Location of the spectrum of operator matrices which are associated to second and higher order equations

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Contents

Abstract

This thesis is based on the paper "Location of the spectrum of operator matrices which are associated to second order equations" by Birgit Jacob and Carsten Trunk.[1]
In this thesis we consider second order equations of the form $\ddot{z}(t)+D\dot{z}(t)+A_0z(t) = 0$, where z is a function ranging in the infinite dimensional Hilbert space H, A_0 is a self adjoint, positive definite linear operator which is possibly unbounded, and D is an unbounded operator on H into H but bounded on $H_{\frac{1}{2}}$ into $H_{-\frac{1}{2}}$. We describe the location of the spectrum and the essential spectrum of the semigroup generator A associated to this equation under various conditions on the damping operator D. In addition we shall extend some results to higher order equations.

[1]Birgit Jacob is associate professor of mathematics at Delt University of Technology and Casten Trunk is associate professor University of Ilmenau.

Introduction

The aim of this master thesis is to study the location of the spectrum of second order equation of the form

$$\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0,$$

which is based on the paper of B.Jacob and C. Trunk. We treat also the higher order equations of the form

$$z^{(n)}(t) + A_{n-1}z^{(n-1)}(t) + \cdots + A_1 z^{(1)}(t) + A_0 z(t) = 0,$$

where $A_{n-1}, \cdots, A_0$ are operators on a Hilbert space.

In Section 1 we introduce the framework. First, we begin with some definitions and properties of functional analysis. Neverthless, is assumed that the reader knows fundamental functional analysis. These notions are briefly: Resolution of identity, spectrum of bounded and unbounded self adjoint operators, scale of Hilbert space, Krein space, and spectrum of a closed linear operators.

Section 2 returns to the main problem. First of all we investigate the location of the spectrum of the second order equation of the form

$$\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0,$$

where $A_0 : H \longrightarrow H$ is possibly an unbounded positive operator on a Hilbert space H which is assumed to be boundedly invertible, and $D : H \longrightarrow H$ the damping operator is an unbounded operator and $D : H_{\frac{1}{2}} \longrightarrow H_{-\frac{1}{2}}$ is bounded such that $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is a bounded self adjoint operator on H. To find the location of the spectrum of A first we prove that this second order o.d.e can be written as a first order equation $\dot{x}(t) = Ax(t)$, where x is a vector ranging in the Hilbert space $H \times H$ and

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix}.$$

This block matrix has been studied in the literature for more than 20 years. Interest in this part of research is motivated by various problems such as stabilization (see [6], [20]). To see the solution of this equation and the relationship of the spectrum of its operator pencil with the spectrum of A when D is a self adjoint operator and A_0 is positive and invertible see [7], [19]. Namely, in this section we prove that the spectrum of A lies in the left half plane and it's symmetric with regard to x-axis in the Krein space $H_{\frac{1}{2}} \times H$.

In Section 3 we explore the location of spectrum of A under some conditions, then we generalize it to fourth order o.d.e. In other research one can see also that if $\beta > 0$, where

$$\beta = \inf_{z \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{||z||_H^2}$$

then A generates an exponentially stable semigroup on $H_{\frac{1}{2}} \times H$, (see, [4], [15]). In particular it's shown that there exists a constant $\omega < 0$ such that $\sigma(A) \subset \{s \in \mathbb{C} | Res \leq \omega\}$. For the constant ω there are quite a few upper bounds available. For example in [8] is shown that

$$\omega \leq max\left\{-\frac{\beta}{2}, max\left\{Res|s \in \sigma(A)\right\}\right\},$$

and in [5] is shown that

$$\omega \leq max\left\{-\frac{\beta}{2}, -||A^{-1}||^{-1}\right\}.$$

We improve these estimates in Theorem 3.2.

Section 4 examines the location of the essential spectrum of A, where the operator $A_0^{-1} : H \longrightarrow H$ is a compact and not compact. Moreover, we extend some result to higher order equations.

Finally, in Section 5 we determine intervals of the real axis which do not contain accumulation points of the non-trivial spectrum. In particular we show that if A_0 has a compact resolvant, then the spectrum of A cannot accumulate to the real axis. We then consider a special case in which the spectrum of A has no accumulation point in $\mathbb{C}$.

Finally, I desired to make this thesis as accessible as possible. I have also attempted to make the proofs of all theorems as detailed as possible and generalized from second order o.d.e to higher order one. It must be also emphasized that in no case does the absence of a reference imply any claim to originality on my part.

1 Preliminary results

The validity of many important facts of the following sections depend on Section 1. In order to emphasize the role played by the following section, some facts of this section are stated in a little more generality than is usually needed.

In the following we assume that H is a Hilbert space and T is a linear operator.
First of all some definitions.

Definition 1.1 *Assume $T : H \longrightarrow H$ bounded. We say that T is self adjoint if $T = T^*$ where T^* is the adjoint of T.* (See, [12, chapter II.12]).

Definition 1.2 *Assume $T : \mathcal{D}(T) \subseteq H \longrightarrow H$ and $\mathcal{D}(T)$ is dense in H, i.e; $\overline{\mathcal{D}(T)} = H$. We say that T is self adjoint if $T = T^*$.* (See, [12, chapter VI.4]).

Remark 1.1 *The density of $\mathcal{D}(T)$ in the above definition is necessary to have a unique adjoint operator for T.* (See, [12, chapter VI.4]).

Definition 1.3 *Assume $T : \mathcal{D}(T) \subseteq H \longrightarrow H$ is a bounded or unbounded operator. We say that T is positive; resp. positive definite on H if $\langle Tx, x\rangle \geq 0$ resp. $\langle Tx, x\rangle > 0$ for each $x \in \mathcal{D}(T)$ and $x \in \mathcal{D}(T) \setminus \{0\}$, respectively.* (See, [12, chapter IV.9]).

Our aim is now to introduce the scale of Hilbert space. First, we have to give the spectral representation for self adjoint operators.

1.1 Spectrum of bounded self adjoint operators

We have seen in functional analysis that the entire spectrum of a self adjoint linear operator T is contained in $[-||T||, ||T||\ ]$. We would like a tighter bound on the spectrum. For this, we have seen before that the spectrum is contained in $[m, M]$, i.e; $\sigma(T) \subseteq [m, M]$, where $m = \inf_{||x||=1} \langle Tx, x\rangle$, and $M = \sup_{||x||=1} \langle Tx, x\rangle$. (See, [3, chapter 25.2]).
Also by a result in functional analysis we know that m and M belong to $\sigma(T)$, which means that the bound $[m, M]$ for $\sigma(T)$ can not be tightened. (See, [3, chapter 25.2]).

1.2 Resolution of identity

We recall that our present aim is a representation of bounded self adjoint linear operators on Hilbert space in terms of very simple operators (projections) whose properties can be readily investigated in order to obtain information about those more complicated operators. Such a representation will be called a *spectral representation* of the operator concerned. Suppose a bounded self adjoint operator $T : H \longrightarrow H$ is given. We are going to give a motivation and definition of the concept of *resolution of identity.* This motivation of spectral family can be obtained from the finite dimensional case as follows. Let $T : H \longrightarrow H$ be a self adjoint linear operator on the space $H = \mathbb{C}^n$, then T is a bounded operator, and we can choose a basis for H and represent T by a Hermitian matrix. In this case the spectrum of the operator consist of the eigenvalues of that matrix, which are real. For simplicity, let us assume that the matrix T has n different eigenvalues, $\lambda_1 < \lambda_2 < \cdots < \lambda_n$, then T has an orthonormal set of n eigenvectors $x_1, x_2, \cdots, x_n$, where x_j corresponds to λ_j. As we know the set of all x is a basis for H, hence for each $x \in H$ we can write

$$x = \sum_{j=1}^{n} \langle x, x_j \rangle \cdot x_j. \tag{1}$$

The essential fact in (1) is that x_j is an eigenvector of T, so that we have $Tx_j = \lambda_j x_j$. If we now apply T to (1) we simply obtain,

$$Tx = \sum_{j=1}^{n} \lambda_j \langle x, x_j \rangle \cdot x_j. \tag{2}$$

Thus, whereas T acts on x in a complicated way, it acts on each term of the sum in (1) in a very simple fashion.
Looking at (1) more closely, we see that we can define an operator $P_j : H \longrightarrow H$ by $P_j x = \langle x, x_j \rangle x_j$. Obviously, P_j is an orthogonal projection of H on to the eigenspace of T corresponding to λ_j.

Formula (1) can now be rewritten by,

$$x = \sum_{j=1}^{n} P_j x \qquad \text{hence} \qquad I = \sum_{j=1}^{n} P_j, \tag{3}$$

where I is the identity operator on H. Formula (2) becomes,

$$Tx = \sum_{j=1}^{n} \lambda_j P_j x \qquad \text{hence} \qquad T = \sum_{j=1}^{n} \lambda_j P_j. \tag{4}$$

This is a representation of T in terms of projections. It follows that the spectrum of T is employed to get a representation of T in terms of very simple operators.
The use of the projections P_j seems quite natural and easy. Unfortunately, our present formulas would not be suitable for immediate generalization to infinite dimensional Hilbert spaces, since in that case spectra of bounded self adjoint linear operators may be more complicated.

We have now enough motivation to give the definition of resolution of identity for our purpose, that is, to represent T in an infinite dimensional Hilbert space as in (4), when T is a self adjoint operator.

Definition 1.4 *A real resolution of identity (or real spectral family, or real decomposition of unity) is a one-parameter family $(E_\lambda)_{\lambda\in\mathbb{R}}$ of orthogonal projections E_λ defined on a Hilbert space H (of any dimension) which depends on a real parameter λ and is such that,* (see, [18, chapter 9.7])

$$\text{for} \quad \lambda_1 \leq \lambda_2, \qquad E_{\lambda_1}E_{\lambda_2} = E_{\lambda_2}E_{\lambda_1},$$
$$lim_{\lambda\to-\infty}E_\lambda x = 0,$$
$$lim_{\lambda\to+\infty}E_\lambda x = x,$$
$$lim_{\mu\downarrow\lambda}E_\mu x = E_\lambda x.$$

We see from this definition that a real resolution of identity can be regarded as a mapping, $\mathbb{R} \longrightarrow B(H,H)$, such that $\lambda \longmapsto E_\lambda$, that is, to each $\lambda \in \mathbb{R}$ there corresponds a projection $E_\lambda \in B(H)$. (See, [18, chapter 9.7]).

In particular $(E_\lambda)_{\lambda\in\mathbb{R}}$ is called a resolution of identity on an interval $[a,b]$ if, (see, [18, chapter 9.7])

$$E_\lambda = 0 \quad \text{for} \quad \lambda < a, \qquad E_\lambda = I \quad \text{for} \quad \lambda \geq b.$$

Such families will be of particular interest to us since the spectrum of a bounded self adjoint linear operator lies in a finite interval on the real line.

We have also the following properties for a resolution of identity. (See, [10, chapter 7.1, 7.2]).

1. From the property $E_\lambda \leq E_\mu$ for $\lambda < \mu$ follows that $||E_\lambda x||^2$ is an monotonically increasing function.
2. We have $E_\lambda^2 = E_\lambda$, since E_λ is an orthonormal projection.
3. E_λ is self adjoint.

The families E_λ is called the spectral resolution associated with the operator T on the interval $[a,b]$ if

$$T = \int_a^b \lambda \, dE_\lambda.$$

1.3 Spectral representation of bounded self adjoint operators

Definition 1.5 *Let $T \in B(H)$, and T is self adjoint. Set $M = \sup_{||x||=1}\langle Tx, x\rangle$, and $m = \inf_{||x||=1}\langle Tx, x\rangle$. For $a < m$, $b \geq M$ we define*

$$T^\alpha = \int_a^b \lambda^\alpha \, dE_\lambda \quad \text{for} \quad \alpha \geq 0,$$

where $(E_\lambda)_\lambda$ is the resolution of identity associated with T. (See, [18, chapter 9.9]).

1.4 Spectral representation of unbounded self adjoint operators

Before we give an equivalent definition to the Definition 1.5 for unbounded self adjoint operators we would like first to adjust the definitions of M and m for the unbounded case. Indeed,

$$m = \inf_{||x||=1} \langle Tx, x\rangle \quad \text{and} \quad M = \sup_{||x||=1} \langle Tx, x\rangle\,,$$

and we have to realize that we may have $m = -\infty$ and $M = +\infty$, or both. In other words, since T is not bounded we cannot say that its spectrum is contained in some closed bounded interval.
The Definition 1.5 is very useful for bounded self adjoint operators. The analogue for the unbounded case reads as follows.

Definition 1.6 *Suppose* $T : \mathcal{D}(T) \subseteq H \longrightarrow H$ *unbounded and self adjoint, then we define* T^α *for* $\alpha \geq 0$ *by* (see, [3, chapter 29.2])

$$T^\alpha = \int_{-\infty}^{+\infty} \lambda^\alpha \, dE_\lambda,$$

where

$$\mathcal{D}(T^\alpha) = \left\{ x \in H | \int_{-\infty}^{+\infty} \lambda^{2\alpha} \, d||E_\lambda x||^2 < \infty \right\},$$

and where $(E_\lambda)_\lambda$ is the resolution of the identity associated with T.

1.5 Spectral representation of bounded and unbounded positive self adjoint operators

We know from functional analysis, that if T is a bounded or unbounded postive (or positive definite) self adjoint operator then $\sigma(T) \subseteq [0, \infty)$. (See, [22, chapter 12.32, chapter 13.24]). This leads us to formulate the Definitions 1.5 and 1.6 for bounded and unbounded operators. If T is a bounded positive self adjoint operator, for $b \geq M$ we define (see, [3, chapter 25.1], [23, chapter 13.4, 13.5])

$$T^\alpha = \int_0^b \lambda^\alpha \, dE_\lambda,$$

where $\alpha > 0$.

If T is an unbounded self adjoint positive operator, for $\alpha \geq 0$ we define (see, [3, chapter 29.2], [23, chapter 13.5, 13.6])

$$T^\alpha = \int_0^{+\infty} \lambda^\alpha \, dE_\lambda,$$

where

$$\mathcal{D}(T^\alpha) = \left\{ x \in H | \int_0^{+\infty} \lambda^{2\alpha} \, d||E_\lambda x||^2 < \infty \right\}.$$

1.6 Scale of Hilbert space

Throughout this work we assume that $A_0 : \mathcal{D}(A_0) \subseteq H \longrightarrow H$ is a self adjoint, positive definite linear operator such that $0 \in \rho(A_0)$, that is, A_0^{-1} exists and is bounded.

We consider the case when A_0 is an unbounded operator. We claim that $\mathcal{D}(A_0^\alpha)$ is a linear subspace of H. Since $\mathcal{D}(A_0^\alpha) \subset H$ we have to prove only that $\forall x, y \in \mathcal{D}(A_0^\alpha)$ and $\forall c \in \mathbb{C}$ (or $\mathbb{R}$) we have the following.

1. $x + y \in\mathcal{D}(A_0^\alpha)$,
2. $cx \in\mathcal{D}(A_0^\alpha)$.

Prove of the first point:
Since $x, y \in \mathcal{D}(A_0^\alpha)$, then

$$\int_0^\infty \lambda^{2\alpha} d||E_\lambda x||^2 < \infty \quad \text{and} \quad \int_0^\infty \lambda^{2\alpha} d||E_\lambda x||^2 < \infty.$$

We will show that

$$\int_0^\infty \lambda^{2\alpha}\, d||E_\lambda(x+y)||^2 < \infty.$$

We write,
$||E_\lambda(x+y)|| = ||E_\lambda x+E_\lambda y|| \leq ||E_\lambda x||+||E_\lambda y||$, hence $||E_\lambda(x+y)||^2 \leq (||E_\lambda x|| + ||E_\lambda y||)^2$.The right hand side of the last inequality can be written as $||E_\lambda x + E_\lambda y||^2 = ||E_\lambda x||^2 + ||E_\lambda y||^2 + 2||E_\lambda x||||E_\lambda y||$. We know also that $2||E_\lambda x||||E_\lambda y|| \leq ||E_\lambda x||^2 + ||E_\lambda y||^2$, hence

$$||E_\lambda(x+y)||^2 \leq 2||E_\lambda x||^2 + 2||E_\lambda y||^2. \tag{5}$$

From the inequality (5) we conclude that

$$\int_0^\infty \lambda^{2\alpha}\, d||E_\lambda(x+y)||^2 \leq 2\int_0^\infty \lambda^{2\alpha}\, d||E_\lambda x||^2 + 2\int_0^\infty \lambda^{2\alpha}\, d||E_\lambda y||^2 < \infty.$$

Hence we obtained what we wished. Now,

$$\int_0^\infty \lambda^{2\alpha}\, d||E_\lambda cx||^2 = c\int_0^\infty \lambda^{2\alpha}\, d||E_\lambda x||^2 < \infty.$$

So the second part is also satisfied.
For convenience first we set $H_\alpha = \mathcal{D}(A_0^\alpha)$ and then we define the inner product on each H_α by

$$\langle x, y \rangle_{H_\alpha} = \langle A_0^\alpha x, A_0^\alpha y \rangle_H \,, \qquad \forall x, y \in \mathcal{D}(A_0^\alpha) \tag{6}$$

and the norm

$$||x||_{H_\alpha} = ||A_0^\alpha x||_H.$$

To see that (6) indeed defines an inner product we only have to prove that $\langle x, x \rangle_{H_\alpha} = 0$ implies $x = 0$. This is an easy consequence of injectivity of A_0^α. Indeed, if $\alpha = 0$, then we have the identity operator and this is clearly injective.

To prove that A_0^α is injective for $\alpha > 0$, we use the following property. $Kern(A_0^\alpha) = Kern(A_0)$, thus $kern(A_0^\alpha) = 0$. We point out that in the last part of the proof we have used the following theorem, which says,
If $\varphi : (0, \infty) \to \mathbb{R}$ and $\varphi > 0$ a.e then $kern\varphi(A_0) = kern(A_0)$.

We remark that one could prove this directly for each $\alpha > 0$ saying that

$$0 = ||A_0^\alpha x||^2 = \int_0^\infty \lambda^{2\alpha} d \langle E_\lambda x, x \rangle = \int_0^\infty \lambda^{2\alpha} ||E_\lambda x||^2$$

and concluding that $||E_\lambda x|| = 0$ for almost all λ. From this $E_\lambda x = 0$ a.e (read almost every where). But, $||E_\lambda x||$ is monotonically increasing function, hence $E_\lambda x = 0$ for each $x \in kern(A_0)$, thus $x = 0$.

We have to mention too, that in the above proofs we have used the equality $\langle E_\lambda x, x \rangle = ||E_\lambda x||^2$. This is true, since E_λ is a self adjoint projection, saying that, $||E_\lambda x||^2 = \langle E_\lambda x, E_\lambda x \rangle = \langle E_\lambda^2 x, x \rangle = \langle E_\lambda x, x \rangle$.

Now we do what we had promised, that is, if $\langle x, x \rangle_{H_\beta} = 0$ then $x = 0$. Since $\langle x, x \rangle_{H_\beta} = \left\langle A_0^{\frac{\beta}{2}}, A_0^{\frac{\beta}{2}} \right\rangle$, then $||A_0^{\frac{\beta}{2}}||_H = 0$ and since $||.||_H$ is a norm on H, then we get, $A_0^{\frac{\beta}{2}} x = 0$. But by the above argument A_0^α is injective for $\alpha \geq 0$, hence $x = 0$.

So, we have an inner product, moreover a norm on H_α for $\alpha \geq 0$. To see that H_α is complete with this norm we argue as follows:
Suppose $\{x_n\}_{n \in \mathbb{N}} \subset H_\alpha$ is a Cauchy sequence for a fixed $\alpha > 0$. We first claim that $\{A_0^\alpha x_n\}$ is a Cauchy sequence in H. We have in fact,

$$||A_0^\alpha x_n - A_0^\alpha x_m||_H = ||x_n - x_m||_{H_\alpha} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Thus $\{A_0^\alpha x_n\}$ is a Cauchy sequence in H. Since H is Hilbert space, it is complete, hence there exists a $y \in H$ such that $A_0^\alpha x_n \longrightarrow y$. Since $A_0^{-\alpha}$ is bounded we have $x_n \longrightarrow A_0^{-\alpha} y$. We have also $A_0^{-\alpha} y \in H_\alpha$, writing $x = A_0^{-\alpha} y$ we have $y = A_0 x$. By these we obtain $A_0^\alpha x_n \longrightarrow A_0^\alpha x$. Again,

$$||x_n - x||_{H_\alpha} = ||A_0^\alpha x_n - A_0^\alpha x||_H = ||A_0^\alpha x_n - y||_H \longrightarrow 0, \text{ as } \; n \longrightarrow \infty.$$

Thus H_α is a Hilbert space for $\alpha > 0$. Since A_0^α are injective for $\alpha \geq 0$, the operators $A_0^\alpha : H_\alpha \to Im(A_0^\alpha) \subset H$ are invertible and we denote the inverse by $A_0^{-\alpha} : Im(A_0^\alpha) \subset H \to H_\alpha$. We use this to make a norm on $H_{-\alpha}$. This is a space which we need in the future.

We are now going to investigate what the relation is between H_α, H, when $\alpha \geq 0$. We claim that if $0 \leq \alpha \leq \beta$, then $H_\beta \subseteq H_\alpha$. Suppose $x \in H_\beta$, hence

$$\int_0^{+\infty} \lambda^{2\beta}\, d\left\langle E_\lambda x, x\right\rangle < \infty.$$

Also for $\lambda \geq 1$ we have $\lambda^\alpha < \lambda^\beta$, if $0 \leq \alpha \leq \beta$. We can also write,

$$\int_0^{+\infty} \lambda^{2\alpha}\, d\left\langle E_\lambda x, x\right\rangle = \int_0^{1} \lambda^{2\alpha} d\left\langle E_\lambda x, x\right\rangle + \int_1^{\infty} \lambda^{2\alpha} d\left\langle E_\lambda x, x\right\rangle. \tag{7}$$

The first integral in the right hand side of (7) is bounded.

The second integral is also bounded, since for $0 \leq \alpha < \beta$ and $\lambda > 1$ we have $\lambda^{2\alpha} < \lambda^{2\beta}$ and we can conclude from this, that,

$$\int_1^{+\infty} \lambda^{2\alpha} d\left\langle E_\lambda x, x\right\rangle \leq \int_1^{+\infty} \lambda^{2\beta} d\left\langle E_\lambda x, x\right\rangle,$$

but by definition of $\mathcal{D}(T^\beta)$, and $\forall x \in H$ we have

$$\int_1^{+\infty} \lambda^{2\beta} d\left\langle E_\lambda x, x\right\rangle < \infty,$$

thus the second integral of (7) is also bounded, consequently,

$$\int_0^{+\infty} \lambda^{2\alpha} d\left\langle E_\lambda x, x\right\rangle < \infty.$$

Thus $x \in H_\alpha$, and we obtain, $H_\beta \subset H_\alpha$.

The situation is a little bit more complicated when $\alpha < 0$. First of all we have to introduce the concept of *rigged Hilbert space.* (See [25]). A rigged Hilbert space is a pair (H, V), with H a Hilbert space and V a dense subspace.

We consider the dual space H^* and V^*. The latter dual is taken with regard to the pivot sapce H. That means that V^* consists of linear functionals of the form $f_y : V \longrightarrow \mathbb{C}$, defined by $f_y(v) = \langle y, v\rangle$, for each $v \in V$, and for a fixed $y \in H$. It is clear that if y varies in H then we get all linear functionals of this form.
We point out here, that since H is a Hilbert space then by Riesz representation theorem $H = H^*$, but since the dual to V is taken with regard to H, we *do not* have $V = V^*$.
We are now going to investigate the relation of H^* and V^* with the knowledge that $V \subset H$. We claim that if $V \subset H$ then $H^* \subset V^*$. To do this we take an element in H^*, i.e; f. By the Riesz representation theorem there exists a unique $y \in H$ such that for each $x \in H$ we have,

$$f(x) = \langle x, y\rangle_H. \tag{8}$$

To show specificly that the linear functional f is related to the fixed y in H we denote it by f_y. Since eq.(8) is valid for each $x \in H$, hence it is valid $\forall v \in V$, because $v \in V \subset H$. Hence we can write eq.(8) as $f_y(v) = \langle v, y \rangle_H$. Hence by definition of V^* we have $f = f_y \in V^*$. Thus we have proved that, $H^* \subset V^*$.
Hence we have obtained,

$$V \subset H = H^* \subset V^*. \tag{9}$$

Analogous to the above proof, we can show that if U and V are subspaces of Hilbert space H and $U \subset V$, then $V^* \subset U^*$.

It's now time to apply the formula (9) for H_α for $\alpha \geq 0$. So $H_\alpha \subset H = H^* \subset (H_\alpha)^*$. To make a better and a convenient notation we write $(H_\alpha)^* = H_\alpha^*$, hence (9) can be written as

$$H_\alpha \subset H = H^* \subset H_\alpha^*. \tag{10}$$

But this leads to the question: What is H_α^* equal to? For this we give the following definition.

Definition 1.7 *Let $\alpha \geq 0$, and let H_α have the same meaning as before. Then we define $H_\alpha^* = H_{-\alpha}$.*

By Definition 1.7, we write (9) as follows.

$$H_\alpha \subset H = H^* \subset H_{-\alpha} \quad \text{for} \quad \alpha \geq 0. \tag{11}$$

We have just proved that for $0 \leq \alpha < \beta$ we have $H_\beta \subset H_\alpha$. The latter helps us to prove that also for $\alpha < \beta < 0$ we have $H_\beta \subset H_\alpha$. This can be shown by the following.
Since $\alpha < \beta < 0$, thus $0 < -\beta < -\alpha$. Hence $H_{-\alpha} \subset H_{-\beta}$. Hence by the argument in the last paragraph we will have, $(H_{-\beta})^* \subset (H_{-\alpha})^*$ or $H_\beta \subset H_\alpha$ by Definition 1.7.
In our work we use also the operator $A_0 : H_\alpha \to H_{\alpha-1}$. No confusing! This operator is the restriction and extension of the original operator A_0, but we use the same notation. The operator $A_0 : H_\alpha \to H_{\alpha-1}$ maps H_α isometricly on $H_{\alpha-1}$. Moreover A_0^{-1} is bounded.In our work we use generally $\alpha = \frac{1}{2}$. This is the operator $A_0 : H_{\frac{1}{2}} \to H_{-\frac{1}{2}}$.

Now it's the time to make from $H_{-\alpha}$ a Hilbert space and consequently a normed space. For each $z \in Im(A_0^\alpha)$ we define

$$\langle z, z \rangle_{H_{-\alpha}} = \left\langle A_0^{-\alpha} z, A_0^{-\alpha} z \right\rangle_H$$

and the norm

$$||z||_{H_{-\alpha}} = ||A_0^{-\alpha} z||_H.$$

$H_{-\alpha}$ is the completion of H with regard to the above norm. We point out that H is not complete with regard to the above norm as the following example shows. In fact, since $(H, \langle ., . \rangle_{H_{-\frac{1}{2}}})$ is an inner product space, thus $(H_{-\frac{1}{2}}, \langle ., . \rangle_{H_{-\frac{1}{2}}})$ is a Hilbert space.
The following example cannot be found in the main paper.

Example 1.1 *Suppose $H = l_2$ and define the operator $A_0 : H \to H$ by $A_0 z = \sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n$ where $\{e_n\}_{n=1}^{\infty}$ is an orthonormal basis for H. Then H is not complete with the norm $||z||_{H_{-\frac{1}{2}}} = ||A_0^{-\frac{1}{2}} z||_H$.*

Solution: From the definition of A_0 we have $A_0^{-\frac{1}{2}} z = \sum_{n=1}^{\infty} n^{-1} \langle z, e_n \rangle e_n$ and consequently $||A_0^{-\frac{1}{2}} z||_{l_2}^2 = \sum_{n=1}^{\infty} n^{-1} | \langle z, e_n \rangle |^2$. We define now, $||z||_{H_{-\frac{1}{2}}}^2 = \sum_{n=1}^{\infty} n^{-1} | \langle z, e_n \rangle |^2$ and claim that l_2 is not complete with this norm. Suppose $x_n = \left\{ \underbrace{1, 1, \cdots, 1}_{n}, 0, 0, \cdots \right\}$, then clearly $x_n \in l_2$. Suppose $m > k$, then $||x_m - x_k||_{H_{-\frac{1}{2}}} \to 0$ as $k, m \to \infty$. So $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence in $H_{-\frac{1}{2}}$. But the limit in $H_{-\frac{1}{2}}$ is the sequence $\{1, 1, \cdots\}$ which is not in l_2.

Now we are going to see the relationship between $\langle z', z \rangle_{H_{-\alpha} \times H_{\alpha}}$ and $\langle z', z \rangle_H$ for $\alpha > 0$ and for each $z' \in H$ and $z \in H_{\alpha}$.
Suppose that $(z', z) \in H \times H_{\alpha}$. We claim that $\langle z', z \rangle_{H_{-\alpha} \times H_{\alpha}} = \langle z', z \rangle_H$ for $\alpha > 0$.
To do this end assume that $v \in H_{-\alpha}$ and $u \in H_{\alpha}$ for $\alpha > 0$, then $\langle v, u \rangle_{H_{-\alpha} \times H_{\alpha}} = v(u)$. Thus, if $v \in H$, then $\langle v, u \rangle_H = v(u)$. This shows that for each $v \in H$, $u \in H_{\alpha}$ when $\alpha > 0$ we have $\langle v, u \rangle_{H_{-\alpha} \times H_{\alpha}} = \langle v, u \rangle_H$.

1.7 Krein space

In traditional linear algebra the concepts of length, angle, and orthogonality are defined by a definite inner product. In Krein space, the definite inner product is replaced by an indefinite one and this produces substantial changes in the spaces.

Definition 1.8 *Suppose V is a complex vector space. A function $[.,.] : V \times V \longrightarrow \mathbb{C}$ is called indefinite inner product in V if the following axiomas are satisfied:* (See, [14, chapter 2.1]).

1. *linearity in the first argument*

   $[\alpha x_1 + \beta x_2, y] = \alpha [x_1, y] + \beta [x_2, y_2], \ \forall x_1, x_2, y \in V \quad \text{and} \quad \forall \alpha, \beta \in \mathbb{C},$

2. *antisymmetry*

   $[x, y] = \overline{[y, x]}, \ \forall x, y \in V,$

3. *nondegeneracy*

   if $[x, y] = 0$ for all $y \in V$, then $x = 0$.

Thus, the function $[.,.]$ satisfies all the properties of a standard inner product with the possible exception that $[x,x]$ may be nonpositive, hence the above definition yields three possibilities. (See, [7, chapter I.2]).

- $[x,x] > 0$, in this case we say that x is a positive element of V,
- $[x,x] < 0$, in this case we say that x is negative element of V,
- $[x,x] = 0$, in this case we say that x is a neutral element of V.

Definition 1.9 *Let V be an indefinite inner product space. We define* (see, [7, chapter I.2])

- $\mathcal{B}^{++} = \{x \in V | \, [x,x] > 0\} \bigcup \{0\}$,
- $\mathcal{B}^{--} = \{x \in V | \, [x,x] < 0\} \bigcup \{0\}$.

Definition 1.10 *Let V be a vector space. The inner product $[.,.]$ on V is said to be* (see, [7, chapter I.2])

- *definite, if $[x,x] = 0$ implies $x = 0$,*
- *positive definite, if $[x,x] > 0$ for $x \neq 0$,*
- *negative definite, if $[x,x] < 0$, for $x \neq 0$.*

Consequently, positive definite, and negative definite inner product spaces are vector spaces with an inner product of the respective kind. (See, [7, chapter I.2]).

We shall denote positive definite, and negative definite subspaces of the inner product space V by V^+ and V^-, respectively.

Definition 1.11 *Let $[.,.]$ be an indefinite inner product on the vector space V, and $[x,y] = 0$ or equivalently $[y,x] = 0$, then we say that x and y are orthogonal to each other. Two sets are said to be orthogonal, writing $U \, [\perp] \, W$, if $x \, [\perp] \, y$ for each $x \in U$ and $y \in W$, with regard to $[.,.]$.* (See, [7, chapter I.3]).

Definition 1.12 *If V is the direct sum of pairwise orthogonal subspaces V_j, $j = 1, 2, \cdots, n$, we say that V is the orthogonal direct sum of subspaces V_j, and we write it by* (see, [7, chapter I.3])

$$V = V_1[\dot{+}]V_2[\dot{+}] \cdots [\dot{+}]V_n.$$

We restrict now our attention to fundamental decomposition.

Definition 1.13 *We say that $V = V^+[\dot{+}]V^-$, where $V^+ \subset \mathcal{B}^{++}$ and $V^- \subset \mathcal{B}^{--}$ is a fundamental decomposition of the inner product space V if V^+ and V^- are orthogonal to each other with regard to inner product on V.* (See, [7, chapter II.10]).

Consider now the fundamental projections P^+ and P^- belonging to a fundamental decomposition of V, this is,

$$P^+ \, V^- = 0 \quad \text{and} \quad P^- \, V^+ = 0,$$

$$P^+ x = x^+, \quad P^- x = x^-, \ \forall x \in V,$$

where $x = x^+ + x^-$, $x^+ \in V^+$, and $x^- \in V^-$. (See, [7, chapter II.10]).

We set $J = P^+ - P^-$ and say that J is the fundamental symmetry belonging to the fundamental decompostion in the Definition 1.13. (See, [7, chapter II]).
Just for a moment we accept without proof that J is a symmetry operator. Since J is symmetric the formula

$$[x, y] = \langle Jx, y \rangle \, , \ \forall x, y \in V \tag{12}$$

is an inner product on V. (See, [7, chapter II]). Where the definite inner product is given by

$$\langle x, y \rangle = \left[P^+ x, P^+ y \right] - \left[P^- x, P^- y \right] .$$

We prove now that J is symmetric. We have $\langle Jx, y \rangle = \langle x^+ - x^-, y^+ + y^- \rangle = \langle x^+, y^+ \rangle + \langle x^+, y^- \rangle - \langle x^-, y^+ \rangle - \langle x^-, y^- \rangle$. The second and third inner products are zero, since V^+ and V^- are orthogonal to each other by definition, hence, $\langle Jx, y \rangle = \langle x^+, y^+ \rangle - \langle x^-, y^- \rangle$. Since $\langle x^+, -y^- \rangle = \langle x^-, y^+ \rangle = 0$ we can write this as $\langle Jx, y \rangle = \langle x^+, y^+ \rangle - \langle x^-, y^- \rangle + \langle x^+, -y^- \rangle + \langle x^-, y^+ \rangle$ or $\langle Jx, y \rangle = \langle x^+, y^+ \rangle + \langle x^+, -y^- \rangle + \langle x^-, y^+ \rangle + \langle x^-, -y^- \rangle = \langle x^+, y^+ - y^- \rangle + \langle x^-, y^+ - y^- \rangle = \langle x^+ + x^-, y^+ - y^- \rangle = \langle x, Jy \rangle$.

We are now ready to define the Krein space.

Definition 1.14 *Let $\wp$ be a definite subspace of the inner product space V. Then the notation $||x|| = | \langle x, x \rangle |^{\frac{1}{2}}$ for $x \in \wp$ defines a norm on $\wp$, and we call $||.||$ the intrinsic norm on $\wp$.* (See, [7, chapter III.8]).

Definition 1.15 *Suppose V is an inner product space. We say that V is a Krein space if,*

1. *V admits a fundamental decomposition, i.e; $V = V^+ (\dot{+}) V^-$, where $V^+ \subset \mathcal{B}^{++}$, and $V^- \subset \mathcal{B}^{--}$.*
2. *V^+ and V^- are intrinsically complete.* (See, [7, chapter V.I]).

In this case, V with $\langle ., . \rangle$ is a Hilbert space.

1.8 Spectrum of a closed linear operator

Let $S : \mathcal{D}(S) \subseteq X \longrightarrow X$ be a densely defined closed linear operator. A complex number $\lambda \in \mathbb{C}$ is called a resolvent point of S if the inverse $(S - \lambda I)^{-1}$ exists and is a bounded operator, (see, [10, chapter 5]) and we denote the set of regular point by $\rho(S)$, i.e;

$$\rho(S) = \left\{ \lambda \in \mathbb{C} | \, (S - \lambda I)^{-1} \text{exists and bounded} \right\}.$$

If λ is not a resolvent point, we call it a spectral point. The spectrum of an operator S can be classified as follows, where S is the same operator as above and also assumed be closed. (See,[10, chapter 5]).

- The point spectrum: (See, [10, chapter 5])
  It's the set of the eigenvalues of S, and we denote it by $\sigma_p(S)$, thus $\lambda \in \sigma_p(S)$ if and only if there exists an $x \in \mathcal{D}(S)$ such that $Sx = \lambda x \, (i.e; kern \, (S - \lambda I) \neq 0)$, equivalently,

$$\sigma_p(S) = \{ \lambda \in \mathbb{C} | S - \lambda I \;\; \text{is not injective} \}.$$

- The continuous spectrum: (See, [10, chapter 5])
  We denote the continuous spectrum of S by $\sigma_c(S)$. By definition, $\lambda \in \sigma_c(S)$ if and only if $\lambda \in \sigma(S) \setminus \sigma_p(S)$ and $Im\,(S - \lambda I)$ is dense in X, i.e; $\overline{Im\,(S - \lambda I)} = X$, but $Im\,(S - \lambda I) \neq X$. Of course $S - \lambda I$ is defined on $\mathcal{D}(S)$.

- The residual spectrum: (See, [10, chapter 5])
  The set $\sigma_r(S) = \sigma(S) \setminus (\sigma_p(S) \cup \sigma_c(S))$ is the residual spectrum of S, equivalently, $\lambda \in \sigma_r(S)$ if $kern\,(S - \lambda I) = 0$ and $\overline{Im\,(S - \lambda I)} \neq X$.

- Approximate point spectrum: (See, [11, chapter IV])
  The approximate point operator of S is the set

$$\sigma_{ap}(S) = \{ \lambda \in \mathbb{C} | S - \lambda I \;\; \text{is not injective or } Im\,(S - \lambda I) \;\; \text{is not closed in } X \}.$$

The reason for the above definition of "approximate point spectrum" can be seen in the next lemma.

Lemma 1.1 *For a closed operator $S : \mathcal{D}(S) \subseteq X \longrightarrow X$ and a number $\lambda \in \mathbb{C}$, one has $\lambda \in \sigma_{ap}(S)$ if and only if there exists a sequence $\{x_n\}_{n \in \mathbb{N}} \subset \mathcal{D}(S)$, such that $||x_n|| = 1$ and $lim_{n \to \infty} ||Sx_n - \lambda x_n|| = 0$.* (See, [11, chapter IV]).

Proof: We only have to consider the case which $S - \lambda$ is injective. We denote by X_1 the domain of S. Then the inverse $(S - \lambda)^{-1} : Im\,(S - \lambda) \longrightarrow X_1$ exists and, by the closed graph theorem, is unbounded if and only if $Im\,(S - \lambda)$ is not closed. On the other hand, if $(S - \lambda)^{-1} : X \longrightarrow X$ is bounded, the closedness of S implies the closedness of $Im\,(S - \lambda)$. Hence $(S - \lambda)^{-1} : X \longrightarrow X_1$ is unbounded if and only if $(S - \lambda)^{-1} : X \longrightarrow X$ is unbounded, and this property can be expressed by the condition above. (See [11, chapter IV]).

Lemma 1.2 *For a closed operator S, we have $\sigma_p \subset \sigma_{ap}(S)$, and $\sigma_c(S) \subset \sigma_{ap}(S)$.*

Proof: Suppose that $\lambda \in \sigma_p(S)$, hence $S - \lambda I$ is not injective by definition. Hence $\lambda \in \sigma_{ap}(S)$. Thus $\sigma_p \subset \sigma_{ap}(S)$. Now, suppose that $\lambda \in \sigma_c(S)$. By definition $\lambda \notin \sigma_p(S)$ and $\overline{Im(S - \lambda I)} = X$ but $Im(S - \lambda I) \neq X$. Since $\lambda \notin \sigma_p(S)$, hence $S - \lambda I$ is injective, in order to have $\lambda \in \sigma_{ap}(S)$, we have to prove that $Im(S - \lambda I)$ is not closed. If it is, then $Im(S - \lambda I) = \overline{Im(S - \lambda I)} = X$, hence $\lambda \notin \sigma_c(S)$ and this is a contradiction. Thus $Im(S - \lambda I)$ is not closed, and $\lambda \in \sigma_{ap}(S)$.

Lemma 1.3 *For a closed operator S, we have $\sigma_{ap}(S) = \sigma_p(S) \cup \sigma_c(S)$.*

Proof: Suppose that $\lambda \in \sigma_{ap}$. Then by definition $S - \lambda I$ is not injective or $Im(S - \lambda I)$ is not closed. In the first case we have $\lambda \in \sigma_p(S)$ and in the second case $Im(S - \lambda I)$ cannot be the whole space but $\overline{Im(S - \lambda I)}$ is the whole space, which shows that $Im(S - \lambda I)$ is not closed, thus $\lambda \in \sigma_c(S)$. This argument with the Lemma 1.2 shows that $\sigma_{ap}(S) = \sigma_p(S) \cup \sigma_c(S)$.

Definition 1.16 *A point $\mu \in \mathbb{C}$ is called a regular point for S if $\mu \notin \sigma_{ap}(S)$. We denote the set of all* ***regular points*** *of S by $r(S)$. Thus $r(S)$ is,*

$$r(S) = \{\lambda \in \mathbb{C} | \lambda \notin \sigma_{ap}(S)\}.$$

Remark 1.2 *If $\mu \in r(S)$, then $\mu \notin \sigma_{ap}(S)$, hence $Im(S - \lambda I)$ is closed in X by definition of $\sigma_{ap}(S)$.*

Lemma 1.4 *Suppose that $\mu \in r(S)$. Then there exists $M > 0$ such that for each $x \in \mathcal{D}(S)$ we have*

$$||(S - \mu)x|| \geq M||x||.$$

Proof: Suppose for each $M > 0$, there exists $x \in \mathcal{D}(S)$ such that $||(S - \mu)x|| < Mx$. Let $M = \frac{1}{n}$ for $n = 1, 2, 3, \cdots$ and denoted by x_n the corresponding $x \in \mathcal{D}(S)$ for which $||(S - \mu)x_n|| < \frac{1}{n}$. Set $u_n = \frac{x_n}{||x_n||}$, then $||u_n|| = 1$ and $||(S - \mu)u_n|| < \frac{1}{n}$. So the condition of Lemma 1.1 is satisfied, hence $\mu \in \sigma_{ap}(S)$.

We introduce here another type of operator called *Fredholm operator*.

Definition 1.17 *A linear operator S (bounded or unbounded) is called a* ***Fredholm operator*** *if the numbers $n(S) = dim\ kern(S)$ and $d(S) = codim\ ImS$ are both finite. In this case the number $ind\ S = n(S) - d(S)$ is called the* ***index*** *of S.* (See, [12, chapter XV.1]).

Example 1.2 *Let $S : l_2 \longrightarrow l_2$ be the linear operator and defined as follows: $S(\alpha_1, \alpha_2, \cdots) = (\underbrace{0, 0, \cdots, 0}_{r}, \alpha_{k+1}, \alpha_{k+2}, \cdots)$, then S is a Fredholm operator, since $kern\ S = \{e_1, e_2, \cdots, e_k\}$ and $l_2 = ImS \oplus sp\{e_1, e_2 \cdots, e_r\}$, where $e_1, e_2, \cdots$ is the standard orthonormal basis in l_2. Thus $n(S) = k$, $d(S) = r$, and $ind\ S = k - r$.* (See, [12, chapter XV.1]).

We give a definition which is related to Fredholm operators.

Definition 1.18 *We define the essential spectrum of S by*

$$\sigma_{ess}(S) = \{\lambda \in \mathbb{C} | S - \lambda I \text{ is not Fredholm}\}.$$

2 Second order o.d.e and main problem

2.1 Operator polynomial

As we promised before, we are now going to show that the equation

$$\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0, \tag{13}$$

can be rewritten as the first order system

$$\dot{x}(t) = A \cdot x(t), \tag{14}$$

where $A : \mathcal{D}(A) \subset \mathcal{D}(A_0^{\frac{1}{2}}) \times H \longrightarrow \mathcal{D}(A_0^{\frac{1}{2}}) \times H$ is given by

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix}.$$

Here

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} z \\ w \end{pmatrix} \in \mathcal{D}(A_0^{\frac{1}{2}}) \times \mathcal{D}(A_0^{\frac{1}{2}}) | A_0 z + Dw \in H \right\},$$

where $A_0 : H \longrightarrow H$ and $D : H \longrightarrow H$ are unbounded operators.

Remark 2.1 *Throughout this work we assume that the operator $D : H_{\frac{1}{2}} \longrightarrow H_{-\frac{1}{2}}$ which is called the damping operator is a bounded operator such that $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is a bounded self adjoint operator in H satisfying*

$$\langle Dz, z \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 0, \quad \forall z \in H_{\frac{1}{2}}.$$

More generally, consider the equation

$$L_l z^{(l)}(t) + L_{l-1} z^{(l-1)}(t) + \cdots + L_1 z^{(1)}(t) + L_0 z(t) = 0, \tag{15}$$

where $L_0, L_1, \cdots, L_{l-1}, L_l$ are operators on the Hilbert space H, and L_l is invertible on H, also the indices on z denote the derivatives with respect to variable t. Define functions $z_0(t), z_1(t), \cdots, z_{l-1}(t)$ in terms of a solution function $z(t)$ by

$$z_0(t) = z(t), z_1(t) = z^{(1)}(t), \cdots, z_{l-1}(t) = z^{(l-1)}(t). \tag{16}$$

Then eq.(15) takes the form

$$L_l \dot{z}_{l-1}(t) + L_{l-1} z_{l-1}(t) + \cdots + L_0 z_0(t) = 0. \tag{17}$$

If we now define

$$x(t) = \begin{pmatrix} z_0(t) \\ z_1(t) \\ \vdots \\ z_{l-1}(t) \end{pmatrix}, \tag{18}$$

then the eqs.(16) and (17) can be condensed to the form

$$\dot{x}(t) = C_l \cdot x(t), \tag{19}$$

where

$$C_l = \begin{pmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & I \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \\ -\hat{L}_0 & -\hat{L}_1 & \cdots & -\hat{L}_{l-1} \end{pmatrix},$$

where $\hat{L}_j = L_l^{-1} \cdot L_j$ for $j = 0, 1, \cdots, l-1$. (See, [19, chapter 14]).
We note that the operator C_l called the companion matrix for eq.(8).
Thus the problem is reduced from order l to order 1 in the derivatives.

We come back now to the eq.(13). Comparing this equation with eq.(15) we see that $l = 2$, $L_0 = -A_0$, $L_1 = D$, and $j = 0, 1$. Thus the companion matrix for eq.(6) is,

$$C_l = \begin{pmatrix} 0 & I \\ -\hat{L}_0 & -\hat{L}_1 \end{pmatrix},$$

where $\hat{L}_0 = -L_2^{-1} \cdot L_0 = -A_0$, and $\hat{L}_1 = -L_2^{-1} \cdot L_1 = -D$.
In our notation we will use in the future A in place of C_l.

2.2 Some results on the space $H_{\frac{1}{2}} \times H$

Definition 2.1 *Let $\left(H_1, \langle ., . \rangle_{H_1}\right), \left(H_2, \langle ., . \rangle_{H_2}\right)$ be two inner product spaces. On the cartesian product $H_1 \times H_2$ the map $\langle (x_1, y_1), (x_2, y_2) \rangle_{H_1 \times H_2} = \langle x_1, x_2 \rangle_{H_1} + \langle y_1, y_2 \rangle_{H_2}$ is an inner product. If $\left(H_1, \langle ., . \rangle_{H_1}\right)$ and $\left(H_2, \langle ., . \rangle_{H_2}\right)$ are Hilbert spacses, then $(H_1 \times H_2, \langle ., . \rangle)$ is a Hilbert space called the Hermitian product of H_1 and H_2.* (See, [12, chapter 6.3]).

Remark 2.2 *By the Definition 1.7 the space $H_{\frac{1}{2}} \times H$ is a Hilbert space.*

Lemma 2.1 *The operator JA is self adjoint in the Hilbert space $H_{\frac{1}{2}} \times H$, where $J = \begin{pmatrix} I_{H_{\frac{1}{2}}} & 0 \\ 0 & -I_H \end{pmatrix}$, and the domain of JA is the same as the domain of the operator A.*

Proof: To prove that JA is self adjoint we have to prove that $JA = (JA)^*$.

$$\left\langle JA \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H} = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, (JA)^* \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H}.$$

The left hand side of the above equality can be written as

$$\langle A_0 x_1, y_2 \rangle_H + \langle A_0 y_1, x_2 \rangle_H + \langle D y_1, y_2 \rangle_H. \tag{20}$$

Also, by the Remark 2.1 and that $\langle x, y \rangle_{H_{-\alpha} \times H_\alpha} = \langle x, y \rangle_H$ for each $x \in H$, $y \in H_\alpha$, the operator $D : H_{\frac{1}{2}} \subset H \longrightarrow H$ is symmetric.

Thus we can write (20) as follows

$$\langle A_0 x_1, y_2 \rangle_H + \langle A_0 y_1, x_2 \rangle_H + \langle y_1, D y_2 \rangle_H.$$

Writing the last one as matrix form, we will get

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} y_2 \\ A_0 x_2 + D y_2 \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H} \quad \text{or} \quad \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, (JA) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H}.$$

Till now we have shown that JA is symmetric. To prove that JA is self adjoint we show that $\mathcal{D}(JA) = \mathcal{D}(JA)^*$. Since $JA = (JA)^* = \begin{pmatrix} 0 & I \\ A_0 & D \end{pmatrix}$. This shows that $\mathcal{D}(JA) = \mathcal{D}(JA)^*$.

We note also that a simple calculation of block matrix multiplication gives us

$$A^* = JAJ.$$

We claim now that the operator A is not self adjoint in the space $H_{\frac{1}{2}} \times H$. To this end, we consider the general form of a 2×2 block matrix , i.e;

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}$$

and we find its adjoint, i.e; B^*. Thus,

$$\left\langle B \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H} = \left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, B^* \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H}.$$

The left hand side of the above equality is equal to

$$\left\langle \begin{pmatrix} B_{11}\ x_1 + B_{12}\ y_1 \\ B_{21}\ x_1 + B_{22}\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H}. \tag{21}$$

Using the Definition 1.7, this becomes, $\langle B_{11} x_1 + B_{12} y_1, x_2 \rangle_{H_{\frac{1}{2}}} + \langle B_{21} x_1 + B_{22} y_1, y_2 \rangle_H$.

The first inner product in the last statement is on $H_{\frac{1}{2}}$, we change it to an inner product in H using the definition given by $\langle x, y\rangle_{H_\alpha} = \langle A_0^\alpha x, A_0^\alpha y\rangle_H$. Hence we obtain

$$\left\langle A_0^{\frac{1}{2}} \left(B_{11}x_1 + B_{12}y_1\right), A_0^{\frac{1}{2}} x_2 \right\rangle_H + \langle B_{21}x_1 + B_{22}y_1, y_2\rangle_H . \tag{22}$$

Now we write (22) as follows,

$$\left\langle A_0^{\frac{1}{2}} \left(B_{11}A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} x_1 + B_{12}\right), A_0^{\frac{1}{2}} x_2 \right\rangle_H + \left\langle B_{21}A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} x_1 + B_{22}y_1, y_2\right\rangle_H . \tag{23}$$

This can be rewritten as

$$\left\langle A_0^{\frac{1}{2}} B_{11}A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} x_1, A_0^{\frac{1}{2}} x_2\right\rangle_H + \left\langle B_{21}A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} x_1, y_2\right\rangle_H + \left\langle A_0^{\frac{1}{2}} B_{12}y_1, A_0^{\frac{1}{2}} x_2\right\rangle_H + \langle B_{22}y_1, y_2\rangle_H . \tag{24}$$

Since A_0 is self adoint $A_0^{\frac{1}{2}}$ is self adjoint too, hence the first two inner products of this can be written as,

$$\left\langle B_{11}A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} x_1, A_0x_2\right\rangle_H + \left\langle B_{21}A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} x_1, y_2\right\rangle_H = \left\langle A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} x_1, B_{11}^* A_0 x_2\right\rangle_H + \left\langle A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} x_1, B_{21}^* y_2\right\rangle_H .$$

Since A_0 is self adjoint A_0^{-1} is self adjoint too, consequently is $A_0^{-\frac{1}{2}}$, hence the right hand side of the last statement can be changed to

$$\left\langle A_0^{\frac{1}{2}} x_1, A_0^{-\frac{1}{2}} B_{11}^* A_0 x_2\right\rangle_H + \left\langle A_0^{\frac{1}{2}} x_1, A_0^{-\frac{1}{2}} B_{21}^* y_2\right\rangle_H ,$$

and this is equal to

$$\left\langle A_0^{\frac{1}{2}} x_1, A_0^{-\frac{1}{2}} B_{11}^* A_0 x_2 + A_0^{-\frac{1}{2}} B_{21}^* y_2\right\rangle_H .$$

Finally, this can be written as

$$\left\langle A_0^{\frac{1}{2}} x_1, A_0^{\frac{1}{2}} \left(A_0^{-1} B_{11}^* A_0 x_2 + A_0^{-1} B_{21}^* y_2\right)\right\rangle_H ,$$

now by the definition, $\langle x, y\rangle_{H_\alpha} = \langle A_0^\alpha x, A_0^\alpha\rangle_H$ we can again write this as

$$\left\langle x_1, A_0^{-1} B_{11}^* A_0 x_2 + A_0^{-1} B_{21}^* y_2\right\rangle_{H_{\frac{1}{2}}} . \tag{25}$$

Let's turn to the last inner products of the second part of (24). This can be written as

$$\begin{aligned}\left\langle A_0^{\frac{1}{2}} B_{12}y_1, A_0^{\frac{1}{2}} x_2\right\rangle_H + \langle B_{22}y_1, y_2\rangle_H = \langle B_{12}y_1, A_0x_2\rangle_H + \langle B_{22}y_1, y_2\rangle_H = \\ \langle y_1, B_{12}^* A_0 x_2\rangle_H + \langle y_1, B_{22}^* y_2\rangle_H = \langle y_1, B_{12}^* A_0 x_2 + B_{22}^* y_2\rangle_H .\end{aligned} \tag{26}$$

Note that the first equality in (26) can be concluded, since $A_0^{\frac{1}{2}}$ is self adjoint in H, and $A_0^{\frac{1}{2}}$ is self adjoint since A_0 is self adjoint on H.

Adding (25) and the last part of (26) we get

$$\left\langle x_1, A_0^{-1}B_{11}^*A_0x_2 + A_0^{-1}B_{21}^*y_2\right\rangle_{H_{\frac{1}{2}}} + \langle y_1, B_{12}^*A_0x_2 + B_{22}^*y_2\rangle_H \, ,$$

and this is equal to

$$\left\langle \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} A_0^{-1}B_{11}^*A_0 & A_0^{-1}B_{21}^* \\ B_{12}^*A_0 & B_{22}^* \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H} . \tag{27}$$

Calculating A^* where

$$A = \begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix}$$

we will have

$$A^* = \begin{pmatrix} 0 & A_0^{-1}(-A_0) \\ IA_0 & -D \end{pmatrix} \quad \text{or} \quad A^* = \begin{pmatrix} 0 & -I \\ A_0 & -D \end{pmatrix},$$

which shows that $A \neq A^*$.

We just proved that the operator A is not a self adjoint operator in the Hilbert space $H_{\frac{1}{2}} \times H$. What we have done, was not futile. In fact the reason that we mentioned this, was to introduce another space than Hilbert space in which the operator A is self adjoint, this is the *Krein space.*

Lemma 2.2 *$H_{\frac{1}{2}} \times H$ is a Krein space with the inner product $[.,.]$ defined by* $\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right] = \left\langle J\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right\rangle, \forall \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in H_{\frac{1}{2}} \times H,$ *where* $J = \begin{pmatrix} I_{H_{\frac{1}{2}}} & 0 \\ 0 & -I_H \end{pmatrix}.$

Proof: We identify $H_{\frac{1}{2}} \times H$ with $H_{\frac{1}{2}} \oplus H$, since, firstly we have an isomorphism between the two sets $H_{\frac{1}{2}} \times H = \left\{(v, w) \,|v \in H_{\frac{1}{2}}, w \in H\right\}$ and $H_{\frac{1}{2}} + H = \left\{\begin{pmatrix} v \\ w \end{pmatrix} |v \in H_{\frac{1}{2}}, w \in H\right\}$, and secondly we can write $\begin{pmatrix} v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ w \end{pmatrix}$ in a unique manner.

We see also that, $\left(H_{\frac{1}{2}}, \langle .,. \rangle_{H_{\frac{1}{2}}}\right)$ is a positive definite subspace of $H_{\frac{1}{2}} \times H$, since $H_{\frac{1}{2}}$ is a Hilbert space by our traditional inner product space. For the same reason $(H, -\langle .,. \rangle_H)$ is a negative definite subspace of $H_{\frac{1}{2}} \times H$. To treat the part 1 of definition of Krein space for the space $H_{\frac{1}{2}} \times H$ it remains to prove that $\oplus$ can be replaced by $[\dot{+}]$, or equivalently, the sum is not only a direct sum but also an orthogonal direct sum with regard to $[.,.]$. To this end, we use that there exists an isomorphism between $H_{\frac{1}{2}}$ and the set $M = \left\{ \begin{pmatrix} v \\ 0 \end{pmatrix} | v \in H_{\frac{1}{2}} \right\}$. Also there is an isomorphism between H and $N = \left\{ \begin{pmatrix} 0 \\ w \end{pmatrix} | w \in H \right\}$. Now it's easy to see that $M \perp N$, since $\forall \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \in H_{\frac{1}{2}} \times H$ we have

$$\left[\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] = \langle x_1, x_2 \rangle_{H_{\frac{1}{2}}} - \langle y_1, y_2 \rangle_H , \tag{28}$$

replacing $\begin{pmatrix} v \\ 0 \end{pmatrix}$, and $\begin{pmatrix} 0 \\ u \end{pmatrix}$ in (28) we obtain, $\left[\begin{pmatrix} v \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u \end{pmatrix} \right] = \langle v, 0 \rangle - \langle 0.v \rangle = 0$. Hence, $H_{\frac{1}{2}}$ and H are othogonal with regard to $[.,.]$.
We treat now the second part of Definition 1.14.
This part is easy to see, since $(H_{\frac{1}{2}}, \langle .,. \rangle_{H_{\frac{1}{2}}})$ and $(H, \langle .,. \rangle_H)$ are Hilbert spaces, thus they are complete with the intrinsic norm on them.

Remark 2.3 *In the above lemma J is defined by $J = \begin{pmatrix} I_{H_{\frac{1}{2}}} & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & I_H \end{pmatrix}$. The first matrix on $H_{\frac{1}{2}} \times H$ is a projection on $H_{\frac{1}{2}} \times H$ to $H_{\frac{1}{2}}$ and the second one a projection on $H_{\frac{1}{2}} \times H$ to H, hence, $J = P^+ - P^-$, and J is in fact a fundamental symmetry belonging to $H_{\frac{1}{2}} \times H = H_{\frac{1}{2}} (\dot{+}) H$, hence it's a symmetry operator by the proof on page 16.*

We have seen before that A is not a self adjoint operator on the Hilbert space $H_{\frac{1}{2}} \times H$ with regard to the inner product $\langle .,. \rangle_{H_{\frac{1}{2}} \times H}$.

Lemma 2.3 *A is a self adjoint operator with regard to $[.,.]$ in the Krein space $H_{\frac{1}{2}} \times H$.*

Proof: For the proof see the proof of Lemma 2.1.

Lemma 2.4 *The operator A is closed.*

Proof: We have to prove that if $\left\{ \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}_{n \in \mathbb{N}} \in \mathcal{D}(A)$, $\left\{ \begin{pmatrix} u_n \\ v_n \end{pmatrix} \right\}_{n \in \mathbb{N}} \to \begin{pmatrix} u \\ v \end{pmatrix}$ and $A \begin{pmatrix} u_n \\ v_n \end{pmatrix} \to \begin{pmatrix} z \\ k \end{pmatrix}$, then $\begin{pmatrix} u \\ v \end{pmatrix} \in \mathcal{D}(A)$ and $A \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} z \\ k \end{pmatrix}$.

We know, $u_n \longrightarrow u$ in $H_{\frac{1}{2}}$ where $u_n \in \mathcal{D}(A_0)$. Since $||x||^2_{H_{\frac{1}{2}}} = ||A_0^{\frac{1}{2}} x||^2_H$, thus $A_0^{\frac{1}{2}} u_n \longrightarrow A_0^{\frac{1}{2}} u$ in H. Since $A \begin{pmatrix} u_n \\ v_n \end{pmatrix} \to \begin{pmatrix} z \\ k \end{pmatrix}$, then $v_n \longrightarrow v$ in H and $v_n \longrightarrow z$ in $H_{\frac{1}{2}}$. So $A_0^{\frac{1}{2}} v_n \longrightarrow A_0^{\frac{1}{2}} z$ in H. By this and boundedness of $A_0^{-\frac{1}{2}}$ we get $v_n \longrightarrow z$ in H and by the uniqueness of limit we have $z = v$. Since $z \in H_{\frac{1}{2}}$, we get $v \in H_{\frac{1}{2}}$ and $A_0^{\frac{1}{2}} v_n \longrightarrow A_0^{\frac{1}{2}} v$ in H. Further we know that $-(A_0 u_n + D v_n) \longrightarrow k$ in H. By this and the boundedness of $A_0^{-\frac{1}{2}}$ we get $-A_0^{\frac{1}{2}} u_n - A_0^{-\frac{1}{2}} D v_n \longrightarrow A_0^{-\frac{1}{2}} k$ in H. On the other hand $A_0^{\frac{1}{2}} u_n \longrightarrow A_0^{\frac{1}{2}} u$ and $A_0^{-\frac{1}{2}} D v_n = A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} v_n$ which converges to $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}} A_0^{\frac{1}{2}} v = A_0^{-\frac{1}{2}} D v$, since $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is a bounded operator by our assumption. So $-A_0^{-\frac{1}{2}} k = A_0^{\frac{1}{2}} u + A_0^{-\frac{1}{2}} D v$. Since $-A_0^{-\frac{1}{2}} k \in \mathcal{D}(A_0^{\frac{1}{2}}) = H_{\frac{1}{2}}$, thus $A_0^{\frac{1}{2}} u + A_0^{-\frac{1}{2}} D v \in \mathcal{D}(A_0^{\frac{1}{2}})$. Applying $A_0^{\frac{1}{2}}$ to both sides we obtain $-k = A_0 u + D v$. This proves the second part.

Before begining the following proposition, we note that if $T : X \longrightarrow Y$ is bounded and X and Y are Banach spaces, then $\lambda \in \rho(T)$ if and only if $kern\,(T - \lambda I) = \{0\}$ and $Im\,(T - \lambda I) = Y$, and the same is true when T is an unbounded closed operator.

The reader can recognize now why we gave the Lemma 1.4. We prove now an important proposition.

Proposition 2.1 *The spectrum of the operator A is symmetric with respect to the real line.* (See, [7, chapter VI.6]). *Moreover, A has a bounded inverse.*

Proof: The first statement in the proposition is equivalent to $\lambda \in \sigma(A)$ if and only if $\bar{\lambda} \in \sigma(A)$, and this is equivalent to $\lambda \in \rho(A)$ if and only if $\bar{\lambda} \in \rho(A)$. We are going to prove the last statement.
We have shown already that $A^{[*]} = JAJ$, hence $JA = A^{[*]} J^{-1} = A^{[*]} J$. But $J(A - \lambda I) = A^{[*]} - \lambda I J = \left(A^{[*]} - \lambda I\right) J = \left(A - \overline{\lambda} I\right)^{[*]} J$, thus $\left(A - \overline{\lambda} I\right)^{[*]} = J(A - \lambda I) J^{-1}$. Let $\lambda \in \rho(A)$. Since all elements in the right hand side are invertible, $\left(A - \overline{\lambda} I\right)^{[*]}$ is invertible, consequently $A - \overline{\lambda} I$ is invertible and $\overline{\lambda} \in \rho(A)$. The converse is proved in the same manner.

The operator A has inverse if $AA^{-1} = I_{H_{\frac{1}{2}} \times H}$ and $A^{-1} A = I_{\mathcal{D}(A)}$. Thus if $A^{-1} = \begin{pmatrix} X & Y \\ Z & K \end{pmatrix}$ we find X, Y, Z, and K.

$$\begin{pmatrix} 0 & I \\ -A_0 & -D \end{pmatrix} \begin{pmatrix} X & Y \\ Z & K \end{pmatrix} = \begin{pmatrix} I_{H_{\frac{1}{2}}} & 0 \\ 0 & I_H \end{pmatrix}.$$

Multiplying the above matrices and solving the above equation we obtain $X = A_0^{-1} D, Y = A_0^{-1}, Z = I_{H_{\frac{1}{2}}}, K = 0$. Thus,

$$A^{-1} = \begin{pmatrix} -A_0^{-1} D & -A_0^{-1} \\ I_{H_{\frac{1}{2}}} & 0 \end{pmatrix},$$

where $A_0^{-1} D$ is an operator acting on $H_{\frac{1}{2}}$ to $H_{\frac{1}{2}}$.
We claim now that A^{-1} is a bounded operator, in other words A has a bounded inverse.

Thus we have to show that there exists an $M > 0$ such that $||A^{-1}X||_{H_{\frac{1}{2}}\times H} \le M||X||_{H_{\frac{1}{2}}\times H}$, where $X = \begin{pmatrix} x \\ y \end{pmatrix} \in H_{\frac{1}{2}} \times H$. Calculating $||A^{-1}X||^2_{H_{\frac{1}{2}}\times H}$ we will have

$$||A^{-1}X||^2_{H_{\frac{1}{2}}\times H} = \left\langle \begin{pmatrix} -A_0^{-1}Dx - A_0^{-1}y \\ x \end{pmatrix}, \begin{pmatrix} -A_0^{-1}Dx - A_0^{-1}y \\ x \end{pmatrix} \right\rangle_{H_{\frac{1}{2}}\times H} =$$

$$\left\langle -A_0^{-1}Dx - A_0^{-1}y, -A_0^{-1}Dx - A_0^{-1}y \right\rangle_{H_{\frac{1}{2}}} + \langle x, x\rangle_H = ||-A_0^{-1}Dx - A_0^{-1}y||^2_{H_{\frac{1}{2}}} + \langle x, x\rangle_H \le$$

$$||A_0^{-1}Dx||^2_{H_{\frac{1}{2}}} + ||A_0^{-1}y||^2_{H_{\frac{1}{2}}} + \langle x, x\rangle_H .$$

So it sufficies to show that $A_0^{-1}D : H_{\frac{1}{2}} \longrightarrow H_{\frac{1}{2}}$, $A_0^{-1} : H \longrightarrow H_{\frac{1}{2}}$, and $I : H_{\frac{1}{2}} \longrightarrow H$ are bounded operators. First we begin with the operator I. Recall that $H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = Im(A_0^{-\frac{1}{2}})$. So, for each $x \in H_{\frac{1}{2}}$, there exists a $y \in H$ such that $x = A_0^{-\frac{1}{2}}y$. Then, since $A_0^{-\frac{1}{2}}$ is bounded and $||x||_H = ||A_0^{-\frac{1}{2}}y||_H$, $||x||_{H_{\frac{1}{2}}} = ||A_0^{\frac{1}{2}}x||_H = ||y||_H$ we have

$$||x||_H = ||A_0^{-\frac{1}{2}}y||_H \le ||A_0^{-\frac{1}{2}}|| \cdot ||y||_H = ||A_0^{-\frac{1}{2}}|| \cdot ||x||_{H_{\frac{1}{2}}}.$$

So $I : H \longrightarrow H_{\frac{1}{2}}$ is a bounded operator.

Now, we are going to show that $A_0^{-1}D$ is bounded on $H_{\frac{1}{2}}$.

The boundedness of $A_0^{-1}D$ on $H_{\frac{1}{2}}$ to $H_{\frac{1}{2}}$ can be shown as follows.
We have to find an $M > 0$ such that $||A_0^{-1}Dx||_{H_{\frac{1}{2}}} \le M||x||_{H_{\frac{1}{2}}}$ for each $x \in H_{\frac{1}{2}}$. We write, $||A_0^{-1}Dx||_{H_{\frac{1}{2}}} = ||A_0^{\frac{1}{2}}A_0^{-1}Dx||_H = ||A_0^{-\frac{1}{2}}Dx||_H$. Since $x \in H_{\frac{1}{2}} = \mathcal{D}(A_0^{\frac{1}{2}}) = \mathcal{R}(A_0^{-\frac{1}{2}})$, thus there exists an $z \in H$ such that $x = A_0^{-\frac{1}{2}}z$ or $z = A_0^{\frac{1}{2}}x$. Hence we can write

$$||A_0^{-1}Dx||_{H_{\frac{1}{2}}} = ||A_0^{-\frac{1}{2}}Dx||_H = ||A_0^{-\frac{1}{2}}DA_0^{-\frac{1}{2}}z||_H \le M||z||_H = M||A_0^{\frac{1}{2}}x||_H = M||x||_{H_{\frac{1}{2}}},$$

and the last inequality is valid since the operator $A_0^{-\frac{1}{2}}DA_0^{-\frac{1}{2}}$ is a bounded operator on H.

It remains to show that $A_0^{-1} : H \longrightarrow H_{\frac{1}{2}}$ is bounded. We have

$$||A_0^{-1}x||_{H_{\frac{1}{2}}} = ||A_0^{-\frac{1}{2}}x|| \le ||A_0^{-\frac{1}{2}}|| \cdot ||x||_H.$$

In the following lemma we are going to show the relation of spectrum, point spectrum, and essential spectrum of an operator and its inverse of operator. We do need this in the proof of a proposition related to operator pencil.

Lemma 2.5 *Suppose T is an invertible, closed linear operator. Then we have the following statements for $\lambda \neq 0$:*

- $\lambda \in \sigma(T) \Leftrightarrow \lambda^{-1} \in \sigma(T^{-1})$,
- $\lambda \in \sigma_p(T) \Leftrightarrow \lambda^{-1} \in \sigma_p(T^{-1})$,
- $\lambda \in \sigma_{ess}(T) \Leftrightarrow \lambda^{-1} \in \sigma_{ess}(T^{-1})$.

Proof: For the first part we claim that

$$\lambda \in \rho(T) \Longleftrightarrow \lambda^{-1} \in \rho(T^{-1}).$$

For this we use the following equality

$$T^{-1} - \lambda^{-1} I = (\lambda I - T)\lambda^{-1} T^{-1}. \tag{29}$$

Suppose $\lambda \in \rho(T)$, then $(\lambda I - T)^{-1}$ exists and is bounded. We show that $\lambda^{-1} \in \rho(T^{-1})$, or $(\lambda^{-1} I - T^{-1})^{-1}$ exists and is bounded. First we show that $(\lambda^{-1} I - T^{-1})^{-1}$ exists. Equivalently it's injection and onto. Suppose $x \in kern(T^{-1})$. Then $T^{-1}x = 0$. Hence $T(T^{-1})x = T(0) = 0$ or $x = 0$. Hence T^{-1} is injective. Also by our assumption $\lambda I - T$ is injective, hence by (29) $T^{-1} - \lambda^{-1} I$ is injective. Manipulating the formula (29) we have

$$\lambda^{-1}(\lambda I - T) = (T^{-1} - \lambda^{-1} I)T. \tag{30}$$

Since $\lambda I - T$ is onto, hence by (30) $(T^{-1} - \lambda^{-1} I)T$ is onto. Thus $\lambda T^{-1} - \lambda^{-1} I$ must be onto. This shows that $(\lambda^{-1} I - T^{-1})^{-1}$ exists. Now we claim that $(T^{-1} - \lambda^{-1} I)^{-1}$ is bounded. Again we use (29). We claim in fact that $T(\lambda I - T)^{-1}$ is bounded. For this suppose $\{x_n\}$ be a sequence in the domain of $(\lambda I - T)^{-1}$. Since $(\lambda I - T)^{-1}$ is continuous we have

$$(\lambda I - T)^{-1} x_n \longrightarrow (\lambda I - T)^{-1} x.$$

Since T is a closed operator, from $T(\lambda I - T)^{-1} x_n \longrightarrow y$ we will have $T(\lambda I - T)^{-1} x = y$. Hence $T(\lambda I - T)^{-1} x_n \longrightarrow T(\lambda I - T)^{-1} x$. Thus $T(\lambda I - T)^{-1}$ is a bounded operator and consequently $(T^{-1} - \lambda^{-1} I)^{-1}$ is bounded. Now assume that $\lambda^{-1} \in \rho(T^{-1})$, hence $(\lambda^{-1} I - T^{-1})^{-1}$ exists and is bounded. We claim that $(\lambda I - T)^{-1}$ exists and bounded. We use the formula (30). Since $\lambda^{-1} I - T^{-1}$ and T are injective operators, hence by (30) $\lambda I - T$ is injective. Since $T^{-1} - \lambda^{-1} I$ is onto, then the right hand side of (29) is onto. Thus $\lambda I - T$ must be onto. Using (30) and the assumptions that T^{-1} and $(T^{-1} - \lambda^{-1} I)^{-1}$ are bounded operators, we conclude that $(\lambda I - T)^{-1}$ is a bounded operators. This prove our first part.
The prove of second part we have already shown in the first part.
For the third part we prove that

$$\lambda I - T \text{ is Fredholm} \Longleftrightarrow \lambda^{-1} I - T^{-1} \text{ is Fredholm}.$$

Suppose $\lambda^{-1} I - T^{-1}$ is Fredholm. Using (30) and the fact that T is Fredholm we will have $\lambda I - T$ is Fredholm.
Converse, suppose that $\lambda I - T$ is Fredholm. By (30) $(T^{-1} - \lambda^{-1} I)T$ is Fredholm. Since T is

Fredholm and $T^{-1} - \lambda^{-1}I$ is densely defined closed operator, then $T^{-1} - \lambda^{-1}I$ is Fredholm. (See, [23, chapter 7]).

We associate now with the block matrix A the operator pencil as follows

$$L(s) = s^2 A_0^{-1} + sA_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}} + I, \quad s \in \mathbb{C}.$$

Proposition 2.2 *Let $s \in \mathbb{C}$. The range of $A - sI$ is closed if and only if $L(s)$ has a closed range. Moreover, we have*

$$\sigma(A) = \{s \in \mathbb{C} |\ 0 \in \sigma(L(s))\},$$
$$\sigma_p(A) = \{s \in \mathbb{C} |\ 0 \in \sigma_p(L(s))\},$$
$$\sigma_{ess}(A) = \{s \in \mathbb{C} |\ 0 \in \sigma_{ess}(L(s))\}.$$

And if $s \in \mathbb{C} \setminus \sigma_{ess}(A)$, then

$$dim\ kern(A - sI) = dim\ kern L(s) \quad \text{and} \quad codim\ Im(A - sI) = codim\ Im(L(s)).$$

Proof: First for the case $s = 0$. By the Proposition 2.1 A is invertible, hence A is onto. Thus $Im(A) = H_{\frac{1}{2}} \times H$ which is a closed set. Also $L(0) = I$, thus $Im(I) = H$. Hence $Im(L(0))$ is also closed. Thus the statement $Im(A)$ is closed if and only $Im(L(0))$ is closed is always true. To deal with the relationship of the spectrums we argue as follows.
By the Proposition 2.1 $0 \notin \sigma(A)$. Since $L(0) = I$, thus $0 \notin \sigma(L(0))$.

We now prove all parts of this proposition for $s \neq 0$. First we introduce the following operator pencil

$$M(s) = -A_0^{-1}D - sI - s^{-1}A_0, \quad s \in \mathbb{C} \setminus \{0\},$$

where $M(s)$ is considerd as a bounded operator on $H_{\frac{1}{2}}$. We write now the operator $A^{-1} - sI$ as the follwing matrix form.

$$A^{-1} - sI = \begin{pmatrix} I & s^{-1}A_0^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} M(s) & 0 \\ 0 & -sI \end{pmatrix} \begin{pmatrix} I & 0 \\ -s^{-1} & I \end{pmatrix}. \tag{31}$$

We note that the first and third matrix in (31) considerd as operators acting from $H_{\frac{1}{2}} \times H$ in to $H_{\frac{1}{2}} \times H$ are isomorphisms. From the equation (31) we see that

$$s^{-1} \in \sigma(A^{-1}) \Longleftrightarrow 0 \in \sigma(M(s^{-1})). \tag{32}$$

But, what is the relationship between $M(s^{-1})$ and $L(s)$? One can see that we have

$$L(s) = -sA_0^{\frac{1}{2}} M(s^{-1}) A_0^{-\frac{1}{2}}, \quad s \in \mathbb{C} \setminus \{0\}. \tag{33}$$

But, $A_0^{\frac{1}{2}}$ is an isometry from $H_{\frac{1}{2}}$ on to H, hence by (33) we have

$$0 \in \sigma(L(s)) \Longleftrightarrow 0 \in \sigma(M(s^{-1})). \tag{34}$$

From (34) and (32) we have

$$s^{-1} \in \sigma(A^{-1}) \Longleftrightarrow 0 \in \sigma(L(s)). \tag{35}$$

Also, by the Lemma 2.5 first part we have

$$s \in \sigma(A) \Longleftrightarrow s^{-1} \in \sigma(A^{-1}).$$

This and (35) show us,

$$s \in \sigma(A) \Longleftrightarrow 0 \in \sigma(L(s)).$$

The relationship between point and essential spectrum of these operators can be done in the same manner. For the last part we use the following theorem. Suppose $T = UVW$ where T, U, V and W are linear operators such that U and W are isometricly onto. Then

$$dim\ kern(T) = dim\ kern(V) \quad \text{and} \quad codim\ Im(T) = codim\ Im(V).$$

Our claim is that,

$$dim\ kern(A - sI) = dim\ kern(A^{-1} - s^{-1}I) = dim\ kern(M(s^{-1})) = dim\ kern(L(s)), \tag{36}$$

$$codim\ Im(A - sI) = codim\ Im(A^{-1} - s^{-1}I) = codim\ Im(M(s^{-1})) = codim\ Im(L(s)). \tag{37}$$

To do this end we write,

$$A - sI = A(I - sA^{-1}s) = -sA(A^{-1} - s^{-1}I).$$

Since A is one-to-one and onto by the Proposition 2.1, thus by the theorem mentioned above we have

$$dim\ kern(A - sI) = dim\ kern(A^{-1} - s^{-1}I) \quad \text{and} \quad codim\ Im(A - sI) = codim\ Im(A^{-1} - s^{-1}I). \tag{38}$$

Thus we have proved the first equations in (36) and (37). For the second equation in (36) and (37) we use the equation (31). Since the first and third matrices in (31) are isomorphisms, hence by the mentioned theorem in above

$$dim\ kern(A^{-1} - s^{-1}I) = dim\ kern \begin{pmatrix} M(s^{-1}) & 0 \\ 0 & -s^{-1}I \end{pmatrix}, \tag{39}$$

$$codim\ Im(A^{-1} - s^{-1}I) = codim\ Im \begin{pmatrix} M(s^{-1}) & 0 \\ 0 & -s^{-1}I \end{pmatrix}. \tag{40}$$

We claim now that

$$dim\ kern(A^{-1} - s^{-1}I) = dim\ kern(M(s^{-1})). \tag{41}$$

In fact, $kern \begin{pmatrix} M(s^{-1}) & 0 \\ 0 & -s^{-1}I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 0$, where $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{\frac{1}{2}} \times H$ if and only if $M(s^{-1})x = 0$ and $s^{-1}y = 0$ and this if and only if $x \in M(s^{-1})$ and $y = 0$. This and (39) prove (41). We claim now that

$$codim\ Im(A^{-1} - s^{-1}I) = codim\ Im(M(s^{-1})). \tag{42}$$

To do this suppose $M_1 = Im(M(s^{-1}))$, $X_1 = H_{\frac{1}{2}}$ and $M_2 = Im \begin{pmatrix} M(s^{-1}) & 0 \\ 0 & -s^{-1}I \end{pmatrix}$, $X_2 = H_{\frac{1}{2}} \times H$ and we claim that

$$codim\ M_1 = codim\ M_2. \tag{43}$$

This is equivalent to show that

$$dim\ \frac{X_1}{M_1} = dim\ \frac{X_2}{M_2}. \tag{44}$$

Thus, let $\begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} M(s^{-1}) & 0 \\ 0 & -s^{-1}I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in \frac{X_2}{M_2}$. This is equal to

$$\begin{pmatrix} M(s^{-1}) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -s^{-1}I \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in Im \begin{pmatrix} M(s^{-1}) & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & -s^{-1}I \end{pmatrix},$$

and this is equal to $\begin{pmatrix} M(s^{-1}) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \in Im \begin{pmatrix} M(s^{-1}) & 0 \\ 0 & 0 \end{pmatrix}$ or equivalently $M(s^{-1})x \in Im(M(s^{-1}))$. This shows (44) and consequently we have (43).

This, (38) and (39) prove the second equality of (37).

To pove the last equality of (36) and (37) we use (33), the fact that $A_0^{\frac{1}{2}} : H_{\frac{1}{2}} \longrightarrow H$ is isometrically onto H and the theorem mentioned above. These show us that,

$$dim\ kern(L(s)) = dim\ kern(M(s^{-1})), codim\ Im(L(s)) = codim\ Im(M(s^{-1})).$$

These equalities with (42), (41), and (38) show the last equality of (36) and (37). Now we claim that for $s \neq 0$, $Im(A - sI)$ is closed if and only if $Im(L(s))$ is closed. Since $A_0^{\frac{1}{2}}$ and $A_0^{-\frac{1}{2}}$ are isometries, then by (33) $ImL(s)$ is closed if and only if $ImM(s^{-1})$ is closed. Using (31) and the fact that the first and third matrices in (31) are invertible we have $Im(M(s^{-1}))$ is closed if and only if $Im(A^{-1} - s^{-1}I)$ is closed. If we prove that $Im(A^{-1} - s^{-1}I) = Im(A - sI)$ then we are done. This can be proeved since $A^{-1} - s^{-1}I = (sI - A)s^{-1}A^{-1}$, thus $Im(A^{-1} - s^{-1}I) \subset Im(sI - A)$, and since $sI - A = (A^{-1} - s^{-1}I)sA$, thus $Im(sI - A) \subset Im(A^{-1} - s^{-1}I)$.

3 The location of spectrum of A

Definition 3.1 *The family* $(T(t))_{t\geq 0}$ *of bounded linear operators on a Banach space* X $(T(t) : X \longrightarrow X)$ *is called a one parameter semigroup on* X *or a semigroup on* X *if it satisfies the following functional equation* (see, [11, chapter 1])

$$\begin{cases} T(t+s) = T(t)T(s) \quad \text{for each } t, s \geq 0; \\ T(0) = I. \end{cases}$$

Example 3.1 *Suppose* $X = \mathbb{C}^n$, $T : \mathbb{R}_+ \longrightarrow L(\mathbb{C}^n) \cong M_{n\times n}(\mathbb{C})$*, and* T *defined by* $T(t) = e^{tB}$*, for each* $B \in M_{n\times n}(\mathbb{C})$*. Then* $(T(t))_{t\geq 0}$ *is a semigroup on Banach space* $\mathbb{C}^n$*.* (See, [11, chapter 1]).

Definition 3.2 *A family* $(T(t))_{t\geq 0}$ *of bounded linear operators is called a strongly continuous semigroup if it's a semigroup and the map* $\xi_x : \mathbb{R}_+ \longrightarrow X$ *defined by* $\xi_x(t) = T(t)x$ *be continuous.* (See, [11, chapter 1]).

Definition 3.3 *The generator* $B : \mathcal{D}(B) \subseteq X \longrightarrow X$ *of a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* X *is the operator* (see, [11, chapter 2])

$$Bx = \lim_{h\to 0^+} \frac{1}{h}(T(h)x - x)$$

defined for every x *in its domain*

$$\mathcal{D}(B) = \{x \in X : \xi_x \text{ is differentiable}\}.$$

The following theorem is well known. Before we prove this theorem we have to define two notions and an important theorem called Lumer-Phillips theorem. (See, [21]). First, the notions.
Let $(H, \langle ., . \rangle)$ be a Hilbert space and $T : \mathcal{D}(T) \subset H \longrightarrow H$ a densely defined linear operator. We say that T is dissipative if $Re\langle Tx, x\rangle \leq 0$ for each $x \in \mathcal{D}(T)$.
A semigroup $(T(t))_{t\geq 0}$ is called a contraction semigroup if $||T(t)|| \leq 1$.
Now, the statement of Lumer-Phillips theorem.
If $T : \mathcal{D}(T) \subset H \longrightarrow H$ be a densely defined closed linear operator and if both T and T^* are dissipative, then T generates a strongly continuous semigroup of contraction operators. (See, [21]).

Theorem 3.1 *The operator* A *is the generator of a strongly continuous semigroup* $(T(t))_{t\geq 0}$ *of contractions on the state space* $H_{\frac{1}{2}} \times H$*.*

Proof: First we mention that just as complex numbers we can write a linear operator $T : H \longrightarrow H$ as $T = T_1 + iT_2$ where $T_1 = \frac{T+T^*}{2}$, $T_2 = \frac{T-T^*}{2i}$. Hence we can write $ReT = \frac{T+T^*}{2}$. Applying this to the operator A and for each $\begin{pmatrix} x \\ y \end{pmatrix} \in H_{\frac{1}{2}} \times H_{\frac{1}{2}}$ we obtain,

$$2Re\left\langle A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} = \left\langle A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} + \left\langle A^*\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} =$$

$$\left\langle A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} + \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, A\begin{pmatrix} x \\ y \end{pmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} =$$

$$\left\langle \begin{pmatrix} y \\ -A_0x - Dy \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} + \left\langle \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} y \\ -A_0x - Dy \end{pmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} =$$

$$\left\langle A_0^{\frac{1}{2}}y, A_0^{\frac{1}{2}}x\right\rangle_H + \langle -A_0x - Dy, y\rangle_H + \left\langle A_0^{\frac{1}{2}}x, A_0^{\frac{1}{2}}y\right\rangle_H + \langle y, -A_0x - Dy\rangle_H =$$

$$\langle A_0y, x\rangle_H - \langle A_0x, y\rangle_H - \langle Dy, y\rangle_H + \langle A_0x, y\rangle_H - \langle y, A_0x\rangle_H - \langle y, Dy\rangle_H =$$

$$-\langle Dy, y\rangle_H - \langle y, Dy\rangle_H = -2\langle Dy, y\rangle_H,$$

where the last equality is valid since it has been already proved that $D : H \longrightarrow H$ is a symmetric operator. Also, since $y \in H_{\frac{1}{2}}$ and by our assumption $\langle Dy, y\rangle_H \geq 0$ for each $y \in H_{\frac{1}{2}}$ we will have $Re\left\langle A\begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix}\right\rangle_{H_{\frac{1}{2}}\times H} \leq 0$. In the same manner we can prove that A^* is a dissipative operator. As a consequence of this theorem we have that $\sigma(A)$ is contained in the closed left half plane. (See [13, chapter I]).

It can happen that the spectrum of A is an unbounded region in the left half plane, in fact the following example shows this property.

Example 3.2 *Let $H = L_2(0,\infty)$, and suppose $\epsilon > 0$ is given. Let $\{q_j\}_{j\in\mathbb{N}} \subset \mathbb{R}$ be a sequence satisfying $\{q_j\}_{j\in\mathbb{N}} = \mathbb{Q}$. Define $a : (0,\infty) \longrightarrow \mathbb{R}$ and $d : (0,\infty) \longrightarrow [0,\infty)$ by*

$$a(x) = q_j \quad \text{if} \quad j-1 < x \leq j, \quad \text{for} \quad j \in \mathbb{N},$$

and

$$d(x) = \begin{cases} \frac{1}{x-j+1} - 1 & \text{if } j-1 < x \leq j \text{ and } |q_j| \geq \epsilon; \\ \frac{1}{x-j+1} - 1 + \sqrt{\epsilon^2 - q_j^2} & \text{if } j-1 < x \leq j \text{ and } |q_j| < \epsilon. \end{cases}$$

The function $a_0 : (0,\infty) \longrightarrow (0,\infty)$ is defined by $a_0(x) = a(x)^2 + d(x)^2$, for each $x \in (0,\infty)$. We shall show that the $\sigma(A) = \mathbb{C}^- \setminus \{0\}$.
For each $x \in (0,\infty)$ we have

$$a_0(x) = a(x)^2 + d(x)^2 \geq q_j^2 \geq \epsilon^2. \tag{45}$$

The reason can be seen by considering the cases $|q_j| \geq \epsilon$, and $|q_j| < \epsilon$.
If $|q_j| \geq \epsilon$, then we have

$$a_0(x) = a(x)^2 + d(x)^2 = q_j^2 + d(x)^2 \geq q_j^2 \geq \epsilon^2.$$

If $|q_j| < \epsilon$, then we have

$$a_0(x) = a(x)^2 + d(x)^2 = q_j^2 + \left(\frac{1}{x-j+1} - 1 + \sqrt{\epsilon^2 - q_j^2} \right)^2 =$$

$$q_j^2 + \left(\frac{1}{x-j+1} - 1 \right)^2 + \epsilon^2 - q_j^2 + 2 \left(\frac{1}{x-j+1} \right) \left(\sqrt{\epsilon^2 - q_j^2} \right).$$

If $|q_j| < \epsilon$, then since $j-1 < x \le j$, thus $\frac{1}{x-j+1}$ attains it's minimum at $x = j$, and it's maximum is 1. Also $|q_j| < \epsilon$, hence the last part in above is always positive. Hence if $|q_j| \ge \epsilon$ or $|q_j| < \epsilon$ we will have

$$a_0(x) = a(x)^2 + d(x)^2 \ge \epsilon^2. \tag{46}$$

If $d(x) \ge 2$, then we have $2d(x) \le d(x)^2$. But we have also $d(x)^2 \le d(x)^2 + a(x)^2$. From the last two equations we obtain

$$2d(x) \le a(x)^2 + d(x)^2, \tag{47}$$

and if $d(x) < 2$ it follows from (46) that

$$2d(x) \le \frac{4}{\epsilon^2} \left(a(x)^2 + d(x)^2 \right). \tag{48}$$

We set $\mathcal{D}(A_0)$=$\{f \in H | \; a_0 f \in H\}$. The operators $A_0 : \mathcal{D}(A_0) \subset H \longrightarrow H$ and $D : H_{\frac{1}{2}} \longrightarrow H_{-\frac{1}{2}}$, defined by

$$\left(A_0 f \right)(x) = a_0(x) f(x), \quad x \in (0, \infty), f \in \mathcal{D}(A_0),$$

$$\left(D g \right)(x) = 2d(x) g(x), \quad x \in (0, \infty), g \in H_{\frac{1}{2}},$$

satisfy the following statements. Firstly, A_0 is positive definite, self adjoint, and A_0^{-1} exists and bounded. Secondly, the operator $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is bounded and self adjoint. Thirdly, D is a positive operator.
A_0 is positive definite, since

$$\langle A_0 f, f \rangle_H = \int_0^\infty a_0(x) f(x) \overline{f(x)} \, dx = \int_0^\infty a_0(x) |f(x)|^2 \, dx.$$

By (31), $a_0(x) > 0$, and we know also that $|f(x)|^2 > 0$, $\forall f \neq 0$. Hence we will have, $\langle A_0 f, f \rangle_H > 0$.
A_0 is self adjoint. This can be seen by,

$$\langle A_0 f, g \rangle_H = \int_0^\infty a_0 f(x) g(x) \, dx = \int_0^\infty f(x) \overline{a_0(x) g(x)} \, dx = \langle f, A_0 g \rangle_H.$$

A_0^{-1} exists and is bounded, since for each f we have $(A_0^{-1}f)(x) = \frac{f(x)}{a_0(x)}$, and by (31) $a_0(x) \neq 0$. Also, the inverse is bounded, because $||(A_0^{-1}f)(x)|| = |\frac{f(x)}{a_0}| \leq \frac{f(x)}{\epsilon^2}$.
Now, for convenience we set $T = A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$. We claim that T is self adjoint and bounded. Note that $Tf(x) = \frac{2d(x)f(x)}{a_0(x)}$. The self adjointness of T can be shown by,

$$\langle Tf, g\rangle_H = \int_0^\infty \frac{2d(x)f(x)}{a_0(x)}\overline{g(x)}\, dx = \int_0^\infty f(x)\overline{\frac{2d(x)g(x)}{a_0(x)}}\, dx = \langle f, Tg\rangle_H .$$

The boundedness of T must be done carefully by taking $d(x) \geq 2$ and $d(x) < 2$.
Suppose $d(x) \geq 2$. Then by $Tf(x) = \frac{2d(x)f(x)}{a(x)^2+d(x)^2}$, and by (48) we have, $||Tf(x)|| = |\frac{2d(x)f(x)}{a(x)^2+d(x)^2}| \leq |\frac{2d(x)f(x)}{2d(x)} = |f(x)|$. Hence in this case T is bounded. Now assume $d(x) < 2$. Then we obtain, $||Tf(x)|| = |\frac{2d(x)f(x)}{a(x)^2+d(x)^2}| \leq 2\frac{|f(x)|}{\epsilon^2}$. Thus in this case is T bounded. Hence T is a bounded operator.

Next, we show that for the operator $D : H_{\frac{1}{2}} \longrightarrow H_{-\frac{1}{2}}$ we have $\langle Dg, g\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}$ for each $g \in H_{\frac{1}{2}}$. For this we prove first that $ImD \subset H$. Assume $g \in H_{\frac{1}{2}} = ImA_0^{-\frac{1}{2}}$. Thus $g = A_0^{-\frac{1}{2}}y$ for some $y \in H$. Hence we have

$$Dg = \frac{2d(x)y(x)}{\sqrt{a_0(x)}} = \frac{2d(x)y(x)}{\sqrt{a(x)^2 + d(x)^2}} \leq 2y(x),$$

which shows that $ImD \subset H$. Therefore we have $\langle Dg, g\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \langle Dg, g\rangle_H$ for each $g \in H_{\frac{1}{2}}$, and we can write

$$\langle Dg, g\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \int_0^\infty 2d(x)g(x)\overline{g(x)}\, dx = \int_0^\infty 2d(x)|g(x)|^2\, dx.$$

Since $d(x) \geq 0$, and $|g(x)|^2 \geq 0$, thus $\langle Dg, g\rangle_H \geq 0$.
One can also show that $D : H_{\frac{1}{2}} \longrightarrow H_{-\frac{1}{2}}$ is a bounded operator.
Also, by Proposition 2.2

$$L(s) = ((s^2A_0^{-1} + sA_0^{-\frac{1}{2}}DA_0^{-\frac{1}{2}} + I)f)(x) = \frac{s^2f(x)}{a(x)^2 + d(x)^2} + 2\frac{sd(x)f(x)}{a(x)^2 + d(x)^2} + f(x). \tag{49}$$

Now, to be not distracted from our idea, we just forget for a moment the details in the following proof.
Our idea is to prove $\sigma(A) = \sigma_{ess}(A)=\mathbb{C}^- \setminus \{0\}$. Set $M = \{-d(x) \pm ia(x)|\ x \in [0,\infty)\}$. We will see that $L(-d(x) \pm ia(x))$ is not Fredholm. For a moment assume that we know this. Then this means that $0 \in \sigma_{ess}(L(-d(x) \pm ia(x))$ or by Proposition 2.2, $-d(x) \pm ia(x) \in \sigma_{ess}(A)$, which shows that $M \subset \sigma_{ess}(A)$. From this $\overline{M} \subset \sigma_{ess}(A)$. Since $\overline{M} =$

$\{-\xi+i\eta|\xi\geq 0,\eta\in\mathbb{R}\}$ and $\sigma_{ess}(A)\subseteq\mathbb{C}^-\setminus\{0\}$ we must have $\overline{M}=\sigma_{ess}(A)$. Since $\sigma(A)\subseteq\mathbb{C}^-\setminus\{0\}$ and $\sigma_{ess}(A)\subseteq\sigma(A)$ we have also $\sigma(A)=\mathbb{C}^-\setminus\{0\}$.

But we remember now the details. In the proof we used that $(L(-d(x)\pm ia(x))$ is not Fredholm. Calculating $L(-d(x)\pm ia(x))$ is not difficult, we do this for example for the plus sign. In fact this is,

$$L(-d(x)+ia(x))=\frac{(-d(x)+ia(x))^2+2(-d(x)+ia(x))d(x)+(-d(x)+ia(x))(-d(x)-ia(x))}{(-d(x)+ia(x))(-d(x)-ia(x))}.$$

After simple calculation we will have, $L(-d(x)+ia(x))=0$. Since $H=L^2(0,\infty)$, thus H is infinite dimensional, thus $dim\ Kern(0)=\infty$, where $0:H\longrightarrow H$. This shows that $L(-d(x)+ia(x))$ is not Fredholm.

We know from Theorem 3.1 that the spectrum of the operator A lies in the left half plane. But there arises a question. Can we give conditions under which we can determine the region of spectrum of A more accuratly? For this we need first some definitions. Theorem 3.2 gives the answer to this question.

Definition 3.4 *Define the following.*

1. $\beta=\inf_{z\in H_{\frac{1}{2}}\setminus\{0\}}\frac{\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{||z||_H^2}$.
2. $\gamma=\sup_{z\in H_{\frac{1}{2}}\setminus\{0\}}\frac{\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{||z||_H^2}$.
3. $\delta=\inf_{z\in H_{\frac{1}{2}}\setminus\{0\}}\frac{\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{||z||_{H_{\frac{1}{2}}}^2}$.

Since $\langle Dz,z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\geq 0$ we conclude that $\beta,\gamma,\delta\in[0,\infty)$. It's clear by the definition of supremum and infimum that $\beta\leq\gamma$. To see that $\delta\leq\beta$ we use the following argument. We need only to prove that there exists a $c>0$ such that for each $z\in H_{\frac{1}{2}}$ we have $||z||_H\leq c||z||_{H_{\frac{1}{2}}}$. To this end we note that by definition $\langle x,y\rangle_{H_{\frac{1}{2}}}=\langle A_0x,y\rangle_H$ for each $x,y\in H_{\frac{1}{2}}$ for some $\epsilon>0$. Since A_0 is positive definite and $0\in\rho(A_0)$ then for some $\epsilon>0$ we have

$$\langle A_0x,x\rangle_H\geq\frac{1}{\epsilon^2}||x||_H^2.$$

Also, by self adjointness of A_0 we have

$$\langle A_0x,x\rangle_H=\left\langle A_0^{\frac{1}{2}}x,A_0^{\frac{1}{2}}x\right\rangle_H.$$

From these two last inequalities we obtain
$||x||^2\leq\epsilon^2\langle A_0x,x\rangle_H=\epsilon^2\left\langle A_0^{\frac{1}{2}}x,A_0^{\frac{1}{2}}\right\rangle_H=\epsilon^2\langle x,x\rangle_{H_{\frac{1}{2}}}=\epsilon^2||x||_{H_{\frac{1}{2}}}$. Hence $||x||_H\leq\epsilon^2||x||_{H_{\frac{1}{2}}}$.

Theorem 3.2 *Suppose δ, β, γ are as in the Definition 3.4, then we have*

1. *If $\beta > 0$, then $\left\{\lambda \in \mathbb{C} | \; Re\lambda > -\frac{\beta}{2}, Im\lambda \neq 0\right\} \subset \rho(A)$.*
2. *If $\gamma < \infty$, then $\left\{\lambda \in \mathbb{C} | \; Re\lambda < -\frac{\gamma}{2}, Im\lambda \neq 0\right\} \subset \rho(A)$.*
3. *If $\delta > 0$, then $\sigma(A) \subset \left\{\lambda \in \mathbb{C} | \; |Im\lambda|^2 \leq -\frac{2}{\delta} Re\lambda - (Re\lambda)^2\right\} \cup (-\infty, 0)$.*
4. *If $\langle Dz, z\rangle^2_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \geq 4||z||^2_{H_{\frac{1}{2}}} \cdot ||z||^2_H$, then $\sigma(A) \subset (-\infty, 0)$.*

Corollary 3.1 *If $D = 0$, then $\sigma(A) = \sigma_{ess}(A) \subset i\mathbb{R}$. Moreover, for $\eta \in \mathbb{R}$ we have, $i\eta \in \sigma(A)$ if and only if $\eta \in \sigma(A_0)$.*

The proof of the above theorem and corollary depends on the results of the following lemma. Also, the proof of part 4 of the Theorem 3.2 is not correct in the original paper, that's why we give another proof for this part.

Lemma 3.1 *Let $\lambda = \mu + i\sigma$ with $\sigma \in \mathbb{R}$, $\mu \leq 0$ and $\lambda \neq 0$. Assume that there exists a sequence $\left\{\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\}_{H_{\frac{1}{2}} \times H} \in \mathcal{D}(A)$ with*

$$\left\|\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\|_{H_{\frac{1}{2}} \times H} = 1 \quad \text{and} \quad \lim_{n\to\infty} \left\|(\lambda I - A)\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\|_{H_{\frac{1}{2}} \times H} = 0. \tag{50}$$

Then we have

1. *$||y_n - \lambda x_n||_{H_{\frac{1}{2}}} \to 0$, as $n \to \infty$.*
2. *$liminf_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}} > 0$.*
3. *If $\sigma \neq 0$, then we have*

$$\left\langle A_0^{-1} D x_n, x_n\right\rangle_{H_{\frac{1}{2}}} + \frac{2\mu}{\mu^2 + \sigma^2} ||x_n||^2_{H_{\frac{1}{2}}} \longrightarrow 0, \text{ as } n \to \infty, \tag{51}$$

$$\left\langle A_0^{-1} D x_n, x_n\right\rangle_{H_{\frac{1}{2}}} + 2\mu ||x_n||^2_H \to 0, \text{ as } n \to \infty, \tag{52}$$

$$\langle A_0 x_n, x_n\rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} - (\mu^2 + \sigma^2)||x_n||^2_H \longrightarrow 0, \text{ as } n \to \infty. \tag{53}$$

If $\sigma = 0$, then we have

$$||x_n||^2_{H_{\frac{1}{2}}} + \mu \left\langle A_0^{-1} D x_n, x_n\right\rangle_{H_{\frac{1}{2}}} + \mu^2 ||x_n||^2_H \to 0, \text{ as } n \to \infty. \tag{54}$$

Proof: We write,

$$\left\|(\lambda I - A)\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\|_{H_{\frac{1}{2}}\times H} = \left\|\begin{pmatrix} \lambda x_n - y_n \\ A_0 x_n + D y_n + \lambda y_n \end{pmatrix}\right\|_{H_{\frac{1}{2}}\times H}.$$

From (50) we have

$$\left(||\lambda x_n - y_n||^2_{H_{\frac{1}{2}}} + ||A_0 x_n + D y_n + \lambda y_n||^2_H \to 0\right)^{\frac{1}{2}}, \text{ as } n \to \infty.$$

Thus,

$$||y_n - \lambda x_n||_{H_{\frac{1}{2}}} \to 0. \tag{55}$$

This proves 1.
For part 2, suppose that $\liminf_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}} = 0$. Thus, there exists a $\{x_{k_n}\}$ subsequence of $\{x_n\}$ such that $x_{k_n} \to 0$, as $n \to \infty$. Also, (55) can be written as, $y_n = \lambda x_n + r_n$, where $r_n \to 0$ as $n \to \infty$. Thus, $y_{k_n} = \lambda x_{k_n} + r_{k_n}$, where $r_{k_n} \to 0$ as $n \to \infty$. Since $x_{k_n} \to 0$, also $y_{k_n} \to 0$, hence $\left\|\begin{pmatrix} x_{k_n} \\ y_{k_n} \end{pmatrix}\right\|_{H_{\frac{1}{2}}\times H} \neq 1$, and this contradicts our assumption. This proves part 2.

We also have

$$||A_0 x_n + D y_n + \lambda y_n||_H \to 0, \text{ as } n \to \infty. \tag{56}$$

We now take the inner product of the vector $A_0 x_n + D y_n + \lambda y_n$ with x_n:

$$\left|\langle A_0 x_n + D y_n + \lambda y_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\right| \le \|A_0 x_n + D y_n + \lambda y_n\|_{H_{-\frac{1}{2}}} \cdot ||x_n||_{H_{\frac{1}{2}}} \to 0, \text{ as } n \to \infty. \tag{57}$$

Since $|| \langle A_0 x_n + D y_n + \lambda y_n, x_n\rangle_{H_{-\frac{1}{2}}} || = || \left\langle A_0^{-\frac{1}{2}}(A_0 x_n + D y_n + \lambda y_n), x_n \right\rangle_H ||$ and $||x_n||_{H_{\frac{1}{2}}} \le 1$, we have

$$\left|\langle A_0 x_n + D y_n + \lambda y_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}\right| \to 0, \text{ as } n \to \infty. \tag{58}$$

We now expand the inner product to obtain

$$\langle A_0 x_n + D y_n + \lambda y_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \langle A_0 x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \langle D y_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \langle \lambda y_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}. \tag{59}$$

In (59) we change the second and third elements in the right hand side. By (55) $y_n = \lambda x_n + r_n$, where $r_n \to 0$ as $n \to \infty$. Hence

$$\langle D y_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \langle D(\lambda x_n + r_n), x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \lambda \langle D x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \langle D r_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}.$$

Also,

$$\langle \lambda y_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \langle \lambda(\lambda x_n + r_n), x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \lambda^2 \langle x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \lambda \langle r_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}.$$

By (58), (59) and the above arguments we have

$$
\begin{aligned}
|\langle A_0x_n + Dy_n + \lambda y_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}| = |\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \lambda\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \\
\lambda^2\langle x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \langle Dr_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \langle r_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}| \to 0, \text{ as } n \to \infty.
\end{aligned}
$$

It is also clear that $\langle r_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \to 0$, since $r_n \to 0$ as $n \to \infty$. We are going now to show that $\langle Dr_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \to 0$, as $n \to \infty$.

Since $D : H_{\frac{1}{2}} \longrightarrow H_{-\frac{1}{2}}$ is a bounded operator by our assumption, $r_n \in H_{\frac{1}{2}}$ and $||x_n||_{H_{\frac{1}{2}}} \leq 1$ we have

$$
\begin{aligned}
|\langle Dr_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}| \leq ||Dr_n||_{H_{-\frac{1}{2}}} \cdot ||x_n||_{H_{\frac{1}{2}}} \leq \\
||D|| \cdot ||r_n||_{H_{\frac{1}{2}}}|| \cdot ||x_n||_{H_{\frac{1}{2}}} \leq ||D|| \cdot ||r_n||_{H_{\frac{1}{2}}} \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

Hence

$$
\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu + i\sigma)\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu + i\sigma)^2\langle x_n, x_n\rangle_H \to 0, \text{ as } n \to \infty. \quad (60)
$$

We point out that in the last statement of (60) we have used the property $\langle z', z\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \langle z', z\rangle_H$, for each $(z', z) \in H \times H_\alpha$, when $\alpha \geq 0$.

To treat the part 3, first we find the imaginary and real part of (60).
Doing a simple algebraic calculation on (60) we obtain

$$
\begin{aligned}
\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu^2 - \sigma^2)\langle x_n, x_n\rangle_H + \mu\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \\
i\sigma\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + i(2\mu\sigma)\langle x_n, x_n\rangle_H \longrightarrow 0, \text{ as } n \longrightarrow \infty.
\end{aligned}
$$

The imaginary part tends to zero, this is,

$$
\sigma\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + 2\mu\sigma\langle x_n, x_n\rangle_H \to 0, \text{ as } n \to \infty.
$$

Since $\sigma \neq 0$ we can devide the above limit to σ and obtain

$$
\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + 2\mu\langle x_n, x_n\rangle_H \to 0, \quad \text{as } n \to \infty. \quad (61)
$$

Also, the real part tends to zero, this is,

$$
\langle A_0x_n, x_n\rangle + \mu\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + (\mu^2 - \sigma^2)\langle x_n, x_n\rangle_H \to 0, \text{ as } n \to \infty. \quad (62)
$$

We are now ready to prove (51), (52), (53), (54).

We begin with (52). By (61) it's enough to prove that $\left\langle A_0^{-1}Dx_n, x_n\right\rangle_{H_{\frac{1}{2}}} = \langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}$. To prove this observe

$$\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \langle Dx_n, x_n\rangle_H = \left\langle A_0^{-\frac{1}{2}}Dx_n, A_0^{-\frac{1}{2}}x_n\right\rangle_{H_{\frac{1}{2}}} = \left\langle A_0^{-1}Dx_n, x_n\right\rangle_{H_{\frac{1}{2}}}. \tag{63}$$

We point out that the last equality above is valid, since $A_0^{-\frac{1}{2}}$ is a self adjoint operator. This proves (52).

We prove now (53). Multiplying (61) by μ, we have

$$\mu\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + 2\mu^2\,\langle x_n, x_n\rangle_H \to 0,\ \text{as}\ \ n\to\infty. \tag{64}$$

Substracting (62) from (64) we get

$$\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - (\mu^2+\sigma^2)\,\langle x_n, x_n\rangle_H \to 0,\ \text{as}\ \ n\to\infty.$$

This proves (53).
Finally, we prove (51).
Multiplying (53) by $\frac{2\mu}{\mu^2+\sigma^2}$, we have

$$\frac{2\mu}{\mu^2+\sigma^2}\,\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} - 2\mu||x_n||^2 \to 0,\ \text{as}\ \ n\to\infty. \tag{65}$$

Adding (65) to (52) we have

$$\left\langle A_0^{-1}Dx_n, x_n\right\rangle_{H_{\frac{1}{2}}} + \frac{2\mu}{\mu^2+\sigma^2}\,\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \to 0,\ \text{as}\ \ n\to\infty. \tag{66}$$

But,

$$\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \left\langle A_0^{\frac{1}{2}}x_n, A_0^{\frac{1}{2}}x_n\right\rangle_H = \langle x_n, x_n\rangle_{H_{\frac{1}{2}}} = ||x_n||^2_{H_{\frac{1}{2}}}. \tag{67}$$

Using (67) in (66) we have (51). Thus (51) is also proved. The proof of (54) can be done as follows. In (60) we set $\sigma = 0$, then we get

$$\langle A_0x_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu\,\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} + \mu^2\,\langle x_n, x_n\rangle \to 0,\ \text{as}\ \ n\to\infty. \tag{68}$$

Using (63) and (67) in (68) we have (54).

Now we prove the Theorem 3.2. As promised, we do this by using the results of Lemma 3.1.

Proof: Suppose $\lambda = \mu + i\sigma$ with $\mu \leq 0$ and $\sigma \neq 0$. For part (1) and (2) we obtain a contradiction when $\mu > -\frac{\beta}{2}$ and $\mu < -\frac{\gamma}{2}$, respectively. By the begining of the proof of Proposition 2.1 we have $\overline{\lambda} \in \sigma(A)$. We have already shown in the Lemma 2.2 that

$\left(H_{\frac{1}{2}} \times H, [.,.]\right)$ is a Krein space, and in the Lemma 2.3 we have proved that A is a self adjoint operator on the Krein space $H_{\frac{1}{2}} \times H$. It follows that at least one of the points λ and $\overline{\lambda}$ belongs to $\sigma_{ap}(A)$. The reason of this is easy to see, since $\sigma = \sigma_p \cup \sigma_c \cup \sigma_r$ and we have $\sigma_p \cup \sigma_c = \sigma_{ap}$, (see Lemma 1.3), hence $\sigma = \sigma_{ap} \cup \sigma_r$. If $\lambda \in \sigma_{ap}(A)$ then we are done. If not, then $\lambda \in \sigma_r(A)$. This implies $\overline{\lambda} \in \sigma_p(A)$. But by the Lemma 1.2 $\sigma_p \subset \sigma_{ap}(A)$, hence $\lambda \in \sigma_{ap}(A)$. Thus by the definition of σ_{ap} there exists a sequence, $\left\{\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\}_{n\in\mathbb{N}} \in \mathcal{D}(A)$ such that

$$\left\|\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\|_{H_{\frac{1}{2}}\times H} = 1 \quad \text{and} \quad \lim_{n\to\infty}\left\|(\lambda I - A)\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\|_{H_{\frac{1}{2}}\times H} = 0.$$

Thus we have the conditions of the Lemma 3.1. So $\liminf_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}} > 0$ and also (61) is valid.
We claim now that if $\mu > -\frac{\beta}{2}$, then

$$\lim_{n\to\infty} \langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = \lim_{n\to\infty} \langle x_n, x_n\rangle_H = 0. \tag{69}$$

For proving this we consider the following possibilities,

1. $\lim_{n\to\infty} \langle x_n, x_n\rangle_H \neq 0$ but $\lim_{n\to\infty} \langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = 0$.
2. $\lim_{n\to\infty} \langle x_n, x_n\rangle_H = 0$ but $\lim_{n\to\infty} \langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \neq 0$.
3. $\lim_{n\to\infty} \langle x_n, x_n\rangle_H \neq 0$ and $\lim_{n\to\infty} \langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} \neq 0$.
4. $\lim_{n\to\infty} \langle x_n, x_n\rangle_H = 0$ and $\lim_{n\to\infty} \langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}} = 0$.

The first and second one can be eliminated directly, since they contradict (61). Thus, we do begin with part 3. By Definition 3.4 we have $\beta \leq \frac{\langle Dx,x\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\langle x,x\rangle_H}$, for each $x \in H_{\frac{1}{2}} \setminus \{0\}$, hence for the sequence $\{x_n\}$ we will have, $-\frac{\beta}{2} \geq -\frac{\langle Dx_n,x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{2\langle x_n,x_n\rangle_H}$. If $\mu > -\frac{\beta}{2}$, then $\mu > -\frac{\langle Dx_n,x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{2\langle x_n,x_n\rangle_H}$, or $2\mu\langle x_n, x_n\rangle_H > -\langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}$. Taking limit we have, $2\mu \lim_{n\to\infty} \langle x_n, x_n\rangle_H \geq -\lim_{n\to\infty} \langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}$.

We note that we can never have $2\mu \lim_{n\to\infty} \langle x_n, x_n\rangle_H = -\lim_{n\to\infty} \langle Dx_n, x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}$ if $\mu > -\frac{\beta}{2}$. If we would have this, then $-\lim_{n\to\infty} \frac{\langle Dx_n,x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{2\langle x_n,x_n\rangle_H} > -\frac{\beta}{2}$, or $\frac{\langle Dx_n,x_n\rangle_{H_{-\frac{1}{2}}\times H_{\frac{1}{2}}}}{\langle x_n,x_n\rangle_H} < \frac{\beta}{2}$ for each $n \in \mathbb{N}$, and this contradics the Definition 3.4 part 1.

Hence the only possibility is the part 4. If we now set (69) in (62) we will obtain $\lim_{n\to\infty} \langle A_0 x_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = 0$. This and (67) give us $\lim_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}} = 0$.
Thus $\liminf_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}} = 0$ and this contradicts the Lemma 3.1 part 2.

For proving part 2 of the Theorem 3.2 we can use the same method.

Next, we prove part 3.
By Definition 3.4 we have

$$\delta \le \frac{\langle Dx, x \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{||x||^2_{H_{\frac{1}{2}}}}, \quad \forall x \in H_{\frac{1}{2}} \setminus \{0\}\,. \tag{70}$$

Also, by (63) we can write (66) as follows,

$$\langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} + \frac{2\mu}{\mu^2 + \sigma^2} ||x_n||^2_{H_{\frac{1}{2}}} \to 0, \quad as \quad n \to \infty.$$

Dividing the last one by $||x_n||^2_{H_{\frac{1}{2}}}$ (we can do this, since $x_n \in H_{\frac{1}{2}} \setminus \{0\}$) we obtain

$$\frac{\langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{||x_n||^2_{H_{\frac{1}{2}}}} + \frac{2\mu}{\mu^2 + \sigma^2} \to 0, \quad \text{as} \quad n \to \infty. \tag{71}$$

From (70) and (71) we have

$$\delta + \frac{2\mu}{\mu^2 + \sigma^2} \le \frac{\langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{||x||^2_{H_{\frac{1}{2}}}} + \frac{2\mu}{\mu^2 + \sigma^2} \to 0, \quad \text{as} \quad n \to \infty.$$

Hence, $\mu^2 + \sigma^2 \le -\frac{2\mu}{\delta}$, or $(Re\lambda)^2 + (Im\lambda)^2 \le -\frac{2}{\delta} Re\lambda$.

If $\sigma = 0$, then $\lambda = \mu$. Since $\mu \le 0$ we will have $\lambda \le 0$. Since $0 \in \rho(A)$, $\lambda < 0$, and we have done part 3 too.

Finally we prove part 4. We give here another proof than the original one, since the original proof is not correct.
By the fact that $\liminf_{n\to\infty} ||x_n||^2_{H_{\frac{1}{2}}} > 0$ we can divide (52) by $||x_n||^2_{H_{\frac{1}{2}}}$. So, we obtain, $\mu = -\lim_{n\to\infty} \frac{\langle A_0^{-1} Dx_n, x_n \rangle_{H_{\frac{1}{2}}}}{2||x_n||^2_{H_{\frac{1}{2}}}}$ or by (63) $\mu = -\lim_{n\to\infty} \frac{\langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}}}{2||x||^2_{H_{\frac{1}{2}}}}$. Since D is a positive operator we have $\mu \le 0$, but $0 \in \rho(A)$, so $\mu < 0$.
We are now going to prove that $\sigma = 0$. If not we have $\lambda = \mu + i\sigma$.
We write,

$$4||x_n||^2_H ||x_n||^2_{H_{\frac{1}{2}}} \le \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \times \langle Dx_n, x_n \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} = \langle A_0^{-1} Dx_n, x_n \rangle_{H_{\frac{1}{2}}} \times \langle A_0^{-1} Dx_n, x_n \rangle_{H_{\frac{1}{2}}}, \tag{72}$$

where the last equality is valid by (63). Also by (51) and (52) we will have respectively,

$$\left\langle A_0^{-1} D x_n, x_n \right\rangle_{H_{\frac{1}{2}}} = -\frac{2\mu}{\mu^2+\sigma^2} ||x_n||^2_{H_{\frac{1}{2}}} + \epsilon_n, \tag{73}$$

where $\epsilon_n \to 0$ as $n \to \infty$, and

$$\left\langle A_0^{-1} D x_n, x_n \right\rangle_{H_{\frac{1}{2}}} = -2\mu ||x_n||^2 + \eta_n, \tag{74}$$

where $\eta_n \to 0$ as $n \to \infty$. Setting (73) and (74) in the right hand side of (72) we obtain

$$4||x_n||^2_H \cdot ||x_n||^2_{H_{\frac{1}{2}}} \leq \frac{4\mu^2}{\mu^2+\sigma^2} ||x_n||^2_H \cdot ||x_n||^2_{H_{\frac{1}{2}}} - 2\mu\epsilon_n ||x_n||^2_H - \frac{2\mu\eta_n}{\mu^2+\sigma_2} ||x_n||^2_{H_{\frac{1}{2}}} + \epsilon_n \eta_n, \tag{75}$$

where $\epsilon_n \to 0$ and $\eta_n \to 0$ as $n \to \infty$. So

$$4\left(1 - \frac{\mu^2}{\mu^2+\sigma^2}\right) ||x_n||^2_H \cdot ||x_n||^2_{H_{\frac{1}{2}}} \leq \epsilon_n\eta_n - 2\mu\epsilon_n ||x_n||^2_H - \frac{2\mu\eta_n}{\mu^2+\sigma^2} ||x_n||^2_{H_{\frac{1}{2}}} \leq$$
$$|\epsilon_n\eta_n| - 2\mu|\epsilon_n| \cdot ||x_n||^2_H - \frac{2\mu|\eta_n|}{\mu^2+\sigma^2} ||x_n||^2_{H_{\frac{1}{2}}}. \tag{76}$$

From (76) we have

$$4\left(1 - \frac{\mu^2}{\mu^2+\sigma^2}\right) \liminf_{n\to\infty} ||x_n||^2_H \cdot ||x_n||^2_{H_{\frac{1}{2}}} \leq$$
$$\liminf_{n\to\infty} \left(|\epsilon_n\eta_n| - 2\mu|\epsilon_n| ||x_n||^2_H - \frac{2\mu|\eta_n|}{\mu^2+\sigma^2} ||x_n||^2_H \right) \leq$$
$$\lim_{n\to\infty} \left(|\epsilon_n\eta_n| - 2\mu|\epsilon_n| \cdot ||x_n||^2_H - \frac{2\mu|\eta_n|}{\mu^2+\sigma^2} ||x_n||^2_{H_{\frac{1}{2}}} \right) = 0. \tag{77}$$

But, $(\liminf_{n\to\infty} ||x_n||^2_H)(\liminf_{n\to\infty} ||x||^2_{H_{\frac{1}{2}}}) \leq \liminf_{n\to\infty}(||x_n||^2_H ||x_n||^2_{H_{\frac{1}{2}}})$. This and (77) give us,

$$4\left(1 - \frac{\mu^2}{\mu^2+\sigma^2}\right) \liminf_{n\to\infty} ||x_n||^2_H \liminf_{n\to\infty} ||x_n||^2_{H_{\frac{1}{2}}} \leq 0. \tag{78}$$

Since in (78) we have $\liminf_{n\to\infty} ||x_n||^2_H > 0, \liminf_{n\to\infty} ||x_n||^2_{H_{\frac{1}{2}}} > 0$, thus we must have $4\left(1 - \frac{\mu^2}{\mu^2+\sigma^2}\right) \leq 0$, so $\sigma = 0$. Hence $\lambda = \mu$ and since $\mu < 0$, thus $\lambda < 0$, or $\sigma(A) \subset (-\infty, 0)$. This proves the part 4 of this theorem.

We point out that we have used the fact that $\liminf_{n\to\infty} ||x_n||_H > 0$. We can prove this in the following manner. Suppose $\liminf_{n\to\infty} ||x_n||_H = 0$. Then there exists a subsequence $\{x_{n_k}\}$ such that $||x_{n_k}||_H \longrightarrow 0$ as $k \longrightarrow \infty$. By (53) we have $\langle A_0 x_{n_k}, x_{n_k} \rangle_{H_{-\frac{1}{2}} \times H_{\frac{1}{2}}} \longrightarrow 0$ as $k \longrightarrow \infty$. Using (67) it follows that $||x_{n_k}||_{H_{\frac{1}{2}}} \longrightarrow 0$ which contradicts part 2 of Lemma 3.1.

Proof of the Corollary 3.1: If $D = 0$ then by the Definition 3.4 $\gamma = 0$. Hence by Theorem 3.2 $M = \{\lambda \in \mathbb{C} | \; Re\lambda < 0, Im\lambda \neq 0\} \subset \rho(A)$. By this and Theorem 3.1 (since this guarantees that the spectrum of A is containd in the closed left half plane) we have $\sigma(A) \subset i\mathbb{R} \cup (-\infty, 0)$. We now prove that $\sigma(A) \subset i\mathbb{R}$. If not then there exists a $\lambda \in \sigma(A)$ such that $\lambda = \mu$, that means that $\sigma = 0$. But if $\sigma = 0$ then by Theorem 3.2 part 4 we have (54). Using $D = 0$ in (54) we obtain

$$\lim_{n\to\infty} \left(||x_n||^2_{H_{\frac{1}{2}}} + ||x_n||^2_H \right) = 0,$$

or $\lim_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}} = \lim_{n\to\infty} ||x_n||_H = 0$, hence $\liminf_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}} = \liminf_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}} = 0$. And this is a contradiction since we have already shown that they can never be zero.

Now we prove that $\sigma(A) = \sigma_{ap}(A)$. We know that $\sigma_{ap}(A) \subset \sigma(A)$. It remains to prove that $\sigma(A) \subset \sigma_{ap}(A)$. To this end, we take an arbitrary element in $\sigma(A)$, i.e; λ. Since $\sigma(A) \subset i\mathbb{R}$ we have $\lambda \in i\mathbb{R}$. But $i\mathbb{R}$ is the boundary of $\sigma(A)$ and by Lemma 2.4 A is a closed operator, hence the boundary of $\sigma(A)$ which we denote by $\partial\sigma(A)$ must be containd in the approximate point spectrum of A, this is, $\partial\sigma(A) \subset \sigma_{ap}(A)$. (See, [11, chapter IV.1]). Thus $\sigma(A) = \sigma_{ap}(A)$.

Now, assume that $i\eta \in \sigma(A)$. Since $\sigma(A) = \sigma_{ap}(A)$, thus $i\eta \in \sigma_{ap}(A)$. Hence there exists a sequence, $\left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\}_{n\in\mathbb{N}}$ in $\mathcal{D}(A)$ such that

$$\left\| \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\|_{H_{\frac{1}{2}} \times H} = 1 \quad \text{and} \quad \left\| (A - i\eta) \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\|_{H_{\frac{1}{2}} \times H} \to 0, \quad \text{as} \quad n \to \infty,$$

or $\begin{pmatrix} y_n - i\eta x_n \\ -A_0 x_n - i\eta_n \end{pmatrix} \to 0$, as $n \to \infty$. From this, $y_n = i\eta x_n + \epsilon_n$, where $\epsilon_n \to 0$, as $n \to \infty$ and $-A_0 x_n - i\eta y_n \to 0$. Replacing y_n in the last formula by $i\eta x_n + \epsilon_n$ we will have $-A_0 x_n + \eta^2 x_n \to 0$, as $n \to \infty$, and from this we obtain $\eta^2 \in \sigma(A_0)$.

We prove now the converse. Suppose $\eta^2 \in \sigma(A_0)$, then we prove that $i\eta \in \sigma(A)$. Since A_0 is self adjoint and $\eta^2 \in \sigma(A_0)$ then there exists a sequence such as $\{x_n\}_{n\in\mathbb{N}} \in H_{\frac{1}{2}}$ such that $(A_0 - \eta^2)x_n \to 0$, as $n \to \infty$. We define $y_n = i\eta x_n$, then for the sequence $\left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\}_{n\in\mathbb{N}} \in H_{\frac{1}{2}} \times H$ we have

$$(A - i\eta I) \begin{pmatrix} x_n \\ y_n \end{pmatrix} = \begin{pmatrix} -i\eta x_n + y_n \\ -A_0 x_n - i\eta y_n \end{pmatrix} = \begin{pmatrix} 0 \\ -A_0 x_n + \eta^2 x_n \end{pmatrix}.$$

Since $(-A_0 + \eta^2 I)x_n \longrightarrow 0$, we have $(A - i\eta I) \begin{pmatrix} x \\ y \end{pmatrix} \longrightarrow 0$, hence $i\eta \in \sigma(A)$.

In the following example we show that the first two statements of the Theorem 3.2 in general cannot be improved.

Example 3.3 *Let $0 < \beta \leq \gamma < \infty$ be arbitrary. We define the function $d : (0, \infty) \to [0, \infty)$ by*

$$d(x) = \frac{1}{2}(\beta + (\gamma - \beta)(x - j + 1)) \text{ if } j - 1 < x \leq j, j \in \mathbb{N}.$$

Assume also that H, a, a_0, A_0 and D be defined as in Example 3.2. Show that the first two statements of the Theorem 3.2 cannot be improved.

Solution: In Example 3.2 we have shown that A_0 is a positive definite self adjoint operator and A_0^{-1} exists and bounded. We also proved that the operator $A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}$ is bounded and self adjoint, and also proved that D is a positive operator. We claim now that

$$\beta = \inf_{g \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle Dg, g \rangle_H}{||g||_H^2} \quad , \quad \gamma = \sup_{g \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle Dg, g \rangle_H}{||g||_H^2}.$$

To do this we say,

$$\langle Dg, g \rangle_H = \int_0^\infty 2d(x)g(x)\overline{g(x)}\, dx = \sum_{j=1}^\infty \int_{j-1}^j (\beta + (\gamma - \beta)(x - j + 1))\, |g(x)|^2\, dx,$$

since $j - 1 < x \leq j$ and d is an increasing function then we have

$$\beta \sum_{j=1}^\infty \int_{j-1}^j |g(x)|^2 \leq \sum_{j=1}^\infty \int_{j-1}^j (\beta + (\gamma - \beta)(x - j + 1))\, |g(x)|^2 \leq \gamma \sum_{j=1}^\infty \int_{j-1}^j |g(x)|^2\, d(x),$$

that is

$$\beta \int_0^\infty |g(x)|^2\, dx \leq \langle Dg, g \rangle_H \leq \gamma \int_0^\infty |g(x)|^2\, dx,$$

or

$$\beta ||g||_H^2 \leq \langle Dg, g \rangle_H \leq \gamma ||g||_H^2,$$

dividing by $||g||_H^2$ we have

$$\beta \leq \frac{\langle Dg, g \rangle_H}{||g||_H^2} \leq \gamma.$$

We claim now that γ and β are supremum and infimmum respectively. This is clear, since $d(x)$ is an increasing function for $j - 1 < x \leq j$. Then as in Example 3.2 we have

$$\left\{ \lambda \in \mathbb{C} | Im\lambda \neq 0, -\frac{\gamma}{2} \leq Re\lambda \leq -\frac{\beta}{2} \right\} \subset \sigma(A).$$

The following example cannot be foud in the main paper.
We are going to illustrate the following example for the second order o.d.e in the special bounded cases. We had of course assumed that A_0 and D are unbounded operators but nothing is here an obstacle to have this special cases. First of all a definition.

Definition 3.5 *For the operator A_0 we define*

$$\gamma_0 = \inf_{x \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle A_0 x, x \rangle_H}{||x||_H}.$$

Example 3.4 *Suppose $\lambda = \mu + i\sigma$ and $\sigma \neq 0$. Then*

1. *If $A_0 = D = I$, then*
$$\sigma(A) \subset \left\{ -\frac{1}{2} + i\frac{\sqrt{3}}{2} \right\} \cup (-\infty, 0).$$

2. *If $A_0 = I$, then*
$$\sigma(A) \subset \left\{ \lambda \in \mathbb{C} | \mu^2 + \sigma^2 = 1, \mu \leq 0, \sigma \neq 0 \right\} \cup (-\infty, 0).$$

3. *Suppose that δ has the same meaning as in Definition 3.4, then for $\delta > 0$ we have*
$$\sigma(A) \subset \{\lambda \in \mathbb{C} | |\lambda| = 1, Re\lambda \leq 0\} \cap \left\{ \lambda \in \mathbb{C} |\ |Im\lambda|^2 \leq -(Re\lambda)^2 - \frac{2}{\delta} \right\} \cup (-\infty, 0).$$
Moreover, if $\delta \geq 2$, then
$$\sigma(A) \subset (-\infty, 0).$$

4. *If $\gamma_0 = \frac{1}{4}$ and $D = I$, then*
$$\sigma(A) \subset \left\{ \lambda \in \mathbb{C} | \mu = -\frac{1}{2}, \sigma \in \mathbb{R} \setminus \{0\} \right\} \cup (-\infty, 0).$$

Proof: First part 1. Using $A_0 = D = I$ in (60) and by a simple calculation we have

$$(1 + \mu + \mu^2 - \sigma^2) \langle x_n, x_n \rangle_H + i\sigma(1 + 2\mu) \langle x_n, x_n \rangle_H \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

From this we have $1 + \mu + \mu^2 - \sigma^2 = 0$ and $1 + 2\mu = 0$. Thus $\mu = -\frac{1}{2}$ and $\sigma = \pm\frac{\sqrt{3}}{2}$. This proves part 1.
For part 2 we use again (60) when $A_0 = I$. By a simple calculation we get

$$\langle x_n, x_n \rangle_H + \mu \langle Dx_n, x_n \rangle_H + \left(\mu^2 - \sigma^2\right) \langle x_n, x_n \rangle_H + i\sigma \langle Dx_n, x_n \rangle_H + 2i\mu\sigma \langle x_n, x_n \rangle_H \longrightarrow 0, \text{ as } n \longrightarrow \infty.$$

Thus,

$$(\mu^2 - \sigma^2 + 1) \langle x_n, x_n \rangle_H + \mu \langle Dx_n, x_n \rangle_H \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty, \tag{79}$$

$$2\mu \langle x_n, x_n \rangle_H + \langle Dx_n, x_n \rangle_H \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{80}$$

Multiplying (80) with $-\mu$ and adding with (79) we have

$$(-\mu^2 - \sigma^2 + 1) \langle x_n, x_n \rangle_H \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Since $\liminf_{n\to\infty} ||x_n||_H > 0$ we have $\mu^2 + \sigma^2 = 1$. This proves part 2.
For part 3 we use part 2 of this example and part 3 of Theorem 3.2 we obtain the first part of our claim. For the second part one can see that for $\delta \geq 2$ these two circles intersect each other only at the point -1 which belongs to $(-\infty, 0)$.

For part 4. Using (60) when $D = I$ we will get after algebraic calculation the following

$$\langle A_0 x_n, x_n\rangle_H + (\mu + \mu^2 - \sigma^2)\langle x_n, x_n\rangle_H + i\sigma(1+2\mu)\langle x_n, x_n\rangle_H \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Thus

$$\langle A_0 x_n, x_n\rangle_H + (\mu + \mu^2 - \sigma^2)\langle x_n, x_n\rangle_H \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty, \tag{81}$$

$$(1+2\mu)\langle x_n, x_n\rangle_H \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{82}$$

By (82) we have $\mu = -\frac{1}{2}$. Using this and $\gamma_0 = \frac{1}{4}$ in (81) we obtain $\sigma^2 \geq 0$. Since $\sigma \neq 0$, thus this is true for each $\sigma \in \mathbb{R} \setminus \{0\}$. This proves part 3.

3.1 Generalization to fourth order o.d.e

This section does not exist in the original paper.
We are now going to find an analogue result for fourth order o.d.e for the Theoreom 3.2 part 3.

Consider the following o.d.e,

$$z^{(4)}(t) + A_3 z^{(3)}(t) + A_2 z^{(2)}(t) + A_1 z^{(1)}(t) + A_0 z(t) = 0. \tag{83}$$

Suppose $A_i : H \longrightarrow H$ are self adjoint operators and possibly unbounded, $A_i : H_{\frac{1}{2}} \longrightarrow H_{-\frac{1}{2}}$ are bounded operators for $1 \leq i \leq 3$ and $\langle A_i z, z\rangle_{H'} \geq 0$ for each $z \in H_{\frac{1}{2}}$ where $H' = H_{-\frac{1}{2}} \times H_{\frac{1}{2}}$. Also, assume that the operators $A_0^{-\frac{1}{2}} A_i A_0^{-\frac{1}{2}}$ are bounded and self adjoint for $1 \leq i \leq 3$. We assume also that A_0 is the same operator as before.
The above o.d.e can be written as $\dot{x}(t) = A \cdot x(t)$ where

$$A : \mathcal{D}(A) \subset H^3_{\frac{1}{2}} \times H \longrightarrow H^3_{\frac{1}{2}} \times H,$$

$$A = \begin{pmatrix} 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \\ -A_0 & -A_1 & -A_2 & -A_3 \end{pmatrix}$$

and

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} | x, y, z, w \in H_{\frac{1}{2}}, A_0 x + A_1 y + A_2 z + A_3 w \in H \right\}$$

and its inverse is given by

$$A^{-1} = \begin{pmatrix} -A_0^{-1}A_1 & -A_0^{-1}A_2 & -A_0^{-1}A_3 & -A_0^{-1} \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \end{pmatrix},$$

and is a bounded operator.
We point out too, that $H_{\frac{1}{2}} \times H_{\frac{1}{2}} \times H_{\frac{1}{2}} \times H$ is a Krein space with the inner product defined by

$$\left[\begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} \right] = \left\langle J \begin{pmatrix} x_1 \\ y_1 \\ z_1 \\ w_1 \end{pmatrix}, \begin{pmatrix} x_2 \\ y_2 \\ z_2 \\ w_2 \end{pmatrix} \right\rangle,$$

where J is the following symmetric matrix

$$J = \begin{pmatrix} 0 & 0 & I & 0 \\ 0 & I & \tilde{A}_1 & 0 \\ I & \tilde{A}_1 & \tilde{A}_2 & 0 \\ 0 & 0 & 0 & -I \end{pmatrix},$$

such that $\tilde{A}_i = A_0^{-1}A_i : H_{\frac{1}{2}} \longrightarrow H_{\frac{1}{2}}$ for each $1 \leq i \leq 3$. One can see that J is a bounded self adjoint matrix and $(H_{\frac{1}{2}})^3 \times H$ is a Krein space with the indefinite inner product given above. We now find an analogue of Lemma 3.1. But first of all we assume that the spectrum of A lies in the left half plane, as an argument like using Lumer-Pillips does not seem to work in this case.

Lemma 3.2 *Let $\lambda = \mu + i\sigma$ with $\sigma \in \mathbb{R} \setminus \{0\}$, $\mu \leq 0$. Assume also that there exists a sequence* $\left\{ \begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix} \right\}_{n\in\mathbb{N}}$ *in $\mathcal{D}(A)$ such that*

$$\left\| \begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix} \right\|_{H''} = 1 \quad \text{and} \quad \lim_{n\to\infty} \left\| (\lambda I - A) \begin{pmatrix} x_n \\ y_n \\ z_n \\ w_n \end{pmatrix} \right\|_{H''} = 0, \tag{84}$$

where $H'' = H_{\frac{1}{2}} \times H_{\frac{1}{2}} \times H_{\frac{1}{2}} \times H$, then we have

1. *The real part tend to zero, that is*

$$\begin{aligned} &\langle A_0 x_n, x_n \rangle_{H'} + \lambda \langle A_1 x_n, x_n \rangle_{H'} + \lambda^2 \langle A_2 x_n, x_n \rangle_{H'} + \\ &\lambda^3 \langle A_3 x_n, x_n \rangle_{H'} + \lambda^4 \langle x_n, x_n \rangle_{H'} \to 0, \quad \text{as } n \to \infty, \end{aligned} \tag{85}$$

2. If $\sigma \neq 0$, then the imaginary part tends to zero, that is

$$\left(3\mu^2 - \sigma^2\right) \langle A_3 x_n, x_n \rangle_{H'} + 2\mu \langle A_2 x_n, x_n \rangle_{H'} + \langle A_1 x_n, x_n \rangle_{H'} + (4\mu^3 - 4\mu\sigma^2) \langle x_n, x_n \rangle_{H'} \to 0. \tag{86}$$

Proof: Immitating the same method of the proof of Lemma 3.1 part 1 we obtain

$$||y_n - \lambda x_n||_{H_{\frac{1}{2}}} \to 0,\ ||z_n - \lambda y_n||_{H_{\frac{1}{2}}} \to 0,\ ||w_n - \lambda z_n||_H \to 0,\ \text{as } n \to \infty, \tag{87}$$

$$||A_0 x_n + A_1 y_n + A_2 z_n + \lambda w_n + A_3 w_n||_H \to 0,\ \text{as } n \to \infty. \tag{88}$$

From (87) and (88) we have

$$\langle A_1 y_n, x_n \rangle_{H'} = \lambda \langle A_1 x_n, x_n \rangle_{H'} + \langle A_1 r_n, x_n \rangle_{H'}, \tag{89}$$

$$\langle A_2 z_n, x_n \rangle_{H'} = \lambda^2 \langle A_2 x_n, x_n \rangle_{H'} + \lambda \langle A_2 r_n, x_n \rangle_{H'}, \tag{90}$$

$$\langle A_3 w_n, x_n \rangle_{H'} = \lambda^3 \langle A_3 x_n, x_n \rangle_{H'} + \lambda^2 \langle A_3 r_n, x_n \rangle_{H'},\ r_n \to 0\ \text{ as }\ n \to \infty. \tag{91}$$

But

$$\begin{aligned} &|\langle A_0 x_n + A_1 y_n + A_2 z_n + \lambda w_n + A_3 w_n, x_n \rangle|_{H'} \le \\ &||A_0 x_n + A_1 y_n + A_2 z_n + \lambda w_n + A_3 w_n||_{H_{-\frac{1}{2}}} \cdot ||x_n||_{H_{\frac{1}{2}}}. \end{aligned} \tag{92}$$

By (88), (89), (90), (91), and (92) we have

$$\begin{aligned} |\langle A_0 x_n + A_1 y_n + A_2 z_n + \lambda w_n + A_3 w_n, x_n \rangle_{H'}| = |\langle A_0 x_n, x_n \rangle_{H'} + \lambda \langle A_1 x_n, x_n \rangle_{H'} + \\ \lambda^2 \langle A_2 x_n, x_n \rangle_{H'} + \lambda^3 \langle A_3 x_n, x_n \rangle_{H'} + \lambda^4 \langle x_n, x_n \rangle_{H'} + \langle A_1 r_n, x_n \rangle_{H'} + \lambda \langle A_2 r_n, x_n \rangle_{H'} + \\ \lambda^2 \langle A_3 r_n, x_n \rangle_{H'}| \to 0,\ \text{as } n \to \infty. \end{aligned}$$

We can prove that $\langle A_1 r_n, x_n \rangle_{H'} \to 0$, $\langle A_2 r_n, x_n \rangle_{H'} \to 0$ and $\langle A_3 r_n, x_n \rangle_H \to 0$ as $r \to 0$ for $n \to \infty$. This proves part 1.

If in the part 1 we use $\lambda = \mu + i\sigma$ for $\sigma \neq 0$ we obtain part 2 after simple algebraic calculations.

Definition 3.6 *Suppose A_1 has the same meanings as above. We define*

$$\eta_1 = \inf_{x \in H_{\frac{1}{2}} \setminus \{0\}} \frac{\langle A_1 x, x \rangle_{H'}}{||x||_H}.$$

Theorem 3.3 *Assume A_1 unbounded, A_2 bounded and $A_3 = I$. Then*

$$\sigma(A) \subset \Big\{\lambda \in \mathbb{C} | (3\mu^2 - \sigma^2) + (4\mu^3 - 4\mu\sigma^2) + 2\mu||A_2|| + \eta_1 \leq 0\Big\} \cup (-\infty, 0),$$

where $\lambda = \mu + i\sigma$ with $\mu \leq 0$ and $\sigma \neq 0$.
Moreover, if $||A_2|| = \eta_1 = 1$, then we have

$$\sigma(A) \subset \Big\{\lambda \in \mathbb{C} |\ \sigma^2 \geq \frac{4\mu^3 + 3\mu^2 + 2\mu + 1}{1 + 4\mu}\ if \mu > -\frac{1}{4}\ and\ \sigma^2 \leq \frac{4\mu^3 + 3\mu^2 + 2\mu + 1}{1 + 4\mu}\ if\ \mu < -\frac{1}{4}\Big\} \cup (-\infty, 0).$$

Proof: Dividing (86) by $\langle x_n, x_n \rangle_{H'}$ and using $A_3 = I$ we have

$$(3\mu^2 - \sigma^2) + (4\mu^3 - 4\mu\sigma^2) + 2\mu||A_2|| + \eta_1 \leq 0. \tag{93}$$

This proves first part. For the second part we use $\eta_1 = ||A_2|| = 1$ in (93). We get then $(-1 - 4\mu)\sigma^2 + (4\mu^3 + 3\mu^2 + 2\mu + 1) \leq 0$, from which we have two possibilities. If $\mu > -\frac{1}{4}$ then $\sigma^2 \geq \frac{4\mu^3+3\mu^2+2\mu+1}{1+4\mu}$ and if $\mu < -\frac{1}{4}$ then $\sigma^2 \leq \frac{4\mu^3+3\mu^2+2\mu+1}{1+4\mu}$. If $\sigma = 0$ then we have $\lambda = \mu$, since $\mu \leq 0$ thus $\lambda \leq 0$. But $0 \in \rho(A)$, thus we must have $\lambda < 0$.

Theorem 3.4 *Suppose γ_0 has the same meaning as in Definition 3.5. If $A_1 = A_2 = A_3 = 0$, then*

$$\sigma(A) \subset \Big\{\lambda \in \mathbb{C} | \mu = \sigma, \mu \leq -(\frac{\gamma_o}{4})^{\frac{1}{4}}\Big\} \cup \Big\{\lambda \in \mathbb{C} | \mu = -\sigma, \mu \leq -(\frac{\gamma_o}{4})^{\frac{1}{4}}\Big\} \cup (-\infty, 0),$$

where $\lambda = \mu + i\sigma$ with $\mu \leq 0$ and $\sigma \neq 0$.

Proof: Using (85) and $A_1 = A_2 = A_3 = 0$ the real part will be

$$\langle A_0 x_n, x_n \rangle_{H'} + (\mu^4 - 6\mu^2\sigma^2 + \sigma^4) \langle x_n, x_n \rangle_{H'} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{94}$$

Also, by (86) when $A_1 = A_2 = A_3 = 0$ we have

$$(4\mu^3 - 4\mu\sigma^2) \langle x_n, x_n \rangle_{H'} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty,$$

since $\liminf_{n\to\infty} ||x_n||_{H'} > 0$, thus $4\mu(\mu^2 - \sigma^2) = 0$. Hence $\mu = 0$ or $\mu = \pm\sigma$. If $\mu = 0$, then (94) will be

$$\langle A_0 x_n, x_n \rangle_{H'} + \sigma^4 \langle x_n, x_n \rangle_{H'} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{95}$$

Using (95) and the Definition 3.5 we have $\sigma^4 \leq -\gamma_0$. This cannot happen since $\sigma^4 \neq 0$ for $\sigma \neq 0$. Thus we have only the possibility $\mu = \pm\sigma$. Setting $\mu = \pm\sigma$ in (94) we will have

$$\langle A_0 x_n, x_n \rangle_{H'} - 4\mu^4 \langle x_n, x_n \rangle_{H'} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

From this we obtain $\mu^4 \geq \frac{\gamma_0}{4}$. Hence $\mu \geq (\frac{\gamma_0}{4})^{\frac{1}{4}}$ or $\mu \leq -(\frac{\gamma_0}{4})^{\frac{1}{4}}$. Since $\mu \leq 0$, thus the possibility $\mu \geq (\frac{\gamma_0}{4})^{\frac{1}{4}}$ cannot happen. Thus we have only $\mu \leq -(\frac{\gamma_0}{4})^{\frac{1}{4}}$. This proves our theorem.

Theorem 3.5 *If $A_2 = A_3 = 0$ and γ_0 has the same meanning as in Definition 3.5, then*

$$\sigma(A) \subset \left\{\lambda \in \mathbb{C} | \gamma_0 \leq 3\mu^4 + 2\mu^2\sigma^2 - \sigma^4\right\} \cup (-\infty, 0),$$

where $\lambda = \mu + i\sigma$ with $\sigma \neq 0$. Specially, when $\gamma_0 = 0$, then

$$\sigma(A) \subset \left\{\lambda \in \mathbb{C} | \frac{|\sigma|}{\sqrt{3}} \leq |\mu|\right\} \cup (-\infty, 0).$$

Proof: Assume $\lambda = \mu + i\sigma$ with $\sigma \neq 0$, $\lambda \in \sigma(A)$. Using (86) when $A_2 = A_3 = 0$ we have

$$\langle A_1 x_n, x_n \rangle_{H'} + \left(4\mu^3 - 4\mu\sigma^2\right) \langle x_n, x_n \rangle_{H'} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{96}$$

Also, using (85) when $A_2 = A_3 = 0$ we obtain

$$\langle A_0 x_n, x_n \rangle_{H'} + \mu \langle A_1 x_n, x_n \rangle_{H'} + (\mu^4 - 6\mu^2\sigma^2 + \sigma^4) \langle x_n, x_n \rangle_{H'} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{97}$$

Multiplying (96) by $-\mu$ and adding with (97) we obtain

$$\langle A_0 x_n, x_n \rangle_{H'} - (3\mu^4 + 2\mu^2\sigma^2 - \sigma^4) \langle x_n, x_n \rangle_{H'} \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

From this and Definition 3.5 we have $\gamma_0 \leq 3\mu^4 + 2\mu^2\sigma^2 - \sigma^4$. This proves first part. For the second part we take $\gamma_0 = 0$, then $3\mu^4 + 2\mu^2\sigma^2 - \sigma^4 \geq 0$. From $3\mu^4 + 2\mu^2\sigma^2 - \sigma^4 = 0$ we have $\mu = \pm\frac{\sigma}{\sqrt{3}}$. A simple calculation shows that the acceptable region is $\frac{|\sigma|}{\sqrt{3}} \leq |\mu|$. This proves the theorem.

4 Location of the essential spectrum of A

In this section we assume that A_0^{-1} is compact from H to H. We are going to investigate the essential spectrum of A with this condition. First we give two lemma's which we use in the proof of Theorem 4.1.

Lemma 4.1 *If A_0^{-1} is compact from H to H then $A_0^{-\frac{1}{2}}$ is compact.*

Proof: Since A_0 is self adjoint, A_0^{-1} is also self adjoint, (see, [12, chapter VI]), consequently $A_0^{-\frac{1}{2}}$ is self adjoint. Assume $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence in H, we claim that $\left\{A_0^{-\frac{1}{2}} x_n\right\}_{n\in\mathbb{N}}$ has a convergent subsequence in H. First we prove the following inequality

$$||A_0^{-\frac{1}{2}} x||^2 = \left\langle A_0^{-\frac{1}{2}} x, A_0^{-\frac{1}{2}} \right\rangle = \left\langle A_0^{-1} x, x \right\rangle \leq ||A_0^{-1} x|| \cdot ||x||. \tag{98}$$

The last equality is valid since $A_0^{-\frac{1}{2}}$ is self adjoint and the inequality is based on Cauchy Schwartz inequality. Since A_0^{-1} is compact, $\left\{A_0^{-1} x_n\right\}_{n\in\mathbb{N}}$ has a subsequence, i.e; $\{x_{k_n}\}_{n\in\mathbb{N}}$ such that $\left\{A_0^{-1} x_{k_n}\right\}_{n\in\mathbb{N}}$ is convergent.

Suppose $\epsilon > 0$ is given. Then for sufficiently large m and n

$$||A_0^{-\frac{1}{2}}x_{k_n} - A_0^{-\frac{1}{2}}x_{k_m}||^2 = ||A_0^{-\frac{1}{2}}(x_{k_n} - x_{k_m})||^2 \leq ||A_0^{-1}(x_{k_n} - x_{k_m})|| \cdot ||x_{k_n} - x_{k_m}|| < 2\epsilon M \quad (99)$$

and the first inequality is by (98) and the second inequality is valid by the following. First use that $\{A_0^{-1}x_{k_n}\}$ is convergent, thus it's a Cauchy sequence, hence $||A_0^{-1}(x_{k_n} - x_{k_m})|| < \epsilon$ for the sufficiently large m and n. Secondly, since $\{x_n\}_{n\in\mathbb{N}}$ is a bounded sequence, there exists an $M > 0$ sucht that $||x_{k_n} - x_{k_m}|| \leq ||x_{k_n}|| + ||x_{k_m}|| \leq 2M$.
This shows that $\left\{A_0^{-\frac{1}{2}}x_{k_n}\right\}_{n\in\mathbb{N}}$ is a Cauchy sequence in H. Since H is a Hilbert space, it's complete, hence $\left\{A_0^{-\frac{1}{2}}x_{k_n}\right\}_{n\in\mathbb{N}}$ is convergent. So $A_0^{-\frac{1}{2}}$ is compact.

Lemma 4.2 *Let A_0^{-1} be a compact operator from H to H, then the identity operator $I : H_{\frac{1}{2}} \to H$ is compact.*

Proof: This time, in place of using the definition of compactness which is related to sequences we use the following definition which says, that if X and Y are Banach spaces and U is the open unit ball in X, then linear operator $T : X \to Y$ is compact if the closure of $T(U)$ is compact in Y.
Thus, using the above definition we have to prove that the set $\overline{B_{H_{\frac{1}{2}}}} = \overline{\left\{x \in H_{\frac{1}{2}} |\ ||x||_{H_{\frac{1}{2}}} < 1\right\}}$ is compact in H where $B_{H_{\frac{1}{2}}}$ is the unit ball in $H_{\frac{1}{2}}$.
Since $x \in H_{\frac{1}{2}}$, thus we can write $x = A_0^{-\frac{1}{2}}y$ when $y \in H$. Also,

$$\langle y, y\rangle_H = \left\langle A_0^{\frac{1}{2}}A_0^{-\frac{1}{2}}y, A_0^{\frac{1}{2}}A_0^{-\frac{1}{2}}y\right\rangle_H = \left\langle A_0^{\frac{1}{2}}x, A_0^{\frac{1}{2}}x\right\rangle_H = \langle x, x\rangle_{H_{\frac{1}{2}}} < 1,$$

hence y lies in the open unit ball of H; i.e, $y \in B_H$. So we have $\overline{B_{H_{\frac{1}{2}}}} \subset \overline{A_0^{-\frac{1}{2}}(B_H)}$. Since A_0^{-1} is compact, by Lemma 4.1 $A_0^{-\frac{1}{2}}$ is compact. By the definition of the compact operators given at the beginning of the proof the set $\overline{A_0^{-\frac{1}{2}}(B_H)}$ is a compact set. Since $\overline{B_{H_{\frac{1}{2}}}}$ is a closed set and we know that each closed subset of a compact set is compact, we conclude that $\overline{B_{H_{\frac{1}{2}}}}$ is compact.

Lemma 4.3 $\sigma_{ess}(A_0^{-1}D) \subset [0, \infty)$ *and* $\sigma_{ess}(-A_0^{-1}D) \subset (-\infty, 0]$, *where* $A_0^{-1}D : H_{\frac{1}{2}} \longrightarrow H_{\frac{1}{2}}$.

Proof: Consider $\lambda I_H + A_0^{-\frac{1}{2}}DA_0^{-\frac{1}{2}} : \mathcal{D}(A_0^{\frac{1}{2}}) \subset H \longrightarrow H$. Here we view $A_0^{-\frac{1}{2}}DA_0^{-\frac{1}{2}}$ as a closed operator on H, since by assumption it's bounded on H. Since $A_0^{-\frac{1}{2}}DA_0^{-\frac{1}{2}}$ is positive this operator is boundedly invertible for $\lambda \notin (-\infty, 0]$. Now, since $||A_0^{\frac{1}{2}}x||_H = ||x||_{H_{\frac{1}{2}}}$, thus $A_0^{\frac{1}{2}} : H_{\frac{1}{2}} \longrightarrow H$ is an isometry onto H, hence its inverse exists and denoted by $A_0^{-\frac{1}{2}} : H \longrightarrow$

$H_{\frac{1}{2}}$. Then for $\lambda \notin (-\infty, 0]$ the operator $\lambda I_{H_{\frac{1}{2}}} + A_0^{-1}D = A_0^{-\frac{1}{2}}(\lambda I_H + A_0^{-\frac{1}{2}} D A_0^{-\frac{1}{2}}) A_0^{\frac{1}{2}} : H_{\frac{1}{2}} \longrightarrow H_{\frac{1}{2}}$ is the composition of three boundedly invertible operators and consequently boundedly invertible.

Theorem 4.1 *If the operator A_0^{-1} is compact from H to, then*

$$\sigma_{ess}(A) = \left\{ \lambda \in \mathbb{C} \setminus \{0\} \,|\, \frac{1}{\lambda} \in \sigma_{ess}(-A_0^{-1}D) \right\} \subset (-\infty, 0)\,.$$

We note that here $A_0^{-1}D$ is considered as an operator in $H_{\frac{1}{2}}$.

Proof: By the Proposition 2.1 $0 \in \rho(A)$, hence A^{-1} exists (also bounded). To prove the theorem we use the last part of Lemma 2.5. In other words, we prove that $\sigma_{ess}(A_0^{-1}) \setminus \{0\} = \sigma_{ess}(-A_0^{-1}D) \setminus \{0\}$.

We write the operator A^{-1} given in the Proposition 2.1 as follows,

$$A^{-1} = \begin{pmatrix} -A_0^{-1}D & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -A_0^{-1} \\ I & 0 \end{pmatrix}. \tag{100}$$

We just for a moment consider only the second matrix in (100). The operator $-A_0^{-1} : H \to H$ is a compact operator by the hypothesis of theorem, thus it's a compact operator from H to $H_{\frac{1}{2}}$, which we show at the end of the proof. Also, by the Lemma 4.2 the operator $I : H_{\frac{1}{2}} \to H$ is a compact operator. Thus the entries of the second matrix are compact operators. Also, if $T : H \longrightarrow H$ is a bounded operator and $K : H \longrightarrow H$ compact, then $\sigma_{ess}(T + K) = \sigma_{ess}(T)$ (see, [12, chapter 8]), thus we will have

$$\sigma_{ess}(A^{-1}) = \sigma_{ess} \begin{pmatrix} -A_0^{-1}D & 0 \\ 0 & 0 \end{pmatrix}. \tag{101}$$

But the essential spectrum of the matrix in (101) is equal to $\sigma_{ess}(-A_0^{-1}D) \cup \{0\}$. Since $0 \in \rho(A)$ we know that $0 \notin \sigma_{ess}(A)$. This and (101) shows that $\sigma_{ess}(A^{-1}) = \sigma_{ess}(-A_0^{-1}D) \setminus \{0\}$, equivalently $\sigma_{ess}(A^{-1}) \setminus \{0\} = \sigma_{ess}(-A_0^{-1}D) \setminus \{0\}$. We claim that $A_0^{-1} : H \longrightarrow H_{\frac{1}{2}}$ is compact. Indeed, let $\{x_n\}_{n\in\mathbb{N}}$ be a bounded sequence in H. We show that $\left\{A_0^{-1}x_n\right\}_{n\in\mathbb{N}}$ has a convergent subsequence in $H_{\frac{1}{2}}$. The reason is, since

$$||A_0^{-1}x_n||_{H_{\frac{1}{2}}} = ||A_0^{\frac{1}{2}} A_0^{-1} x_n||_H = ||A_0^{-\frac{1}{2}} x_n||_H,$$

and we have already shown that $A_0^{-\frac{1}{2}}$ is a compact operator on H.

Corollary 4.1 *Assume that A_0^{-1} is a compact operator and that the operator D is bounded acting on $H_{\frac{1}{2}}$ into H_α for some $\alpha > -\frac{1}{2}$. Then $\sigma_{ess}(A) = \emptyset$.*

Proof: By our assumption the operator $A_0^{-1} : H \longrightarrow H$ is compact. We write $A_0^{-1} = A_0^{-\alpha} A_0^{-1} A_0^{\alpha}$, where in the right hand side of the equality we have $A_0^{-1} : H \longrightarrow H$, $A_0^{\alpha} : H_\alpha \longrightarrow H$, and $A_0^{-\alpha} : H \longrightarrow H_\alpha$. Thus in the left hand side of the equality we have $A_0^{-1} : H_\alpha \longrightarrow H_\alpha$.

First we claim that this operator is compact. We claim that $A_0^\alpha : H_\alpha \longrightarrow H$ is a bounded operator for $\alpha \in \mathbb{R}$. The case $\alpha = 0$ is clear, since $A_0^0 = I$. Assume $\alpha > 0$. Then by definition we have $||A_0^\alpha x||_H = ||x||_{H_\alpha}$ for each $x \in H_\alpha$. Hence $||A_0^\alpha|| = 1$. Thus $A_0 : H_\alpha \to H$ is a bounded operator for $\alpha \geq 0$. In the same manner we can show that for the operator $A_0^\beta : H_\beta \to H$ we have $||A_0^\beta|| = 1$ where $\beta < 0$. Hence the operator $A_0^\beta : H_\beta \to H$ is a bounded operator for $\beta < 0$. Also by our assumption $A_0^{-1} : H \longrightarrow H$ is compact and $A_0^{-\alpha} : H \longrightarrow H_\alpha$ is bounded, hence $A_0^{-1} : H_\alpha \longrightarrow H_\alpha$ is a compact operator. Since by our assumption $D : H_{\frac{1}{2}} \longrightarrow H_\alpha$ is bounded, thus $A_0^{-1}D : H_{\frac{1}{2}} \longrightarrow H_\alpha$ will be a compact operator for $\alpha > -\frac{1}{2}$, consequently the operator $A_0^{-1}D : H_{\frac{1}{2}} \longrightarrow H_{\frac{1}{2}}$ will be compact. Thus $\sigma_{ess}(A_0^{-1}D) = \{0\}$ or $\sigma_{ess}(A_0^{-1}D) = \emptyset$. We claim now that we cannot have the possibility $\sigma_{ess}(A_0^{-1}D) = \{0\}$. If so, then $\sigma_{ess}(A_0^{-1}D) \setminus \{0\} = \emptyset = \sigma_{ess}(A^{-1}) \setminus \{0\}$. But, $\emptyset = \sigma_{ess}(A^{-1}) \setminus \{0\}$ if and only if $\emptyset = \sigma_{ess}(A) \setminus \{0\}$. Since $0 \in \rho(A)$, thus $\sigma_{ess}(A) \setminus \{0\} = \sigma_{ess}(A)$. Hence $\sigma_{ess}(A) = \emptyset$.

Remark 4.1 We point out here that A^{-1} is not necessary compact if A_0^{-1} is compact. To show this we use the Theorem 4.1 and take $A_0 = D$. In this case we have

$$\sigma_{ess}(A) = \left\{\lambda \in \mathbb{C}| \ \frac{1}{\lambda} \in \sigma_{ess}(-A_0^{-1}A_0)\right\} = \left\{\lambda \in \mathbb{C}| \ \frac{1}{\lambda} \in \sigma_{ess}(-I)\right\}.$$

But $\frac{1}{\lambda} \in \sigma_{ess}(-I)$ if and only if $\frac{1}{\lambda}I + I$ is not Fredholm or $\frac{\lambda+1}{\lambda}I$ not Fredholm, for each $\lambda \in \mathbb{C} \setminus \{0\}$. If $\lambda \neq -1$ then then $\frac{\lambda+1}{\lambda}$ is invertible and surely Fredholm. If $\lambda = -1$ this will be the zero operator and it's not Fredholm since $dim\ Im(0) = 0$ and $\infty = dim\ H = dim\ Im(0) + codim\ Im(0)$, hence $codim\ Im(0) = \infty$. This shows that $\sigma_{ess}(A) = -1$ and by the last part of Lemma 2.5, $-1 \in \sigma_{ess}(A^{-1})$. This shows that A^{-1} is not compact.

We note here something about the last part of the above remark. If an operator is compact then its essential spectrum is zero or empty, (see, [13, Chapter XI]). Since $-1 \in \sigma_{ess}(A)$, A cannot be a compact operator. We point out here that without assuming that A_0^{-1} is compact the essential spectrum of A is quite arbitrary in the closed half plane as we see in the Example 3.2.

The following theorem states that the non-real essential spectrum of A is located in a certain strip parallel to the imaginary axis. We have to define some notions which we need for the proof of the following theorem.

Suppose V is a linear space. A non zero vector $v \in V$ is called a root vector of the operator T if $(T - \lambda_0 I)^n v = 0$ for some positive integer n. The set of all root vectors of the operators T, corresponding to one and the same eigenvalue λ_0, together with the vector $v = 0$, forms a linear space V_{λ_0}, which is called the root linear space. The dimension of V_{λ_0} denoted by $dim\ V_{\lambda_0} = \nu_{\lambda_0}$, is called the algebraic multiplicity of the eigenvalue λ_0.

An operator T is called semi-Fredholm if at least $kern(T - \lambda I)$ or $Im(T - \lambda I)$ is finite dimensional.

Theorem 4.2 *Set* $\alpha_1 = \infty$ *and* $\gamma_1 = 0$ *when* $\sigma_{ess}(A_0^{-1}D) = \emptyset$, *and set* $\gamma_1 = \infty$ *if* $0 \in \sigma_{ess}(A_0^{-1}D)$. *Otherwise, let*

$$\alpha_1 = \frac{1}{2||A_0^{-1}||} min\left\{s \in \mathbb{R}| \ s \in \sigma_{ess}(A_0^{-1}D)\right\},$$

and

$$\gamma_1 = 2\frac{1}{min\left\{s\in\mathbb{R}|\ s\in\sigma_{ess}(A_0^{-1}D)\right\}}.$$

In the formula above $||A_0^{-1}||$ *is the operator norm of* A_0^{-1} *acting on* H *and* $A_0^{-1}D$ *is considered as an operator acting on* $H_{\frac{1}{2}}$*. Then*

1. $\sigma_{ess}(A)\subset(-\infty,0)\cup\{\lambda\in\mathbb{C}|\ Re\lambda\leq-\alpha_1\}$.
2. *If* $\rho(A)\cap\{\lambda\in\mathbb{C}|\ Re\lambda<-\gamma_1, Im\lambda\neq 0\}\neq\emptyset$*, then*

$$\sigma_{ess}(A)=(-\infty,0)\cup\{\lambda\in\mathbb{C}|\ -\gamma_1\leq Re\lambda\leq 0\}.$$

Proof: First of all we mention that in the theorem the term minimum is used in place of infimum. The reason is that the essential spectrum of a bounded operator is closed. The fact that $A_0^{-1}D$ is a bounded operator on $H_{\frac{1}{2}}$ to $H_{\frac{1}{2}}$ was observed in the proof of Proposition 2.1. Also the reason that $\alpha_1\geq 0$ and $\gamma_1\geq 0$ is by the Lemma 4.3. We have also $\sigma_{ess}(A)\subset\{s\in\mathbb{C}|\ Res\leq 0\}$, by Theorem 3.1.
1. Assume $\alpha_1>0$. Set

$$\mathcal{U}=\{\lambda\in\mathbb{C}|-\alpha_1<Re\lambda\leq 0, Im\lambda\neq 0\},$$

$$G_\lambda = span\left\{x\in H_{\frac{1}{2}}|A_0^{-1}Dx=\nu x,\nu\leq -2\mu||A_0^{-1}||\right\}. \tag{102}$$

Suppose $\lambda\in\mathcal{U}$, $\lambda=\mu+i\sigma$. Since $Re\lambda=\mu$ hence by the definition of $\mathcal{U}$ we will obtain $-\alpha_1<\mu$, thus $-2\mu||A_0^{-1}||<min\left\{s\in\mathbb{R}|\ s\in\sigma_{ess}(A_0^{-1}D)\right\}$.

What we are going to prove is:

1. For any $\lambda\in\mathcal{U}$ there exists a finite dimensional subspace G_λ and a constant c_λ such that for each $\begin{pmatrix} x\\ y\end{pmatrix}\in\mathcal{D}(A)\cap(G_\lambda\times G_\lambda)^\perp$ we have

$$||(A-\lambda I)\begin{pmatrix} x\\ y\end{pmatrix}||_{H_{\frac{1}{2}}\times H}\geq c_\lambda||\begin{pmatrix} x\\ y\end{pmatrix}||_{H_{\frac{1}{2}}\times H}, \tag{103}$$

2. $\mathcal{U}\setminus E\subset\rho(A)$, where E is the set of isolated eigenvalues of A with finite algebraic multiplicity.

We deal with the first point. Firstly, we claim that G_λ is a finite dimesional subspace of $H_{\frac{1}{2}}$. That this is a linear subspace of $H_{\frac{1}{2}}$ is clear. Suppose G_λ is infinite dimensional. Then, there must exist infinite eigenvalues for $A_0^{-1}D$. We denote these eigenvalues by ν_i. These eigenvalues satisfy the inequality $\nu_i\leq -2\mu||A_0^{-1}||$. So, we must have at least one accumulation point in the interval $\left[0,-2\mu||A_0^{-1}||\right]$, i.e; denoting this accumulation point by ν_a we have $\nu_a\in\left[0,-2\mu||A_0^{-1}||\right]$. On the other hand $A_0^{-1}D$ is a self adjoint operator on $H_{\frac{1}{2}}$, thus its all accumulation points must belong to $\sigma_{ess}(A_0^{-1}D)$. Hence $\nu_a\in\sigma_{ess}(A_0^{-1}D)$.

But, This is a contradiction, since by the inequality $-2\mu||A_0^{-1}|| < min\ \sigma_{ess}(A_0^{-1}D)$ we have $\left[0, -2\mu||A_0^{-1}||\right] \cap \sigma_{ess}(A_0^{-1}D) = \emptyset$.

Secondly, assume that there exists a sequence $\left\{\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\} \in \mathcal{D}(A) \cap (G_\lambda \times G_\lambda)^\perp$ (we have to take this sequence also in domain of A to assure that we can act with A). If $\begin{pmatrix} x_n \\ y_n \end{pmatrix} \in (G_\lambda \times G_\lambda)^\perp$ then by the definition of perpendicularity for each $\begin{pmatrix} x \\ y \end{pmatrix} \in G_\lambda \times G_\lambda$ we have $\left\langle \begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle_{H_{\frac{1}{2}} \times H_{\frac{1}{2}}} = 0$. This gives us, $\langle x_n, x\rangle_{H_{\frac{1}{2}}} + \langle y_n, y\rangle_{H_{\frac{1}{2}}} = 0$. Since the last equation is valid for each $x, y \in G_\lambda$, we can take $y = 0$. In this case we will have $\langle x_n, x\rangle_{H_{\frac{1}{2}}} = 0$. Thus $x_n \in G_\lambda^\perp$.

Thirdly, by part 3 of Lemma 3.1

$$\left\langle A_0^{-1}Dx_n, x_n\right\rangle_{H_{\frac{1}{2}}} + 2\mu \langle x_n, x_n\rangle_H \to 0, \text{ as } n \to \infty, \tag{104}$$

also, since $A_0^{-1}D$ a self adjoint operator on $H_{\frac{1}{2}}$, the essential spectrum is the set of non isolated points of spectrum on $H_{\frac{1}{2}}$. Hence we can write,

$$\sigma(A_0^{-1}D|_{H_{\frac{1}{2}}}) = \{v_1, v_2, \cdots, v_{k+1}, \cdots\} \cup \sigma_{ess}(A_0^{-1}D).$$

Since G_λ is finite dimensional, the set of eigenvalues of $A_0^{-1}D$ on G_λ must be finite and it's equal to $\sigma(A_0^{-1}D|_{G_\lambda})$. Suppose $\sigma_{ess}(A_0^{-1}D|_{G_\lambda}) = \{v_1, v_2, \cdots, v_k\}$. Thus $v_k \leq -2\mu||A_0^{-1}||$ and $min\ \sigma(A_0^{-1}D|_{G_\lambda^\perp}) = v_{k+1}$. Hence there exists an $\eta > 0$ such that $-2\mu||A_0^{-1}|| + \eta = v_{k+1}$. Take now $\eta = \delta||A_0^{-1}||$ for some $\delta > 0$. Hence,

$$-2\mu||A_0^{-1}|| + \delta||A_0^{-1}|| = v_{k+1}. \tag{105}$$

We know also from functional analysis that for a linear operator $T : H \longrightarrow H$ where H is a Hilbert space and $\sigma(T) \subset [m, \infty)$, then $\langle Tx, x\rangle_H \geq m||x||_H^2$, for each $x \in H$. Applying this to $A_0^{-1}D$ on $H_{\frac{1}{2}}$ for each $x_n \in G_\lambda^\perp$ and (105) we obtain

$$\left\langle A_0^{-1}Dx_n, x_n\right\rangle_{H_{\frac{1}{2}}} \geq -2\mu||A_0^{-1}|| \cdot ||x_n||_{\frac{1}{2}}^2 + \delta||A_0^{-1}|| \cdot ||x_n||_{H_{\frac{1}{2}}}^2 \geq -2\mu||x_n||_H^2 + \delta||A_0^{-1}|| \cdot ||x_n||_H^2, \tag{106}$$

where the last inequality is valid by the following,

$$\langle x_n, x_n\rangle_H = \left\langle A_0^{-\frac{1}{2}}x_n, A_0^{-\frac{1}{2}}x_n\right\rangle_{H_{\frac{1}{2}}} = \left\langle A_0^{-1}x_n, x_n\right\rangle_{H_{\frac{1}{2}}} \leq ||A_0^{-1}|| \cdot ||x_n||_{H_{\frac{1}{2}}}^2.$$

Also from (105) and by Lemma 3.1 part 2 we have

$$\liminf_{n\to\infty} \left(\left\langle A_0^{-1}Dx_n, x_n\right\rangle_{H_{\frac{1}{2}}} + 2\mu||x_n||_H^2\right) \geq \delta||A_0^{-1}|| \liminf_{n\to\infty} ||x_n||_{H_{\frac{1}{2}}}^2 > 0.$$

This shows that $\lim_{n\to\infty}\left(\left\langle A_0^{-1}Dx_n, x_n\right\rangle_{H_{\frac{1}{2}}} + 2\mu||x_n||_H^2\right) > 0$, which contradics (104).

We begin now to prove point 2. We begin to prove that $A-\lambda I$ is semi-Fredholm on $H_{\frac{1}{2}}\times H$. We claim that $kern(A-\lambda I)$ is finite dimensional in $H_{\frac{1}{2}}\times H$. Since G_λ is finite dimensional, $G_\lambda\times G_\lambda$ is finite dimensional too. Then $kern(A-\lambda I)$ must be finite dimensional in $G_\lambda\times G_\lambda$. Also if $(A-\lambda I)\begin{pmatrix} x \\ y \end{pmatrix} = 0$ for some $x, y\in (G_\lambda\times G_\lambda)^\perp\cap\mathcal{D}(A)$ then by (103) $||\begin{pmatrix} x \\ y \end{pmatrix}||_{H_{\frac{1}{2}}\times H} = 0$. Thus $||x||_{H_{\frac{1}{2}}} + ||y||_H = 0$. This gives us $x = y = 0$. Hence $Kern(A-\lambda I) = 0$ in $(G_\lambda\times G_\lambda)^\perp$. Hence $A-\lambda I$ is a semi-Fredholm operator on $H_{\frac{1}{2}}\times H$. Our aim is to show that $\mathcal{U}\cap\sigma_{ess}(A) = \emptyset$. For this we set $\mathcal{U} = \mathcal{U}_1\cup\mathcal{U}_2$, where

$$\mathcal{U}_1 = \{\lambda\in\mathbb{C}| -\alpha_1\le Re\lambda\le 0, Im\lambda > 0\}\,,\ \mathcal{U}_2 = \{\lambda\in\mathbb{C}| -\alpha_1\le Re\lambda\le 0, Im\lambda < 0\}\,.$$

We prove that $\mathcal{U}_1\cap\sigma_{ess}(A) = \emptyset$ and $\mathcal{U}_2\cap\sigma_{ess}(A) = \emptyset$. Since $A-\lambda I$ is semi-Fredholm on $H_{\frac{1}{2}}\times H$ for each $\lambda\in\mathcal{U}$, thus it is semi-Fredholm on $H_{\frac{1}{2}}\times H$ for each $\lambda\in\mathcal{U}_1$ and $\lambda\in\mathcal{U}_2$. We have to mention here the following theorem which we use in our proof. If $T-\lambda I$ is a semi-Fredholm operator for each $\lambda\in\Omega$ where Ω is a connected set, then $ind(T-\lambda I)$ remains unchanged for each $\lambda\in\Omega\setminus\Delta$, where Δ is the set of isolated eigenvalues of T in Ω with finite algebraic multiplicity. Applying this theorem to $T = A$ and $\Omega = \mathcal{U}_1$, $\Delta = E_1$ we conclude that $ind(A-\lambda I)$ remains unchanged in $\mathcal{U}_1\setminus E_1$, where E_1 is the set of isolated eigenvalues of A with finite algebraic multiplicity in $\mathcal{U}_1$. To find the value of $ind(A-\lambda I)$ for $\lambda\in\mathcal{U}_1\setminus E_1$ we use our assumption $0\in\rho(A)$. Since $\rho(A)$ is an open set we can conclude that there exists an open nbhd of zero, i.e; $B_\epsilon(0)$ such that $A-\lambda I$ is invertible in this nbhd with $ind(A-\lambda I) = 0$. Thus $A-\lambda I$ is a semi-Fredholm operator on $H_{\frac{1}{2}}\times H$ for each λ in this nbhd with $ind(A-\lambda I) = 0$. Since $ind(A-\lambda I)$ must be unchanged in $\mathcal{U}_1\setminus E_1$ by the theorem mentioned above, we must have $ind(A-\lambda I) = 0$ for each $\lambda\in\mathcal{U}_1\setminus E_1$. Now we claim that $dim\ kern(A-\lambda I) = 0$ and $codim\ Im(A-\lambda I) = 0$ for each $\lambda\in\mathcal{U}_1\setminus E_1$. For this we take two arbitrary elements μ and η_1 in $\mathcal{U}_1\setminus E_1$ such that $\eta_1\in\partial B_\epsilon(0)$ and such that they are connected by a smooth curve C. Then there exists an $\epsilon_1 > 0$ such that for each $\xi\in B_{\epsilon_1}(\eta_1)$ we have $dim\ kern(A-\xi I) = 0$. Take now $\eta_2\in\partial B_{\epsilon_1}(\eta_1)\cap C$. Then there exists an $\epsilon_2 > 0$ such that for each $\xi\in B_{\epsilon_2}(\eta_2)$ we have $dim\ kern(A-\xi I) = 0$. Continuing this and using Heine-Borel theorem we reach μ by finite choices of η_n's. Hence we will have $dim\ kern(A-\mu I) = 0$ for each $\mu\in\mathcal{U}_1\setminus E_1$. Since $ind(A-\mu I) = 0$ we conclude that $codim\ Im(A-\mu I) = 0$. Hence $\mathcal{U}_1\setminus E_1\subset\rho(A)$. But what about E_1? Can we prove that $E_1\cap\sigma_{ess}(A) = \emptyset$? If we do this then we are done. This can be shown by the following theorem. Any point of spectrum of a closed operator not belonging to its essential spectrum is an isolated eigenvalue with finite algebraic multiplicity. (See, [17, chapter IV]). Since by Lemma 2.4 A is a closed operator, $A-\lambda I$ is a closed operator too. Applying this theorem to $A-\lambda I$ we obtain $E_1\cap\sigma_{ess}(A) = \emptyset$. We do the same proof $\mathcal{U}_2$. This proves part 1.

Now the proof of part 2. This part is so similar with part 1, therefore we prove or explain the points which have little or no similarities.
We define

$$\mathcal{U} = \{\lambda\in\mathbb{C}|Re\lambda < -\gamma_1, Im\lambda\neq 0\}\,.$$

Let $\lambda \in \mathcal{U}$, $\lambda = \mu + i\sigma$. By Lemma 4.3 $\sigma_{ess}(A_0^{-1}D) \subset [0, \infty)$, hence $\mu < 0$ by the definition of $\mathcal{U}$. Since $\mu < 0$ we have $-\frac{2\mu}{\mu^2+\sigma^2} < -\frac{2}{\mu}$. This and the definition of $\mathcal{U}$ give us $-\frac{2\mu}{\mu^2+\sigma_2} < min \left\{ s \in \mathbb{R} | s \in \sigma_{ess}(A_0^{-1}D) \right\}$. We define now

$$G_\lambda = span \left\{ x \in H_{\frac{1}{2}} | A_0^{-1}Dx = \nu x, \nu \leq -\frac{2\mu}{\mu^2 + \sigma^2} \right\}.$$

In the same manner as in part 1 we can show that G_λ is a finite dimensional subspace of $H_{\frac{1}{2}}$. Assume now that there exists a sequence $\left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\} \in \mathcal{D}(A) \cap (G_\lambda \times G_\lambda)^\perp$ which satisfies (50). Hence $x_n \in G_\lambda^\perp$. Then for each such $x_n \in G_\lambda^\perp$ we have

$$\left\langle A_0^{-1}Dx_n, x_n \right\rangle_{H_{\frac{1}{2}}} + \frac{2\mu}{\mu^2 + \sigma^2} ||x_n||^2_{H_{\frac{1}{2}}} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{107}$$

Since $A_0^{-1}D$ is a self adjoint operator on $H_{\frac{1}{2}}$, the essential spectrum of $A_0^{-1}D$ is the set of non-isolated points of $\sigma(A_0^{-1}D)$.(See, [9, chapter XIII]). Hence $\sigma(A_0^{-1}D|_{G_\lambda^\perp}) \subset (-\frac{2\mu}{\mu^2+\sigma^2}, \infty)$. So there exists a $\delta > 0$ such that $\sigma(A_0^{-1}D|_{G_\lambda^\perp}) \subset [-\frac{2\mu}{\mu^2+\sigma^2} + \delta, \infty)$. Thus by the same property for self adjoint operators which is mentioned in the proof of part 1 we will have

$$\left\langle A_0^{-1}Dx_n, x_n \right\rangle_{H_{\frac{1}{2}}} \geq (-\frac{2\mu}{\mu^2 + \sigma^2} + \delta) ||x_n||^2_{H_{\frac{1}{2}}}. \tag{108}$$

From Lemma 3.1 part 2 and (108) we have

$$\liminf_{n \to \infty} \left(\left\langle A_0^{-1}Dx_n, x_n \right\rangle_{H_{\frac{1}{2}}} + \frac{2\mu}{\mu^2 + \sigma^2} ||x_n||^2_{H_{\frac{1}{2}}} \right) \geq \delta \liminf_{n \to \infty} ||x_n||^2_{H_{\frac{1}{2}}} > 0.$$

This contradicts (107). Therefore for every $\lambda \in \mathcal{U}$ there exists a finite dimensional subspace G_λ and a constant $c_\lambda > 0$ such that for all $\begin{pmatrix} x \\ y \end{pmatrix} \in (G_\lambda \times G_\lambda)^\perp$ we have

$$||(A - \lambda I) \begin{pmatrix} x \\ y \end{pmatrix} ||_{H_{\frac{1}{2}} \times H} \geq c_\lambda || \begin{pmatrix} x \\ y \end{pmatrix} ||_{H_{\frac{1}{2}} \times H}.$$

Consider $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$, where

$$\mathcal{U}_1 = \{ \lambda \in \mathbb{C} | Re\lambda < -\gamma_1, Im\lambda > 0 \}, \mathcal{U}_2 = \{ \lambda \in \mathbb{C} | Re\lambda < -\gamma_1, Im\lambda < 0 \}.$$

One can see that $\mathcal{U}_1$ and $\mathcal{U}_2$ are connected in $\mathbb{C}$ and $\mathcal{U}_1 \cap \rho(A) \neq \emptyset$, $\mathcal{U}_2 \cap \rho(A) \neq \emptyset$. This follows from the fact that $\mathcal{U} \cap \rho(A) \neq \emptyset$ and $\rho(A)$ is symmetric with regard to real axis.
Since $\mathcal{U}_1 \cap \rho(A) \neq \emptyset$, there exists a $\lambda_0 \in \mathcal{U}_1 \cap \rho(A)$. Hence $A - \lambda_0 I$ is invertible, consequently Fredholm with $ind(A - \lambda_0 I) = 0$. For sufficiently small $\lambda \in \mathcal{U}_1$, $A - \lambda I$ is Fredholm and $ind(A - \lambda I) = 0$. Hence $A - \lambda I$ is a semi-Fredholm operator for these λ's with $ind(A - \lambda I) = 0$. The rest of the proof we take from part 1 without any change. That means that using part

of the proof of part 1 we can prove that $\mathcal{U} \setminus E \subset \rho(A)$ and no point of E is a point of $\sigma_{ess}(A)$. This proves part 2.

In the Theorem 4.2 we have found the essential spectrum of the operator A by means of the operator $A_0^{-1}D$. We are now going to find another set which contains the essential spectrum of A, and which is described in terms of the operator $A_0^{\frac{1}{2}}$. The following theorem cannot be found in tha main paper.

Theorem 4.3 *If $\sigma_{ess}(A_0^{\frac{1}{2}}) = \emptyset$ then we set $\theta = \infty$. Otherwise, we define*

$$\theta = \frac{min\left\{s \in \mathbb{R} | s \in \sigma_{ess}(A_0^{\frac{1}{2}})\right\}}{\sqrt{2}}.$$

Then we have

$$\sigma_{ess}(A) \subset (-\infty, 0) \cup \left\{\lambda \in \mathbb{C} | \mu \leq -\frac{\theta}{\sqrt{2}}\right\} \cup \left\{\lambda \in \mathbb{C} ||\sigma| > |\mu| \ if \ 0 > \mu > -\frac{\theta}{\sqrt{2}}\right\},$$

where $\lambda = \mu + i\sigma$.

Proof: We set,

$$\mathcal{U} = \left\{\lambda \in \mathbb{C} | \ 0 \geq \mu > -\frac{\theta}{\sqrt{2}}, |\sigma| \leq |\mu|, \sigma \neq 0\right\}.$$

Let $\lambda \in \mathcal{U}$. Then $-\sqrt{2}\mu < \theta$. Define

$$G_\lambda = span\left\{x \in H_{\frac{1}{2}} | A_0^{\frac{1}{2}} x = \nu x, \nu \leq -\sqrt{2}\mu\right\}.$$

Then G_λ is a finite dimensional subspace of $H_{\frac{1}{2}}$ by the same reason as given in the Theorem 4.2.

Assume that there exists a sequence $\left\{\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\} \in \mathcal{D}(A) \cap (G_\lambda \times G_\lambda)^\perp$ which satisfies (50). Hence $x_n \in G_\lambda^\perp$. Then for each such $x_n \in G_\lambda^\perp$ we have

$$\langle A_0 x_n, x_n \rangle_H - (\mu^2 + \sigma^2)||x_n||_H^2 \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty. \tag{109}$$

Hence there exists a $\delta > 0$ such that

$$\langle A_0 x_n, x_n \rangle_H \geq (2\mu^2 + \delta^2)||x_n||_H^2. \tag{110}$$

From (110) and $\liminf_{n\to\infty} ||x_n||_H > 0$ we have

$$\liminf_{n\to\infty} \left(\langle A_0 x_n, x_n \rangle_H - 2\mu^2 ||x_n||_H^2\right) > 0. \tag{111}$$

Since $\lambda \in \mathcal{U}$, thus $|\sigma| \leq |\mu|$, thus by (109) we have

$$0 \leq \langle A_0 x_n, x_n \rangle_H - 2\mu^2||x_n||_H^2 \leq \langle A_0 x_n, x_n \rangle_H - (\mu^2 + \sigma^2)||x_n||_H^2 \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty, \tag{112}$$

where the first inequality is valid by (110). Also, by (112) we have

$$\langle A_0 x_n, x_n \rangle_H - 2\mu^2 ||x_n||_H^2 \longrightarrow 0, \quad \text{as} \;\; n \longrightarrow \infty. \tag{113}$$

Then, (113) contradicts (111). Therefore for every $\lambda \in \mathcal{U}$ there exists a finite dimensional subspace G_λ and a constant $c_\lambda > 0$ such that for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(A) \cap (G_\lambda \times G_\lambda)^\perp$ we have

$$||(A - \lambda I) \begin{pmatrix} x \\ y \end{pmatrix} ||_{H_{\frac{1}{2}} \times H} \geq c_\lambda || \begin{pmatrix} x \\ y \end{pmatrix} ||_{H_{\frac{1}{2}} \times H}. \tag{114}$$

Again, assume that E is the set of all points of $\mathcal{U}$ with finite algebraic multiplicity. Using (114) and the same argument which we have used in the Theorem 4.2 we can prove that $\mathcal{U} \setminus E \subset \rho(A)$ and no point of E is a point of $\sigma_{ess}(A)$. This proves our theorem.

Corollary 4.2 *If in the Theorem 4.2* $\alpha_1 > \gamma_1$*, then* $\sigma_{ess}(A) \subset (-\infty, 0)$.

Proof: Set $M = \{\lambda \in \mathbb{C} | Re\lambda \leq -\alpha_1\}$, $N = \{\lambda \in \mathbb{C} | 0 \geq Re\lambda \geq -\gamma_1\}$. Then $\sigma_{ess}(A) \subset (-\infty, 0) \cup M$ and $\sigma_{ess}(A) \subset (-\infty, 0) \cup N$. Since $\alpha_1 > \gamma_1$ by our assumption then $-\alpha_1 < -\gamma_1$, then by the definitions of M and N we have $M \cap N = \emptyset$. Also,

$$\sigma_{ess}(A) \subset ((-\infty, 0) \cup M) \cap ((-\infty, 0) \cup N) = (-\infty, 0) \cup (M \cap N) = (-\infty, 0).$$

The following corollary does not exist in the original work.

Corollary 4.3 *Suppose* $\sigma_{ess}(A) = \{\lambda \in \mathbb{C} | Re\lambda \leq -\alpha_1\}$ *and* $\Omega = \{\lambda \in \mathbb{C} | -\alpha_1 < Re\lambda < \epsilon\}$ *for some* $\epsilon > 0$*. Then* $\sigma(A) \setminus \sigma_{ess}(A)$ *has no accumulation point in* Ω.

Proof: To prove this we use the following result:
Let $T : X \longrightarrow X$ be a possibly unbounded operator and X a Banach space. Suppose Ω is an open connected subset of $\mathbb{C} \setminus \sigma_{ess}(T)$. If $\Omega \cap \rho(T) \neq \emptyset$, then $\sigma(T) \cap \Omega$ has no accumulation point in Ω. (See, [13, chapter XVII]). First of all we see that $\rho(A) \neq \emptyset$ by Proposition 2.1. Since by this proposition the spectrum lies in the left half plane, consequently $\rho(A)$ must contain the right half plane, thus it's not empty. Also, Ω is a connected set and $\Omega \subset \mathbb{C} \setminus \sigma_{ess}(A)$. Moreover, by the definition of Ω we have $\Omega \cap \rho(A) \neq \emptyset$. Thus by the result just mentioned $\sigma(A) \cap \Omega$ has no accumulation point in Ω. But $\sigma(A) \cap \Omega = \sigma(A) \setminus \sigma_{ess}(A)$. Hence $\sigma(A) \setminus \sigma_{ess}(A)$ has no accumulation point in Ω.

4.1 Generalization to n-th order o.d.e

One cannot find this section in the original work.
Before beginning the next section we are going to give a generalization of Theorem 4.1.

Consider the following o.d.e.

$$z^{(n)}(t) + A_{n-1} z^{(n-1)}(t) + \cdots + A_1 z^{(1)}(t) + A_0 z(t) = 0,$$

where $A_i : H \longrightarrow H$ are possibly unbounded operators for $1 \leq i \leq n-1$ and A_0 invertible. Assume also that $A_0^{-\frac{1}{2}} A_i A_0^{-\frac{1}{2}} : H \to H$ are bounded operators for $1 \leq i \leq p$. This guarantees that the operators $A_0^{-1} A_i : H_{\frac{1}{2}} \to H_{\frac{1}{2}}$ are bounded operators for $1 \leq i \leq p$. Suppose also that the operators $A_0^{-\frac{1}{2}} A_i : H \to H$ are bounded operators for $i \geq p+1$ which guarantees that the operators $A_0^{-1} A_i : H \to H_{\frac{1}{2}}$ are bounded for $i \geq p+1$. The above o.d.e can be written as $\dot{x}(t) = A \cdot x(t)$, where

$$A : \mathcal{D}(A) \subset (\underbrace{H_{\frac{1}{2}} \times \cdots \times H_{\frac{1}{2}}}_{p-times}) \times (\underbrace{H \times \cdots \times H}_{(n-p)-times}) \longrightarrow (\underbrace{H_{\frac{1}{2}} \times \cdots \times H_{\frac{1}{2}}}_{p-times}) \times (\underbrace{H \times \cdots \times H}_{(n-p)-times})$$

is given by

$$A = \begin{pmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & I \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \\ -A_0 & -A_1 & \cdots & -A_{n-1} \end{pmatrix},$$

$$\mathcal{D}(A) = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \mid x_i \in H_{\frac{1}{2}} \quad for \quad 1 \leq i \leq p, x_i \in H \quad for \quad i \geq p+1, \sum_{i=1}^{n} A_i x_i \in H \right\}.$$

Theorem 4.4 *If A_0^{-1} compact, then*

$$\sigma_{ess}(A) \setminus \{0\} = \sigma_{ess} \begin{pmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & I \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \\ -A_0 & -A_1 & \cdots & -A_p \end{pmatrix} \setminus \{0\}.$$

Proof: We consider first the inverse of A. This is

$$A^{-1} = \begin{pmatrix} -A_0^{-1} A_1 & -A_0^{-1} A_2 \cdots - A_0^{-1} A_{n-1} & & & -A_0^{-1} \\ I & 0 & \cdots & 0 & 0 \\ 0 & I & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & I & 0 \end{pmatrix}.$$

In the above matrix we have the operator I with different domain and range. There are $I : H_{\frac{1}{2}} \longrightarrow H_{\frac{1}{2}}$, $I : H \longrightarrow H$, and $I : H_{\frac{1}{2}} \longrightarrow H$. The first two identity operators are not compact. But by the Lemma 4.2 the identity operator $I : H_{\frac{1}{2}} \longrightarrow H$ is compact. We separate A^{-1} into two matrices, just as we have done in the proof of Theorem 4.1. The non-compact operator plus a compact operator. We write thus,

$$A^{-1} = \begin{pmatrix} -A_0^{-1}A_1 \cdots - A_0^{-1}A_p \cdots - A_0^{-1}A_{n-1} & & & & & -A_0^{-1} \\ I & \cdots & 0 & \cdots & 0 & 0 \\ 0 & I \; \cdots & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & & \cdots & & I & 0 \end{pmatrix} + \begin{pmatrix} 0 & & \cdots & & 0 \\ 0 & & & \cdots & 0 \\ 0 & \cdots & I & \cdots & 0 \\ \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & & \cdots & & 0 \end{pmatrix}$$

The second matrix is compact by the Lemma 4.2. Hence the essential spectrum of A^{-1} is equal to the essential spectrum of the first matrix for the same reason as in the proof of the Theorem 4.1.

We write now the first matrix as a block matrix $\begin{pmatrix} X & Y \\ Z & W \end{pmatrix}$, where

$$X = \begin{pmatrix} -A_0^{-1}A_1 & \cdots & -A_0^{-1}A_p \\ I & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & I \quad 0 \end{pmatrix}, Z = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix},$$

$$Y = \begin{pmatrix} -A_0^{-1}A_{p+1} \cdots - A_0^{-1}A_1 & & & -A_0^{-1} \\ 0 & \cdots & 0 & 0 \\ \vdots & \cdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 \end{pmatrix}, W = \begin{pmatrix} 0 & \cdots & 0 \\ I & \cdots & 0 \\ \vdots & \cdots & \vdots \\ 0 & \cdots \; I & 0 \end{pmatrix}.$$

Thus we have

$$\sigma_{ess}(A^{-1}) = \sigma_{ess} \begin{pmatrix} X & Y \\ 0 & W \end{pmatrix} = \sigma_{ess}(X) \cup \sigma_{ess}(W) = \sigma_{ess}(X) \cup \{0\},$$

since $\sigma_{ess}(W) = \{0\}$, thus we must have $\sigma_{ess}(A^{-1}) = \sigma_{ess}(X) \setminus \{0\}$. We know also that $0 \in \rho(A)$ thus we can say also, $\sigma_{ess}(A^{-1}) \setminus \{0\} = \sigma_{ess}(X) \setminus \{0\}$.
Finally, using the Lemma 2.5 we have

$$\sigma_{ess}(A) \setminus \{0\} = \sigma_{ess} \begin{pmatrix} 0 & I & \cdots & 0 \\ 0 & 0 & I \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & I \\ -A_0 & -A_1 & \cdots & -A_p \end{pmatrix} \setminus \{0\}.$$

We point out here that in the special case $n = 2$ we have then Theorem 4.1.

5 Accumulation points of the non-real spectrum of A

In this section we introduce a theorem which shows that certain intervals do not contain any accumulation point of the non-real spectrum of A. First of all we give a definition which we use in the proof of the theorem.

Definition 5.1 *Suppose T is a self adjoint operator in a Krein space K. A point $\lambda_0 \in \sigma(T)$ is called a spectral point of type π_+(type π_-) of T, denoted by σ_{π^+} (σ_{π^-}) respectively, if there exists a linear space $K_0 \subset K$ with codim $K_0 < \infty$ such that for every sequence $\{x_n\} \subset K_0 \cap \mathcal{D}(T)$ with*

$$||x_n|| = 1 \quad \text{and} \quad \lim_{n\to\infty} ||(T - \lambda I)x_n|| = 0, \tag{115}$$

we have

$$\liminf_{n\to\infty}[x_n, x_n] > 0 \quad (resp, \limsup_{n\to\infty}[x_n, x_n] < 0). \tag{116}$$

Theorem 5.1 *Set $\alpha_2 = \infty$ if $\sigma_{ess}(A_0^{\frac{1}{2}}) = \emptyset$. Otherwise set $\alpha_2 = min\left\{s \in \mathbb{R} | s \in \sigma_{ess}(A_0^{\frac{1}{2}})\right\}$. Assume that α_1 has the same meaning as in the Theorem 4.2. Set $\alpha = max\{\alpha_1, \alpha_2\}$. Then no point of the interval $(-\alpha, 0)$ is an accumulation point of the non-real spectrum of A.*

Proof: We split the proof into two parts. First assume that $max\{\alpha_1, \alpha_2\} = \alpha_1$, then $\alpha = \alpha_1$. We prove that no point of $(-\alpha_1, 0)$ is an accumulation point of non-real spectrum.
Let $\lambda \in (-\alpha_1, 0) \cap \sigma(A)$. We claim that $\lambda \in \sigma_{\pi^+(A)}$. To prove this we use the definition of $\sigma_{\pi_+(A)}$. Suppose $\left\{\begin{pmatrix} x_n \\ y_n \end{pmatrix}\right\} \in \mathcal{D}(A) \cap (G_\lambda \times G_\lambda)^\perp$ where G_λ is given by (102) and the sequence satisfies (50). Then

$$\liminf_{n\to\infty}\left[\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix}\right] = \liminf_{n\to\infty}\left(\langle x_n, y_n\rangle_{H_{\frac{1}{2}}} - \langle y_n, y_n\rangle_H\right), \tag{117}$$

where $[.,.]$ is the inner product defined on the Krein space $H_{\frac{1}{2}} \times H$, and the equality in (117) is valid by (28). Consider the right hand side of (117). By Lemma 3.1, part 1 we have $||y_n - \lambda x_n||_{H_{\frac{1}{2}}} \longrightarrow 0$ as $n \longrightarrow \infty$. Thus we can write $y_n = \lambda x_n + \epsilon_n$ where $\epsilon_n \longrightarrow 0$ as $n \longrightarrow \infty$. Also,

$$||y_n||^2 = \langle y_n, y_n\rangle_H = \langle \lambda x_n + \epsilon_n, \lambda x_n + \epsilon_n\rangle_H = \lambda^2||x_n||_H^2 + \gamma_n,$$

where

$$\gamma_n = 2\lambda Re\,\langle \epsilon_n, x_n\rangle_H + ||\epsilon_n||_H^2.$$

Then $\lim_{n\to\infty}\gamma_n = 0$ as $n \longrightarrow \infty$. Thus, $||x_n||^2_{H_{\frac{1}{2}}} - ||y_n||_H^2 = ||x_n||^2_{H_{\frac{1}{2}}} - ||x_n||_H^2 - \gamma_n$. Since $\gamma_n \longrightarrow 0$ as $n \longrightarrow \infty$ we obtain

$$\liminf_{n\to\infty}\left(\langle x_n, x_n\rangle_{H_{\frac{1}{2}}} - \langle y_n, y_n\rangle_H\right) = \liminf_{n\to\infty}\left(\langle x_n, x_n\rangle_{H_{\frac{1}{2}}} - \lambda^2\langle x_n, x_n\rangle_H\right).$$

Also, by (54) we have

$$||x_n||^2_{H_{\frac{1}{2}}} + \lambda^2||x_n||^2_H = -\lambda \left\langle A_0^{-1}Dx_n, x_n \right\rangle_{H_{\frac{1}{2}}} + \eta_n, \eta_n \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

Hence for n sufficiently great we have

$$||x_n||^2_{H_{\frac{1}{2}}} - \lambda^2||x_n||^2_H = -\lambda \left\langle A_0^{-1}Dx_n, x_n \right\rangle_{H_{\frac{1}{2}}} - 2\lambda^2||x_n||^2_H. \tag{118}$$

Setting (118) in the right hand side of (117), by (105) and (106) we have

$$\liminf_{n\to\infty} \left[\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right] = -\lambda \liminf_{n\to\infty} \left(\left\langle A_0^{-1}Dx_n, x_n \right\rangle_{H_{\frac{1}{2}}} + 2\lambda||x_n||^2_H \right) \geq$$
$$-\lambda\delta||A_0^{-1}|| \liminf_{n\to\infty} ||x_n||^2_H > 0,$$

for some $\delta > 0$.
Now we claim that $\lambda \in \sigma_{ap}(A)$. This is easy to show, since by our assumption the sequence must satisfy (50).
Assume now $K_0 = (G_\lambda \times G_\lambda)^\perp$ in the Definition 5.1. Then $(G_\lambda \times G_\lambda)^\perp$ is a linear subspace of $H_{\frac{1}{2}} \times H$ with $codim\,(G_\lambda \times G_\lambda)^\perp < \infty$, since $(G_\lambda \times G_\lambda)$ is a finite dimensional space. This shows us that $(-\alpha_1, 0) \cap \sigma(A) \subset \sigma_{\pi_+}(A)$ or $[-\alpha_1, 0] \cap \sigma(A) \subset \sigma_{\pi_+}(A)$. Now we use the following theorem which says that if $\rho(A) \neq \emptyset$ and $[a, b]$ is a closed bounded interval such that $[a, b] \cap \sigma(A) \subset \sigma_{\pi_+}(A)$ and that each point of $[a, b]$ is an accumulation point of $\rho(A)$, then there exists an open nbhd Ω in $\mathbb{C}$ of $[a, b]$ such that $\Omega \setminus \mathbb{R} \subset \rho(A)$. (See [2, Theorem 18]).
We are now checking the conditions of Theorem 18 of [2]. By the Proposition 2.1 $\rho(A) \neq \emptyset$. We have also shown that $[-\alpha_1, 0] \cap \sigma(A) \subset \sigma_{\pi_+}(A)$. Also, if $x \in [-\alpha_1, 0]$ and $B_r(x)$ is an arbitrary nbhd of x in $\mathbb{C}$, then $B_r(x) \cap \rho(A) \neq \emptyset$, hence each point of $[-\alpha_1, 0]$ is an accumulation point of $\rho(A)$. Thus we have all conditions of the Theorem 18 of [2]. Hence there exists an open nbhd Ω in $\mathbb{C}$ of $[-\alpha_1, 0]$ such that $\Omega \setminus \mathbb{R} \subset \rho(A)$. If Ω is an unbounded nbhd of $[-\alpha_1, 0]$, then we choose $\Delta \subset \Omega$ which is bounded and open nbhd of $[-\alpha_1, 0]$. Suppose Δ intersects the x axis at $-\alpha_1$ and 0, then $\Delta \setminus [-\alpha_1, 0] = \Delta \setminus \mathbb{R} \subset \rho(A)$. Thus no point of $[-\alpha_1, 0]$ is an accumulation point of non-real spectrum of A. Indeed, if not then for each $x \in [-\alpha_1, 0]$ and for each nbhd of x; i.e, $B_r(x)$ which would be small enough had to have an intersection with non-real spectrum of A. We take r small enough such that $B_r(x) \subset \Delta$, then $B_r(x) \setminus [-\alpha_1, 0] \subset \Delta \setminus [-\alpha_1, 0] \subset \rho(A)$. Hence $B_r(x) \setminus [-\alpha_1, 0]$ has no intersection with non-real spectrum of A.
Next, assume $max\,\{\alpha_1, \alpha_2\} = \alpha_2$. We choose now $\mu \in (-\alpha_2, 0)$ and set $\lambda = \mu + i\sigma$ for some $\sigma \neq 0$. Define

$$G_\lambda = span \left\{ x \in H_{\frac{1}{2}} | A_0^{\frac{1}{2}} x = vx, v \leq -\mu \right\}. \tag{119}$$

We can show in the same manner as before that G_λ is a finite dimensional subspace of $H_{\frac{1}{2}}$. Assume that there exists a sequence $\left\{ \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right\} \in \mathcal{D}(A) \cap (G_\lambda \times G_\lambda)^\perp$ which satisfies

(50). Then by (53),

$$\langle A_0 x_n, x_n \rangle_H - (\mu^2 + \sigma^2) \langle x_n, x_n \rangle_H \longrightarrow 0, \quad \text{as} \quad n \longrightarrow \infty.$$

We can show again, analoguously to the proof of the Theorem 4.2 that $x_n \in G_\lambda^\perp$ for each $n \in \mathbb{N}$. Since A_0 is a self adjoint operator on $H_{\frac{1}{2}}$, $\sigma_{ess}(A_0)$ is the set of non-isolated points of $\sigma(A_0)$. (See, [9, chapter XIII]). Also, since $A_0^{\frac{1}{2}} x = \nu x$, $\nu \leq -\mu$, we have $A_0 x = \nu^2 x$, $\nu^2 \leq \mu^2$. Thus there exists a $\delta > 0$ such that $\sigma(A_0|_{G_\lambda^\perp}) \subset [\mu^2 + \delta^2, \infty]$. Hence, by the same property for the self adjoint operators which is mentioned in the proof of the Theorem 4.2 that for each $x_n \in G_\lambda$ we will have

$$\langle A_0 x_n, x_n \rangle_H \geq (\mu^2 + \sigma^2) ||x_n||_H^2. \tag{120}$$

If we choose $|\sigma| < \delta$, then

$$\langle A_0 x_n, x_n \rangle_H - \left(\mu^2 + \sigma^2\right) ||x_n||_H^2 \geq \left((\mu^2 + \delta^2) - (\mu^2 + \sigma^2)\right) ||x_n||_H^2,$$

or

$$\liminf_{n\to\infty} \left(\langle A_0 x_n, x_n \rangle_H - (\mu^2 + \sigma^2) ||x_n||_H^2\right) \geq (\delta^2 - \sigma^2) \liminf_{n\to\infty} ||x_n||_H^2,$$

since $\liminf_{n\to\infty} ||x_n||_H^2 > 0$, hence

$$\lim_{n\to\infty} \left(\langle A_0 x_n, x_n \rangle_H - (\mu^2 + \sigma^2) ||x_n||_H^2\right) > 0, \tag{121}$$

and this contradicts (53).
Hence there exists an open nbhd Λ in $\mathbb{C}$ of $(-\alpha_2, 0)$ such that for every $\lambda \in \Lambda \setminus (-\alpha_2, 0)$ there exists a finite dimensional subspace G_λ and a constant $c_\lambda > 0$ such that for all $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{D}(A) \cap (G_\lambda \times G_\lambda)^\perp$ the inequality (103) holds. Since A has a bounded inverse, thus A is Fredholm at 0 with $ind(A) = 0$. Thus there exists a nbhd of 0 on which $A - \lambda I$ is Fredholm for each λ in this nbhd and $ind(A - \lambda I) = 0$. The nbhd of 0 on which $A - \lambda I$ is Fredholm intersects also Λ. Thus we have found a region in Λ on which $A - \lambda I$ is Fredholm with $ind(A - \lambda I) = 0$. Using the same proof as in part 1 of the Theorem 4.2 we can show that $(\Lambda \setminus (-\alpha_2, 0)) \setminus \tilde{E} \subset \rho(A)$, where $\tilde{E}$ is the set of all isolated points of $\sigma(A)$ in $\Lambda \setminus (-\alpha_2, 0)$ with finite algebraic multiplicity. This shows that no point of $(-\alpha_2, 0)$ can be an accumulation point for non-real spectrum of A except for the set $\tilde{E}$. Now we prove that no point of $(-\alpha_2, 0)$ is an accumulation point of $\tilde{E}$. If we do this then the proof will be complete. Consider the Krein space $H_{\frac{1}{2}} \times H$ with the inner product $[.,.]$. Let $\lambda \in (-\alpha_2, 0)$ and define G_λ as in (119). For every sequence which satisfies (50) we have

$$\liminf_{n\to\infty} \left[\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right] = \liminf_{n\to\infty} \left(\langle x_n, x_n \rangle_{H_{\frac{1}{2}}} - \langle y_n, y_n \rangle_H \right), \tag{122}$$

where the last equality is valid by (28).
By the arguments at the beginning of this proof and also by $\langle A_0 x_n, x_n\rangle_H = \left\langle A_0^{\frac{1}{2}} x_n, A_0^{\frac{1}{2}} x_n \right\rangle_H = \langle x_n, x_n\rangle_{H_{\frac{1}{2}}}$ the equality (122) can be written as

$$\liminf_{n\to\infty} \left(\langle x_n, x_n\rangle_{H_{\frac{1}{2}}} - \langle y_n, y_n\rangle_H \right) = \liminf_{n\to\infty} \left(\langle A_0 x_n, x_n\rangle_H - \lambda^2 \langle x_n, x_n\rangle_H \right).$$

Also, since $\lambda \in (-\alpha_2, 0) \cap \sigma(A)$, then $\sigma = 0$. Using this in (121) we obtain

$$\liminf_{n\to\infty} \left(\langle A_0 x_n, x_n\rangle_H - \lambda^2 \langle x_n, x_n\rangle_H \right) > 0.$$

This and (122) give us

$$\liminf_{n\to\infty} \left[\begin{pmatrix} x_n \\ y_n \end{pmatrix}, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right] > 0,$$

where $x_n \in G_\lambda^\perp$, $n \in \mathbb{N}$.
Following the same reasoning as at the beginning of the proof of this theorem we can show that $(-\alpha_2, 0] \cap \sigma(A) \subset \sigma_{\pi^+}(A)$. Again, immitating the same proof as at the beginning of this theorem we can show that no point of the interval $(-\alpha_2, 0)$ is an accumulation point of $\tilde{E}$. Consequently no point of the interval $(-\alpha_2, 0)$ is an accumulation point of non-real spectrum of A.

Corollary 5.1 *Suppose A_0^{-1} is compact from H to H. Then no point from $\sigma_{ess}(A)$ is an accumulation point of the non-real spectrum.*

Proof: By Lemma 2.5 we have

$$s^{-1} \in \sigma_{ess}(A_0^{-\frac{1}{2}}) \Longleftrightarrow s \in \sigma_{ess}(A_0^{\frac{1}{2}}),$$

where $s \neq 0$. By assumption A_0^{-1} is compact, hence by the Lemma 4.1 $A_0^{-\frac{1}{2}}$ is compact. Thus $\sigma_{ess}(A_0^{-\frac{1}{2}}) = \emptyset$ or $\sigma_{ess}(A_0^{-\frac{1}{2}}) = \{0\}$. If $\sigma_{ess}(A_0^{-\frac{1}{2}}) = \emptyset$, then $\sigma_{ess}(A_0^{\frac{1}{2}}) = \emptyset$. Thus we have $\alpha_2 = \infty$. Hence $\alpha = max\{\alpha_1, \alpha_2\} = \infty$. But by Theorem 5.1 no point of $(-\alpha, 0) = (-\infty, 0)$ is an accumulation point of non-real spectrum of A. Also, by the Theorem 4.1 $\sigma_{ess}(A) \subset (-\infty, 0)$. Thus no point of $\sigma_{ess}(A)$ is an accumulation point of non-real spectrum of A. If $\sigma_{ess}(A_0^{-\frac{1}{2}}) = \{0\}$, then $s = \infty$. Thus $\alpha_2 = \infty$, and following the above argument we can prove our aim. But, assume now that $\sigma_{ess}(A_0^{\frac{1}{2}}) = \{0\}$. This cannot be true since by our assumption $0 \in \rho(A_0)$, consequently $0 \in \rho(A_0^{\frac{1}{2}})$.

The following corollary cannot be found in the original paper.

Corollary 5.2 *Suppose A_0^{-1} is compact from H to H and $\sigma_{ess}(A) = (-\infty, \beta]$. Then the non-real spectrum of A has no accumulation point in $\mathbb{C}$.*

Proof: Suppose N is the real spectrum of A and let N' denote the set of accumulation points of N. We use the following theorem to prove. Let $T : X \longrightarrow X$ be an operator with a nonempty resolvent set. Assume that Ω is an open connected subset of $\mathbb{C} \setminus \sigma_{ess}(A)$ such that $\Omega \cap \rho(A) \neq \emptyset$, then $\sigma(A) \cap \Omega$ has no accumulation point in Ω. (See [13, chapter XVII]). Setting $\Omega = \mathbb{C} \setminus \sigma_{ess}(A)$ and using the mentioned theorem, no point of Ω will be an accumulation point of $\sigma(A) \setminus \sigma_{ess}(A)$. Since $\sigma_{ess}(A) \subset N$, thus $\sigma(A) \setminus N \subset \sigma(A) \setminus \sigma_{ess}(A)$. Hence

$$(\sigma(A) \setminus N)' \cap \Omega \subset (\sigma(A) \setminus \sigma_{ess}(A))' \cap \Omega = \emptyset,$$

thus $(\sigma(A) \setminus N)' \cap \Omega = \emptyset$, or no point of Ω is an accumulation point of $\sigma(A) \setminus N$.
Also, no point of $\sigma_{ess}(A)$ is an accumulation point of $\sigma(A) \setminus N$ by the Corollary 5.1. Thus no point of $(\mathbb{C} \setminus \sigma(A)) \cup \sigma_{ess}(A)$ is an accumulation point of $\sigma(A) \setminus N$, or $\sigma(A) \setminus N$ has no accumulation point in $\mathbb{C}$.

In the Definition 3.4 we have seen what the relationship was among δ, β, and γ . We had in fact

$$\delta \leq \beta \leq \gamma.$$

The following examples show us that there is no such relationship among α_1, α_2, β, and γ_1.

Example 5.1 *Let H be an infinite dimensional Hilbert space, $D = 0$ and $A_0 = I$. Then we have*

$$\alpha_1 = 0, \ \alpha_2 = 1, \ \beta = 0, \ \text{and } \gamma_1 = \infty.$$

Solution: For finding all these quantities we use their definitions. We have, $s \in \sigma_{ess}(A_0^{-1}D)$ if and only if $sI - A_0^{-1}D$ is not Fredholm, equivalently sI is not Fredholm. But this is only possible if $s = 0$. Hence $\sigma_{ess}(A_0^{-1}D) = 0$. Using this in the formula of α_1 and γ_1 we find $\alpha_1 = 0$, $\gamma_1 = \infty$.
Using the definition of β we find $\beta = 0$, since $D = 0$ by the assumption of this example.
It remains to compute α_2. We just denote $I|_{H_{\frac{1}{2}}}$ (read restriction of I on $H_{\frac{1}{2}}$) by I. We claim that $A_0^{\frac{1}{2}} - sI : H_{\frac{1}{2}} \longrightarrow H$ is not Fredholm for $s = 1$. Since $A_0^{\frac{1}{2}} = I$ we can write

$$A_0^{\frac{1}{2}} - sI = A_0^{\frac{1}{2}}(1 - s) = I(1 - s).$$

Thus $A_0^{\frac{1}{2}} - sI$ is not Fredholm for $s = 1$, or $\sigma_{ess}(A_0^{\frac{1}{2}}) = \{1\}$. Using this in the definition of α_2, we obtain $\alpha_2 = 1$. Thus,

$$\alpha_1 = \beta < \alpha_2 < \gamma_1.$$

Example 5.2 *Let H be an infinite dimensional Hilbert space with orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$. Define the operators A_0 and D in $L(H)$ as follows,*

$$A_0 z = 9 \sum_{n=1}^{\infty} (1 + n^{-1}) \langle z, e_n \rangle e_n, \quad Dz = 9 \sum_{n=2}^{\infty} (1 + n^{-1}) \langle z, e_n \rangle .$$

Then we have $\alpha_1 = 9$, $\alpha_2 = 3$, $\beta = 0$, and $\gamma_1 = 2$.

Solution: First, we are going to write the inverse of A_0. This is in fact,

$$A_0^{-1}z = \frac{1}{9}\sum_{n=1}^{\infty}\frac{1}{1+n^{-1}}\langle z, e_n\rangle e_n. \tag{123}$$

One can see by an elementary calculation that A_0^{-1} given above is the inverse of A_0 by showing that $A_0A_0^{-1} = I$ and $A_0^{-1}A_0 = I$. By (123) we have

$$||A_0^{-1}z|| = \frac{1}{9}||\sum_{n=1}^{\infty}\frac{1}{1+n^{-1}}\langle z, e_n\rangle e_n|| \leq \frac{1}{9}||\sum_{n=1}^{\infty}\langle z, e_n\rangle e_n|| = \frac{1}{9}||z||,$$

hence $||A_0^{-1}|| \leq \frac{1}{9}$. On the other hand we have $||A_0^{-1}|| = \sup_{||z||=1}||A_0^{-1}z||$. Taking $z = e_n$ we obtain

$$||A_0^{-1}|| = \sup_{||z||=1}||A_0^{-1}z|| \geq ||A_0^{-1}e_n|| = \frac{1}{9}\frac{n}{n+1}.$$

Taking the limit in the right hand side we have $||A_0^{-1}|| \geq \frac{1}{9}$. Hence we obtain $||A_0^{-1}|| = \frac{1}{9}$.

We now find the essential spectrum of the operator $A_0^{-1}D$. To find this we use the matrix representation of A_0^{-1} and D. A_0^{-1} is a diagonal matrix with diagonal the sequence $\left\{\frac{1}{9}\frac{n}{n+1}\right\}_{n=1}^{\infty}$. And D is also a diagonal matrix with diagonal the sequence $\{9(1+n^{-1})\}_{n=2}^{\infty}$. Thus $A_0^{-1}D$ will be a diagonal matrix with diagonal the sequence $\left\{\frac{3}{4}, \frac{8}{9}, \frac{15}{16}, \frac{24}{25}, \frac{35}{36}, \frac{48}{49}, \cdots\right\}$ which converges to 1. Since $A_0^{-1}D$ is a self adjoint operator its essential spectrum consists only of the accumulation points of the spectrum. Hence $\sigma_{ess}(A_0^{-1}D) = \{1\}$. Using this and $||A_0^{-1}|| = \frac{1}{9}$ in the definition of α_1 and γ_1 we find $\alpha_1 = \frac{9}{2}$ and $\gamma_1 = 2$.

For finding β we use its definition. Setting $z = e_1$ in the definition of D we have $De_1 = 0$. Putting this in the definition of β we obtain $\beta = 0$.

For α_2 we do as follows: $A_0^{\frac{1}{2}}$ has the matrix representation which is diagonal and its diagonal is the sequence $\left\{3\sqrt{\frac{n+1}{n}}\right\}_{n=1}^{\infty}$. Hence $\sigma_{ess}(A_0^{\frac{1}{2}}) = \{3\} \cup \left\{3\sqrt{\frac{n+1}{n}}\right\}_{n=1}^{\infty}$. We have to mention here that 3 is also an element of the spectrum since it is the accumulation point of the sequence $\left\{3\sqrt{\frac{n+1}{n}}\right\}_{n=1}^{\infty}$ and the spectrum is a closed set.

Since $A_0^{\frac{1}{2}}$ is a self adjoint operator, thus the essential spectrum of $A_0^{\frac{1}{2}}$ must contain the non-isolated points of $\sigma(A_0^{\frac{1}{2}})$. (See [9, chapter XIII]). But the only non-isolated point of $\sigma(A_0^{\frac{1}{2}})$ is 3. Thus $\sigma_{ess}(A_0^{\frac{1}{2}}) = \{3\}$ and we find $\alpha_2 = 3$. Thus,

$$\beta < \gamma_1 < \alpha_2 < \alpha_1.$$

Example 5.3 *Let H and e_n as in previous example. Define A_0 and D in $L(H)$ as follows.*

$$A_0z = \langle z, e_1\rangle e_1 + 9\sum_{n=2}^{\infty}\left(1+n^{-1}\right)\langle z, e_n\rangle e_n, \quad Dz = 9\sum_{n=1}^{\infty}\left(1+n^{-1}\right)\langle z, e_n\rangle e_n.$$

Then we have $\alpha_1 = \frac{1}{2}$, $\alpha_2 = 3$, $\beta = 9$, and $\gamma = 2$.

Solution: First of all we find $||A_0^{-1}||$.

$$||A_0^{-1}z|| = || \langle z, e_1 \rangle e_1 + \frac{1}{9} \sum_{n=2}^{\infty} \langle z, e_n \rangle e_n|| = || \langle z, e_1 \rangle e_1 + \frac{1}{9} \left(\sum_{n=1}^{\infty} \langle z, e_n \rangle e_n - \langle z, e_1 \rangle e_1 \right) || =$$

$$||\frac{8}{9} \langle z, e_1 \rangle e_1 + \frac{1}{9} \sum_{n=1}^{\infty} \langle z, e_n \rangle e_n|| \leq ||\frac{8}{9} \sum_{n=1}^{\infty} \langle z, e_n \rangle e_n + \frac{1}{9} \sum_{n=1}^{\infty} \langle z, e_n \rangle e_n|| = || \sum_{n=1}^{\infty} \langle z, e_n \rangle e_n|| = ||z||.$$

Hence $||A_0^{-1}z|| \leq ||z||$, thus $||A_0^{-1}|| \leq 1$.
Also, $||A_0^{-1}|| = \sup_{||z||=1} ||A_0^{-1}z|| \geq ||A_0^{-1}z_0||$ for each $z_0 \in H$ such that $z_0 = e_1$. Take $z_0 = e_1$, then $A_0^{-1}e_1 = \langle e_1, e_1 \rangle + \frac{1}{9} \sum_{n=2}^{\infty} \frac{n}{n+1} \langle e_1, e_n \rangle e_n$. Thus $A_0^{-1}e_1 = e_1 + 0$ and we obtain $A_0^{-1}z_0 = z_0$. Thus $||A_0^{-1}|| \geq ||z|| = 1$. Hence we have $||A_0^{-1}|| = 1$.
We now find $\sigma_{ess}(A_0^{-1}D)$. The matrix representation of A_0^{-1} is a diagonal matrix which has as its diagonal the sequence $\{1\} \cup \left\{9\left(\frac{n+1}{n}\right)\right\}_{n=2}^{\infty}$ and D is a diagonal matrix with diagonal the sequence $\left\{9\left(\frac{n+1}{n}\right)\right\}_{n=1}^{\infty}$. Hence $A_0^{-1}D$ has the matrix representation which its diagonal is the sequence $\{18\} \cup \{1, 1, \cdots\}$. Thus, we will have $\sigma(A_0^{-1}D) = \{18, 1\}$. But what is $\sigma_{ess}(A_0^{-1}D)$? This must be a subset of the spectrum of $A_0^{-1}D$ or equal with it. We claim that $\sigma_{ess}(A_0^{-1}D) = \{1\}$. First we show that $18 \notin \sigma_{ess}(A_0^{-1}D)$. To show this we use the formula (see, [23, chapter 7])

$$\sigma_{ess}(A_0^{-1}D) = \cap \left\{\sigma(A_0^{-1}D + K) | K \ compact\right\}.$$

Take K_1 as zero matrix and K_2 as a diagonal, of which the first entry is -18 and the rest is zero. Thus, $\sigma(A_0^{-1}D + K_1) = \{1, 18\}$ and $\sigma(A_0^{-1}D + K_2) = \{0, 1\}$. This shows that

$$\sigma(A_0^{-1}D + K_1) \cap \sigma(A_0^{-1}D + K_2) = \{1\}.$$

Is $\sigma_{ess}(A_0^{-1}D) = \{1\}$? To answer this we show that $A_0^{-1}D - I$ is not Fredholm. We have in fact, $kern(A_0^{-1}D - I) = \left\{(x_1, x_2, \cdots) \in H | (A_0^{-1}D - I)(x_1, x_2, \cdots) = 0\right\}$, from this we obtain $x_1 = 0$. Thus,

$$kern(A_0^{-1}D - I) = \left\{(0, x_2, x_3, \cdots) | (A_0^{-1}D - I)(0, x_2, x_3, \cdots) = 0\right\}.$$

Thus $kern(A_0^{-1}D - I)$ is an infinite dimensional space. So $A_0^{-1}D$ is not Fredholm for $\lambda = 1$ or $\sigma_{ess}(A_0^{-1}D) = \{1\}$. Using this information in the definitions of α_1 and γ_1 we obtain $\alpha_1 = \frac{9}{2}$ and $\gamma_1 = 2$.

To find α_2 we use the matrix representation of $A_0^{\frac{1}{2}}$. This is a matrix which is diagonal and its diagonal is the sequence $\{1, 3\} \cup \left\{3\sqrt{\frac{n+1}{n}}\right\}_{n=2}^{\infty}$. Hence $\sigma(A_0^{\frac{1}{2}}) = \{1, 3\} \cup \left\{3\sqrt{\frac{n+1}{n}}\right\}_{n \in \mathbb{N}}$. By the same argument as in the previous example we find $\sigma_{ess}(A_0^{\frac{1}{2}}) = \{3\}$. Using this in the definition of α_2 we find $\alpha_2 = 3$.

For β we just use again its definition. First we compute $\langle Dz, z\rangle_H$. This is,

$$\langle Dz, z\rangle = \left\langle 9\sum_{n=1}^{\infty}\left(1+n^{-1}\right)\langle z, e_n\rangle e_n, z\right\rangle = 9\left\langle 2\langle z, e_1\rangle e_1 + \frac{3}{2}\langle z, e_2\rangle e_2 + \cdots, z\right\rangle =$$

$$9\left(2\langle z, e_1\rangle\langle e_1, z\rangle + \frac{3}{2}\langle z, e_2\rangle\langle e_2, z\rangle + \cdots\right) = 9\sum_{n=1}^{\infty}\left(\frac{n+1}{n}\right)|\langle z, e_n, e_n\rangle|^2 \geq$$

$$9\sum_{n=1}^{\infty}|\langle z, e_n\rangle|^2.$$

Suppose now that $z = \sum_{i=1}^{\infty}\alpha_i e_i$, then we have $\langle z, e_i\rangle = \alpha_i$. Using this in the above inequality we have

$$\langle Dz, z\rangle \geq 9\sum_{n=1}^{\infty}|\alpha_i|^2. \tag{124}$$

Also,

$$||z||_H^2 = \langle z, z\rangle_H = \langle \alpha_1 e_1 + \alpha_2 e_2 + \cdots, \alpha_1 e_1 + \alpha_2 e_2 + \cdots\rangle_H =$$
$$|\alpha_1|^2\langle e_1, e_1\rangle + |\alpha_2|^2\langle e_2, e_2\rangle + \cdots.$$

Since e_n is an orthonormal set, thus we have $||z||_H^2 = \sum_{i=1}^{\infty}|\alpha_i|^2$. By this and (124) we obtain

$$\frac{\langle Dz, z\rangle_H}{\langle z, z\rangle_H} \geq 9, \quad \text{or} \quad \inf_{z\in H_{\frac{1}{2}}\setminus\{0\}}\frac{\langle Dz, z\rangle_H}{\langle z, z\rangle_H} \geq 9.$$

We claim now that this infimum is equal to 9. For this suppose that $z = e_n$, then $\langle De_n, e_n\rangle = 9\left(1+n^{-1}\right)$. Taking the limit as n goes to infinity we see that, $\langle De_n, e_n\rangle \longrightarrow 9$. This shows that the above infimum is equal to 9. Thus,

$$\alpha_1 < \gamma_1 = \alpha_2 < \beta.$$

Conclusion

In this master thesis we have studied the location of spectrum of second order o.d.e, $\ddot{z}(t) + D\dot{z}(t) + A_0 z(t) = 0$, where A_0 and D were operators on a Hilbert space. We have also studied the essential spectrum of this o.d.e where A_0^{-1} were compact and not compact. We also generalized our discussion to higher order o.d.e. The idea of higher order equation was almost the same as second order, but we had to change some essential points in our discusions and proofs.

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