gamma(W_- \otimes L) \oplus \tilde{\Omega}^0(M) \oplus \{0\}\Big)$ is perpendicular to the image of L, we have $D_A^-\sigma = 0$ and $(ia \cdot \psi, \sigma) = (\delta a, u)$ for all $a \in \Omega^1(M)$. This means in particular that $(ib \cdot \psi, \sigma) = 0$ for all b such that $\delta b = 0$. Now define such a b by $\langle b, a\rangle = \langle ia \cdot \psi, \sigma\rangle$. This is such a b, since we have because D_A^+ is the formal adjoint of D_A^-, we have

$$\langle D_A^+(\psi), \sigma\rangle(p) - \langle \psi, D_A^-(\sigma)\rangle(p) = \delta b$$

and $D_A^+(\psi) = D_A^-(\sigma) = 0$. We then get $\langle b, b\rangle = \langle ib \cdot \psi, \sigma\rangle = 0$ hence $b = 0$. Now by assumption, $\psi \neq 0$, so we must have $\sigma = 0$ on an open set, hence by unique continuation $\sigma = 0$ on M and so $(\delta a, u) = 0$ for all $a \in \Omega^1(M)$. This can only mean that $u = 0$ and so equation 2.2.8

Now we can calculate the dimension of our submanifold. Choose $\phi \in \Omega_+^2(M)$ to be a regular value of π. The dimension of the submanifold $\pi^{-1}(\phi)$ is the index of L, which is the index of $L_0 = D_A^+ \oplus \delta \oplus d^+$. The complex index of D_A^+ is given by Atiyah-Singer index theorem, so the real index is

$$-\frac{1}{4}\tau(M) + \int_M c_1(L)^2$$

The index of $\delta \oplus d^+$ was calculated in equation 1.2.4 and was $b_1 - b_2^+$ so the total dimension is

$$-\frac{1}{4}\tau(M) + \int_M c_1(L)^2 - b_2^+ + b_1$$

Orientation If $F : V \to W$ is a Fredholm map. Let K be the kernel of F with dimension k and L the cokernel with dimension l. Then we can define the determinant bundle of F as

$$\det(F) = \bigwedge^k K \otimes \bigwedge^l L$$

If T is a smooth compact manifold, and $\{F_t\}_{t\in T}$ a continuous family of Fredholm operators, there is a natural way to describe $\det(F_t)$ as a line bundle over T [FM94].

Now if F_s and $F_{s's}$ are two homotopic families of Fredholm operators, we can look at the product $S \times \{0, 1\}$ and see from the continuously varying part that they must define two isomorphic line bundles. Now define the following family of vector bundles:

$$L_t(a, \psi') = (D_A^+\psi' - ita \cdot \psi, \delta a, (da)^+ - 2t\sigma(\psi, \psi'))$$

If $t = 1$, the determinant of this bundle is $\bigwedge^d(\ker(L) = \bigwedge^d(\tilde{\mathcal{M}}_\phi$, so the existence of a nowhere zero section of this determinant proves that $\tilde{\mathcal{M}}_\phi$ is orientable. But all the determinants of L_t are isomorphic, so we can restrict ourselves to the simpler case of $\det(L_0) = \det(D_A^+) \otimes \det(\delta \oplus d^+)$.

We can set an orientation on the first factor by observing that it is a complex linear operator. The kernel and cokernel of a complex linear operator are complex spaces which can be given a orientation by complex multiplication. For the second factor, we have to choose an orientation for the group $\mathcal{H}^2_+(M)$ by the fundamental complex 1.2.3. This gives an nowhere section on $\det(\delta \oplus d^+)$ We then have an orientation of $\det(L_0)$ hence of $\det(L_1)$ thus of the entire moduli space. $\square$

2.2.5 The invariants

Now with all this machinery established, we can define the Seiberg-Witten invariants as follows:

For a 4-dimensional Riemannian manifold M with $b_2^+ > 1$, choose a spin^c structure L on it, and fix a Riemannian metric g. Since $b_2^+ > 1$ we have that for a generic choice of π the moduli space $\mathcal{M}_{L,\phi}$ is a smooth submanifold with dimension $d = 2(\text{complex index of } D_A^+) - b_2^+ + b_1 - 1$. We have a principal S^1 bundle $\tilde{\mathcal{B}}^*$ over the space $\tilde{\mathcal{B}}$, which descends to the moduli space. Define c_1 to be the Chern class associated with this bundle.

Definition 2.2.19. *The* Seiberg-Witten invariant *of M for a generic metric h is*

$$SW(M) = \int_{\mathcal{M}_{\mathcal{L},\phi}} c_1^{d/2}$$

if d is even and 0 *otherwise.*

This definition might seem to be dependent on the choice of g and ϕ, but we will see that because $b_2^+ > 1$, this dependence vanishes.

Lemma 2.2.20. *If for M, $b_2^+ > 1$, then the Seiberg-Witten invariants of M are independent of g and ϕ*

Proof. Pick two metrics g_0 and g_1 and generic self-dual two forms ϕ_0, ϕ_1 so that theorem 2.2.18 holds for the spaces $\tilde{\mathcal{M}}_{\phi_0}$ and $\tilde{\mathcal{M}}_{\phi_1}$ with metrics g_0 and g_1 respectively. Now set $g(t)$ to be a smooth path in the space of all metrics such that $g(0) = g_0$ and $g(1) = g_1$. We want to establish the existence of a smooth path $\phi(t)$ from ϕ_0 to ϕ_1 such that the space $\tilde{\mathcal{M}}_{\phi(t)} \subset \tilde{\mathcal{B}} \times [0,1]$ such that

$$\begin{aligned} F_A^{+_t} &= \sigma(\psi) + \phi(t) \\ D^+_{A,g_t}\psi &= 0 \end{aligned}$$

where the $+_t$ means that the splitting into self-dual and anti-self-dual forms by the Hodge star is metric dependent and D^+_{A,g_t} varies due to the changing Levi-Civita connection on the spinor bundle. Consider the space $\mathcal{P}(\phi_0, \phi_1)$ of smooth paths $\eta : [0,1] \to \Omega^2_+(M)$ such that $\eta(0) = \phi_0$ and $\eta(1) = \phi_1$. This space has tangent space at a point η consisting of all maps $[0,1] \to Omega^2_+(M)$ vanishing at the endpoints.

Just as in the proof of the transversality in theorem 2.2.18 we want to prove that the differential of

$$F : (\tilde{\mathcal{B}} \times I \times \mathcal{P}) \to \Gamma(W_- \otimes L) \times \Omega^2_+(M)$$

given by

$$F(A, \psi, t, \phi) = (D^+_{A,g_t}, F_A^{+_t} - \sigma(\psi) - \phi(t))$$

is surjective for all points where F vanishes. Now on $t = 0$ and $t = 1$ this is the case, due to the choice of ϕ_0 and ϕ_1. When $0 < t < 1$ we know that due to the same argumentation as in theorem 2.2.18 the differential is surjective when restricted to $T\mathcal{P}$ hence by (again) the same argument in 2.2.18 we see that DF is surjective. Now consider the space $\mathscr{M}_L = F^{-1}(0)/\mathcal{G}$. By the transversality of F, this is a smooth manifold with boundary. The projection of $\mathscr{M}_L$ to $\mathcal{P}$ is smooth so by the Sard-Smale theorem 2.2.17 the fiber over a generic η is a smooth manifold which is precisely the subset of irreducible solutions in $\tilde{\mathcal{M}}_{L,\eta}$. Now if η is a generic path, there will be no reducible solutions in $\mathcal{M}_{L,\eta}$ since by lemma 2.2.13 these form a codimension $b_2^+ > 1$ subspace which a path will generically miss. $\tilde{\mathcal{M}}_{L,\eta}$ is also compact, because for every t the $\tilde{\mathcal{M}}_{L,\eta(t)}$ is compact. The boundary consists of $\tilde{\mathcal{M}}_{L,\phi_0}$ and $\tilde{\mathcal{M}}_{L,\phi_1}$.

It is oriented by a choice of orientation on the boundary and an orientation on $I = [0,1]$. Thus we have a cobordism of $\tilde{\mathcal{M}}_{L,\phi_0}$ and $\tilde{\mathcal{M}}_{L,\phi_1}$, which means that the cohomology classes of c_1 defined in 2.2.19 are the same, hence the Seiberg-Witten invariants are invariant under a change of metric and a generic perturbation. □

We also have by theorem 2.2.10

Corollary 2.2.21. *If there is a strictly positive metric on M, $SW(M) = 0$.*

The Seiberg-Witten invariants can be used to establish the existence of infinitely many non-diffeomorphic smooth structures on $n\mathbb{CP}^2\#k\bar{\mathbb{CP}}^2$ for many n and k. Normally, these manifolds with their standard smooth structure admit a metric with strictly positive curvature, so their Seiberg-Witten invariants are

0. However there exist "exotic" smooth structures on these manifolds which do not admit a positive metric:[PSS05]

Theorem 2.2.22. *There exist infinitely many pairwise non-diffeomorphic* 4*manifolds all homeomorphic to* $\mathbb{CP}^2\#5\bar{\mathbb{CP}}^2$.

The same can be established for $\mathbb{CP}^2\#k\bar{\mathbb{CP}}^2$ for $k = 6, 7, 8$ and 9, see the references in [PSS05]. While this was already know using Donaldson theory for $k = 8$ and $k = 9$, for $k = 5, 6$ and 7, Seiberg-Witten invariants were instrumental.

Chapter 3

Instantons

In this final chapter of the thesis, we explore the relation between the Seiberg-Witten theory and Donaldson theory following the approach by Nekrasov in [Nek03]. First we will define instantons and give their construction as a hyperkähler quotient. We will see that this possible in two ways, first as a hyperkähler quotient from the infinite dimensional space of all connections, and also via the ADHM construction as a hyperkähler quotient of a finite dimensional vector space. These approaches give the same answer.

Then, we want to calculate the equivariant volume of the instanton moduli space. For this, we need to generalize the localization theorems of chapter 1. The most powerful tool we use is a generalization of the Jeffrey-Kirwan localization theorem. The Jeffrey-Kirwan theorem is gives the equivariant volume of a symplectic quotient in terms of residues of integrals over fixed point sets. An essential part of this theorem is the Kirwan map, which relates the equivariant cohomology of the symplectic quotient with that of the original space. The residue operation on differential forms is introduced in the next section.

Then the generalized Jeffrey-Kirwan theorem is applied to the instanton moduli space given by the ADHM construction. Finally, the generating function of the total equivariant volume of the instanton moduli space is analyzed, and the relation with the Seiberg-Witten theory investigated.

3.1 Instantons and the ADHM construction

The Donaldson invariants describe similar invariants of four-manifolds as the Seiberg-Witten invariants. In fact, one of Witten's conjectures when writing down the Seiberg-Witten equations was that they are in fact the same, modulo some numerical constants. This was of course motivated by the procedure where Seiberg and Witten obtained their monopole equations from the quantum field theory describing the Donaldson instantons. The correspondence however is still a conjecture from the mathematical point of view: the derivation by Seiberg and Witten uses many concepts (Feynman path integral, electric-magnetic duality) which are not rigorously defined. In the case of instantons over $\mathbb{R}^4$, however, recent progress has been made in this direction by Nekrasov et. al [Nek03],[NS04] and [NO06]. This goes via the so called Atiyah-Drinfel'd-Hitchin-Manin, ADHM, construction of instantons.

3.1.1 Instantons

Just as the monopoles are defined as the solutions of certain gauge invariant differential equations in equation 2.2.1, instantons are connections which satisfy the differential equation

$$F_A^+ = 0 \tag{3.1.1}$$

with F_A the curvature of a $SU(2)$ connection A on a vector bundle, and F_A^+ the self-dual part of the curvature 2-form, as defined in 1.2.2a. This partial differential equation is also called the *anti-self-dual equation* or shortly the ASD equation.

Now using the theory of characteristic classes from section 1.3.4, we can write for a connection A on a vector bundle E:

$$\left[\frac{1}{2\pi}\mathrm{Tr}(F_A)\right] = c_1(E)$$

$$\left[\frac{1}{8\pi^2}\mathrm{Tr}(F_A^2)\right] = c_2(E) - \frac{1}{2}c_1(E)^2$$

with c_1 and c_2 the Chern classes.

The term $\mathrm{Tr}(F \wedge \star F)$ is known in physics literature as the *Yang-Mills Lagrangian.* The integral over M of this Lagrangian $\int_M \mathrm{Tr}(F \wedge \star F)$ is called the *Yang-Mills action.* If we assume $c_1(E) = 0$, then the second Chern class $c_2(E)$ is the absolute minimum for a connection on a vector bundle E, as we can see from the following computation:

$$\begin{aligned} 0 &\leq \int_M [\mathrm{Tr}(F_A - \star F_A) \wedge \star(F_A - \star F_A)] \\ &= \int_M \mathrm{Tr} F_A \wedge \star F_A - F_A \wedge F_A - \star F_A \wedge \star F_A + \star F_A \wedge F_A \\ &= \int_M \mathrm{Tr} 2F_A \wedge \star F_A - 2F_A \wedge F_A \end{aligned}$$

Since the second Chern class of the vector bundle E is $c_2(E) = \frac{1}{8\pi^2}\mathrm{Tr}F \wedge F$ we see that the minimum of the Yang-Mills action is $8\pi^2 \int_M c_2(E)$. This minimum is attained when $F_A = \pm \star F_A$, especially, $F_A = -\star F_A$ when $c_2(E) < 0$.

We define the *charge* of the instanton to be $c_2(E)$.

When we compactify $\mathbb{R}^4$ to $\mathbb{CP}^2$ we set a framing, that is we set a certain trivialization of the vector bundle on which the connection lives, at the line at infinity, $[0 : z_1 : z_2]$. This means the first Chern class is zero, and such instantons are called *framed instantons*.

The instanton moduli space is well-known to be non-compact. In order to calculate the Donaldson invariants, we need to integrate over the moduli space of instantons, and this is problematic when the space is non-compact, so we need to compactify the moduli space. There are several ways to do this, we will briefly describe the most commonly used one, the *Uhlenbeck compactification*

The Uhlenbeck compactification works set-theoretically as follows. Let M be the manifold over which we want to describe the moduli space of instantons. We then have sequence of moduli spaces M_k of instantons with charge $k \geq 0$. Then define *ideal instantons* to be:

Definition 3.1.1. *An* ideal instanton *of charge k over M is a pair:*

$$([A], (x_1, \dots, x_l))$$

where $[A]$ is a point in M_{k-1} and $(x_1, \dots, x_l)$ is an unordered l-tuple of points of M.

The curvature density of this ideal instanton is the measure

$$|F_{[A]}|^2 + 8\pi^2 \sum_{r=1}^{l} \delta_{x_r}$$

with δ_{x_r} the point measure at position x_r

These can be viewed as "point-like instantons", where all the curvature of the connection A is concentrated at the points $(x_1, \dots, x_l)$. We can then embed M_k in the space of all ideal instantons of charge k:

$$IM_k = M_k \cup M_{k-1} \times M \cup M_{k-2} \times s^2(M) \cup \dots$$

where $s^l(M)$ is the l times symmetric product of M. Now we need a topology on the moduli space.

Definition 3.1.2. *Let $\{A_i\}$ be a sequence of $SU(2)$ connections on a $SU(2)$ fiber bundle P_k of second Chern class k. A sequence of gauge equivalence classes $\{[A_i]\}_{i \in \mathbb{N}}$* converges weakly *to an ideal instanton $([A], (x_1, \dots, x_l))$ if*

1. *The curvature densities converge as measure, that is for any continuous function f:*

$$\lim_{i \to \infty} \int_M f \cdot |F_{A_i}|^2 d\mu = \int_M f \cdot |F_{[A]}|^2 + 8\pi^2 \sum_{r=1}^{l} f(x_r)$$

2. *There are bundle maps $\rho_i : P_l|_{M \setminus \{x_1, \dots, x_l\}} \to P_k|_{M \setminus \{x_1, \dots, x_l\}}$ such that $\rho_i^*(A_i)$ converges to A.*

This notion of convergence can of course be extended to ideal instantons. This then defines a topology on IM_k and so we can define the closure of M_k in IM_k.

Theorem 3.1.3 (Uhlenbeck compactification). *The closure of M_k in IM_k, denoted by $\bar{M}_k$, is compact.*

For a proof of this theorem, see for example [DK90]. Though this space is compact, it has many singularities, making it cumbersome to deal with.

3.1.2 ADHM construction

We want to construct instantons over $\mathbb{R}^4$. The anti-self dual equations 3.1.1 are elliptic partial differential equation, which can be very hard to solve. Fortunately, the solutions to this particular equation on $\mathbb{R}^4$ can be constructed entirely by linear algebra as follows[AHDM78]:

Theorem 3.1.4 (ADHM construction of instantons). *Fix a complex structure I on $\mathbb{R}^4$ with coordinates (z_1, z_2). There is a one-to-one correspondence between framed $SU(n)$-instantons of charge k and matrices*

$$(\alpha_1, \alpha_2, a, b) \in M_{k\times k}(\mathbb{C}) \times M_{k\times k}(\mathbb{C}) \times M_{k\times n}(\mathbb{C}) \times M_{n\times k}(\mathbb{C})$$

such that $(\alpha_1, \alpha_2, a, b)$ satisfy the ADHM equations:

1. $[\alpha_1, \alpha_2] + ba = 0$
2. $[\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a = 0$
3. *The map* $\begin{pmatrix} \alpha_1 - \lambda_1 Id_{k\times k} \\ \alpha_2 - \lambda_2 Id_{k\times k} \\ a \end{pmatrix}$ *from $\mathbb{C}^k$ to $\mathbb{C}^{2k+n}$ is injective for all $\lambda_1, \lambda_2 \in \mathbb{C}$*
4. *The map* $\begin{pmatrix} -\alpha_2 + \lambda_2 Id_{k\times k} & \alpha_1 - \lambda_1 Id_{k\times k} & b \end{pmatrix}$ *from $\mathbb{C}^{2k+n}$ to $\mathbb{C}^k$ is surjective for all $\lambda_1, \lambda_2 \in \mathbb{C}$.*

modulo the action of $U(k)$ on this space given by:

$$(\alpha_1, \alpha_2, a, b) \mapsto (g\alpha_1 g^{-1}, g\alpha_2 g^{-1}, ag^{-1}, gb)$$

For a proof see for example [DK90].

We can write these equations as a monad, see definition 1.3.10:

$$\mathbb{C}^k \xrightarrow{L} \mathbb{C}^k \otimes \mathbb{C}^k \oplus \mathbb{C}^n \xrightarrow{R} \mathbb{C}^k \tag{3.1.2}$$

where $L = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ a \end{pmatrix}$ and $R = \begin{pmatrix} -\alpha_2 & \alpha_1 & b \end{pmatrix}$. The equation in 1 is then equivalent to saying that that 3.1.2 defines a complex. If we now replace α_i by $\alpha_i - \lambda_i \mathrm{Id}_{k\times k}$ we get a family of parametrized by $\mathbb{C}^2$, still equivalent to 1:

$$\mathbb{C}^k \xrightarrow{\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ a \end{pmatrix}} \mathbb{C}^k \otimes \mathbb{C}^2 \oplus \mathbb{C}^n \xrightarrow{\begin{pmatrix} -\alpha_2 & \alpha_1 & b \end{pmatrix}} \mathbb{C}^k \tag{3.1.3}$$

If we now also demand that this must hold for any complex structure on $\mathbb{R}^4$, we get condition 2.

The vector bundle on which the ASD connection works is given by the cohomology class, as in lemma 1.3.11:

$$E_x = \frac{\ker \begin{pmatrix} -\alpha_2 + z_2 & \alpha_1 - z_1 & b \end{pmatrix}}{\mathrm{Im} = \begin{pmatrix} \alpha_1 - z_1 \\ \alpha_2 - z_2 \\ a \end{pmatrix}}$$

and the connection A is given by projecting the flat connection on $\mathbb{R}^4 \times \mathbb{C}^k \times \mathbb{C}^k \times \mathbb{C}^n$ to E.

The Uhlenbeck compactification of the moduli space can also be viewed in the ADHM construction, as follows. Modify the matrices α_1, α_2, a, b as follows:

$$(\alpha_1, \alpha_2, a, b) \times y \mapsto \left[\begin{pmatrix} \alpha_1 & 0 \\ 0 & \mathrm{diag}(y_1^0, \dots, y_l^0) \end{pmatrix}, \begin{pmatrix} \alpha_2 & 0 \\ 0 & \mathrm{diag}(y_1^1, \dots, y_l^1) \end{pmatrix}, \begin{pmatrix} a & 0 \end{pmatrix}, \begin{pmatrix} b \\ 0 \end{pmatrix} \right]$$

with (y_i^0, y_i^1) are complex coordinates of $\mathbb{R}^4$ and the coordinates $(y_1^0, y_1^1, \dots, y_l^0, y_l^1)$ are taken to be in $S^l\mathbb{R}^4$, the l-times symmetric product of $\mathbb{R}^4$.

Now following [Mac91] we will establish a hyperkähler isometry between the space given by the ADHM construction, and the moduli space of instantons, thus proving

Theorem 3.1.5. *The ADHM correspondence is a hyperkähler isometry.*

Proof. Fix a complex structure I on $\mathbb{R}^4$. Let ω denote the flat Kähler form on $\mathbb{R}^4$ defined by I. Let $\mathcal{A}_k$ be the moduli space of $SU(n)$ connections with charge k. The symplectic form ω induces a symplectic form Ω on $\mathcal{A}_k$: $\Omega(u, v) = \int \mathrm{Tr}(u \wedge v) \wedge \omega / (2\pi^2)$, where $u, v \in T_A\mathcal{A}_k$, with the tangent space identified with the space of one-forms on $\mathbb{R}^4$ with values in $\mathfrak{su}(n)$.

Lemma 3.1.6. *The function*

$$K(A) = \frac{1}{16\pi^2} \int_{R^4} |x|^2 \, TrF_A^2$$

is a Kähler potential for the Kähler form Ω *on* $\mathcal{M}_{k,n}$

Proof. We can construct on E a holomorphic structure compatible with A, so that the covariant derivative splits :

$$d_A = \partial_A + \bar{\partial}_A$$

Pick $\alpha, \bar{\alpha} \in T_{[A]}\mathcal{M}_{k,n} \otimes \mathbb{C}$ such that $\partial_A \bar{\alpha} = \bar{\partial}_A \alpha = 0$. Also, if $t > 0$ and real, then $F_{A+ta}^2 = F_A^2 + 2tF_A \wedge d_A a + \mathcal{O}(t^2)$ as can be computed by using the chain rule, so we have

$$
\begin{aligned}
-\frac{1}{2}i\frac{\partial}{\partial\bar{\alpha}}\frac{\partial}{\partial\alpha}K(A) &= \frac{1}{4\pi^2}\frac{\partial}{\partial\bar{\alpha}}\int\frac{-i}{4}|x|^2\mathrm{Tr}F_A\wedge\partial_A\alpha \\
&= \frac{1}{2\pi^2}\int\frac{-i}{4}|x|^2\mathrm{Tr}\left[(\bar{\partial}_A\bar{\alpha}\wedge\partial_A\alpha)+F_A\wedge\alpha\wedge\bar{\alpha}\right] \\
&= \frac{1}{2\pi^2}\int\frac{-i}{4}|x|^2\mathrm{Tr}\left[(\bar{\partial}_A\bar{\alpha}\wedge\partial_A\alpha)+\bar{\partial}_A\partial_A\alpha\wedge\bar{\alpha}\right]
\end{aligned}
$$

Now use that the Kähler potential for ω is given by $\frac{1}{2}|x|^2$, so $\omega=\frac{-i}{4}\partial\bar{\partial}|x|^2$ and so we see by using Stokes's theorem twice that

$$
-\frac{1}{2}i\frac{\partial}{\partial\bar{\alpha}}\frac{\partial}{\partial\alpha}K(A)=\frac{1}{2\pi^2}\int\omega\wedge\mathrm{Tr}(\alpha\wedge\bar{\alpha})
$$

and so we get $-\frac{1}{2}i\bar{\partial}\partial K(A)(\alpha,\beta)=\Omega(\alpha,\beta)$ □

If we define three complex structures I_1, I_2, I_3 on $\mathbb{R}^4$ satisfying the hyperkähler condition 1.2.7, we see that this actually proves the moduli space $\mathcal{M}_{k,n}$ is actually a hyperkähler manifold, with hyperkähler potential $K(A)$.

Now to prove that the hyperkähler potential on the ADHM moduli space is the same, we first note that the hyperkähler structure on the space of matrices (α_1,α_2,a,b) is simpler since the flat Kähler structure on this space is invariant under the action of $U(k)$ given in theorem 3.1.4.

There is an obvious Kähler potential on the space of matrices, given by the norm squared of the matrices:

$$
\frac{1}{2}\left(||\alpha_1||^2+||\alpha_2||^2+||a||^2+||b||^2\right)
$$

Denote by p the ADHM correspondence, we then need to prove

$$
\frac{1}{2}||p(A)^2||=K(A)
$$

Now, while it is possible to calculate directly the Kähler potential $K(A)$ in terms of matrices $p(A)$ in the case that $k=1$, but this becomes much harder when k increases. Instead, we will use a formula due to Osborn [Osb79]:

$$
\star\mathrm{Tr}F_A^2=-\frac{1}{2}\Delta\Delta\mathrm{logdet}L(x) \tag{3.1.4}
$$

where $L(x)=(\alpha_1-z_1)^*(\alpha_1-z_1)+(\alpha_2-z_2)^*(\alpha_2-z_2)+a^*a$ for $(\alpha_1,\alpha_2,a,b)=p(A)$ and Δ the standard Euclidean Laplacian on $\mathbb{R}^4$.

We can now integrate $K(A)$ by parts:

$$
K(A)=\frac{1}{32\pi^2}\mathrm{lim}_{R\to\infty}\int_{S^3(R)}\left[R^4xd\Delta\log\det L-2R^4\Delta\log\det L+8R^2xd\log\det L\right]dS \tag{3.1.5}
$$

with $S^3(R)$ the 3-sphere of radius R with measue R^3dS Now we rewrite L in terms of the vector of matrices γ given by:

$$\gamma_1 = \frac{1}{2}(\alpha_1 + \alpha_1^*) \qquad \gamma_2 = \frac{i}{2}(\alpha_1 - \alpha_1^*)$$
$$\gamma_3 = \frac{1}{2}(\alpha_2 + \alpha_2^*) \qquad \gamma_4 = \frac{i}{2}(\alpha_2 - \alpha_2^*)$$

This gives for the derivatives of L

$$L_i' := \frac{\partial}{\partial x_i} L = 2(x_i - \gamma_i)$$

Now substituting this into equation 3.1.5:

$$\mathrm{Tr}\left[16R^2\langle\gamma, x\rangle L^2(L^{-1})^3 - 24R^4(R^2 - \langle\gamma, x\rangle)L(L^{-1})^3 \right.$$
$$\left. + 2R^4(L + 2R^2 - 2\langle\gamma, x\rangle)(R^2(L^{-1})^2 + \langle L^{-1}\gamma, L^{-1}\gamma\rangle - 2\langle\gamma, x\rangle(L^{-1})^2)L^{-1}\right] dS$$

where we have cyclically permuted the matrices because of the properties of the trace. Also we have left explicit factors of L^{-1} since in limit of $R \to \infty$ this matrix tends to $R^{-2}I_{k\times k}$ and so we can neglect terms which are of order less than 1 in terms of R. Now we substitute

$$L = \alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 + R^2 - 2\langle\gamma, x\rangle + a^*a$$

and we obtain

$$\mathrm{Tr}\left[\left(-16R^6(\alpha_1^*\alpha_1 + \alpha_2^*\alpha_2 + a^*a) - 24R^6\langle\gamma, x\rangle + 80R^4\langle\gamma, x\rangle^2\right)(L^{-1})^3 \right.$$
$$\left. + 24R^6\langle L^{-1}\gamma, L^{-1}\gamma\rangle L^{-1}\right] dS$$

We use the $x \mapsto -x$ symmetry of all the terms to calculate the $24R^6\langle\gamma, x\rangle(L^{-1})^3$ terms which seems at first glance to diverge.

$$\begin{aligned}
\int_{S^3(R)} \mathrm{Tr} 24R^6\langle\gamma, x\rangle(L^{-1})^3 dS &= 12\int_{S^3(R)} R^6\langle\gamma, x\rangle(L(-x)^3 - L(x)^3)M(x)^3M(-x)^3 \\
&= 144\int_{S^3(R)} \mathrm{Tr} R^{10}\langle\gamma, x\rangle^2 M(x)^3M(-x)^3 dS \\
&\approx 144\int_{S^3(R)} \mathrm{Tr}\frac{\langle\gamma, x\rangle}{R^2} dS \quad \text{in the larg } R \text{ limit} \\
= 72\pi^2\left(||\alpha_1||^2 + ||\alpha_2||^2\right)
\end{aligned}$$

Now we can calculate:

$$\begin{aligned}
K(A) &= \frac{-1}{32\pi^2}\left[-32\pi^2\left(||\alpha_1||^2 + ||\alpha_2||^2 + ||a||^2\right) + (-72 + 40 + 48)\pi^2\left(||\alpha_1||^2 + ||\alpha_2||^2\right)\right] \\
&= \frac{1}{2}\left(||\alpha_1||^2 + ||\alpha_2||^2 + ||a||^2 + ||b||^2\right)
\end{aligned}$$

which is precisely the Kähler potential for the ADHM construction. □

This proof together with the formula for the Kähler potential highlights also that the ADHM construction is a hyperkähler quotient, with $[\alpha_1, \alpha_2] + ba$ and $[\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a$ as moment maps on $M_{k\times k}(\mathbb{C}) \times M_{k\times k}(\mathbb{C}) \times M_{k\times r}(\mathbb{C}) \times M_{r\times k}(\mathbb{C}) \simeq \mathbb{C}^{2k^2+2kr}$ divided out by the action of $U(r)$ on $\mathbb{C}^{2k^2+2kr}$.

3.1.3 The Donaldson invariants

We will sketch how the Donaldson invariants are defined, in order to relate them to the Seiberg-Witten invariants via the conjecture of Witten in [Wit94]. We follows [Mor98].

Let M be a 4-dimensional manifold with $b_2^+ > 1$ and odd, and set P to be a principal $SU(2)$ bundle over M with $c_2(P) > 0$. Then the dimension of the moduli space of instantons on P can be calculated by using the Atiyah-Singer index theorem to be

$$d'(P) = 8c_2(P) - 3(b_0(M) - b_1(M) + b_2^+(M)$$

This number is even if M is simply connected and $b_2^+(M)$ is odd, and since we will be integrating 2-forms over this manifold, we will demand it is even and define d to be half the dimension:

$$d(P) = 4c_2(P) - \frac{3}{2}(b_0(M) - b_1(M) + b_2^+(M) \qquad (3.1.6)$$

The Donaldson polynomial invariant associated to a $SU(2)$-bundle P is a symmetric multilinear function of degree d on $H_2(M,\mathbb{Z})$ with values in $\mathbb{Q}$:

$$\gamma_d(M) : \underbrace{H_2(M,\mathbb{Z}) \otimes \ldots \otimes H_2(M,\mathbb{Z})}_{d \text{ times}} \to \mathbb{Q}$$

There exists a principal $SO(3)$ bundle

$$\mathcal{A}^*(P) \times_{\mathcal{G}(P)} P \to \mathcal{B}^* \times M$$

where $\mathcal{A}^*(P)$ is the space of all irreducible $SU(2)$ connections on P, $\mathcal{B}^*$ the space of irreducible $SU(2)$ connections modulo the gauge group, and $\mathcal{A}^*(P) \times_{\mathcal{G}(P)} P$ the associated vector bundle under the action of the gauge group $\mathcal{G}(P)$ on P. This is a $SO(3)$ principal fiber bundle by the identification of $\mathrm{Ad}(SO(3)) = \mathrm{Ad}(SU(2))$ since $SU(2)$ is a double cover of $SO(3)$. We call the total space of the fiber bundle ξ. Now we use the *slant product*

$$/ : H^n(X \times Y; R) \times H_j(Y; R) \to H^{n-j}(X; R)$$

with the Pontrjagin class, defined in 1.3.9 in cohomology to define a homomorphism:

$$H_2(M;\mathbb{Z}) \to H^2(\mathcal{B}^*(P);\mathbb{Z}) : x \mapsto p_1(\xi)/x$$

Restrict this map to the moduli space of irreducible instantons $\mathcal{M}^*(P) \subset \mathcal{B}^*(P)$ to define the map

$$\mu' : H_2(M;\mathbb{Z}) \to H^2(\mathcal{M}^*(P);\mathbb{Z})$$

Using the properties of the Pontrjagin class we can see that this map is divisible by 4 and so define the μ-map:

$$\mu : H_2(M;\mathbb{Z}) \to H^2(\mathcal{M}^*(P);\mathbb{Z}) \quad \mu(x) = \frac{-p_1(\xi)/x}{4} \qquad (3.1.7)$$

Similarly, there is a map

$$\mu : H_0(M;\mathbb{Z}) \to H^4(\mathcal{B}^*;\mathbb{Z})$$

given by sending $1 \in H_0(M;\mathbb{Z})$ to $(-1/4)p_x(\xi)$

These maps can be extended to maps $\bar{\mu}$ over the Uhlenbeck compactification of the moduli space of instantons with charge $k : \overline{\mathcal{M}}(P_k)$, as in theorem 3.1.3. Now for technical reasons, we need each part of the space of ideal instantons $\mathcal{M}_{k-l} \times S^l(M)$ to be of dimension two less than $\mathcal{M}_k$. This is automatic for all $l < k$, since then $\dim\big(\mathcal{M}_{k-l} \times S^l(M)\big) = 2d(P_k) - 4l$, but for $l = k$ we get the moduli space of the trivial bundle, which is a single point, but with formal dimension due to equation 3.1.6 of $-3(1 + b_2^+(M))$, so we demand that k is in the *stable range*, that is:

$$4k \geq 3b_2^+(M) + 5$$

Definition 3.1.7. *For classes $x_1, \ldots x_a \in H_2(M;\mathbb{Z})$ and $e_1, \ldots, e_b \in H_0(M;\mathbb{Z})$ with $a+2b = d(P)$ and $4a \geq 3b_2^+ + 5$ we define the* stable Donaldson polynomial invariant

$$\gamma_d(B) : \underbrace{H_2(M;\mathbb{Z}) \otimes \ldots \otimes H_2(M;\mathbb{Z})}_{a\ times} \otimes \underbrace{H_0(M;\mathbb{Z}) \otimes \ldots \otimes H_0(M;\mathbb{Z})}_{b\ times} \to \mathbb{Z}$$

as

$$\gamma_d(x_1, \ldots, x_a, e_1 \ldots e_b) = \int_{\overline{\mathcal{M}}(P)} \bar{\mu}(x_1) \ldots \bar{\mu}(x_a)\bar{\mu}(e_1) \ldots \bar{\mu}(e_b) \qquad (3.1.8)$$

where $\bar{\mu}$ is the extension to the Uhlenbeck compactification of the map μ defined in equation 3.1.7

Using essentially the same type of arguments as in lemma 2.2.20 we can see that these polynomial invariants are invariant under a C^∞ change of metric.

This definition can be extended to the unstable range by considering blowups.

Theorem 3.1.8. *Consider a 4-manifold M, and assume $b_2^+(M) > 1$. Let P be a principal $SU(2)$-bundle over M for which half the formal dimension of the moduli space of instantons, given by 3.1.6 is given by d. Then there is a $b > 0$ such that $d + b$ is in the stable range for the b-fold blowup M' of M. Let $e_1, \ldots, e_b \in H_2(M';\mathbb{Z})$ be the classes represented by the exceptional curves in M'. For any classes $x_1, \ldots, x_d \in H_2(M;\mathbb{Z})$ consider*

$$\left(\frac{-1}{2}\right)^b \gamma_{d+b}(M')(x_1, \ldots x_d, e_1, \ldots, e_b)$$

where each e_i occurs 4 times. This result is independent of b if $d + b$ lies in the stable range for M and defines a multilinear symmetric function

$$\gamma_d(B) : \underbrace{H_2(M;\mathbb{Z}) \otimes \ldots \otimes H_2(M;\mathbb{Z})}_{d\ times} \to \mathbb{Q}$$

where the image is contained

$$\left(\frac{1}{2}\right)^{b_0}\mathbb{Z}\subset\mathbb{Q}$$

where b_0 is the minimal integer b such that $d+b$ lies in the stable range.

This can be extended to maps

$$\gamma_d(B):\underbrace{H_2(M;\mathbb{Z})\otimes\ldots\otimes H_2(M;\mathbb{Z})}_{a\text{ times}}\otimes\underbrace{H_0(M;\mathbb{Z})\otimes\ldots\otimes H_0(M;\mathbb{Z})}_{b\text{ times}}\to\mathbb{Q}$$

which is called the *full Donaldson polynomial invariant.*

This is only defined when $d\equiv 3(1+b_2^+(M))/2(\text{mod}4)$ due to equation 3.1.6. This can be extended to $d\equiv 3(1+b_2^+(M))/2(\text{mod}2)$ by defining

$$\gamma_d(M)(x_1,\ldots,x_d)=\frac{1}{2}\gamma_{d+2}(M)(x_1,\ldots,x_d,1)$$

where $1\in H_0(M;\mathbb{Z})$ is the generator of the orientation, if $d\equiv 3(1+b_2^+(M))/2(\text{mod}2)$ but not $d\equiv 3(1+b_2^+(M))/2(\text{mod}4)$. We can then define the *Donaldson series* as a formal power series:

$$D_M(x)=\sum_{d=0}^{\infty}\frac{\gamma_d(M)(x)}{d!}$$

There is no general proof of convergence of this series, but in all known examples this does converge. An important simplification in the calculation of Donaldson invariants is when the manifold is of *simple type*. This definition is due to Kronheimer and Mrowka. A manifold is said to be of simple type when

$$\gamma_d(\alpha_1,\ldots,\alpha_t,1,1)=4\gamma_d(\alpha_1,\ldots,\alpha_t)$$

for all classes $\alpha_i\in H_2(M;\mathbb{Z})$ and with $1\in H_0(M;\mathbb{Z})$ the generator of the orientation.

Now we can state the conjecture of Witten regarding Donaldson and Seiberg-Witten invariants

Conjecture 3.1.9. *Consider a 4-manifold M of simple type, with $b_2^+>1$. Let $SW(x)$ denote the Seiberg-Witten invariants for each isomorphism class x of spin^c structures on M. Introduce formal variables $q_1,\ldots,q_{b_2},\lambda$ and write the generating function for the Donaldson series:*

$$D_M\left(e^{\sum_a q_a\alpha_a+\lambda 1}\right)=\sum_{a_i,b}\frac{D_M\left((q_1\alpha_1)^{a_1},\ldots,(q_{b_2}\alpha_{b_2})^{a_{b_2}},(\lambda 1)^b\right)}{a_1!\cdots a_{b_2}!b!}$$

Set $v=\sum_a q_a\alpha_a$, then the Witten conjecture is

$$D_M\left(e^{\sum_a q_a\alpha_a+\lambda 1}\right)=$$

$$2^{1+\frac{1}{4}(7\xi+11\sigma)}\cdot\left[e^{\left(\frac{v^2}{2}+2\lambda\right)}\sum_x SW(x)\cdot e^{v\cdot x}+i^{\frac{\xi+\sigma}{4}}\sum_x SW(x)\cdot e^{-iv\cdot x}\right]$$

where ξ *is the Euler class of* M *and* σ *the signature.*

3.1.4 The Seiberg-Witten prepotential

One of the most important objects in Seiberg-Witten theory is the prepotential, which Seiberg and Witten used to determine the low energy limit of the Donaldson theory, leading them to the $U(1)$ monopole equations. This prepotential is given as the generator of periods of a family of curves, the *Seiberg-Witten curves*. We follow [NY04].

Definition 3.1.10. *Let* $\vec{u} = (u_1, \dots, u_r)$ *be global complex coordinates on a special Kähler manifold, called the* u-plane. *The* Seiberg-Witten curves *is the family of curves parametrized by* $\vec{u}$*:*

$$C_{\vec{u}} : \Lambda^r \left(w + \frac{1}{w} \right) = P(z) = z^r + u_2 z^{r-2} + u_3 z^{r-3} + \ldots + u_r$$

For generic u these are curves of genus $r-1$. If we project $(w,z) \in C_{\vec{u}}$ to $z \in \mathbb{CP}^1$, we see that they have the structure of hyperelliptic curves, with hyperelliptic involution $\iota : w \mapsto \frac{1}{w}$.

We can write $y = \Lambda^r \left(w - \frac{1}{w} \right)$, then the equations become

$$y^2 = (P(z) - 2\Lambda^r)(P(z) + 2\Lambda^r) \tag{3.1.9}$$

The parameter Λ is called the *renormalization scale* in physics.

Let $\{z_1, \dots, z_r\}$ be the solutions of $P(z) = 0$. First assume that we work in the sector $\mathcal{U}_\infty$ of the u-planet such that $|z_\alpha - z_\beta|$ and $|z_\alpha|$ are much larger than $|\Lambda|$, in particular, the z_α are distinct. By Vieta's formula for the roots of a polynomial, we have $\sum_\alpha z_\alpha = 0$, so these form $r-1$ independent functions of u. In $\mathcal{U}_\infty$, we can consider the z_α as local coordinates. From equation 3.1.9 we see that near each z_α there are solutions $z_\alpha^\pm$ of the equation $P(z_\alpha^\pm) = \pm 2\Lambda^r$ when $|z_\alpha| \gg |\Lambda|$. By the definition of y, we see that these are the points where $w = \frac{1}{w}$, these are called the *branch points* of the projection $C_{\vec{u}}$ to $\mathbb{CP}^1$. The genus of $C_{\vec{u}}$ is then $r-1$ by the Riemann-Hurwitz formula.

Since the curve $C_{\vec{u}}$ is hyperelliptic, we can view it as being made up of two copies of the Riemann sphere, glued along cuts through z_α^+ and z_α^-. Now define the cycles A_α as the cycles encircling the cut between z_α^+ and z_α^-. Now choose cycles B_α such that $\{A_\alpha, B_\alpha\}$ form a symplectic basis of $H_1(C_{\vec{u}}, \mathbb{Z})$, that is $A_\alpha \cdot A_\beta = B_\alpha \cdot B_\beta = 0$ and $A_\alpha \cdot B_\beta = \delta_{\alpha\beta}$. Note that these are not global cycles on the u-plane.

Definition 3.1.11. *The* Seiberg-Witten differential *is given by*

$$dS = -\frac{1}{2\pi} z \frac{dw}{w} = -\frac{1}{2\pi} \frac{zP'(z)dz}{\sqrt{P(z)^2 - 4\Lambda^{2r}}} = -\frac{1}{2\pi} \frac{zP'(z)dz}{y}$$

where $P(z)$ *as defined in 3.1.10.*

We can differentiate the Seiberg-Witten differential by setting w to be constant

$$\left. \frac{\partial}{\partial u_p} dS \right|_{w=\text{constant}} = \frac{1}{2\pi} \frac{z^{r-p}}{P'(z)} \frac{dw}{w} = \frac{1}{2\pi} \frac{z^{r-p} dz}{y}$$

These form a basis of the holomorphic differentials due to[GH78]

Lemma 3.1.12. *Let S be a hyperelliptic Riemann surface of genus g, given by the equation $y^2 = g(z)$ with $g(z)$ a polynomial of degree $2g+2$. The differentials*

$$\left\{ \frac{dz}{y}, z\frac{dz}{y}, \dots, z^{g-1}\frac{dz}{y} \right\}$$

form a basis of holomorphic differentials on S

Proof. The hyperelliptic involution $\iota : y \mapsto -y$ is of order 2, and so the induced transformation $\iota^* : H^0(S, \Omega^1) \to H^0(S, \Omega^1)$ has eigenvalues ± 1 and we obtain a decomposition of $H^0(S, \Omega^1)$ into eigenspaces with eigenvalues ± 1. The $+1$ eigenspace is trivial, since given a holomorphic one-form ω with $\iota^*\omega = \omega$ this would descend to a holomorphic one-form on $\mathbb{CP}^1$ which does not exist. So $\iota^*\omega = -\omega$ for $\omega \in H^0(S, \Omega^1)$. Now start with the one-form $\omega_0 = \frac{dz}{y}$. This is holomorphic and nonzero away from the points above ∞. Now if ω is any other holomorphic 1-form on S we must have $\omega = h\omega_0$ with h a meromorphic function, holomorphic away from ∞. But since ω, ω_0 both lie in the -1 eigenspace for ι^* we must have $\iota^* h = h$, so h is a function of z alone, hence a polynomial in z. If h is of degree d, it has $2d$ zeroes away from ∞ and so a pole of order d at each of the points at ∞. Now ω_0 has total degree $2g-2$ and must have the same order of poles or zeroes at each of the points at ∞, so it must have a zero of oder $g-1$ at each of these points. In order for $h\omega_0$ to be holomorphic, we then need deg$h \leq g-1$, hence the 1-forms in the lemma form a basis. □

This means the Seiberg-Witten differential is a "potential" for holomorphic differentials.

Now define function a_α and a^D_β on the $\mathcal{U}_\infty$ part of the u-plane by

$$a_\alpha = \int_{A_\alpha} dS \quad a^D_\beta = 2\pi i \int_{B_\beta} dS \tag{3.1.10}$$

Let $\sigma_{\alpha p}$ be the matrix defined as

$$\sigma_{\alpha p} = \frac{\partial a_\alpha}{\partial u_p}$$

we can then set

$$\omega_\beta = \frac{\partial}{\partial a_\beta} dS \bigg|_{w=\text{constant}} = \frac{1}{2\pi} \sum_p \sigma^{-1}_{\alpha p} \frac{z^{r-p} dz}{y}$$

This holomorphic one form is normalized: $\int_{A_\alpha} \omega_\beta = \delta_{\alpha\beta}$. This means it can be used to calculate the *period matrix*

$$\tau_{\alpha\beta} = \int_{B_\alpha} \omega_\beta = \frac{1}{2\pi i} \frac{\partial a^D_\alpha}{\partial a_\beta} \tag{3.1.11}$$

Because of the first Riemann bilinear equation [GH78], this function is symmetric, and so a^D_α must be the the derivative of locally defined holomorphic function $\mathscr{F}$, called the *prepotential*.

Definition 3.1.13. *The* Seiberg-Witten prepotential *is the holomorphic function $\mathscr{F}$, locally defined as*

$$a_\alpha^D = -\frac{\partial \mathscr{F}}{\partial a_\alpha}$$

3.2 Equivariant localization of instantons

3.2.1 Kirwan map

We want to compute equivariant integrals over spaces which are given as symplectic quotients of other spaces. For this, we need a map from the equivariant cohomology of the original space to the normal cohomology of the reduced space. Here, we follow [JK05].

Recall our definitions of equivariant cohomology, definitions 1.4.1 and 1.4.4. If T acts freely on M, we have

$$H_T^*(M) = H^*(M/T) \tag{3.2.1}$$

On the other hand, if T acts trivially on M, then

$$H_T^*(M) = H^*(M) \otimes S(\mathfrak{t}^*) = H^*(M) \otimes H_T^*(\{pt\}) \tag{3.2.2}$$

Now let $p \in \mathfrak{t}^*$ be a regular value of the moment map μ associated to T and denote by $i_p : \mu^{-1}(p) \hookrightarrow M$ the canonical inclusion. Then T acts locally free on $\mu^{-1}(p)$. This induces a map $i_p^* : H_T^*(M) \to H_T^*(\mu^{-1}(p))$. Then, by equation 3.2.1, we have

$$H_T^*(\mu^{-1}(p)) \simeq H^*(\mu^{-1}(p)/T) = H^*(M/\!\!/_p T)$$

Combining this with the canonical inclusion and we get the *Kirwan map*

$$\kappa_p : H_T^*(M) \to H^*(M/\!\!/_p T) \tag{3.2.3}$$

This map is a surjection if M is compact [Kir84]. This procedure also works if M is the symplectic quotient of a space by another (commuting) action, so we get the *equivariant Kirwan map*, where we split a torus T into the direct product of two tori: $T = R \times S$.

$$p \in \mu(M) \quad \kappa_p : H_{R\times S}^*(M) \to H_R^*(\mu^{-1}(p)/S) \tag{3.2.4}$$

with μ the moment map of R. This map is also a surjection [Gol02]. In particular, if $p \in M^R$, the fixed points of M under the action of R, we get the surjection

$$p \in \mu(M^R) \quad \kappa_p^R : H_{R\times S}^*(M^R) \to H_R^*(M^R /\!\!/_p S)$$

Now apply this to the procedure of the symplectic cut. Given a cone Σ_β, the product $T \times T$ acts on $M \times \mathbb{C}_\beta$ and the cut M_Σ is given by reducing $M \times \mathbb{C}_\beta$ at 0. We We can think of $T \times T$ as the product of $T \times e$ and ΔT, and the equivariant version of the Kirwan map 3.2.4 gives a map

$$\kappa_\Sigma^T : H_{T\times T}^*(M \times \mathbb{C}_\beta) \to H_{T\times e}^*(M_\Sigma) = H_T^*(M_\Sigma)$$

which is surjective.

The torus T_Σ, orthogonal to Σ actually acts trivial on M_Σ, hence we get by equation 3.2.2

$$H_T^*(M_\Sigma) = H_{T/T_\Sigma}^*(M_\Sigma) \otimes H_{T_\Sigma}(pt)$$

Now, for $\eta \in H_T^*(M)$ denote by the η_Σ the class $\kappa_\Sigma(\eta \otimes 1) \subset H_T^*(M_\Sigma)$. Now if F_i is a connected component of the fixed point set of both M and M_Σ with $\mu(F_i) = \mu_\Sigma(F_i) \in \mathrm{Int}(\Sigma)$ then

$$\iota_{F_i}^* \eta = \iota_{F_i}^* \eta_\Sigma$$

The equivariant Kirwan map gives rise to a map of equivariant Euler classes [JK05]

Lemma 3.2.1. *Decompose a torus action T as $T = S \times H$. Let $\pi : \mathfrak{t}^* \to \mathfrak{s}^*$ be the natural projection, $p \in \mu(M^H)$, and $q = \pi(p)$. Then $\kappa_p^H : H_T^*(M^H) \to H_H^*(M^H /\!\!/_p S)$ takes the T-equivariant Euler class of the normal bundle of M^H onto the H-equivariant Euler class of the normal bundle of $M^H /\!\!/_p S$ in $M /\!\!/_p S$*

Proof. Set $Z = \mu^{-1}(p) \cap M^H$ with μ the moment map for T, and let ρ be the projection from Z to $M^H /\!\!/_p S$. Let $i : Z \hookrightarrow M^H$ be the natural inclusion. Then $i^* \nu(M^H) = \rho^* \nu(M^H /\!\!/_p S)$, and since by theorem 1.3.5 characteristic classes are natural with respect to pull-backs, we get the asked mapping. □

3.2.2 Residues

We will define the residues needed for the calculation of the Jeffrey-Kirwan localization formula by iterating the one-dimensional case.

Let $f(z)$ be a meromorphic function on the Riemann-sphere with values in a vector space V which is of the following form

$$f(z) = \sum_j g_j(z) e^{\lambda_j z}$$

a finite sum with the $g_j(z)$ rational functions of z and $\lambda_j \in \mathbb{R}$. Then we define the *residue* of the form $f dz$ as:

$$\mathrm{res}^+(f dz) = \sum_{\lambda_j \geq 0} \sum_{b \in \mathbb{C}} \mathrm{res}_b g_j(z) e^{\lambda_j z} \tag{3.2.5}$$

Now if all λ_j are zero, we see that this is just the standard residue, and we see that if f is holomorphic

$$\mathrm{res}^+(f dz) = -\mathrm{res}_{z=\infty}(f dz)$$

and from this we conclude that $\mathrm{res}^+ \frac{p(z)}{q(z)} = 0$ if $\deg(p(x)) + 1 \neq \mathrm{def}(q(x))$ for polynomials p and q.

The higher dimensional residue needed in the Jeffrey-Kirwan formula can be obtained from this one-dimensional residue by iterating this procedure. Consider functions f which can be written as linear combinations of the following functions on $\mathfrak{t} \otimes \mathbb{C}$ taking values in a vector space V:

$$h(X) = \frac{q(X) e^{\lambda}(X)}{\prod_{j=1}^k \alpha_j(X)} \tag{3.2.6}$$

with $q(X)$ a polynomial on $\mathfrak{t}_\mathbb{C}$, $\lambda \in \mathfrak{t}^*$ and $\alpha_j \in \mathfrak{t}^* - \{0\}$. Now choose a connected component Λ of the set

$$\{X \in \mathfrak{t} : \alpha_j(X) \neq 0 \forall \alpha_j\}.$$

The set Λ is a cone. Now we choose a linear coordinate system $X_1, \ldots X_n$ on $\mathfrak{t}$ and write $\text{res}^+_{X_i}$ for the residue with respect to i-th coordinate by treating the other coordinates as constants. The resulting functions has as remaining variables the original ones with X_i left out. Now we define the residue with respect to a cone as follows:

$$\text{JKRes}^\Lambda(h[dx]) = \Delta \text{res}^+_{X_1} \ldots \text{res}^+_{X_n} h(X_1, \ldots, X_n)[dX]^n_1 \tag{3.2.7}$$

with Δ the determinant of a $n \times n$ matrix with columns defining an orthonormal basis of $\mathfrak{t}$ with the same orientation as the coordinate system and $[dX]^n_l$ stands for the form $dX_l \wedge \ldots \wedge dX_n$. This definition only depends on the choice of the cone and the inner product on $\mathfrak{t}$, not on the coordinates chosen, as shown in [JK97]. There also some properties which uniquely define this residue are determined. They are the following:

1. Let $\alpha_1, \ldots, \alpha_v \in \Lambda^*$ be vectors in the dual cone. Assume that λ is not in any cone of dimension $m-1$ or less spanned by a subset of these vectors. Then if $J = (j_1, \ldots, j_m)$ is a multi-index and $X^J = X_1^{j_1} \ldots X_m^{j_m}$ we have

$$\text{JKRes}^\Lambda \left(\frac{X^J e^{i\lambda(X)}[dX]}{\prod_{i=1}^v \alpha_i(X)} \right) = 0$$

   unless all of the following are true:

   (a) the vectors $\{\alpha_i\}_{i=1}^v$ span $\mathfrak{t}^*$ as vector space.

   (b) $v - \sum_{i=1}^m j_i \geq m$

   (c) λ lies in the positive span of the vectors $\{\alpha_i\}$.

2. If the properties 1 a) - c) are satisfied then

$$\text{JKRes}^\Lambda \left(\frac{X^J e^{i\lambda(X)}[dX]}{\prod_{i=1}^v \alpha_i(X)} \right) = \sum_{k \geq 0} \lim_{s \to 0^+} \text{JKRes}^\Lambda \left(\frac{X^J (i\lambda(X))^k e^{is\lambda(X)}[dX]}{k! \prod_{i=1}^v \alpha_i(X)} \right)$$

   and only the term where $k = v - \sum_{j=1}^m j_i - m$ is non-zero.

3. If properties 1 a) -c) are satisfied with $\alpha_1, \ldots, \alpha_m$ are linearly independent, then

$$\text{JKRes}^\Lambda \left(\frac{e^{i\lambda(X)}[dX]}{\prod_{i=1}^v \alpha_i(X)} \right) = \frac{1}{\det(\bar{\alpha})}$$

   with $\bar{\alpha}$ the non-singular matrix whose columns are the coordinates of $\alpha_1, \ldots, \alpha_m$ with respect to an orthonormal basis defining the same orientation. Thus the residue is not identically 0.

Lemma 3.2.2. *Let $f(X_1, \ldots, X_M)$ be a linear combination of functions on $\mathfrak{t}$ given by equation 3.2.6 such that for every set of values $(a_1, \ldots, a_{m-1})$, the function $f(a_1, \ldots, a_{m_1}, z)$ is holomorphic. Set Λ to be a cone such that $(0, \ldots, 1) \in \Lambda$*

and $Y_1, \ldots, Y_m$ *to be coordinates such that* $(0, \ldots, 1) \in -\Lambda$. *Then*

$$JKRes^{\Lambda}(f[dX]) = JKRes^{-\Lambda}(f[dY]) \tag{3.2.8}$$

Proof. Pick a $a_1, \ldots a_{m-1}$. Then

$$f(a_1, \ldots, a_{m-1}, z) = \sum_j f_j(z) e^{i\lambda_j z}$$

and define

$$\text{res}^-(f(a_1, \ldots, a_{m-1}, z)dz) = \sum_{\lambda_j < 0} \sum_{b \in \mathbb{C}} \left(\text{res} f_j(z) e^{i\lambda_j z}; z = b\right)$$

Since $f(a_1, \ldots, a_m, z)$ is holomorphic, its total residue is zero,

$$\text{res}^-(f(a_1, \ldots, a_{m-1}, z)dz) = -\text{res}^+(f(a_1, \ldots, a_{m-1}, z)dz)$$

Now we can choose $Y_1, \ldots, Y_m$ arbitrary such that $(0, \ldots, 0, 1) \in -\Lambda$ since the residue is independent of the choice of coordinates, we choose $Y_i = X_i$ for $i < m$ and $Y_m = -X_m$. Then fixing the first $m - 1$ coordinates, we calculate

$$\begin{aligned} \text{res}^+(f(Y_1, \ldots, Y_m)dY_m) &= \text{res}^+(-f(X_1, \ldots, X_{m-1}, -X_m)dX_m) \\ &= \text{res}^-(-f(X_1, \ldots, X_{m-1}, X_m)dX_m) \\ &= \text{res}^+(f(X_1, \ldots, X_{m-1}, X_m)dX_m) \end{aligned}$$

Where the last line makes use of the fact that the total residue is zero. Then by equation 3.2.7, we see that since the other coordinates are unchanged, formula 3.2.8 holds. □

With the help of this lemma, we can prove the Jeffrey-Kirwan localization. This theorem is a first step to the full equivariant localization theorem for hyperkähler quotients.

3.2.3 Equivariant volumes

The Berline-Vergne, Duistermaat-Heckman and Prato-Wu localization theorems all describe general symplectic manifolds. However, given a special manifold, such as a manifold constructed as a symplectic quotient, one can strengthen these theorems, in order to simplify the computation of the fixed-point contribution. Jeffrey-Kirwan localization is one possible strengthening of localization theorems. First proven in [JK95], we will instead follow the approach of [JK05], since this approach easily generalizes to equivariant actions on symplectic quotients.

Theorem 3.2.3 (Jeffrey-Kirwan localization). *Let* M *be a manifold with a symplectic form* ω *and a compact connected torus* T *which acts Hamiltonian on* M. *Then for any* $\eta \in H_T^*(M)$ *we have:*

$$\int_{M/\!/_0 T} \kappa_0(\eta) e^{\omega} = \sum_{F \subset M^T} \frac{1}{vol(T)} JKRes^{\Lambda} \int_F \frac{i_F^* \eta e^{\omega + \mu_T}}{e_T(\nu_F)} \tag{3.2.9}$$

With κ_p the Kirwan map $H^*_T(M) \to H^*(M/\!\!/T)$. Furthermore $e_T(\nu_F)$ is the equivariant Euler class of the normal bundle of the fixed point set F.

Proof. We prove this theorem by considering the integral $\int_{M_\Sigma} \eta_\Sigma e^{\tilde{\omega}_\epsilon}$ for a suitably chosen cone Σ, taking the residue and using the various properties of the residue to reduce this to the asked formula.

First we consider the choice of a cone Σ inside Λ with respect to which we take the symplectic cut. The torus T is an effective Hamiltonian torus action on M. Let $\{\alpha_i\}$ be the set of all weights of the isotropy representation of T at the fixed points. Choose the cone Λ to be a connected component of the set $\{\xi \in \mathfrak{t} | \alpha_i(\xi) \neq 0\}$. The dual cone Λ^* is given by $\{X \in \mathfrak{t}^* | X(\xi) \geq 0 \text{ for all} i\}$.

Now in [JK05] it is proven we can pick a cone Σ transverse to $\mu(M)$ spanned by $\dim \mathfrak{t}$ weights $\{\beta_1, \ldots, \beta_m\}$ such that $\Lambda^* \subseteq \Sigma$ and for every wall W of $\mu(M)$ such that $W \cap \Sigma$ is not empty, there exists a $\xi_W \in \Sigma^*$ such that the maximum of $q(\xi_W) \in W \cap \Sigma$ is attained at a vertex of W.

Define $\{\gamma_i\}$ to be the set of all weights appearing as weighs of isotropy representations at fixed points of the symplectic cut M_Σ. Now define a cone $\bar{\Lambda}$ to be a connected component of the set $\Sigma^* \cap \{\xi \in \mathfrak{t} | \gamma_i(\xi) \neq 0 \text{ for all } i\}$ For each connected component F of M_Σ^T let $\{\gamma_j^F\}$ be the set of weights of the isotropy representation of T at F. We now polarize the weights $\{\gamma_j^F\}$ by the rule that if $\gamma_j^F(\xi) < 0$ for all $\xi \in \bar{\Lambda}$ then $\bar{\gamma}_j^F = \gamma_j^F$ and $\bar{\gamma}_j^F = -\gamma_j^F$ otherwise. We then define another cone C_F associated to each connected component of the fixed point set of M_Σ to be the cone containing all points of the form

$$\mu_\Sigma(F) + \sum s_j \bar{\gamma}_j^F$$

with s_j nonnegative real numbers.

Then in [JK05] the following useful fact is proven about these C_F

Lemma 3.2.4. *Let F be a new connected component of the fixed point set M_Σ^T. Then the cone C_F does not intersect the interior of Σ.*

Also, if F is an old connected component, then $\bar{\Lambda} \subseteq \Lambda$ and they define the same polarization. Lastly, if $F = M_0$ then $C_F = -\Sigma$.

Now take these cones Λ, $\bar{\Lambda}$ and Σ. Consider the symplectic cut with respect to the cone Σ. M_Σ. For technical reasons, we will not pick the standard moment map for Σ, but slightly modify it. Let $p \in \Sigma$ be a point close to the origin. Then $\mu_\epsilon = \mu_\Sigma - \epsilon p$ is a moment map on M for $\epsilon > 0$. Define $\tilde{\omega}_\epsilon = \omega_\Sigma + i\mu_\epsilon$, the equivariant symplectic form on M_Σ. Now we have an action of T_G on M_Σ, so we look at the fixed points of this action, $M_\Sigma^{T_G}$. There are three different cases we must consider. The "old" fixed points, the image of all F under μ_ϵ in Σ, the symplectic reduction at 0, M_0 and the new fixed points F'. We apply the Berline-Vergne localization theorem 1.4.15 to the integral over M_Σ of the corresponding form $\eta_\Sigma \in H^*_T(M_\Sigma)$:

$$\int_{M_\Sigma} \eta_\Sigma e^{\tilde{\omega}_\epsilon} = \int_{M/\!\!/_0 T} \frac{e^{\tilde{\omega}_\epsilon} i^* \eta_\Sigma}{e_T(\nu_{M/\!\!/_0 T}} + \sum_{F_i} \int_{F_i} \frac{e^{\tilde{\omega}_\epsilon} i^*_{F_i} \eta_\Sigma}{e_T(\nu_{F_i})} + \sum_{F'_j} \int_{F'_j} \frac{e^{\tilde{\omega}_\epsilon} i^*_{F'_j} \eta_\Sigma}{e_T(\nu_{F'_j})} \quad (3.2.10)$$

We can now take the residue of the left hand side. By lemma 3.2.2 we see that it does not matter if we take the residue with respect to Λ or $-\Lambda$. We take the residues with respect to $\bar{\Lambda}$ $-\bar{\Lambda}$ and set these equal, leading to six terms. However, four of them will actually vanish. By lemma 3.2.4 the cones $C_{F'}$ and F_{M_0} do not contain ϵp. Then by property 1 we get

$$\mathrm{JKRes}^{\bar{\Lambda}}\left(\int_{M/\!/_0 T}\frac{e^{\tilde{\omega}_\epsilon}i^*\eta_\Sigma}{e_T(\nu_{M/\!/_0 T}}\right)=\mathrm{JKRes}^{\bar{\Lambda}}\left(\sum_{F'_j}\int_{F'_j}\frac{e^{\tilde{\omega}_\epsilon}i^*_{F'_j}\eta_\Sigma}{e_T(\nu_{F'_j})}\right)=0$$

Now if we choose ϵ small enough, the cones $-C_{F'_j}$ and $-C_{F_i}$ will not contain ϵp hence again by property 1:

$$\mathrm{JKRes}^{-\bar{\Lambda}}\int_{F_i}\frac{e^{\tilde{\omega}_\epsilon}i^*_{F_i}\eta_\Sigma}{e_T(\nu_{F_i})}=\mathrm{JKRes}^{-\bar{\Lambda}}\sum_{F'_j}\int_{F'_j}\frac{e^{\tilde{\omega}_\epsilon}i^*_{F'_j}\eta_\Sigma}{e_T(\nu_{F'_j})}=0$$

This means we can summarize equation 3.2.10 and the operations after it as:

$$\mathrm{JKRes}^{-\bar{\Lambda}}\int_{M/\!/_{\epsilon p} T}\frac{e^{\tilde{\omega}_\epsilon}i^*\eta_\Sigma}{e_T(\nu_{M/\!/_\epsilon T})}=\mathrm{JKRes}^{\bar{\Lambda}}\sum_{F_i}\int_{F_i}\frac{e^{\tilde{\omega}_\epsilon}i^*_{F_i}\eta_\Sigma}{e_T(\nu_{F_i})}\tag{3.2.11}$$

Finally, we want to take the limit $\epsilon\to 0$ of both sides. By the splitting principle, the normal bundle $\nu(M_0)$ splits a sum of line bundle $\oplus_i L_i$ whose Euler class is given by $e(\nu(M_0))=\prod(\tilde{\beta}_i+c_1(L_1))$ with $\tilde{\beta}_i$ weights of the action of T/T^Σ on M_0 and $c_1(L_i)$ the first Chern class of L_i. Then by property 2 of the residue, we see that

$$\lim_{\epsilon\to 0^+}\mathrm{JKRes}^{-\bar{\Lambda}}\int_{M/\!/_{\epsilon p} T}\frac{e^{\tilde{\omega}_\epsilon}i^*\eta_\Sigma}{e_T(\nu_{M/\!/_\epsilon T})}=c'\int_{M_0}\kappa_0(\eta e^\omega)$$

for some constant c'. The limit of the right-hand side of 3.2.11 is simple, just $\mathrm{JKRes}^{\bar{\Lambda}}\sum_{F_i}\int_{F_i}\frac{e^{\tilde{\omega}}i^*_{F_i}\eta_\Sigma}{e_T(\nu_{F_i})}$ hence we have

$$\int_{M_0}\kappa_0(\eta e^\omega)=c\mathrm{JKRes}^{\bar{\Lambda}}\sum_{F_i}\int_{F_i}\frac{e^{\tilde{\omega}}i^*_{F_i}\eta_\Sigma}{e_T(\nu_{F_i})}$$

To conclude the proof, note that F_i are the old connected components of the fixed point set M^T_Σ, so $\mu(F_i)=\mu_\Sigma(F_i)$, and so $\iota^*_{F_i}(\eta_\Sigma e^\omega)$ is the restriction of $\eta e^\omega\in H^*_T(M)$ to F_i. Also, if for a $F_i\subset M^T$ the image of the moment map μ is not inside Σ and the cone C_{F_i} does not contain the origin and thus by property 1 of the residue of this term is 0. The constant c can be proven to be $\frac{1}{\mathrm{vol}T}$, [JK95]. □

In [Mar00] the following abelianization theorem was proven, which allows us to generalize the Jeffrey-Kirwan localization to group actions which are not abelian.

Theorem 3.2.5 (Abelianization theorem). *Let M be as in theorem 3.2.3 and let G be a compact connected Lie group which acts Hamiltonian on M. Set T_G to be a maximal torus of G, $\alpha\in H^*_{G\times H}(M)$, and $\kappa_G: H^*_G(M)\to H^*(M/\!/G)$*

the equivariant Kirwan map. Then

$$\int_{M/\!\!/G} \kappa_G(\alpha) = \frac{1}{W_G} \int_{M/\!\!/T_G} \kappa_{T_G}(\varpi^2 \alpha)$$

where ϖ is the product of the positive roots of the Lie algebra of G, defined in section 1.1.2.

This means we have as a corollary of 3.2.3 and 3.2.5:

Corollary 3.2.6. *Let G and M be as in theorem 3.2.5, then:*

$$\int_{M/\!\!/G} \kappa_G(\alpha) e^{\omega} = \sum_{F \subset M^{T_G}} \frac{(-1)^{n_+}}{vol(T_G)|W_G|} JKRes^{\Lambda} \varpi^2 \int_F \frac{i_F^* \eta e^{\omega + \mu_{T_G}}}{e_{T_G}(\nu_F)} \tag{3.2.12}$$

with n_+ the number of positive roots.

In [Mar08], the Jeffrey-Kirwan localization theorem is generalized to include equivariant actions.

Theorem 3.2.7 (Equivariant Jeffrey-Kirwan localization). *Let G and H be two compact connected Lie groups with commuting Hamiltonian actions on a compact symplectic manifold M with symplectic form ω. Then the action of H descends to the symplectic quotient $M/\!\!/G$. Also, with equivariant Kirwan map $\kappa : H^*_{G\times H}(M) \to H^*_H(M/\!\!/G)$ we have for any $\eta \in H^*_{G\times H}(M)$ we have:*

$$\int_{M/\!\!/G} \kappa(\eta) e^{\omega+\mu_H} = \sum_{F \subset M^{T_G}} \frac{-1^{n_+}}{vol(T_G)|W_G|} JKRes^{\Lambda} \varpi^2 \int_F \frac{i_F^* \eta e^{\omega+\mu_{T_G}+\mu_H}}{e_{T_G}(\nu_F)} \tag{3.2.13}$$

with the same notation as in corollary 3.2.6.

The proof goes via successive approximations of the Borel model of equivariant cohomology by successive spaces $(E_i H \times M)/H$, see the notes after definition 1.4.1. These are not symplectic, but are built out of symplectic leaves, which make it possible to generalize the localization theorems to those spaces. For the full proof, see [Mar08].

3.2.4 Quotients by linear actions

Actually, the theory described so far is a bit overkill for the problem at hand, the calculation of the volume of the equivariant instanton moduli space. This space is given as the hyperkähler quotient of a vector space V by a linear group action G, see definition 1.2.8.

$$V/\!\!/\!\!/_{(\zeta_1,\zeta_2,\zeta_3)} G = \left(\mu_1^{-1}(\zeta_1) \cap \mu_2^{-1}(\zeta_2) \cap \mu_3^{-1}(\zeta_3)\right)/G$$

In fact, the space $\left(\mu_{\mathbb{C}}^{-1}(0)\right)_{\zeta_1} = \mu_{\mathbb{C}}^{-1}(0)/\!\!/_{\zeta_1} G$ can be viewed as a zero set of a section of a T-equivariant vector bundle. The moment map $\mu_{\mathbb{C}}$ is quadratically homogeneous, and V_λ, being the symplectic cut of a vector space with respect to a circle action with constant weight, is the projective space $\mathbb{P}V$. Thus the zeroes of $\mu_{\mathbb{C}}$ coincide with the zeroes of a section of $\mathcal{O}(2)$, the tensor product of the tautological line bundle over $\mathbb{P}V$ with itself. This vector bundle is also clearly equivariant.

We will call this section, by abuse of notation, also $\mu_{\mathbb{C}}$

This allows us, by using the equivariant Thom class of theorem 1.4.12 to express integration over $(V/\!/\!/_{\zeta_1,0,0}G)_\lambda$ in terms of integration over $(V/\!/_{\zeta_1}G)_\lambda$ as follows:

$$\int_{(V/\!/\!/G)_\lambda} e^{\omega+\mu_T} = \int_{(V)_\lambda/\!/\!/G} e^{\omega+\mu_T} \tag{3.2.14a}$$

$$= \int_{(V)_\lambda/\!/G} e^{\omega+\mu_T} e_T(\mathcal{O}(2)\otimes\mathfrak{g}_{\mathbb{C}}) \tag{3.2.14b}$$

since for an equivariant vector bundle $E \to M$ and a submanifold N of M, we have $\int_M e_G(E)\wedge\alpha = \int_M i^*(\tau_G(E))\wedge\alpha = \int_N i^*\alpha$.

Roughly speaking, a hyperkähler quotient gives an extra Euler class in the enumerator of the integral 3.2.3. By further abuse of notation, we call the Euler class associated to the vector bundle $\mathcal{O}(2)\otimes\mathfrak{g}_{\mathbb{C}}$ at a fixed point $x \in V_\lambda$ $e_{T\times T_G}(\mu^x_{\mathbb{C}})$.

In practice, the procedure for the symplectic cut is a bit cumbersome. Since we know already how the symplectic cut V_λ is going to look after taking the limit $\lambda \to \infty$, $V \sqcup \mathbb{P}V$, we can try to calculate the residue immediately. Since V is a vector space and $T \times T_G$ acts linearly by assumption, we see that there can be only one fixed points in V, 0. The other fixed points must lie within $\mathbb{P}V = V/\!/_\lambda U(1)$.

First we will make some assumptions on the action of $T \times T_G$ that will simplify our calculations. These assumptions are valid for the equivariant ADHM construction, so will not restrict us in the calculation of the equivariant volume.

We will assume the fixed points of $T \times T_G$ are isolated. Also assume the group G acts tri-Hamiltonian on the hyperkähler vector space V, with complex moment map $\mu_{\mathbb{C}}$. Also assume that inside the torus T there is a circle acting with global weight 1, so we are in a situation where the symplectic cut makes sense.

Now we will analyze when a term in the equivariant localization formula of theorem 3.2.7 will be non-trivial.

Let ρ_i be the weights of the torus action of $T \times T_G$ on V. Now consider the fixed point in $\mathbb{P}V$, the symplectic reduction $\mu^{-1}_{T_G}(\lambda)/U(1)$. Since the fixed points are assumed to be isolated, we see that they must lie within a weight space V_{ρ_i} and at most one can lie within each weight space. The image of such a fixed point under $\mu_{T\times T_G}$ is then $\lambda\rho_i$. The weights of the isotropy representation at a fixed point in V_{ρ_i} can easily be calculated, since it is all just torus actions. They are:

$$\rho_i^1 = \rho_1 - \rho_i, \rho_i^2 = \rho_2 - \rho_i, \ldots, \rho_i^i = -\rho_i, \ldots$$

With some abuse of notation, we call the fixed point in V_{ρ_i} ρ_i. By the notation above we will call the associated equivariant Euler class $e_{T\times T_G}(\mu^{\rho_i}_{\mathbb{C}})$. The Euler class of the fixed point 0 we denote simply by $e_{T\times T_G}(\mu_{\mathbb{C}})$.

The ρ_i^j are functions in the linear coordinate system $(X_1, \ldots, X_n)$ on V in which the cone Λ is described. Now we write, for a function in these coordinates, with the q_l linear functions

$$\mathrm{res}_j^{X_i} \frac{p}{\prod q_l}$$

for taking the residue with respect to the j-th term of the denominator in the variable X_i. This is a function with term less in the denominator in the coordinate system

$$(X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n)$$

.

Now at a fixed point ρ_i we see that the contribution to the equivariant localization due to theorem 3.2.7 is:

$$\mathrm{JKRes}^{\Lambda} \varpi^2 e_{T \times T_G}(\mu_{\mathbb{C}}^{\rho_i}) \frac{e^{\lambda \rho_i}}{e_{T \times T_G}(\nu_{\rho_i})} = \sum \mathrm{res}_{j_1}^{X_1} \ldots \mathrm{res}_{j_n}^{X_n} \frac{\varpi^2 e_{T \times T_G}(\mu_{\mathbb{C}}^{\rho_i}) e^{\lambda \rho_i}}{\prod_j \rho_j^i}$$

There is no integral, since the fixed points are assumed to be isolated. Since the ρ_i^j are sums of weights expressed in the coordinate system $(X_1, \ldots, X_n)$, they are linear in that coordinate system.

We want to find out which terms on the left hand side contribute, in other words, in which order we must take the iterated residues. First, notice that each pole ρ_i^j and each variable X_l can only occur once. Secondly, if ρ_i^i is not in the list, we still end up with a non-trivial term in the exponent, which, when taking the limit $\lambda \to \infty$, will suppress the entire term. Likewise, if the pole ρ_i^i is evaluated, the next step in the iterated residue will be 0, since the poles $\prod_j \rho_i^j$ will be trivial then. So ρ_i^i must be evaluated last. Then there is a third, more complicated condition. Recall the definition of the one-dimensional residue 3.2.5. We see that for each variable X_l, the term in the exponent $\lambda \rho_i$ must be ≥ 0. So the condition is that before taking each residue with respect to a variable X_l, the l-th term of ρ_i must be ≥ 0.

Reiterating, we have three conditions in order for a term in the iterated residue to be non-zero:

1. All X_l and ρ_i^j occur at most once
2. The last residue step must be $\mathrm{res}_i^{X_1}$
3. Before taking the residue with respect to X_l, we need the l-the component of $\rho_i \geq 0$

This may seem complicated, but these conditions can be used simplify all the calculations. In fact, in [Mar08] with some bookkeeping and simple linear algebra, the following is proven:

Lemma 3.2.8. *An iterated residue*

$$res_i^{X_1} res_{j_2}^{X_2} \ldots res_{j_n}^{X_n} \varpi^2 e_{T \times T_G}(\mu_{\mathbb{C}}^{\rho_i}) \frac{e^{\lambda \rho_i}}{\prod_j \rho_i^j}$$

satisfies the above conditions if and only if at the beginning the X_n component of $\rho_i \geq 0$ and then at step l we have the X_{l-1} component of $\rho_{l-1} \geq 0$. Also, the *residue above can also be computed as*

$$(-1)^n res_{j_2}^{X_1} \dots res_{j_n}^{X_{n-1}} res_i^{X_n} \varpi^2 \frac{e_{T\times T_G}(\mu_{\mathbb{C}})}{\prod_j \rho_i^j}$$

This gives us the following, relatively easy, way of calculating equivariant integrals over manifolds given by a hyperkähler quotient

Theorem 3.2.9. *Let G be compact connected Lie group with maximal torus T_G, acting tri-Hamiltonian on a hyperkähler vector space V, and let T be commuting torus action on V such that it acts Hamiltonian with respect to one of the kähler structures on V. Denote by ω_T and μ_T the corresponding kähler form and moment map on $V/\!/\!/G$. We then have*

$$\int_{V/\!/\!/G} e^{\omega_T+\mu_T} = \frac{(-1)^{n_+}\cdot(-1)^n}{Vol(T_G)|W_G|} Res_+^{X_1,\dots,X_n} \frac{\varpi^2 e_{T\times T_G}(\mu_C)}{\prod_j \rho_j^i}$$

with the notation as as in theorem 3.2.7, ρ_i^j the $\rho_i - \rho_j$ for the weights ρ_i and $Res_+^{X_1,\dots,X_n}$ defined as the iterated residue:

$$Res_+^{X_1,\dots,X_n} \frac{p}{\prod_j \rho_j^i} = res_+^{X_1} \dots res_+^{X_n} \frac{p}{\prod_j \rho_j^i}$$

and each step defined as

$$res_+^X \frac{p}{\prod_j \rho_j^i} = \sum res_i^X \frac{p}{\prod_j \rho_j^i}$$

where the sum is over all i such that X-coefficient of $\rho_i \geq 0$.

3.2.5 Localization of instantons

Now, we are ready to connect this work with the work of Nekrasov in [Nek03]. He constructed a partition function, based in part on integrals over moduli spaces of instantons, that described the whole $\mathcal{N}=2$ supersymmetric Yang-Mills theory. We will use the techniques above to construct this partition function, and extract some information on the supersymmetric Yang-Mills theory from it.

In the case of the ADHM construction, we set G to be the action of $SU(k)$, with k the instanton charge, on the $2k^2+2kr$ dimensional vectorspace $V = (\alpha_1, \alpha_2, a, b)$, consisting of the matrices α_1, α_2, a, b as in theorem 3.1.4. We have two moment maps

$$\mu_{\mathbb{R}} = [\alpha_1, \alpha_1^*] + [\alpha_2, \alpha_2^*] + bb^* - a^*a$$
$$\mu_{\mathbb{C}} = [\alpha_1, \alpha_2] + ba$$

by theorem 3.1.5 and $\mu_1 = \mu_{\mathbb{R}}$ and $\mu_2 + i\mu_3 = \mu_{\mathbb{C}}$. We can then think of hyperkähler quotients as being the symplectic quotient

$$\mu_{\mathbb{C}}^{-1}(\zeta_2 + i\zeta_3)/\!/_{\zeta_1} G$$

The action of $U(k)$ on this vector space is given in 3.1.4 and is

$$(\alpha_1, \alpha_2, a, b) \mapsto (g\alpha_1 g^{-1}, g\alpha_2 g^{-1}, ag^{-1}, gb)$$

From the definition of Donaldson invariants in section 3.1.3 , we know that the Donaldson invariants can be viewed as gauge invariants polynomials and integrals over cycles of such cycles. However, on $\mathbb{R}^4$, all such cycles are homologically trivial, and we end up with trivial invariants. A key idea of Nekrasov in [Nek03] was to not look at normal instantons, but at *equivariant* ones. In particular, let us fix a complex structure $\mathbb{C}^2$ on $\mathbb{R}^4$. We demand that our instanton structure is equivariant with respect to an action of $U(2)$ acting on this complex space, acting with weights ϵ_1 and ϵ_2. The moment map $\mu_{U(2)}$ is given by

$$\mu_{U(2)} = \epsilon_1 |z_1|^2 + \epsilon_2 |z_2|^2$$

We then look at equivariant instantons, calculate their prepotential and then look at the limit $\epsilon_1, \epsilon_2 \to 0$.

Now we look at the action of the torus $T^2 \times T_{U(n)}$ on this vector space. This was calculated in [NY05a] and is:

$$(e_1, e_2, t)(\alpha_1, \alpha_2, a, b) = (e_1\alpha_1, e_2\alpha_2, at^{-1}, e_1 e_2 tb) \tag{3.2.15}$$

where e_1 and e_2 are in T^2 and $t \in T_{U(n)}$.

The torus action we are considering for the equivariant ADHM construction, $T^2 \times T_{U(n)}$, does not actually preserve the hyperkähler structure, but since $\mu_{\mathbb{C}} = [\alpha_1, \alpha_2] + ba$ is quadratic homogeneous, the action of $T^2 \times T_{U(n)}$ does preserve the level set $\mu_{\mathbb{C}}^{-1}(0)$.

Nekrasov sought to calculate the integral

$$\int_{\mathcal{M}_{n,k}} e^{\omega + \mu_T}$$

where μ_T is the moment map belonging to the action of $T_{U(n)} \times T^2$ where $T_{U(n)}$ is the action of a maximal torus of $U(n)$, given by rotation at infinity and T^2 is a rescaling of $\mathbb{C}^2 \subset \mathbb{CP}^2$. The action of this torus on $\mathbb{M}_{n,k}$ is calculated in [NY05a] and is as follows:

$$(t, e_1, e_2)(\alpha_1, \alpha_2, a, b) \mapsto (e_1\alpha_1, e_2\alpha_2, at^{-1}, e_1 e_2 tb)$$

with $t \in T_{U(n)}$ and $(e_1, e_2) \in T^2$

Normally $\mathcal{M}_{n,k}$ is non-compact, however we can take the Uhlenbeck compactification to fix this. The Uhlenbeck space is however not a proper manifold, but a stratified space, given by components of varying dimensions given by the various ideal instantons one is adding. One approach to fix this is by not taking the hyperkähler quotient at 0, but by considering $(\mu_{\mathbb{R}}^{-1}(\zeta \mathrm{Id}) \cap \mu_{\mathbb{C}}^{-1}(0))/U(n)$. This space has several different names, the Gieseker compactification, the space of instantons over non-commutative $\mathbb{R}^4$ [NS98], or the moduli space of torsion-free sheaves[OSS80]. Alternatively, we can view it as the blow-up of the singular points of the Uhlenbeck space.

We must take care that we still calculate the same quantity however. Denote by $\mathcal{M}^u_{k,n}$ the Uhlenbeck compactification and by $\mathcal{M}^g_{k,n}$ the Gieseker compactification. We can view the symplectic form on $\mathcal{M}^g_{k,n}$ as the pull-back of the symplectic form on $\mathcal{M}^u_{k,n}$ via the blow-up map. On the degenerate set of this map, the symplectic form degenerates, however, on the original space $\mathcal{M}_{n,k}$ this map stays proper, and one can still speak of moment maps associated to this symplectic form. Finally, we note that the set $\mathcal{M}^g_{k,n} \backslash \mathcal{M}_{n,k}$ has measure zero, so the equivariant volume is the same.

Now theorem 3.2.9 shows us how to calculate this integral:

$$\int_{\mathcal{M}_{n,k}} e^{\omega+\mu_T} = \frac{K \cdot (-1)^k}{|W_G|} \mathrm{Res}_+^{\sigma_1,\ldots,\sigma_k} \frac{\bar{\omega}^2 e_{T\times T_G}(\mu_C)}{\prod_j \rho_j} \tag{3.2.16}$$

we can now calculate what the various terms on the right hand side are. The Weyl group W_G of $SU(k)$ is the permutation group of S_k, which has $k!$ elements. Positive roots of G are given by $\sigma_i - \sigma_j$ for $i < j$, which gives $\varpi = \prod_{i<j}(\sigma_i - \sigma_j)$. The number of positive roots $n_+ = k(k-1)/2$ can be absorbed into the second ϖ factor, to give $(-1)^{n_+}\varpi^2 = \prod_{i\neq j}(\sigma_i - \sigma_j)$. The total number of roots n is even. The volume of T_G can be chosen to be 1.

The action of $T_{SU(k)} \times T^2 \times T_{U(n)}$ on $\mu_{\mathbb{C}}$ is given by

$$\begin{aligned}
&(s,(e_1,e_2),t)([\alpha_1,\alpha_2]+ba)\\
&= se_1\alpha_1 s^{-1}e_2 s\alpha_2 s^{-1} - e_2 s\alpha_2 s^{-1} se_1\alpha_1 s^{-1} + sat^{-1}e_1e_2tbs^{-1}\\
&= se_1\left([\alpha_1,\alpha_2]+ab\right)e_2 s^{-1}
\end{aligned}$$

Where we denote $s = \mathrm{diag}(e^{\sigma_1},\ldots,e^{\sigma_k}) \in T_{U(k)}$, $(e_1,e_2) = (e^{\epsilon_1},e^{\epsilon_2})$ and $t = \mathrm{diag}(e^{\tau_1},\ldots,e^{\tau_n})$. The result of this action on the tangent space of $\mu_{\mathbb{C}}^{-1}(0)$, gives rise to an action with weights $\sigma_i + \epsilon_1 + \epsilon_2 - \sigma_j$. This determines the factors in the numerator. For the denominator, we need to calculate the weights of the action of $T_{U(n)} \times T^2 \times T_{SU(k)}$ on the tangent space of a fixed point. We see that $\prod_j \rho_j$ is the product of the weights of the action on α_1, α_2, a and b, so we get the following expression:

$$\int_{\mathcal{M}_{n,k}} e^{\omega+\mu_T} = \frac{1}{k!}\mathrm{Res}_+^{\sigma_i} \frac{\prod\limits_{e\neq f}(\sigma_e-\sigma_f) \prod\limits_{1\leq g,h\leq k}(\epsilon_1+\epsilon_2+\sigma_g-\sigma_h)}{\prod\limits_{\substack{1\leq i,j\leq k\\ l\in\{1,2\}}}(\epsilon_l+\sigma_i-\sigma_j) \prod\limits_{\substack{1\leq m\leq k\\ 1\leq o\leq n}}(\sigma_m-\tau_o) \prod\limits_{\substack{1\leq p\leq k\\ 1\leq q\leq n}}(\epsilon_1+\epsilon_2-\sigma_p+\tau_q)} \tag{3.2.17}$$

3.3 From instanton counting to Seiberg-Witten theory

3.3.1 The free energy

While equation 3.2.17 already is a big step forward in the calculation of instanton corrections of $\mathcal{N} = 2$ supersymmetric Yang-Mills theory, Nekrasov and collaborators in [NS04] and [NO06] went further, and extracted information on the $\epsilon_1, \epsilon_2 \to 0$ limit of the whole partition function

$$Z_{\text{inst}}(\tau, \epsilon_1, \epsilon_2, \Lambda) = \sum_{k=1}^{\infty} \Lambda^{2nk} \int_{\mathcal{M}_{n,k}} e^{\omega + \mu_T} \tag{3.3.1}$$

We follow [Sha04].

In order to extract this information, we will first do some simple manipulations on equation 3.2.17. We first look at

$$\frac{\prod_{e \neq f} (\sigma_e - \sigma_f) \prod_{1 \leq g,h \leq k} (\epsilon_1 + \epsilon_2 + \sigma_g - \sigma_h)}{\prod_{\substack{1 \leq i,j \leq k \\ l \in \{1,2\}}} (\epsilon_l + \sigma_i - \sigma_j)}$$

We can write this as

$$\frac{(\epsilon_1 + \epsilon_2)^k}{(\epsilon_1 \epsilon_2)^k} \prod_{i \neq j} \frac{(\sigma_i - \sigma_j)(\sigma_i - \sigma_j + \epsilon_1 + \epsilon_2)}{(\sigma_i - \sigma_j + \epsilon_1)(\sigma_i - \sigma_j + \epsilon_2)}$$

by extracting the $i = j$ terms. We see that this is equal to

$$\frac{(\epsilon_1 + \epsilon_2)^k}{(\epsilon_1 \epsilon_2)^k} \prod_{i<j} \frac{(\sigma_i - \sigma_j)^2 \left((\sigma_i - \sigma_j)^2 - (\epsilon_1 + \epsilon_2)^2\right)}{((\sigma_i - \sigma_j)^2 - \epsilon_1^2))\,((\sigma_i - \sigma_j)^2 - \epsilon_2^2))}$$

by multiplying the (i, j) term with the (j, i) terms. We see that we can extract an equal number of -1 terms in the numerator and the denominator.

Now define $\Delta(t) = \prod_{i<j} \left((\sigma_i - \sigma_j)^2 - t^2\right)$. We see that $\Delta'(t)|_{t=0} = 0$, so

$$\frac{d^2}{dt^2} \log \Delta(t)\bigg|_{t=0} = \lim_{\epsilon_1, \epsilon_2 \to 0} \frac{\log(\Delta(\epsilon_1 + \epsilon_2) - \log(\Delta(\epsilon_1)) - \log(\Delta(\epsilon_2)) + \log(\Delta(0))}{\epsilon_1 \epsilon_2}$$

and we can write

$$\frac{\Delta(0)\Delta(\epsilon_1 + \epsilon_2)}{\Delta(\epsilon_1)\Delta(\epsilon_2)} = \exp\left(\epsilon_1 \epsilon_2 \frac{d^2}{dt^2} \log \Delta(t)\bigg|_{t=0} + \text{higher order terms in } \epsilon\right)$$

Now we want to convert the other part of equation 3.2.17 into a similar expression. Note that $\prod(\sigma_m - \tau_o) \prod(\epsilon_1 + \epsilon_2 - \sigma_p + \tau_q)$ can be written as

$$\exp\left(\sum \log(\sigma_m - \tau_o) + \sum \log(\epsilon_1 + \epsilon_2 - \sigma_p + \tau_q)\right)$$

Again, we are only interested in the limit $\epsilon_1, \epsilon_2 \to 0$ and the term $\sum \log(\epsilon_1 + \epsilon_2 - \sigma_p + \tau_q)$ can be written as a series in $\epsilon_1 + \epsilon_2$, of which then only the first term $\sum \log(\sigma_p + \tau_q)$ survives. Together with the Λ^{2n}, we get thus a term

$$2 \sum_{i=1}^{k} \log\left(\frac{\prod_{j=1}^{n} (\sigma_i - \tau_n)}{\Lambda^N}\right)$$

in the exponent. We expect that in the limit $\epsilon_1, \epsilon_2 \to 0$, the term $k \sim \frac{1}{\epsilon_1\epsilon_2}$ gives the main contribution. In the limit $k \to \infty$, the number of σ_i becomes infinite, and we can introduce the density

$$\rho_k(x) = \epsilon_1\epsilon_2 \sum_{i=1}^{k} \delta(x - \sigma_i)$$

with $\delta(x)$ the Dirac distribution. This is k dependent, when $k = \frac{c}{\epsilon_1\epsilon_2} \to \infty$ with c some constant, this becomes a smooth function, still normalized due to the $\epsilon_1\epsilon_2$ term. We can then write

$$Z^k_{\text{inst}}(\tau, \epsilon_1, \epsilon_2, \Lambda) = \exp\left(\frac{-1}{\epsilon_1\epsilon_2} E_\Lambda[\rho]\right)$$

with

$$E_\Lambda[\rho] = \int_{x\neq y} dxdy \frac{\rho(x)\rho(y)}{(x-y)^2} - 2\int dx\rho(x)\log\left(\frac{\prod_{j=1}^{n}(\phi_i - \tau_j)}{\Lambda^n}\right)$$

Since we are interested in the limit $\epsilon_1, \epsilon_2 \to 0$, we see that we must find the $\rho(x)$ such that $E_\Lambda[\rho]$ is minimal.

Nekrasov then argues that in order to get a physical system, and to reproduce the results of 3.1.4, we need to add a term independent of ρ, that diverges logarithmically with Λ, which has an interpretation from physics as the perturbative (in contrast to the instanton) contribution to the prepotential

$$\frac{1}{2}\sum_{k\neq l}(\tau_k - \tau_l)^2 \log\left(\left(\frac{|\tau_l - \tau_k|}{\Lambda}\right) - \frac{3}{2}\right) \tag{3.3.2}$$

This gives the free energy

$$E_\Lambda[\rho] = -\frac{1}{2}\sum_{k\neq l}(\tau_k - \tau_l)^2 \log\left(\left(\frac{|\tau_l - \tau_k|}{\Lambda}\right) - \frac{3}{2}\right)$$
$$+ \int_{x\neq y} dxdy \frac{\rho(x)\rho(y)}{(x-y)^2} - 2\int dx\rho(x)\log\left(\frac{\prod_{j=1}^{n}(z - \tau_j)}{\Lambda^n}\right) \tag{3.3.3}$$

which we wish to minimize.

3.3.2 Profile function

We wish to include equation 3.3.2 into our expression for Z. In order to do this, we introduce the *profile function* $f(x)$ and derive some of its properties. The name profile function comes from the derivation in [NO06], where the profile function is first introduced as the profile of a random partition.

Definition 3.3.1. *The* profile function *is the following function*

$$f(x) = \sum_{l=1}^{n} |x - \tau_l| - 2\rho(x)$$

where $\rho(x)$ is the density of eigenvalues defined above.

We derive some properties of this profile function. Note that the $\rho(x)$ have compact support, which can be chosen to be n disjoint intervals $[\alpha_l^-, \alpha_l^+]$ such that $\tau_l \in [\alpha_l^-, \alpha_l^+]$. Also $f(x)$ behaves like $n|x|$ for $x \to \pm\infty$. The second derivative $f''(x)$ also has support inside $[\alpha_l^-, \alpha_l^+]$. We have

$$\int_{\alpha_l^-}^{\alpha_l^+} f''(x)dx = 2\int_{\alpha_l^-}^{\alpha_l^+} \left(\delta(x-\tau_l) - \rho''(x)\right) dx = 2 \tag{3.3.4}$$

since $\rho'(x) = 0$ on the boundaries α_l^- and α_l^+. From this it follows

$$\int_{\mathbb{R}} f''(x)dx = 2n \tag{3.3.5}$$

Also

$$\int_{\alpha_l^-}^{\alpha_l^+} xf''(x)dx = 2\int_{\alpha_l^-}^{\alpha_l^+} x\left(\delta(x-\tau_l) - \rho''(x)\right) = 2\tau_l - 2\left(x\rho'(x) - \rho(x)\right)\Big|_{\alpha_l^-}^{\alpha_l^+} = 2\tau_l$$

and since $\sum_{l=1}^n \tau_l = 0$ because the τ_l are the weights of a torus in $SU(n)$ we have

$$\int_{\mathbb{R}} xf''(x)dx = 0 \tag{3.3.6}$$

Finally we have

$$\begin{aligned}\int_{\mathbb{R}} x^2 f''(x)dx &= 2\int_{\mathbb{R}} x^2\left(\delta(x-\tau_l) - \rho''(x)\right) dx \\ &= 2\sum_l \tau_l^2 - 4\int_{\mathbb{R}} \rho(x)dx = 2\sum_l \tau_l^2 - 4\epsilon_1\epsilon_2 k \end{aligned} \tag{3.3.7}$$

3.3.3 The minimizer

With this profile function, we can write the free energy 3.3.3 as

$$E_\Lambda[f] = -\frac{1}{4}\int f''(x)f''(y)\frac{1}{2}\sum_{k\neq l}(x-y)^2 \log\left(\left(\frac{|x-y|}{\Lambda}\right) - \frac{3}{2}\right) dxdy \tag{3.3.8}$$

In order to study the minimum of this functional, we first introduce Lagrange multipliers $\xi_1, \dots, \xi_n$:

$$\begin{aligned} L[f,\xi] &= E_\Lambda[f] + \sum_{l=1}^n \xi_n \left(\frac{1}{2}\int_{\alpha_l^-}^{\alpha_l^+} xf''(x)dx - \tau_l\right) \\ &= S[f,\xi] - \sum_{l=1}^n \xi_l\tau_l \end{aligned} \tag{3.3.9}$$

with

$$S[f,\xi] = E_\Lambda[f] + \sum_{l=1}^n \xi_n \frac{1}{2}\int_{\alpha_l^-}^{\alpha_l^+} xf''(x)dx \tag{3.3.10}$$

Now if we have found the minimizer f_* we should now also check that this is a stationary point with respect to the ξ_l:

$$\left.\frac{\partial S[f_*, \xi]}{\partial \xi_l}\right|_{\text{keep } f_* \text{ constant}} = \tau_l$$

We can neglect the ξ_l dependence of f_* since it is a minimizer of $S[f_*, \xi_l]$ hence the derivative of S is zero with respect to f at f_*. This gives the ξ_l as functions of τ_l. In order to finally recover the prepotential we should plug back these functions in equation 3.3.9.

We note an apparent problem: the last term in 3.3.10 requires information on the support of f_* which is needed to solve the equation. However, we know that, due to equation 3.3.4

$$f'(\alpha_l^+) - f'(\alpha_l^-) = \int_{\alpha_l^-}^{\alpha_l^+} f''(x)dx = 2$$

This means we can introduce a piecewise linear function $\sigma(t)$ such that $\sigma'(t) = \xi_l$ when $t = f'(x)$ with $x \in [\alpha_l^-, \alpha_l^+]$, or equivalently $t \in (-n + 2(l-1), -n + 2l)$. Then the last term in equation 3.3.10 becomes

$$-\frac{1}{2}\sigma(f'(x))dx \tag{3.3.11}$$

Now we are ready to implement the Euler-Lagrange program. The Euler-Lagrange equation for the functional 3.3.10 is

$$2\frac{\delta S[f,\xi]}{\delta f'(x)} = \int dy f''(y)(y-x)\left(\log\left(\frac{|x-y|}{\Lambda}\right) - 1\right) - \sigma'(f'(x)) = 0 \tag{3.3.12}$$

when $x \in [\alpha_l^-, \alpha_l^+]$ we know that $\sigma'(f'(x)) = \xi_l$. Outside this interval, we do not know what the value of σ' is, except that it must lie between ξ_l and ξ_{l+1} if x between α_l^+ and α_{l+1}^-. Now taking the derivative with respect to x we get

$$\int dy f''(y) \log\left|\frac{x-y}{\Lambda}\right| = 0 \quad \text{when } x \in [\alpha_l^-, \alpha_l^+] \tag{3.3.13}$$

Define

$$F(z) = \frac{1}{4\pi i}\int_{\mathbb{R}} dy f''(y) \log\left(\frac{z-y}{\Lambda}\right)$$

Then we have the equation

$$F(x) = \phi(x)$$

where the complex map $\phi(x)$ is holomorphic, and maps the upper half-plane to the domain $[0, n/2] \times i\mathbb{R}_+$. Now suppose $|\tau_l - \tau_m| \gg \Lambda$ if $l \neq m$, then the map $\phi(z)$ can be reconstructed by using that $\phi(z) = 0$ if $z \in [\alpha_l^-, \alpha_l^+]$:

$$\phi(z) = \frac{1}{2\pi}\arccos\frac{P(z)}{2\Lambda^n}$$

with $P(x) = \prod_{l=1}^n (x - \tau_l)$. This reproduces the Seiberg-Witten theory on Riemann surfaces as described in section 3.1.4.

Chapter 4

Conclusions and outlook

We have seen that one the one hand, the Donaldson invariants are related to the Seiberg-Witten invariants by the conjecture of Witten 3.1.9 which he argued by using indirect properties of the prepotential. On the other hand, Nekrasov et. al. established a procedure to calculate all terms of the prepotential. This is a huge breakthrough in the field of gauge theories, since it establishes the validity of the approach of Seiberg and Witten with mathematical sound arguments. This also could shed new light on the analysis of integrable systems based on Seiberg-Witten theory, especially on the geometric side.

On the other hand, the equivariant localization approach has some limitations. There is still no direct calculation of either the Donaldson or Seiberg-Witten invariants from the prepotential, leaving all the arguments of their equivalence indirect. Some progress has been made in [GNY06], where the higher order terms of the $\lim_{\epsilon_1,\epsilon_2\to 0}$ of the equivariant prepotential were related to the wall-crossing terms of manifolds with $b_2^+ = 1$. Also, the use of the ADHM construction of the instantons seems limited to hyperkähler manifolds, and the extra torus action needed for the equivariant localization further restricts the class of manifolds for which this approach might work. Maybe an infinite dimensional generalization can be found, which allows us to utilize an equivariant volume directly from the total space of connections.

The equivariant localization approach also lends itself to generalization, from gauge group $SU(2)$ to higher rank bundles. In [Kro05] the corresponding Donaldson invariants were introduced. It seems worthwhile to further investigate the relation with the Nekrasov approach. Finally, another generalization comes to mind, one where not only the classical groups $SU(n)$, $SO(n)$ and $Sp(n)$ are consider but also the gauge theory with the exceptional groups E_6, E_7, E_8, F_4 and G_2. This might give some clues on how a possible classification of simply connected smooth 4 dimensional manifolds up to diffeomorphisms might look like.

Bibliography

[AHDM78] M. F. Atiyah, N. J. Hitchin, V. G. Drinfel′d, and Yu. I. Manin. Construction of instantons. *Phys. Lett. A*, 65(3):185–187, 1978.

[Aro56] Nachman Aronszajn. *A unique continuation theorem for solutions of elliptic partial differential equations or inequalities of second order*, volume 16 of *Studies in eigenvlaue problems*. Lawrence, Kansas : Department of mathematics of the Univ. of Kansas, 1956.

[Aub98] Thierry Aubin. *Some nonlinear problems in Riemannian geometry*. Springer monographs in mathematics. Springer, 1998.

[BE06] A. Braverman and P. Etingof. Instanton counting via affine Lie algebras. II. From Whittaker vectors to the Seiberg-Witten prepotential. In *Studies in Lie theory*, volume 243 of *Progr. Math.*, pages 61–78. Birkhäuser Boston, Boston, MA, 2006.

[Bra04] Alexander Braverman. Instanton counting via affine Lie algebras. I. Equivariant J-functions of (affine) flag manifolds and Whittaker vectors. In *Algebraic structures and moduli spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 113–132. Amer. Math. Soc., Providence, RI, 2004.

[BV82] Nicole Berline and Michèle Vergne. Classes caractéristiques équivariantes. Formule de localisation en cohomologie équivariante. *C. R. Acad. Sci. Paris Sér. I Math.*, 295(9):539–541, 1982.

[Car51] Henri Cartan. La transgression dans un groupe de Lie et dans un espace fibré principal. In *Colloque de topologie (espaces fibrés), Bruxelles, 1950*, pages 57–71. Georges Thone, Liège, 1951.

[DH82] J. J. Duistermaat and G. J. Heckman. On the variation in the cohomology of the symplectic form of the reduced phase space. *Invent. Math.*, 69(2):259–268, 1982.

[DK90] S.K. Donaldson and P.B. Kronheimer. *The Geometry of Four-Manifolds*. Oxford Mathematical Monographs. Oxford University Press, 1990.

[Don83] S. K. Donaldson. An application of gauge theory to four-dimensional topology. *J. Differential Geom.*, 18(2):279–315, 1983.

[FL98] Paul M. N. Feehan and Thomas G. Leness. PU(2) monopoles. I. Regularity, Uhlenbeck compactness, and transversality. *J. Differential Geom.*, 49(2):265–410, 1998.

[FL01] Paul M. N. Feehan and Thomas G. Leness. PU(2) monopoles and links of top-level Seiberg-Witten moduli spaces. *J. Reine Angew. Math.*, 538:57–133, 2001.

[FM94] R. Friedman and J. Morgan. *Smooth Four-Manifolds and Complex Surfaces*. Ergebnisse der Mathematik und ihrer Grensgebiete. Springer-Verlag, New York, 1994.

[Fre99] Daniel S. Freed. Special Kaehler manifolds. *Commun. Math. Phys.*, 203:31–52, 1999.

[GH78] P. Griffiths and J. Harris. *Principles of Algebraic Geometry*. Wiley, 1978.

[GNY06] Lothar Gttsche, Hiraku Nakajima, and Kota Yoshioka. Instanton counting and donaldson invariants. *arXiv.org*, 2006.

[Gol02] R. F. Goldin. An effective algorithm for the cohomology ring of symplectic reductions. *Geom. Funct. Anal.*, 12(3):567–583, 2002.

[GS99] Victor W. Guillemin and Shlomo Sternberg. *Supersymmetry and equivariant de Rham theory*. Springer, 1999.

[HM89] Jr. H. Blaine Lawson and Marie-Louise Michelsohn. *Spin Geometry*, volume 38 of *Princeton Mathematical Series*. Princeton University Press, 1989.

[JK95] L.C. Jeffrey and F.C. Kirwan. Localization for nonabelian group actions. *Topology*, 34:291–327(37), April 1995.

[JK97] Lisa C. Jeffrey and Frances C. Kirwan. Localization and the quantization conjecture. *Topology*, 36(3):647–693, 1997.

[JK05] Lisa C. Jeffrey and Mikhail Kogan. Localization theorems by symplectic cuts. In *The breadth of symplectic and Poisson geometry*, pages 303–326. Birkhauser Boston, 2005.

[Kir84] F. Kirwan. *Cohomology of quotients in symplectic and algebraic geometry*. Princeton University Press, 1984.

[Kro05] P.B. Kronheimer. Four-manifold invariants from higher-rank bundles. *Journal of Differential Geometry*, 70(1):59–112, 2005.

[Lan72] Serge Lang. *Differential manifolds*. Addison-Wesley Publishing Co., Inc., Reading, Mass.-London-Don Mills, Ont., 1972.

[Ler95] Eugene Lerman. Symplectic cuts. *Math. Res. Lett.*, 2(3):247–258, 1995.

[LMTW98] Eugene Lerman, Eckhard Meinrenken, Sue Tolman, and Chris Woodward. Nonabelian convexity by symplectic cuts. *Topology*, 37(2):245–259, 1998.

[Mac91] Antony Maciocia. Metrics on the moduli spaces of instantons over euclidean 4-space. *Communications in Mathematical Physics*, 135(3):467–482, 1991.

[Mar00] Shaun Martin. Symplectic quotients by a nonabelian group and by its maximal torus, 2000.

[Mar08] Johan Martens. Equivariant volumes of non-compact quotients and instanton counting. *Communications in Mathematical Physics*, 281(3):827–857, 2008.

[Mor96] John W. Morgan. *The Seiberg-Witten equations and applications to the topology of smooth four-manifolds*, volume 44 of *Mathematical Notes*. Princeton University Press, 1996.

[Mor98] John W. Morgan. An introduction to gauge theory. In *Gauge theory and the topology of four-manifolds (Park City, UT, 1994)*, volume 4 of *IAS/Park City Math. Ser.*, pages 51–143. Amer. Math. Soc., Providence, RI, 1998.

[MQ86] Varghese Mathai and Daniel Quillen. Superconnections, Thom classes, and equivariant differential forms. *Topology*, 25(1):85–110, 1986.

[Nek03] Nikita A. Nekrasov. Seiberg-Witten prepotential from instanton counting. *Adv. Theor. Math. Phys.*, 7(5):831–864, 2003.

[NO06] Nikita A. Nekrasov and Andrei Okounkov. Seiberg-Witten theory and random partitions. In *The unity of mathematics*, volume 244 of *Progr. Math.*, pages 525–596. Birkhäuser Boston, Boston, MA, 2006.

[NS98] Nikita Nekrasov and Albert Schwarz. Instantons on noncommutative $\mathbb{R}^4$, and $(2,0)$ superconformal six-dimensional theory. *Comm. Math. Phys.*, 198(3):689–703, 1998.

[NS04] Nikita A. Nekrasov and Sergei Shadchin. Abcd of instantons. *Commun. Math. Phys.*, 252:359–391, 2004.

[NY04] H. Nakajima and K. Yoshioka. Lectures on instanton counting. In *Algebraic structures and moduli Spaces*, volume 38 of *CRM Proc. Lecture Notes*, pages 31–101. Amer. Math. Soc., Providence, RI, 2004.

[NY05a] Hiraku Nakajima and Kōta Yoshioka. Instanton counting on blowup. I. 4-dimensional pure gauge theory. *Invent. Math.*, 162(2):313–355, 2005.

[NY05b] Hiraku Nakajima and Kota Yoshioka. Instanton counting on blowup. ii. k-theoretic partition function. *Transformation Groups*, 10:489–519, 2005.

[Osb79] H. Osborn. Calculation of multi-instanton determinants. *Nuclear Physics B*, 159(3):497–511, 1979.

[OSS80] C. Okonek, M. Schneider, and H. Spindler. *Vector bundles on complex projective spaces*, volume 3 of *Progress in Mathematics*. Birkhäuser Boston, 1980.

[PSS05] Jongil Park, András I. Stipsicz, and Zoltán Szabó. Exotic smooth structures on $\mathbb{CP}^2 \# 5\overline{\mathbb{CP}^2}$. *Math. Res. Lett.*, 12(5-6):701–712, 2005.

[PW94] Elisa Prato and Siye Wu. Duistermaat-Heckman measures in a non-compact setting. *Compositio Math.*, 94(2):113–128, 1994.

[Rud86] Walter Rudin. *Functional analysis*. Tata McGraw-Hill, 1986.

[Sch00] Martin Schechter. *Principles of Functional Analysis*, volume 36 of *Graduate Studies in Mathematics*. American Mathematical Society, 2nd edition, 2000.

[Sha04] Sergey Shadchin. Saddle point equations in seiberg-witten theory. *JHEP*, 0410:033, 2004.

[SW94] Nathan Seiberg and Edward Witten. Monopole condensation, and confinement in n=2 supersymmetric yang-mills theory. *Nucl. Phys. B*, 426:19–52, 1994.

[Tau83] C. Taubes. Self-dual connection on 4-manifolds with indefinite intersection matrix. *Journal of Differential Geometry*, 19(2):517–560, 1983.

[Wit88] Edward Witten. Topological quantum field theory. *Comm. Math. Phys.*, 117(3):353–386, 1988.

[Wit94] Edward Witten. Monopoles and four-manifolds. *Math. Res. Lett.*, 1(6):769–796, 1994.