The Segal–Bargmann transform and its generalizations

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Abstract

This paper gives an introduction to the Segal–Bargmann transform and its generalizations. The classical Segal–Bargmann transform is a unitary transform between the Schrődinger and Fock representations of quantum mechanics on Euclidean space. Hall has generalized this transform to include the case of compact Lie groups. His transform consists of two steps: First take the convolution with the heat kernel and then take the analytic continuation of the resulting function. We will prove the unitarity of this transform by showing that both of these steps are unitary. In the last part we will try to generalize this method to include the case of noncompact symmetric spaces.

Revised Edition

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1 Introduction

The Segal–Bargmann transform is an important mathematical tool in quantum mechanics and quantum field theory. It is named after Irving Segal and Valentine Bargmann, who have discovered this transform about fifty years ago. Inspired by the work of Leonard Gross, Brian Hall has revived the interest in the Segal–Bargmann transform by defining the Segal–Bargmann transform on a compact Lie group. It turns out that this generalization has an application in geometric quantization and so the transform has found its way to modern physics.

This generalized Segal–Bargmann transform is also of purely mathematical interest. It provides an equivalence between square integrable functions on a space and certain holomorphic functions on the complexification of that space. The interesting point is that this equivalence is given by convolution with the heat kernel.

Building forth on the work of Brian Hall, Matthew Stenzel has given a straightforward generalization of the Segal–Bargmann transform to a special class of compact Riemannian manifolds, including compact symmetric spaces. In the past ten years several authors have tried to find a Segal–Bargmann transform for noncompact symmetric spaces. Although more than one solution has been proposed, the problem has not yet been settled.

In this paper I will introduce the classical Segal–Bargmann transform and Hall's generalization to compact Lie groups in sections 2 and 3. In section 4 I will describe the heat transform and show how this transform can be used to prove that the Segal–Bargmann transform is isometric. Then I will make a detour to symmetric spaces and give an overview of some recent results on the Segal–Bargmann transform.

The heart of this paper is section 6, where I apply the theory of the heat transform to the hyperbolic plane and identify the image of this transform. Unfortunately time was too short to find the proper complexification of the hyperbolic space and show that the functions in the image of the heat transform can be extended to this complexification. Instead the reader will find the outline of a strategy that could achieve this. I have collected a number of forthcoming questions in the last section.

I would like to thank my supervisor, Erik van den Ban, for his patience and his criticism. With his hints, many of my arguments have been improved. I am grateful to Joop Kolk, Hans Duistermaat and all my other teachers, for teaching me the necessary techniques to complete this thesis. Special thanks are due to Thomas Rot and all other students with whom I have discussed my ideas. I will feel sad when I will leave my beautiful graduation chamber with the splendid view of the botanical gardens ...

2 The Segal–Bargmann transform

The Segal–Bargmann transform is named after Irving Segal (1918–1998) and Valentine Bargmann (1908–1989). Segal has discovered an infinite-dimensional version of this transform [Seg62, Seg63], while Bargmann has considered the transformation in the finite-dimensional case [Bar61]. For a detailed description of its history, see §6.4 of [Hal00].

According to Bargmann, the purpose of his paper is "to study in greater detail the function space on which Fock's solution is realized, and its connection with the conventional Hilbert space of square integrable functions." We will study these notions before the Segal–Bargmann transform is introduced in section 2.4. A short introduction to quantum mechanics is compiled from [Dir58].

2.1 States and observables

In quantum mechanics a dynamical system is described by its states and dynamical variables and the relation between them. A *state* is a motion of the system that is consistent with the laws of physics. A *dynamical variable* is an operator acting on the space of states. The most important dynamical variables are the *observables*, which are selfadjoint dynamical variables that have a complete set of eigenstates. The physical significance of an observable is that it corresponds to a quantity that can be measured.

The states of a system are represented by unit vectors in a complex Hilbert space, called *state space*. The dynamical variables are represented by linear operators on that space. The relations between the dynamical variables are given in the form of commutation relations. For quantum systems that have a classical analogue, the commutation relations reduce to the Poisson bracket of the corresponding classical system in the classical limit. (The Poisson bracket for a classical system determines the time evolution and can thus be seen as a law of motion.)

Two of such representations are Schrődinger's representation and Fock's representation, which will be studied next.

2.2 Schrödinger's representation

The Schrődinger representation is applicable to a system with n degrees of freedom that has a classical analogue. Having n degrees of freedom means that the classical state of the system is fixed by n positions and n momenta. In the quantum picture these variables are represented by observables, Q_r denoting the corresponding position operators and P_r the momentum operators, where $1 \leq r \leq n$. According to Dirac the canonical commutation relations between these observables are

$$[Q_r, Q_s] = 0 \qquad [Q_r, P_s] = i\hbar\delta_{rs} \qquad [P_r, P_s] = 0, \tag{2.1}$$

with $1 \leq r, s \leq n$, the bracket denoting the commutator and $\hbar$ a value determined by experiment.

It is known that in this case the position operators Q_r form a complete set of commuting observables. That is, the state $|\psi\rangle$ is uniquely determined by its spectral decomposition in terms of these operators, up to a phase factor of absolute value one. Therefore we may represent the state as a function on the space of all possible positions, called *configuration space*. If we denote this space by X, the state space is given by $L^2(X)$, which is technically a space of half-densities [Ash80, §2]. For simplicity we will consider this as a function space with a fixed measure on X. The states are then represented by *wave functions* $\psi \in L^2(X)$ with norm one.

Schrődinger found that the linear operator $-i\hbar\frac{\partial}{\partial q_r}$ acting on a dense subset of the state space satisfies the canonical commutation relations (2.1) by

$$\begin{aligned}\left[q_r, -i\hbar\frac{\partial}{\partial q_r}\right]\psi &= -i\hbar\left(q_r\frac{\partial\psi}{\partial q_r} - \frac{\partial}{\partial q_r}q_r\psi\right)\\ &= -i\hbar\left(q_r\frac{\partial\psi}{\partial q_r} - q_r\frac{\partial\psi}{\partial q_r} - \psi\right)\\ &= i\hbar\psi.\end{aligned}$$

Note that the observable Q_r acts by multiplication with q_r in this representation. This gives that $P_r + i\hbar\frac{\partial}{\partial q_r}$ commutes with all the Q_r. Since the Q_r form a complete set, this sum is a function of the Q_r [Dir58, p. 78]. The Schrődinger representation now is a representation in which this sum vanishes, so that $P_r = -i\hbar\frac{\partial}{\partial q_r}$. This can always be arranged, using the freedom to multiply states with a phase factor of absolute value one.

Segal and Bargmann considered the configuration space $X = \mathbb{R}^n$. In fact, Segal is mainly interested in the case $n \to \infty$, but we will not deal with that limit in this paper. The wave functions are taken to be square integrable complex-valued functions on configuration space, equipped with the usual inner product. In other words, states are represented by functions in $L^2(X)$. This Hilbert space is the one that is commonly used in quantum mechanics, though there are some nasty domain issues: Neither Q_r nor P_r is defined on all of $L^2(X)$. This problem can be solved, but we will not try to do that here. The theory presented in this paper is quite independent of that problem.

2.3 Fock's representation

As in the Schrődinger representation we start with a system with n degrees of freedom, observables Q_r, P_r and the relations (2.1). As before we take the index $r \in \{1, \ldots, n\}$. We form the observables

$$H_r = \frac{1}{2m}\left(P_r^2 + m^2\omega^2 Q_r^2\right), \tag{2.2}$$

which you may recognize as the Hamiltonian of a harmonic oscillator. Otherwise you could just ignore the constants m and ω. The operators H_r are selfadjoint operators on the state space and they form a complete set of commuting observables with eigenvalues $(k_r + \frac{1}{2})\hbar\omega$, $k_r \in \mathbb{Z}_{\geq 0}$. The corresponding eigenstates will be denoted by $|k_1, \ldots, k_n\rangle$.

A derivation of this can be found in §34 of [Dir58] and most other textbooks on quantum mechanics. In the Schrődinger representation these eigenstates correspond to Hermite functions, as we will see in the next section.

Introducing the complex dynamical variables

$$\eta_r = (2m\hbar\omega)^{-\frac{1}{2}}(P_r + im\omega Q_r) \tag{2.3a}$$

$$\bar{\eta}_r = (2m\hbar\omega)^{-\frac{1}{2}}(P_r - im\omega Q_r), \tag{2.3b}$$

we can easily check that they satisfy

$$[\eta_r, \eta_s] = 0 \qquad [\bar{\eta}_r, \eta_s] = \delta_{rs} \qquad [\bar{\eta}_r, \bar{\eta}_s] = 0, \tag{2.4}$$

using (2.1), and this gives that

$$[\bar{\eta}_r, \eta_r^l] = l\eta_r^{l-1}, \qquad l \in \mathbb{Z}_{\geq 0}.$$

The Hamiltonian (2.2) can be written in terms of these variables, giving

$$H_r = \hbar\omega(\eta_r\bar{\eta}_r + \tfrac{1}{2}).$$

The operators η_r and $\bar{\eta}_r$ are usually called *ladder operators*, where one is the *raising operator* and the other the *lowering operator*.

The representation we want to consider now is one in which the H_r are diagonal. The nontrivial result from Fock [Foc28] is that there is such a representation in which the eigenstates of H_r are represented as holomorphic functions of the complex variables η_r. To see how this works, we consider the *ground state* $|0, \ldots, 0\rangle$, or just $|0\rangle$ for short, with eigenvalue $\frac{1}{2}\hbar\omega$ for all H_r. Then $\eta_r^l|0\rangle$ is an eigenstate for H_r with eigenvalue $\hbar\omega(l + \frac{1}{2})$ by

$$\begin{aligned} H_r\eta_r^l|0\rangle &= [H_r, \eta_r^l]|0\rangle + \eta_r^l H_r|0\rangle \\ &= \hbar\omega[\eta_r\bar{\eta}_r, \eta_r^l]|0\rangle + \eta_r^l(\tfrac{1}{2}\hbar\omega)|0\rangle \\ &= \hbar\omega\left(\eta_r[\bar{\eta}_r, \eta_r^l] + \tfrac{1}{2}\eta_r^l\right)|0\rangle \\ &= \hbar\omega\left(\eta_r l\eta_r^{l-1} + \tfrac{1}{2}\eta_r^l\right)|0\rangle \\ &= \hbar\omega\left(l + \tfrac{1}{2}\right)\eta_r^l|0\rangle. \end{aligned}$$

Using that the set of eigenstates is complete, we may decompose an arbitrary state $|\psi\rangle$ as

$$|\psi\rangle = \sum_{k_1,\ldots,k_n} c(k_1, \ldots, k_n)\eta_1^{k_1} \cdots \eta_n^{k_n}|0\rangle. \tag{2.5}$$

It can be shown that when $|\psi\rangle = f(\eta_1, \ldots, \eta_r)|0\rangle$ as above, the function f is holomorphic. In this way we can represent states as holomorphic functions. The space of these holomorphic functions was named *Diracschen Raum* by Fock. Bargmann however denoted it with $\mathfrak{F}$, probably in honor of Fock and/or Fischer, while afterwards it was often named after

Bargmann and Segal. Note that in physics the Fock space usually describes the number of particles instead of the energy level.

Before turning to the Segal–Bargmann transform I would like to make two more remarks on this model and make a small computation. First of all we note that the η_r are not selfadjoint but rather $\eta_r^* = \bar{\eta}_r$. This is because P_r and Q_r are selfadjoint. Secondly the commutation relations (2.4) have the operator solution $\bar{\eta}_r = \frac{\partial}{\partial \eta_r}$. Third of all, as a preparation for the Segal–Bargmann transform, we will compute the state $|0\rangle$ in the Schrődinger representation.

The ground state $|0\rangle$ is the state with the lowest eigenvalue. States with higher eigenvalues are obtained by applying the operators η_r to this state. On the other hand, if we apply $\bar{\eta}_r$ to the zero state, we find a state satisfying

$$\begin{aligned} H_r\bar{\eta}_r|0\rangle &= \hbar\omega[\eta_r\bar{\eta}_r, \bar{\eta}_r]|0\rangle + \bar{\eta}_r(\tfrac{1}{2}\hbar\omega)|0\rangle \\ &= \hbar\omega\left(-\bar{\eta}_r + \tfrac{1}{2}\bar{\eta}_r\right)|0\rangle \\ &= -\tfrac{1}{2}\hbar\omega\bar{\eta}_r|0\rangle. \end{aligned}$$

Since $\frac{1}{2}\hbar\omega$ is the lowest eigenvalue, this gives that $\bar{\eta}_r|0\rangle$ must vanish for all $1 \leq r \leq n$. By (2.3b) this means that $(P_r - im\omega Q_r)\,|0\rangle = 0$. Let us now suppose that the configuration space is $\mathbb{R}^n$. In the Schrődinger representation we find that $P_r = -i\hbar\frac{\partial}{\partial q_r}$ and $Q_r = q_r$, so that the wave function ψ_0 of the ground state $|0\rangle$ in this representation satisfies

$$-i\hbar\frac{\partial\psi_0}{\partial q_r} - im\omega q_r\psi_0 = 0.$$

The solution to this system is

$$\psi_0(q_1,\ldots,q_n) = C\exp\left(-\frac{m\omega}{2\hbar}(q_1^2 + \cdots + q_n^2)\right), \tag{2.6}$$

with $C = (m\omega/\pi\hbar)^{\frac{n}{4}}$ if we want the state ψ_0 to be normalized in $L^2(\mathbb{R}^n)$.

2.4 The Bargmann space and transform

In his 1961 paper Bargmann studies a space of holomorphic functions and the transformation between this space and L^2 space. I will first state Bargmann's definition of the holomorphic function space and then argue why this would be the correct space. After a few preliminaries we are ready to prove the unitarity of the classical version of the Segal–Bargmann transform.

Definition 2.1 (Bargmann space)**.** *The* Bargmann space *of index n is a subspace of the space $\mathcal{O}(\mathbb{C}^n)$ of holomorphic functions on $\mathbb{C}^n$, given by*

$$\mathfrak{F}_n = \{f \in \mathcal{O}(\mathbb{C}^n) \mid (f,f) < \infty\},$$

where the pairing $(\cdot,\cdot)$ *is defined by*

$$(f,g) = \frac{1}{\pi^n}\int_{\mathbb{C}^n} \overline{f(z)}g(z)e^{-|z|^2}\,\mathrm{d}z \tag{2.7}$$

for all $f,g \in \mathcal{O}(\mathbb{C}^n)$ *and Lebesgue measure* $\mathrm{d}z$ *on* $\mathbb{C}^n$.

In the previous section we have already seen that in Fock's model states are represented by holomorphic functions. The new ingredient here is the pairing, which makes this space of holomorphic functions into a Hilbert space:

Lemma 2.2. *The Bargmann space is a Hilbert space.*

Proof. It is easlily verified that the pairing defined on the Bargmann space is an inner product. Therefore we just have to show that the space $\mathfrak{F}_n$ is complete. To do this we will use a theorem of Weierstrass to show that any Chauchy sequence in $\mathfrak{F}_n$ has a limit $f \in \mathcal{O}(\mathbb{C}^n)$. Then it is a small step to check that f is in $\mathfrak{F}_n$ and indeed is the limit of the Cauchy sequence in the norm $\|\cdot\|$ of $\mathfrak{F}_n$ induced by the inner product.

Before turning to the proof itself, we will first show that any Cauchy sequence in $\mathfrak{F}_n$ is also Cauchy in the space of continuous functions $C(\mathbb{C}^n)$ endowed with the *topology of compact convergence*. In this topology a sequence of functions $\{f_j\}_{j=1}^\infty$ converges if and only if for every compact set $K \subset \mathbb{C}^n$ the sequence converges uniformly on K.

The goal is to find for any K as above an estimate of the form

$$\sup_{z\in K}|f_j(z) - f_k(z)| \le c_K\,\|f_j - f_k\|, \tag{2.8}$$

where c_K is a constant depending on the compact set K. Once we have this, we find for each K a uniform bound and hence the sequence $\{f_j\}_{j=1}^\infty$ is Cauchy in $C(\mathbb{C}^n)$ if it is Cauchy in $\mathfrak{F}_n$.

Let $K \subset \mathbb{C}^n$ be compact. To simplify some computations choose $R > 0$ such that K is contained in B_R, the ball with radius R around the origin. Then pick any smooth function with compact support $\chi \in C_c^\infty(\mathbb{C}^n)$ such that $\chi(x) = 1$ for all $x \in \partial B_{R+1}$. Here and in the following, B_R denotes the sphere of radius R. Set $U = \operatorname{supp}\chi \setminus B_{R+1}$. This is arranged so that U is a compact set with distance one to B_R and χ vanishes on all parts of the boundary of U except on ∂B_{R+1}, where it is one. Since $K \subset B_R$ we have

$$\sup_{z\in K}|f_j(z) - f_k(z)| \le \sup_{z\in B_R}|f_j(z) - f_k(z)|$$

and by the Bochner-Martinelli generalization of Cauchy's integral theorem this equals

$$\sup_{z\in B_R}\left|\frac{(n-1)!}{(2\pi i)^n}\int_{\partial B_{R+1}}(f_j(\zeta) - f_k(\zeta))\sum_{r=1}^n(-1)^{r-1}\frac{\bar\zeta_r - \bar z_r}{|\zeta - z|^{2n}}\,\mathrm{d}\bar\zeta[r]\wedge\mathrm{d}\zeta\right|,$$

with $\mathrm{d}\zeta = \mathrm{d}\zeta_1\wedge\ldots\wedge\mathrm{d}\zeta_n$ and $\mathrm{d}\bar\zeta[r] = \mathrm{d}\bar\zeta_1\wedge\ldots\wedge\mathrm{d}\bar\zeta_{r-1}\wedge\mathrm{d}\bar\zeta_{r+1}\wedge\ldots\wedge\mathrm{d}\bar\zeta_n$.

By the properties of χ and U this is equal to

$$\sup_{z\in B_R}\left|\frac{(n-1)!}{(2\pi i)^n}\int_{\partial U}(f_j(\zeta)-f_k(\zeta))\,\chi(\zeta)\,\sum_{r=1}^{n}(-1)^{r-1}\frac{\bar{\zeta}_r-\bar{z}_r}{|\zeta-z|^{2n}}\,\mathrm{d}\bar{\zeta}[r]\wedge\mathrm{d}\zeta\right|.$$

Applying Stokes' theorem with $\mathrm{d}f=\sum_{r=1}^{n}\frac{\partial f}{\partial z_r}\mathrm{d}z_r+\frac{\partial f}{\partial \bar{z}_r}\mathrm{d}\bar{z}_r$ the above is precisely

$$\sup_{z\in B_R}\left|\frac{(n-1)!}{(2\pi i)^n}\int_{U}(f_j(\zeta)-f_k(\zeta))\,\sum_{r=1}^{n}\frac{\partial}{\partial\bar{\zeta}_r}\left[\chi(\zeta)\,\frac{\bar{\zeta}_r-\bar{z}_r}{|\zeta-z|^{2n}}\right]\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta\right|.$$

We recognize $(2i)^{-n}\mathrm{d}\bar{\zeta}\wedge\mathrm{d}\zeta$ as Lebesgue measure on $\mathbb{C}^n$ and apply Schwarz' inequality in $L^2(U)$ to find that

$$\sup_{z\in K}|f_j(z)-f_k(z)|\leq\|f_j-f_k\|_{L^2(U)}\sup_{z\in B_R}\|\omega_z\|_{L^2(U)},$$

where ω_z replaces all other factors not yet dealt with. A small inspection of this term shows that ω_z is composed of factors χ and its partial derivatives, which are all C_c^∞ functions, ζ_r-z_r and its complex conjugate, which are bounded because U is compact and finally negative powers of $|\zeta-z|$, which are bounded below by one since the distance from B_R to U is at least one by construction. Therefore the supremum of $\|\omega_z\|_{L^2(U)}$ may be estimated by a constant.

On the other hand, for every bounded set U we find that the norm in Lebesgue measure $\mathrm{d}z$ is equivalent to the norm in *Gauss measure* $\mu(z)=e^{-|z|^2}\mathrm{d}z$. Thus

$$\|f_j-f_k\|_{L^2(U)}\leq c\,\|f_j-f_k\|_{L^2(U,\mu)}\leq c\,\|f_j-f_k\|$$

for some constant c depending on U, where the rightmost norm is the norm of $\mathfrak{F}_n$. Putting all this together we arrive at the desired estimate (2.8). Notice that the constant depends on R and on U, but it is readily seen that R and U depend on K only.

Now we are ready to start with the proof itself. Let $\{f_j\}_{j=1}^{\infty}$ be a Cauchy sequence in $\mathfrak{F}_n$. Then by the above result it is a Cauchy sequence in $C(\mathbb{C}^n)$ with the topology of compact convergence. Since this space is complete, there exists a limit function $f\in C(\mathbb{C}^n)$. By a theorem of Weierstrass (see for example [Lan99, Ch. V]) this function is holomorphic.

The sequence $\{f_j\}_{j=1}^{\infty}$ is Cauchy in $\mathfrak{F}_n$, so that the $\|f_j\|$ are bounded by a constant C. From the uniform convergence on compacta it follows that for any compact set $K\subset\mathbb{C}^n$

$$\int_K|f(z)|^2\,\mu(z)=\lim_{j\to\infty}\int_K|f_j(z)|^2\,\mu(z)\leq\lim_{j\to\infty}\|f_j\|^2\leq C^2$$

and we see that f is square integrable on every compact subset. By monotone convergence we find that $\|f\|\leq C$, so $f\in\mathfrak{F}_n$. Using the triangle inequality we see that $\|f-f_j\|\leq 2C$ for all $j\in\mathbb{Z}_{>0}$. Thus we may apply Fubini to obtain

$$\lim_{j\to\infty}\|f-f_j\|^2=\lim_{j\to\infty}\sup_K\int_K|f-f_j|^2\mu=\sup_K\lim_{j\to\infty}\int_K|f-f_j|^2\mu,$$

which is zero because the f_j converge uniformly to f on each compact set K. We conclude that $\|f-f_j\|\to 0$ as $j\to\infty$, so $f_j\to f$ in $\mathfrak{F}_n$. This completes the proof. □

The Gaussian measure that is used in the definition of the inner product (2.7) follows from the properties of the operators η_r, $\bar{\eta}_r$:

Suppose we have a space of holomorphic functions in the variables $z = (z_1, \ldots, z_r)$, multiplication operators $\eta_r : f(z) \mapsto z_r f(z)$ and differential operators $\bar{\eta}_r : f(z) \mapsto \frac{\partial f(z)}{\partial z_r}$, so that the commutation relations (2.4) are satisfied. Furthermore, suppose that there exists an inner product on this space of holomorphic functions that is of the form

$$(f, g) = \int_{\mathbb{C}^n} \overline{f(z)} g(z) \rho(z, \bar{z}) \, \mathrm{d}z$$

for some weight function ρ. The requirement that $\eta_r^* = \bar{\eta}_r$ then gives that $(\eta_r f, g) = (f, \bar{\eta}_r g)$, or

$$\int_{\mathbb{C}^n} \overline{z_r f(z)} g(z) \rho(z, \bar{z}) \, \mathrm{d}z = \int_{\mathbb{C}^n} \overline{f(z)} \frac{\partial g(z)}{\partial z_r} \rho(z, \bar{z}) \, \mathrm{d}z.$$

The right hand side is equal to

$$\int_{\mathbb{C}^n} \frac{\partial}{\partial z_r} \left(\overline{f(z)} g(z) \rho(z, \bar{z}) \right) - \frac{\partial \overline{f(z)}}{\partial z_r} g(z) \rho(z, \bar{z}) - \overline{f(z)} g(z) \frac{\partial \rho(z, \bar{z})}{\partial z_r} \, \mathrm{d}z.$$

The first term of the integrand vanishes if we assume that the inner product between f and g is finite, so that $\overline{f} g \rho \to 0$ sufficiently fast as $|z| \to \infty$. The second term also vanishes, because f is holomorphic, so that $\bar{f}$ is anti-holomorphic and hence $\frac{\partial \bar{f}}{\partial z_r} = 0$. This gives

$$\int_{\mathbb{C}^n} \overline{z_r f(z)} g(z) \rho(z, \bar{z}) + \overline{f(z)} g(z) \frac{\partial \rho(z, \bar{z})}{\partial z_r} \, \mathrm{d}z = 0,$$

which is solved for arbitrary f and g if $\bar{z}_r \rho(z, \bar{z}) + \frac{\partial \rho(z, \bar{z})}{\partial z_r} = 0$, giving

$$\rho(z, \bar{z}) = C \exp(-|z|^2).$$

The constant C is chosen to be π^{-n}, so that the norm of the constant function $z \mapsto 1$ is one.

This explains why the space of holomorphic functions is equipped with a Gaussian measure. In a similar way the form of the Segal–Bargmann transform can be derived. The Segal–Bargmann transform describes a connection between the Schrődinger representation and the Fock representation. In fact these two representations are unitarily equivalent and the Segal–Bargmann transform is the unitary transformation between them.

The ladder operators were defined in terms of observables. It is clear that observables are independent of the chosen representation. (They act on the system of states and not on the representing space.) For this reason the Segal–Bargmann transform has to intertwine with all observables and functions of them, in particular with η_r and $\bar{\eta}_r$. This property turns out to be sufficient to give an explicit form of this transform:

Suppose the transformation $\mathbf{A} \colon L^2(\mathbb{R}^n) \to \mathfrak{F}_n$ is of the form

$$\mathbf{A}\psi = \int_{\mathbb{R}^n} A(\cdot, q) \psi(q) \, \mathrm{d}q$$

for some smooth function $A : \mathbb{C}^n \times \mathbb{R}^n \to \mathbb{R}$.

The intertwining property imposes the conditions that

$$\begin{aligned}\mathbf{A}(\eta_r\psi) &= \eta_r(\mathbf{A}\psi)\\ \mathbf{A}(\bar{\eta}_r\psi) &= \bar{\eta}_r(\mathbf{A}\psi)\end{aligned}$$

for all $1 \leq r \leq n$ and for all ψ for which both sides are defined. Remember that in the Schrődinger representation on $L^2(\mathbb{R}^n)$ the ladder operators act as

$$\eta_r(\psi(q)) = (2m\hbar\omega)^{-\frac{1}{2}} \left(-i\hbar\frac{\partial}{\partial q_r} + im\omega q_r\right)\psi(q) \tag{2.9a}$$

$$\bar{\eta}_r(\psi(q)) = (2m\hbar\omega)^{-\frac{1}{2}} \left(-i\hbar\frac{\partial}{\partial q_r} - im\omega q_r\right)\psi(q), \tag{2.9b}$$

while in the Fock representation we have

$$\eta_r(f(z)) = z_r f(z) \tag{2.10a}$$

$$\bar{\eta}_r(f(z)) = \frac{\partial f}{\partial z_r}(z). \tag{2.10b}$$

If we set $f = \mathbf{A}\psi$, we find from (2.10a) that

$$\eta_r f(z) = z_r f(z) = \int_{\mathbb{R}^n} z_r A(z,q)\psi(q)\,\mathrm{d}q,$$

while the transform of (2.9a) equals the above by the intertwining property and is given by

$$\begin{aligned}\int_{\mathbb{R}^n} A(z,q)(\eta_r\psi(q))\,\mathrm{d}q &= \int_{\mathbb{R}^n} A(z,q)(2m\hbar\omega)^{-\frac{1}{2}}\left(-i\hbar\frac{\partial}{\partial q_r} + im\omega q_r\right)\psi(q)\,\mathrm{d}q\\ &= \int_{\mathbb{R}^n}\left((2m\hbar\omega)^{-\frac{1}{2}}\left(i\hbar\frac{\partial}{\partial q_r} + im\omega q_r\right)A(z,q)\right)\psi(q)\,\mathrm{d}q,\end{aligned}$$

where in the last line the boundary term is assumed to vanish. Therefore we expect $A(z,q)$ to satisfy the differential equation

$$z_r A(z,q) = (2m\hbar\omega)^{-\frac{1}{2}}\left(i\hbar\frac{\partial}{\partial q_r} + im\omega q_r\right)A(z,q)$$

and in a similar way we find for $\bar{\eta}_r$ the differential equation

$$\frac{\partial A(z,q)}{\partial z_r} = (2m\hbar\omega)^{-\frac{1}{2}}\left(i\hbar\frac{\partial}{\partial q_r} - im\omega q_r\right)A(z,q).$$

Rewriting the first line and substituting this in the second line gives

$$\begin{aligned}\frac{\partial A(z,q)}{\partial q_r} &= \left(-iz(2\tfrac{m\omega}{\hbar})^{\frac{1}{2}} - \tfrac{m\omega}{\hbar}q\right)A(z,q)\\ \frac{\partial A(z,q)}{\partial z_r} &= \left(-iq(2\tfrac{m\omega}{\hbar})^{\frac{1}{2}} + z\right)A(z,q),\end{aligned}$$

with solution

$$A(z,q) = C \exp\left(\tfrac{1}{2}z^2 - i(2\tfrac{m\omega}{\hbar})^{\frac{1}{2}} zq - \tfrac{1}{2}\tfrac{m\omega}{\hbar} q^2\right).$$

Note that the constants m, ω and $\hbar$ only appear in a particular combination. In the following the parameter $t = \frac{\hbar}{2m\omega}$ is used instead. It is a small positive number. I will suppress the dependence of t in the notation, since it is a fixed number. Bargmann chose $t = \frac{1}{2}$, and moreover his choice of ladder operators (2.3) is slightly different, which boils down to replacing z with iz in the above exponential. Taking $C = (2\pi t)^{-\frac{n}{4}}$ turns out to be the most convenient choice.

Theorem 2.3. *The Segal–Bargmann transform* $\mathbf{A}\colon L^2(\mathbb{R}^n) \to \mathfrak{F}_n : \psi \mapsto f$ *defined by*

$$f(z) = \int_{\mathbb{R}^n} A(z,q)\psi(q)\, dq \tag{2.11}$$

with

$$A(z,q) = (2\pi t)^{-\frac{n}{4}} \exp\left(\tfrac{1}{2}z^2 - it^{-\frac{1}{2}} zq - \tfrac{1}{4t} q^2\right)$$

is a surjective isometry.

The proof of this theorem consists of three steps. First we will show that for any $\psi \in L^2(\mathbb{R}^n)$ the image $\mathbf{A}\psi$ is a holomorphic function. Secondly we will introduce the complete orthogonal sequences $\{\psi_k\}$ in $L^2(\mathbb{R}^n)$ and $\{z^k\}$ in $\mathfrak{F}_n$ and show that $\mathbf{A}\psi_k = z^k$. Lastly we gather all the data obtained so far and conclude that $\mathbf{A}$ maps into $\mathfrak{F}_n$ isometrically and surjectively.

Proof. We start by showing that $\mathbf{A}\psi(z)$ is holomorphic for all $z \in \mathbb{C}^n$. Using theorem 25 of [vdB05] we are able to show that $\mathbf{A}\psi$ is holomorphic on every bounded open set $\Omega \subset \mathbb{C}^n$, which implies the former statement. The theorem states that if the following conditions are satisfied:

(a) $q \mapsto A(z,q)\psi(q)$ is integrable for every $z \in \Omega$;
(b) $z \mapsto A(z,q)\psi(q)$ is complex differentiable on Ω for almost every $q \in \mathbb{R}^n$; and
(c) there exists an integrable real-valued function g such that $|A(z,q)\psi(q)| \leq g(q)$ for all $z \in \Omega$ and almost every $q \in \mathbb{R}^n$,

then $z \mapsto \int_{\mathbb{R}^n} A(z,q)\psi(q)\mathrm{d}q$ is holomorphic on Ω and we may differentiate under the integral sign.

Let us check these properties. It will be sufficient to check this in one variable only, since $A(z,q) = A(z_1,q_1)\cdots A(z_n,q_n)$ and by Hartog's theorem a function that is holomorphic in each variable is holomorphic. The first point follows once we see that $A(z,q)$ is bounded and rapidly decreasing as $|q| \to \infty$ for all $z \in \mathbb{C}$, so that $A(z,\cdot) \in L^2(\mathbb{R})$. By the Schwarz inequality it follows then that

$$\left|\int_{\mathbb{R}} A(z,q)\psi(q)\, \mathrm{d}q\right| \leq \|A(z,\cdot)\|_{L^2} \|\psi\|_{L^2} < \infty.$$

In fact $A(z,\cdot)$ is an element of the Schwartz space of rapidly decreasing functions. A detailed analysis of this space and the action of $\mathbf{A}$ on it is given in [Bar67]. The second point is immediate, since $A(z,q)$ is a holomorphic function in z and $\psi(q)$ is independent of z. Only the third point needs some more attention and we need to use the boundedness of Ω several times. For simplicity assume that Ω is (contained in) the ball of radius R around the origin. Then we have

$$\begin{aligned}
|A(z,q)| &= (2\pi t)^{-\frac{1}{4}} \left|\exp\left(\tfrac{1}{2}z^2 - it^{-\frac{1}{2}}zq - \tfrac{1}{4t}q^2\right)\right| \\
&= (2\pi t)^{-\frac{1}{4}} \exp\left(\tfrac{1}{2}((\Re z)^2 - (\Im z)^2) + t^{-\frac{1}{2}}\Im zq - \tfrac{1}{4t}q^2\right) \\
&= (2\pi t)^{-\frac{1}{4}} \exp\left(-(q/\sqrt{4t} - \Im z)^2 + \tfrac{1}{2}|z|^2\right) \\
&\le C \exp\left(-(q/\sqrt{4t} - \Im z)^2\right),
\end{aligned} \tag{2.12}$$

with $C = (2\pi t)^{-\frac{1}{4}} \exp(\frac{1}{2}R^2)$. Taking $\omega(q)$ the function which is one on the interval $[-\sqrt{4t}R, \sqrt{4t}R]$ and decays like $\exp(-(q/\sqrt{4t} \pm R)^2)$ for $q < -R$ and $q > R$ respectively, we find a sharp bound $C\omega(q)$ of the function (2.12) on Ω. Comparing with item (a) we see that $g(q) = C\,\omega(q)\,|\psi(q)|$ satisfies all the conditions of item (c). We find that $\mathbf{A}\psi$ is holomorphic on all of $\mathbb{C}$ for all $\psi \in L^2(\mathbf{R})$.

Next we will consider orthogonal systems in $L^2(\mathbb{R}^n)$ and $\mathfrak{F}_n$. Define for all $j \in \mathbb{Z}_{\ge 0}$

$$\psi_j(x) = (2\pi t)^{-\frac{1}{4}}\,(-i\sqrt{2})^{-j}\,H_j\left(\frac{x}{\sqrt{2t}}\right), \tag{2.13}$$

with $x \in \mathbb{R}$ and $H_j(x)$ the j-th *Hermite function*. According to [Ste93] these functions are given by

$$H_j(x) = (-1)^j e^{\frac{1}{2}x^2} \frac{\mathrm{d}^j}{\mathrm{d}x^j} e^{-x^2}.$$

They form a complete orthogonal system in $L^2(\mathbb{R})$ with norms $\|H_j\|_{L^2}^2 = \sqrt{\pi}\,2^j\,j!$.

Lemma 2.4. *The system*

$$\psi_k(q) = \psi_{k_1}(q_1)\cdots\psi_{k_n}(q_n), \qquad k \in \mathbb{Z}^n_{\ge 0} \tag{2.14}$$

forms a complete orthogonal system in $L^2(\mathbb{R}^n)$.

Proof. Since the Hermite functions $\{H_j \,|\, j \in \mathbb{Z}_{\ge 0}\}$ form a complete orthogonal squence in $L^2(\mathbb{R})$, so do the functions $\{\psi_j \,|\, j \in \mathbb{Z}_{\ge 0}\}$. Let $\varphi \in L^2(\mathbb{R}^n)$. The space $L^2(\mathbb{R}^n)$ is the Hilbert completion of $\bigotimes_{r=1}^n L^2(\mathbb{R})$, so there exists a sequence

$$\varphi_l(q) = \varphi_l^{(1)}(q_1)\cdots\varphi_l^{(n)}(q_n),$$

with $\varphi_l^{(r)} \in L^2(\mathbb{R})$ for all $1 \le r \le n$ and $l \in \mathbb{Z}_{\ge 0}$, converging to φ as $l \to \infty$. Now we can construct for $J \in \mathbb{Z}_{\ge 0}$ and $1 \le r \le n$ the series

$$s_{lJ}^{(r)}(q_r) = \sum_{j=0}^{J} a_{lj}^{(r)}\,\psi_j(q_r)$$

so that

$$s_{lJ}^{(r)}(q_r) \to \varphi_l^{(r)}(q_r) \qquad \text{as } J \to \infty.$$

If we set

$$s_{lJ}(q) = s_{lJ}^{(1)}(q_1) \cdots s_{lJ}^{(n)}(q_n),$$

we find that $s_{lJ} \to \varphi_l$ as $J \to \infty$ and moreover, the diagonal s_{ll} converges to φ as $l \to \infty$. This shows that the system (2.14) is complete. Orthogonality follows trivially from orthogonality of the Hermite functions. □

The norms of the ψ_k are given by

$$\|\psi_k\|_{L^2}^2 = (2\pi t)^{-\frac{n}{2}}\, 2^{-k} \prod_{r=1}^{n} \int_{\mathbb{R}} |H_{k_r}((2t)^{-\frac{1}{2}} q_r)|^2 \mathrm{d}q_r = \pi^{-\frac{n}{2}}\, 2^{-k} \prod_{r=1}^{n} \|H_{k_r}\|_{L^2}^2 = k!.$$

The second ingredient we will use is a basis of $\mathfrak{F}_n$:

Lemma 2.5. *The system $\{z^k \mid k \in \mathbb{Z}^n_{\geq 0}\}$ forms a complete orthogonal system in $\mathfrak{F}_n$.*

Proof. Following the argument of [Hal00, §3], we will first show that the z^k are orthogonal:

$$\begin{aligned}
(z^k, z^{k'}) &= \pi^{-n} \int_{\mathbb{C}^n} \bar{z}^k z^{k'} e^{-|z|^2}\, \mathrm{d}z \\
&= (2\pi)^{-n} \int_0^{2\pi} e^{i(k'-k)\phi} \mathrm{d}\phi \int_0^{\infty} r^{k+k'} e^{-r^2}\, \mathrm{d}(r^2) \\
&= \delta_{k,k'} \int_0^{\infty} s^k e^{-s} \mathrm{d}s \\
&= \delta_{k,k'} \Gamma(k+1).
\end{aligned}$$

This gives that the system is orthogonal with the norm of each z^k given by $\|z^k\|^2 = k!$. In this computation the substitutions $z = re^{i\phi}$ and $s = r^2$ have been made.

Let $f \in \mathfrak{F}_n$. Then f is holomorphic and we can write it as a power series $\sum_{k \in \mathbb{Z}^n_{\geq 0}} a_k z^k$, converging uniformly on compact subsets of $\mathbb{C}^n$. Suppose that $(z^{k'}, f) = 0$ for all $k' \in \mathbb{Z}^n_{\geq 0}$. Using first dominated convergence and then the uniform convergence on compact sets, we find that

$$\begin{aligned}
(z^{k'}, f) &= \pi^{-n} \int_{\mathbb{C}^n} \bar{z}^{k'} \sum_k a_k z^k e^{-|z|^2}\, \mathrm{d}z \\
&= (2\pi)^{-n} \lim_{R \to \infty} \int_0^R \int_0^{2\pi} \sum_k a_k e^{i(k-k')\phi}\, r^{k+k'} e^{-r^2}\, \mathrm{d}\phi \mathrm{d}(r^2) \\
&= \lim_{R \to \infty} \sum_k a_k \delta_{k,k'} \int_0^R s^k e^{-s} \mathrm{d}s \\
&= a_{k'} \Gamma(k'+1)
\end{aligned}$$

and hence $a_{k'} = 0$ for all $k' \in \mathbb{Z}^n_{\geq 0}$, giving $f = 0$. This shows the completeness of the system. □

Now we will show that the Segal–Bargmann transform maps ψ_k isometrically to z^k. Setting $x = q/\sqrt{2t}$ and thereby $\mathrm{d}q = (2t)^{\frac{n}{2}}\mathrm{d}x$ we compute

$$\begin{aligned}
\mathbf{A}\psi_k(z) &= (2\pi t)^{-\frac{n}{2}}(-i\sqrt{2})^{-k}(-1)^k \int_{\mathbb{R}^n} e^{\frac{1}{2}z^2 - i\sqrt{2}zx - \frac{1}{2}x^2} e^{\frac{1}{2}x^2} \frac{\mathrm{d}^k}{\mathrm{d}x^k} e^{-x^2} (2t)^{\frac{n}{2}}\mathrm{d}x \\
&= \pi^{-\frac{n}{2}}(-i\sqrt{2})^{-k} \int_{\mathbb{R}^n} \frac{\mathrm{d}^k}{\mathrm{d}x^k}\left[e^{\frac{1}{2}z^2 - i\sqrt{2}zx}\right] e^{-x^2}\mathrm{d}x \\
&= \pi^{-\frac{n}{2}} z^k \int_{\mathbb{R}^n} e^{\frac{1}{2}z^2 - i\sqrt{2}zx - x^2}\mathrm{d}x \\
&= \pi^{-\frac{n}{2}} z^k \int_{\mathbb{R}^n} e^{-(x+iz/\sqrt{2})^2}\mathrm{d}x. \qquad (2.15)
\end{aligned}$$

In the first line I did integrate by parts, which simplifies the computation considerably. For purely imaginary z we recognize equation (2.15) as a standard Gaussian integral and find $\mathbf{A}\psi_k(z) = z^k$. However, we have just shown that $\mathbf{A}\psi_k$ is holomorphic and since the imaginary axis is a totally real set, we find that $\mathbf{A}\psi_k = z^k$ on all of $\mathbb{C}^n$ (see for a proof [Lan99, Ch. III]).

By linearity of $\mathbf{A}$ it follows that it is isometric on all finite linear combinations of the ψ_k. We will show now that $\mathbf{A}$ is isometric on all of $L^2(\mathbb{R}^n)$, following the argument of Bargmann [Bar61, §2c].

Let $\varphi \in L^2(\mathbb{R}^n)$. Then there exists a sequence φ_l, $l \in \mathbb{Z}_{\geq 0}$, such that each φ_l is a finite linear combination of the ψ_k and $\varphi_l \to \varphi$ in $L^2(\mathbb{R}^n)$ as $l \to \infty$. Then $\mathbf{A}\varphi_j$ is a Cauchy sequence in $\mathfrak{F}_n$, so it has a limit f in $\mathfrak{F}_n$. In the proof of lemma 2.2, we have shown that convergence in $\mathfrak{F}_n$ implies convergence in $\mathcal{O}(\mathbb{C}^n)$, so in particular $\mathbf{A}\varphi_l \to f$ pointwise as $l \to \infty$.

Also, for all $z \in \mathbb{C}^n$, we have the estimate

$$|\mathbf{A}\varphi(z) - \mathbf{A}\varphi_l(z)| = |\mathbf{A}(\varphi - \varphi_l)(z)| \leq \|A(z,\cdot)\|_{L^2}\,\|\varphi - \varphi_l\|_{L^2},$$

where we compute $\|A(z,\cdot)\|_{L^2}$ by

$$\begin{aligned}
\|A(z,\cdot)\|_{L^2}^2 &= \int_{\mathbb{R}^n} A(z,q)\overline{A(z,q)}\,\mathrm{d}q \\
&= (2\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left(\tfrac{1}{2}(z^2+\bar{z}^2) - it^{-\frac{1}{2}}(z-\bar{z})q - \tfrac{1}{2t}q^2\right)\mathrm{d}q \\
&= (2\pi t)^{-\frac{n}{2}} \int_{\mathbb{R}^n} \exp\left(-\tfrac{1}{2t}(q + i\sqrt{t}(z-\bar{z}))^2 + z\bar{z}\right)\mathrm{d}q \\
&= \exp(z\bar{z}).
\end{aligned}$$

It follows that $\mathbf{A}\varphi_l \to \mathbf{A}\varphi$ pointwise as $l \to \infty$ and we conclude that $\mathbf{A}\varphi = f$. From the above considerations we find that

$$\|\mathbf{A}\varphi\|_{\mathfrak{F}_n} = \|f\|_{\mathfrak{F}_n} = \lim_{l\to\infty}\|\mathbf{A}\varphi_l\|_{\mathfrak{F}_n} = \lim_{l\to\infty}\|\varphi_l\|_{L^2} = \|\varphi\|_{L^2},$$

which shows that $\mathbf{A}$ extends to an isometry on all of $L^2(\mathbb{R}^n)$.

The only thing left to show is that $\mathbf{A}$ is surjective. Let $f \in \mathfrak{F}_n$ and write $f(z) = \sum_{k \in \mathbb{Z}^n_{\geq 0}} a_k z^k$. Set $\psi = \sum_{k \in \mathbb{Z}^n_{\geq 0}} a_k \psi_k$. Using the fact that $\mathbf{A}$ is isometric, it is readily shown that $\psi \in L^2(\mathbb{R}^n)$ and $\mathbf{A}\psi = f$. This completes the proof. $\square$

Remark 2.6. The choice of the ψ_k in (2.13) was rather ad hoc. The justification is that by this definition the zeroth term agrees with equation (2.6) and moreover we have $\psi_k = \eta^k \psi_0$, with η as in (2.9a). This is easy to verify by direct computation. For the evaluation of $\bar{\eta}_r \psi_j(q_r)$, $j \in \mathbb{Z}_{\geq 0}$, we can use the identification $H_j(x) = \exp(-\frac{1}{2}x^2) h_j(x)$, with h_j the j-th *Hermite polynomial*. By the differential relation $h_j' = 2j h_{j-1}$ [AS64, §22.8] we find that $\bar{\eta}_r \psi_j(q_r) = j \psi_{j-1}(q_r)$.

Comparing with (2.10) we find that the operators η and $\bar{\eta}$ intertwine with the transform $\mathbf{A}$, at least for all finite linear combinations the orthogonal systems. The point of this remark is that the intertwining property holds on a dense subset. This is the most we could hope for, since the operators involved are unbounded.

This remark concludes the introduction on quantum mechanics and the clasical Segal–Bargmann transform. In the following section we will turn our attention to Hall's generalization of this transform to compact Lie groups.

3 Generalizations of the Segal–Bargmann transform

Different kinds of Segal–Bargmann transforms have appeared in the fields of harmonic analysis, mathematical physics, geometry and stochastic analysis. Therefore it is difficult to give an overview of the applications and generalizations of this transform. I will mention a few results and then concentrate on the generalization of the Segal–Bargmann transform from Hall, which was first published in [Hal94].

3.1 Appearances of the Segal–Bargmann transform

The first and oldest application I want to mention is the method of *coherent states*, which originates from the dissertation of John Klauder and his subsequent article [Kla60]. Coherent states are eigenfunctions of the Hamiltonian and those are of great interest for physicists. This subject dwells on the border of physics and mathematical physics. However it has strong links to analysis, the strongest of which is the connection with the Segal–Bargmann transform. The (earlier) results from Klauder are after translation from physical lingo even equivalent to those of Bargmann and Segal. Two books on the methods of coherent states are [KS85, Per86].

As discovered by Segal in [Seg62, Seg63] the Segal–Bargmann transformation can be used in infinite dimensions. This enables the step from quantum mechanics to quantum field theory. The key concept here is the Gaussian measure in infinite dimensions. The general picture and the supporting theorems can be found in the book [BSZ92].

The relation between (smooth) functions on a manifold X and holomorphic functions on the complexification of that manifold is a more general principle that has been studied by Guillemin and others. The 'average' result of these papers is that for every compact smooth manifold X there exist a complexification and a neighbourhood of X in its complexification such that the smooth functions on X are in one-to-one correspondence with holomorphic functions on this neighbourhood. The inversion is given by integration over the fibers.

The study of complex neighbourhoods, called *Grauert tubes*, is described in [GS91, GS92]. In the second article the *Gysin map* appears, which is just integration over the fibers. As we will see later, Hall's inversion formula for the Segal–Bargmann transform is also of this form. Related to this is the conjecture of Boutet de Monvel and Guillemin [BdMG81], which states that the holomorphic functions on a Grauert tube around X are in one-to-one correspondence to smooth functions on X. Here X is a again a compact smooth manifold. The proof of this conjecture is written down in [EM90], but this article has never been published. The isomorphism involved is called *Tőplitz correspondence*. In the same spirit is defined the *Fourier–Bros–Iagolnitzer transform*. An overview of the FBI transform and references to relevant articles can be found in [WZ01].

A nice way to look at the Segal–Bargmann transform is given in [Has00]. The Schrődinger and Fock representations are in fact irreducible unitary representations of the Heisenberg group [Fol89]. In this setting the computations from the previous section can be phrased in the much more elegant language of representation theory.

The last application of the Segal–Bargmann transform I want to mention here is in the field of geometric quantization. We start with a classical system, modelled by a symplectic manifold called *phase space*. Prequantization then gives a space of functions on this manifold. By choosing a polarization we restrict to a subspace of these functions. The map from one of these subspaces to another is called *pairing map*. The pairing map between the vertical and Kåhler polarized subspace now is exactly a constant multiple of the Segal–Bargmann transform. A detailed description of this procedure and a list of references are given in [Hal02].

3.2 Hall's generalization

After 30 years of evolution the theory of Segal and Bargmann had produced many descendants, but most of them only vaguely related. Therefore the 1994 paper of Hall [Hal94] was a very important contribution to this family. Not only has it provided links between several members of the family, but it has also generalized the Segal–Bargmann transform from $\mathbb{R}^n$ to compact Lie groups. Furthermore this paper made it possible to link the theory directly with quantization [Hal00, Hal02]. I will first mention the three most important discoveries and then describe these in more detail.

The heat kernel is the fundamental solution to the *heat equation*

$$\frac{\partial u}{\partial t} = \frac{1}{2}\Delta u,$$

with Δ the Laplace operator on $\mathbb{R}^n$. The fundamental solution at the identity is given by

$$\rho_t(x) = (2\pi t)^{-n/2} \exp\left(-\tfrac{1}{2t}x^2\right). \tag{3.1}$$

The first step in the generalization of the Segal–Bargmann transform is to rewrite the integral kernel of the Segal–Bargmann transform (see theorem 2.3 on page 12) in terms of this ρ_t, giving

$$A(z,q) = \frac{\rho_t\left(-i\sqrt{t}z - q\right)}{\sqrt{\rho_t(q)}}, \tag{3.2}$$

where the function in the enumerator is to be interpreted as the analytic continuation of the fundamental solution to $\mathbb{C}^n$.

Any compact Lie group K has a bi-invariant Riemannian structure, so we can form the Laplace–Beltrami operator, which is a generalization of the Laplace operator. The correspondingly generalized heat equation has a fundamental solution as before. Now we may replace $\mathbb{R}^n$ by the compact Lie group K and the fundamental solution by its Lie group version to obtain the generalized Segal–Bargmann transform

$$(\mathbf{A}\psi)(\zeta) = \int_K \psi(x) \frac{\rho_t\left(\zeta x^{-1}\right)}{\sqrt{\rho_t(x)}}\,\mathrm{d}x, \tag{3.3}$$

with ζ an element of the complexification $K_{\mathbb{C}}$ of K. This complexification exists and Hall has shown that the heat kernel ρ_t has a holomorphic extension to $K_{\mathbb{C}}$. Notice that the transform has exactly the same form as (2.11), with the integral kernel A given by (3.2). The only difference is the substitution $-i\sqrt{t}z \rightarrow \zeta$. This is necessary because scalar multiplication on elements of a general Lie group is not defined. The consequence of this substitution is that the Bargmann space is replaced by a new space of holomorphic functions on $K_{\mathbb{C}}$, with an inner product that depends on the parameter t.

A new transform is found if we drop the scaling by the square root of the heat kernel. The generalized Segal–Bargmann transform (3.3) then reduces to a simple convolution

$$(\mathbf{C}_t\psi)(\zeta) = \int_K \psi(x)\rho_t\left(\zeta x^{-1}\right)\,\mathrm{d}x. \tag{3.4}$$

This transform still maps $L^2(K, \mathrm{d}x)$ to the space of holomorphic functions on the complexified space, and if we modify the measure on the complexified space this transform too will be a surjective isometry. In section 3.5 we will study the technical background of this transform.

3.3 The heat kernel

The aim of this subsection is to establish the following result: The heat operator has a smooth integral kernel for positive time. We will reach this result in a few steps, jumping and dodging on our way. This text is loosely based on sections 2.1–3.2 of the review article [Gri06]. More details on heat kernels can be found in the book of Davies [Dav89] or the article of Gross [Gro01] for the special case of Lie groups.

In order to understand the heat equation on a Riemannian manifold, we have to find an analogue of the Laplace operator. Remember that the Laplace operator on $\mathbb{R}^n$ is given by composition of the gradient and the divergence operators; for a scalar function f and a vectorfield X these operators read

$$\operatorname{grad} f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{pmatrix} \qquad\qquad \operatorname{div} X = \sum_{i=1}^{n} \frac{\partial X_i}{\partial x_i}$$

so that the Laplace operator is given by

$$\Delta f = \operatorname{div}(\operatorname{grad} f) = \sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}.$$

Let us define the gradient, divergence and Laplace operator on a smooth, connected Riemannian manifold (M, g) of dimension n. We will follow the definitions of [Cha84]. Let f be a smooth function on M. Then the *gradient* $\operatorname{grad} f$ is the vectorfield on M such that $g(\operatorname{grad} f, X) = X(f)$ for all smooth vectorfields X on M.

The *divergence* of a smooth vectorfield X in $m \in M$ is defined by $(\operatorname{div} X)(m) = \operatorname{tr}(\xi \mapsto \nabla_\xi X)$. Here ξ ranges over $T_m M$ and ∇ denotes the *Levi–Civita connection* on M.

Now let f be a smooth function on M. Then the *Laplace–Beltrami operator* Δ is defined by $\Delta f = \operatorname{div}(\operatorname{grad} f)$ as in the Euclidean case.

We will also define the *Riemannian measure*. Let $\{x_\alpha : U_\alpha \to \mathbb{R}^n \mid \alpha \in I\}$ be a cover of M by charts with subordinate partition of unity $\{\phi_\alpha \mid \alpha \in I\}$. Then the Riemannian measure $\mathrm{d}V$ is defined by

$$\mathrm{d}V = \sum_{\alpha \in I} \phi_\alpha \sqrt{|g_\alpha|}\, \mathrm{d}x_\alpha.$$

Here $\mathrm{d}x_\alpha = \mathrm{d}x_{\alpha,1} \cdots \mathrm{d}x_{\alpha,n}$ is the density of Lebesgue measure on $x_\alpha(U_\alpha) \subset \mathbb{R}^n$ and $|g_\alpha|$ is the determinant of the metric in the coordinates x_α, which we will define below. This density is independent of the choice of coordinates on the U_α.

Another way to define the Riemannian measure is given in the proof of lemma 1.2 in [Hel00, Ch. I]. This proof shows that the Riemannian measure is a uniquely defined volume form on M if M is oriented. It is given by

$$\mathrm{d}V = \sqrt{|g_\alpha|}\, \mathrm{d}x_\alpha \tag{3.5}$$

on U_α, assuming that the charts are compatible with the orientation. The Riemannian measure is sometimes also called Riemannian volume or density.

Let us choose an $\alpha \in I$ and study the metric and the differential operators defined above in the local coordinates $x_\alpha : U_\alpha \to \mathbb{R}^n$. First of all we can identify the tangent bundle over U_α with $x_\alpha(U_\alpha) \times \mathbb{R}^n$ using the local coordinates $\frac{\partial}{\partial x_\alpha}$ induced by the x_α. The metric $g(m)$ is a nondegenerate bilinear map $T_m M \times T_m M \to \mathbb{R}$. In the induced coordinates on the tangent space this is an invertible matrix. Therefore the metric in local coordinates is given by a map $g_\alpha : U_\alpha \to GL(n, \mathbb{R})$ and $|g_\alpha|$ is just the determinant applied to this function.

Furthermore we find that the gradient and divergence operators in local coordinates are given by

$$\operatorname{grad} f = g_\alpha^{-1} \cdot \begin{pmatrix} \frac{\partial f}{\partial x_{\alpha,1}} \\ \vdots \\ \frac{\partial f}{\partial x_{\alpha,n}} \end{pmatrix} \qquad\qquad \operatorname{div} X = |g_\alpha|^{-\frac{1}{2}} \sum_{i=1}^{n} \frac{\partial}{\partial x_{\alpha,i}} \left(|g_\alpha|^{\frac{1}{2}} X_i \right)$$

for any smooth function f and any smooth vectorfield X on M.

Having described the Laplace operator on a Riemannian manifold, we may return to the heat equation. In the normalization used by Hall it reads

$$\frac{\partial u}{\partial t} = \frac{1}{2} \Delta u, \tag{3.6}$$

with $u : \mathbb{R}_+ \times M \to \mathbb{C}$ and Δ the Laplace–Beltrami operator on M. If we specify the initial value of u by $u(0, x) = f(x)$ for a function f in say $L^2(M)$ we find the corresponding initial value problem.

A function $p_t(x,y)$ on $\mathbb{R}_+ \times M \times M$ is called a *fundamental solution* of the heat equation if at every point $x \in M$, $p_t(x,y)$ is smooth and satisfies (3.6) as function of t and y and moreover

$$\lim_{t\downarrow 0} \int_M p_t(x,y)\varphi(y)\,\mathrm{d}V(y) = \varphi(x) \tag{3.7}$$

for all $\varphi \in C_c^\infty(M)$, the space of smooth functions with compact support. This last condition means that $\lim_{t\downarrow 0} p_t(x,y) = \delta_{x-y}$ in distributional sense. If in addition this function is positive and $\int_M p_t(x,y)\,\mathrm{d}V(y) \leq 1$ it is called *regular*.

Since the Laplace operator is non-positive definite and selfadjoint, we can define its *heat semigroup* $e^{t\Delta/2}$ for all $t \geq 0$. This is a family of positive definite bounded selfadjoint operators in $L^2(M)$. We find [Gri06, Theorem 3.1] that for all $f \in L^2(M)$ there exists a smooth function $u(t,x)$, equal to $e^{t\Delta/2}f$ almost everywhere, that satisfies the heat equation and converges to f in L^2 as t approaches zero from above, and the function u is bounded below and above by the essential extrema of the function f.

Now we have the following theorem, which is a classical result. I will state the version of Dodziuk [Dod83] here. For more references see theorem 3.3 from Grigor'yan [Gri06].

Theorem 3.1. *Let (M,g) be a Riemannian manifold. Then there exists a unique smooth function $p_t(x,y)$ on $\mathbb{R}_+ \times M \times M$ such that for all $\psi \in L^2(M)$, $t > 0$ and $x \in M$*

$$\left(e^{t\Delta/2}\psi\right)(x) = \int_M p_t(x,y)\psi(y)\,\mathrm{d}V(y). \tag{3.8}$$

This function is called the heat kernel. *It is a regular fundamental solution to the heat equation, symmetric in x and y and satisfies for $t,s > 0$ the semigroup identity*

$$p_{t+s}(x,y) = \int_M p_t(x,z)p_s(z,y)\,\mathrm{d}V(z). \tag{3.9}$$

Example 3.2. On $\mathbb{R}^n$ we find that the heat kernel $p_t(x,y)$ is given by $\rho_t(x-y)$, with ρ_t as in equation (3.1). This function has an analytic continuation to all of $\mathbb{C}^n$.

Example 3.3. For a general connected Lie group G the heat kernel with respect to left invariant Haar measure is $p_t(x,y) = \rho_t(y^{-1}x)$. With respect to right invariant Haar measure it is $p_t(x,y) = \rho_t(y^{-1}x)\,\Delta(y)$, where $\Delta(y)$ is the *modular function* on G. In both cases ρ_t is the fundamental solution at the identity $e \in G$ of the heat equation determined by the left invariant operator Δ. See for the definition and properties of a modular function for example [Kna96]. The proof of this is given in the book of Robinson [Rob91].

Remark 3.4. In this paper, we will use the term *heat kernel* also for the heat kernel at the identity $\rho_t(x) = p_t(x,e)$. It should be clear from the context and the notation whether the heat kernel p_t or the heat kernel at the identity ρ_t is meant.

3.4 Compact Lie groups

From the expression for the integral kernel (3.2) of the classical Segal–Bargmann transform in theorem 2.3, we find that the transform consists of three steps:

I. Scale $\psi \in L^2$ with the square root of the heat kernel;
II. Apply the heat operator cf. equation (3.8) and example 3.2; and
III. Analytically continue the resulting expression to the complex plane.

This description of the Segal–Bargmann transform is from Hall [Hal94] and enables the generalization to compact Lie groups described in that article. We will consider this generalization now.

Theorem 3.1 leads to the generalization of step I and II once we have rescaled $-i\sqrt{t}z \to \zeta$. The consequence of this scaling for the Bargmann space (see definition 2.1) is that the measure $\pi^{-n}e^{-|z|^2}\,\mathrm{d}z$ is replaced by $(\pi t)^{-n}e^{-|\zeta|^2/t}\,\mathrm{d}\zeta$. This follows easily by working out the substitution in the real and imaginary parts separately. Comparing with equation (3.1) we see that the weight function of this new measure is exactly the fundamental solution $\rho_{t/2}(\zeta)$, viewed as function on $\mathbb{C}^n \simeq \mathbb{R}^{2n}$.

Let us rephrase the Segal–Bargmann transform from theorem 2.3 in these new terms. We recognize $(\mathbb{R}^n, +)$ as a Lie group and call it M. Equipped with the standard inner product it is also a Riemannian manifold and therefore it has a heat kernel $p_t^{\mathbb{R}}(x,y)$. Since M is abelian it is unimodular and following example 3.3 we write $\rho_t(x) = p_t^{\mathbb{R}}(x,e)$, with $e \in M$ the identity element. The natural complexification of M is of course $(\mathbb{C}^n, +)$ and we call it $M_{\mathbb{C}}$. If $\langle\cdot,\cdot\rangle$ is denotes the inner product on M, (3.11) defines an inner product on $M_{\mathbb{C}}$. Hereby $M_{\mathbb{C}}$ becomes a Riemannian manifold and we can find a heat kernel $p_t^{\mathbb{C}}(\zeta,\eta)$. We write $\sigma_t(\zeta) = p_t^{\mathbb{C}}(\zeta,e)$, where now e is the identity of $M_{\mathbb{C}}$. Then theorem 2.3 tells us that $\mathbf{A}$ maps $L^2(M)$ isometrically onto $\mathcal{O}(M_{\mathbb{C}}) \cap L^2(M_{\mathbb{C}}, \sigma_{t/2}(\zeta)\mathrm{d}V(\zeta))$ and is given by

$$(\mathbf{A}\psi)(\zeta) = \int_M p_t^{\mathbb{R}}(\zeta,y)\frac{\psi(y)}{\sqrt{\rho_t(y)}}\,\mathrm{d}V(y). \tag{3.10}$$

The variable ζ lies in $M_{\mathbb{C}}$ and therefore $p_t^{\mathbb{R}}$ has to be interpreted as its analytic continuation from M to $M_{\mathbb{C}}$. By the *analytic continuation* of $p_t^{\mathbb{R}}$ I mean the holomorphic function on $M_{\mathbb{C}}$ which restricts to $p_t^{\mathbb{R}}$ on M. Observe that this continuation is different from the heat kernel on $M_{\mathbb{C}}$, although they have the same restriction to M: The analytic continuation of ρ_t is like $\exp(-\zeta^2)$, while σ_t is like $\exp(-|\zeta|^2)$.

In order to generalize this procedure to the case of a compact Lie group K, we have to find

(a) a Riemannian structure on K,
(b) a complexification $K_{\mathbb{C}}$ of K,
(c) a Riemannian structure on $K_{\mathbb{C}}$, and
(d) the analytic continuation of the heat kernel from K to $K_{\mathbb{C}}$.

From points (a) and (c) we can find the heat kernels $p_t^{\mathbb{R}}$ and $p_t^{\mathbb{C}}$ respectively and by point (d) $p_t^{\mathbb{R}}$ extends to $K_{\mathbb{C}}$. Once we have all this, we may apply the transform $\mathbf{A}$ as in equation (3.10), with M replaced by K, and of course $M_{\mathbb{C}}$ by $K_{\mathbb{C}}$. We will construct these four objects below.

For every compact Lie group K there exists an $\mathrm{Ad}(K)$-invariant inner product on its Lie algebra $\mathfrak{k} = T_eK$ [Kna96, Prop. 4.24]. Choose such an inner product and denote it by $\langle\cdot,\cdot\rangle$. By left translation we may extend it to all of TK, giving a bi-invariant Riemannian structure on K.

A complexification of K is a continuous homomorphism $i : K \to K_{\mathbb{C}}$ such that any continuous homomorphism from $i(K)$ into a complex Lie group H can be extended uniquely to a holomorphic homomorphism from $K_{\mathbb{C}}$ into H. This definition is from Hochschild [Hoc65] and he calls it the *universal complexification*. For a compact Lie group Hochschild finds the following results:

(a) A complexification exists and is unique up to isomorphism;
(b) The Lie algebra of $K_{\mathbb{C}}$ is the complexification of the Lie algebra of K;
(c) The homomorphism $i : K \to K_{\mathbb{C}}$ is injective; and
(d) $i(K)$ is a maximal compact subgroup of $K_{\mathbb{C}}$.

It follows that K is embedded in its complexification and we can identify K with $i(K)$. Point (a) guarantees that we can find a complexification, with the one as good as the other.

The Riemannian structure on $K_{\mathbb{C}}$ can be defined using property (b). This says that the Lie algebra of $K_{\mathbb{C}}$ is given by $\mathfrak{k}_{\mathbb{C}} = \mathfrak{k} \oplus i\mathfrak{k}$, with $\mathfrak{k}$ the Lie algebra of K. Remember that $\mathfrak{k}$ has an inner product $\langle \cdot, \cdot \rangle$. The inner product on $\mathfrak{k}_{\mathbb{C}}$ then is defined by

$$\langle V + iW, X + iY \rangle_{\mathbb{C}} = \langle V, X \rangle + \langle W, Y \rangle. \tag{3.11}$$

Using the left action of $K_{\mathbb{C}}$, we extend this inner product to a Riemannian structure on $K_{\mathbb{C}}$. It is $\mathrm{Ad}(K)$-invariant, but only left invariant under the action of $K_{\mathbb{C}}$.

Stein [Ste70] expresses the heat kernel of a compact Lie group in terms of its characters, which can be analytically continued by the defining property of the complexification. Hall shows that this defines a unique analytic continuation for the heat kernel [Hal94, §4].

Having defined the Riemannian structure on K, the complexification with its Riemannian structure and the analytic continuation of the heat kernel, we may ask whether the transform (3.10) makes sense. Hall shows that it does: This generalization of the Bargmann transform is isometric and surjective for any compact and connected Lie group.

3.5 The K-averaged transform

The name of the K-averaged transform refers to the construction of the new, K-invariant measure on $K_{\mathbb{C}}$. In contrast to the original Segal–Bargmann transform, where the measure on $K_{\mathbb{C}}$ is the heat kernel measure centered at the identity, the new measure on $K_{\mathbb{C}}$ is the average of the old measure over K. Remember that we can always view K as a subgroup of $K_{\mathbb{C}}$, defining an action of K on $K_{\mathbb{C}}$. In formulas this has the following form:

The measure on the image space of the transform $\mathbf{A}$ is given by the heat kernel σ on $K_{\mathbb{C}}$ as $\sigma_{t/2}(\zeta)\, \mathrm{d}\zeta$, with $\mathrm{d}\zeta$ as usual the Riemannian volume measure. For the new transform the measure on $K_{\mathbb{C}}$ is given by $\nu_t(\zeta)\, \mathrm{d}\zeta$, with ν_t defined by

$$\nu_t(\zeta) = \int_K \sigma_{t/2}(x^{-1}\zeta)\, \mathrm{d}x. \tag{3.12}$$

For comparison I will now formulate the generalized Segal–Bargmann transform and the K-averaged version, in slightly adapted notation from [Hal94] and with K a connected, compact Lie group.

Theorem 3.5 (Hall). *The transform* $\mathbf{A}_t : L^2(K, \mathrm{d}x) \to \mathcal{O}(K_{\mathbb{C}}) \cap L^2(K_{\mathbb{C}}, \sigma_{t/2}(\zeta)\,\mathrm{d}\zeta)$ *given by*

$$(\mathbf{A}_t f)(\zeta) = \int_K f(x)\, \frac{\rho_t(x^{-1}\zeta)}{\sqrt{\rho_t(x)}}\, \mathrm{d}x$$

is a surjective isometry for all $t > 0$*. Here* $f \in L^2(K, \mathrm{d}x)$*,* $\zeta \in K_{\mathbb{C}}$ *and* ρ_t*,* σ_t *are the heat kernels at the identity for* K *and* $K_{\mathbb{C}}$ *respectively.*

Theorem 3.6 (Hall). *The transform* $\mathbf{C}_t : L^2(K, \mathrm{d}x) \to \mathcal{O}(K_{\mathbb{C}}) \cap L^2(K_{\mathbb{C}}, \nu_t(\zeta)\,\mathrm{d}\zeta)$ *given by*

$$(\mathbf{C}_t f)(\zeta) = \int_K f(x)\, \rho_t(x^{-1}\zeta)\, \mathrm{d}x$$

is a surjective isometry for all $t > 0$*. Here* $f \in L^2(K, \mathrm{d}x)$*,* $\zeta \in K_{\mathbb{C}}$*,* ρ_t *is the heat kernel at the identity for* K *and* ν_t *is given by equation* (3.12).

Note that this last transform is basically the *heat transform*, which is convolution with the heat kernel. The two features that are added are first of all the analytic continuation of the heat kernel from K to $K_{\mathbb{C}}$ and secondly the characterization of the image of the transform, namely those holomorphic functions $F \in \mathcal{O}(K_{\mathbb{C}})$ satisfying

$$\int_{K_{\mathbb{C}}} |F(\zeta)|^2\, \nu_t(\zeta)\, \mathrm{d}\zeta < \infty.$$

There is yet another version of this transform, denoted $\mathbf{B}_t$. This version is a convenient rescaling of the transform $\mathbf{A}_t$. Instead of taking $f \in L^2(K)$, we consider the function $g = f/\sqrt{\rho_t} \in L^2(K, \rho_t(x)\,\mathrm{d}x)$. Then $\mathbf{B}_t : L^2(K, \rho_t(x)\,\mathrm{d}x) \to \mathcal{O}(K_{\mathbb{C}}) \cap L^2(K_{\mathbb{C}}, \sigma_{t/2}(\zeta)\,\mathrm{d}\zeta)$ given by

$$(\mathbf{B}_t g)(\zeta) = \int_K g(x)\, \rho_t(x^{-1}\zeta)\, \mathrm{d}x \tag{3.13}$$

is a surjective isometry for all $t > 0$. We see that the transform $\mathbf{A}_t$ is just the composition of the isometry $I : L^2(K) \to L^2(K, \rho_t(x)\,\mathrm{d}x) : f \mapsto f/\sqrt{\rho_t}$ and $\mathbf{B}_t$. The isometry I is called the *ground state transform*, because we divide by the ground state (2.6). This relation makes that we may use these two transforms interchangeably.

Remark 3.7. It is a classical result that these theorems also hold for Euclidean space. This has been reproven by Hall and Driver [Hal94, Dri95]. We will prove theorem 3.6 in the Euclidean case in section 4.2.

Remark 3.8. I have called the K-averaged version of the Segal–Bargmann a new transform. In fact this transform is not at all new. It was already found by Segal and Bargmann and is a well-known version of the classical Segal–Bargmann transform; in $\mathbb{R}^n$ the transforms $\mathbf{A}_t$ and $\mathbf{C}_t$ are related by a rescaling of the variables.

3.6 The Segal–Bargmann transform on a circle

In this section I will describe the Segal–Bargmann transform on the circle. This example illustrates the generalization of the Segal–Bargmann transform to compact Lie groups as described in section 3.4. It does only cover the case of abelian compact Lie groups.

As a model we will take the unit circle S in $\mathbb{C}$. The Riemannian metric is inherited from $\mathbb{C}$, seen as the Euclidean plane, so that the distance between two points of S is equal to the shortest angle between these points as usual. The Lie group action is rotation, which coincides with complex multiplication.

To determine the complexification of the circle, we first find its Lie algebra $\mathrm{Lie}(S) = T_eS = \mathbb{R}$. The complexification of this Lie algebra is $\mathbb{R} + i\mathbb{R} = \mathbb{C}$, which is equal to the Lie algebra $\mathfrak{gl}(1, \mathbb{C})$ of complex 1×1 matrices. The corresponding connected Lie group is given by $GL(1, \mathbb{C})$, the set of complex numbers with multiplication. This group is the complexification of S. It contains S as a maximal compact subgroup as expected.

The Riemannian structure on $GL(1, \mathbb{C})$ comes from the inner product on the real part of its Lie algebra, $\mathrm{Lie}(S)$. This is just $\mathbb{R}$ with the standard Euclidean inner product, so that $\mathfrak{gl}(1, \mathbb{C})$ also has the standard Euclidean inner product. We will denote the norm by the usual absolute value

$$|\varphi + i\chi|^2 = \varphi^2 + \chi^2$$

for all points $\varphi + i\chi \in \mathfrak{gl}(1, \mathbb{C})$ with $\varphi, \chi \in \mathbb{R}$. We extend this inner product to a Riemannian structure on $GL(1, \mathbb{C})$ using the left action. Note that in this case the Riemannian structure is $\mathrm{Ad}(GL(1, \mathbb{C}))$-invariant, since this Lie group is abelian.

The exponential map of S at $\alpha \in S$ is given by $\exp_\alpha : T_\alpha S \to S : \varphi \mapsto \alpha \cdot e^{i\varphi}$. This map can be analytically continued to the tangent bundle of $GL(1, \mathbb{C})$ and we find the function

$$\exp_\alpha : T_\alpha GL(1, \mathbb{C}) \to GL(1, \mathbb{C}) : \varphi + i\chi \mapsto \alpha \cdot e^{i(\varphi + i\chi)}, \tag{3.14}$$

now with $\alpha \in GL(1, \mathbb{C})$. Remember that the tangent space over every point α is just $\mathbb{C}$, so that we can write points of it as $\varphi + i\chi$, with $\varphi, \chi \in \mathbb{R}$. Since $\{1, i\}$ is an orthonormal basis of $T_\alpha GL(1, \mathbb{C})$, (3.14) defines geodesic normal coordinates on a neighbourhood of α.

The Laplace–Beltrami operator on S is given by the second order differential operator

$$\Delta_S = \frac{\mathrm{d}^2}{\mathrm{d}\varphi^2} \tag{3.15}$$

if we consider the functions on the circle as periodic functions of $\varphi \in \mathbb{R}$. In other words, we write functions on the circle as $f(e^{i\varphi})$. The Laplacian on the complexified space is given by the familiar expression

$$\Delta_{\mathbb{C}} = \frac{\partial^2}{\partial\varphi^2} + \frac{\partial^2}{\partial\chi^2}.$$

In fact the choice of coordinates in the previous paragaph is such that the Laplacian reduces to this simple expression. We could also have computed the metric g from section 3.3 in the geodesic normal coordinates and find that it is the identity matrix. In still other terms the vectorfields $\frac{\partial}{\partial\varphi}, \frac{\partial}{\partial\chi}$ form an orthonormal frame in the tangent bundle. If we would express $\Delta_{\mathbb{C}}$ in terms of the usual polar coordinates, we find by $\frac{\partial}{\partial\chi} = -r\frac{\partial}{\partial r}$ the expression $\frac{\partial^2}{\partial\varphi^2} + r\frac{\partial}{\partial r} + r^2\frac{\partial^2}{\partial r^2}$, which is just r^2 times the planar Laplacian.

The fundamental solution of the heat equation can be easily computed with the help of *Fourier series*. The Fourier series of a smooth function f on the circle is given by

$$\sum_{k\in\mathbb{Z}} \mathcal{F}_k(f)\, e^{ik\varphi}, \tag{3.16}$$

where $\mathcal{F}_k(f)$ denotes the k-th *Fourier coefficient* of f defined as

$$\mathcal{F}_k(f) = \frac{1}{2\pi}\int_S f(\varphi)\, e^{-ik\varphi}\,\mathrm{d}\varphi. \tag{3.17}$$

The series (3.16) converges absolutely and uniformly to the function f. From the expression of the Fourier coefficients (3.17) we find by partial integration the well known expression

$$\mathcal{F}_k\left(\frac{\mathrm{d}f}{\mathrm{d}\varphi}\right) = ik\,\mathcal{F}_k(f).$$

The fundamental solution $\rho_S(t,\varphi)$ on $\mathbb{R}_+ \times S$ satisfies the heat equation

$$\frac{\partial \rho_S(t,\varphi)}{\partial t} = \frac{1}{2}\Delta_S \rho_S(t,\varphi)$$

and converges to the Dirac delta at the identity as $t \downarrow 0$. In the following, both the Fourier transform and the Laplace operator act only with respect to the variable $\varphi \in S$. To solve this equation we compare the Fourier coefficients on both sides:

$$\frac{\partial}{\partial t}\left(\mathcal{F}_k\rho_S\right) = \mathcal{F}_k\left(\frac{\partial \rho_S}{\partial t}\right) = \mathcal{F}_k\left(\frac{1}{2}\Delta_S\rho_S\right) = \mathcal{F}_k\left(\frac{1}{2}\frac{\partial^2\rho_S}{\partial\varphi^2}\right) = -\frac{1}{2}k^2\,\mathcal{F}_k\rho_S. \tag{3.18}$$

This tells us that the Fourier coefficients of ρ_S are of the form $c_k\, e^{-\frac{1}{2}k^2 t}$ with c_k a constant. The values of these constants follow if we impose that ρ_S converges to the Dirac delta at 0, so that

$$\lim_{t\downarrow 0}\int_S \rho_S(t,\varphi)e^{-im\varphi}\,\mathrm{d}\varphi = e^{-im0} = 1$$

for all $m \in \mathbb{Z}$. Substituting the series (3.16) for ρ_S, interchanging the order of summation and integration, evaluating the integral and taking the limit then give that $2\pi c_m = 1$. Therefore all the coefficients c_k are equal to $1/2\pi$ and the fundamental solution to the heat equation is

$$\rho_S(t,\varphi) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}} e^{ik\varphi-\frac{1}{2}k^2 t}. \tag{3.19}$$

In the language of Stein [Ste70], we would say that ρ_S is a weighted sum of the multiplicative characters $e^{ik\varphi}$ of finite dimensional representations of the circle. In each of these representations the Laplacian acts as the scalar $-k^2$, so the weight $e^{-\frac{1}{2}k^2 t}$ is a solution to the heat equation in this representation. The factor of 2π arises because the Riemannian measure we are using is 2π times the normalized Haar measure.

The fundamental solution on the complexified Lie group is found through decomposition. We have that $GL(1,\mathbb{C}) \cong S \times \mathbb{R}_+$ as Lie groups, with $\mathbb{R}_+$ the set of real numbers with multiplication. From equation (3.11) we see that the Riemannian structure on $GL(1,\mathbb{C})$ is equal to the product of the Riemannian structures on S and $\mathbb{R}_+$. This establishes $GL(1,\mathbb{C})$ as a Riemannian product. By [Gri06, §4.1] the fundamental solution on this space is just the product of the fundamental solutions of the heat equations on S and $\mathbb{R}_+$.

Parametrizing the points r of $\mathbb{R}_+$ by $r = e^{-\chi}$ as before, the fundamental solution of the heat equation $\tilde{\tau}(t,r)$ on $\mathbb{R}_+$ satisfies

$$\frac{\partial}{\partial t}\tilde{\tau}(t,r) = \frac{1}{2}\frac{\partial^2}{\partial \chi^2}\tilde{\tau}(t,r).$$

Writing $\tau(t,\chi) = \tilde{\tau}(t,e^{-\chi})$ we see that this equation reduces to the usual heat equation on $\mathbb{R}$. From (3.1) we read off that

$$\tau(t,\chi) = \frac{1}{\sqrt{2\pi t}}\, e^{-\frac{1}{2t}\chi^2} \tag{3.20}$$

and the original solution $\tilde{\tau}(t,r)$ is then of course given by $\tilde{\tau}(t,r) = \tau(t,-\log r)$.

We conclude that the fundamental solution at the identity of the heat equation on $GL(1,\mathbb{C})$ is given by the product of equations (3.19) and (3.20). In our preferred set of coordinates it reads

$$\sigma(t,\varphi,\chi) = \frac{1}{\sqrt{8\pi^3 t}}\, e^{-\frac{1}{2t}\chi^2} \sum_{l\in\mathbb{Z}} e^{il\varphi - \frac{1}{2}l^2 t}. \tag{3.21}$$

The Segal–Bargmann transform may now be computed explicitly. I will show that this transform is isometric for each $t > 0$. I will give only a few steps, since the computation is quite straightforward. All sums that appear converge absolutely and uniformly on compact sets. I will do the computation for the $\mathbf{B}_t$ version of the transform.

To start with we take any smooth function on S. Its Fourier series is given by (3.16) and its norm in $L^2(S,\rho_S(\varphi)\,\mathrm{d}\varphi)$ is computed as

$$\begin{aligned}\|f\|^2_{\rho_S} &= \int_S |f(\varphi)|^2\,\rho_S(t,\varphi)\,\mathrm{d}\varphi \\ &= \sum_k \sum_{k'} \overline{\mathcal{F}_k(f)}\,\mathcal{F}_{k'}(f)\; e^{-\frac{1}{2}(k-k')^2 t},\end{aligned}$$

the sums ranging over $\mathbb{Z}$. The transform of f is given by convolution with the analytic continuation of ρ_S. This works out as

$$(\mathbf{B}_t f)(\varphi + i\chi) = \sum_k \mathcal{F}_k(f)\; e^{ik(\varphi+i\chi)}\, e^{-\frac{1}{2}k^2 t}.$$

The norm of this function in $L^2(GL(1,\mathbb{C}), \sigma(t/2,\varphi,\chi)\, \mathrm{d}\varphi \mathrm{d}\chi)$ then is

$$
\begin{aligned}
\|\mathbf{B}_t f\|^2 &= \int_S \int_{\mathbb{R}} |\mathbf{B}_t f(\varphi + i\chi)|^2\, \sigma(t/2,\varphi,\chi)\, \mathrm{d}\varphi \mathrm{d}\chi \\
&= \sum_k \sum_{k'} \sum_l \overline{\mathcal{F}_k(f)}\, \mathcal{F}_{k'}(f)\, \frac{1}{2\pi} \int_S \exp\left(i(k'-k+l)\varphi\right)\, \mathrm{d}\varphi \\
&\quad \times \frac{1}{\sqrt{\pi t}} \int_{\mathbb{R}} \exp\left(-\frac{1}{t}\chi^2 - (k+k')\chi - \frac{1}{2}\left(k^2 + k'^2 + \tfrac{1}{2} l^2\right) t\right)\, \mathrm{d}\chi \\
&= \sum_k \sum_{k'} \overline{\mathcal{F}_k(f)}\, \mathcal{F}_{k'}(f)\, e^{-\frac{1}{2}(k-k')^2 t}.
\end{aligned}
$$

In the second line I have just substituted the expressions for $\mathbf{B}_t f$ and σ and rearranged the terms in a convenient way. This is possible using the uniform convergence on compact sets and the way to deal with a noncompact set used in the second part of the proof of lemma 2.5. The integral in the second line is zero unless $l = k - k'$ and the integral in the third line is just a Gaussian integral. We find that $\|\mathbf{B}_t f\|^2 = \|f\|^2_{\rho_S}$ as expected.

4 Isometricity and the heat transform

In this section we will study the heat transform and analytic continuation of functions in the image of the heat transform separately. We will find an explicit expression for the norm on the image of the heat transform that makes this transform isometric. Following Saitoh [HS90] and Driver [Dri95] we will study an isometric transform between this image and the image of the Segal–Bargmann transform for Euclidean space and compact Lie groups.

4.1 The heat transform

Let (M, g) be a smooth, connected Riemannian manifold and $p_t(x, y)$ its heat kernel as in section 3.3. We fix $t > 0$ for the rest of this paper. Then the *heat transform* $\Gamma_t : L^2(M) \to L^2(M)$ is defined by

$$\Gamma_t\psi(x) = \int_M p_t(x, y)\psi(y)\,\mathrm{d}V(y).$$

To show that the heat transform is well defined on its domain $L^2(M)$, we prove the following lemma. Just a few words on notation first: When the measure on the space is understood, as in the case of the Riemannian measure on a Riemannian manifold, I will write $L^2(M)$ instead of $L^2(M, \mathrm{d}V)$. The inner product on $L^2(M)$ is given by

$$\langle \varphi, \psi \rangle = \int_M \overline{\varphi(x)}\psi(x)\,\mathrm{d}V(x)$$

and we can write $\Gamma_t\psi(x) = \langle p_t(x, \cdot), \psi \rangle$ since p_t is real-valued. For brevity, if $p_t(x, y)$ is considered as function of y, we will write $p_t(x) = p_t(x, \cdot)$.

Lemma 4.1. *Let (M, g) and p_t be as above. Then $p_t(x) \in L^2(M)$ for any $x \in M$.*

Proof. By the semigroup identity (3.9) we have

$$p_{2t}(x, x) = \int_M p_t(x, y)p_t(y, x)\,\mathrm{d}V(y).$$

Since p_t is real-valued and symmetric, we find that $p_{2t}(x, x) = \langle p_t(x), p_t(x) \rangle$. Using that p_{2t} is a (smooth) function on $M \times M$ then gives that $\langle p_t(x), p_t(x) \rangle < \infty$, so $p_t(x) \in L^2(M)$. □

The image of the heat transform is of course the space $\Gamma_t(L^2(M))$. I will denote this space by $H(t, M)$. There is a natural norm on H defined by

$$\|f\|_H^2 = \inf\left\{ \langle \psi, \psi \rangle \mid \psi \in L^2(M) \text{ and } f = \Gamma_t\psi \right\}. \tag{4.1}$$

This norm induces an inner product on H, which I will denote by $(\cdot, \cdot)_H$. From basic functional analysis we find that H with this inner product is a Hilbert space. Moreover the transform Γ_t is an isometry from $L^2(M)$ onto H if and only if the kernel of Γ_t is trivial.

The above can be nicely formulated in terms of integral transforms and reproducing kernels. In fact it is just a particular example of the rich theory developed in [Sai97].

Remark 4.2. I will only consider the K-averaged transform from theorem 3.6. This is precisely the heat transform followed by analytic continuation. The other two versions of the transform can only be recovered by choosing a basepoint $m \in M$. Moreover the formulas are slightly more complicated, so I will not deal with them in the next few sections. However the **B**-form might have slight advantages, as it has in the infinite dimensional case. See for a short discussion of this topic page 60.

4.2 The heat transform on Euclidean space

In this section, X will denote n-dimensional Euclidean space and Γ_t will be the heat transform on X. Moreover I will write $\mathrm{d}x$ instead of $\mathrm{d}V(x)$, since the Riemannian measure of Euclidean space is just Lebesgue measure. From example 3.2 we find that the heat kernel equals

$$p_t(x,y) = \rho_t(x-y) = (2\pi t)^{-n/2} \exp\left(-\tfrac{1}{2t}(x-y)^2\right).$$

We see from the above discussion that Γ_t is an isometry from $L^2(X)$ onto $H(t,X)$ if and only if Γ_t is injective. Indeed by definition of the norm (4.1) this follows immediately. To show that Γ_t is isometric, it will thus be sufficient to prove the following lemma:

Lemma 4.3. *Γ_t is injective on $L^2(X)$.*

Proof. In this proof we will use the *Fourier transform*, which is a linear, isometric and surjective map from $L^2(X)$ onto itself. In fact, the Fourier transform and its inverse are defined as

$$\mathcal{F}^{\pm 1}(\psi)(\xi) = (2\pi)^{-n/2} \int_X \psi(x) \exp\left(\mp i\xi \cdot x\right) \mathrm{d}x$$

on the Schwartz space of rapidly decreasing functions $\mathcal{S}(X)$. Since this is a dense subspace of $L^2(X)$ and the Fourier transform is continuous, linear and isometric on this subspace, it extends to a surjective isometry on $L^2(X)$. See for the proofs and details on the Fourier transform and Schwartz space chapter 14 of [DK09].

Furthermore we need that the Fourier transform of $p_t(x)$ is given by

$$\begin{aligned}\mathcal{F}(p_t(x))(\xi) &= (4\pi^2 t)^{-n/2} \int_X \exp\left(-\tfrac{1}{2t}(x-y)^2 - i\xi\cdot y\right) \mathrm{d}y \\ &= (2\pi)^{-n/2} \exp\left(-\tfrac{t}{2}\xi^2 - i\xi \cdot x\right).\end{aligned}$$

Then we compute for $\psi \in L^2(\mathbb{R}^n)$, with $p_t(x) \in L^2(X)$ by lemma 4.1,

$$\begin{aligned}\Gamma_t\psi(x) &= \langle p_t(x), \psi\rangle = \langle \mathcal{F}(p_t(x)), \mathcal{F}(\psi)\rangle \\ &= (2\pi)^{-n/2} \int_X \exp\left(-\tfrac{t}{2}\xi^2 + i\xi\cdot x\right) \mathcal{F}(\psi)(\xi)\,\mathrm{d}\xi \\ &= \mathcal{F}^{-1}\left(\xi \mapsto \exp\left(-\tfrac{t}{2}\xi^2\right) \mathcal{F}(\psi)(\xi)\right)(x). \qquad (4.2)\end{aligned}$$

We see that Γ_t is the composition of three functions: A Fourier transform, multiplication with a nonzero function and an inverse Fourier transform. Since multiplication with a nonzero function is injective and the Fourier transform is also injective because it is an isometry, we find that Γ_t is injective. Notice that we have multiplied with a bounded function, so that the result is again in $L^2(X)$. □

In the following we will need some more theory on Fourier transforms, in particular the results that for any $\psi \in \mathcal{S}(X)$ and any multi-index $j \in \mathbb{Z}^n_{\geq 0}$ we have

$$(i\xi)^j \mathcal{F}(\psi)(\xi) = \mathcal{F}\left(\frac{\partial^j \psi}{\partial x^j}\right)(\xi). \tag{4.3}$$

The image of Γ_t is still quite mysterious. However a complete description of all functions in that space is contained in formula (4.2). The above formula implies that $\Delta\psi = \mathcal{F}^{-1}(\xi \mapsto -\xi^2 \mathcal{F}(\psi)(\xi))$. Formally equation (4.2) then gives that

$$\Gamma_t \psi = \exp\left(\tfrac{t}{2}\Delta\right)\psi$$

as we would have expected from equation (3.8).

As a motivation for the next theorem, we will perform a formal computation of the norm on $H(t,X)$. Let $f \in H(t,X)$. Then

$$\begin{aligned}\|f\|_H^2 &= \|e^{t\Delta/2}e^{-t\Delta/2}f\|_H^2 &&= \|e^{-t\Delta/2}f\|^2 \\ &= \langle e^{-t\Delta/2}f, e^{-t\Delta/2}f\rangle &&= \langle e^{-t\Delta}f, f\rangle \\ &= \sum_{j=0}^{\infty} \frac{t^j}{j!}\langle(-\Delta)^j f, f\rangle.\end{aligned}$$

Partial integration of this last term gives (4.4) below. Notice that $\exp(-t\Delta/2)$ is symmetric on $H(t,X)$ with respect to the L^2 inner product, because $\exp(t\Delta/2)$ is injective and selfadjoint. The problem here is that $\exp(-t\Delta/2)f$ does not belong to $H(t,X)$.

Theorem 4.4. *Let $X = \mathbb{R}^n$, Γ_t the heat transform on X and set $H(t,X) = \Gamma_t(L^2(X))$. Then $H(t,X) \subset C^\infty(X)$ and for all $f \in H(t,X)$ we have $\|f\|_H^2 = N(f)$, where $\|\cdot\|_H$ is the norm* (4.1) *and $N(f)$ is defined by*

$$N(f) = \sum_{j \in \mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!}\int_X \left|\frac{\partial^j f}{\partial x^j}(x)\right|^2 \mathrm{d}x. \tag{4.4}$$

Proof. We will show first that all functions $f \in H(t,X)$ are smooth. Let $f \in H(t,X)$. Then $f = \Gamma_t\psi$ with $\psi \in L^2(X)$ and $\|f\|_H = \|\psi\|$ as a consequence of lemma 4.3. The smoothness of f follows if we write

$$f = \Gamma_t\psi = \rho_t * \psi.$$

It is easy to see that $\rho_t \in \mathcal{S}(X)$ and $\psi \in \mathcal{S}'(X)$, the space of tempered distributions. It follows that f is smooth. This can be proven with the help of the theory developed in chapter 11 of [DK09].

Now we will show that the norm on $H(t,X)$ is given by (4.4). It is convenient to do this on the dense subspace of $H(t,X)$ given by $\Gamma_t(\mathcal{S}(X))$: Suppose that $f = \Gamma_t\psi$ with $\psi \in \mathcal{S}(X)$. Then we have that

$$\begin{aligned}
\|f\|_H^2 &= \|\psi\|^2 = \|\mathcal{F}(\psi)\|^2 \\
&= \int_{\mathbb{R}^n} |\mathcal{F}(\psi)(\xi)|^2 \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \left|\exp\left(-\tfrac{t}{2}\xi^2\right)\mathcal{F}(\psi)(\xi)\right|^2 \exp\left(t\xi^2\right) \, \mathrm{d}\xi.
\end{aligned}$$

Taking the Fourier transform of equality (4.2), we find that this equals

$$\begin{aligned}
&= \int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|^2 \exp\left(t\xi^2\right) \, \mathrm{d}\xi \\
&= \int_{\mathbb{R}^n} \sum_{j\in\mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} |\xi^j \mathcal{F}(f)(\xi)|^2 \, \mathrm{d}\xi.
\end{aligned}$$

Since $\|f\|_H^2 < \infty$, we can apply Fubini's theorem and continue with

$$\begin{aligned}
&= \sum_{j\in\mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} \int_{\mathbb{R}^n} |\xi^j \mathcal{F}(f)(\xi)|^2 \, \mathrm{d}\xi \\
&= \sum_{j\in\mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} \int_{\mathbb{R}^n} \left|\mathcal{F}\left(\frac{\partial^j f}{\partial x^j}\right)(\xi)\right|^2 \mathrm{d}\xi \\
&= \sum_{j\in\mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} \int_X \left|\frac{\partial^j f}{\partial x^j}(x)\right|^2 \mathrm{d}x.
\end{aligned}$$

Since this last expression equals $\|f\|_H^2$ on $\Gamma_t(\mathcal{S}(X))$, it is clearly continuous in the norm of $H(t,X)$ and so the equality extends to all of $H(t,X)$, proving the theorem. □

The inclusion mentioned in the theorem is an equality, which is shown in the following lemma.

Lemma 4.5. *The space $H(t,X)$ is equal to $\{f \in C^\infty(X) \mid N(f) < \infty\}$.*

Proof. Let us start with a technicality. Suppose that $f \in C^\infty$ and $N(f) < \infty$. Then $\frac{\partial^j f}{\partial x^j} \in L^2(X)$ for all $j \in \mathbb{Z}^n_{>0}$, so $\mathcal{F}\left(\frac{\partial^j f}{\partial x^j}\right) \in L^2(\mathbb{R}^n)$ for all these j. Viewing these last functions as tempered distributions, we find by [DK09, Ch. 14] the equality

$$\mathcal{F}\left(\frac{\partial^j f}{\partial x^j}\right)(\xi) = (i\xi)^j \mathcal{F}(f)(\xi),$$

which holds in $\mathcal{S}'(\mathbb{R}^n)$, the space of tempered distributions. Since the left hand side is in $L^2(\mathbb{R}^n)$ and the embedding from $L^2(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is linear and injective, it follows that the right hand side is contained in $L^2(\mathbb{R}^n)$ and the equality holds in $L^2(\mathbb{R}^n)$. We will use this in the computation below.

Let f be as before. Then from the proof of theorem 4.4 we find that

$$\begin{aligned} N(f) &= \sum_{j\in\mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} \int_{\mathbb{R}^n} \left| \mathcal{F}\left(\frac{\partial^j f}{\partial x^j}\right)(\xi)\right|^2 \mathrm{d}\xi \\ &= \sum_{j\in\mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} \int_{\mathbb{R}^n} |\xi^j \mathcal{F}(f)(\xi)|^2 \,\mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} |\mathcal{F}(f)(\xi)|^2 \exp\left(t\xi^2\right) \mathrm{d}\xi \\ &= \int_{\mathbb{R}^n} \left|\exp\left(\tfrac{t}{2}\xi^2\right) \mathcal{F}(f)(\xi)\right|^2 \mathrm{d}\xi. \end{aligned}$$

This is finite, so there is a $\psi \in L^2(X)$ such that

$$\psi = \mathcal{F}^{-1}\left(\xi \mapsto \exp\left(\tfrac{t}{2}\xi^2\right)\mathcal{F}(f)(\xi)\right).$$

Then $f = \mathcal{F}^{-1}\left(\xi \mapsto \exp\left(-\frac{t}{2}\xi^2\right)\mathcal{F}(\psi)(\xi)\right)$, so by equation (4.2) we find that $f = \Gamma_t\psi$, hence $f \in H(t,X)$. This shows that $\{f \in C^\infty(X) \mid N(f) < \infty\} \subset H(t,X)$. The reverse inclusion has already been shown in theorem 4.4, so this concludes the proof of the lemma. □

The above theorem and lemma establish the image of the heat transform as a Hilbert space $H(t,X)$ with norm squared $\|\cdot\|_H^2$ equal to (4.4). This description is quite different from the characterization of the image of the Segal–Bargmann transform given in theorem 3.6. To match these results, we will use part B of lemma 1 from [HS90]. Let us introduce some notation first. If C is a complex manifold and μ a measure on C, then let

$$\mathfrak{F}(C,\mu) = \mathcal{O}(C) \cap L^2(C,\mu)$$

denote the space of square integrable, holomorphic functions on C. The inner product on $\mathfrak{F}(C,\mu)$ is written

$$(f,g) = \int_C \overline{f(z)} g(z)\mu(z). \tag{4.5}$$

In our case the complex manifold is the complexification of X, and we write $X_{\mathbb{C}} = \mathbb{C}^n$. The measure on $X_{\mathbb{C}}$ is the one used in theorem 3.6 and given by equation (3.12). The heat kernel σ_t on $\mathbb{C}^n$ is given by

$$\sigma_t(z) = (2\pi t)^{-n} \exp\left(-\tfrac{1}{2t}|z|^2\right)$$

so we find

$$\begin{aligned}\nu_t(z) &= (\pi t)^{-n} \int_X \exp\left(-\tfrac{1}{t}|z-x|^2\right)\,\mathrm{d}x \\ &= (\pi t)^{-\frac{n}{2}} \exp\left(\tfrac{1}{4t}\left[(z+\bar{z})^2 - 4z\bar{z}\right]\right) \\ &= (\pi t)^{-\frac{n}{2}} \exp\left(-\tfrac{1}{t}(\Im z)^2\right).\end{aligned}$$

In this notation the lemma reads:

Lemma 4.6. *Let $f \in \mathfrak{F}(X_{\mathbb{C}}, \nu_t(z)\mathrm{d}z)$. Then the series for $N(f)$ converges and $(f,f) = N(f)$. Conversely, if $f \in C^\infty(X)$ and $N(f) < \infty$, then f has an analytic continuation $\tilde{f}$ such that $\tilde{f} \in \mathfrak{F}(X_{\mathbb{C}}, \nu_t(z)\mathrm{d}z)$ and $(\tilde{f},\tilde{f}) = N(f)$.*

Hayashi and Saitoh only give the proof for $n = 1$. Because of the finiteness assumptions, it is an easy exercise to show that the theorem holds for arbitrary dimension n using Fubini's theorem.

Theorem 4.4 together with the lemmas 4.5 and 4.6 show that the Segal–Bargmann transform as given in theorem 3.6 is a surjective isometry on Euclidean space. This means that we can use the characterization of the image of the heat transform to show that the Segal–Bargmann transform on Euclidean space is isometric. In the rest of this paper we will study whether this method can be generalized to other spaces.

4.3 The heat transform on compact Lie groups

The goal of this section is to follow the steps outlined in the previous section for a compact Lie group. Using the Fourier transform for compact Lie groups we will describe an alternative characterization of the range of the heat transform on a compact Lie group. Then we will show that this space is equivalent to the range of Hall's generalized Segal–Bargmann transform. This section is divided in four parts: First of all the theory of Fourier transforms on compact Lie groups, followed by a presentation of the Laplace–Beltrami operator, then the computation of the heat kernel and finally the application of the steps mentioned above. The theory for the first part follows closely Helgason's book [Hel00].

The Fourier transform on a compact Lie group K decomposes a function ψ into functions of the form $\mathrm{tr}(\pi(\cdot)\mathcal{F}(\psi)(\pi))$, where tr stands for the trace operator and π is a unitary, finite-dimensional irreducible representation of K. Let us unwrap this statement.

A *finite-dimensional representation* of a compact Lie group K is a continuous homomorphism π of K into the group $GL_{\mathbb{C}}(V_\pi)$ of invertible linear transformations of a finite-dimensional complex vector space V_π. An *invariant subspace* for the representation π on V_π is a vector subspace $U \subset V_\pi$ such that $\pi(k)(U) \subset U$ for all $k \in K$. The representation π is called *irreducible* if if has no invariant subspaces other than $\{0\}$ and V_π.

Furthermore π is called *unitary* if, for a given Hermitean inner product on V_π, $\pi(k)$ is unitary for all $k \in K$. If K is compact there always exists a Hermitean inner product on V_π such that π is unitary, because π is finite-dimensional. Two representations π, π' are called *equivalent* if there exists a linear surjective homeomorphism $A : V_\pi \to V_{\pi'}$ such that $A\pi(k) = \pi'(k)A$ for all $k \in K$. The set of all equivalence classes of finite-dimensional irreducible representations is denoted by $\hat{K}$. If I write $\pi \in \hat{K}$ it means that π is a representative of an equivalence class, and we will always assume that V_π is endowed with a Hermitean inner product such that π is unitary.

Let $\mathrm{d}x$ denote *normalized Haar measure* on K. Since the Haar measure and the Riemannian measure are both invariant, it follows that

$$\mathrm{d}V(x) = \mathrm{vol}(K)\,\mathrm{d}x,$$

where $\mathrm{vol}(K)$ is the Riemannian volume $\int_K \mathrm{d}V(x)$. We write $L^2(K)$ for the space of L^2 functions on K with norm and inner product

$$\|\psi\|^2 = \langle \psi, \psi \rangle = \int_K |\psi(x)|^2\,\mathrm{d}x.$$

I have switched to Haar measure because this measure is used everywhere in the literature on Lie groups. We see from the above that this measure is equivalent to Riemannian measure, in the sense that we can change back to Riemannian measure at the cost of multiplication with a constant. In any case the heat kernel is normalized by its boundary condition (3.7), so this constant is absorbed in the definition of the heat kernel.

A result of the Peter-Weyl theorem [Hel00, Ch. IV, §1] is that the space $L^2(K)$ can be expressed as the Hilbert direct sum

$$L^2(K) = \widehat{\bigoplus_{\pi \in \hat{K}}} H_\pi, \tag{4.6}$$

with H_π the space of continuous functions on K of the form

$$H_\pi = \{x \mapsto \mathrm{tr}(\pi(x)C) \mid C \in \mathrm{Hom}(V_\pi, V_\pi)\}.$$

We equip the space of linear operators $\mathrm{Hom}(V_\pi, V_\pi)$ with the norm

$$\|C\|_\pi^2 = \dim V_\pi \ \mathrm{tr}(C^*C), \tag{4.7}$$

which is a scalar multiple of the usual Hilbert–Schmidt norm on matrices.

The Fourier transform $\mathcal{F}(\psi)$ associates to every equivalence class of representations $\pi \in \hat{K}$ the linear operator $\mathcal{F}(\psi)(\pi) \in \mathrm{Hom}(V_\pi, V_\pi)$ so that

$$\psi(x) = \sum_{\pi \in \hat{K}} \dim V_\pi\, \mathrm{tr}(\pi(x)\mathcal{F}(\psi)(\pi)),$$

according to the decomposition (4.6). Notice that the summands are orthogonal in L^2 sense, because of Schur orthogonality. I have adjusted the norm (4.7) so that $\|\psi\|^2$ is equal to

$$\sum_{\pi \in \hat{K}} \|\mathcal{F}(\psi)(\pi)\|_\pi^2. \tag{4.8}$$

The Fourier transform on K is thus a map

$$\mathcal{F} : L^2(K) \to \prod_{\pi \in \hat{K}} \mathrm{Hom}(V_\pi, V_\pi) : \psi \mapsto \mathcal{F}(\psi)$$

in the sense that $\mathcal{F}(\psi)(\pi) \in \mathrm{Hom}(V_\pi, V_\pi)$. It is readily seen that the Fourier transform is linear. The norm on the range of $\mathcal{F}$ is of course given by the square root of (4.8). By Schur orthogonality we may check that this norm is equal to the norm $\|\psi\|$, so that the Fourier transform is isometric. By linearity it then follows that it is injective. The Fourier transform is also surjective, see for example [Hel00, Ch. V, §2], and the *Fourier coefficients* $\mathcal{F}(\psi)(\pi)$ of $\psi \in L^2(K)$ are given by

$$\mathcal{F}(\psi)(\pi) = \int_K \psi(x)\pi\left(x^{-1}\right)\,\mathrm{d}x.$$

The conclusion for now is that the Fourier transform on compact Lie groups has all the properties we have used in the previous section. I have restricted my attention to compact Lie groups, but most of the above results can be generalized to other spaces. For example, Helgason covers the case of a compact symmetric space in chapter V, §4. I expect that almost all of the analysis in this section carries over to that case. In fact the generalization of Hall's theorem 3.6 to compact symmetric spaces has been proven by Stenzel [Ste99].

The Laplace–Beltrami operator on a compact Lie group is a bi-invariant second order differential operator. We will introduce it following the definition from [Far08, p. 162]. Remember that for every compact Lie group there exists an $\mathrm{Ad}(K)$-invariant inner product on its Lie algebra $\mathfrak{k}$ (p. 22). Choose such an inner product and let $\{X_r \mid 1 \le r \le n\}$ be an orthonormal basis of $\mathfrak{k}$ with respect to this inner product. The X_r can be extended to left-invariant vector fields $\tilde{X}_r$ by setting

$$\left(\tilde{X}_r f\right)(k) = \frac{\mathrm{d}}{\mathrm{d}t} f(k\exp(tX_r))\bigg|_{t=0}$$

for any smooth function f on K and $k \in K$. Define the Laplace–Beltrami operator Δ by

$$\Delta = \sum_{r=1}^{n} \tilde{X}_r^2. \tag{4.9}$$

Then Δ does not depend on the choice of orthonormal basis and is left invariant by construction. Since the inner product is Ad-invariant, Δ is also right invariant. Moreover it is a symmetric and nonpositive operator. This follows by the usual Green formulas

$$\int_K \overline{\Delta f} f' \,\mathrm{d}x = -\sum_{r=1}^{n} \int_K |\tilde{X}_r f|^2 \,\mathrm{d}x = \int_K \overline{f} \Delta f' \,\mathrm{d}x$$

for any two smooth function f, f' on K.

The heat kernel on a compact Lie group can be computed in roughly the same way as the heat kernel on the circle was computed on page 26. As a preliminary exercise we have to show that the heat kernel ρ_t is conjugation invariant. This was obvious in the abelian case, since then every function is invariant under conjugation.

Lemma 4.7. *The heat kernel ρ_t on a compact Lie group K is conjugation invariant.*

Proof. The heat kernel ρ_t is defined by the equation $\rho_t(y^{-1}x) = p_t(x,y)$, with $p_t(x,y)$ the heat kernel on K. To show that ρ_t is conjugation invariant it is thus sufficient to show that $p_t(xk,yk) = p_t(x,y)$ for arbitrary $k \in K$. Now we fix any k and set for $f \in L^2(K)$

$$\begin{aligned} u(t,x) &= \int_K p_t(x,y)f(y)\,\mathrm{d}y \\ v(t,x) &= \int_K p_t(xk,yk)f(y)\,\mathrm{d}y. \end{aligned}$$

Then u satisfies the heat equation (3.6) and the boundary condition (3.7) by definition of $p_t(x,y)$. But

$$v(t,xk^{-1}) = \int_K p_t(x,yk)f(y)\,\mathrm{d}y = \int_K p_t(x,z)f(zk^{-1})\,\mathrm{d}z,$$

with $z \mapsto f(zk^{-1}) \in L^2(K)$ by invariance of the measure, so that $v(t,xk^{-1})$ also satisfies the heat equation. Using the right invariance of the Laplace-Beltrami operator Δ it follows that $v(t,x)$ satisfies the heat equation. Moreover $v(t,x)$ satisfies the boundary condition by

$$\lim_{t\downarrow 0} v(t,x) = \lim_{t\downarrow 0}\int_K p_t(xk,z)f(zk^{-1})\,\mathrm{d}z = f(xkk^{-1}) = f(x).$$

By uniqueness of the heat kernel we find that $p_t(x,y) = p_t(xk,yk)$ and we conclude that ρ_t is invariant under conjugation. □

By Schur's lemma the only functions in H_π that are conjugation invariant are scalar multiples of the *character* $\chi_\pi(x) = \mathrm{tr}(\pi(x))$. Combining this with the property that the spaces H_π are invariant under left and right translation, that is if $f \in H_\pi$, then $x \mapsto f(kx)$ and $x \mapsto f(xk)$ are in H_π for all $k \in K$, we find that we can write any square integrable conjugation invariant function as a sum of characters:

$$\rho_t(x) = \sum_{\pi\in\hat{K}} c_\pi(t)\,\chi_\pi(x), \tag{4.10}$$

with $c_\pi : \mathbb{R}_+ \to \mathbb{C}$ coefficients depending on the time t and the equivalence class π. Since ρ_t is smooth this sum is absolutely and uniformly convergent [Hel00, Ch. V, Cor. 3.6]. Proceeding in exactly the same way as (3.18) we find that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}c_\pi(t) = \frac{\mathrm{d}}{\mathrm{d}t}\langle\rho_t,\chi_\pi\rangle \qquad &= \frac{1}{2}\int_K \Delta\rho_t(x)\,\mathrm{tr}(\pi(x))\,\mathrm{d}x \\ = \frac{1}{2}\int_K \rho_t(x)\,\mathrm{tr}(\Delta\pi(x))\,\mathrm{d}x \quad &= -\frac{\lambda_\pi}{2}\int_K \rho_t(x)\,\mathrm{tr}(\pi(x))\,\mathrm{d}x \\ = -\frac{\lambda_\pi}{2}\langle\rho_t,\chi_\pi\rangle \qquad &= -\frac{\lambda_\pi}{2}c_\pi(t). \end{aligned}$$

Here we have used the symmetry of Δ and lemma 1.6 from [Hel00, Ch. V]. This lemma states that if π is an irreducible representation of K and D a bi-invariant differential operator, then $D\pi = c\,D$ for some scalar c; it is an application of Schur's lemma. In this case we have that Δ acts as the scalar $-\lambda_\pi$ on the space H_π. The minus sign is chosen so that $\lambda_\pi \geq 0$. This follows by

$$-\lambda_\pi \int_K |\chi_\pi|^2 \,\mathrm{d}x = \int_K \overline{\Delta\chi_\pi}\chi_\pi \,\mathrm{d}x = -\sum_{r=1}^{n} \int_K |\tilde{X}_r\chi_\pi|^2 \,\mathrm{d}x \leq 0.$$

Notice that the Lie group K is compact, so its Lie algebra is the direct sum of an abelian and a semisimple part. Therefore it is sufficient to verify the above in the abelian and semisimple case. The abelian case is covered by the example of the circle in section 3.6.

Returning to the coefficients of the heat kernel, we have found that the coefficients c_π satisfy the differential equation

$$\frac{\mathrm{d}c_\pi}{\mathrm{d}t} = -\frac{1}{2}\lambda_\pi c_\pi. \tag{4.11}$$

Computing the inner product of $\chi_\pi(x)$ with (4.10) for some $\pi \in \hat{K}$ and taking the limit as $t \downarrow 0$ says that

$$c_\pi(0) = \operatorname{tr}(\pi(e)) = \operatorname{tr}(I) = \dim V_\pi.$$

Solving the differential equation (4.11) with this boundary condition then gives the heat kernel as

$$\rho_t(x) = \sum_{\pi\in\hat{K}} \dim V_\pi \, \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \, \chi_\pi(x). \tag{4.12}$$

For more details I refer to the original construction in [Ste70].

The heat transform for a compact connected Lie group has the same form as (4.2): We have

$$\mathcal{F}(\Gamma_t\psi)(\pi) = \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \, \mathcal{F}(\psi)(\pi). \tag{4.13}$$

This is easily computed in a few steps, using that $\operatorname{tr} AB = \operatorname{tr} BA$ holds for all $A, B \in \operatorname{Hom}(V_\pi, V_\pi)$.

$$\begin{aligned}
\Gamma_t\psi(w) &= \int_K \psi(x)\rho_t(x^{-1}w)\,\mathrm{d}x \\
&= \int_K \psi(x) \sum_{\pi\in\hat{K}} \dim V_\pi \, \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \, \operatorname{tr}(\pi(x^{-1})\pi(w))\,\mathrm{d}x \\
&= \sum_{\pi\in\hat{K}} \dim V_\pi \operatorname{tr}\left(\pi(w) \, \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \, \int_K \psi(x)\pi(x^{-1})\,\mathrm{d}x\right) \\
&= \sum_{\pi\in\hat{K}} \dim V_\pi \operatorname{tr}\left(\pi(w) \, \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \, \mathcal{F}(\psi)(\pi)\right).
\end{aligned}$$

To show that we can interchange the order of summation and integration, I will show that

$$\sum_{\pi\in\hat{K}} \int_K \left|\dim V_\pi \, \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \, \psi(x) \operatorname{tr}(\pi(w)\pi(x^{-1}))\right| \,\mathrm{d}x$$

is finite, so that we can apply Fubini's theorem and interchange the order of integration and summation. Let us rewrite the above in a more convenient way and compute

$$\begin{aligned}
&\sum_{\pi\in\hat{K}} \dim V_\pi \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \langle|\psi|, |x\mapsto \chi_\pi(x^{-1}w)|\rangle \\
&\qquad\leq \sum_{\pi\in\hat{K}} \dim V_\pi \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \|\psi\| \, \|x\mapsto \chi_\pi(x^{-1}w)\| \\
&\qquad= \|\psi\| \sum_{\pi\in\hat{K}} \dim V_\pi \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \\
&\qquad\leq \|\psi\| \sum_{\pi\in\hat{K}} \dim V_\pi \exp\left(-\tfrac{t}{2}\lambda_\pi\right) \operatorname{tr}(\pi(e)) \\
&\qquad= \|\psi\| \, \rho_t(e),
\end{aligned}$$

which is finite since $\psi \in L^2(K)$ and ρ_t is a smooth function. Note that the norm of $\chi_\pi(x^{-1}w) = \overline{\chi_\pi(w^{-1}x)}$ is one, because the norm of any character is one and left translation is an isometry. With (4.13) replacing (4.2) we can immediately adapt lemma 4.3 to the case of compact Lie groups. Our next task will be to find an analogue of theorem 4.4. This is given by the following theorem.

Theorem 4.8. *Let K be a compact connected Lie group and Δ as in* (4.9)*. Then $H(t,K) \subset C^\infty(K)$ and for all $f \in H(t,K)$ we have $\|f\|_H^2 = N'(f)$, where $\|\cdot\|_H$ is the norm* (4.1) *and $N'(f)$ is defined by*

$$N'(f) = \sum_{j\in\mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} \int_K |\tilde{X}^j f(x)|^2 \,\mathrm{d}x. \tag{4.14}$$

Proof. The smoothness of functions in $H(t,K)$ follows in exactly the same way as before. Also we find from page 32 that we can compute the norm of $f = \Gamma_t\psi$ as

$$\|f\|_H^2 = \sum_{\pi\in\hat{K}} e^{\lambda_\pi t} \|\mathcal{F}(f)(\pi)\|_\pi^2 = \sum_{\pi\in\hat{K}} \sum_{j=0}^{\infty} \frac{t^j}{j!} \lambda_\pi^j \|\mathcal{F}(f)(\pi)\|_\pi^2. \tag{4.15}$$

Here we have used that Γ_t and $\mathcal{F}$ are isometric, together with relation (4.13). By Fubini we may interchange the order of summation, since $\|f\|_H^2 < \infty$. Let $\langle\cdot,\cdot\rangle_\pi$ denote the inner product corresponding to the norm (4.7). Then

$$\begin{aligned}
\|f\|_H^2 &= \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{\pi\in\hat{K}} \langle \lambda_\pi^j \mathcal{F}(f)(\pi), \mathcal{F}(f)(\pi)\rangle_\pi \\
&= \sum_{j=0}^{\infty} \frac{t^j}{j!} \sum_{\pi\in\hat{K}} \langle \mathcal{F}((-\Delta)^j f)(\pi), \mathcal{F}(f)(\pi)\rangle_\pi \\
&= \sum_{j=0}^{\infty} \frac{t^j}{j!} \langle (-\Delta)^j f, f\rangle \\
&= \sum_{j\in\mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} \int_K |\tilde{X}^j f|^2 \,\mathrm{d}x.
\end{aligned}$$

Here we have used that the Laplace operator acts as the scalar $-\lambda_\pi$ in the representation π. □

As a counterpart to lemma 4.5, we have:

Lemma 4.9. *The space $H(t,K)$ is equal to $\{f \in C^\infty(K) \mid N'(f) < \infty\}$.*

Proof. We have to show that $\{f \in C^\infty(K) \mid N'(f) < \infty\} \subset H(t,K)$, since the opposite inclusion is already given by the preceding theorem. Let $f \in C^\infty(K)$ and suppose that $N'(f) < \infty$. Since on a compact Lie group the Fourier transform of a smooth function converges absolutely and uniformly, we find by the proof of theorem 4.8 that

$$N'(f) = \sum_{\pi \in \hat{K}} \| e^{\lambda_\pi t/2} \mathcal{F}(f)(\pi) \|_\pi^2,$$

which is finite by assumption. Setting

$$\psi = \mathcal{F}^{-1}\left(e^{\lambda_\pi t/2} \mathcal{F}(f)(\pi) \right)$$

this gives that $\psi \in L^2(K)$ and we have

$$f = \mathcal{F}^{-1}\left(e^{-\lambda_\pi t/2} \mathcal{F}(\psi)(\pi) \right) = \Gamma_t \psi$$

by equation 4.13. This shows that $f \in H(t,K)$. □

Now I will present the result of Driver, by which I mean theorem 4.4 from [Dri95]. It would be worthwile to investigate his proof of the theorem. We will work in the notation established in section 3 for compact Lie groups. That is, K is a compact Lie group, $K_{\mathbb{C}}$ its complexification and ρ_t, σ_t the heat kernels of these groups. Define in accordance with equation (4.14) the function

$$N'(f)(x) = \sum_{j \in \mathbb{Z}^n_{\geq 0}} \frac{t^j}{j!} |\tilde{X}^j f(x)|^2, \tag{4.16}$$

so that for $f \in H(t,K)$ we have $\int_K N'(f)(x)\,\mathrm{d}x = N'(f)$.

Although the setting of Driver's paper is a little different, a slight modification of his result will prove useful. The original result is that for a complex connected Lie group G the Bargmann space maps isometrically to a subspace of the dual of the universal enveloping algebra. If G is simply connected, this map is surjective. See also [Gro01] for an overview of these results. This map is sometimes called *Taylor map*. We will use the following:

Theorem 4.10. *Let $f \in \mathfrak{F}(K_{\mathbb{C}}, \sigma_{t/2}(z)\mathrm{d}z)$ and denote the inner product on this space by $(\cdot,\cdot)_e$. Then $(f,f)_e = N'(f)(e)$ with $e \in K$ the identity.*

If we replace the heat kernel at the identity $\sigma_{t/2}(z)$ by the heat kernel at $k \in K$, we can define the inner product

$$(f, f)_k = \int_{K_{\mathbb{C}}} |f(z)|^2 \sigma_{t/2}(k^{-1}z)\,\mathrm{d}z.$$

Application of Lebesgue dominated convergence gives that this inner product is continuous in k. As a variant of the theorem we find that $(f, f)_k = N'(f)(k)$ and we compute

$$\begin{aligned}\int_K (f, f)_k\,\mathrm{d}k &= \int_K \int_{K_{\mathbb{C}}} |f(z)|^2 \sigma_{t/2}(k^{-1}z)\,\mathrm{d}z\,\mathrm{d}k \\ &= \int_{K_{\mathbb{C}}} |f(z)|^2 \int_K \sigma_{t/2}(k^{-1}z)\,\mathrm{d}k\,\mathrm{d}z \\ &= \int_{K_{\mathbb{C}}} |f(z)|^2\, \nu_t(z)\,\mathrm{d}z\end{aligned}$$

for $f \in \mathfrak{F}(K_{\mathbb{C}}, \sigma_{t/2}(z)\mathrm{d}z)$ and ν_t as in equation (3.12). On the other hand we have arranged that $\int_K N'(f)(x)\,\mathrm{d}x = N'(f)$, so we find that if $g \in \mathfrak{F}(K_{\mathbb{C}}, \nu_t(z)\mathrm{d}z)$, then $(g, g) = N'(g)$. Here $(\cdot, \cdot)$ is the inner product in $\mathfrak{F}(K_{\mathbb{C}}, \nu_t(z)\mathrm{d}z)$.

The conclusion is that there is an isometric map from $\mathfrak{F}(K_{\mathbb{C}}, \nu_t(z)\mathrm{d}z)$ to $H(t, K)$, given by restriction to K. On the other hand, Hall has shown that the heat kernel of K can be analytically continued, so the functions in a dense subset of $H(t, K)$ have analytic continuations belonging to $\mathfrak{F}(K_{\mathbb{C}}, \nu_t(z)\mathrm{d}z)$. We find that restriction from $K_{\mathbb{C}}$ to K and analytic continuation from K to $K_{\mathbb{C}}$ extend to surjective isometries between $H(t, K)$ and $\mathfrak{F}(K_{\mathbb{C}}, \nu_t(z)\mathrm{d}z)$.

5 Symmetric spaces

In the previous sections we have seen that the classical Segal–Bargmann transform has a counterpart on compact Lie groups. A few years after this discovery, Mathew Stenzel has constructed a further generalization to Riemannian globally symmetric spaces of Helgason's compact type [Ste99]. Since then, several papers have appeared dealing with the problem of finding a Segal–Bargmann transform for Riemannian globally symmetric spaces of the noncompact type. In this section we will define these spaces first and then review some of the results from the literature.

5.1 Compact and noncompact symmetric spaces

Recall that an analytic Riemannian manifold (M, g) is an analytic manifold with an analytic Riemannian structure. Following [Hel78, Ch. IV,§3] we define

Definition 5.1 (Riemannian globally symmetric space)**.** *A* Riemannian globally symmetric space *is an analytic Riemannian manifold* (M, g) *with the property that each* $m \in M$ *is an isolated fixed point of an involutive isometry* s_m *of* M*.*

An *involutive isometry* is an isometry that is its own inverse. The definition may be better understood if we know that the isometry s_m from the definition is unique and in local coordinates is given by the geodesic symmetry: If $\gamma(t)$ is a unit speed geodesic with $\gamma(0) = m$, then $s_m(\gamma(t)) = \gamma(-t)$. This is defined for all $t \in \mathbb{R}$ since the space is complete. In the following *symmetric space* will mean Riemannian globally symmetric space.

A symmetric space may be characterized in a completely different way. Let $I(M)$ denote the group of isometries of M. It is a Lie group and has a natural, differentiable action on M. Let G denote the identity component of $I(M)$. If we pick any $m \in M$ we may consider the subgroup K of G such that $k \cdot m = m$ for all $k \in K$. It is a compact subgroup of the connected group G and G/K is analytically diffeomorphic to M via the map $gK \mapsto g \cdot m$.

We have $s_m \in K$ and the map $\sigma : G \to G : g \mapsto s_m g s_m$ is involutive. Let K_σ denote the closed group of all fixed points of σ, i.e. $K_\sigma = \{g \in G \,|\, \sigma(g) = g\}$. Then K is contained in K_σ and contains the identity component of K_σ, so K is an open subgroup of K_σ.

The Lie algebra $\mathfrak{g}$ of G decomposes as a direct sum $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, with $\mathfrak{k}$ the Lie algebra of K and at the same time the $+1$ eigenspace of $T_e\sigma$, and $\mathfrak{p}$ its -1 eigenspace. Let π denote the natural action of G on M. Then $T_e\pi$ sends $\mathfrak{k}$ to $\{0\}$ and identifies $\mathfrak{p}$ with $T_m M$. The subspace $\mathfrak{p} \cong T_{eK} G/K$ will play an important role in the next section.

Since K is a compact subgroup of G, we can find an $\mathrm{Ad}(K)$-invariant inproduct on $\mathfrak{p}$. We can extend it by the G action to all of G/K, giving a well-defined Riemannian structure on G/K by the $\mathrm{Ad}(K)$-invariance. We will always assume that G/K is equipped with such a Riemannian structure. Using this structure, we can define the Laplace operator and hence the heat kernel on G/K.

It will be convenient to assume that the Riemannian space we are dealing with is simply connected in order to define a complexification. Otherwise we could take the universal cover and prove that the construction still holds after taking the quotient. This procedure will be beyond the scope of this paper. So let us assume that the symmetric space M is simply connected. Then by proposition 4.2 of [Hel78, Ch. V] we find that M is given by the product $M = M_0 \times M_+ \times M_-$, where M_0 is a Euclidean space and M_+ and M_- are symmetric spaces of the *compact* and *noncompact type* respectively. The type corresponds to the sign of the sectional curvature, which is nonnegative on the space M_+ of compact type and similarly for the other two types. Let us also suppose that the spaces of compact and noncompact type are irreducible, which means that they cannot be written as the product of two of such spaces. We will sometimes refer to a (non)compact symmetric space, meaning a symmetric space of the (non)compact type.

A slight refinement of this classification is the following classification: There are five types of symmetric spaces, called I–IV and Euclidean space. The types I and II are compact symmetric spaces and the types III and IV noncompact symmetric spaces dual to the spaces of type I and II respectively. The compact symmetric spaces of type II are of the form $U \times U/\Delta(U)$, with U a compact Lie group and $\Delta(U)$ the diagonal subgroup, see [Hel78, Ch. VII,§4]. The noncompact symmetric spaces of type IV are called *complex*. The heat kernel for the types II and IV is much better understood than for the types I and III.

5.2 The complexification of a symmetric space

Now that we have defined symmetric spaces of the compact and noncompact type, we will consider their complexifications. Remember that we have assumed that the symmetric spaces we are dealing with are simply connected and irreducible.

The complexification of a compact symmetric space is not too difficult to find, since any compact symmetric space can be written as $X = U/K$, with U and K compact Lie groups. By the assumption that U/K is simply connected, we find that U is simply connected and that K is connected by the proof of proposition 4.2, [Hel78, Ch. V]. We can form the complexification $U_\mathbb{C}$ of U and the connected Lie subgroup $K_\mathbb{C}$ of $U_\mathbb{C}$ with Lie algebra $\mathfrak{k}_\mathbb{C}$. Then the complexification $X_\mathbb{C}$ is given by $U_\mathbb{C}/K_\mathbb{C}$.

On the other hand, let $X = G/K$ be a symmetric space of the noncompact type. Since it is simply connected, we find that G and K are connected and G has trivial center. Following [LGS96, §8], we find that G is a linear group, i.e. that it can be identified by an injective, continuous homomorphism with a subgroup of $GL(N, \mathbb{R})$ for some $N \in \mathbb{Z}_{\geq 0}$. Since this group is contained in the complex group $GL(N, \mathbb{C})$, this identifies G as a subgroup of a complex group, thereby defining the universal complexification $G_\mathbb{C}$ of G as in [Hoc65, XVII.5]. Since K is a subgroup of G, we take for $K_\mathbb{C}$ the connected subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{k}_\mathbb{C}$. This gives us a complexification $X_\mathbb{C} = G_\mathbb{C}/K_\mathbb{C}$ of X.

The duality between symmetric spaces of the compact and the noncompact type can be established through their complexification: Let U/K be a compact symmetric space. Then the Lie algebra of $U_\mathbb{C}$ is given by $(\mathfrak{k} + \mathfrak{p})_\mathbb{C}$. Let G be the connected subgroup of $U_\mathbb{C}$ with Lie algebra $\mathfrak{k} + i\mathfrak{p}$. Then G/K is the noncompact symmetric space dual to U/K. This construction also works the other way round, starting with a noncompact symmetric space.

This duality has an interesting consequence for the exponential map. Let $o = eK$ be the identity coset of U/K and $\exp_o : \mathfrak{p} \to U/K$ the Riemannian exponential map of U/K. Then the analytic continuation of $\exp_o$ to a neighbourhood $N \subset \mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}$ coincides with the exponential map of G/K on $N \cap i\mathfrak{p}$. Therefore, if $X \in \mathfrak{p}$, we will view $\exp_o(iX)$ as an element of G/K. Notice that the Riemannian exponential map is equal to the exponential map of the Lie group U, followed by the natural projection from U to U/K.

In order to state the next theorem we will need one more construction. In the notation of [Hal06], let U/K be a compact symmetric space and G/K its noncompact dual space as before. Let ν_t be the heat kernel on G/K and $\nu_t(y)\,\mathrm{d}V(y)$ the *heat kernel measure* on G/K. This object is $\mathrm{Ad}(K)$-invariant, hence left-K-invariant. Since G/K is noncompact, it is diffeomorphic to $i\mathfrak{p}$, which is the tangent space at the identity coset [Hel78, Ch. VI, thm. 1.1]. Using this diffeomorphism, followed by the identification $i\mathfrak{p} \leftrightarrow \mathfrak{p}$, we lift the heat kernel measure on G/K to a measure on $\mathfrak{p} \cong T_o U/K$. Since this measure is still left-K-invariant, we can use the left action of U to obtain a measure on each tangent space $T_x U/K$, $x \in U/K$. In the following, this measure is denoted by $\nu_t(iY) j(iY)\,\mathrm{d}(iY)$, where $Y \in T_x U/K$. If we work this out, we find that iY determines a left-K-orbit in G/K and, by left-invariance of the heat kernel measure on G/K, this expression is single-valued. Note that $j(iY)\,\mathrm{d}(iY)$ is just the pullback of the Riemannian measure $\mathrm{d}V$ on G/K by the exponential map at the identity coset of G/K. This defines the *Jacobian determinant* $j(iY)$.

Remark 5.2. In the above construction, we have identified G/K with its tangent space at the identity coset. I think we should not use this identification, but work directly with the heat flow on the tangent space, as in section 6.2. This could be particularly useful in the case described in section 5.5, see also the discussion on duality in [Hal06, §5].

5.3 The transform on a compact symmetric spaces

Let us start with stating the result from Stenzel as given by Hall in [Hal06, §4].

Theorem 5.3 (Stenzel, 1999)**.** *Let X be a symmetric space of the compact type, Γ_t the heat transform on $L^2(X)$ and $\psi \in L^2(X)$. Then $\Gamma_t\psi$ has an analytic continuation to $X_{\mathbb{C}}$ and we have the isometry*

$$\|\psi\|^2 = \int_{TX} |\Gamma_t\psi(\exp_x(iY))|^2\, \nu_{2t}(2iY)\, j(2iY)\, \mathrm{d}x\, \mathrm{d}(2iY). \tag{5.1}$$

Here the complexified space $X_{\mathbb{C}}$ is identified with the tangent bundle via the diffeomorphism

$$\Phi : TX \to X_{\mathbb{C}} : (x, Y) \mapsto \exp_x(iY), \tag{5.2}$$

where $\exp_x$ is the analytic continuation of the Riemannian exponential map at $x \in X$. More precisely we have $\exp_x(iY) = x \exp_o(iY)$ with $\exp_o : i\mathfrak{p} \mapsto G/K : iY \to \exp_o(iY)$ the exponential of G/K at the identity. This also explains why the Jacobian determinant appears in the isometry formula.

The complex structure on TX is the adapted complex structure defined in [LS91], see the next section for an explanation. In a subsequent paper [Sző91] it is shown that this adapted complex structure is defined on all of TX if X is a compact symmetric space.

The duality between compact and noncompact symmetric spaces has inspired several authors to look for a Segal–Bargmann transform on a noncompact symmetric space. We will review some of these articles below. It is important to note that, although Stenzel proves his theorem for compact symmetric spaces, his method is valid for any compact analytic Riemannian manifold with an adapted complex structure defined on its tangent bundle. This adapted complex structure will play an important role in the noncompact case, so we will devote the next section to it.

5.4 The Akhiezer-Gindikin crown domain

The next three papers we will discuss are [KS05, KÓS05, Ste06]. The interesting point is that they use the *Akhiezer–Gindikin crown domain* from [AG90] as a complexification. Another point they have in common is that the results are valid on general noncompact symmetric spaces. We will study the crown domain in this section.

In the compact case the complexification $X_{\mathbb{C}}$ can be identified with the tangent bundle TX globally. Also the Riemannian metric and the heat kernel can be analytically continued to all of $X_{\mathbb{C}}$. However, in the noncompact case the Riemmannian metric degenerates and the heat kernel develops singularities outside a certain open neighbourhood of $X \subset X_{\mathbb{C}}$. Moreover, the identification of TX with $X_{\mathbb{C}}$ breaks down outside a neighbourhood of the zero section. To avoid all these problems, Krőtz, Ólafsson and Stanton, followed by Stenzel, consider the Segal–Bargmann transform on the Akhiezer–Gindikin crown domain instead of the full complexification $X_{\mathbb{C}}$.

Akhiezer and Gindikin [AG90] pointed out that the action of G on $X_{\mathbb{C}}$, extending the action of G on X, is not proper, so the orbit space $X_{\mathbb{C}}/G$ is not Hausdorff. That means that if we expect the complexification $X_{\mathbb{C}}$ to have the same symmetry as X itself, the orbits of $X_{\mathbb{C}}/G$ are not even manifolds. To solve this problem, Akhiezer and Gindikin constructed a domain $\Xi \subset X_{\mathbb{C}}$ with the following properties:

(a) Ξ is invariant under the action of G and this action is proper;

(b) $X \subset \Xi$ is embedded as a totally real submanifold;

(c) Ξ is a Stein domain and open in $X_{\mathbb{C}}$.

This domain is commonly referred to as *crown domain* or *Akhiezer–Gindikin domain*. Fifteen years later, it was shown by Krőtz and Stanton in [KS04, KS05] that all eigenfunctions of the G-invariant differential operators on X can be holomorphically extended to Ξ. They used this to show that the heat kernel can be holomorphically extended [KS05, thm. 6.1], writing the heat kernel as a superposition of eigenfunctions of the Laplace operator, in fact the inverse Fourier transform of equation (6.6).

Another construction of a complex neighbourhood of X in $X_{\mathbb{C}}$ was given by Guillemin and Stenzel [GS91, GS92] and Lempert and Szőke [LS91]. In terms of noncompact symmetric spaces, the first two authors find that there exists a *Grauert tube*

$$T^R X = \{(x, Y) \in TX \mid \|Y\| < R\}$$

of positive radius $R > 0$ with a Kăhler metric. The square root of the associated Kăhler potential satisfies the homogeneous Monge–Ampère equation and the Kăhler metric restricts to the Riemannian metric on X as desired. The construction of Lempert and Szőke is quite different in nature. They consider geodesics $\gamma : \mathbb{R} \to X$ on X and extend them to

$$\tilde{\gamma} : T\mathbb{R} \to TX : (t,s) \mapsto (\gamma(t), s\dot{\gamma}(t)).$$

Here $T\mathbb{R} \cong \mathbb{C}$ as usual ($(t,s) \leftrightarrow t + is$) and TX plays the role of the complexification of X as before. The complex structure on a sufficiently small Grauert tube $T^R X$ then is induced by the complex structure of $\mathbb{C}$. It was shown in [LGS96] that both constructions give the same complex structure on $T^R X$, called *adapted complex structure*, and that this is the only complex structure for which the map (5.2) is biholomorphic.

In 2003 it was shown by Burns, Halverscheid and Hind [BHH03] that the crown domain is biholomorphic to a *maximal Grauert domain*, which is the largest connected domain in TM containing M for which the adapted complex structure exists. When $\operatorname{rank} X = 1$ the maximal Grauert domain is a tube.

We conclude that we may use the three models described above interchangeably, since we can map holomorphic functions on one domain to holomorphic functions on any of the others. There is one seemingly strange point, namely that the crown domain did not make use of the Riemannian structure of X, which was crucial in the other descriptions. However, this model uses natural extensions of the Killing form, which is equal to the metric up to a constant, and the Levi–Civita connection [KS05, §4]. The analytic continuation of the group theoretic exponential map to the complexification of the Lie algebra is in fact very similar to the continuation of the geodesic curves performed by Lempert and Szőke.

There are also other extensions of the metric on X to a neighbourhood of X in $X_{\mathbb{C}}$, but not at all so dominantly present in the literature as the one described above. For example Alekseevskiĭ [Ale79] finds that when a Lie group G acts properly, smoothly and effectively on a manifold M, there exists a complete G-invariant metric on M. Unfortunately this article is in Russian, so about the only thing I could understand was that there is a 'theorema' in it. Somewhat more accessible is the presentation in paragraph 8 from [KS05]. They identify the symmetric space with a symmetric cone (see [FK94]) and find that the polarization of the metric on the cone gives a complete metric on a subdomain of Ξ. This domain is denoted by $\Xi^{1/2}$ and consequently called the *square root domain*. It is comparable to a Grauert tube of half the maximal radius.

5.5 Transforms on a noncompact symmetric space

After this slightly involved introduction to the crown domain, we are ready to state a negative and a positive result on the generalization of the Segal–Bargmann transform to a noncompact symmetric space.

In [KÓS05] the authors give a description of the image of the heat transform, followed by analytic continuation to the crown domain. The resulting expression is quite involved and not like the ones we have seen before. Another result is that they show that there is no G-equivariant weight function w such that we have

$$\|\psi\|^2 = \int_{\Xi} |\Gamma_t \psi(z)|^2 \, w(z) \, \mathrm{d}z,$$

with $\psi \in L^2(X)$ and Γ_t the heat transform as before. It seems then that we cannot hope for a nice Segal–Bargmann transform on the crown domain, nor that it is possible to go beyond this domain.

A workaround to this problem has been found by Stenzel [Ste06], using the local properties of analytic Riemannian manifolds from [LGS96]. The result of Stenzel is an inversion formula for the heat transform on a noncompact symmetric space X, at least for sufficiently nice $\psi \in L^2(X)$. We will study only the basics of this article. The precise conditions and other preliminaries can be found in his paper.

Let $T^R X \subset TX$ be a Grauert tube with the adapted complex structure defined on the whole tube. As explained in [BHH03], this Grauert tube is biholomorphic to a subset of Ξ by the diffeomorphism Φ from equation (5.2). On the other hand, the fibers of TX may be locally identified with the compact symmetric space dual to X as in the compact case described at the end of section 5.2. This gives us a Riemannian structure on the fibers of $T^R X$ when R is sufficiently small. We transport this Riemannian structure on $T^R X$ to $\Phi(T^R X)$ and use this last structure to define a Riemannian volume form $\mathrm{d}V(Y)$, a heat kernel $\sigma_t(Y)$ and a gradient operator grad_Y on the fibers of Ξ. The heat kernel measure $\sigma_t(Y)\mathrm{d}V(Y)$ is comparable to the measure $\nu_t(iY)j(iY)\mathrm{d}(iY)$ we had before, but is only defined on a neighbourhood of the zero section. Now we may state the inversion formula from Stenzel:

Theorem 5.4 (Stenzel, 2006)**.** *Let X be a noncompact symmetric space, $x \in X$ and $\psi \in L^2(X)$ sufficiently regular. In the notation defined above, we have*

$$\psi(x) = \int_{T_x^R X} \Gamma_t \psi(\exp_x(iY)) \, \sigma_t(Y) \, \mathrm{d}V(Y) + \int_0^t \int_{\partial T_x^R X} \iota(W_s) \, \mathrm{d}V(Y) \mathrm{d}s,$$

with

$$W_s = \sigma_s(Y) \, \mathrm{grad}_Y \left(\Gamma_s \psi(\exp_x(iY))\right) - \Gamma_s \psi(\exp_x(iY)) \, \mathrm{grad}_Y \left(\sigma_s(Y)\right)$$

and $\iota(W_s) \, \mathrm{d}V(Y)$ the contraction of $\mathrm{d}V$ *with W_s.*

We see that the function ψ, hence also the norm of ψ, may be recovered by integration over the fibers and adding a boundary term. Together with the result from Krőtz, Ólafsson and Stanton, this says that we can not recover ψ by integrating $\Gamma_t \psi$ over the crown domain alone, but that we also need to incorporate the behaviour of $\Gamma_t \psi$ at the boundary of the crown domain. The problem is that we are missing data, namely the heat flow leaving Ξ. This phenomenon also accounts for the boundary term in theorem 5.4, the cause of this problem being the fact that the extended metric on Ξ is not complete.

On the other hand the square root domain introduced in the previous section is complete, which gives us precisely the opposite situation: Since the heat flow can not leave the domain $\Xi^{1/2}$, all possible data is contained in this domain. However this domain is contained in the domain Ξ, so in this generality we have arrived at a contradiction.

5.6 The complex case

We will consider now the results from Ólafsson and Schlichtkrull [ÓS07] and Hall and Mitchell [HM08], since these are the main results in the complex case. Other articles of the same authors with related results are [HM05, HM, ÓS08].

Let U be a compact Lie group. Then U can be viewed as a compact symmetric space and we can apply theorem 5.3. Let X be the dual noncompact symmetric space to U, so that X is of complex type. Then the heat kernel $\nu_t(W)$ on $i\mathfrak{p} \cong T_oX$ is given by

$$\nu_t(W) = (2\pi t)^{-\frac{n}{2}} e^{-\|\rho\|^2 t/2} e^{-\|W\|^2/2t} j(W)^{-\frac{1}{2}}, \qquad W \in i\mathfrak{p} \tag{5.3}$$

conform [Gan68]. Here n is the dimension of X and ρ is the sum of positive roots of X, see for example §1 of [AO03] for a short exposition. Thus the isometry (5.1) reduces to

$$\|\psi\|^2 = (\pi t)^{-\frac{n}{2}} e^{-\|\rho\|^2 t} \int_{TU} |\Gamma_t \psi(\exp_u(W))|^2 e^{-\|W\|^2/t} j(2W)^{\frac{1}{2}} \, \mathrm{d}u \, \mathrm{d}W. \tag{5.4}$$

Hall and Mitchell [HM05, HM08] have reversed the roles of U and X in the above formula to obtain an isometry for noncompact symmetric spaces:

Theorem 5.5 (Hall and Mitchell, 2008)**.** *Let X be a noncompact symmetric space of the complex type, Γ_t the heat transform on $L^2(X)$ and $\psi \in L^2(X)$. Then the integral*

$$I_\psi(R) = (\pi t)^{-\frac{n}{2}} e^{\|\rho\|^2 t} \int_{T^R X} |\Gamma_t \psi(\exp_x(iW))|^2 e^{-\|iW\|^2/t} j(2iW)^{\frac{1}{2}} \, \mathrm{d}x \, \mathrm{d}(iW)$$

is defined and finite for sufficiently small R. Moreover the function $I_\psi(R)$ is analytic and can be extended to $(0,\infty)$. This extension satisfies

$$\lim_{R\to\infty} I_\psi(R) = \|\psi\|^2.$$

Notice that $\exp : T^R X \to X_{\mathbb{C}} : (x, W) \mapsto \exp_x(iW)$ is a diffeomorphism onto its image for small R, and the pushforward of the measure $(\pi t)^{-\frac{n}{2}} e^{\|\rho\|^2 t} e^{-\|iW\|^2/t} j(2iW)^{\frac{1}{2}} \, \mathrm{d}(iW)$ under $\exp_o : \mathfrak{p} \to U : iW \mapsto \exp_o(iW)$ is just the heat kernel density on U.

A way to interpret these results is that the integration over TX cancels the singularities of the analytic continuation of $\Gamma_t\psi$. An extended discussion of this type is contained in [HM08]. A more precise statement is developed in [HM05, ÓS07] for K-invariant functions on noncompact symmetric spaces of the complex type. There it is shown that if $\psi \in L^2(X)$ is invariant under the left action of K, then $\Gamma_t\psi$ has a meromorphic extension and the product of it with a certain Jacobian is even holomorphic. However it is not clear how this result fits into the above theorem, since the theorem involves the square root of a Jacobian, whereas for K-invariant functions we need the Jacobian itself. These articles make clear that the analytic continuation does not behave very badly when we venture outside the realms of the crown domain.

6 Isometry for the hyperbolic plane

In sections 4.2 and 4.3 we have seen that the image of the heat equation is a space of smooth functions. We have constructed a norm on this image, which makes the transform isometric. Using that norm, Saitoh and Driver were able to show that the Segal–Bargmann transform is isometric on Euclidean space, respectively on a compact Lie group. In this section I will describe a similar construction for the hyperbolic plane, one of the simplest noncompact symmetric spaces. I have chosen this space to be able to give explicit formulas and computations. However I expect that the construction holds for arbitrary noncompact symmetric spaces. The tricky point is to generalize properly to symmetric spaces with rank greater than one.

In the following, we will consider the Poincaré model of the hyperbolic plane and call it D. For a description of this space as the Poincaré disc, see for example the introduction of [Hel00]. It has an analytic Riemannian structure, denoted by g, and we form the associated Laplace–Beltrami operator Δ and Riemannian measure $\mathrm{d}V(x)$. The distance on D will always be the Riemannian distance, unless stated otherwise.

6.1 The image of the heat equation

In this subsection I will present Helgason's Fourier transform on the hyperbolic plane and use this transform to find an expression for the norm on the space $H(t,D)$. Remember that this space is the image of the heat transform

$$\Gamma_t : L^2(D) \to L^2(D) : \psi \mapsto \int_D p_t(\cdot,y)\psi(y)\,\mathrm{d}V(y), \tag{6.1}$$

with $p_t(x,y)$ the heat kernel on D. The norm on $H(t,D)$ is induced by the norm on $L^2(D)$ by imposing that Γ_t be isometric, so $\|\Gamma_t\psi\|_H = \|\psi\|$ for all $\psi \in L^2(D)$.

The Fourier transform on the hyperbolic plane extends to an isometry

$$\mathcal{F} : L^2(D) \to L^2(\mathbb{R}_{>0} \times B, \lambda/(2\pi)\tanh(\pi\lambda/2)\,\mathrm{d}\lambda\mathrm{d}b),$$

where B is the boundary of D, with normalized Haar measure $\mathrm{d}b$, and $\mathrm{d}\lambda$ is the usual Lebesgue measure on $\mathbb{R}$. The transform is given by

$$\mathcal{F}(f)(\lambda,b) = \int_D f(x)e^{(1-i\lambda)\langle x,b\rangle}\mathrm{d}V(x) \tag{6.2}$$

and has the following inverse transform:

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}_{>0}}\int_B \mathcal{F}(f)(\lambda,b)e^{(1+i\lambda)\langle x,b\rangle}\,\lambda\tanh\frac{\pi\lambda}{2}\,\mathrm{d}\lambda\mathrm{d}b. \tag{6.3}$$

Here $\langle x,b\rangle$ is the distance from the origin to the circle that is tangent to B at b and passes through x. In fact we have the equality

$$e^{2\langle x,b\rangle} = \frac{1-|x|^2}{|x-b|^2}, \tag{6.4}$$

with $|\cdot|$ the Euclidean distance on D, viewed as the unit disc in $\mathbb{R}^2$. For more information on this transform I refer to the thorough treatment in [Hel00, Intro. §4].

To compute an explicit expression for the norm on $H(t, D)$, we will need the following three lemmas.

Lemma 6.1. *Let $\psi \in C_c^\infty(D)$, $\lambda \in \mathbb{R}_{>0}$, $b \in B$ and let ρ_t be the heat kernel on D. Then*

$$\mathcal{F}(\psi * \rho_t)(\lambda, b) = \mathcal{F}(\psi)(\lambda, b) \cdot \mathcal{F}(\rho_t)(\lambda, b).$$

Proof. In order to show this, we adopt the proof following equation (42) from [Hel00, p. 48] to this situation. By Young's inequality we have that the convolution $\rho_t * \psi$ is square integrable, since $\rho_t \in L^1(D)$ and $\psi \in L^2(D)$, so both sides of the expression are well defined. Expanding the left hand side gives

$$\mathcal{F}(\psi * \rho_t)(\lambda, b) = \int_D \int_G \psi(y \cdot o)\rho_t(y^{-1} \cdot x)\, \mathrm{d}y\, e^{(1-i\lambda)\langle x,b\rangle}\, \mathrm{d}V(x), \tag{6.5}$$

with $y \in G = SU(1,1)$, $K = SO(2)$, $o = eK$ and $y^{-1} \cdot x$ the action of G on $D \cong G/K$. The measure $\mathrm{d}y$ on G is Haar measure normalized by

$$\int_G \psi(y \cdot o)\, \mathrm{d}y = \int_D \psi(x)\, \mathrm{d}V(x).$$

Let us suppose for the moment that the double integral (6.5) converges absolutely. We will show this later. Then by Fubini's theorem we can interchange the order of integration and find that (6.5) is equal to

$$\int_G \psi(y \cdot o) \int_D \rho_t(x)\, e^{(1-i\lambda)\langle y \cdot x, b\rangle}\, \mathrm{d}V(x)\, \mathrm{d}y.$$

We use the identity $\langle y \cdot x, y \cdot b\rangle = \langle y \cdot o, y \cdot b\rangle + \langle x, b\rangle$ from the proof of lemma 4.7 in [Hel00, Intro], and rewrite the above to

$$\int_G \psi(y \cdot o)\, e^{(1-i\lambda)\langle y \cdot o, b\rangle} \int_D \rho_t(x)\, e^{(1-i\lambda)\langle x, y^{-1} \cdot b\rangle}\, \mathrm{d}V(x)\, \mathrm{d}y.$$

It is readily seen that $y^{-1} \cdot b \in B$, so that the inner integral is just $\mathcal{F}(\rho_t)(\lambda, y^{-1} \cdot b)$. However, since ρ_t is radial (see for example [AO03]), this expression is invariant under the action of G on B and we find that the inner integral equals $\mathcal{F}(\rho_t)(\lambda, b)$. Finally we recognize the outer integral as the Fourier transform of ψ by the normalization of the Haar measure on G and the lemma is proved.

The only thing left to show is that the double integral in (6.5) converges absolutely. Choose $R, C > 0$ so that $|\psi| \leq C \cdot 1_R$, where 1_R denotes the characteristic function of B_R, the ball of radius R in D centered at the origin. Since $\langle x, b\rangle$ is real, the absolute value of the integrand of (6.5) is dominated by C times

$$1_R(y \cdot o)\, \rho_t(y^{-1} \cdot x)\, e^{\langle x,b\rangle}$$

and we will show that this expression is integrable on $G \times D$. The problem is that $e^{\langle x,b\rangle}$ has a singularity as $x \longrightarrow b$, but this is easily compensated for by the rapid decrease of ρ_t as we will see below.

The geodesic distance $\|x\| = d(o,x)$ from the origin $o = eK \in D$ is related to the Euclidean distance $|x|$ as

$$|x| = \tanh\|x\|.$$

From (6.4) we see that $e^{2\langle x,b\rangle}$ is bounded by

$$\frac{1-\tanh^2\|x\|}{(1-\tanh\|x\|)^2} = \frac{1+\tanh\|x\|}{1-\tanh\|x\|} = \frac{\cosh\|x\|+\sinh\|x\|}{\cosh\|x\|-\sinh\|x\|} = e^{2\|x\|}.$$

Notice that this is a very bad estimate, since $e^{\langle x,b\rangle}$ has a singularity as $x \longrightarrow b$ only, whereas $e^{\|x\|}$ is unbounded as x approaches any point of B.

We also have a bound on the heat kernel of a hyperbolic space. According to theorem 5.7.2 in [Dav89] we have, for some constant $C(t) > 0$, the estimate

$$\rho_t(y^{-1}\cdot x) \le C(t)\,(1+\|y^{-1}\cdot x\|)\,\exp\left(-(\|y^{-1}\cdot x\|+t)^2/2t\right).$$

Then we find that

$$\begin{aligned}&\int_D\int_G 1_R(y\cdot o)\,\rho_t(y^{-1}\cdot x)\,e^{\langle x,b\rangle}\,\mathrm{d}y\,\mathrm{d}V(x) \le \\ &C(t)\int_D\int_G 1_R(y\cdot o)\,(1+\|y^{-1}\cdot x\|)\,\exp\left(-(\|y^{-1}\cdot x\|+t)^2/2t\right)\,e^{\|x\|}\,\mathrm{d}y\,\mathrm{d}V(x).\end{aligned}$$

If we integrate over any ball of finite radius in D, we find that this integral is finite, since we integrate a finite quantity over relatively compact sets in D and G. Therefore we may assume that $\|x\|$ is large, in particular larger than R. We use that the action of G on D is isometric to find that

$$\|y^{-1}\cdot x\| = d(o,y^{-1}\cdot x) = d(y\cdot o,x) \ge d(x,o) - d(y\cdot o,o) \ge d(x,o) - R$$

when $y\cdot o \in B_R$. Similarly we find the upper bound $\|y^{-1}\cdot x\| \le d(x,o)+R$ and so

$$\begin{aligned}&\int_G 1_R(y\cdot o)\,(1+\|y^{-1}\cdot x\|)\,\exp\left(-(\|y^{-1}\cdot x\|+t)^2/2t\right)\,\mathrm{d}y \le \\ &\qquad \operatorname{vol}B_R\,(1+\|x\|+R)\,\exp\left(-(\|x\|-R+t)^2/2t\right),\end{aligned}$$

with $\operatorname{vol}B_R = \int_D 1_R(x)\,\mathrm{d}V(x)$ by the normalization of measure. The last integral to deal with is now fairly simple, being given by

$$\begin{aligned}&\int_{D\setminus B_R}(1+\|x\|+R)\,\exp\left(-(\|x\|-R+t)^2/2t\right)\,e^{\|x\|}\,\mathrm{d}V(x) = \\ &\qquad \int_R^\infty(1+r+R)\,\exp\left(-(r-R+t)^2/2t\right)\,e^r\,2\pi\sinh(2r)\,\mathrm{d}r.\end{aligned}$$

Here $2\pi\sinh(2r)$ is the surface area of the sphere in D with radius r according to [Hel00, p. 153]. It is easy to find an upper bound for this integral. It follows that the double integral in (6.5) converges absolutely. $\square$

Lemma 6.2. *Let $\psi \in C_c^\infty(D)$, $\lambda \in \mathbb{R}_{>0}$, $b \in B$ and Γ_t as in equation* (6.1)*. Then*

$$\mathcal{F}(\Gamma_t\psi)(\lambda,b) = e^{-\left(1+\lambda^2\right)t/2}\,\mathcal{F}\psi(\lambda,b).$$

Proof. First of all we use the symmmetry of the hyperbolic plane to rewrite the heat equation to

$$\Gamma_t\psi = \psi * \rho_t$$

as in [KS05, §6]. This is actually the same form as we had before in equation (3.4). We may apply lemma 6.1 to find that

$$\mathcal{F}(\Gamma_t\psi)(\lambda,b) = \mathcal{F}\rho_t(\lambda,b)\cdot\mathcal{F}\psi(\lambda,b)$$

To complete the proof we note that

$$\mathcal{F}\rho_t(\lambda,b) = e^{-\left(1+\lambda^2\right)t/2} \tag{6.6}$$

by equation (6.7) from [KS05]. This expression is found similar to formula (4.12) in the case of a compact Lie group. Notice that the factor two appears because of the different normalization of the heat equation (3.6). ☐

Lemma 6.3. *Let $\psi \in C_c^\infty(D)$, $\lambda \in \mathbb{R}_{>0}$, $b \in B$ and $k \in \mathbb{Z}_{\geq 0}$. Then*

$$\mathcal{F}(\Delta^k\Gamma_t\psi)(\lambda,b) = \left(-\left(1+\lambda^2\right)\right)^k\mathcal{F}(\Gamma_t\psi)(\lambda,b).$$

Proof. Since ρ_t is smooth, we find that

$$\Delta^k\Gamma_t\psi = \Delta^k(\psi*\rho_t) = (\Delta^k\psi)*\rho_t,$$

see for example chapter 11 of [DK09]. Now since $\exp\left((1+i\lambda)\langle x,b\rangle\right)$ is an eigenfunction of Δ with eigenvalue $-(1+\lambda^2)$ ([Hel00, p. 32]) and $\Delta^l\psi \in C_c^\infty(D)$ for any $l \in \mathbb{Z}_{\geq 0}$, we find by Green's formula that

$$\begin{aligned}\mathcal{F}(\Delta^k\psi)(\lambda,b) &= \int_D \Delta^k\psi(x)e^{(1+i\lambda)\langle x,b\rangle}\,\mathrm{d}V(x)\\ &= \int_D \psi(x)\Delta^k e^{(1+i\lambda)\langle x,b\rangle}\,\mathrm{d}V(x)\\ &= \left(-1-\lambda^2\right)^k\mathcal{F}(\psi)(\lambda,b).\end{aligned}$$

Applying lemma 6.1 twice then gives that

$$\mathcal{F}(\Delta^k\Gamma_t\psi) = \mathcal{F}(\Delta^k\psi)\cdot\mathcal{F}(\rho_t) = \left(-1-\lambda^2\right)^k\mathcal{F}(\psi)\cdot\mathcal{F}(\rho_t) = \left(-1-\lambda^2\right)^k\mathcal{F}(\Gamma_t\psi),$$

which completes the proof. ☐

Now that we have these lemmas, we may follow again the steps outlined on page 32 to give an explicit expression for the norm on $H(t,D)$.

Theorem 6.4. *Let Γ_t be the heat transform on the hyperbolic space D and $\psi \in L^2(D)$. Then $\|\psi\| = \|\Gamma_t\psi\|_H$, where $\|\cdot\|_H$ is defined by*

$$\|f\|_H^2 = \sum_{m=0}^{\infty} \frac{t^{2m}}{(2m)!} \int_D |\Delta^m f(x)|^2 \mathrm{d}V(x) + \frac{t^{2m+1}}{(2m+1)!} \int_D g\left(\operatorname{grad}\Delta^m f(x), \overline{\operatorname{grad}\Delta^m f(x)}\right) \mathrm{d}V(x).$$

Proof. Let us first suppose that $\psi \in C_c^\infty(D)$. Then by the isometricity of the Fourier transform we find that

$$\begin{aligned}\|\psi\|^2 &= \frac{1}{2\pi}\int_{\mathbb{R}_{>0}}\int_B |\mathcal{F}\psi(\lambda,b)|^2\,\lambda\tanh\frac{\pi\lambda}{2}\,\mathrm{d}\lambda\mathrm{d}b\\ &= \frac{1}{2\pi}\int_{\mathbb{R}_{>0}}\int_B e^{\left(1+\lambda^2\right)t}\left|e^{-\left(1+\lambda^2\right)t/2}\mathcal{F}\psi(\lambda,b)\right|^2\lambda\tanh\frac{\pi\lambda}{2}\,\mathrm{d}\lambda\mathrm{d}b.\end{aligned}$$

Using respectively lemma 6.2, Fubini's theorem, lemma 6.3 and again isometricity of the Fourier transform we continue with

$$\begin{aligned}\|\psi\|^2 &= \frac{1}{2\pi}\int_{\mathbb{R}_{>0}}\int_B e^{\left(1+\lambda^2\right)t}|\mathcal{F}(\Gamma_t\psi)|^2\,\lambda\tanh\frac{\pi\lambda}{2}\,\mathrm{d}\lambda\mathrm{d}b\\ &= \sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{1}{2\pi}\int_{\mathbb{R}_{>0}}\int_B\left(1+\lambda^2\right)^k|\mathcal{F}(\Gamma_t\psi)|^2\,\lambda\tanh\frac{\pi\lambda}{2}\,\mathrm{d}\lambda\mathrm{d}b\\ &= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\frac{1}{2\pi}\int_{\mathbb{R}_{>0}}\int_B \mathcal{F}(\Delta^k\Gamma_t\psi)\overline{\mathcal{F}(\Gamma_t\psi)}\,\lambda\tanh\frac{\pi\lambda}{2}\,\mathrm{d}\lambda\mathrm{d}b\\ &= \sum_{k=0}^{\infty}\frac{(-t)^k}{k!}\int_D \Delta^k(\Gamma_t\psi)(x)\,\overline{\Gamma_t\psi(x)}\,\mathrm{d}V(x).\end{aligned} \tag{6.7}$$

From the proof it follows that for even k we may express these summands also as

$$\frac{t^k}{k!}\int_D \Delta^{k/2}(\Gamma_t\psi)(x)\,\overline{\Delta^{k/2}(\Gamma_t\psi)(x)}\,\mathrm{d}V(x),$$

while for odd k we find

$$-\frac{t^k}{k!}\int_D \Delta^{(k+1)/2}(\Gamma_t\psi)(x)\,\overline{\Delta^{(k-1)/2}(\Gamma_t\psi)(x)}\,\mathrm{d}V(x).$$

The function $\Delta^{(k-1)/2}(\Gamma_t\psi)$ decays rapidly enough to show that the above expression is equal to

$$\frac{t^k}{k!}\int_D g\left(\operatorname{grad}\Delta^{(k-1)/2}(\Gamma_t\psi)(x), \overline{\operatorname{grad}\Delta^{(k-1)/2}(\Gamma_t\psi)(x)}\right)\,\mathrm{d}V(x).$$

There is a quite lengthy proof of this by Prof. Van den Ban, that uses Harish-Chandra's Schwartz functions. This last step is only needed to obtain a nice symmetric formula, so the proof will be left out. It turns out that we will only need expression (6.7) in the rest of this paper. In the following, we will denote the above integrals for odd and even k by the heuristic $\|\operatorname{grad}^k \Gamma_t \psi\|^2$.

This proves the theorem for $\psi \in C_c^\infty$. The only thing left to show now is that $\psi \mapsto \|\Gamma_t\psi\|_H$ is continuous on C_c^∞ with respect to the L^2-norm. This follows immediately by the equality $\|\Gamma_t\psi\|_H = \|\psi\|$. Therefore the identity $\|\Gamma_t\psi\|_H = \|\psi\|$ extends by continuity to all of $L^2(D)$ and the theorem is proved. □

Notice that because of the theorem, the norm $\|\cdot\|_H$ coincides with the norm on $H(t, D)$ defined in (4.1).

Remark 6.5. If we would replace the Fourier transform by an expansion in terms of the eigenfunctions of the Laplace–Beltrami operator, we could try to reprove the theorem for an arbitrary Riemannian manifold. I expect that on a suitable space of functions we will find exactly the same expression as (6.7). The corresponding result in the compact case is found in any textbook on Riemannian geometry and heat kernels.

Remark 6.6. The natural step following theorem 6.4 woud be to start using

$$\|f\|_H^2 = \sum_{k=0}^{\infty} \frac{t^k}{k!} \|\operatorname{grad}^k f\|^2 \tag{6.8}$$

I had expected that this equation would recover the results from Hall and Stenzel, but I did not succeed in finding a proper Laplacian after partial integration of grad and $\overline{\text{grad}}$. The problem is that I do not know how to continue the metric $\mathrm{d}V(x)$ to a complex neighbourhood. First I have tried to use the analytic continuation of $\sqrt{\det g(x)}$. This gives a 'Laplace operator' with mixed holomorphic and anti-holomorphic second order derivations and holomorphic first order derivatives. Obviously both the approach and the result lack the symmetry between holomorphic and antiholomorphic functions, which is present in the definition of the norm. I tried a second approach, which was taking the absolute value of the analytically continued measure $|\sqrt{\det g(z)}|$. Partial integration then gives a nice, symmetric operator, but unfortunately it has a nonzero constant term. I dropped this attempt and switched to formula (6.7). This formula was much more promising, as we will see in the next section.

6.2 Isometry for noncompact symmetric spaces

In this section I will sketch the steps that could be taken to recover that the Segal–Bargmann transform is isometric. We will study this in the setting of theorem 5.4 and theorem 5.5 respectively, starting from the above description of the image of the heat transform.

The general procedure is as follows:

(a) Identify a complex neighbourhood Z of D;
(b) Let $f \in H(t, D)$ and extend f to Z;

(c) Extend the Riemannian metric g to Z and let $\mathrm{d}V(z)$ denote the corresponding measure (similar to (3.5));
(d) Extend the Laplace operator to an operator $\Box$ on $C^\infty(Z)$, selfadjoint with respect to $\mathrm{d}V(z)$; and
(e) Find a sequence $\varphi_j \in C_c^\infty(Z)$, $j \in \mathbb{Z}_{\geq 0}$, such that

$$\lim_{j\to\infty} \int_Z F(z)\, \varphi_j(z)\, \mathrm{d}V(z) = \int_D F(x)\, \mathrm{d}V(x)$$

for all functions F that can be written as $F(z) = \Box^k f(z)\, \overline{f(z)}$ with $k \in \mathbb{Z}_{\geq 0}$ and $f, \Box$ as above. We will write $\lim_{j\to\infty} \varphi_j = \delta_D$ in distributional sense.

Of course most of these points do not make sense in this generality, but we will come to that after giving the motivation for the introduction of these 'orphans'. We will make one more assumption, namely that the extended Laplacian commutes with $\bar{f}$. If we take the *holomorphic extension* in the steps (b)–(d), this says that the holomorphic extension of a differential operator commutes with antiholomorphic functions.

Disregarding all problems we compute

$$\begin{aligned}
\|f\|_H^2 &= \sum_{k=0}^\infty \frac{(-t)^k}{k!} \int_D \Delta^k f(x)\overline{f(x)}\, \mathrm{d}V(x) \\
&= \sum_{k=0}^\infty \frac{(-t)^k}{k!} \lim_{j\to\infty} \int_Z \Box^k f(z)\overline{f(z)}\varphi_j(z)\, \mathrm{d}V(z) \\
&= \sum_{k=0}^\infty \frac{(-t)^k}{k!} \lim_{j\to\infty} \int_Z |f(z)|^2\, \Box^k \varphi_j(z)\, \mathrm{d}V(z) \\
&= \int_Z |f(z)|^2 e^{-t\Box}\delta_D(z)\, \mathrm{d}V(z).
\end{aligned}$$

We interpret $e^{-t\Box}\delta_D$ as the fundamental solution at time $2t$ of the heat equation on Z

$$\begin{gathered}
\frac{\partial u(t,z)}{\partial t} = -\frac{1}{2}\Box u(t,z) \\
\forall \psi \in C_c^\infty(Z) \text{ and } \forall x \in D: \lim_{t\downarrow 0} \int_{U_x} u(t,z)\psi(z)\, \mathrm{d}V(Y) = \psi(x),
\end{gathered} \tag{6.9}$$

where $z = \exp_x(iY)$, $\mathrm{d}V(Y)$ as on page 47, $Y \in U_x$ and $U_x \subset T_xD$ any neighbourhood of $0 \in T_xD$ such that $\exp_x(iU_x) \subset Z$. In other words, we want that $\lim_{t\downarrow 0} u(t, \exp_x(iY)) = \delta$, the Dirac delta distribution at the origin of T_xD, so that $\lim_{t\downarrow 0} u(t,z) = \delta_D$. Below it is explained why there appears a minus sign in the heat equation.

This computation gives us an idea why the space $H(t,D)$ is important, what we can expect of the image of the Segal–Bargmann transform and how we could use the norm on $H(t,D)$ to show that the Segal–Bargmann transform is isometric.

Following Stenzel, we take (a) the crown domain Ξ as the complexification of D. We identify this domain diffeomorphically with a subset U of the tangent bundle TD by

$$\Phi : U \to \Xi : (x, Y) \mapsto \exp_x(iY)$$

and we will abbreviate $z = \exp_x(iY)$ as before. Then by [KS05] we find that (b) the functions $f \in H(t, D)$ can be analytically continued to Ξ. Following the construction of [LGS96, §1], the Riemannian structure on D can be analytically continued to the fibers of TD: If g is the Riemannian structure on D, let g^+ denote the analytic continuation of g to Ξ and g_x the restriction of g^+ to the fiber above x in Ξ. This construction gives us on each fiber (c) the holomorphic extension g_x of the metric and also (d) the holomorphic extension $\Box_x$ of the Laplacian with respect to the complex structure on Ξ.

For $U \subset TD$, let $\rho_x(U) = \{u \in U \,|\, \pi(u) = x\}$ with $\pi : TD \to D$ projection to the basepoint. Then in theorem 8.5 of [LGS96] it is shown that $(\rho_x(\Phi^{-1}(\Xi)), -g_x)$ is a Riemannian manifold isometric to a neighbourhood of the identity coset in the compact symmetric space dual to D. The Laplacian corresponding to this metric is given by $-\Box_x$, which is thus a negative definite operator. This explains the minus sign in equation (6.9).

Please note that the paper [LGS96] does not contain exactly the statements we need, but only serves as an illuminated path to the results described here. One mismatch is that we may have to restrict to a smaller neighbourhood $Z \subset \Xi$, but this does not affect our computation. The justification of the final result is contained in [Ste06], but we are mainly interested in the method to derive this result.

Remark 6.7. From the paper of Golse, Leichtnam and Stenzel we see that we could only expect to find a heat kernel on the fibers if the space is locally symmetric. That is, by proposition 1.17 of their paper we find that the metric g_x is real if and only if the space is locally symmetric. If this is not the case, g_x and $\Box_x$ will be complex-valued and we are facing additional problems.

Let us continue with the last condition (e): it is actually quite weak and we will take the existence of such a sequence for granted. Now that we have given a meaning to the elements of our formula, we will restart the computation: Let $f \in H(t, D)$ and $z = \Phi(x, Y)$. Then starting with (6.7) we find that

$$\begin{aligned}
\|f\|_H^2 &= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \lim_{j\to\infty} \int_Z \Box^k f(z)\overline{f(z)}\varphi_j(z)\,\mathrm{d}V(z) \\
&= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \lim_{j\to\infty} \int_Z \Box^k |f(z)|^2 \varphi_j(z)\,\mathrm{d}V(z) \\
&= \sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \lim_{j\to\infty} \int_D \int_{\rho_x(\Phi^{-1}(\Xi))} \Box_x^k |f(z)|^2 \varphi_j(z)\, j_x(iY)\,\mathrm{d}(iY)\,\mathrm{d}V(x).
\end{aligned}$$

Here $j_x(iY)\,\mathrm{d}(iY)\,\mathrm{d}V(x)$ is the analytic continuation of the measure $\mathrm{d}V(x)$ from D to TD and $\mathrm{d}V(z)$ is its pushforward by the holomorphic diffeomorphism Φ. The differentiation $\Box$ on Z goes over to differentiation in the fiberwise directions if we take the functions φ_j

constant in the direction of D. This can be arranged if we replace the compact support of the $\varphi_j \in C^\infty(Z)$ by support in a tubular neighbourhood of D and choose these functions invariant under the action of G, where $D \cong G/K$. Note that this approach seems independent of the size of the neighbourhood Z, as long as it is contained in the crown domain Ξ.

Since the φ_j have compact support in each fiber, we can apply the Green function from [LGS96] to show that the above formula equals

$$\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \lim_{j\to\infty} \int_D \int_{\rho_x(\Phi^{-1}(\Xi))} |f(z)|^2 \Box_x^k \varphi_j(z)\, j_x(iY)\, \mathrm{d}(iY)\, \mathrm{d}V(x). \tag{6.10}$$

The problem is of course to deal with the limits $j, k \to \infty$. I expect that this limit gives rise to the appearance of boundary terms, but I do not know exactly why they will appear. Except for a small difference in notation, we could expect that the above formula gives an isometry for Stenzel's inversion formula from theorem 5.4. An isometry formula with boundary terms is in perfect accordance with the remark in [KÓS05] that there is no weight on the crown domain such that the Segal–Bargmann transform on this domain is isometric.

Hall and Mitchell have approached the problem from a different point of view. We will use the notation developed above and see if we could recover the result from theorem 5.5. They are working with symmetric spaces of type IV, also called complex type. It seems that their result also holds for symmetric spaces of rank one. The hyperbolic plane is about the simplest example of a symmetric space of rank one.

We have seen that $(\rho_x(\Phi^{-1}(\Xi)), -g_x)$ is isometric to a neighbourhood of the zero coset of the compact symmetric space dual to D. Let us denote this dual space by S, because in this case it is the 2-sphere. Since $\rho_x(\Phi^{-1}(\Xi)) \subset T_x D$, we can identify it by the action of $G = SU(1,1)$ with a subset of $\mathfrak{p} = T_o S$ as described in section 5. Call this pair (Σ, g_Σ), with $\exp_o(\Sigma)$ an open subset of S and g_Σ a metric on Σ isometric by $\exp_o$ to the metric of S. The measure on Σ is well known and given by $j(W)\mathrm{d}W$ in geodesic normal coordinates, where j denotes the Jacobian of the exponential $\exp_o$ on $T_o S$. This measure coincides exactly with the measure on $T_x D$ employed in formula (6.10). In full detail we have that

$$\int_{i\Sigma} f(iY) j_x(iY)\, \mathrm{d}(iY) = \int_\Sigma f(W) j(W)\, \mathrm{d}W.$$

Using this identification, we may extend (Σ, g_Σ) to a larger open neighbourhood $(\Sigma', g_{\Sigma'})$, with the property that $\exp_o(\Sigma')$ is dense in S. Then we find that the derivative of the exponential map vanishes at the boundary of Σ', so that $j(W) = 0$ on $\partial\Sigma'$. Therefore the metric $g_{\Sigma'}$ degenerates at these points. This means that the leading coefficients of the Laplacian blow up to infinity exactly as fast as $g_{\Sigma'}$ goes to zero as we approach the boundary of Σ'. The problem is that in all formulas there appear arbitrary high powers of the Laplacian.

However, following the geodesic flow, we arrive in safe waters again beyond $\partial\Sigma'$. The idea now is to extend formula (6.10) to Σ', obtain boundary conditions C at $\partial\Sigma'$ and continue with exactly the same formula beyond $\partial\Sigma'$, imposing boundary conditions at the other side that are opposite to the conditions C. If this would work, we get rid of the boundary conditions of Stenzel and find a single boundary condition at infinity. At this point one more assumption would do no harm, so suppose that this boundary condition vanishes. Then we should find a Segal–Bargmann transform from the formula

$$\sum_{k=0}^{\infty} \frac{(-t)^k}{k!} \lim_{j\to\infty} \int_D \int_{T_xD} |f(x,W)|^2 \Box_x^k \varphi_j(W)\, j(W)\, \mathrm{d}W\, \mathrm{d}V(x). \tag{6.11}$$

There is still a lot to show before we could hope to recover the results of Hall and Mitchell. I will summarize some of the remaining questions in the next section. To finish this section, I will work out some of the quantities defined above in the case of the hyperbolic plane.

Example 6.8. Let D be the hyperbolic plane. We view it as the unit ball in $\mathbb{R}^2$, so that we have global coordinates $x_{1,2} : D \to \mathbb{R}^2$. Let $|\cdot|$ be the distance on D induced by the Euclidean distance on $\mathbb{R}^2$: For $x \in D$ we set $|x|^2 = x_1(x)^2 + x_2(x)^2$. Consequently we will denote the point $x \in D$ such that $|x| = 0$ with o. The Riemannian measure on D in these coordinates is given by

$$\mathrm{d}V(x) = \frac{1}{(1-|x|^2)^2}\, \mathrm{d}x_1 \mathrm{d}x_2,$$

with $\mathrm{d}x_1\mathrm{d}x_2$ the measure on D corresponding to Lebesgue measure on $\mathbb{R}^2$. In terms of the Riemannian distance $d(o,x) = \|x\|$, defined by $\tanh \|x\| = |x|$, it is given by

$$\mathrm{d}V(x) = \cosh^2 \|x\|\, \mathrm{d}x_1\mathrm{d}x_2. \tag{6.12}$$

Let $\mathrm{d}W_1\mathrm{d}W_2$ denote Lebesgue measure on T_oD. Then the analytic continuation of $\mathrm{d}V(x)$ restricted to T_oD is given by

$$\mathrm{d}V(W) = \cos^2 \|W\|\, \mathrm{d}W_1\mathrm{d}W_2.$$

This is most easily seen if we use that the analytic continuation of the metric on D is related to the metric on S by

$$-\|ix\|_D^2 = \|W\|_S^2$$

when $x = \exp_o(W)$. We find that the weight $\cos^2 \|W\|$ is positive outside a set of circles in T_oS.

It seems then that, to make life easier, we should consider the analytic continuation to TD instead of $D_{\mathbb{C}}$, see also the remark at the end of section 1.3 in [HM08]. As a consequence, we obtain a kind of fiberwise Segal–Bargmann transform on the fibers of the tangent bundle.

7 Further research

In this section I want to present some topics for further research on the Segal–Bargmann transform.

7.1 The heat transform and analytic continuation

In sections 4.2 and 4.3 we have studied the two phases of the Segal–Bargmann transform, the heat transform and the subsequent analytic continuation, for Euclidean space and compact Lie groups. In section 6 we attacked the Segal–Bargmann transform on the hyperbolic plane in the same way. We were able to identify the image of the heat transform using Helgason's Fourier transform. This raises immediately the question whether we are able to do this on a general globally symmetric space of the noncompact type:

> *Question 1.a:* Can we identify the image of the heat transform for any noncompact symmetric space?

I expect that when the noncompact symmetric space has rank one, the proof of theorem 6.4 will carry over almost literally. However when the rank is greater than one, there might arise some difficulties. Partly because the Fourier transform becomes more complicated, partly because the heat kernel is relatively unknown. Following remark 6.5 we could also ask the same question for a Riemannian manifold, say connected and complete:

> *Question 1.b:* Does the spectral decomposition of the Laplace–Beltrami operator give a characterization of the image of the heat transform on a Riemannian manifold?

To recover a Segal–Bargmannlike transform, we have to make the procedure sketched in section 6.2 rigorous. I do not know how to do this, so I phrase it as a question here:

> *Question 1.c:* Is it possible to arrive at an isometry formula from formula (6.11)?

If we have a closer look at formula (6.10), it suggests that the isometry does not depend on the analytic continuation to the complexification, but only on the behaviour of the derivatives at the real locus. In that sense it is not strange that the isometry formula is independent of the chosen domain Z in the complexification, as discussed on page 57. This theory is also supported by the result from Hall and Mitchell from theorem 5.5, where they find an isometry formula starting on a Grauert tube of arbitrary small radius. I do not see why this approach should be confined to the complex case, as in their result.

7.2 The relation with the classical Segal–Bargmann transform

In section 2.4 we have seen the derivation of the classical Segal–Bargmann transform. The main property was the intertwining property of certain operators on the L^2 space with other operators on the holomorphic function space. Hall shows that the Segal–Bargmann transform on compact Lie groups also has this intertwining property. It would be interesting to know which of the proposed transforms on symmetric spaces have a similar property.

> *Question 2.a:* Is there a generalization of the intertwining property on symmetric spaces and do the proposed generalizations of the Segal–Bargmann transform possess this property?

This question is interesting in order to investigate a possible generalization of the article [Hal02], see section 7.3.

Another interesting comparison with the classical Segal–Bargmann transform is the original version of the transform found by Segal, which is of the form (3.13). In the previous sections we have almost always identified all tangent spaces with the tangent space at the identity coset. An alternative approach is to consider the heat flow on Z of a unit of heat starting at the point o. The question is:

> *Question 2.b:* Does the **B**-form of the Segal–Bargmann transform generalize to symmetric spaces?

The drawback of this approach is that the construction depends on the basepoint. However this provides us with a preferred choice of geodesic coordinates and the identification of the tangent spaces may become more natural.

7.3 Geometric quantization

The application of the generalized Segal–Bargmann transform on compact Lie groups is only mentioned briefly in this paper, but it was the starting point of my thesis. To be more precise, the following remark from Hall in [Hal02, p. 245] was the beginning of this paper:

> *I do not know whether the geometric quantization pairing map is unitary in the case of general compact symmetric spaces X. There is, however, a unitary Segal–Bargmanntype transform, given in terms of heat kernels and described in [Ste99].*

The first problem that puzzled me was

> *Question 3.a:* Is Stenzel's construction on symmetric spaces of the compact type a straightforward generalization of Hall's construction on compact Lie groups?

It took me quite a while to see that the answer was affirmative. It turns out that the Riemannian structure imposed by Hall (3.11) is the same as the analytic continuation of the Riemannian structure to the fibers, followed by a change of sign as performed by Stenzel.

The following rather disappointing discovery is that the answer to the following question is already contained in the paragraphs preceeding the above citation.

> *Question 3.b:* Is the Segal–Bargmann transform on compact symmetric spaces equivalent to a pairing map in geometric quantization?

Remember that the classical Segal–Bargmann transform was a unitary transform between the Schrődinger representation and the Fock representation (sections 2.2–2.4). A similar thing holds for the generalization to compact Lie groups, only now it is phrased in the terms of geometric quantization. If we start with a Riemannian globally symmetric space X, we find that the prequantization of this space is $L^2(T^*X)$, where we integrate against the Liouville measure ε of the symplectic space T^*X.

In order to obtain a quantization we have to choose a polarization, which defines a subspace of the prequantized space $L^2(T^*X)$. The most easy one is the vertical polarization, which is the subspace of functions constant on the fibers of T^*X. Thus the vertically polarized space can be identified with $L^2(X)$, taking for example Riemannian measure on this space. The other choice of polarization discussed here is the Kăhler polarization. There are two flavors of this polarization: With and without half form correction. Without, the polarized space is the subspace of holomorphic functions in $L^2(T^*K, e^{-\|Y\|^2/t}\varepsilon)$, which does not give any result. The suggestion of Dan Freed was that Hall should include the half-form correction, giving the space

$$\mathcal{O}(T^*K) \cap L^2(T^*K, e^{-\|Y\|^2/t} j(Y)^{\frac{1}{2}} \varepsilon).$$

This space has the same measure as employed in equation (5.4), up to a constant multiple. Notice that this is special for the case of a compact Lie group amongst all compact symmetric spaces. The complex structure used is the same as before, only now transported from the tangent bundle to the cotangent bundle by the Riemannian structure.

The conclusion of Hall is that the map between these two polarizations, arising naturally from geometric quantization, is up to this constant multiple the same as the Segal–Bargmann transform on a compact Lie group. Therefore a scalar multiple of this first map is unitary. The reason why this does not hold for a compact symmetric space is simply that the heat kernel is not of the form (5.3). However, the heat kernel for three dimensional hyperbolic space is of this form, according to [Cha84, p. 150], which leads us to the question:

> *Question 3.c:* Does the Segal–Bargmann transform agree with the pairing map from geometric quantization on three dimensional hyperbolic space?

If we still want to reconcile the Segal–Bargmann transform on a compact symmetric space with geometric quantization, there is only one way out:

> *Question 3.d:* Is there a sensible alternative to the half-form correction, so that the pairing map from geometric quantization agrees with the Segal–Bargmann transform on a compact symmetric space?

I have to admit that this is rather a wild guess and that I expect that this approach will not be viable. It is quite well possible that we will not be able to detect positive powers of Planck's constant t by geometric quantization. This could also be the reason why the constant $e^{\|\rho\|^2 t}$ is missing. Unfortunately I do not know enough of geometric quantization to work this out. I have stated this merely because I have nowhere found a hint in this direction.

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