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Lyapunov-Schmidt Reduction for Singular Perturbations: Wave Trains in Reaction-Diffusion Systems

by

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in the
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Declaration of Authorship

I, Yudistira Lesmana, have read and declare that this thesis titled, 'Lyapunov-Schmidt Reduction for Singular Perturbations: Wave Trains in Reaction-Diffusion Systems' in my opinion is fully adequate in scope and quality as a dissertation for the degree Master of Science.

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"May there be good intentions in the mind, so that you may perform noble deeds"

The Upanishads

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Abstract

Faculty of Sciences
Department Of Mathematical Sciences

Master of Science

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We discuss wave trains in reaction-diffusion system on the real line. We also discuss the singular perturbation theory. The framework of Lyapunov-Schmidt reduction method is discuss in detail and we also show that wave trains exist for zero diffusion coefficients.

Keywords: Wave Trains; Reaction-Diffusion Equations; Singular Perturbation.

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Abbreviations

inf	: Infimum (greatest lower bound)
sup	: Supremum (least upper bound)
ker	: Kernel
dim	: Dimension
w.r.t	: with respect to
bdd	: bounded
Ran (Rg)	: Range
Re	: Real
Im	: Imaginary
IFT	: Implicit Function Theorem

Symbols

$\mathbb{N}$	: set of Natural numbers
$\mathbb{R}$	: set of Real numbers
$\mathbb{C}$	: set of Complex numbers
$\mathbb{Z}$	: set of Integers
$\mathbb{R}^n$	: n dimensional Euclidean space
$\lvert x \rvert$	: the absolute value of x
X, Y, Z	: (real) Banach spaces
$C^k[a, b]$	: k times continuously differentiable on $[a, b]$
$C[a, b]$	: space of continuous functions on $[a, b]$
∂_x	: partial derivative with respect to x
∂_t	: partial derivative with respect to t
l^p	: $\{f : \sum \lvert f_j \rvert^p < \infty\}$
$H^{k,p}; W^{k,p}$	: Sobolev spaces

Dedicated to my parents

and

uncle Unnikrishnan on the occassion of his 59th birthday

Chapter 1

Introduction

Many physical processes with applications in engineering, biology, chemistry, physics and other fields can be modelled by studying partial differential equations (PDEs). An important class of nonlinear parabolic PDE's are reaction-diffusion equations. The main motivation for studying wave-type solutions is physical phenomena such as wall propagation in liquid crystals, nucleation kinetics, neutron action in the reactor, nerve impulse propagation in nerve fibres and general pattern formation in dissipative systems [3, 10, 12, 20].

Some of the dominant patterns exhibited by these systems are homogeneous or uniform steady state, travelling waves, spiral waves, Turing patterns, localised structures and spatio-temporal chaos. As with general solutions to nonlinear systems, these are often treated by perturbation or numerical methods since it is difficult to demonstrate the existence analytically.

In this thesis, we are interested in spatio-temporally periodic solutions in spatially one-dimensional reaction-diffusion systems. These are wave trains whose profile depends periodically on a single-phase variable in which space x and time t are coupled through wave number k and frequency ω by $(kx - \omega t)$ (Figure 1.1). A concrete example where wave trains are discussed in detail can be found in [18]. Specifically, we analyze the cases when some of the diffusion coefficients or the wave number are close to zero.

1.1 Examples

1. The Oregonator model

   This model (Figure 1.2(a)), was first introduced by Field, Koros and Noyes in 1974. It has been derived as a model of the Belusov-Zhabotinsky (BZ) (Figure

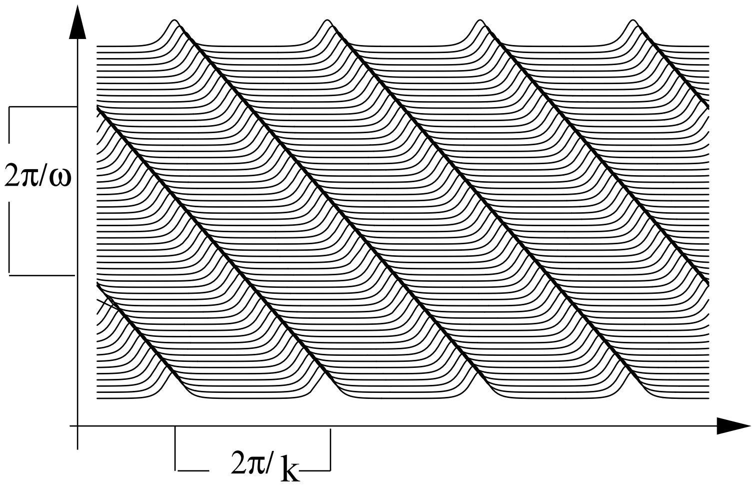
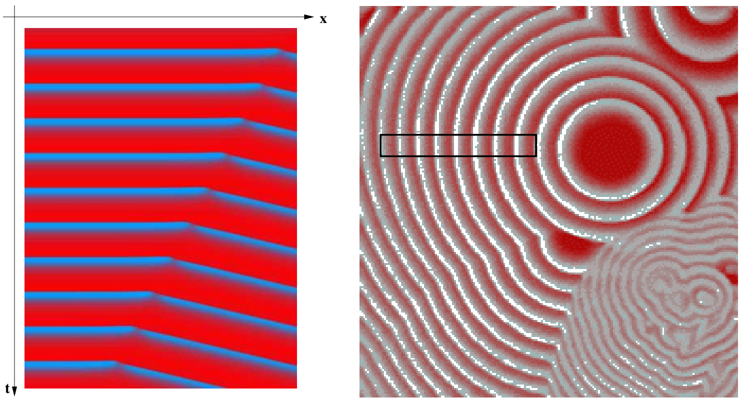


FIGURE 1.1: Sketch of space-time plot



(a) (b)

FIGURE 1.2: (a) Space-time plot of a simulation of the oregonator model that show homogeneous oscillation (horizontal blue stripes) and wave trains (sloped blue stripes). The red colour indicates the low values of w and blue stripes indicate the high values; (b) Experiment of the Belousov-Zhabotinsky reaction [18]. The dynamics inside the box show an approximate wave train for this model and circles represent target patterns in 2-D. Picture (b) is taken from [4]

1.2(b)), pattern forming chemical reaction that involve three main active species, namely: bromous acid $HBRO_2$ acts as an autocatalyst, bromide ions Br^- act as inhibitor and the oxidised form of the catalyst M_{ox}. BZ models can be modified in two or three variable version. Here, we will give the three-variable version of the Oregonator model [20]

$$\varepsilon\partial_t u = u(1-u) - w(u-q) + \Delta u$$
$$\partial_t v = u - v$$
$$\varepsilon'\partial_t w = \Phi + fv - w(u+q) + D_w\Delta w,$$

where u,v and w essentially correspond to the concentration of Bromous acid, Bromide ions and oxidised catalyst respectively with parameters are $f = 1.5$, $D_v = 1.12$, $q = 0.01$, $\varepsilon = 0.09$, $\varepsilon' = 0.011$, and $\Phi = 0.0002$ [17]. There exists a homogeneous oscillation through a Hopf bifurcation [15, 16, 20] and a family of large amplitude wave trains near a pulse. Note that v does not diffuse.

2. The FitzHugh-Nagumo Equation

   The model has been widely used to study various phenomena in neurophysiology and cardiophysiology [10, 12]. The FitzHugh-Nagumo equations read [12]

$$\partial_t u = \partial_{xx} u + f(u) - w$$
$$\partial_t w = \varepsilon(u - \gamma w),$$

   where u and w correspond to membrane potential and recovery respectively, ε and γ are positive and f has the form $u(u-a)(1-u)$ where $0 < a < 1$. As for the Oregonator model, a Hopf bifurcation occurs and there exist wave train near a pulse [8]. Note that w does not diffuse.

There are many more examples such as the Brusselator model and the Lotka-Volterra model that have the same properties.

1.2 Singular Perturbation

In both examples, a diffusion coefficient is set to zero since it is very small in the underlying experiments. The question arises whether or not setting these quantities to zero is justified. This is not obvious since it is a so-called singular perturbation for the equations that determine the spatial profile. To see this, first consider the abstract form of reaction-diffusion equations on the real line $\mathbb{R}$

$$\partial_t u(x,t) = D\partial_{xx} u(x,t) + f(u(x,t),\mu) \tag{1.1}$$

where D is a diagonal matrix with non-negative entries, i.e , $d_i \geq 0$ for $i = 1, ..., N$, $x \in \mathbb{R}$; $\mu \in \mathbb{R}^\ell$; $u \in \mathbb{R}^N$ and the nonlinearity f describes the kinetics for various reactions. The patterns we are concerned with are travelling waves that are equilibria in a co-moving variable $y = x - ct$. This ansatz in (1.1) yields

$$0 = D\partial_{yy} u + c\partial_y u + f(u,\mu) \tag{1.2}$$

which is an Ordinary Differential Equations (ODEs) in y; clearly letting $d_j \to 0$ constitute a singular perturbation.

The main result in this thesis shows how to regularise this problem for wave trains. In order to compare this with classical geometric singular perturbation [13], we next discuss this approach.

Consider the simplest case of a scalar equation

$$\varepsilon u'' + cu' + f(u) = 0 \tag{1.3}$$

with small parameter ε, $0 < \varepsilon \ll 1$. Equation (1.3) can also be written as

$$u'' + \frac{1}{\varepsilon}(cu' + f(u)) = 0. \tag{1.4}$$

Next, we want to find/analyze the slow manifold and slow-fast flow. This can be done by writing (1.4) as a first-order system, i.e. ,

$$\begin{aligned} u' &= v \\ \varepsilon v' &= -(cv + f(u)). \end{aligned} \tag{1.5}$$

By time rescaling $t = \varepsilon\tau$

$$\begin{aligned} \dot{u} &= \varepsilon v \\ \dot{v} &= -(cv + f(u)). \end{aligned} \tag{1.6}$$

As $\varepsilon \to 0$ in (1.3), we obtain $cv = -f(u)$ which is the slow manifold with slow flow $cu' = -f(u)$, for an illustration see Figure 1.3(a). On the other hand, as $\varepsilon \to 0$ in (1.4) and (1.6) we obtain the fast flow as given in Figure 1.3(b), where the slow manifold is a manifold of equilibria.

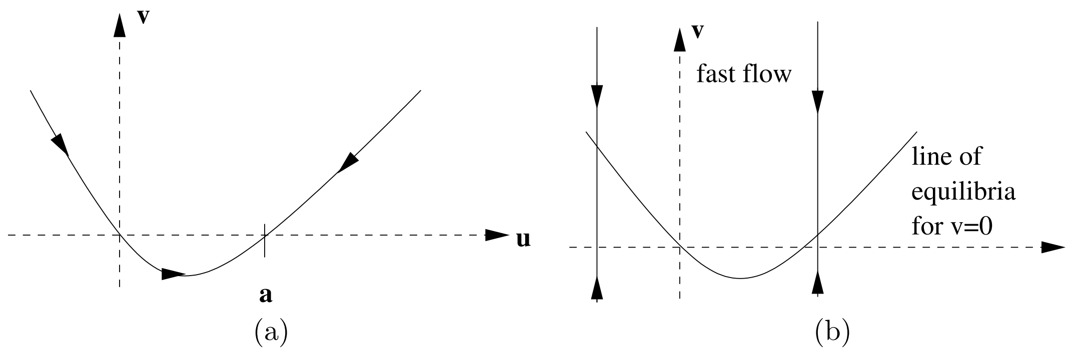


FIGURE 1.3: (a) The slow-manifold graph which is given by $cv = -f(u)$ herewith $f(u) = u(u - a)$; (b) The fast flow of (1.3) obtained by setting $\varepsilon = 0$ in (1.5)

Concerning the stability of the slow manifold we compute the eigenvalues of the linearization of system (1.6),

$$\lambda_{\pm} = -\frac{c^2}{2} \pm \sqrt{\frac{c^2}{4} - \varepsilon f'(u,\mu)}. \tag{1.7}$$

At $\varepsilon = 0$ we obtain $\lambda_+ = 0$ and $\lambda_- = -c$. Evaluated on the manifold of equilibira, we obtain the corresponding eigenvector $e_+ = (c, f')$ and $e_- = (0, 1)$, where e_+ is the direction of the manifold of equilibira and e_- the direction of the fast flow.

By normal hyperbolicity for $c \neq 0$, Fenichel theory applies; see, e.g., [13]. This implies persistence of the slow manifold for $\varepsilon > 0$ which justifies setting $\varepsilon = 0$ if it is small.

This analysis carries over to (1.2) for zero diffusion; see [17]. A condition for the persistence of a wave train is that it is hyperbolic, i.e, it has Floquet multipliers off the unit circle, i.e, away from the unit circle, e.g [17].

1.3 Reaction-Diffusion Equations

The standing assumption we make here is $f \in C^2$. We assume that (1.1) at $\mu = \mu_*$ has a wave-train solution $u(x,t) = u_{\mathrm{wt}}(k_* x - \omega_* t)$ with $u_{\mathrm{wt}}(\xi + 2\pi) = u_{\mathrm{wt}}(\xi)$, where k_* is the wave number and ω_* is the frequency.

Wave trains $u_{\mathrm{wt}}(kx - \omega t)$ are travelling wave that move with constant speed $\frac{\omega}{k}$ without changing shape, i.e, equilibria in the co-moving variable $y = x - \frac{\omega}{k}t$. Indeed, if u is travelling wave with velocity $\frac{\omega}{k}$ then $u(x,t) = \tilde{u}(x - \frac{\omega}{k}t) = \tilde{u}(y)$ so that

$$D\tilde{u}_{yy} + \frac{\omega}{k}\tilde{u}_y + f(\tilde{u},\mu) = 0. \tag{1.8}$$

On the other hand, for $u = u_{\mathrm{wt}}(kx - \omega t)$ we have $\tilde{u}(y) = u_{\mathrm{wt}}(ky)$ and therefore a wave train at parameter value μ is a travelling wave solving

$$F(u;\omega,k,\mu) = k^2 D u'' + \omega u' + f(u,\mu) = 0 \;\; ; \; ' = \frac{d}{d\xi} \;\; ; \; u(2\pi) = u(0). \tag{1.9}$$

Next, we will observe two special cases, namely $k = 0$ and $\omega = 0$. In the case $k = 0$, the wave train is in fact a spatially homogenenous oscillation (Figure 1.4). Thus the solution is independent of the space variable and solves the ODE of the reaction kinetics. Finding wave trains near $k = 0$ therefore shows how patterns arise near a periodic solution of the reaction kinetics that itself does not generate a spatial pattern. The case $\omega = 0$ corresponds to spatially periodic stationary solutions which are also called Turing patterns [19].

We emphasize that the case $k \sim 0$ for $\omega \sim \omega_* \neq 0$ is different from the classical approach to the onset of pattern formation from a homogeneous steady state undergoing an instability. The latter leads to patterns like those depicted in Figure 1.5 whose amplitude is near zero.

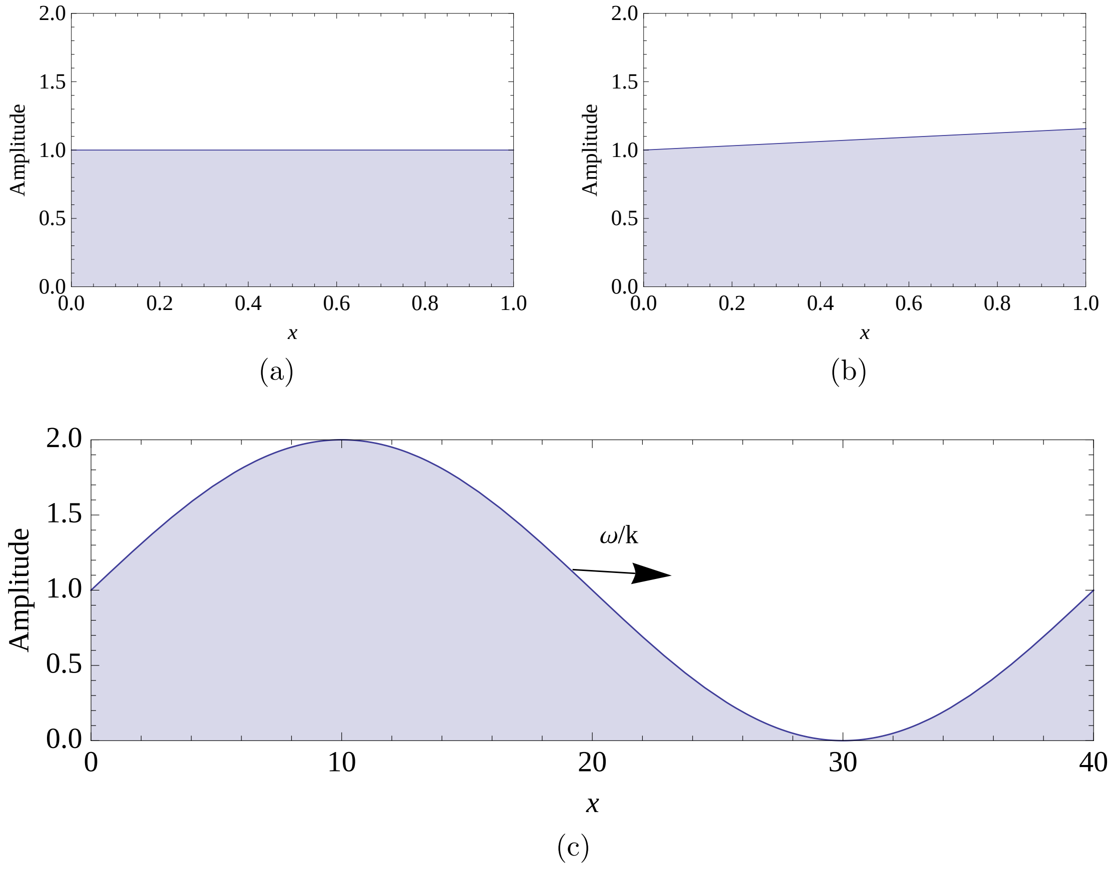


FIGURE 1.4: Snapshots of wave train profiles (a) An illustration for $k = 0$; (b) and (c) Illustration for $k \sim 0$. Note that the amplitude of the oscillation is not small

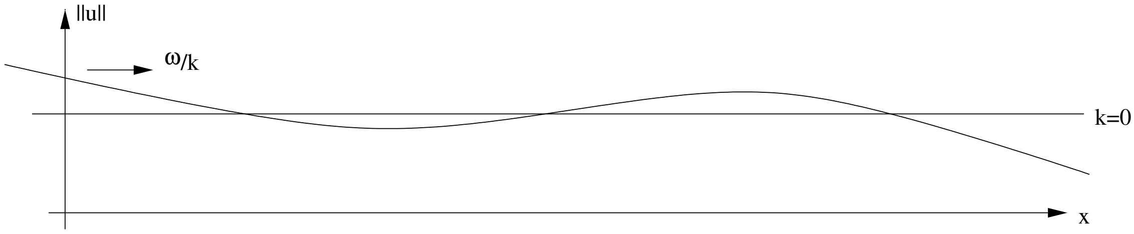


FIGURE 1.5: An illustration for $k = 0$ and $k \sim 0$

It was shown in [19] that for $k_* \neq 0$ or $\omega_* \neq 0$ there exists a family of wave trains with a wave number $k \sim k_*$ and frequency $\omega(k), \omega(k_*) = \omega_*$. The first part of the thesis gives all details of the proof, which were not given in [19]. In the second part, this result is extended to the case of vanishing diffusion coefficients. Recall that this frequently occurs in models see section 1.1.

In the case $k \neq 0$, $D > 0$ these results are regular perturbation problem that can be treated by Implicit Function Theorem and for $k = 0$ these are singular perturbation problem that can be derived using geometric singular perturbation theory. However, here we will prove them using Lyapunov-Schmidt reduction as in [19].

In the following we will simply use u to denote wave trains.

Chapter 2

Lyapunov Schmidt Reduction for Near-Homogeneous Wave Trains

2.1 Continuation in Wave Number

This section will discuss how to apply the Lyapunov-Schmidt reduction method to wave trains. In oder to do so, we require F to be a nonlinear Fredholm operator with index zero. In particular, the linearisation $\mathcal{L}$ of F with respect to u in u_* which is formally given by:

$$\mathcal{L}u := k^2 Du'' + \omega u' + \partial_u f(u_{wt}, \mu)u, \tag{2.1}$$

should be a Fredholm operator with index zero. We define $H^j = \left(H^j_{\text{per}}([0, 2\pi])\right)^N$, $j = 1, 2$ and $L^2 = \left(L^2_{\text{per}}([0, 2\pi])\right)^N$.

For the case $k \neq 0$, we have $\mathcal{L} : H^2 \to L^2$, $F : H^2 \to L^2$ and for $k = 0$, we have $\mathcal{L} : H^1 \to L^2$, $F : H^1 \to L^2$. In order to see this, first we consider F and note that we need to show that $f : H^1 \to L^2$ as $H^2 \subset H^1$. This follows since H^1 is a Banach algebra, i.e, $f : H^1 \to H^1$; see Theorem 5.23 with $n = 1$ and $p = 2$ in [1].

In the case $f : \mathbb{R}^N \to \mathbb{R}^N$ is C^2, we also have $\partial_u f : H^1 \to H^1$ [Appendix A]. In particular $\partial_u f : H^2 \to L^2$ is well defined.

Note that $\partial_\xi : H^2 \to L^2$ is not Fredholm because the co-kernel is infinite dimensional. Note that the domain is $H^2 \subset H^1$.

Definition 2.1. Let A be a linear operator. Let $N(A)$ and $R(A)$ denote kernel and range respectively. Then

The kernel of A is the set $N(A) = \{x \in \mathbb{R}^n : Ax = 0\}$.

The Range of A is the set $R(A) = \{v\ \exists x : Ax = v\}$

Remark 1. *The spatial translation symmetry implies that for a wave train u we have u' is always in the kernel of $\mathcal{L}$. To see that directly we compute $\mathcal{L}u' = 0$, i.e ,*

$$\begin{aligned} k^2Du''' + \omega u'' + \partial_u f(u,\mu)u' = 0 \\ \Leftrightarrow \partial_\xi(k^2D\partial_{\xi\xi} + \omega\partial_\xi)u + \partial_u f(u;\mu)u' = 0 \\ \Leftrightarrow \frac{d}{d\xi}[k^2Du'' + \omega u' + f(u,\mu)] = 0 \\ \Leftrightarrow \frac{d}{d\xi}F(u,\mu) = 0, \end{aligned}$$

which is true since $F(u,\mu) = 0$. Note that for $f \in C^1$ it follows that $u \in C^3$.

We assume that $\mathcal{L}$ has one-dimensional kernel that does not lie in its range so that 0 is a simple eigenvalue of algebraic multiplicity 1. Note that for this we view $\mathcal{L}$ as an unbounded operator from $H^2 \subset L^2 \to L^2$.

Why Lyapunov Schmidt Reduction?

Due to the kernel by spatial translation symmetry the Implicit Function Theorem (IFT) does not apply. However, since it is simply a spatial translation, one may view it as IFT after all by adding a constraint which also reduces the problem to one-dimensional equation which is trivial in this case since simply mean spatial translation. Note that the present setup allows to treat higher dimensional kernels, see [18].

The following result is the basic Lyapunov-Schmidt reduction for our purposes.

Theorem 2.2. *Let X and Z be Banach spaces. Let $u_* \in X$; $\omega_* \in \mathbb{R}$ and $\lambda_* \in \mathbb{R}^d$, and $U \subset X$ and $V \subset \mathbb{R}$ denote open sets containing u_* and ω_*, respectively, and $W \in \mathbb{R}^d$ a neighbourhood of the parameters λ_*. Let $\mathcal{F} : U \times V \times W \to Z$ be given with $\mathcal{F}(u_*;\omega_*,\lambda_*) = 0$. Assume that:*

(i) $\mathcal{F} : U \times V \times W \to Z$ is continuously differentiable

(ii) dim $N(\partial_u\mathcal{F}(u_*;\omega_*,\lambda_*)) = 1$ and the Fredholm index of $\partial_u\mathcal{F}(u_*,\omega_*,\lambda_*)$ is zero, i.e, codim $R(\partial_u\mathcal{F}(u_*,\omega_*,\lambda_*)) = 1$.

(iii) $\partial_\omega\mathcal{F}(u_*,\omega_*,\lambda_*) \notin R(\partial_u\mathcal{F}(u_*,\omega_*,\lambda_*))$.

Then there are continuously differentiable solutions given by

$$u(s,\lambda),\omega(s,\lambda), s\in(-\delta,\delta),\lambda\in\widetilde{W}\subset W; \qquad (2.2)$$
$$(u(0,\lambda_*),\omega(0,\lambda_*))=(u_*,\omega_*),$$

such that $\mathcal{F}(u,\omega,\lambda)=0$ *for* $s\in(-\delta,\delta);\lambda\in\widetilde{W}\subset W$ *and all solutions of* $\mathcal{F}(u,\omega,\lambda)=0$ *in a neighbourhood of* (u_*,ω_*) *and* $\lambda_*\in\widetilde{W}$ *belong to* (2.2).

Note that, for the application to wave train s plays the role of spatial translation in here, so that we can take $s\in(0,2\pi]$.

Proof. We slightly extend Theorem I.4.1 p. 12 in [14] by adding parameters λ in the application of the IFT, see [11]. More precisely, using the notation in [14], we take neighbourhoods $\tilde{U}\subset U_1$ of zero in $N(\mathcal{L})$, V_{ω_*} for values ω near ω_*, I_{λ_*} for values λ near λ_*. Let Z_0 be a complement of Z defined as

$$Z=R(\partial_u F(u;\omega,\lambda))\oplus Z_0.$$

The extended bifurcation function corresponding to equation (I.4.9) in [14] maps

$$\Phi:\tilde{U}\times V_{\omega_*}\times I_{\lambda_*}\to Z_0,$$

and we can apply the IFT here, since $D_\omega\Phi\neq 0$ still holds. Thus the solution of $F(u;\omega,\lambda)=0$ can be found by solving $\Phi(u;\omega,\lambda)$ near $(u_*;\omega_*,\lambda_*)$. □

The basic result for continuation of wave train is

Lemma 2.3. *Assume that* (1.9) *has a wave train solution* u_* *for some* $(k,\omega,\mu)=(k_*,\omega_*,\mu_*)$ *where* $k_*\neq 0$ *or* $\omega_*\neq 0$. *Suppose that* $\gamma=0$ *is an eigenvalue of* $\mathcal{L}$ *of algebraic multiplicity 1. Then there exists a local family of wave trains* $u(\xi,k,\mu)$ *with frequencies* $\omega(k,\mu)$ *for* $(k,\mu)\sim(k_*,\mu_*)$ *which are continuously differentiable in* (k,μ) *with* $u_*(\xi,k_*,\mu_*)=u_*$ *and* $\omega(k_*,\mu_*)=\omega_*$. *Every solution* $(u,\omega)\sim(u_*,\omega_*)$ *and* $(k,\mu)\sim(k_*,\mu_*)$ *to* (1.9) *belong to this family of wave trains and frequencies up to spatial translation.*

Proof of Lemma 2.3

To emphasize the differences between $k_*\neq 0$ and $k_*=0$ and to prepare $k_*=0$, we discuss $k_*\neq 0$.

Step 1: $k_* \neq 0$

We will apply Theorem 2.2 with $X = H^2$, $Z = L^2$ and $\mathcal{F} = F$, $W = \mathbb{R}^{1+\ell}$, $\lambda = (k, \mu)$.

First the condition for $\mathcal{F}(u_*, \omega_*, \lambda_*) = 0$ holds by assumption that a wave train exists. Second, we need to verify if all assumptions (i), (ii) and (iii) of Theorem 2.2 are satisfied. The verification is the following:

(i) Since F is linear in ω and k^2, $\partial_\omega F = u'_*$, $\partial_{k^2} F = Du''_* \in L^2$ since $u_* \in H^2$. The differentiability of f with respect to μ and u are proven in Appendix A.

(ii) Lemma 2.3 assumes $N(\mathcal{L})$ is one dimensional. Since $\partial_u f(u_*, \mu) : H^2 \to H^2$ and H^2 is compact in L^2, it follows that $\partial_u f(u_*, \mu)$ is a compact perturbation. Therefore it is irrelevant for Fredholm properties. Since

$$\widetilde{\mathcal{L}} \equiv k^2 D\partial_{\xi\xi} + \omega\partial_\xi : H^2 \to L^2 \tag{2.3}$$

is Fredholm, see Theorem 2.3.3 in [7], the claim follows from the fact that the adjoint $(-1)^j \partial_\xi^j : H^2 \subset L^2 \to H^2$ of ∂_ξ^j has the same kernel dimension.

(iii) We compute $\partial_\omega F(u_*, k_*, \omega_*, \mu_*)$ gives u'_*. Using the argument in Remark 1 we have $u'_* \in \ker(\partial_u \mathcal{F}(u_*, k_*, \omega_*, \mu_*))$ but since we have assumed that 0 of $\mathcal{L}$ is of algebraic multiplicity 1, u'_* does not lie in the range of $\mathcal{L}$. Hence $\partial\omega F(u_*, k_*, \omega_*, \mu_*) \notin R(\partial_u \mathcal{F}(u_*, k_*, \omega_*, \mu_*)$.

Therefore, from Theorem 2.2, we obtain the solutions (2.2). The parameter s corresponds to the spatial translations of u_* so that in fact we can take $s \in (0, 2\pi]$. Hence, in $\omega(s, k, \mu)$ the s- dependence is trivial (it is just a phase shift), thus we get $\omega(k, \mu)$ as claimed.

For the case $k_* \neq 0$ **:**

We have the linearisation $\mathcal{L} = \omega\partial_\xi + f(u_*, \mu)$ where we view $\mathcal{L} : H^1 \to L^2$. By our assumption its kernel is spanned by u'_*. Note that the operator $\mathcal{L} : H^2 \to L^2$ is not Fredholm operator with index zero, so Theorem 2.2 cannot be applied as in the case $k \neq 0$. Equation (1.9), i.e. ,

$$F(u) = k^2 D\partial_{\xi\xi} u + \omega\partial_\xi u + f(u, \mu) = 0, \tag{2.4}$$

can be written as

$$F(u) = \Big(\frac{k^2}{\omega} D\partial_\xi + 1\Big)\omega\partial_\xi u + f(u, \mu) = 0. \tag{2.5}$$

Suppose the operator $\mathcal{N} := \left(\frac{k^2}{\omega}D\partial_\xi + 1\right) : H^1 \to L^2$ is invertible; we can multiply (2.5) by its inverse which gives

$$G(u;,k,\omega,\mu) := \omega\partial_\xi u + \left(\frac{k^2}{\omega}D\partial_\xi + 1\right)^{-1} f(u,\mu) = 0, \tag{2.6}$$

where the highest order derivative is non-zero at $k = 0$. We have $\mathcal{N}^{-1} : L^2 \to H^1$ is well defined only for $k \neq 0$, because at $k = 0$, $\mathcal{N}$ is the identity on L^2 and H^1. Hence $\mathcal{N}^{-1}$ is well defined as a densely defined unbounded operator or $\mathcal{N}^{-1} : L^2 \to L^2$ or $\mathcal{N}^{-1} : H^1 \to H^1$ as a bounded operator. For that reason and to obtain differentiability in $K = \frac{k^2}{\omega}$, we define $\mathcal{M} := (KD\partial_\xi + 1)^{-1} : H^1 \to L^2$. To justify this we need the following lemma.

Lemma 2.4. *The operator $\mathcal{M}$ is bounded from $L^2 \to L^2$ and $||\mathcal{M}|| \leq 1$.*

Proof. First we take $u \in C^1$ so that u has a uniformly convergent Fourier series $\{c_n\} \in l^1$ and also $\{c_n\} \in l^2$ (see Appendix B). We compute

$$(\mathcal{M}u)(x) = \sum_{n\in\mathbb{Z}} c_n(KD\partial_x + 1)^{-1}e^{inx} = \sum_{n\in\mathbb{Z}} c_n(inKD + 1)^{-1}e^{inx}. \tag{2.7}$$

The function $n \mapsto |KDin + 1|$ attains its minimum value at 1 when $n = 0$, i.e, $|KDin + 1|^{-1} \leq 1$, so from here we compute

$$\begin{aligned}
\|\mathcal{M}u\|_2^2 = \int_0^{2\pi} |\mathcal{M}u(x)|^2 dx &= \int_0^{2\pi} \Big|\sum_{n\in\mathbb{Z}} c_n(KDin + 1)^{-1}e^{inx}\Big|^2 dx \\
&\leq \int_0^{2\pi} \sum_{n\in\mathbb{Z}} |c_n|^2 |KDin + 1|^{-2} dx \\
&\leq \int_0^{2\pi} \sum_{n\in\mathbb{Z}} |c_n|^2 dx \\
&= 2\pi \sum_{n\in\mathbb{Z}} |c_n|^2 dx \\
&\leq \|u\|_2^2 \text{ (by Plancherel Theorem).}
\end{aligned}$$

Thus we have shown that $\mathcal{M} : C^1 \subset L^2 \to L^2$ is bounded operator and it is possible to extend it to $\mathcal{M} : L^2 \to L^2$ since C^1 is dense in L^2. □

In order to apply Theorem 2.2 to (2.6), we need $G = \mathcal{M}F \in C^1$. In particular, we will show that $\partial_K\mathcal{M} : H^1 \to L^2$ is also bounded and well defined.

In (2.7), we recognise the Fourier symbol computed from the Fourier transform namely:

$$s(\kappa, K) = (KDi\kappa + 1)^{-1}$$

for which $\kappa \in \mathbb{R}$ is evaluated at $\kappa = n \in \mathbb{Z}$.

Lemma 2.5. *For all K, $\partial_K \mathcal{M} : H^1 \to L^2$ is bounded, $\|\partial_K \mathcal{M}\| \leq |D|$.*

Proof. First we compute

$$\frac{\partial s}{\partial K}(\kappa, K) = Di\kappa(KDi\kappa + 1)^{-2}.$$

Assuming $u \in C^1$, we have Fourier coefficients $(c_n) \in l^1$. If the sum

$$\sum_{n\in\mathbb{Z}} c_n Din(KDin + 1)^{-2} e^{inx} \tag{2.8}$$

converges uniformly then the formula for $\partial_K \mathcal{M}u$ is given by (2.8).

Next we show that the sum converges for $K \neq 0$. Using

$$|KDi\kappa + 1|^2 = 1 + |KD\kappa|^2$$

we obtain

$$|Di\kappa(KDi\kappa + 1)^{-2}| = \frac{|D\kappa|}{|KD\kappa|^2 + 1}.$$

Now, let $\tilde{f}(\kappa) := \frac{|D\kappa|}{|KD\kappa|^2+1}$. In order to estimate $\tilde{f}(\kappa)$, we observe that $\tilde{f}(\kappa) \to 0$ as $\kappa \to \pm\infty$, and $\tilde{f}(-\kappa) = \tilde{f}(\kappa)$. In addition $\tilde{f}'(\kappa) = 0$ has a unique positive solution $\kappa = \kappa_{\max} := \frac{1}{|KD|}$ and therefore $\tilde{f}(\kappa) \leq \tilde{f}(\kappa_{\max}) = \frac{1}{2|K|}$ for $\kappa \in \mathbb{R}$. Hence the sum in (2.8) converges absolutely for $K \neq 0$.

The bound being unbounded as $K \to 0$ indicates that the operator is not differentiable in K from L^2 to L^2 at $K = 0$.

For the case $K \sim 0$, we assume that the Fourier coefficients c_n of the functions to which we apply $\mathcal{M}$ decay like n^{-3} which holds for $u \in C^3$ [Appendix B Theorem B.1], so that the sum in (2.8) converges absolutely. We now compute for $u \in C^3$ that

$$\begin{aligned}\left\|\partial_K \mathcal{M}u\right\|_{L^2}^2 &= \int_0^{2\pi} \Big|\sum_{n\in\mathbb{Z}} c_n Din(KDin + 1)^{-2} e^{inx}\Big|^2 dx \\ &\leq \int_0^{2\pi} \sum_{n\in\mathbb{Z}} |nc_n|^2 |D|^2 |(KDin + 1)|^{-4} dx \\ &\leq 2\pi |D|^2 \sum_{n\in\mathbb{Z}} |nc_n|^2 \leq |D|^2 ||u||_{H^1}^2,\end{aligned}$$

where the Sobolev-H^1 norm is

$$||u||_{H^1}^2 = 2\pi \sum_{n\in\mathbb{Z}} (1 + n^2) c_n^2$$

Since C^3 is dense in H^1 then $\frac{\partial \mathcal{M}}{\partial K} : C^3 \subset H^1 \to L^2$ is bounded operator and can be extended to $\frac{\partial \mathcal{M}}{\partial K} : H^1 \to L^2$. □

We next apply Theorem 2.2 with $X = H^1$, $Z = L^2$, $\mathcal{F} = \mathcal{M}F$, $W = \mathbb{R}^{1+\ell}$ and $\lambda = (k, \mu)$. Again, we have to verify all assumptions of Theorem 2.2. Note that; the condition $\mathcal{F}(u_*, \omega_*, \lambda_*) = 0$ holds by assumption in Lemma 2.3. Next, the verification of assumptions are follows:

(i) $G \in C^1$ w.r.t μ and u follows since $\mathcal{M}$ is linear in u and independent of μ, and f is C^1 in (u, μ), see Appendix A. The differentiability w.r.t ω and k^2 follows from $\mathcal{M} \in C^1$ w.r.t $K = \frac{k^2}{\omega}$.

(ii) We need to show $\partial_u G(u_*; 0, \omega_*, \mu_*)$ is Fredholm index zero. This can be seen from

$$G(u; 0, \omega_*, \mu_*) \equiv \mathcal{M}F(u_*; 0, \omega_*, \mu_*) = \omega_* \partial_\xi u + f(u; \mu_*).$$

Hence $\partial_u G(u_*; 0, \omega_*, \mu_*) = \omega_* \partial_\xi u + \partial_u f(u_*, \mu_*)$, where $\partial_\xi : H^1 \to L^2$ is Fredholm index zero and since $\partial_u f$ is a compact perturbation this also hold for $\partial_u G$.

(iii) We need to show $\partial_\omega G(u_*; 0, \omega_*, \mu_*) \notin R\partial_u G(u_*; 0, \omega_*, \mu_*)$.

From (2.6), we compute $\partial_\omega G(u_*; 0, \omega_*, \mu_*) = u_*'$. Now $u_*' \in \ker \partial_u F(u_*; 0, \omega_*, \mu_*)$ therefore $u_*' \in \partial_u G(u_*; 0, \omega_*, \mu_*)$.

Since all assumptions of Theorem 2.2 have been verified thus obtain a solution (2.2) given by $\omega(k_*, \mu_*)$ as in the case $k \neq 0$. This completes the proof of Lemma 2.3.

2.2 Continuation in Diffusion Coefficients

In this chapter, we are interested in investigating the special case of vanishing diffusion coefficients in the diagonal matrix D. We will show that for a wave train, it is typically justified to set such coefficients to zero where they are "small", which has been done in the examples in §1.1.

In this case we write system in (1.9) as a coupled system in the form

$$\mathbf{F}(u) = \begin{pmatrix} k^2 \widehat{D}\widehat{u}'' + \omega \widehat{u}' + \widehat{f}(u, \mu) \\ k^2 \widetilde{D}\widetilde{u}'' + \omega \widetilde{u}' + \widetilde{f}(u, \mu) \end{pmatrix} = 0, \tag{2.9}$$

where $\widehat{D} = \mathrm{diag}(d_1, ..., d_M)$, $\widetilde{D} = \mathrm{diag}(d_{M+1}, ..., d_N)$, $\widehat{u} = (u_1, ..., u_M)$, $\widehat{f} = (f_1, ..., f_M)$, $\widetilde{u} = (u_{M+1}, ..., u_N)$ and $\widetilde{f} = (f_{M+1}, ..., f_N)$.

Assuming a wave train $u(\mu_*, k_*)$ at $\widehat{D} = 0$, we are looking for wave train solutions $u(\widehat{D}, \mu, k)$ with frequency $\omega(\widehat{D}, k, \mu)$, such that $\omega(\widehat{D}, k_*, \mu_*) = \omega_*$ with $d_j \in [0, d_j^*]$, $j = 1, ..., M$ and (k, μ) in a neighbourhood of (k_*, μ_*). Further, we are only interested in the case for $k \neq 0$, as for $k = 0$ the result follows in combination with Lemma 2.3.

The linearization $\mathbf{L}$ of $\mathbf{F}(u)$ w.r.t u in u_* is given by:

$$\mathbf{L}u := \begin{pmatrix} \omega_* \widehat{u}' + \partial_u \widehat{f}(u_*, \mu_*) u \\ k^2 \widetilde{D} \widetilde{u}'' + \omega_* \widetilde{u}' + \partial_u \widetilde{f}(u_*, \mu_*) u \end{pmatrix}, \tag{2.10}$$

where $\widehat{D} = 0$ and $\widetilde{D} = \mathrm{diag}(d_{M+1}, ..., d_N)$ with $d_{M+1}, ..., d_N > 0$.

For this particular case, we define the Sobolev spaces to be $H = \left(H^1_{\mathrm{per}}(0, 2\pi)\right)^M \times \left(H^2_{\mathrm{per}}(0, 2\pi)\right)^{N-M}$ so that we have $\mathbf{L} : H \to L^2$. In this setting $\mathbf{L}$ is Fredholm index zero if $k \neq 0$ see §2.1.

Since in this particular case we want $\widehat{D}$ to vary, Lemma 2.3 can be reformulated as follows.

Lemma 2.6. *Assume that (2.9) has a wave train solution u_* for some $(\widehat{D}, k, \omega, \mu) = (0, k_*, \omega_*, \mu_*)$ where $k_* \neq 0$ and $\omega_* \neq 0$. Suppose that $\gamma = 0$ is an eigenvalue of $\mathbf{L}$ of algebraic multiplicity 1. Then there exists a local family of wave trains $u(\widehat{D}, k, \mu)$ with frequencies $\omega(\widehat{D}, k, \mu)$ for $(\widehat{D}, k, \mu) \sim (0, k_*, \mu_*)$ which are continuously differentiable in $(\widehat{D}, k, \mu)$ with $u(\widehat{D}, k_*, \mu_*) = u_*$ and $\omega(0, k_*, \mu_*) = \omega_*$. Every solution $(u, \omega) \sim (u_*, \omega_*)$ and $(\widehat{D}, k, \mu) \sim (0, k_*, \mu_*)$ to (2.9) belong to this family of wave trains and frequencies up to spatial translation.*

Proof. In analogy to the case $k = 0$ in §2.1 we define

$$\mathbf{M}u := \begin{pmatrix} \left(\frac{k^2}{\omega} \widehat{D} \partial_\xi + 1\right)^{-1} \widehat{u} \\ \widetilde{u} \end{pmatrix} \tag{2.11}$$

and replace $\mathbf{F}(u) = 0$ by $\mathbf{MF}(u) = 0$. This is justified since $\mathbf{M} : H \to L^2$ is bounded (see Lemma 2.2) and differentiable in $\hat{D}$, which follows from Lemma 2.3 replacing k with $\hat{D}$.

Next, in order to verify our claim, we show that all assumptions (i)-(iii) in Theorem 2.2 are satisfied for $X = H$, $Z = L^2$, $\mathcal{F} = \mathbf{MF}$, $W = \mathbb{R}^{1+M+\ell}$ and $\lambda = (\widehat{D}, k, \mu)$. The assumption that a wave train exists; means $\mathcal{F}(u_*, \omega_*, \lambda_*) = 0$. Now, the verification of the assumptions (i)-(iii) is the following:

(i) Since $\widetilde{\mathbf{F}} = (F_{M+1}, ..., F_N)$ is linear in ω, k, we immediately obtain $\partial_\omega \widetilde{F} = \widetilde{u}'_*$, $\partial_{k^2} \widetilde{F} = \widetilde{D}\widetilde{u}''_* \in L^2$ since $\widetilde{u}_* \in H^2$. The differentiability of $\widehat{F} = (F_1, ..., F_M)$, the proof of Lemma 2.3 for $k = 0$ applies. Next,

$$f(u) := \begin{pmatrix} \widehat{f}(u, \mu) \\ \widetilde{f}(u, \mu) \end{pmatrix} : H \to L^2. \tag{2.12}$$

Since $H \subset H^1$ and we know from §2.1 that $f : H^1 \to L^2$ is differentiable for $f \in C^2(\mathbb{R}^N, \mathbb{R}^N)$, it follows that $f : H \to L^2$ is also differentiable. The differentiability in μ follows from $\partial_\mu f : H^1 \to H^1$ [Appendix A].

(ii) This follows as in the discussion of (2.3).

(iii) At the bifurcation point, i.e, $\hat{D} = 0$, $\mathbf{M}$ is an identity on L^2. We compute $\partial_\omega \mathbf{MF}(u_*; 0, k_*, \omega_*, \mu_*)$ gives u'_*. Using argument in Remark 1, u'_* is always in the kernel of $\mathbf{L}$ but since we have assumed that the eigenvalue 0 of $\mathbf{L}$ is of algebraic multiplicity 1, u'_* does not lie in the range of $\mathbf{L}$. Hence $\partial_\omega \mathbf{MF} \notin R\partial_u \mathbf{ML}$.

Therefore, in applying Theorem 2.2 we obtain a solution (2.2). Thus we get $\omega(\widehat{D}, k, \mu)$ as claimed. □

Note that; in this result it is possible to have negative diffusion coefficients. However, in practice this is rather meaningless since the reaction diffusion equation does not give a well posed initial value problem. However for the wave train equation (1.9), they are still proper solutions.

2.3 Discussion

In this thesis we are interested in spatio-temporally periodic solutions in spatially one dimensional reaction-diffusion systems, particularly when some diffusion coefficients or wave number are close to zero. We start out by giving examples in §1 for which this situation occurs. In both examples there exists a Hopf bifurcation implies wave trains with wave number zero. Moreover there exist wave trains near pulses providing example for wave trains with zero diffusion coefficient.

In Chapter 2, we start out by explaining the framework for Fredholm analysis by choosing suitable Sobolev spaces for two cases, namely $k \neq 0$ and $k = 0$. Next, we explain the technical method of Lyapunov-Schmidt reduction. Also, the complete detail proof of Lemma Continuation of Wave Trains is discussed in detail where the Lyapunov-Schmidt reduction play a major role.

Next, the new result in this thesis, i.e, Continuation in Diffusion Coefficients is obtained. The main technical difficulty here is to obtain or set up the right function spaces which is not completely obvious. Again, the proof is given in detail by applying the Lyapunov-Schmidt reduction. In both cases, the result shows that there exists a solution curve parametrised by parameter s and λ which given by Theorem 1.

Note that, the method we apply in §2.1 and §2.2 does not apply in general case. For example, consider the following

$$K(u) := \varepsilon u_{xxxx} + u_{xx} + f(u) = 0 \tag{2.13}$$

The problem occurs when we need to show the identity element of K which given by

$$\mathcal{K} = (\varepsilon \partial_{xx} + 1)^{-1}, \tag{2.14}$$

is bounded and C^2 differentiable by applying the Fourier coefficient as in the case $k = 0$ in §2.1. We compute

$$(\mathcal{K}u)(x) = \sum_{n\in\mathbb{Z}} c_n(\varepsilon\partial_{xx} + 1)^{-1} e^{inx} = \sum_{n\in\mathbb{Z}} c_n(-\varepsilon n^2 + 1)e^{inx} \tag{2.15}$$

The last element of (2.15) can lead the denominator to zero when $\varepsilon = \frac{1}{n^2}$. Thus, we obtain a so-called resonance.

Appendix A

Smoothness and Differentiability

Claim 1. *If $f : \mathbb{R}^N \to \mathbb{R}^N$ is C^2 in u then $\mathcal{F} : H^j \to L^2$, $j = 1, 2$ is differentiable in u with derivative $\mathcal{L}$*

Proof. We have $\widetilde{\mathcal{L}} \equiv k^2 D u'' + \omega u' : H^j \to L^2$ is bounded; hence it remains to show $f : H^j \to L^2$ is C^1. Recall that $f : H^j \to L^2$ is well defined; see §2.1. For $v \in H^j$, we need to show that

$$\tau(v) := ||f(u+v) - f(u) - f'(u)v||_{L^2}, \tag{A.1}$$

satisfies $\frac{\tau(v)}{||v||_{H^j}} \to 0$ as $||v||_{H^j} \to 0$.

We compute

$$\tau^2(v) = \int_0^{2\pi} \underbrace{|f(u(x)+v(x)) - f(u(x)) - f'(u(x)v(x))|^2}_{\leq \sup_{s\in[0,1]} |f''(u(x)+sv(x))|^2 |v^2(x)|^2} dx \leq \int_0^{2\pi} \Big(g(v(x))(v(x))^2 \Big)^2 dx,$$

where $g : \mathbb{R}^N \to \mathbb{R}, v \mapsto \sup_{s\in[0,1], x\in[0,2\pi]} |f''(u(x)+sv)|^2$. Since f'' is continous and $u \in H^j \hookrightarrow \left(C^{j-1}_{per}[0,2\pi]\right)^N$, by Sobolev embedding, u is bounded so that g is well defined. For same reason $\|v\|_{H^j} \to 0$ implies $\sup |v(x)| \to 0$. Therefore for each C_1 there exists C_2 such that if $\|v\|_{H^j} \leq C_1$ then $g(v) \leq C_2$. Notice that f'' is continuous. Hence we obtain $\tau(v) \leq C\|v^2\|_{L^2}$ for C independent of v.

Let $v = h\hat{v}$ with $||\hat{v}||_{H^j} = 1$ and $h \in \mathbb{R}$, then

$$||v^2||^2_{L^2} = h^4 ||\hat{v}^2||^2_{L^2} \leq h^4 ||\hat{v}^2||^2_{H^j} \leq h^4 ||\hat{v}||^4_{H^j} = h^4,$$

where H^2 is also an algebra, see Theorem 5.22 in [1]. Combining this with the estimate of τ yields

$$\frac{\tau(v)}{||v||_{H^j}} \leq \frac{Ch^2}{h} = Ch \to 0 \text{ as } h \to 0$$

Hence $f : H^j \to L^2$ is C^1 and it follows that $\mathcal{F} : H^j \to L^2$ is. Therefore the formula for $\partial_u \mathcal{F}$ is given by $\mathcal{L}$ □

Next, we want to show the differentiability of f w.r.t μ.

Claim 2. *$f : \mathbb{R}^N \times \mathbb{R}^\ell \to \mathbb{R}^N$ is C^2 in μ, then $f : H^j \times \mathbb{R}^\ell \to L^2$ is differentiable in μ*

Proof. We want to show $f : H^j \times \mathbb{R}^\ell \to L^2$, $j = 1, 2$ is C^1 in μ. For $\alpha \in \mathbb{R}^\ell$, we need to show that

$$\sigma(\alpha) := \|f(u(x); \mu + \alpha) - f(u(x); \mu) - \partial_\mu f(u(x); \mu)\alpha)\|_{L^2} \tag{A.2}$$

satisifies $\frac{\sigma(\alpha)}{\|\alpha\|_{\mathbb{R}^\ell}} \to 0$ as $\|\alpha\|_{\mathbb{R}^\ell} \to 0$.

We compute,

$$\begin{aligned}
\sigma^2(\alpha) &= \int_0^{2\pi} |f(u(x); \mu + \alpha) - f(u(x); \mu) - \partial_\mu f(u(x); \mu)\alpha)|^2 dx \qquad (A.3)\\
&\leq \int_0^{2\pi} \sup_{s\in[0,1]} |\partial_{\mu\mu} f(u(x); \mu + s\alpha)|^2 \|\alpha\|^4 dx \\
&= \|\alpha\|^4 \int_0^{2\pi} \sup_{s\in[0,1]} |\partial_{\mu\mu} f(u(x); \mu + s\alpha)|^2 dx \\
&\leq C^2 \|\alpha\|^4.
\end{aligned}$$

For some $C > 0$ because u is bounded (see proof of claim 1) and $\partial_{\mu\mu} f$ is continuous. Thus $\sigma(\alpha) \leq C\|\alpha\|^2$ and therefore

$$\frac{\sigma(\alpha)}{\|\alpha\|} \leq C\|\alpha\| \to 0 \text{ as } \|\alpha\| \to 0$$

□

Appendix B

Fourier Series Representation

The Fourier series and its coefficients are defined as follows:

$$u(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} \tag{B.1}$$

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} u(x)e^{-inx}dx \tag{B.2}$$

The integral in (B.2) exists for u in L^2 because $L^2 \subset L^1$ whereas the sum in (B.1) does not converge for general series $(c_n) \in l^2$ but it is well defined for $(c_n)_{n\in\mathbb{Z}} \in l^1$, which is true if $u \in C^1$.

Further, if f is 2π periodic, the following theorem holds

Theorem B.1 (10, Theorem 2.6)**.** *Suppose f is 2π periodic. If f is of class C^{k-1} and f^{k-1} is piecewise smooth (thus f^k exists except at finitely many points in each bounded interval and is piecewise continuous) then the Fourier coefficients of f satisfy*

$$\sum |n^k c_n|^2 < \infty$$

In particular, as $n \to \infty$,

$$n^k c_n \to 0$$

On the other hand, suppose the Fourier coefficients c_n for $n \neq 0$ satisfy $|c_n| \leq C|n|^{-(k+\alpha)}$ for some $C > 0$ and $\alpha > 1$ then f is of class C^k.

In particular $f \in C^k$ implies $c_n \leq Cn^{-k}$.

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