Infinitesimal Deformation Theory of Algebraic Structures

Master Thesis

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Contents

Introduction

The aim of this thesis is to develop a deformation theory which classifies infinitesimal deformations of multicategories. In order to achieve this, methods from homological algebra will be used: a cochain complex will be constructed such that infinitesimal deformations up to equivalence correspond to elements of the second cohomology group. The thesis starts with presenting the classical result of deformations of associative algebras. On the way we also encounter deformations of Lie algebras and of categories.

In chapter 1 we present the deformation theory of associative algebras which goes back to the article "On the deformations of rings and algebras" by Murray Gerstenhaber in 1964 (see [11]) preceded by a warming up "The cohomology of an associative ring" in 1963 (see [10]) of the same author. An associative algebra is a vector space with an associative multiplication. We want to perturb this multiplication. A deformation of an associative algebra is an associative algebra structure on an extension of the underlying vector space such that it reduces to the original algebra when the extended vector space is reduced to the original one. This means the following. Take the underlying vector space of an associative algebra and extend the scalars by some ring R, i.e. if the k-vector space is A then the extension of scalars is given by the tensor product $A \otimes_k R$. This is now both a k-vector space and an R-module. An infinitesimal deformation is a special extension, namely one where R is $k[t]/(t^2)$. The motivation for the name comes from synthetic differential geometry: $A \otimes_k k[t]/(t^2)$ is isomorphic to $A \oplus At$ and it holds that $(At)^2 = 0$, which is called an infinitesimal object. It follows that an infinitesimal deformation of an associative algebra (A, m) is another associative algebra $(A \oplus At, m_t)$ such that for $t = 0$ we have $(A, m_0) \cong (A, m)$.

We want to classify all these deformations up to equivalence. For this purpose Gerstenhaber describes in [11] a method to construct a cochain complex bringing us into the arena of homological algebra. The idea of homological algebra is to assign to an object a cochain complex, i.e. a sequence of modules such that the composition of any two consecutive maps is zero. This complex gives rise to abelian groups called cohomology groups, defined by the quotient of the kernel of a map divided by the image of the preceding map. Throughout all of mathematics cohomology groups give all kind of interesting invariants. In our case, the second cohomology group corresponds to all infinitesimal deformations up to equivalence. Now let us see how a cochain complex is associated to an associative algebra. We want a bilinear map $m_t : (A \oplus At) \otimes_{k \oplus kt} (A \oplus At) \to (A \oplus At)$ to be associative. First note that for m_t there are maps $m_0, m_1 \in Hom_k(A \otimes A, A)$ such that the linear extension of $m_0 + m_1 t$ is exactly m_t. In what follows, we will only consider $m_0 + m_1 t$. Note that since m_t evaluated in $t = 0$ must be m we have $m_0 = m$, hence our candidates for infinitesimal deformations are in $Hom_k(A \otimes A, A)$. One can write down the conditions which m_1 has to satisfy in order for $m + m_1 t$ to be associative. This condition gives rise to a map $d : Hom_k(A \otimes A, A) \to Hom_k(A \otimes A \otimes A, A)$ such that if $d(m_1) = 0$ then the condition is satisfied. It is possible to define a map between any $Hom_k(A^{\otimes n}, A) \to Hom_k(A^{\otimes n+1}, A)$ for all $n \geq 0$ such that $d \circ d = 0$. The complex obtained in this way is the Hochschild complex introduced in the article "On the cohomology groups of an associative algebra" by G. Hochschild in 1945. The cohomology associated to this complex is the Hochschild cohomology. It is then shown that the second cohomology group classifies all infinitesimal deformations.

Another central idea in the formal deformation theory is the extension of deformations. Suppose we are given an associative algebra $(\oplus_{k=0}^n At^k, m_t)$ such that for $t = 0$ it reduces to (A, m). Such a deformation is called an n-deformation. In the same way, as for infinitesimal deformations, m_t corresponds to a family of multiplications $m_i \in Hom_k(A \otimes A, A)$ for $i = 0, \ldots, n$. Suppose this family satisfies a condition such that m_t is associative. Now take another $m_{n+1} \in Hom_k(A \otimes A, A)$. The question is now, when is $m_0 + m_1 t + \ldots + m_n t^n + m_{n+1} t^{n+1}$ an associative multiplication given that $m_0 + \ldots + m_n t^n$ is one. This problem is called an extension problem. There will be a certain term, called obstruction, which, if it vanishes, will allow such an extension. As a last step it will be shown that obstructions are in the third cohomology group.

The same techniques can also be applied to Lie algebras as is pointed out in [11] and will be done in chapter 2. For a given Lie algebra there is an underlying vector space. This vector space will be extended, where a bracket will be defined such that the Lie algebra conditions are satisfied. The result is called a deformation of a Lie algebra if, when reducing the vector space to the original one, gives back the original Lie algebra structure. Since one of the conditions is that the bracket should be anti-symmetric we will consider alternating multilinear forms with values in itself. The only condition which remains is the Jacobi identity. The differential will again be defined such that if the image of the differential is zero then that alternating 2-form satisfies the Jacobi-identity. In this way we obtain a cochain complex which was introduced in "Cohomology theory of Lie groups and Lie algebras" in 1948 by Chevalley-Eilenberg. The second cohomology group classifies all infinitesimal deformations of Lie algebras and the third cohomology group contains the obstructions extending n-deformations to $(n+1)$-deformations for all $n \geq 0$.

A first step in the generalization is the deformation theory of categories which will be described in chapter 3. A (linear) category consists of a set of objects and for each pair of objects a vector space where the vectors are called arrows. The first object of the pair is called the domain of the arrow and the second the codomain. Arrows may be composed and for each object there is an identity arrow. The composition should be associative and the identity arrows should be compatible with the composition. To get a better feeling let us see how this generalizes an associative algebra. A category with only one object is an associative algebra. There is just one pair hence just one vector space, implying that all arrows are composable. The composition is thus an associative multiplication. Now we want to deform categories. This will be done in exactly the same way as for associative algebras. First the underlying collection will be extended and then on the extended arrows we ask for a composition giving rise to a category structure such that if reduced gives back the original category. In the infinitesimal case, the composition can be written in terms of a family of maps such that certain conditions are satisfied. These conditions are exactly the same as for associative algebras with the only difference that the composition is not defined for any pair of arrows (only for the composable ones). Hence it should come as no surprise that we will see the Hochschild complex appearing again. The second cohomology group classifies all infinitesimal deformations of a category composition. A good reference for this theory is the paper "On Deformations of Pasting Diagrams" by D. N. Yetter in 2009 (see [34]).

So far the examples. Let us do some serious work and try to deform multicategories. This is done in chapter 4. A multicategory has, like a category, objects but instead of having just arrows with one input and one output they may have several, possibly zero, inputs. Composing arrows can be done in two ways. Either we take an arrow and choose for each input another arrow and compose them to form a new arrow or we just choose two arrows and specify where the arrow should be inserted into the other one. Even though the first composition might seem more natural at first sight it turns out that the second gives rise to much nicer structures. Therefore we will work with the second one. In order to do this in a systematic way one observes that multicategories conform to the dynamics of trees. To see this, represent arrows by trees. The composition is then the grafting of trees such that the order of successive graftings does not matter. Apart from the compositions already the collection can be elegantly described by trees. A collection is a functor assigning to each corolla a vector space and to each isomorphism of trees an isomorphism between vector spaces. Thus the functor transfers the dynamics of the category of trees to the category of vector spaces.

A deformation of multicategories will then be, again, an extension of the underlying collection with a multicategory structure on it, which reduces to the original multicategory for the original collection. From here everything can be developed as before; express the composition as a family of compositions and determine the conditions on this family. Then define a differential such that if the differential is zero the conditions are satisfied. It will be shown that in this way a complex is obtained for which the second cohomology group classifies all infinitesimal deformations of multicategories. This complex is a generalization of the Hochschild complex, meaning that if only trees with one input are considered one obtains the Hochschild complex for linear categories. The main result of this thesis is the demonstration that the complex exhibits a differential graded Lie algebra (dg-Lie algebra for short) structure. For this dg-Lie algebra the differential is equal to the graded Lie bracket curried by the composition of the original multicategory. From here it can easily be shown that the differential is indeed a differential, i.e. the composition of two successive differentials is zero. It further holds that this structure reduces to the dg-Lie algebra structures for categories and associative algebras. The same way as before, the obstructions are in the third cohomology group.

Finally it is also possible to deform the morphisms of each structure, which is done at the end of each chapter. The desire to deform the morphisms comes from the fact that representations of multicategories and especially of operads, multicategories with one object, are known algebras. The essence of a representation of an object of some category is to choose another object in that category which is well-known and study the induced dynamics of the original object in the well-known object. In case of multicategories, the well-known object is the endomorphism multicategory **End**. A representation of a multicategory $\mathcal{M}$ is then just a multifunctor from $\mathcal{M}$ to **End**. By the hom-tensor adjunction a representation is the same as an algebra. There exists for example an operad whose representations are associative algebras, or an operad whose representations are Lie algebras. Some examples of representations of multicategories are categories, operads and even multicategories themselves. Knowing how to deform a multifunctor, a representation can be deformed. It is then shown that infinitesimal deformations of representations give infinitesimal deformations of the algebras.

In the epilogue some topics will be described which might be worth considering after reading this thesis. Some of the topics are the following. The extensions, as they have been described here, turn out to be 1-extensions. As the name suggest there exist higher extensions too, so one might wonder whether they have deformation theoretic interpretations. A different topic is about deformation functors, which were developped by Grothendieck, Artin and Schlessinger. A deformation functor is a functor from the category of local Artinian rings over some field k of characteristic zero. The local Artinian rings are used since Grothendieck noticed that they have similar behavior as jets in differential geometry. A deformation functor is supposed to send the ground field to the original structure and for any other ring the deformations of the structure over that ring. In the thesis such a deformation functor is implicitly constructed in the section of Maurer-Cartan elements of each chapter.

In appendix A the end and the coend are described. Both are notions from category theory. It is a limit and colimit respectively in which several diagrams are simultaneously satisfied. These constructions are described since they are used in the free multicategory and, more importantly, in the definition of the cochain complex for multicategories. In appendix B the deformation theory for multicategories with a full composition is given. Remember that it was said, that the composition might be given in two ways, where in the exposition of chapter 4 the second has been chosen. Here the theory is developed for the first choice. This leads naturally to the consideration of layered trees, for which most of the constructions of chapter 4 go through, mutatis mutandis. Yet, there are two reasons why this appears only in the appendix. The first is that no dg-Lie algebra structure could be found on the cochain complex described with the help of layered trees. This is not too much of a problem at the beginning: it can be shown that the second cohomology group classifies all infinitesimal deformations of the full composition. The lack of a dg-Lie algebra structure is especially annoying at the moment one has to show that the obstructions are cocycles. With two definitions of compositions and two complexes around it is naturally to ask how they relate. Here appears the other problem. Two maps can be constructed in an attempt to relate them, but they turn out not to commute with the differential. Hence the more modest place in this thesis.

As a final word, I would like to take this opportunity to thank all the people involved in the creation of this thesis for their stimulating discussions, brilliant suggestions and for their incredible support.

Chapter 1

Deformation Theory of Associative Algebras

Deformation theory is concerned with variations of structures such that the deformed structure is still of the same type as the original one. In this chapter deformations of associative algebras will be considered. The main idea is to augment the underlying field of an associative algebra, thus obtaining a unital ring R. An R-algebra B is a deformation of an algebra A if it is associative and the reduction is isomorphic to A. In order to get richer structures, certain restrictions will be imposed on the deformations. These make it possible to write the deformed multiplication m on B as a sum of multiplications m_k defined over A. The associativity of m then translates into the condition that all the m_k-s have to satisfy some equations. In particular, an infinitesimally deformed multiplication has to satisfy the following equation:

$$am_1(b \otimes c) - m_1(ab \otimes c) + m_1(a \otimes bc) - m_1(a \otimes b)c = 0,$$

for all $a, b, c \in A$. This is where homological algebra enters the picture. The essential step is to interpret $m_1 \in Hom_{Vec}(A \otimes A, A)$ as a so-called Hochschild 2-cocycle. In fact, it will be shown that the second Hochschild cohomology group coincides with infinitesimal deformations, up to equivalence. Also, the extensions of n-deformations to $(n+1)$-deformations are controlled by the third cohomology group. With the Gerstenhaber bracket the deformation complex carries a differential graded Lie algebra structure which turns the Hochschild cohomology into a graded Lie algebra. The Hochschild complex and the Hochschild cohomology form even a homotopy Gerstenhaber algebra and a Gerstenhaber algebra respectively. Solutions to the Maurer-Cartan equation will be related to deformations; they give rise to infinitesimal deformations. Furthermore, obstructions to extensions of deformations are described using the Maurer-Cartan equation. This chapter finishes with the deformation theory of algebra morphisms. This will enable the study of representations, relevant for later chapters.

1.1 Deformation

1.1.1 Associative algebras

Let R be a commutative ring. An associative algebra (A, m) consists of an R-module A and a binary operation $m : A \otimes_R A \to A$, such that m is bilinear and satisfies

$$m(m(x, y), z) = m(x, m(y, z)).$$

for all $x, y, z \in A$.

In some literature an algebra is already considered to be associative. In this thesis this is not the case. Note that if R is a field then A becomes a vector space. The associative algebra is not supposed to have a unit unless explicitly stated.

A morphism $f : (A, m_A) \to (B, m_B)$ between associative algebras is an R-module morphism compatible with the multiplications, i.e. the following diagram commutes:

$$\begin{array}{ccc} A \otimes_R A & \xrightarrow{m_A} & A \\ \downarrow{\scriptstyle f\otimes f} & & \downarrow{\scriptstyle f} \\ B \otimes_R B & \xrightarrow[m_B]{} & B. \end{array}$$

The associative algebras together with the morphisms define a category **Ass** where the composition is just the composition of functions.

Example 1.1.1.

$Mat_n(k)$*: The* $n \times n$*-matrices with values in a field* k *together with the matrix product.*

$\mathbb{R}[X]$*: Polynomials with coefficients in* $\mathbb{R}$ *together with the usual product of polynomials.*

$\mathbb{C}$*: The complex numbers together with their multiplication.*

$U(\mathfrak{g})$*: The universal enveloping algebra of a finite dimensional Lie algebra* $\mathfrak{g}$*.*

Throughout this chapter A will be an associative algebra over a field k of characteristic zero, unless otherwise stated.

1.1.2 Augmentation

An augmentation of k by an ideal I of R is a $k-$module R together with a $k-$module morphism $\varepsilon : R \to k$ such that the exact sequence

$$0 \longrightarrow I \longrightarrow R \xrightarrow{\varepsilon} k \longrightarrow 0$$

splits with splitting $\omega : k \to R$. That the exact sequence splits just means that $\varepsilon \circ \omega = id_k$. Note that in this case $R = I \oplus k$.

Example 1.1.2.

For all $n \in \mathbb{N}$ *the module* $k[t]/(t^n)$ *is an augmentation of* k *by* (t) *with augmentation map* $\varepsilon(\sum_i r_i t^i) = r_0$*, i.e. evaluating the polynomials in zero. The splitting is just the inclusion of* k *into* $k[t]/(t^n)$*.*

$k[[t]]$ *is an augmentation of* k *by* (t) *with augmentation map evaluation in* $t = 0$ *and splitting the inclusion.*

Note that a field, K, can not be a non-trivial augmentation, because the only ideal of K extending k is the trivial one, thus by the exactness $K/(0) = K$ is equal to k. Therefore the only augmentation of a field is the field itself. Given an augmentation (R, ε) it then follows that R is a k-module by the use of ε and k an R-module by using ω.

Let (A, m) be an associative $k-$algebra and let (R, ε) be an augmentation of k. Then A can be augmented to an $R-$module by $A \otimes_k R$. The multiplication can be extended bilinearly giving rise to an augmented algebra $(A \otimes_k R, m)$.

1.1.3 Deformations

With the help of the previous notions deformations of associative algebras can be defined.

Definition 1.1.1. *Given an algebra* A *and an augmented unital ring* R*, an* R*-deformation* (B, α) *of* A *is an associative* R*-algebra* B *together with a* k*-algebra isomorphism* $\alpha : B \otimes_R k \to A$*.*

In case the multiplication m of B needs to be stressed, a deformation will be denoted by (B, m, α). Certain R-deformations carry a name, the following are the most important ones:

A formal deformation is a $k[[t]]$-deformation.

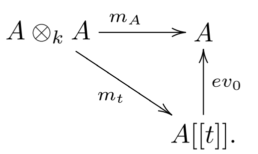
An n-deformation is a $k[t]/(t^{n+1})$-deformation.

An infinitesimal deformation is a $k[t]/(t^2)$-deformation.

Note that an infinitesimal deformation is a special case of a n-deformation. In the general form not much can be said about deformations, so certain restrictions will be imposed on B, R and α. In this chapter B will be chosen to be of the form $A \otimes_k R$ and only formal or n-deformations will be considered. The isomorphism α will be chosen to be $can : (A \otimes_k R) \otimes_R k \to A$ given by $(a \otimes 1) \otimes 1 \mapsto a$. This choice is not very restrictive since it will be shown in 1.1.5 that $(A \otimes_k R, can)$ is equivalent to an arbitrary $(A \otimes_k R, \alpha)$.

There is a more convenient way of writing $A \otimes_k k[[t]]$ using the isomorphism $A \otimes_k k[[t]] \cong A[[t]]$. Thus the elements of $A[[t]]$ are formal power series $\sum_k a_k t^k$, where $a_k \in A$. Using a similar construction elements of n-deformations can be written as $\sum_{k=0}^{n} a_k t^k$, with $a_k \in A$.

For formal there is an alternative definition of a deformation. Instead asking for an $k[[t]]$-algebra morphism m_t, one ask for a map $m_t : A \otimes_k A \to A[[t]]$ such that the diagram



commutes. By linearity this map extends to $m_t : A[[t]] \otimes_{k[[t]]} A[[t]] \to A[[t]]$ giving rise to a formal deformation in the previous sense. The same holds for n-deformations. In what follows this definition will be used.

1.1.4 Family of operations

The next lemma provides another description of deformations, namely as a family of multiplications. Remember that only formal or n-deformations are considered.

Lemma 1.1.1. *A formal deformation is equivalent to a family of multiplications*

$$\{m_i : A \otimes_k A \to A \mid i \in \mathbb{N}_0\},$$

such that

$$(D_k) : \quad \sum_{i+j=k, i,j \geq 0} m_i(m_j(a \otimes b) \otimes c) - m_i(a \otimes m_j(b \otimes c)) = 0$$

holds for all $a, b, c \in A$ and for all $k \in \mathbb{N}_0$. Furthermore $m_0 = m_A$.

Proof. Consider a formal deformation $(A \otimes_k k[[t]], m, can)$. Since $A \otimes_k k[[t]] \cong \bigoplus_{i \geq 0} At^i$ and the Hom-functor preserves limits it follows that

$$Hom_k(A \otimes_k A, \bigoplus_{i \geq 0} At^i) \cong \bigoplus_{i \geq 0} Hom_k(A \otimes_k A, A).$$

More explicitly, consider two elements $a, b \in A$. Since $m(a \otimes b) \in A[[t]]$, there exists a sequence $(c_k)_{k \in \mathbb{N}_0}$ of elements of A, such that $m(a \otimes b) = \sum_k c_k t^k$. Let $m_k(a \otimes b) := c_k$. In this way a multiplication $m_k : A \otimes A \to A$ is obtained for each $k \in \mathbb{N}_0$. Therefore m gives rise to a family of multiplications

$$\{m_i : A \otimes_k A \to A \mid i \in \mathbb{N}_0\}.$$

Thus m has the form $m = m_0 + m_1 t^1 + m_2 t^2 + \ldots$. Since $(A \otimes_k k[[t]], m, can)$ is a deformation, m is associative, therefore

$$
\begin{aligned}
0 =& m(m(a \otimes b) \otimes c) - m(a \otimes m(b \otimes c)) \\
=& \sum_{i,j} \left(m_i(m_j(a \otimes b) \otimes c) - m_i(a \otimes m_j(b \otimes c))\right) t^{i+j}.
\end{aligned}
$$

This sum is zero exactly if the coefficient of t^k is zero for all k. These coefficients are

$$
\sum_{i+j=k} \left(m_i(m_j(a \otimes b) \otimes c) - m_i(a \otimes m_j(b \otimes c))\right),
$$

which are exactly (D_k). That m_0 is equal to the multiplication on A will follow from the fact that $ev_0 \circ m = m_A \circ (ev_0 \otimes ev_0)$. Let $a, b \in A$. Then $ev_0 \circ m(a \otimes b) = ev_0(\sum_{i \geq 0} m_i(a \otimes b) t^i) = m_0(a \otimes b)$. On the other hand $m_A(ev_0(a) \otimes ev_0(b)) = m_A(a \otimes b)$. Hence the desired equality.

To prove the other implication consider a family of multiplications $\{m_i\}_{i \in \mathbb{N}_0}$ satisfying (D_k). Define $m := \sum_k m_k t^k$. This is a multiplication on $A[[t]]$. It has to be proved that m is associative, but this is straightforward by the previous calculation. Start with (D_k) and derive $m(m \otimes id) - m(id \otimes m) = 0$. Therefore $(A \otimes_k k[[t]], m, can)$ gives rise to a $k[[t]]$-deformation. □

Similar results may be obtained for n-deformations. Using the same reasoning it can be shown that an n-deformed multiplication m is given by a family of multiplications $\{m_i\}_{0 \leq i \leq n}$ satisfying (D_k) up to $k = n$. Thus m on $A \otimes A$ can be written as $\sum_{k=0}^{n} m_k t^k$. Also in this case m_0 is the original multiplication on A.

For an infinitesimal deformation, being a 2-deformation, a multiplication m working on $A \otimes A$ is thus given by $m = m_0 + m_1 t$. In order for m to be associative (D_0) and (D_1) have to be satisfied.

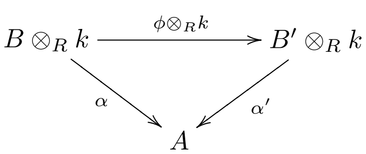
If the family of a deformation should be stressed the deformation will be written as $(A[[t]], \{m_i\}_{i=1}^{\infty})$ or $(A[[t]]/(t^n), \{m_i\}_{i=1}^{n})$ in case of a formal or $n-$deformation respectively.

1.1.5 Equivalence of deformations

In this section it will be explained what it means for two R-deformations to be equivalent. With this notion a trivial deformation can be defined. It will be shown that an algebra morphism is equivalent to a family of automorphisms in a similar way as deformations were characterized by a family of multiplications.

Finally it will be shown that the deformation of unital algebras is equivalent to the deformation of an algebra where the unit is not deformed. This proves that no special attention has to be given to unital algebras with respect to their deformation theory.

Definition 1.1.2. *Two R-deformations (B, α) and (B', α') are said to be equivalent, $(B, \alpha) \sim (B', \alpha')$, if there exists an R-algebra isomorphism $\phi : B \to B'$ such that*



commutes.

It was said that any R-deformation $(A \otimes_k R, \alpha)$ is equivalent to $(A \otimes_k R, can)$. This means that for each α there exists an R-algebra isomorphism. This isomorphism is given by $\phi_\alpha(a \otimes r \otimes v) := \alpha(a \otimes r \otimes v) \otimes 1 \otimes 1$. It is obviously an isomorphism since α is one and $\alpha = can \circ \phi_\alpha$.

Two formal deformations m_t and m'_t are equivalent if there exists a formal $k[[t]]$-algebra isomorphism ϕ_t satisfying the previous commutative diagram. As for formal deformations, there is an alterate way of defining an equivalence. Two formal deformations are equivalent if there is an isomorphism $\phi_t : A \to A[[t]]$ such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{id_A} & A \\ & \underset{\phi_t}{\searrow} & \uparrow{\scriptstyle ev_0} \\ & & A[[t]] \end{array}$$

commutes. In this case, ϕ_t corresponds to a family of morphisms $(\phi_k)_{k\in\mathbb{N}_0}$ such that

$$\phi_t = \sum_k \phi_k t^k.$$

It follows that $\phi_0 = id_A$.If ϕ_t is an isomorphism then so are the ϕ_k-s. Similar results are obtained by considering equivalences of n-deformations.

An R-deformation (B, α) is trivial if $(B, \alpha) \sim (A, id)$. In case of a formal deformation a trivial deformation is of the form $m_0(a, b) + \sum_k 0t^k$ for all $a, b \in A$.

So far no special attention has been given to the question what happens to the unit in a unital associative algebra. The following theorem justifies this lack of attention.

Theorem 1.1.1. *Let $(A[[t]], m', u')$ be a deformation of the associative unital algebra (A, m_A, u_A) where $u' = \sum_i u'_i t^i$ is the deformed unit. Then there exists a deformation $(A[[t]], m'', u_A)$ equivalent to $(A[[t]], m', u')$.*

Proof. Define $\alpha : A[[t]] \to A[[t]]$ by $x \mapsto m'(x, u_A)$. Note that the unit is preserved since $\alpha(u') = m'(u', u_A) = u_A$. The inverse to α is given by $\alpha^{-1}(y) := m'(y, u_A^{-1})$ where the components of u_A^{-1} are recursively defined by the property $\sum_{i=0}^k m'_i((u_A^{-1})_{k-i}, u_A) = u'_k$. Define the multiplication m'' to be

$$m''(x, y) := m'(m'(x, id_A^{-1}), y),$$

for all $x, y \in A$. The associativity follows from the associativity of m'. That α is compatible with m' and m'' follows from

$$\begin{aligned} \alpha(m'(x, y)) =& m'(m'(x, y), u_A) \\ =& m'(x, m'(y, u_A)) \\ =& m'(m'(x, u'), \alpha(y)) \\ =& m'(m'(x, m'(u_A, u_A^{-1})), \alpha(y)) \\ =& m'(m'(m'(x, u_A), u_A^{-1}), \alpha(y)) \\ =& m'(m'(\alpha(x), u_A^{-1}), \alpha(y)) \\ =& m''(\alpha(x), \alpha(y)). \end{aligned}$$

It remains to check that α gives rise to the commutative square in the definition of an equivalence of deformations. Let $\sum_{i\geq 0} x_i t^i \in A[[t]]$.

$$\begin{aligned} ev_0 \circ \alpha(\sum_{i\geq 0} x_i t^i) =& ev_0 \left(\sum_{i,j\geq 0} m'_j(x_i, u_A) t^{i+j} \right) \\ =& m'_0(x_0, u_0) \\ =& x_0. \end{aligned}$$

This proves the commutativity, showing that $(A[[t]], m', u')$ is equivalent to $(A[[t]], m'', u)$. $\square$

1.1.6 Extension and obstructions

Extensions and obstructions will now be defined. To shorten the notation the multiplication $m(a \otimes b)$ will be denoted by $m(a, b)$ or even by ab.

Definition 1.1.3. *An extension of a n-deformation* $\{m_1, \dots, m_n\}$ *is* $\{m_1, \dots, m_n\} \cup \{m_{n+1}\}$ *such that* (D_{n+1}) *is satisfied.*

The equation (D_{n+1}) holds if

$$\sum_{i+j=n+1, i,j \geq 0} m_i(m_j(a, b), c) - m_i(a, m_j(b, c)) = 0.$$

This can be rewritten as

$$\begin{aligned} 0 &= \sum_{i+j=n+1, i,j \geq 0} (m_i(m_j(a, b), c) - m_i(a, m_j(b, c))) \\ &= (m_{n+1}(a, b)c - am_{n+1}(b, c) + m_{n+1}(ab, c) - m_{n+1}(a, bc)) + \sum_{\substack{i+j=n+1 \\ i,j>0}} (m_i(m_j(a, b), c) - m_i(a, m_j(b, c))) \,. \end{aligned}$$

The first part, $m_{n+1}(a, b)c - am_{n+1}(b, c) + m_{n+1}(ab, c) - m_{n+1}(a, bc)$, will later be recognized as a cocycle.

Definition 1.1.4. *The term*

$$\mathfrak{O}_n := \sum_{\substack{i+j=n+1 \\ i,j>0}} m_i(a, m_j(b, c)) - m_i(m_j(a, b), c)$$

in (D_{n+1}) *is called obstruction.*

Consider the following example. Let a 2-deformation be given, it will be extended by introducing a multiplication m_3. An explicit calculation of $m(m(a, b), c) - m(a, m(b, c))$ for $m = m_0 + m_1 t + m_2 t^2 + m_3 t^3$ arranged by the t^k-levels gives:

$$t^0: \quad (ab)c - a(bc) = 0 \qquad (D_0)$$

$$t^1: \quad -am_1(b, c) + m_1(ab, c) - m_1(a, bc) + m_1(a, b)c = 0 \qquad (D_1)$$

$$\begin{aligned} t^2: \quad & -am_2(b, c) + m_2(ab, c) - m_2(a, bc) + m_2(a, b)c + \\ & + (m_1(m_1(a, b), c) - m_1(a, m_1(b, c))) = 0 \qquad (D_2) \end{aligned}$$

$$\begin{aligned} t^3: \quad & -am_3(b, c) + m_3(ab, c) - m_3(a, bc) + m_3(a, b)c + \\ & m_2(m_1(a, b), c) + m_1(m_2(a, b), c) - m_2(a, m_1(b, c)) - m_1(a, m_2(b, c)) \qquad (D_3) \end{aligned}$$

It has been used that (D_k) holds up to $k = 2$ (because of the associativity of the 2-deformation). What is left is again a cocycle minus $\mathfrak{O}_2$. Note that the term $m_1(m_1(a, b), c) - m_1(a, m_1(b, c)$ in the coefficient of t^2 would exactly be the obstruction $\mathfrak{O}_1$ of extending a 1-deformation to a 2-deformation.

1.2 Hochschild Complex

Given an infinitesimal deformation, the multiplication $m : A \to A[t]/(t^2)$ is of the form $m = m_0 + m_1 t$. Writing out explicitly the condition for m to be associative gives, collected by the corresponding t^k coefficients:

$$t^0 : \qquad (ab)c - a(bc) = 0$$

$$t^1 : \qquad -am_1(b,c) + m_1(ab,c) - m_1(a,bc) + m_1(a,b)c = 0.$$

The aim is now to interpret m_0 and m_1 as 2-cochains such that $d(m_1) = -am_1(b,c) + m_1(ab,c) - m_1(a,bc) + m_1(a,b)c$. In order for $m_0 + m_1 t$ to be associative m_1 has to be a cocycle. The cochain complex providing this setting for associative algebras is the Hochschild complex. The cohomology over this complex is the Hochschild cohomology which has deformation theoretical interpretations.

1.2.1 Hochschild Complex

The Hochschild complex is constructed as follows. Let $n \in \mathbb{Z}$ and let A be an associative algebra. Define

$$C^n_{\text{Hoch}}(A,A) := \begin{cases} Hom_{\mathbf{Vec}_k}(A^{\otimes n}, A) & n \geq 0 \\ 0 & n < 0, \end{cases}$$

where A^0 is defined to be k and let $d^n_{\text{Hoch}} : C^n_{\text{Hoch}}(A,A) \to C^{n+1}_{\text{Hoch}}(A,A)$ be given by

$$(d^n_{\text{Hoch}} f)(a_1, \ldots, a_n, a_{n+1}) := \begin{cases} 0 & , n < 0 \\ (af - fa) & , n = 0 \\ a_1 f(a_2, \ldots, a_{n+1}) + & , n > 0 \\ \sum_{i=1}^{n} (-1)^i f(a_1, \ldots, a_{i-1}, (a_i a_{i+1}), a_{i+2}, \ldots, a_{n+1}) + & \\ (-1)^{n+1} f(a_1, \ldots, a_n) a_{n+1}. & \end{cases}$$

In case $n = 0$, the isomorphism $Hom_{\mathbf{Vec}_k}(k, A) \cong A$ allows the interpretation of f as an element of A. As it is done often in homological algebra, the map d^n_{Hoch} will simply be written d_{Hoch} or even just d, for all $n \in \mathbb{Z}$.

The proof that d is a differential, i.e. $d^2 = 0$, will be postponed until lemma 1.2.1. For now the result will be assumed.

Definition 1.2.1. *The positive cochain complex $(C^\bullet_{Hoch}(A,A), d)$, i.e.*

$$\cdots \xrightarrow{d} 0 \xrightarrow{d} C^0_{Hoch}(A,A) \xrightarrow{d} C^1_{Hoch}(A,A) \xrightarrow{d} C^2_{Hoch}(A,A) \xrightarrow{d} \cdots$$

is called the Hochschild complex.

To endow the Hochschild complex with additional structures a degree shift will be necessary. The k^{th} degree shift of a complex $C^\bullet$, denoted by $C^\bullet[k]$, is defined to be $(C^\bullet[k])^n := C^{n+k}$. The dimension of an element $f \in C^n_{\text{Hoch}}[k](A,A)$ is defined to be $dim(f) := n + k$ and the degree $deg(f) := n$.

The Hochschild complex gives rise to a differential graded module over the field k by defining

$$C_{\text{Hoch}}(A,A) := \bigoplus_{n \in \mathbb{Z}} C^n_{\text{Hoch}}(A,A).$$

1.2.2 Graded pre-Lie algebra

The Hochschild complex can be given the structure of a pre-Lie system by providing circle-i operations. These circle-i operations allow the definition of a circle operation giving rise to a graded pre-Lie algebra.

Definition 1.2.2. *A pre-Lie system consists of a collection of R-modules $\{A^n\}_{n\geq 0}$ together with binary operations $\circ_i : A^m \otimes_k A^n \to A^{m+n}$, such that for $a \in A^m, b \in A^n$ and $c \in A^p$*

$$(a \circ_i b) \circ_j c = \begin{cases} a \circ_i (b \circ_{j-i+1} c) & i \leq j \leq i+n-1 \\ (a \circ_{j-i} c) \circ_i b & i+n \leq j \leq m+n-1 \\ (a \circ_j c) \circ_{i+p-1} b & 1 \leq j \leq i-1 \end{cases}$$

is satisfied for all i and j in $\mathbb{Z}$.

Take $C_{\text{Hoch}}(A,A)[1]^n$ as a sequence of k-modules and define the circle-i operations for all $f \in C_{\text{Hoch}}(A,A)[1]^m$ and $g \in C_{\text{Hoch}}(A,A)[1]^n$ as:

$$f \circ_i g(a_0, \ldots, a_{m+n}) := f(a_1 \otimes \ldots a_{i-1} \otimes g(a_i \otimes \ldots \otimes a_{i+n}) \otimes a_{i+n+1} \otimes \ldots \otimes a_{m+n}).$$

Note that the complex has been shifted by one in order for the circle-i operations to respect the grading on the Hochschild complex. In case of the unsuspended sequence the degree is $deg(f \circ_i g) = m+n-1$ and not $deg(f) + deg(g) = m+n$. On the other hand, in the shifted case, $deg(f \circ_i g) = m+n = deg(f) + deg(g)$. It is a straightforward verification that the axioms of a pre-Lie system are satisfied by $C_{\text{Hoch}}(A,A)[1]$ with the above defined $\circ_i$-operations.

Now it will be shown that the complex can be endowed with a right pre-Lie algebra structure. A right pre-Lie algebra is a graded module $A = \bigoplus_{n\in\mathbb{Z}} A^n$ together with a binary operation $\circ : A \otimes_k A \to A$ respecting the grading, such that

$$(a \circ b) \circ c - a \circ (b \circ c) = (-1)^{deg(b)deg(c)} ((a \circ c) \circ b - a \circ (c \circ b))$$

for all homogeneous a, b and c in A.

Theorem 1.2.1. *Let $(V_n, \circ_i)_{n,i}$ be a pre-Lie system and $f \in V[1]_m$ and $g \in V[1]_n$. Define*

$$f \circ g := \sum_{i=1}^{m+1} (-1)^{(i-1)n} f \circ_i g.$$

Then the shifted vector space $V[1] = \bigoplus_{n\geq 0} V[1]_n$ together with $\circ$ forms a right algebra.

For a prove see [10]. A corollary is that $(C_{\text{Hoch}}(A,A)[1], \circ)$ is a graded pre-Lie algebra.

Remark 1.2.1. *The sign $(-1)^{n(i-1)}$ can nicely be interpreted as coming from the permutations. Let $x \otimes y = (-1)^{|x||y|} y \otimes x$ then the sign of $f \circ_i g$ is obtained by the permutation*

$$f \otimes g \otimes x_1 \otimes \ldots x_{m+n-1} \mapsto f \otimes x_1 \ldots x_{i-1} \otimes g \otimes x_i \otimes \ldots \otimes x_{m+n-1}$$

with $|x_i| = 1$.

1.2.3 Graded Lie algebra

Now the Hochschild complex will be endowed with a graded Lie algebra structure.

Definition 1.2.3. *A graded Lie algebra $(A, [-,-])$ is a graded vector space A together with a bracket $[-,-]$ respecting the grading, such that the following two axioms hold:*

Graded antisymmetry: $[b,a] = -(-1)^{deg(a)deg(b)}[a,b]$

Graded Jacobi identity: $(-1)^{deg(c)deg(a)}[a,[b,c]] + (-1)^{deg(a)deg(b)}[b,[c,a]] + (-1)^{deg(b)deg(c)}[c,[a,b]] = 0$.

Define the Gerstenhaber bracket on $C_{\text{Hoch}}(A,A)[1]$ as follows. For $f \in C_{\text{Hoch}}[1]^m(A,A)$ and $g \in C_{\text{Hoch}}[1]^n(A,A)$ let

$$[f,g] := f \circ g - (-1)^{mn} g \circ f.$$

Lemma 1.2.1. *The Gerstenhaber bracket $[-,-]$ is a graded Lie bracket on $C_{Hoch}(A,A)[1]$.*

Proof. The grading is respected by the bracket since it is respected by $\circ$.

Let $f \in C^m_{\text{Hoch}}[1](A,A), g \in C^n_{\text{Hoch}}[1](A,A)$ and $h \in C^p_{\text{Hoch}}[1](A,A)$. Then

$$\begin{aligned}[f,g] =& f \circ g - (-1)^{mn} g \circ f \\ =& -(-1)^{mn}(g \circ f - (-1)^{mn} f \circ g) \\ =& -(-1)^{mn}[g,f],\end{aligned}$$

i.e. the graded antisymmetry is satisfied. Now the graded Jacobi identity has to be checked.

$$\begin{aligned}&[f,[g,h]] + (-1)^{m(n+p)}[g,[h,f]] + (-1)^{p(m+n)}[h,[f,g]] = \\ =& \Big(f \circ (g \circ h - (-1)^{np} h \circ g) - (-1)^{m(n+p)}(g \circ h - (-1)^{np} h \circ g) \circ f\Big) \\ &+ (-1)^{m(n+p)} \Big(g \circ (h \circ f - (-1)^{pm} f \circ h) - (-1)^{n(p+m)}(h \circ f - (-1)^{pm} f \circ h) \circ g\Big) \\ &+ (-1)^{p(m+n)} \Big(h \circ (f \circ g - (-1)^{mn} g \circ f) - (-1)^{p(m+n)}(f \circ g - (-1)^{mn} g \circ f) \circ h\Big) \\ =& \Big(f \circ g \circ h - (-1)^{np} f \circ h \circ g - (-1)^{m(n+p)} g \circ h \circ f + (-1)^{(np)+m(n+p)} h \circ g \circ f\Big) \\ &+ \Big((-1)^{m(n+p)} g \circ h \circ f - (-1)^{2mp+mn} g \circ f \circ h - (-1)^{2mn+(m+n)p} h \circ f \circ g + (-1)^{2(mn+mp)+np} f \circ h \circ g\Big) \\ &+ \Big((-1)^{p(m+n)} h \circ f \circ g - (-1)^{(mn)+p(m+n)} h \circ g \circ f - (-1)^{2p(m+n)} f \circ g \circ h + (-1)^{(mn)+2p(m+n)} g \circ f \circ h\Big) \\ =& 0.\end{aligned}$$

Thus the Gerstenhaber bracket is a graded Lie bracket on $(C_{\text{Hoch}}(A,A)[1],[-,-])$. ☐

1.2.4 Differential graded Lie algebra

The Hochschild complex carries even more structure, namely that of a differential graded Lie algebra.

Definition 1.2.4. *Given a graded Lie algebra $(\mathfrak{g},[-,-])$ and a map $d : \mathfrak{g} \to \mathfrak{g}$ such that $d^2 = 0$ and d is a derivation of degree 1, then $(\mathfrak{g},[-,-],d)$ is said to be a differential graded (dg) Lie algebra.*

In order to show that $(C_{\text{Hoch}}(A,A)[1],[-,-])$ together with d_{Hoch} forms a dg-Lie algebra, the following lemma will be useful.

Lemma 1.2.2. *Let $m_0 \in C^1_{Hoch}[1](A,A)$ and $f \in C^n_{Hoch}[1](A,A)$. Then $d_{Hoch}(f) = [m_0,f]$.*

Proof. This follows from an easy computation.

$$\begin{aligned}[m_0,f] =& m_0 \circ f - (-1)^{1n} f \circ m_0 \\ =& \sum_{i=1}^{2}(-1)^{(i+1)n} m_0 \circ_i f - (-1)^n \sum_{i=1}^{n+1}(-1)^{(i+1)1} f \circ_i m_0 \\ =& (-1)^n \left((-1)^n m_0 \circ_1 f + m_0 \circ_2 f + \sum_{i=1}^{n+1}(-1)^i f \circ_i m_0\right) \\ =& (-1)^n \left(m_0 \circ_2 f + \sum_{i=1}^{n+1}(-1)^i f \circ_i m_0 - (-1)^{n+1} m_0 \circ_1 f\right) = d(f).\end{aligned}$$

☐

Now it is easy to prove that $d^2 = 0$ (which is quite a job to check directly from the definition).

Proposition 1.2.1. *The Hochschild map d_{Hoch} is a differential and a degree 1 derivation.*

Proof. For $f \in C^m_{\text{Hoch}}[1](A, A)$

$$\begin{aligned} d^2 f =& (-1)^{2m}[m_0, [m_0, f]] && (df = [m_0, f]) \\ =& -(-1)^{1+m}[m_0, [f, m_0]] - (-1)^{m(1+1)}[f, [m_0, m_0]] && \text{(graded Jacobi identity)} \\ =& -(-1)^m(-1)[m_0, [f, m_0]] && ([m_0, m_0] = 0) \\ =& -[m_0, [m_0, f]], && \text{(graded antisymmetry)} \end{aligned}$$

thus $d^2 f = 0$, and therefore proves that d is a differential.

For $f \in C^m_{\text{Hoch}}[1](A, A)$ and $g \in C^n_{\text{Hoch}}[1](A, A)$ the following holds:

$$\begin{aligned} [df, g] +& (-1)^m[f, dg] = \\ =& [[m_0, f], g] + (-1)^m[f, [m_0, g]] && (d(f) = [m_0, f]) \\ =& [[m_0, f], g] + (-1)^m\left(-(-1)^{m(1+n)}[m_0, [g, f]] - (-1)^{n(m+1)}[g, [f, m_0]]\right) && \text{(graded Jacobi identity)} \\ =& [[m_0, f], g] - (-1)^{mn}[m_0, [g, f]] - [[m_0, f], g] && \text{(graded antisymmetry)} \\ =& [m_0, [f, g]] && \text{(graded antisymmetry)} \\ =& d([f, g]). && (d(f) = [m_0, f]) \end{aligned}$$

This shows that d is a derivation of degree 1. □

Therefore $(C_{\text{Hoch}}(A, A)[1], [-, -], d_{\text{Hoch}})$ is a dg-Lie algebra.

1.2.5 Homotopy Gerstenhaber algebra

The Hochschild cochain complex $C_{\text{Hoch}}(A, A)$ has the structure of a homotopy Gerstenhaber algebra (c.f. [18]).

Definition 1.2.5. *A homotopy Gerstenhaber-algebra, or homotopy G-algebra for short, is a quadruple $(V, \{\}, \cdot, d)$, where*

V is a graded vector space: $V = \bigoplus_n V_n$

$\{\}$ is a collection of braces degree $-k$: $-\{-, \ldots, -\} : V_m \times V_{n_1} \times \ldots \times V_{n_k} \to V_{(m-k)+\sum_i n_i}$

$\cdot$ is a dot product of degree 0: $\cdot : V_m \times V_n \to V_{m+n}$

d is a differential degree 1: $d : V_m \to V_{m+1}, \; d^2 = 0.$

such that

1. *$(V, \cdot, d)$ is a associative dg-algebra*
2. *composition of braces:*

$$\begin{aligned} &x\{y_1, \ldots, y_m\}\{z_1, \ldots, z_n\} = \\ &= \sum_{0 \le i_1 \le \ldots \le i_m \le n} (-1)^{\left(\sum_{p=1}^{m}(deg(y_p)-1)\right)\left(\sum_{q=1}^{i_p}(deg(z_q)-1)\right)} x\{z_1, \ldots, y_{i_1}, y_1\{z_{i_1+1}, \ldots, y_{j_1}\}, y_{j_1+1}, \ldots, z_n\} \end{aligned}$$

3. $\cdot$ *must be compatible with the braces:*

$$(x_1 \cdot x_2)\{y_1, \ldots, y_n\} = \sum_{k=0}^{n} (-1)^{deg(x_2) \sum_{p=1}^{k}(deg(y_p)-1)} x_1\{y_1, \ldots, y_n\} \cdot x_2\{y_1, \ldots, y_n\}$$

4. d *must be compatible with the braces:*

$$\begin{aligned} d(x\{y_1, \ldots, y_{n+1}\}) - (dx)\{y_1, \ldots, y_n\} - (-1)^{deg(x)-1} \sum_{i=1}^{n+1} (-1)^{\sum_j^{i-1} deg(y_j)-i-1} x\{y_1, \ldots, d(y_i), \ldots, y_{n+1}\} = \\ = -(-1)^{deg(x)(deg(y_1)-1)} y_1 \cdot (x\{y_2, \ldots, y_{n+1}\}) + \\ -(-1)^{deg(x)} \sum_{i=1}^{n} (-1)^{\sum_{j=1}^{i} deg(y_j)-i} x\{y_1, \ldots, (y_i \cdot y_{i+1}), \ldots, y_{n+1}\} + \\ + (-1)^{deg(x)+\sum_{j=1}^{n} deg(x_j)-n} x\{y_1, \ldots, y_n\} \cdot y_{n+1}. \end{aligned}$$

If $(V, \{\})$ satisfies only (2) then it is called a brace algebra (c.f. [20]). Every brace algebra and hence every homotopy G-algebra gives rise to a dg-Lie algebra $(V, [-,-], d)$ by defining $[x, y] = x\{y\} - (-1)^{deg(x)deg(y)} y\{x\}$. The graded commutativity of the dot product and the graded Leibniz rule of the bracket only hold up to coboundaries:

$$x \cdot y - (-1)^{deg(x)deg(y)} y \cdot x = (-1)^{deg(x)-1} \Big(d(x \circ y) - d(x) \circ y - (-1)^{deg(x)-1} x \circ d(y) \Big)$$

$$\begin{aligned} &[x, y \cdot z] - [x, y] \cdot z - (-1)^{(deg(x)-1)deg(y)} y \cdot [x, z] = \\ &= (-1)^{(deg(x)-1)deg(y)} (d(x\{y, z\}) - d(x)\{y, z\} - (-1)^{deg(x)-1} x\{d(y), z\} - (-1)^{(deg(x)-1)+(deg(y)-1)} x\{y, d(z)\}) \end{aligned}$$

for all x, y and z in V. In cohomology the coboundaries vanish and will give rise to a Gerstenhaber algebra as will be seen later.

Remark 1.2.2. *The name homotopy comes from the fact that an algebra over a cofibrant replacement of an operad in the category of differential graded operads is called a homotopy algebra. The homotopy Gerstenhaber algebras are algebras over a cofibrant replacement of the Gerstenhaber-operad.*

To show that the Hochschild cochain complex $C_{\text{Hoch}}(A, A)$ has the structure of a homotopy G-algebra, note that $C_{\text{Hoch}}(A, A)$ is a graded vector space and define the following operations:

Braces: Let $x, y_1, \ldots, y_n \in C^{\bullet}_{\text{Hoch}}(A, A)$.

$$\begin{aligned} x\{y_1, \ldots, y_n\}(a_1, \ldots, a_{n_i}) := \sum_{1 \le i_1 \le i_1+deg(y_1) \le i_2 \le \ldots \le i_n+deg(y_n) \le m} (-1)^{\sum_{p=1}^{n}(deg(y_p)-1)i_p} \\ x(a_1, \ldots, a_{i_1}, y_1(a_{i_1+1}, \ldots), \ldots a_{i_n}, y_2(a_{i_n+1}, \ldots), \ldots, a_m) \end{aligned}$$

where the summation runs over all possible ways of composing $y_1, \ldots, y_n$ with x in such a way that it respects the order of $\{y_1, \ldots, y_n\}$ and $i_1, \ldots, i_n \in \{1, \ldots, n\}$. By convention define $x\{\} := x$ and $x \circ y := x\{y\}$. In case x is a unitary map, $\circ$ reduces to the ordinary composition since there is just one way to compose y with x.

Dot product: $\cdot : C^m_{Hoch} \otimes C^n_{Hoch} \to C^{m+n}_{\text{Hoch}}, \quad x \otimes y(v_1, \ldots, v_{m+n}) \mapsto x(v_1, \ldots, v_m) y(v_{m+1}, \ldots, v_n)$.

Differential: Let d be the Hochschild differential.

From the definition of the dot product it can readily be seen that it respects the grading of $C_{\text{Hoch}}(A, A)$. A straightforward calculation shows that the dot product is associative. Since d is the Hochschild differential, it follows that it is a differential and a degree one derivation. It remains to check the axioms 2 till 4. The proofs are long calculations and can be found in [18] or [10]. With these operations $(C_{\text{Hoch}}(A, A), \{\}, \cdot, d)$ is a homotopy G-algebra.

1.3 Hochschild Cohomology

1.3.1 Hochschild cohomology

In the previous section the Hochschild complex has been introduced. Cohomology can now be taken over this cochain complex giving rise to the Hochschild cohomology.

Definition 1.3.1. *The n^{th} Hochschild cohomology group $H^n_{Hoch}(A,A)$ is defined by*

$$\begin{aligned} H^n_{Hoch}(A,A) :=& H^n(C^\bullet_{Hoch}(A,A)) \\ =& Ker(d_{Hoch})/Im(d_{Hoch}). \end{aligned}$$

The degree shift of the Hochschild cohomology is defined in the same way as for the Hochschild complex. It will be used to endow the Hochschild cohomology with a differential graded Lie algebra structure.

1.3.2 Differential graded Lie algebra

Let $H_{\mathrm{Hoch}}[1](A,A) := \bigoplus_{m\in\mathbb{Z}} H^m_{\mathrm{Hoch}}[1](A,A)$.

Proposition 1.3.1. *The dg-Lie algebra $(C_{Hoch}(A,A)[1],[-,-],d_{Hoch})$ induces a graded Lie algebra structure on $(H_{Hoch}[1](A,A),[-,-]')$, where $[-,-]'$ is induced by the Gerstenhaber bracket.*

Proof. For $[f],[g] \in H_{\mathrm{Hoch}}[1](A,A)$ the bracket is defined to be $[[f],[g]]' = [[f,g]]$. It has to be shown that it is well-defined and that it is a Lie bracket preserving the grading on $H_{\mathrm{Hoch}}[1](A,A)$.

For the well-definedness, let $f \in Ker(d_{m+1}) \subset C^m_{\mathrm{Hoch}}[1](A,A)$ and $g \in Ker(d_{n+1}) \subset C^n_{\mathrm{Hoch}}[1](A,A)$. Then $[f,g] \in C^{m+n}_{\mathrm{Hoch}}[1](A,A)$, and since d is a derivation of degree 1,

$$\begin{aligned} d([f,g]) =& [df,g] + (-1)^m[f,dg] && \text{(derivation)} \\ =& 0. && \text{(elements of the kernel of } d) \end{aligned}$$

Thus $[f,g] \in Ker(d_{m+n+1})$. This shows that $\bigoplus_{m\in\mathbb{Z}} Ker(d_{m+1})$ is a subring of $C^{\bullet+1}_{\mathrm{Hoch}}(A,A)$. Now it will be shown that $\bigoplus_{m\in\mathbb{Z}} Im(d_{m+1})$ form a two-sided ideal of $\bigoplus_{m\in\mathbb{Z}} Ker(d_{m+1})$. For $f \in Ker(d_{m+1})$ and $g \in Im(d_{n+1})$, there exists an $h \in C^{(n-1)}_{\mathrm{Hoch}}[1](A,A)$, such that $g = d(h)$. Then

$$\begin{aligned} [f,g] =& [f,d(h)] \\ =& -(-1)^m d([f,h]) + (-1)^m[df,h] \\ =& -(-1)^m d([f,h]). \end{aligned}$$

Therefore $[f,g]$ is the image of $-(-1)^m[f,h] \in C^{m+(n-1)}_{\mathrm{Hoch}}[1](A,A)$, showing that $[f,g] \in Im(d_{m+n+1})$ and proving that $\bigoplus_{m\in\mathbb{Z}} Im(d_{m+1})$ is a left ideal of $\bigoplus_{m\in\mathbb{Z}} Ker(d_{m+1})$. The right ideal property follows immediately from the graded antisymmetric property of the Gerstenhaber bracket.

Now it will be shown that this bracket is actually a graded Lie bracket. The graded antisymmetry follows:

$$\begin{aligned} [f+Im(d), f'+Im(d)] =& [f,f'] + [f,Im(d)] + [Im(d),f'] + [Im(d),Im(d)] \\ =& [f,f'] + Im(d) && (Im(d) \text{ is a two-sided ideal}) \\ =& -(-1)^{mn}[f',f] + Im(d) && \text{(Graded antisymmetry)} \\ =& -(-1)^{mn}[f'+Im(d), f+Im(d)]. \end{aligned}$$

To prove the graded Jacobi identity, similar calculations together with the two-sided ideal property of $Im(d[1])$ give the desired result.

It is easily seen that $[-,-]'$ preserves grading, therefore $(H_{\mathrm{Hoch}}[1](A,A),[-,-])$ is a graded Lie algebra. □

To endow the Hochschild cohomology with a differential the trivial map 0 will be used. This is trivially a differential and a degree one derivation, thus $(H_{\mathrm{Hoch}}[1](A,A),[-,-],0)$ is a dg-Lie algebra.

1.3.3 Gerstenhaber algebra

The homotopy G-structure of the Hochschild complex induces a Gerstenhaber algebra structure on $H_{\mathrm{Hoch}}(A, A)$.

Definition 1.3.2. *A G-algebra is a triple* $(V, \cdot, [-,-])$ *where*

V *is a graded vector space.*

$\cdot$ *is a dot product of degree* 0*.*

$[-,-]$ *is a bracket of degree* -1*.*

These operations are subject to the following axioms:

$(V, \cdot)$ *is an associative and graded commutative algebra, i.e.*

$$\begin{aligned}(x \cdot y) \cdot z =& x \cdot (y \cdot z)\\ x \cdot y =& (-1)^{deg(x)deg(y)} y \cdot x\end{aligned}$$

$(V, [-,-])$ *is a graded Lie algebra, i.e.*

$$\begin{aligned}[x, y] =& -(-1)^{(deg(x)-1)(deg(y)-1)}[y, x]\\ 0 =& (-1)^{(deg(x)-1)(deg(z)-1)}[x, [y, z]] + (-1)^{(deg(y)-1)(deg(x)-1)}[y, [z, x]] + (-1)^{(deg(z)-1)(deg(y)-1)}[z, [x, y]]\end{aligned}$$

the $\cdot$ *and* $[-,-]$ *are related by the graded Leibniz rule:*

$$[x, y \cdot z] = [x, y] \cdot z + (-1)^{(deg(x)-1)deg(y)} y \cdot [x, z]$$

for all homogenous $x, y, z \in V$*.*

Note that the Lie bracket is of degree -1 rather than 0. This is needed since the cohomology has been taken over the non-shifted Hochschild complex. Calculations show (see [18]) that:

$$x \cdot y - (-1)^{deg(x)deg(y)} y \cdot x = (-1)^{deg(x)-1} \Big(d(x \circ y) - d(x) \circ y - (-1)^{deg(x)-1} x \circ d(y)\Big)$$

and

$$\begin{aligned}&[x, y \cdot z] - [x, y] \cdot z - (-1)^{(deg(x)-1)deg(y)} y \cdot [x, z] =\\ &\quad = (-1)^{(deg(x)-1)+(deg(y)-1)+1}(\\ &\qquad d(x\{[y, z]\}) - (dx)\{y, z\} - (-1)^{(deg(x)-1)} x\{d(y), z\} - (-1)^{(deg(x)-1)+(deg(y)-1)} x\{y, d(z)\})\end{aligned}$$

for all x, y and z in V. In case of the Hochschild cohomology the right hand side of the previous equations vanishes and thus the equations become:

$$x \cdot y - (-1)^{deg(x)deg(y)} y \cdot x = 0$$

and

$$[x, y \cdot z] - [x, y] \cdot z - (-1)^{(deg(x)-1)deg(y)} y \cdot [x, z] = 0.$$

This shows that that the Hochschild cohomology forms a graded commutative algebra and satisfies the graded Leibniz rule. It follows that $(H_{\mathrm{Hoch}}(A, A), \cdot, [-,-])$ is a Gerstenhaber algebra.

1.4 Classification

The Hochschild cohomology groups have a deformation theoretic interpretation.

1.4.1 Center

Let $a, b \in C^0_{\text{Hoch}}(A,A) = A$. Then the kernel of $d : C^0_{\text{Hoch}}(A,A) \to C^1_{\text{Hoch}}(A,A)$ is:

$$(da)(b) = ab - ba = 0$$

for all a, b. Thus $H^0_{\text{Hoch}}(A,A) = \{b \in A \mid ab - ba = 0 \text{ for all } a \in A\}$ is exactly the center of A. Maps $d_b \in C^1_{\text{Hoch}}(A,A)$ of the form $d_b(a) = ab - ba$ are called inner derivations.

1.4.2 Outer derivations

Let $f \in C^1_{\text{Hoch}}(A,A)$. The kernel of $d : C^1_{\text{Hoch}}(A,A) \to C^2_{\text{Hoch}}(A,A)$ is:

$$df(a,b) = -\,af(b) + f(ab) - f(a)b = 0$$

for all $a, b \in A$. Thus $f(ab) = f(a)b - af(b)$ demonstrating that f is a derivation. Therefore $H^1_{\text{Hoch}}(A,A) = \frac{\{\text{derivations}\}}{\{\text{inner derivations}\}}$. Elements in the quotient are called outer derivations.

1.4.3 Infinitesimal deformations

Let (A, m_0) be an associative algebra. An infinitesimal deformation gives rise to a family of multiplications $\{m_0, m_1\}$. This family satisfies D_0 and D_1 since $m_0 + m_1 t$ is associative. That D_1 is satisfied implies that $d(m_1) = 0$, thus m_1 is a cocycle. It follows that all cocycles give rise to an infinitesimal deformation. The following theorem says that equivalent deformations differ by a coboundary, thus $H^2_{\text{Hoch}}(A,A)$ classifies infinitesimal deformations up to equivalence.

Theorem 1.4.1. *Let (A, m_0) be an associative algebra. Two infinitesimal deformations $(A[[t]]/(t^2), \{m_0, m_1\})$ and $(A[[t]]/(t^2), \{m_0, m_1'\})$ are equivalent if and only if $[m_1] = [m_1']$ in $H^2_{Hoch}(A,A)$.*

Proof. The fact that $(A[[t]]/(t^2), \{m_0, m_1\})$ and $(A[[t]]/(t^2), \{m_0, m_1'\})$ are equivalent implies that there exists an algebra isomorphism $\phi : A[[t]]/(t^2) \to A[[t]]/(t^2)$. Now ϕ is completely determined by its action on A, so there exists a ϕ_1 such that $\phi = id_A + \varphi t$. Since ϕ is an algebra morphism it is compatible with the multiplications, i.e.

$$\phi \circ m = m' \circ (\phi \otimes \phi).$$

Evaluating the left hand gives

$$\begin{aligned}
\phi(m(a \otimes b)) =& \phi(m_A(a \otimes b) + m_1(a \otimes b)t) \\
=& m_A(a \otimes b) + \varphi(m_A(a \otimes b))t + m_1(a \otimes b)t + \varphi(m_1(a \otimes b)t^2 \\
=& m_A(a \otimes b) + \varphi(m_A(a \otimes b))t + m_1(a \otimes b)t
\end{aligned}$$

and the right hand side

$$\begin{aligned}
m'(\phi(a) \otimes \phi(b)) =& m_A(\phi(a) \otimes \phi(b)) + m_1'(\phi(a) \otimes \phi(b))t \\
=& m_A((a + \varphi(a)t) \otimes (b + \varphi(b)t)) + m_1'((a + \varphi(a)t) \otimes (b + \varphi(b)t))t \\
=& m_A(a \otimes b) + m_A(a \otimes \varphi(b))t + m_A(\varphi(a) \otimes b)t + m_A(\varphi(a) \otimes \varphi(b))t^2 + \\
&+ m_1'(a \otimes b)t + m_1'(a \otimes \varphi(b))t^2 + m_1'(\varphi(a) \otimes b)t^2 + m_1'(\varphi(a) \otimes \varphi(b))t^3 \\
=& m_A(a \otimes b) + m_A(a \otimes \varphi(b))t + m_A(\varphi(a) \otimes b)t + m_1'(a \otimes b)t.
\end{aligned}$$

Since the infinitesimal deformations are equivalent, the two sides should be equal, thus

$$\begin{aligned}
m_A(a \otimes b) + \varphi(m_A(a \otimes b))t + m_1(a \otimes b)t =& m_A(a \otimes b) + m_A(a \otimes \varphi(b))t + m_A(\varphi(a) \otimes b)t + m_1'(a \otimes b)t \\
m_1(a \otimes b) =& (m_A(a \otimes \varphi(b)) - \varphi(m_A(a \otimes b)) + m_A(\varphi(a) \otimes b)) + m_1'(a \otimes b) \\
=& d(\varphi)(a \otimes b) + m_1'(a \otimes b).
\end{aligned}$$

Since $m_1, m_1' \in Ker(d) \subseteq C^2_{\text{Hoch}}(A, A)$ it follows that

$$[m_1] = [m_1'] \in H^2_{\text{Hoch}}(A, A).$$

Now the other way around. Suppose $[m_1] = [m_1']$ in $H^2_{\text{Hoch}}(A, A)$. Note that m_1 and m_1' are both 2-cocycles, thus satisfy (D_1). This implies that they give rise to infinitesimally deformed multiplications.
The equality of classes means there exists a $\theta \in C^1_{\text{Hoch}}(A, A)$, such that

$$m_1 = d(\theta) + m_1'.$$

By the previous calculation it follows that there is an algebra morphism with an action on $A \otimes A$ given by $\phi = id_A + \theta t$. The action on $A[[t]]/(t^2)$ is then given by

$$\phi(a + bt) = \phi(a) + \phi(b)t = (a + \theta(a)t) + (b + \theta(b)t)t = a + (\theta(a) + b)t.$$

It remains to check that ϕ is an isomorphism. The injectivity of ϕ follows immediately. For the surjectivity, let $x + yt$ be arbitrary, where $x, y \in A$. Then $x + yt = \phi(x + (y - \theta(x)t))$ gives the desired result. ☐

1.4.4 Obstructions

Let $\{m_i\}_{i=1,\ldots,n}$ be a family of multiplications on A. The obstruction extending this family to an $(n+1)$-deformation is given by

$$\mathfrak{O}_n = \sum_{\substack{i+j=n+1\\ i,j>0}} m_i\{m_j\} = \sum_{\substack{i+j=n+1\\ i,j>0}} m_i \circ m_j.$$

The obstruction $\mathfrak{O}_n$ is a cocycle, thus $[\mathfrak{O}_n]$ is an element of $H^3_{\text{Hoch}}(A, A)$. The idea of the proof is to use the fact that D_k holds for one till n, thus $d(m_k) + \mathfrak{O}_{k-1} = 0$, the rest is just calculations (see 1.5.1). In case $\mathfrak{O}_n$ is a coboundary then there exists a multiplication extending the n-deformation to an $(n+1)$-deformation. This is the content of the following theorem.

Theorem 1.4.2. *For an associative algebra A, an n-deformation extends to an $(n+1)$-deformation if and only if $[\mathfrak{O}_n] = [0] \in H^3_{Hoch}(A, A)$.*

Proof. Suppose an n-deformation extends to an $(n+1)$-deformation. In that case D_{n+1} holds showing that $\mathfrak{O}_n = d(m_{n+1})$, thus $[\mathfrak{O}_n] = [0] \in H^3_{\text{Hoch}}(A, A)$.

Now suppose that $[\mathfrak{O}_n] = [0] \in H^3_{\text{Hoch}}(A, A)$. It follows that there exists a $m_{n+1} \in C^2_{\text{Hoch}}(A, A)$ such that $\mathfrak{O}_n = d(m_{n+1})$. Suppose the n-deformation is equivalent to the family $\{m_1, \ldots, m_n\}$. Extend this set of multiplications with m_{n+1} to $\{m_1, \ldots, m_n\} \cup \{m_{n+1}\}$. Since $\mathfrak{O}_n = d(m_{n+1})$, it follows that D_{n+1} holds, thus proving that $\{m_1, \ldots, m_{n+1}\}$ is an $(n+1)$-deformation. ☐

An immediate consequence of the previous theorem is, that if $H^3_{\text{Hoch}}(A, A) = 0$, then all deformations extend to formal deformations.

1.5 Maurer-Cartan equation

The Lie bracket and the associativity of an operation are related. A multiplication $m \in C^1_{\text{Hoch}}[1](A, A)$ is associative if and only if $[m, m] = 0$. First assume associativity of m.

$$\begin{aligned} 0 =& m(m \otimes id) - m(id \otimes m) && \text{(associativity)}\\ =& m \circ_1 m + (-1) m \circ_2 m && \text{(definition of } \circ_i)\\ =& m \circ m. && \text{(definition of } \circ) \end{aligned}$$

If $m \circ m = 0$ then clearly $[m, m] = m \circ m - (-1)^{1 \cdot 1} m \circ m = 0$.

On the other hand, if $[m,m]=0$ it follows that $0=[m,m]=m\circ m-(-1)^{1\cdot 1}m\circ m=2(m\circ m)$ thus $m\circ m=0$ and by the previous computation m is associative.

Consider now the case where m_t is of the form $m_t:=\sum_{k\geq 0}m_k t^k$. From the previous result it holds that m_t is associative if $[m_t,m_t]=0$. It follows that

$$\begin{aligned}0=&[m_t,m_t]\\ =&[\sum_{i\geq 0}m_i t^i,\sum_{j\geq 0}m_j t^j]\\ =&\sum_{k\geq 0}\sum_{i+j=k}[m_i,m_j]t^k\\ =&\sum_{k\geq 0}2\left(d(m_k)+\sum_{\substack{i+j=k\\ i,j\geq 1}}\frac{1}{2}[m_i,m_j]\right)t^k\end{aligned}$$

or rather for all $k\geq 0$ it must hold that

$$(MC_k):\quad d(m_k)+\sum_{\substack{i+j=k\\ i,j\geq 1}}\frac{1}{2}[m_i,m_j]=0.$$

Here the Maurer-Cartan equation can be recognized.

Let $x=\sum_{i\geq 1}m_i t^i$. Then $m_t=m+x$. Thus m_t is associative if

$$0=[m_t,m_t]=[m+x,m+x]=[m,m]+[m,x]+[x,m]+[x,x]=2(d(x)+\frac{1}{2}[x,x]).$$

Therefore define a set

$$MC_{C_{\text{Hoch}}(A,A)[1]}(k[[t]])=\{x\in C_{\text{Hoch}}(A,A)[1]\otimes_k (t)\ \mid\ d(x)+\frac{1}{2}[x,x]=0\},$$

where (t) is the maximal ideal of $k[[t]]$.

Note that any $x\in MC_{C_{\text{Hoch}}(A,A)[1]}(k[t]/(t^2))$ is an infinitesimal deformation since $m_t=m+x$ is associative. Further it holds that $MC_{C_{\text{Hoch}}(A,A)[1]}(k[[t]])$ are all formal deformations. Note that it is possible that two deformations are equivalent. It is possible, as will be seen later, to define an equivalence relation modding out the equivalent deformations.

There is an intimate relation between MC_k and D_k, namely they are equivalent for all $k\geq 0$.

$$\begin{aligned}d(m_k)+\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}[m_i,m_j]=&d(m_k)+\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}(m_i\circ m_j+m_j\circ m_i)\\ =&d(m_k)+\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}m_i\circ m_j+\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}m_j\circ m_i\\ =&d(m_k)+\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}m_i\circ m_j+\frac{1}{2}\sum_{\substack{i+j=k\\ i,j>0}}m_i\circ m_j\\ =&d(m_k)+\sum_{\substack{i+j=k\\ i,j>0}}m_i\circ m_j\\ =&d(m_k)+\sum_{\substack{i+j=k\\ i,j>0}}(m_i\circ_2 m_j-m_i\circ_1 m_j)\\ =&d(m_k)+\mathfrak{O}_{k-1}.\end{aligned}$$

It follows that $\frac{1}{2} \sum_{i+j=k, i,j>0} [m_i, m_j]$ is exactly the obstruction $\mathfrak{O}_{k-1}$.

Proposition 1.5.1. *The obstruction $\mathfrak{O}_n$ is a cocycle.*

Proof. The result is trivial for $\mathfrak{O}_0$ and $\mathfrak{O}_1$ since the obstructions are equal to zero. The first non-trivial obstruction occurs for $\mathfrak{O}_2 = \frac{1}{2}[m_1, m_1]$. In this case the obstruction $\mathfrak{O}_2$ can be seen to be a cocycle by the following computation.

$$\begin{aligned} 2d(\mathfrak{O}_2) =& d([m_1, m_1]) \\ =& [d(m_1), m_1] - [m_1, d(m_1)] \\ =& [0, m_1] - [m_1, 0] \\ =& 0 \end{aligned}$$

where the fact $d(m_1) + \mathfrak{O}_1 = 0$ has been used. Therefore $\mathfrak{O}_2$ is a cocycle. The same approach will be used in general. Assume that MC_n is satisfied for $n = 0, \dots, k-1$. Consider now the obstruction $\mathfrak{O}_k$.

$$\begin{aligned} 2d(\mathfrak{O}_k) =& \sum_{\substack{i+j=k \\ i,j>0}} d([m_i, m_j]) \\ =& \left(\sum_{\substack{i+j=k \\ i,j>0}} [d(m_i), m_j] \right) - \left(\sum_{\substack{i+j=k \\ i,j>0}} [m_i, d(m_j)] \right) \\ =& 2 \sum_{\substack{i+j=k \\ i,j>0}} [d(m_i), m_j] \\ =& 2 \sum_{\substack{i+j=k \\ i,j>0}} [\frac{1}{2} \sum_{\substack{p+q=i \\ p,q>0}} [m_p, m_q], m_j] \\ =& \sum_{\substack{p+q+j=k \\ p,q,j>0}} [[m_p, m_q], m_j]. \end{aligned}$$

Note that any cyclic permutation of a fixed p, q and j appears in the sum. In case k is even there is just one term for $p = q = j = k/2$, but the Jacobi identity in this case gives $3[[m_{k/2}, m_{k/2}], m_{k/2}] = 0$. Thus $[[m_{k/2}, m_{k/2}], m_{k/2}] = 0$. In all the other cases the sum of the corresponding three terms obtained by a cyclic permutation are zero by the Jacobi identity. Therefore $d(\mathfrak{O}_k) = 0$. Since $\mathfrak{O}_k \in C_{\text{Hoch}}(A, A)[1]^2$, it follows that $\mathfrak{O}_k$ is a cocycle. ☐

1.5.1 Gauge Action on the Maurer-Cartan Elements

If a Lie algebra is nilpotent then it is possible to construct a Lie group to the Lie algebra. Using local Artinian rings a nilpotent Lie algebra will be constructed from a dg-Lie algebra. A local ring is one with a unique maximal ideal and the prefix Artinian means that any decreasing chain stabilizes. For more information on this topic the reader is referred to [9].

Let $\mathfrak{g}$ be a dg-Lie algebra and let R be a local Artinian ring with maximal ideal $\mathfrak{m}$. Define a new dg-Lie algebra L by $L^n := \mathfrak{g}^n \otimes_k \mathfrak{m}$ with $[x \otimes r, y \otimes s] := [x, y] \otimes rs$ and $d(x \otimes r) := d(x) \otimes r$. In this case $MC_{\mathfrak{g}}(R) \subseteq L^1$. The maximal ideal $\mathfrak{m}$ is nilpotent and hence is the Lie algebra L^0.

The aim is to construct a group which acts on the Maurer-Cartan elements in such a way that elements in the same orbit correspond to equivalent deformations. Define $G_{\mathfrak{g}}(R) := (exp(L^0), \cdot)$ where $\cdot$ is given by the Baker-Campbell-Hausdorff-Dynkin formula:

$$x \cdot y := \sum_{n \geq 0} \frac{(-1)^{n-1}}{n} \sum_{\substack{r_i+s_i>0 \\ 1\leq i\leq n}} \frac{1}{\sum_{i=1}^{n}(r_i+s_i)} \frac{1}{r_1!s_1!\dots r_n!s_n!}[x^{r_1}y^{s_1}x^{r_2}y^{s_2}\dots x^{r_n}y^{s_n}]$$

where $[x^{r_1}y^{s_1}\dots x^{r_n}y^{s_n}] := [x,[x,\dots[x,[y,\dots[y,\dots[x,\dots[x,[y,[y,\dots[y,y]]\dots]$. Note that if $s_n \geq 1$ or $s_n = 0$ together with $r_n \geq 1$ then the previous term is zero.

Lemma 1.5.1. *$(G_{\mathfrak{g}}(R), \cdot)$ is a group, called the gauge group.*

Proof. Note that $\mathfrak{m}$ is nilpotent and hence L^0 is nilpotent. The nilpotency assures the convergency of the BCHD-formula and therefore the closure of the multiplication. The identity element is given by e^0. Note that 0 commutes with all elements of $\mathfrak{g}^0$ hence the BCHD-formula reduces to the sum and adding zero gives the identity, hence the identity is well-defined. The inverse of e^X is given by e^{-X} since X and $-X$ commute, i.e. $[X,-X]=0$ and therefore $e^X e^{-X} = e^{X+(-X)} = e^0$. Finally the associativity follows from the associativity of the addition on $\mathfrak{g}^0$. This shows that G is a group. □

The gauge group acts on L^1. In order to define this action to $(L^\bullet, [-,-], d)$ yet another dg-Lie algebra will be constructed. Define a graded vector space $L_d^\bullet$ by

$$L_d^i = \begin{cases} L^1 \oplus k \cdot d & i = 1 \\ L^i & \text{otherwise.} \end{cases}$$

The bracket and the differential are defined by $[x+ad, y+bd]_d := [x,y] + ad(y) - (-1)^{|v|}bd(x)$ and $d_d(x+ad) := [0+1d, x+ad]_d = d(x)$ for all $x, y \in L^i$ and $a, b \in k$. Consider the map

$$\phi : L^1 \to L_d^1, \quad x \mapsto x + d.$$

It then holds that Maurer-Cartan equation is satisfied, i.e. $d(x) + \frac{1}{2}[x,x] = 0$, if and only if $[\phi(x), \phi(x)]_d = 0$.

Define an action $G_{\mathfrak{g}}(R) \times L^1 \to L^1$ of $G_{\mathfrak{g}}(R)$ on L^1 by

$$e^{X\otimes r} \cdot (v \otimes s) := \phi^{-1} \circ e^{[X\otimes r,-]_d} \circ \phi(v \otimes s),$$

or in a more explicit form:

$$\begin{aligned} \phi^{-1} \circ e^{[X\otimes r,-]_d} \circ \phi(v\otimes s) =& e^{[X\otimes r,-]_d}(v\otimes s + d) - d \\ =& \sum_{j\geq 0} \frac{[X\otimes r,-]_d^j}{j!}(X\otimes r + d) - d \\ =& X\otimes r + \sum_{j\geq 1} \frac{[X\otimes r,-]_d^j}{j!}(X\otimes r + d) \\ =& X\otimes r + \sum_{j\geq 0} \frac{[X\otimes r,-]_d^{j+1}}{(j+1)!}(X\otimes r + d) \\ =& X\otimes r + \sum_{j\geq 0} \frac{[X\otimes r,-]_d^j}{(j+1)!}([X,v]\otimes rs - d(X)\otimes r). \end{aligned}$$

Recall that $MC_{\mathfrak{g}}(R) \subseteq L^1$.

Proposition 1.5.2. *The group $G_{\mathfrak{g}}(R)$ acts on the Maurer-Cartan elements $MC_{\mathfrak{g}}(R)$ with the previously defined action.*

In the proof, due to [15], of this proposition the newly defined dg-Lie algebra really pays off.

Proof. Let $x \in L^0$ and $v \in MC_{\mathfrak{g}}(R)$ then

$$d(e^x \cdot v) = d_d(e^{[x,-]_d}(v+d) - d) = [d, e^{[x,-]_d}(v+d) - d]_d = [d, e^{[x,-]_d}(v+d)]_d.$$

On the other hand

$$\begin{aligned}[e^x \cdot v, e^x \cdot v] =& [e^{[x,-]_d}(v+d) - d, e^{[x,-]_d}(v+d) - d]_d \\ =& [e^{[x,-]_d}(v+d), e^{[x,-]_d}(v+d)]_d - 2[d, e^{[x,-]_d}(v+d)]_d \\ =& e^{[x,-]_d}[\phi(v), \phi(v)]_d - 2[d, e^{[x,-]_d}(v+d)]_d \\ =& -2[d, e^{[x,-]_d}(v+d)]_d.\end{aligned}$$

Hence

$$d(e^x \cdot v) + \frac{1}{2}[e^x \cdot v, e^x \cdot v] = [d, e^{[x,-]_d}(v+d)]_d + \frac{1}{2}\left(-2[d, e^{[x,-]_d}(v+d)]_d\right) = 0.$$

This proves that $e^x \cdot v$ is again a Maurer-Cartan element. □

Chose now $R = k[t]/(t^2)$ and $\mathfrak{g} = C^\bullet[1]_{\text{Hoch}}(A, A)$, then $MC_{\mathfrak{g}}(R)$ are infinitesimal deformations and $G_{\mathfrak{g}}(R)$ acts on them.

Lemma 1.5.2. *Maurer-Cartan elements in the same orbit give rise to equivalent deformations.*

Proof. Let m_t and m'_t be two equivalent infinitesimal deformation where $m_t = m + xt$ and $m'_t = m + yt$. Then there exists a $w \in C^1(A, A)$ such that $y = x + d(w)$ or rather $yt = xt + d(w)t$. Thus it holds that

$$y \otimes t = x \otimes t + d(w) \otimes t = e^{w \otimes t} \cdot x \otimes t$$

hence $x \otimes t$ and $y \otimes t$ are in the same orbit. The proof is complete with the observation that infinitesimal deformations $m + xt$ and $m + yt$ of m imply that $x \otimes t$ and $y \otimes t$ are Maurer-Cartan elements and vice versa. □

Thus the quotient of $MC_{\mathfrak{g}}(R)$ by $G_{\mathfrak{g}}(R)$ gives all infinitesimal deformations up to equivalence. This quotient is denoted by $\text{Def}_{\mathfrak{g}}(R) := MC_{\mathfrak{g}}(R)/G_{\mathfrak{g}}(R)$. The following theorem comes then as no surprise.

Theorem 1.5.1. $Def_{C_{Hoch}[1]^\bullet(A,A)}(k[[t]]/(t^2)) \cong H^2(A, A)$.

1.6 Deformations of Morphisms

Now the deformation theory of algebra morphisms will be investigated. This enables to study deformations of representations of algebras. This will be of importance in chapter 4 and chapter 5 where representations of operads will be in fact algebras.

1.6.1 Deformation

The deformation theory of an algebra morphism will be developed in the same way as for algebras.

Definition 1.6.1. *An R-deformation of a morphism $f : A \to B$ between two associative k-algebras is an R-algebra morphism $f_t : A_t \to B_t$ between two R-deformations (A_t, μ_t) and (B_t, ν_t) of A and B, respectively, such that the diagram*

$$\begin{array}{ccc} A_t \otimes_R k & \xrightarrow{f_t \otimes_R k} & B_t \otimes_R k \\ \cong \downarrow & & \downarrow \cong \\ A & \xrightarrow[f]{} & B \end{array}$$

commutes.

For a formal deformation $f_t : A \otimes_k k[[t]] \to B \otimes_k k[[t]]$ the following properties should hold:

$$\begin{aligned} f_t(rx + sy) =& r f_t(x) + s f_t(y) \qquad\qquad r, s \in k[[t]], x, y \in A \otimes_k k[[t]] \\ f_t \circ m_t =& \nu_t \circ (f_t \otimes f_t) \end{aligned}$$

Substituting f_t with $\sum_{k \geq 0} f_k t^k$ and likewise μ_t and ν_t gives

$$\begin{aligned} f_k(rx + sy) =& r f_k(x) + s f_k(y) \qquad\qquad r, s \in k[[t]], x, y \in A \otimes_k k[[t]] \\ \sum_{\substack{l+p=q \\ l,p \geq 0}} f_l \circ \mu_p -& \sum_{\substack{k+i+j=q \\ k,i,j \geq 0}} \nu_k \circ (f_i \otimes f_j). \end{aligned}$$

The first equation shows that each f_i should be a linear map. The second suggest that the differential should be given by

$$d((\mu_1, \nu_1, f_1)) := \nu_1 \circ (f \otimes f) + \nu \circ (f \otimes f_1) + \nu \circ (f_1 \otimes f) - f_1 \circ \mu - f \circ \mu_1 \tag{1.1}$$

$$= (\nu \circ (f \otimes f_1) - f_1 \circ \mu + \nu \circ (f_1 \otimes f)) + (-f \circ \mu_1 + \nu_1 \circ (f \otimes f)) . \tag{1.2}$$

Equivalence of deformations Two R-deformations f_t^1 and f_t^2 of a morphism $f : A \to B$ are equivalent if there is a pair of R-algebra automorphism (ϕ_t^A, ϕ_t^B) such that the diagram

$$\begin{array}{ccc} A_t & \xrightarrow{\phi_t^A} & A_t \\ {\scriptstyle f_t^1}\downarrow & & \downarrow{\scriptstyle f_t^2} \\ B_t & \xrightarrow[\phi_t^B]{} & B_t \end{array}$$

commutes.

In the infinitesimal case, substituting ϕ_t^A with $1_A + \phi_A t$ and ϕ_t^B with $1_B + \phi_B t$ results in the commutativity of the square:

$$f_1' = f_1 + f \circ \phi_1^A - \phi_1^B \circ f.$$

Thus $f \circ \phi_1^A - \phi_1^B \circ f$ should become a coboundary for an appropriate complex.

1.6.2 Cochain complex

Fix a algebra morphism $f : A \to B$. The first term in equation (1.1) is

$$\nu \circ (f \otimes f_1) - f_1 \circ \mu + \nu \circ (f_1 \otimes f)$$

with $f_1 \in C^1_{\text{Hoch}}(A, B)$ and $C^n(A, B) := Hom(A^{\otimes n}, B)$. Define an (A, A)-bimodule structure on B by

$$\begin{aligned} A \otimes B \otimes A &\xrightarrow{\cdot} B \\ a \otimes b \otimes a' &\mapsto f(a) b f(a'). \end{aligned}$$

The map

$$d_f(f_1) := \nu \circ (f \otimes f_1) - f_1 \circ \mu + \nu \circ (f_1 \otimes f)$$

defines a differential on $C^\bullet(A, B)$, hence it is a cochain complex. If $B = A$ and $f = id_A$ then $d_{id_A} = d_{\text{Hoch}}$.

Using this differential the expression in (1.2) can be described by

$$- (f \circ \mu_1 - \nu_1 \circ (f \otimes f) - d_f(f_1)) .$$

This motivates the construction of the following complex for a morphism $f : A \to B$:

$$\begin{aligned}C^0(f) :=&0,\\ C^n(f) :=&C^n_{\text{Hoch}}(A, A) \times C^n_{\text{Hoch}}(B, B) \times C^{n-1}(A, B),\end{aligned}$$

together with the differential

$$d(\alpha, \beta, \varphi) := (d_{\text{Hoch}}(\alpha), d_{\text{Hoch}}(\beta), f \circ \alpha - \beta \circ f^{\otimes n} - d_f(\varphi)).$$

Each of the sets becomes a module by pointwise addition and multiplication.

Remark 1.6.1. *On $C^\bullet(A, B)$ on $C^\bullet(f)$ no dg-Lie algebra structure could be found.*

The cohomology groups associated to a morphism are defined to be the cohomology groups over the cochain complex $C^\bullet(f)$.

1.6.3 Classification

Infinitesimal deformations

Lemma 1.6.1. *Equivalent infinitesimal deformations differ by a coboundary.*

Proof. Let $f : A \to B$ be a fixed algebra morphism. Let f_t and f'_t be two infinitesimal deformations of f. Consider the two elements (α, β, f_1) and (α', β', f'_1) in $C^2(f)$.

Suppose f_t and f'_t are equivalent then they differ by the term $f \circ \phi^A_1 - \phi^B_1 \circ f$ for some automorphisms (ϕ^A_t, ϕ^B_t). From equation (1.1) it follows that ϕ^A_1 en ϕ^B_1 have to be cochains. The equivalence

$$f'_1 = f_1 + f \circ \phi^A_1 - \phi^B_1 \circ f$$

can then be expressed in $C^2(f)$ as

$$\begin{aligned}(0, 0, f'_1) =&(0, 0, f_1 + f \circ \phi^A_1 - \phi^B_1 \circ f)\\ =&(0, 0, f_1) + (0, 0, f \circ \phi^A_1 - \phi^B_1 \circ f)\\ =&(0, 0, f_1) + (d(\phi^A_1), d(\phi^B_1), f \circ \phi^A_1 - \phi^B_1 \circ f) && \text{(using that } \phi^{A,B}_1 \text{ are coboundaries)}\\ =&(0, 0, f_1) + d(\phi^A_1, \phi^B_1, 0) && \text{(definition of the differential)}\end{aligned}$$

Therefore equivalent deformations differ by a coboundary. □

Theorem 1.6.1. *The second cohomology group $H^2(C^\bullet(f))$ classifies all infinitesimal deformations of f up to equivalence.*

Proof. Let $f : (A, \mu) \to (B, \nu)$ and let $f_1, f'_1 \in C^1(A, B)$. In order to guarantee that $f + f_1 t$ and $f + f'_1 t$ are algebra morphisms f_1 and f'_1 have to be cochains because of (1.1). This shows that f_1 and f'_1 are in $H^2(C^\bullet(f))$. Lemma 1.6.1 shows that equivalent deformations are in the same cohomology class, i.e. $(0, 0, f'_1) \in [(0, 0, f_1)]$.

On the other hand let $[(\alpha, \beta, \varphi)] \in H^2(C^\bullet(f))$. It follows that α and β have to be cochains. For φ it holds that $0 = f \circ \alpha - \beta \circ f^{\otimes 2} - d(\varphi)$. This shows that $f + \varphi t$ is a deformation of f between $(A \otimes_k k[t]/(t^2), \mu + \alpha t)$ and $(B \otimes_k k[t]/(t^2), \nu + \beta t)$. □

Obstructions Analogously to the associative algebra case, the condition that a map is an algebra morphism is

$$\sum_{\substack{k+i+j=q\\ k,i,j\geq 0}} \nu_k \circ (f_i \otimes f_j) - \sum_{\substack{l+p=q\\ l,p\geq 0}} f_l \circ \mu_p = 0.$$

With the help of the differential the equation can be rewritten as

$$
\begin{aligned}
-d((\mu_q,\nu_q,f_q)) + \sum_{\substack{k+i+j=q\\ k,i,j>0}} \nu_k\circ(f_i\otimes f_j) + \sum_{\substack{k+i+j=q\\ k,i>0,j=0}} \nu_k\circ(f_i\otimes f_j) +\\
+ \sum_{\substack{k+i+j=q\\ k,j>0,i=0}} \nu_k\circ(f_i\otimes f_j) + \sum_{\substack{k+i+j=q\\ i,j>0,k=0}} \nu_k\circ(f_i\otimes f_j) +\\
- \sum_{\substack{l+p=q\\ l,p>0}} f_l\circ\mu_p.
\end{aligned}
$$

Definition 1.6.2. *The obstruction is defined to be*

$$
\begin{aligned}
\mathcal{O}_{q-1} := \sum_{\substack{k+i+j=q\\ k,i,j>0}} \nu_k\circ(f_i\otimes f_j) + \sum_{\substack{k+i+j=q\\ k,i>0,j=0}} \nu_k\circ(f_i\otimes f_j) +\\
+ \sum_{\substack{k+i+j=q\\ k,j>0,i=0}} \nu_k\circ(f_i\otimes f_j) + \sum_{\substack{k+i+j=q\\ i,j>0,k=0}} \nu_k\circ(f_i\otimes f_j) +\\
- \sum_{\substack{l+p=q\\ l,p>0}} f_l\circ\mu_p.
\end{aligned}
$$

Since there is no dg-Lie algebra structure on deformation complex of a morphism it is not possible to give an elegant proof that the obstructions are cocycles. The proof has to be done in a straight forward way and can be adopted from the proof in [24]. As a consequence it holds that $[\mathcal{O}_q] \in H^3(f)$.

Chapter 2

Deformation Theory of Lie Algebras

In this chapter deformations of Lie algebras will be defined and the infinitesimal deformations will be classified up to equivalence. The theory will be developed in the same way as for associative algebras. The role of the Hochschild complex will be played by the Chevalley-Eilenberg complex.

2.1 Deformation

For completeness the definition of a Lie algebra and a Lie algebra morphism will now be given.

2.1.1 Lie algebras

Let V be a finite dimensional vector space over a field k with characteristic zero. A Lie algebra $\mathfrak{g}$ is a vector space V together with a binary operation $[-,-]$ satisfying:

- Bilinearity: $[ax+by,z]=a[x,z]+b[y,z]$ and $[x,ay+bz]=a[x,y]+b[x,z]$.
- Antisymmetry: $[y,x]=-[x,y]$
- Jacobi identity: $[x,[y,z]]+[y,[z,x]]+[z,[x,y]]=0$

for all $a,b\in k$ and $x,y,z\in\mathfrak{g}$.

Let $(\mathfrak{g},[-,-])$ and $(\mathfrak{g}',[-,-]')$ be two Lie algebras. A Lie algebra morphism is a linear map $f:\mathfrak{g}\to\mathfrak{g}'$, such that f is compatible with the products, i.e. the diagram

$$\begin{array}{ccc} \mathfrak{g}\times\mathfrak{g} & \xrightarrow{[-,-]} & \mathfrak{g} \\ {\scriptstyle f\times f}\downarrow & & \downarrow{\scriptstyle f} \\ \mathfrak{g}'\times\mathfrak{g}' & \xrightarrow[{[-,-]'}]{} & \mathfrak{g}' \end{array}$$

is commutative. The Lie algebras together with the Lie algebra morphisms form a category: **Lie**.

Example 2.1.1.

$(\mathbb{R}^3,\times)$: *3-dimensional Euclidean space with the cross product on the vectors.*

$(\mathfrak{gl}(n))$: *All $n\times n-$matrices with the commutator as Lie bracket, i.e. for A,B invertible, $[A,B]:=AB-BA$.*

$Lie(G)$: Left-invariant vector fields on a Lie group G together with the bracket defined by $[A,B]:=ad(A)(B)$ where ad is the adjoint representation of $Lie(G)$.

2.1.2 Augmentation

An augmentation (R, ε) of the ring k is a ring R together with a ring morphism $\varepsilon : R \to k$, such that the sequence

$$0 \to Ker(\varepsilon) \to R \to k \to 0$$

is exact. The pair $(R, Ker(\varepsilon))$ is called a base. Note that $R/Ker(\varepsilon) = k$, because by exactness $Ker(\varepsilon)$ is a maximal ideal of R. On the other hand, a ring R together with a maximal ideal m_R gives rise to an augmentation $(R, R \twoheadrightarrow R/m_R)$.

Let $\mathfrak{g}$ be a Lie algebra over a field k then $\mathfrak{g} \otimes_k R$ is an R-module which can be turned into a Lie algebra by extending the Lie bracket of $\mathfrak{g}$ bilinearly. This Lie algebra is called the augmented Lie algebra of $\mathfrak{g}$.

2.1.3 Deformations

Let $\mathfrak{g}$ be a Lie algebra over a field k and let R be an augmentation of k.

Definition 2.1.1. *A deformation of $\mathfrak{g}$ with base (R, m_R) is a Lie algebra $\mathfrak{h}$ over R together with a k-isomorphism $\alpha : \mathfrak{h} \otimes_R k \to \mathfrak{g}$.*

For different bases the corresponding deformations have special names.

A deformation with base $(k[[t]], (t))$ is called formal.

A deformation with base $(k[t]/(t^{n+1}), (t))$ is called an n-deformation for all $n \in \mathbb{N}$.

A deformation with base $(k[t]/(t^2), (t))$ is called infinitesimal.

For a commutative ring the resulting deformation is called global.

In case of a formal deformation it is generally easier to use the isomorphism $\mathfrak{g}[[t]] \cong \mathfrak{g} \otimes_k k[[t]]$. To simplify the analysis, only deformations with base $(k[[t]], (t))$ or $(k[[t]]/(t^n), (t))$ will be considered. Note that ε in both of the previous augmentations is given by the evaluation map $ev_0 : \mathfrak{g}[[t]] \to \mathfrak{g}$ setting t to zero.

2.1.4 Family of operations

A formal deformation $(g[[t]], \rho)$ is equivalent to a family of operations $\{\rho_i\}_{i=1}^{\infty}$ where $\rho = \sum_i \rho_i t^i$ and ρ_0 is the original Lie bracket on $\mathfrak{g}$. The Lie bracket ρ is antisymmetric and satisfies the Jacobi identity. The antisymmetry is satisfied if $\rho_i(y \otimes x) = -\rho(x \otimes y)$ for all $i \geq 0$. The antisymmetry can be guaranteed by considering maps not in $Hom(T^2(\mathfrak{g}), \mathfrak{g})$ but in $Hom(\bigwedge^2(\mathfrak{g}), \mathfrak{g})$ instead. In the first Hom-set the $T(\mathfrak{g})$ denotes the tensor algebra defined by $\bigotimes_{n\geq 0} T^n(\mathfrak{g})$ where $T^k(\mathfrak{g}) := \mathfrak{g} \otimes \ldots \otimes \mathfrak{g}$ and the multiplication is defined by

$$(x_1 \otimes \ldots \otimes x_m) \cdot (y_1 \otimes \ldots \otimes y_n) := x_1 \otimes \ldots \otimes x_m \otimes y_1 \otimes \ldots \otimes y_n$$

for $x_1 \otimes \ldots \otimes x_m \in T^m(\mathfrak{g})$ and $y_1 \otimes \ldots \otimes y_n \in T^n(\mathfrak{g})$. The $\bigwedge(\mathfrak{g})$ is the exterior algebra on $\mathfrak{g}$ which is defined as a quotient of the tensor algebra. Let I denote the ideal generated by elements of the form $x \otimes x$ for all $x \in \mathfrak{g}$ then $\bigwedge(\mathfrak{g}) := T(\mathfrak{g})/I$. The wedge product is defined to be the tensor product modulo I. The antisymmetry in characteristic zero follows since for $x, y \in \mathfrak{g}$ it follows that $x + y \in \mathfrak{g}$ and thus

$$\begin{aligned} 0 =& (x + y) \wedge (x + y) \\ =& x \wedge x + x \wedge y + y \wedge x + y \wedge y \\ =& x \wedge y + y \wedge x. \end{aligned}$$

The graded anticommutativity $y \wedge x = (-1)^{mn} x \wedge y$ for $x \in \bigwedge^m(\mathfrak{g})$ and $y \in \bigwedge^n(\mathfrak{g})$ follows since $x \wedge y$ can be obtained from $y \wedge x$ by sequentially interchanging one element each time introducing a minus sign.

Now the Jacobi identity is satisfied if

$$\begin{aligned} 0 &= \sum_{\sigma \in Sh(1,2)} \rho(\rho(g_{\sigma(1)} \wedge g_{\sigma(2)}) \wedge g_{\sigma(3)}) \\ &= \sum_{\sigma \in Sh(1,2)} \left(\sum_i \rho_i t^i \right) (\left(\sum_j \rho_j t^j \right) (g_{\sigma(1)} \wedge g_{\sigma(2)}) \wedge g_{\sigma(3)}) \\ &= \sum_{\sigma \in Sh(1,2)} \sum_{i,j} \rho_i(\rho_j(g_{\sigma(1)} \wedge g_{\sigma(2)}) \wedge g_{\sigma(3)}) t^{i+j} \end{aligned}$$

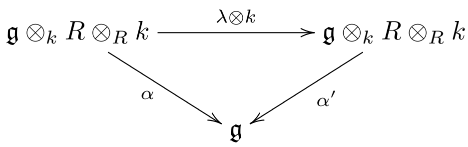
for $g_i \in \mathfrak{g}$. Therefore ρ satisfies the Jacobi identity if and only if for each k and all $g_i \in \mathfrak{g}$ the equation

$$D_k : \quad \sum_{\sigma \in Sh(1,2)} \sum_{\substack{i+j=k \\ i,j \geq 0}} \rho_i(\rho_j(g_{\sigma(1)} \wedge g_{\sigma(2)}) \wedge g_{\sigma(3)}) = 0,$$

holds.

2.1.5 Equivalence of deformations

Two deformations $(\mathfrak{g} \otimes_k R, \rho, \alpha)$ and $(\mathfrak{g} \otimes_k R, \rho', \alpha')$ of a Lie algebra $\mathfrak{g}$ with the same base (R, m) are equivalent if there exists a Lie algebra isomorphism $\lambda : \mathfrak{g} \otimes_k R \to \mathfrak{g} \otimes_k R$, such that the diagram



commutes.

In case of two infinitesimal deformations, $\rho_t = \rho_0 + \rho_1 t$ and $\rho'_t = \rho_0 + \rho'_1 t$ this just means that there exists an isomorphism of the form $\varphi_t = id_{\mathfrak{g}} + \varphi_1 t$ such that

$$\varphi_t \circ \rho'_t = \rho_t \circ (\varphi_t \wedge \varphi_t)$$

for all t. Thus

$$\rho'_1(h \wedge h') = \rho_1(h \wedge h') + (\rho_0(g_1(h) \wedge h') + \rho_0(h \wedge g_1(h')) - g_1(\rho_0(h \wedge h')))$$

for all $h, h' \in \mathfrak{g}$.

2.2 Chevalley-Eilenberg Complex

The aim is, analogous to the deformation of associative algebras, to construct a cochain complex such that for a two-form ρ the differential $d(\rho)$ is equal to the coefficient of t^1 in D_1. The cochain complex providing this structure will be the Chevalley-Eilenberg complex. It will also be shown that this complex can be endowed with a dg-Lie algebra structure.

2.2.1 Chevalley-Eilenberg complex

Let $\mathfrak{g}$ be a Lie algebra. The Chevalley-Eilenberg complex is constructed as follows.

$$C^n(\mathfrak{g}, \mathfrak{g}) := \begin{cases} Hom_{\mathbf{Vec}}(\bigwedge^n \mathfrak{g}, \mathfrak{g}) & n \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

and the differential $d_{CE} : C^n(\mathfrak{g}, \mathfrak{g}) \to C^{n+1}(\mathfrak{g}, \mathfrak{g})$ is given by:

$$d_{CE}(\omega)(g_1, \ldots, g_{n+1}) := \sum_{1 \le i \le n+1} (-1)^i [g_i, \omega(g_1, \ldots, \hat{g}_i, \ldots, g_{n+1})] + \sum_{1 \le i < j \le n+1} (-1)^{i+j-1} \omega([g_i, g_j], g_1, \ldots, \hat{g}_i, \ldots, \hat{g}_j, \ldots, g_{n+1})$$

for all $g_1, \ldots, g_{n+1} \in \mathfrak{g}$.

For finite dimensions and non-negative n the module $C^n(\mathfrak{g}, \mathfrak{g})$ is isomorphic to $\bigwedge^n(\mathfrak{g}^*) \otimes_k \mathfrak{g}$. The differential on $\bigwedge^n(\mathfrak{g}^*) \otimes_k \mathfrak{g}$ corresponding to d_{CE} is defined as follows.

- For $g \in \mathfrak{g}$ define $d(g)(g_1) := -[g_1, g] = [g, g_1] = ad_g(g_1)$.
- For $\omega \in \mathfrak{g}^*$ define $d(\omega)(g_1, g_2) := \omega([g_1, g_2]) - [\omega(g_1), g_2] - [g_1, \omega(g_2)]$.
- For $\omega \in \bigwedge^m(\mathfrak{g}^*)$ and $\eta \in \bigwedge^n(\mathfrak{g}^*)$ let $d(\omega \wedge \eta) := d(\omega) \wedge \eta + (-1)^{deg(\omega)} \omega \wedge d(\eta)$.
- Finally for $\omega \otimes g \in \bigwedge^n(\mathfrak{g}^*) \otimes_k \mathfrak{g}$ define $d(\omega \otimes g) := d(\omega) \otimes g + (-1)^{deg(\omega)} \omega \otimes d(g)$.

Define $C(\mathfrak{g}, \mathfrak{g}) := \bigoplus_{n \ge 0} C^n(\mathfrak{g}, \mathfrak{g})$ then

Lemma 2.2.1. *$(C(\mathfrak{g}, \mathfrak{g}), d_{CE})$ is a complex, i.e. $d_{CE}^2 = 0$.*

Proof. Here the characterization of d on $\bigwedge(\mathfrak{g}^*) \otimes_k \mathfrak{g}$ is useful. First it will be shown that d^2 is zero on $\mathfrak{g}$:

$$\begin{aligned} d^2(g)(g_1, g_2) =& dg([g_1, g_2]) - [dg(g_1), g_2] - [g_1, dg(g_2)] \\ =& [g, [g_1, g_2]] - [[g, g_1], g_2] - [g_1, [g, g_2]] \\ =& -[g_1, [g_2, g]] - [g_2, [g, g_1]] + [g_2, [g, g_1]] + [g_1, [g_2, g]] \\ =& 0 \end{aligned}$$

for all $g, g_1, g_2 \in \mathfrak{g}$. By the derivation property of d this extends to $\bigwedge^n(\mathfrak{g}^*)$ and then to $\bigwedge^n(\mathfrak{g}^*) \otimes_k \mathfrak{g}$. Thus d_{CE} is a differential on $C(\mathfrak{g}, \mathfrak{g})$. □

2.2.2 Pre-Lie algebra

The Chevalley-Eilenberg complex $(C(\mathfrak{g}, \mathfrak{g}), d_{CE})$ can be endowed with a pre-Lie algebra. Remember $(C(\mathfrak{g}, \mathfrak{g})[1], \circ)$ forms a right pre-Lie algebra if

$$(f \circ g) \circ h - f \circ (g \circ h) = (-1)^{|g||h|} ((f \circ h) \circ g - f \circ (h \circ g))$$

is satisfied for all homogeneous $f, g, h \in C(\mathfrak{g}, \mathfrak{g})[1]$. Define a circle operation $\circ : C(\mathfrak{g}, \mathfrak{g})[1]^m \times C(\mathfrak{g}, \mathfrak{g})[1]^n \to C(\mathfrak{g}, \mathfrak{g})[1]^{m+n}$ to be

$$f \circ g(v_1, \ldots, v_{m+n+1}) := \sum_{\sigma \in Sh(n+1, m)} (-1)^{mn + deg(\sigma)} f(g(v_{\sigma(1)}, \ldots, v_{\sigma(n+1)}), v_{\sigma(n+2)}, \ldots, v_{\sigma(n+m+1)}).$$

With the above definition it holds that

Lemma 2.2.2. *$(C(\mathfrak{g}, \mathfrak{g})[1], \circ)$ is a graded pre-Lie algebra.*

Proof. For $f \in C(\mathfrak{g}, \mathfrak{g})[1]^m$ and $g \in C(\mathfrak{g}, \mathfrak{g})[1]^n$ the circle operation applied to f and g is an element of $C(\mathfrak{g}, \mathfrak{g})[1]^{m+n}$ thus $\circ$ respects the grading. It has to be shown that for $h \in C(\mathfrak{g}, \mathfrak{g})[1]^l$ the relation $(h \circ f) \circ g - (-1)^{nl}(h \circ g) \circ f = h \circ f \circ g - (-1)^{mn} g \circ f)$ holds.

$$
\begin{aligned}
&h \circ (f \circ g - (-1)^{mn} g \circ f)(v_1, \ldots, v_{m+n+l+3}) = \\
=&\sum_{\eta} (-1)^{deg(\sigma)} h(f \circ g - (-1)^{mn} g \circ f(v_{\eta(1)}, \ldots, v_{\eta(m+n+1)}), v_{\eta(m+n+3)}, \ldots, v_{\eta(m+n+l+3)}) \\
=&\sum_{\sigma} (-1)^{deg(\eta)+deg(\sigma)} h(f(g(v_{\sigma\circ\eta(1)}, \ldots, v_{\sigma\circ\eta(n+1)}), \ldots, v_{n+m+1})+ \\
&- (-1)^{mn} g(f(v_{\sigma\circ\eta(1)}, \ldots, v_{\sigma\circ\eta(m+1)}), \ldots, v_{n+m+2}), \ldots, v_{\sigma\circ\eta(n+m+l+3)}) \\
=&\sum_{\sigma} (-1)^{deg(\eta)+deg(\sigma)} h(f(g(v_{\sigma\circ\eta(1)}, \ldots, v_{\sigma\circ\eta(n+1)}), \ldots, v_{n+m+1}), \ldots, v_{n+m+2}), \ldots, v_{\sigma\circ\eta(n+m+l+1)})+ \\
&- (-1)^{mn} \sum_{\sigma} (-1)^{deg(\eta)+deg(\sigma)} g(f(v_{\sigma\circ\eta(1)}, \ldots, v_{\sigma\circ\eta(m+1)}), \ldots, v_{n+m+2}), \ldots, v_{\sigma\circ\eta(n+m+l+1)}) \\
=&(h \circ f) \circ g - (-1)^{mn} (h \circ g) \circ f(v_1, \ldots, v_{m+n+l+1}).
\end{aligned}
$$

□

2.2.3 Graded Lie algebra

Define a bracket $[f, g] := f \circ g - (-1)^{mn} g \circ f$ for all $f \in C(\mathfrak{g}, \mathfrak{g})[1]^m$ and $g \in C(\mathfrak{g}, \mathfrak{g})[1]^n$. By the following proposition the graded pre-Lie algebra structure $(C(\mathfrak{g}, \mathfrak{g})[1], \circ)$ with the above bracket leads to a graded Lie algebra structure on $C(\mathfrak{g}, \mathfrak{g})[1]$.

Proposition 2.2.1. *Let $A = \oplus_{n\in\mathbb{Z}} A^n$ and let $(A, \circ)$ be a graded pre-Lie algebra. For $a \in A^m$ and $b \in A^n$ define $[a, b] := a \circ b - (-1)^{mn} b \circ a$. Then $(A, [-, -])$ is a graded Lie algebra.*

Proof. That $[-, -]$ respects the grading follows immediately from the fact that $\circ$ does. So the two graded Lie algebra properties have to be checked.

Graded anti-symmetry:

$$
\begin{aligned}
[g, f] =& g \circ f - (-1)^{mn} f \circ g \\
=& - (-1)^{mn} (f \circ g - (-1)^{mn} g \circ f) \\
=& - (-1)^{mn} [f, g].
\end{aligned}
$$

Graded Jacobi-identity:

$$
\begin{aligned}
(-1)^{ml}[[f, g], h] +& (-1)^{nm}[[g, h], f] + (-1)^{ln}[[h, f], g] = \\
=& (-1)^{ml}[f \circ g, h] - (-1)^{ml+mn}[g \circ f, h]+ \\
&+ (-1)^{nm}[g \circ h, f] - (-1)^{nm+nl}[h \circ g, f]+ \\
&+ (-1)^{ln}[h \circ f, g] - (-1)^{ln+lm}[f \circ h, g] \\
=& (-1)^{ml}(f \circ g) \circ h - (-1)^{ml+(m+n)l} h \circ (f \circ g)+ \\
&- (-1)^{ml+mn}(g \circ f) \circ h + (-1)^{ml+mn+(n+m)l} h \circ (g \circ f)+ \\
&+ (-1)^{nm}(g \circ h) \circ f - (-1)^{nm-(n+l)m} f \circ (g \circ h)+ \\
&- (-1)^{nm+nl}(h \circ g) \circ f + (-1)^{nm+nl+(l+n)m} f \circ (h \circ g)+ \\
&+ (-1)^{ln}(h \circ f) \circ g - (-1)^{ln+(l+m)n} g \circ (h \circ f)+ \\
&- (-1)^{ln+lm}(f \circ h) \circ g + (-1)^{ln+lm+(m+l)n} g \circ (f \circ h) \\
=& 0.
\end{aligned}
$$

This shows that $(A, [-, -])$ is a graded Lie-algebra. □

2.2.4 Differential graded Lie algebra

Note that for the graded Lie-algebra $(C(\mathfrak{g},\mathfrak{g})[1],[-,-])$ the differential d can be written in the more convenient form

$$\begin{aligned}d_{CE}(f) =&\rho_0 \circ f - (-1)^n f \circ \rho_0\\ =&[\rho_0, f]\end{aligned}$$

for all $f \in C^n[1](\mathfrak{g},\mathfrak{g})$. Since $(C(\mathfrak{g},\mathfrak{g})[1],[-,-])$ is a graded Lie-algebra and d_{CE} is a differential and a degree one derivation, it follows that $(C(\mathfrak{g},\mathfrak{g})[1],[-,-],d_{CE})$ is a dg-Lie algebra.

2.3 Lie algebra cohomology

In this section the cohomology over the Chevalley-Eilenberg complex will be taken. The resulting cohomology is called the Lie algebra cohomology. The cohomology groups can be endowed with the structure of a dg-Lie algebra.

2.3.1 Lie algebra cohomology

The Lie algebra cohomology groups of $\mathfrak{g}$ with values in $\mathfrak{g}$ is defined by the cohomology over the Chevalley-Eilenberg complex, i.e.:

$$H^n(\mathfrak{g},\mathfrak{g}) := \frac{Ker(d : C^n(\mathfrak{g},\mathfrak{g}) \to C^{n+1}(\mathfrak{g},\mathfrak{g}))}{Im(d : C^{n-1}(\mathfrak{g},\mathfrak{g}) \to C^n(\mathfrak{g},\mathfrak{g})}.$$

2.3.2 Differential graded Lie algebra

Define $H_{CE}(\mathfrak{g},\mathfrak{g}) := \bigoplus_{n\in\mathbb{Z}} H^n(\mathfrak{g},\mathfrak{g})$ then $H_{CE}(\mathfrak{g},\mathfrak{g})$ is a graded vector space. Together with the induced Lie bracket of the Chevalley-Eilenberg complex gives

Lemma 2.3.1. $(H_{CE}[1](\mathfrak{g},\mathfrak{g}),[-,-])$ *is a graded Lie algebra.*

Proof. The following four properties have to be checked:

1. $[-,-]$ is well-defined: Let $f \in C^m[1](\mathfrak{g},\mathfrak{g}) \cap Ker(d)$ and $g \in C^n[1](\mathfrak{g},\mathfrak{g}) \cap Ker(d)$. Then
$$d([f,g]) = [d(f),g] + (-1)^m[f,d(g)] = 0.$$
Let $h \in d(C^{m-1}[1](\mathfrak{g},\mathfrak{g}))$, so there exists an $h' \in C^{m-1}[1](\mathfrak{g},\mathfrak{g})$, such that $h = d(h')$.
$$\begin{aligned}[f,h] =&[f,d(h')]\\ =&(-1)^m d([f,g]) - (-1)^m[d(f),h']\\ =&(-1)^m d([f,g]).\end{aligned}$$
Thus $[f,h] \in d(C^{m+n}[1](\mathfrak{g},\mathfrak{g}))$. This proves that $[-,-]$ is well-defined on $H_{CE}[1](\mathfrak{g},\mathfrak{g})$.
2. The bracket preserves grading: Let $f \in H^m_{CE}[1](\mathfrak{g},\mathfrak{g})$ and $g \in H^n_{CE}[1](\mathfrak{g},\mathfrak{g})$. Since $[f,g] \in C^{m+n}[1](\mathfrak{g},\mathfrak{g})$, by the previous result $[f,g] \in C^{m+n}[1](\mathfrak{g},\mathfrak{g}) \cap Ker(d)$ it follows that $[f,g] \in H^{m+n}_{CE}[1](\mathfrak{g},\mathfrak{g})$.
3. Graded anti-symmetry:
$$\begin{aligned}[g+Im(d), f+Im(d)] =&[g,f] + [g,Im(d)] + [Im(d),f] + [Im(d),Im(d)]\\ =&[g,f] + Im(d)\\ =&-(-1)^{mn}[f,g] + Im(d).\end{aligned}$$
4. Graded Jacobi-identity: With similar calculations it can be seen that the graded Jacobi identity holds.

Therefore $(H_{CE}[1](\mathfrak{g},\mathfrak{g}),[-,-])$ is a graded Lie algebra. □

Note that with the trivial differential the graded Lie algebra becomes a dg-Lie algebra.

2.4 Classification

In this section the Lie algebra cohomology groups will be determined up to $n = 3$. It will be shown that the second cohomology group classifies infinitesimal deformations and obstructions are in the third group.

2.4.1 Center

For $x, y \in C^0(\mathfrak{g}, \mathfrak{g})$, it holds that

$$d_{CE}(x)(y) = -[y, x] = ad(x)(y).$$

Thus $H^0(\mathfrak{g}, \mathfrak{g}) = \{x \in \mathfrak{g} \mid [\mathfrak{g}, x] = 0\}$. The map $ad(x) : \mathfrak{g} \to \mathfrak{g}$ is called an inner derivation.

2.4.2 Outer derivations

For $x, y \in C^1(\mathfrak{g}, \mathfrak{g})$, it holds that

$$\begin{aligned} 0 =& d_{CE}(\omega)(x, y) \\ =& -\alpha([x, y]) \\ =& -[x, \omega(y)] + [y, \omega(x)] + \omega([x, y]). \end{aligned}$$

Therefore $\omega([x, y]) = [\omega(x), y] + [x, \omega(y)]$, proving that ω is a derivation. The image of $\alpha \in Im(d)$ gives inner derivations. Derivations which are not inner are called outer. Thus

$$H^1(\mathfrak{g}, \mathfrak{g}) = \frac{\{\text{derivations }\}}{\{\text{inner derivations}\}} = \{\text{outer derivations}\}.$$

2.4.3 Infinitesimal deformations

For $n = 2$ the group elements are infinitesimal deformations in the following way. For $x, y, z \in \mathfrak{g}$

$$\begin{aligned} 0 =& d_{CE}(\omega)(x, y, z) \\ =& [x, \omega(y, z)] - [y, \omega(x, z)] + [z, \omega(x, y)] + \omega([x, y], z) - \omega([x, z], y) + \omega([y, z], x) \\ =& [\omega(x, y), z] + [\omega(y, z), x] + [\omega(z, x), y] + \omega([x, y], z) + \omega([z, x], y) + \omega([y, z], x). \end{aligned}$$

Let ρ_0 be the Lie bracket of $\mathfrak{g}$ and $\rho_1 = \omega$. Then the last expression can be written as

$$\sum_{\sigma} \sum_{i+j=1, i,j \geq 0} \rho_i(\rho_j(x, y), z) = 0.$$

Thus ω satisfies D_1, and $\rho_0 + \omega t$ is a Lie bracket giving rise to an infinitesimal deformation.

Theorem 2.4.1. *Let $(\mathfrak{g}, \rho_0)$ be a Lie algebra. Two infinitesimal deformations $\rho := \rho_0 + \rho_1 t$ and $\rho' := \rho_0 + \rho_1' t$ are equivalent if and only if $[\rho_1] = [\rho_1']$ in $H^2(\mathfrak{g}, \mathfrak{g})$.*

Proof. The fact that ρ and ρ' are equivalent implies there exists an isomorphism $\phi : \mathfrak{g}[[t]] \to \mathfrak{g}[[t]]$ of the form $\phi = id_{\mathfrak{g}} + \phi_1 t$ such that $\rho'(t) = \phi^{-1} \circ \rho(\phi \otimes \phi)$. Evaluating both sides gives for all $a, b \in \mathfrak{g}$:

$$\begin{aligned} \rho_0(a \wedge b) + \rho_1'(a \wedge b)t =& (id_{\mathfrak{g}} + g_1 t)\left((\rho_0 + \rho_1 t)(a + g_1(a)t \wedge b + g_1(b)t)\right) \\ =& \rho_0(a \wedge b) + \left(\rho_0(a \wedge g_1(b)) + \rho_0(g_1(a) \wedge b) - g_1 \circ \rho_0(a \wedge b)\right)t + \rho_1(a \wedge b)t. \end{aligned}$$

It follows that

$$\rho_1'(a \wedge b) = d(g_1)(a \wedge b) + \rho_1(a \wedge b).$$

Since ρ_1 is a cocycle then we have $[\rho_1'] = [\rho_1]$ in $H^2(\mathfrak{g}, \mathfrak{g})$.

Now the converse statement. Let $[b], [b'] \in H^2(\mathfrak{g}, \mathfrak{g})$ such that $[b'] = [b]$. By definition there exists an $h \in C(\mathfrak{g}, \mathfrak{g})[1]^0$ such that $b'(a \wedge b) = d(h)(a \wedge b) + b(a \wedge b)$ for all $a, b \in \mathfrak{g}$. By reversing the previous deduction it follows that $\rho_0 + b't = (id_{\mathfrak{g}} - ht) \circ (\rho_0 + bt)(id_{\mathfrak{g}} + ht \wedge id_{\mathfrak{g}} + ht)$. Thus h defines an equivalence between $\rho := \rho_0 + bt$ and $\rho' := \rho_0 + b't$. □

An immediate corollary is that if $H^2(\mathfrak{g}, \mathfrak{g}) = [0]$ then all infinitesimal deformations are trivial.

2.4.4 Obstructions

As an analogy to the associative case it will be investigated when an $n-$deformation extends to an $(n+1)-$deformation. The problem will be approached in the same way as in the associative case, i.e. D_k will be split into a coboundary part and a remaining part which will be called obstruction. The next step is to show that this obstruction is actually in $H^3(\mathfrak{g}, \mathfrak{g})$.

An extension of an $n-$deformation $\rho = \sum_{k=0}^{n} \rho_k t^k$ to an $(n+1)-$deformation is an operation $\rho' = \rho + \rho_{n+1} t^{n+1}$ such that ρ' satisfies D_{n+1} for Lie algebras. The operation ρ' satisfies D_{n+1} if for all $a, b, c \in \mathfrak{g}$, it holds that

$$
\begin{aligned}
\sum_{\sigma} \sum_{i+j=n+1, i,j \geq 0} \rho_i(\rho_j(a \wedge b) \wedge c) = & \sum_{i+j=k, i,j \geq 0} \rho_i(\rho_j(a \wedge b) \wedge c) + \rho_i(\rho_j(c \wedge a) \wedge b) + \rho_i(\rho_j(b \wedge c) \wedge a) \\
= & - \rho(a \wedge \rho_{n+1}(b \wedge c)) + \rho(b \wedge \rho_{n+1}(a \wedge c)) - \rho(c \wedge \rho_{n+1}(a \wedge b)) + \\
& - \rho_{n+1}(a \wedge \rho(b \wedge c)) + \rho_{n+1}(b \wedge \rho(a \wedge c)) - \rho_{n+1}(c \wedge \rho(a \wedge b)) + \\
& + \sum_{i+j=k, i,j > 0} \rho_i(\rho_j(a \wedge b) \wedge c) + \rho_i(\rho_j(c \wedge a) \wedge b) + \rho_i(\rho_j(b \wedge c) \wedge a) \\
= & - \rho(a \wedge \rho_{n+1}(b \wedge c)) + \rho(b \wedge \rho_{n+1}(a \wedge c)) - \rho(c \wedge \rho_{n+1}(a \wedge b)) + \\
& + \rho_{n+1}(\rho(a \wedge b) \wedge c) - \rho_{n+1}(\rho(a \wedge c) \wedge b) - \rho_{n+1}(\rho(b \wedge c) \wedge a) + \\
& + \sum_{i+j=k, i,j > 0} \rho_i(\rho_j(a \wedge b) \wedge c) + \rho_i(\rho_j(c \wedge a) \wedge b) + \rho_i(\rho_j(b \wedge c) \wedge a) \\
= & d(\rho_{n+1})(a \wedge b \wedge c) + \\
& + \sum_{i+j=k, i,j > 0} \rho_i(\rho_j(a \wedge b) \wedge c) + \rho_i(\rho_j(c \wedge a) \wedge b) + \rho_i(\rho_j(b \wedge c) \wedge a).
\end{aligned}
$$

So D_{n+1} can be written as a coboundary plus another term. Define

$$
\mathfrak{O}_n(a \wedge b \wedge c) := \sum_{i+j=k, i,j > 0} \rho_i(\rho_j(a \wedge b) \wedge c) + \rho_i(\rho_j(c \wedge a) \wedge b) + \rho_i(\rho_j(b \wedge c) \wedge a)
$$

for all $a, b, c \in \mathfrak{g}$ to be the obstruction of order n. Obstructions can be seen to be cocycles in the same way as in the associative case (see 2.5), thus $[\mathfrak{O}_n]$ is an element of $H^3(\mathfrak{g}, \mathfrak{g})$. This leads to the following theorem.

Theorem 2.4.2. *For any Lie algebra $\mathfrak{g}$, an n-deformation can be extended to an $(n+1)$-deformation if and only if $[\mathfrak{O}_n] = [0] \in H^3(\mathfrak{g}, \mathfrak{g})$.*

Proof. Let ρ be an n-deformation and ρ' be an extension of ρ. Then there exists ρ_{n+1} such that $\rho' = \rho_0 + \rho_{n+1} t^{n+1} = \left(\sum_{k=0}^{n} \rho_k t^k\right) + \rho_{n+1} t^{n+1}$. Since ρ' is an extension ρ_{n+1} satisfies D_{n+1} which means $d(\rho_{n+1}) + \mathfrak{O}_n = 0$. This shows that $[\mathfrak{O}_n] = [0]$ in $H^3(\mathfrak{g}, \mathfrak{g})$.

Suppose $[\mathfrak{O}_n] = [0]$ in $H^3(\mathfrak{g}, \mathfrak{g})$. Then there exists a $\varphi \in \bigwedge^2(\mathfrak{g}, \mathfrak{g})$ such that $d(\varphi) = 0$. This φ will be the candidate multiplication extending ρ. In order to satisfy the Jacobi identity φ has to satisfy D_{n+1}. Using the fact that the equation D_{n+1} can be written as $d(\varphi) + \mathfrak{O}_n = 0$ it is clear that φ satisfies D_{n+1}. Therefore define $\rho' := \rho + \varphi t^{p+1}$. This operation is an $(n+1)$-deformation of ρ. □

If $H^3_{CE}(\mathfrak{g}, \mathfrak{g}) = 0$, then there are no obstructions implying that each infinitesimal deformation extends to a formal deformation.

2.5 Maurer-Cartan Equation

As in the associative case it can be shown that an element $\rho \in C[1]^1(\mathfrak{g},\mathfrak{g})$ satisfies the Jacobi identity if and only if $[\rho,\rho]=0$. Suppose $\rho \in C[1]^1(\mathfrak{g},\mathfrak{g})$ satisfies the Jacobi identity. It follows that

$$\begin{aligned} 0 =& 2\left(\rho(\rho(v_1,v_2),v_3)+\rho(\rho(v_3,v_1),v_2)+\rho(\rho(v_2,v_3),v_1)\right)\\ =& 2(\rho\circ\rho)(v_1,v_2,v_3)\\ =& [\rho,\rho](v_1,v_2,v_3). \end{aligned}$$

For the other direction suppose that $[\rho,\rho]=0$. Then

$$\begin{aligned} 0 =& [\rho,\rho](v_1,v_2,v_3)\\ =& \rho\circ\rho(v_1,v_2,v_3)-(-1)^{1\cdot 1}\rho\circ\rho(v_1,v_2,v_3)\\ =& 2(\rho\circ\rho)(v_1,v_2,v_3)\\ =& 2\left(\sum_{\sigma\in S_3}(-1)^{deg(\sigma)}\rho(\rho(v_{\sigma(1)},v_{\sigma(2)}),v_{\sigma(3)})\right)\\ =& 2\left(\rho(\rho(v_1,v_2),v_3)+\rho(\rho(v_2,v_3),v_1)+\rho(\rho(v_3,v_1),v_2)\right), \end{aligned}$$

thus ρ satisfies the Jacobi identity.

In case ρ can be written as $\sum_{k=0}^{\infty}\rho_k t^k$ with $\rho_0=\rho$, then ρ satisfying the Jacobi identity means:

$$0=\sum_{\sigma}\rho(\rho(v_1,v_2),v_3)=\sum_{\sigma}\sum_{i=0}^{\infty}\rho_i t^i(\sum_{j=0}^{\infty}\rho_j t^j(v_1,v_2),v_3)=\sum_{\sigma}\sum_{i,j}\rho_i(\rho_j(v_1,v_2),v_3)t^{i+j},$$

where the first sums run over all cyclic permutations of S_3. Thus each t-level has to be zero, and so

$$D_k:\ \sum\sum_{i+j=k,i,j\geq 0}\rho_i(\rho_j(v_1,v_2),v_3)=0.$$

On the other hand $\left[\sum_{i=0}^{\infty}\rho_i t^i,\sum_{j=0}^{\infty}\rho_j t^j\right]=0$ and using the fact that $d(\rho_k)=[\rho_0,\rho_k]$, it gives:

$$0=\left[\sum_{i=0}^{\infty}\rho_i t^i,\sum_{j=0}^{\infty}\rho_j t^j\right]=\sum_{k=0}^{\infty}\sum_{i+j=k}[\rho_i,\rho_j]t^k=2d(\rho_k)+\sum_{k=1}^{\infty}\sum_{\substack{i+j=k\\ i,j>0}}[\rho_i,\rho_j]t^k.$$

Thus the condition for an infinitesimal deformation $\rho=\rho_0+\rho_1 t$ to satisfy the Jacobi identity translates into the requirement that

$$d(\rho_1)+\frac{1}{2}[\rho_1,\rho_1]=0$$

should hold, which is the Maurer-Cartan equation.

Note that it also holds that $\frac{1}{2}\sum_{i+j=k,\ i,j>0}[\rho_i,\rho_j]$ gives the k^{th} obstruction $\mathfrak{O}_k$. It immediately follows from 1.5.1 that $\mathfrak{O}_k$ is a cocycle since in the prove only the dg-Lie algebra structure has been used. The shifted Chevalley-Eilenberg complex forms a dg-Lie algebra hence the result.

In the Lie algebra case the group $G_{\mathfrak{g}}(R)$ acts on $MC_{\mathfrak{g}}(R)$ for $R=k[t]/(t^2)$ and $\mathfrak{g}=C^1_{CE}[1](\mathfrak{g},\mathfrak{g})$. Thus the following theorem holds.

Theorem 2.5.1. *$Def_{\mathfrak{g}}(R)\cong H^2_{CE}(A,A)$.*

2.6 Relation between Lie and associative algebras

Given an associative algebra it is possible to define a Lie algebra structure on the underlying vector space by defining the bracket to be the commutator. The other way around, the universal enveloping algebra of a Lie algebra gives an associative algebra. Note that a morphism between associative algebras respects the commutator property, thus $f([a,b]) = [f(a), f(b)]$ giving rise to a Lie algebra morphism. The other way around, a Lie algebra morphism gives rise to an associative algebra morphism which is the unique morphism obtained by the universal property of the enveloping algebras. In fact this defines a functor $\mathcal{L}$ from the category of associative algebras to the category of Lie algebras and a functor $\mathcal{U}$ the other way around. These functors are adjoint, i.e.

$$Hom_{\mathbf{Ass}}(\mathcal{U}(\mathfrak{g}), A) \cong Hom_{\mathbf{Lie}}(\mathfrak{g}, \mathcal{L}(A))$$

which follows immediately from the universality of the universal enveloping algebra.

There is an even further reaching relation: in [29] it has been shown that $H_{CE}(\mathfrak{g},\mathfrak{g})$ is isomorphic to $H_{\mathrm{Hoch}}(\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g}))$. Thus every deformation theory of Lie algebras gives rise to a deformation theory of their universal enveloping algebras and vice versa.

2.7 Deformation of Lie algebra morphisms

The deformation of a Lie algebra morphism $f : (\mathfrak{g}, [-,-]) \to (\mathfrak{h}, [-,-]')$ is defined to be a morphism $f_t : (\mathfrak{g}_t, [-,-]_t) \to (\mathfrak{h}_t, [-,-]'_t)$ between two deformations of $\mathfrak{g}$ and $\mathfrak{h}$, respectively. The family of morphism associated to a formal deformation of a Lie algebra morphism have to satisfy:

$$\begin{aligned} f_k(rx+sy) &= rf_k(x) + sf_k(y) \\ \sum_{\substack{k+i+j=q \\ k,i,j\geq 0}} [f_i(x), f_j(y)]'_k - \sum_{\substack{l+p=q \\ l,p\geq 0}} f_l([x,y]_p) &= 0. \end{aligned}$$

There is no fundamental difference between deforming Lie algebra morphisms and associative algebra morphisms. In fact everything can be developed in exactly the same way.

The deformation complex for a Lie algebra morphism $f : \mathfrak{g} \to \mathfrak{h}$ is given by

$$C^n(f) := C^n(\mathfrak{g},\mathfrak{g}) \times C^n(\mathfrak{h},\mathfrak{h}) \times C^{n-1}(\mathfrak{g},\mathfrak{h})$$

with the differential

$$d((\rho_1, \varrho_1, f_1)) := (d_{CE}(\rho_1), d_{CE}(\varrho_1), f \circ [-,-] - [-,-]' \circ f^n - d_{CE}(f_1)).$$

Chapter 3

Deformation Theory of Linear Categories

An associative algebra can be considered to be a category with one object by interpreting the elements of the algebra as morphisms between some objects. This suggests a way to deform categories. In order to do this the Hom-sets should form vector spaces, therefore linear categories will be introduced. The deformation of an arbitrary locally small category can be obtained by linearizing the category and then use the deformation theory of linear categories as will be developed now. The deformation of functors will be done in more generality in chapter 5.

3.1 Deformation

In this section the language of deformation theory will be developed in the context of categories. First categories and linear categories will be introduced and then deformations of linear categories will be defined. Everything will be done completely analogous to the case of deformations of associative algebras.

3.1.1 Categories

For completeness the definition of a category will be given.

Definition 3.1.1. *A category $\mathcal{C}$ consists of a class of objects Obj($\mathcal{C}$), a class of morphisms Mor($\mathcal{C}$), two maps dom, cod* $: Mor(\mathcal{C}) \to Obj(\mathcal{C})$ *assigning to each morphism a domain and codomain, for each object A in Obj($\mathcal{C}$) there exists a morphism id_A in Mor($\mathcal{C}$) called the identity morphism, and a composition $\circ : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ which is associative and respects the identity.*

Denote the morphisms with domain A and codomain B by $Hom_{\mathcal{C}}(A, B)$. A category is locally small if $Hom_{\mathcal{C}}(A, B)$ forms a set for all objects A and B. If in addition the objects form a set, then the category is called small.

Example 3.1.1.

Set: *Objects are sets and morphisms are functions. The composition is the ordinary composition of functions.*

*R-**Mod***: *Objects are $R-$modules and morphisms are $R-$module morphisms. The composition is the composition of functions.*

uAss$_k$: *Objects are unital associative k-algebras and morphisms are k-algebra morphisms. The composition is the composition of functions.*

***Lie**: Objects are Lie-algebras and the morphisms are Lie-algebra morphisms. The composition is the composition of functions.*

***Cat**: Objects are small categories and morphisms are functors. The composition is given by the composition of functors.*

3.1.2 Linear Categories

An R-linear category $\mathcal{C}$ is a category where the Hom-sets are R-modules and the composition is a bilinear map. Let $\mathcal{C}$ and $\mathcal{D}$ be two R-linear categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is R-linear if $F(Hom_{\mathcal{C}}(A, B))$ is an R-module, for all A and B in $\mathcal{C}$. The category of all R-linear small categories is denoted by R-**Cat**.

An arbitrary locally small category $\mathcal{C}$ can be linearized by extending each set of morphism with formal k-linear combinations of morphism and extending the composition bilinearly. For small categories this gives an adjunction: $L : \mathbf{Cat} \rightleftarrows \text{k-}\mathbf{Cat} : U$ where U is the obvious forgetful functor.

Another way of describing linear categories, which will be of importance later, is as follows. Define a functor $\mathbb{S} : \mathbf{Set} \to \mathbf{Cat}$ which sends a set S to the discrete category whose objects are $S \times S$. Let $\mathcal{E}$ be a symmetric monoidal category. A collection in $\mathcal{E}$ is defined to be

$$\mathbf{Coll}(\mathcal{E}, S) := [\mathbb{S}(S), \mathcal{E}].$$

A functor $C \in \text{Coll}(\mathcal{E}, S)$ picks for each pair of objects a Hom-object in $\mathcal{E}$.

Definition 3.1.2. *An $\mathcal{E}$-category is a set of objects S and a collection $C : \mathbb{S}(S) \to \mathcal{E}$, together with maps $I \to C((x, x))$ and arrows in $\mathcal{E}$*

$$\circ_{x,y,z} : C((x, y)) \otimes_{\mathcal{E}} C((y, z)) \to C((x, z)),$$

such that they are associative and satisfy the identity law.

A k-linear category is then just a $\mathbf{Mod}_k$-category. Note that this definition coincides with the above one. The advantage of this description is the more transparent role of the 'hosting' category $\mathcal{E}$.

3.1.3 Augmentation

Let (R, ε) be an augmentation of k and let $C : \mathbb{S}(S) \to \mathbf{Mod}_k$ be a collection. By post-composition with the functor $- \otimes_k R : \mathbf{Mod}_k \to \mathbf{Mod}_R$ a new, augmented, collection $C \otimes_k R : \mathbb{S}(S) \to \mathbf{Mod}_R$ is obtained. An augmented category is a category structure on the augmented collection. This assignment of a k-linear category to an R-linear one is functorial and gives rise to an adjunction $- \otimes_k R : k\text{-}\mathbf{Cat} \rightleftarrows R\text{-}\mathbf{Cat} : - \otimes_R k$.

3.1.4 Deformations

Definition 3.1.3. *Let (R, ε) an augmentation of k. A deformation of a k-linear category $\mathcal{C}$ is an R-linear category $\mathcal{D}$ together with an isomorphism $\mathcal{D} \otimes_R k \xrightarrow{\alpha} \mathcal{C}$. An R-deformation of $\mathcal{C}$ is a category $\mathcal{C} \otimes_k R$ together with a functor $can : (\mathcal{C} \otimes_k R) \otimes_R k \to \mathcal{C}$.*

In case of a local ring $(R, \mathfrak{m})$ an R-deformation is an R-linear category $\mathcal{C} \otimes_k R$, such that modulo the maximal ideal $\mathfrak{m}$ the composition and identity of $\mathcal{C}$ are obtained. In case of the rings $k[[t]]$ or $k[t]/(t^n)$, with $n \in \mathbb{N}_0$, the corresponding deformation is called a formal or an n-deformation, respectively.

If R is the module $k[[t]]$, then the morphisms in $Hom_k(A, B) \otimes_k k[[t]]$ can be written as $f \otimes \sum_i a_i t^i$, or in case $Hom_k(A, B)$ has a finite dimension, as $\sum_i f_i t^i$. As for associative algebras, a composition

$$m_t^{A,B,C} : Hom(A, B)[[t]] \otimes_{k[[t]]} Hom(B, C)[[t]] \to Hom(A, C)[[t]]$$

gives rise to a family of maps

$$m_i^{A,B,C} : Hom(A,B) \otimes_k Hom(B,C) \to Hom(A,C)$$

making the diagram

$$\begin{array}{ccc} Hom(A,B) \otimes_k Hom(B,C) & \xrightarrow{m_i^{A,B,C}} & Hom(A,C) \\ \downarrow{\scriptstyle \iota\otimes\iota} & \searrow{\scriptstyle m_t|_{Hom(A,B)\otimes Hom(B,C)}} & \uparrow{\scriptstyle ev_0 \circ t^{-k}} \\ Hom(A,B)[[t]] \otimes_{k[[t]]} Hom(B,C)[[t]] & \xrightarrow[m_t^{A,B,C}]{} & Hom(A,C)[[t]] \end{array}$$

commute, for all $k \geq 0$ and for all objects A, B, C. On the other hand, given a family of operations, in order for the sum $m_t := \sum_{i\geq 0} m_i t^i$ to be a composition of a category the operations have to satisfy certain conditions, namely:

$$\text{reduction to original composition: } m_0^{A,B,C} = \circ_{A,B,C}$$

$$\text{left identity axiom: } 0 = \sum_{\substack{i+j=k \\ i,j\geq 0}} m_i^{A,A,B}(id_A, f) \qquad , k \geq 1$$

$$\text{right identity axiom: } 0 = \sum_{\substack{i+j=k \\ i,j\geq 0}} m_i^{A,B,B}(f, id_B) \qquad , k \geq 1$$

$$\text{associativity: } 0 = \sum_{\substack{i+j=k \\ i,j\geq 0}} \Big(m_i^{A,B,D}(f, m_j^{B,C,D}(g,h)) - m_i^{A,C,D}(m_j^{A,B,C}(f,g),h) \Big) \qquad , m \geq 1$$

for all objects A, B, C, D of $\mathcal{C}$.

3.1.5 Equivalence of deformations

Two $(R, \varepsilon)-$deformations of a $k-$linear category are equivalent if there is an isomorphism between them which reduces modulo $\mathfrak{m} = \text{Kern}(\varepsilon)$ to the identity functor. With the notion of equivalences of deformations it is possible to state a similar result of theorem 1.1.1 for categories.

Theorem 3.1.1. *Let $(\mathcal{C}[[t]], \circ_t, id_t)$ be a deformation of the linear category $(\mathcal{C}, \circ, id)$, where id_t is the deformed identity. Then there exists a deformation $(\mathcal{C}[[t]], \circ', id)$ which is equivalent to $(\mathcal{C}[[t]], \circ_t, id_t)$.*

The analogue of α in the proof of theorem 1.1.1 is given by the functor $A : \mathcal{C}[[t]] \to \mathcal{C}[[t]]$ acting as an identity on the objects and sending a morphism f to $f \circ_t id_{dom(f)}$. The composition $\circ'$ is defined to be $f \circ' g := (f \circ_t id^{-1}_{dom(f)}) \circ_t g$. With these choices the proof of this theorem is done in exactly the same way as for algebras and will thus be omitted.

A deformation with the composition given by $\circ + \sum_{i\geq 1} 0t^i$ is called a trivial deformation. A deformation equivalent to the trivial deformation is also called trivial.

3.2 Hochschild Complex

The similarity of deformations of linear categories to associative algebras suggests that we can use the same deformation complex, i.e. the Hochschild complex, to classify infinitesimal deformations.

Let $\mathcal{C}$ be an k-linear category. The Hochschild cochain complex is defined by the k-modules

$$C^0_{\text{Hoch}}(\mathcal{C}) := \coprod_{A\in Obj(\mathcal{C})} \mathcal{C}(A,A)$$
$$C^n_{\text{Hoch}}(\mathcal{C}) := \coprod_{A_0,\dots,A_n\in Obj(\mathcal{C})} Hom_{k\text{-}\mathbf{Mod}}\left(\mathcal{C}(A_0,A_1)\otimes_k \dots \otimes_k \mathcal{C}(A_{n-1},A_n), \mathcal{C}(A_0,A_n)\right).$$

Let $C_{\text{Hoch}}(\mathcal{C})$ be the direct sum of these modules. The differential $d : C^n(\mathcal{C}) \to C^{n+1}(\mathcal{C})$ is defined by:

$$\begin{aligned} d(\varphi)(f_{0,1}\otimes\dots\otimes f_{n,n+1}) =& m_{0,1,n+1}(f_{0,1},\varphi(f_{1,2}\otimes\dots\otimes f_{n,n+1}))+ \\ &+\sum_{i=1}^{n}(-1)^i\varphi(f_{0,1}\otimes\dots\otimes m_{i-1,i,i+1}(f_{i-1,i},f_{i,i+1})\otimes\dots\otimes f_{n,n+1})+ \\ &+(-1)^{n+1}m_{0,n,n+1}(\varphi(f_{0,1}\otimes\dots\otimes f_{n-1,n}),f_{n,n+1}). \end{aligned}$$

An alternative description of the Hochschild complex (making later generalizations easier) is by the use of the nerve functor:

$$N : \mathbf{Cat}_k \to \mathbf{Mod}_k^{\Delta^{op}},$$

sending a category to the simplicial object $N(\mathcal{C}) : \Delta^{op} \to \mathbf{Mod}_k$. Here Δ denotes the simplex category. The simplicial object $N(\mathcal{C})$ sends totally ordered finite sets $[n]$ to $N(\mathcal{C})_n$, the $k-$module of $n-$composable arrows. By definition the zero-composable arrows $N(\mathcal{C})_0$ are the objects of $\mathcal{C}$. The face maps $d_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n-1}$ are defined by:

$$d_i : \left(A_0 \to \dots \to A_{i-1} \xrightarrow{f_{i-1}} A_i \xrightarrow{f_i} A_{i+1} \to \dots \to A_n\right) \mapsto \left(A_0 \to \dots \to A_{i-1} \xrightarrow{f_i\circ f_{i-1}} A_{i+1} \to \dots \to A_n\right)$$

and the degeneracy maps $s_i : N(\mathcal{C})_n \to N(\mathcal{C})_{n+1}$ by:

$$s_i : (A_0 \to \dots \to A_{i-1} \to A_i \to A_{i+1} \to \dots \to A_n) \mapsto \left(A_0 \to \dots \to A_{i-1} \to A_i \xrightarrow{id_{A_i}} A_i \to A_{i+1} \to \dots \to A_n\right).$$

The Hochschild complex becomes then

$$C^n(\mathcal{C}) = \{\varphi \in Hom_{\mathbf{Mod}_k}(N\mathcal{C}_n, N\mathcal{C}_1) \mid \forall\theta\in N\mathcal{C}_n : dom(\varphi(\theta)) = dom(\theta) \wedge cod(\varphi(\theta)) = cod(\theta)\}$$

with

$$d_{\text{Hoch}}\varphi = d_1\circ(id\otimes\varphi) + \sum_{i=1}^{n}(-1)^i\varphi\circ d_i + (-1)^{n+1}d_1\circ(\varphi\otimes id).$$

Note that this complex for an associative algebra viewed as a category with one object coincides with the Hochschild complex for associative algebras. Further note that the Hochschild complex does not come from a simplicial object, i.e. there is no simplicial object where the alternating sum of the face maps gives the Hochschild complex, because of the first and last term in the differential.

3.2.1 Differential graded Lie algebra structure

Because of the similarity to the associative case it comes as no surprise that the Hochschild complex for linear categories gives rise to a preLie system and preLie algebra, a dg-Lie algebra and even a brace algebra.

The circle i-operations are again defined by

$$\begin{aligned} &\theta\circ_i\psi(f_{01}\otimes\dots\otimes f_{m+n-2m+n-1}) := \\ &\quad\theta(f_{01}\otimes\dots\otimes f_{i-2i-1}\otimes\psi(f_{i-1i}\otimes\dots\otimes f_{i+n-2i+n-1})\otimes f_{i+n-1i+n}\otimes\dots\otimes f_{m+n-2m+n-1}). \end{aligned}$$

A straightforward calculation shows that this indeed defines a preLie-system on $C[1](\mathcal{C})$. It follows that $(C[1](\mathcal{C}), \circ)$ with $\circ := {}_i\circ_i$ is a preLie algebra and thus the Gerstenhaber bracket turns $C[1](\mathcal{C})$ into a graded Lie algebra. This is the analogue of the Gerstenhaber bracket. For this bracket it also holds that d_{Hoch} is equal to $[\circ, -]$ and the cochain complex $(C_{\text{Hoch}}(\mathcal{C})[1], [-,-], [\circ, -])$ forms a dg-Lie algebra. Consequently the cohomology $(H(C_{\text{Hoch}}(\mathcal{C})[1], d_{\text{Hoch}}), [-,-])$ forms a graded Lie algebra. Note that in case a category consists of just one object, all the constructions give rise to the structures as defined for associative algebras.

3.2.2 Normalized Hochschild Complex

Candidates of infinitesimal deformations are given by $C[1](\mathcal{C})^1$ in the sense that an element $m \in C[1](\mathcal{C})^1$ together with the original composition $\circ$ give a deformed composition $\circ + mt$, which is associative. In order to be a composition of a category it should also satisfy the identity axioms.

The left and right identity condition in (3.2) show that as soon as one identity morphism appears in the composition the differential should be zero. In order to account for this property, the normalized Hochschild complex (see [34]) will be considered.

Define an k-normalized cochain complex $C_k[1](\mathcal{C})$ to consist of cochains of $C[1](\mathcal{C})$ satisfying the condition that if at least one of the first k composable arrows is an identity, the result becomes zero. It is easily seen that $C_k[1](\mathcal{C})^n \subseteq C(\mathcal{C})[1]^n$. To check that the differential of $C[1](\mathcal{C})$ restricted to the k-cochains is again a differential it has to be checked that for a k-cochain ψ it holds that $d\psi$ is again a k-cochain.

$$\begin{aligned} d\psi(id \otimes f_{12} \otimes \ldots \otimes f_{nn+1}) =& id \circ \psi(f_{12} \otimes \ldots \otimes f_{nn+1}) - \psi(id \circ f_{12} \otimes \ldots \otimes f_{nn+1}) + \\ &+ \sum_{i=2}^{n} (-1)^i \psi(id \otimes \ldots f_i \circ f_{i+1} \otimes \ldots \otimes f_{nn+1}) + \\ &+ (-1)^{n+1} \psi(id \otimes f_{12} \otimes \ldots \otimes f_{n-1n}) \circ f_{nn+1} \\ =& \psi(f_{12} \otimes \ldots \otimes f_{nn+1}) - \psi(f_{12} \otimes \ldots \otimes f_{nn+1}) \\ =& 0. \end{aligned}$$

If the identity appears between 2 and $k-1$ then everything is just zero. In case it appears at the k^{th} place the terms

$$(-1)^n \psi(f_{01} \otimes \ldots \otimes f_{n-1n} \circ f_{nn+1}) + (-1)^{n+1} \psi(f_{01} \otimes \ldots \otimes f_{n-1n}) \circ f_{nn+1}$$

cancel each other, thus are zero too. This shows that $(C_k[1](\mathcal{C}), d)$ is a complex. It holds that $C_0[1](\mathcal{C}) \subseteq C[1](\mathcal{C})$ and $C_{k+1}(\mathcal{C})[1] \subseteq C_k(\mathcal{C})[1]$. Define $\overline{C}[1](\mathcal{C}) := \bigcap_{k \geq 0} C_k[1](\mathcal{C})$. This complex is called the normalized Hochschild complex.

Proposition 3.2.1. *The complex $\overline{C}[1](\mathcal{C})$ is a chain deformation retraction of $C[1](\mathcal{C})$.*

Proof. In order to prove that $\overline{C}[1](\mathcal{C})$ is a chain deformation retraction of $C[1](\mathcal{C})$, it has to be shown that there exist maps $f : \overline{C}[1](\mathcal{C}) \to C[1](\mathcal{C})$ and $g : C[1](\mathcal{C}) \to \overline{C}[1](\mathcal{C})$, such that $id_{C[1](\mathcal{C})} \sim f \circ g$ and $g \circ f = id_{\overline{C}[1](\mathcal{C})}$. This will be done by proving that for each i the i-normalized complex is a deformation retraction of the $(i-1)-$normalized complex. The proof concludes then by the composition of the deformation retractions which is again a deformation retraction.

Define $s_k : C[1](\mathcal{C})^{n-1} \to C(\mathcal{C})[1]^{n-2}$ by

$$s_k(\psi)(f_{01} \otimes \ldots \otimes f_{n-2n-1}) := \begin{cases} \psi(f_{01} \otimes \ldots \otimes 1 \otimes \ldots \otimes f_{n-2n-1}) & k \leq n \\ 0 & k > n. \end{cases}$$

Now define $h_k : C(\mathcal{C})[1]^{n-1} \to C[1](\mathcal{C})^{n-1}$ by

$$h_k = id_{C[1](\mathcal{C})} - d \circ s_k - s_k \circ d.$$

It will be proved that $h_k : C[1](\mathcal{C}) \to C_0[1](\mathcal{C})$. Suppose that ψ does not vanish for any identity. The aim is to show that $h_0(\psi)$ is a 0-normalized cochain.

$$
\begin{aligned}
&h_0(\psi)(id, f_{12}, \dots, f_{n-1n}) = \psi(id, f_{12}, \dots, f_{n-1n}) - d(\psi)(id, id, f_{12}, \dots, f_{n-1n}) - s_0 \circ d(\psi)(id, f_{12}, \dots, f_{n-1n}) \\
&=\psi(id, f_{12}, \dots, f_{n-1n}) \\
&\quad \begin{cases} -\psi(id, f_{12}, \dots, f_{n-1n}) \\ +\psi(id, f_{12}, \dots, f_{n-1n}) - \psi(id, f_{12}, \dots, f_{n_1 n}) + \sum_{i=3}^{n} (-1)^{i+1} \psi(id, id, f_{12}, \dots, (f_{i-1i} \circ f_{ii+1}), \dots f_{n-1n}) \\ +(-1)^n \psi(id, f_{12}, \dots, f_{n-2n-1}) f_{n-1n} \end{cases} \\
&\quad \begin{cases} -\psi(id, f_{12}, \dots, f_{n-1n}) \\ +\psi(id, f_{12}, \dots, f_{n-1n}) + \sum_{i=2}^{n-1} (-1)^{i+1} \psi(id, id, f_{12}, \dots, (f_{i-1i} \circ f_{ii+1}), \dots f_{n-1n}) \\ +(-1)^{n+1} \psi(id, f_{12}, \dots, f_{n-2n-1}) f_{n-1n} \end{cases} \\
&=0.
\end{aligned}
$$

A similar calculation shows that $h_k : C_l[1](\mathcal{C}) \to C_{l+1}[1](\mathcal{C})$ for an arbitrary l. Consider $id_{C[1](\mathcal{C})} - \iota_0 \circ h_0$. Then

$$
id_{C(\mathcal{C})}[1](\psi) - \iota_0 \circ h_0(\psi) = \psi - (\psi - d \circ s_0(\psi) - s_0 \circ d(\psi)) = d \circ s_0(\psi) - s_0 \circ d(\psi),
$$

and therefore s_0 is indeed a chain homotopy between $id_{C[1](\mathcal{C})}$ and $\iota_0 \circ h_0$. On the other hand consider $h_0 \circ \iota_0$. Note that $s_0(\iota_0(\psi)) = 0$ and $s_0(d(\psi)) = 0$, thus $h_0 \circ \iota_0(\psi) = \psi - d \circ s_0(\psi) - s_0(d(\psi)) = \psi$. This proves that $C[1](\mathcal{C})$ is a chain deformation retraction onto $C_0[1](\mathcal{C})$.

That $C_k[1](\mathcal{C})$ is a chain deformation retraction of $C_{k-1}[1](\mathcal{C})$ with chain homotopy s_k is shown in analogous manner. It is easily seen that $C_1[1](\mathcal{C})$ is a chain deformation retraction of $C[1](\mathcal{C})$, since $s_0 + \iota_0 \circ s_1 \circ h_0$ is a chain homotopy between id and $\iota_0 \circ \iota_1 \circ h_1 \circ h_0$ and $h_1 \circ h_0 \circ \iota_0 \circ \iota_1 = id$. Define $\bar{\iota} := \iota_0 \circ \iota_1 \circ \dots$ then $\overline{h} := \dots \circ h_1 \circ h_0$ and $\overline{s} := s_0 + s_1 \circ h_1 + s_2 \circ h_1 \circ h_0 + \dots$. This gives then the desired chain deformation retraction between $\overline{C}[1](\mathcal{C})$ and $C[1](\mathcal{C})$. □

Since cohomology can not distinguish between homotopic complexes the cohomology of the normalized is isomorphic to the cohomology of the unnormalized Hochschild complex. Therefore in the classification the unnormalized chain complex can be used.

3.3 Classification

3.3.1 Center

Let $\varphi \in C^0(\mathcal{C})$ and $f_{0,1} \in \mathcal{C}(A_0, A_1)$.

$$
d\varphi(f_{0,1}) = m_{0,0,1}(\varphi, f_{0,1}) - m_{0,1,1}(f_{0,1}, \varphi)
$$

where the first $\varphi \in \mathcal{C}(A_0, A_0)$ and the second $\varphi \in \mathcal{C}(A_1, A_1)$. Remember that φ is in fact a sum of morphisms $\varphi_A \in \mathcal{C}(A, A)$ over all objects. The only non-zero morphisms are the composable ones hence the result.

3.3.2 Infinitesimal deformations

Let $\varphi \in C^2(\mathcal{C})$. Then

$$
\begin{aligned}
d\varphi(f_{0,1} \otimes f_{1,2} \otimes f_{2,3}) =& m_{0,1,3}(f_{0,1}, \varphi(f_{1,2} \otimes f_{2,3})) - \\
&- \varphi(m_{0,1,2}(f_{0,1}, f_{1,2}) \otimes f_{2,3}) + \varphi(f_{0,1} \otimes m_{1,2,3}(f_{1,2}, f_{2,3})) - \\
&- m_{0,2,3}(\varphi(f_{0,1} \otimes f_{1,2}), f_{2,3}).
\end{aligned}
$$

Proposition 3.3.1. *The infinitesimal deformations of a category $\mathcal{C}$ are classified up to equivalence by the second Hochschild cohomology group $H^2(\mathcal{C})$.*

Proof. The proof is basically the same as in the associative case. Suppose two deformations are equivalent, i.e. $[c'] = [c] \in H^2(\mathcal{C})$. Then they differ by a coboundary which vanishes in cohomology. Thus in cohomology they are in the same class.

On the other hand, suppose that $[c] \in H^2(\mathcal{C})$. Define $m_{A,B,C}(f,g) + c_{A,B,C}(f,g)t$ for all $f \in \mathcal{C}(A,B)$ and $g \in \mathcal{C}(B,C)$ and for all $A, B, C \in Obj(\mathcal{C})$. The composition $m + ct$ is associative if it satisfies condition D_0 and D_1. D_0 is always satisfied since m is the original composition and D_1 is satisfied if $d(c) = 0$, but this is true since c is a cocycle. □

3.3.3 Obstructions

The obstructions for extending an n-deformation to an $(n+1)$-deformation is given by

$$\mathcal{O}_n = \sum_{\substack{i+j=n \\ i,j>0}} m_i \circ m_j.$$

It follows that $\mathcal{O}_n = \frac{1}{2} \sum_{\substack{i+j=n \\ i,j>0}} [m^i, m^j]$. The Hochschild complex for categories is a dg-Lie algebra thus by proposition 1.5.1 and the fact that in its proof only the dg-Lie algebra structure has been used, it follows that the obstruction is a cocycle. It should then come as no surprise that a similar result as theorem 1.4.2 holds for categories.

Theorem 3.3.1. *For any linear category $\mathcal{C}_k$, an $n-$deformation can be extended to an $(n+1)-$deformation if and only if $[\mathcal{O}_n] = [0] \in H^3(\mathcal{C})$.*

The proof of this theorem is done in exactly the same way as for associative algebras.

3.4 Comparison

It is possible to turn a category into a unital associative algebra by taking the direct sum of the Hom-sets. On the other hand, to a unital associative algebra a category can be associated. This relationship is functorial and forms even an adjunction. Using these functors, the deformation of a category will be compared to the deformation of a category as an algebra.

Adjunction Define a functor $A : \mathbf{Cat}_k \to \mathbf{uAss}_k$ by

$$(\mathcal{C}, \circ) \mapsto (\coprod_{A,B} \mathcal{C}(A,B), m, \coprod_{A} id_A)$$

where

$$m(f,g) := \begin{cases} g \circ f & cod(f) = dom(g) \\ 0 & \text{otherwise.} \end{cases}$$

This multiplication is associative since $\circ$ is associative. The functor A acts as the identity on the morphisms.

On the other hand, define a functor $C : \mathbf{uAss}_k \to \mathbf{Cat}_k$ by associating to any monoid (M, m, u) the category $C(M)$ with the idempotents of M as objects and the arrows given by

$$Hom(a,b) := \{x \in M | m(b,x) = x = m(x,a)\}.$$

The identity is given by $(id_a : a \mapsto a) \in Hom(a,a)$ and the composition of $x \in Hom(a,b)$ and $y \in Hom(b,c)$ is given by $m(y,x) \in Hom(a,c)$. A morphism $f : M \to N$ between two monoids defines a functor between $C(M)$ and $C(N)$ since idempotents are mapped to idempotents and $Hom(a,b) \xrightarrow{f} Hom(f(a), f(b))$.

Remark 3.4.1. *The unit u is always an idempotent and $Hom(u,u) = M$. The zero-object is also an idempotent but $Hom(0,0) = 0$ and $Hom(a,0) = 0 = Hom(0,b)$.*

Note that the category $C(A)$ is equivalent to the Karubi envelope (c.f. [1]) of the one object category associated to the unital associative algebra. The Karubi envelope of a category $\mathcal{C}$ is the category with objects pairs consisting of an object of $\mathcal{C}$ and an idempotent on that object. An arrow $f : (A,a) \to (B,b)$ is a morphism $f : A \to B$ in $\mathcal{C}$ such that $b \circ f = f = f \circ a$. This category is denoted by $\mathbf{Split}(\mathcal{C})$. The Karubi envelope of a category has the property that every idempotent splits and in fact is the smallest with this property in the sense that there is an obvious inclusion functor $\iota : \mathcal{C} \to \mathbf{Split}(\mathcal{C})$. For any other functor $F : \mathcal{C} \to \mathcal{D}$ there is a unique functor $\bar{F} : \mathbf{Split}(\mathcal{C}) \to \mathcal{D}$ defined by $\bar{F}((A,a) \xrightarrow{f} (B,b)) := F(A) \xrightarrow{F(f)} F(B)$.

Lemma 3.4.1. *The functor $A : \boldsymbol{Cat}_k \to \boldsymbol{uAss}_k$ is left adjoint to $C : \boldsymbol{uAss}_k \to \boldsymbol{Cat}_k$.*

Proof. The adjunction will proved by constructing the unit and counit of the adjunction and prove that they satisfy the triangular identities. Consider the following two natural transformations:

$$\begin{array}{ccc} AC(M) & \xrightarrow{\exists! \varepsilon_M} & M \\ \uparrow{\scriptstyle \iota_{u,u}} & \nearrow\!\!\!\!= & \\ M = Hom(u,u) & & \end{array}$$

where ε_M is obtained by the universality of the coproduct, and $\eta_{\mathcal{C}} : \mathcal{C} \to CA(\mathcal{C})$ defined by sending an object to the identity arrow of it and a morphism $f : X \to Y$ to the same morphism $f : id_X \to id_Y$. Basically only the objects change in appearance.

It holds that

$$\begin{aligned} id_{A(\mathcal{C})} =& \varepsilon_{A(\mathcal{C})} \circ A\eta_{\mathcal{C}} : A(\mathcal{C}) \to ACA(\mathcal{C}) \to A(\mathcal{C}), \\ id_{C(M)} =& C\varepsilon_M \circ \eta_{C(M)} : C(M) \to CAC(M) \to C(M), \end{aligned}$$

showing that A is left adjoint to C. □

Deformation of a category $\mathcal{C}$ versus $A(\mathcal{C})$ Deforming a category $\mathcal{C}$ as described in this chapter will be compared to the deformation of the associated algebra $A(\mathcal{C})$. Let $R = k[[t]]$ be an augmentation of k. Consider the following two formal deformations, $A(\mathcal{C}) \otimes_k R$ and $\mathcal{C} \otimes_k R$. Note that

$$A(\mathcal{C}) \otimes_k R = \left(\coprod_{X,Y} \mathcal{C}(X,Y) \right) \otimes_k R \cong \coprod_{X,Y} (\mathcal{C}(X,Y) \otimes_k R) = A(\mathcal{C} \otimes_k R)$$

hence, using the unit of the adjunction,

$$\eta_{\mathcal{C} \otimes_k R} : \mathcal{C} \otimes_k R \to CA(\mathcal{C} \otimes_k R).$$

Note that both $A(\mathcal{C} \otimes_k R)$ and $A(\mathcal{C}) \otimes_k R$ have the same idempotents and equally trivial contain the idempotents of $A(\mathcal{C})$. It also holds that $C^n(\mathcal{C}) \subseteq C^n_{\text{Hoch}}(A(\mathcal{C}), A(\mathcal{C}))$ since

$$\begin{array}{ccc} A(\mathcal{C}) \otimes_k A(\mathcal{C}) & \xrightarrow{\exists!} & A(\mathcal{C}) \\ \uparrow{\scriptstyle \iota_{X,Y,Z}} & & \uparrow{\scriptstyle \iota_{X,Z}} \\ Hom(X,Y) \otimes_k Hom(Y,Z) & \longrightarrow & Hom(X,Z) \end{array}$$

commutes. Since both use the Hochschild differential, it follows that $C^\bullet(\mathcal{C})$ is a subcomplex of $C^\bullet_{\text{Hoch}}(A(\mathcal{C}), A(\mathcal{C}))$. In general it just forms a subcomplex since the candidates of multiplications when restricted to $Hom(X,Y) \otimes_k Hom(Y,Z)$ end up in $A(\mathcal{C})$ and not necessarily in $Hom(X,Z)$ which would be needed in order to be a candidate for the composition.

Lemma 3.4.2. *For a linear category $\mathcal{C}$ it holds that $H^n(\mathcal{C}) \subseteq H^n_{Hoch}(A(\mathcal{C}), A(\mathcal{C}))$.*

Proof. The complex $C^\bullet(\mathcal{C})$ is a subcomplex of $C^\bullet_{\text{Hoch}}(A(\mathcal{C}), A(\mathcal{C}))$, as was seen before, by defining the maps to be zero on non-composable arrows. This map, denoted by ι, is a chain-map. By the functoriality of the n^{th} cohomology group functor a morphism $H^n(\iota) : H^n(\mathcal{C}) \to H^n(A(\mathcal{C}), A(\mathcal{C}))$ is obtained, it is given by $[\psi] \mapsto [\iota(\psi)]$. It remains to check the injectivity of this map.

Let $\psi, \varphi \in C^n(A(\mathcal{C}), A(\mathcal{C}))$. Suppose $[\iota(\psi)] = [\iota(\varphi)]$. Then there exists an $\alpha \in C^{n-1}(A(\mathcal{C}), A(\mathcal{C}))$ such that $\iota(\psi) = \iota(\varphi) + d(\alpha)$. If these maps are applied to a non-composable arrow, then both $\iota(\psi)$ and $\iota(\varphi)$ are zero, hence forcing $d(\alpha)$ to be zero. The equation forces $\alpha(f_1 \otimes \ldots \otimes f_n)$ to have the same domain as f_1 and the same codomain as f_n, where $f_i \in A(\mathcal{C})$. This shows that $d(\alpha)$ is in the image of ι, hence there exists an α' such that $\iota(\alpha') = \alpha$. Further that $\psi = \varphi + d(\alpha')$ holds. Since ι is an embedding, it follows that $[\psi] = [\varphi]$ completing the proof that $H^n(\iota)$ is an injective group morphism. □

3.5 Maurer-Cartan Elements

The Maurer-Cartan elements for $k[t]/(t^2)$ based on the dg-Lie algebra $(C[1](\mathcal{C}), d_{\text{Hoch}}, [-,-])$ are infinitesimal deformations. Let $x \otimes t \in MC_{C[1](\mathcal{C})}(k[t]/(t^2))$. Note that in this case $x \in C[1](\mathcal{C})^1$ so it really is a candidate for an infinitesimal deformation. Then $x \otimes t$ is a cocycle since $d(x \otimes t) = -[x \otimes t, x \otimes t] = [x, x] \otimes t^2 = 0$. By definition $0 = d(x \otimes t) = d(x) \otimes t$ thus $d(x) = 0$, implying that x is a cocycle. On the other hand, let x be a representative of $[x] \in H^2(\mathcal{C})$. It has to be checked that $x \otimes t$ satisfies the Maurer-Cartan equation.

$$d(x \otimes t) + \frac{1}{2}[x \otimes t, x \otimes t] = d(x) \otimes t + \frac{1}{2}[x, x] \otimes t^2 = 0.$$

Thus $x \in MC_{C[1](\mathcal{C})}(k[t]/(t^2))$. The only difference between $H^2(\mathcal{C})$ and $MC_{C[1](\mathcal{C})}(k[t]/(t^2))$ is that the latter contains also equivalent deformations which are modded out in the cohomology case. It is also true that the Maurer-Cartan elements in the same orbit of the $G_{C[1](\mathcal{C})}(k[t]/(t^2))$ action are equivalent, hence $\text{Def}_{C[1](\mathcal{C})}(k[t]/(t^2)) \cong H^2(\mathcal{C})$.

Chapter 4

Deformation Theory of Linear Multicategories

In a category all arrows have one input. The same way as the notion of multivariable functions generalizes that of functions, multicategories generalize categories: in multicategories arrows are allowed to have several inputs. A special case consists of multicategories with only one object, these are called operads. An important role is played by the endomorphism multicategory **End** for which the n-ary operations are defined to be linear maps from a tensor product of n vector spaces to some other vector space. With the help of this multicategory it is possible to define $\mathcal{M}$-algebras, i.e. algebras modeled by a multicategory, these are just multifunctors $\rho : \mathcal{M} \to \mathbf{End}$. They are also called representations of $\mathcal{M}$. For example, the representations of the operad **Ass** gives rise to associative algebras, **Lie** gives rise to Lie algebras and **Cat**-algebras are exactly categories. The aim is develop deformation theory for multicategories and multifunctors. This in particular enables the study of the deformation of representations of multicategories.

4.1 Multicategories

Multicategories will be introduced in two ways. As a slogan, the two ways correspond to multicategories given with full versus partial compositions. The first definition is the classical way of introducing multicategories and it is given here because of its accessible nature while the second definition uses the language of trees which is more suited for the deformation theory. Using the trees a collection is defined. A multicategory is then a collection with extra structure. As an example, among others, the endomorphism multicategory will be described, enabling the definition of representations and algebras. Categories, operads and even multicategories can be obtained as representations over suitably chosen multicategories.

4.1.1 Multicategories, a first definition

Multicategories come in two flavors: as symmetric and non-symmetric ones.

Definition 4.1.1. *A multicategory $\mathcal{M}$ consists of a class of objects $Obj(\mathcal{M})$ and for $n \geq 0$ and objects $X_1, \dots, X_n, X$ a class of n-ary operations from $X_1, \dots, X_n$ to X, denoted by $\mathcal{M}(X_1, \dots, X_n|X)$. For each object X there is an identity arrow $id_X \in \mathcal{M}(X|X)$. Finally there is a family of compositions, denoted by $\circ$,*

$$\mathcal{M}(X_1, \dots, X_n|X) \times (\mathcal{M}(X_{1,1}, \dots, X_{1,m_1}|X_1) \times \dots \times \mathcal{M}(X_{n,1}, \dots, X_{n,m_n}|X_n))$$
$$\downarrow$$
$$\mathcal{M}(X_{1,1}, \dots, X_{n,m_n}|X),$$

such that the compositions satisfies the associativity and unit laws.

Definition 4.1.2. *A symmetric multicategory is a multicategory $\mathcal{M}$ with a right action of the permutation group S_n on the class of n-ary operations given by*

$$- \cdot \sigma : \mathcal{M}(X_1, \ldots, X_n|Y) \to \mathcal{M}(X_{\sigma(1)}, \ldots, X_{\sigma(n)}|Y)$$

such that

$$\psi \cdot 1_{S_n} = \psi$$
$$(\psi \cdot \sigma) \cdot \rho = \psi \cdot (\sigma\rho)$$

and it is compatible with the composition, i.e.:

$$(\theta \cdot \sigma) \circ (\psi_{\sigma(1)} \cdot \rho_{\sigma(1)}, \ldots, \psi_{\sigma(n)} \cdot \rho_{\sigma(n)}) = (\theta \circ (\psi_1, \ldots, \psi_n)) \cdot (\sigma \circ (\rho_{\sigma(1)}, \ldots, \rho_{\sigma(n)})).$$

Note that a multicategory with just unary operations is a category and in that case the symmetric structure on the multicategory becomes trivial. A (symmetric) multicategory is called small if the objects and the n-ary operations form sets.

A map between multicategories, called a multifunctor, sends objects to objects and $n-$ary operations to $n-$ary operations such that the composition and the identities are respected. A symmetric multifunctor is in addition compatible with the symmetric action on the $n-$ary operations. All small multicategories together with the multifunctors form a category **Multicat**, and the symmetric ones form $\mathbf{Multicat}_S$.

Definition 4.1.3. *A (symmetric) operad is a (symmetric) multicategory with one object.*

Since operads have just one object, say $*$, the class of n-ary operations $\mathcal{M}(*, \ldots, *|*)$ is simply denoted by $\mathcal{M}(n)$ for each $n \geq 0$. The operads form a full subcategory of the (symmetric) multicategories denoted by **Operad** and $\mathbf{Operad}_S$ respectively.

Enrichment Let $\mathcal{E}$ be a symmetric monoidal category. An $\mathcal{E}$-enriched multicategory $\mathcal{M}$ is a multicategory where each set of n-ary operations is an object of the category $\mathcal{E}$ and both the composition and the units are morphisms in $\mathcal{E}$. A multifunctor $F : \mathcal{M} \to \mathcal{N}$ between $\mathcal{E}$-enriched multicategories is $\mathcal{E}$-enriched if all the maps

$$F_{X_1,\ldots,X_n|X} : \mathcal{M}(X_1, \ldots, X_n|X) \to \mathcal{N}(F_0(X_1), \ldots, F_0(X_n)|F_0(X))$$

are morphisms in $\mathcal{E}$. The category of $\mathcal{E}$-enriched multicategories and $\mathcal{E}$-enriched symmetric multicategories are denoted by $\mathbf{Multicat}(\mathcal{E})$ and $\mathbf{Multicat}_S(\mathcal{E})$, respectively. In particular for multicategories with one object, the above construction gives $\mathcal{E}$-enriched operads.

Note that **Set**-enriched multicategories are ordinary multicategories.

Definition 4.1.4. *A linear multicategory is a multicategory enriched over $\boldsymbol{Mod}_k$.*

4.1.2 Trees

A tree is defined to be a finite, simply connected graph without cycles and a chosen outer edge called the root. An edge is outer if it is connected to only one vertex. To a tree T the following sets can be associated. The set of all edges denoted by $edges(T)$, the set of outer edges $outer(T)$, the set of inputs $in(T)$ formed by outer edges except the root, and the set of inner edges $inner(T)$ consisting of those edges which are not outer. All these notions may be defined for a vertex of T as well. By convention a tree is depicted by drawing for each vertex the output under the inputs.

The category of planar trees $\mathbf{\Omega}_p$ is given as follows. The objects of $\mathbf{\Omega}_p$ are all planar trees. In order to describe the arrows, a tree T will be interpreted as a multicategory $\Omega(T)$ in the sense of chapter 4. The objects of $\Omega(T)$ are the edges of T. Every subtree $T' \subseteq T$ gives an operation in $\Omega(T)(in(T')|root(T'))$. Note that each Hom-set has at most one element. The arrows between trees T and T' are then defined to be the multifunctors, again in the sense of chapter 4, from $\Omega(T)$ to $\Omega(T')$. The composition in $\mathbf{\Omega}_p$ is given by composition of multifunctors.

In $\mathbf{\Omega}_p$ there are special arrows which are the dendroidal analogs of the face and degeneracy maps of the simplicial case. Given a tree T with an inner edge e denote by T/e the tree obtained from T by contracting the edge e. Then a face map $d_e : T/e \to T$ is defined by sending the edges of T/e to the corresponding ones in T. The same happens for the vertices, except for the vertex v in T/e which corresponds to the contracted part of T. The vertex v is sent to the unique element of $\Omega(T)(in(v)|root(v))$. Let v be an outer vertex of T, then T/v is the tree obtained from T by removing v, i.e. all the edges in $in(v)$ and the vertex v are deleted. The face map $d_v : T/v \to T$ is just the inclusion. To describe the degeneracy maps suppose T contains a subtree t_1. Denote the two edges and the vertex of t_1 by e_1, e_2 and v, respectively. Let $T\backslash v$ be the tree obtained from T by replacing t_1 with a single edge e. Then a degeneracy $s_v : T \to T\backslash v$ is defined by sending e_1 and e_2 to e, v to id_e and acts as the identity on the remaining edges and vertices. The face and degeneracy maps satisfy similar relations to the simplicial identities, called dendroidal identities, which may be found in [27].

The edges of a tree may be labeled by the elements of some set S, resulting in a labeled tree. Explicitly, a labeled tree is a pair (T, l) consisting of a tree T and a function $l : edges(T) \to S$ called the labeling of T.

The following category will play an important role in the generalization of collections. Define the category $\mathbb{T}(S)$ to have objects all S-labeled trees and arrows all isomorphisms of trees preserving the labeling, i.e.

$$(T, l) \xrightarrow{\phi} (T', l') \iff \phi(T) \cong T' \text{ and } l' \circ \phi|_{\text{edges}(T)} = l.$$

If the labeling of $\mathbb{T}(S)$ is removed one obtains a subcategory of $\mathbf{\Omega}_p$. The assignment of S to $\mathbb{T}(S)$ is functorial: the functor $\mathbb{T} : \mathbf{Set} \to \mathbf{Cat}$ maps $f : S \to S'$ to $\mathbb{T}(f) : \mathbb{T}(S) \to \mathbb{T}(S')$, which is given by $(T, l) \mapsto (T, f \circ l)$ and $\phi \mapsto \phi$.

There are full subcategories $\mathbb{T}(S)_{n,m}$ of $\mathbb{T}(S)$ consisting of trees with n vertices and m inputs. The categories $\mathbb{T}(S)_{n,\bullet} := \coprod_{m\geq 0} \mathbb{T}(S)_{n,m}$ and $\mathbb{T}(S)_{\bullet,m} := \coprod_{n\geq 0} \mathbb{T}(S)_{n,m}$ are subcategories of $\mathbb{T}(S)$ as well. In fact all these categories are groupoids. Define a functor $\partial : \mathbb{T}(S)_{n,\bullet} \to \mathbb{T}(S)_{1,\bullet}$ for $n \geq 1$ by sending a tree (T, l) to $(T/\text{inner}(T), l|_{\text{outer}(T)})$ and an isomorphism ϕ to $\phi|_{\text{outer}(T)}$. For $n = 0$ define $\partial : \mathbb{T}(S)_{0,\bullet} \to \mathbb{T}(S)_{1,\bullet}$ by sending a tree $(|, l)$ to a tree with one vertex and one input such that both edges have the same label as $|$.

Sometimes the degeneracies should also be taken into account. Therefore define yet another category $\mathbb{D}(S)$ by extending $\mathbb{T}(S)$ with all the degeneracies preserving the labeling. Note that degeneracies leave the number of inputs invariant but decrease the number of vertices by one, thus $\mathbb{D}(S)$ can not be graded by the number of vertices. Define the subcategories $\mathbb{D}(S)_{n,m}$ by first adding to $\mathbb{T}(S)_{n,m}$ the degeneracies whose domain is in $\mathbb{T}(S)_{n,m}$. Then also add the codomains of these degeneracies together with their identities. Note that $\mathbb{D}(S)_{1,1}$ contains both $\mathbb{T}(S)_{1,1}$ and $\mathbb{T}(S)_{0,1}$, since $\mathbb{T}(S)_{0,1}$ is a discrete category and for any tree (t_1, l) where the labels of the two edges coincide there exists a degeneracy $\sigma : (t_1, l) \to (|, l)$.

4.1.3 Collections

Let $\mathcal{E}$ be a symmetric monoidal category and let S be an arbitrary set. Define $\mathbb{T}(S)_1 := \mathbb{T}(S)_{1,\bullet} \cup \mathbb{T}(S)_{0,1}$ and $\mathbb{D}(S)_1 := \coprod_{m\geq 0, m\neq 1} \mathbb{T}(S)_{1,m} \cup \mathbb{D}(S)_{1,1}$.

Definition 4.1.5. *For S define the category of symmetric collections and symmetric pointed collections in $\mathcal{E}$ to be*

$$Coll_{\mathbb{S}}(\mathcal{E}, S) := [\mathbb{T}(S)_1^{op}, \mathcal{E}] \quad \text{and} \quad Coll^{\bullet}_{\mathbb{S}}(\mathcal{E}, S) := [\mathbb{D}(S)_1^{op}, \mathcal{E}],$$

such that any collection M sends $(|, l)$ to the tensor unit I. Non-symmetric collections and non-symmetric pointed collections are defined similarly after removing the isomorphisms from $\mathbb{T}(S)_1$ and $\mathbb{D}(S)_1$, respectively. The categories of the non-symmetric versions are denoted by $Coll(\mathcal{E}, S)$ and $Coll^{\bullet}(\mathcal{E}, S)$.

Let M be a collection. Then the image of (T, l) in $\mathcal{E}$ is denoted by $M((T, l)) = M(l(e_1), \dots, l(e_n)|l(e))$, where $in(T) = \{e_1, \dots, e_n\}$ and $e = root(T)$. From now on by a collection a symmetric collection will be meant, unless otherwise specified.

Remark 4.1.1. *The definition of pointed and unpointed collections gives back the definition as stated in [2] and [3], respectively.*

To see this first consider the unpointed collections. If S is a singleton $*$ then $\mathbb{T}(*)_{1,m}$ consists of one object, namely the m-corolla, and the morphisms are all permutations of S_m. It follows that $\mathbb{T}(*)_{1,m} \cong S_m$ as categories. Thus

$$\begin{aligned} Coll_{\mathbb{S}}(\mathcal{E}, *) =&[\mathbb{T}(*)^{op}_{1,\bullet} \cup \mathbb{T}(*)^{op}_{0,1}, \mathcal{E}] = [\mathbb{T}(*)^{op}_{1,\bullet}, \mathcal{E}] \times [\mathbb{T}(*)^{op}_{0,1}, \mathcal{E}] \cong \prod_{m \geq 0} [\mathbb{T}^{op}_{1,m}, \mathcal{E}] \times [\mathbb{T}(*)^{op}_{0,1}, \mathcal{E}] \\ \cong& Coll_{\mathbb{S}}(\mathcal{E}) \times [\mathbb{T}(*)^{op}_{0,1}, \mathcal{E}] \end{aligned}$$

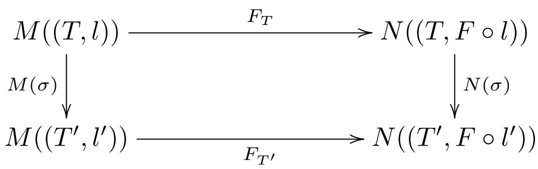
The last part is completely determined since a collection has to send $(|, l)$ to the tensor unit.

Now consider the pointed collection for a set S. It holds that

$$Coll^{\bullet}_{\mathbb{S}}(\mathcal{E}, S) = \prod_{\substack{m \geq 0 \\ m \neq 1}} [\mathbb{T}(S)^{op}_{1,m}, \mathcal{E}] \times [\mathbb{D}(S)^{op}_{1,1}, \mathcal{E}].$$

Note that a functor K in $[\mathbb{D}(S)^{op}_{1,1}, \mathcal{E}]$ is equivalent to give a functor K in $[\mathbb{T}(S)^{op}_{1,1} \cup \mathbb{T}(S)^{op}_{0,1}, \mathcal{E}]$ together with the specification where the degeneracies are mapped to, i.e. $K(\sigma) : K((|, l)) \to K((t_1, l))$, but that implies that giving a collection K in $Coll^{\bullet}_{\mathbb{S}}(\mathcal{E}, S)$ is exactly the same as giving a collection K in $Coll_{\mathbb{S}}(\mathcal{E}, S)$ together with maps $K_s : I \to K(s|s)$.

A morphism between collections is a natural transformation $F : M \to N$, i.e.



commutes for all T, T' and $\sigma : T' \to T$. In other words, a map between collections is a family of equivariant morphisms in $\mathcal{E}$.

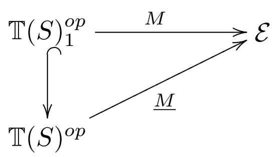
Each collection could be extended to all of $\mathbb{T}(S)$ in the following way. Let M be a collection then define $\underline{M}$ to agree with M on the corollas. Let (T, l) be an arbitrary tree in $\mathbb{T}(S)$. Note that each tree T can be written as $T = t_n \circ (T_1, \dots, T_n)$, for some n, where $\circ$ denotes the grafting of trees. Now define $\underline{M}$ recursively by

$$\underline{M}((T, l)) := M((t_n, l|_{t_n})) \otimes_{\mathcal{E}} (\underline{M}(T_1, l|_{T_1}) \otimes_{\mathcal{E}} \dots \otimes_{\mathcal{E}} \underline{M}(T_n, l|_{T_n})).$$

An isomorphism ϕ decomposes in the same way into $\sigma \circ (\phi_1, \dots, \phi_n)$, define

$$\underline{M}(\phi) := M(\sigma) \otimes_{\mathcal{E}} (\underline{M}(\phi_1) \otimes_{\mathcal{E}} \dots \otimes_{\mathcal{E}} \underline{M}(\phi_n).$$

In this way a functor $\underline{M} : \mathbb{T}(S)^{op} \to \mathcal{E}$ is obtained such that the diagram



commutes.

4.1.4 Multicategories with partial compositions

Now multicategories can be described in terms of partial compositions.

Definition 4.1.6. *A symmetric multicategory $\mathcal{M}$ consists of a set of objects S and a collection $\mathcal{M} : \mathbb{T}(S)_1^{op} \to \mathcal{E}$ together with a family of compositions*

$$\circ_T : \underline{\mathcal{M}}(T) \to \mathcal{M} \circ \partial(T)$$

for all $T \in \mathbb{T}(S)_{2,\bullet}$ and units $1_s : I \to \mathcal{M}(s,s)$ for all objects $s \in S$ such that:

***Associativity**: All trees with three vertices have two outer vertices denoted by v_1 and v_2. For all $T \in \mathbb{T}(S)_{3,\bullet}$ the diagram*

$$\begin{array}{ccc} \underline{\mathcal{M}}(T) & \xrightarrow{1\otimes\circ_{d_{v_2}(T)}} & \underline{\mathcal{M}}(d_{e_2}(T)) \\ {\scriptstyle 1\otimes\circ_{d_{v_1}(T)}}\downarrow & & \downarrow{\scriptstyle \circ_{d_{v_1}(T)}} \\ \underline{\mathcal{M}}(d_{e_1}(T)) & \xrightarrow[\circ_{d_{v_2}(T)}]{} & \mathcal{M} \circ \partial(T) \end{array}$$

commutes.

***Unit axiom**: The diagram below commutes for all objects $A, B \in S$.*

$$\begin{array}{ccccc} I \otimes \mathcal{M}(B|A) & \overset{\sim}{=\!=} & \mathcal{M}(B|A) & \overset{\sim}{=\!=} & \mathcal{M}(B|A) \otimes I \\ {\scriptstyle 1_A\otimes 1}\downarrow & \nearrow{\scriptstyle \circ} & & \nwarrow{\scriptstyle \circ} & \downarrow{\scriptstyle 1\otimes 1_B} \\ \mathcal{M}(A|A) \otimes \mathcal{M}(B|A) & & & & \mathcal{M}(B|A) \otimes \mathcal{M}(B|B) \end{array}$$

***Equivariance**: For all $\phi : T' \to T$ in $\mathbb{T}(S)_{2,\bullet}$ the diagram*

$$\begin{array}{ccc} \underline{\mathcal{M}}(T) & \xrightarrow{\circ_T} & \mathcal{M} \circ \partial(T) \\ {\scriptstyle \underline{\mathcal{M}}(\phi)}\downarrow & & \downarrow{\scriptstyle \mathcal{M}\circ\partial(\phi)} \\ \underline{\mathcal{M}}(T') & \xrightarrow[\circ_{T'}]{} & \mathcal{M} \circ \partial(T'). \end{array}$$

commutes.

Instead of specifying the units, a pointed collection can be used. The unpointed collection is used because of clarity. The definition of a non-symmetric multicategory is obtained by dropping the equivariance condition.

Remark 4.1.2. *In this setting it also makes sense to talk about operads, i.e. multicategories with one object. If the units are removed from the definition of a multicategory one obtains the notion of a pseudo-multicategory, a generalization of a pseudo-operad (c.f. [25]). The partial compositions of pseudo-operads are called $\circ_i$-compositions. This motives the terminology $\circ_T$-compositions in the many object case.*

Comparison In this paragraph multicategories given with the full composition will be compared to multicategories given with the $\circ_T$-composition. In order to describe the full composition in such a way that it can be conveniently compared, the category of layered trees will be introduced. A layered tree is a tree where the first layer consists of one corolla, the second layer of corollas grafted into the input of the previous layer for all inputs, etc. Define the category $\mathbb{L}(S)$ to have objects S-labeled layered trees and isomorphisms of layered trees as arrows. The composition is given by the composition of isomorphisms.

Lemma 4.1.1. *The category $\mathbb{L}(S)$ is isomorphic to the category $\mathbb{T}(S)$.*

Proof. Every layered tree is in particular a tree in the sense of $\mathbb{T}(S)$ hence every arrow in $\mathbb{L}(S)$ is an arrow in $\mathbb{T}(S)$. On the other hand, each tree in $\mathbb{T}(S)$ can be extended to a layered tree by grafting of the unit tree $|$. Since the unit is sent to the tensor unit the isomorphism obviously extends to the layered tree. Thus there are functors $\iota : \mathbb{L}(S) \rightleftarrows \mathbb{T}(S) : \varepsilon$. Obviously $\varepsilon \circ \iota = id_{\mathbb{L}(S)}$ and $\iota \circ \varepsilon = id_{\mathbb{T}(S)}$, since $|$ is the unit of the grafting operation. □

The category $\mathbb{L}(S)$ can be graded by the number of layers and the number of inputs. It is even possible to grade with respect to the number of vertices. Thus there are full subcategories $\mathbb{L}(S)_{l,m,n}$, where l, m and n denote the number of layers, inputs and vertices, respectively. It is easily seen that $\mathbb{T}(S)_{n,m}$ and $\mathbb{L}(S)_{\bullet,m,n}$ are isomorphic too. Note that

$$\mathbb{L}(S)_1 := \mathbb{L}(S)_{1,\bullet,\bullet} \cong \mathbb{T}(S)_{1,\bullet} \cup \mathbb{T}(S)_{0,1} = \mathbb{T}(S)_1.$$

Thus $[\mathbb{L}(S)_1^{op}, \mathcal{E}] \cong [\mathbb{T}(S)_1^{op}, \mathcal{E}]$.

It is possible to extend a functor $M \in [\mathbb{L}(S)_1^{op}, \mathcal{E}]$ to a functor $\underline{M}^L : \mathbb{L}(S)^{op} \to \mathcal{E}$ by applying M to each component in a layer from left to right starting with the first layer.

The full composition of a multicategory $\mathcal{M}$ is given by maps

$$\circ_T^L : \underline{\mathcal{M}}^L(T) \to \mathcal{M} \circ \partial(T),$$

for all $T \in \mathbb{L}(S)_{2,\bullet,\bullet}$. The aim is now to compare these maps to the $\circ_T$-compositions.

Given a full composition $\circ^L$ then define $\circ_T$ for $T \in \mathbb{T}(S)_{2,\bullet}$ by

$$\begin{array}{ccc} \underline{\mathcal{M}}(T) & \xrightarrow{\circ_T} & \mathcal{M} \circ \partial(T) \\ \text{shuffle} \downarrow & & \| \\ \underline{\mathcal{M}}^L(\varepsilon(T)) & \xrightarrow[\circ^L_{\varepsilon(T)}]{} & \mathcal{M} \circ \partial(\varepsilon(T)). \end{array}$$

This just means that $\circ_T = \circ^L_{t_n \circ (1,\ldots,1,t_m,1,\ldots,1)}$, where $T = t_n \circ t_m$. On the other hand, given $\circ_T$-compositions construct a full composition as follows. Note that $\iota(T)$ for $T \in \mathbb{L}(S)_{2,\bullet,\bullet}$ does not have to be in $\mathbb{T}(S)_{2,\bullet}$. Therefore define T_i to be the subtree of T with two vertices containing the inner edge $e_i \in inner(T)$. Then the full composition is given by

$$\circ_T^L : \underline{\mathcal{M}}(T) \xrightarrow{\circ_{T_1}} \underline{\mathcal{M}}(T/e_1) \xrightarrow{\circ_{T_2}} \cdots \xrightarrow{\circ_{T_{e-n}}} \underline{\mathcal{M}}(T/\{e_1, \ldots, e_{n-1}\}) = \mathcal{M} \circ \partial(T).$$

This $\circ^L$ is well-defined since the $\circ_T$-compositions are associative.

Lemma 4.1.2. *Thus multicategories defined using the full composition are equivalent to multicategories defined by $\circ_T$-compositions.*

This generalizes the observation that each operad gives rise to a pseudo-operad and each pseudo-operad with unit an operad.

4.1.5 Multifunctors

Recall that a function $f : S \to S'$ induces a functor $\mathbb{T}(f) : \mathbb{T}(S) \to \mathbb{T}(S')$. Using $\mathbb{T}(f)$ a functor $f^* : [\mathbb{T}(S')_1, \mathcal{E}] \to [\mathbb{T}(S)_1, \mathcal{E}]$ between collections can be constructed by

$$f^* : M \mapsto M \circ \mathbb{T}(f).$$

A symmetric multifunctor between symmetric multicategories $\mathcal{M}$ and $\mathcal{N}$ is a pair (F^O, F) consisting of a function $F^O : Obj(\mathcal{M}) \to Obj(\mathcal{N})$ and a natural transformation $F : \mathcal{M} \Longrightarrow \mathcal{N} \circ \mathbb{T}(F^O)$ between the collections respecting the composition and satisfying the unit axiom

$$\begin{array}{ccc} I & \xrightarrow{1_A} & \mathcal{M}(A|A) \\ {\scriptstyle 1_{F(A)}}\downarrow & \swarrow{\scriptstyle F} & \\ \mathcal{N}(F(A)|F(A)) & & \end{array}$$

for all objects A in $\mathcal{M}$. Note that F can be extended to a map from $\underline{M}$ to $\underline{N \circ \mathbb{T}(F^O)} = \underline{N} \circ \mathbb{T}(F^O)$ again denoted by F. Then the compatibility with the compositions can be expressed by the commutativity of the following diagram

$$\begin{array}{ccc} \underline{\mathcal{M}}(T) & \xrightarrow{\circ_{\mathcal{M}}} & \mathcal{M} \circ \partial(T) \\ {\scriptstyle F_T}\downarrow & & \downarrow{\scriptstyle F \circ \partial(T)} \\ \underline{\mathcal{N}} \circ \mathbb{T}(F^O)(T) & \xrightarrow[\circ_{\mathcal{N}} \circ \mathbb{T}(F^O)]{} & \mathcal{N} \circ \mathbb{T}(F^O) \circ \partial(T) \end{array}$$

for all $T \in \mathbb{T}(S)_{2,\bullet}$ and the equivariance by

$$\begin{array}{ccc} \underline{\mathcal{M}}(T) & \xrightarrow{F_T} & \underline{\mathcal{N}} \circ \mathbb{T}(F^O)(T) \\ {\scriptstyle \underline{\mathcal{M}}(\phi)}\downarrow & & \downarrow{\scriptstyle \underline{\mathcal{N}} \circ \mathbb{T}(F^O)(\phi)} \\ \underline{\mathcal{M}}(T') & \xrightarrow[F_{T'}]{} & \underline{\mathcal{N}} \circ \mathbb{T}(F^O)(T') \end{array}$$

for any $\phi : T' \to T$.

Definition 4.1.7. *An isomorphism between two multicategories $\mathcal{M}$ and $\mathcal{N}$ is a functor F such that F^O is a bijection and F a natural isomorphism on the collections respecting the compositions and the equivariance condition.*

4.1.6 Category of multicategories

All small symmetric multicategories and all symmetric multifunctors form the category $\mathbf{Multicat}_{\mathbb{S}}$. The small $\mathcal{E}$-enriched multicategories together with the symmetric multifunctors form a subcategory denoted by $\mathbf{Multicat}_{\mathbb{S}}(\mathcal{E})$. Note that for each multicategory the objects are fixed, hence the category $\mathbf{Multicat}_{\mathbb{S}}(\mathcal{E})$ is fibred over $\mathbf{Set}$, i.e.

$$\mathbf{Multicat}_{\mathbb{S}}(\mathcal{E}) = \coprod_{S \in Obj(\mathbf{Set})} \mathbf{Multicat}_{\mathbb{S}}(\mathcal{E}, S).$$

In the same way all small non-symmetric multicategories form a category with the non-symmetric multifunctors

$$\mathbf{Multicat}(\mathcal{E}) = \coprod_{S \in Obj(\mathbf{Set})} \mathbf{Multicat}(\mathcal{E}, S).$$

4.1.7 Examples of multicategories

Categories and operads Note that a multicategory with one object is an operad and a multicategory with only unitary operations is a category. A unitary associative algebra can be interpreted as a special operad or as a special category, hence it is a special multicategory as well. All these structures may be arranged into the following diagram.

$$\begin{array}{ccccc} & & \mathbf{Cat} & & \\ & \nearrow & & \searrow & \\ \mathbf{uAss} & & & & \mathbf{Multicats} \\ & \searrow & & \nearrow & \\ & & \mathbf{Operads} & & \end{array}$$

Permutation operad The operad S is defined by $\mathrm{S}(n) := S_n$ the permutation group of n elements, for all n. The composition $\sigma \circ (\sigma_1, \ldots, \sigma_n)$ is defined by block permutations.

Initial multicategory Suppose $\mathcal{E}$ has an initial object, then there is an initial object I in $\mathbf{Multicat}(\mathcal{E}, S)$, the constant multicategory I induced by the tensor unit I of $\mathcal{E}$. Explicitly, the collection is given by $I(A|A) := I$, for all $A \in S$, and all the other Hom-objects are defined to be the initial object in $\mathcal{E}$. The composition is given by the isomorphism $I \otimes_{\mathcal{E}} I \cong I$. The units $I \xrightarrow{1_s} I(s,s)$ are given by the identity id_I.

Free (symmetric) multicategory There are forgetful functors $U : \mathbf{Multicat}_{\mathbb{S}}(\mathcal{E}, S) \to Coll^{\bullet}_{\mathbb{S}}(\mathcal{E}, S)$ and $U : \mathbf{Multicat}(\mathcal{E}, S) \to Coll^{\bullet}(\mathcal{E}, S)$ sending a multicategory to its underlying collection. The free (symmetric) multicategory in $\mathcal{E}$ over S is defined to be the left adjoint to the forgetful functor, i.e.

$$Coll^{\bullet}(\mathcal{E}, S) \underset{U}{\overset{\mathcal{F}}{\rightleftarrows}} \mathbf{Multicat}(\mathcal{E}, S) \quad \text{and} \quad Coll^{\bullet}_{\mathbb{S}}(\mathcal{E}, S) \underset{U}{\overset{\mathcal{F}}{\rightleftarrows}} \mathbf{Multicat}_{\mathbb{S}}(\mathcal{E}, S).$$

Intuitively $\mathcal{F}$ assigns to a collection over S a new collection consisting of trees whose edges are labeled by S and the vertices are labeled by elements of the collection. For an explicit description consider a collection M in $\mathcal{E}$ over S. It was shown that M extends to $\mathbb{T}(S)$. In order to extend $\underline{M}$ to $\mathbb{D}(S)$ it has to be specified what $\underline{M}$ does with the degeneracies. Let $s : T \to T'$ be a degeneracy then define $\underline{M}(s) : \underline{M}(T') \to \underline{M}(T)$ with the help of the units $\eta_s : I \to M(s|s)$ to be

$$M(t_n) \otimes \ldots \otimes M(t_m) \cong M(t_n) \otimes \ldots \otimes I \otimes \ldots \otimes M(t_m) \xrightarrow{1 \otimes .. \otimes \eta_s \otimes .. \otimes 1} M(t_n, l) \otimes \ldots \otimes M(s|s) \otimes \ldots \otimes M(t_m).$$

These degeneracies have been introduced in order to assure that the free multicategory satisfies the unit axiom.

The free non-symmetric multicategory over a collection $M \in Coll^{\bullet}(\mathcal{E}, S)$ is given by a new collection

$$\mathcal{F}(M)((t_n, l)) := \coprod_{\substack{(T,l') \in \mathbb{T}(S) \\ l'|_{outer(T)} = l}} \underline{M}(T),$$

and the compositions are defined in the following way. Let $(T, l) \in \mathbb{T}(S)_{2,\bullet}$. Then there are trees (t_n, l_n) and (t_m, l_m) such that T is obtained from grafting t_m into t_n. Since in a closed symmetric monoidal category the tensor product preserves colimits it suffices to specify the composition on $\underline{M} \otimes \underline{M}$, i.e.

$$\begin{array}{ccccc} \underline{\mathcal{F}(M)}(T) & = & \mathcal{F}(M)(t_n) \otimes \mathcal{F}(M)(t_m) & \overset{\exists! \circ_T}{\dashrightarrow} & \mathcal{F}(M)(\partial T) \\ & & \uparrow & & \uparrow \\ & & \underline{M}(T_n) \otimes \underline{M}(T_m) & = & \underline{M}(T_n \circ T_m), \end{array}$$

whenever the inclusion makes sense.

Now let $M \in Coll^{\bullet}_{\mathbb{S}}(\mathcal{E}, S)$ be a symmetric collection. Define a functor $\lambda : \mathbb{D}(S) \to \mathbf{Set} \to \mathcal{E}$ by the composition of the following maps. First send a tree (T, l) to the set of all bijections between $\{1, \dots, n\}$ and $in(T)$. Then take the copower (see A) over that set to obtain an object in $\mathcal{E}$. An isomorphism $\phi : T \to T'$ is mapped to the post composition of a bijection with $\phi|_{in(T)}$ and a degeneracy is mapped to the identity.

Define $\mathbb{D}((t_n, l))$ to be the full subcategory of $\mathbb{D}(S)$ consisting of the trees (T, l') with n inputs for which $l'(in(T)) \cong l(in(t_n))$ and $l'(root(T)) = l(root(t_n))$. The collection for the free symmetric multicategory is then given by the coend (see A)

$$\mathcal{F}(M)((t_n, l)) := \int^{(T,l) \in \mathbb{D}((t_n,l))} \underline{M}(T) \otimes \lambda(T).$$

The composition is defined in the same way as for the free non-symmetric multicategory.

Multicategories in terms of generators and relations A right ideal $\mathcal{I}$ of a multicategory $\mathcal{M}$ in $\mathcal{E}$ over S is a collection $\mathcal{I} : \mathbb{T}(S)_1^{op} \to \mathcal{E}$ such that $\mathcal{I}(t_n, l) \rightarrowtail \mathcal{M}(t_n, l)$ for all $(t_n, l) \in \mathbb{T}(S)_1$ satisfying

$$\mathcal{M}(t_n) \otimes \mathcal{I}(t_m) \xrightarrow{\circ_{t_n \circ t_m}} \mathcal{I}(t_n \circ t_m)$$

for all $(t_n, l), (t_m, l') \in \mathbb{T}_{1,\bullet}$.

Similarly a left ideal is as a right ideal except it satisfies

$$\mathcal{I}(t_n) \otimes \mathcal{M}(t_m) \xrightarrow{\circ_{t_n \circ t_m}} \mathcal{I}(t_n \circ t_m)$$

for all $(t_n, l), (t_m, l') \in \mathbb{T}_{1,\bullet}$. An ideal is a left and a right ideal.

The quotient $\mathcal{M}/\mathcal{I}$ of a multicategory $\mathcal{M}$ by an ideal $\mathcal{I}$ is a collection $\mathcal{M}/\mathcal{I} : \mathbb{T}(S)_1^{op} \to \mathcal{E}$ defined by

$$\mathcal{M}/\mathcal{I}(t_n) := \mathcal{M}(t_n)/\mathcal{I}(t_n)$$

for all $(t_n, l) \in \mathbb{T}(S)_1$. The composition of $\mathcal{M}$ induces a composition on $\mathcal{M}/\mathcal{I}$.

Definition 4.1.8. *Let E be a collection in $Coll(\mathcal{E}, S)$ and R a collection such that $R(t_n, l) \subset F(E)(t_n, l)$ for all $(t_n, l) \in \mathbb{T}(S)_1$. The multicategory generated by E with relations R is given by*

$$\mathcal{F}(E)/(R)$$

where (R) is the ideal generated by R.

Quadratic Multicategories Certain algebraic structures can be modeled by a multicategory in terms of generators and relations that involve just the composition of two operations.

Definition 4.1.9. *A quadratic data (E, R) consists of a collection E in $\boldsymbol{Coll}(\mathcal{E}, S)$, such that $E_0 = 0$, and a collection R consisting of $R_n \subseteq \mathcal{F}(E)^{(2)}(n)$ where $\mathcal{F}(E)^{(2)}$ is the free multicategory with generators E consisting of all possible compositions of two operations. The quadratic data is called binary if the only non-zero object in E is E_2.*

Definition 4.1.10. *Let (E, R) be a quadratic data. The multicategory $\mathcal{F}(E)/(R)$ is called quadratic and is denoted by $\mathcal{M}(E, R)$.*

Given a binary quadratic multicategory $\mathcal{M}$ the quadratic data can be extracted by $E_2 := \mathcal{M}(2)$ and $R := Ker(\pi : \mathcal{F}(E_2)(3) \twoheadrightarrow \mathcal{M}(3))$.

Lemma 4.1.3. *Let $\mathcal{M}(E_2, R)$ be a binary quadratic multicategory and let $\mathcal{Q}$ be an arbitrary multicategory. A morphism $\rho : \mathcal{M}(E_2, R) \to \mathcal{N}$ is completely determined by a morphism $\beta : E_2 \to \mathcal{N}(2)$ satisfying $\beta(R) = 0$.*

Proof. To simplify notation let $M := \mathcal{M}(E_2, R)$. Note that the fact that $\mathcal{M}$ is binary quadratic implies that it is of the form $\mathcal{F}(E_2)/R$. First it will be shown that β gives rise to a map between $\mathcal{F}(E_2)$ and $\mathcal{N}$. If this map respects the relations R then it gives a morphism between $\mathcal{M}$ and $\mathcal{N}$.

To show that β determines an operad morphism the adjunction

$$Hom_{\mathbf{Multicat}(\mathcal{E},S)}(\mathcal{F}(E), \mathcal{N}) \cong Hom_{Coll(\mathcal{E},S)}(E, U(\mathcal{N}))$$

will be used. Here E is a collection, but E is concentrated in degree two thus $\beta : E_2 \to \mathcal{N}(2)$ determines a morphism $\rho : E \to \mathcal{N}$ and therefore by the adjunction a morphism $\gamma : \mathcal{F}(E_2) \to \mathcal{N}$.

It follows that if γ respects the relations R, i.e. $\gamma(R) = 0$, then γ is well-defined on $\mathcal{M}$ and therefore provides an operad morphism $\gamma : \mathcal{M} \to \mathcal{N}$. □

Associative operad There is a symmetric and a non-symmetric operad, both denoted by **Ass**, whose representations are associative algebras. The non-symmetric one is defined by the collection

$$\mathbf{Ass}(n) := k,$$

for all $n \geq 0$. The collection for the symmetric one is given by:

$$\mathbf{Ass}(0) := k,$$
$$\mathbf{Ass}(n) := k[S_n],$$

for all $n \geq 1$. It is also possible to set $\mathbf{Ass}(0) = 0$ but then the unit of the algebra is lost. The identity and the composition are in both cases given by the identity map on k and by block permutations respectively.

A different description of **Ass** is given by generators and relations. The quadratic data for the non-symmetric version is given by $E_2 := k[\mu]$ and $E_n := 0$ for all $n \neq 2$ and with the relation $\mu \circ (\mu, 1) - \mu \circ (1, \mu)$. For the symmetric one E is defined by the graded vector space concentrated in degree two with $E_2 = k[S_2]$, and relation $1_2 \circ (1_2, 1_1) - 1_2 \circ (1_1, 1_2)$. Note that both are binary quadratic operads.

Lie operad The **Lie** operad will be given in terms of generators and relations. The quadratic data for **Lie** is given by the graded vector space E concentrated in degree zero with $E_2 = k[\mu]$ and the relations are given by

$$\mu + \sigma_{(12)}\mu$$
$$\mu \circ (\mu, 1) + \sigma_{(123)}\mu \circ (\mu, 1) + \sigma_{(132)}\mu \circ (\mu, 1).$$

These relations describe anti-symmetry and the Jacobi relation. Also this operad is an example of a binary quadratic operad.

Endomorphism multicategory There is an analog of the endomorphism operad for multicategories. Let S be a set and $\varphi : S \to Obj(\mathcal{E})$ a function. The objects of $\mathbf{End}_\varphi$ are the elements of S. The collection $\mathbf{End}_\varphi : \mathbb{T}(S)_1^{op} \to \mathcal{E}$ is defined by

$$\mathbf{End}_\varphi((t_n, l)) := \underline{Hom}_{\mathcal{E}}(\varphi \circ l(e_1) \otimes_{\mathcal{E}} \ldots \otimes_{\mathcal{E}} \varphi \circ l(e_n) \mid \varphi \circ l(e)),$$

where $in(t_n) = \{e_1, \ldots, e_n\}$ and e is the root of t_n. The composition is given by the obvious composition of arrows.

If S is the singleton $*$ then it suffices to specify $V := \varphi(*)$ and one writes $\mathbf{End}_V((t_n, l))$. In this way, the endomorphism operad is obtained, where $\mathbf{End}_V(t_n) = \underline{Hom}_k(V^{\otimes n}, V)$.

In order to describe the endomorphism multicategory in a nicer way, define the functors $\mathcal{X}_{I,\varphi} : \mathbb{T}(S)_1^{op} \to \mathcal{E}^{op}$ and $\mathcal{X}_{O,\varphi} : \mathbb{T}(S)_1^{op} \to \mathcal{E}$ by

$$\mathcal{X}_{I,\varphi}((t_n, l)) := \varphi \circ l(e_1) \otimes_{\mathcal{E}} \ldots \otimes_{\mathcal{E}} \varphi \circ l(e_n), \quad (4.1)$$
$$\mathcal{X}_{O,\varphi}((t_n, l)) := \varphi \circ l(e) \quad (4.2)$$

on the objects. On the arrows of $\mathbb{T}(S)_1$ they are defined to be $\mathcal{X}_{I,\varphi}(\phi)(X_1\otimes\ldots\otimes X_n) := X_{\phi^{-1}(1)}\otimes\ldots\otimes X_{\phi^{-1}(n)}$ and $\mathcal{X}_{O,\varphi}(\phi)(X) = id_X$. Then the endomorphism operad is given by the composition

$$\mathbf{End}_\varphi : \mathbb{T}(S)_1^{op} \xrightarrow{\Delta} \mathbb{T}(S)_1^{op} \times \mathbb{T}(S)_1^{op} \xrightarrow{\mathcal{X}_{I,\varphi}\times\mathcal{X}_{O,\varphi}} \mathcal{E}^{op} \times \mathcal{E} \xrightarrow{\underline{Hom}} \mathcal{E}.$$

It follows that for some isomorphism ϕ, $\mathbf{End}_\varphi(\phi)$ is given by

$$\mathbf{End}_\varphi(\phi) = \underline{Hom}(\mathcal{X}_{I,\varphi}(\phi)|\mathcal{X}_{I,\varphi}(\phi)) = \underline{Hom}(\mathcal{X}_{I,\varphi}(\phi)|id).$$

4.1.8 Representations and algebras of a multicategory

The interest in multicategories comes from the fact that they encode algebraic information, i.e. representations of multicategories give rise to algebras and vice versa. The essence of a representation of an object of some category is to chose another object in that category which is well-known and study the induced dynamics of the original object in the well-known object. In case of multicategories, the well-known object is the endomorphism multicategory.

Definition 4.1.11. *A representation of a (symmetric) multicategory $\mathcal{M}$ over S in $\mathcal{E}$ is a (symmetric) multifunctor $\rho : \mathcal{M} \to \boldsymbol{End}_\varphi$, for some set $\varphi : Obj(\mathbf{End}) \to Obj(\mathcal{E})$.*

Definition 4.1.12. *Let $\mathcal{M}$ be a non-symmetric multicategory. An $\mathcal{M}$-algebra is a family of morphisms indexed by $\mathbb{T}(S)_1$, where for each (t_n, l) there is a morphism given by*

$$\rho_{(t_n,l)} : \mathcal{M}((t_n, l)) \otimes_\mathcal{E} \mathcal{X}_{I,\varphi} \circ \mathbb{T}(\rho^O)((t_n, l)) \to \mathcal{X}_{O,\varphi} \circ \mathbb{T}(\rho^O)((t_n, l)),$$

with $\rho^O : S \to Obj(\boldsymbol{End})$ a function and $\chi_{I,\varphi}$ and $\chi_{O,\varphi}$ are defined in (4.1) *and* (4.2)*, respectively.*

For a symmetric multicategory $\mathcal{M}$ an $\mathcal{M}$-algebra is a family of morphisms

$$\rho_{(t_n,l)} : \mathcal{M}((t_n, l)) \otimes_{\mathbb{T}(S)_{n,\bullet}^{op}} \mathcal{X}_{I,\varphi} \circ \mathbb{T}(\rho^O)((t_n, l)) \to \mathcal{X}_{O,\varphi} \circ \mathbb{T}(\rho^O)((t_n, l)).$$

Note that $\mathbb{T}(\rho^O)(t_n, l) = (t_n, \rho^O \circ l)$. In case S is a singleton, i.e. an operad, and $\rho^O : * \to \{V\}$ then the maps ρ_{t_n} become

$$\rho_{t_n} : \mathcal{M}(n) \otimes_{S_n} V^{\otimes n} \to V.$$

This is the definition of an algebra over an operad (see [25]).

Proposition 4.1.1. *Let $\mathcal{M}$ be a (symmetric) multicategory. A representation of $\mathcal{M}$ is equivalent to an $\mathcal{M}$-algebra.*

Proof. First consider the non-symmetric case. In a closed monoidal category the following adjunction holds:

$$- \otimes_\mathcal{E} E : \underset{\perp}{\rightleftarrows} \underline{Hom}_\mathcal{E}(E, -)$$

for all objects E in $\mathcal{E}$, in particular for $E := \mathcal{X}_{I,\varphi}((t_n, \rho^O \circ l))$. Hence

$$Hom_\mathcal{E}(\mathcal{M}((t_n, l)) \otimes E, \mathcal{X}_{O,\varphi}((t_n, l))) \cong Hom_\mathcal{E}(\mathcal{M}((t_n, l)), \underline{Hom}_\mathcal{E}(E, \mathcal{X}_{O,\varphi}((t_n, l)))),$$

showing the equivalence of the two notions.

Now consider the symmetric case. Let $\rho : \mathcal{M} \to \mathbf{End}_\varphi$ be a representation. Then ρ is a symmetric multifunctor. By definition the maps $\rho_{(t_n,l)} : \mathcal{M}((t_n, l)) \to \underline{Hom}(X_{I,\varphi}((t_n, \rho^O \circ l)), X_{O,\varphi}((t_n, \rho^O \circ l)))$ for all $(t_n, l) \in \mathbb{T}(S)_1$ are equivariant, i.e.

$$\begin{array}{ccc} \underline{\mathcal{M}}((t_n, \phi \circ l)) & \xrightarrow{\rho_{(t_n,\phi\circ l)}} & \underline{\mathbf{End}_\varphi}((t_n, \rho^O \circ \phi \circ l)) \\ \downarrow{\scriptstyle \underline{\mathcal{M}}(\phi)} & & \downarrow{\scriptstyle \underline{\mathbf{End}_\varphi}(\rho^O\circ\phi)} \\ \underline{\mathcal{M}}((t_n, l)) & \xrightarrow[\rho_{(t_n,l)}]{} & \underline{\mathbf{End}_\varphi}((t_n, \rho^O \circ l)) \end{array}$$

for any isomorphism $\phi : (t_n, l) \to (t_n, \phi \circ l)$. By the adjunction any $\rho_{(t_n,l)}$ corresponds to some

$$\alpha_{(t_n,l)} : \mathcal{M}((t_n, l)) \otimes_{\mathcal{E}} X_{I,\varphi}((t_n, \rho^O \circ l)) \to X_{O,\varphi}((t_n, \rho^O \circ l)).$$

The aim is to show that the maps $\alpha_{(t_n,l)}$ are equivariant. This is done by first applying the functor $- \otimes_{\mathcal{E}} \mathcal{X}_{I,\varphi}$ to the above commutative diagram. Then apply the counit to the right side of the diagram to obtain the commutative diagram

$$\begin{array}{ccc}
\mathcal{M}((t_n, l)) \otimes_{\mathcal{E}} X_{I,\varphi}((t_n, \rho^O \circ \phi \circ l)) & \xrightarrow{\alpha_{(t_n,\phi\circ l)}} & X_{O,\varphi}((t_n, \rho^O \circ \phi \circ l)) \\
\downarrow{\scriptstyle \mathcal{M}(\phi)\otimes\mathcal{X}_{I,\varphi}(\rho^O\circ\phi)} & & \| {\scriptstyle id} \\
\mathcal{M}((t_n, l)) \otimes_{\mathcal{E}} X_{I,\varphi}((t_n, \rho^O \circ l)) & \xrightarrow[\alpha_{(t_n,l)}]{} & X_{O,\varphi}((t_n, \rho^O \circ l)).
\end{array}$$

Hence the $\alpha_{(t_n,l)}$ for all $(t_n, l) \in \mathbb{T}(S)$ are equivariant. Finally, by construction of the coend it follows that the maps $\alpha_{(t_n,l)}$ for $(t_n, l) \in \mathbb{T}(S)_{n,\bullet}$ factor through $\mathcal{M} \otimes_{\mathbb{T}(S)^{op}_{n,\bullet}} X_{I,\varphi}$.

The other direction follows immediately from the adjunction too and clearly they are inverses to each other. This shows the equivalence of the two notions for the symmetric case. □

Note that $\mathcal{M} \otimes_{\mathbb{T}(S)^{op}_{n,\bullet}} X_{I,\varphi}$ in the proof above reduces to $\mathcal{M}(n) \otimes_{S_n} V^{\otimes n}$ in the operad case with $V := \varphi(*)$.

4.1.9 Examples of representations

Lie algebras Given a representation for the **Lie** operad over V, i.e. $\rho : \mathbf{Lie} \to \mathbf{End}_V$. The **Lie** operad is binary quadratic therefore by (4.1.3) ρ is completely determined by $\rho^2 : \mathbf{Lie}(2) \to \mathbf{End}_V(2)$. Remember that $\mathbf{Lie}(2) = E_2 = k[S_2] \cdot \mu$ thus ρ^2 is completely determined by where μ is mapped to. Define $m := \rho^2(\mu)$. The aim is to show that (V, m) is a Lie algebra. It has to be shown that m is antisymmetric and satisfies the Jacobi identity. For the antisymmetry consider the following calculation.

$$0 = \rho^2(0) = \rho^2(\mu + \sigma_{(12)}\mu) = m + \sigma_{(12)}m$$

and for the Jacobi identity

$$\begin{aligned}
0 = \rho^2(0) =& \rho^2(\mu \circ (\mu, 1) + \sigma_{(123)}\mu \circ (\mu, 1) + \sigma_{(132)}\mu \circ (\mu, 1)) \\
=& m \circ (m, 1) + \sigma_{(123)}m \circ (m, 1) + \sigma_{(132)}m \circ (m, 1).
\end{aligned}$$

Thus m is a Lie bracket on V.

Associative algebras Note that associative algebras can be interpreted as a special category or as a special operad. Both categories and operads can be obtained as representations, as will be seen shortly, it follows that associative algebras can be obtained as representations as well.

Categories The multicategory whose representations are categories over a fixed set of objects S will now be described. Denote this multicategory by $\mathbf{Cat}_S$ or simply by $\mathbf{Cat}$ if the set S is clear from the context and there is no danger to confuse it with the category of small categories. The objects of $\mathbf{Cat}_S$ are $S \times S$ and the n-ary operations are defined to be

$$\begin{aligned}
\mathbf{Cat}(\cdot|(A, A)) :=& *, \qquad \forall A \in S \\
\mathbf{Cat}((A, B)|(A, B)) :=& *, \qquad \forall (A, B) \in S \times S \\
\mathbf{Cat}((A_1, A_2), (A_3, A_4), \dots, (A_{2n-3}, A_{2n-2}), (A_{2n-1}, A_{2n})|(A_1, A_{2n})) :=& \begin{cases} *, & \forall 1 \leq i \leq n-1, A_{2i} = A_{2i+1}, \\ \emptyset, & \text{otherwise.} \end{cases}
\end{aligned}$$

Now define a representation $\rho : \mathbf{Cat} \to \mathbf{End}_\varphi$ by a function $\varphi : (A,B) \mapsto \varphi((A,B)) =: Hom(A,B)$ and

$$\begin{aligned} \mathbf{Cat}(\cdot|(A,A)) \to & Hom_{\mathbf{Set}}(I, Hom(A,A)) & \rho(*) =& id_A \\ \mathbf{Cat}((A,B)|(A,B)) \to & Hom_{\mathbf{Set}}(Hom(A,B), Hom(A,B)) & \rho(*) =& id_{Hom(A,B)} \\ \mathbf{Cat}((A,B),(B,C)|(A,C)) \to & Hom_{\mathbf{Set}}(Hom(A,B) \times Hom(B,C), Hom(A,C)) & \rho(*) =& \circ_{A,B,C} \end{aligned}$$

It has to be shown that a **Cat**-algebra $(\mathcal{C}, \circ)$ is indeed a category, i.e. the composition is associative and respects the identity. The associativity follows from the diagram below

$$\begin{array}{ccccc} *,(*,*) & \longrightarrow & * & \longleftarrow & *,(*,*) \\ \downarrow{\scriptstyle \rho_{ACCD}\otimes\rho_{ABBC}\otimes\rho_{CD}} & & & & {\scriptstyle \rho_{ABBD}\otimes\rho_{AB}\otimes\rho_{BCCD}}\downarrow \\ \circ_{A,C,D},(\circ_{A,B,C}, id_{C,D}) & & \downarrow{\scriptstyle \rho_{AB}} & & \circ_{A,B,D},(id_{A,B}, \circ_{B,C,D}) \\ & \searrow & & \swarrow & \\ & & \circ_{A,C,D} \circ (\circ_{A,B,C}, id_{C,D}) = \circ_{A,B,D} \circ (id_{A,B}, \circ_{B,C,D}) & & \end{array}$$

and the unit axiom from

$$\begin{array}{ccccc} *,(*,*) & \longrightarrow & * & \longleftarrow & *,(*,*) \\ {\scriptstyle \rho_{ABBB}\otimes\rho_{B}\otimes\rho_{AB}}\downarrow & & \downarrow{\scriptstyle \rho_{AB}} & & \downarrow{\scriptstyle \rho_{AAAB}\otimes\rho_{AB}\otimes\rho_{A}} \\ \circ_{A,B,B},(id_B, id_{Hom(A,B)}) & \longrightarrow & id_{Hom(A,B)} & \longleftarrow & \circ_{A,A,B},(id_{Hom(A,B)}, id_A). \end{array}$$

The category of all small categories **Cat** is fibred over the category **Set** with fibers $\mathbf{Cat}(S) = [\mathbf{Cat}_S, \mathbf{End}]$.

Operads The multicategory **Operad**, due to [3], will now be constructed whose representations are symmetric operads. The objects of **Operad** are the natural numbers including zero. The Hom-set $\mathbf{Operad}(n_1, \dots, n_k \mid n)$ consist of equivalence classes of tuples (T, σ, τ), where T is a planar tree with k vertices and n inputs, a bijection $\sigma : \{1, \dots, k\} \to \text{vertices}(T)$ such that the number of inputs of the i^{th} vertex is n_i and a bijection $\tau : \{1, \dots, n\} \to in(T)$. Two such tuples are equivalent if there exists a planar isomorphism compatible with the bijections. Note that $\mathbf{Operad}(0 \mid 1)$ contains just the unit tree, i.e. the tree with one input and no vertices. Finally set $\mathbf{Operad}(\cdot \mid 1) := I$. The idea behind this construction is, that the compositions of an operad are indexed by trees, hence the planar trees. The σ is needed to be able to describe the symmetric action on the collection **Operad** and the τ is used to describe the symmetry of the collection for the operad.

The unit $I \to \mathbf{Operad}(n \mid n)$ is given by the equivalence class of the n-corolla with τ the increasing ordering.

The composition is defined as follows. Let $T \in \mathbb{T}(\mathbb{N})_{2,\bullet}$ of the shape $T = t_n \circ (1, \dots, 1, t_m, 1, \dots, 1)$. Then the composition $\circ_T : \underline{\mathbf{Operad}}(T) \to \mathbf{Operad} \circ \partial(T)$ is defined by replacing the vertex in the tree of the equivalence class $[(t_n, \sigma_n, \tau_n)]$ corresponding to the inner edge of T by the tree of the equivalence class $[(t_m, \sigma_m, \tau_m)]$.

For the symmetry let $\pi \in S_k$. Then the map

$$\pi^* : \mathbf{Operad}(n_1, \dots, n_k \mid n) \to \mathbf{Operad}(n_{\pi(1)}, \dots, n_{\pi(k)} \mid n)$$

is given by $[(T, \sigma, \tau)] \mapsto [(T, \sigma \circ \pi, \tau)]$.

So far this gives operads in **Set**. To obtain operads in an arbitrary monoidal category copowering can be used, i.e. the collection is given by

$$\mathbf{Operad}(n_1, \dots, n_k \mid n) := \coprod_{[(T,\sigma,\tau)]} I.$$

Given a representation $F : \mathbf{Operad} \to \mathbf{End}$ it will be shown that indeed operads are obtained. First a collection will be constructed. Define $F : \mathbb{T}(\star)_1^{op} \to \mathbf{Set}$ by $t_n \mapsto F^O(n)$ and $| \mapsto I$. The symmetry on this collection is obtained from the fact that $\mathbf{Operad}(n \mid n) \cong S_n^{op}$. Hence the representation gives a map

$$S_n^{op} \cong \mathbf{Operad}(n \mid n) \to Hom(F^O(n) \mid F^O(n)).$$

By the Hom-tensor adjunction a map $S_n^{op} \times F^O(n) \to F^O(n)$ is obtained giving the desired symmetric action.

For the composition consider a tree $T \in \mathbb{T}(\star)_{2,\bullet}$ obtained from grafting an m-corolla on top of an n-corolla. Then $[(T, \sigma, \tau)] \in \mathbf{Operad}(n, m \mid n + m - 1)$, where σ sends $\sigma(1) = t_n$ and $\sigma(2) = t_m$ and τ the increasing order in the planar representation. The composition is then defined to be $c_T := c_{F([(T,\sigma,tau)])}$. Note that $F([(T, \sigma, \tau)] \in Hom(F^O(n) \times F^O(m) \mid F^O(n + m - 1)) = Hom(\underline{F^O}(T) \mid F^O \circ \partial(T))$.

The unit is determined by the single element $[|]$ in $\mathbf{Operad}(\cdot \mid 1)$ and it is given by $F(*) \in Hom(I \mid F^O(1))$.

It remains to check the axioms. The unit axiom is derived from the functor property together with the equality of the following diagram

$$\begin{array}{ccccc}
 & & \mathbf{Operad}(1, n \mid n) \times \mathbf{Operad}(n \mid n) & \overset{\sim}{=\!=} & \mathbf{Operad}(1, n \mid n) \times \mathbf{Operad}(- \mid 1) \times \mathbf{Operad}(n \mid n) \\
 & & \downarrow{\scriptstyle \text{composition}} & & \\
I & \xrightarrow{\text{unit}} & \mathbf{Operad}(n \mid n) & & \\
 & & \uparrow{\scriptstyle \text{composition}} & & \\
 & & \mathbf{Operad}(n, 1 \mid n) \times \mathbf{Operad}(n \mid n). & &
\end{array}$$

The associativity is derived as follows. Let $T \in \mathbb{T}(\star)_{3,\bullet}$ be the tree obtained from grafting an m- and a k-corolla on top of an n-corolla. Then the equality of the following diagram together with the functoriality of F gives the associativity.

$$\begin{array}{ccc}
\mathbf{Operad}(n, (m + k - 1) \mid n + m + k - 2) \times \mathbf{Operad}(m, k \mid m + k - 1) & & \\
 & \searrow & \\
 & & \mathbf{Operad}(n, m, k \mid n + m + k - 2) \\
 & \nearrow & \\
\mathbf{Operad}((n + m - 1), k \mid n + m + k - 2) \times \mathbf{Operad}(n, m \mid n + m - 1). & &
\end{array}$$

The compatibility of the symmetry with the composition is derived by the diagram:

$$\begin{array}{ccc}
\mathbf{Operad}(n, m \mid n + m - 1) \times \mathbf{Operad}(n \mid n) \times \mathbf{Operad}(m \mid m) & & \\
 & \searrow & \\
 & & \mathbf{Operad}(n, m \mid n + m - 1) \\
 & \nearrow & \\
\mathbf{Operad}(n, m \mid n + m - 1) \times \mathbf{Operad}(n \mid n) \times \mathbf{Operad}(m \mid m). & &
\end{array}$$

In words the diagram says that if nothing is permuted, i.e. if the $\tau's$ are all just increasing in the planar representation, then both sides are the same. If on the other hand the trees of $\mathbf{Operad}(n \mid n)$ and $\mathbf{Operad}(m \mid m)$ are permuted and the tree $\mathbf{Operad}(n, m \mid n+m-1)$ is not, then a certain tree is obtained. This same tree can be obtained by permuting the tree $\mathbf{Operad}(n, m \mid n+m-1)$ and the other tree have the increasing order. This exactly means that the composition of the operad is compatible with the symmetric action.

Multicategories There is even a multicategory, $\mathbf{Mult}_S$, which has as representations symmetric multicategories over a fixed set S. The objects of $\mathbf{Mult}_S$ are tuples $(c_1, \dots, c_k|c)$ with $c_i, c \in S$.

The collection is given by $\mathbf{Mult}_S((c_{1,1}, \dots, c_{1,n_1}|c_1), \dots, (c_{k,1}, \dots, c_{k,n_k}|c_k) \mid (c_1, \dots, c_k|c))$ consisting of the equivalence classes $[(T, l, \sigma, \tau)]$ as before with in addition a labeling $l : \text{edges}(T) \to S$ and besides the matching number of inputs of $\sigma(i)$ now also the labels have to match. The remaining constructions are done in exactly same way as for **Operad**. This construction generalizes **Operad** and **Cat**. Since the objects of small multicategories form a set it follows that **Multicat** is fibred over **Set** with fibers $\mathbf{Multicat}(S) := [\mathbf{Mult}(S), \mathbf{End}]$.

4.2 Deformation of Multicategories

The deformation theory of multicategories differs from a straightforward generalization of the deformation theory of operads by the use of $\circ_T$-operations. All notions needed for the deformation of multicategories, like augmentations, equivalence and extensions of deformations, can be formulated with respect to the $\circ_T$-operations. The result is a deformation complex on which a differential graded Lie algebra structure can be defined. Hence obstructions can be conveniently classified, solving the problem described in **??**.

4.2.1 Augmentation

The underlying collection of a multicategory can be augmented the same way as the underlying module of an algebra. Recall that there are functors

$$- \otimes_k R : \mathbf{Mod}_k \longleftrightarrow \mathbf{Mod}_R : - \otimes_R k.$$

Using these functors by post composition with a collection the following functors between collections are obtained:

$$- \otimes_k R : Coll(\mathbf{Mod}_k, S) \longleftrightarrow Coll(\mathbf{Mod}_R, S) : - \otimes_R k.$$

More generally, given two collections, F and G, in $\mathbf{Mod}_k$ it is possible to define a new collection $F \otimes G : \mathbb{T}(S)_1^{op} \to \mathbf{Mod}_k$ by

$$\begin{aligned}(F \otimes G)\,((t_n, l)) :=& F((t_n, l)) \otimes_k G((t_n, l)),\\ (F \otimes G)\,(\sigma) :=& F(\sigma) \otimes G(\sigma).\end{aligned}$$

The previous augmentation can then be interpreted by tensoring with the constant collection R.

4.2.2 Deformation

Let R be a local Artinian ring with residual field k and maximal ideal $\overline{m}$. A multicategory $\mathcal{M}$ in the symmetric monoidal category $\mathbf{Mod}_k = (\mathbf{Mod}_k, \otimes_k, k, \tau)$ is called linear.

Definition 4.2.1. *An R-deformation of $\mathcal{M}$ is a multicategory $\mathcal{N}$ in $\mathbf{Mod}_R$ such that there is a natural isomorphism $\alpha : \mathcal{N} \otimes_R k \to \mathcal{M}$.*

Only multicategories over the collection $\mathcal{M} \otimes_k R$ for some R will be considered rather than an arbitrary multicategory $\mathcal{N}$ in $\mathbf{Mod}_R$. In that case the canonical isomorphism

$$can : (\mathcal{M} \otimes_k R) \otimes_R k \cong \mathcal{M}$$

can be used. A formal deformation is a $k[[t]]$-deformation and an n-deformation is a $k[t]/(t^{n+1})$-deformation.

Let R be an augmentation of k, i.e. the sequence

$$0 \to \overline{m} \xrightarrow{\iota} R \xrightarrow{p} k \to 0$$

is short exact. Let $\mathcal{N}$ be an R-deformation of $\mathcal{M}$. If the functor $\mathcal{N} \otimes_R -$ is flat one obtains a short exact sequence

$$0 \to \mathcal{N} \otimes_R \overline{m} \xrightarrow{\iota} \mathcal{N} \otimes_R R \xrightarrow{p} \mathcal{N} \otimes_R k \to 0.$$

Note that this is equivalent to the pullback diagram

$$\begin{array}{ccccccc} Ker(p) & \overset{\sim}{=\!=} & \mathcal{N} \otimes_R \overline{m} & \xrightarrow{\iota} & \mathcal{N} \otimes_R R & \overset{\sim}{=\!=} & \mathcal{N} \\ & & \downarrow & & \downarrow p & & \\ & & 0 & \longrightarrow & \mathcal{N} \otimes_R k & \underset{\alpha}{\overset{\sim}{=\!=}} & \mathcal{M}. \end{array}$$

If the functor $\mathcal{M} \otimes_k -$ is flat then an augmentation can be used to construct a deformation of $\mathcal{M}$ by applying that functor to the sequence. In that case one obtains the pullback diagram

$$\begin{array}{ccccccc} Ker(p) & \overset{\sim}{=\!=} & \mathcal{M} \otimes_k \overline{m} & \xrightarrow{\iota} & \mathcal{M} \otimes_k R & & \\ & & \downarrow & & \downarrow p & & \\ & & 0 & \longrightarrow & \mathcal{M} \otimes_k k & \overset{\sim}{=\!=} & \mathcal{M}. \end{array}$$

Consider the case for $R = k[t]/(t^2)$. Note that $\mathcal{M} \otimes_R -$ is flat and note that there is an obvious section $\delta : \mathcal{M} \to \mathcal{M} \otimes_k R$ defined by sending θ to $\theta \otimes 1$. Hence the sequence is split exact and one obtains the diagram

$$\begin{array}{ccc} \mathcal{M}t & \xrightarrow{\iota} & \mathcal{M} \oplus \mathcal{M}t \\ \downarrow & & \downarrow p \\ 0 & \longrightarrow & \mathcal{M} \end{array}$$

where the composition $\circ_T$ on $\mathcal{M} \oplus \mathcal{M}t$ for any $T \in \mathbb{T}(S)_{2,\bullet}$ is given by

$$\circ_T : (\theta + \theta_1 t \otimes \gamma + \gamma_1 t) \quad \mapsto \quad (\theta \circ_T \gamma) + (\theta \circ_T \gamma_1 + \theta_1 \circ_T \gamma)t.$$

Note that $(\mathcal{M}t)^2 = 0$, hence $\mathcal{M} \oplus \mathcal{M}t$ is called the square-zero extensions of $\mathcal{M}$.

Now consider formal deformations. By linearity a composition on the collection $\bigoplus_{k\geq 0} \mathcal{M}t^k$ is completely determined by maps $m : \mathcal{M} \otimes_k \mathcal{M} \to \bigoplus_{k\geq 0} \mathcal{M}t^k$, which is equivalent to a family of compositions $\{m^k : \mathcal{M} \otimes_k \mathcal{M} \to \mathcal{M}\}_{k\geq 0}$. Since the composition should be associative and equivariant, there will be some conditions on the family of compositions. These conditions are given by

$$m^k(\alpha \otimes 1) = 0, \qquad \text{(left unit axiom)}$$

$$m^k(1 \otimes \alpha) = 0, \qquad \text{(right unit axiom)}$$

$$m^k_i(\sigma_1(\alpha) \otimes \sigma_2(\beta)) = (\sigma_1 \circ_i \sigma_2) m^k_{\sigma(i)}(\alpha \otimes \beta), \qquad \text{(equivariance)}$$

$$0 = \begin{cases} \sum_{k+l=q} m^k_i(m^l_j(\alpha \otimes \beta) \otimes \gamma) - m^k_{j+p-1}(m^l_i(\alpha \otimes \gamma) \otimes \beta), & 1 \leq i \leq j-1 \\ \sum_{k+l=q} m^k_i(m^l_j(\alpha \otimes \beta) \otimes \gamma) - m^k_j(\alpha \otimes m^l_{i-j+1}(\beta \otimes \gamma)), & j \leq i \leq j+n-1 \\ \sum_{k+l=q} m^k_i(m^l_j(\alpha \otimes \beta) \otimes \gamma) - m^k_j(m^l_{i-n+1}(\alpha \otimes \gamma) \otimes \beta), & j+n \leq i \end{cases} \qquad \text{(associativity)}$$

where $\sigma_1 \circ_i \sigma_2$ is the block permutation (see 4.1.7) and m_i^k is the k^{th} composition in the family grafting the given operations on the i^{th} place. The associativity condition could have been given with the help of the functor $\underline{\mathcal{M}}$. Note that the three cases of the associativity correspond to the three different ways of grafting three corollas into each other.

Equivalence of deformations

Definition 4.2.2. *Two R-deformations $(\mathcal{N}, \alpha)$ and $(\mathcal{N}', \alpha')$ of $\mathcal{M}$ are equivalent if there is a natural isomorphism $\phi : \mathcal{N} \to \mathcal{N}'$ such that the diagram*

$$\begin{array}{ccc} \mathcal{N} \otimes_R k & \xrightarrow{\phi \otimes_R k} & \mathcal{N} \otimes_R k \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & \mathcal{M} & \end{array}$$

commutes.

This is equivalent to asking for the existence of an isomorphism ϕ in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{N} \otimes_R \overline{m} & \xrightarrow{\iota} & \mathcal{N} \otimes_R R & \xrightarrow{p} & \mathcal{N} \otimes_R k & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\phi} & & \Vert{\scriptstyle\sim} & & \\ 0 & \longrightarrow & \mathcal{N}' \otimes_R \overline{m} & \xrightarrow[\iota']{} & \mathcal{N}' \otimes_R R & \xrightarrow[p']{} & \mathcal{N}' \otimes_R k & \longrightarrow & 0. \end{array}$$

The following lemma is one of the main reasons to use local Artinian rings.

Lemma 4.2.1. *Let R be a local Artinian ring with maximal ideal $\overline{m}$. Then any R-linear multifunctor $\phi : \mathcal{N} \to \mathcal{N}'$ such that*

$$\begin{array}{ccc} \mathcal{N} \otimes_R k & \xrightarrow{\phi \otimes_R k} & \mathcal{N}' \otimes_R k \\ & {\scriptstyle\alpha}\searrow \quad \swarrow{\scriptstyle\alpha'} & \\ & \mathcal{M} & \end{array}$$

is an equivalence of deformations.

In case of infinitesimal deformations it holds that two deformations are equivalent if ϕ in

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{M}t & \xrightarrow{\iota} & \mathcal{M} \oplus \mathcal{M}t & \xrightarrow{p} & \mathcal{M} & \longrightarrow & 0 \\ & & \downarrow & & \downarrow{\scriptstyle\phi} & & \Vert & & \\ 0 & \longrightarrow & \mathcal{M}t & \xrightarrow{\iota} & \mathcal{M} \oplus \mathcal{M}t & \xrightarrow{p} & \mathcal{M} & \longrightarrow & 0 \end{array}$$

is an isomorphism. Using the canonical reductions, one finds that ϕ reduced to $\mathcal{M}$ is indeed the identity. This is true for n-deformations and formal deformations as well. Thus an arrow ϕ has the form $1 + \phi_{(t)}$ where (t) is the maximal ideal of the local Artinian algebra. It follows by the previous lemma that it suffices to consider $\phi_{(t)}$, since any such morphism induces an arrow ϕ and hence an equivalence of deformations.

Extensions and obstructions Extensions of deformations of multicategories can be defined too.

Definition 4.2.3. *An extension of an n-deformation $\mathcal{N}$ to an $(n+1)$-deformation is an $k[t]/(t^{n+1})$-deformation $\hat{\mathcal{N}}$ such that $\hat{\mathcal{N}} \otimes_{k[t]/(t^{n+1})} k[t]/(t^n) \cong \mathcal{N}$.*

Since the deformations are supposed to have a special shape an extension of an n-deformation to an $(n+1)$-deformation amounts to add a composition m^{n+1} to the family of the n-deformation such that a similar condition as before is satisfied. The obstruction to a q-extension is then given by:

$$\mathcal{O}_q(\alpha\otimes\beta\otimes\gamma)=\begin{cases}\sum\limits_{\substack{k+l=q+1\\k,l>0}} m_i^k(m_j^l(\alpha\otimes\beta)\otimes\gamma)-m_{j+p-1}^k(m_i^l(\alpha\otimes\gamma)\otimes\beta), & 1\le i\le j-1\\ \sum\limits_{\substack{k+l=q+1\\k,l>0}} m_i^k(m_j^l(\alpha\otimes\beta)\otimes\gamma)-m_j^k(\alpha\otimes m_{i-j+1}^l(\beta\otimes\gamma)), & j\le i\le j+n-1\\ \sum\limits_{\substack{k+l=q+1\\k,l>0}} m_i^k(m_j^l(\alpha\otimes\beta)\otimes\gamma)-m_j^k(m_{i-n+1}^l(\alpha\otimes\gamma)\otimes\beta), & j+n\le i.\end{cases}$$

4.2.3 Deformation Complex

The deformation complex of linear multicategories will be defined by first specifying the modules of the cochain complex. Before defining the differential it will be shown that these modules carry a graded Lie-algebra structure. The differential will then be defined with the help of the Lie bracket. Let $\mathcal{M}$ be a linear multicategory. Then there is a functor $\underline{\mathcal{M}}:\mathbb{T}(Obj(\mathcal{M}))^{op}\to\mathbf{Mod}_k$. Denote the restriction of $\underline{\mathcal{M}}$ to the category $\mathbb{T}(S)_{n,\bullet}$ by $\underline{\mathcal{M}}_{n,\bullet}$.

Define $C^n(\mathcal{M},\mathcal{M})$ to be the end (see A) of the functor $\underline{Hom}_k(\underline{\mathcal{M}}_{n,\bullet},\underline{\mathcal{M}}_{1,\bullet}\circ\partial)$, i.e.

$$C^n(\mathcal{M},\mathcal{M}):=\int_{\mathbb{T}(S)^{op}_{n,\bullet}}\underline{Hom}_k(\underline{\mathcal{M}}_{n,\bullet},\underline{\mathcal{M}}_{1,\bullet}\circ\partial)$$

for all $n\ge 0$. In other words $C^n(\mathcal{M},\mathcal{M})$ consists of natural transformations $m:\underline{\mathcal{M}}_{n,\bullet}\to\underline{\mathcal{M}}_{1,\bullet}\circ\partial$, i.e the diagram

$$\begin{array}{ccc}\underline{\mathcal{M}}_{n,\bullet}(T) & \xrightarrow{m_T} & \underline{\mathcal{M}}_{1,\bullet}\circ\partial(T)\\ {\scriptstyle \underline{\mathcal{M}}_{n,\bullet}(\phi)}\downarrow & & \downarrow{\scriptstyle \underline{\mathcal{M}}_{1,\bullet}\circ\partial(\phi)}\\ \underline{\mathcal{M}}_{n,\bullet}(T') & \xrightarrow[m_{T'}]{} & \underline{\mathcal{M}}_{1,\bullet}\circ\partial(T')\end{array}$$

commutes for all T,T' and ϕ. Note that these modules are constructed such that the $\circ_T$-operations of the multicategory are elements in $C^2(\mathcal{M},\mathcal{M})$.

Graded Lie algebra structure on $C^\bullet[1](\mathcal{M},\mathcal{M})$ Define an operation

$$f\lhd g:\underline{\mathcal{M}}_{n+m-1,\bullet}\longrightarrow\underline{\mathcal{M}}_{1,\bullet}\circ\partial$$

for $f\in C^n(\mathcal{M},\mathcal{M})$ and $g\in C^m(\mathcal{M},\mathcal{M})$ by

$$(f\lhd g)_T:=\sum_{\substack{T'\subset T\\T'\in\mathbb{T}(S)_m}}(-1)^{\varepsilon}(f\lhd_{T'}g),$$

where $f\lhd_{T'}g$ is obtained as follows. First extend the map $g_{T'}:\underline{\mathcal{M}}(T')\to\underline{\mathcal{M}}\circ\partial(T')$ to a map $g'_T:\underline{\mathcal{M}}(T)\to\underline{\mathcal{M}}(T/\text{inner}(T'))$ by defining g'_T to act as the identity on the components of $\underline{\mathcal{M}}(T)$ that do not belong to $\underline{\mathcal{M}}(T')$. Then define $f\lhd_{T'}g$ to be the composition

$$\underline{\mathcal{M}}(T')\xrightarrow{g_{T'}}\underline{\mathcal{M}}(T/\text{inner}(T'))\xrightarrow{f}\underline{\mathcal{M}}(T/\text{inner}(T))=\underline{\mathcal{M}}\circ\partial(T).$$

To obtain ε, let b be the number of components in $\underline{\mathcal{M}}(T)$ before the first component of $\underline{\mathcal{M}}(T')$. Let s be the number of transposition needed to obtain an uninterrupted sequence of components of $\underline{\mathcal{M}}(T')$ such that the component corresponding to the root vertex of T' stays fixed and the order of the components is preserved. Then $\varepsilon := b|g| + s$. Note that in the case of linear trees this exactly gives back the classical sign $|g|(i-1)$, where the root vertex of the subtree T' appears at the i^{th} place.

Proposition 4.2.1. *The operation $\triangleleft$ defines a right pre-Lie algebra structure on $C^\bullet[1](\mathcal{M},\mathcal{M})$.*

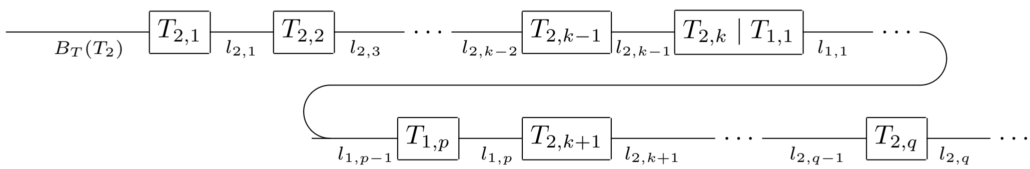
Proof. It has to be shown that the operation $\triangleleft$ satisfies

$$(f \triangleleft g) \triangleleft h - f \triangleleft (g \triangleleft h) = (-1)^{|g||h|} \left((f \triangleleft h) \triangleleft g - f \triangleleft (h \triangleleft g)\right).$$

For a subtree $T' \subseteq T$ define the set $B_T(T')$ to consist of those vertices of T which are before the first vertex of T' with respect to the evaluation order of $\underline{\mathcal{M}}$. For a subtree $T' \subseteq T$ define $l_k(T')$ to be the set of vertices appearing on top of the k^{th} input of T' in T.

Each term of the equation above is a double sum. Suppose h is acting on the subtree T_1 of T with n vertices and g is on the subtree T_2 of T with m vertices. If $T_1 \cup T_2$ is a subtree with $(n+m-1)$ vertices then $g \triangleleft h$ and $h \triangleleft g$ may act on it. In that case $(f \triangleleft_{T_2} g) \triangleleft_{T_1} h = f \triangleleft_{T_1 \cup T_2} (g \triangleleft_{T_1} h)$ and $(f \triangleleft_{T_1} h) \triangleleft_{T_2} g = f \triangleleft_{T_1 \cup T_2} (h \triangleleft_{T_2} g)$. If, on the other hand, T_1 and T_2 are disjoint, then $(f \triangleleft_{T_2} g) \triangleleft_{T_1} h = (f \triangleleft_{T_1} h) \triangleleft_{T_2} g$. Note that these two cases take into account all terms of the double sums. It will be shown that the signs of the corresponding terms are equal. To do this in each of the above cases two further subcases have to be distinguished.

Case Ia: Suppose T_1 is on top of T_2, i.e. the root vertex of T_1 is also a vertex of T_2. In other words: $T_1 - \text{root}(T_1) \subset l_k(T_2)$. The following profile illustrates how $\underline{\mathcal{M}}(T)$ is built up.



Here $T_{i,j}$ stands for the j^{th} uninterrupted sequence of components of $\underline{\mathcal{M}}(T_i)$ in $\underline{\mathcal{M}}(T)$ and $l_{i,j}$ the uninterrupted sequence of the components of $l_j(T_i)$. By definition $T_{2,k}$ contains the root of $T_{1,1}$. First b and then s will be calculated. Then

$$\begin{aligned}\varepsilon_h^1 =&(m-1)\underbrace{\left(B_T(T_2) + (T_{2,1} + \ldots + T_{2,k-1} + (T_{2,k} - 1)) + (l_{2,1} + \ldots + l_{2,k-1})\right)}_{b} + \\ &+ \underbrace{T_{1,2} l_{1,1} + \ldots + T_{1,p}(l_{1,1} + \ldots + l_{1,p-1})}_{s},\end{aligned}$$

$$\begin{aligned}\varepsilon_g^1 =&(n-1)\left(B_T(T_2)\right) + \\ &+ T_{2,2} l_{2,1} + \ldots + T_{2,k}(l_{2,1} + \ldots + l_{2,k-1}) + \\ &+ T_{2,k+1}((l_{2,1} + \ldots + l_{2,k-1}) + (l_{1,1} + \ldots + l_{1,p})) + \\ &+ \ldots + T_{2,q}((l_{2,1} + \ldots + l_{2,k-1}) + (l_{1,1} + \ldots + l_{1,p}) + (l_{2,k+1} + \ldots + l_{2,q-1})).\end{aligned}$$

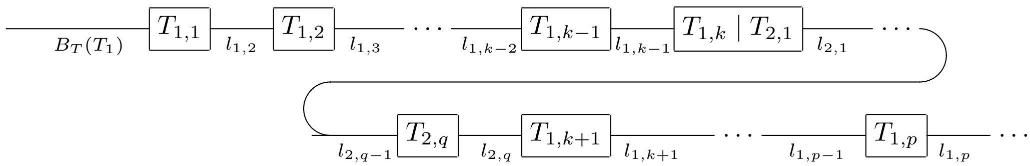
On the other hand $\varepsilon^2_{g\triangleleft h}$ and ε^2_h are

$$\begin{aligned}\varepsilon^2_{g\triangleleft h} =&(m+n-2)\,(B_T(T_2))+\\ &+T_{2,2}l_{2,1}+\ldots+T_{2,k-1}(l_{2,1}+\ldots+l_{2,k-2}+(T_{2,k}+T_{1,1})(l_{2,1}+\ldots+l_{2,k-1})+\\ &+T_{1,2}((l_{2,1}+\ldots+l_{2,k-1})+l_{1,1})+\\ &+\ldots+T_{1,p}((l_{2,1}+\ldots+l_{2,k-1})+(l_{1,1}+\ldots+l_{1,p-1})+\\ &+T_{2,k+1}((l_{2,1}+\ldots+l_{2,k-1})+(l_{1,1}+\ldots+l_{1,p})+\\ &+\ldots+T_{2,q}((l_{2,1}+\ldots+l_{2,k-1})+(l_{1,1}+\ldots+l_{1,p})+(l_{2,k+1}+\ldots+l_{2,q-1})),\end{aligned}$$

$$\varepsilon^2_h =(n-1)\,(T_{2,1}+\ldots+T_{2,k-1}+(T_{2,k}-1))\,.$$

Comparing the terms it follows that $(\varepsilon^1_h+\varepsilon^1_g)-(\varepsilon^2_{g\triangleleft h}+\varepsilon^2_h)=0$ (mod2). Therefore the corresponding terms cancel each other.

Case Ib: In this case the role of T_1 and T_2 are switched, i.e. $T_2-\mathrm{root}(T_2)\subset l_k(T_1)$. The difference with case Ia is, that this time the contraction of T_1 has effect on T_2. The profile of $\underline{\mathcal{M}}(T)$ is:



This time $T_{1,k}$ is defined to contain root(T_2). Then ε^1_h and ε^1_g are given by

$$\begin{aligned}\varepsilon^1_h =&(n-1)\,(B_T(T_1))+\\ &+T_{1,2}l_{1,1}+\ldots+T_{1,k}(l_{1,1}+\ldots+l_{1,k-1})+\\ &+T_{1,k+1}\,((l_{1,1}+\ldots+l_{1,k-1})+(l_{2,1}+\ldots+l_{2,q})+(T_{2,1}+\ldots+T_{2,q}))+\\ &+T_{1,k+2}\,((l_{1,1}+\ldots+l_{1,k-1})+(l_{2,1}+\ldots+l_{2,q})+(T_{2,1}+\ldots+T_{2,q})+l_{1,k+1})+\\ &+\ldots+T_{1,p}\,((l_{1,1}+\ldots+l_{1,k-1})+(l_{2,1}+\ldots+l_{2,q})+(T_{2,1}+\ldots+T_{2,q})+(l_{1,k+1}+\ldots+l_{1,p-1}))\,,\end{aligned}$$

$$\begin{aligned}\varepsilon^1_g =&(m-1)\,(B_T(T_1))+\\ &+T_{2,1}\,((l_{1,1}+\ldots+l_{1,k-1}))+\\ &+T_{2,2}\,((l_{1,1}+\ldots+l_{1,k-1})+l_{2,1})+\\ &+\ldots+T_{2,q}\,((l_{1,1}+\ldots+l_{1,k-1})+(l_{2,1}+\ldots+l_{2,q-1}))\,.\end{aligned}$$

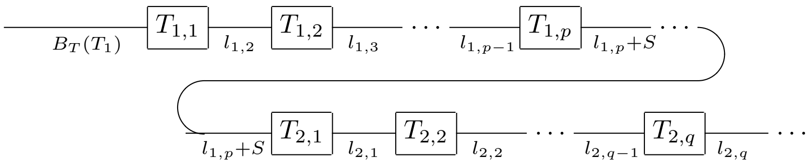
On the other hand:

$$\begin{aligned}\varepsilon^2_{g\triangleleft h} =&(m+n-2)\,(B_T(T_2))+\\ &+T_{1,2}l_{1,1}+\ldots+T_{1,k-1}(l_{1,1}+\ldots+l_{1,k-2})+(T_{1,k}+T_{2,1})(l_{1,1}+\ldots+l_{1,k-1})+\\ &+T_{2,2}\,((l_{1,1}+\ldots+l_{1,k-1})+l_{2,1})+\\ &+\ldots+T_{2,q}\,((l_{1,1}+\ldots+l_{1,k-1})+(l_{2,1}+\ldots+l_{2,q-1}))+\\ &+T_{1,k+1}\,((l_{1,1}+\ldots+l_{1,k-1})+(l_{2,1}+\ldots+l_{2,q-1})+l_{2,q})+\\ &+\ldots+T_{1,p}\,((l_{1,1}+\ldots+l_{1,k-1})+(l_{2,1}+\ldots+l_{2,q-1})+l_{2,q}+(l_{1,k+1}+\ldots+l_{1,p-1}))\,,\end{aligned}$$

$$\begin{aligned}\varepsilon^2_h =&(n-1)\,(0)+\\ &+T_{1,k+1}\,(T_{2,1}+\ldots+T_{2,q})+\\ &+\ldots+T_{1,p}\,(T_{2,1}+\ldots+T_{2,q})\,.\end{aligned}$$

Note that $T_{2,1}+\ldots+T_{2,q}=T_2=m$. Matching the terms one finds again that $(\varepsilon_h^1+\varepsilon_g^1)-(\varepsilon_{g\lhd h}^2+\varepsilon_h^2)=0$ (mod2).

Case IIa: Suppose that T_1 and T_2 are disjoint and suppose that T_1 appears completely before T_2 in $\underline{\mathcal{M}}(T)$. In other words: $T_1 \subseteq B_T(T_2)$. Then the profile looks like



where S is some fixed set of vertices between $l_{1,p}$ and $T_{2,1}$. Then

$$\begin{aligned}\varepsilon_h^1 =&(n-1)\,(B_T(T_1))+\\ &+T_{1,2}l_{1,1}+\ldots+T_{1,p}\left((l_{1,1}+\ldots+l_{1,k-1})+(l_{1,k+1}+\ldots+l_{1,p-1})\right),\end{aligned}$$

$$\begin{aligned}\varepsilon_g^1 =&(m-1)\,(B_T(T_1)+1+(l_{1,1}+\ldots+l_{1,p})+S)+\\ &+T_{2,2}l_{2,1}+\ldots+T_{2,q}(l_{2,1}+\ldots+l_{2,q-1}),\end{aligned}$$

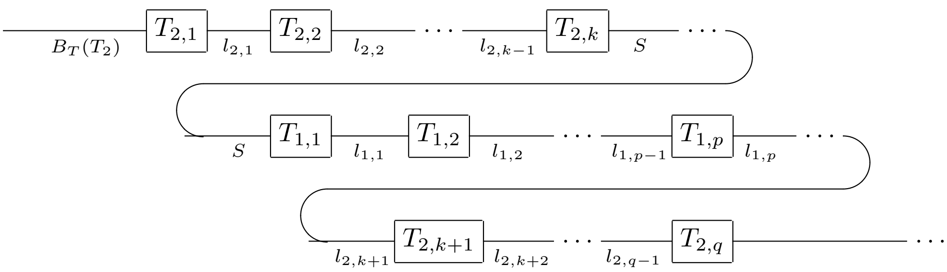
and

$$\begin{aligned}\varepsilon_g^2 =&(m-1)\,(B_T(T_1)+(T_{1,1}+\ldots+T_{1,p})+(l_{1,1}+\ldots+l_{1,p})+S)+\\ &+T_{2,2}l_{2,1}+\ldots+T_{2,q}(l_{2,1}+\ldots+l_{2,q-1}),\\ \varepsilon_h^2 =&(n-1)\,(B_T(T_1))+\\ &+T_{1,2}l_{1,1}+\ldots+T_{1,p}\left((l_{1,1}+\ldots+l_{1,k-1})+(l_{1,k+1}+\ldots+l_{1,p-1})\right).\end{aligned}$$

The aim is to show that $(\varepsilon_h^1+\varepsilon_g^1)=(\varepsilon_h^2+\varepsilon_g^2)+(m-1)(n-1)$. Note that $\varepsilon_h^1=\varepsilon_h^2$. Subtracting the second sum of the first gives:

$$\begin{aligned}(\varepsilon_h^1+\varepsilon_g^1)-(\varepsilon_h^2+\varepsilon_g^2)+(m-1)(n-1) =&\varepsilon_g^1-\varepsilon_g^2-(m-1)(n-1)\\ =&(m-1)-(m-1)(T_{1,1}+\ldots+T_{1,p})-(m-1)(n-1)\\ =&(m-1)(1-n-(n-1))\\ =&0 \text{ (mod2)}.\end{aligned}$$

Case IIb: T_1 and T_2 are again disjoint, but this time T_1 appears on top of the k^{th} input of T_2 for some k, i.e. $T_1 \subset l_k(T_2)$. The profile is then:



for some fixed set S. Then

$$\begin{aligned}\varepsilon_h^1 =&(n-1)\left(B_T(T_2)+(T_{2,1}+\ldots+T_{2,k})+(l_{2,1}+\ldots+l_{2,k-1})+S\right)+\\ &+T_{1,2}l_{1,1}+\ldots+T_{1,p}\left((l_{1,1}+\ldots+l_{1,k-1})+(l_{1,k+1}+\ldots+l_{1,p-1})\right),\end{aligned}$$

$$\begin{aligned}\varepsilon_g^1 =&(m-1)\left(B_T(T_2)\right)+\\ &+T_{2,2}l_{2,1}+\ldots+T_{2,k}(l_{2,1}+\ldots+l_{2,k-1})+\\ &+T_{2,k+1}\left((l_{2,1}+\ldots+l_{2,k-1})+(l_{1,1}+\ldots+l_{1,p})+S+1\right)+\\ &+\ldots+T_{2,q}\left((l_{2,1}+\ldots+l_{2,k-1})+(l_{1,1}+\ldots+l_{1,p})+S+1+(l_{2,k+1}+\ldots+l_{2,q-1})\right),\end{aligned}$$

and

$$\begin{aligned}\varepsilon_g^2 =&\varepsilon_g^1+\\ &+T_{2,k+1}(T_{1,1}+\ldots+T_{1,p})+\ldots+T_{2,q}(T_{1,1}+\ldots+T_{1,p}).\\ \varepsilon_h^2 =&(n-1)\left(B_T(T_2)+1+(l_{2,1}+\ldots+l_{2,k-1})+S\right)+\\ &+T_{1,2}l_{1,1}+\ldots+T_{1,p}\left((l_{1,1}+\ldots+l_{1,k-1})+(l_{1,k+1}+\ldots+l_{1,p-1})\right).\end{aligned}$$

It follows that $(\varepsilon_h^1+\varepsilon_g^1)=(\varepsilon_h^2+\varepsilon_g^2)+(m-1)(n-1)$.

This completes the proof. □

Hence the bracket

$$[\varphi,\gamma]:=\varphi\triangleleft\gamma-(-1)^{|\varphi||\gamma|}\gamma\triangleleft\varphi$$

defines a graded Lie algebra structure on $C^\bullet[1](\mathcal{M},\mathcal{M})$.

Differential graded Lie algebra structure on $C^\bullet[1](\mathcal{M},\mathcal{M})$ There is a special element in $C^1[1](\mathcal{M},\mathcal{M})$ namely the composition $\circ$ of $\mathcal{M}$. Define the differential $d: C^n[1](\mathcal{M},\mathcal{M})\to C^{n+1}[1](\mathcal{M},\mathcal{M})$ to be

$$d(\varphi):=[\circ,\varphi].$$

Together with this differential $C^\bullet[1](\mathcal{M},\mathcal{M})$ forms a complex and furthermore a differential graded Lie algebra.

Consider the case for $n=0$, and let $\varphi\in C^0(\mathcal{M},\mathcal{M})$. Then $\varphi_{(|,l)}:\underline{\mathcal{M}}((|,l))\to\mathcal{M}\circ\partial((|,l))$ is a map $I\to\mathcal{M}(A\mid A)$, where $A:=l(|)$. Therefore

$$C^0(\mathcal{M},\mathcal{M})\cong\coprod_{A\in Obj(\mathcal{M})}\mathcal{M}(A\mid A).$$

The differential $d(\varphi)$ is given by

$$d(\varphi)_T=[\circ,\varphi]_T=\coprod_{\substack{T'\subseteq T\\ T'\in\mathbb{T}(S)_{0,\bullet}}}(-1)^{\varepsilon}\circ\triangleleft_{T'}\varphi+\coprod_{\substack{T'\subseteq T\\ T'\in\mathbb{T}(S)_{2,\bullet}}}(-1)^{\varepsilon'}\varphi\triangleleft_{T'}\circ=\coprod_{\substack{T'\subseteq T\\ T'\in\mathbb{T}(S)_{0,\bullet}}}(-1)^{\varepsilon}\circ\triangleleft_{T'}\varphi$$

for all $T\in\mathbb{T}(S)_{1,\bullet}$. Thus $d(\varphi)_T(\theta)$ is obtained by the sum of all possible compositions of φ with θ.

Examples Let $T := t_2(BC|A) \circ_1 t_2(DE|B) \circ_1 t_2(FG|E) \circ_2 t_3(HIJ|C) \in \mathbb{T}(S)_{4,6}$ and let $\theta_1 \otimes \theta_2 \otimes \theta_3 \otimes \theta_4 \in \underline{\mathcal{M}}(T) = \mathcal{M}(BC|A) \otimes \mathcal{M}(DE|B) \otimes \mathcal{M}(FG|E) \otimes \mathcal{M}(HIJ|C)$. Then

$$\begin{aligned} d(\varphi)(\theta_1 \otimes \theta_2 \otimes \theta_3 \otimes \theta_4) =& (-1)^{0+0}\varphi(\theta_1 \otimes \theta_2 \otimes \theta_4) \circ \theta_3 + \\ &+ (-1)^{0+1}\varphi(\theta_1 \otimes \theta_2 \otimes \theta_3) \circ \theta_4 - \\ &- (-1)^{0+0}\varphi((\theta_1 \circ \theta_2) \otimes \theta_3 \otimes \theta_4) - \\ &- (-1)^{|\circ|1+0}\varphi(\theta_1 \otimes (\theta_2 \circ \theta_3) \otimes \theta_4) - \\ &- (-1)^{0+2}\varphi((\theta_1 \circ \theta_4) \otimes \theta_2 \otimes \theta_3). \end{aligned}$$

Note that the leading minus sign of the last three terms comes from $d(\varphi) = [\circ, \varphi] = \circ \lhd \varphi - (-1)^{1\cdot 2}\varphi \lhd \circ$.

Consider now an example where T is linear. Let $T := t_1(A|B) \circ_1 t_1(B|C) \circ_1 t_1(C|D)$ and $\theta_1 \otimes \theta_2 \otimes \theta_3 \in \underline{\mathcal{M}}(T)$. Let $\varphi \in C^2(\mathcal{M}, \mathcal{M})$.

$$\begin{aligned} d(\varphi)(\theta_1 \otimes \theta_2 \otimes \theta_3) =& (-1)^{|\varphi|1+0}\theta_1 \circ \varphi(\theta_2 \otimes \theta_3) + \varphi((\theta_1 \circ \theta_2) \otimes \theta_3) + \\ &+ (-1)^{|\circ|1+0}\varphi(\theta_1 \otimes (\theta_2 \circ \theta_3)) + \varphi(\theta_1 \otimes \theta_2) \circ \theta_3. \end{aligned}$$

In this case the ordinary Hochschild differential is obtained.

Finally consider the example where $\varphi \in C^0(\mathcal{M}, \mathcal{M})$. Let (t_3, l) such that $\underline{\mathcal{M}}((t_3, l)) = \mathcal{M}(A, B, C|D)$. The subtrees of (t_3, l) with zero vertices are $|A, |B, |C$ and $|D$, where the edges have been decorated by the label. Then

$$\begin{aligned} (-1)^{\varepsilon_1}(\circ \lhd_{|A} \varphi) : \mathcal{M}(A, B, C|D) \cong& \mathcal{M}(A, B, C|D) \otimes I \xrightarrow{1\otimes\varphi} \mathcal{M}(A, B, C|D) \otimes \mathcal{M}(A|A) \xrightarrow{\circ} \mathcal{M}(A, B, C|D) \\ (-1)^{1(-1)+0}(\circ \lhd_{|A} \varphi) =& - f \circ_1 \varphi(1). \\ (-1)^{\varepsilon_2}(\circ \lhd_{|B} \varphi) : \mathcal{M}(A, B, C|D) \cong& \mathcal{M}(A, B, C|D) \otimes I \xrightarrow{1\otimes\varphi} \mathcal{M}(A, B, C|D) \otimes \mathcal{M}(B|B) \xrightarrow{\circ} \mathcal{M}(A, B, C|D) \\ (-1)^{1(-1)+0}(\circ \lhd_{|B} \varphi) =& - f \circ_2 \varphi(1). \\ (-1)^{\varepsilon_3}(\circ \lhd_{|C} \varphi) : \mathcal{M}(A, B, C|D) \cong& \mathcal{M}(A, B, C|D) \otimes I \xrightarrow{1\otimes\varphi} \mathcal{M}(A, B, C|D) \otimes \mathcal{M}(C|C) \xrightarrow{\circ} \mathcal{M}(A, B, C|D) \\ (-1)^{1(-1)+0}(\circ \lhd_{|C} \varphi) =& - f \circ_3 \varphi(1). \end{aligned}$$

$$\begin{aligned} (-1)^{\varepsilon_4}(\circ \lhd_{|D} \varphi) : \mathcal{M}(A, B, C|D) \cong& I \otimes \mathcal{M}(A, B, C|D) \xrightarrow{\varphi\otimes 1} \mathcal{M}(D|D) \otimes \mathcal{M}(A, B, C|D) \xrightarrow{\circ} \mathcal{M}(A, B, C|D) \\ (-1)^{0+0}(\circ \lhd_{|D} \varphi) =& \varphi(1) \circ_1 f. \end{aligned}$$

Therefore the differential is given by

$$d(\varphi)_{(t_3,l)}(f) = \varphi(1) \circ_1 f - f \circ_1 \varphi(1) - f \circ_2 \varphi(1) - f \circ_3 \varphi(1).$$

In the algebra case this just means $d(\varphi)(\theta) = \varphi \circ \theta - \theta \circ \varphi$ and for operads $d(\varphi)(\theta) = \varphi \circ \theta - \theta \circ_1 \varphi - \ldots - \theta \circ_n \varphi$ if the arity of θ is n.

Normalized Complex As noticed in 4.2.2, the unit axiom forces φ to be zero. In order to ensure this property for categories and associative algebras it was shown that the normalized cochain complex was a deformation retraction. In the proof the notion of an i-cochain was used, so the question is what the equivalent statement is for general trees and not just linear ones. The definition that as soon as one of the first i vertices, with respect to the order imposed by $\underline{\mathcal{M}}$, is an identity the image should be zero turns out to be too naive. Everything works fine, except for the case that the identity appears at an input vertex. In that case the sign will depend on the number of vertices under and left of the identity which depends on the shape of the tree and one can easily construct a counter example. Hence another way to normalize the complex has to be found.

The dual Dold-Kan correspondence (see [32]) states that there is an equivalence of categories between $\mathbf{Ch}^+(\mathcal{A})$ and $[\mathbf{\Delta}, \mathcal{A}]$ where $\mathcal{A}$ is any abelian category. The equivalence is given by the Moore complex functor N and an inverse K. These two functors will be described below.

Consider the functor $N : [\mathbf{\Delta}, \mathcal{A}] \to \mathbf{Ch}^+(\mathcal{A})$. For a cosimplicial object A then the cochain complex $(N(A)^\bullet, \partial)$ is defined as follows. The modules $N(A)^n$ are given by

$$N(A)^n := A^n \big/ _{(\coprod_{i=1}^{n} d_i(A^{n-1}))} \cong \bigcap_{i=0}^{n-1} Ker(s_i)$$

and the differentials by

$$\partial := \sum_{i=0}^{n} (-1)^i d_i.$$

An inverse to N is given by $K : \mathbf{Ch}^+(\mathcal{A}) \to [\mathbf{\Delta}, \mathcal{A}]$ sending a cochain complex $C^\bullet$ to the cosimplicial object $K(C^\bullet) : \mathbf{\Delta} \to \mathcal{A}$. The object $K(C^\bullet)_n$ of n-cosimplices is defined to be

$$K(C^\bullet)_n := \coprod_{p=0}^{n} \binom{n}{p} C^p.$$

Note that $\binom{n}{p}$ is the number of epimorphisms from $[n]$ to $[p]$ in $\mathbf{\Delta}$. Let $\alpha : [m] \to [n]$ be an arrow in $\mathbf{\Delta}$. The map $K(\alpha)$ will now be described by specifying which components C^p and C^q in $K(C^\bullet)_m$ and $K(C^\bullet)_n$ will be identified, respectively. Suppose C^p corresponds to the epimorphism $\eta : [m] \to [p]$ and C^q to $\eta' : [n] \to [q]$. Consider the diagram

$$\begin{array}{ccc} [m] & \xrightarrow{\alpha} & [n] \\ {\scriptstyle \eta}\downarrow & & \downarrow{\scriptstyle \eta'} \\ [p] & \underset{\varepsilon}{\dashrightarrow} & [q]. \end{array}$$

If $\varepsilon = id_{[q]}$ makes the diagram commute then C^p is identified with C^q by the identity map. If $\varepsilon = \delta_q$ with $q = p + 1$ makes the diagram commute then C^p and C^q are identified by $C^p \xrightarrow{d} C^{p+1} = C^q$. In all other cases there is no identification.

The coface and codegeneracy maps can be derived from the general case by $d_i := K(\delta_i)$ and $s_i := K(\sigma_i)$. The cosimplicial algebra $K(C^\bullet)$ may be depicted as

$$C^0 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow}\; C^0 \oplus C^1 \;\substack{\longrightarrow \\ \longleftarrow \\ \longrightarrow \\ \longleftarrow \\ \longrightarrow}\; C^0 \oplus C^1 \oplus C^1 \oplus C^2 \;\substack{\dashrightarrow \\ \dashrightarrow \\ \dashrightarrow \\ \dashrightarrow}\; \cdots$$

where the first five face and three degeneracy maps are:

$$\begin{aligned} d_0(x) &= x + 0 & d_0(x + y) &= x + 0 + y + 0 \\ d_1(x) &= x + d(x) & d_1(x + y) &= x + y + y + 0 \\ s_0(x + y) &= x & d_2(x + y) &= x + y + d(x) + d(y) \\ & & s_0(x + y + z + w) &= x + z \\ & & s_1(x + y + z + w) &= x + y. \end{aligned}$$

Since $C^\bullet(\mathcal{M}, \mathcal{M})$ is a cochain complex and $\mathbf{Mod}_k$ is abelian, $N \circ K(C^\bullet(\mathcal{M}, \mathcal{M}))$ is naturally isomorphic to $C^\bullet(\mathcal{M}, \mathcal{M})$. It follows that $N \circ K(C^\bullet(\mathcal{M}, \mathcal{M}))$ inherits a dg-Lie algebra structure. So the deformation complex of $\mathcal{M}$ with values in itself is defined to be $N \circ K(C^\bullet(\mathcal{M}, \mathcal{M}))$.

Cohomology of multicategories The cohomology of multicategories is defined to be the cohomology of the cochain complex $(C^\bullet(\mathcal{M},\mathcal{M}),d)$, i.e.

$$H^n(\mathcal{M},\mathcal{M}) := H^n(C^\bullet(\mathcal{M},\mathcal{M}),d).$$

Note that in cohomology the normalized and non-normalized deformation complex can not be distinguished therefore it is sufficient to consider the non-normalized one.

4.2.4 Classification

Derivations Let $\delta \in C^1(\mathcal{M},\mathcal{M})$ be a cocycle, i.e. for all $T = T_1 \circ_e T_2 \in \mathbb{T}_{2,\bullet}$ it holds that

$$0 = d(\delta)(\theta_1 \otimes \theta_2) = \theta_1 \circ_{T/e} \delta_{T_2}(\theta_2) - \delta_{T/e}(\theta_1 \circ_T \theta_2) + \delta_{T_1}(\theta_1) \circ_{T/e} \theta_2,$$

where e is the inner edge of T. In case of linear trees with a singleton set, i.e. in the associative case, this exactly defines a derivation of δ. On the other hand note that $d(\phi)$ gives rise to an inner derivation, where $\phi \in C^0(\mathcal{M},\mathcal{M})$. It follows that $H^1(\mathcal{M},\mathcal{M})$ consists of the outer derivations.

Infinitesimal Deformations Let $\mathcal{M}$ be a linear multicategory with composition $\circ$. By construction it holds that if $\varphi \in C^2(\mathcal{M},\mathcal{M})$ is a cocycle then the sum $\circ + \varphi t$ defines a multicategory structure on the extended collection $\mathcal{M} \otimes_k k[t]/(t^2)$. On the other hand, let $(\mathcal{M}_t, m_t)$ and $(\mathcal{M}_t, m'_t)$ be two equivalent deformations of $\mathcal{M}$ with equivalence ϕ_t. The compatibility with the composition means that the following equation must hold: $m'_t \circ (\phi_t \otimes \phi_t) = \phi_t \circ m_t$. By substituting ϕ_t with $1 + \phi_1 t$, m_t with $m_0 + m_1 t$ and m'_t with $m_0 + m'_1 t$, it follows that m'_1 and m_1 differ by $d(\phi_1)$. This shows that equivalent infinitesimal deformations differ by a coboundary. Therefore

Theorem 4.2.1. $H^2(\mathcal{M},\mathcal{M})$ *classifies all infinitesimal deformations up to equivalence.*

Obstructions Using the differential graded Lie algebra structure on $C^\bullet[1](\mathcal{M},\mathcal{M})$ the obstructions are given by

$$\mathcal{O}_q = \frac{1}{2} \sum_{\substack{k+l=q+1 \\ k,l>0}} [m_k, m_l]$$

for all $m_k, m_l \in C^1[1](\mathcal{M},\mathcal{M})$. It follows from 1.5.1 that $\mathcal{O}_q$ is a cocycle, hence $[\mathcal{O}_q] \in H^3(\mathcal{M},\mathcal{M})$.

4.3 Examples of deformations of multicategories

4.3.1 Associative algebras

It will be shown that in the case of a unital associative algebra the complex defined for multicategories reduces to the Hochschild complex. Let (A, m, u) be a unital associative algebra over k. The collection $A : \mathbb{T}(*)_{1,\bullet} \to \mathbf{Mod}_k$ of A is given by $A(t_1) := A$ and all the other Hom-sets are 0. The composition is given by the multiplication m. Then

$$\begin{aligned} C^n(A,A) &= \int_{\mathbb{T}(*)_{n,1}} \underline{Hom}_k(\underline{A}_{n,1}, \underline{A}_{1,1} \circ \partial) \\ &= \int_{\mathbb{T}(*)_{n,1}} \underline{Hom}_k(A^{\otimes n}, A) && (\underline{A}(t_1 \circ \ldots \circ t_1) = A \otimes \ldots \otimes A) \\ &= \underline{Hom}_k(A^{\otimes n}, A) && \text{(the symmetric action is trivial)} \\ &= C^n_{\mathrm{Hoch}}(A,A). \end{aligned}$$

The differential reduces to the Hochschild differential, therefore the deformation complex coincides with the Hochschild complex.

4.3.2 Categories

A category is also a special case of a multicategory: the underlying collection is trivial for all trees with $k \neq 1$ inputs. The symmetric action is trivial, therefore the end vanishes and the natural transformation reduces to a family of maps between the objects indexed by all trees with n vertices and one input. The differential coincides with the Hochschild differential for categories, therefore the complexes are equal.

The deformation complex of categories is also related to a particular case of the Baues-Wirsching complex, defined as follows. First the notion of a natural system has to be introduced. Define the category of fractions $\mathbb{F}\mathcal{C}$ to have all arrows of $\mathcal{C}$ as objects and arrows are pairs (α, β) of arrows in $\mathcal{C}$ such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{\alpha} & B \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ A & \xrightarrow[\beta]{} & A' \end{array}$$

commutes in $\mathcal{C}$. A natural system is then a functor $D : \mathbb{F}\mathcal{C} \to \mathbf{Ab}$. An example of such a natural system is the composite $\mathbb{F}\mathcal{C} \xrightarrow{\pi} \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{Hom_k} \mathbf{Ab}$, where the first functor is given by $f \mapsto (dom(f), cod(f))$ and $(\alpha, \beta) \mapsto (\beta, \alpha)$. Note that here the linearity of the category has been used in order to land in $\mathbf{Ab}$ instead of $\mathbf{Set}$. The complex is then defined to be

$$\begin{aligned} C^0_{BW}(\mathcal{C}, D) :=& \coprod_{A \in \mathcal{C}_0} D_{id_A} \\ C^n_{BW}(\mathcal{C}, D) :=& \{\varphi \in Hom_k(N\mathcal{C}_n, \coprod_{g \in \mathcal{C}_1} D_g) \mid \varphi(\theta_1 \otimes \ldots \otimes \theta_n) \in D_{\theta_1 \circ \ldots \circ \theta_n}\} \end{aligned}$$

and the differential is given by

$$\begin{aligned} d(\varphi)(\theta_1) :=& D(\theta_1, 1)(\varphi(dom(f))) - D(1, \theta_1)(\varphi(cod(f))) \\ d(\varphi)(\theta_1 \otimes \ldots \otimes \theta_n) :=& D(\theta_1, 1)\varphi(\theta_2 \otimes \ldots \otimes \theta_n) + \\ &+ \sum_{i=1}^{n-1} (-1)^i \varphi(\theta_1 \otimes \ldots \otimes (\theta_i \circ \theta_{i+1}) \otimes \ldots \otimes \theta_n) + \\ &+ (-1)^n D(1, \theta_n)\varphi(\theta_1 \otimes \ldots \otimes \theta_{n-1}). \end{aligned}$$

Now fix the natural system given in the example and denote by S the objects of $\mathcal{C}$. Let $\varphi \in C^n(\mathcal{C}, \mathcal{C})$, then there are morphisms $\varphi_T : \underline{\mathcal{C}}(T) \to \mathcal{C}(T/_{\text{inner}(T)})$. Note that

$$N\mathcal{C}_n = \coprod_{T \in \mathbb{T}(S)_{n,1}} \underline{\mathcal{C}}(T) \quad \text{and} \quad N\mathcal{C}_1 = \coprod_{T \in \mathbb{T}(S)_{1,\bullet}} \mathcal{C} \circ \partial.$$

Further, for the given natural system it holds that $\coprod_{g \in \mathcal{C}_1} (Hom_k \circ \pi)_g = N\mathcal{C}_1$. Let $\theta_1 \otimes \ldots \otimes \theta_n \in \underline{\mathcal{C}}(T)$, i.e. $\theta_1 \otimes \ldots \otimes \theta_n$ is a sequence of composable arrows. Note that $\varphi_T(\theta_1 \otimes \ldots \otimes \theta_n) \in \mathcal{C}(T/_{\text{inner}(T)})$, showing that $\varphi_T(\theta_1 \otimes \ldots \otimes \theta_n) \in D_{theta_1 \circ \ldots \circ \theta_n} \hookrightarrow \coprod_{g \in \mathcal{C}_1} D_g$. Then it holds that

$$\begin{array}{ccc} \coprod_{T \in \mathbb{T}(S)_{n,1}} \underline{\mathcal{C}}(T) & \overset{\exists!}{\dashrightarrow} & \coprod_{T \in \mathbb{T}(S)_{1,1}} \underline{\mathcal{C}}(T) \\ \uparrow & & \uparrow \\ \underline{\mathcal{C}}(T) & \xrightarrow[\varphi_T]{} & \mathcal{C}(T/_{\text{inner}(T)}) \end{array}$$

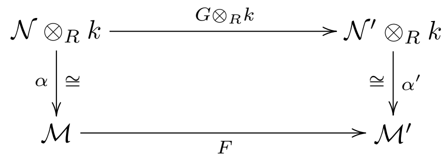
for all $T \in \mathbb{T}(S)_{n,1}$. Therefore $\varphi \in C^n_{BW}(\mathcal{C}, Hom_k \circ \pi)$.

On the other hand, let $\varphi \in C^n_{BW}(\mathcal{C}, Hom_k \circ \pi)$. It follows that $\varphi : N\mathcal{C}_n \to N\mathcal{C}_1$. Let $\theta_1 \otimes \ldots \otimes \theta_n \in N\mathcal{C}_n$, then $\theta_1 \otimes \ldots \otimes \theta_n \in \underline{\mathcal{C}}(T)$ for some $T \in \mathbb{T}(S)_{n,\bullet}$. Define $\varphi_T := \varphi|_{\underline{\mathcal{C}}(T)}$. Then φ defines a multifunctor from $\underline{\mathcal{C}}_n$ to $\mathcal{C} \circ \partial$, since there is just a trivial symmetric action. Therefore $C^n(\mathcal{C}, \mathcal{C})$ and $C^n_{BW}(\mathcal{C}, Hom_k \circ \pi)$ are equal. This suggests to generalize natural systems to multicategories, but it is not clear what that should be if it makes sense at all.

4.4 Deformations of Multifunctors

Let $F : \mathcal{M} \to \mathcal{M}'$ be a multifunctor.

Definition 4.4.1. *A deformation of F is a multifunctor $G : \mathcal{N} \to \mathcal{N}'$ between deformations $\mathcal{N}$ and $\mathcal{N}'$ of $\mathcal{M}$ and $\mathcal{M}'$, respectively, such that the diagram*

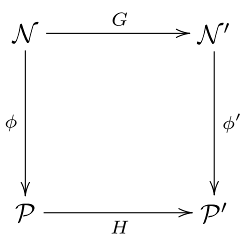


commutes.

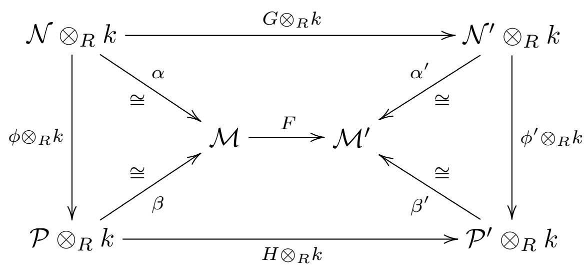
A functor G between formal deformations of $\mathcal{M}$ and $\mathcal{M}'$ can be written as $\sum_{k\geq 0} G_k t^k$. Since the reduction is given by evaluation t in zero, it follows that $G_0 = F$.

An equivalence between two deformations is defined as follows.

Definition 4.4.2. *Let $G : \mathcal{N} \to \mathcal{N}'$ and $H : \mathcal{P} \to \mathcal{P}'$ be two deformations of $F : \mathcal{M} \to \mathcal{M}'$. G and H are equivalent if there exist two isomorphisms $\phi : \mathcal{N} \to \mathcal{P}$ and $\phi' : \mathcal{N}' \to \mathcal{P}'$ such that the diagrams*



and



commute.

Denote the respective compositions of $\mathcal{M} \otimes_k k[[t]]$ and $\mathcal{M}' \otimes_k k[[t]]$ by $\sum_{i\geq 0} \alpha_i t^i$ and $\sum_{j\geq 0} \beta_j t^j$. Then $\sum_{k\geq 0} F_k t^k$ is a multifunctor between the formal deformations if and only if the F_k's satisfy the following conditions:

$$F_k(id) = 0, \qquad \text{(unit axiom)}$$

$$\sum_{\substack{i+j=q\\ i,j\geq 0}} F_i \circ \alpha_j - \sum_{\substack{p+l+r=q\\ k,l,r\geq 0}} \beta_r \circ (F_p \otimes F_l) = 0, \qquad \text{(compatibility with the compositions)}$$

for all $q \geq 0$ and all $k \geq 1$.

4.4.1 Modules

In order to define the deformation complex, modules over a multicategory have to be introduced.

Definition 4.4.3. *A left $\mathcal{M}$-module is a collection N together with maps*

$$m_{t_n \circ t_m} : N(t_n) \otimes \mathcal{M}(t_m) \to N \circ \partial(t_n \circ t_m),$$

for all $t_n \circ t_m \in \mathbb{T}(S)_{2,\bullet}$ satisfying an obvious associativity law.

A right $\mathcal{M}$-module is a collection N together with maps

$$m_{t_n \circ t_m} : \mathcal{M}(t_n) \otimes N(t_m) \to N \circ \partial(t_n \circ t_m),$$

for all $t_n \circ t_m \in \mathbb{T}(S)_{2,\bullet}$ again satisfying an associativity law.

An $(\mathcal{M}, \mathcal{M})$-bimodule N is a left and right $\mathcal{M}$-module.

An example of a module which will play an important role in the following construction is as follows. Let $F : \mathcal{M} \to \mathcal{N}$ be a multifunctor. Then F gives rise to an $(\mathcal{M}, \mathcal{M})$-bimodule structure on $\mathcal{N}$ by the composition

$$\circ_T^F : \mathcal{N}(t_n) \otimes \mathcal{M}(t_m) \xrightarrow{1 \otimes F} \mathcal{N}(t_n) \otimes \mathcal{N}(t_m) \xrightarrow{\circ_T^{\mathcal{N}}} \mathcal{N} \circ \partial(T),$$
$$\circ_T^F : \mathcal{M}(t_n) \otimes \mathcal{N}(t_m) \xrightarrow{F \otimes 1} \mathcal{N}(t_n) \otimes \mathcal{N}(t_m) \xrightarrow{\circ_T^{\mathcal{N}}} \mathcal{N} \circ \partial(T).$$

The deformation complex of a multicategory $\mathcal{M}$ may be generalized to take values in an arbitrary $(\mathcal{M}, \mathcal{M})$-bimodule N by

$$C^n(\mathcal{M}, N) := \int_{\mathbb{T}(S)^{op}_{n,\bullet}} \underline{Hom}_k(\underline{M}_{n,\bullet}, N \circ \partial)$$

with differential $d(\varphi) := m \triangleleft \varphi - (-1)^{|m||\varphi|} \varphi \triangleleft \circ_{\mathcal{M}}$.

Given a multifunctor $F : \mathcal{M} \to \mathcal{N}$ between two multicategories. Then by the previous example $\mathcal{N}$ is endowed with an $(\mathcal{M}, \mathcal{M})$-bimodule structure. The differential of $C^n(\mathcal{M}, \mathcal{N})$ is denoted by

$$d_F(\varphi) = \circ_{\mathcal{N}}^F \triangleleft \varphi - (-1)^{|\varphi|} \varphi \triangleleft \circ_{\mathcal{M}}.$$

Remark 4.4.1. *Note that there is no Lie algebra structure on $C^\bullet(\mathcal{M}, N)$.*

4.4.2 Deformation complex

Consider the equation

$$\sum_{\substack{i+j=q \\ i,j \geq 0}} F_i \circ \alpha_j - \sum_{\substack{p+l+r=q \\ k,l,r \geq 0}} \beta_r \circ (F_p \otimes F_l) = 0$$

for $q = 1$, i.e.

$$F \circ \alpha_1 + (F_1 \circ \alpha - \beta \circ (F \otimes F_1) - \beta \circ (F_1 \otimes F)) - \beta_1 \circ (F \otimes F) = F \circ \alpha_1 - \beta_1 \circ (F \otimes F) - d_F(F_1).$$

This motivates the following definition. The deformation complex of a functor $F : \mathcal{M} \to \mathcal{M}'$ is defined to be

$$C^n(F) := C^n(\mathcal{M}, \mathcal{M}) \times C^n(\mathcal{N}, \mathcal{N}) \times C^{n-1}(\mathcal{M}, \mathcal{N}),$$

for $n \geq 1$ and the differential

$$d(\alpha, \beta, \varphi) := \left(d(\alpha), d(\beta), F \circ \alpha - \beta \circ F^{\otimes n} - d_F(\varphi)\right).$$

The set $C^n(F)$ is endowed with a k-module structure by pointwise addition and multiplication. The module $C^0(F)$ is defined to be trivial, i.e. $C^0(F) := 0$. Note that this complex is normalized if each of the constituting chains is. The cohomology of F is then the cohomology of the deformation complex.

4.4.3 Classification

First cohomology group The first cohomology group is the quotient of the kernel by the image of the following maps.

$$0 \xrightarrow{0} C^1(F) \xrightarrow{d} C^2(F).$$

Thus $H^1(F) = Ker(d)$. Let $\alpha \otimes \beta \otimes \gamma \in C^1(F)$. Then

$$(0,0,0) = d(\alpha,\beta,\gamma) = \left(d(\alpha), d(\beta), F \circ \alpha - \beta \circ F - d_F(\varphi)\right).$$

This shows that α and β have to be cocycles. If both α and β are 0 then it follows that γ is a cocycle in $C^\bullet(\mathcal{M},\mathcal{M}')$.

Infinitesimal deformations The motivation for the complex comes from the following. Let $F_t : \mathcal{M}_t \to \mathcal{M}'_t$ be an infinitesimal deformation of $F : \mathcal{M} \to \mathcal{M}'$. Then it holds that for some $T \in \mathbb{T}(S)_{2,n+m-1}$, where $T = t_n \circ t_m$, the diagram

$$\begin{array}{ccc}
(\mathcal{M}(t_n) \oplus \mathcal{M}(t_n)t) \otimes (\mathcal{M}(t_m) \oplus \mathcal{M}(t_m)t) & \xrightarrow{\circ_T^M + \alpha_T^1 t} & \mathcal{M}(\partial(T)) \oplus \mathcal{M}(\partial(T))t \\
\downarrow {\scriptstyle F_{t_n} + F^1_{t_n} t \otimes F_{t_m} + F^1_{t_m} t} & & \downarrow {\scriptstyle F_{\partial(T)} + F^1_{\partial(T)} t} \\
(\mathcal{M}'(t_n) \oplus \mathcal{M}'(t_n)t) \otimes (\mathcal{M}'(t_m) \oplus \mathcal{M}'(t_m)t) & \xrightarrow[\circ_T^N + \beta_T^1 t]{} & \mathcal{M}'(\partial(T)) \oplus \mathcal{M}'(\partial(T))t
\end{array}$$

commutes. Take care not to confuse the formal parameter t with a tree. This shows that

$$\begin{aligned}
\circ_T^N \circ (F_{t_n} \otimes F_{t_m}) &= F_{\partial(T)} \circ \circ_T^M \\
\circ_T^N \circ (F^1_{t_n} \otimes F_{t_m}) + \circ_T^N \circ (F_{t_n} \otimes F^1_{t_m}) + \beta^1_T \circ (F_{t_n} \otimes F_{t_m}) &= F^1_{\partial(T)} \circ \circ_T^M + F_{\partial(T)} \circ \alpha^1_T.
\end{aligned}$$

The first equation is satisfied since F is a multifunctor and the second equation is exactly $d(\alpha^1 \otimes \beta^1 \otimes F^1)$. Therefore:

Lemma 4.4.1. *Cocycles in $C^2(F)$ correspond to deformations of the source and target category and deformations of multifunctors.*

On the other hand:

Lemma 4.4.2. *Equivalent infinitesimal deformations differ by a coboundary.*

Proof. Suppose G and H are equivalent infinitesimal deformations of F. Then the diagram

$$\begin{array}{ccc}
\mathcal{M}(T) \oplus \mathcal{M}(T)t & \xrightarrow{F_T + G^1_T t} & \mathcal{M}'(T) \oplus \mathcal{M}'(T)t \\
\downarrow {\scriptstyle 1_T + \varphi^1_T t} & & \downarrow {\scriptstyle 1_T + \phi^1_T t} \\
\mathcal{M}(T) \oplus \mathcal{M}(T)t & \xrightarrow[F_T + H^1_T t]{} & \mathcal{M}'(T) \oplus \mathcal{M}'(T)t
\end{array}$$

commutes. Thus

$$H^1_T \circ 1_T + F_T \circ \varphi^1_T = \phi^1_T \circ F_T + 1_T \circ G^1_T.$$

This shows that $(0,0,H^1) = (0,0,G^1) + d(\varphi^1, \phi^1, 0)$, which gives the desired result. □

Note that in the last step it has been used that $C^\bullet(F)$ is a product of the constituent cochains rather than a tensor product. It follows that:

Theorem 4.4.1. *Infinitesimal deformations of F are classified by $H^2(F)$.*

Extensions Let F_t be an n-deformation of F, i.e. $F_t : \bigoplus_{k=0}^{n} \underline{\mathcal{M}}t^k \to \bigoplus_{k=0}^{n} \underline{\mathcal{M}'}t^k$. Given a functor F^{n+1}, $F + \sum_{k=1}^{n} F^k t^k + F^{n+1}t^{n+1}$ is an $(n+1)$-deformation of F if and only if the following conditions are satisfied:

$$\begin{aligned} F^k \circ \phi &= \phi \circ F^k, && \text{(equivariance)} \\ F^k(1) &= 0, && \text{(unit axiom)} \\ \sum_{\substack{i+j_1+j_2=n+1 \\ i,j_1,j_2 \geq 0}} \beta^i \circ (F^{j_1} \otimes F^{j_2}) - F^{(j_1+j_2)} \circ \alpha^i &= 0, && \text{(associativity)} \end{aligned}$$

for all $k \geq 0$. Suppose that either i, j_1 or j_2 is equal to $(n+1)$, then

$$-F \circ \alpha^{n+1} + \beta^{n+1} \circ (F \otimes F) + \beta \circ (F^{n+1} \otimes F) + \beta \circ (F \otimes F^{n+1}) - F^{n+1} \circ \alpha = d(\alpha^{n+1}, \beta^{n+1}, F^{n+1}).$$

Define the n^{th} obstruction to be

$$\begin{aligned} \mathfrak{O}_n := & \sum_{\substack{i+j_1+j_2=n+1 \\ i,j_1,j_2 > 0}} \beta^i \circ (F^{j_1} \otimes F^{j_2}) - F^{(j_1+j_2)} \circ \alpha^i + \\ & + \sum_{\substack{i+j_1+j_2=n+1 \\ i=0,j_1,j_2 \geq 0}} \beta \circ (F^{j_1} \otimes F^{j_2}) - F^{(j_1+j_2)} \circ \alpha + \\ & + \sum_{\substack{i+j_1+j_2=n+1 \\ j_1=0,i,j_2 \geq 0}} \beta^i \circ (F \otimes F^{j_2}) - F^{j_2} \circ \alpha^i + \\ & + \sum_{\substack{i+j_1+j_2=n+1 \\ j_2=0,i,j_1 \geq 0}} \beta^i \circ (F^{j_1} \otimes F) - F^{j_1} \circ \alpha^i . \end{aligned}$$

It follows that

$$\sum_{\substack{i+j_1+j_2=n+1 \\ i,j_1,j_2 \geq 0}} \beta^i \circ (F^{j_1} \otimes F^{j_2}) - F^{(j_1+j_2)} \circ \alpha^i = d(\alpha^{n+1}, \beta^{n+1}, F^{n+1}) + \mathfrak{O}_n.$$

Unfortunately the obstruction can not be written in terms of a Lie bracket, since no differential graded Lie algebra structure could be found on $C^\bullet(F)$.

4.5 Examples of deformations of multifunctors

4.5.1 Associative algebras

Let (A, m) be a unital associative algebra and $\rho : \mathbf{Ass} \to \mathbf{End}_A$ the corresponding representation, i.e. $\rho(2) : \mathbf{Ass}(2) \to Hom(A^{\otimes 2}, A)$ is defined by $\rho(2)(id) = m$.

Lemma 4.5.1. *A infinitesimal deformation of (A, m) corresponds to an infinitesimal deformation of ρ and vice versa.*

Proof. Suppose $(A \oplus At, m_t)$ is an infinitesimal deformation of (A, m). Then there is a representation $\chi : \mathbf{Ass} \to \mathbf{End}_{A\oplus At}$ with $\chi_2(1) = m_t$. Since $\mathbf{End}_{A\oplus At} \cong \mathbf{End}_A \oplus \mathbf{End}_A t$ there is a representation $\rho + \gamma t : \mathbf{Ass} \to \mathbf{End}_A \oplus \mathbf{End}_A t$. It holds that $\gamma \in C^1(\mathbf{Ass}, \mathbf{End}_A)$. Since $\rho + \gamma t$ is a representation which reduces to ρ when evaluated in $t = 0$, it follows that $\rho + \gamma t$ is an infinitesimal deformation of ρ, hence $[\gamma] \in H^2(\rho)$.

Suppose $\rho + \gamma t : \mathbf{Ass} \to \mathbf{End}_A \oplus \mathbf{End}_A t$ is an infinitesimal deformation of $\rho : \mathbf{Ass} \to \mathbf{End}_A$. Again using the isomorphism $\mathbf{End}_{A\oplus At} \cong \mathbf{End}_A \oplus \mathbf{End}_A t$ it follows that there is a representation $\chi : \mathbf{Ass} \to \mathbf{End}_{A\oplus At}$. Define $m_t := \chi_2(1)$. Then $(A \oplus At, m_t)$ is an associative algebra which reduces to (A, m) and hence an infinitesimal deformation of (A, m). Note that m_t is completely determined by $\rho_2(1) + \gamma_2(1)t = m + m_1 t$ for some $m_1 \in C^2_{\text{Hoch}}(A, A)$. If follows that $[m_1] \in H^2_{\text{Hoch}}(A, A)$. □

In order to relate the cohomology groups $H^2(\rho)$ and $H^2_{\text{Hoch}}(A,A)$ it remains to check what happens for equivalent deformations.

Lemma 4.5.2. *Equivalent infinitesimal deformations of* (A,m) *give rise to an equivalence of infinitesimal deformations of* ρ*.*

Proof. Suppose $(A\oplus At, m_t)$ and $(A\oplus At, m'_t)$ are equivalent infinitesimal deformations of (A,m). Then there are representations $\chi : \mathbf{Ass} \to \mathbf{End}_{A\oplus At}$ and $\chi' : \mathbf{Ass} \to \mathbf{End}_{A\oplus At}$ with $\chi_2(1) = m_t$ and $\chi'_2(1) = m'_t$. By the isomorphism $\mathbf{End}_{A\oplus At} \cong \mathbf{End}_A \oplus \mathbf{End}_A t$ there are correpsonding representations $\rho+\gamma t$ and $\rho+\gamma' t$. The aim is to show that $\gamma' \in [\gamma] \in H^2(\rho)$.

The fact that m_t and m'_t are equivalent implies that there is an isomorphism $\phi : A \to A$ such that $m'_1 = m_1 + d(\phi)$, where m_t and m'_t are identified with $m+m_1t$ and $m+m'_1t$, respectively and $\phi \in C^1_{\text{Hoch}}(A,A)$. Hence $\gamma'_2(1) = m'_1 = \gamma_2(1) + d(\phi)$.

Define $\varphi_n : \mathbf{End}_A(n) \to \mathbf{End}_A(n)$ by

$$f \mapsto [f,\phi]_{\text{Hoch}}$$

for all $n \geq 0$. Note that $\varphi_2(m) = [m,\phi] = d(\phi)$. Thus it holds that $\gamma = \gamma' - \varphi \circ \rho$ and therefore

$$(0,0,\gamma) = (0,0,\gamma') + d(0,\varphi,0).$$

This shows that $\rho+\gamma t$ and $\rho+\gamma'$ are equivalent infinitesimal deformations of ρ. □

This shows that there is a well-defined map $H^2_{\text{Hoch}}(A,A) \to H^2(\rho)$. Define $H^2_{01}(\rho)$ to be the group of all $[\gamma] \in H^2(\rho)$ such that $\rho + \gamma t : \mathbf{Ass} \to \mathbf{End}_A \oplus \mathbf{End}_A$ is an infinitesimal deformation of ρ.

Proposition 4.5.1. $H^2_{\textit{Hoch}}(A,A) \cong H^2_{01}(\rho)$*.*

Proof. The previous construction assigns to an infinitesimal deformation of (A,m) an infinitesimal deformation in $H^2_{01}(\rho)$ of ρ. There is also an assignment of any element $H^2_{01}(\rho)$ to an element in $H^2_{\text{Hoch}}(A,A)$. It remains to check that this map is well-defined, i.e. equivalent deformations in $H^2_{01}(\rho)$ give equivalent deformations in $H^2_{\text{Hoch}}(A,A)$.

Let $\rho+\gamma t$ and $\rho+\gamma' t$ be equivalent deformations of ρ. Then there are α and β such that

$$(0,0,\gamma) = (0,0,\gamma') + d(\alpha,\beta,0).$$

Since **Ass** is just the trivial deformation it follows that $\alpha = 0$. Thus it holds for $m_1 := \gamma_2(1)$ and $m'_1 := \gamma'_2(1)$ that

$$m_1 = m'_1 - \beta_2(m).$$

If $\beta_2(m)$ is a coboundary then the result follows. Since $m+m_1t$ and $m+m'_1t$ are infinitesimal deformations of m it follows that $m'_1 \in [m_1] \in H^2_{\text{Hoch}}(A,A)$. Therefore $[m_1] = [m'_1] = [m_1 + \beta_2(m)] = [m_1] + [\beta_2(m)]$. This shows that $\beta_2(m)$ is a coboundary for some $\phi \in C^1_{\text{Hoch}}(A,A)$. Therefore m_1 and m'_1 differ by a coboundary, hence equivalent. □

Let $(\oplus^n_{k=0}, m_t)$ be an n-deformation of (A,m) and $\chi^n : \mathbf{Ass} \to \mathbf{End}_{\oplus^n_{k=0}}$ the corresponding representation. Then there exists a representation $\sum^n_{k=0} \rho_k t^k : \mathbf{Ass} \to \oplus^n_{k=0}\mathbf{End}_A t^k$ such that m_t corresponds to $m + m_1 t + \ldots + m_n t^n$. Suppose $(\oplus^n_{k=0}, m_t)$ extends to an $(n+1)$-deformation $(\oplus^{n+1}_{k=0}, m'_t)$. Then there is a representation $\chi^{n+1} : \mathbf{Ass} \to \mathbf{End}_{\oplus^{n+1}_{k=0}}$ such that $\chi^{n+1}_2(1) = m'_t$. Hence there is a representation $\sum^{n+1}_{k=0} \rho_k t^k : \mathbf{Ass} \to \oplus^{n+1}_{k=0}\mathbf{End}_A t^k$ such that m'_t corresponds to $m + m_1 t + \ldots + m_n t^n + m_{n+1} t^{n+1}$. This is clearly an $(n+1)$-deformation of ρ, hence the vanishing of the Hochschild obstruction $\mathcal{O}_n$ implies that the obstruction $\mathcal{O}^{\mathbf{Ass}}_n$ vanishes. The converse statement, if the representation extends from an n-deformation to an $(n+1)$-deformation then $\mathcal{O}_n$ is zero, is similarly derived.

4.5.2 Lie algebras

The previous constructions hold equally well for the Lie case since no special use of the multicategory in question has been made except in the definition of φ which becomes $\varphi_n := [-, \phi]_{CE}$. Therefore

Proposition 4.5.2. $H^2(\mathfrak{g}, \mathfrak{g}) \cong H^2_{01}(\rho : \boldsymbol{Lie} \to \boldsymbol{End}_{\mathfrak{g}})$.

and an extension of a representation implies an extension of multiplications and vice versa.

4.5.3 Categories

With the same reasoning it holds that

Proposition 4.5.3. $H^2(\mathcal{C}) \cong H^2_{01}(\rho : \boldsymbol{Cat} \to \boldsymbol{End})$

for a linear category $\mathcal{C}$ with objects S. Here the objects of **End** are chose to be $Obj(\mathbf{Mod}_k)$, i.e. $\varphi = id$.

Let $\rho : \mathbf{Cat} \to \mathbf{End}$ be the corresponding representation, with $\rho^O : S \times S \to \mathbf{Mod}_k$ defined by $(A, B) \mapsto \mathcal{C}(A, B)$. The idea is that an infinitesimal deformation $(\mathcal{C} \oplus \mathcal{C}t, m_t)$ of $(\mathcal{C}, \circ)$ corresponds to a representation $\chi : \mathbf{Cat} \to \mathbf{End}$ with $\chi^O(A, B) = \mathcal{C}(A, B) \oplus \mathcal{C}(A, B)t$. Then $\chi_2^{ABC}(*) = m_t^{ABC} \in Hom(\mathcal{C}(A, B) \oplus \mathcal{C}(A, B)t \otimes \mathcal{C}(B, C) \oplus \mathcal{C}(B, C)t, \mathcal{C}(A, C) \oplus \mathcal{C}(A, C)t)$. But then there is a representation $\rho + \gamma t : \mathbf{Cat} \to \mathbf{End}$ with $\rho_{ABC}(*) + \gamma_{ABC}(*)t = \circ_{ABC} + m_{ABC}t$ and $\gamma^O := \rho^O$. Since this is a representation which reduces to ρ when evaluated in $t = 0$ it follows that $\rho + \gamma t$ is an infinitesimal deformation of ρ.

4.5.4 Multicategories

Everything goes through the same way for operads and for multicategories as for categories, showing that $H^2(\mathcal{P}, \mathcal{P})$ and $H^2(\mathcal{M}, \mathcal{M})$ are mapped to $H^2(\rho : \rho : \mathbf{Oper} \to \mathbf{End})$ and $H^2(\rho : \mathbf{Mult} \to \mathbf{End})$, respectively.

Chapter 5

Epilogue

So far the formal deformation theory of algebras, Lie algebras, categories, operads and multicategories have been described. From here there are several ways to go. On one hand, it is possible to consider structures on the higher extensions of the underlying collection. In this way it is possible to give a deformation theoretic interpretation to the higher cohomology groups. Instead of looking at formal deformations one might study other sorts of deformations, like continuous ones or the deformations of structure constants. On the other hand the structures at hand may be varied. It would be interesting to consider a deformation theory of monoid objects in some monoidal category, since all of the structures mentioned earlier are monoid objects in appropriate categories. Some of these topics will be in short described in this chapter.

5.1 Higher Extensions

Recall that an augmentation of a k-module V by a ring R is a short exact sequence $0 \to V \otimes_k \mathfrak{m} \to V \otimes_k R \to V \otimes_k k \cong V \to 0$. This corresponds to the notion of a 1-extension where an n-extension is defined as follows.

Definition 5.1.1. *Let $\mathcal{C}$ be an abelian category. An n-extension of A by B in $\mathcal{C}$ is an exact sequence*

$$0 \to B \to X_n \to \ldots \to X_1 \to A \to 0.$$

Two n-extensions X and Y are said to be equivalent if there exist $f_i : X_i \to Y_i$ for all $1 \leq i \leq n$ such that the diagram

$$\begin{array}{ccccccccccccc} 0 & \longrightarrow & B & \longrightarrow & X_n & \longrightarrow & \cdots & \longrightarrow & X_1 & \longrightarrow & A & \longrightarrow & 0 \\ & & \| & & \downarrow{\scriptstyle f_n} & & & & \downarrow{\scriptstyle f_1} & & \| & & \\ 0 & \longrightarrow & B & \longrightarrow & Y_n & \longrightarrow & \cdots & \longrightarrow & Y_1 & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

commutes. The set of all equivalence classes of n-extensions of A by B is denoted by $\mathbf{Ext}^n_{\mathcal{C}}(A, B)$. The Baer sum defines a group structure on $\mathbf{Ext}^n_{\mathcal{C}}(A, B)$, it is given by

$$X \oplus Y := 0 \to B \to X_n \cup_B Y_n \to X_{n-1} \oplus Y_{n-1} \to \ldots \to X_2 \oplus Y_2 \to X_1 \times_A Y_1 \to A \to 0,$$

where $X_n \cup_B Y_n$ denotes the pushout and $X_1 \times_A Y_1$ the pullback.

Let X be an element of $\mathbf{Ext}^m_{\mathcal{C}}(A, B)$ and Y an element of $\mathbf{Ext}^n_{\mathcal{C}}(B, C)$ then the Yoneda composite $X \circ Y$, defined by

$$X \circ Y := 0 \to C \to Y_n \to \ldots \to Y_1 \xrightarrow{\iota_X \circ \pi_Y} X_n \to \ldots \to X_1 \to A \to 0,$$

where $\pi_Y : X_1 \to B$ and $\iota_X : B \to X_n$, gives an $(m+n)$-extension $\mathbf{Ext}^{m+n}_{\mathcal{C}}(A, C)$.

This Yoneda composite defines a (**Grp**-enriched) category structure on the collection $\mathbf{Ext}^{\bullet}_{\mathcal{C}} : \mathbb{T}(Obj(\mathcal{C}))^{op}_1 \to$ **Grp**, where $\mathbf{Ext}^{\bullet}_{\mathcal{C}}$ is given by

$$(t_0, l) \mapsto 1, \qquad (t_1, l) \mapsto \coprod_{n\geq 0} \mathbf{Ext}^n_{\mathcal{C}}(l \circ in(t_1)|l \circ root(t_1)), \qquad (t_n, l) \mapsto 0,$$

for all $n \geq 2$.

Associative case Let A be a k-algebra and M an (A, A)-bimodule. Then it holds that

$$\mathbf{Ext}^{\bullet}_{\mathbf{Mod}_{A\otimes_k A^{op}}}(A, M) \cong H^{\bullet+1}_{\text{Hoch}}(A, M).$$

In other words, the extensions of A by an (A, A)-bimodule M correspond to higher Hochschild cohomology groups.

To see how this works consider the case $n = 1$. Let $[f] \in H^2(A, M)$. Define a multiplication on $A \oplus M$ by

$$(x, y)(z, w) := (xy, xw + yz + f(x, z)).$$

There are obvious inclusion and projection maps $M \xrightarrow{\iota} A \oplus M$ and $A \oplus M \xrightarrow{\pi} A$. It follows that

$$0 \to M \xrightarrow{\iota} A \oplus M \xrightarrow{\pi} A \to 0$$

is a 1-extension and hence the corresponding equivalence class is in $\mathbf{Ext}^1_{A\otimes A^{op}-\mathbf{Mod}}(A, M)$.

On the other hand, let $[0 \to M \xrightarrow{\iota} X \xrightarrow{\pi} A \to 0] \in \mathbf{Ext}^1_{A\otimes_k A^{op}-\mathbf{Mod}}(A, M)$. Choose a section $\sigma : A \to X$, which can be done since π is a surjection. This implies that $X \cong A \oplus M$. Then it is possible to endow $A \oplus M$ with a multiplication defined by

$$(x, y)(z, w) := (xy, xw + yz + f(x, z)),$$

where $f : A \otimes_k A \to M$ is a k-linear map. The associativity of the just defined multiplication implies that the following equation must hold:

$$-xf(z, u) + f(xz, u) - f(x, zu) + f(x, z)u = 0.$$

Thus $d_{\text{Hoch}}(f) = 0$ and hence $[f] \in H^2(A, M)$. These two constructions are inverses to each other. For a more detailed treatment see [19].

Lie case Consider the universal enveloping algebra of a Lie group $\mathfrak{g}$. Then it holds that

$$\mathbf{Ext}^{\bullet}_{\mathcal{U}(\mathfrak{g})-\mathbf{Mod}}(\mathcal{U}(\mathfrak{g}), M) \cong H^{\bullet+1}_{CH}(\mathfrak{g}, M),$$

for a $\mathcal{U}(\mathfrak{g})$-module. In this way the Lie algebra cohomology is obtained.

It would be interesting to give also to the higher extensions a deformation theoretic interpretation and to extend the isomorphisms to the case of categories, operads and multicategories.

5.2 Continuous Deformations

Recall that a formal deformation is of the form $\sum_{k\geq 0} m_k t^k$ with $m_0 = m$ where both m and the sum are of the same structure. Instead of formal deformations now continuous deformations will be considered. A deformation is continuous if the assignment of a structure to some parameter, say t, is continuous and for a particular point, $t = 0$, the original, undeformed, structure is obtained. This implies that the set of structures has to be endowed with a topology.

As an example, consider the case of associative k-algebras. Let (A, m) be an associative k-algebra. In order to describe the set of all structures, the multiplication on a vector space is expressed in terms of structure constants. This works only if the underlying k-vector space is finite dimensional or such that certain sums converge. For simplicity assume it to be finite dimensional. Denote the category of finite dimensional unital associative k-algebras by $\mathbf{fuAss}_k$.

Suppose A is an n-dimensional vector space with basis $e_1, \ldots, e_n$. Then a multiplication m gives rise to n^3 constants $(c_{i,j}^k)_{1 \geq i,j,k, \geq n} \subset k^{n^3}$ by

$$m(e_i, e_j) = \sum_{k=1}^{n} c_{i,j}^k e_k.$$

These constants are called structure constants. Note that these constants still make sense if the basis is infinite and the above sum converges. Not all vectors in k^{n^3} give rise to associative multiplications. The condition on such a vector is

$$\sum_{p,r=1}^{n} c_{i,j}^p c_{p,k}^r - \sum_{q,r=1}^{n} c_{j,k}^q c_{i,q}^r = 0.$$

This equation is obtained from writing out the equation $m(m(e_i, e_j), e_k) = m(e_i, m(e_j, e_k))$ in terms of the structure constants. Define linear maps $p_{i,j,k} : k^{n^3} \to k$ by

$$p_{i,j,k}(v) = \sum_{p,q,r=1}^{n} v_{i,j}^p v_{p,k}^r - v_{j,k}^q v_{i,q}^r.$$

Thus $\{v \in k^{n^3} \mid p_{i,j,k}(v) = 0, \forall 1 \leq i,j,k \leq n\}$ is isomorphic to all associative k-algebras of dimension n. If the underlying field k has characteristic zero, then the norm $||x|| := \sqrt{\langle x, x \rangle}$, where $\langle x, y \rangle := \sum_{i=1}^{n}(x_i y_i)$, induces a topology on k^{n^3}. The subset of k^{n^3} of all associative k-algebras of dimension n can be endowed with the subspace topology.

Definition 5.2.1. *A continuous deformation of an associative k-algebra (A, m) of dimension n is a path in $\{v \in k^{n^3} \mid p_{i,j,k}(v) = 0, \forall 1 \geq i,j,k \geq n\}$, i.e. a continuous map*

$$\gamma : I \to \{v \in k^{n^3} \mid p_{i,j,k}(v) = 0, \forall 1 \leq i,j,k \leq n\},$$

such that $0 \in I$ and $\gamma(0) = m$.

The theory of continuous deformations is described in [4].

5.3 Deformation in terms of the structure constants

In the previous section structure constants have been introduced. Now another way of looking at structure constants will be given bringing the deformation problem into the arena of algebraic geometry. Note that the functions $p_{i,j,k}^{<n>} : k^{n^3} =: \mathbb{A}^{n^3} \to k$ are polynomials in $k[\{v_{i,j}^k\}_{i,j,k}] =: R$. It was shown that the set $\{v \in k^{n^3} \mid p_{i,j,k}(v) = 0, 1 \leq i,j,k \leq n\}$ corresponds to all n-dimensional associative k-algebras. Hence all associative multiplications are given by

$$Ass(A) := Z\left(\coprod_{n \geq 0} \{p_{i,j,k}^{<n>}\}_{1 \leq i,j,k \leq n}\right),$$

where $Z(S) := \{p \in \mathbb{A}^{n^3} \mid f(p) = 0, \forall f \in S\}$. In order to apply Hilberts root theorem (Nullstellensatz) the field k is supposed to be algebraically closed. Then it follows that $Ass(A)$ is an algebraic set and hence closed with respect to the Zariski topology. This shows that the set $Ass(A)$ is algebraic. It is even a scheme.

Deformations A deformation of a scheme is like a bundle with a special fibre isomorphic to the scheme to be deformed. The other fibers are then deformations of the special fibre. In order to have a good notion of such a bundle flat families of schemes have to be discussed. A family of schemes is a scheme whose fibers are again schemes. The prefix flat implies that the bundle map is flat.

Definition 5.3.1. *Let $f : X \to Y$ be a morphism of schemes and F be an $\mathcal{O}_X$-module, then F is flat over Y at $x \in X$ if F_x, the stalk of F at x, is a flat $\mathcal{O}_{f(x),Y}$-module. The family F is called flat over Y if it is flat over all points of X. X is said to be flat over Y if $\mathcal{O}_X$ is flat over Y.*

Remember that a module M is flat if the associate functor $M \otimes_k -$ is also left exact.

Definition 5.3.2. *Let X be flat over T w.r.t. $f : X \to T$, then X is a global deformation of X_0 over T if there exists an arrow $t : Spec(k) \to T$ such that $f^{-1}(t) \cong X_0$, i.e. the diagram*

$$\begin{array}{ccc} X_0 & \longrightarrow & X \\ \downarrow & \lrcorner & \downarrow f \\ Spec(k) & \xrightarrow[t]{} & T \end{array}$$

is a pullback square. An infinitesimal deformation of X_0 is a deformation of X_0 over $Spec(k[t]/(t^2))$. Further a deformation of X_0 over T is trivial if X is isomorphic to the product scheme $X \times T$.

It can be shown that for an algebraically closed field k all infinitesimal deformation of a nonsingular scheme X_0 are classified by the first sheaf cohomology group $H^1(X, \mathcal{T}_X)$, where $\mathcal{T}_X$ is the tangent sheaf. For more information see [14].

5.4 Deformation Functors

There is yet another approach to deformation theory, namely that by deformation functors. A deformation functor is a covariant functor from the category of local Artinian rings with residual field k, i.e. for such a local Artinian ring R with maximal ideal $\mathfrak{m}$ it holds that $R/\mathfrak{m} = k$, and ring morphisms to the category of sets. Such a functor is supposed to satisfy the property that $F(k)$ is a singleton. This singleton can be interpreted as the structure to be deformed and the elements of $F(R)$, for any local Artinian ring R, as deformations of that structure.

Let $\mathfrak{g}$ be a dg-Lie algebra. Then $\mathbf{Def}_{\mathfrak{g}} : \mathbf{Artin}_k \to \mathbf{Set}$ is an example of a deformation functor. It sends an Artinian ring R to the set $MC_{\mathfrak{g}}(R)/G_{\mathfrak{g}}(R)$ and an arrow $f : R \to S$ to the arrow defined by $\mathbf{Def}_{\mathfrak{g}}(f)(x \otimes r) := x \otimes f(r)$. To show that $\mathbf{Def}_{\mathfrak{g}}(f)$ is indeed a morphism between Maurer-Cartan elements consider:

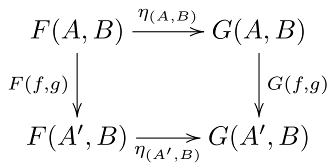
$$\begin{aligned} d(x \otimes f(r)) + \frac{1}{2}[x \otimes f(r), x \otimes f(r)] =& d(x) \otimes f(r) + \frac{1}{2}[x, x] \otimes f(r)f(r) \\ =& f(d(x) \otimes r + \frac{1}{2}[x, x] \otimes r^2) = f(0) = 0. \end{aligned}$$

Each of the cochain complexes with a dg-Lie algebra structure described in the previous chapters gives rise to such a deformation functor. The representability of a deformation functor, i.e. whether there exists an Artinian ring R such that the deformation functor is isomorphic to $Hom_{\mathbf{Artin}_k}(R, -)$, plays an important role in further analyses of the resulting deformations (see [28])

Appendix A

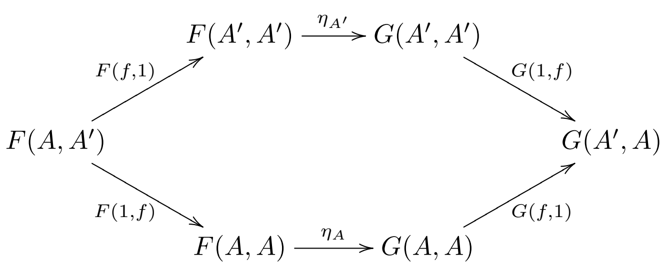
Ends and Coends

Let $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ be two functors. A natural transformation $\eta : F \Longrightarrow G$ is just a family of maps $\eta_{(A,B)} : F(A,B) \to G(A,B)$ for every pair of objects (A,B) in $\mathcal{C}^{op} \times \mathcal{C}$, such that for any morphism $(f,g) : (A',B) \to (A,B')$ the diagram



commutes. Note that $\eta_{(A,A')}$ is completely determined by $\eta_{(A,A)}$. In this case there is a more efficient way of defining a natural transformation leading to the notion of a dinatural transformation.

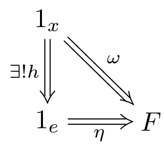
Definition A.0.1. *A dinatural transformation η between two functors $F, G : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is defined by a map $\eta_A : F(A,A) \to G(A,A)$ for all objects A of $\mathcal{C}$ such that for all morphisms $f : A' \to A$ the diagram*



commutes.

End Let e be an object in $\mathcal{D}$, then the constant functor $1_e : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ sends every object to e and every arrow to id_e.

Definition A.0.2. *An end of a functor $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ is an object e in $\mathcal{D}$ and a dinatural transformation $\eta : 1_e \Longrightarrow F$ such that for any other dinatural transformation $\omega : 1_x \Longrightarrow F$ with x an object in $\mathcal{D}$ there exists a unique arrow $h : x \to e$ making the diagram*

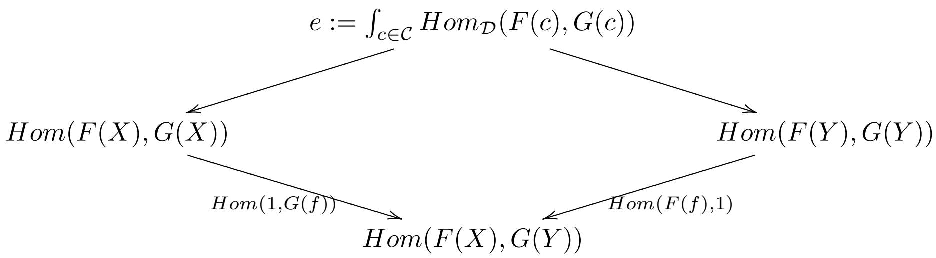


commute. For an end the object e is denoted by $\int_{c \in \mathcal{C}} F(c,c)$.

Example A.0.1. *Natural transformations form an example of an end. Consider the functors* $F, G : \mathcal{C} \to \mathcal{D}$*. Then another functor*

$$Hom_{\mathcal{D}}(F(-), G(-)) : \mathcal{C}^{op} \times \mathcal{C} \xrightarrow{F \times G} \mathcal{D}^{op} \times \mathcal{D} \xrightarrow{Hom} \textbf{\textit{Set}}$$

can be constructed. Then the diagram



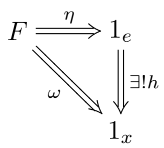
is a pullback square for any arrow $f : X \to Y$ *simultaneously. In other words, let* $\eta \in e$ *and denote the images of* η *in* $Hom(F(X), G(X))$ *and* $Hom(F(Y), G(Y))$ *by* η_X *and* η_Y *respectively then it holds that*

$$\eta_Y \circ F(f) = G(f) \circ \eta_X.$$

Hence η *is a natural transformation between* F *and* G *and* e *can be identified with* $Nat(F, G)$*.*

Coend Similarly the notion of a coend of a functor is defined as follows.

Definition A.0.3. *A coend of a functor* $F : \mathcal{C}^{op} \times \mathcal{C} \to \mathcal{D}$ *is an object* e *in* $\mathcal{D}$ *and a dinatural transformation* $\eta : F \Longrightarrow 1_e$ *such that for any other dinatural transformation* $\omega : F \Longrightarrow 1_x$ *with* x *an object in* $\mathcal{D}$ *there exists a unique arrow* $h : e \to x$ *making the diagram*

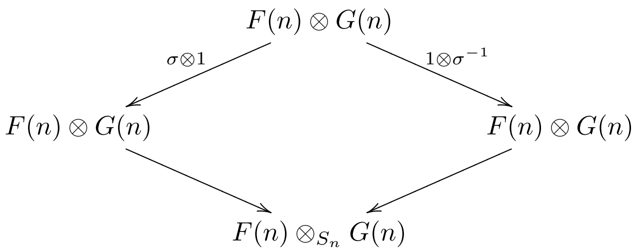


commute. For a coend the object e *is denoted by* $\int^{c \in \mathcal{C}} F(c, c)$*.*

Example A.0.2. *An example of a coend is the symmetric tensor product of two functors* $F : S_n^{op} \to \mathcal{E}$ *and* $G : S_n \to \mathcal{E}$*, whit* $\mathcal{E}$ *a monoidal category. Then it is possible to define a functor*

$$F \otimes G : S_n^{op} \times S_n \xrightarrow{F \times G} \mathcal{E} \times \mathcal{E} \xrightarrow{\otimes} \mathcal{E}.$$

The coend $\int^{n \in S_n} F(n) \otimes G(n)$ *of this functor is* $F(n) \otimes_{S_n} G(n)$ *since for any permutation* $\sigma : n \to n$ *of* S_n *the diagram*



is a pushout diagram. In other words, let $x \in F(n)$ *and* $y \in G(n)$ *then in* $F(n) \otimes_{S_n} G(n)$ *it holds that* $x.\sigma \otimes y = x \otimes \sigma^{-1}.x$*.*

Copower Let $\mathcal{E} = (\mathcal{E}, \otimes, I)$ be a symmetric monoidal category with coproducts. Given a set-valued functor $F : \mathcal{C} \to \mathbf{Set}$ it is possible to construct a functor $F' : \mathcal{C} \to \mathcal{E}$ by defining

$$F'(A) := \coprod_{x \in F(A)} I.$$

Appendix B

Multicategories using layered trees

Instead of deforming the partial compositions it is also possible to deform the full one. Recall that a multicategory $\mathcal{M}$ with full compositions is defined to be a collection $\mathcal{M} : \mathbb{T}(S)_1^{op} \to \mathcal{E}$ together with maps

$$\circ_T^L : \underline{\mathcal{M}}^L(T) \to \mathcal{M} \circ \partial(T)$$

for all $T \in \mathbb{L}(S)_{2,\bullet}$ satisfying some obvious axioms. The problem with this deformation theory is, that no differential graded Lie algebra structure could be found on the deformation complex, which will be described in a moment. Because of this the obstructions can not elegantly be described. Despite the lack of a dg-Lie algebra structure the complex classifies all infinitesimal deformations. There is a complex for the layered multicategories and one for the non-layered ones. It is natural to ask how they are related, and how the cohomology groups relate. Two maps will be constructed relating the complexes, but these maps turn out not to commute with the differentials hence it is not clear how to compare.

A deformation of a multicategory is again a multicategory structure on the extended collection which reduces to the original structure when the collection is reduced to the original collection. All the other notions can be stated as for multicategories with partial compositions using layered trees instead. The deformation complex is given by

$$C_L^m(\mathcal{M}, \mathcal{M}) := \int_{\mathbb{L}(S)_{m,\bullet}} \underline{Hom}_k(\underline{\mathcal{M}}^L, \mathcal{M} \circ \partial),$$

with differential

$$\begin{aligned} & d(\gamma)(\theta \otimes \overrightarrow{\theta}^1 \otimes \ldots \otimes \overrightarrow{\theta}^n) = \\ & \quad = \sum_{j=1}^{m} \theta \circ (\theta_1 \circ \overrightarrow{\theta}_1^2 \circ \ldots \circ \overrightarrow{\theta}_1^n, \ldots, \gamma(\theta_j \otimes \overrightarrow{\theta}_j^2 \otimes \ldots \otimes \overrightarrow{\theta}_j^n), \ldots, \theta_m \circ \overrightarrow{\theta}_m^2 \circ \ldots \circ \overrightarrow{\theta}_m^n) + \\ & \quad + \sum_{i=1}^{n} (-1)^i \gamma(\theta \otimes \overrightarrow{\theta}^1 \otimes \ldots \otimes (\overrightarrow{\theta}^{i-1} \circ \overrightarrow{\theta}^i) \otimes \ldots \otimes \overrightarrow{\theta}^n) + \\ & \quad + (-1)^{n+1} \gamma(\theta \otimes \overrightarrow{\theta}^1 \otimes \ldots \otimes \overrightarrow{\theta}^{n-1}) \circ \overrightarrow{\theta}^n. \end{aligned}$$

It can be checked by long and tedious calculation, see the next page, that d is indeed a differential, i.e. $d^2 = 0$.

To see that everything becomes zero in that calculation note that (4.1) cancels (4.4), (4.2) cancels (4.5), (4.3) cancels (4.8), (4.6) cancels itself and (4.9 + 4.10) cancels with (4.7). Therefore $(C(\mathcal{M}, \mathcal{M}), d)$ is a cochain complex. In order to obtain the left and right unit condition, the Dold-Kan correspondence is used again.

$$
\begin{aligned}
d^2(\gamma)(\theta\otimes\overrightarrow{\theta}^1\otimes\ldots\otimes\overrightarrow{\theta}^n)=&\\
&\sum_{j=1}^{m}\theta\circ(\theta_1\circ\overrightarrow{\theta}^1_1\circ\ldots\circ\overrightarrow{\theta}^n_1,\ldots,d(\gamma)(\theta_j\otimes\overrightarrow{\theta}^2_j\otimes\ldots\otimes\overrightarrow{\theta}^n_j),\ldots,\theta_m\circ\overrightarrow{\theta}^1_m\circ\ldots\circ\overrightarrow{\theta}^n_m)+\sum_{i=1}^{n}(-1)^i d(\gamma)(\theta\otimes\overrightarrow{\theta}^1\otimes\ldots\otimes(\overrightarrow{\theta}^{i-1}\circ\overrightarrow{\theta}^i)\otimes\ldots\otimes\overrightarrow{\theta}^n)+\\
&+(-1)^{n+1}d(\gamma)(\theta\otimes\overrightarrow{\theta}^1\otimes\ldots\otimes\overrightarrow{\theta}^{n-1})\circ\overrightarrow{\theta}^n\\
=\;&\sum_{j=1}^{m}\sum_{k=1}^{m_2}\theta\circ(\theta_1\circ\overrightarrow{\theta}^2_1\circ\ldots\circ\overrightarrow{\theta}^n_1,\ldots,\theta_j\circ(\overrightarrow{\theta}^2_{j,1}\circ\ldots\circ\overrightarrow{\theta}^n_{j,1},\ldots,\gamma(\overrightarrow{\theta}^2_{j,k}\otimes\ldots\otimes\overrightarrow{\theta}^n_{j,k}),\ldots,\overrightarrow{\theta}^2_{j,m_2}\circ\ldots\circ\overrightarrow{\theta}^n_{j,m_2}),\ldots,\theta_m\circ\overrightarrow{\theta}^2_m\circ\ldots\circ\overrightarrow{\theta}^n_m) && \text{(B.1)}\\
&+\sum_{j=1}^{m}\sum_{l=2}^{n}(-1)^{l-1}\theta\circ(\theta_1\circ\overrightarrow{\theta}^1_1\circ\ldots\circ\overrightarrow{\theta}^n_1,\ldots,\gamma(\overrightarrow{\theta}^1_j\otimes\ldots\otimes(\overrightarrow{\theta}^{l-1}_j\circ\overrightarrow{\theta}^l_j)\otimes\ldots\otimes\overrightarrow{\theta}^n_j),\ldots,\theta_m\circ\overrightarrow{\theta}^2_m\circ\ldots\circ\overrightarrow{\theta}^n_m) && \text{(B.2)}\\
&+\sum_{j=1}^{m}(-1)^m\theta\circ(\theta_1\circ\overrightarrow{\theta}^1_1\circ\ldots\circ\overrightarrow{\theta}^n_1,\ldots,\gamma(\theta_j\otimes\overrightarrow{\theta}^2_j\otimes\ldots\otimes\overrightarrow{\theta}^{n-1}_j)\circ\overrightarrow{\theta}^n_j,\ldots,\theta_m\circ\overrightarrow{\theta}^2_m\circ\ldots\circ\overrightarrow{\theta}^n_m)+ && \text{(B.3)}\\
&-\sum_{r=1}^{n-1}(-1)^r\theta\circ\overrightarrow{\theta}^1\circ(\overrightarrow{\theta}^2_1\circ\ldots\circ\overrightarrow{\theta}^n_1,\ldots,\gamma(\overrightarrow{\theta}^2_r\otimes\ldots\otimes\overrightarrow{\theta}^n_r),\ldots,\overrightarrow{\theta}^1_n\circ\ldots\circ\overrightarrow{\theta}^n_m) && \text{(B.4)}\\
&+\sum_{i=2}^{n}\sum_{r=1}^{n-1}(-1)^i\theta\circ(\overrightarrow{\theta}^1_1\circ\ldots\circ\overrightarrow{\theta}^n_1,\ldots,\gamma(\overrightarrow{\theta}^1_r\otimes\ldots\otimes(\overrightarrow{\theta}^{i-1}_r\circ\overrightarrow{\theta}^i_r)\otimes\ldots\otimes\overrightarrow{\theta}^n_r),\ldots,\overrightarrow{\theta}^1_n\circ\ldots\circ\overrightarrow{\theta}^n_m) && \text{(B.5)}\\
&+\sum_{i=1}^{n}\sum_{s=1}^{n-1}(-1)^{i+s}\gamma(\theta\otimes\overrightarrow{\theta}^1\otimes\ldots\otimes(\overrightarrow{\theta}^{s-1}\circ\overrightarrow{\theta}^s)\otimes\ldots\otimes(\overrightarrow{\theta}^{i-1}\circ\overrightarrow{\theta}^i)\otimes\ldots\otimes\overrightarrow{\theta}^n) && \text{(B.6)}\\
&+\sum_{i=1}^{n-1}(-1)^i\gamma(\theta\otimes\overrightarrow{\theta}^1\otimes\ldots\otimes(\overrightarrow{\theta}^{i-1}\circ\overrightarrow{\theta}^i)\otimes\ldots\otimes\overrightarrow{\theta}^{n-1})\circ\overrightarrow{\theta}^n+(-1)^n\gamma(\theta\otimes\overrightarrow{\theta}^1\otimes\ldots\otimes\overrightarrow{\theta}^{n-2})\circ\overrightarrow{\theta}^{n-1}\circ\overrightarrow{\theta}^n && \text{(B.7)}\\
&+\sum_{u=1}^{v}(-1)^{n+1}\theta\circ(\theta_1\circ\overrightarrow{\theta}^1_1\circ\ldots\circ\overrightarrow{\theta}^{n-1}_1,\ldots,\gamma(\theta_j\otimes\overrightarrow{\theta}^2_j\otimes\ldots\otimes\overrightarrow{\theta}^{n-1}_j),\ldots,\theta_m\circ\overrightarrow{\theta}^1_m\circ\ldots\circ\overrightarrow{\theta}^{n-1}_m)\circ\overrightarrow{\theta}^n && \text{(B.8)}\\
&+\sum_{w=1}^{q}(-1)^{n+1+w}\gamma(\theta\otimes\overrightarrow{\theta}^1\otimes\ldots\otimes(\overrightarrow{\theta}^{w-1}\circ\overrightarrow{\theta}^w)\otimes\ldots\otimes\overrightarrow{\theta}^{n-1})\circ\overrightarrow{\theta}^n && \text{(B.9)}\\
&+(-1)^{n+1+q+1}\gamma(\theta\otimes\overrightarrow{\theta}^1\otimes\ldots\otimes\overrightarrow{\theta}^{n-2})\circ\overrightarrow{\theta}^{n-1}\circ\overrightarrow{\theta}^n && \text{(B.10)}\\
=\;&0.
\end{aligned}
$$

Comparison Comparing the layered complex with the non-layered one boils down to comparing the deformation of the full composition versus the deformation of the $\circ_T$-compositions.

Given a functor $\gamma \in \underline{Hom}_k(\underline{\mathcal{M}}_{2,\bullet}, \mathcal{M} \circ \partial)$ a similar trick as in 4.1.4 can be applied to obtain a functor $\nu(\gamma) \in \underline{Hom}_k(\underline{\mathcal{M}}^L_{2,\bullet}, \mathcal{M} \circ \partial)$. On the other hand, given a functor $\varphi \in \underline{Hom}_k(\underline{\mathcal{M}}^L_{2,\bullet}, \mathcal{M} \circ \partial)$ it is possible to obtain a functor $\xi(\varphi) \in \underline{Hom}_k(\underline{\mathcal{M}}_{2,\bullet}, \mathcal{M} \circ \partial)$ by $\xi(\varphi)_T := \varphi_{\varepsilon T}$.

This construction can be generalized to obtain maps $\xi : C^n_L(\mathcal{M}, \mathcal{M}) \leftrightarrows C^n_M(\mathcal{M}, \mathcal{M}) : \nu$. Let φ be a multifunctor in $\underline{Hom}_k(\underline{\mathcal{M}}^L_{m,\bullet}, \mathcal{M} \circ \partial)$ and $T \in \mathbb{T}(S)_{n,\bullet}$. Note that $\varepsilon(T)$ has at most n layers. If it has less, add as many layers consisting of identities as needed. Denote this extension of ε by $\hat{\varepsilon}$. Then $\xi(\varphi)_T := \varphi_{\hat{\varepsilon}(T)}$ gives the desired functor. On the other hand, let $\gamma \in \underline{Hom}_k(\underline{\mathcal{M}}_{m,\bullet}, \mathcal{M} \circ \partial)$. For a tree $T \in \mathbb{L}(S)_{m,\bullet,\bullet}$ there are three cases, $|T| = n, |T| < n$ or $|T| > n$. In the first case, define $\hat{\gamma}_T := \gamma_T$. In the second case, there are not enough vertices to apply γ to. This can be resolved by adding identities and then apply γ. In the third case apply γ repeatedly till there are less than n vertices left, then add identities and apply γ once more. This defines a functor $\nu(\gamma) \in \underline{Hom}_k(\underline{\mathcal{M}}^L_{n,\bullet}, \mathcal{M} \circ \partial)$. Unfortunately, these maps are not chain maps. Therefore it is not clear how to compare the complexes and their cohomology groups.

Origin Originally this complex in case of operads was inspired by the nerve construction used for categories. Define the 'nerve' of an operad by

$$N\mathcal{P}^i_n = \coprod_{T \in \mathbb{L}_{n,i}} \underline{\mathcal{P}}^L(T).$$

Using this description the deformation complex can be defined by

$$C^m_O(\mathcal{P}, \mathcal{P}) = \coprod_{i \geq 0} \underline{Hom}_{[S_i, \mathbf{Mod}_k]}(N\mathcal{P}^i_m, N\mathcal{P}^i_1)$$

together with the same differential as above.

This complex can be rewritten as

$$\begin{aligned}
C^m_O(\mathcal{P}, \mathcal{P}) &= \coprod_{i \geq 0} \underline{Hom}_{[S_i, \mathbf{Mod}_k]}(N\mathcal{P}^i_m, N\mathcal{P}^i_1) \\
&= \coprod_{i \geq 0} \underline{Hom}_{[S_i, \mathbf{Mod}_k]}(\coprod_{T \in \mathbb{L}_{m,i}} \underline{\mathcal{P}}^L(T), \underline{\mathcal{P}}(t_i)) \\
&\cong \coprod_{i \geq 0} \prod_{T \in \mathbb{L}_{m,i}} \underline{Hom}_{[S_i, \mathbf{Mod}_k]}(\underline{\mathcal{P}}^L(T), \underline{\mathcal{P}}(t_i)) \\
&= \coprod_{i \geq 0} \prod_{T \in \mathbb{L}_{m,i}} \underline{Hom}_{[S_i, \mathbf{Mod}_k]}(\underline{\mathcal{P}}^L(T), \underline{\mathcal{P}} \circ \partial(T)) \qquad (\partial(T) = t_i) \\
&= \int_{T \in \mathbb{L}_{m,\bullet}} \underline{Hom}_k(\underline{\mathcal{P}}^L(T), \mathcal{P} \circ \partial(T)) \\
&= C^m_L(\mathcal{P}, \mathcal{P}).
\end{aligned}$$

The problems with the layered complex $C^\bullet_O(\mathcal{P}, \mathcal{P})$ led me to consider the complex $C^\bullet_M$ for multicategories. Retrospectively it was then recognized that $C^\bullet_O$ is just $C^\bullet_L$, which is $C^\bullet_M$ for layered trees.

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