Master's thesis

Ergodic theorems for amenable actions

Twan Dollevoet
Department of Mathematics
Utrecht University

Supervisor: Dr. K. Dajani

20th August 2007

Abstract

Classical ergodic theory deals with the action of $\mathbf{N}$ on an arbitrary finite measure space (X, μ). It is long recognized that many of the results in ergodic theory can be generalized to more general acting groups. The appropriate class of groups has turned out to be the class of *amenable* groups. It has taken remarkably long to extend the pointwise ergodic theorem to the action of any amenable group. Although this result was extended to many particular subclasses of amenable groups, the full result was established only in [12]. The result relies on a weak maximal inequality for the action of G.
The extension of the Cesàro averages to amenable groups can be viewed as a function that is similar to the maximal function, introduced by Hardy and Littlewood. This induces an operator A^*, that assigns this maximal function to $f \in L^p(X)$. We will show that this operator is bounded, whenever $1 < p < \infty$. To do so, we will first prove the statement for the action of G on itself. Thereafter, we derive it for the action of G on every measure space, using a technique known as transference.

Acknowledgements

This master thesis marks the end of my studies Physics and Mathematics in Utrecht. There are many people who have helped me during my education. I would like to take this opportunity to thank these people.

First of all, I am very glad that I have chosen Karma Dajani as my supervisor both for my bachelor's and for my master's thesis. After a pleasant cooperation during my bachelor's thesis, working together on this project has also been very fruitful. I am especially grateful for her patience -when I was busy doing numerous other things- and for her confidence that this research would come to a good end. Without her confidence I would have lost courage long ago.

There are several co-students who have helped me with the completion of this text. In the beginning, Marco has helped me with a proof for Theorem 2.1.14. Imke, although not a mathematics student, has read the first chapters and identified many linguistic errors. Job has read later chapters and focused more on the mathematical correctness. I am also thankful to Johnson, for providing me with a copy of the book of Adams and to Manouk, for a template of this document. I cannot list all the parts that Bart has helped me with. No matter how busy he was with his own thesis, he has always agreed to help me when I had a question. It has always been very clarifying to discuss my problems with him.

I would also like to thank my other fellow students for our time during classes. Without them my time at the Minnaert Building would not have that enjoyable. I am especially thankful to Hugo, Marc, Pieter and Willem, with whom I have had much contact outside our study hours as well. My friends from my home town, Jeroen, Lonneke, Ruud, Roel, Joran, Lydia, Rudi, Sylvia and Wilbert made sure I had enough distraction during the weekends. Having fun with you all was much more pleasant than doing my homework on Saturday nights.

The unceasing support of my parents has made it possible to follow my education without many problems. Although I extended the date of my graduation many times, they always let me take my time and do things my own way. Without them, and my sister Renske, I would never had achieved this.

Thank you all,

Twan

Contents

1 Introduction **1**

2 Prerequisites **4**
2.1 Topology 4
2.1.1 Topological Properties 4
2.1.2 The weak and weak-$*$ topology 7
2.1.3 Nets 10
2.1.4 Topological Groups 11
2.2 Haar measure on locally compact groups 12
2.2.1 Classification of Haar measures 15
2.3 Functional Analysis 18
2.3.1 Functional analysis on a topological group 18
2.3.2 Functional analysis on a finite measure space 24
2.4 Probability Theory 25

3 Amenability **30**
3.1 Introduction 30
3.2 Characterizations of amenability 32
3.3 Invariant means 32
3.4 Asymptotical invariance 39
3.5 Følner's conditions 51

4 The maximal inequality **67**
4.1 Introduction 67
4.2 The maximal inequality for the action of $\mathbf{R}^n$ 68
4.3 A covering lemma for discrete groups 72
4.4 A covering lemma for non-discrete groups 78
4.5 The weak maximal inequality 82

5 Ergodic theorems **86**
5.1 The pointwise ergodic theorem 86
5.2 Strong type maximal inequalities 91

Chapter 1

Introduction

An invertible, measure-preserving transformation T, defined on a probability space $(X, \mathcal{F}, \mu)$, induces an action of $\mathbf{Z}$ on X, that is measure-preserving. These measure-preserving transformations were first studied in the context of statistical physics. If T represents the transformation from one physical state to the other, the average value of a function f on X can be computed as

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f\left(T^n x\right).$$

These averages are the basic concept in ergodic theory. Inspired by the questions raised in statistical physics, the problem whether these averages converge or not is the central subject of ergodic theory.

The mean ergodic theorem of von Neumann solves such a problem. It states that there exists for every $f \in L^2(X, \mu)$ a function $\bar{f} \in L^2(X, \mu)$, such that

$$\lim_{N \to \infty} \left\| \frac{1}{N} \sum_{n=1}^{N} f\left(T^n x\right) - \bar{f} \right\|_2 = 0.$$

The pointwise ergodic theorem, first proven by Birkhhoff, states that for every $f \in L^1(X, \mu)$, there exists a T-invariant function $\bar{f} \in L^1(X, \mu)$, such that

$$\lim_{n \to \infty} \frac{1}{N} \sum_{n=1}^{N} f\left(T^n x\right) = \bar{f}(x),$$

for μ-almost every $x \in X$. The first proofs of these convergence theorems were based on the maximal inequality. Later proofs were based on more specific properties of $\mathbf{Z}$, in particular the fact that the acting group is essentially the same as the set in which the limit is taken. In [6], one can find such a modern proof.

The maximal inequality is originally proven for $\mathbf{R}^n$. This observation has made it possible to generalize both the mean and the pointwise ergodic theorem to the action of $\mathbf{R}^n$ on a probability space $(X, \mathcal{F}, \mu)$. The next step now is to generalize ergodic theory to the action of more general groups. The natural class of groups turned out to be the class of *amenable* groups. Although many of the deep results from ergodic theory were generalized to the action of any amenable group, the pointwise ergodic theorem could only be proven in full generality in [12]. The main result here is the maximal inequality for general amenable groups, from which the pointwise ergodic theorem can be derived easily.

In this thesis we will present the pointwise ergodic theorem for amenable actions. Restricting to the action of $\mathbf{R}^n$ on itself, the Cesàro averages induce a function on X, defined by

$$A^*f(x) = \sup_{n\in\mathbf{N}} \frac{1}{\mu_L(B(x,n))} \int_{B(x,n)} f(y)d\lambda(y).$$

This function is similar to the maximal function of Hardy and Littlewood. We denote by A^* the mapping, that assigns this maximal function A^*f to $f \in L^p(X, \mu)$. The maximal inequality will be applied, to show that this mapping A^* is bounded as a sublinear operator on $L^p(X, \mu)$, whenever $1 < p \leq \infty$. For a proof of this theorem for the action of $\mathbf{R}^n$ on itself, see for example [17]. We will generalize this theorem to the action of a general amenable group.

In the next section, we will first give an introduction of amenability. There are many equivalent characterizations of amenability. The ones that will be used to prove the maximal inequality are very different from the original definition of an amenable group. We will therefore show that the properties that will be used indeed characterize amenability.

With a good understanding of amenability, we will present the maximal inequality, as proven in [12]. We will prove a covering lemma and then generalize the maximal inequality to the action on an arbitrary σ-finite measure space. This generalizes the statement in [12], that only deals with *finite* measure spaces.

In the final section, the maximal inequality will be applied to present the pointwise ergodic theorem. Furthermore, we will prove that the maximal operator $A^* : f \mapsto A^*f$ is bounded when the action of an amenable group on itself is considered. Finally, we will use a technique called *transference* to lift the statement for the action of a group on itself to the action of the group on any measure space.

Notation and conventions

Most of the notation in this text is adopted from [12]. Throughout, we denote by $(G, \mathcal{B}, m_L)$ and $(G, \mathcal{B}, m_R)$ a locally compact, Hausdorff, topological group, equipped with the left and right Haar measure, respectively, on the Borel algebra $\mathcal{B} = \mathcal{B}(\mathcal{T})$ generated by the compact sets. We will often assume furthermore that this topological group is second countable and amenable. In that case, the group admits an exhaustion by compact sets, that is generically denoted by $\{K_n\}_{n\in\mathbf{N}}$. Amenability of G is equivalent to the existence of a (tempered) Følner sequence, which will be denoted by $\{F_n\}_{n\in\mathbf{N}}$. $(X, \mathcal{F}, \mu)$ is a measure space, which is acted upon by G. All other non-standard notation is clarified at the point where it appears first.

Chapter 2

Prerequisites

In this section we will repeat some mathematical issues that will be important in later sections. Most of these subjects will be familiar for most readers. We will sometimes only state the results that are needed, giving references where one can find more details. In other places, where the material is more specific, more background and proofs will be included.

2.1 Topology

2.1.1 Topological Properties

We will restrict attention to topological spaces that are both locally compact and second countable. Furthermore, every topological space is assumed to be Hausdorff. We will now define local compactness and second countability. We will use the following definition of a neighborhood[1].

Definition 2.1.1 *Let $(X, \mathcal{T})$ be a topological space and A a subset of X. Then a neighborhood of A in X is a set B, such that $A \subset B \subset X$ and such that there exists an open set U for which $A \subset U \subset B$.*

The above definition applies to singletons also. By a neighborhood of $x \in X$, we will mean a neighborhood of $\{x\} \subset X$. Hence, a neighborhood of x in X is a set $V \ni x$, that contains an open set U that contains x. Note that every open set is a neighborhood, but not every neighborhood is open.

Local compactness is usually defined by the property that every point has a compact neighborhood. As we assume every topological space to be Hausdorff, we can use the following definition, that is equivalent for a Hausdorff space. [13, Theorem 29.2]

[1] Note that this definition is different from the one used in [13], that is used in many courses on topology.

Definition 2.1.2 *A topological space $(X, \mathcal{T})$ is called locally compact if for every neighborhood $V \subset X$ of every point $x \in X$ there exists a compact neighborhood K of x in X, such that $x \in K \subset V$.*

The second countability axiom in some sense puts restrictions on the number of open sets in a topology.

Definition 2.1.3 *A topological space $(X, \mathcal{T})$ is called second countable if there exists a countable basis for the topology $\mathcal{T}$.*

Example $\mathbf{R}^n$, equipped with the Euclidean topology is both locally compact and second countable. To show the former, take an arbitrary neighborhood V of an arbitrary point $x \in \mathbf{R}^n$. By definition, there exists an open ball $U = B(x, \epsilon)$ such that $x \in U \subset V$. The closed ball $\overline{B(x, \frac{\epsilon}{2})}$ is compact and contained in U, and so also in V. It is a neighborhood of x as it contains the open set $B(x, \frac{\epsilon}{2})$. This proves that $\mathbf{R}^n$ is locally compact. To show it is second countable, take as a basis the collection $\{B(q, \frac{1}{n}) : q \in \mathbf{Q}, n \in \mathbf{N}\}$.

Example $\mathbf{R}^n$, equipped with the discrete topology is not second countable. To show this, note that $\{x\} \in \mathcal{T}$ for all $x \in \mathbf{R}^n$. Therefore, a basis for $\mathcal{T}$ should contain, for every $x \in \mathbf{R}^n$, a set B_x such that $x \in B_x \subset \{x\}$. It follows a basis should contain all sets $\{x\}$ and thus cannot be countable. The topology $\mathcal{T}$ is therefore not second countable.

We already said that second countability in some sense reduces the number of open sets in a topological space X. It also implies that X admits an exhaustion by compact sets.

Definition 2.1.4 *Let $(X, \mathcal{T})$ be a topological space. An exhaustion by compact sets for X is a sequence of compact sets $\{K_n\}_{n \in \boldsymbol{N}}$, such that*

$$X = \bigcup_{n \in \boldsymbol{N}} K_n$$

and $K_n \subset \mathit{int}\, K_{n+1}$ for all $n \in \boldsymbol{N}$.

To show that a second countable, locally compact, topological space admits an exhaustion by compact sets, we will use the following equivalent characterization of local compactness.

Lemma 2.1.5 *Let $(X, \mathcal{T})$ be a Hausdorff topological space. Then X is locally compact if and only if for every $x \in X$ and every open neighborhood U of x, there exists an open neighborhood V of x, such that the closure $\overline{V}$ of V is compact and contained in U.*

Proof Assume X is locally compact. Select an arbitrary element $x \in X$ and an open neighborhood U of x. By definition of local compactness, there exists a compact neighborhood K of x in U. By definition, there exist an open set V and a compact set K, such that

$$x \in V \subset K \subset U.$$

Now, as X is Hausdorff, compactness of K implies that K is closed. $V \subset K$ now implies that $\overline{V} \subset \overline{K} = K$. We conclude, that $\overline{V}$ is a closed set in the compact set K, hence it is compact. The set V now satisfies the claims. The reverse implication holds by definition. $\square$

Theorem 2.1.6 *If $(X, \mathcal{T})$ is a locally compact, second countable, topological space, then X admits an exhaustion by compact sets.*

Proof Let $\mathcal{U}$ be a countable basis for the topology of X, that exists as X is second countable. Define the subcollection $\mathcal{V} \subset \mathcal{U}$, that consists of the sets $U \in \mathcal{U}$, for which there exists an open set V, such that $\overline{V}$ is compact and $U \subset V$. We will show that $\mathcal{V}$ is a basis also. Select therefore $x \in X$ and an open neighborhood A of x arbitrarily. As X is locally compact, the previous lemma gives us an open set V, such that $\overline{V}$ is compact and $\overline{V} \subset A$. As the set $\mathcal{U}$ is a basis for X, there exists an $U \in \mathcal{U}$, such that $x \in U_n \subset V$. Therefore, $V \in \mathcal{V}$. We conclude that $\mathcal{V}$ is a basis for the topology of X. As a subcollection of the countable collection $\mathcal{U}$, it is countable as well.
Enumerate now the countable basis $\mathcal{V}$ as $\{V_n\}_{n \in \mathbf{N}}$. We will define the exhaustion inductively. Define $K_1 = \overline{V}_1$. As $\mathcal{V}$ is a basis, $\mathcal{V}$ covers K_1. By compactness, there exists an N_1, such that the subcollection $\{V_n\}_{n=1}^{N_1}$, covers K_1. Define now

$$K_2 = \overline{V_1 \cup \ldots \cup V_N} = \overline{V_1} \cup \ldots \cup \overline{V_N}.$$

By construction, K_2 is compact and $K_1 \subset \text{int } K_2$. By compactness of K_2, we can select an $N_2 > N_1$, such that $\{V_n\}_{n=1}^{N_2}$ covers K_2. Define K_3 as the closure of the union of this finite collection. Then K_3 is compact and K_2 is contained in its interior. Continuing in this fashion we find the exhaustion by compact sets for X. $\square$

The exhaustion by compact sets has a very important property. By definition, as every K_n is contained in *int* K_{n+1}, the sets *int* K_{n+1} define a countable collection of open sets, that cover X. This implies, for every compact set $K \subset X$, that there exists an $N \in \mathbf{N}$, such that $K \subset$ *int* K_{N+1}. By definition, K is contained in the compact set K_{N+1}. It follows that for every compact set K, there exists a set K_N from the exhaustion, that contains K. Note that the condition that $K_n \subset$ *int* K_{n+1} is really necessary here.

For arbitrary topological spaces, this exhaustion by compact sets is stronger than the existence of a sequence of compact sets, that only cover X. However, in the case of a locally compact group, these are equivalent. [5, Proposition I.9.15]

Theorem 2.1.7 *Let $(X, \mathcal{T})$ be a locally compact topological space. Then the existence of an exhaustion by compact sets is equivalent to the existence of a sequence of compact sets $\{K_n\}_{n \in \boldsymbol{N}}$, that cover X.*

A locally compact group that admits a covering sequence of compact sets is called $\sigma-$locally compact. By the above theorem, σ-locally compact groups admit an exhaustion by compact sets. Furthermore, every second countable, locally compact, topological group is σ-locally compact.

2.1.2 The weak and weak-$*$ topology

We will now consider topological vector spaces. Recall that a vector space over $\mathbf{R}$ is a linear space, in which elements can be multiplied by every $r \in \mathbf{R}$. The obvious definition for a topological vector space is the following. One could say, that the topology is compatible with the linear structure.

Definition 2.1.8 *Let E be a vector space over $\boldsymbol{R}$. Then the topological space $(E, \mathcal{T})$ is a topological vector space, if*

- *the mapping $E \times E \to E : (e, e') \mapsto e + e'$ is continuous,*
- *for every $r \in \boldsymbol{R}$, the mapping $E \to E : e \mapsto re$ is continuous.*

On a topological vector space $(E, \mathcal{T})$, one can define the weak topology. The weak topology depends on the original topology via the topological dual space. The topological dual space X^* of X consists of all continuous linear functionals on X. The weak topology is by definition the smallest topology on E, for which every functional from the dual space (with respect to the original topology) remains continuous.

Definition 2.1.9 *Let E be a topological vector space over $\boldsymbol{R}$. Let $F = E^*$ be the topological dual of E, that is the space of continuous, linear functionals from E to $\boldsymbol{R}$. Then the weak topology $\mathcal{W}$ on E is the weakest topology on E, such that every $f \in F$ is continuous.*

Remark that the weak topology can be constructed only on a *topological* vector space. This is because one needs to know which linear functionals are continuous. By definition, the weak topology is contained in the original topology on E, which is therefore conventionally called the strong topology.

Theorem 2.1.10 *Let $(E, \mathcal{T})$ be a topological vector space. Denote $(E, \mathcal{W})$ for E equipped with the weak topology. Then*

- *the topology $\mathcal{W}$ is generated by the open sets*
$$U(e, \epsilon, f_1, \ldots, f_n) = \{e' \in E \,:\, |f_i(e') - f_i(e)| < \epsilon, 1 \le i \le n\},$$
- *a sequence $\{e_i\}$ in E converges to e in $\mathcal{W}$ if and only if for all $f \in F$*
$$\lim_{i \to \infty} f(e_i) = f(e),$$

- *$(E, \mathcal{W})$ is a locally convex, topological vector space.*

By a similar construction it is possible to define a topology on the dual of a topological vector space.

Definition 2.1.11 *Let E be a topological vector space over $\boldsymbol{R}$. Denote $F = E^*$ for the dual space of E, consisting of all continuous linear functionals from E to $\boldsymbol{R}$. By setting*

$$e(f) = f(e) \quad \forall e \in E, f \in F,$$

one associates to every element $e \in E$ a linear functional on F. The weak-$*$ topology is the weakest topology on F for which every $e \in E$ is continuous.

Note that the dual space of a topological Banach space is naturally equipped with the topology induced by the operator norm. In that case every element $e \in E$ is continuous. To see this, recall that the operator norm of $f \in F^*$ is given by

$$\|f\|_{\mathrm{op}} = \sup\{f(e) : e \in E, \, \|e\| \leq 1\}.$$

By linearity, it follows that if $\|f\|_{\mathrm{op}} \leq 1$, then

$$f(e) = \|e\| f\left(\frac{e}{\|e\|}\right) \leq \|e\|$$

for all $e \in E$. Consider now the element $e \in E$ as a linear function on F. Its operator norm is then given by

$$\begin{aligned} \|e\|_{\mathrm{op}} &= \sup\{e(f) : f \in F, \, \|f\|_{\mathrm{op}} \leq 1\} \\ &= \sup\{f(e) : f \in F, \, \|f\|_{\mathrm{op}} \leq 1\} \leq \|e\|. \end{aligned}$$

We see that every $e \in E$ is bounded as a functional on F and hence continuous. Therefore, the weak-$*$ topology is weaker than the topology induced by the norm. The latter is therefore also called the strong topology.

Theorem 2.1.12 *Let $F = E^*$ be the continuous dual space of a topological vector space E. Equip F with the weak-$*$ topology. Then*

- *the weak-$*$ topology is generated by the open sets*

$$U(f, \epsilon, e_1, \ldots, e_n) = \{f' \in F : |e_i(f') - e_i(f)| < \epsilon, 1 \leq i \leq n\},$$

- *a sequence $\{f_i\}$ converges to f in F, if and only if for every $e \in E$, it holds that*

$$\lim_{i \to \infty} f_i(e) = f(e),$$

- *F is a locally convex, topological vector space.*

When the dual space F of a topological vector space E is equipped with the weak-$*$ topology, the dual space of F is simply equal to E.

Theorem 2.1.13 *If $\lambda : F = E^* \to \boldsymbol{R}$ is linear and continuous with respect to the weak-$*$ topology, it can be identified with $e \in E$.*

Proof Let $0 < \epsilon < 1$ be given. By definition of the weak-$*$ topology, there exist $\delta > 0$ and $\{e_i\}_{i=1}^n \subset E$, such that

$$\lambda\big(\{f \in F \,:\, |f(e_i)| \leq \delta, 1 \leq i \leq n\}\big) \subset B(0, \epsilon). \tag{2.1}$$

Define now, for these $\{e_i\}$, the linear functional

$$T : F \to \mathbf{R}^{n+1} : f \mapsto (\lambda(f), f(e_1), \ldots, f(e_n)).$$

As F is a linear space, $T(F)$ is a linear subspace of $\mathbf{R}^{n+1}$. As $\mathbf{R}^{n+1}$ is finite dimensional, it follows $T(F)$ is closed. In particular it is a Hilbert space. Assume now that $l \in F$ satisfies $l(e_i) = 0$ for all i. Then by (2.1) it holds that $|\lambda(l)| < \epsilon < 1$. Therefore, the vector $\hat{r}_1 = (1, 0, \ldots, 0) \notin T(F)$. As $T(F)$ is closed, the distance from $\hat{r}_1$ to $T(F)$ is positive. By the Perpendicular Principle [20, Theorem 2D] we have a vector $\alpha \in \mathbf{R}^{n+1}$, such that $\alpha \cdot T(F) = 0$. We can choose α such that $\hat{r}_1 \cdot \alpha \neq 0$. To see this, assume the contrary. Then, for every $\alpha \in T(F)^\perp$, we have $\hat{r}_1 \perp \alpha$. This implies $\hat{r}_1 \in (T(F)^\perp)^\perp = T(F)$, as $T(F)$ is closed. This contradiction proves we can choose a vector $\alpha = (\alpha_0, \ldots, \alpha_n)$ perpendicular to $T(F)$, for which $\hat{r}_1 \cdot \alpha \neq 0$. This implies $\alpha_0 \neq 0$. It follows, for all $f \in F$, that

$$0 = \alpha \cdot T(f) = \alpha_0 \lambda(f) + \sum_{i=1}^{n} \alpha_i f(e_i).$$

We conclude that

$$\lambda(f) = f\left(-\sum_{i=1}^{n} \frac{\alpha_i}{\alpha_0} e_i\right).$$

This proves the claim. □

We can also consider the dual space of a product of topological vector spaces. Let therefore I be an arbitrary index set, and E_i be a topological vector space for each $i \in I$. Then $E = \prod_{i \in I} E_i$ is naturally equipped with the product topology, that is defined as the weakest topology, such that every projection $\pi_i : E \to E_i$ is continuous. We have the following about the topological dual space of E.

Theorem 2.1.14 *Let I be an arbitrary index set and E_i a topological vector space for every $i \in I$. Define $E = \prod_{i \in I} E_i$. As a set, we have $E^* = \bigoplus_{i \in I} (E_i)^*$. Furthermore, the product topology on E equals the weak topology on E.*

Proof We will first prove that the direct sum of the individual dual spaces is indeed the topological dual of E. This holds independent of the topology on the individual coordinate spaces E_i. Note that the product topology on E assures that every projection mapping $\pi_i : E \to E_i$ is continuous. Choose now an arbitrary index i and an element $f \in E_i^*$. We can extend f to the whole of E, by setting $\tilde{f} : E \to \mathbf{R} : e \mapsto f(e_i)$. Then $\tilde{f} = f \circ \pi_i$, which proves that $\tilde{f}$ is continuous. Taking finite sums of such elements $\tilde{f}$ shows us that $\bigoplus_{i\in I}(E_i)^*$ is contained in E^*. Assume now that $\bigoplus_{i\in I} E_i^* \setminus E^* \neq \emptyset$. Choose f from the difference. Then there is an infinite set $\tilde{I} \subset I$, such that $f \circ \pi_i$ is non-zero on E_i, for all $i \in \tilde{I}$. This implies one can choose in every coordinate space E_i an element e_i, such that $f(e_i) = 1$. As $\tilde{I}$ is infinite, there is a injection $\iota : \mathbf{N} \to \tilde{I}$. Define now the sequence $\{\tilde{e}_n\}_{n\in\mathbf{N}}$, by setting

$$\tilde{e}_n = \begin{cases} e_i & i = \iota(n) \\ 0 & \text{otherwise.} \end{cases}$$

Then the sequence $\tilde{e}_n$ converges to 0 in the product topology. However, by construction, $f(\tilde{e}_n) = 1$ for all $n \in \mathbf{N}$. This proves that f is not continuous. We conclude $E^* = \bigoplus_{i\in I} E_i^*$.

We will now proceed with the second statement. Assume first that E is equipped with the weak topology. Recall that this is the weakest topology, such that every $f \in E^*$ is continuous. Choose an index $i \in I$ and $f \in E_i^*$. Extend f to the whole of E, by setting again $\tilde{f} : e \mapsto f(e_i)$. Then $\tilde{f} = f \circ \pi_i$. As both f and $\tilde{f}$ are continuous, so is π_i. This proves that every projection mapping is continuous. As the product topology is the weakest topology with this property, the weak topology on E contains the product of the topologies. This implies, that the product topology is weaker than the weak topology. As by construction the weak topology is weaker than the product topology, the two topologies must be equal. □

For a sequence to converge in the weak topology it is sufficient to converge pointwise. This makes the weak topology more tractable than stronger topologies. We will apply this feature of the weak topologies often in the next chapter.

A classical result from Functional Analysis on the weak-$*$ topology is the following. It is due to Alaoglu. [10, Theorem B.25]

Theorem 2.1.15 *Let E be a normed vector space. Equip E with the topology induced by the norm. Then the operator norm is a norm on the dual space $F = E^*$. The unit ball B in this dual is weakly-$*$ compact.*

2.1.3 Nets

In metric spaces, one can find for every point in the closure of a subset a sequence that converges to that point. If the space is not metrizable, there is in general no such sequence. However, generalizing the notion of sequences to nets, such statements do hold. To define nets, directed sets are needed.

Definition 2.1.16 *A directed set is a nonempty set I with a binary relation $\leq$, such that for all $x, y, z \in I$,*

- *$x \leq x$ (reflexivity),*
- *$x \leq y$ and $y \leq z$ imply $x \leq z$ (transitivity),*
- *there exists $w \in I$, such that $x \leq w$ and $y \leq w$ (directedness).*

Example The set of natural numbers, $\mathbf{N}$, with the ordinary relation $\leq$ is a directed set. The same holds for any other totally ordered set.

Example For a point x in a topological space X, the set of open sets containing x constitute a directed set by reverse inclusion. Directedness follows from the fact that the intersection of two open sets containing x is again an open set containing x.

Directed sets are used in the definition of a net.

Definition 2.1.17 *A net in a topological space X is a function from some directed set to X.*

As said before, a net generalizes the notion of a sequence in a metric space (or general first countable space) to an arbitrary topological space. We denote $\overline{K}$ for the closure of K. The following can be found in [8] [2].

Theorem 2.1.18 *Let X be a topological space. Then we have the following.*

- *If $k \in \overline{K} \setminus K$, then there exists a net $\{k_i\}$ in K converging to k.*
- *If K is a compact space, every net in K has a convergent subnet.*

2.1.4 Topological Groups

A group is the simplest algebraic structure. A topological group is a group with a topology that respects the algebraic properties.

Definition 2.1.19 *A topological space $(G, \mathcal{T})$ is a topological group if G is a group and the multiplication map $G \times G \to G$ and the inversion map $G \to G$ are continuous.*

Example $\mathbf{R}^n$ is both a topological group under addition and a topological vector space over $\mathbf{R}$. Recall that the topology on $\mathbf{R}^n$ is defined by the Euclidean norm. Take now an arbitrary open, non-empty set U. Denote the addition mapping by $(+)$ and choose an arbitrary point $(x, y) \in (+)^{-1}(U)$. As U is open, there exists an open ball $B(x + y, \epsilon) \subset U$. By the triangle inequality $(+)(B(x, \frac{\epsilon}{2}) \times B(y, \frac{\epsilon}{2})) = B(x, \frac{\epsilon}{2}) + B(y, \frac{\epsilon}{2}) \subset B(x+y, \epsilon) \subset U$. This proves $(+)$ is continuous. Choose now $r \in \mathbf{R}$. Choose $x \in (r)^{-1}(U)$. Then there exists a ball $B(rx, \epsilon) \subset U$. This implies that $r(B(x, \frac{\epsilon}{r})) \subset B(rx, \epsilon) \subset U$. This proves

[2] The first property is exactly Theorem 1.5.2. The second statement can be obtained from Theorems 1.5.1 and 3.1.12.

that every mapping (r) is continuous. Applying this to $r = -1$ proves that the inversion map is continuous. This proves all claims.

For the sake of simplicity, we will show now that there exist an exhaustion by compact sets for the topological group G if it is locally compact and second countable, such that every set F_n in the exhaustion contains the identity. This can easily be established by shifting i, such that $F_1 \neq \emptyset$ and redefining $F'_n = f^{-1}F_n$, for some $f \in F_1$.

Theorem 2.1.20 *Let G be a locally compact, second countable, topological group. Then G admits an exhaustion by compact sets $\{K_n\}_{n\in\boldsymbol{N}}$, that all contain the identity.*

The following theorems summarize some important topological properties of a topological groups G. For the proofs we refer to [10]

Theorem 2.1.21 *[10, Theorem 4.4] Let G be a topological group. Then the product of two compacta in G is compact.*

Theorem 2.1.22 *[10, Theorems 4.5 and 4.6] Let G be a topological group and W' an open neighborhood of e in G. Then there exists an open neighborhood W of e, such that $WW^{-1} \subset W'$.*

Theorem 2.1.23 *[10, Theorem 4.9] Let G be a topological group, U an open neighborhood of e in G and K a compact subset of G. Then there exists a neighborhood V of e in G, such that $xVx^{-1} \subset U$ for all $x \in K$.*

Theorem 2.1.24 *Let G be a topological group. C be a compact set and $U \supset C$ an open set. Then there exists a symmetric, open set V, containing the identity, such that $CV \subset U$.*

Proof Let the sets C and U be given. Define the directed set $\mathcal{V}$ of all open sets that contain the identity. Assume that there is no open set V, such that $CV \subset U$. In particular, it then holds for all $V \in \mathcal{V}$, that $CV \setminus U \neq \emptyset$. Select for every $V \in \mathcal{V}$ an element $c(V)v(V)$ from this difference. By construction, the net $v(V)$ converges to e. Furthermore, the net $c(V)$ has a converging subnet, as C is compact. Let c be the limit of this subnet. Then $c = ce = \lim_{V\in\mathcal{V}} c(V)v(V)$. As every point $c(V)v(V)$ is in the closed set $G \setminus U$, so must be the limit c. This contradicts the fact that $C \subset U$. The claim follows if we take $V' = V \cap V^{-1}$. □

The properties listed here will be used very often in the rest of this text. Therefore, we will mostly omit explicit references.

2.2 Haar measure on locally compact groups

An important property of locally compact, topological groups is the existence of a measure, that is compatible with the topology.

Definition 2.2.1 *Let $(G, \mathcal{T})$ be a topological space. We define the σ-algebra $\mathcal{B}$ as the smallest σ-algebra that contains all compact subsets of X. This σ-algebra is called the Borel-algebra. The sets in this algebra are called Borel sets.*

We chose the compact sets to generate the Borel algebra, because the compact sets are the natural starting point to define the Haar measure that will be introduced later. This implies, that there could be open sets that are not Borel. Therefore, generally not every open set $U \in \mathcal{T}$ is measurable. As every set $\{g\}$ is compact, every singleton is measurable. These problems occur even when the group is assumed to be locally compact.

Fortunately, when the group G is assumed to be σ-locally compact, every open set belongs to the Borel algebra [9, Theorem 51.A]. Note that a locally compact, second countable group is σ-locally compact. We conclude, that in our case every open set is a Borel set. In a Hausdorff topological space, every compact set is closed, and hence the complement of an open set. This shows that every compact set is contained in the σ-algebra generated by the open sets. We conclude, that for σ-locally compact groups, it holds that

$$\sigma(\mathcal{T}) = \mathcal{B}(\mathcal{T}).$$

We will continue to work with the Borel algebra as generated by the compact sets, but the reader should recall that in the cases we consider, this Borel algebra equals the σ-algebra generated by the open sets.

Alfred Haar showed that every locally compact group G admits a left invariant measure on the Borel-algebra. He also showed, that this measure is unique up to multiplication, when it behaves sufficiently regular.

Definition 2.2.2 *Let $(G, \mathcal{B}, m)$ be a measure space, where X is a locally compact topological group and $\mathcal{B}$ the corresponding Borel-algebra. The measure m is outer regular on $B \in \mathcal{B}$, if*

$$m(B) = \inf\{m(U) : U \supset B, U \text{ open and Borel}\},$$

and inner regular on $B \in \mathcal{B}$, if

$$m(B) = \sup\{m(K) : K \subset B, K \text{ compact}\}.$$

Haar measure on G is essentially the only measure, that is sufficiently regular and left-invariant [9, Section 58].

Theorem 2.2.3 *Let $(G, \mathcal{T})$ be a locally compact group. Define the Borel σ-algebra $\mathcal{B} = \mathcal{B}(\mathcal{T})$. Then there is, up to multiplication, a unique measure m_L on $(X, \mathcal{B})$ for which*

- *m_L is outer regular on all Borel sets $B \in \mathcal{B}$*
- *m_L is inner regular on all Borel sets $B \in \mathcal{B}$*

- *$m_L(K) < \infty$ for all compact sets $K \subset G$*
- *m_L is left translation invariant, that is, for all $g \in G$ and $B \in \mathcal{B}$, it holds that $m_L(gB) = m_L(B)$.*

Such a measure m_L is called left Haar measure. Right Haar measures are defined similarly.

Example The restriction of the Lebesgue measure to the Borel sets on $\mathbf{R}^n$ is a left Haar measure as well as a right Haar measure.

Example Counting measure on an arbitrary group with the discrete topology is both a left and a right Haar measure.

The Haar measure induces a homomorphism of G into $\mathbf{R}_+$ under multiplication. This homomorphism is called the modular function.

Definition 2.2.4 *Let m_L be a left Haar measure on a locally compact, topological space $(G, \mathcal{T})$. Then for every $t \in G$, $m_L^t : A \mapsto m_L(At)$ is also a left Haar measure. This implies $m_L^t = \Delta(t)m_L$, for some constant $\Delta(t)$. We define $\Delta : t \mapsto \Delta(t)$ as the modular function on G. We say G is unimodular if the modular function is identically equal to 1.*

We have defined the modular function from a left Haar measure, but a right Haar measure could have been used also. All give rise to the same modular function. To see this, let m_L and m_R be a left and right Haar measure on a locally compact, topological group $(G, \mathcal{T})$. Define $m_R^{-1}(S) = m_R(S^{-1})$. Then m_R^{-1} is a left translation invariant Haar measure, and hence $m_R^{-1} = km_L$, for some positive constant k. By definition of the modular function, the following equality holds.

$$\begin{aligned} m_R(A) &= m_R^{-1}\left(A^{-1}\right) \\ &= km_L(A^{-1}) \\ &= k\Delta(t)^{-1}m_L(t^{-1}A^{-1}) \\ &= k\Delta(t)^{-1}m_L((At)^{-1}) \\ &= \Delta(t)^{-1}m_R(At). \end{aligned}$$

It follows, that

$$m_L(At) = \Delta(t)m_L(A) \quad \text{and} \quad m_R(tA) = \Delta(t)^{-1}m_R(A).$$

Theorem 2.2.5 *Let $(G, \mathcal{T})$ be a locally compact, topological group. Then the modular function Δ is a continuous homomorphism.*

Proof As every Haar measure is a positive function, the modular function must be positive as well. We will first show that it is a homomorphism. That $\Delta(e) = 1$ is obvious. Let now $g, g' \in G$ and $A \in \mathcal{B}(\mathcal{T})$ be given. Then

$$\Delta(gg')m_L(A) = m_L(gg'A) = \Delta(g)m_L(g'A) = \Delta(g)\Delta(g')m_L(A).$$

This proves that the modular function respects the algebraic structure. We conclude it is a homomorphism. To show that it is continuous, let an arbitrary compact set $C \subset G$ and $\epsilon > 0$ be given. By outer regularity, there exists an open set $C \subset U \subset G$, such that $m_L(U) \leq (1+\epsilon)m_L(C)$. By Theorem 2.1.24, there exists an symmetric open set V, such that $CV \subset U$. This implies, for every $x \in V$, that

$$\Delta(x)m_L(C) = m_L(Cx) \leq m_L(U) \leq (1+\epsilon)m_L(C)$$
$$\frac{m_L(C)}{\Delta(x)} = m_L\left(Cx^{-1}\right) \leq m_L(U) \leq (1+\epsilon)m_L(C).$$

We conclude that

$$\frac{1}{1+\epsilon} \leq \Delta(x) \leq (1+\epsilon).$$

This proves that Δ is continuous at the identity. Let now an arbitrary element $g \in G$ be given. To show that Δ is continuous at g, let $\epsilon > 0$ be given. Define $\epsilon' = \frac{\epsilon}{\Delta(g)}$. Let V be a set such that $\Delta(v) < \epsilon'$ for all $v \in V$. Then it holds for every $v \in V$, that

$$|\Delta(gv) - \Delta(g)| = \Delta(g)|\Delta(v) - \Delta(e)| < \Delta(g)\epsilon' = \epsilon.$$

This proves that Δ is continuous. □

We will mostly consider locally compact, second countable groups G with left Haar measure m_L. Among the set of right Haar measures, we will denote from now m_R for that right Haar measure, that satisfies $m_R(A) = m_L(A^{-1})$. Put differently, we will conventionally choose left and right Haar measure m_L and m_R, such that

$$m_L(A) = m_R(A^{-1}), \quad m_R(A) = m_L(A^{-1}), \quad \text{for all} \quad A \in \mathcal{B}(\mathcal{T}).$$

It can easily be seen that $\Delta = 1$ for Abelian groups. It holds for general unimodular groups G, that $m_L = m_R$. To show this, remark first that by definition of the modular function the measures m_L and m_R are both left and right invariant. By uniqueness of the Haar measure, $m_L = km_R$. Consider now any compact, symmetric neighborhood U in G. By construction $0 < \mu(U) < \infty$. It holds that

$$m_L(U) = km_R(U) = km_L(U^{-1}) = km_L(U).$$

We conclude that $k = 1$, so that $m_L = m_R$. Summarizing, in a unimodular group, every Haar measure is both left and right invariant, and $m_L(U) = m_L(U^{-1})$. A measure that satisfies the last equation is said to be inversion invariant.

2.2.1 Classification of Haar measures

For a locally compact, second countable, topological group G, any left Haar measure on the Borel algebra will turn G into a measure space $(G, \mathcal{B}(\mathcal{T}), m_L)$.

The existence of an exhaustion by compact sets implies that the measure m_L is σ-finite. We will show in this section that the measures m_L so obtained are either counting measures or non-atomic measures.

To classify the measure spaces, we will distinguish the cases where singleton have positive or zero measure. We will need that every non-empty open Borel set has positive measure. To show this, let an arbitrary non-empty open Borel set U be given, and assume that $m_L(U) = 0$. Fix $u \in U$ and choose any compact set K. As multiplication is a continuous function, the set gU is an open Borel set, with measure $m_L(gU) = m_L(U) = 0$, for all $g \in G$. By compactness of K, the cover

$$\{gU\}_{g \in G}$$

of K has a finite subcover. Let this subcover be indexed by $\tilde{G}$. Then it holds that

$$m_L(K) \leq m_L\left(\bigcup_{g \in \tilde{G}} gU\right) \leq \sum_{g \in \tilde{G}} m_L(gU) = 0.$$

As every compact set has zero measure, inner regularity now implies that $m_L = 0$. The property of a Haar measure that it is non-zero implies that every non-empty open Borel set has positive measure.

Assume now first that $\mu(\{g\}) > 0$ for some g. Then, by left invariance, every singleton has positive measure. Denote $\epsilon = m_L(\{g\})$. For every countable set $H \subset G$, it holds that

$$m_L(H) = \sum_{g_i \in H} m_L(\{g_i\}) = \epsilon \sum_{g_i \in H} 1 = \epsilon|H|,$$

where $|H|$ denotes the number of elements of H. It follows m_L/ϵ is the counting measure on G. Choose now $g \in G$ arbitrarily and assume that $\{g\}$ is not open. Then every open set that contains g has at least measure 2ϵ. This would contradict outer regularity of the Haar measure, which implies that $\{g\}$ is open. This implies that every subset of G is open, so that G is equipped with the discrete topology. As any base for this topology should contain all singletons, second countability implies G is countable. Remark that the counting measure is unimodular, and therefore both left and right invariant.

Assume now that $m_L(\{g\}) = 0$ for all $g \in G$. We showed that every open set has positive measure. Outer regularity of the set $\{g\}$ for the left Haar measure now implies that for every $n \in \mathbf{N}$, there exists an open Borel set $U_n^g \ni g$, such that

$$0 < m_L(U_n^g) \leq \frac{1}{n}. \tag{2.2}$$

We will prove that this implies that m_L is non-atomic.

To do so, assume the contrary, namely that μ is an atomic measure. Then there exists an atom A. By definition, A is measurable set, such that $\alpha = m_L(A) > 0$ and every subset of A has measure 0 or α. Take now an exhaustion by compact sets $\{K_n\}_{n\in\mathbf{N}}$ for G. Such a sequence exists by Theorem 2.1.20. Then the sequence $\{K_n \cap A\}_{n\in\mathbf{N}}$ forms an increasing sequence of measurable sets that converge to A. Therefore,

$$\lim_{n\to\infty} m_L(K_n \cap A) = m_L(A) = \alpha.$$

As every subset of A has measure 0 or α, there must be an $N \in \mathbf{N}$, such that for all $n \geq N$, $m_L(K_n \cap A) = \alpha$. Define $K = K_N$ and $B = \mathit{int}\, K_{N+1}$. As the sequence $\{K_n\}_{n\in\mathbf{N}}$ forms an exhaustion, it holds that $K \subset B \subset K_{N+1}$. It follows B has finite measure $m_L(B) = \beta$.

We will now prove that for any $M \in \mathbf{N}$, there exists a non-empty, open set U, such that B contains M disjoint left translates of U in B. Without loss of generality, we can assume $e \in B$. Select a natural number $n > M/m_L(B)$ and define the set $U_0 = U_n^e \cap B$ from (2.2). By local compactness, there exists an open set U_1 and a compact set K', such that $e \in U_1 \subset K' \subset U_0$. For $m \leq M$, we will recursively define non-empty open sets $U_m \subset U_1$ that contain e and admit m disjoint left translates in B. For $m = 1$ we just take U_1. Suppose now that for $m < M$ an open neighborhood $U_m \subset U_1$ of e is given, such that B contains m disjoint left translates $\{g_i U_m\}_{i=1}^m$. We will construct a smaller set U_{m+1} that allows $m+1$ disjoint translates in B. As $K' \subset U_0 = U_n^e$ and $m \leq M < m_L(B)n$, it holds that

$$m_L\left(\bigcup_{i=1}^m g_i K\right) \leq \sum_{i=1}^m m_L(g_i K) = m m_L(K) \leq m m_L(U_0) < \frac{m}{M} m_L(B) < m_L(B).$$

Define the set

$$C = B \setminus \bigcup_{i=1}^m g_i K.$$

As the complement of a finite union of closed sets, the set C is open. As its measure is positive, it is non-empty. By definition, there exists a $c \in C$. Define now $U_{m+1} = U_m \cap c^{-1}C$. It is easily seen to be an open set that contains e. By construction the sets $\{g_i U_{m+1}\}_{i=1}^m$ are mutually disjoint. Furthermore, we have for $1 \leq i \leq m$, that $g_i U_{m+1} \subset g_i K$, while $cU_{m+1} \subset C$. As the sets $g_i K$ are disjoint from C by definition, the claim holds for $m+1$, if we define $g_{m+1} = c$.

We will now return to the proof of non-atomicity. Recall that we have an atom $A \cap K$, contained in the compact set K and that K is contained in the open set B. Choose any $M > \beta/\alpha$. By the above, there exists an open set $U \ni e$, such that B contains M disjoint, left translates of U. The sets $\{gU\}_{g\in G}$ form an open covering of the compact set K. Therefore, there exists a finite subcovering $\{g_i U\}_{i\in I}$, that cover K. As $A \cap K$ is an atom, there must be at least one $i^* \in I$, such that $A \cap g_{i^*}U$ has measure α. This implies that the set U has measure $m_L(U) \geq \alpha$. As B admits M disjoint left translates of U, we have

$$m_L(B) \geq M m_L(U) > \frac{\beta}{\alpha} m_L(U) \geq \beta = m_L(B).$$

This contradiction proves that the measure m_L cannot have an atom. By definition, m_L is non-atomic.

We showed in this section that the Haar measure spaces $(G, \mathcal{B}, m_L)$ can be classified in two classes. On the one hand, if m_L is the counting measure on G, then G is a countable group and $\mathcal{B} = \mathcal{P}(G)$. On the other hand, if m_L is not a counting measure, then m_L is non-atomic.

2.3 Functional Analysis

2.3.1 Functional analysis on a topological group

Let G be a second countable, locally compact group. The left Haar measure turns G into a measure space. Therefore we can define the Lebesgue integral on G and consider the vector spaces $L^p(G) = L^p(G, \mathcal{B}, m_L)$ of functions for which the $p-$th power of the absolute value is integrable. The mapping

$$\|f\|_p = \left(\int_G |f|^p(g) dm_L(g) \right)^{\frac{1}{p}}$$

defines a semi-norm on $L^p(G)$. By taking the Kolmogorov quotient, that identifies functions f and f' when $\|f - f'\|_p = 0$, one obtains a normed vector space. We will from now always this Kolmogorov quotient by $L^p(G)$, Notice that $L^p(G)$ is a Banach space over $\mathbf{R}$ [10, Chapter 12].

We will put some further restrictions on f. A function $f \in L^p(G)$ is called positive, written as $f \geq 0$, if

$$\int_A f(g) dm_L(g) \geq 0, \quad \text{for all } A \in \mathcal{B}(G).$$

We denote $P^1(G)$ as the set of all positive functions in $L^1(G)$ that have unit norm,

$$P^1(G) = \left\{ f \in L^1(G) \,:\, f \geq 0, \int_G f(g) dm_L(g) = 1 \right\}.$$

The modular function of G induces a norm-preserving automorphism on $L^1(G)$ and $P^1(G)$. This automorphism is given by

$$f \mapsto f^* = \left(g \mapsto f\left(g^{-1}\right) \Delta\left(g^{-1}\right)\right).$$

It is easily verified that this automorphism respects the vector space structure of $L^1(G)$. To prove it is measure-preserving, we will show that the modular function is in fact the Radon-Nikodym derivative.

Theorem 2.3.1 *Let G be a locally compact, second countable group. Then the modular function on G equals the Radon-Nikodym derivative:*

$$\Delta(g) = \frac{dm_L(g)}{dm_R(g)}.$$

Proof The modular function induces a measure on $(G, \mathcal{B})$, by

$$\tilde{m_L} : A \mapsto \int_G \mathbf{1}_A(g)\Delta(g)dm_R(g).$$

This measure is easily seen to be left invariant. Let $a \in G$ be given. Then

$$\begin{aligned}\tilde{m_L}(aA) &= \int_G \mathbf{1}_{aA}(g)\Delta(g)dm_R(g) \\ &= \int_G \mathbf{1}_A\left(a^{-1}g\right)\Delta(g)dm_R(g) \\ &= \int_G \mathbf{1}_A(g)\Delta(ag)\,dm_R(ag) = \int_G \mathbf{1}_A(g)\Delta(g)dm_R(g).\end{aligned}$$

We will now prove that $\tilde{m_L}$ is regular. To do so, let $\epsilon > 0$ and $A \in \mathcal{B}$ be given. Let $\{K_n\}_{n\in\mathbf{N}}$ be an exhaustion of G by compact sets. Define for $n \in \mathbf{N}$ the set $A_n = A \cap K_n$. The modular function Δ on G is continuous by Theorem 2.2.5. Therefore, its maximal value on every compact set is bounded. We can thus define, for $n \in \mathbf{N}$,

$$\epsilon_n = \frac{\epsilon}{2^n(\max\{\Delta(f) : f \in K_{n+1}\})} > 0.$$

As the measure m_R is outer regular, there exists for every $n \in \mathbf{N}$ a set $B'_n \supset A_n$, such that $m_R(B'_n \setminus A_n) < \epsilon_n$. Defining $B_n = B'_n \cap \operatorname{int} K_{n+1}$, we have that $A_n \subset B_n$ and $B_n \subset K_{n+1}$. This implies, that

$$\begin{aligned}\tilde{m_L}(B_n \setminus A_n) &= \int_{K_{n+1}} \mathbf{1}_{B_n\setminus A_n}(g)\Delta(g)dm_R(g) \\ &\le m_R(B_n \setminus A_n)\max_{x\in K_{n+1}}\Delta(x) < \epsilon_n\left(\max_{x\in K_{n+1}}\Delta(x)\right) \le \frac{\epsilon}{2^n}.\end{aligned}$$

Denoting $B = \bigcup_{n\in\mathbf{N}} B_n$, B is an open set that contains A. By construction, $B \setminus A \subset \bigcup_{n\in\mathbf{N}} B_n \setminus A_n$. Therefore, it holds that

$$\tilde{m_L}(B \setminus A) \le \sum_{n=1}^{\infty} \tilde{m_L}(B_n \setminus A_n) \le \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon.$$

By definition, $\tilde{m_L}$ is outer regular. The argument can easily be adapted to deal with inner regularity also. Indeed, by inner regularity of m_R, there is for each $n \in \mathbf{N}$ a compact set C_n, such that $m_R(A_n \setminus C_n) < \epsilon_n$. This implies again, that

$$m_L(A_n \setminus C_n) < \frac{\epsilon}{2^n}.$$

Define now for every $N \in \mathbf{N}$,

$$C'_N = \bigcup_{n=1}^{N} C_n.$$

Then the sets $\{C'_N\}_{N\in\mathbf{N}}$ form an increasing sequence of compact sets, all of them contained in A. Furthermore,

$$\inf_{N\in\mathbf{N}} \tilde{m_L}(A \setminus C'_N) \leq \lim_{N\to\infty} \tilde{m_L}(A \setminus C'_N) = \epsilon.$$

As this holds for all $\epsilon > 0$, the measure $\tilde{m_L}$ is inner regular. As Δ is finite on every compact set K, and $m_R(K) < \infty$, we conclude that $\tilde{m_L}(K) < \infty$. We have obtained that $\tilde{m_L}$ is a left Haar measure, and therefore $\tilde{m_L} = cm_L$. We will prove that $c = 1$. Assume therefore the contrary. Then there exists $\epsilon > 0$, such that

$$c \notin \left(\frac{1}{1+\epsilon}, 1+\epsilon\right) = I(\epsilon),$$

where the last equation defines $I(\epsilon)$. By continuity of Δ, there exists an open set U, such that $\Delta(U) \subset I(\epsilon)$. As Δ is a homomorphism, the set U^{-1} then has the same property. This implies, that the symmetric set $U' = U \cup U^{-1}$ satisfies the conditions also. Furthermore

$$cm_L(U) = \tilde{m_L}(U) = \int_U \Delta(g) dm_R(g) \in I(\epsilon) m_R(U) = I(\epsilon) m_L(U).$$

The last equality holds as U is symmetric. This contradiction proves that $c = 1$. We have thus proven, that for every $B \in \mathcal{B}$, it holds that

$$\int_G \mathbf{1}_B(g) dm_L(g) = m_L(B) = \tilde{m_L}(B) = \int_G \mathbf{1}_B(g)\Delta(g) dm_R(g).$$

This implies that the modular function Δ is indeed the Radon-Nikodym derivative. □

Theorem 2.3.2 *For every $f \in L^1(G)$ on a locally compact group G, we have*

$$\int_G f(g) dm_L(g) = \int_G f^*(g) d\mu_L(g) = \int_G f\left(g^{-1}\right)\Delta\left(g^{-1}\right) dm_L(g).$$

Proof As Δ is the Radon-Nikodym derivative, it holds for every $f \in L^1(G)$, that

$$\int_G f(g) dm_L(g) = \int_G f(g)\Delta(g) dm_R(g).$$

Writing g^{-1} for g, this implies, that

$$\begin{aligned}\int_G f(g) dm_L(g) &= \int_G f\left(g^{-1}\right)\Delta\left(g^{-1}\right) dm_R\left(g^{-1}\right) \\ &= \int_G f\left(g^{-1}\right)\Delta\left(g^{-1}\right) dm_L(g).\end{aligned}$$

This proves the theorem. □

From the theorem, we see that $f \mapsto f^*$ is a norm-preserving mapping from $L^1(G)$ onto itself or from $P^1(G)$ onto itself. As it obviously preserves the linear structure of these spaces, this mapping is a unitary automorphism.

Every function $\phi \in P^1(G)$ defines a measure on G, by

$$\mathcal{B} \ni A \mapsto \int_A \phi(g) dm_L(g),$$

that is absolutely continuous with respect to μ and that satisfies $\mu(G) = 1$. Denoting $M^1(G)$ for the set of all *bounded* measures on the Borel algebra $\mathcal{B}$ of G, we find an embedding $\phi \mapsto \mu_\phi$ of $P^1(G)$ into $M^1(G)$.

The convolution puts more structure on the function spaces. For $f \in L^1(G)$, $1 \leq p \leq \infty$ and $g \in L^p(G)$ the convolution

$$f * g(x) = \int_G f(y) g\left(y^{-1}x\right) dm_L(y)$$

exists for almost every $x \in G$ and can be considered as an element in $L^p(G)$. Moreover, we have for $1 \leq p \leq \infty$, that [10, Theorem 20.14]

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p. \tag{2.3}$$

For every $\phi \in P^1(G)$, this reduces to

$$\begin{aligned} \phi * f(x) &= \int_G \phi(y) f\left(y^{-1}x\right) dm_L(x) \\ &= \int_G f\left(y^{-1}x\right) \phi(y) dm_L(y) = \int_G f\left(y^{-1}x\right) d\mu_\phi(x). \end{aligned}$$

By Theorem 2.3.2, we also have, for $f \in L^1(G)$,

$$\begin{aligned} f * \phi(x) &= \int_G f(y) \phi\left(y^{-1}x\right) dm_L(y) \\ &= \int_G f\left(xy\right) \phi\left(y^{-1}\right) dm_L(y) \\ &= \int_G f\left(xy^{-1}\right) \Delta\left(y^{-1}\right) \phi\left(y\right) dm_L(y) \\ &= \int_G f\left(xy^{-1}\right) \Delta\left(y^{-1}\right) d\mu_\phi(y). \end{aligned}$$

In view of the embedding $P^1(G) \hookrightarrow M^1(G)$, it is therefore natural to define for $f \in L^1(G)$ and $\mu \in M^1(G)$

$$\mu * f(x) = \int_G f\left(y^{-1}x\right) d\mu(y) \quad \text{and} \quad f * \mu(x) = \int_G f\left(xy^{-1}\right) \Delta\left(y^{-1}\right) d\mu(y).$$

The number $\mu * f(x)$ exists m_L-almost everywhere and defines a function in $L^p(G)$, whenever $f \in L^p(G)$, for $1 \leq p \leq \infty$ [10, Thereom 20.12]. Similarly, $f * \mu$ defines a function in $L^1(G)$ for every $f \in L^1(G)$ [10, Theorem 20.13].

The convolution has some nice properties on uniform continuity.

Definition 2.3.3 *Let Φ be a mapping from a topological group G to a normed space X. Φ is called right uniformly continuous, if for every $\epsilon > 0$, there exists an open set U, such that*

$$\|\Phi(x) - \Phi(x')\| < \epsilon,$$

whenever $x'x^{-1} \in U$. Similarly, Φ is called left uniformly continuous, if for every $\epsilon > 0$, there exists an open set U, such that

$$\|\Phi(x) - \Phi(x')\| < \epsilon,$$

whenever $x^{-1}x' \in U$. If Φ is both right and left uniformly continuous, then Φ is called uniformly continuous.

The properties on uniform continuity are summarized in the following theorems. For the proofs, we refer to [10, Theorem 20.16]

Theorem 2.3.4 *Let $f \in L^1(G)$ and $g \in L^\infty(G)$. Then $f * g$ is right uniformly continuous: for every $\epsilon > 0$, there exists an open neighborhood U of e in G, such that*

$$\|f * g(x) - f * g(x')\|_\infty < \epsilon,$$

for all $x, x' \in G$, for which $x'x^{-1} \in U$. If $f \in L^\infty(G)$ and $x \mapsto g(x^{-1}) \in L^1(G)$, then $f * g$ is left uniformly continuous: for every $\epsilon > 0$, there exists an open neighborhood U of e in G, such that

$$\|f * g(x) - f * g(x')\|_\infty < \epsilon,$$

for all $x, x' \in G$, for which $x^{-1}x' \in U$.

From this theorem one can easily deduce the following.

Theorem 2.3.5 *Let $\phi \in P^1(G)$ and $f \in L^\infty(G)$. When f is left uniformly continuous, then the convolution $\phi * f$ is uniformly continuous.*

Proof By definition, there exists an open set U, such that $|f(x) - f(x')| < \epsilon$, for all $x^{-1}x' \in U$. Select now $g \in G$. For those x, x', it holds that $\left(y^{-1}x\right)^{-1} y^{-1}x' = x^{-1}yy^{-1}x' = x^{-1}x' \in U$. This implies, that

$$\begin{aligned} |(\phi * f)(x) - (\phi * f)(x')| &= \left| \int_G \phi(y) \left(f\left(y^{-1}x\right) - f\left(y^{-1}x'\right)\right) dm_L(y) \right| \\ &\leq \int_G |\phi(y)| \left| f\left(y^{-1}x\right) - f\left(y^{-1}x'\right)\right| dm_L(y) < \epsilon. \end{aligned}$$

By definition $\phi * f$ is left uniformly continuous. As $\phi * f$ is also right uniformly continuous by the previous theorem, we conclude that $\phi * f$ is uniformly continuous. □

We will now define a left action of G on $L^p(G)$. This action has the property that it is right uniformly continuous. It is given by

$$(L_g\phi)(x) = \phi(gx).$$

As the measure m_L is left invariant, the mapping $\phi \mapsto L_g\phi$ is measure preserving. The following theorem tells it is right uniformly continuous. [10, Theorem 20.4]

Theorem 2.3.6 *Let $f \in L^p(G)$ where $1 \leq p < \infty$ and consider the mapping $z \mapsto L_z\phi$. Then for every $\epsilon > 0$, there is an open neighborhood V of e in G, such that*

$$\|L_z f - L_{z'} f\|_p < \epsilon,$$

for every $z, z' \in G$, for which $z'z^{-1} \in U$. The mapping $z \mapsto L_z\phi$ is therefore called right uniformly continuous.

A similar mapping is defined by

$$(R_g\phi)(x) = \Delta\left(g^{-1}\right)\phi\left(xg^{-1}\right).$$

Integrating this equation and using that $\Delta\left(g^{-1}\right)m_L(A) = m_L\left(Ag^{-1}\right)$ for every $A \in \mathcal{B}$, we find that

$$\int_G (R_g\phi)(g')dm_L(g') = \Delta\left(g^{-1}\right)\int_G \phi\left(g'g^{-1}\right)dm_L(g') = \int_G \phi(g')dm_L(g').$$

We conclude that the mapping $\phi \mapsto R_g\phi$ is also measure-preserving. Furthermore, for $g_1, g_2 \in G$, it holds that

$$\begin{aligned} R_{g_1}R_{g_2}\phi(x) &= \Delta\left(g_1^{-1}\right)\left(R_{g_2}\phi\left(xg_1^{-1}\right)\right. \\ &= \Delta\left(g_1^{-1}\right)\Delta\left(g_2^{-1}\right)\phi\left(xg_2^{-1}g_1^{-1}\right) \\ &= \Delta\left((g_1g_2)^{-1}\right)\phi\left(x\left(g_1g_2\right)^{-1}\right) = R_{g_1g_2}\phi(x). \end{aligned}$$

This shows that the mapping $g \mapsto R_g$ defines an action of G on $L^p(G)$ also.

A useful theorem is the following.

Theorem 2.3.7 *Let $(G, \mathcal{T})$ be a second countable, locally compact group. Let $\phi \in L^1(G)$ and $\delta > 0$ be given. Then there exists a compact set $K \subset G$, such that*

$$\int_{G\setminus K} |\phi(g)|dm_L(g) < \delta.$$

Proof Let $\{K_n\}_{n\in\mathbf{N}}$ be an exhaustion by compact sets for G. The sequence $\{\mathbf{1}_{K_n}|\phi|\}_{n\in\mathbf{N}}$ converges pointwise to $|\phi|$. By Lebesgue's monotone convergence theorem, it holds that

$$\lim_{n\to\infty}\int_G \mathbf{1}_{K_n}|\phi(g)|dm_L(g) = \int_G |\phi(g)|dm_L(g) < \infty.$$

By definition, there must be for every δ an $N \in \mathbf{N}$, such that for all $n > N$, we have

$$\int_{G\setminus F_n} |\phi(g)|dm_L(g) = \int_G (\mathbf{1}_G(g) - \mathbf{1}_{F_n}(g))\,|\phi(g)|dm_L(g) < \delta.$$

This proves the claim. □

2.3.2 Functional analysis on a finite measure space

We will now turn to measure spaces $(X, \mathcal{F}, \mu)$ that have the property that the meausure μ is finite. The following is an easy consequence of Hölder's inequality.

Theorem 2.3.8 *If $(X, \mathcal{F}, \mu)$ is a finite measure space, then every space $L^p(X)$ is contained in $L^1(X)$. Furthermore, the norm $\|\cdot\|_1$ is bounded by $\|\cdot\|_p$.*

Proof Let $1 < p < \infty$ and $f \in L^P(X)$ be given. Note that for a finite measure space, it holds that $1 \in L^q(X)$, where q is defined by $q = \frac{p}{p-1}$. Then Hölder's inequality [10, Theorem 12.4] states that

$$\|f\|_1 = \|1\cdot f\|_1 \le \|1\|_q\|f\|_p = (\mu(X))^{\frac{p}{p-1}}\,\|f\|_p.$$

It follows that $\|f\|_1 < \infty$ whenever $\|f\|_p < \infty$. This implies that $L^p(X) \subset L^1(X)$. For $p = \infty$, it follows by definition of the supremum norm, that

$$\int_X |f(x)|d\mu(x) \le \int_X \|f(x)\|_\infty d\mu(x) = \|f(x)\|_\infty \mu(X) < \infty.$$

This proves the claim. □

Simple functions are functions that take only a finite set of values. On a finite measure space, the set of simple functions is contained in $L^p(X)$, for $1 \le p \le \infty$. As the Lebesgue integral is defined by a sequence of integrals of simple functions, the simple functions are by construction dense in $L^1(X)$. One obtains that the simple functions are dense in $L^p(X)$ for every $1 \le p \le \infty$ from Lieb's theorem [18, Theorem 6.2.4]. We only need the following.

Theorem 2.3.9 *The space S of measurable, simple functions is dense in $L^1(X)$, for all finite measure spaces $(X, \mathcal{F}, \mu)$.*

Let now $(X, \mathcal{F}, \mu)$ be a finite measure space. The above theorems then imply

$$S \subset L^2 \subset L^1 \quad\Rightarrow\quad L^1 = \overline{S} \subset \overline{L^2} \subset L^1.$$

Note that we mean the L^1 closure in the right hand side. It follows that the closure in $L^1(X)$ of the subspace $L^2(X)$ is equal to $L^1(X)$.

2.4 Probability Theory

We will use two distributions from probability theory. The first one is the Bernoulli random variable.

Definition 2.4.1 *The random variable that takes the value 1 with probability p and the value 0 with probability $1 - p$ is called the Bernoulli random variable with parameter p. We denote this random variable by B_p.*

Example Let X be the outcome of throwing a fair coin. Identify head with 1, and tail with 0. Then X is a Bernoulli random number with parameter $\frac{1}{2}$.

Example Let X be the outcome of throwing a die. Identify rolling six pips with 1, and rolling anything else with 0. Then X is a Bernoulli random number with parameter $\frac{1}{6}$.

The binomial distribution $B(n, p)$ is defined as the sum of n Bernoulli variables with parameter p. The possible outcomes are the numbers between 0 and n. The probability that one finds $B(n, p) = r$ is given by

$$\mathbb{P}(B(n,p) = r) = \frac{n!}{r!(n-r)!} p^r (1-p)^{n-r}, \quad 0 \leq r \leq n.$$

The expectation of $B(n, p)$ can easily be computed as np. The following is an easy probabilistic lemma.

Lemma 2.4.2 *Let $B = B(n, p) = \sum_{i=1}^{n} (B_p)_i$ be the sum of n independent identically distributed Bernoulli random variables that are 1 with probability p. Then it holds that*

$$\mathbb{E}(B|B \geq 1) \leq 1 + (n-1)p. \tag{2.4}$$

Proof We will prove the statement by induction on n. For $n = 1$, (2.4) holds trivially. Assume now that it holds for $n - 1$. Let $(B_p)_i$ for $1 \leq i \leq n$ be a Bernoulli random variables with probability p. Define the Bernoulli random variables A_i for $1 \leq i \leq n$ as

$$A_i = (B_p)_i \,|B \geq 1 \ .$$

We will now condition on the value of A_n. If $A_n = 1$, what happens with some probability p', B is strictly positive, so we find

$$\mathbb{E}\left(B|B \geq 1, A_n = 1\right) = \mathbb{E}\left(B|A_n = 1\right) = 1 + (n-1)p.$$

If $A_n = 0$, with probability $1 - p'$ we conclude by induction hypothesis (2.4), that

$$\mathbb{E}\left(B|B \geq 1, A_n = 0\right) = \mathbb{E}\left(\sum_{i=1}^{n-1} (B_p)_i \middle| \sum_{i=1}^{n-1} (B_p)_i \geq 1\right) \leq 1 + (n-2)p.$$

Combining these steps, we find

$$\begin{aligned}\mathbb{E}\left(B|B \geq B\right) &= \mathbb{E}\left(\mathbb{E}\left(B|B \geq 1, A_n\right)\right)\\ &= p'(1 + (n-1)p) + (1-p')(1 + (n-2)p)\\ &= 1 + (n-1)p - (1-p')p \leq 1 + (n-1)p.\end{aligned}$$

This proves the claim. □

Consider now a set of N independent and identically distributed Bernoulli random variables $\{(B_p)_n\}_{n=1}^N$, defined on some probability space $(\Omega, \mathbb{P})$. For a given subset $A \subset \{1, \ldots, N\}$, define the random variable

$$B_A^p : \Omega \to \mathbf{N} : \omega \mapsto B_A^p(\omega) = \sum_{n \in A} (B_p(\omega))_n.$$

As the Bernoulli random variables $(B_p)_n$ are independent, so are the random variables B_A^p and $B_{A'}^p$, if the sets A and A' are disjoint. The same observation holds for any finite collection of sets A_i. This suggests, that

$$B^p_{\bigcup_i A_i} = \sum_i B^p_{A_i}, \quad \text{if } i \neq j \Rightarrow A_i \cap A_j = \emptyset.$$

As an empty summation is zero by definition, it also holds that

$$B_\emptyset^p = 0.$$

Fixing now $\omega \in \Omega$, it follows that he mapping $A \mapsto B_A^p(\omega)$ is a measure on the power set $\mathcal{P}(\{1, \ldots, N\})$. On the other hand, fixing the subset $A \subset \{1, \ldots, N\}$, the random variable B_A^p is binomially distributed. Summarizing, a set of N Bernoulli variables induces for every $A \in \{1, \ldots, N\}$ a random variable, with the following properties.

- If a collection of sets $A_i \subset \{1, \ldots, N\}$ is mutually disjoint, then the random variables
$$B_{A_i}^p$$
are independent.
- For each set $A \subset \{1, \ldots, N\}$, the random variable
$$B_A^p = \sum_{n \in A} (B_p)_n$$
is binomially distributed. The parameters are the cardinality of the set A and p. If μ denotes the counting measure on $\{1, \ldots, N\}$, the parameters are given by $\mu(A)$ and p.

To show the relation of the Bernoulli variable with the Poisson process, define for $\omega \in \Omega$, the set-valued random variable $B(\omega) \subset \{1, \ldots, N\}$ as the set of Bernoulli random variables which take the value 1 for ω. More precisely, define

$$B : \Omega \to \mathcal{P}(\{1, \ldots, N\}) : \omega \mapsto \{n \in \{1, \ldots, N\} : (B_p(\omega))_n = 1\}.$$

Then the random variables $\mu(B \cap A_i)$ are independent when the sets A_i are disjoint, and the random variable $\mu(B \cap A)$ is binomially distributed with parameters $\mu(A)$ and p.

The Poisson process is a generalization of the Bernoulli random variable. We will consider it in the most general form, a Poisson process on a σ-finite measure space. Recall that an infinite sum of Bernoulli variables, where the parameter p is chosen such that the mean np stays constant in the limit, converges to a Poisson distribution.

Definition 2.4.3 *Let $(X, \mathcal{B}, \mu)$ be a σ-finite measure space. Then a Poisson process with intensity ρ is a set-valued random variable Σ on a probability space $(\Omega, \mathbb{P})$, with the following properties.*

- *$\Sigma(\omega)$ is a countable subset of X for every $\omega \in \Omega$.*
- *For any collection $\{B_j\} \subset \mathcal{B}$ that is mutually disjoint, it holds that the random variables $|\Sigma \cap B_j|$ are independent.*
- *$|\Sigma \cap B|$ has expected value $\rho\mu(B)$, for any $B \in \mathcal{B}$,*

$$\mathbb{E}(|\Sigma \cap B|) = \int_\Omega |\Sigma(\omega) \cap B| d\mathbb{P}(\omega) = \rho\mu(B). \tag{2.5}$$

Here $|\cdot|$ is the counting measure on the set Σ.

Example The one dimensional Poisson process with parameter λ on $\mathbf{R}^n$. The independent increments are the key characteristic of this process. It is known that independent increments imply that the number of events in any interval (a, b) is exponentially distributed with mean $\lambda(b - a)$.

Under suitable conditions, one can derive that the number of events $|\Sigma \cap B|$ is Poisson distributed, for every set B, when the measure μ is non-atomic. The only thing of importance for us is however, that there *exists any* Poisson process, for which the random numbers $|\Sigma \cap B|$ are Poisson distributed, therefore, we will omit the details on this issue.

If the measure is σ-finite and non-atomic there exists a Poisson process for which the random numbers $|\Sigma \cap B|$ are Poisson distributed. For a finite measure space X this is quite straightforward. The random set Σ is given by the first N points of the countable set $\{x_n\}_{n\in\mathbf{N}}$, where each x_n is uniformly distributed over the set X and N is a Poisson distributed random variable. The reason that we need the measure to be non-atomic, is to ensure that the event where one element x is selected twice has zero probability. When one element is selected twice, the set Σ has only $N - 1$ elements, when N is the realization of the Poisson distrbution. We want this rarity to have zero probability. When the measure is σ-finite, one takes a Poisson process on each finite component, and takes the union of the random sets. This intuitive approach indeed works, as can be seen in [11, Section 2.5].

Consider now a Poisson process Σ with intensity ρ on $(G, \mathcal{B}(\mathcal{T}), \mu)$. Then for any $\omega \in \Omega$, the mapping $\mu_\omega : A \mapsto |A \cap \Sigma(\omega)|$ defines a measure on G. This shows how a Poisson process induces a Poisson random measure. For every measurable set $A \in \mathcal{B}$, equation (2.5) implies that the expectation of $\omega \mapsto \mu_\omega(A)$ equals $\rho\mu(B)$. Extending from subsets to arbitrary functions, we get the following.

Theorem 2.4.4 *Let Σ be a Poisson process on $(G, \mathcal{B}(\mathcal{T}), \mu)$ with intensity α. Then for any $f \in L^1(G)$, it holds that*

$$\mathbb{E}\left(\sum_{g\in\Sigma} f(g)\right) = \alpha \int_G f(g) d\mu(g).$$

Proof We will prove the equality first for indicator functions. Consider therefore a measurable set $A \subset G$. By definition of the Poisson process, it holds for all $\omega \in \Omega$, that

$$\int_G \mathbf{1}_A d\mu_\omega(g) = \mu_\omega(A) = |A \cap \Sigma(\omega)| = \sum_{g\in\Sigma(\omega)} \mathbf{1}_A(g).$$

Taking the expectation of this expression, the last equality implies, that

$$\mathbb{E}\left(\sum_{g\in\Sigma} \mathbf{1}_A(g)\right) = \mathbb{E}(|A \cap \Sigma|) = \alpha\mu(A).$$

Let now $h = \sum_i \beta_i \mathbf{1}_{B_i}$ be a simple function. As the expectation is linear, it holds that

$$\mathbb{E}\left(\sum_{g\in\Sigma} h(g)\right) = \sum_i \beta_i \mathbb{E}\left(\sum_{g\in\Sigma} \mathbf{1}_{B_i}(g)\right) = \alpha \sum_i \beta_i \mu(B_i) = \alpha \int_G h(g) d\mu(g).$$

A standard argument now suffices to prove the above for any function $f \in L^1(G)$. Let $f \in L^1(G)$ be a positive function and f_n a sequence of positive, simple functions that converges pointwise to f. As the expectation is just an integration with respect to the probability measure, we conclude from the Beppo-Levi theorem [4, Theorem 16.2], that

$$\begin{aligned}\mathbb{E}\left(\sum_{g\in\Sigma} f(g)\right) &= \lim_{n\to\infty} \mathbb{E}\left(\sum_{g\in\Sigma} f_n(g)\right)\\ &= \lim_{n\to\infty} \alpha \int_G f_n(g) d\mu(g) = \alpha \int_G f(g) d\mu(g).\end{aligned}$$

Decomposing an arbitrary $f \in L^1(G)$ in a positive and a negative part now completes the argument. $\square$

The following lemma is a generalization of Lemma 2.4.2. Recall that we assume that the random numbers $|\Sigma \cap B|$ are Poisson distributed.

Lemma 2.4.5 *Let Σ be a Poisson process on a σ-finite, non-atomic measure space $(X, \mathcal{B}, \mu)$ with intensity 1. Then, for every compact subset A of X, it holds that*

$$\mathbb{E}(|\Sigma \cap A| \,|\, |\Sigma \cap A| \geq 1) \leq 1 + \mu(A).$$

Proof By definition of the Poisson process, $|\Sigma \cap A|$ is Poisson distributed with parameter $\mu(A)$. Denote $P(a)$ for this Poisson distributed random variable, with parameter $a = \mu(A)$. Assume that $a > 0$. Then we have

$$\begin{aligned}
\mathbb{E}(|\Sigma \cap A| \,|\, |\Sigma \cap A| \geq 1) &= \mathbb{E}(P(a) | P(a) \geq 1) \\
&= \sum_{n=1}^{\infty} n \frac{\mathbb{P}(P(a) = n)}{\mathbb{P}(P(a) \geq 1)} \\
&= \frac{\sum_{n=1}^{\infty} n \mathbb{P}(P(a) = n)}{1 - \mathbb{P}(P(a) = 0)} = \frac{\mathbb{E}(P(a))}{1 - \exp(-a)} = \frac{a}{1 - \exp(-a)}.
\end{aligned}$$

It follows we should prove, that

$$\frac{a}{1 - \exp(-a)} \leq 1 + a.$$

Multiplying both sides by $1 - \exp(-a)$ and rearranging terms, we find

$$a \leq 1 + a - (1 + a)\exp(-a) \Leftrightarrow (1 + a)\exp(-a) \leq 1 \Leftrightarrow 1 + a \leq \exp(a).$$

The last equality follows easily by expanding the right hand side as its Taylor series. $\square$

Chapter 3

Amenability

3.1 Introduction

The notion of amenability is closely related to the existence of a *mean*. In normal language, the mean of a set of numbers is the average value of these numbers. Our intuitive idea of what an average should be, indicates that the mean $M(f)$ of a function f on S should have the following properties.

- The mean of a function should be somewhere between its lowest and its highest value,
$$\inf_{s\in S} f(s) \leq M(f) \leq \sup_{s\in S} f(s).$$
- The mean of the function that is always 1 should be one, $M(\mathbf{1}_G) = 1$.
- The mean of a sum of functions should be equal to the sum of the means, $M(f_1 + f_2) = M(f_1) + M(f_2)$.

A natural question in this respect is whether a given set S allows a mean or not. For the case of *measurable* functions on a *probability* space (X, μ), this question is easily solved. It can be seen immediately that the integral with respect to the measure μ induces a mean on $L^\infty(X, \mu)$. The observation that the Lebesgue integral on $\mathbf{R}^n$ is invariant under isometries raised the question whether a mean on $\mathbf{R}^n$ should be invariant under some group that acts upon it.

Definition 3.1.1 *Let X be a set and let L be some set of real-valued functions on X. Then a mean on L is a linear mapping from L to $\boldsymbol{R}$, such that*

$$\inf_{x\in X} f(x) \leq M(f) \leq \sup_{x\in X} f(x).$$

If G is a group that acts on X, then the mean M on L is said to be G-invariant if for all $f \in L$ and $g \in G$, it holds that

$$M\left(x \mapsto f(x)\right) = M\left(x \mapsto f(gx)\right) = M(L_g f).$$

The Banach-Tarksi paradox is the key results that states that in the case where all functions on $\mathbf{R}^n$ are considered, it is impossible to construct a mean that is invariant under the isometries. The proof of this remarkable result indicates that the existence of a mean, that is invariant under the action of a group, is really a property of the group that acts [19]. The study of means therefore mainly concerns groups. For a group, the natural definition of a mean is the following.

Definition 3.1.2 *Let G be a group, and let L be some set of functions on G. Then a left invariant mean M on L is a linear mapping from L to $\boldsymbol{R}$, such that $M(L_g f) = M(f)$ and*

$$\inf_{g \in G} f(g) \leq M(f) \leq \sup_{g \in G} f(g),$$

for any $f \in L$ and $g \in G$.

In the case where G is a locally compact group, there exists a left Haar measure m_L on G. In that case, every $\phi \in P^1(G)$ induces a mean on $L^\infty(G)$, by

$$M(f) = \int_G f(g)\phi(g)dm_L(g).$$

Although the measure m_L is left invariant, the mean M induced by ϕ will only be invariant when ϕ is. Unless the measure m_L is finite, there exists no G-invariant function $\phi \in L^1(G)$.

When we consider a locally compact group $(G, \mathcal{B}, m_L)$, equipped with the left Haar measure, the natural set of functions to consider is $L^\infty(G)$. The study of means on $L^\infty(G)$ has led to many characterizations that are equivalent to the existence of a mean. In the next section we will introduce some of these characterizations. The nice property that these characterization imply for G, makes them the deserve the name as which they are known now: a locally compact group G that admits a mean on $L^\infty(G)$ is called *amenable*.

Definition 3.1.3 *A locally compact group G is called amenable, if there exists a left invariant mean on $L^\infty(G, \mathcal{B}, m_L)$.*

Most of the characterizations of amenability presented in the next section are described in [15]. The proofs of the equivalences are in most cases only sketches. One exception should be remarked. The equivalence of Følner's weak condition and Leptin's condition, implied by Lemma 3.5.8, is the most difficult one to prove. Fortunately, the proof of that lemma was included with more details.

The characterizations of amenability that we use in the next sections are the structural properties, that at first sight seem very different from the original ones. For this reason we have included the proofs in full detail. The last characterization, the one that implies the existence of a Følner sequence, does not appear in the standard literature. Therefore, we had to come up with a proof of the last equivalence ourselves.

3.2 Characterizations of amenability

Let $(G, \mathcal{T})$ be a $\sigma-$locally compact, topological group with left invariant Haar measure m_L, defined on the Borel σ-algebra $\mathcal{B} = \mathcal{B}(\mathcal{T})$. We define the following conditions for G.

1. G is amenable: there exists a left invariant mean on the space $L^\infty(G)$ of essentially bounded functions on G.

2. Reiter's condition: For every compact subset $K \subset G$, there exists a net $\{\phi_i\}$ in $P^1(G)$ such that

$$\lim_{i\to\infty} \|L_a\phi_i - \phi_i\|_1 = 0,$$

   for all $a \in K$.

3. Følner's condition: For every finite subset $K \subset G$, and every $\epsilon > 0$, there exists a compact set F, such that, for every $k \in K$

$$m_L(kF\Delta F) < \epsilon m_L(F).$$

4. Leptin's condition: For every compact set $K \subset G$ and every $\epsilon > 0$, there exists a compact set U, such that

$$\frac{m_L(KU)}{m_L(U)} < 1 + \epsilon.$$

5. Følner's strong condition: For every compact subset $K \subset G$, and every $\epsilon > 0$, there exists a compact set F, such that

$$m_L(KF\Delta F) < \epsilon m_L(F).$$

6. Følner sequence: There exists a Følner sequence, a sequence $\{F_n\}_{n\in\mathbf{N}}$ such that for every compact set K and $\epsilon > 0$, there is an $N \in \mathbf{N}$, such that for every $n > N$,

$$\frac{m_L(F_nK\Delta K)}{m_L(F_n)} < \epsilon.$$

In the rest of this chapter we will show that these conditions are equivalent. A σ-locally compact group G is therefore amenable if any of the above conditions is satisfied.

3.3 Invariant means

Definition 3.3.1 *Let G be a locally compact group and assume that M is a mean on $L^\infty(G)$. The mean M is said to be topologically left invariant, if for every $\phi \in P^1(G)$ it holds that*

$$M(\phi * f) = M(f).$$

We will now show that every topologically left invariant mean M is also left invariant.

Theorem 3.3.2 *Let G be a locally compact group. Every topologically left invariant mean M on $L^\infty(G)$ is left invariant.*

Proof Let $f \in L^\infty(G)$ and $a \in G$ be given. Choose an arbitrary $\phi \in P^1(G)$. By definition of the convolution and the modular function and by left invariance of the measure m_L, we obtain

$$\begin{aligned}
(\phi * L_a f)(x) &= \int_G \phi(y)(L_a f)\left(y^{-1}x\right) dm_L(y) \\
&= \int_G \phi(y) f\left(ay^{-1}x\right) dm_L(y) \\
&= \int_G \phi\left(ya^{-1}a\right) f\left(\left(ya^{-1}\right)^{-1} x\right) dm_L\left(ya^{-1}a\right) \\
&= \int_G \phi(ya) f\left(y^{-1}x\right) dm_L(ya) \\
&= \int_G \Delta(a)\phi(ya) f\left(y^{-1}x\right) dm_L(y) = ((R_{a^{-1}}\phi) * f)(x).
\end{aligned}$$

The mapping $R_{a^{-1}}$ is measure-preserving. Therefore, $R_{a^{-1}}\phi \in P^1(G)$ whenever ϕ is. It is easy to see that $L_a f \in L^\infty(G)$. Using now that the mean M is topologically invariant, we find

$$\begin{aligned}
|M(L_a f - f)| &= |M(L_a f - \phi * L_a f + (R_{a^{-1}}\phi) * f - f)| \\
&\leq |M(\phi * (L_a f) - (L_a f))| + |M((R_{a^{-1}}\phi) * f - f)| = 0.
\end{aligned}$$

We conclude M is left invariant. □

The converse of this theorem does not hold in general. However, restricting the mean to the space of uniformly continuous functions, a left invariant mean does define a topologically left invariant mean. To show this, we will need the following lemma.

Lemma 3.3.3 *Let f be a uniformly continuous function on a locally compact group G and $\phi \in P^1(G)$. Then, for every $\epsilon > 0$, there exists a finite set of tuples $\{(a_i, \alpha_i) : 1 \leq i \leq n\}$, such that $\sum_i \alpha_i = 1$, and*

$$\left\|\phi * f - \sum \alpha_i L_{a_i} f\right\|_\infty < \epsilon.$$

Proof By linearity of the convolution and the norm, we can restrict attention to uniformly continuous functions f with unit norm. Choose $\epsilon > 0$ arbitrarily. Define $\delta = \frac{\epsilon}{3}$. By Theorem 2.3.7 there exists a compact set $K \subset G$, such that

$$\int_{G \setminus K} |\phi(x)| dm_L(x) < \delta.$$

As f is right uniformly continuous, there exists an open neighbourhood V in G of the unity e, such that $|f(x) - f(x')| < \delta$, for every $x'x^{-1} \in V$. For every y and z, such that $y \in zV$, it holds that

$$\|L_{y^{-1}} f - L_{z^{-1}} f\|_\infty = \operatorname*{ess\,sup}_{g \in G} \left| f(y^{-1}g) - f(z^{-1}g) \right| < \delta, \tag{3.1}$$

because $y \in zV$ implies that $(z^{-1}g)(y^{-1}g)^{-1} = z^{-1}y \in V$.

Cover now the set K by the sets $\{gV\}_{g \in G}$. By compactness, there exists a finite subset $\{g_1, \ldots, g_n\} \subset G$ such that

$$K \subset \bigcup_{i=1}^{n} g_i V.$$

We will extract from these sets a partition of G. Define $g_0 = e$, $V_0 = G \setminus K$ and, inductively, $V_i = g_i V \setminus \bigcup_{j=0}^{i-1} V_j$. Set $\alpha_i = \int_{V_i} \phi(y) dm_L(y)$. As $\phi \in P^1(G)$, we have $\sum_{i=0}^{n} \alpha_i = 1$. The following holds for every $x \in G$.

$$\begin{aligned} \left| (\phi * f)(x) - \sum_{i=0}^{n} \alpha_i L_{g_i^{-1}} f(x) \right| &= \left| \int_G \phi(g) f\left(g^{-1}x\right) dm_L(g) - \sum_{i=0}^{n} \alpha_i f\left(g_i^{-1}x\right) \right| \\ &= \left| \sum_{i=0}^{n} \int_{V_i} \phi(g) \left(f\left(g^{-1}x\right) - f\left(g_i^{-1}x\right)\right) dm_L(g) \right| \\ &\le \sum_{i=0}^{n} \int_{V_i} |\phi(g)| \left| f(g^{-1}x) - f(g_i^{-1}x) \right| dm_L(g). \end{aligned}$$

Note that $|f(g^{-1}x) - f(g_i^{-1}x)| \le 2$, because f has unit norm. Furthermore, for $1 \le i \le n$ and $g \in V_i \subset g_i V$, we have $|f(g^{-1}x) - f(g_i^{-1}x)| \le \delta$, by (3.1). Taking now the essential supremum of the above expression, we find

$$\begin{aligned} \left\| \phi * f - \sum_{i=1}^{n} \alpha_i L_{g_i^{-1}} f \right\|_\infty &\le \operatorname*{ess\,sup}_{x \in G} \sum_{i=0}^{n} \int_{A_i} |\phi(g)| \left| f(g^{-1}x) - f(g_i^{-1}x) \right| dm_L(g) \\ &\le 2 \int_{A_0} |\phi(g)| dm_L(g) + \delta \sum_{i=1}^{n} \int_{A_i} |\phi(g)| dm_L(g) \\ &\le 2 \int_{G \setminus K} |\phi(g)| dm_L(g) + \delta \int_G |\phi(g)| dm_L(g) \le 3\delta = \epsilon. \end{aligned}$$

This proves the claim. □

Lemma 3.3.4 *Let f be a uniformly continuous function, and M a left invariant mean. It then holds for every $\phi \in P^1(G)$, that*

$$M(\phi * f - f) = 0.$$

Proof Let $\phi \in P^1(G)$ be given and $\epsilon > 0$ be arbitrary. From Lemma 3.3.3 we obtain a finite set of tuples (a_i, α_i), such that

$$\left\| \phi * f - \sum_{i=1}^{n} \alpha_i L_{a_i} f \right\|_\infty < \epsilon.$$

and $\sum_i \alpha_i = 1$. Using that the mean M is left invariant, we obtain

$$M(f) = \sum_{i=1}^{n} \alpha_i M\left(L_{a_i} f\right) = M\left(\sum_{i=1}^{n} \alpha_i L_{a_i} f\right).$$

It now follows easily, that

$$\begin{aligned} |M(\phi * f) - M(f)| &= \left| M\left(\phi * f\right) - M\left(\sum_{i=1}^{n} \alpha_i L_{a_i} f\right) \right| \\ &= \left| M\left(\phi * f - \sum_{i=1}^{n} \alpha_i L_{a_i} f\right) \right| \leq \epsilon, \end{aligned} \tag{3.2}$$

where we used in the last equation that $|M(f)| \leq \|f\|_\infty$, which follows from $\|M\| = 1$. To prove that $\|M\| \leq 1$, note that $-\|f\|_\infty \leq f \leq \|f\|_\infty$. By definition of a mean, it holds that

$$-\|f\|_\infty \leq \text{ess sup} f \leq M(f) \leq \|f\|_\infty.$$

We conclude that indeed $|M(f)| \leq \|f\|_\infty$. Taking now the limit for ϵ to zero in the (3.2) proves the statement. □

We will now show that when a left invariant mean M on $L^\infty(G)$ exists, there exists also a mean M' that is topologically left invariant. Let therefore M be a left invariant mean on $L^\infty(G)$. By restriction, we obtain a topologically left invariant mean $\tilde{M}$ on the space of uniformly continuous functions. This mean $\tilde{M}$ can be extended to a topologically left invariant mean on $L^\infty(G)$. We will need the following lemma.

Lemma 3.3.5 *Let $\phi_1, \phi_2 \in P^1(G)$ be given. Then there exists a function $\psi \in P^1(G)$, such that*

$$\|\psi * \phi_i - \phi_i\|_1 < \epsilon,$$

for $i = 1, 2$.

Proof Choose $\epsilon > 0$ arbitrarily. As the mapping $x \mapsto L_x \phi_i$ is right uniformly continuous by Theorem 2.3.6, there exist open sets $U_i \subset G$, such that

$$\left\| L_{y^{-1}} \phi_i - \phi_i \right\|_1 = \left\| L_{y^{-1}} \phi_i - L_e \phi_i \right\|_1 < \epsilon,$$

for every y, such that $y = e(y^{-1})^{-1} \in U_i$. As G is locally compact, there exists a compact neighborhood U of e that is contained in $U_1 \cap U_2$. U contains an non-empty, open set, which implies that $m_L(U) > 0$. Define $\psi = m_L(U)^{-1}\mathbf{1}_U$. Then it holds for $i = 1, 2$,

$$\begin{aligned}
\|\psi * \phi_i - \phi_i\|_1 &= \int_G |\psi * \phi_i(x) - \phi_i(x)| dm_L(x) \\
&= \int_G \left| \int_G \psi(y)\phi_i(y^{-1}x) dm_L(y) - \int_G \psi(y) dm_L(y)\phi_i(x) \right| dm_L(x) \\
&\leq \int_G \int_U |\psi(y)| \left| L_{y^{-1}}\phi_i(x) - \phi_i(x) \right| dm_L(y) dm_L(x) \\
&= \int_U \left(\int_G \left| L_{y^{-1}}\phi_i(x) - \phi_i(x) \right| dm_L(x) \right) \psi(y) dm_L(y) \\
&= \int_U \left\| L_{y^{-1}}\phi_i - \phi_i \right\|_1 \psi(y) dm_L(y) \leq \epsilon \int_U \psi(y) dm_L(y) = \epsilon,
\end{aligned}$$

where we used Fubini's Theorem in the fourth equality. We conclude the function ψ satisfies the conditions. □

The above lemma can be used to construct a sequence of approximate units for a function $\phi \in P^1(G)$. For every $\epsilon_n = \frac{1}{n}$, there exists a function ψ_n, such that

$$\|\psi_n * \phi - \phi\|_1 < \frac{1}{n}.$$

By definition, it holds that $\psi_n * \phi \to \phi \in P^1(G)$. This observation will be used in the following lemma.

Lemma 3.3.6 *Let $\phi_1, \phi_2 \in P^1(G)$ be given, let f be a left uniformly continuous function and let M be a topologically left invariant mean on the set of uniformly continuous functions. Then we have*

$$M(\phi_1 * f) = M(\phi_2 * f).$$

Proof By Theorem 2.3.4, $\phi * f$ is uniformly continuous whenever f is left uniformly continuous and $\phi \in L^1(G)$. Therefore, the expression makes sense. By Theorem 2.3.2, it holds for $i = 1, 2$ that $\phi_i^* : x \mapsto \phi_i(x^{-1})\Delta(x^{-1}) \in P^1(G)$, whenever $\phi_i \in P^1(G)$. Therefore, by the previous lemma, there exists a sequence of functions $\psi_n \in P^1(G)$, such that

$$\|\psi_n * \phi_i^* - \phi_i^*\| < \frac{1}{n}.$$

Define now $\psi_n^* : x \mapsto \psi_n(x^{-1})\Delta(x^{-1})$. Then it follows, that

$$\begin{aligned}
|\phi_i * \psi_n^*(x) - \phi_i(x)| &= \left| \int_G \phi_i(y)\psi_n^*(y^{-1}x) dm_L(y) - \phi_i(x) \right| \\
&= \left| \int_G \phi_i(y)\Delta(x^{-1}y)\psi_n(x^{-1}y) dm_L(y) - \phi_i(x) \right| \\
&= \left| \int_G \phi_i(xy)\Delta(y)\psi_n(y) dm_L(y) - \phi_i(x) \right|.
\end{aligned}$$

Integrating this expression now over $x \in G$, and using Theorem 2.3.2, we find,

$$\begin{aligned}
&\|\phi_i * \psi_n^*(x) - \phi_i(x)\|_1 \\
&= \int_G \left| \int_G \phi_i(xy)\Delta(y)\psi_n(y)dm_L(y) - \phi_i(x) \right| dm_L(x) \\
&= \int_G \Delta(x^{-1}) \left| \int_G \phi(x^{-1}y)\Delta(y)\psi_n(y)dm_L(y) - \phi_i(x^{-1}) \right| dm_L(x) \\
&= \int_G \left| \int_G \phi(x^{-1}y)\Delta(x^{-1}y)\psi_n(y)dm_L(y) - (\Delta\phi_i)(x^{-1}) \right| dm_L(x) \\
&= \int_G \left| \int_G \psi_n(y)(\Delta\phi_i)(x^{-1}y)dm_L(y) - (\Delta\phi)(x^{-1}) \right| dm_L(x) \\
&= \int_G \left| \int_G \psi_n(y)\phi_i^* \left(y^{-1}x\right) dm_L(y) - \phi_i^* \left(x\right) \right| dm_L(x) \\
&= \|\psi_n * \phi_i^* - \phi_i^*\|_1 < \frac{1}{n}.
\end{aligned}$$

By (2.3), we have

$$\lim_{n\to\infty} \|\phi_i * \psi_n * f - \phi_i * f\|_\infty \leq \lim_{n\to\infty} \|\phi_i * \psi_n - \phi_i\|_1 \|f\|_\infty = 0.$$

It now holds, as M is topologically left invariant, that

$$M(\phi_i * f) = \lim_{n\to\infty} M(\phi_i * \psi_n * f) = \lim_{n\to\infty} M(\psi_n * f).$$

As this holds for both $i = 1$ and $i = 2$, and by uniqueness of the limit, we obtain

$$M(\phi_1 * f) = M(\phi_2 * f).$$

This proves the claim. □

Theorem 3.3.7 *Let G be a locally compact group and $\tilde{M}$ a topologically left invariant mean on the space of uniformly continuous functions. Then for every non-empty, open, symmetric, relatively compact set $U \subset G$, the linear functional*

$$N : L^\infty(G) \to \boldsymbol{R} : f \mapsto \frac{1}{m_L(U)^2}\tilde{M}(\boldsymbol{1}_U * f * \boldsymbol{1}_U) = \tilde{M}\left(\xi_U * f * \xi_U\right)$$

defines a topologically left invariant mean on $L^\infty(G)$, where

$$\xi_U = \frac{\boldsymbol{1}_U}{m_L(U)}.$$

Proof As U is symmetric, $\mathbf{1}_U(g) = \mathbf{1}_U\left(g^{-1}\right)$. From Theorem 2.3.4, we see that $f * \mathbf{1}_U$ is left uniformly continuous for all $f \in L^\infty(G)$. By Theorem 2.3.5, we conclude that $\mathbf{1}_U * f * \mathbf{1}_U$ is uniformly continuous. As U is non-empty, open and contained in a compact set, we have $0 < m_L(U) < \infty$, therefore the expression makes sense. It follows easily that $\mathbf{1}_U * \mathbf{1}_G * \mathbf{1}_U = m_L(U)^2\mathbf{1}_G$. This implies $N(\mathbf{1}_G) = 1$. Also, when $f \geq 0$, we obviously have $\mathbf{1}_U * f * \mathbf{1}_U \geq 0$. Hence, N is positive. It remains to show that N is topologically left invariant.

To do so, recall that $f * \xi_U$ is left uniformly continuous. As $\phi \in P^1(G)$, we easily find

$$\begin{aligned}
\|\mathbf{1}_U * \phi\|_1 &= \int_G \int_G \mathbf{1}_U(y)\phi(y^{-1}x)dm_L(y)dm_L(x) \\
&= \int_G \mathbf{1}_U(y) \int_G \phi(y^{-1}x)dm_L(x)dm_L(y) \\
&= \int_G \mathbf{1}_U(y) \int_G \phi(x)dm_L(x)dm_L(y) \\
&= m_L(U) \int_G \phi(x)dm_L(x) = m_L(U).
\end{aligned}$$

Hence $\xi_U * \phi \in P^1(G)$. By the previous lemma, we conclude that

$$\begin{aligned}
N(\phi * f) &= M(\xi_U * \phi * f * \xi_U) \\
&= M((\xi_U * \phi) * f * \xi_U) \\
&= M(\xi_U * f * \xi_U) = N(f).
\end{aligned}$$

It follows N is topologically left invariant. □

We conclude that indeed every topologically left invariant mean on the space of uniformly continuous function can be extended to the space $L^\infty(G)$. Therefore, for every left invariant mean M, there exists a mean $\tilde{M}$ that is topologically left invariant. The conditions on the mean $\tilde{M}$ can even be strengthened further. In view of the embedding $P^1(G) \hookrightarrow M^1(G)$, the following holds.

Theorem 3.3.8 *Let G be a locally compact group. G admits a topologically left invariant mean, if and only if there exists a mean M, for which $M(\mu * f) = M(f)$ for all $\mu \in M^1(G)$ and $f \in L^\infty(G)$.*

Proof Let M be a topologically left invariant mean. Choose $\mu \in M^1(G)$ arbitrarily. It then holds that $\mu * f \in L^\infty(G)$. Furthermore, for every $\phi \in P^1(G)$ we have, $\phi * \mu \in P^1(G)$. By topological invariance, it follows that

$$M(\mu * f) = M(\phi * \mu * f) = M((\phi * \mu) * f) = M(f).$$

Let now a mean M be given, for which $M(f) = M(\mu * f)$ for all $\mu \in M^1(G)$. By the embedding $P^1(G) \hookrightarrow M^1(G)$, we can view $\phi \in P^1(G)$ as an element m_ϕ

of $M^1(G)$. The equality

$$\begin{aligned}(\phi * f)(y) &= \int_G \phi(y) f(y^{-1}x) dm_L(y) \\ &= \int_G f(y^{-1}x)\phi(y) dm_L(y) \\ &= \int_G f\left(y^{-1}x\right) dm_\phi(y) = (m_\phi * f)(y)\end{aligned}$$

then implies that the mean M is also topologically left invariant. This proves the claim. □

The following corollary summarizes the above results.

Corollary 3.3.9 *Let G be a locally compact group. Then the following conditions are equivalent.*

- *There exists a left invariant mean, that is a mean M on $L^\infty(G)$ for which $M(f) = M(L_a f)$ for all $a \in G$ and $f \in L^\infty(G)$.*
- *There exists a topologically left invariant mean, that is a mean M on $L^\infty(G)$ for which $M(f) = M(\phi * f)$ for all $\phi \in P^1(G)$ and $f \in L^\infty(G)$.*
- *There exists an $M^1(G)$-invariant mean, that is a mean M on $L^\infty(G)$ for which $M(f) = M(\mu * f)$ for all $\mu \in M^1(G)$ and $f \in L^\infty(G)$.*

3.4 Asymptotical invariance

In the previous section we encountered the embedding $P^1(G) \hookrightarrow M^1(G)$, that maps ϕ to m_ϕ, defined by

$$m_\phi : \mathcal{B} \to \mathbf{R} : A \mapsto \int_A \phi(x) dm_L(x).$$

We will now show that actually $P^1(G)$ is dense in the set of all means on $L^\infty(G)$. Therefore, every mean M on $L^\infty(G)$ can be approximated by a net in $P^1(G)$. This leads to asymptotical characterizations of amenability, first introduced by Day.

Density of $P^1(G)$ is a consequence of the Hahn-Banach theorem. The Hahn-Banach theorem is a classical result in Functional Analysis. It is currently known in several equivalent formulations. We will use it in the original form [16, Theorem 2.5].

Theorem 3.4.1 *Let M be a subspace of a vector space X. Let $p : X \to \boldsymbol{R}$ satisfy*

$$\begin{aligned}p(x+y) &\leq p(x) + p(y) && \forall x, y \in X, \\ p(\alpha x) &= \alpha(x) && \forall x \in X, \alpha > 0,\end{aligned}$$

and let f be a linear functional on M that is bounded by p on M. Then there exists a linear functional F on X, that extends f and is bounded by p. That is, F satisfies

$$\begin{aligned} F(x) &= f(x) \quad \forall x \in M \\ F(x) &\leq p(x) \quad \forall x \in X. \end{aligned}$$

The following theorem is an important consequence of the Hahn-Banach theorem. It states that convex sets can be separated with a linear functional.

Theorem 3.4.2 *Let $(X, \mathcal{T})$ be a topological vector space. Assume there exists a metric on X such that the metric topology on X is stronger than $\mathcal{T}$. Let B and Y be disjoint, non-empty, convex subsets of X. Assume furthermore that B is open. Then there is a continuous, linear functional Λ on X and a constant c, such that*

$$\Lambda(b) < c \leq \Lambda(y) \quad \forall b \in B, y \in Y.$$

Proof Choose $b_0 \in B$ and $y_0 \in Y$ arbitrarily. As B is open, also $B - x$ is open, for all $x \in X$. Therefore, the set

$$\begin{aligned} U &= (B - b_0) - (Y - y_0) \\ &= \{(b - b_0) - (y - y_0) : b \in B, y \in Y\} \\ &= \bigcup_{y \in Y} B - (b_0 - y + y_0) \end{aligned}$$

is an open set that contains 0. From convexity of B and Y, we conclude for $u, u' \in U$,

$$\begin{aligned} \lambda u + (1-\lambda)u' &= \lambda((x - x_0) - (y - y_0)) + (1-\lambda)((x' - x_0) - (y' - y_0)) \\ &= (\lambda x + (1-\lambda)x') - x_0 - ((\lambda y + (1-\lambda)y') - y_0) \in U. \end{aligned}$$

It follows the set U is convex. As U is open in $\mathcal{T}$, it is open in the metric topology. Therefore $U \ni 0$ contains an open ball $B(0, \epsilon)$ around the origin. It follows, we can define the so called Minkowski functional p on X, by

$$p : X \to \mathbf{R} : x \mapsto \inf\{\alpha^{-1} : \alpha > 0, \alpha x \in U\}.$$

We will now show p satisfies the hypotheses of the Hahn-Banach Theorem. Let therefore $x, y \in X$ be given. Assume $\alpha x, \beta y \in U$. Then, by convexity of U, also

$$\frac{x + y}{\alpha^{-1} + \beta^{-1}} = \frac{\alpha^{-1}\alpha x + \beta^{-1}\beta y}{\alpha^{-1} + \beta^{-1}} \in U.$$

It follows $p(x + y) \leq \alpha^{-1} + \beta^{-1}$. Since this equality holds for all $\alpha, \beta > 0$, for which $\alpha x \in U$ and $\beta y \in U$, it follows, that

$$p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X.$$

Let now $\alpha > 0$ and $x \in X$ be given. We then have

$$\begin{aligned} p(\alpha x) &= \inf\{\beta^{-1} : \beta\alpha x \in U, \beta > 0\} \\ &= \inf\{\alpha(\alpha\beta)^{-1} : (\alpha\beta)x \in U, (\alpha\beta) > 0\} \\ &= \alpha \inf\{\gamma^{-1} : \gamma x \in U, \gamma > 0\} = \alpha p(x). \end{aligned}$$

We conclude the functional p satisfies the hypotheses of the Hahn-Banach Theorem. Define now $z_0 = y_0 - b_0$. As B and Y are disjoint, $z \notin U$. It follows $p(z_0) \geq 1$. Define a linear functional λ on $\mathbf{R}z_0$, by

$$\lambda : \mathbf{R}z_0 \to \mathbf{R} : tz_0 \mapsto t.$$

For every $t > 0$, it holds that

$$\lambda(tz_0) = t \leq tp(z_0) = p(tz_0).$$

For every $t \leq 0$, we have

$$\lambda(tz_0) = t \leq 0 \leq p(tz_0).$$

Combining these inequalities, it follows for every $t \in \mathbf{R}$, that

$$\lambda(tz_0) \leq p(tz_0).$$

By the Hahn-Banach theorem, there exists a linear extension Λ on X of λ. By definition of p, it holds that $\Lambda \leq 1$ on U. By linearity, $\Lambda \geq -1$ on $-U$. We conclude that $\|\Lambda\| \leq 1$ on the open set $\tilde{U} = U \cap -U$ that contains 0. We will prove that this implies that Λ is continuous. Let $B(0, \epsilon)$ be an arbitrary ball in $\mathbf{R}$. By linearity, it holds for $x \in (\epsilon\tilde{U})$, that $\|\Lambda\| < \epsilon$. As $\epsilon\tilde{U}$ is open, Λ is continuous at 0. Take now an arbitrary ball $B(y, \epsilon)$ in $\mathbf{R}$. By linearity, there exists an $x \in X$, such that $\Lambda(x) = y$. Obviously $x + \epsilon\tilde{U}$ is open. It holds that

$$\Lambda(x + \epsilon\tilde{U}) = \Lambda(x) + \Lambda(\epsilon\tilde{U}) \subset y + B(0, \epsilon) = B(y, \epsilon).$$

By definition, the inverse image of every ball in $\mathbf{R}$ is open. We conclude Λ is continuous.

As Λ is bounded by p, we have, for every $b \in B, y \in Y$,

$$\Lambda(b) - \Lambda(y) + 1 = \Lambda(b - y + z_0) \leq p(b - y + z_0) < 1,$$

as $b - y + z_0 = (b - b_0) - (y - y_0) \in U$. This implies $\Lambda(b) < \Lambda(y)$ for all $b \in B$, $y \in Y$. As a linear functional maps convex sets to convex sets, $\Lambda(B)$ and $\Lambda(Y)$ are disjoint intervals. As Λ is continuous and surjective, and B is open, $\Lambda(B)$ is an open interval by the Open Mapping Theorem [16, Corollary 3.19]. Therefore, we have

$$\Lambda(b) < \sup_{b \in B} \Lambda(b) \leq \Lambda(y),$$

for all $b \in B$ and $y \in Y$. This proves the claim. □

The separating linear functional, defined as a consequence of the Hahn-Banach theorem, can be used to show that the set $P^1(G)$ is weakly-$*$ dense in the set of all means, as a subspace of $(L^\infty(G))^*$.

Theorem 3.4.3 *The set $P^1(G)$ is dense in the set of all means on $L^\infty(G)$ with respect to the weak-$*$ topology.*

Proof The embedding $P^1(G) \hookrightarrow M^1(G)$ implies that every $\phi \in P^1(G)$ can be viewed as a measure on G. By definition it holds that

$$\int_G \mathbf{1}_G(y)\phi(y)dm_L(y) = \int_G \phi(y)dm_L(y) = 1$$

and

$$\int_G f(y)\phi(y)dm_L(y) \geq 0$$

for every $0 \leq f \in L^\infty(G)$. By linearity of the integral, ϕ induces a mean on $L^\infty(G)$. It follows $P^1(G)$ can be viewed as a subset of the set of means on $L^\infty(G)$.

It remains to show that $P^1(G)$ is dense in the set of means. Consider therefore the topological vector space $X = L^\infty(G)^*$, with the weak-$*$ topology. Recall that the weak-$*$ topology is weaker than the metric topology on X, induced by the operator norm. Note that $P^1(G)$, and therefore also its weak-$*$closure, are convex. Define now $Y = \overline{P^1(G)}$ as this closure of $P^1(G)$. By definition, the complement of Y is open. Suppose now that there exists a mean M on $L^\infty(X)$ in the complement $X \setminus Y$. As X is locally convex with respect to the weak-$*$ topology, there exists a convex open set B, that contains M and has empty intersection with Y. Theorem 3.4.2 gives us a linear functional $\Lambda \in X^*$, and a constant c, such that

$$\Lambda(M) < c \quad \text{and} \quad \Lambda(N) \geq c,$$

for all $N \in Y$. By Theorem 2.1.13, there exits $f \in L^\infty(G)$, such that $x(f) = \Lambda(x)$, for all $x \in X$. It follows, for $f' = \frac{f}{c} \in L^\infty(G)$, that

$$M(f') = M\left(\frac{f}{c}\right) = \frac{\Lambda(M)}{c} < 1 \quad \text{and} \quad N(f') = N\left(\frac{f}{c}\right) = \frac{\Lambda(N)}{c} \geq 1, \quad (3.3)$$

for all $N \in Y$. For every $f' \in L^\infty(G)$ it holds, by definition, that $(\operatorname{ess\,inf} f')\mathbf{1}_G \leq f'$. As every mean M is positive, this implies

$$\operatorname{ess\,inf} f' = M\left((\operatorname{ess\,inf} f')\mathbf{1}_G\right) \leq M(f') < 1.$$

Again by definition there exists a measurable set A with finite measure, for which

$$\int_A f'(x)dm_L(x) < m_L(A).$$

This implies, for $\phi = m_L(A)^{-1}\mathbf{1}_A \in P^1(G) \subset Y$, that

$$\phi(f') = \int_G f'(x)\phi(x)dm_L(x) = \frac{1}{m_L(A)}\int_A f'(x)dm_L(x) < 1.$$

As this contradicts (3.3), we see such a mean M can not exist. This implies every mean M is in the weak-$*$ closure of $P^1(G)$. □

The previous lemma shows that $P^1(G)$ is weak-$*$ dense in the set of all means. Therefore, we can find a net $\{\phi_i\}$ in $P^1(G)$, that converges to any given mean M.

We will now apply these theorems to an $M^1(G)$-invariant mean M on $L^\infty(G)$. The existence of such a mean guarantees the existence of a net $\{\phi_i\}$, that is asymptotically invariant.

Theorem 3.4.4 *Let G be a locally compact group. Then there exists an $M^1(G)$-invariant mean on $L^\infty(G)$, if and only if there exists a net $\{\phi_i\}$ in $P^1(G)$, such that for every $\mu \in M^1(G)$, the net $(\mu * \phi_i - \phi_i)$ converges pointwise to 0.*

Proof Let M be an $M^1(G)$-invariant mean on $L^\infty(G)$. As $P^1(G)$ is dense in the set of all means on $L^\infty(G)$, there exists a net $\{\phi_i\}$ in $P^1(G)$, that converges pointwise to M,

$$\lim_i \phi_i(f) = M(f) \quad \forall f \in L^\infty(G).$$

Let now $\mu \in M^1(G)$ be arbitrary. Then $\tilde{\mu} : A \mapsto \mu(A^{-1})$ is a bounded mean also. Therefore, for every $f \in L^\infty(G)$, we have $\tilde{\mu} * f \in L^\infty(G)$. This implies

$$\begin{aligned}
(\mu * \phi_i)(f) &= \int_G \int_G \phi_i(y^{-1}z)d\mu(y)f(z)dm_L(z) \\
&= \int_G \int_G \phi_i(z)f(yz)d\mu(y)dm_L(z) \\
&= \int_G \int_G \phi_i(z)f(y^{-1}z)d\mu(y^{-1})dm_L(z) \\
&= \int_G \phi_i(z)\int_G f(y^{-1}z)d\tilde{\mu}(y)dm_L(z) \\
&= \int_G \phi_i(z)(\tilde{\mu} * f)(z)dm_L(z) = \phi_i(\tilde{\mu} * f).
\end{aligned}$$

Subtracting from both sides $\phi_i(f)$, we find

$$(\mu * \phi_i - \phi_i)(f) = \phi_i(\tilde{\mu} * f - f).$$

Taking the limit on both sides we obtain, using $M^1(G)$-invariance of the mean

$$\lim_{i\to\infty}(\mu * \phi_i - \phi_i)(f) = \lim_{i\to\infty}\phi_i(\tilde{\mu} * f - f) = M(\tilde{\mu} * f - f) = 0.$$

We conclude the net $\{\mu * \phi_i - \phi_i\}$ converges pointwise to 0, that is,

$$\lim_i(\mu * \phi_i - \phi_i)(f) = 0,$$

for all $f \in L^\infty(G)$.

Let now a net $\{\phi_i\}$ in $P^1(G) \subset (L^\infty(G))^*$ be given, for which $\{\mu * \phi_i - \phi_i\}$ converges pointwise to 0 for every $\mu \in M^1(G)$. As the unit ball is weak-$*$ compact due to Alaoglu's Theorem 2.1.15, and the set of means is the weak-$*$ closure of $P^1(G)$, it follows the set of means is compact in $L^\infty(G)^*$. As a net in the set of means, the net $\{\phi_i\}$ must therefore have a converging subnet $\{\phi_{i_j}\}$, indexed by j. Denote M for the limit of this subnet. We then have for every $\mu \in M^1(G)$ and $f \in L^\infty(G)$, as above

$$M(\mu * f - f) = \lim_{j\to\infty} \phi_{i_j}(\mu * f - f) = \lim_{j\to\infty} (\tilde{\mu} * \phi_{i_j} - \phi_{i_j})(f) = 0.$$

Therefore, the subnet converges to a mean M that is $M^1(G)$-invariant. □

The theorems so far only deal with pointwise, or weak, convergence. We would like to conclude the existence of similar nets that converge in the normed topology. The following lemma tells us such conclusions can be drawn.

Lemma 3.4.5 *Let $(E, \mathcal{T})$ be a locally convex topological space. Then a convex subset $K \subset E$ is closed if and only if it is weakly closed.*

Proof By construction the weak topology is weaker than the original topology, therefore $\mathcal{W} \subset \mathcal{T}$. This proves that the subset K is closed when it is weakly closed. To prove the other direction, let $x \in E \setminus K$ be given. As E locally convex, there exists a convex open set U that contains X. By the Hahn-Banach theorem 3.4.2, there exists $y \in F = E^*$, such that

$$y(x) < c \le y(k), \quad \forall k \in K.$$

With this linear operator y, we define the set

$$\{x' \in X : |y(x') - y(x)| < c - y(x)\},$$

that is a weakly open set, disjoint from K and containing x. We conclude K is weakly closed. □

For σ-finite groups G, the dual of $L^1(G)$ is given by $L^\infty(G)$. For $G = \mathbf{R}^n$, this is proven in [1, Theorem 2.34]. The proof given there extends straightforwardly to the more general case where G is a σ-finite measure space. When we use this in combination with the previous lemma, we get the following.

Theorem 3.4.6 *Let G be a σ-locally compact group. Let $\{\phi_i\}$ be a net, such that for all $\mu \in M^1(G)$, the net $\{\mu * \phi_i - \phi_i\}$ converges pointwise to zero in $L^\infty(G)^*$. Then there exists a net ψ_i, such that, for every $\mu \in M^1(G)$,*

$$\lim_i \|\mu * \psi_i - \psi_i\|_1 = 0.$$

Proof We first remark that the net $\{\mu * \phi_i - \phi_i\}$ can be viewed as a net in $L^1(G)$. As $(\mu * \phi_i - \phi_i)(f) \to 0$ for all $f \in L^\infty(G) = L^1(G)^*$, we have that the net $\{\mu * \phi_i - \phi_i\}$ converges to 0 in the weak topology of $L^1(G)$ for every $\mu \in M^1(G)$.

Consider now the product space $(L^1(G))^{M^1(G)}$ with the subset

$$N\left(P^1(G)\right) = \{(\mu * \phi - \phi)_{\mu \in M^1(G)} \,:\, \phi \in P^1(G)\} \subset (L^1(G))^{M^1(G)}.$$

Convexity of this set follows easily because $P^1(G)$ is convex. Furthermore, by assumption, $\mu * \phi_i - \phi_i$ converges weakly to 0 in every coordinate space, so that by definition in the product of weak topologies

$$(\mu * \phi_i - \phi_i)_{\mu \in M^1(G)} \to 0 \in (L^1(G))^{M^1(G)}.$$

By Theorem 2.1.14, the weak topolgy on a product space equals the product of the weak topologies, so the convergence also holds in the weak topology on $(L^1(G))^{M^1(G)}$. It follows 0 is an element in the weak closure of $N\left(P^1(G)\right)$. As the weak and strong closure are equal for convex sets by Lemma 3.4.5, it follows 0 is also in the strong closure of $N\left(P^1(G)\right)$. (The strong topology is the product of the normed topologies.) We conclude there exists a net $\{\psi_i\}$ that converges to 0 in the strong topology. By definition of the product topology, in every coordinate space we have

$$\lim_i \|\mu * \psi_i - \psi_i\| = 0.$$

This proves the claim. □

Restricting the set of measures from $M^1(G)$ to the set of measures whose support is contained in some compact set K, one can choose the sequence such that the convergence holds uniformly.

Theorem 3.4.7 *Let G be a σ-locally compact group. Let $\{\phi_i\}$ be a sequence, such that $\|\mu * \phi_i - \phi_i\|_1$ converges to zero, for all $\mu \in M^1(G)$. Then there exists a sequence $\{\psi_i\}$ in $P^1(G)$, such that for every compact subset K of G, it holds that*

$$\lim_{i \to \infty} \|\mu * \psi_i - \psi_i\|_1 = 0,$$

uniformly for every $\mu \in M^1(G)$, for which the support of μ is contained in K.

Proof Let the net $\{\phi_i\}$ be given, such that $\|\mu * \phi_i - \phi_i\|$ converges to zero for every $\mu \in M^1(G)$. Choose $\phi \in P^1(G)$ arbitrarily and define $\psi_i = \phi * \phi_i$. By construction we have

$$\begin{aligned}\|\mu * \psi_i - \psi_i\|_1 &= \|\mu * \phi * \phi_i - \phi_i + \phi_i - \phi * \phi_i\|_1 \\ &\leq \|(\mu * \phi) * \phi_i - \phi_i\|_1 + \|\phi * \phi_i - \phi_i\|_1.\end{aligned}$$

It follows $\|\mu * \psi_i - \psi_i\|_1 \to 0$. Let now ϵ be given. As the mapping $z \mapsto L_z\phi$ is right uniformly continuous for every $\phi \in P^1(G)$ by Theorem 2.3.6, there exists a neighborhood U of e in G, such that

$$\|L_z\phi - \phi\|_1 < \epsilon$$

for all $z \in U$. It follows, for every i, that

$$\begin{aligned}
\|L_z\psi_i - \psi_i\|_1 &= \left\| x \mapsto \int_G \phi(y)\phi_i(y^{-1}zx)dm_L(y) - \int_G \phi(y)\phi_i(y^{-1}x)dm_L(y) \right\|_1 \\
&= \left\| x \mapsto \int_G \phi(zy)\phi_i(y^{-1}x)dm_L(y) - \int_G \phi(y)\phi_i(y^{-1}x)dm_L(y) \right\|_1 \\
&= \left\| x \mapsto \int_G (\phi(zy) - \phi(y))\phi_i(y^{-1}x)dm_L(y) \right\|_1 \\
&= \|(L_z\phi - \phi) * \phi_i\|_1 \leq \|L_z\phi - \phi\|_1 \|\phi_i\|_1 < \epsilon.
\end{aligned}$$

Let now a compact set $K \subset G$ be given. Cover K by the open sets $\{a^{-1}U^{-1}\}_{a\in G}$. By compactness, there is a finite set $\{a_1, \ldots, a_n\}\subset G$, such that $K \subset \bigcup_{j=1}^n a_j^{-1}U^{-1}$. As $\|\mu * \psi_i - \psi_i\|$ converges to zero for every μ, there is an index $i_0 \in I$, such that for every $1 \leq j \leq n$ and every $i > i_0$

$$\left\|L_{a_j}\psi_i - \psi_i\right\|_1 = \left\|\delta_{a_j^{-1}} * \psi_i - \psi_i\right\|_1 \leq \epsilon.$$

This implies, for every $i > i_0$, $1 \leq j \leq n$ and $z \in U$, that

$$\begin{aligned}
\|L_{za_j}\psi_i - \psi_i\|_1 &= \left\|L_z\left(L_{a_j}\psi_i - \psi_i\right) + L_z\psi_i - \psi_i\right\|_1 \\
&\leq \|L_{a_j}\psi_i - \psi_i\|_1 + \|L_z\psi_i - \psi_i\|_1 \leq 2\epsilon.
\end{aligned}$$

For every $y \in K$ there exist $z \in U$ and $1 \leq j \leq n$, such that $y^{-1} = za_j$. Therefore we have $\|L_{y^{-1}}\psi_i - \psi_i\| < 2\epsilon$ for every $y \in K$. This implies for every $\mu \in M^1(G)$ such that the support of μ is contained in K, that

$$\begin{aligned}
\|\mu * \psi_i - \psi_i\|_1 &= \int_G \left| \int_G \psi_i\left(y^{-1}x\right) d\mu(y) - \psi_i(x) \right| dm_L(x) \\
&= \int_G \left| \int_G \psi_i\left(y^{-1}x\right) d\mu(y) - \psi_i(x) \int_G d\mu(y) \right| dm_L(x) \\
&\leq \int_G \int_G \left|\psi_i\left(y^{-1}x\right) - \psi_i(x)\right| d\mu(y)dm_L(x) \\
&\leq \int_K \int_G \left|\psi_i\left(y^{-1}x\right) - \psi_i(x)\right| dm_L(x)d\mu(y) \\
&= \int_K \|L_{y^{-1}}\psi_i - \psi_i\|_1 d\mu(y) \leq 2\epsilon m_L(K).
\end{aligned}$$

By definition, the net converges uniformly for every μ whose support is contained in K. □

A consequence of this theorem is the following. It follows easily from the embedding $a \mapsto \delta_a$ of G into $M^1(G)$, as by definition it holds that $L_a\phi = \delta_{a^{-1}} * \phi$. This condition is known in literature as Reiter's condition.

Corollary 3.4.8 *Let G be a σ-locally compact group. If G is amenable, then for every compact set $K \subset G$ and every $\epsilon > 0$, there exists $\phi \in P^1(G)$, such that $\|L_a\phi - \phi\|_1 < \epsilon$ for all $a \in K$.*

The converse of this corollary also holds. In fact it is even true for locally compact groups G that are not σ-finite.

Theorem 3.4.9 *Let G be a locally compact group. If for every compact set $K \subset G$ and every $\epsilon > 0$ there exists $\phi \in P^1(G)$, such that $\|L_a\phi - \phi\| < \epsilon$ for every $a \in K$, then there exists a left invariant mean on G.*

Proof We define, for each compact set $K \subset G$ and every $\epsilon > 0$, the set

$$A_{K,\epsilon} = \{\phi \in P^1(G) \,:\, \forall a \in K \quad \|L_a\phi - \phi\|_1 < \epsilon\} \subset P^1(G).$$

By assumption, this set is non-empty for every $\epsilon > 0$ and every compact subset K. Equip $L^\infty(G)$ with the weak-$*$ topology. Then the closure of $P^1(G)$ equals the set of all means by Theorem 3.4.3, and this implies that the closure of each $A_{K,\epsilon}$ is contained in the set of means. Consider now the following collection of closed subsets in the set of means on $L^\infty(G)$,

$$\left\{\overline{A_{K,\epsilon}} : K \subset G \text{ compact}, \epsilon > 0\right\}.$$

Choose a finite numbers of elements in this collection. Then, as the finite union of compact sets is again compact, and the minimum of a finite number of positive numbers is again positive, this finite collection has a non-empty intersection. By definition this collection of closed sets has the Finite Intersection Property. As the set of means is weakly-$*$ compact by Alaoglu's Theorem 2.1.15, we obtain that the intersection

$$\bigcap\{\overline{A_{K,\epsilon}} : K \subset G \text{ compact}, \epsilon > 0\} \neq \emptyset.$$

Now select an arbitrary element ϕ from this intersection. Then for every $f \in L^\infty(G)$, $a \in G$ and $\epsilon > 0$, we have, as $\phi \in \overline{A_{\{a\},\epsilon}}$, that there exists a net $\{\phi_i\} \in A_{\{a\},\epsilon}$, that converges weakly-$*$ to ϕ. For every i we have

$$\begin{aligned}
\phi_i(L_{a^{-1}}f - f) &= \int_G \left(f\left(a^{-1}x\right) - f(x)\right)\phi_i(x)dm_L(x) \\
&= \int_G f(a^{-1}x)\phi_i(x)dm_L(x) - \int_G f(x)\phi_i(x)dm_L(x) \\
&= \int_G f(x)\phi_i(ax)dm_L(x) - \int_G f(x)\phi_i(x)dm_L(x) \\
&= \int_G f(x)(L_a\phi_i(x) - \phi_i(x))dm_L(x).
\end{aligned}$$

Taking now the absolute value of this expression, we find for every i

$$\begin{aligned}|\phi_i(L_{a^{-1}}f - f)| &\leq \int_G |f(x)(L_a\phi_i(x) - \phi_i(x))|\, dm_L(x) \\ &= \|f(x)(L_a\phi_i(x) - \phi_i(x)\|_1 \leq \|f\|_\infty \|L_a\phi_i - \phi_i\|_1 = \epsilon\|f\|_\infty,\end{aligned}$$

using Hölder's inequality. By definition of weak-$*$ convergence, we have

$$\phi\left(L_{a^{-1}}f - f\right) = \lim_i \phi_i\left(L_{a^{-1}}f - f\right).$$

Using that the norm on $\mathbf{R}$ is continuous, it follows

$$|\phi\left(L_{a^{-1}}f - f\right)| = \lim_i |\phi_i\left(L_{a^{-1}}f - f\right)| \leq \epsilon\|f\|_\infty.$$

As this holds for all ϵ, we see $\phi(L_{a^{-1}}f) = \phi(f)$. By definition the mean ϕ is a left invariant mean. □

Reiter's condition can even be weakened. If it holds for finite subsets F, we already obtain that G is amenable. We will need the following lemma.

Lemma 3.4.10 *Let K be a subset of a locally compact group G and let $\epsilon > 0$ be given. Then for every $\phi \in P^1(G)$ there exists $\psi \in P^1(G)$, such that $\|\psi * L_b\phi - L_b\phi\|_1 < \epsilon$ for every $b \in K$. Furthermore, there exists an open neighborhood W of e in G, such that $\|\psi * \delta_{ab^{-1}} - \psi\|_1 < \epsilon$, for all $a, b \in W$.*

Proof Let ϵ be given. As the mapping $x \mapsto L_x\phi$ is right uniformly continuous by Theorem 2.3.6, there exists an open neighborhood U of e in G, such that $\|L_{x^{-1}}\phi - \phi\| < \epsilon$ for all $x \in U$. By compactness of K and Theorem 2.1.23 there exists a neighborhood V such that $bVb^{-1} \subset U$ for all $b \in K$. Define now $\psi = m_L(V)^{-1}\mathbf{1}_V$. Then for all $b \in K$, it follows that

$$\begin{aligned}\|\psi * L_b\phi - L_b\phi\|_1 &= \int_G \left| \int_G \psi(z)\phi(bz^{-1}x)dz - \int_G \psi(z)\phi(bx)dm_L(z) \right| dm_L(x) \\ &\leq \int_G \psi(z) \int_G \left|\phi(bz^{-1}b^{-1}x) - \phi(x)dm_L(x)\right| dm_L(z) \\ &= \int_V \psi(z)\|L_{bz^{-1}b^{-1}}\phi - \phi\|_1 dm_L(z) < \epsilon,\end{aligned}$$

because for all $z \in V$, it holds that $bz^{-1}b^{-1} \in U$. Define now the function $\psi^* : x \mapsto \psi(x^{-1})\Delta(x^{-1})$. By Theorem 2.3.2, it follows that $\psi^* \in P^1(G)$. Again by right uniform continuity of $x \mapsto L_x\psi^*$, there exists a compact neighborhood W' of e in G, such that $\|L_x\psi^* - \psi^*\|_1 < \epsilon$ for all $x \in W'$. Let now W be a open neighborhood of e such that $ab^{-1} \in W'$ for all $a, b \in W$. Such a neighborhood exists by Theorem 2.1.22. Select $a, b \in W$. We then find, with $w = ab^{-1} \in W'$, that

$$\begin{aligned}\|\psi * \delta_w - \psi\|_1 &= \int_G \left| \int_G \psi\left(xz^{-1}\right) \Delta\left(z^{-1}\right) d\delta_w(z) - \psi(x) \right| dm_L(x) \\ &= \int_G \left|\psi(xw^{-1})\Delta(w^{-1}) - \psi(x)\right| dm_L(x) \\ &= \int_G \left|\psi\left(x^{-1}w^{-1}\right) \Delta\left(x^{-1}w^{-1}\right) - \psi\left(x^{-1}\right) \Delta\left(x^{-1}\right)\right| dm_L(x) \\ &= \int_G \left|\psi^*(wx) - \psi^*(x)\right| dm_L(x) = \|L_w\psi^* - \psi^*\|_1 < \epsilon.\end{aligned}$$

This proves the claims. □

Using this lemma, we can prove that Reiter's weakened condition implies amenability.

Theorem 3.4.11 *Let G be a locally compact group. If for every finite set $F \subset G$ and every $\epsilon > 0$ there exists $\phi \in P^1(G)$, such that $\|L_a\phi - \phi\|_1 < \epsilon$ for every $a \in F$, then G is amenable.*

Proof Let a compact subset $K \subset G$ and $\epsilon > 0$ be given. Define the compact subset $\tilde{K} = K \cup K^{-1} \cup \{e\}$ of G. Select $\phi \in P^1(G)$ arbitrarily. By Theorem 2.3.7, there exists a compact set $C' \subset G$, such that

$$\int_{G \setminus C'} \phi(x) < \epsilon.$$

Define now the set $C = \tilde{K}C'$. Then for every $b \in \tilde{K}$, we have $bC = b\tilde{K}C' \supset C'$. Hence, for every $b \in \tilde{K}$,

$$\int_{G\setminus C} \phi(bx) dm_L(x) = \int_{G\setminus bC} \phi(x) dm_L(x) \le \int_{G \setminus C'} \phi(x) dm_L(x) < \epsilon.$$

By Theorem 2.1.21, the set C is compact. Consider now the function $\psi \in P^1(G)$ and the open neighborhood W of e given by the previous lemma. Cover C by the sets $\{Wc\}_{c\in C}$. By compactness there is a finite subset $\{c_1, \dots, c_n\} \subset C$, such that $C \subset \bigcup_{i=1}^n Wc_i$. Defining inductively $B_i = C \cap \left(Wc_i \setminus \bigcup_{j=1}^{i-1} B_j\right)$, we obtain a finite partition of C. Pick an element $b_i \in B_i$ for all $1 \le i \le n$. As $B_i \subset Wc_i$, every $x \in B_i$ can be written as $x = wc_i$ for some $w \in W$. As also $b_i = w'c_i$ for some $w' \in W$, it follows that $xb_i^{-1} = ww'^{-1}$. By the previous lemma, it follows that

$$\|\psi * \delta_x - \psi * \delta_{b_i}\|_1 = \left\| \left(\psi * \delta_{xb_i^{-1}} - \psi\right) * \delta_{b_i} \right\|_1 = \left\| \psi * \delta_{xb_i^{-1}} - \psi \right\|_1 < \epsilon. \quad (3.4)$$

By assumption, there exists $\phi' \in P^1(G)$, such that

$$\|\delta_{b_i} * \phi' - \phi'\|_1 < \epsilon, \quad (3.5)$$

for all $1 \leq i \leq n$. By the first conclusion of the previous lemma, we then find, for every $b \in \tilde{K}$, that

$$\begin{aligned} \|L_b(\phi * \phi') - \psi * L_b(\phi * \phi')\|_1 &= \|(L_b\phi) * \phi' - \psi * (L_b\phi) * \phi'\|_1 \\ &\leq \|\psi * L_b\phi - L_b\phi\|_1 \|\phi'\|_1 < \epsilon. \end{aligned} \tag{3.6}$$

By definition of the convolutions we also have

$$\begin{aligned} &(\psi * (L_b\phi) * \phi')(z) \\ &= \int_G (\psi * (L_b\phi))(y)\phi'\left(y^{-1}z\right) dm_L(y) \\ &= \int_G \int_G \psi(x)(L_b\phi)\left(x^{-1}y\right)\phi'\left(y^{-1}z\right) dm_L(x)dm_L(y) \\ &= \int_G \int_G \psi(yx)(L_b\phi)\left(x^{-1}\right)\phi'\left(y^{-1}z\right) dm_L(y)dm_L(x) \\ &= \int_G \int_G \psi\left(yx^{-1}\right)\Delta\left(x^{-1}\right)(L_b\phi)(x)\phi'\left(y^{-1}z\right) dm_L(y)dm_L(x) \\ &= \int_G \int_G \int_G \psi\left(yw^{-1}\right)\Delta\left(w^{-1}\right) d\delta_x(w)\phi'\left(y^{-1}z\right) dm_L(z)dm_L(y)(L_b\phi)(x)dm_L(x) \\ &= \int_G \int_G (\psi * \delta_x)(y)\phi'\left(y^{-1}z\right) dm_L(y)(L_b\phi)(x)dm_L(x) \\ &= \int_G (\psi * \delta_x * \phi')(z)(L_b\phi)(x)dm_L(x). \end{aligned} \tag{3.7}$$

Combining now the above equations, for every $b \in \tilde{K}$, we conclude that

$$\begin{aligned} &\|\psi *(L_b\phi) * \phi' - \psi * \phi'\| \\ &\quad= \int_G |(\psi * (L_b\phi) * \phi')(z) - \psi * \phi'(z)|\, dm_L(z) \\ &\quad= \int_G \left| \int_G [(\psi * \delta_x * \phi')(z)(L_b\phi)(x) - (L_b\phi)(x)(\psi * \phi')(z)]\, dm_L(x) \right| dm_L(z) \\ &\quad= \int_G \int_G |(L_b\phi)(x)\left((\psi * \delta_x * \phi')(z) - (\psi * \phi')(z)\right)| \, dm_L(z)dm_L(x) \\ &\quad= \int_G (L_b\phi)(x)\|\psi * \delta_x * \phi' - \psi * \phi'\|_1 dm_L(x), \end{aligned}$$

where we used Fubini's Theorem. Splitting now the integration domain in the compact set C and its complement, it follows, that

$$\begin{aligned}
&\|\psi * (L_b\phi) \ *\phi' - \psi * \phi'\| \\
&\qquad = \int_{G\setminus C} (L_b\phi)(x)\|\psi * \delta_x * \phi' - \psi * \phi'\|_1 dm_L(x) \\
&\qquad\quad + \sum_{i=1}^{n} \int_{B_i} (L_b\phi)(x)\|\psi * \delta_x * \phi' - \psi * \phi'\|_1 dm_L(x) \\
&\qquad \leq (\|\psi * \delta_x * \phi'\|_1 + \|\psi * \phi'\|_1) \int_{G\setminus C} (L_b\phi)(x) dm_L(x) \\
&\qquad\quad + \sum_{i=1}^{n} \int_{B_i} (L_b\phi)(x)\|\psi * \delta_x - \psi * \delta_{b_i}\|_1 \|\phi'\|_1 dm_L(x) \\
&\qquad\quad + \sum_{i=1}^{n} \int_{B_i} (L_b\phi)(x)\|\psi\|_1 \|\delta_{b_i} * \phi' - \phi'\|_1 dm_L(x) \leq 2\epsilon + \epsilon + \epsilon = 4\epsilon.
\end{aligned}$$

Define now $\phi'' = \phi * \phi' \in P^1(G)$. For every $a \in K \subset \tilde{K}$, combining the above equations implies, that

$$\begin{aligned}
\|L_a(\phi'') - \phi''\|_1 &\leq \|L_a(\phi * \phi') - \psi * L_a(\phi * \phi')\|_1 + \|\psi * (L_a\phi) * \phi' - \psi * \phi'\|_1 \\
&\quad + \|\psi * \phi' - \psi * (L_e\phi) * \phi'\|_1 + \|\psi * \phi * \phi' - \phi * \phi'\|_1 \\
&< \epsilon + 4\epsilon + 4\epsilon + \epsilon = 10\epsilon.
\end{aligned}$$

It follows the locally compact group G satisfies the hypotheses of Theorem 3.4.9. We conclude G is amenable. □

The following corollary summarizes the results of this section.

Corollary 3.4.12 *Let G be a locally compact group. Then G is amenable if and only if there exists a net $\{\phi_i\}$ in $P^1(G)$, such that for all $\mu \in M^1(G)$, it holds that $\{\mu * \phi_i - \phi_i\}$ converges to 0 in the weak-$*$ topology of $L^\infty(G)^*$.*
If moreover G is σ-compact, then G is amenable if and only if for every finite or compact set $K \subset G$ and every $\epsilon > 0$, there exists a $\phi \in P^1(G)$, such that $\|L_a\phi - \phi\|_1 < \epsilon$, for all $a \in K$.

3.5 Følner's conditions

The conditions in the previous sections all dealt with invariance of functions on G. In this section, we will consider subsets of G that are invariant. This leads to Følner's characterizations of amenability. We will apply the following lemma.

Lemma 3.5.1 *Let G be a locally compact group and let $\psi \in P^1(G)$ be a simple function. Then there exist measurable subsets $A_n \subset \ldots \subset A_1$ of G with finite measure and numbers $\alpha_i > 0$, that add to one, such that*

$$\psi = \sum_{i=1}^{n} \frac{\alpha_i}{m_L(A_i)} \mathbf{1}_{A_i}.$$

Furthermore, for $a \in G$,

$$\|L_{a^{-1}}\psi - \psi\|_1 = \sum_{i=1}^{n} \alpha_i \frac{m_L(aA_i \Delta A_i)}{m_L(A_i)}.$$

Proof As a simple function, ψ takes only a finite number of values $\beta_1, \ldots, \beta_m$. By relabeling, one can assume these values to be increasing. Define A_i for $1 \leq i \leq m$ to be the set of elements $x \in G$ for which $\psi(x) \geq \beta_i$. Then the sets A_i are decreasing. Define $\alpha_1 = \beta_1 m_L(A_1)$ and, for $1 < i \leq n$, $\alpha_i = (\beta_i - \beta_{i-1}) m_L(A_i)$. Then it holds, that

$$\psi = \sum_{i=1}^{m} \frac{\alpha_i}{m_L(A_i)} \mathbf{1}_{A_i}.$$

Let now $a \in G$ be given. Let $i, j \in \{1, \ldots, n\}$ be given. If $i \geq j$, so that $A_i \subset A_j$, then $aA_i \subset aA_j$ which implies that $aA_i \setminus A_i \subset aA_j$. Also if $i < j$, so that $A_j \subset A_i$, then $A_j \setminus aA_j \subset A_i$. In both cases it follows, that

$$(aA_i \setminus A_i) \cap (A_j \setminus aA_j) = \emptyset. \tag{3.8}$$

Define $A = \bigcup_{i=1}^{n} (A_i \setminus aA_i) \subset G$ and choose now $x \in A$. Then there exists i, such that $x \in A_i \setminus aA_i$. By definition it holds that $\mathbf{1}_{A_i}(x) - \mathbf{1}_{aA_i}(x) = 1$. By (3.8), $x \notin aA_j \setminus A_j$, for all $1 \leq j \leq n$, so that $\mathbf{1}_{A_j}(x) - \mathbf{1}_{aA_j}(x) \geq 0$. Now pick $x \in G \setminus A$. By definition, $x \notin A_i \setminus aA_i$, for all $1 \leq i \leq n$, so that $\mathbf{1}_{aA_i}(x) - \mathbf{1}_{A_i}(x) \geq 0$. Combining these results, we obtain, that

$$\begin{aligned}
\|L_{a^{-1}}\psi - \psi\|_1 &= \int_G \left|\psi\left(a^{-1}x\right) - \psi(x)\right| dm_L(x) \\
&= \int_G \left|\sum_{i=1}^{n} \frac{\alpha_i}{m_L(A_i)} \left(\mathbf{1}_{A_i}\left(a^{-1}x\right) - \mathbf{1}_{A_i}(x)\right)\right| dm_L(x) \\
&= \int_A \sum_{i=1}^{n} \frac{\alpha_i}{m_L(A_i)} \left(\mathbf{1}_{A_i}(x) - \mathbf{1}_{aA_i}(x)\right) dm_L(x) \\
&\quad + \int_{G\setminus A} \sum_{i=1}^{n} \frac{\alpha_i}{m_L(A_i)} \left(\mathbf{1}_{aA_i}(x) - \mathbf{1}_{A_i}(x)\right) dm_L(x) \\
&= \sum_{i=1}^{n} \frac{\alpha_i}{m_L(A_i)} \int_G \left|\mathbf{1}_{aA_i}(x) - \mathbf{1}_{A_i}(x)\right| dm_L(x) \\
&= \sum_{i=1}^{n} \alpha_i \frac{m_L(aA_i \Delta A_i)}{m_L(A_i)}.
\end{aligned}$$

This proves the claim. □

We will now apply this lemma to prove that Reiter's condition and Følner's conditions are equivalant. To do so, we first prove that every amenable group satisfies a weaker version of Følner's condition.

Theorem 3.5.2 *Let G be a locally compact group. Assume for every $\epsilon' > 0$ and compact set $K' \subset G$, there exists $\phi \in P^1(G)$, such that $\|L_a\phi - \phi\|_1 < \epsilon'$ for all $a \in K'$. Then, for every compact set $K \subset G$ and $\epsilon, \delta > 0$, there exists a measurable subset U of G with finite measure and a subset N of U, with measure $m_L(N) < \delta$, such that $m_L(aU\Delta U) < \epsilon m_L(U)$, for all $a \in K \setminus N$.*

Proof Let K and $\delta, \epsilon > 0$ be given. If $m_L(K) = 0$, one can take $N = K$. Assume now that $m_L(K) \neq 0$. Define $\alpha^{-1} = 1 + 3m_L(K)(\delta\epsilon)^{-1} > 1$. By assumption, there exists a $\phi \in P^1(G)$, such that $\|L_{x^{-1}}\phi - \phi\| < \alpha$, for every $x \in K$. By definition of the Lebesgue intergral, there exists a simple function ψ, such that $\|\phi - \psi\|_1 < \alpha$. As

$$1 = \|\phi\|_1 \leq \|\phi - \psi\|_1 + \|\psi\|_1 = \alpha + \|\psi\|_1,$$

we conclude, that $\|\psi\|_1 > 1 - \alpha > 0$. Define the simple function $\psi' = \psi/\|\psi\|_1 \in P^1(G)$. Then we have, for every $x \in K$, that

$$\begin{aligned}\|L_{x^{-1}}\psi' - \psi'\|_1 &= \frac{\|L_{x^{-1}}\psi - \psi\|_1}{\|\psi\|_1} \\ &= \frac{\|L_{x^{-1}}(\psi - \phi)\|_1 + \|L_{x^{-1}}\phi - \phi\|_1 + \|\phi - \psi\|_1}{\|\psi\|_1} < \frac{3\alpha}{1-\alpha}.\end{aligned}$$

Denoting now $a = 3m_L(K)(\delta\epsilon)^{-1}$, so that $1 - \alpha = 1 - 1/(1 + a) = a/(1 + a)$, we conlude, that

$$\|L_{x^{-1}}\psi' - \psi'\|_1 < \frac{3\alpha}{1-\alpha} = \frac{3(1+a)}{a(1+a)} = \frac{\delta\epsilon}{m_L(K)}.$$

The previous lemma, applied to the simple function ψ' gives us sets $A_i \subset G$ and numbers $\alpha_i > 0$, for which

$$\sum_{i=1}^{n} \alpha_i \frac{m_L(xA_i\Delta A_i)}{m_L(A_i)} = \|L_{x^{-1}}\psi' - \psi'\|_1 < \frac{\delta\epsilon}{m_L(K)}.$$

After integrating this equation over K, we obtain

$$\sum_{i=1}^{n} \alpha_i \left(\int_K \frac{m_L(xA_i\Delta A_i)}{m_L(A_i)} dm_L(x) \right) < \delta\epsilon.$$

As the sum in this equality is a convex combination, smaller than some constant. there must be a summand that is smaller than that constant. Let i' be the index of that summand. Then we have

$$\int_K \frac{m_L(xA_{i'}\Delta A_{i'})}{m_L(A_{i'})} dm_L(x) < \delta\epsilon.$$

Choosing now $U = A_{i'}$ and

$$N = \left\{ x \in K : \quad \frac{m_L(xA_{i'}\Delta A_{i'})}{m_L(A_{i'})} \geq \epsilon \right\},$$

shows the condition is satisfied. □

Although this condition seems to be weaker than Følner's original condition, the equivalence follows quite easily.

Theorem 3.5.3 *Let G be a locally compact group. Assume that for every compact set $K \subset G$ and $\epsilon, \delta > 0$, there exists a measurable set $U \subset G$ with finite measure and a measurable set $N \subset G$ with measure $m_L(N) < \delta$, such that $m_L(aU\Delta U) < \epsilon m_L(U)$, for all $a \in K \setminus N$. Then the set N may be chosen to be the empty set.*

Proof Let a compact set $K \subset G$ and $\epsilon > 0$ be given. By local compactness, there exists a compact neighborhood V' of e in G. Define $V = K \cup V'$ and $W = V^2$. Then V and W are compact. For every $a \in K \subset V$ and $x \in V$, $ax \in W \cap aW$. This implies

$$m_L(W \cap aW) \geq m_L(aV) = m_L(V) > 0,$$

as V contains an open set. Set $\delta = \frac{1}{2}m_L(V)$. By assumption, there exist subsets U and N of G, such that $m_L(N) < \delta$, $N \subset W$ and $m_L(wU\Delta U) < \frac{1}{2}\epsilon m_L(U)$ for all $w \in W \setminus N$. Then, by the above inequality, for every $a \in K$, it holds that

$$\begin{aligned}
2\delta &= m_L(V) \\
&\leq m_L(W \cap aW) \\
&\leq m_L((W \setminus N) \cap a(W \setminus N)) + m_L(N) + m_L(aN) \\
&< m_L\left((W \setminus N) \cap a(W \setminus N)\right) + 2\delta.
\end{aligned}$$

It follows $(W \setminus N) \cap a(W \setminus N)$ has positive measure and is therefore non-empty. Hence, there exist $y, z \in W \setminus N$, such that $a = yz^{-1}$. This implies, that

$$\begin{aligned}
m_L(aU\Delta U) &= m_L(yz^{-1}U\Delta U) \\
&= m_L(z^{-1}U\Delta y^{-1}U) \\
&\leq m_L(z^{-1}U\Delta U) + m_L(U\Delta y^{-1}U) \\
&= m_L(U\Delta zU) + m_L(yU\Delta U) = \epsilon m_L(U),
\end{aligned}$$

using a triangular-like inequality for $A, B \mapsto m_L(A\Delta B)$. The set U now satisfies the claims. □

The previous theorem shows that for every compact set K of an amenable group and every $\epsilon > 0$, there exists a set U, that is almost invariant under multiplication by every $k \in K$. We will now show that the set U can be chosen to be compact.

Theorem 3.5.4 *Let G be an amenable group and $K \subset G$ a compact set. For every $\epsilon > 0$, there exists a compact set $U \subset G$ with finite measure, such that $m_L(aU\Delta U) < \epsilon m_L(U)$ for every $a \in K$.*

Proof Let $K \subset G$ and $\epsilon > 0$ be given. By the previous theorem, there exists a measurable set $U \subset G$ such that $m_L(aU\Delta U) < (\epsilon/6)m_L(U)$ for all $a \in K$. As Haar measure is inner regular, there exists a compact set $U' \subset U$, for which

$$m_L(U \setminus U') < \min\left\{\frac{m_L(U)}{2}, \frac{m_L(U)}{6}\epsilon\right\}.$$

As $m_L(U \setminus U') < m_L(U)/2$ and U' is compact, it follows, that $0 < m_L(U') < \infty$. It also holds, that

$$m_L(U) = m_L(U') + m_L(U \setminus U') < m_L(U') + \frac{m_L(U)}{2}.$$

We conclude that $2m_L(U') > m_L(U)$. For every $a \in K$, it now holds, that

$$\begin{aligned}\frac{m_L(aU'\Delta U')}{m_L(U')} &\leq \frac{m_L(U)}{m_L(U')}\left(\frac{m_L(aU'\Delta aU)}{m_L(U)} + \frac{m_L(aU\Delta U)}{m_L(U)} + \frac{m_L(U\Delta U')}{m_L(U)}\right) \\ &\leq 2\left(\frac{m_L(a(U\setminus U'))}{m_L(U)} + \frac{m_L(aU\Delta U)}{m_L(U)} + \frac{m_L(U\setminus U')}{m_L(U)}\right) \\ &\leq 2\cdot 3\cdot\frac{\epsilon}{6} = \epsilon.\end{aligned}$$

The set U' now satisfies all the claims. □

The last theorem gives us for every compact set K and every ϵ a compact set U, such that U is almost invariant under the action of every $a \in K$. However, by Følner's strong condition, there exists a set U, that is invariant under the action of the whole of K. To prove this, we will need some covering lemma's.

Lemma 3.5.5 *Let V be an open, symmetric neighborhood of e in a locally compact group G, such that $\overline{V}$ is compact. Define $H = \bigcup_{i=1}^{\infty} V^n$ as the subgroup generated by V. Then there exists a subset T of H and $k \in \boldsymbol{N}$, such that*

$$H = \bigcup_{t\in T} tV,$$

and such that the intersection of more than k sets from $\{tV : t \in T\}$ is empty.

Proof One concludes easily that H is a subgroup of G. As all sets V^n are open, also H is open. Let y be an element in the complement H^c of H. As $e \in V \subset H$, $y \in yH$. Assume $yH \cap H \neq \emptyset$. Then there exist $h, h' \in H$, such that $h = yh'$. This implies $y = {h'}^{-1}h \in H$. From this contradiction we see, that every point $y \in H^c$ is contained in an open set yH, disjoint from H. It follows H is closed.

When H is compact, the statements in the lemma follow easily, as the sets $\{gV : g \in H\}$ constitute an open cover of H. Suppose now that H is not compact. We will need that for all $n \in \mathbf{N}$, it holds that $\overline{V}^n \not\subset \bigcup_{i=1}^{n-1} V^n$. To show this, assume the contrary. Then $V^n \subset \bigcup_{i=1}^{n-1} V^i$ and also

$$V^{n+1} = VV^n \subset V \bigcup_{i=1}^{n-1} V^i = \bigcup_{i=2}^{n} V^i \subset \bigcup_{i=1}^{n-1} V^i.$$

By induction we conclude that $V^m \subset \bigcup_{i=1}^{n-1} V^i$ for every $m \geq n$, so that

$$H = \bigcup_{i=1}^{\infty} V^n = \bigcup_{i=1}^{n-1} V^i.$$

As $V \subset \overline{V}$, $H \subset \bigcup_{i=1}^{n-1} \overline{V}^i$. As the product of two compacta is again compact, $\bigcup_{i=1}^{n-1} \overline{V}^i$ is compact also. As H is a closed subset of the compact set $\bigcup_{i=1}^{n-1} \overline{V}^i$, H itself is compact. This contradiction implies that for every $n \in \mathbf{N}$, we have $\overline{V}^n \not\subset \bigcup_{i=1}^{n-1} V^i$.

Let now $n \in \mathbf{N}$ be given and define $t_1 = e$. We will show there is no infinite sequence t_m, such that $t_m \in \overline{V}^n \setminus \bigcup_{j=1}^{i-1} t_j V$ for all $m \in \mathbf{N}$. To do so, assume that such a sequence does exist. Select $m^* \in \mathbf{N}$ arbitrarily. By definition, $t_{m^*} \notin t_m V$ for all $m < m^*$. For $m > m^*$, as V is symmetric, $t_m \notin t_{m^*} V$ implies $t_{m^*} \notin t_m V$. This proves $t_{m^*} \notin t_m V$ for all $m \neq m^*$. Therefore the sets $\{t_m V\}_{m \in \mathbf{N}}$ constitute a cover of the compact set $\overline{V}^n$ that does not admit a finite subcover. This contradiction proves there can only be a finite number of t_m that satisfy the claim.

Using these observation, we can construct a countable subset T, for which the conditions hold. Define $t_1 = e$ and select inductively for $1 \leq m \leq n_1$ elements $t_m \in \overline{V}^2 \setminus \bigcup_{i=1}^{m-1} t_i V$, until $\overline{V}^2 \setminus \bigcup_{i=1}^{n_1} t_i V$ is empty. Note that by the above observation, such an n_1 will exist. Continue now with $\overline{V}^3$, by selecting inductively for $t_1 + 1 \leq m \leq t_2$, elements t_m for which $t_m \in \overline{V}^3 \setminus \bigcup_{i=1}^{m-1} t_i V$. Again this will define a finite number of t_m. Repeating this argument will give an infinite sequence of t_m, as by the first observation, every $\overline{V}^n$ will give at least one new t_m. We will show the set $T = \{t_1, \ldots\}$ satisfies the hypothesis. By definition of the t_m it holds that $H = TV$. To prove the existence of $k \in \mathbf{N}$ satisfying the conditions, consider an open symmetric neighborhood of e in G, such that $W^2 \subset V$. Then, for all $i, j \in \mathbf{N}$, $y \in t_i W \cap t_j W$ implies that $y = t_i w = t_j w'$. It follows $t_i = t_j w' w^{-1} \subset t_j V$. This implies $i = j$, which shows that the sets $\{t_m W\}_{m \in \mathbf{N}}$ are pairwise disjoint. Let now S be a subset of T, such that $\bigcap_{s \in S} sV \neq \emptyset$. Choose x from this intersection. Then, for every $s \in S$, it holds that $s \in xV$, as V is symmetric. This implies $sW \subset xVW$, so that $SW \subset xVW$. As the sets $\{sW\}_{s \in S}$ are pairwise disjoint, this implies

$$|S| m_L(W) = m_L(SW) \leq m_L(xVW) = m_L(VW),$$

where $|S|$ denotes the cardinality of S. It follows, that

$$|S| \leq \frac{m_L(VW)}{m_L(V)} < \infty.$$

Therefore, the set T satisfies the conclusions of the lemma, for any $k \in \mathbf{N}$, such that $k > m_L(VW)/m_L(V)$. $\square$

The previous lemma gives a cover only of the subgroup generated by the subset V. We will now show there exists also a cover of the whole of G, with the same property.

Lemma 3.5.6 *Let V be an open, symmetric, relatively compact neighborhood of e in a locally compact group G. Then there exists a subset T of G and a number $k \in \mathbf{N}$, such that $G = TV$ and such that the intersection of more than k sets from $\{tV : t \in T\}$ is empty.*

Proof Let V be an open, symmetric, relatively compact neighborhood of the identity. Define $H = \bigcup_{i=1}^{\infty} V^n$ as the subgroup generated by V. The previous lemma gives us a set $T \subset H$ and a number $k \in \mathbf{N}$, such that $H = TV$ and at most k of the sets $\{tV\}_{t \in T}$ have a non-empty intersection. For every $a \in G$ it holds, that $aH = (aT)V$. Furthermore, at most k of the sets $\{(at)V\}_{t \in T}$ have a non-empty intersection. Remark now that H is a subgroup and consider the left cosets $gH \subset G$. As the relation

$$x \sim y \Leftrightarrow xH = yH$$

is an equivalence relation, there exists a set of representatives R, such that the sets $\{rH\}_{r \in R}$ form a partition of G. Define now $T' = RT$. We have $G = RH = (RT)V$. As the sets $\{rH\}_{r \in R}$ are pairwise disjoint, at most k of the sets $\{rtV : r \in R, t \in T\}$ have a non-empty intersection. $\square$

Actually, there exists a covering of the set G for every relatively compact neighborhood of every point in G.

Lemma 3.5.7 *Let G be a locally compact group and V a relatively compact set that contains an open set. Then there exists a set $T \subset G$ and a $k \in \mathbf{N}$, such that $G = TV$ and such that the intersection of more than k sets from $\{tV : t \in T\}$ is empty.*

Proof Let V be a relatively compact neighborhood of some point $g \in G$. Select an open, symmetric, relatively compact neighborhood W of e. As V contains some open set and $\overline{W}$ is compact, there exists a finite set $\{a_1, \ldots a_n\} \subset G$, such that $W \subset \overline{W} \subset \bigcup_{i=1}^{n} a_i V$. By the previous lemma, there exist a subset T of G and a number $k \in \mathbf{N}$, such that $G = TW$ and at most k of the sets $\{tW : t \in T\}$ have a non-empty intersection. Define $T' = T\{a_1, \ldots, a_n\}$. Then $G = TW \subset T\{a_1, \ldots, a_n\}V = T'V$. It follows $G = T'V$. We will show that for $k' = kmn$, no more than k' of the sets $\{tV : t \in T'\}$ can have a non-empty intersection. Define therefore $V' = \bigcup_{i=1}^{n} a_i V$. As V is relatively compact, so is V'. Therefore, there exists a finite set $\{b_1, \dots b_m\} \subset G$, such that $V' \subset \bigcup_{j=1}^{m} V b_j$. Assume now that there is a subset S of T', with cardinality larger than kmn, such that $x \in \bigcap_{s \in S} sV$. By definition there are more than kmn tupels (a_i, t_i), such that $x \in \bigcap_i t_i a_i V$. It follows there exists an $1 \leq i^* \leq n$, such that $x \in t a_{i^*} V$ for more than km elements $t \in T$. As $a_{i^*} V \subset V'$, there is a set $S' \subset T$ with cardinality larger than km, such that $x \in \bigcap_{s \in S'} sV'$. Every element $v \in V'$ must be contained in $V b_j$ for some $1 \leq j \leq m$, therefore, there exists for every $s \in S'$ a $1 \leq j_s \leq n$, such that $x \in sV j_s$. As there are only m different elements j_s, there must be a j^*, such that $x \in sV b_{j^*}$ for more than k elements $s \in S'$. This implies $x b_{j^*}^{-1} \in sV$ for more than k elements $s \in S' \subset T$. As this contradicts the conditions on the covering TV, we conclude there cannot be more than kmn sets that have a non-empty intersection. This proves the claim. □

The previous lemma showed that for every relatively compact neighborhood V in G, there is a cover of G with left translates of V that is nearly disjoint. We will apply this cover in the following lemma.

Lemma 3.5.8 *Let G be an amenable, locally compact group G and let K be a compact neighborhood of e in G. If there exists a sequence of compact subsets U_n of G with finite measure, such that for all $a \in K' = KKK^{-1}$,*

$$\frac{m_L(aU_n \Delta U_n)}{m_L(U_n)} < \frac{1}{n},$$

then for every $\epsilon > 0$ there exists a sequence of compact sets $V_n \subset U_n$ with finite measure, such that $m_L(U_n \setminus V_n) < \epsilon/(1+\epsilon) m_L(U_n)$ and

$$\limsup_{n \to \infty} \frac{m_L(KV_n)}{m_L(V_n)} \leq 1 + \epsilon.$$

Proof Let the compact set $K \subset G$ and $\epsilon > 0$ be given. The set $K' = KKK^{-1}$ is compact, as a product of compact sets. Define $\delta = \epsilon/(2+2\epsilon)$. Suppose that there exists a sequence U_n, such that $0 < m_L(U_n) < \infty$ and such that

$$\frac{m_L(aU_n \Delta U_n)}{m_L(U_n)} < \frac{1}{n},$$

for all $a \in K'$. We will construct the sequence V_n. To do so, define for all $x \in G$, $n \in \mathbf{N}$ and $\alpha > 0$, the following measurable sets.

$$\begin{aligned} A(x,n) &= K' \setminus U_n x^{-1} \\ B(n,\alpha) &= \{x \in U_n : m_L(A(x,n)) > \alpha\}. \end{aligned}$$

As a subset of the compact set K', the set $A(x,n)$ has bounded measure for every $x \in G$ and $n \in \mathbf{N}$. Also, as a subset of the compact set U_n, $B(n,\alpha)$ has bounded measure. Furthermore, by definition we have $B(n,\alpha) \subset B(n,\alpha')$

whenever $\alpha > \alpha'$. Therefore, $m_L(B(n,\alpha))$ decreases when α increases. This allows us to define

$$\alpha_n = \inf \left\{\alpha > 0 : m_L(B(n,\alpha)) \leq \delta m_L(U_n)\right\}.$$

We will now show that the sequence α_n converges to zero. To do so, assume the contrary. Then there is a $\beta > 0$, such that for every $M \in \mathbf{N}$ there exists an $m_M > M \in \mathbf{N}$, such that $\alpha_{m_M} > \beta$. By definition this implies for all $M \in \mathbf{N}$, that

$$m_L\left(B(m_M,\beta)\right) > \delta m_L\left(U_{m_M}\right).$$

Define now for $y \in K'$ and $M \in \mathbf{N}$ the set

$$C(y,M) = \{x \in B(m_M,\beta) : y \in A(x,m_M)\} \subset U_n.$$

Then Fubini's theorem implies, as all sets involved have a finite measure, that for all $M \in \mathbf{N}$,

$$\begin{aligned}
\int_{B(m_M,\beta)} m_L(A(x,m_M))dm_L(x) &= \int_{B(m_M,\beta)}\int_{K'} \mathbf{1}_{A(x,m_M)}(y)dm_L(y)dm_L(x) \\
&= \int_{K'}\int_{B(m_M,\beta)} \mathbf{1}_{A(x,m_M)}(y)dm_L(x)dm_L(y) \\
&= \int_{K'}\int_{B(m_M,\beta)} \mathbf{1}_{C(y,M)}(x)dm_L(x)dm_L(y) \\
&= \int_{K'} m_L(C(y,M))dm_L(y). \qquad (3.9)
\end{aligned}$$

Select now $y \in K'$ and consider $x \in C(y,m)$. By definition, it then holds that $y \notin U_{m_M}x^{-1}$. It follows $yx \notin U_{m_M}$, so that, $yC(y,M) \cap U_{m_M} = \emptyset$. We can thus write

$$(yU_{m_M}) \cap U_{m_M} = y(U_{m_M} \setminus C(y,M)) \cap U_{m_M}.$$

However, by the conditions on the sequence U_n we have for all $y \in K'$, that

$$\lim_{M\to\infty} \frac{m_L(yU_{m_M} \cap U_{m_M})}{m_L(U_{m_M})} = 1.$$

We conclude, that

$$\begin{aligned}
1 &= \liminf_{M\to\infty} \frac{m_L(y(U_{m_M} \setminus C(y,M)) \cap U_{m_M})}{m_L(U_{m_M})} \\
&\leq \liminf_{M\to\infty} \frac{m_L(y(U_{m_M} \setminus C(y,M)))}{m_L(U_{m_M})} \\
&= \liminf_{M\to\infty} \frac{m_L(U_{m_M}) - m_L(C(y,M))}{m_L(U_{m_M})} = 1 - \limsup_{M\to\infty} \frac{m_L(C(y,M))}{m_L(U_{m_M})}.
\end{aligned}$$

It follows, that

$$\limsup_{M\to\infty} \frac{m_L(C(y,M))}{m_L(U_{m_M})} = 0,$$

which implies the limit exists and is equal to zero. By Lebesgue's dominated convergence theorem and (3.9), we then have, that

$$\begin{aligned} 0 &= \lim_{M\to\infty} \int_{K'} \frac{m_L(C(y,M))}{m_L(U_{m_M})} dm_L(y) \\ &= \lim_{M\to\infty} \frac{1}{m_L(U_{m_M})} \int_{B(m_M,\beta)} m_L(A(x,m_M)) dm_L(x) \\ &> \liminf_{M\to\infty} \frac{1}{m_L(U_{m_M})} \beta\delta m_L(U_{m_M}) = \beta\delta, \end{aligned}$$

using that $x \in B(m_M,\beta)$ implies that $m_L(A(x,m_M)) > \delta$. This contradiction shows that $\lim_{n\to\infty} \alpha_n = 0$.

Consider the compact set K^{-1}. By the previous lemma, there exists a set $T \subset G$ and a number $k \in \mathbf{N}$, such that $G = TK^{-1}$ and at most k sets from $\{tK^{-1} : t \in T\}$ admit a non-empty intersection. This implies that $G = KT^{-1}$, and that at most k of the sets $\{Kt^{-1} : t \in T\}$ have a non-empty intersection. In this way G is covered by right translates of K. Define now for $n \in \mathbf{N}$, $\beta_n = \alpha_n + \frac{1}{n}$, $W_n = U_n \setminus B(n,\beta_n)$ and $T_n = \{t \in T : Kt \cap W_n \neq \emptyset\}$. To every $t \in T_n$, we associate $z_t \in Kt \cap W_n$. Then there exists a $k^* \in K$, such that $z_t = k^*t$. For every $k \in K$, it holds that

$$kt = kk^{*-1}k^*t = kk^{*-1}z_t \in KK^{-1}z_t. \tag{3.10}$$

This implies, for every $t \in T_n$, as $e \in K$, that

$$Kt = eKt \subset KKt \subset KKK^{-1}z_t = K'z_t. \tag{3.11}$$

By definition of T_n, it holds that $W_n \subset KT_n$. This implies, using (3.10), that

$$KW_n \subset KKT_n = \bigcup_{t\in T_n} KKt \subset \bigcup_{t\in T_n} KKK^{-1}z_t = \bigcup_{t\in T_n} K'z_t. \tag{3.12}$$

By definition there exists $k \in K$, such that $z_t = kt$. Therefore, $Kz_t = Kkt$, so that

$$\frac{m_L(Kz_t)}{m_L(K)} = \frac{m_L(Kkt)}{m_L(K)}.$$

We conclude, for the modular function Δ, that

$$\Delta(z_t) = \Delta(kt) \leq \sup_{k\in K} \Delta(kt) = \sup_{k\in K} \Delta(k)\Delta(t). \tag{3.13}$$

Select now $w \in W_n$. Then $w \notin B(n,\beta_n)$, which implies that $m_L(A(w,n)) \leq \beta_n$. By definition of $A(w,n)$, we have

$$m_L(K'w\backslash U_n) = \Delta(w)m_L(K'\backslash U_n w^{-1}) = \Delta(w)m_L(A(w,n)) \leq \beta_n\Delta(w). \tag{3.14}$$

By the conditions on the cover KT, it holds that $\sum_{t\in T} \mathbf{1}_{Kt} \leq k$. As $T_n \subset T$, this implies for every $n \in \mathbf{N}$, that

$$\begin{aligned}\sum_{t\in T_n} m_L(Kt\cap U_n) &= \sum_{t\in T_n}\int_G \mathbf{1}_{U_n}(y)\mathbf{1}_{Kt}(y)dm_L(y)\\ &= \int_G \mathbf{1}_{U_n}(y)\left(\sum_{t\in T_n}\mathbf{1}_{Kt}(y)\right)dm_L(y) \leq km_L(U_n). \qquad (3.15)\end{aligned}$$

We will now combine the equalities above to prove the statement. For $n \in \mathbf{N}$ and $t \in T_n$, we conclude from (3.11), (3.14) and (3.13), that

$$m_L(Kt\setminus U_n) \leq m_L(K'z_t\setminus U_n) \leq \beta_n\Delta(z_t) \leq \beta_n\Delta(t)\sup_{k\in K}\Delta(k) = \beta_n\Delta(t)\gamma, \qquad (3.16)$$

where the last equation defines γ. Combining this to (3.15), we conclude that

$$\begin{aligned}km_L(U_n) &\geq \sum_{t\in T_n} m_L(Kt\cap U_n)\\ &= \sum_{t\in T_n} m_L(Kt) - m_L(Kt\setminus U_n) \geq (m_L(K)-\beta_n\gamma)\sum_{t\in T_n}\Delta(t).\end{aligned}$$

By definition of β_n, it holds that $\lim_n \beta_n = 0$. As $m_L(K) > 0$, there must be $N \in \mathbf{N}$, such that $m_L(K) - \beta_n\gamma > 0$ for all $n > N$. For those n, we then conclude, that

$$\sum_{t\in T_n}\Delta(t) \leq \frac{km_L(U_n)}{m_L(K)-\beta_n\gamma}.$$

By inner regularity of the Haar measure, for every set W_n, there exists a compact set $V_n \subset W_n$, such that $m_L(W_n\setminus V_n) < \delta m_L(U_n)$. By (3.12) and (3.16) it follows for $n > N$, that

$$\begin{aligned}m_L(KV_n) &\leq m_L\left(\bigcup_{t\in T_n} K'z_t\right) \qquad (3.17)\\ &\leq \left(\sum_{t\in T_n} m_L(K'z_t\setminus U_n)\right) + m_L(U_n)\\ &\leq \left(\beta_n\gamma\sum_{t\in T_n}\Delta(t)\right) + m_L(U_n)\\ &\leq \left(\frac{k\beta_n\gamma}{m_L(K)-\beta_n\gamma}+1\right)m_L(U_n). \qquad (3.18)\end{aligned}$$

By definition of β_n, we have $\beta_n > \alpha_n$, so that, by construction $m_L(B(n,\beta_n)) \leq \delta m_L(U_n)$. This implies

$$\begin{aligned}m_L(W_n) &= m_L(U_n) - m_L(B(n,\beta_n)) \geq (1-\delta)m_L(U_n),\\ m_L(V_n) &= m_L(W_n) - m_L(W_n\setminus V_n) > (1-2\delta)m_L(U_n) > 0,\end{aligned}$$

as $\delta < \frac{1}{2}$ by definition. As $W_n \subset U_n$, it holds that $U_n \setminus V_n = U_n \setminus W_n \cup W_n \setminus V_n$. From this it follows easily, that

$$m_L(U_n \setminus V_n) \leq m_L(B(n, \beta_n)) + \delta m_L(U_n) = 2\delta m_L(U_n).$$

Taking now the superior limit in (3.17), we finally find, that

$$\limsup_{n\to\infty} \frac{m_L(KV_n)}{m_L(V_n)} \leq \limsup_{n\to\infty} \left(\frac{k\beta_n\gamma}{m_L(K) - \beta_n\gamma} + 1 \right) \frac{1}{1-2\delta} = \frac{1}{1-2\delta} = 1 + \epsilon.$$

This proves the statements. □

Applying this lemma, it can easily be shown that Følners condition implies the following condition, that is due to Leptin.

Theorem 3.5.9 *Let G be an amenable, locally compact group. Then, for every compact set $K \subset G$ and every $\epsilon > 0$, there exists a compact measurable set $U \subset G$, such that*

$$\frac{m_L(KU)}{m_L(U)} < 1 + \epsilon.$$

Proof Let K be a compact subset in G. As G is locally compact, there exists a compact, symmetric neighborhood $\hat{K}$ of e. Denote the compact set $\tilde{K} = K \cup K^{-1} \cup \hat{K}$ and consider the set $K' = \tilde{K}\tilde{K}\tilde{K}^{-1}$. By Theorem 3.5.4, for every $n \in \mathbf{N}$, there exists a set U_n, such that

$$\frac{m_L(kU_n \Delta U_n)}{m_L(U)} < \frac{1}{n},$$

for every $k \in K'$. By Lemma 3.5.8, there exists a sequence V_n of compact sets, such that

$$\limsup_{n\to\infty} \frac{m_L(\tilde{K}V_n)}{m_L(V_n)} < 1 + \frac{\epsilon}{4}.$$

By definition, there exists $N \in \mathbf{N}$, such that

$$\frac{m_L(\tilde{K}V_N)}{m_L(V_N)} \leq 1 + \frac{\epsilon}{2}.$$

Note now that $e \in \tilde{K}$. Therefore,

$$\frac{m_L(\tilde{K}V_N \Delta V_N)}{m_L(V_N)} = \frac{m_L(\tilde{K}V_N \setminus V_N)}{m_L(V_N)} = \frac{m_L(\tilde{K}V_N)}{m_L(V_N)} - 1 = \frac{\epsilon}{2}.$$

Choose now $k \in K$. By construction, $k^{-1} \in \tilde{K}$ and $K \subset \tilde{K}$. This implies, that

$$\begin{aligned} KV_N \Delta V_N &= KV_N \setminus V_N \cup V_N \setminus KV_N \\ &\subset \tilde{K}V_N \setminus V_N \cup V_N \setminus kV_N \subset \tilde{K}V_N \setminus V_N \cup k(\tilde{K}V_N \setminus V_N). \end{aligned}$$

This implies, that

$$\frac{m_L(KV_N\Delta V_N)}{m_L(V_N)} \leq \frac{m_L(\tilde{K}V_N \setminus V_N)}{m_L(V_N)} + \frac{m_L(k(\tilde{K}V_N \setminus V_N))}{m_L(V_N)} = 2\frac{m_L(\tilde{K}V_N \setminus V_N)}{m_L(V_N)} \leq \epsilon.$$

The set V_N satisfies the claims. □

Using Leptin's condition, it is easy to show that every amenable group satisfies Følner's strong condition.

Theorem 3.5.10 *Let G be an amenable, locally compact group. Then, for every compact set $K \subset G$ and every $\epsilon > 0$, there exists a compact set $U \subset G$, such that*

$$\frac{m_L(KU\Delta U)}{m_L(U)} < \epsilon.$$

Proof Let a compact set K and ϵ be given. Define $K' = K^{-1} \cup K \cup \{e\}$. By Leptin's condition, there exists a compact set $U \subset G$, such that

$$\frac{m_L(K'U)}{m_L(U)} < 1 + \epsilon.$$

As $K \subset K'$, it holds for any $k \in K$, that

$$\begin{aligned} KU\Delta U &= KU \setminus U \cup U \setminus KU \\ &\subset KU \setminus U \cup kk^{-1}U \setminus kU \subset K'U \setminus U \cup k(K'U \setminus U). \end{aligned}$$

Taking the measure of the above expression, we find

$$\begin{aligned} m_L(KU\Delta U) &\leq m_L(K'U \setminus U) + m_L(k(K'U \setminus U)) \\ &= 2m_L(K'U \setminus U) = 2(m_L(K'U) - m_L(U)). \end{aligned}$$

By Leptin's condition, we easily obtain

$$\frac{m_L(KU\Delta U)}{m_L(U)} \leq 2\frac{m_L(K'U) - m_L(U)}{m_L(U)} = 2\left(\frac{m_L(K'U)}{m_L(U)} - 1\right) < 2\epsilon.$$

By definition, G satisfies Følner's strong condition. □

We have shown that every locally compact, amenable group satisfies Følner's conditions. The converse of this statement follows easily, even for a weaker version of Følner's condition.

Theorem 3.5.11 *Let G be a locally compact group. Assume for every finite set $F \subset G$ and $\epsilon > 0$, there exists a measurable set U with finite measure, such that $m_L(aU\Delta U) < \epsilon m_L(U)$ for all $a \in F$. Then G is amenable.*

Proof Let F be a finite subset of G and $\epsilon > 0$. By assumption, there exists a measurable set U of G with finite measure, such that $m_L(aU\Delta U) < \epsilon m_L(U)$ for all $a \in F$. It then also follows, with $\psi = \mathbf{1}_U / m_L(U)$, that

$$\|L_a\psi - \psi\|_1 = \frac{m_L(a^{-1}U\Delta U)}{m_L(U)} = \frac{m_L(U\Delta aU)}{m_L(U)} < \epsilon.$$

By Theorem 3.4.11, G is amenable. □

Assume now furthermore that G is second countable and locally compact. As told in Theorem 2.1.20, the set G admits an exhaustion, that is, a sequence $K_n \subset G$ of compact sets, such that

$$G = \bigcup_{n=1}^{\infty} K_n \text{ and } K_n \subset \text{int } K_{n+1}.$$

We will now show that this implies the existence of a so-called Følner sequence.

Definition 3.5.12 *Let G be a locally compact group. A sequence U_n of compact subsets of G is called a Følner sequence if the following holds. For every compact set K and $\epsilon > 0$, there exists an $N \in \boldsymbol{N}$, such that*

$$m_L(KU_n\Delta U_n) < \epsilon m_L(U_n)$$

for all $n > N$.

The existence of a Følner sequence follows easily from the exhaustion and Følner's strong condition. We will apply the following lemma.

Lemma 3.5.13 *Let K and U be given, such that $e \in K$ and*

$$m_L(KU\Delta U) < \epsilon m_L(U).$$

Then for any set $K' \subset K$ that contains e, it holds that

$$m_L(K'U\Delta U) < \epsilon m_L(U).$$

Proof Note that both K and K' contain the identity. This implies that

$$K'U\Delta U = K'U \setminus U \subset KU \setminus U = KU\Delta U.$$

It follows easily, that

$$m_L(K'U\Delta U) \leq m_L(KU\Delta U) < \epsilon m_L(U).$$

This proves the claim. □

The construction of a Følner sequence is now quite straightforward.

Theorem 3.5.14 *Let G be an amenable, locally compact, second countable group. Then G admits a Følner sequence.*

Proof Consider the exhaustion $\{K_n\}_{n\in\mathbf{N}}$ given by Theorem 2.1.20. Note that every set K_n contains the identity. By Følner's strong condition there exists a sequence U_n of compact subsets of G, such that

$$m_L(K_nU_n\Delta U_n) < \frac{1}{n}m_L(U_n).$$

We will now show that this sequence is a Følner sequence. Let therefore a compact set $K \subset G$ and $\epsilon > 0$ be given. As $K_n \subset \text{int } K_{n+1}$ for every $n \in \mathbf{N}$, the sets int K_n cover G. They therefore cover $K \cup K^{-1}$ also. Compactness of $K \cup K^{-1}$ implies the existence of an $N_1 \in \mathbf{N}$, such that $K \cup K^{-1} \subset \bigcup_{n=1}^{N_1} K_n = K_{N_1}$. Select N_2 such that $\frac{1}{N_2} < \frac{\epsilon}{2}$ and define $N = \max\{N_1, N_2\}$. Select $k \in K$. Then we have, for $n > N$, that

$$\begin{aligned} KU_n\Delta U_n &\subset KU_n \setminus U_n \cup U_n \setminus kU_n \\ &\subset KU_n \setminus U_n \cup k(KU_n \setminus U_n) \subset K_NU_n \setminus U_n \cup k(K_NU_n \setminus U_n). \end{aligned}$$

Applying the previous lemma, with $K' = K_N \subset K_n$, we conclude, that

$$m_L(KU_n\Delta U_n) \leq 2m_L(K_NU_n\Delta U_n) \leq 2m_L(K_nU_n\Delta U_n) \leq \frac{2}{N} < \epsilon.$$

The sequence U_n therefore constitutes a Følner sequence. □

The definition of a Følner sequence that we use puts the strongest conditions on the sequence of sets U_n. One could also define a Følner sequence as a sequence such that the statements holds only for sets K that are singletons. In that case, for any $g \in G$, there must be an $N \in \mathbf{N}$, such that for all $n > N$, it holds that

$$m_L(gU_n\Delta U_n) < \epsilon m_L(U_n).$$

It can be shown that this condition is equivalent to the following, that states that the N can be chosen uniformly over a compact set K. More precisely, for every compact set K, there exists an $N \in \mathbf{N}$, such that for all $n > N$ and $k \in K$, it holds that [7, Theorem 3]

$$m_L(kU_n\Delta U_n) < \epsilon m_L(U_n).$$

However, the formally weakest condition already implies that for every finite set $F \subset G$ and every $\epsilon > 0$, there exists a measurable set U, such that $m_L(aU\Delta U) < \epsilon m_L(U)$ for all $a \in F$. Theorem 3.5.11 now implies that G is amenable. This shows that all conditions are equivalent.

We finally want to put some restrictions on the growth of the Følner sequence $\{F_n\}_{n\in\mathbf{N}}$. The following condition is introduced by Shulman.

Definition 3.5.15 *A Følner sequence $\{F_n\}_{n\in\boldsymbol{N}}$ is a tempered Følner sequence, if there exists $C>0$, such that for all $n\in\boldsymbol{N}$, it holds that*

$$m_L\left(\bigcup_{k=1}^{n-1} F_k^{-1}F_n\right)\leq Cm_L\left(F_n\right).$$

The restriction for a tempered Følner sequence is mild enough to ensure that every amenable group admits one.

Theorem 3.5.16 *Let G be a second countable, locally compact group. Then G is amenable if and only if G admits a tempered Følner sequence.*

Proof As G is amenable, there exists a Følner sequence $\{F_n\}_{n\in\mathbf{N}}$ for G. We will define a subsequence $\{n_m\}_{m\in\mathbf{N}}$ such that the sequence

$$\{F_{n_m}\}_{m\in\mathbf{N}}$$

forms a tempered Følner sequence. Define $n_1=1$. We will define n_m for $m\geq 2$ inductively. Assume that $n_1,\dots n_M$ are defined. The set

$$F=\bigcup_{m=1}^{M} F_{n_m}^{-1}$$

is a compact set. By definition of a Følner sequence, there exists $n_{M+1}>M+1$, such that

$$m_L(FF_n\Delta F_n)<m_L(F_n)$$

for all $n\geq n_{M+1}$. In particular, this implies, that

$$\begin{aligned} m_L\left(FF_{n_{M+1}}\right)&=m_L\left(FF_{n_{M+1}}\setminus F_{n_{M+1}}\right)+m_L\left(F_{n_{M+1}}\right)\\ &\leq m_L\left(FF_{n_{M+1}}\Delta F_{n_{M+1}}\right)+m_L\left(F_{n_{M+1}}\right)\leq 2m_L\left(F_{n_{M+1}}\right). \end{aligned}$$

The sequence $\{F_{n_m}\}_{m\in\mathbf{N}}$ forms a tempered Følner sequence. This proves the claim. □

This section contains all the characterizations of amenability that will be used in the rest of this text. We want to note one more characterization. The Følner sequence implied by Theorem 3.5.14 can be used to define a finitely additive, left translation invariant measure on the power set of G. This construction uses ultra limits. As these limits are not familiar for most readers, and as the characterization is not needed in later sections, we have omitted the proof. From the finitely additive measure, one can construct a G-invariant mean on $L^\infty(G)$, restricting the supremum over *measurable* simple functions and using that the supremum is bounded for $f\in L^\infty(G)$. This shows that a locally compact group G is amenable if and only if there exists a finitely additive, left translation invariant mean on the power set $\mathcal{P}(G)$. This characterization is used in [19], in relation to the famous Banach-Tarski paradox.

Chapter 4

The maximal inequality

4.1 Introduction

In real harmonic analysis, one often encounters the Hardy-Littlewood maximal function $M(f) : \mathbf{R}^n \to \mathbf{R}$, of a function $f \in L^p(\mathbf{R}^n, \lambda)$, defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y) d\lambda(u).$$

In the study of these maximal functions, one interesting problem is the question whether the mapping $M : f \mapsto M(f)$ is bounded. It can be proven that for $p > 1$, this is indeed the case. For $p = 1$, the mapping is not bounded. However, one can derive the following weak maximal inequality

$$\lambda\left(\left\{x \in \mathbf{R}^n : M(f)(x) = \sup_{r>0} \frac{1}{\lambda(B(x,r))} \int_{B(x,r)} f(y) d\lambda(y) > \alpha\right\}\right) \leq C \frac{\|f\|_1}{\alpha}.$$

Consider now the situation where the group $\mathbf{R}$ acts on itself. When one compares the maximal function $M(f)$ to the ergodic averages

$$\lim_{R\to\infty} \frac{1}{\lambda([-R,R])} \int_{-R}^{R} f(x+y) d\lambda(y),$$

the similarity is immediate. In particular, the pointwise ergodic theorem, that states that the ergodic averages converge almost everywhere, follows immediately from the maximal inequality. It is therefore interesting to generalize the weak maximal inequality to the situation where $\mathbf{R}^n$ acts on a more general measure space $(X, \mathcal{F}, \mu)$. These generalizations are used in the first classical proofs of the pointwise ergodic theorem.

Inspired by this generalization, in [12] it is proven that the maximal inequality can be generalized to the action of any locally compact *amenable* group G on a measure space $(X, \mathcal{F}, \mu)$. In the next section, we will first give the generalization where $\mathbf{R}^n$ is the acting group. The proof is based on a covering lemma, that is implied by the Vitalí covering lemma. [12] generalizes this covering lemma, and obtains the maximal inequality from this generalized covering lemma. In ergodic theory, one usually studies the action of a *finite* measure space. Contrary to this, we will use a weak maximal inequality for the action of an amenable group on a σ-finite measure space. We therefore have adjusted the proof of the maximal inequality to this more general situation.

To be able to define these generalizations, the notion of an action is needed.

Definition 4.1.1 *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space. An action of a group G on X is a homomorphism Φ, that assigns to each $g \in G$ an automorphism on X. More precisely, for each $g \in G$, $\Phi(g)$ is a measurable bijection from X to itself and the mapping Φ is a group-homomorphism.*

An action of G on X is said to be measure-preserving, if for all $g \in G$, it holds that

$$\mu(g^{-1}A) = \mu(A),$$

for all $A \in \mathcal{F}$.

4.2 The maximal inequality for the action of $\mathbf{R}^n$

In order to generalize the maximal inequality to general amenable actions, we will first demonstrate the maximal inequality for the action of $\mathbf{R}^n$ on a measurable space $(X, \mathcal{F}, \mu)$. Normally one considers the action of $\mathbf{R}^n$ on itself, but we will here give a prove that is valid for the action on general spaces X. To do so, denote $(\mathbf{R}^n, \mathcal{B}, \lambda)$ for the Borel measure space $\mathbf{R}^n$.

Consider a probability space $(X, \mathcal{F}, \mu)$. Assume $\mathbf{R}^n$ acts measure-preservingly on X. We denote $rx = \Phi(r)(x)$ for the image of $x \in X$ under the homomorphism associated to $r \in \mathbf{R}^n$. Consider a function $f \in L^1(X, \mathcal{F}, \mu)$. We then naturally find, for every $x \in X$, the function $\mathbf{R}^n \to \mathbf{R} : r \mapsto f(rx)$.

We will prove the existence of a constant C, independent of $f \in L^1(X, \mathcal{F}, \mu)$, for which the weak (type 1-1) maximal inequality

$$\mu\left(\left\{x \in X : \sup_{R>0} \frac{1}{\lambda(B_r)} \int_{B_R} f(rx) d\lambda(r) > \alpha\right\}\right) \leq \frac{C}{\alpha} \|f\|_1$$

holds. To do so, we will need the famous Vitalí covering lemma, that states that from any collection of balls with bounded radius, one can choose a finite subcollection of disjoint balls, that covers a given portion of the original balls.

Lemma 4.2.1 ***Vitalí covering lemma*** *Let $\mathcal{U}$ be an arbitrary collection of balls in $\boldsymbol{R}^n$, contained in some compact set K. Then there is a finite subcollec-*

tion $\hat{\mathcal{U}}$ of disjoint balls, such that

$$\frac{1}{2}\frac{1}{3^n}\lambda\left(\bigcup_{B\in\mathcal{U}} B\right) \leq \lambda\left(\bigcup_{B\in\hat{\mathcal{U}}} B\right)$$

Proof Let an arbitrary collection $\mathcal{U}$ be given. As the balls are all contained in the compact set K, their union has finite measure. This implies, by inner regularity of Lebesgue measure, that there exists a compact set $K' \subset \bigcup_{B\in\mathcal{U}} B$, for which

$$\lambda(K') \geq \frac{1}{2}\lambda\left(\bigcup_{B\in\mathcal{U}} B\right). \tag{4.1}$$

The collection $\mathcal{U}$ forms a cover of K', which implies there is a finite subcover $\mathcal{U}'$, that covers K' also. To find the subcollection $\hat{\mathcal{U}}$, apply the following procedure. Choose iteratively the ball $B \in \mathcal{U}'$ with the largest radius and put it in $\hat{\mathcal{U}}$. Delete this ball and all balls it intersects from $\mathcal{U}'$. Repeat this procedure until $\mathcal{U}'$ is empty. We will show the collection $\hat{\mathcal{U}}$ obtained in this way satisfies the conditions. Disjointness follows easily by definition. To prove the second condition, consider a ball $B \in \mathcal{U}' \setminus \hat{\mathcal{U}}$. By construction, there is a ball $B' \in \mathcal{U}$, with a larger radius, that intersects B. It follows B is contained in the ball B', when the radius of B' is tripled. As this holds for any ball $B \in \mathcal{U}' \setminus \hat{\mathcal{U}}$, we conclude

$$K' \subset \bigcup_{B\in\mathcal{U}'} B \subset \bigcup_{B\in\hat{\mathcal{U}}} \tilde{B},$$

where $\tilde{B}$ is the ball centered at the center of B, but with thrice that radius. Taking measures and substituting (4.1), we conclude

$$\lambda\left(\bigcup_{B\in\mathcal{U}} B\right) \leq 2\lambda(K') \leq 2\lambda\left(\bigcup_{B\in\mathcal{U}'} B\right) \leq 2\times 3^n\lambda\left(\bigcup_{B\in\hat{\mathcal{U}}} B\right).$$

This implies the second condition on the cover $\hat{\mathcal{U}}$. □

Although the Vitalí covering lemma is easily proven, it is strong enough to prove the maximal inequality. However, we need a covering theorem for closed balls with integral radii. This can easily be derived from the Vitalí covering lemma.

Lemma 4.2.2 *Let $\mathcal{U}$ be an arbitrary collection of closed balls in $\boldsymbol{R}^n$ with a radius in $\{1,\ldots,N\}$, contained in some compact set K. Define, for every $n \in \{1,\ldots,N\}$, the set A_n of centers of balls with radius n. Assume these sets A_n are measurable. Then there is a finite subcollection $\hat{\mathcal{U}} \subset \mathcal{U}$ of disjoint balls, such that*

$$\frac{1}{4}\frac{1}{3^n}\lambda\left(\bigcup_{n=1}^{N} A_n\right) \leq \lambda\left(\bigcup_{B\in\hat{\mathcal{U}}} B\right)$$

Proof For any closed ball $B \in \mathcal{U}$, define the open ball B' with the same center, but whose radius is $\sqrt[n]{2}$ times that of B. The union of these balls will still be contained in the compact set $K' = \sqrt[n]{2}K$, as compactness in $\mathbf{R}^n$ is equivalent to boundedness and closedness. Consider now the finite subcollection $\hat{\mathcal{U}}$ of the collection $\{B'\}_{B\in\mathcal{U}}$ from the Vitalí covering lemma. It holds, that

$$\bigcup_{n=1}^{N} A_n \subset \bigcup_{B\in\mathcal{U}} B \subset \bigcup_{B\in\mathcal{U}} B'.$$

Taking measures and applying the conclusions of the Vitalí covering lemma, we have

$$\frac{1}{4}\frac{1}{3^n}\lambda\left(\bigcup_{n=1}^{N} A_n\right) \le \frac{1}{4}\frac{1}{3^n}\lambda\left(\bigcup_{B\in\mathcal{U}} B'\right) \le \frac{1}{2}\lambda\left(\bigcup_{B'\in\hat{\mathcal{U}}} B'\right) = \lambda\left(\bigcup_{B\in\hat{\mathcal{U}}} B\right),$$

where the factor $\frac{1}{2}$ disappears because the volume of any ball B' is twice that of the ball B. □

Theorem 4.2.3 *Let $(X, \mathcal{F}, \mu)$ be a probability space on which $\boldsymbol{R}^n$ acts measure-preservingly. Then there exists a constant C, independent of $(X, \mathcal{F}, \mu)$, such that for every $f \in L^1(X, \mathcal{F}, \mu)$, the following holds*

$$\mu\left(\left\{x \in X : \sup_{n\in\boldsymbol{N}} \frac{1}{\lambda(B_n)}\int_{B_n} f(rx)d\lambda(r) > \alpha\right\}\right) \le \frac{C}{\alpha}\|f\|_1.$$

Proof Let $f \in L^1(X, \mathcal{F}, \mu)$ and $\epsilon > 0$ be given. As μ is a probability measure, the set

$$D' = \left\{x \in X : \sup_{n\in\mathbf{N}} \frac{1}{\lambda(B_n)}\int_{B_n} f(rx)d\lambda(r) > \alpha\right\}$$

has finite measure. It follows, there is an $N \in \mathbf{N}$, such that the set

$$D = \left\{x \in X : \sup_{n\in\{1,\ldots,N\}} \frac{1}{\lambda(B_n)}\int_{B_n} f(rx)d\lambda(r) > \alpha\right\},$$

has measure $\mu(D) > \mu(D') - \epsilon$. Define $K = \bigcup_{n=1}^{N} \bar{B}_n = \bar{B}_N$. Then K is compact, and as $\mathbf{R}^n$ is amenable, there exists by Leptin's condition in Theorem 3.5.9, a compact set U, such that $\lambda(KU) \le (1+\epsilon)\lambda(U)$.

Choose now $x \in X$. Define, for $n \in \{1, \ldots, N\}$, the set

$$A_n = \left\{g \in U : \frac{1}{\lambda(B_n)}\int_{B_n} f(rgx)d\lambda(r) > \alpha\right\}.$$

Consider an element $g \in \bigcup_{n=1}^{N} A_n \subset U$. By definition, there exists an n, such that

$$\frac{1}{\lambda(B_n)}\int_{B_n} f(rgx)d\lambda(r) > \alpha,$$

from which it follows, that $gx \in D$. Reversing the arguments, we conclude, that

$$\{g \in U : gx \in D\} = \bigcup_{n=1}^{N} A_n.$$

Taking measures, it follows, that

$$\int_U \mathbf{1}_D(gx) d\lambda(g) = \lambda\left(\bigcup_{n=1}^{N} A_n\right). \tag{4.2}$$

By construction, for any $a \in A_n$, it holds that $\bar{B}_n a \subset KU$. Therefore the collection

$$\mathcal{U} = \{\bar{B}_n a : n \in \{1, \ldots, N\}, a \in A_n\}$$

is a collection of closed balls contained in the compact set KU. Define $\hat{\mathcal{U}}$ as the finite subcollection of $\mathcal{U}$ from Lemma 4.2.2. It induces a counting function

$$\mathbf{1}_{\hat{\mathcal{U}}} = \sum_{B \in \hat{\mathcal{U}}} \mathbf{1}_B.$$

By definition of A_n, it holds, for every $a \in A_n$, that

$$\int_{B_n} f(rax) d\lambda(r) > \alpha\lambda(B_n).$$

Substituting $r \to r' = ra$, this implies

$$\alpha\lambda(B_n) < \int_{B_n a} f(r'x) d\lambda\left(r'a^{-1}\right) = \Delta\left(a^{-1}\right) \int_{B_n a} f(r'x) d\lambda(r'),$$

where $\Delta = 1$ is the modular function. Ignoring for the moment that $\mathbf{R}^n$ is uni-modular, and using that Δ is a homomorphism, we find

$$\alpha\lambda\left(B_n a\right) = \alpha\Delta(a)\lambda\left(B_n\right) < \int_{B_n a} f(rx) d\lambda(r).$$

From this we conclude

$$\begin{aligned}\int_{KU} \mathbf{1}_{\hat{\mathcal{U}}}(r) f(rx) d\lambda(r) &= \sum_{B_n a \in \hat{\mathcal{U}}} \int_{B_n a} f(rx) d\lambda(r) \\ &> \sum_{B_n a \in \hat{\mathcal{U}}} \alpha\lambda(B_n a) \geq \alpha \frac{1}{4} \frac{1}{3^n} \lambda\left(\bigcup_{n=1}^{N} A_n\right),\end{aligned}$$

where the last inequality follows from Lemma 4.2.2. To finish the argument, we now remark that the right hand side of the above equation is non-negative, which implies the left hand side is positive. Therefore

$$\begin{aligned}\alpha \frac{1}{4} \frac{1}{3^n} \lambda\left(\bigcup_{i=1}^{N} A_n\right) &< \left|\int_{KU} \mathbf{1}_{\hat{\mathcal{U}}}(r) f(rx) d\lambda(r)\right| \\ &\leq \int_{KU} \mathbf{1}_{\hat{\mathcal{U}}}(r) |f(rx)| d\lambda(r) \leq \int_{KU} |f(rx)| d\lambda(r).\end{aligned} \tag{4.3}$$

In the last step we used that the collection $\hat{\mathcal{U}}$ is disjoint, such that the counting function $\mathbf{1}_{\hat{\mathcal{U}}} \leq 1$. Combining the steps, we now have

$$\begin{aligned}
\mu(D') - \epsilon \leq \mu(D) &= \frac{1}{\lambda(U)} \int_U \int_X \mathbf{1}_D(x) d\mu(x) d\lambda(g) \\
&= \frac{1}{\lambda(U)} \int_U \int_X \mathbf{1}_D(gx) d\mu(x) d\lambda(g) \\
&= \frac{1}{\lambda(U)} \int_X \int_U \mathbf{1}_D(gx) d\lambda(g) d\mu(x) \\
&= \frac{1}{\lambda(U)} \int_X \lambda\left(\bigcup_{n=1}^{N} A_n\right) d\mu(x) \\
&\leq \frac{4 \times 3^n}{\alpha\lambda(U)} \int_X \int_{KU} |f(rx)| d\lambda(r) d\mu(x) \\
&\leq \frac{4 \times 3^n(1+\epsilon)}{\alpha\lambda(KU)} \int_{KU} \int_X |f(rx)| d\mu(x) d\lambda(r) \\
&= \frac{4 \times 3^n(1+\epsilon)}{\alpha\lambda(KU)} \int_{KU} \int_X |f(x)| d\mu(x) d\lambda(r) \\
&= \frac{4 \times 3^n(1+\epsilon)}{\alpha} \|f\|_1.
\end{aligned}$$

The above expression holds for any $\epsilon > 0$. It should therefore also hold in the limit $\epsilon \rightarrow 0$. This implies, that

$$\mu\left(\left\{x \in X : \sup_{n \in \mathbf{N}} \frac{1}{\lambda(B_n)} \int_{B_n} f(rx) d\lambda(r) > \alpha\right\}\right) = \mu(D) \leq \frac{4 \times 3^n}{\alpha} \|f\|_1.$$

This proves the maximal inequality. □

It should be noted that the constant C from the maximal inequality does not depend on the probability space X. Another important observation is that we only used the fact that $\mathbf{R}^n$ is amenable and the properties of the covering described in Lemma 4.2.2. If we could prove a similar lemma for arbitrary amenable groups, it should be possible to generalize the maximal inequality. We will do so in the next sections, although the generality comes at a price.

4.3 A covering lemma for discrete groups

In this section we will generalize Lemma 4.2.2 to discrete amenable groups. This lemma was first presented in [12]. Recall that a discrete topological group is a countable group with the discrete topology. Haar measure on a discrete group corresponds to the counting measure. To deal with the generality of amenable groups, a *random* subcover will be constructed, that has the desired properties *on average*. This means we will construct a probability space $(\Omega, \mathcal{B}, \mathbb{P})$, and associate to every $\omega \in \Omega$ a random subcollection. Before going into the details, we will first introduce some notation.

Let $\bar{\mathcal{F}}$ be a collection of right translates of compact sets $F_n \subset G$. This implies that there exist sets A_n, such that

$$\bar{\mathcal{F}} = \{F_n a : n \in \{1, \dots, N\}, a \in A_n\}.$$

Let now a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and a mapping

$$\mathcal{F} : \Omega \to \mathcal{P}(\bar{\mathcal{F}}) : \omega \mapsto \mathcal{F}(\omega)$$

be given. Each subcollection $\mathcal{F}(\omega) \subset \bar{\mathcal{F}}$ induces a counting function on G, given by

$$\Lambda^\omega = \mathbf{1}_{\mathcal{F}(\omega)} = \sum_{B \in \mathcal{F}(\omega)} \mathbf{1}_B.$$

If the subcollection $\mathcal{F}(\omega)$ is disjoint, then $\Lambda^\omega \leq 1$, but in general this will not be the case. Therefore, the notation Λ^ω is preferred. Viewing ω as a variable also, we have the function-valued random variable

$$\Lambda : \Omega \times G \to \mathbf{N} : (\omega, g) \mapsto \Lambda^\omega(g).$$

We are now in a position to state the generalization of Lemma 4.2.2. Recall that the Haar measure on a discrete topological group equals the counting measure. As this measure is bi-invariant, we denote it by m.

Lemma 4.3.1 *Let G be a discrete, topological group with bi-invariant Haar measure m. Let $\{F_n\}_{n=1}^N$ be a collection of compact subsets of G, satisfying*

$$m\left(\bigcup_{m=1}^{n-1} F_m^{-1} F_n\right) < Cm(F_n), \tag{4.4}$$

for any $n \in \{2, \dots, N\}$ and some constant C, independent of n. Let $F \subset G$ be compact and assume that a collection

$$\bar{\mathcal{F}} = \{F_n a : n \in \{1, \dots, N\}, a \in A_n\}$$

of subsets of F is given. Then there exists a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and a mapping $\mathcal{F} : \Omega \to \mathcal{P}(\mathcal{F})$ as above, such that

$$1) \quad \mathbb{E}(\Lambda(g) | \Lambda(g) \geq 1) \leq (1 + \delta) \text{ for all } g \in G,$$

$$2) \quad \mathbb{E}\left(\int_F \Lambda(g) dm(g)\right) \geq \gamma(\delta, C) m\left(\bigcup_{n=1}^N A_n\right),$$

where $\gamma(\delta, C) = \delta/(1 + C\delta)$ and $\Lambda : \Omega \times G \to \boldsymbol{N}$ is the counting function.

Condition (4.4) implies that the sets $\{F_n\}_{n=1}^N$ form part of a tempered sequence. The first condition on Λ shows that the subcollection $\mathcal{F}(\omega)$ is almost disjoint on average. The second condition shows that it covers a given part of the original cover.

Proof From $f_n A_n \subset F$, we obtain $A_n \subset f_n^{-1} F$. As F is compact, so is $f_n^{-1} F$. In the discrete topology any set is closed. We conclude that A_n is a closed set in a compact set. It follows, that every set A_n is compact, and thus finite.

Let now $(\Omega, \mathbb{P})$ be a probability space, on which one can define for each $1 \leq n \leq N$ and $a \in A_n$ a Bernoulli random variable $\epsilon(n, a)$ with parameter $p_n = \delta/m(F_n)$. The existence of such a probability space is implied by the existence of a space on which one can define one Bernoulli random variable [3, Section 5.4]. One can also construct it manually. Put therefore

$$k = N \max_{1 \leq n \leq N} m(A_n)$$

and define $\Omega = \{1, \ldots, 2^k\}$. Then $(\Omega, \mathcal{P}(\Omega))$ can be made into a probability space by given each element $\omega \in \Omega$ probability 2^{-k}. The Bernoulli random variables $\epsilon(n, a)$ can then be defined by mapping $\epsilon(n, a)(\omega)$ to p_n times the $(Na + n)$-th digit in the binary representation of ω.

Now that we have specified the probability space, we will define the mapping $\mathcal{F}$. Let therefore an element $\omega \in \Omega$ be given. The following procedure describes how the random collection $\mathcal{F}(\omega)$ is constructed inductively.

1. Start with $n = N$ and $\{A_{n|N+1}(\omega)\}_{n=1}^N = \{A_n\}_{n=1}^N$.
2. Define the set of $a \in A_{n|n+1}(\omega)$ that are selected

   $$\Sigma_n(\omega) = \{a \in A_{n|n+1}(\omega) : \epsilon(n, a) = 1\}$$

   and the subcollection

   $$\mathcal{F}_n(\omega) = \{F_n a : a \in \Sigma_n(\omega)\}.$$

3. Define, for $m < n$, the set $A_{m|n}(\omega) \subset A_{m|n+1}(\omega)$ as

   $$A_{m|n}(\omega) = \{a \in A_{m|n+1}(\omega) : F_m a \cap F_n \Sigma_n(\omega) = \emptyset\}.$$

4. If $n \geq 2$, set $n = n - 1$ and return to 2.
5. Define $\mathcal{F}(\omega) = \bigcup_{n=1}^N \mathcal{F}_n(\omega)$.

This inductive procedure resembles the selection of the balls in Lemma 4.2.2. After selecting the sets $F_n a$, this algorithm also gives the sets $A_{m|n}(\omega)$, which consist of those $a \in A_m$, for which $F_n a$ is disjoint from any set that is selected in steps $n, \ldots, N$. One can derive that once the subcollection $\mathcal{F}_N(\omega)$ is constructed, the subcollection $\bigcup_{n=1}^{N-1} \mathcal{F}_n(\omega)$ has the same distribution as the one obtained when the algorithm is applied to the sets F_n and $A_{n|N}$, for $n \in \{1, \dots, N-1\}$. More precisely, for the distribution of the random collections $\mathcal{F}_n$, where $1 \leq n < N$, knowing the random sets $A_{n|N}$ gives the same information as knowing the random collection $\mathcal{F}_N$. We will make this more precise. The sets A_n are external to the algorithm. Note that the random collections $\mathcal{F}_n(\omega)$ are fully determined by the sets $\Sigma_n(\omega)$. For the random collections $\mathcal{F}_n(\omega)$, the outcome of the random variables $\epsilon(n,a)$ for $a \in A_n \setminus A_{n|n+1}$ is irrelevant. To determine the random collections $\mathcal{F}_n(\omega)$ for $1 \leq n < N$, one only needs to know the sets $\Sigma_n(\omega)$ for $1 \leq n < N$. One can get the relevant information about $\Sigma_N(\omega)$ both from $A_{n|N}(\omega)$ and from $\Sigma_N(\omega)$ and from $\mathcal{F}_N(\omega)$. This shows, that for $1 \leq n < N$, we indeed have

$$\mathbb{P}\left(\Sigma_n | A_{n|N}\right) = \mathbb{P}\left(\Sigma_n | \mathcal{F}_N\right) = \mathbb{P}\left(\Sigma_n | \Sigma_N\right) = \mathbb{P}\left(\Sigma_n | \{\epsilon(N,a) : a \in A_N\}\right).$$

We will now show that the mapping $\mathcal{F}$ has the desired properties. Define therefore the counting functions

$$\Lambda_n^\omega(g) = \sum_{B \in \mathcal{F}_n(\omega)} \mathbf{1}_B(g).$$

By construction, for every $\omega \in \Omega$, elements of the collections $\mathcal{F}_n(\omega)$ and $\mathcal{F}_{n'}(\omega)$ are disjoint, when $n \neq n'$. Therefore, we only need to deal with one function Λ_n^ω. Assume the set $A_{n|n+1}(\omega)$ is given. This corresponds to knowledge of all random variables $\epsilon(m,a)$ with $m > n$. We then have

$$\begin{aligned}
\Lambda_n^\omega(g) &= \sum_{B \in \mathcal{F}_n(\omega)} \mathbf{1}_B(g) \\
&= \sum_{F_n a \in \mathcal{F}_n(\omega)} \mathbf{1}_{F_n a}(g) \\
&= \sum_{a \in A_{n|n+1}} \epsilon(n,a)(\omega) \mathbf{1}_{F_n a}(g) \\
&= \sum_{a \in A_{n|n+1}} \epsilon(n,a)(\omega) \mathbf{1}_{F_n^{-1} g}(a) = \sum_{a \in A_{n|n+1}(\omega) \cap F_n^{-1} g} \epsilon(n,a)(\omega).
\end{aligned}$$

It follows, that $\Lambda_n^\omega(g)$ is the sum of $m(A_{n|n+1}(\omega) \cap F_n^{-1} g) \leq m(F_n)$ independent Bernoulli random variables. For any realization of the sets $A_{n|n+1}(\omega)$, it holds by Lemma 2.4.2, that

$$\mathbb{E}(\Lambda_n(g) | \Lambda_n(g) \geq 1, A_{n|n+1}) \leq 1 + m(F_n) p_n = 1 + \delta.$$

We will apply the law of minimal information[1] to get rid off the dependence on $A_{n|n+1}$. This shows, that

$$\mathbb{E}(\Lambda_n(g) | \Lambda_n(g) \geq 1) \leq 1 + \delta.$$

[1] $\mathbb{E}(\mathbb{E}(X|Y,Z)|Y) = \mathbb{E}(X|Y)$,

As there is only one $n \in \{1, \dots, N\}$ for each $g \in G$ and $\omega \in \Omega$, for which $\Lambda_n^\omega(g)$ is positive, this implies, that

$$\mathbb{E}(\Lambda(g)|\Lambda(g) \geq 1) \leq 1 + \delta.$$

To prove the second condition on the counting function Λ, we will exploit the recursive character of the algorithm. We will show by induction, that

$$\mathbb{E}\left(\sum_{g \in F} \Lambda(g)\right) = \mathbb{E}\left(\sum_{n=1}^{N} \sum_{g \in F} \Lambda_n(g)\right) \geq \gamma(\delta, C)\mu\left(\bigcup_{n=1}^{N} A_n\right), \tag{4.5}$$

where we explicitly stated that the integral over a discrete group is equal to a summation. As we start with a finite collection of right translates of the finite sets F_n, it is immediate that the summation will converge. For the counting function Λ_N, the following holds.

$$\begin{aligned}
\mathbb{E}\left(\sum_{g \in F} \Lambda_N(g)\right) &= \mathbb{E}\left(\sum_{g \in F_N} \sum_{F_N a \in \mathcal{F}} \mathbf{1}_{F_N a}(g)\right) \\
&= \mathbb{E}\left(\sum_{g \in F} \sum_{a \in A_N} \epsilon(N, a) \mathbf{1}_{F_N a}(g)\right) \\
&= \mathbb{E}\left(\sum_{a \in A_N} \epsilon(N, a) \sum_{g \in F} \mathbf{1}_{F_n a}(g)\right) \\
&= m(A_N) p_N m(F_N) = \delta m(A_N)
\end{aligned} \tag{4.6}$$

For $N = 1$, this implies (4.5) as $\delta > \gamma(\delta, C)$. Assume now that (4.5) holds for $N = N' - 1$. We will show it holds for N' also. Recall therefore that the random collections $\mathcal{F}_n$ for $n \leq N'$ have the same distribution as when the algorithm had been applied to the sets $A_{n|N'}(\omega)$. Assume now that the random variables $\epsilon(N', a)(\omega)$ are known. We will again apply the law of minimal information, to split up the expectation as

$$\mathbb{E}\left(\sum_{g \in F} \Lambda(g)\right) = \mathbb{E}\left(\sum_{g \in F} \Lambda_{N'}(g)\right) + \mathbb{E}\left(\mathbb{E}\left(\sum_{n=1}^{N'-1} \sum_{g \in F} \Lambda_n(g) \middle| \epsilon(N', a)\right)\right). \tag{4.7}$$

We will now focus on the second term in the right hand side of this expression. We will apply that F_n is the beginning of a *tempered* sequence. By definition of $A_{n|N'}(\omega)$, we have for every $\omega \in \Omega$,

$$A_{n|N'}(\omega) = A_n \setminus F_n^{-1} F_{N'} \Sigma_{N'}(\omega).$$

This implies, taking the union and suppressing the dependence on ω, that

$$\bigcup_{n=1}^{N'-1} A_{n|N'} = \bigcup_{n=1}^{N'-1} (A_n \setminus F_n^{-1} F_{N'} \Sigma_{N'}) \supset \bigcup_{n=1}^{N'-1} A_n \setminus \left(\bigcup_{n=1}^{N'-1} F_n^{-1} F_{N'}\right) \Sigma_{N'}.$$

Note that for any product of sets AB in a *discrete* topological group, we have

$$m(AB) = m\left(\bigcup_{a\in A} aB\right) \leq \sum_{a\in A} m(aB) = m(A)m(B).$$

This implies, by assumption on the sets F_n, that

$$\begin{aligned} m\left(\bigcup_{n=1}^{N'-1} A_{n|N'}(\omega)\right) &\geq m\left(\bigcup_{n=1}^{N'} A_n\right) - m\left(\left(\bigcup_{n=1}^{N'} F_n^{-1}F_{N'}\right)\Sigma_{N'}(\omega)\right) \\ &\geq m\left(\bigcup_{n=1}^{N'-1} A_n\right) - C\mu(F_{N'})m(\Sigma_{N'}(\omega)). \end{aligned}$$

Using this equation, we can evaluate the second term in (4.7). Note that the counting functions Λ_n are fully determined by $\mathcal{F}_n$. Furthermore, the information about $\mathcal{F}_n$ for $1 \leq n < N'$, captured in $\epsilon(N', a)$ is equal to the information captured in the sets $A_{n|N'}$. By the induction hypothesis (4.5), we therefore have

$$\begin{aligned} &\mathbb{E}\left(\mathbb{E}\left(\sum_{n=1}^{N'-1}\sum_{g\in F}\Lambda_n(g)\middle|\epsilon(N',a)\right)\right) \\ &\geq \mathbb{E}\left(\gamma(\delta,C)m\left(\bigcup_{n=1}^{N'-1} A_{n|N'}\right)\right) \\ &\geq \mathbb{E}\left(\gamma(\delta,C)m\left(\bigcup_{n=1}^{N'-1} A_n - Cm(F_{N'})m(\Sigma_{N'})\right)\right) \\ &= \gamma(\delta,C)m\left(\bigcup_{n=1}^{N'-1} A_n\right) - \gamma(\delta,C)m(F_{N'})\mathbb{E}(\Sigma_{N'}). \\ &= \gamma(\delta,C)m\left(\bigcup_{n=1}^{N'-1} A_n\right) - \gamma(\delta,C)m(F_{N'})\delta m(A_{N'}). \end{aligned}$$

Substituting this equation and (4.6) into (4.7), we conclude, that

$$\begin{aligned}\mathbb{E}\left(\sum_{g\in F}\Lambda(g)\right) &= \mathbb{E}\left(\sum_{g\in F}\Lambda_{N'}(g)\right) + \mathbb{E}\left(\mathbb{E}\left(\sum_{n=1}^{N'-1}\sum_{g\in F}\Lambda_n(g)\middle|\epsilon(N',a)\right)\right)\\ &\geq \delta\mu(A_{N'}) + \gamma(\delta,C)m\left(\bigcup_{n=1}^{N'-1}A_n\right) - \gamma(\delta,C)C\delta m(A_{N'})\\ &= \gamma(\delta,C)\left(m\left(\bigcup_{n=1}^{N'-1}A_n\right) + \left(\frac{\delta}{\gamma(\delta,C)} - C\delta\right)m(A_{N'})\right)\\ &\geq \gamma(\delta,C)m\left(\bigcup_{n=1}^{N'}A_n\right).\end{aligned}$$

This proves the statements on the counting function $\Lambda(g)$. □

4.4 A covering lemma for non-discrete groups

Any second countable, locally compact group G that is non-discrete has a σ-finite Haar measure which is non-atomic. Therefore, we can construct a Poisson process on the measurable space $(G, \mathcal{B})$. We will apply this Poisson process to prove the following analog of Lemma 4.3.1. Also this lemma was originally proven in [12].

Lemma 4.4.1 . *Let G be a non-discrete, topological group with left Haar measure m_L. Let $\{F_n\}_{n=1}^N$ be a collection of compact subsets of G, satisfying*

$$m_L\left(\bigcup_{m=1}^{n-1}F_m^{-1}F_n\right) < Cm_L(F_n), \tag{4.8}$$

for any $n \in \{2,\dots,N\}$ and some constant C, independent of n. Let $F \subset G$ be compact and assume that a collection

$$\bar{\mathcal{F}} = \{F_n a : n \in \{1,\dots,N\}, a \in A_n\}$$

of subsets of F is given. Then there exists a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and a mapping $\mathcal{F}$, such that

$$\begin{aligned}&1)\quad \mathbb{E}(\Lambda^\epsilon(g)|\Lambda^\epsilon(g)\geq 1) \leq (1+\delta) \text{ for all } g\in G,\\ &2)\quad \mathbb{E}\left(\int_F \Lambda^\epsilon(g)dm_L(g)\right) \geq \gamma(\delta,C)m_L\left(\bigcup_{n=1}^{N}A_n\right),\end{aligned}$$

where $\gamma(\delta, C) = \delta/(1 + C\delta)$ and $\Lambda : \Omega \times G \to \boldsymbol{N}$ is the counting function.

Proof Let the measurable space $(G, \mathcal{B}(\mathcal{T}))$ be given. Equipped with the *right* Haar measure, $(G, \mathcal{B}(\mathcal{T}), m_R)$ is a non-atomic measure space. For $1 \leq n \leq N$, let Υ'_n be a Poisson process on this measure space with intensity $\alpha_n = \delta / m_L(F_n)$. By definition, there exist probability spaces $(\Omega_n, \mathbb{P}_n)$ on which these Poisson processes are defined. Let $(\Omega, \mathbb{P})$ be the product of these probability spaces. By putting $\Upsilon_n = \Upsilon'_n \circ \pi_n$, the Poisson processes Υ_n are independent random variables on Ω [3, Section 5.4]. Given the random variables Υ_n, we will now describe the mapping $\mathcal{F}$. Let $\omega \in \Omega$ be given. Again the random subcollection $\mathcal{F}(\omega)$ will be described inductively.

1. Start with $n = N$ and $\{A_{n|n+1}(\omega)\}_{n=1}^N = \{A_n\}_{n=1}^N$.
2. Define the set of $a \in A_{n|n+1}(\omega)$ that are selected
$$\Sigma_n(\omega) = A_{n|n+1}(\omega) \cap \Upsilon_n(\omega)$$
and the subcollection
$$\mathcal{F}_n(\omega) = \{F_n a : a \in \Sigma_n(\omega)\}.$$
3. Define, for $m < n$, the set $A_{m|n}(\omega) \subset A_{m|n+1}(\omega)$ as
$$A_{m|n}(\omega) = \{a \in A_{m|n+1}(\omega) : F_m a \cap F_n \Sigma_n(\omega) = \emptyset\}.$$
4. If $n \geq 2$, set $n = n - 1$ and return to 2.
5. Define $\mathcal{F}(\omega) = \bigcup_{n=1}^N \mathcal{F}_n(\omega)$.

Note the resemblance of this procedure with the one for discrete groups. Again the subcollection $\bigcup_{n=1}^{N-1} \mathcal{F}_n(\omega)$ has the same distribution as when the the algorithm would have been applied to the sets F_n and $A_{n|N}(\omega)$, for $1 \leq n \leq N-1$. To show this mapping $\mathcal{F}$ satisfies the conditions, define the counting functions

$$\Lambda_n^\omega(g) = \sum_{B \in \mathcal{F}_n(\omega)} \mathbf{1}_B(g).$$

Assume that the set $A_{n|n+1}(\omega)$ is given. Then it holds that

$$\begin{aligned}\Lambda_n^\epsilon(g) &= \sum_{F_n a \in \mathcal{F}_n(\omega)} \mathbf{1}_{F_n a}(g) \\ &= \sum_{a \in A_{n|n+1}(\omega) \cap \Upsilon_n(\omega)} \mathbf{1}_{F_n a}(g) = \sum_{a \in A_{n|n+1}(\omega) \cap \Upsilon_n(\omega) \cap F_n^{-1} g} 1.\end{aligned}$$

It follows that $\Lambda_n^\omega(g)$ counts the number of elements from $\Upsilon_n(\omega)$ in $A_{n|n+1}(\omega) \cap F_n^{-1} g$. By definition of the Poisson process, $\Lambda_n^\omega(g) = |\Upsilon(\omega) \cap (A_{n|n+1}(\omega) \cap F_n^{-1} g)|$

is Poisson distributed with parameter $\alpha_n m_R(A_{n|n+1}(\omega) \cap F_n^{-1}g)$. Lemma 2.4.5 now implies, for all n, that

$$\begin{aligned}\mathbb{E}(\Lambda_n(g)|\Lambda_n(g) \geq 1, A_{n|n+1}) &\leq 1 + \alpha_n m_R(A_{n|n+1} \cap F_n^{-1}g) \\ &\leq 1 + \frac{\delta}{m_L(F_n)} m_R(F_n^{-1}g) \\ &= 1 + \delta \frac{m_R(F_n^{-1})}{m_L(F_n)} = 1 + \delta \frac{m_L(F_n)}{m_L(F_n)} = 1 + \delta.\end{aligned}$$

The first equality is allowed as the measure m_R is right invariant, which explains why we constructed a Poisson process with that measure. Taking now the expectation of this expression, one finds, as the functions Λ_n^ϵ have disjoint supports, that

$$\mathbb{E}(\Lambda(g)|\Lambda(g) \geq 1) \leq 1 + \delta.$$

It remains to show that

$$\mathbb{E}\left(\int_F \Lambda(g) dm_L(g)\right) = \mathbb{E}\left(\sum_{n=1}^N \int_F \Lambda_n(g) dm_L(g)\right) \geq \gamma(\delta, C) m_L\left(\bigcup_{n=1}^N A_n\right). \tag{4.9}$$

We will prove this again by induction. We will first show that for any N, it holds that

$$\mathbb{E}\left(\int_F \Lambda_N(g) dm_L(g)\right) \geq \gamma(\delta, C) m_L(A_N).$$

To do so, we will apply Theorem 2.4.4 with $f = \mathbf{1}_{A_N}\Delta$. By definition of the Poisson process, it holds that

$$\begin{aligned}\mathbb{E}\left(\int_F \Lambda_N dm_L(g)\right) &= \mathbb{E}\left(\sum_{F_N a \in \mathcal{F}_N} \int_F \mathbf{1}_{F_N a}(g) dm_L(g)\right) \\ &= \mathbb{E}\left(\sum_{a \in \Sigma_N} m_L(F_N a)\right) \\ &= m_L(F_N)\mathbb{E}\left(\sum_{a \in \Upsilon_N} \Delta(a) \mathbf{1}_{A_N}(a)\right) \\ &= \alpha_N m_L(F_N) \int_F \mathbf{1}_{A_N}(g)\Delta(g) dm_R(g) = \delta m_L(A_N) \quad (4.10)\end{aligned}$$

The last equality follows as Δ is the Radon-Nikodym derivative, as proven in Theorem 2.3.1. For $N = 1$, this proves (4.9). Assume now the statement holds for $N = N' - 1$. We will show that it holds for $N = N'$ also. We will use that given the subcollection $\mathcal{F}_{N'}$, the random subcollections $\mathcal{F}_n$ for $n < N'$ have the same distribution as when the algorithm was executed with the sets $A_{n|N'}$. Note that the random collection $\mathcal{F}_{N'}$ is determined by the Poisson process $\Upsilon_{N'}$.

Using the law of minimal information, we can express the expectation as follows.

$$\begin{aligned}\mathbb{E}\left(\int_F \Lambda(g)dm_L(g)\right) =&\mathbb{E}\left(\int_F \Lambda_{N'}(g)dm_L(g)\right)\\ &+\mathbb{E}\left(\mathbb{E}\left(\sum_{i=1}^{N'-1}\int_G \Lambda_n(g)dm_L(g)\middle|\Upsilon_{N'}\right)\right).\end{aligned}$$

We will compute the expectation of both terms on the right hand side. For the first one, equality (4.10) shows that

$$\mathbb{E}\left(\int_F \Lambda(g)dm_L(g)\right) = \mathbb{E}\left(\sum_{a\in\Sigma_{N'}}\int_F \mathbf{1}_{F_{N'}a}(g)dm_L(g)\right) = \delta m_L(A_{N'}).$$

For the second one, we will use that the sets F_N form the beginning of a *tempered* sequence. By construction of the algorithm, it holds that

$$A_{n|N'}(\omega) = A_n \setminus F_n^{-1}F_{N'}\Sigma_{N'}(\omega),$$

which implies that the union

$$\bigcup_{n=1}^{N'-1} A_{n|N'}(\omega) \supset \bigcup_{n=1}^{N'-1} A_n \setminus \left(\bigcup_{n=1}^{N'-1} F_n^{-1}F_{N'}\right)\Sigma_{N'}(\omega).$$

By definition of the Poisson process, the set $\Sigma_{N'}\subset\Upsilon_{N'}$ is countable. Therefore, it holds that

$$\begin{aligned}m_L\left(\left(\bigcup_{n=1}^{N'-1} F_n^{-1}F_{N'}\right)\Sigma_{N'}(\omega)\right) &\leq \sum_{g\in\Sigma_{N'}(\omega)} m_L\left(\bigcup_{n=1}^{N'-1} F_n^{-1}F_{N'}\right)\\ &= |\Sigma_{N'}(\omega)| m_L\left(\bigcup_{n=1}^{N'-1} F_n^{-1}F_{N'}\right)\\ &\leq |\Sigma_{N'}(\omega)| C m_L\left(F_{N'}\right).\end{aligned}$$

This now easily shows that

$$m_L\left(\bigcup_{n=1}^{N'-1} A_{n|N'}(\omega)\right) \geq m_L\left(\bigcup_{n=1}^{N'-1} A_n\right) - C\mu(F_{N'})|\Upsilon_{N'}(\omega)\cap A_{N'}|.$$

Taking now the expectation, we conclude by definition of the Poisson process, that

$$\mathbb{E}\left(m_L\left(\bigcup_{n=1}^{N'-1} A_{n|N'}\right)\right) \geq m_L\left(\bigcup_{n=1}^{N'-1} A_n\right) - C\delta\alpha_{N'}\mu(A_{N'})$$

A similar derivation as in the discrete case now proves the statement for N'. This proves the claim. □

4.5 The weak maximal inequality

With the generalization of Lemma 4.2.2, we can derive the maximal inequality for general amenable actions. The only difference with Section 4.2 is that we have to incorporate the fact that we now have a probabilistic subcollection of right translates of the sets F_n, that form the beginning of a tempered Følner sequence. Besides that, we will consider a more general setting, where the measure μ is σ-finite. This generalizes the maximal inequality as it is proven in [12], where the maximal inequality is only derived for *finite* measure spaces.

Theorem 4.5.1 *Let $(X, \mathcal{F}, \mu)$ be a σ-finite measure space on which an amenable, locally compact, second countable group G acts measure-preservingly. Let F_n be a tempered Følner sequence in G. Then there exists a constant C, independent of $(X, \mathcal{B}, \mu)$, such that for every $f \in L^1(X, \mathcal{F}, \mu)$, the following holds.*

$$\mu\left(\left\{x \in X : \sup_{n\in\mathbf{N}} \frac{1}{m_L(F_n)} \int_{F_n} f(gx)dm_L(g) > \alpha\right\}\right) \leq \frac{C}{\alpha}\|f\|_1.$$

Proof Let $f \in L^1(X, \mathcal{F}, \mu)$ and $\epsilon > 0$ be given. As the measure μ is σ-compact, there exists a sequence of sets $\{C_n\}_{n\in\mathbf{N}}$, that all have positive measure and such that $X = \bigcup_{n\in\mathbf{N}} C_n$. Defining

$$C'_n = \bigcup_{m=1}^{n} C_m,$$

we obtain an increasing sequence of sets with finite measure, whose union covers X. Define now the set

$$D' = \left\{x \in X : \sup_{n\in\mathbf{N}} \frac{1}{m_L(F_n)} \int_{F_n} f(gx)dm_L(g) > \alpha\right\} \subset X.$$

Choose $M \in \mathbf{N}$. By compactness of C'_M, the set $D' \cap C'_M$ has finite measure. Therefore, we can select $N \in \mathbf{N}$, such that the set

$$D = \left\{x \in X : \sup_{n\in\{1,\ldots,N\}} \frac{1}{m_L(F_n)} \int_{F_n} f(gx)dm_L(g) > \alpha\right\}$$

has measure $\mu(D) > \mu(D' \cap C'_M) - \epsilon$. Define $K = \bigcup_{n=1}^{N} F_n$. As a finite union of compact sets, K is compact. By amenability of G, there exists by Leptin's condition in Theorem 3.5.9 a compact set U, such that $m_L(KU) \leq (1+\epsilon)m_L(U)$.

Choose now $x \in X$. Define, for every $n \in \{1, \ldots, N\}$, the set

$$A_n = \left\{g' \in U : \frac{1}{m_L(F_n)} \int_{F_n} f(gg'x)dm_L(g) > \lambda\right\}.$$

Then for every $g' \in \bigcup_{n=1}^{N} A_n \subset U$, there exists an $n \in \{1, \ldots, N\}$, such that

$$\frac{1}{m_L(F_n)} \int_{F_n} f(gg'x)dm_L(g) > \alpha,$$

which implies that $g'x \in D$. Reversing this argument, we see that

$$\{g' \in U : g'x \in D\} = \bigcup_{n=1}^{N} A_n.$$

Taking measures, the above equality implies that

$$\int_U \mathbf{1}_D(g'x)dm_L(g') = m_L\left(\bigcup_{n=1}^{N} A_n\right). \tag{4.11}$$

For every $a \in A_n$, it holds that $F_n a \subset KU$. The collection

$$\overline{\mathcal{F}} = \{F_n a : n \in \{1, \ldots, N\}, a \in A_n\}$$

is therefore a collection of right translates of the F_n, all contained in the compact set KU. As the sets $\{F_n\}_{n\in\mathbf{N}}$ form a tempered Følner sequence, there exists a constant C', such that

$$m_L\left(\bigcup_{m=1}^{n-1} F_m^{-1}F_n\right) < C' m_L(F_n).$$

Let now $\delta = 1$. By Lemmas 4.3.1 and 4.4.1 respectively, there exists a probability space $(\Omega, \mathcal{B}, \mathbb{P})$ and a mapping $\mathcal{F}$ that assigns to each $\omega \in \Omega$ a subcollection of $\mathcal{F}(\omega) \subset \overline{\mathcal{F}}$. We will now evaluate

$$\mathbb{E}\left(\int_{KU} \Lambda(g)f(gx)dm_L(g)\right)$$

in two ways, where $\Lambda : \Omega \times G \to \mathbf{N}$ is the counting function. First, as $a \in A_n$ implies that

$$\int_{F_n} f(gax)dm_L(g) > \alpha m_L(F_n),$$

we find by substituting $g \to g' = ga$, that

$$\alpha m_L(F_n) < \int_{F_n a} f(g'x)dm_L(g'a^{-1}) = \Delta(a^{-1})\int_{F_n a} f(g'x)dm_L(g'),$$

with Δ the modular function. As it is a homomorphism, we find

$$\alpha m_L(F_n a) = \alpha\Delta(a)m_L(F_n) < \int_{F_n a} f(gx)dm_L(g).$$

By the second statement in Lemmas 4.3.1 and 4.4.1, this implies

$$\begin{aligned}\mathbb{E}\left(\int_{KU}\Lambda(g)f(gx)dm_L(g)\right) &= \mathbb{E}\left(\sum_{B\in\mathcal{F}}\int_B f(gx)dm_L(g)\right)\\ &\geq \alpha\mathbb{E}\left(\sum_{F_na\in\mathcal{F}} m_L(F_na)\right)\\ &= \alpha\mathbb{E}\left(\sum_{F_na\in\mathcal{F}}\int_{KU}\mathbf{1}_{F_na}(g)dm_L(g)\right)\\ &= \alpha\mathbb{E}\left(\int_{KU}\Lambda(g)dm_L(g)\right)\\ &\geq \alpha\gamma(1,C')m_L\left(\bigcup_{n=1}^N A_n\right). \qquad (4.12)\end{aligned}$$

On the other hand, by the first conclusion on the counting function from Lemmas 4.3.1 and 4.4.1, it holds that $\mathbb{E}(\Lambda(g)) \leq 2$ for every $g \in G$. Therefore, interchanging the expectation and the integration by virtue of Fubini's theorem, it holds that

$$\begin{aligned}\mathbb{E}\left(\int_{KU}\Lambda(g)f(gx)dm_L(g)\right) &= \int_{KU}\mathbb{E}(\Lambda(g))f(gx)dm_L(g) \qquad (4.13)\\ &\leq 2\int_{KU}f(gx)dm_L(g) \leq 2\int_{KU}|f(gx)|dm_L(g).\end{aligned}$$

Combining now the equations (4.11) – (4.13), it follows that

$$2\int_{KU}|f(gx)|dm_L(g) \geq \alpha\gamma(1,C')\int_U \mathbf{1}_D(gx)dm_L(g).$$

It is now easy to deduce the maximal inequality. The line of reasoning is exactly that of Theorem 4.2.3.

$$\begin{aligned}\mu(D'\cap C'_M) - \epsilon &\leq \mu(D)\\ &= \frac{1}{m_L(U)}\int_X\int_U \mathbf{1}_D(gx)d\mu(x)dm_L(g)\\ &\leq \frac{2(1+\epsilon)}{\alpha\gamma(1,C')m_L(KU)}\int_X\int_{KU}|f(gx)|dm_L(g)d\mu(x)\\ &= \frac{2(1+\epsilon)}{\alpha\gamma(1,C')m_L(KU)}\int_{KU}\int_X|f(x)|d\mu(x)dm_L(g)\\ &= \frac{2(1+\epsilon)}{\alpha\gamma(1,C')m_L(KU)}\int_{KU}\|f\|_1 dm_L(g) = \frac{2(1+\epsilon)\|f\|_1}{\alpha\gamma(1,C')}.\end{aligned}$$

As the expression holds for every $\epsilon > 0$, it should also hold in the limit where $\epsilon \to 0$. Therefore

$$\mu(D'\cap C'_M) \leq \frac{2\|f\|_1}{\alpha\gamma(1,C')}.$$

This inequality holds for every $M \in \mathbf{N}$. Taking the limit, the left hand side converges to $\mu(D')$, as the union of the sets C'_M equals the whole of X. Taking now the limit for $M \to \infty$, we conclude that

$$\mu\left(\left\{x \in X : \sup_{n\in\mathbf{N}} \frac{1}{m_L(F_n)} \int_{F_n} f(gx)dm_L(g) > \alpha\right\}\right) = \mu(D') \leq \frac{C}{\alpha}\|f\|_1,$$

where $C = 2/\gamma(1, C')$. This proves the maximal inequality. □

The maximal inequality can be applied to prove several ergodic theorems. In the next chapter we will demonstrate a number of such results.

Chapter 5

Ergodic theorems

5.1 The pointwise ergodic theorem

The first proofs of the classical pointwise ergodic theorem were based on the maximal inequality. Later proofs were constructed that use the more specific properties of $\mathbf{N}$ and $\mathbf{R}$. As these properties are not valid for general amenable groups, the maximal inequality is the obvious point to generalize the pointwise ergodic theorem. In [14], the maximal inequality is used for the classical pointwise ergodic theorem. We will now generalize it to all amenable actions. In [12] only a sketch of the proof was given for the case of a finite measure space. For completeness we include the full proof here.

Theorem 5.1.1 *Let $(G, \mathcal{B}, m_L)$ be a locally compact, second countable, topological group, that acts measure-preservingly on a finite measure space $(X, \mathcal{F}, \mu)$. Let the sequence $\{F_n\}_{n\in\boldsymbol{N}}$ be a tempered Følner sequence. Then for every $f \in L^1(X)$, there exists a G-invariant $\bar{f} \in L^1(X)$, such that for μ-almost every $x \in X$,*

$$\lim_{n\to\infty} \frac{1}{m_L(F_n)} \int_{F_n} f(gx) dm_L(g) \to \bar{f}(x).$$

For ease of notation, we will denote the ergodic averages

$$\frac{1}{m_L(F_n)} \int_{F_n} f(gx) dm_L(g)$$

by $(A_n f)(x)$ for $x \in X$, $f \in L^1(X)$ and $n \in \mathbf{N}$.

The proof of the pointwise ergodic theorem relies on four basic steps. We will present them separately. The first step is that for coboundaries the pointwise ergodic theorem holds, where the G invariant function is equal to 0.

Definition 5.1.2 *Let $f \in L^\infty(G)$ and $g \in G$ be given. Then the function*

$$f_g : X \to \boldsymbol{R} : x \mapsto f(x) - f(gx),$$

is the coboundary defined by f and g.

Lemma 5.1.3 *Let $1 \leq p \leq \infty$ and a coboundary in $L^p(x)$ be given. For μ-almost every $x \in X$, it holds, that*

$$\lim_{n\to\infty} A(f_g, n)(x) = \lim_{n\to\infty} \frac{1}{m_L(F_n)} \int_{F_n} f_g(g'x)dm_L(g') = 0.$$

Proof Let a coboundary $f_g : x \mapsto f(x) - f(gx)$ be given. For every $n \in \mathbf{N}$, it holds that

$$\begin{aligned}
\int_{F_n} f_g(g'x)dm_L(g') &= \int_{F_n} f(g'x)dm_L(g') - \int_{F_n} f(gg'x)dm_L(g') \\
&= \int_{F_n} f(g'x)dm_L(g') - \int_{gF_n} f(g'x)dm_L(g') \\
&= \int_{F_n\setminus gF_n} f(g'x)dm_L(g') - \int_{gF_n\setminus F_n} f(g'x)dm_L(g'). \quad (5.1)
\end{aligned}$$

Let now $A \subset G$ be an arbitrary subset. Assume that the set

$$D = \left\{x \in X : \int_A f(gx)dm_L(g) > m_L(A)\|f\|_p\right\}$$

has positive measure. We will show that

$$\int_D \int_A f(gx)dm_L(g)d\mu(x) > \mu(D)m_L(A)\|f\|_p. \quad (5.2)$$

To do so, assume the contrary. Then it holds, that

$$\int_G \left(\mathbf{1}_D(x) \int_A f(gx)dm_L(g) - \mathbf{1}_D(x)m_L(A)\|f\|_p\right) d\mu(x) = 0.$$

But this says that the integral of a positive function vanishes. By definition of the L^p-norm, this implies for almost every $x \in G$, that

$$\int_A f(gx)dm_L(g) = m_L(A)\|f\|_p.$$

However, we assumed that the above equality is invalid for every $x \in D$ and that $\mu(D) > 0$. This contradiction proves the claim. Applying now Fubini's theorem to (5.2) shows that

$$\begin{aligned}
\mu(D)m_L(A)\|f\|_p &< \int_D \int_A |f(gx)|dm_L(g)d\mu(x) \\
&= \int_A \int_D |f(gx)|d\mu(x)dm_L(g) \\
&= \int_A \int_D |f(x)|d\mu(x)dm_L(g) \\
&= m_L(A) \int_D |f(x)|d\mu(x).
\end{aligned}$$

This implies that $\int_D |f(x)| dm_L(x) > \|f\|_p$, which is impossible by definition of the norm in $L^p(X)$. Therefore, for every subset $A \subset G$, we have for μ-almost every $x \in X$,

$$\int_A |f(gx)| dm_L(g) \leq m_L(A) \|f\|_p.$$

We will now return to (5.1). Taking the absolute value, the above implies for μ-almost every $x \in X$, that

$$\begin{aligned}\left| \int_{F_n} f_g(g'x) dm_L(g') \right| &\leq \int_{F_n \setminus gF_n} |f(g'x)| dm_L(g') + \int_{gF_n \setminus F_n} |f(g'x)| dm_L(g') \\ &= \int_{F_n \Delta gF_n} |f(g'x)| dm_L(g') \leq m_L(F_n \Delta gF_n) \|f\|_p.\end{aligned}$$

Now, as the sequence $\{F_n\}_{n \in \mathbf{N}}$ is a Følner sequence, it holds for every $g \in G$, that

$$\lim_{n \to \infty} \frac{m_L(F_n \Delta gF_n)}{m_L(F_n)} \to 0.$$

This proves the claim. □

The next lemma determines the orthogonal complement of the set of coboundaries, as a subspace of $L^2(X)$.

Lemma 5.1.4 *Every function $f \in L^2(X)$ that is orthogonal to every coboundary in $L^2(X)$, is G-invariant.*

Proof Let $f \in L^2(X)$ be orthogonal to every coboundary and let $g \in G$ be given. Take an arbitrary simple function $h \in L^1(X)$. Then it holds by assumption, that

$$\int_X f(x)(h(x) - h(gx)) d\mu(x) = 0.$$

This implies, as G acts measure-preservingly, that

$$\begin{aligned}0 &= \int_X f(x)h(x) d\mu(x) - \int_X f(x) h\left(gx\right) d\mu(x) \\ &= \int_X f(x)h(x) d\mu(x) - \int_X f\left(g^{-1}x\right) h(x) d\mu(x) \\ &= \int_X \left(f(x) - f\left(g^{-1}x\right)\right) h(x) d\mu(x)\end{aligned}$$

It follows, that the function $x \mapsto f(x) - f(g^{-1}x)$ is orthogonal to every simple function. As the inner product is continuous and the simple functions are dense in $L^2(X)$, it follows $x \mapsto f(x) - f(g^{-1}x)$ is orthogonal to the whole of $L^2(X)$. As the orthogonal complement of $L^2(X)$ is only the zero function, the statement follows. □

Define now $\mathcal{D}_1$ as the set of coboundaries in $L^2(X)$, and $\overline{\mathcal{D}_1}^2$ as its closure in $L^2(X)$. By the orthogonal principle [20, Theorem 2D], every $f \in L^2(X)$ can be written as $f = f_1 + f_2$, where f_1 is in $\overline{\mathcal{D}_1}^2$ and $f_2 \perp \overline{\mathcal{D}_1}^2$. By the previous lemma, f_2 is G-invariant. This implies that $L^2(X) = \overline{\mathcal{D}_1}^2 \oplus \mathcal{D}_2$, where we defined $\mathcal{D}_2$ as the subspace of $L^2(X)$ of G-invariant functions.

We will now show that if the sequence converges almost everywhere, then there is a G-invariant function to which it converges almost everywhere.

Lemma 5.1.5 *Let $f \in L^1(X)$ be given. If the sequence $A(f,n)(x)$ of ergodic averages converges for almost every $x \in X$, then there exists a G-invariant function $\bar{f}$, such that for almost every $x \in X$,*

$$\lim_{n\to\infty} A(f,n)(x) = \bar{f}.$$

Proof Let $f \in L^1(X)$ and $g \in G$ be given, such that the ergodic averages converge for almost every $x \in X$. Then the same holds for $x \mapsto f(g^{-1}x)$. This implies for almost every $x \in X$, that

$$\begin{aligned}\bar{f}(x) - \bar{f}(gx) &= \lim_{n\to\infty} \frac{1}{m_L(F_n)} \left(\int_{F_n} f(g'x) dm_L(g') - \int_{F_n} f(g'gx) dm_L(g') \right) \\ &= \lim_{n\to\infty} \frac{1}{m_L(F_n)} \int_{F_n} (f(g'x) - f(g'gx)) dm_L(g') = \lim_{n\to\infty} A(f_g, n)(x).\end{aligned}$$

By Lemma 5.1.3, the limit converges to zero for almost every $x \in X$. It follows that $\bar{f} = \bar{f} \circ g$ almost everywhere. This proves that $\bar{f}$ is G-invariant. □

It is now easy to show that the set of functions that satisfy the pointwise ergodic theorem is closed.

Lemma 5.1.6 *Define $\mathcal{C}$ as the set of functions that satisfy the pointwise ergodic theorem:*

$$\mathcal{C} = \left\{ f \in L^1(X) : \exists \bar{f} \in L^1(X) : \lim_{n\to\infty} \frac{1}{m_L(F_n)} \int_{F_n} f(gx) dm_L(g) = \bar{f}(x) \right\}.$$

Then the set $\mathcal{C}$ is closed in $L^1(X)$.

Proof To show the set $\mathcal{C}$ is closed, let $\{f_m\}_{m\in\mathbf{N}}$ be a converging sequence in $L^1(X)$. By definition there exists an $f \in L^1(X)$, such that

$$\|f - f_m\|_1 \to 0.$$

We will show that this implies that $f \in \mathcal{C}$. By definition of the set $\mathcal{C}$, for every $m \in \mathbf{N}$, the ergodic averages

$$A(f_m, n)(x) = \frac{1}{m_L(F_n)} \int_{F_n} f_m(gx) dm_L(g)$$

converge for almost every $x \in X$. We will show the ergodic averages $A(f,n)(x)$ converge for almost every $x \in X$. By the previous lemma, the function f satisfies the pointwise ergodic theorem, and hence belongs to $\mathcal{C}$.

To prove that the ergodic averages converge for almost every $x \in X$, we will show that the sequence form a Cauchy sequence in $\mathbf{R}$ for almost every x. Let therefore $n, n' \in \mathbf{N}$ be given. Then it holds that

$$\begin{aligned}|A(f,n)(x) - A(f,n')(x)| \leq &|A(f_m,n)(x) - A(f_m,n')(x)| \\ &+ |A(f - f_m, n)(x)| + |A(f - f_m, n')(x)|.\end{aligned}$$

Taking now the limit for $n, n' \to \infty$, the first term in the right hand side converges to 0 for almost every $x \in X$, by definition of $\mathcal{C}$. Therefore,

$$\begin{aligned}&\mu\left(\left\{x \in X : \limsup_{n,n'\to\infty} |A(f,n)(x) - A(f,n')(x)| > \alpha\right\}\right) \\ &\quad \leq \mu\left(\left\{x \in X : 2\sup_{n\in\mathbf{N}} |A_n(f - f_m)(x)| > \alpha\right\}\right) \leq \frac{2C}{\alpha}\|f - f_m\|_1,\end{aligned}$$

where the last equation follows from the maximal inequality, and C is the constant involved in the maximal inequality. As this holds for every $m \in \mathbf{N}$, taking the limit for $m \to \infty$, it follows, that

$$\mu\left(\left\{x \in X : \limsup_{n,n'\to\infty} |A(f,n)(x) - A(f,n')(x)| > \alpha\right\}\right) = 0.$$

Taking now the limit for $\alpha \to 0$, we find that $A(f,n)(x)$ is a Cauchy sequence in $\mathbf{R}$ for almost every $x \in X$. This implies the ergodic averages for f converge almost everywhere. We conclude $f \in \mathcal{C}$. It follows $\mathcal{C}$ is closed. □

Using these lemmas, the pointwise ergodic theorem can easily be derived.

Proof of Theorem 5.1.1 The statement of the theorem is equivalent to $\mathcal{C} = L^1(X)$. Define

$$\mathcal{D}_1 = \{h_g : h \in L^\infty(X), g \in G\} \cap L^2(X)$$

and

$$\mathcal{D}_2 = \{f \in L^2(X) : f \text{ is } G-\text{invariant}\}.$$

By Lemma 5.1.3, $\mathcal{D}_1 + \mathcal{D}_2 \subset \mathcal{C}$. This implies

$$\mathcal{D}_1 + \mathcal{D}_2 \subset \mathcal{C} \subset L^1(X).$$

Let now $\{f_m\}_{m\in\mathbf{N}}$ be a sequence in $\mathcal{D}_1$, that converges in $L^2(X)$. As $\|\cdot\|_1 \leq c\|\cdot\|_2$ by Theorem 2.3.8, it follows that the sequence converges in $L^1(X)$ also. Lemma 5.1.4 now implies, that

$$L^2(X) = \overline{\mathcal{D}_1}^2 + \mathcal{D}_2 \subset \overline{\mathcal{D}_1}^1 + \mathcal{D}_2 \subset \overline{\mathcal{D}_1 + \mathcal{D}_2}^1,$$

where the superscripts indicate the closure in $L^1(X)$, $L^2(X)$ respectively. Recall that $\mathcal{D}_1 + \mathcal{D}_2 \subset \mathcal{C}$, and that the closure of $L^2(X)$ in $L^1(X)$ is $L^1(X)$, as $L^2(X)$ contains all simple functions. We now have, as $\mathcal{C}$ is closed by Lemma 5.1.6, that

$$S \subset L^2(X) \subset \overline{\mathcal{D}_1 + \mathcal{D}_2}^1 \subset \overline{\mathcal{C}}^1 = \mathcal{C} \subset L^1(X).$$

This implies that

$$S \subset \mathcal{C} \subset L^1(X).$$

Taking the closure in $L^1(X)$ of this expression yields that $\mathcal{C} = L^1(X)$. □

5.2 Strong type maximal inequalities

Considered as an operator on $L^p(X, \mathcal{F}, \mu)$, where $p > 1$, the mapping A^*, defined by

$$(A^* f)(x) = \sup_{n \in \mathbf{N}} \frac{1}{m_L(F_n)} \int_{F_n} f(gx) dm_L(g),$$

is strongly bounded. By strong boundedness, we mean that the operator norm of the mapping is finite.

Definition 5.2.1 *Let A be an operator from $L^p(X, \mathcal{F}, \mu)$ to itself. Then A is strongly bounded, if there is a constant C, such that for all $f \in L^p(X, \mathcal{F}, \mu)$, it holds that*

$$\|Af\|_p \leq C\|f\|_p.$$

Mappings that satisfy the condition in this definition are normally just called bounded. The reason that we call them *strongly* bounded, is to distinguish them explicitly from *weakly* bounded operators.

Definition 5.2.2 *Let A be an operator from $L^p(X, \mathcal{F}, \mu)$ to itself. Then A is weakly bounded, if there is a constant C, such that for all $f \in L^p(X, \mathcal{F}, \mu)$, it holds that*

$$\mu\left(\{x \in X : (Af)(x) > \alpha\}\right) \leq \frac{C}{\alpha}\|f\|_p.$$

Applying this to the operator A^* on $L^p(X, \mathcal{F}, \mu)$, the maximal inequality 4.5.1 shows that A^* is weakly bounded on $L^1(X, \mathcal{F}, \mu)$. It will be the subject of this section to prove that A^* is strongly bounded when $p > 1$. To do so, we will first show the statement for $X = G$. Then we will lift the statement to arbitrary sets X on which G acts. This technique is known as *transference* [2]

Consider now first the situation where G acts on itself by left translation. This action is measure-preserving by definition of the Haar measure. For $p = \infty$, the statement now follows easily, as the next lemma shows.

Lemma 5.2.3 *Define, for every $f \in L^\infty(G)$, the maximal function*

$$A^*f : G \to \boldsymbol{R} : g \mapsto \sup_{n\in\boldsymbol{N}} \frac{1}{m_L(F_n)} \int_{F_n} f(g'g)dm_L(g').$$

Then $A^*f \in L^\infty(G)$ and $\|A^*f\|_\infty \le \|f\|_\infty$.

Proof As every F_n has positive measure in G, it holds by definition of the supremum norm, that

$$\begin{aligned}\int_{F_n} f(g'g)dm_L(g') &= \int_{F_n g} f(g')dm_L(g'g^{-1}) \\ &\le \Delta\left(g^{-1}\right) m_L(F_n g)\|f\|_\infty = \mu(F_n)\|f\|_\infty.\end{aligned}$$

Therefore, for every $n \in \mathbf{N}$ and $g \in G$, we have $A_n f(g) \le \|f\|_\infty$. Taking now the supremum over n, we conclude for all $g \in G$, that

$$A^*f(g) = \sup_{n\in\mathbf{N}} A_n f(g) \le \|f\|_\infty.$$

By definition of the supremum norm, $\|A^*f\|_\infty \le \|f\|_\infty$. This proves both claims. □

The previous lemma and the weak type maximal inequality in Lemma 4.5.1 prove the propositions for the operator A^* for the boundary cases $p = 1$ and $p = \infty$. It is a common technique to use some sort of interpolation for the other values of p. This is applied in the Theorems of Marcinkiewicz and Riesz-Thorin, for example, for more general operators on L^p. We will prove a generalization of [17, Chapter II, Theorem 3.7]. To do so, we need the following lemma.

Lemma 5.2.4 *Let $p > 1$ and $f \in L^p(G)$ be given. Then*

$$\|f\|_p^p = 2^p p \int_0^\infty s^{p-1} m_L(B(f, 2s))d\lambda(s),$$

where $B(f, 2s) = \{g \in G : |f(g)| > 2s\}$.

Proof Assume first that f is a simple function. Then f can be decomposed as

$$f = \sum_{i=1}^n \mathbf{1}_{B_i}\beta_i,$$

where the sets $\{B_i\}_{i=1}^n$ are mutually disjoint. The norm of f can be computed from

$$\|f\|_p^p = \int_G \sum_{i=1}^n \mathbf{1}_{B_i}(g)|\beta_i|^p dm_L(g) = \sum_{i=1}^n \int_{B_i} |\beta_i|^p dm_L(g) = \sum_{i=1}^n m_L(B_i)|\beta_i|^p.$$

On the other hand, it holds by definition, that

$$B(f, 2s) = \bigcup_{i:|\beta_i|>2s} B_i, \quad \Rightarrow \quad m_L(B(f, 2s)) = \sum_{i:|\beta_i|>2s} m_L(B_i).$$

This implies

$$\begin{aligned}\int_0^\infty s^{p-1} m_L(B(f,2s))d\lambda(s) &= \int_0^\infty s^{p-1} \sum_{i:|\beta_i|>2s} m_L(B_i)d\lambda(s) \\ &= \int_0^\infty s^{p-1} \sum_{i=1}^n \mathbf{1}_{(0,\frac{1}{2}|\beta_i|)}(s) m_L(B_i)d\lambda(s) \\ &= \sum_{i=1}^n m_L(B_i) \int_0^{\frac{1}{2}|\beta_i|} s^{p-1} d\lambda(s) \\ &= \sum_{i=1}^n m_L(B_i) \frac{1}{p} \frac{|\beta_i|^p}{2^p} = \frac{1}{p2^p} \sum_{i=1}^n m_L(B_i)|\beta_i|^p.\end{aligned}$$

Multiplying this equation by $p2^p$, we find

$$p2^p \int_0^\infty s^{p-1} m_L(B(f,2s))d\lambda(s) = \sum_{i=1}^n m_L(B_i)|\beta_i|^p = \|f\|_p^p. \tag{5.3}$$

Let now an arbitrary, positive function $f \in L^p(G)$ be given. By definition of the Lebesgue integral, there exists a sequence of simple functions $\{f_n\}_{n\in\mathbf{N}}$, such that $f_n \leq |f|^p$ and

$$\int_G |f|^p(g)dm_L(g) = \lim_{n\to\infty} \int_G f_n(g)dm_L(g).$$

Define now $f_n' = \sqrt[p]{f_n}$ for all $n \in \mathbf{N}$. It holds that $f_n' \leq |f|$ for all n. Furthermore, as $f_n' \to f$ pointwise, it holds that

$$m_L(B(f,2s)) = \lim_{n\to\infty} m_L(B(f_n',2s)).$$

Furthermore, the sequence $m_L(B(f_n',2s))$ constitute an increasing sequence for every $s \in \mathbf{R}_+$. Multiplying this equation by s^{p-1}, and integrating it over $\mathbf{R}_+$, we therefore find from the Beppo-Levi theorem [4, Chapter 16.2], that

$$\begin{aligned}p2^p &\int_0^\infty s^{p-1} m_L(B(f,2s))d\mu(s) \\ &= \lim_{n\to\infty} p2^p \int_0^\infty s^{p-1} m_L(B(f_n',2s))d\lambda(s) = \lim_{n\to\infty} \|f_n'\|_p^p = \|f\|_p^p,\end{aligned}$$

where the last equality follows by definition of f_n'. This proves the claim. □

The interpolation theorems work only when the operator is sub-additive. Fortunately, the mapping $f \mapsto A^*f$ is sub-additive. Let $f, h \in L^p(G)$ be given. For any n, it holds that

$$\begin{aligned} A_n(f+h)(g) =& \frac{1}{m_L(F_n)} \int_{F_n} (f+h)(g'g)dm_L(g') \\ =& \frac{1}{m_L(F_n)} \int_{F_n} f(g'g)dm_L(g') \\ &+ \frac{1}{m_L(F_n)} \int_{F_n} h(g'g)dm_L(g') = A_n f(g) + A_n h(g). \end{aligned}$$

Taking now the supremum over $n \in \mathbf{N}$, it follows immediately, that

$$\begin{aligned} A^*(f+h) &= \sup_{n\in\mathbf{N}} A_n(f+h) \\ &= \sup_{n\in\mathbf{N}} A_n f + A_n h \\ &\leq \sup_{n\in\mathbf{N}} A_n f + \sup_{n\in\mathbf{N}} A_n h = A^* f + A^* h. \end{aligned} \tag{5.4}$$

We will apply this to prove that the operator A^* is strongly bounded on $L^p(G)$, when $p > 1$. The following can be seen as a generalization of the Theorem of Hardy and Littlewood on maximal functions, that states that the operator, that maps a function to its Hardy-Littlewood maximal function, is strongly bounded [17, Chapter II, Theorem 3.7].

Theorem 5.2.5 *The operator $A^* : L^p(G) \to L^p(G) : f \mapsto A^* f$ is strongly bounded when $p > 1$. This means, that there exists a $C = C(p)$, such that it holds for all $f \in L^p(G)$, that*

$$\|A^* f\|_p \leq C\|f\|_p.$$

Proof Let an arbitrary $f \in L^p(G)$ be given. Define, for any $s > 0$, the functions

$$f_s^1(g) = \begin{cases} f(g) & : |f(g)| > s \\ 0 & : |f(g)| \leq s, \end{cases}$$

and

$$f_s^\infty(g) = \begin{cases} 0 & |f(g)| > s \\ f(g) & |f(g)| \leq s. \end{cases}$$

By definition, $f_s^\infty(g) \leq s$, which implies that $f_s^\infty \in L^\infty(G)$. Furthermore, for every $g \in G$,

$$A^* f_s^\infty(g) = \sup_{n\in\mathbf{N}} \frac{1}{m_L(F_n)} \int_{F_n} f_s^\infty(g'g)dm_L(g') \leq \|f_s^\infty\|_\infty \leq s.$$

This implies, that the set

$$B^*(f_s^\infty, s) = B(A^* f_s^\infty, s) = \emptyset.$$

Consider now f_s^1. By construction, if $f_s^1(g) \neq 0$, it holds that

$$\frac{1}{s}|f_s^1(g)| > 1 \quad \Rightarrow \quad \left(\frac{|f_s^1(g)|}{s}\right) < \left(\frac{|f_s^1(g)|}{s}\right)^p = \left(\frac{|f(g)|}{s}\right)^p.$$

As $f \in L^p(G)$, $\|f^p\|_1 < \infty$. The above equality therefore implies that $f_s^1 \in L^1(G)$.

We will now apply the weak type maximal inequality from Theorem 4.5.1. First, assume that $A^* f_s^1(g) \leq s$. Then, by sub-additivity of A^*, it follows, that

$$A^* f(g) \leq A^* f_s^1(g) + A^* f_s^\infty(g) \leq s + s = 2s.$$

This implies, defining the sets

$$B^*(h, s) = B(A^* h, s) = \{g \in G : |A^* h(g)| > s\}$$

for any measurable function h, that $B^*(f, 2s) \subset B^*(f_s^1, s)$, so that $m_L(B^*(f, 2s)) \leq m_L(B^*(f_s^1, s))$. Note that the weak type maximal inequality from Theorem 4.5.1 tells us, that

$$m_L(B^*(f_s^1, s)) \leq C \frac{\|f_s^1\|_1}{s} = \frac{C}{s} \int_G |f_s^1(g)| dm_L(g).$$

We will now finish the argument, combining all the above steps. It is clear that $A^* f$ is a measurable function, as the supremum of a countable set of measurable functions. Using Lemma 5.2.4, it follows that

$$\begin{aligned}
\|A^* f\|_p^p &= p 2^p \int_0^\infty s^{p-1} m_L(B^*(f, 2s)) d\lambda(s) \\
&\leq p 2^p \int_0^\infty s^{p-1} m_L(B^*(f_s^1, s)) d\lambda(s) \\
&\leq p 2^p C \int_0^\infty s^{p-2} \int_G |f_s^1(g)| dm_L(g) d\lambda(s) \\
&= p 2^p C \int_0^\infty \int_{\{g \in G : |f(g)| > s\}} s^{p-2} |f(g)| dm_L(g) d\lambda(s) \\
&= p 2^p C \int_0^\infty \int_G s^{p-2} \mathbf{1}_{(s,\infty)}(|f(g)|) |f(g)| dm_L(g) d\lambda(s) \\
&= p 2^p C \int_G \int_0^\infty \mathbf{1}_{(0,|f(g)|)}(s) s^{p-2} d\lambda(s) |f(g)| dm_L(g) \\
&= p 2^p C \int_G \int_0^{|f(g)|} s^{p-2} d\lambda(s) |f(g)| dm_L(g) \\
&= p 2^p C \int_G \frac{1}{p-1} |f(g)|^{p-1} |f(g)| dm_L(g) \\
&= \frac{p}{p-1} 2^p C \int_G |f(g)|^p dm_L(g) = \frac{p}{p-1} 2^p C \|f\|_p^p.
\end{aligned}$$

This proves the claim. □

We now want to extend the above theorem to the action of G on an arbitrary measure space $(X, \mathcal{F}, \mu)$. This can indeed by done, which tells us that characteristics of amenable actions rely on the group G that acts really. Lifting statements from the acting group to the group upon which it acts is known as transference.

Theorem 5.2.6 *Let* $(G, \mathcal{B}, m_L)$ *be an amenable group, that acts measure-preservingly on a finite measure space* $(X, \mathcal{F}, \mu)$*. Let* $\{F_n\}_{n\in\boldsymbol{N}}$ *be a Følner sequence for* G*. Then the mapping*

$$A^* : L^p(X) \to L^p(X) : f(x) \mapsto \sup_{n\in\boldsymbol{N}} \frac{1}{m_L(F_N)} \int_{F_n} f(g'x)dm_L(g')$$

is strongly bounded for $p > 1$ *and weakly bounded for* $p = 1$*. In particular, for* $p > 1$*, there exists a constant* $C(p)$*, such that for every* $f \in L^p(X)$*,*

$$\|A^* f\|_p \leq C(p)\|f\|_p.$$

Proof For $p = 1$, the ergodic theorem 5.1.1 shows that the mapping is weakly bounded. We will show that for $p > 1$, there exists a constant $C(p)$, such that for every $f \in L^p(X)$,

$$\|A^* f\|_p^p \leq C(p)^p \|f\|_p^p.$$

Then A^* is bounded by definition. To do so, select $p > 1$. Recall from Lemma 5.2.5, that there exists a constant C, such that for any $h \in L^p(G)$, we have

$$\|A^* h\|_p^p \leq C^p \|h\|_p^p.$$

Let now $0 \leq a' \in L^p(G)$ and $M, N \in \mathbf{N}$ be given. Defining $F = F_N F_M \cap F_M F_N \cap F_M$, it holds that $a = a'\mathbf{1}_F \in L^p(G)$ also. The above equality then boils down to the following.

$$\int_G [A^* a(g)]^p \, dm_L(g) \leq C^p \int_G a^p(g) dm_L(g) = C^p \int_F a^p(g) dm_L(g).$$

From this it follows, as a is assumed positive and $F_M \subset G$, that

$$\begin{aligned}
\int_{F_M} \left[\sup_{1\leq n\leq N} A_n a(g)\right]^p dm_L(g) &\leq \int_G \left[\sup_{1\leq n\leq N} A_n a(g)\right]^p dm_L(g) \\
&\leq \int_G \left[\sup_{n\in\mathbf{N}} A_n a(g)\right]^p dm_L(g) \\
&\leq C^p \int_F a^p(g) dm_L(g). \qquad (5.5)
\end{aligned}$$

Let now $0 \leq f \in L^p(X)$ be given. Then $f^p \in L^1(X)$. Define for any $x \in X$ the function $a' : G \to G$ by

$$a'(g) = \begin{cases} f(gx) & : g \in F \\ 0 & : g \in G \setminus F. \end{cases}$$

Recall that $f \geq 0$ and $F \subset F_M$. The weak type maximal inequality now implies, that the set

$$\left\{x \in X : \int_F f^p(gx) dm_L(g) > \lambda m_L(F_M)\right\}$$

has measure

$$\begin{aligned}
&\mu\left(\left\{x\in X : \int_F f^p(gx)dm_L(g) > \lambda m_L(F_M)\right\}\right)\\
&\leq \mu\left(\left\{x\in X : \int_{F_M} f^p(gx)dm_L(g) > \lambda m_L(F_M)\right\}\right)\\
&\leq \mu\left(\left\{x\in X : \sup_{n\in\mathbf{N}} \frac{1}{m_L(F_n)}\int_{F_n} f^p(gx)dm_L(g) > \lambda\right\}\right) \leq \frac{C'}{\lambda}\|f\|_p^p.
\end{aligned}$$

We conclude that

$$\int_G a'^p(gx)dm_L(g) = \int_F f^p(gx)dm_L(g)$$

exists for μ-almost every $x \in X$. Therefore, $a' \in L^p(G)$ for μ-almost every $x \in X$. Substituting the above equality into equation (5.5), we find

$$\int_{F_M}\left[\sup_{1\leq n\leq N}\frac{1}{m_L(F_n)}\int_{F_n} f(g'gx)dm_L(g)\right]^p dm_L(g) \leq C^p\int_F f^p(gx)dm_L(g)$$

for μ-almost every $x \in X$. We will now integrate this equation over X. For the right hand side we find

$$\begin{aligned}
\int_X\int_F f^p(gx)dm_L(g)d\mu(x) &= \int_F\int_X f^p(gx)d\mu(x)dm_L(g)\\
&= \int_F\int_X f^p(x)d\mu(x)dm_L(g)\\
&= m_L(F)\int_X f^p(x)d\mu(x).
\end{aligned}$$

Similarly, integrating the left hand side over X implies, that

$$\begin{aligned}
&\int_X\int_{F_M}\left[\sup_{1\leq n\leq N} A_n f(gx)\right]^p dm_L(g)d\mu(x)\\
&= \int_{F_M}\int_X\left[\sup_{1\leq n\leq N} A_n f(gx)\right]^p d\mu(x)dm_L(g)\\
&= \int_{F_M}\int_X\left[\sup_{1\leq n\leq N} A_n f(x)\right]^p d\mu(x)dm_L(g)\\
&= m_L(F_M)\int_X\left[\sup_{1\leq n\leq N} A_n f(x)\right]^p d\mu(x).
\end{aligned}$$

Combining the above equalities and dividing by $m_L(F_M)$, we conclude that

$$\int_X\left[\sup_{1\leq n\leq N} A_n f(x)\right]^p d\mu(x) \leq C^p\frac{m_L(F)}{m_L(F_M)}\int_X f^p(gx)d\mu(x). \tag{5.6}$$

We will now exploit that the sets $\{F_n\}_{n\in\mathbf{N}}$ form a Følner sequence. By construction of F, we have

$$F \subset F_N F_M = (F_N F_M \setminus F_M) \cup F_M \subset (F_N F_M \Delta F_M) \cup F_M.$$

The defining property of a Følner sequence now implies that

$$\frac{m_L(F)}{m_L(F_M)} \leq \frac{m_L(F_N F_M \Delta F_M)}{m_L(F_M)} + \frac{m_L(F_M)}{m_L(F_M)} = \frac{m_L(F_N F_M \Delta F_M)}{m_L(F_M)} + 1.$$

Taking the limit for $M \to \infty$, we conclude that

$$\lim_{L\to\infty} \frac{m_L(F)}{m_L(F_M)} = 1.$$

We will now take the limit for $M \to \infty$ in (5.6). It follows, that

$$\int_X \left[\sup_{1\leq n\leq N} A_n f(x)\right]^p d\mu(x) \leq C^p \int_X f^p(gx) d\mu(x) = C^p \|f\|_p^p.$$

Taking now the limit for $N \to \infty$ proves our claim. □

Bibliography

[1] Robert A. Adams. *Sobolev Spaces.* Number 22 in Pure and Applied Mathematics. Academic Press, fourth print edition, 1973.

[2] Idris Assani. *Wiener Wintner Ergodic Theorems.* World Scientific, 2003.

[3] Heinz Bauer. *Probability Theory and Elements of Measure Theory.* International Series in Decision Processes. Holt, Rinehart and Winston, 1972.

[4] Patrick Billingsley. *Probability and Measure.* Wiley Series in Probability and Mathematical Statistics. John Wiley and Sons, 1979.

[5] N. Bourbaki. *Topologie Générale*, volume II of *Éléments de Mathematique.* Hermann, 1971.

[6] Karma Dajani. Ergodic theory (lecture notes), 2004.

[7] William R. Emerson. Ratio properties in locally compact amenable groups. *Transactions of the American Mathematical Society*, 133(1):179–204, August 1968.

[8] R. Engelking. *Outline of general topology.* North-Holland publishing company, 1968.

[9] Paul R. Halmos. *Measure Theory.* Princeton, 1950.

[10] Edwin Hewitt and Kenneth A. Ross. *Abstract Harmonic Analysis*, volume 1 of *Die Grundlehren der Mathematischen Wissenschaften.* Springer Verlag, 1963.

[11] J. F. C. Kingman. *Poisson processes.* Number 3 in Oxford studies in probability. Oxford University Press, 1993.

[12] Elon Lindenstrauss. Pointwise theorems for amenable groups. *Inventiones Mathematicae*, 146:259–295, 2001.

[13] James R. Munkres. *Topology.* Prentice Hall, second edition, 2000.

[14] Karl Petersen. *Ergodic Theory.* Number 2 in Cambridge studies in advanced mathematics. Cambridge University Press, 1982.

[15] Jean-Paul Pier. *Amenable locally compact groups.* Pure and Applied Mathematics. John Wiley and Sons, 1984.

[16] Martin Schechter. *Principles of Functional Analysis.* Number 36 in Graduate Studies in Mathematics. Springer Verlag, second edition, 1995.

[17] Elias M. Stein and Guido Weiss. *Fourier Analysis on Euclidean spaces.* Number 32 in Princeton Mathematical Series. Princeton University Press, 1971.

[18] Daniel W. Stroock. *A concise Introduction to the Theory of Integration.* Birkhauser, third edition, 1999.

[19] Stan Wagon. *The Banach-Tarski Paradox.* Number 24 in Encyclopedia of Mathematics and its Applications. Cambridge University Press, paperback edition, 1993.

[20] Eberhard Zeidler. *Applied Functional Analysis.* Number 108 in Applied Mathematical Sciences. American Mathematical Society, 2002.

Index

$L^p(G)$, 18
L_g, 23
$M^1(G)$, 21
$P^1(G)$, 18
R_g, 23
X^*, 7
$\mathcal{B} = \mathcal{B}(\mathcal{T})$, 13
$\mu * f$, 21
$\phi * f$, 21
$f * \mu$, 21
$f * g$, 21
f^*, 18
$*$, 21

action, 68
action of a group, 68
Aloaglu's Theorem, 10
amenable, 31, 32
atomic measure, 17

Bernoulli random variable, 25
Borel-algebra, 13

coboundary, 87
convolution, 21
counting measure, 14

dual space, 7

exhaustion by compact sets, 5, 12

Følner sequence, 32
Følner's condition, 32
Følner's strong condition, 32

G-invariant mean, 30

Haar measure, 14

inner regular, 13

left Haar measure, 14
left invariant mean, 31
left translation invariant, 14
left uniformly continuous, 22
Leptin's condition, 32
locally compact, 5

maximal inequality, 67
mean, 30
measure preserving action, 68
modular function, 14

neighborhood, 4
net, 11

outer regular, 13

Poisson process, 27
product topology, 9

Reiter's condition, 32
right Haar measure, 14
right translation invariant, 14
right uniformly continuous, 22

$\sigma-$locally compact, 7
second countable, 5
simple function, 24
strong topology on X, 7
strong topology on X^*, 8
strongly bounded, 91

tempered Følner sequence, 66
topological dual space, 7
topological group, 11
topological vector space, 7

topologically left invariant, 32
translation invariant, 14

uniformly continuous, 22
unimodular, 14

weak maximal inequality, 67
weak topology, 7
weak-$*$ topology, 8
weakly bounded, 91