Stochastic comparison of Markov queueing networks using coupling

Master's thesis

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Abstract

Stochastic comparison is a method to prove bounds on performance metrics of stochastic models. Here, coupling can be used to define two processes on a common probability space, which makes it possible to compare the steady-state distributions of the processes. Two processes are stochastically related if their steady-state distributions satisfy a certain comparison relation. Such a stochastic relation can be more general than a stochastic order. In this thesis, a thorough description of stochastic comparison using coupling for the probability kernels of Markov processes is presented. Necessary and sufficient conditions for the stochastic comparison of stochastically related Markov queueing networks are given, in particular for the coordinate-wise and the summation relation. Also, an example of a Jackson network with breakdowns is studied, and an explicit coupling which preserves a subrelation of the coordinate-wise order relation is constructed. This allows to conclude that the steady-state distributions of the breakdown models are coordinate-wise comparable.

Keywords: coupling, stochastic comparison, stochastic order, stochastic relation, Strassen's theorem, Markov queueing network, Jackson network, probability kernel

Contents

Chapter 1

Introduction

Stochastic queueing networks are networks of multiple interconnected service stations. Customers arrive in some random manner at a station in the system, where they may have to wait until they are served. After service, the customers jump randomly from one server to another, or leave the system. Queueing networks are used to model a great variety of applications. A few examples where queueing networks can be used in mathematical modeling are:

- call centers, where calls arrive and are put through to different service desks,
- manufacturing models, where requests or materials have to move along different production facilities or employees,
- the internet, where data packets are routed from source to end,
- many more applications in physics, biology and economy.

More on the variety and applications of queueing networks can be found in [Serfozo, 1999], [Kelly, 1985] and [Kelly, 1991].

Our principal aim is to obtain information about *system quantities*. For queueing models we can think, for example, of the number of customers in the system, the waiting time, the loss probability, or the busy period of a service station. System quantities can be means, but also distributions, such as the waiting time distribution at a certain server. The behaviour of a stochastic model is completely determined when we know the model dynamics and the initial distribution of the network. In practice, distribution functions may not be known, or their full dependence structure may be unknown.

Often, queueing models are too complex to analyze explicitly, but if it is not possible to compute the probability distributions, it may still be possible to provide bounds on the system quantities of the model. For example, by comparing the model to a simplified model. Possible ways to prove the validity of the bounds are (i) simulation, which is useful in practice because it is easy to implement, but only gives results for certain parameters of the model; (ii) mean value analysis, for example by using the Markov reward approach [van Dijk, 1998]; and (iii) stochastic comparison, which provides approximations for the distributions of the processes. In this thesis, stochastic comparison of Markov queueing networks is considered.

Stochastic orders are frequently used for the derivation of comparison results for stochastic models. Formally, *stochastic ordering* is a partial ordering on the state space of probability measures. The term *order* refers to the *strong* or *usual stochastic order*. However, in this context we require a partial order on the state space where both processes live. In [Müller and Stoyan, 2002] and [Shaked and Shanthikumar, 2007] the reader can find an overview on stochastic orders. In [Kamae et al., 1977], characterizations for the partial ordering of probability measures are given. [Massey, 1987] and [Whitt, 1981] extended these characterizations for the stochastic domination of continuous-time Markov processes, and [Last and Brandt, 1995] extends it to more general jump processes. In [Leskelä, 2010], the notion of stochastic orders is extended by defining *stochastic relations*. Unlike orders, relations do not have to be reflexive or transitive. Also, a relation between two processes is defined as an arbitrary subset of the product space in which the two processes live. Therefore, the two processes do not have to take values in a common ordered space, and the spaces do not have to be the same.

Coupling is a method for proving stochastic comparability. The main idea of coupling is the joint construction of two random elements on a common probability space, while adding dependencies on the processes. Adding dependencies allows comparison of the distributions of the elements [Lindvall, 1992], [Thorisson, 2000]. In [Szekli, 1995] and [Last and Brandt, 1995], several coupling constructions are described. Stochastic comparison using coupling is often based on Strassen's theorem, which ensures the existence of an order-preserving coupling [Strassen, 1965]. Strassen's theorem remains valid for relations, which allows us to use coupling methods to analyze stochastic relations. Leskelä presents a characterization of stochastic relations and gives if and only if conditions for the stochastic comparison with respect to relations. The existence of a coupling which preserves some subrelation is sufficient to lead to strong comparison results.

Jackson networks are one of the simplest classes of Markov queueing networks to study. Sufficient conditions for the stochastic comparison of Jackson networks with identical routing probabilities are given in [Lindvall, 1992]. In [López et al., 2000], necessary and sufficient conditions for the stochastic comparison of Jackson networks with increasing service rates are derived, by constructing an explicit coupling. In [Economou, 2003], sufficient conditions are stated for the coordinate-wise ordering without assumptions on the service rates. However, these conditions are not sharp. If and only if conditions for the same problem are presented in [Delgado et al., 2004].

In this thesis, the theory of stochastic comparison with coupling is applied to Markov queueing networks. A detailed introduction to couplings and stochastic relations is presented. In most applications, the queueing models consist of a finite number of service stations and the state spaces are countable. These assumptions allow simplification of the theory of stochastic comparison. The model dynamics of Markov processes are usually given by transition probability matrices. New notions of *coupling of transition matrices* and *coupling of transition rate matrices* are defined. Although several comparability conditions and characterizations have already been given in terms of probability kernels [Kamae et al., 1977] and [Leskelä, 2010], couplings of transition matrices have not been defined before. The definition of coupling of bounded transition rate matrices for continuous-time Markov processes follows naturally from the discrete-time definition. We prove that the definitions of a coupling of transition probability matrices coincide with coupling of the related discrete-time Markov processes. The same result for a coupling of transition rate matrices and continuous-time Markov processes is given, by writing continuous-time Markov processes in discrete-time using the uniformization method as in [Ross, 2007].

For Markov queueing networks on countable state spaces, we derive necessary and sufficient conditions for the stochastic comparison of these networks, using a comparison result from [Leskelä, 2010]. We work out these conditions for two relations. In particular: the *coordinate-wise order relation* compares the number of customers in the networks at each service station, while the *summation relation* compares the total number of customers in the entire system. The results obtained for the coordinate-wise order relation are complementary to the results in [Delgado et al., 2004]. Also, we consider a Jackson network where breakdowns can occur. In this example, it is not possible to give a coupling for which the required comparison relation is invariant, but still, the stochastic comparison relation can be proved by defining a subrelation of the comparison relation. We give an explicit coupling which preserves a subrelation of the coordinate-wise order relation, and illustrate that this indeed leads to a strong stochastic comparison result.

The thesis is outlined as follows. We start in Chapter 2 with an example of a coupling of two single-server queueing systems. Then, after some basic definitions and measure theoretic revisions, the notion of coupling is defined. For Markov processes, the coupling of probability (rate) matrices is studied. Stochastic comparison of random elements and processes is discussed in Chapter 3. In Chapter 4 a basic Markov queueing network is presented and the necessary and sufficient conditions for the stochastic comparison of two of those Markov queueing networks are proved for two different types of relations. For the coordinate-wise order relation, a new characterization for sharp comparison conditions of general Markov queueing networks is stated. The conditions of this theorem are analytically easy to verify. Chapter 5 illustrates how strong comparison results follow from a coupling which preserves some subrelation. Chapter 6 concludes the thesis.

Chapter 2

Coupling

Coupling is a possible way to compare two or more random variables or processes. By coupling two processes we establish a joint construction on a common probability space. This chapter gives an introduction to the coupling approach. We start by giving a basic example of the coupling of two single-server queues in Section 2.1. In the second section the formal definition of the coupling of stochastic measures, elements and processes is given. In Section 2.3 and 2.4 we consider the coupling of respectively discrete-time and continuous-time Markov processes and we express coupling in terms of the transition probability kernels of the processes.

2.1 Single-server queue

We consider two single-server exponential queueing systems X and Y. In both systems customers arrive at the service station in accordance with a Poisson process having rate λ. That is, all inter-arrival times are independent and exponentially distributed with mean $1/\lambda$. The service times are also independent and exponentially distributed with mean $1/\mu$ for queueing system X, and mean $1/\mu'$ for system Y. As state description we use the number of customers in the system, denoted by n with $n \in \mathbb{N}$ and $0 \leq n \leq N$. The notation $\mathbb{N}$ is used for all positive integers where we adopt the convention that $0 \in \mathbb{N}$. Furthermore, we assume that the systems have a capacity constraint of N on the total number of customers in the system. If $n \geq N$, arriving customers are rejected and lost.

$$0 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} 1 \underset{\mu}{\overset{\lambda}{\rightleftarrows}} \cdots \underset{\mu}{\overset{\lambda}{\rightleftarrows}} N$$

Figure 2.1: Single-server queue $(M/M/1)$

The two systems differ only in the service rate: the service rate μ in system X is assumed to be bigger than the service rate μ' in system Y. So we expect that on average there are less customers in system X, because the waiting and service times of the customers in this system are shorter. We expect the loss rate of system X to be smaller than the loss rate of system Y. In this simple example we can easily compute the steady-state distribution, for example, by solving the equation $\pi = \pi P$ for the transition probability matrix P [Ross, 2007]. In this example the state space is one-dimensional. In more dimensional state spaces, however, we do not want to compute these steady-state distributions, because it is too difficult or time-intensive.

The goal is to prove that process X is *stochastically smaller* than process Y in some sense. If we do not want to compute the steady-state distributions we can use other approaches, such as coupling. Formal definitions will be given in Section 3.1 and 3.2.

2.1.1 Three couplings

In this single-server example, X and Y are real-valued and live in a one-dimensional space. A random variable $\hat{X}$ is a *copy* or *representation* of X if $\hat{X}$ and X have the same distribution.

$$\hat{X} \stackrel{d}{=} X \text{ if and only if } \mathbb{P}(X \leq s) = \mathbb{P}(\hat{X} \leq s) \quad \text{for all } s \in \mathbb{R}. \tag{2.1}$$

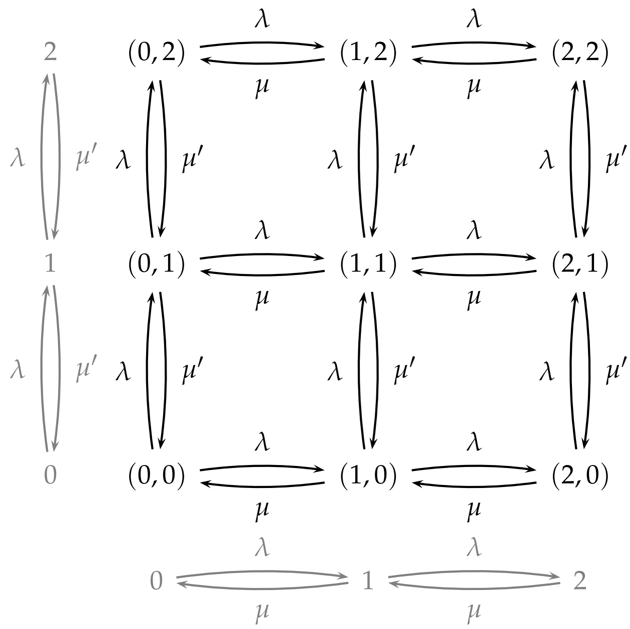

Figure 2.2: The trivial coupling of X and Y.

A coupling of two processes X on S_1 and Y on S_2 is a third process $Z = (\hat{X}, \hat{Y})$ on the product space $S_1 \times S_2$ such that $\hat{X}$ is a copy of X, and $\hat{Y}$ is a copy of Y. So the marginals of the new process coincide with the original processes. For the two single-server queues X and Y described above we can have for example the simplest coupling, called the *trivial coupling*, shown in Figure 2.2. In this picture we can see that in any state (x, y) in the product space, the marginal outgoing rates behave just the same as the outgoing rates from state x of X and state y of Y. So a coupling $(\hat{X}, \hat{Y})$ is a process on the product space such that the marginal distributions behave just like X and Y.

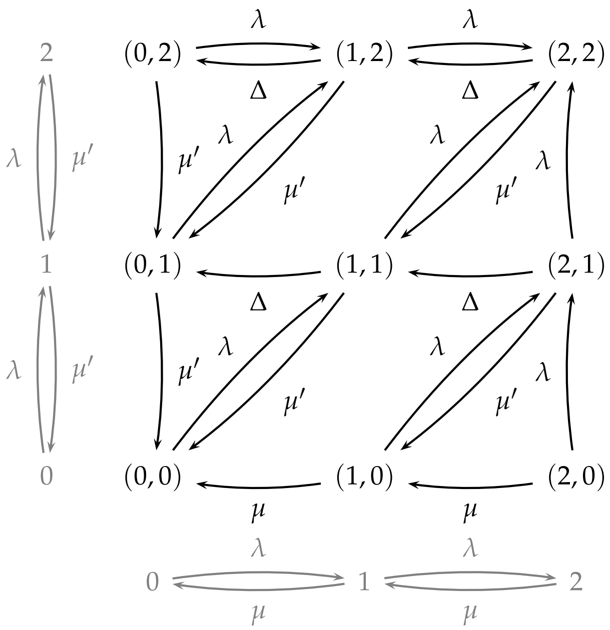

Figure 2.3: Maximal coupling of X and Y.

The trivial coupling always exists and does not give us any extra insight in the behaviour of the two original processes. Typically we want to include a certain amount of dependency in the coupling (that is dependency between $\hat{X}$ and $\hat{Y}$), although of course the marginal distributions have to remain the same.

For our example, define $\Delta := \mu - \mu'$. Then we can adapt the coupling such that whenever a departure occurs at process Y, there will also be a departure in process X, and thus we have a joint departure rate of μ'. The arrival rates are also coupled. There exists more couplings of X and Y. Figure 2.3 and Figure 2.4 gives two other couplings of X and Y.

Summarized, we start with two processes X and Y, and we create a coupling of them $(\hat{X}, \hat{Y})$ such that $\hat{X} \stackrel{d}{=} X$ and $\hat{Y} \stackrel{d}{=} Y$, and such that $\hat{X}$ and $\hat{Y}$ have an interesting dependence structure. Now let us consider what we actually want to achieve. We want to prove that X is stochastically less than Y. For random variables this means

$$\mathbb{P}(X \leq s) \geq \mathbb{P}(Y \leq s), \text{ for all } s \in \mathbb{R}. \tag{2.2}$$

$\hat{X} \stackrel{d}{=} X$ and $\hat{Y} \stackrel{d}{=} Y$, and (2.2) is equivalent to

$$\mathbb{P}(\hat{X} \leq \hat{Y}) = 1. \tag{2.3}$$

Consider Figure 2.4. The dotted area in this figure is exactly the area where $\hat{X} \leq \hat{Y}$ holds. This is also the area where the relation $R = \{(x, y) : x \leq y\}$ is true. So, all we have to do is to prove that in steady-state the process $(\hat{X}, \hat{Y})$ is inside the dotted area R. The dotted area is *invariant* (sometimes also called *absorbing*) for the process $(\hat{X}, \hat{Y})$. That is, once entered into this area the process will never leave the dotted area anymore. So if we start in this

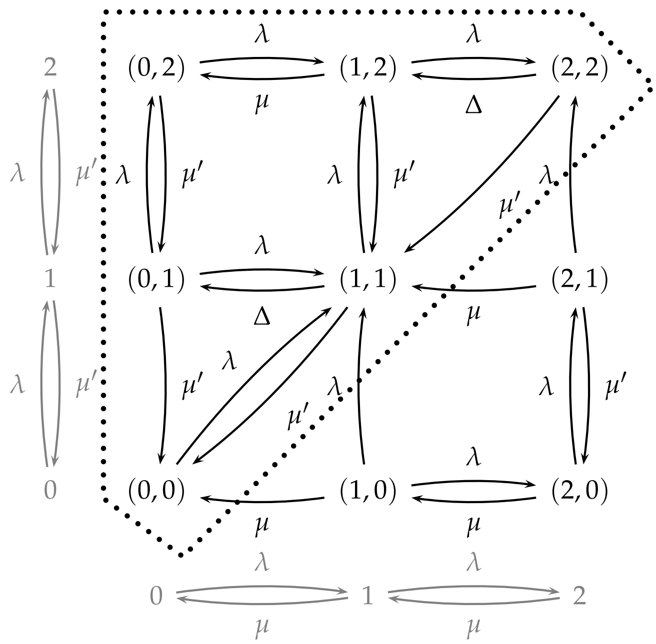

Figure 2.4: A third coupling of X and Y.

area, we will stay there forever, but also when we do not start in this area, the probability that we never enter it is zero. So we found a coupling process for which in steady-state the relation R is true with probability one.

Note that to compute a coupling for which the relation R is invariant, we can start with the trivial coupling and add dependencies only in those states at the boundary of relation R. That is the dotted line in Figure 2.4.

2.1.2 Stochastic Comparison

Figures 2.3 and 2.4 both present couplings for which the relation R is invariant. Now that we found such couplings, we can prove that $X \leq_{st} Y$. Assume $x \leq y$, that is $(x,y) \in R$. Let the initial states be $X_0 = x$ and $Y_0 = y$. Then for all $s \in \mathbb{R}$:

$$\begin{aligned}
& \mathbb{P}\Big(X_t(x) > s\Big) \\
& \qquad \textit{$(\hat{X}_t)_{t\in\mathbb{R}_+}$ is a copy of $(X_t)_{t\in\mathbb{R}_+}$} \\
= \; & \mathbb{P}\Big(\hat{X}_t((x,y)) > s\Big) \\
& \qquad \textit{due to the fact that } \mathbb{P}(\hat{X}_t((x,y)) > s,\ \hat{X}_t((x,y)) > \hat{Y}_t((x,y))) = 0 \\
= \; & \mathbb{P}\Big(\hat{X}_t((x,y)) > s,\ \hat{X}_t((x,y)) \leq \hat{Y}_t((x,y))\Big) \\
& \qquad \textit{because event } (\hat{Y}_t((x,y)) > s) \textit{ is a subset of } (\hat{X}_t((x,y)) > s,\ \hat{X}_t((x,y)) \leq \hat{Y}_t((x,y))) \\
\leq \; & \mathbb{P}\Big(\hat{Y}_t((x,y)) > s\Big)
\end{aligned}$$

and, because $(\hat{Y}_t)_{t\in\mathbb{R}_+}$ *is a copy of* $(Y_t)_{t\in\mathbb{R}_+}$

$$= \mathbb{P}\Big(Y_t(y) > s\Big).$$

Thus, $X_t(x) \leq_{st} Y_t(y)$ for $x \leq y$ for all t. Taking limits at both sides where we use π_X to denote the limit distribution of X, using Theorem 3.14 we get that the limit distributions of X and Y exist and do not depend on the initial states. By Theorem 3.15, which ensures us that the probability to stay out of the invariant subspace R is zero, we have $\pi_X \leq_{st} \pi_Y$. We call the processes X and Y *stochastically related* with respect to relation R.

Usually, the explicit coupling is never given. Existence of a coupling for which a (sub-relation of) relation R is invariant is sufficient to prove the strong stochastic comparison, as we conclude at the end of the next chapter. Another explicit coupling of Markov queueing systems is presented in Chapter 5, and more examples can be found, for example, in [Jonckheere and Leskelä, 2008] or [Chen, 2005].

2.2 Coupling

In this section, we introduce some basic concepts required for the rest of this thesis. After a brief revision of the concepts of *measurability*, *random elements*, *stochastic processes*, and the definition of the *distribution* of a random element, we give the definition of *coupling* for probability measures and random elements.

2.2.1 Stochastic process

We start by introducing the necessary measure-theoretic terminology and notation. A *σ-algebra* on S is a non empty subset of the power set 2^S which is closed under complements and countable unions. A *measurable space* is a pair $(S, \mathcal{S})$ where S is a set, and $\mathcal{S}$ is a σ-algebra on S. A *probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ is a measurable space with a probability measure. A mapping X between two measurable spaces $(S, \mathcal{S})$ and $(S', \mathcal{S}')$ is called *measurable* if $X^{-1}(A') \in \mathcal{S}$ for all $A' \in \mathcal{S}'$.

A *random element* in a measurable space $(S, \mathcal{S})$, defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, is a measurable mapping $X : (\Omega, \mathcal{F}, \mathbb{P}) \to (S, \mathcal{S})$, and we denote

$$X^{-1}(A) = \{X \in A\} := \{\omega \in \Omega : X(\omega) \in A\} \text{ for } A \in \mathcal{S}.$$

A *random variable* is a random element in $(\mathbb{R}, \mathcal{B})$, where $\mathbb{R}$ denotes the real numbers and $\mathcal{B}$ denotes the Borel subsets of $\mathbb{R}$ [Schilling, 2005].

A *stochastic process* with index set T is a family $X = (X_t)_{t\in T}$ where all X_t are random elements defined on a common probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $(S, \mathcal{S})$. We can think of X as a mapping $X : \Omega \to U \subseteq S^T$, or equivalently we can see the process X as a collection of random elements X_t in the state space S with $X_t : \Omega \to S$ for all $t \in T$ (see [Kallenberg, 2002], Lemma 3.1). In this work the index set T denotes the time line, which is either discrete ($T = \mathbb{N}$) or continuous ($T = \mathbb{R}_+$). The *paths* of a stochastic process X are realizations $X(\omega) = (X_t(\omega))_{t\in T}$, $\omega \in \Omega$. When we speak about a path, no randomness is involved anymore ($\omega \in \Omega$ is fixed) because the random elements $(X_t)_{t\in T}$ are fixed. The *path space* U is the subset of all paths $X(\omega)$. Thus, X is a stochastic process on Ω with paths in U. For discrete-time stochastic processes, $U = S^{\mathbb{N}}$, where

$$S^{\mathbb{N}} = \{\text{functions from } \mathbb{N} \text{ into } S\}.$$

For continuous-time processes, the path space U is denoted by $D = D(\mathbb{R}_+, S)$, where

$$D(\mathbb{R}_+, S) = \{\text{functions from } \mathbb{R}_+ \text{ into } S \text{ which are right-continuous and have left limits}\}.$$

Here, we restrict $D \subseteq S^{\mathbb{R}}$ to the set of functions which are right-continuous and have left limits to ensure measurability [Leskelä, 2010].

2.2.2 Coupling of probability measures

Let $\mathbb{P}$ be a probability measure on $S_1 \times S_2$. The *marginals* of $\mathbb{P}$ are defined by

$$[\mathbb{P}]_1(A_1) := \mathbb{P}(A_1 \times S_2) \text{ for all } A_1 \in \mathcal{S}_1, \text{ and}$$

$$[\mathbb{P}]_2(A_2) := \mathbb{P}(S_1 \times A_2) \text{ for all } A_2 \in \mathcal{S}_2.$$

Definition 2.1. *[Coupling of probability measures]*
Let $\mathbb{P}_X$ and $\mathbb{P}_Y$ be probability measures in $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$, respectively. A probability measure $\mathbb{P}: \ S_1 \times S_2 \to [0, 1]$ is a *coupling* of $\mathbb{P}_X$ and $\mathbb{P}_Y$ if the marginals of $\mathbb{P}$ equal $\mathbb{P}_X$ and $\mathbb{P}_Y$. This means, $\mathbb{P}$ is a coupling of $\mathbb{P}_X$ and $\mathbb{P}_Y$ if

$$[\mathbb{P}]_1(A_1) = \mathbb{P}_X(A_1) \text{ for all } A_1 \in \mathcal{S}_1,$$

and

$$[\mathbb{P}]_2(A_2) = \mathbb{P}_Y(A_2) \text{ for all } A_2 \in \mathcal{S}_2.$$

2.2.3 Distribution

The *distribution* of a random element X (under measure $\mathbb{P}$) is the probability measure on $(S, \mathcal{S})$ induced by X, namely $\mathbb{P}X^{-1}$, where

$$\mathbb{P}X^{-1}(A) = \mathbb{P}(X \in A) = \mathbb{P}(\{\omega \in \Omega : X(\omega) \in A\}) \text{ for } A \in \mathcal{S}.$$

A random element X' in $(S, \mathcal{S})$ is a *copy* or *representation* of X if the distributions of X and X' are equal, so

$$\mathbb{P}(X \in A) = \mathbb{P}(X' \in A) \text{ for all } A \in \mathcal{S}.$$

We write $X' \stackrel{d}{=} X$ to indicate that X' is a copy of X.

2.2.4 Coupling of random elements

Coupling is the joint construction of two (or more) random elements on one common probability space. The idea of this probability space is that it is a common probability space on which both random elements are defined.

Let X and Y be random elements in the spaces $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$, respectively. We introduce a third bivariate random element $Z = (\hat{X}, \hat{Y})$ in a space $(S, \mathcal{S})$. The random element Z lives on the product space of $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$. This common probability space is the natural product space

$$(S_1, \mathcal{S}_1) \otimes (S_2, \mathcal{S}_2) = (S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2),$$

where $S_1 \times S_2$ denotes the ordinary Cartesian product of S_1 and S_2, and $\mathcal{S}_1 \otimes \mathcal{S}_2$ is the product σ-algebra generated by $\mathcal{S}_1$ and $\mathcal{S}_2$ (for reference see for example [Schilling, 2005] or [Williams, 1991]). From Section 2.2.4 on, this product space will be denoded simply by $S_1 \times S_2$, leaving the σ-algebra implicit. Also, we do not need to care about measurability of the different sets, because our state spaces are always countable. Measurability also ensures us that all subsets of the state spaces are closed. We denote the first marginal of $\mathbb{P}Z^{-1}$ by $[\mathbb{P}Z^{-1}]_1$ or $\mathbb{P}\hat{X}^{-1}$ and the second marginal by $[\mathbb{P}Z^{-1}]_2$ (or $\mathbb{P}\hat{Y}^{-1}$):

$$[\mathbb{P}Z^{-1}]_1(A_1) := \mathbb{P}Z^{-1}(A_1 \times S_2),$$

$$[\mathbb{P}Z^{-1}]_2(A_2) := \mathbb{P}Z^{-1}(S_1 \times A_2).$$

Now we can give the formal definition of a coupling.

Definition 2.2. *[Coupling of random elements]*
Let X be a random element in $(S_1, \mathcal{S}_1)$ and Y a random element in $(S_2, \mathcal{S}_2)$. A random element $Z = (\hat{X}, \hat{Y})$ in $S_1 \times S_2$ is a *coupling* of X and Y if the marginal distributions of Z equal the distributions of X and Y. This means, $Z = (\hat{X}, \hat{Y})$ is a coupling of X and Y if

$$\left[\mathbb{P}Z^{-1}\right]_1 (A_1) = \mathbb{P}X^{-1}(A_1) \text{ for all } A_1 \in \mathcal{S}_1,$$

and

$$\left[\mathbb{P}Z^{-1}\right]_2 (A_2) = \mathbb{P}Y^{-1}(A_2) \text{ for all } A_2 \in \mathcal{S}_2.$$

In other words: a coupling $Z = (\hat{X}, \hat{Y})$, defined on some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}})$, of random elements X and Y is a measurable mapping $Z : (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{P}}) \to (S_1 \times S_2, \mathcal{S}_1 \otimes \mathcal{S}_2)$ such that $\hat{X} \stackrel{d}{=} X$ and $\hat{Y} \stackrel{d}{=} Y$.

Note that there always exists at least one coupling, namely the so-called *trivial coupling* where $\hat{X}$ and $\hat{Y}$ are independent of each other (Fact 3.1 of Chapter 3 in [Thorisson, 2000]).

2.2.5 Coupling of random variables

Now that coupling is defined for random elements, by the definitions in the beginning of the chapter we can specify the notion of coupling of random variables. A random variable is a random element with values on the real line; in this case the distribution of a random variable X is completely defined by $\mathbb{P}(X \leq s) = \mathbb{P}X^{-1}((-\infty, s])$ for all $s \in \mathbb{R}$. For a bivariate random element Z defined in $\mathbb{R} \times \mathbb{R}$, the marginal distributions are given by

$$\left[\mathbb{P}Z^{-1}\right]_1 ((-\infty, x]) := \mathbb{P}Z^{-1}((-\infty, x], (-\infty, \infty)), \text{ for all } x \in \mathbb{R},$$

$$\left[\mathbb{P}Z^{-1}\right]_2 ((-\infty, y]) := \mathbb{P}Z^{-1}((-\infty, \infty), (-\infty, y]), \text{ for all } y \in \mathbb{R}.$$

Definition 2.3. *[Coupling of random variables]*
Let X and Y be two random variables on $(\mathbb{R}, \mathcal{B})$. A bivariate random element $Z = (\hat{X}, \hat{Y})$ in $\mathbb{R} \times \mathbb{R}$ is a *coupling* of X and Y if the marginal distribution functions of Z are equal to the distribution functions of X and Y. Hence, Z is a coupling of X and Y if

$$\left[\mathbb{P}Z^{-1}\right]_1 ((-\infty, x]) = \mathbb{P}X^{-1}((-\infty, x]) \text{ for all } x \in \mathbb{R},$$

and

$$\left[\mathbb{P}Z^{-1}\right]_2 ((-\infty, y]) = \mathbb{P}Y^{-1}((-\infty, y]) \text{ for all } y \in \mathbb{R}.$$

We make a few remarks on Definitions 2.2 and 2.3. First of all, it is important to remark that for a coupling Z of X and Y, the *marginal* distributions coincide. Often for the original elements X and Y, the joint distribution is not defined or at least not known. As [Thorisson, 2000] says: *$\hat{X}$ and $\hat{Y}$ live together, X and Y do not*. This is exactly where couplings can be useful. When constructing a coupling, only the marginals have to coincide, and we have some freedom in constructing dependence between $\hat{X}$ and $\hat{Y}$. The trick is thus to find a coupling with a nice dependence relation of $\hat{X}$ and $\hat{Y}$, which gives us some comparison properties between X and Y. We study such comparison relations in Chapter 3. In Chapter 3 we will also present some theorems on the joint distribution of couplings. Strassen's Theorem (Section 3.3), for example, ensures the existence of a coupling which preserves a certain ordering. In the next two sections we will consider coupling in a Markov chain setting.

2.3 Coupling of discrete-time Markov processes

This section is focused on coupling of discrete-time Markov processes. A coupling of *transition probability matrices* is defined, and we prove that a coupling of the transition matrices of two processes is the transition matrix of a coupling of the processes.

2.3.1 Discrete-time Markov processes

Consider two discrete-time Markov processes $X = (X_t)_{t\in T}$ and $Y = (Y_t)_{t\in T}$ having countable state spaces S_1 and S_2, where the time horizon is $\mathbb{N}$. So each realization of process X is an infinite path in $S_1^{\mathbb{N}}$. At each time moment $t \in \mathbb{N}$, the process jumps to another state (or remains in the same state) with a certain probability. $X(x,\cdot)$ is a Markov process with transition matrix P_1 on state space S_1 and initial state x. We write $X(x,\cdot)$ for $(X_t(x))_{t\in\mathbb{N}}$ where t stands for time and x denotes the initial state.

Process X has *probability matrix*, or *transition matrix*, P_1 where $P_1(x,x')$ is the probability to jump from state x to state x'. The second process Y, taking values in space S_2, has probability matrix P_2. For all probability transition matrices, we must of course have that all entries are greater or equal than zero and all rows sum to one.

2.3.2 Coupling of transition matrices

Definition 2.4 generalizes the notion of coupling to discrete-time Markov processes determined by the transition probability matrices.

Definition 2.4. *[Coupling of transition matrices]*
Let $\hat{P}$ be a transition matrix on $S_1 \times S_2$. Then matrix $\hat{P}$ is called a *coupling* of P_1 and P_2 if for all $x \in S_1$ and $y \in S_2$,

$$\begin{aligned} P_1(x,x') &= \sum_{y'\in S_2} \hat{P}((x,y),(x',y')) \quad \text{for all } x' \in S_1, \\ &\text{and} \\ P_2(y,y') &= \sum_{x'\in S_1} \hat{P}((x,y),(x',y')) \quad \text{for all } y' \in S_2. \end{aligned}$$

Theorem 2.5. *[Coupling theorem for discrete-time Markov processes]*
Let P_1, P_2 be transition matrices on S_1 and S_2. Let $X(x,\cdot) = (X_t(x))_{t\in\mathbb{N}}$ be a discrete-time

Markov process on S_1 with transition matrix P_1 and initial state x. Similarly, let $Y(y,\cdot) = (Y_t(y))_{t\in\mathbb{N}}$ be a discrete-time Markov process on S_2 with transition matrix P_2 and initial state y. Assume that $\hat{P}$ on $S_1 \times S_2$ is a coupling of P_1 and P_2. Let $Z(z,\cdot) = (\hat{X}(z,\cdot), \hat{Y}(z,\cdot))$ be a Markov process on $S_1 \times S_2$ with transition matrix $\hat{P}$ and initial state $z = (x,y)$. Then the process $Z(z,\cdot)$ is a coupling of the processes $X(x,\cdot)$ and $Y(y,\cdot)$. In particular, $Z(z,t)$ is a coupling of $X_t(x)$ and $Y_t(y)$ for all t.

As we know from the previous section, the realizations of process X are paths in the state space. We are interested in $\mathbb{P}(X(x,\cdot) \in F)$ for subsets of paths $F \subseteq S^{\mathbb{N}}$. For the proof of Theorem 2.5, we use the following lemma which states that the distribution of a process $X = (X_t)_{t\in\mathbb{N}}$ is completely determined by the distribution functions of its finite-dimensional paths.

Lemma 2.6. *[Finite-dimensional distributions]*
Let $X = (X_t)_{t\in\mathbb{N}}$ and $Y = (Y_t)_{t\in\mathbb{N}}$ be processes living in S, with time horizon $\mathbb{N}$ and paths in $S^{\mathbb{N}}$. For each t, X_t and Y_t are random elements in S. Then $X \stackrel{d}{=} Y$ if and only if

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \stackrel{d}{=} (Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) \text{ for all } t_1, \ldots, t_n \in \mathbb{N},\ n \in \mathbb{N}.$$

For the proof of Lemma 2.6 see [Kallenberg, 2002], Proposition 3.2. In other words, Lemma 2.6 states that if we are interested in $\mathbb{P}(X(x,\cdot) \in B)$ for all subsets of paths $B \subset S^{\mathbb{N}}$, it is enough to consider the probability distributions of all finite paths

$$\mathbb{P}\Big(X_{t_0} = a_0; X_{t_1} = a_1; \ldots; X_{t_n} = a_n\Big),$$

for all $t_0, \ldots, t_n \in T$, $n \in \mathbb{N}$ and $a_0, a_1, \ldots, a_n \in S$. In this section, we deal with discrete-time Markov processes which jump at $t = 0, 1, \ldots$ Therefore, in Lemma 2.6 above, we can read $t_i = i$ for all $i \in \mathbb{N}$. But the lemma is also valid for continuous-time processes.

The *indicator function* of a set A is $\mathbb{1}_A := \begin{cases} 1 & \text{if } A \text{ is true,} \\ 0 & \text{otherwise.} \end{cases}$

We denote $X(x,\cdot)$ by $(X_t(x))_{t\in\mathbb{N}}$; $Y(y,\cdot) = (Y_t(y))_{t\in\mathbb{N}}$ and $Z(z,\cdot) = (Z_t(z))_{t\in\mathbb{N}}$ where $Z = (\hat{X}, \hat{Y})$, so $Z_t(z) = (\hat{X}_t, \hat{Y}_t)(z) = (\hat{X}_t(z), \hat{Y}_t(z))$ for all $z = (x,y) \in S_1 \times S_2$. Now, the coupling theorem for discrete-time Markov processes can be proved.

Proof of Theorem 2.5. Fix an arbitrary $n \in \mathbb{N}$ and a finite path $(a_t)_{t=0,\ldots,n}$ in S_1: $(a_0, a_1, \ldots, a_n) \in S_1^{n+1}$. Then, for all initial states b_0 in S_2 we have:

$$\begin{aligned}
& \mathbb{P}\Big(X_0(x) = a_0; X_1(x) = a_1; \ldots; X_n(x) = a_n\Big) \\
= \ & \mathbb{1}_{\{x=a_0\}} \cdot P_1(a_0,a_1) \cdot P_1(a_1,a_2) \cdot \ \ldots \ \cdot P_1(a_{n-1},a_n) \\
= \ & \mathbb{1}_{\{x=a_0\}} \cdot \sum_{b_1\in S_2} \Bigg\{ \hat{P}\Big((a_0,b_0),(a_1,b_1)\Big) \cdot \sum_{b_2\in S_2} \Bigg\{ \hat{P}\Big((a_1,b_1),(a_2,b_2)\Big) \cdot \ \ldots \\
& \qquad \cdot \sum_{b_{n-1}\in S_2} \Bigg\{ \hat{P}\Big((a_{n-2},b_{n-2}),(a_{n-1},b_{n-1})\Big) \cdot \sum_{b_n\in S_2} \Big\{ \hat{P}\Big((a_{n-1},b_{n-1}),(a_n,b_n)\Big) \Big\} \Bigg\} \ldots \Bigg\} \Bigg\} \\
= \ & \mathbb{P}\Big(Z_0((x,y)) \in \{a_0\} \times S_2;\ Z_1((x,y)) \in \{a_1\} \times S_2;\ \ldots;\ Z_n((x,y)) \in \{a_n\} \times S_2\Big) \\
= \ & \mathbb{P}\Big(\hat{X}_0((x,y)) = a_0;\ \hat{X}_1((x,y)) = a_1;\ \ldots;\ \hat{X}_n((x,y)) = a_n\Big),
\end{aligned}$$

for all $y \in S_2$.

In the same way we can fix an arbitrary finite path $(b_t)_{t=0,\dots,n}$ in S_2 and show that for this path also holds, for all initial states $a_0 \in S_1$:

$$\mathbb{P}\Big(Y_0(y) = b_0; Y_1(y) = b_1; \dots; Y_n(y) = b_n\Big) =$$

$$\mathbb{P}\Big(\hat{Y}_0((x,y)) = b_0; \hat{Y}_1((x,y)) = b_1; \dots; \hat{Y}_n((x,y)) = b_n\Big)$$

for all $x \in S_1$.

This holds for all finite paths $(a_t)_{t=0,\dots,n}$ in S_1 and $(b_t)_{t=0,\dots,n}$ in S_2, and for all initial states x, y in S_1 and S_2. Using Lemma 2.6 we can generalize this to infinite paths. Therefore, we conclude that process $\Big(Z_t((x,y))\Big)_{t\in\mathbb{N}} = \Big((\hat{X}_t, \hat{Y}_t)((x,y))\Big)_{t\in\mathbb{N}}$ is a coupling of the processes $(X_t(x))_{t\in\mathbb{N}}$ and $(Y_t(y))_{t\in\mathbb{N}}$ when the transition probability matrix $\hat{P}$ is a coupling of P_1 and P_2.

□

2.4 Coupling of continuous-time Markov processes

In this section we will do the same for continuous-time Markov processes as we did in Section 2.3 for the discrete-time situation. Probability matrices are replaced by *transition rate matrices*. In Section 2.4.2 we present the *uniformization method*, which allows us to express a continuous-time process in a discrete-time way. With Theorem 2.10 we prove that a coupling of transition rate matrices yields a coupling of the associated processes.

2.4.1 Continuous-time Markov processes

Now, suppose we have a continuous-time Markov process with transition rate matrix Q on state space S. During this section we will denote Markov processes in continuous-time (where the time horizon $T = \mathbb{R}_+$) using bold face symbols like $\mathbf{X}$ and $\mathbf{Y}$. For discrete-time processes the time $T = \mathbb{N}$, and these processes are denoted by normal-face symbols as X and Y.

Suppose that a continuous-time process $\mathbf{X}$ on a space S has the *transition rate matrix* (or *probability kernel*) Q where $Q(x,x')$ is the transition rate from state x to state x' for $x \neq x'$. In the continuous case we do not care about the transition rate from x to itself. Usually the entry $Q(x,x)$ is defined as $Q(x,x) = -\sum_{x'\neq x} Q(x,x')$, so that the entries on each row of the transition rate matrix sum up to zero, which is useful for some computations. In this work we will not need it, we can even leave the $Q(x,x)$'s undefined. For simplicity we adopt the convention that $Q(x,x) = 0$ for all $x \in S$.

2.4.2 Uniformization

Define for each state the *outflow rate* $q(x) := \sum_{x'\neq x} Q(x,x')$ of that state. The time that the process remains in state x has an exponential distribution with mean $1/q(x)$. Assume that the outflow rates are bounded: $\sup_{x\in S} q(x) < \infty$. Then there exists a constant $\gamma \in \mathbb{R}_+$ such that $q(x) \leq \gamma$ for all $x \in S$. Fix this *uniformization constant*.

Definition 2.7. *[Uniformization]*

$$P(x,x') := \left(\frac{q(x)}{\gamma}\right)\left(\frac{Q(x,x')}{q(x)}\right)\mathbb{1}_{\{x\neq x'\}} + \left(1 - \frac{q(x)}{\gamma}\right)\mathbb{1}_{\{x=x'\}}. \tag{2.4}$$

We call P the uniformized matrix of the transition rate matrix Q, or more briefly: P is the *uniformized Q-matrix*. Note that this matrix is uniquely determined for a fixed uniformization constant γ.

The uniformization formula (2.4) transforms the collection of transition rates $Q(x, x')$ into transition probabilities $P(x, x')$, where events occur according to a Poisson arrival process with rate γ. Here $\frac{q(x)}{\gamma}$ is the fraction of the events which turn out to true jumps (to another state) of the Markov process, while $1 - \frac{q(x)}{\gamma}$ is the fraction of arrivals where the process 'jumps' into the same state. The fraction $\frac{Q(x,x')}{q(x)}$ is exactly the probability that a true jump in the continuous-time process out of state x goes to state x', so it is the transition probability of the jump process (also called the *embedded Markov process*).

Lemma 2.8. *[Construction of continuous-time Markov processes]*
Let $\mathbf{X}(x, \cdot) = (\mathbf{X}_t(x))_{t\in\mathbb{R}_+}$ be a continuous-time Markov process with bounded transition rate matrix Q, and let P be the uniformized Q-matrix with uniformization constant γ, as defined in Definition 2.7. Given an initial state $x \in S$, let $X(x, \cdot) = (X_t(x))_{t\in\mathbb{N}}$ be the discrete-time Markov process with transition matrix P and let $N = (N_t)_{t\in\mathbb{R}_+}$ be a Poisson process with rate γ. Assume that the discrete-time process $X(x, \cdot)$ and Poisson process N are independent. Then

$$\Big(X_{N(t)}(x)\Big)_{t\in\mathbb{R}_+} \stackrel{d}{=} \Big(\mathbf{X}_t(x)\Big)_{t\in\mathbb{R}_+}$$

is a continuous-time Markov process with transition rate matrix Q.

This lemma follows from Theorem 12.18 of Kallenberg [Kallenberg, 2002], and gives us a way of constructing a continuous-time Markov process. The Poisson process $(N_t)_{t\in\mathbb{R}_+}$ generates events which come independent of each other with inter-arrival times that have exponential distribution with parameter γ. At an arrival instant, the state of the process $N(t)$ jumps from n to $n+1$. N_t is thus a counting process giving for each time t the total number of events which occurred in the time interval $[0, t]$. Any realization of the process N_t corresponds to a random sequence in $\mathbb{R}_+$ of time moments $(t_n)_{n\in\mathbb{N}}$, where $0 = t_0 < t_1 < \ldots$.

Intuitively, we can see the process $(X_{N(t)}(x))_{t\in\mathbb{R}_+}$ as a discrete process, but the jumps do not happen at fixed time points $1, 2, \ldots \in \mathbb{N}$ but at random moments in time $t_1, t_2, \ldots \in \mathbb{R}_+$. At those random time moments, generated by N_t, an alarm clock rings when an event occurs. Every time that this Poisson-bell is ringing, with certain probabilities the process jumps to another state. These probabilities are given by the discretized (uniformized) probability $P(x, x')$ for a jump from state x to state x' (given in Equation (2.4)). We use two different notations: $(X(x, N(t)))_{t\in\mathbb{R}_+} = (X_{N(t)}(x))_{t\in\mathbb{R}_+}$.

2.4.3 Coupling of transition rate matrices

Suppose that a process $\mathbf{X}$ on S_1 has transition rate matrix Q_1 and a process $\mathbf{Y}$ on S_2 has transition rate matrix Q_2.

Definition 2.9. *[Coupling of transition rate matrices]*
Let $\hat{Q}$ be a transition rate matrix on $S_1 \times S_2$. Then $\hat{Q}$ is called a *coupling* of Q_1 and Q_2 if

for all $x \in S_1$ and $y \in S_2$:

$$\begin{aligned} Q_1(x,x') &= \sum_{y' \in S_2} \hat{Q}((x,y),(x',y')) \quad \text{for all } \ x' \in S_1,\ x' \neq x, \\ &\text{and} \\ Q_2(y,y') &= \sum_{x' \in S_1} \hat{Q}((x,y),(x',y')) \quad \text{for all } \ y' \in S_2,\ y' \neq y. \end{aligned}$$

This definition is in line with Definition 2.4, except the fact that the rates from a state to itself $Q_1(x,x)$ and $Q_2(y,y)$ are not defined; they do not make any sense for a continuous-time Markov process.

Theorem 2.10. *[Coupling theorem for continuous-time Markov processes]*
Let Q_1 be a bounded transition rate matrix on S_1 and let $\mathbf{X}(x,\cdot) = (\mathbf{X}_t(x))_{t \in \mathbb{R}_+}$ be a continuous-time Markov process on S_1 with transition rate matrix Q_1, and initial state x. Similarly, let Q_2 be a bounded transition rate matrix on S_2, and $\mathbf{Y}(y,\cdot) = (\mathbf{Y}_t(y))_{t \in \mathbb{R}_+}$ be a continuous-time Markov process on S_2 with transition rate matrix Q_2 and initial state y. Assume that the matrix $\hat{Q}$ on $S_1 \times S_2$ is a coupling of Q_1 and Q_2. Let $\mathbf{Z}(z,\cdot)$ be a continuous-time Markov process on $S_1 \times S_2$ with transition rate matrix $\hat{Q}$ and initial state $z = (x,y)$. Then the process $\mathbf{Z}(z,\cdot)$ is a coupling of the processes $\mathbf{X}(x,\cdot)$ and $\mathbf{Y}(y,\cdot)$.

The proof of this theorem is given in Section 2.4.4. We write $\mathbf{Z}(z,\cdot) = (\mathbf{Z}_t(z))_{t \in \mathbb{R}_+}$ and $\mathbf{Z} = (\hat{\mathbf{X}}, \hat{\mathbf{Y}})$, so $\mathbf{Z}_t(z) = (\hat{\mathbf{X}}_t(z), \hat{\mathbf{Y}}_t(z))$, for $z = (x,y) \in S_1 \times S_2$.

Fix uniformization constant γ such that $q_1(x) \leq \gamma$ and $q_2(y) \leq \gamma$ for all $x \in S_1$ and for all $y \in S_2$. Define the *uniformized matrices* P_1, P_2 and $\hat{P}$ derived from the transition rate matrices (Q_1, Q_2 and $\hat{Q}$) of the processes $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$:

$$\begin{cases} P_1(x,x') := \frac{q_1(x)}{\gamma} \left(\frac{Q_1(x,x')}{q_1(x)} \right) \mathbb{1}_{\{x \neq x'\}} + \left(1 - \frac{q_1(x)}{\gamma}\right) \mathbb{1}_{\{x = x'\}}, \\ P_2(y,y') := \frac{q_2(y)}{\gamma} \left(\frac{Q_2(y,y')}{q_2(y)} \right) \mathbb{1}_{\{y \neq y'\}} + \left(1 - \frac{q_2(y)}{\gamma}\right) \mathbb{1}_{\{y = y'\}}, \\ \hat{P}\,(z,z') := \frac{\hat{q}(z)}{\gamma} \left(\frac{\hat{Q}(z,z')}{\hat{q}(z)} \right) \mathbb{1}_{\{z \neq z'\}} + \left(1 - \frac{\hat{q}(z)}{\gamma}\right) \mathbb{1}_{\{z = z'\}}, \end{cases} \tag{2.5}$$
$$\text{for all } x,\ x' \in S_1, \text{ all } y,\ y' \in S_2 \text{ and for all } z = (x,y),\ z' = (x',y') \in S_1 \times S_2.$$

Lemma 2.11. *[Coupling of the uniformized matrices]*
Let $\hat{Q}$ be a coupling of Q_1 and Q_2 and define $\hat{P}$, P_1 and P_2 as above. Then $\hat{P}$ is a coupling of P_1 and P_2.

Proof. We first look at the outflow rate $\hat{q}(z)$ for all $z = (x, y) \in S_1 \times S_2$.

$$\begin{aligned}
\hat{q}(z) = &\sum_{z' \in S_1 \times S_2:\ z' \neq z} \hat{Q}(z, z') \\
= &\sum_{(x',y') \in S_1 \times S_2:\ (x',y') \neq (x,y)} \hat{Q}((x,y),(x',y')) \\
= &\sum_{y' \in S_2} \sum_{x' \in S_1:\ x' \neq x} \hat{Q}((x,y),(x',y')) + \sum_{y' \in S_2:\ y' \neq y} \hat{Q}((x,y),(x,y')) \\
= &\ q_1(x) + \sum_{y' \in S_2:\ y' \neq y} \hat{Q}((x,y),(x,y')),
\end{aligned}$$

where in the last step we used that
$\sum_{y'} \sum_{x' \neq x} \hat{Q}((x,y),(x',y')) = \sum_{x' \neq x} \sum_{y'} \hat{Q}((x,y),(x',y')) = \sum_{x' \neq x} Q_1(x,x') = q_1(x)$.
From this, it follows that for all $x = x'$ we have:

$$\sum_{y' \in S_2:\ y' \neq y} \hat{Q}((x,y),(x,y')) = \hat{q}((x,y)) - q_1(x). \tag{2.6}$$

We want to prove that for all $x \in S_1$ and for all $y \in S_2$

$$\sum_{y' \in S_2} \hat{P}((x,y),(x',y')) = P_1(x,x') \text{ for every } x' \in S_1.$$

- In the case that we have $x = x'$:

  $\sum_{y' \in S_2} \hat{P}((x,y),(x',y')) =$

  by the definition of the uniformized matrix (Equation 2.5)

  $$\sum_{y' \in S_2} \left[\frac{\hat{q}((x,y))}{\gamma} \frac{\hat{Q}((x,y),(x',y'))}{\hat{q}((x,y))} \mathbb{1}_{\{(x,y) \neq (x',y')\}} + \left(1 - \frac{\hat{q}((x,y))}{\gamma}\right) \mathbb{1}_{\{(x,y) = (x',y')\}} \right]$$

  the summation on y' disappears by the indicator function

  $$= \sum_{y' \in S_2:\ y' \neq y} \left[\frac{\hat{q}((x,y))}{\gamma} \frac{\hat{Q}((x,y),(x',y'))}{\hat{q}((x,y))} \right] + \left(1 - \frac{\hat{q}((x,y))}{\gamma}\right)$$

  because of Equation (2.6)

  $= \frac{\hat{q}((x,y)) - q_1(x)}{\gamma} + \left(1 - \frac{\hat{q}((x,y))}{\gamma}\right) = 1 - \frac{q_1(x)}{\gamma} = P_1(x,x')$.

- For the case that $x \neq x'$:

$$\sum_{y' \in S_2} \hat{P}((x,y),(x',y')) =$$

$\hat{Q}$ is a coupling of Q_1 and Q_2, so by the definition of the uniformized matrix (Equation (2.5)): $Q_1(x,x') = \sum_{y' \in S_2} \hat{Q}((x,y),(x',y'))$ for all $x \neq x'$

$$\sum_{y' \in S_2} \left[\frac{\hat{q}((x,y))}{\gamma} \frac{\hat{Q}((x,y),(x',y'))}{\hat{q}((x,y))} \mathbb{1}_{\{(x,y) \neq (x',y')\}} \right] = \sum_{y' \in S_2} \left[\frac{\hat{Q}((x,y),(x',y'))}{\gamma} \right]$$

by the continuous-time definition of coupling (Definition 2.9)

$$= \frac{Q_1(x,x')}{\gamma}$$

again from definition of the uniformized matrix (2.5)

$$= P_1(x,x').$$

We conclude that we have for all $x \in S_1$ and $y \in S_2$

$$P_1(x,x') = \sum_{y' \in S_2} \hat{P}((x,y),(x',y')) \text{ for all } x' \in S_1.$$

In exactly the same way we can prove that for all $y \in S_2$ and $x \in S_1$

$$P_2(y,y') = \sum_{x' \in S_1} \hat{P}((x,y),(x',y')) \text{ for all } y' \in S_2.$$

These are exactly the conditions of Definition 2.4, so we conclude that $\hat{P}$ is indeed a coupling of P_1 and P_2.

□

2.4.4 Proof of the coupling theorem for continuous-time

Proof of Theorem 2.10. We use the uniformization method of Section 2.4.2 to write the continuous-time processes as a (composed) discrete-time process. Let

$X(x,\cdot) = (X_n(x))_{n \in \mathbb{N}}$ be the discrete-time Markov process on S_1 with initial state x and transition matrix P_1 where P_1 is the uniformized Q_1-matrix with rate γ. Let $N_1(\cdot) = (N_1(t))_{t \geq 0}$ be independent of X.

$Y(y,\cdot) = (Y_n(y))_{n \in \mathbb{N}}$ be the discrete-time Markov process on S_2 with initial state y and transition matrix P_2 where P_2 is the uniformized Q_2-matrix with rate γ. Let $N_2(\cdot) = (N_2(t))_{t \geq 0}$ be independent of Y.

$Z((x,y),\cdot) = \Big(Z_n((x,y))\Big)_{n \in \mathbb{N}} = \Big((\hat{X}_n,\hat{Y}_n)((x,y))\Big)_{n \in \mathbb{N}}$ be the discrete-time Markov process on $S_1 \times S_2$ with initial state (x,y) and transition matrix $\hat{P}$ where $\hat{P}$ is the uniformized $\hat{Q}$-matrix. Let $N(\cdot) = (N(t))_{t \geq 0}$ be independent of $Z = (\hat{X},\hat{Y})$.

For the processes N_1, N_2 and N here we want to use a Poisson process with arrival rate γ. We let them coincide: $(N_1(t))_{t\geq 0} = (N_2(t))_{t\geq 0} = (N(t))_{t\geq 0}$, and furthermore this N is independent of the processes X, Y and Z. Then Lemma 2.8 gives:

$$\begin{cases} \left(X_{N(t)}(x)\right)_{t\in\mathbb{R}_+} \overset{d}{=} \left(\mathbf{X}_t(x)\right)_{t\in\mathbb{R}_+} \text{ is a continuous-time Markov process with transition} \\ \text{rate matix } Q_1. \\ \\ \left(Y_{N(t)}(y)\right)_{t\in\mathbb{R}_+} \overset{d}{=} \left(\mathbf{Y}_t(y)\right)_{t\in\mathbb{R}_+} \text{ is a continuous-time Markov process with transition} \\ \text{rate matix } Q_2. \\ \\ \left(Z_{N(t)}(x,y)\right)_{t\in\mathbb{R}_+} = \left((\hat{X},\hat{Y})_{N(t)}((x,y))\right)_{t\in\mathbb{R}_+} \overset{d}{=} \left(\mathbf{Z}_t((x,y))\right)_{t\in\mathbb{R}_+} = \left((\hat{\mathbf{X}},\hat{\mathbf{Y}})_t((x,y))\right)_{t\in\mathbb{R}_+} \\ \text{is a continuous-time Markov process with transition rate matrix } \hat{Q}. \end{cases}$$

We want to prove that $(\mathbf{Z}_t)_{t\geq 0}$ is a coupling of $(\mathbf{X}_t)_{t\geq 0}$ and $(\mathbf{Y}_t)_{t\geq 0}$. To do this, it is enough to prove the equivalent statement that $(Z_{N(t)}(x,y))_{t\geq 0}$ is a coupling of $(X_{N(t)}(x))_{t\geq 0}$ and $(Y_{N(t)}(y))_{t\geq 0}$. We do this by proving that for all finite paths, Z is a coupling of X and Y, and using Lemma 2.6 we get the general result for the continuous-time.

Claim:
For all $n \in \mathbb{N}$, and for all $0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$ and for all finite paths $(a_0, a_1, \ldots, a_n)$, where $a_i \in S_1$ for all $i = 0, 1, \ldots n$, and for all initial states $z = (x,y) = (a_0, b_0) \in S_1 \times S_2$ we have:

$$\begin{aligned} & \mathbb{P}\left(\hat{\mathbf{X}}_{t_1}(x) = a_1; \hat{\mathbf{X}}_{t_2}(x) = a_2; \ldots; \hat{\mathbf{X}}_{t_n}(x) = a_n\right) \\ = \; & \mathbb{P}\left(\mathbf{X}_{t_1}(x) = a_1; \mathbf{X}_{t_2}(x) = a_2; \ldots; \mathbf{X}_{t_n}(x) = a_n\right). \end{aligned} \tag{2.7}$$

Proof of Claim:
Initialization step
Equation (2.7) holds for $n = 1$:

$$\begin{aligned} \mathbb{P}\left(\hat{\mathbf{X}}_{t_1}((x,y)) = a_1\right) &= \sum_{n=0}^{\infty} \mathbb{P}(N_{t_1} = n)\, \mathbb{P}\left(\hat{X}_n((x,y)) = a_1\right) \\ & \quad \textit{because of Lemma 2.11} \\ &= \sum_{n=0}^{\infty} \mathbb{P}(N_{t_1} = n)\, \mathbb{P}\left(X_n(x) = a_1\right) \\ &= \mathbb{P}\left(\mathbf{X}_{t_1}(x) = a_1\right). \end{aligned}$$

The first and last equality come from the fact that Z and X are both independent of N.

Induction step
Assume (2.7) holds for $n \in \mathbb{N}$. Fix any $0 < t_1 < t_2 < \cdots < t_{n+1}$ and any path

$(a_1, a_2, \ldots, a_{n+1})$ in S_1.

$$\begin{aligned}
& \mathbb{P}\left(\hat{\mathbf{X}}_{t_1}((x,y)) = a_1; \hat{\mathbf{X}}_{t_2}((x,y)) = a_2; \ldots; \hat{\mathbf{X}}_{t_{n+1}}((x,y)) = a_{n+1}\right) \\
& \quad \textit{because of the memoryless property of Markov processes} \\
= \; & \mathbb{P}\left(\hat{\mathbf{X}}_{t_1}(x) = a_1; \ldots; \hat{\mathbf{X}}_{t_n}(x) = a_n\right) \mathbb{P}\left(\hat{\mathbf{X}}_{t_{n+1}-t_n}(a_n) = a_{n+1}\right) \\
& \quad \textit{use the induction hypothesis} \\
= \; & \mathbb{P}\left(\mathbf{X}_{t_1}(x) = a_1; \ldots; \mathbf{X}_{t_n}(x) = a_n\right) \mathbb{P}\left(\hat{\mathbf{X}}_{t_{n+1}-t_n}(a_n) = a_{n+1}\right) \\
& \quad \textit{use the initialization step} \\
= \; & \mathbb{P}\left(\mathbf{X}_{t_1}(x) = a_1; \ldots; \mathbf{X}_{t_n}(x) = a_n\right) \mathbb{P}\left(\mathbf{X}_{t_{n+1}-t_n}(a_n) = a_{n+1}\right) \\
& \quad \textit{because of the memoryless property of Markov processes} \\
= \; & \mathbb{P}\left(\mathbf{X}_{t_1}(x) = a_1;\ \mathbf{X}_{t_2}(x) = a_2;\ \ldots;\ \mathbf{X}_{t_{n+1}}(x) = a_{n+1}\right).
\end{aligned}$$

By induction we proved that the claim holds for all $n \in \mathbb{N}$.

In exactly the same way we can prove the claim for $\hat{Y}$:
For all $n \in \mathbb{N}$, and for all $0 < t_1 < t_2 < \cdots < t_n$ and for all paths $b_0, b_1, \ldots, b_n$ where $b_i \in S_2$ for all $i = 1, \ldots n$, and for all initial distributions $x = (x, y) = (a_0, b_0)$ we have:

$$\begin{aligned}
& \mathbb{P}\left(\hat{\mathbf{Y}}_{t_1}((x,y)) = b_1; \hat{\mathbf{Y}}_{t_2}((x,y)) = b_2; \ldots; \hat{\mathbf{Y}}_{t_n}(x) = b_n\right) \\
& \quad = \mathbb{P}\left(\mathbf{Y}_{t_1}(y) = b_1;\ \mathbf{Y}_{t_2}(y) = b_2;\ \ldots;\ \mathbf{Y}_{t_n}(y) = b_n\right).
\end{aligned} \tag{2.8}$$

Lemma 2.6 gives us the generalization that Equations (2.7) and (2.8) hold for every (infinite) path. And we finally conclude that $\mathbf{Z}((x,y),\cdot)$ is a coupling of $\mathbf{X}(x,\cdot)$ and $\mathbf{Y}(y,\cdot)$.

$\square$

Chapter 3

Stochastic comparison

The final goal is to compare two stochastic processes (or random elements). But to compare the behaviour of two processes, they must somehow be *comparable*. This is basically the purpose of this chapter. We will define the notion of stochastic comparison. In Section 3.1 we consider stochastic ordering of processes which are defined on one and the same ordered state space, by a *partial ordering* on this space. The second section generalizes the notion of a stochastic order. By defining *relations* it is possible to compare processes defined on different probability spaces. In Section 3.3 *Strassen's theorem* gives a nice comparison condition, and in Section 3.4 we look at the stochastic comparison of random processes. Finally, in Section 3.5, we present theorems which give necessary and sufficient conditions for the stochastic comparison of random processes which give us easier conditions to check whether or not two Markov processes are stochastically related to each other with respect to a given relation. As we proved the coupling equivalences between discrete-time and continuous-time processes by uniformization in the previous section, we will not anymore make a distinction in the notation of discrete-time or continuous-time processes. For the rest of this thesis, all processes are assumed to be continuous-time processes unless otherwise specified.

3.1 Stochastic orders

In this section we assume state space S to be equipped with a partial order $\preceq$. A *partial order* $\preceq$ on a set S is a binary relation for which the following three conditions hold:

$$x \preceq x \qquad \forall\, x \in S \qquad \textit{(reflexivity)} \tag{3.1}$$
$$x \preceq y \;\&\; y \preceq z \;\Rightarrow\; x \preceq z \quad \forall\, x, y, z \in S \quad \textit{(transitivity)} \tag{3.2}$$
$$x \preceq y \;\&\; y \preceq x \;\Rightarrow\; x = y \quad \forall\, x, y \in S \quad \textit{(antisymmetry)} \tag{3.3}$$

An order which only satisfies (3.1) and (3.2) is called a *pre-order*. More on stochastic orderings can be found in [Müller and Stoyan, 2002] or [Shaked and Shanthikumar, 2007].

3.1.1 Stochastic domination of probability measures

Definition 3.1. *[Stochastically ordered measures]*
Let $\mathbb{P}_X$ and $\mathbb{P}_Y$ be two probability measures on $(S, \mathcal{S})$ and let $\preceq$ be a partial order in S. We

call the measures $\mathbb{P}_X$ and $\mathbb{P}_Y$ *stochastically ordered with respect to a partial order* $\preceq$ if there exist a coupling $\mathbb{P}$ of $\mathbb{P}_X$ and $\mathbb{P}_Y$ such that

$$\mathbb{P}(\{(x,y):\ x \preceq y\}) = 1.$$

We use the term *stochastic domination* for stochastic comparison in partially ordered spaces. We say that the measure $\mathbb{P}_{\mathbf{X}}$ is stochastically dominated by $\mathbb{P}_{\mathbf{Y}}$ and write $\mathbb{P}_X \preceq_{st} \mathbb{P}_Y$.

Usually, the concept of stochastic domination (or strong stochastic ordering) is defined in terms of expectations of increasing functions. This better known definition can be found in for example in Chapter 4 of [Lindvall, 1992].

A function $f : S \to \mathbb{R}$ is *increasing* with respect to the partial order $\preceq$ if

$$x \preceq y \Longrightarrow f(x) \leq f(y) \text{ for all } x, y \in S.$$

Definition 3.2. *[Alternative definition of stochastic domination]*
Two probability measures $\mathbb{P}_X$ and $\mathbb{P}_Y$ on $(S, \mathcal{S})$ are *stochastically ordered with respect to a partial order* $\preceq$ on S if

$$\int f\, d\mathbb{P}_X \leq \int f\, d\mathbb{P}_Y$$

for all bounded increasing measurable functions $f : S \to \mathbb{R}$.

The assumption that the function f is bounded assures us that the expectations $\int f\, d\mathbb{P}_X$ and $\int f\, d\mathbb{P}_Y$ exist. Definition 3.2 holds if and only if our (alternative) Definition 3.1 holds, this is proved with Strassen's Theorem on stochastic ordering [Strassen, 1965] and follows also from Theorem 2.6.3 in [Müller and Stoyan, 2002].

3.1.2 Stochastic domination of random elements

Definition 3.3. [*Stochastically ordered random elements*]
Let X and Y be random elements on $(S, \mathcal{S})$. A *random element Y dominates X stochastically* with respect to a partial order $\preceq$ (notation $X \preceq_{st} Y$) if the distribution of X is stochastically dominated by the distribution of Y, that is if $\mathbb{P}X^{-1} \preceq_{st} \mathbb{P}Y^{-1}$. Recall that $Z = (\hat{X}, \hat{Y})$ is a coupling of X and Y if the marginal distributions of Z coincide with the distributions of X and Y. Therefore,

$$X \preceq_{st} Y$$

if and only if there exists a coupling $Z = (\hat{X}, \hat{Y})$ of X and Y such that

$$\mathbb{P}(\hat{X} \preceq \hat{Y}) = 1.$$

3.1.3 Stochastic domination of random variables

For real-valued random elements the definitions for the stochastic ordering can be formulated in an intuitively easier way. We denote the natural order on $\mathbb{R}$ by $\leq$, and we write $X \leq_{st} Y$ to denote that X is stochastically smaller than Y with respect to this order.

Stochastically ordered measures on $(\mathbb{R}, \mathcal{B})$
Let $\mathbb{P}_X$ and $\mathbb{P}_Y$ be probability measures on state space $(\mathbb{R}, \mathcal{B})$. For real-valued probability measures, Definition 3.1 becomes

$$\mathbb{P}_X \leq_{st} \mathbb{P}_Y \text{ if and only if } \mathbb{P}_X([s, \infty)) \leq \mathbb{P}_Y([s, \infty)) \text{ for all } s \in \mathbb{R}.$$

See Section 3.2 in [Thorisson, 2000] or [Kamae et al., 1977].

Stochastically ordered random variables
For random variables X and Y, $X \leq_{st} Y$ if there exists a coupling $Z = (\hat{X}, \hat{Y})$ of X and Y with $\mathbb{P}(\hat{X} \geq s) \leq \mathbb{P}(\hat{Y} \geq s)$ for all $s \in \mathbb{R}$. And because $(\hat{X}, \hat{Y})$ is a coupling of X and Y, $\mathbb{P}(\hat{X} \geq s) = \mathbb{P}(X \geq s)$ and $\mathbb{P}(\hat{Y} \geq s) = \mathbb{P}(Y \geq s)$ for all $s \in \mathbb{R}$. So for real-valued random variables we have

$$X \leq_{st} Y \text{ if and only if } \mathbb{P}(X \geq s) \leq \mathbb{P}(Y \geq s) \text{ for all } s \in \mathbb{R}.$$

Intuitively, $X \leq_{st} Y$ means that the probability that the random variable X is big is smaller than the probability that Y is big, or in other words, X assumes small values with a higher probability than Y does.

3.1.4 Strassen's characterization of stochastic orders

A characterization for the existence of a coupling $(\hat{X}, \hat{Y})$ for which $\hat{X} \preceq \hat{Y}$ is true with probability one, which thus implies $X \preceq_{st} Y$, is presented for example in [Kamae et al., 1977]. Kamae, Krengel and O'Brien came up with a characterization in terms of *upper sets* of the state space S.

$A \subseteq S$ is an *upper set* (sometimes also called *increasing set*) if $x \in A$ implies $\{y : x \preceq y\} \subseteq A$. In [Massey, 1987] and [Whitt, 1986] necessary and sufficient conditions are given for the stochastic ordering of continuous-time Markov processes using the notion of upper sets. Note that A is an upper set if and only if $\mathbb{1}_A$ is an increasing function, and this is how Definitions 3.1 and 3.2 in Section 3.1.1 are related.

Theorem 3.4. *[Strassen's characterization of stochastic orders]*
Consider two random elements X and Y living on the same space S, where S is equipped with the partial order $\preceq$. $\mathbb{P}_X \preceq_{st} \mathbb{P}_Y$ if and only if

$$\mathbb{P}_X(A) \leq \mathbb{P}_Y(A) \textit{ for all upper sets } A \in \mathcal{S}.$$

A proof is given, for example in [Müller and Stoyan, 2002], see Theorem 2.6.4. A generalization of Theorem 3.4 for stochastic comparison with respect to relations will be given in Section 3.3, Theorem 3.8.

3.2 Stochastic relations

A *relation* R between state spaces S_1 and S_2 is a measurable subset of the product space $S_1 \times S_2$. We write $x \sim y$ if $(x, y) \in R$, and $R = \{(x, y) : x \sim y\}$.

Stochastic domination with respect to a partial order is a special case of stochastic comparison with respect to a relation. Working with the more general definition of relations instead of orderings gives us the possibility to compare processes on different state spaces, and furthermore, we have the possibility to use relations which are not partial orders, as we will do for example in Section 4.3.

3.2.1 Stochastically related probability measures

Definition 3.5. *[Stochastically related measures]*
We consider probability measures $\mathbb{P}_X$ and $\mathbb{P}_Y$ on $(S_1, \mathcal{S}_1)$ and $(S_2, \mathcal{S}_2)$, respectively. The

measures $\mathbb{P}_X$ *and* $\mathbb{P}_Y$ *are stochastically related* (notation $\mathbb{P}_X \sim_{st} \mathbb{P}_Y$) if there exists a coupling $\mathbb{P}$ of $\mathbb{P}_X$ and $\mathbb{P}_Y$ such that $\mathbb{P}(R) = 1$.

3.2.2 Stochastically related random elements

Definition 3.6. *[Stochastically related random elements]*
Let X and Y be two random elements in S_1 and S_2, and let $R \subset S_1 \times S_2$ be some relation between S_1 and S_2. Then X *is stochastically related to* Y with respect to the relation R if there exists a coupling $Z = (\hat{X}, \hat{Y})$ of X and Y such that

$$\mathbb{P}Z^{-1}(R) = 1.$$

$\mathbb{P}Z^{-1}(R) = 1$ means $\mathbb{P}((\hat{X}, \hat{Y}) \in R) = 1$ or in other words $\hat{X} \sim \hat{Y}$ with probability one. We write $X \sim_{st} Y$ to denote that X is stochastically related to Y.

Recall that a coupling Z of X and Y couples the distributions of X and Y. This implies that $X \sim_{st} Y$) exactly if the distribution of X is stochastically related to the distribution of Y:

$$\mathbb{P}X^{-1} \sim_{st} \mathbb{P}Y^{-1}.$$

3.3 Strassen's characterizations of stochastic relations

Random elements are stochastically related if there exist a coupling of them, for which the relation holds with probability one. In this section we give necessary and sufficient conditions for the stochastic comparison of random elements. We look at a characterization in terms of *relational conjugates* (a kind of upper sets for relations) of the state spaces, and we present necessary and sufficient conditions for the existence of a coupling of random elements, for which the relation is invariant.

3.3.1 Relational conjugates

In Section 3.1.4, Theorem 3.4 gives us a nice characterization for stochastic domination in terms of *upper subsets* of the underlying state space. *Relational conjugates* are an intuitive analogue of upper sets for relations. An analogous characterization for the stochastic comparison of processes X and Y (*not* living in one and the same state space) with respect to a relation can be given, see [Leskelä, 2010]. Instead of upper sets Leskelä introduces *relational conjugates* which are a generalization of upper sets for a relation R of between countable subsets S_1 and S_2.

Definition 3.7. *[Relational conjugates]*
The *right conjugate* of a set $B_1 \subseteq S_1$ with respect to relation R is given by

$$B_1^{\rightarrow} := \bigcup_{x \in B_1} \{y \in S_2 : x \sim y\} \subseteq S_2. \tag{3.4}$$

The *left conjugate* of a set $B_2 \subseteq S_2$ with respect to relation R is given by

$$B_2^{\leftarrow} := \bigcup_{y \in B_2} \{x \in S_1 : x \sim y\} \subseteq S_1. \tag{3.5}$$

3.3.2 Strassen's theorem

Strassen's theorem for stochastic orders (Theorem 3.4) can be extended from orders to relations as defined in the previous section.

Theorem 3.8. *[Strassen's characterization of stochastic relations]*
$\mathbb{P}_X \sim_{st} \mathbb{P}_Y$ *if and only if*

$$\mathbb{P}_X(B_1) \leq \mathbb{P}_Y(B_1^{\rightarrow}) \text{ for all } B_1 \subseteq S_1. \tag{3.6}$$

Leskelä proved that it suffices if Equation (3.6) holds for all compact sets B_1, or in our case where the spaces S_1 and S_2 are countable, $\mathbb{P}_X \sim_{st} \mathbb{P}_Y$ if and only if (3.6) holds for all finite $B_1 \subseteq S_1$ [Strassen, 1965], [Leskelä, 2010].

3.4 Stochastically related stochastic processes

For stochastic processes, we have to take the time aspect into account. Our first task in this section is to extend the relation somehow to the different values of the processes in the time. In the end, we look at the stochastic comparison of stationary distributions and we present a useful theorem which assures us the comparison of stationary distributions (Theorem 3.15).

3.4.1 Stochastic relation on a finite path

Suppose X and Y are finite paths in S_1 and S_2. Relation R slightly changes to a relation on finite paths in $S_1 \times S_2$:

$$R^n := \{(x, y) \in S_1^n \times S_2^n : x_i \sim y_i \text{ for all } i = 1, \ldots, n\}. \quad \textit{[Finite path relation]}$$

So if X and Y are finite paths of length n in S_1 and S_2 ($X \in S_1^n$ and $Y \in S_2^n$) we say that *X is stochastically related to Y* if $(X_{t_i})_{i=1,\ldots,n} \sim_{st} (Y_{t_i})_{i=1,\ldots,n}$ with respect to the path-relation R^n.

3.4.2 Stochastically related random processes

Each realization of a process X is a path $x = (x_t)_{t \in \mathbb{R}_+}$. For a continuous-time process, each realization of process X is an element in $D_1 = D_1(\mathbb{R}_+, S_1) = \{$functions from $\mathbb{R}_+$ into S_1 which are right-continuous and have left limits$\}$. Similarly, the continuous-time process $Y = (Y_t)_{t \in \mathbb{R}_+}$ has its realizations in $D_2(\mathbb{R}_+, S_2)$. The relation $R \subseteq S_1 \times S_2$ generalized to a relation between path spaces D_1 and D_2 is given by

$$R^D := \{(x, y) \in D_1 \times D_2 : x_t \sim y_t \text{ for all } t \in \mathbb{R}_+\}. \quad \textit{[Path relation]}$$

A stochastic process X is stochastically related to Y if $(X_t)_{t \in \mathbb{R}_+} \sim_{st} (Y_t)_{t \in \mathbb{R}_+}$ with respect to the path-relation R^D. Completely in line with earlier definitions, we call two stochastic processes $X = (X_t)_{t \in \mathbb{R}_+}$ and $Y = (Y_t)_{t \in \mathbb{R}_+}$ living in S_1 and S_2 stochastically related if there exists a coupling for which the relation R^D is true with probability one:

Definition 3.9. *[Stochastically related continuous-time processes]*
A *continuous-time process* $(X_t)_{t \in \mathbb{R}_+}$ *is stochastically related to* $(Y_t)_{t \in \mathbb{R}_+}$ if there exists a coupling $(Z_t)_{t \in \mathbb{R}_+} = (\hat{X}_t, \hat{Y}_t)_{t \in \mathbb{R}_+}$ of $(X_t)_{t \in \mathbb{R}_+}$ and $(Y_t)_{t \in \mathbb{R}_+}$ such that

$$\mathbb{P}\big(Z_t \in R \text{ for all } t \in \mathbb{R}_+\big) = 1.$$

Theorem 3.10. *[Stochastically related random sequences]*
Two stochastic processes $X = (X_t)_{t\in\mathbb{R}_+}$ and $Y = (Y_t)_{t\in\mathbb{R}_+}$ are stochastically related with respect to the path-relation R^D if and only if

$$(X_{t_1}, X_{t_2}, \ldots, X_{t_n}) \sim_{st} (Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}) \text{ with respect to relation } R^n \text{ for all } t_1, \ldots, t_n;\ n \in \mathbb{N}.$$

This theorem ensures us that it is sufficient to care about all finite paths of a process, in the same way as Lemma 2.6 helped us in Chapter 2. The proof follows easily from Lemma 2.6.

3.4.3 Relation-preserving Markov processes

We are interested in the steady-state behaviour of the processes under consideration. Therefore we do not need to have that for the whole process the relation is true, as long as for the tail the relation remains true once entered a state in the relation.

Definition 3.11. *[Invariant set]*
A set $R \subseteq S_1 \times S_2$ is *invariant* (or *absorbing*) for a Markov process $(Z_t)_{t\in\mathbb{R}_+}$ in $S_1 \times S_2$, if

$$\mathbb{P}\big(Z_t(z) \in R \text{ for all } t \in \mathbb{R}_+\big) = 1,$$

for all initial states $z \in R$.

In terms of transitions rate matrices, a set R is invariant if the probability to get out of R is zero.

Definition 3.12. *[Invariance for transition rate matrices]*
Let Q be the transition rate matrix of a process X on S. The set $R \subseteq S$ is invariant for Q if for all $x \in R$:

$$Q(x, y) = 0 \text{ for all } y \notin R.$$

Definition 3.13. *[Relation-preserving Markov processes]*
Markov processes $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ stochastically preserve a relation R if for all initial states $(x, y) \in R$:

$$(X_t(x))_{t\in\mathbb{R}_+} \sim_{st} (Y_t(y))_{t\in\mathbb{R}_+} \text{ with respect to the path-relation } R^D.$$

Or, equivalently, if for all initial states $(x, y) \in R$:

$$X_t(x) \sim_{st} Y_t(y) \text{ with respect to relation } R \text{ for all } t \in \mathbb{R}_+.$$

The last equivalence in Definition 3.13 comes from the memoryless property of Markov processes.

3.4.4 Stochastically related stationary distributions

A Markov process is *ergodic* when it is *positive recurrent* and *aperiodic*. A process X in S is *irreducible* if each state is visited with positive probability starting from any other state; and *positive recurrent* if, when starting in a state x, the process will eventually return in x with probability one.

Theorem 3.14. *[Stationary distribution of Markov process]*
For an irreducible ergodic Markov process X, the stationary distribution exists and is independent of the initial state $x \in S$:

$$X_t(x) \xrightarrow{d} \pi_X \text{ for all initial states } x \in S.$$

[Ross, 2007], Theorem 4.1.

Where the notation $X_t \xrightarrow{d} X$ is used to denote convergence in distribution. That is, $\lim_{t\to\infty} \mathbb{P}(X_t(x) \in A) \to \mathbb{P}(X \in A)$ for all initial states x and all subsets A of S.

For continuous-time Markov processes X and Y which are stochastically related and for which the stationary distributions exist, the following theorem gives us that the stationary distributions satisfy the relation.

Theorem 3.15. *[Stochastically related stationary distributions]*
Let $(X_t)_{t\in\mathbb{R}_+}$ and $(Y_t)_{t\in\mathbb{R}_+}$ be continuous-time Markov processes with stationary distributions π_X and π_Y; that is $X_t \xrightarrow{d} \pi_X$ and $Y_t \xrightarrow{d} \pi_Y$.

$$\text{If } (X_t)_{t\in\mathbb{R}_+} \sim_{st} (Y_t)_{t\in\mathbb{R}_+}, \text{ then } \pi_X \sim_{st} \pi_Y.$$

Proof. $(X_t)_{t\in\mathbb{R}_+} \sim_{st} (Y_t)_{t\in\mathbb{R}_+}$, so for all $A_1 \subseteq S_1$

$$\mathbb{P}(X_t \in A_1) \leq \mathbb{P}(Y_t \in A_1^{\rightarrow}).$$

Furthermore,

$$\pi_X(A_1) = \lim_{t\to\infty} \mathbb{P}(X_t \in A_1) \leq \lim_{t\to\infty} \mathbb{P}(T_t \in A_1^{\rightarrow}) = \lim_{t\to\infty} \mathbb{P}(X_t \in A_1),$$

where the first and the last equality is given by the convergence in distribution of X_t and Y_t. So $\pi_X(A_1) \leq \pi_Y(A_1^{\rightarrow})$ for all $A_1 \subseteq S_1$, and Strassen's Theorem 3.8 implies $\pi_X \sim_{st} \pi_Y$. □

3.5 Comparison conditions for Markov processes

In the last section of this chapter, we consider discrete- and continuous-time Markov processes and their probability (rate) matrices. We define when transition matrices and transition rate matrices are stochastically related with respect to a relation, and two comparison theorems which give necessary and sufficient conditions for stochastic comparison of X and Y are presented.

3.5.1 Comparison conditions for discrete-time Markov processes

From Theorem 2.5 we know that a coupling of probability matrices P_1 and P_2 gives a coupling of the processes. This motivates the following definition:

Definition 3.16. *[Stochastically related transition matrices]*
Let P_1 and P_2 be the transition matrices of a discrete-time Markov processes X and Y living on S_1 and S_2, respectively. The *transition matrices are stochastically related* with respect to a relation R if there exists a coupling P of P_1 and P_2 such that relation R is invariant for P. We denote this by $P_1 \sim_{st} P_2$.

Theorem 3.17. *[Stochastic relation conditions for discrete-time Markov processes]*
Let X and Y be two discrete-time Markov processes with probability matrices respectively P_1 and P_2 living in state spaces S_1 and S_2. For a given relation R, the following conditions are equivalent:

(i) X and Y stochastically preserve the relation R.

(ii) There exists a Markovian coupling of X and Y for which the set R is invariant.

(iii) Probability matrix P_1 is stochastically related to P_2 with respect to the relation R:

$$P_1 \sim_{st} P_2.$$

(iv) For all $x \in S_1$ and $y \in S_2$ with $x \sim y$:

$$P_1(x, B) \leq P_2(y, B_1^{\rightarrow}) \text{ for all finite } B_1 \subseteq S_1.$$

3.5.2 Comparison conditions for continuous-time Markov processes

When we consider Markov processes in continuous-time, Theorem 2.10 motivates the following definition:

Definition 3.18. *[Stochastically related transition rate matrices]*
Let Q_1 and Q_2 be bounded transition rate matrices of continuous-time Markov processes X and Y living in S_1 and S_2, respectively. The *transition rate matrices are stochastically related* if there exists a coupling Q of Q_1 and Q_2 such that relation R is invariant for Q. We denote this by $Q_1 \sim_{st} Q_2$.

From Section 2.4 we have that the transition rate matrices Q_1 and Q_2 are stochastically related with respect to the relation R exactly if the uniformized Q-matrices P_1 and P_2 are stochastically related with respect to R.

Theorem 3.19. *[Stochastic relation conditions for continuous-time Markov processes]*
Let X and Y be two continuous-time Markov processes with bounded transition rate matrices respectively Q_1 and Q_2 living in state spaces S_1 and S_2. Furthermore, let P_1 and P_2 be the uniformized Q-matrices of Q_1 and Q_2 according to Definition 2.7. For a given relation R, the following conditions are equivalent:

(i) X and Y stochastically preserve the relation R.

(ii) There exists a Markovian coupling of X and Y for which the set R is invariant.

(iii) The uniformized Q-matrices P_1 and P_2 are stochastically related:

$$P_1 \sim_{st} P_2.$$

(iv) For all $x \sim y$:

$$Q_1(x, B_1) \leq Q_2(y, B_1^{\rightarrow}) \text{ for } x \notin B_1 \text{ and } y \notin B_1^{\rightarrow}, \tag{3.7}$$

$$Q_2(y, B_2) \leq Q_1(x, B_2^{\leftarrow}) \text{ for } x \notin B_2^{\leftarrow} \text{ and } y \notin B_2. \tag{3.8}$$

For the proofs of Theorems 3.17 and 3.19 we refer to [Leskelä, 2010]. Theorem 3.19 gives us simply verifiable conditions to test whether or not processes X and Y are stochastically related to each other.

The conditions in (iv) are intuitively interpretable in the following way: For each pair (x, y) *in* the relation R, we want to construct a coupling so that the rate from (x, y) to (x', y') is zero whenever $(x', y') \notin R$. So for each positive rate from x to x', this rate should be smaller than the total rate from y to any y' for which the relation is preserved.

Chapter 4

Stochastic queueing networks

Queueing networks are networks of multiple interconnected service stations. Customers arrive at a station in the system, where they are served (eventually after a queueing period), and can jump from one station to another after being served. We call such a network a *queueing network*. We describe a general model of Markov queueing networks in terms of continuous-time Markov processes in Section 4.1. In Section 4.2, we derive step-by-step necessary and sufficient conditions for the stochastic comparison of two Markov queueing networks considering the stochastic *coordinate-wise order*. In Section 4.3 we consider the stochastic comparison of those networks under the *summation relation*.

4.1 Stochastic queueing networks

In this section, we describe a class of stochastic queueing networks as continuous-time Markov processes. This class is a generalization of the so-called Jackson networks.

4.1.1 Markov queueing network

Consider a stochastic network with M service stations (servers or machines), where customers receive services at the stations they pass. We write this as a Markov process $X = (X_t)_{t\in\mathbb{R}_+}$ over continuous time. The state of the system is denoted by $\mathbf{x} = (x_1, \ldots, x_M) \in S$, where x_i represents the number of customers at the i-th station, and S is a subset of $\mathbb{N}^M$.

Arrivals and services

Customers (or jobs) arrive at station i according to a time-inhomogeneous Poisson process at rate $\beta_i(X_t)$. The service requirements are exponentially distributed and can differ per station. The service rate at station i is denoted by $\delta_i(X_t)$. The arrival rate and the service rates can both depend on the state of the whole system. If there are no customers at station i the service rate is zero: $\delta_i(\mathbf{x}) = 0$ if $x_i = 0$.

Jumping to another queue

After the service at station i, the customer jumps to another station j with routing probability p_{ij}. We assume $0 \le p_{ij} \le 1$ and $0 \le \sum_{j\neq i} p_{ij} \le 1$. With probability $p_i = 1 - \sum_{j\neq i} p_{ij}$ the customer leaves the system from station i.

Bounded state space
Furthermore, we assume the state space to be bounded. This means that the total number of customers in the system can not be larger than N. If there are more than N jobs in the system upon an arrival instant, this arrival will be rejected (and lost).

All the service requirements, inter arrival times and jump probabilities are assumed to be independent of each other. To summarize, the system can be described by the following three equations, where $\mathbf{e}_i$ stands for the i-th unit vector in $\mathbb{N}^M$:

$$\begin{aligned} q(\mathbf{x}, \mathbf{x} + \mathbf{e}_i) &= \beta_i(\mathbf{x}) \mathbb{1}_{\{|\mathbf{x}|<N\}} && \text{for all } i = 1, 2, \ldots, M && (4.1) \\ q(\mathbf{x}, \mathbf{x} - \mathbf{e}_i + \mathbf{e}_j) &= \delta_i(\mathbf{x}) p_{ij} && \text{for all } i, j = 1, 2, \ldots, M \text{ and } i \neq j && (4.2) \\ q(\mathbf{x}, \mathbf{x} - \mathbf{e}_i) &= \delta_i(\mathbf{x}) p_i && \text{for all } i = 1, 2, \ldots, M. && (4.3) \end{aligned}$$

4.1.2 Two service stations

In the rest of this chapter we derive conditions for the stochastic domination of two M-station Markov queueing processes X and Y. For simplicity we start with $M = 2$. The arrival rates, service rates and jump probabilities of X are denoted by β, δ and p respectively, and we write β', δ' and p' for the same parameters of process Y. A graphical representation of the model is presented in Figure 4.1.

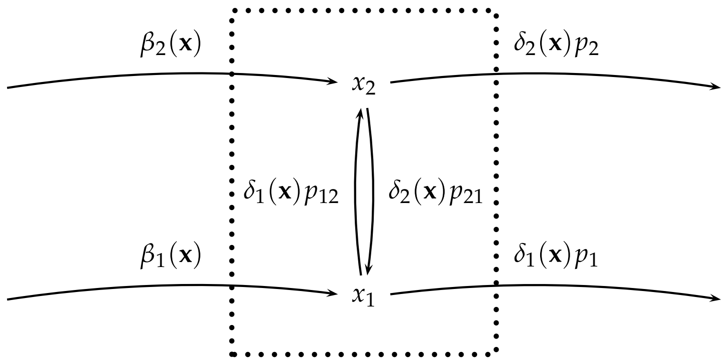

Figure 4.1: A two-station queueing network.

The first process X takes values in state space $S_1 \subseteq \mathbb{N} \times \mathbb{N}$. For each element $\mathbf{x} = (x_1, x_2)$ in S_1, $0 \leq x_1 + x_2 \leq N$. The following formula expresses the total transition rate from state $\mathbf{x}$ to the subset $B_1 \subseteq S_1$, for all $\mathbf{x} \in S_1$ and for any subset B_1 of S_1:

$$\begin{aligned} Q_1(\mathbf{x}, B_1) = \; & \beta_1(\mathbf{x}) \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_1 \in B_1\}} + \beta_2(\mathbf{x}) \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_2 \in B_1\}} + \\ & \delta_1(\mathbf{x}) p_{12} \cdot \mathbb{1}_{\{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2 \in B_1\}} + \delta_2(\mathbf{x}) p_{21} \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2 \in B_1\}} \\ & + \delta_1(\mathbf{x}) p_1 \cdot \mathbb{1}_{\{\mathbf{x}-\mathbf{e}_1 \in B_1\}} + \delta_2(\mathbf{x}) p_2 \cdot \mathbb{1}_{\{\mathbf{x}-\mathbf{e}_2 \in B_1\}}. \end{aligned} \tag{4.4}$$

The second process Y is also a Markov queueing network, and lives in state space $S_2 \subseteq \mathbb{N} \times \mathbb{N}$. State $\mathbf{y} = (y_1, y_2) \in S_2$ if and only if $0 \leq y_1 + y_2 \leq N'$. The state spaces can be the same, in this case when $N = N'$, but this is not needed in general. In practice this can be useful, for example if $N \neq N'$ or if $N' = \infty$, S_2 is not the same as S_1. In this thesis,

$N = N'$ is assumed to keep notations more simple, but this assumption does not effect the presented results.

The arrival rates of process Y are denoted by $\beta'_1(\mathbf{y})$ and $\beta'_2(\mathbf{y})$; service rates $\delta'_1(\mathbf{y})$ and $\delta'_2(\mathbf{y})$; jump probabilities p'_{12}, p'_{21} and departure probabilities p'_1 and p'_2. For all $\mathbf{y} \in S_2$ and for all $B_2 \subseteq S_2$ we have:

$$\begin{aligned} Q_2(\mathbf{y}, B_2) = \beta'_1(\mathbf{y}) \cdot \mathbb{1}_{\{\mathbf{y}+\mathbf{e}_1 \in B_2\}} + \beta'_2(\mathbf{y}) \cdot \mathbb{1}_{\{\mathbf{y}+\mathbf{e}_2 \in B_2\}} + \\ \delta'_1(\mathbf{y}) p'_{12} \cdot \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_1+\mathbf{e}_2 \in B_2\}} + \delta'_2(\mathbf{y}) p'_{21} \cdot \mathbb{1}_{\{\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2) \in B_2\}} \\ + \delta'_1(\mathbf{y}) p'_1 \cdot \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_1 \in B_2\}} + \delta'_2(\mathbf{y}) p'_2 \cdot \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_2 \in B_2\}}. \end{aligned} \tag{4.5}$$

4.1.3 Jackson and Whittle

The queueing network described in this section, is a generalization of the *Jackson network*. In Jackson networks, the arrival rates β_i are positive constants and the service rates at a node depend only on the number of customers at this node: $\delta_i(x_i)$. Jackson networks fulfill the so-called balance equations and have relatively simple, product-form steady-state solutions. The general network that we present in this section are sometimes called *Whittle networks* [Whittle, 1986] if all states satisfy the load balance equations. For a thorough background on the generality and applications of stochastic networks we refer the reader to [Serfozo, 1999]. In the next section we derive necessary and sufficient conditions for the stochastic comparison of these queueing networks under the coordinate-wise order.

4.2 Coordinate-wise coupling of Markov queueing networks

Recall that Theorem 3.19 gives necessary and sufficient conditions for the stochastic comparison of Markov processes. In this section we work out this theorem for the *coordinate-wise order*. We first derive sufficient conditions for the coordinate-wise comparison of 2-station Markov queueing networks. We obtain rate conditions which are sufficient to ensure the existence of an order-preserving coupling, but not necessary. Subsequently, we present *if and only if*-conditions for the M-station case, which are stated in Theorem 4.4. This theorem gives a new characterization of stochastic comparability with respect to the coordinate-wise order. Finally, we show similarity to an alternative characterization presented in [Delgado et al., 2004].

Two vectors x and y in $\mathbb{R}^M$ are *coordinate-wise ordered* if $x_i \leq y_i$ for all $i = 1, \ldots, M$. That is, the coordinates x_i and y_i are ordered with respect to the usual order in $\mathbb{R}$ for every i.

Definition 4.1. *[coordinate-wise order relation]*

$$R^{coord} = \{(\mathbf{x}, \mathbf{y}) : x_i \leq y_i \text{ for all } i = 1, \ldots, M\}. \tag{4.6}$$

Recall Definition 3.13. If we have two Markov queueing networks X and Y, these processes X and Y are comparable with respect to relation R^{coord} if at every station, the number of customers at time t in X is less than or equal to the number of customers in Y. Processes X *and* Y *stochastically preserve the relation* R^{coord} if there exist a coupling of X and Y for which the relation R^{coord} is invariant.

4.2.1 Sufficient two-station comparison conditions

[Right-conjugate conditions]
Recall condition (3.7): For all $\mathbf{x} \in S_1$, $B_1 \subseteq S_1$ and $\mathbf{y} \in S_2$

$$Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow}), \text{ with } \mathbf{x} \sim \mathbf{y}, \mathbf{x} \notin B_1, \ \mathbf{y} \notin B_1^{\rightarrow}. \tag{4.7}$$

Let $B_1 \subseteq S_1$ be arbitrary. Recall that the *right conjugate* of a set $B_1 \subseteq S_1$ is the subset in the second state space for which a jump from $\mathbf{x}$ to B_1 in S_1 can be compensated such that relation R^{coord} remains valid. The right conjugate is given by $B_1^{\rightarrow} \subseteq S_2$, where

$$B_1^{\rightarrow} = \bigcup_{\mathbf{x} \in B_1} \{\mathbf{y} : \mathbf{x} \sim \mathbf{y}\} = \bigcup_{\mathbf{x} \in B_1} \{\mathbf{y} : x_1 \leq y_1 \ \& \ x_2 \leq y_2\}. \tag{4.8}$$

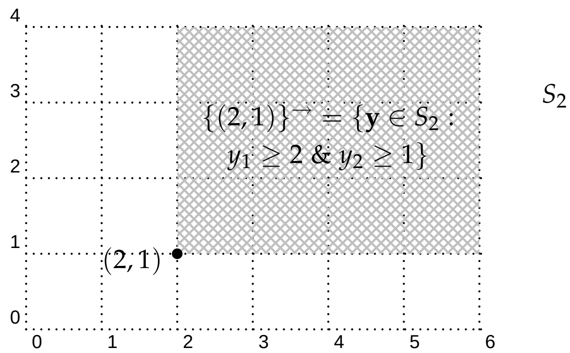

Figure 4.2: The right conjugate of $\mathbf{x} = (2, 1)$ is given by the shadowed area.

For all $\mathbf{x} \in S_1 \setminus B_1$, the following formula gives the value of $Q_1(\mathbf{x}, B_1)$:

$$\begin{aligned} Q_1(\mathbf{x}, B_1) = {} & \beta_1(\mathbf{x}) \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_1 \in B_1\}} + \beta_2(\mathbf{x}) \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_2 \in B_1\}} \\ & + \delta_1(\mathbf{x}) p_{12} \cdot \mathbb{1}_{\{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2 \in B_1\}} + \delta_2(\mathbf{x}) p_{21} \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2 \in B_1\}} \\ & + \delta_1(\mathbf{x}) p_1 \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_1 \in B_1\}} + \delta_2(\mathbf{x}) p_2 \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_2 \in B_1\}}. \end{aligned}$$

We see that the value of $Q_1(\mathbf{x}, B_1)$ depends on whether or not the following elements are in B_1:

$$\mathbf{x} - \mathbf{e}_1; \ \mathbf{x} - \mathbf{e}_2; \ \mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2; \ \mathbf{x} + \mathbf{e}_1 - \mathbf{e}_2; \ \mathbf{x} + \mathbf{e}_1; \text{ and } \mathbf{x} + \mathbf{e}_2.$$

By adding any other element $\tilde{\mathbf{x}}$ which is *not* equal to one of those six elements listed above, to B_1 would not increase $Q_1(\mathbf{x}, B_1)$. However, it would increase the size of $B_1^{\rightarrow}$ and thus could it increase the rate $Q_2(\mathbf{y}, B_1^{\rightarrow})$.

We are looking for the *minimal* conditions such that Equation (4.7) holds; and to break this inequality, we want to choose B_1 such that $Q_1(\mathbf{x}, B_1)$ is as big as possible while for this same B_1, $Q_2(\mathbf{y}, B_1^{\rightarrow})$ is as small as possible. Therefore, without loss of generality, we have to look only at sets for which

$$B_1 \subseteq \{\mathbf{x} - \mathbf{e}_1; \ \mathbf{x} - \mathbf{e}_2; \ \mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2; \ \mathbf{x} + \mathbf{e}_1 - \mathbf{e}_2; \ \mathbf{x} + \mathbf{e}_1; \ \mathbf{x} + \mathbf{e}_2\}.$$

- Suppose $\mathbf{x} - \mathbf{e}_1 \in B_1$. Then $\{\mathbf{x} - \mathbf{e}_1\}^{\rightarrow} = \{\mathbf{y} : x_1 - 1 \leq y_1 \text{ and } x_2 \leq y_2\} \subseteq B_1^{\rightarrow}$. Condition (4.7) must hold for all $\mathbf{y}$ for which $\mathbf{x} \sim \mathbf{y} \ \& \ \mathbf{y} \notin B_1^{\rightarrow}$. But $\{\mathbf{y} : \mathbf{x} \sim \mathbf{y}\} \subseteq \{\mathbf{x} - \mathbf{e}_1\}^{\rightarrow}$, therefore there exists no such $\mathbf{y}$.

- In the same way, suppose $\mathbf{x}-\mathbf{e}_2 \in B_1$. Then $\{\mathbf{y} : \mathbf{x} \sim \mathbf{y}\} \subseteq \{\mathbf{x}-\mathbf{e}_2\}^{\rightarrow} = \{\mathbf{y} : x_1 \leq y_1 \text{ and } x_2 - 1 \leq y_2\} \subseteq B_1^{\rightarrow}$. Hence, there exist no $\mathbf{y}$ such that $\mathbf{x} \sim \mathbf{y}$ and $\mathbf{y} \notin B_1^{\rightarrow}$.

We conclude that it suffices to consider the $B_1 \subseteq S_1$ for which $\mathbf{x}-\mathbf{e}_1,\ \mathbf{x}-\mathbf{e}_2 \notin B_1$:

$$B_1 \subseteq \{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2;\ \mathbf{x}+\mathbf{e}_1-\mathbf{e}_2;\ \mathbf{x}+\mathbf{e}_1;\ \mathbf{x}+\mathbf{e}_2\}.$$

- Let $\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2 \in B_1$, so that $Q_1(\mathbf{x}, B_1) \geq \delta_1(\mathbf{x})p_{12}$. The right conjugate $\{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2\}^{\rightarrow} = \{\mathbf{y} : x_1 - 1 \leq y_1 \ \&\ x_2 + 1 \leq y_2\} \subseteq B_1^{\rightarrow}$. For $\mathbf{y} \in S_2 \setminus B_1^{\rightarrow}$ such that $\mathbf{x} \sim \mathbf{y}$, we must have $x_2 = y_2$. If $x_1 \leq y_1$ and $x_2 = y_2$ and there is an 12-jump in S_1, the only way to maintain the relation $\mathcal{R}^{coord}$ is when the second coordinate y_2 is also increased by one:

  $$Q_2(\mathbf{y}, \{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2\}^{\rightarrow}) = \beta_2'(\mathbf{y}) + \delta_1'(\mathbf{y})p_{12}'.$$

  Note that $\{\mathbf{x}+\mathbf{e}_2\}^{\rightarrow} = \{\mathbf{y} : x_1 \leq y_1 \text{ and } x_2 + 1 \leq y_2\} \subseteq \{\mathbf{y} : x_1 - 1 \leq y_1 \text{ and } x_2 + 1 \leq y_2\} = \{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2\}^{\rightarrow}$. Hence, by adding the state $\mathbf{x}+\mathbf{e}_2$ to B_1 we can increase $Q_1(\mathbf{x}, B_1)$ while $Q_2(\mathbf{y}, B_1^{\rightarrow})$ remains unchanged:

  $$Q_2(\mathbf{y}, \{\mathbf{x}+\mathbf{e}_2\}^{\rightarrow}) = \beta_2'(\mathbf{y}) + \delta_1'(\mathbf{y})\mathbb{1}_{\{x_1<y_1\}}p_{12}',$$

  because

  $$\{\mathbf{x}+\mathbf{e}_2\}^{\rightarrow} \cup \{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2\}^{\rightarrow} = \{\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2\}^{\rightarrow}.$$

We want $Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow})$, and in the worst case for $\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2 \in B_1$ we also have $\mathbf{x}+\mathbf{e}_2 \in B_1$. This gives the following condition:

$$\beta_2(\mathbf{x}) + \delta_1(\mathbf{x})p_{12} \leq \beta_2'(\mathbf{y}) + \delta_1'(\mathbf{y})p_{12}' \text{ for all } x_1 \leq y_1 \ \&\ x_2 = y_2. \tag{4.9}$$

- Suppose $\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2 \in B_1$. With a similar argument as above we have $\{\mathbf{x}+\mathbf{e}_1\}^{\rightarrow} \subseteq \{\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2\}^{\rightarrow}$, and the worst case situation is that whenever $\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2 \in B_1$ we also have $\mathbf{x}+\mathbf{e}_1 \in B_1$ (because this increases Q_1 while it does not increase Q_2) and the condition becomes:

  $$\beta_1(\mathbf{x}) + \delta_2(\mathbf{x})p_{21} \leq \beta_1'(\mathbf{y}) + \delta_2'(\mathbf{y})p_{21}' \text{ for all } x_1 = y_1 \ \&\ x_2 \leq y_2. \tag{4.10}$$

- Suppose now both $\mathbf{x}-\mathbf{e}_1+\mathbf{e}_2$ and $\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2 \in B_1$. Then, the worst case is when all these four elements are in B_1 and we get the combination of the two conditions above:

  $$\beta_1(\mathbf{x}) + \beta_2(\mathbf{x}) + \delta_1(\mathbf{x})p_{12} + \delta_2(\mathbf{x})p_{21} \leq \beta_1'(\mathbf{y}) + \beta_2'(\mathbf{y}) + \delta_1'(\mathbf{y})p_{12}' + \delta_2'(\mathbf{y})p_{21}'$$

  for all $\mathbf{x} = \mathbf{y}$. We omit this condition, because it is less strict than the conditions (4.9) and (4.10) together.

- Now, suppose we have $\mathbf{x}+\mathbf{e}_1 \in B_1$ and $\mathbf{x}+\mathbf{e}_1-\mathbf{e}_2 \notin B_1$. Then $\{\mathbf{y} : x_1 + 1 \leq y_1 \ \&\ x_2 \leq y_2\} \subseteq B_1^{\rightarrow}$, and $\mathbf{x} \sim \mathbf{y}$ combined with $\mathbf{y} \notin B_1^{\rightarrow}$ implies $x_1 = y_1$ and $x_2 \leq y_2$. Relation R^{coord} remains true only if the arrival rate of type-1 customers in S_1 is compensated by a type-1 arrival in S_2. The maximal type-1 arrival rate out of state $\mathbf{y}$ is

  $$Q_2(\mathbf{y}, B_1^{\rightarrow}) = \beta_1'(\mathbf{y}) + \delta_2'(\mathbf{y})p_{21}' \cdot \mathbb{1}_{\{x_2<y_2\}}.$$

  The worst case happens if $x_2 = y_2$, which gives the condition

  $$\beta_1(\mathbf{x}) \leq \beta_1'(\mathbf{y}) \text{ for } \mathbf{x} = \mathbf{y}. \tag{4.11}$$

- Similarly, suppose $\mathbf{x} + \mathbf{e}_2 \in B_1$ and $\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2 \notin B_1$. The worst-case condition leads to

$$\beta_2(\mathbf{x}) \le \beta_2'(\mathbf{y}) \text{ for } \mathbf{x} = \mathbf{y}. \tag{4.12}$$

- Again, conditions (4.11) and (4.12) cover also the case in which states $\mathbf{x} + \mathbf{e}_2$ and $\mathbf{x} + \mathbf{e}_1 \in B_1$, but the states $\mathbf{x} - \mathbf{e}_1 + \mathbf{e}_2$ and $\mathbf{x} + \mathbf{e}_1 - \mathbf{e}_2 \notin B_1$.

[Left-conjugate conditions]
Now, we work out the left-conjugate conditions for the stochastic comparison of two Markov queueing networks with respect to the coordinate-wise order. Recall condition (3.8): For all $\mathbf{x} \in S_1$, $\mathbf{y} \in S_2$ and $B_2 \subseteq S_2$

$$Q_2(\mathbf{y}, B_2) \le Q_1(\mathbf{x}, B_2^{\leftarrow}), \text{ for all } \mathbf{x} \sim \mathbf{y} \text{ with } \mathbf{x} \notin B_2^{\leftarrow},\ \mathbf{y} \notin B_2. \tag{4.13}$$

Let $(\mathbf{x}, \mathbf{y})$ be an element in the product space such that $\mathbf{x} \sim \mathbf{y}$, and let B_2 be an arbitrary subset of S_2 with $\mathbf{y} \notin B_2$. The *left conjugate* $B_2^{\leftarrow} \subseteq S_1$ is the maximal subset of states in S_1 such that for each state $x' \in B_2^{\leftarrow}$ there exists an $\mathbf{y}' \in B_2$ such that $\mathbf{x}' \sim \mathbf{y}'$. That is

$$B_2^{\leftarrow} = \bigcup_{\mathbf{y} \in B_2} \{\mathbf{x} : \mathbf{x} \sim \mathbf{y}\} = \bigcup_{\mathbf{y} \in B_2} \{\mathbf{x} : x_1 \le y_1 \ \&\ x_2 \le y_2\}. \tag{4.14}$$

An example of the left conjugate of a singleton in S_2 with respect to the coordinate-wise order is given in Figure 4.3.

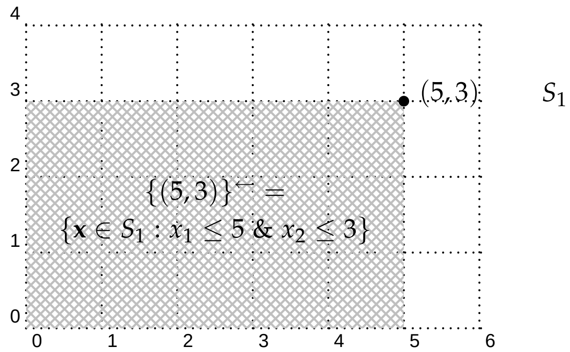

Figure 4.3: The left conjugate of $\mathbf{y} = (5, 3)$ is given by the shadowed area.

The outgoing rate from y to any subset B_2 in S_2 is given by

$$\begin{aligned} Q_2(\mathbf{y}, B_2) = {} & \beta_1'(\mathbf{y}) \cdot \mathbb{1}_{\{\mathbf{y}+\mathbf{e}_1 \in B_2\}} + \beta_2'(\mathbf{y}) \cdot \mathbb{1}_{\{\mathbf{y}+\mathbf{e}_2 \in B_2\}} \\ & + \delta_1'(\mathbf{y}) p_{12}' \cdot \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_1+\mathbf{e}_2 \in B_2\}} + \delta_2'(\mathbf{y}) p_{21}' \cdot \mathbb{1}_{\{\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2 \in B_2\}} \\ & + \delta_1'(\mathbf{y}) p_1' \cdot \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_1 \in B_2\}} + \delta_2'(\mathbf{y}) p_2' \cdot \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_2 \in B_2\}}. \end{aligned}$$

Note that $Q_2(\mathbf{y}, B_2)$ depends on whether or not the following elements are part of B_2:

$$\{\mathbf{y} + \mathbf{e}_1,\ \mathbf{y} + \mathbf{e}_2,\ \mathbf{y} - \mathbf{e}_1 + \mathbf{e}_2,\ \mathbf{y} + \mathbf{e}_1 - \mathbf{e}_2,\ \mathbf{y} - \mathbf{e}_1,\ \mathbf{y} - \mathbf{e}_2\}.$$

The assumption that $\mathbf{x} \notin B_2^{\leftarrow}$ implies that there is no $\mathbf{y}' \in B_2$ such that $x_1 \le y_1'$ and $x_2 \le y_2'$.

- Suppose $\mathbf{y}+\mathbf{e}_1 \in B_2$. Then $\mathbf{x} \in B_2^{\leftarrow}$ because $\{\mathbf{y}\}^{\leftarrow} \subseteq \{\mathbf{y}+\mathbf{e}_1\}^{\leftarrow}$. This contradicts with $\mathbf{x} \notin B_2^{\leftarrow}$, thus, we can assume $\mathbf{y}+\mathbf{e}_1 \notin B_2$.
- With the same argument we have that $\{\mathbf{y}\}^{\leftarrow} \subseteq \{\mathbf{y}+\mathbf{e}_2\}^{\leftarrow}$, therefore, if $\mathbf{y}+\mathbf{e}_2 \in B_2$ then there would not exist such an $\mathbf{x}$ we are looking for outside $B_2^{\leftarrow}$.

We conclude that

$$B_2 \subseteq \{\mathbf{y}-\mathbf{e}_1+\mathbf{e}_2,\ \mathbf{y}+\mathbf{e}_1-\mathbf{e}_2,\ \mathbf{y}-\mathbf{e}_1,\ \mathbf{y}-\mathbf{e}_2\}.$$

- Suppose now $\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2 \in B_2$. $\mathbf{x} \notin B_2^{\leftarrow}$ & $\mathbf{x} \sim \mathbf{y}$ if and only if $x_1 \leq y_1$ & $x_2 = y_2$. The transition rates from $\mathbf{y}$ to $\{\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2\}$ and from $\mathbf{x}$ to $\{\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2\}^{\leftarrow}$ are given by: $Q_1(\mathbf{x}, \{\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2\}^{\leftarrow}) = \delta_2(\mathbf{x})(p_2+p_{21})$ and $Q_2(\mathbf{y}, \{\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2\}) = \delta_2'(\mathbf{y})p_{21}'$. Thus, for any $B_2 \ni \mathbf{y}+\mathbf{e}_1-\mathbf{e}_2$; $x_1 \leq y_1$ & $x_2 = y_2$ we have $Q_2(\mathbf{y}, B_2) = \delta_2'(\mathbf{y})p_{21}' + \delta_2'(\mathbf{y})p_2' \cdot \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_2 \in B_2\}}$. In the worst case $\mathbf{y}-\mathbf{e}_2$ is also in B_2 and we get the condition

  $$\delta_2'(\mathbf{y})(p_2'+p_{21}') \leq \delta_2(\mathbf{x})(p_2+p_{21}) \text{ for } x_1 \leq y_1 \text{ \& } x_2 = y_2.$$

- If $\mathbf{y}-\mathbf{e}_1+\mathbf{e}_2 \in B_2$, in the worst-case situation we also have $\mathbf{y}-\mathbf{e}_1 \in B_2$. The condition becomes now

  $$\delta_1'(\mathbf{y})(p_1'+p_{12}') \leq \delta_1(\mathbf{x})(p_1+p_{12}) \text{ for } x_1 = y_1 \text{ \& } x_2 \leq y_2,$$

  and, because we have $p_1+p_{12} = p_2+p_{21} = p_1'+p_{12}' = p_2'+p_{21}' = 1$, these conditions become together

  $$\delta_2'(\mathbf{y}) \leq \delta_2(\mathbf{x}) \text{ for } x_1 \leq y_1 \text{ \& } x_2 = y_2, \tag{4.15}$$

  and

  $$\delta_1'(\mathbf{y}) \leq \delta_1(\mathbf{x}) \text{ for } x_1 = y_1 \text{ \& } x_2 \leq y_2. \tag{4.16}$$

- Remark that if both $\mathbf{y}-\mathbf{e}_1+\mathbf{e}_2 \in B_2$ and $\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2 \in B_2$ are in B_2, conditions (4.15) and (4.16) together are stricter than the condition

  $$\delta_1'(\mathbf{y}) + \delta_2'(\mathbf{y}) \leq \delta_1(\mathbf{x}) + \delta_2(\mathbf{x}) \text{ for } \mathbf{x} = \mathbf{y}.$$

Finally we look at the departure rates from $\mathbf{y}$.

- Suppose $\mathbf{y}-\mathbf{e}_1 \in B_2$, but $\mathbf{y}-\mathbf{e}_1+\mathbf{e}_2 \notin B_2$. Both $\mathbf{x} \notin B_2^{\leftarrow}$ and $\mathbf{x} \sim \mathbf{y}$ are true if and only if $x_1 = y_1$ & $x_2 \leq y_2$. Thus, $Q_1(\mathbf{x}, \{\mathbf{y}-\mathbf{e}_1\}^{\leftarrow}) = \delta_1(\mathbf{x})p_1$.
- We can hold a similar argument for $\mathbf{y}-\mathbf{e}_2 \in B_2$, but $\mathbf{y}+\mathbf{e}_1-\mathbf{e}_2 \notin B_2$. Then $\mathbf{x} \notin B_2^{\leftarrow}$ & $\mathbf{x} \sim \mathbf{y}$ if and only if $x_1 \leq y_1$ & $x_2 = y_2$. It gives us the following conditions:

  $$\delta_1'(\mathbf{y})p_1' \leq \delta_1(\mathbf{x})p_1 \text{ for } x_1 = y_1 \text{ \& } x_2 \leq y_2, \tag{4.17}$$
  $$\delta_2'(\mathbf{y})p_2' \leq \delta_2(\mathbf{x})p_2 \text{ for } x_1 \leq y_1 \text{ \& } x_2 = y_2. \tag{4.18}$$

All together, the conditions are summarized in the following theorem:

Theorem 4.2. *Let X and Y be two 2-station Markov queueing networks with parameters (β, δ, p) and (β', δ', p'). If for all $\boldsymbol{x}$ and $\boldsymbol{y}$ with $\boldsymbol{x} \sim \boldsymbol{y}$ the following conditions hold:*

$$
\begin{array}{rcl}
\textit{if } x_1 = y_1\textit{:} & & \\
\beta_1(\boldsymbol{x}) & \leq & \beta_1'(\boldsymbol{y}), \\
\beta_1(\boldsymbol{x}) + \delta_2(\boldsymbol{x})p_{21} & \leq & \beta_1'(\boldsymbol{y}) + \delta_2'(\boldsymbol{y})p_{21}', \\
\delta_1'(\boldsymbol{y})p_1' & \leq & \delta_1(\boldsymbol{x})p_1, \\
\delta_1'(\boldsymbol{y})(p_1' + p_{12}') & \leq & \delta_1(\boldsymbol{x})(p_1 + p_{12}), \\
\textit{and if } x_2 = y_2\textit{:} & & \\
\beta_2(\boldsymbol{x}) & \leq & \beta_2'(\boldsymbol{y}), \\
\beta_2(\boldsymbol{x}) + \delta_1(\boldsymbol{x})p_{12} & \leq & \beta_2'(\boldsymbol{y}) + \delta_1'(\boldsymbol{y})p_{12}', \\
\delta_2'(\boldsymbol{y})p_2' & \leq & \delta_2(\boldsymbol{x})p_2, \\
\delta_2'(\boldsymbol{y})(p_2' + p_{21}') & \leq & \delta_2(\boldsymbol{x})(p_2 + p_{21}),
\end{array}
$$

then there exists a coupling of the processes X and Y for which the coordinate-wise relation is invariant. That means precisely that X is stochastically related to process Y in the steady-state.

4.2.2 Sufficient comparison conditions for M-station queueing networks

We can generalize Theorem 4.2 to Markov queueing networks with an arbitrary number of stations. This theorem is stated below, and is a natural extension of Theorem 4.2. The proof is omitted because in Section 4.2.3 a stronger result is proved. Theorem 4.3 below is basically equivalent with the results derived in [Economou, 2003], and it can give the reader more insight in Theorem 4.4.

Theorem 4.3. *Let X and Y be two M-station Markov queueing networks with parameters (β, δ, p) and (β', δ', p'). Suppose for all $\boldsymbol{x}$ and $\boldsymbol{y}$ for which $\boldsymbol{x} \sim \boldsymbol{y}$ conditions*

$$
\beta_i(\boldsymbol{x}) + \sum_{j \in \mathcal{J}} \delta_j(\boldsymbol{x}) p_{ji} \leq \beta_i'(\boldsymbol{y}) + \sum_{j \in \mathcal{J}} \delta_j'(\boldsymbol{y}) p_{ji}', \tag{4.19}
$$

and

$$
\delta_i'(\boldsymbol{y}) \left[p_i' + \sum_{j \in \mathcal{J}} p_{ij}' \right] \leq \delta_i(\boldsymbol{x}) \left[p_i + \sum_{j \in \mathcal{J}} p_{ij} \right], \tag{4.20}
$$

hold, for each $i \in \{1, ..., M\}$ for which $x_i = y_i$, and for all $\mathcal{J} \subseteq \{1, \ldots, M\} \setminus \{i\}$. Then there exists a coupling of the processes X and Y for which the coordinate-wise relation is invariant.

4.2.3 Sharp comparison conditions for M-station queueing networks

Finally, we give necessary and sufficient conditions for the general case where the Markov queueing networks have an arbitrary (but finite) number inter-connected service stations. In [Economou, 2003], necessary conditions and sufficient conditions are separately derived. However, these conditions were not sharp. Theorem 4.4 gives sharp conditions for the stochastic comparison of two M-station queueing networks with respect to the coordinate-wise relation.

Theorem 4.4. *Let X and Y be two M-station Markov queueing networks with parameters (β, δ, p) and (β', δ', p'), respectively. There exist a coupling of the processes X and Y for which the coordinate-wise relation is invariant if and only if for all $\boldsymbol{x}$, $\boldsymbol{y}$ with $\boldsymbol{x} \sim \boldsymbol{y}$:*

$$\beta_i(\boldsymbol{x}) + \sum_{j \in \mathcal{J}} \delta_j(\boldsymbol{x}) p_{ji} \leq \beta_i'(\boldsymbol{y}) + \sum_{j \in \mathcal{J}} \delta_j'(\boldsymbol{y}) p_{ji}' + \sum_{j \notin \mathcal{J}} \mathbb{1}_{\{y_j > x_j\}} \delta_j'(\boldsymbol{y}) p_{ji}', \tag{4.21}$$

and

$$\delta_i'(\boldsymbol{y}) \left[p_i' + \sum_{j \in \mathcal{J}} p_{ij}' \right] \leq \delta_i(\boldsymbol{x}) \left[p_i + \sum_{j \in \mathcal{J}} p_{ij} + \sum_{j \notin \mathcal{J}} \mathbb{1}_{\{x_j < y_j\}} p_{ij} \right], \tag{4.22}$$

for all i such that $x_i = y_i$, and for all $\mathcal{J} \subseteq \{1, \ldots, M\} \setminus \{i\}$.

Proof. PART 1
Let $\mathbf{x} \sim \mathbf{y}$ be arbitrary and suppose that condition (4.21) holds for all i such that $x_i = y_i$, for all $\mathcal{J} \subseteq \{1, \ldots, M\} \setminus \{i\}$. We want to prove that also condition (3.7) is fulfilled.

Let $B_1 \subseteq S_1$ be arbitrary, $\mathbf{x} \notin B_1$ and $\mathbf{y} \notin B_1^{\rightarrow}$. We want to prove $Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow})$. Remark that $Q_1(\mathbf{x}, B_1)$ depends only on the elements where $\mathbf{x}$ can jump to in just one step. That is all elements in $N_{\mathbf{x}}$, *the neighbor set of $\boldsymbol{x}$*, where

$$N_{\mathbf{x}} = \{\mathbf{x} + \mathbf{e}_i,\ \mathbf{x} + \mathbf{e}_i - \mathbf{e}_j,\ \mathbf{x} - \mathbf{e}_i;\ \text{for } i, j = 1, \ldots, M \text{ and } j \neq i\}.$$

Thus, $Q_1(\mathbf{x}, B_1)$ does not increase by adding any elements from $S_1 \setminus N_{\mathbf{x}}$ to B_1, while whenever $B_1 \subseteq \tilde{B}_1$ we have $B_1^{\rightarrow} \subseteq \tilde{B}_1^{\rightarrow}$ and $Q_2(\mathbf{y}, B_1^{\rightarrow}) \leq Q_2(\mathbf{y}, \tilde{B}_1^{\rightarrow})$. In other words, we have for all B_1:

$$Q_1(\mathbf{x}, B_1) = Q_1(\mathbf{x}, (B_1 \cap N_{\mathbf{x}})), \text{ and}$$
$$Q_2(\mathbf{y}, (B_1 \cap N_{\mathbf{x}})^{\rightarrow}) \leq Q_2(\mathbf{y}, B_1^{\rightarrow}).$$

For this reason we can consider without loss of generality only those sets B_1 such that $B_1 \subseteq N_{\mathbf{x}}$.

Now suppose that $\mathbf{x} - \mathbf{e}_i \in B_1$ for a certain i. Then the right conjugate of B_1 contains the set $\{\mathbf{x} - \mathbf{e}_1\}^{\rightarrow} = \{\tilde{\mathbf{y}} : \mathbf{x} - \mathbf{e}_i \sim \tilde{\mathbf{y}}\} = \{\tilde{\mathbf{y}} : x_i - 1 \leq \tilde{y}_i \text{ and } x_j \leq \tilde{y}_j \text{ for all } j \neq i\}$. But this set contains $\mathbf{y}$, which contradicts the assumption that $\mathbf{y} \notin B_1^{\rightarrow}$. We conclude that $\mathbf{x} - \mathbf{e}_i \notin B_1$ for all i, and hence, it is sufficient to study the sets

$$B_1 \subseteq \{\mathbf{x} + \mathbf{e}_i,\ \mathbf{x} + \mathbf{e}_i - \mathbf{e}_j\ ; i, j = 1, \ldots, M \text{ and } j \neq i\}.$$

For all i and for every $\mathbf{x}$, define the subset of neighbors of $\mathbf{x}$ for which the i-th coordinate is increased by one:

$$A^{(i)}(\mathbf{x}) := \{\tilde{\mathbf{x}} : \tilde{x}_i = x_i + 1\}.$$

Define also the intersection

$$B_1^{(i)}(\mathbf{x}) := B_1 \cap A^{(i)}(\mathbf{x}).$$

Hence, only the elements $\mathbf{x} + \mathbf{e}_i$ and $\mathbf{x} + \mathbf{e}_i - \mathbf{e}_j$ (for all $j \neq i$) are possible elements of $B_1^{(i)}(\mathbf{x})$:

$$B_1^{(i)}(\mathbf{x}) = \{\mathbf{x}' \in B_1 : \mathbf{x}' = \mathbf{x} + \mathbf{e}_i \text{ or } \mathbf{x}' = \mathbf{x} - \mathbf{e}_j + \mathbf{e}_i \text{ for some } j\}.$$

Given B_1 and $\mathbf{x}$, define furthermore the index set

$$\mathcal{I}_1 := \{i : B_1^{(i)} \neq \emptyset\},$$

and, for each $i \in \mathcal{I}_1$, define the subsets $\mathcal{J}^{(i)} \subseteq \{1, \dots, M\} \setminus \{i\}$ by

$$\mathcal{J}^{(i)} := \{j : \mathbf{x} + \mathbf{e}_i - \mathbf{e}_j \in B_1\}.$$

We can write B_1 in terms of the $B_1^{(i)}$'s: $B_1 = \bigcup_{i \in \mathcal{I}_1} B_1^{(i)}$.

Observe that

$$\begin{aligned}
Q_1(\mathbf{x}, B_1) &= \sum_i \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_i \in B_1\}} \beta_i(\mathbf{x}) + \sum_{i,\,j} \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_i-\mathbf{e}_j \in B_1\}} \delta_j(\mathbf{x}) p_{ji} \\
&\quad \textit{which can be written in terms of } i \in \mathcal{I}_1 \\
&= \sum_{i \in \mathcal{I}_1} \left[\mathbb{1}_{\{\mathbf{x}+\mathbf{e}_i \in B_1\}} \beta_i(\mathbf{x}) + \sum_{j \in \mathcal{J}^{(i)}} \delta_j(\mathbf{x}) p_{ji} \right] \\
&\le \sum_{i \in \mathcal{I}_1} \left[\beta_i(\mathbf{x}) + \sum_{j \in \mathcal{J}^{(i)}} \delta_j(\mathbf{x}) p_{ji} \right] \\
&\quad \textit{for all } i \in \mathcal{I}_1, \textit{ the set } B_1^{(i)} \textit{ is non-empty. This implies that } \boldsymbol{x} + \boldsymbol{e}_i \in B_1^{\rightarrow} \textit{ and} \\
&\quad \textit{because conditions } \boldsymbol{x} \sim \boldsymbol{y} \textit{ and } \boldsymbol{y} \notin B_1^{\rightarrow}, \textit{ the equality } x_i = y_i \textit{ must be true. Hence,} \\
&\quad \textit{Equation (4.21) holds for all } i \in \mathcal{I}_1. \\
&\le \sum_{i \in \mathcal{I}_1} \left[\beta_i'(\mathbf{y}) + \sum_{j \in \mathcal{J}^{(i)}} \delta_j'(\mathbf{y}) p_{ji}' + \sum_{\substack{k \notin \mathcal{J}^{(i)}, \\ k \neq i}} \mathbb{1}_{\{x_k < y_k\}} \delta_k'(\mathbf{y}) p_{ki}' \right] \\
&= Q_2(\mathbf{y}, B_1^{\rightarrow}).
\end{aligned}$$

On the other hand, suppose $Q_1(\mathbf{x}, B_1) \le Q_2(\mathbf{y}, B_1^{\rightarrow})$ holds for all $\mathbf{x} \sim \mathbf{y}$ such that $\mathbf{x} \notin B_1$ and $\mathbf{y} \notin B_1^{\rightarrow}$. Let $\mathbf{x} \sim \mathbf{y}$ be arbitrary and suppose $x_i = y_i$. Let $\mathcal{J} \subseteq \{1, \dots, M\} \setminus \{i\}$ be arbitrary. Let

$$B_1 = \{\mathbf{x} + \mathbf{e}_i\} \cup \bigcup_{j \in \mathcal{J}} \{\mathbf{x} + \mathbf{e}_i - \mathbf{e}_j\}.$$

Then $B_1 \subseteq S_1$ is such that $\mathbf{x} \notin B_1$ and $\mathbf{y} \notin B_1^{\rightarrow}$, and therefore we have $Q_1(\mathbf{x}, B_1) \le Q_2(\mathbf{y}, B_1^{\rightarrow})$. Hence,

$$Q_1(\mathbf{x}, B_1) = \beta_i(\mathbf{x}) + \sum_{j \in \mathcal{J}} \delta_j(\mathbf{x}) p_{ji}, \text{ and}$$

$$Q_2(\mathbf{y}, B_1^{\rightarrow}) = \beta_i'(\mathbf{y}) + \sum_{j \in \mathcal{J}} \delta_j'(\mathbf{y}) p_{ji}' + \sum_{j \notin \mathcal{J}} \mathbb{1}_{\{y_j > x_j\}} \delta_j'(\mathbf{y}) p_{ji}',$$

which gives the first condition of Theorem 4.4 because the set $\mathcal{J}$ was arbitrary.

Part 2

In this second part we will prove (3.8) $\Leftrightarrow$ (4.22).

Let $\mathbf{x} \sim \mathbf{y}$ be arbitrary and suppose that condition (4.22) is true for all i such that $x_i = y_i$, for all $\mathcal{J} \subseteq \{1, \dots, M\} \setminus \{i\}$. Let $B_2 \subseteq S_2$ be arbitrary such that $\mathbf{y} \notin B_2$ and $\mathbf{x} \notin B_2^{\leftarrow}$. We want to prove (4.22), that is

$$Q_2(\mathbf{y}, B_2) \le Q_1(\mathbf{x}, B_2^{\leftarrow}).$$

As formula (4.5) shows, $Q_2(\mathbf{y}, B_2)$ depends only on the elements from B_2 which are also in the *neighbor set of* $\boldsymbol{y}$ $N_{\mathbf{y}}$ of $\mathbf{y}$, where

$$N_{\mathbf{y}} = \{\mathbf{y} - \mathbf{e}_i,\ \mathbf{y} - \mathbf{e}_i + \mathbf{e}_j,\ \mathbf{y} + \mathbf{e}_i;\ \text{for } i, j = 1, \dots, M \text{ and } j \neq i\}.$$

Without loss of generality, we can assume that $B_2 \subseteq N_{\mathbf{y}}$, because

$$Q_2(\mathbf{y}, B_2) = Q_2(\mathbf{y}, (B_2 \cap N_{\mathbf{y}}))$$

holds, while

$$Q_1(\mathbf{x}, (B_2 \cap N_{\mathbf{y}})^{\leftarrow}) \leq Q_2(\mathbf{x}, B_2^{\leftarrow}),$$

(where in the last inequality we use that $B_2 \cap N_{\mathbf{y}} \subseteq B_2$).

Suppose $\mathbf{y} + \mathbf{e}_i \ \in B_2$ for some i. Then the left conjugate contains $\{\mathbf{y} + \mathbf{e}_i\}^{\leftarrow}$, but then also $\mathbf{y} \in B_2^{\leftarrow}$, and $\mathbf{x} \sim \mathbf{y}$ implies $\mathbf{x} \in B_2^{\leftarrow}$, which is in contradiction with the assumption that $\mathbf{x} \notin B_2^{\leftarrow}$. We conclude that we can restrict us to the sets B_2 for which $\mathbf{y} + \mathbf{e}_i \ \notin B_2$.

$$B_2 \subseteq \{\mathbf{x} - \mathbf{e}_i,\ \mathbf{x} - e_i + \mathbf{e}_j\ ; i, j = 1, \dots, M \text{ and } j \neq i\}.$$

For all $\mathbf{y}$ and for all i, define the subset of neighbors of $\mathbf{y}$ for which the i-th coordinate is decreased by one.

$$A_2^{(i)}(\mathbf{y}) := \{\tilde{\mathbf{y}} : \tilde{y}_i = y_i - 1\}.$$

Define also

$$B_2^{(i)}(\mathbf{y}) := B_1 \cap A_2^{(i)}(\mathbf{y}),$$

hence,

$$B_2^{(i)}(\mathbf{y}) \subseteq \{\mathbf{y}' \in B_2 : \mathbf{y}' = \mathbf{y} - \mathbf{e}_i \text{ or } \mathbf{y}' = \mathbf{y} - \mathbf{e}_i + \mathbf{e}_j \text{ for some } j\}.$$

Define furthermore index sets

$$\mathcal{I}_2 := \{i : B_2^{(i)} \neq \emptyset\},$$

and for all $i \in \mathcal{I}$

$$\mathcal{J}^{(i)} := \{j : \mathbf{y} - \mathbf{e}_i + \mathbf{e}_j \in B_1\} \subseteq \{1, \dots, M\} \setminus \{i\}.$$

We can write B_2 in terms of the $B_2^{(i)}$'s: $B_2 = \bigcup\limits_{i \in \mathcal{I}_2} B_2^{(i)}$.

Now,

$$\begin{aligned}
Q_2(\mathbf{y}, B_2) &= \sum_i \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_i \in B_2\}} \delta_i'(\mathbf{y}) p_i' + \sum_{\substack{i,j \\ j \neq i}} \mathbb{1}_{\{\mathbf{y}-\mathbf{e}_i+\mathbf{e}_j \in B_2\}} \delta_i'(\mathbf{y}) p_{ij}' \\
&\quad \textit{which we can write in terms of } i \in \mathcal{I}_2 \\
&= \sum_{i \in \mathcal{I}_2} \left[\mathbb{1}_{\{\mathbf{y}-\mathbf{e}_i \in B_2\}} \delta_i'(\mathbf{y}) p_i' + \sum_{j \in \mathcal{J}^{(i)}} \delta_i'(\mathbf{y}) p_{ij}' \right] \\
&\le \sum_{i \in \mathcal{I}_2} \left[\delta_i'(\mathbf{y}) p_i' + \sum_{j \in \mathcal{J}^{(i)}} \delta_i'(\mathbf{y}) p_{ij}' \right] \\
&\quad \textit{for all } i \in \mathcal{I}_2,\ B_2^{(i)} \neq \emptyset. \textit{ This implies that } \boldsymbol{y} - \boldsymbol{e}_i \in B_2^{\leftarrow}, \textit{ and because} \\
&\quad \textit{conditions } \boldsymbol{x} \sim \boldsymbol{y} \textit{ and } \boldsymbol{x} \notin B_2^{\leftarrow}, \textit{ we must have } y_i = x_i. \textit{ Hence, (4.22) is true for} \\
&\quad \textit{all } i. \\
&\le \sum_{i \in \mathcal{I}_2} \left[\delta_i(\mathbf{x}) p_i + \sum_{j \in \mathcal{J}^{(i)}} \delta_i(\mathbf{x}) p_{ij} + \sum_{\substack{k \notin \mathcal{J}^{(i)}, \\ k \neq i}} \mathbb{1}_{\{x_k < y_k\}} \delta_i(\mathbf{x}) p_{ik} \right] \\
&= Q_1(\mathbf{x}, B_2^{\leftarrow}).
\end{aligned}$$

We conclude that indeed condition (4.22) implies condition (3.8).

Now we will prove (3.8) $\Rightarrow$ (4.22). Suppose $Q_2(\mathbf{y}, B_2) \le Q_1(\mathbf{x}.B_2^{\leftarrow})$ is true for all $\mathbf{x} \sim \mathbf{y}$ where $\mathbf{y} \notin B_2$ and $\mathbf{x} \notin B_2^{\leftarrow}$. Let $\mathbf{x} \sim \mathbf{y}$ be arbitrary with $x_i = y_i$, for some $i = 1, \ldots, M$ and let $\mathcal{J}$ be an arbitrary subset of $\{1, \ldots, M\} \setminus \{i\}$. Define

$$B_2 = \{\mathbf{x} - \mathbf{e}_i\} \cup \bigcup_{j \in \mathcal{J}} \{\mathbf{x} - \mathbf{e}_i + \mathbf{e}_j\}.$$

Then $B_2 \subseteq S_2$ is such that $\mathbf{y} \notin B_2$ and $\mathbf{x} \notin B_2^{\leftarrow}$. Hence we have $Q_2(\mathbf{y}, B_2) \le Q_1(\mathbf{x}, B_2^{\leftarrow})$. The transition rates are given by

$$Q_2(\mathbf{y}, B_2) = \delta_i'(\mathbf{y}) \left[p_i' + \sum_{j \in \mathcal{J}} p_{ij}' \right], \text{ and}$$

$$Q_1(\mathbf{x}, B_2^{\rightarrow}) = \delta_i(\mathbf{x}) \left[p_i + \sum_{j \in \mathcal{J}} p_{ij} + \sum_{j \notin \mathcal{J}} \mathbb{1}_{\{x_j < y_j\}} p_{ij} \right],$$

which imply exactly condition (4.22) of Theorem 4.4 since i and $\mathcal{J}$ were arbitrary. This concludes the proof.

□

4.2.4 Alternative sharp comparison conditions

In 2004; Delgado, López and Sanz also derived if and only if conditions for the stochastic comparison of M-station queueing networks. The models that the authors use are based on interacting particle systems. In their proof they construct the order-preserving coupling explicitly. The formulation of the theorem in [Delgado et al., 2004] looks more complex, but after some study we can proof in fact their equivalence.

Delgado, López and Sanz use the following notation:

$$\Lambda^{[+]x,y} = \begin{cases} \Lambda & \text{if } x \neq y \\ \Lambda^+ := \max(\Lambda,\, 0) & \text{if } x = y \end{cases} \tag{4.23}$$

Theorem 4.5. *Consider two M-station Markov queueing networks X and Y with rates* (β, δ, p) *and* (β', δ', p')*. There exist a coupling of the processes X and Y for which the coordinate-wise relation is invariant if and only if for all* $\boldsymbol{x}, \boldsymbol{y}$ *with* $\boldsymbol{x} \sim \boldsymbol{y}$*:*

$$\beta_i'(\boldsymbol{y}) - \beta_i(\boldsymbol{x}) \geq \sum_{j \in N_i} \left[\delta_j(\boldsymbol{x}) p_{ji} - \delta_j'(\boldsymbol{y}) p_{ji}'\right]^{[+]x_j,y_j} \tag{4.24}$$

and

$$\delta_i(\boldsymbol{x}) \left[1 - \sum_{j \in N_i} p_{ij}\right] - \delta_i'(\boldsymbol{y}) \left[1 - \sum_{j \in N_i} p_{ij}'\right] \geq \sum_{j \in N_i} \left[\delta_i'(\boldsymbol{y}) p_{ij}' - \delta_i(\boldsymbol{x}) p_{ij}\right]^{[+]x_j,y_j} \tag{4.25}$$

hold for all i such that $x_i = y_i$*.*

In Theorem 4.5, the transition rates β_i and δ_i can also depend on the whole state $\mathbf{x}$, hence, the models are the same as described in Section 4.1. Although formula's (4.24) and (4.25) look more complicated and less intuitive than the conditions of Theorem 4.4, the advantage of Theorem 4.5 is that for a specific example we have to check only two formulas. In Theorem 4.4 we need to check (4.21) and (4.21) *for all* subsets of service stations $\mathcal{J} \subseteq \{1, \ldots, M\}$. Typically in queueing theory the number of stations we work with is not very high. Using then the conditions of Theorem 4.4 can help our intuition. But, for applications such as *interacting particle systems* where we have to deal with a large number of stations, we might prefer Theorem 4.5 because of the computational complexity. Thus, we prefer Theorem 4.4 when checking conditions in an analytic way while Theorem 4.5 gives advantages when we want to check a certain example numerically.

Proof. The proof is divided into four parts, which together prove (4.24) $\Leftrightarrow$ (4.21) and (4.25) $\Leftrightarrow$ (4.22).

Condition (4.24) $\Rightarrow$ condition (4.21):

Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements from the product space with $\mathbf{x} \sim \mathbf{y}$ and suppose $x_i = y_i$; so that (4.24) holds. Let $\mathcal{J} \subseteq \{1, \ldots, M\} \setminus \{i\}$ be arbitrary. We want to prove that Equation (4.21) is true.

By rewriting and working out the function $\Lambda^{[+]x_j,y_j}$ in (4.24) we have the following equivalence:

$$\beta_i'(\mathbf{y}) - \beta_i(\mathbf{x}) \geq \sum_{j \in N_i} \left[\delta_j(\mathbf{x}) p_{ji} - \delta_j'(\mathbf{y}) p_{ji}'\right]^{[+]x_j,y_j}$$

if and only if

$$\beta_i(\mathbf{x}) + \sum_{j:\ x_j < y_j} \delta_j(\mathbf{x}) p_{ji} + \sum_{\substack{j:\ x_j = y_j\ \& \\ \delta_j(\mathbf{x}) p_{ji} \geq \delta_j'(\mathbf{y}) p_{ji}'}} \delta_j(\mathbf{x}) p_{ji}$$
$$\leq \beta_i'(\mathbf{y}) + \sum_{j:\ x_j < y_j} \delta_j'(\mathbf{y}) p_{ji}' + \sum_{\substack{j:\ x_j = y_j\ \& \\ \delta_j(\mathbf{x}) p_{ji} \geq \delta_j'(\mathbf{y}) p_{ji}'}} \delta_j'(\mathbf{y}) p_{ji}'.$$

This inequality remains valid if we add $\delta_j(\mathbf{x})p_{ji}$ on the left and $\delta_j'(\mathbf{y})p_{ji}'$ on the right whenever $\delta_j(\mathbf{x})p_{ji} < \delta_j'(\mathbf{y})p_{ji}'$. We apply this to all $j: \ j \in \mathcal{J}$ and $x_j = y_j$ to conclude that (4.24) implies

$$\begin{aligned}\beta_i(\mathbf{x}) + \sum_{j:x_j<y_j} \delta_j(\mathbf{x})p_{ji} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}\geq\delta_j'(\mathbf{y})p_{ji}'}} \delta_j(\mathbf{x})p_{ji} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}<\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} \\ \leq \ \beta_i'(\mathbf{y}) + \sum_{j:\ x_j<y_j} \delta_j'(\mathbf{y})p_{ji}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}\geq\delta_j'(\mathbf{y})p_{ji}'}} \delta_j'(\mathbf{y})p_{ji}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}<\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j'(\mathbf{y})p_{ji}'.\end{aligned}$$

If $\delta_j(\mathbf{x})p_{ji} \geq \delta_j'(\mathbf{y})p_{ji}'$, the inequality remains valid when we subtract $\delta_j(\mathbf{x})p_{ji}$ on the left and $\delta_j'(\mathbf{y})p_{ji}'$ on the right. We apply it to all $j: \ j \notin \mathcal{J}\ \&\ x_j = y_j$ and get that (4.24) implies

$$\begin{aligned}\beta_i(\mathbf{x}) + \sum_{j:x_j<y_j} \delta_j(\mathbf{x})p_{ji} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}\geq\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}<\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} \\ \leq \ \beta_i'(\mathbf{y}) + \sum_{j:\ x_j<y_j} \delta_j'(\mathbf{y})p_{ji}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}\geq\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j'(\mathbf{y})p_{ji}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}<\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j'(\mathbf{y})p_{ji}'.\end{aligned}$$

The index set $\{j: \ x_j = y_j\ \&\ j \in \mathcal{J}\} \subseteq \{j: \ x_j = y_j\}$ and all rates are positive, thus $\sum_{\substack{j:\ x_j<y_j \\ \&\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} \leq \sum_{j:\ x_j<y_j} \delta_j(\mathbf{x})p_{ji}$. We subtract $\sum_{\substack{j:\ x_j<y_j \\ \&\ j\notin\mathcal{J}}} \delta_j(\mathbf{x})p_{ji}$ only on the left-hand side. We get

$$\begin{aligned}\beta_i(\mathbf{x}) + \sum_{\substack{j:\ x_j<y_j \\ \&\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}\geq\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}<\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} \\ \leq \ \beta_i'(\mathbf{y}) + \sum_{j:\ x_j<y_j} \delta_j'(\mathbf{y})p_{ji}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}\geq\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j'(\mathbf{y})p_{ji}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_j(\mathbf{x})p_{ji}<\delta_j'(\mathbf{y})p_{ji}' \\ \&\ j\in\mathcal{J}}} \delta_j'(\mathbf{y})p_{ji}',\end{aligned}$$

which is equivalent to

$$\begin{aligned}\beta_i(\mathbf{x}) + \sum_{\substack{j:\ x_j<y_j\ \& \\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} + \sum_{\substack{j:\ x_j=y_j\ \& \\ j\in\mathcal{J}}} \delta_j(\mathbf{x})p_{ji} \\ \leq \ \beta_i'(\mathbf{y}) + \sum_{j:\ x_j<y_j} \delta_j'(\mathbf{y})p_{ji}' + \sum_{\substack{j:\ x_j=y_j \\ \&\ j\in\mathcal{J}}} \delta_j'(\mathbf{y})p_{ji}'.\end{aligned}$$

Rewriting the equation above gives

$$\beta_i(\mathbf{x}) + \sum_{j:\ j\in\mathcal{J}} \delta_j(\mathbf{x})p_{ji} \leq \ \beta_i'(\mathbf{y}) + \sum_{j:\ j\in\mathcal{J}} \delta_j'(\mathbf{y})p_{ji}' + \sum_{\substack{j:\ x_j<y_j \\ \&\ j\notin\mathcal{J}}} \delta_j'(\mathbf{y})p_{ji}',$$

which is exactly condition (4.21).

Condition (4.21) $\Rightarrow$ condition (4.24):
Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements from the product space with $\mathbf{x} \sim \mathbf{y}$ and suppose $x_i = y_i$. Suppose we have

$$\beta_i(\mathbf{x}) + \sum_{j:\ j \in \mathcal{J}} \delta_j(\mathbf{x}) p_{ji} \leq \beta_i'(\mathbf{y}) + \sum_{j:\ j \in \mathcal{J}} \delta_j'(\mathbf{y}) p_{ji}' + \sum_{\substack{j:\ x_j < y_j \\ \&\ j \notin \mathcal{J}}} \delta_j'(\mathbf{y}) p_{ji}'$$

for all $\mathcal{J} \subseteq \{1, \ldots, M\} \setminus \{i\}$. Choose $\mathcal{J} \subseteq \{1, \ldots, M\} \setminus \{i\}$ such that

$$\begin{cases} j \in \mathcal{J} & \text{for all } j :\ x_j < y_j, \\ j \in \mathcal{J} & \text{for all } j :\ x_j = y_j \ \&\ \delta_j(\mathbf{x}) p_{ji} \geq \delta_j'(\mathbf{y}) p_{ji}' \text{ and} \\ j \notin \mathcal{J} & \text{for all } j :\ x_j = y_j \ \&\ \delta_j(\mathbf{x}) p_{ji} < \delta_j'(\mathbf{y}) p_{ji}'. \end{cases}$$

This implies that the subset $\{j : x_j < y_j \ \&\ j \notin \mathcal{J}\}$ is empty, as well as $\{j : x_j = y_j \ \&\ \delta_j(\mathbf{x}) p_{ji} \geq \delta_j'(\mathbf{y}) p_{ji}' \ \&\ j \notin \mathcal{J}\} = \emptyset$ and $\{j : x_j = y_j \ \&\ \delta_j(\mathbf{x}) p_{ji} < \delta_j'(\mathbf{y}) p_{ji}' \ \&\ j \in \mathcal{J}\} = \emptyset$. Hence, for this specific $\mathcal{J}$, condition (4.21) tells us

$$\begin{aligned} \beta_i(\mathbf{x}) + \sum_{j:\ x_j < y_j} \delta_j(\mathbf{x}) p_{ji} + \sum_{\substack{j:\ x_j = y_j\ \& \\ \delta_j(\mathbf{x}) p_{ji} \geq \delta_j'(\mathbf{y}) p_{ji}'}} \delta_j(\mathbf{x}) p_{ji} &= \beta_i(\mathbf{x}) + \sum_{j:\ j \in \mathcal{J}} \delta_j(\mathbf{x}) p_{ji} \\ &\leq \beta_i'(\mathbf{y}) + \sum_{j:\ j \in \mathcal{J}} \delta_j'(\mathbf{y}) p_{ji}' + \sum_{\substack{j:\ x_j < y_j \\ \&\ j \notin \mathcal{J}}} \delta_j'(\mathbf{y}) p_{ji}' \\ &= \beta_i'(\mathbf{y}) + \sum_{j:\ x_j < y_j} \delta_j'(\mathbf{y}) p_{ji}' + \sum_{\substack{j:\ x_j = y_j\ \& \\ \delta_j(\mathbf{x}) p_{ji} \geq \delta_j'(\mathbf{y}) p_{ji}'}} \delta_j'(\mathbf{y}) p_{ji}'. \end{aligned}$$

Condition (4.25) $\Rightarrow$ condition (4.22):
Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements from the product space $S_1 \times S_2$, for which $\mathbf{x} \sim \mathbf{y}$ and suppose $x_i = y_i$ and suppose (4.25) holds.

$$\delta_i(\mathbf{x}) \left[1 - \sum_{j \in N_i} p_{ij} \right] - \delta_i'(\mathbf{y}) \left[1 - \sum_{j \in N_i} p_{ij}' \right] \geq \sum_{j \in N_i} \left[\delta_i'(\mathbf{y}) p_{ij}' - \delta_i(\mathbf{x}) p_{ij} \right]^{[+] x_j, y_j}$$

if and only if

$$\begin{aligned} \delta_i(\mathbf{x}) p_i + \sum_{j:\ x_j < y_j} \delta_i(\mathbf{x}) p_{ij} + \sum_{\substack{j:\ x_j = y_j\ \& \\ \delta_i'(\mathbf{y}) p_{ij}' \geq \delta_i(\mathbf{x}) p_{ij}}} \delta_i(\mathbf{x}) p_{ij} & \\ \geq \delta_i'(\mathbf{y}) p_i' + \sum_{j:\ x_j < y_j} \delta_i'(\mathbf{y}) p_{ij}' + \sum_{\substack{j:\ x_j = y_j\ \& \\ \delta_i'(\mathbf{y}) p_{ij}' \geq \delta_i(\mathbf{x}) p_{ij}}} \delta_i'(\mathbf{y}) p_{ij}'. & \end{aligned}$$

Where we use $p_i = (1 - \sum_{j \in N_i} p_{ij})$ and we use Delgado's definition (4.23).

This inequality remains valid if we add $\delta_j(\mathbf{x}) p_{ji}$ on the left and $\delta_j'(\mathbf{y}) p_{ji}'$ on the right when $\delta_i'(\mathbf{y}) p_{ij}' < \delta_i(\mathbf{x}) p_{ij}$. We apply this on all j for which $x_j = y_j$ and $j \in \mathcal{J}$, therefore

(4.25) implies

$$\delta_i(\mathbf{x})p_i + \sum_{j:\ x_j<y_j} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'\geq\delta_i(\mathbf{x})p_{ij}}} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'<\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i(\mathbf{x})p_{ij}$$
$$\geq \delta_i'(\mathbf{y})p_i' + \sum_{j:\ x_j<y_j} \delta_i'(\mathbf{y})p_{ij}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'\geq\delta_i(\mathbf{x})p_{ij}}} \delta_i'(\mathbf{y})p_{ij}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'<\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}'.$$

The inequality remains also valid when we subtract $\delta_j(\mathbf{x})p_{ji}$ on the left and $\delta_j'(\mathbf{y})p_{ji}'$ on the right if $\delta_i'(\mathbf{y})p_{ij}' \geq \delta_i(\mathbf{x})p_{ij}$. We apply this on all $j:\ j\notin\mathcal{J}$ and $x_j = y_j$. Remark that $\{j:\ x_j = y_j\ \&\ \delta_i'(\mathbf{y})p_{ij}' \geq \delta_i(\mathbf{x})p_{ij}\ \&\ j\in\mathcal{J}\}$ is exactly equal to $\{j:\ x_j = y_j\ \&\ \delta_i'(\mathbf{y})p_{ij}' \geq \delta_i(\mathbf{x})p_{ij}\} \setminus \{j:\ x_j = y_j\ \&\ \delta_i'(\mathbf{y})p_{ij}' \geq \delta_i(\mathbf{x})p_{ij}\ \&\ j\notin\mathcal{J}\}$. Hence, (4.24) implies also

$$\delta_i(\mathbf{x})p_i + \sum_{j:\ x_j<y_j} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'\geq\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'<\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i(\mathbf{x})p_{ij}$$
$$\geq \delta_i'(\mathbf{y})p_i' + \sum_{j:\ x_j<y_j} \delta_i'(\mathbf{y})p_{ij}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'\geq\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'<\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}'.$$

The set $\{j:\ x_j = y_j\ \&\ j\in\mathcal{J}\}$ is a subset of $\{j:\ x_j = y_j\}$ and all rates are positive, therefore $\sum_{\substack{j:\ x_j<y_j \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}' \leq \sum_{j:\ x_j<y_j} \delta_i'(\mathbf{y})p_{ij}'$. We subtract $\sum_{\substack{j:\ x_j<y_j \\ \&\ j\notin\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}'$ on the right-hand side of our equation and get

$$\delta_i(\mathbf{x})p_i + \sum_{j:\ x_j<y_j} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'\geq\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'<\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i(\mathbf{x})p_{ij}$$
$$\geq \delta_i'(\mathbf{y})p_i' + \sum_{\substack{j:\ x_j<y_j \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'\geq\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}' + \sum_{\substack{j:\ x_j=y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}'<\delta_i(\mathbf{x})p_{ij} \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}';$$

which is again equal to

$$\delta_i(\mathbf{x})p_i + \sum_{j:\ x_j<y_j} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j=y_j \\ \&\ j\in\mathcal{J}}} \delta_i(\mathbf{x})p_{ij}$$
$$\geq \delta_i'(\mathbf{y})p_i' + \sum_{\substack{j:\ x_j<y_j \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}' + \sum_{\substack{j:\ x_j=y_j \\ \&\ j\in\mathcal{J}}} \delta_i'(\mathbf{y})p_{ij}'.$$

This holds if and only if

$$\delta_i(\mathbf{x})p_i + \sum_{j:\ j\in\mathcal{J}} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j<y_j \\ \&\ j\notin\mathcal{J}}} \delta_i(\mathbf{x})p_{ij} \geq \delta_i'(\mathbf{y})p_i' + \sum_{j:\ j\in\mathcal{J}} \delta_i'(\mathbf{y})p_{ij}',$$

which is exactly condition (4.22).

Condition (4.22) $\Rightarrow$ condition (4.25):
Let $\mathbf{x}, \mathbf{y}$ be arbitrary elements from the product space with $\mathbf{x} \sim \mathbf{y}$ and suppose $x_i = y_i$. Suppose we have, for all $\mathcal{J} \subseteq \{1, \ldots, M\} \setminus \{i\}$

$$\delta_i(\mathbf{x})p_i + \sum_{j:\ j \in \mathcal{J}} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j < y_j \\ \&\ j \notin \mathcal{J}}} \delta_i(\mathbf{x})p_{ij} \geq \delta_i'(\mathbf{y})p_i' + \sum_{j:\ j \in \mathcal{J}} \delta_i'(\mathbf{y})p_{ij}'.$$

Let $\mathcal{J} \subseteq \{1, \ldots, M\} \setminus \{i\}$ be such that

$$\begin{cases} j \in \mathcal{J} & \text{for all } j:\ x_j < y_j \\ j \in \mathcal{J} & \text{for all } j:\ x_j = y_j \ \&\ \delta_i'(\mathbf{y})p_{ij}' \geq \delta_i(\mathbf{x})p_{ij} \\ j \notin \mathcal{J} & \text{for all } j:\ x_j = y_j \ \&\ \delta_i'(\mathbf{y})p_{ij}' < \delta_i(\mathbf{x})p_{ij}. \end{cases}$$

This implies that $\{j : x_j < y_j \ \&\ j \notin \mathcal{J}\} = \emptyset$, $\{j : x_j = y_j \ \&\ \delta_i'(\mathbf{y})p_{ij}' \geq \delta_i(\mathbf{x})p_{ij} \ \&\ j \notin \mathcal{J}\} = \emptyset$ and $\{j : x_j = y_j \ \&\ \delta_i'(\mathbf{y})p_{ij}' < \delta_i(\mathbf{x})p_{ij} \ \&\ j \in \mathcal{J}\} = \emptyset$. Hence, for this specific $\mathcal{J}$

$$\begin{aligned} \delta_i(\mathbf{x})p_i + \sum_{j:\ x_j < y_j} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j = y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}' \geq \delta_i(\mathbf{x})p_{ij}}} \delta_i(\mathbf{x})p_{ij} &= \\ \delta_i(\mathbf{x})p_i + \sum_{j:\ j \in \mathcal{J}} \delta_i(\mathbf{x})p_{ij} + \sum_{\substack{j:\ x_j < y_j \\ \&\ j \notin \mathcal{J}}} \delta_i(\mathbf{x})p_{ij} &\geq \delta_i'(\mathbf{y})p_i' + \sum_{j:\ j \in \mathcal{J}} \delta_i'(\mathbf{y})p_{ij}' \\ &= \delta_i'(\mathbf{y})p_i' + \sum_{j:\ x_j < y_j} \delta_i'(\mathbf{y})p_{ij}' + \sum_{\substack{j:\ x_j = y_j\ \& \\ \delta_i'(\mathbf{y})p_{ij}' \geq \delta_i(\mathbf{x})p_{ij}}} \delta_i'(\mathbf{y})p_{ij}', \end{aligned}$$

which gives us exactly condition (4.25).

□

4.3 Summation coupling of Markov queueing networks

In the previous section we considered the coordinate-wise ordering of Markov queueing networks X and Y. In this section we consider an alternative relation, which is not an order. In many situations, it makes sense to consider the total number of customers in the system. We call this relation the *summation relation*. The total number of customers in the system is important, for example, if there are certain costs per customer in the system — no matter at which service point this customer is situated in the system.

We start again to consider the 2-station Markov queueing network. We now look at a different relation, which keeps track on the total number of jobs (or customers) in the system. We denote $x_1 + x_2 = |\mathbf{x}|$ and $y_1 + y_2 = |\mathbf{y}|$.

Definition 4.6. *[Summation relation]*

$$R^{sum} = \{(\mathbf{x}, \mathbf{y}) : |\mathbf{x}| \leq |\mathbf{y}|\}. \tag{4.26}$$

Note that this relation is an example of a relation which is not a partial order, because it does not satisfy the anti-symmetry condition.

4.3.1 Summation-order comparison of two-server queueing networks

Conditions for $Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow})$
Recall Condition (3.7):

$$Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow}), \text{ for all } B_1 \subseteq S_1 \text{ with } \mathbf{x} \notin B_1,\ \mathbf{y} \notin B_1^{\rightarrow}.$$

The right conjugate $B_1^{\rightarrow}$ of $B_1 \subseteq S_1$ is in this case given by $B_1^{\rightarrow} \subseteq S_2$, where

$$B_1^{\rightarrow} = \bigcup_{\mathbf{x} \in B_1} \{\mathbf{y} : |\mathbf{y}| \geq |\mathbf{x}|\} = \{\mathbf{y} \in S_2 : |\mathbf{y}| \geq \inf_{\mathbf{x} \in B_1} |\mathbf{x}|\}. \tag{4.27}$$

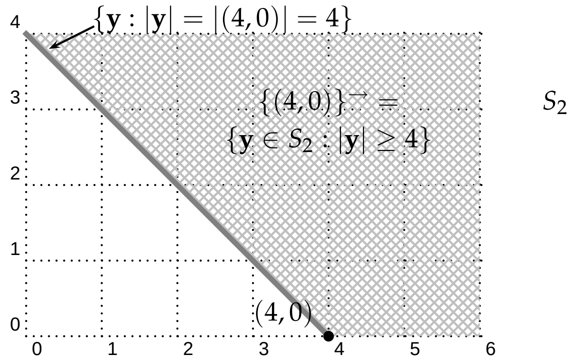

Figure 4.4: The right conjugate of $\mathbf{x} = (4, 0)$ is given by the shadowed area.

Let $\mathbf{x} \sim \mathbf{y}$ be an element from $S_1 \times S_2$ and let B_1 be an arbitrary non-empty subset of S_1 with $\mathbf{x} \notin B_1$. Because $B_1 \neq \emptyset$, $\inf_{\mathbf{x} \in B_1} |\mathbf{x}|$ is well-defined. Then either one of the following three possibilities holds:

- Suppose $\inf_{\mathbf{x}' \in B_1} |\mathbf{x}'| \leq |\mathbf{x}|$. Then $\{\mathbf{y} : |\mathbf{y}| \geq |\mathbf{x}|\} \subseteq B_1^{\rightarrow}$, thus we always have $\mathbf{y} \in B_1^{\rightarrow}$ while the condition of the theorem only holds for $\mathbf{y} \notin B_1^{\rightarrow}$. Thus, this case does not give us any conditions on the different transition rates.

- Suppose $\inf_{\mathbf{x}' \in B_1} |\mathbf{x}'| = |\mathbf{x}| + 1$. This means the only point to which $\mathbf{x}$ can jump into B_1 with positive rate is when a new arrival occurs, provided of course that the specific state is an element of B_1:

$$Q_1(\mathbf{x}, B_1) = \beta_1(\mathbf{x}) \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_1 \in B_1\}} + \beta_2(\mathbf{x}) \cdot \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_2 \in B_1\}}.$$

  On the other hand, $B_1^{\rightarrow} = \{\mathbf{y} : |\mathbf{y}| \geq |\mathbf{x}| + 1\}$. We have $\mathbf{y} \notin B_1^{\rightarrow}$ and thus $|\mathbf{y}| < |\mathbf{x}| + 1$, while $|\mathbf{x}| \leq |\mathbf{y}|$ due to $\mathbf{x} \sim \mathbf{y}$. This implies $|\mathbf{x}| = |\mathbf{y}|$. Hence, $Q_2(\mathbf{y}, B_1^{\rightarrow})$ contains only arrival rates from $\mathbf{y}$:

$$Q_2(\mathbf{y}, B_1^{\rightarrow}) = \beta_1'(\mathbf{y}) + \beta_2'(\mathbf{y}).$$

  To break the condition $Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow})$, $Q_1(\mathbf{x}, B_1)$ has to be as large as possible while $Q_2(\mathbf{y}, B_1^{\rightarrow})$ is as small as possible. Therefore, the worst case scenario happens if $\mathbf{x} + \mathbf{e}_1$ and $\mathbf{x} + \mathbf{e}_2 \in B_1$. And this worst case gives us the condition

$$\beta_1(\mathbf{x}) + \beta_2(\mathbf{x}) \leq \beta_1'(\mathbf{y}) + \beta_2'(\mathbf{y}) \ \forall\, \mathbf{x}, \mathbf{y} \text{ with } |\mathbf{x}| = |\mathbf{y}|. \tag{4.28}$$

- Suppose $\inf_{\mathbf{x}' \in B_1} |\mathbf{x}'| > |\mathbf{x}| + 1$. In this case there are no possibilities for $\mathbf{x}$ to jump to any other state with positive probability:

$$Q_1(\mathbf{x}, B_1) = 0.$$

  In this case we always have $Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow})$ and we do not get any extra conditions on the rates.

Conditions for $Q_2(\mathbf{y}, B_2^{\leftarrow}) \leq Q_2(\mathbf{y}, B_2)$

The second condition of Theorem (3.19) is

$$Q_2(\mathbf{y}, B_2) \leq Q_2(\mathbf{y}, B_2^{\leftarrow}) \text{ for all } \mathbf{x} \sim \mathbf{y},\ \mathbf{y} \notin B_2 \text{ and } \mathbf{x} \notin B_2^{\leftarrow}.$$

Let $(\mathbf{x}, \mathbf{y}) \in R^{sum}$, $B_2 \subseteq S_2$ (non-empty) be arbitrary and $\mathbf{y} \notin B_2$.

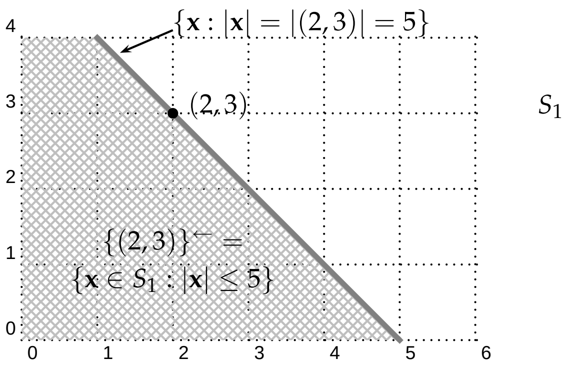

Figure 4.5: The left conjugate of $\mathbf{y} = (2, 3)$ is given by the shadowed area.

The left conjugate $B_2^{\leftarrow} \subseteq S_1$ with respect to relation R^{sum} is equal to

$$B_2^{\leftarrow} = \cup_{\mathbf{y} \in B_2} \{\mathbf{x} \in S_1 : |\mathbf{x}| \leq |\mathbf{y}|\} = \{\mathbf{x} \in B_1 : |\mathbf{x}| \leq \sup_{\mathbf{y} \in S_2} |\mathbf{y}|\}. \quad (4.29)$$

Again, there are three possibilities:

- Suppose $\sup_{\mathbf{y}' \in B_2} |\mathbf{y}'| < |\mathbf{y}| - 1$. Then $Q_2(\mathbf{y}, B_2) = 0$, therefore we have always $Q_2(\mathbf{y}, B_2^{\leftarrow}) \leq Q_2(\mathbf{y}, B_2)$ for $\mathbf{y} \notin B_2$ and $\mathbf{x} \notin B_2^{\leftarrow}$.
- Suppose $\sup_{\mathbf{y}' \in B_2} |\mathbf{y}'| = |\mathbf{y}| - 1$. This implies $B_2^{\leftarrow} = \{x : |\mathbf{x}| \leq |\mathbf{y}| - 1\}$. The conditions $\mathbf{x} \notin B_2^{\leftarrow}$ and $|\mathbf{x}| \leq |\mathbf{y}|$ imply $|\mathbf{x}| = |\mathbf{y}|$. For $Q_2(\mathbf{y}, B_2)$ it means that only a departure can cause any positive outgoing rate. This happens only if $\mathbf{y} - \mathbf{e}_1 \in B_2$ and/or $\mathbf{y} - \mathbf{e}_2 \in B_2$:

$$Q_2(\mathbf{y}, B_2) = \delta_1'(\mathbf{y}) p_1' \cdot \mathbb{1}_{\{\mathbf{y} - \mathbf{e}_1 \in B_2\}} + \delta_2'(\mathbf{y}) p_2' \cdot \mathbb{1}_{\{\mathbf{y} - \mathbf{e}_2 \in B_2\}}.$$

  On the other hand, $\mathbf{x}$ can jump to each state $\mathbf{x}'$ for which $|\mathbf{x}'| \leq |\mathbf{y}| - 1 = |\mathbf{x}| - 1$. Thus, also in this case all rates outgoing from $\mathbf{x}$ are caused by departures

$$Q_1(\mathbf{x}, B_2^{\leftarrow}) = \delta_1(\mathbf{x}) p_1 + \delta_2(\mathbf{x}) p_2.$$

We conclude that in this case the following condition is equivalent with inequality (3.8):

$$\delta_1'(\mathbf{y})p_1' + \delta_2'(\mathbf{y})p_2' \leq \delta_1(\mathbf{x})p_1 + \delta_2(\mathbf{x})p_2 \text{ for all } \mathbf{x}, \mathbf{y} \text{ with } |\mathbf{x}| = |\mathbf{y}|. \tag{4.30}$$

- Suppose $\sup_{\mathbf{y}' \in B_2} |\mathbf{y}'| \geq |\mathbf{y}|$. Now $B_2^{\leftarrow} = \{x : |\mathbf{x}| \leq |\mathbf{y}|\}$, and there is no $\mathbf{x}$ such that $\mathbf{x} \sim \mathbf{y}$ but $\mathbf{x} \notin B_2^{\leftarrow}$. Hence, we do not get any conditions.

The derivations above lead to the following theorem.

Theorem 4.7. *[Coupling of two 2-server Markov queueing networks]*
Let X and Y be two Markov queueing networks with parameters (β, δ, p) and (β', δ', p') on S_1 and S_2. Then there exists a coupling of X and Y for which the relation R^{sum} is invariant if and only if for all $(\boldsymbol{x}, \boldsymbol{y}) \in S_1 \times S_2$ with $|\boldsymbol{x}| \leq |\boldsymbol{y}|$:

$$\beta_1(\boldsymbol{x}) + \beta_2(\boldsymbol{x}) \quad \leq \quad \beta_1'(\boldsymbol{y}) + \beta_2'(\boldsymbol{y}), \tag{4.31}$$

and,

$$\delta_1(\boldsymbol{x})p_1 + \delta_2(\boldsymbol{x})p_2 \quad \geq \quad \delta_1'(\boldsymbol{y})p_1' + \delta_2'(\boldsymbol{y})p_2'. \tag{4.32}$$

An idea of the proof is already given, for the formal proof we refer to the proof of Theorem 4.8.

4.3.2 Summation relation comparison of M-station queueing networks

Theorem 4.8. *[Coupling of two M-station queueing networks]*
Let X and Y be two M-station Markov queueing networks with parameters (β, δ, p) and (β', δ', p') on S_1 and S_2. Then there exists a coupling of X and Y for which the relation R^{sum} is invariant if and only if for all $(\boldsymbol{x}, \boldsymbol{y}) \in S_1 \times S_2$ such that $|\boldsymbol{x}| \leq |\boldsymbol{y}|$:

$$\sum_{i=1}^{M} \beta_i(\boldsymbol{x}) \quad \leq \quad \sum_{i=1}^{M} \beta_i'(\boldsymbol{y}) \tag{4.33}$$

and

$$\sum_{i=1}^{M} \delta_i(\boldsymbol{x})p_i \quad \geq \quad \sum_{i=1}^{M} \delta_i'(\boldsymbol{y})p_i'. \tag{4.34}$$

In words, this theorem states that whenever there is an equal number of customers in both systems, the total arrival rate into the system in process X must be smaller than or equal to the total arrival rate of Y, and the total departure rate of X must be bigger than or equal to the total departure rate of Y. This makes sense intuitively.

Proof. It suffices to prove that conditions (4.33) and (4.34) hold *if and only if* the conditions (3.7) and (3.8) hold for the summation relation. In Part 1 we prove that (4.33) $\Leftrightarrow$ (3.7); and in Part 2 (4.34) $\Leftrightarrow$ (3.8).

PART 1

Suppose that for all $\mathbf{x}, \mathbf{y}$ such that $|\mathbf{x}| = |\mathbf{y}|$ we have $\sum_{i=1}^{M} \beta_i(\mathbf{x}) \leq \sum_{i=1}^{M} \beta_i'(\mathbf{y})$. Let $B_1 \subseteq S_1$ be arbitrary with $\mathbf{x} \notin B_1$ and $\mathbf{y} \notin B_1^{\rightarrow}$. Recall condition (3.7) of Theorem (3.19) is

$$Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow}) \text{ for all } \mathbf{x} \sim \mathbf{y},\ \mathbf{x} \notin B_1 \text{ and } \mathbf{y} \notin B_1^{\rightarrow} \tag{4.35}$$

If $B_1 = \emptyset$ we have $Q_1(\mathbf{x}, B_1) = 0$, which is always less or equal than Q_2. Therefore, assume $B_1 \neq \emptyset$. Then $\inf_{\mathbf{x}' \in B_1} |\mathbf{x}'|$ is well defined and $\inf_{\mathbf{x}' \in B_1} |\mathbf{x}'| \in \mathbb{N}$. The right conjugate is given by (4.27):

$$B_1^{\rightarrow} = \bigcup_{\mathbf{x} \in B_1} \{\mathbf{y} : |\mathbf{y}| \geq |\mathbf{x}|\} = \{\mathbf{y} \in S_2 : |\mathbf{y}| \geq \inf_{\mathbf{x} \in B_1} |\mathbf{x}|\}.$$

There are three possibilities:

- Suppose $\inf_{\mathbf{x}' \in B_1} |\mathbf{x}'| < |\mathbf{x}| + 1$. Then $\{\mathbf{y} : |\mathbf{y}| \geq |\mathbf{x}|\} \subseteq B_1^{\rightarrow}$, hence we always have $\mathbf{y} \in B_1^{\rightarrow}$ while Condition (4.35) of Theorem 3.19 is only required for $\mathbf{y} \notin B_1^{\rightarrow}$. Thus, this case does not satisfy Condition (4.35).

- Suppose $\inf_{\mathbf{x}' \in B_1} |\mathbf{x}'| = |\mathbf{x}| + 1$. This means that the only station to which $\mathbf{x}$ can jump into B_1 with positive rate is when a new arrival occurs, provided of course that the specific state is an element of B_1. Hence, $\mathbf{x}$ can jump to $\mathbf{x} + \mathbf{e}_i$ for each $i = 1, \ldots, M$. This occurs with rate $\beta_i(\mathbf{x})$:

$$Q_1(\mathbf{x}, B_1) = \sum_{i=1}^{M} \mathbb{1}_{\{\mathbf{x}+\mathbf{e}_i \in B_1\}} \beta_i(\mathbf{x}).$$

  On the other hand, $B_1^{\rightarrow} = \{\mathbf{y} : |\mathbf{y}| \geq |\mathbf{x}| + 1\}$. We assumed $\mathbf{y} \notin B_1^{\rightarrow}$, hence $|\mathbf{y}| < |\mathbf{x}| + 1$, while $|\mathbf{x}| \leq |\mathbf{y}|$ due to $\mathbf{x} \sim \mathbf{y}$. This implies $|\mathbf{x}| = |\mathbf{y}|$. Hence,

$$Q_2(\mathbf{y}, B_1^{\rightarrow}) = \sum_{i=1}^{M} \beta_i'(\mathbf{y}).$$

  Thanks to Condition (4.33) $Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow})$ is always true.

- Suppose $\inf_{\mathbf{x}' \in B_1} |\mathbf{x}'| > |\mathbf{x}| + 1$. In this case there are no possibilities for $\mathbf{x}$ to jump to any other state with positive probability, because the probability that two arrivals occur exactly at the same moment in time is zero:

$$Q_1(\mathbf{x}, B_1) = 0.$$

  Therefore, in this case we always have $Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow})$.

We conclude that Condition (4.35) always holds.

To prove the converse, suppose that $Q_1(\mathbf{x}, B_1) \leq Q_2(\mathbf{y}, B_1^{\rightarrow})$ is true for all $\mathbf{x} \sim \mathbf{y}$ such that $\mathbf{x} \notin B_1$ and $\mathbf{y} \notin B_1^{\rightarrow}$. Let $\mathbf{x}, \mathbf{y} \in S_1 \times S_2$ and suppose that $|\mathbf{x}| = |\mathbf{y}|$. Let $B_1 = \bigcup_{i=1,\ldots,M} \{\mathbf{x} + \mathbf{e}_i\}$. Then $B_1 \subseteq S_1$; $\mathbf{x} \notin B_1$ and $\mathbf{y} \notin B_1^{\rightarrow}$, thus we can apply Condition (3.7). As a consequence,

$$Q_1(\mathbf{x}, B_1) = \sum_{i=1}^{M} \beta_i(\mathbf{x})$$

and

$$Q_2(\mathbf{y}, B_1^{\rightarrow}) = \sum_{i=1}^{M} \beta_i'(\mathbf{y}),$$

hence, Equation (4.33) holds.

PART 2

Suppose that Equation (4.34) holds for all $\mathbf{x}, \mathbf{y}$ for which $|\mathbf{x}| = |\mathbf{y}|$. Let $B_2 \subseteq S_2$ be arbitrary such that $\mathbf{y} \notin B_2$ and $\mathbf{x} \notin B_2^{\leftarrow}$. We can assume that $B_2 \subseteq N_{\mathbf{y}}$, because

$$Q_2(\mathbf{y}, B_2) = Q_2(\mathbf{y}, B_2 \cap N_{\mathbf{y}})$$

and

$$Q_1(\mathbf{x}, B_2^{\leftarrow}) \geq Q_1(\mathbf{x}, (B_2 \cap N_{\mathbf{y}})^{\leftarrow}).$$

The second condition of Theorem (3.19) is

$$Q_2(\mathbf{y}, B_2) \leq Q_1(\mathbf{x}, B_2^{\leftarrow}) \text{ for all } \mathbf{x} \sim \mathbf{y},\ \mathbf{y} \notin B_2 \text{ and } \mathbf{x} \notin B_2^{\leftarrow} \tag{4.36}$$

Hence, it is sufficient if for all $B_2 \subseteq N_{\mathbf{y}}$ Condition (4.36) holds.

The left conjugate $B_2^{\leftarrow} \subseteq S_1$ with respect to the summation relation $\mathcal{R}^{sum}$ is equal to:

$$B_2^{\leftarrow} = \bigcup_{\mathbf{y}' \in B_2} \{\mathbf{x} \in S_1 : |\mathbf{x}| \leq |\mathbf{y}'|\} = \{\mathbf{x} \in B_1 : |\mathbf{x}| \leq \sup_{\mathbf{y}' \in S_2} |\mathbf{y}'|\}. \tag{4.37}$$

Furthermore we can assume that $B_2 \neq \emptyset$, so that $\sup\limits_{\mathbf{y}' \in B_2} |\mathbf{y}|$ is well defined. (Remark that if $B_2 = \emptyset$, we have $Q_2(\mathbf{y}, B_2) = 0$, such that then (4.36) is automatically true.) Again, there are three possibilities:

- Suppose $\sup_{\mathbf{y}' \in B_2} |\mathbf{y}'| < |\mathbf{y}| - 1$. Then $Q_2(\mathbf{y}, B_2) = 0$ and we have always $Q_2(\mathbf{y}, B_2^{\leftarrow}) \leq Q_2(\mathbf{y}, B_2)$ for $\mathbf{y} \notin B_2$ and $\mathbf{x} \notin B_2^{\leftarrow}$.

- Suppose $\sup_{\mathbf{y}' \in B_2} |\mathbf{y}'| = |\mathbf{y}| - 1$. This implies $B_2^{\leftarrow} = \{x : |\mathbf{x}| \leq |\mathbf{y}| - 1\}$. The conditions $\mathbf{x} \notin B_2^{\leftarrow}$ and $|\mathbf{x}| \leq |\mathbf{y}|$ imply $|\mathbf{x}| = |\mathbf{y}|$. For $Q_2(\mathbf{y}, B_2)$ it means that only a departure can cause any positive outgoing rate. This happens only if $\mathbf{y} - \mathbf{e}_i \in B_2$:

  $$Q_2(\mathbf{y}, B_2) = \sum_{i=1}^{M} \delta_i'(\mathbf{y}) p_i' \cdot \mathbb{1}_{\{\mathbf{y} - \mathbf{e}_i \in B_2\}}.$$

  On the other hand, $\mathbf{x}$ can jump to each state $\mathbf{x}'$ for which $|\mathbf{x}'| \leq |\mathbf{y}| - 1 = |\mathbf{x}| - 1$. Therefore, also in this case all rates outgoing from $\mathbf{x}$ are caused by departures:

  $$Q_1(\mathbf{x}, B_2^{\leftarrow}) = \sum_{i=1}^{M} \delta_i(\mathbf{x}) p_i.$$

  And because we have $\sum\limits_{i=1}^{M} \delta_i(\mathbf{x}) p_i \geq \sum\limits_{i=1}^{M} \delta_i'(\mathbf{y}) p_i'$ we conclude that (4.36) is true.

- Suppose $\sup_{\mathbf{y}' \in B_2} |\mathbf{y}'| \geq |\mathbf{y}|$. Now, $B_2^{\leftarrow} = \{x : |\mathbf{x}| \leq |\mathbf{y}|\}$, and there is no $\mathbf{x}$ such that $\mathbf{x} \sim \mathbf{y}$ but $\mathbf{x} \notin B_2^{\leftarrow}$, and the condition $Q_2(\mathbf{y}, B_2) \leq Q_1(\mathbf{x}, B_2^{\leftarrow})$ is not violated.

Finally, suppose that the conditions of Theorem 3.19 are true for all $B_2 \subseteq S_2$ with $\mathbf{y} \notin B_2$ and $\mathbf{x} \notin B_2^{\leftarrow}$. We want to prove that $\sum_{i=1}^{M} \delta_i(\mathbf{x}) p_i \geq \sum_{i=1}^{M} \delta_i'(\mathbf{y}) p_i'$ for all $\mathbf{x}, \mathbf{y}$ such that $|\mathbf{x}| = |\mathbf{y}|$.

Let $\mathbf{x}, \mathbf{y}$ be arbitrary such that $|\mathbf{x}| = |\mathbf{y}|$. Define

$$B_2 := \bigcup_{i=1,\ldots,M} \{\mathbf{x} - \mathbf{e}_i\}.$$

Then $B_2 \subseteq S_2$; $\mathbf{y} \notin B_2$ and $\mathbf{x} \notin B_2^{\rightarrow}$ therefore we can apply Condition (4.36).

$$Q_2(\mathbf{y}, B_2) = \sum_{i=1}^{M} \delta_i'(\mathbf{y}) p_i'$$

and

$$Q_1(\mathbf{x}, B_2^{\leftarrow}) = \sum_{i=1}^{M} \beta_i(\mathbf{x}),$$

and because we also have $Q_2(\mathbf{y}, B_2) \leq Q_1(\mathbf{x}, B_2^{\leftarrow})$ hence Condition (4.34) holds.

$\square$

We give a last remark on single-server Markov queueing systems. If $M = 1$, the coordinate-wise order and the summation relation are the same

$$R^{coord} = R^{summation} = \{(x, y) : x \leq y\}.$$

In our notation, since we only have one service station, we ommit the 'subscript-i'. The conditions of Theorems 4.4 and 4.8 are equal, and to ensure the existence of an order preserving coupling we must have

$$\beta(x) \leq \beta'(y)$$

and

$$\delta'(y) \cdot p' \leq \delta(x) \cdot p$$

for all $x = y$. That is, if both systems happen to be in the same state (on the boundary of the relation), the arrival rate in X should not exceed the arrival rate in Y and the departure rate in Y should not exceed the departure rate in X. As we have seen, these conditions hold for the example given in Section 2.1.

Chapter 5

A loss network with breakdowns

In this chapter we study a Jackson network with breakdowns. Having breakdowns in the network complicates the stochastic comparison. We construct an explicit coupling and obtain a stochastic comparison result with respect to the coordinate-wise ordering. The chapter is outlined as follows. In Section 5.1 we introduce the Jackson network with breakdowns. We study the single-server case and present a coupling for this simple case in Section 5.2. In Section 5.3 we derive a coupling for the general multi-server network and give the stochastic comparison result.

5.1 Jackson network with breakdowns

Jackson network

Consider a Markov queueing network X with M service stations and Poisson arrivals with parameter λ. The service requirements are exponential and all service requirements and inter-arrival times are assumed to be independent of each other. At station i, the service rate is $\mu_i(x_i)$ if x_i jobs are present at station i, where we assume $\mu_i(x_i)$ to be increasing in x_i for every i. This is a simplification of the queueing networks described in Section 4.1, and is also known as a Jackson network. After service, the job (or customer) jumps from station i to station j with routing probability p_{ij}, and the probability of leaving the system from station i is $p_i = 1 - \sum_{j=1}^{M} p_{ij}$. Furthermore, the total number of jobs at service station i is bounded by N_i. If there are more than N_i jobs upon an arrival instant at a server i, this arrival will be rejected.

Breakdowns

In addition, each workstation i has a departure channel which is subject to breakdowns. When the departure channel is down, a job attempting to leave station i (either to jump to another station or to leave the system) remains at that station. We denote the state of departure channel of server i by θ_i, where $\theta_i = 0$ if the departure channel is down and $\theta_i = 1$ if the channel is working.

The state space is $S \subseteq \mathbb{N}^M \times \{0,1\}^M$, where $(\mathbf{x}, \boldsymbol{\theta}) \in S$ if and only if $0 \leq x_i \leq N_i$ and $\theta_i \in \{0,1\}$. We denote a specific state in the system by $(\mathbf{x}; \boldsymbol{\theta}) \in S$, where $\mathbf{x} \in \mathbb{N}^M$ represents the number of customers at each station and the vector $\boldsymbol{\theta} \in \{0,1\}^M$ represents if the stations are up or down.

In the system X', which we want to compare to, only the arrival rates are different. The customers join the queue at station i only if the departure channel of that station is up, thus arrivals are rejected if $\theta'_i = 0$. By this modification, the so-called balance equations hold in every state of the system. The balance equations state that for each subset of states in the system, the total inflow rate must equal the outflow rate in steady-state. Then, a product-form solution of the steady state distribution exist. In this sense, the modified system would give us an relatively easier bound for the behaviour of the original system. See also [van Dijk, 1998] and [Kelly, 1979].

As in Section 4.1, the systems can be described as continuous-time Markov processes with the transition rates given in Table 5.1.

Table 5.1: Transition rates of the Jackson network with breakdowns

Original system X $(\mathbf{x};\ \boldsymbol{\theta}) \longmapsto \ldots$		Modified system X' $(\mathbf{x}';\ \boldsymbol{\theta}') \longmapsto \ldots$	
arrival: $(\mathbf{x} + \mathbb{1}_{\{x_i<N_i\}}\mathbf{e}_i;\ \boldsymbol{\theta})$	*at rate* λ_i	*arrival:* $(\mathbf{x}' + \theta'_i \mathbb{1}_{\{x'_i<N_i\}}\mathbf{e}_i;\ \boldsymbol{\theta}')$	*at rate* λ_i
departure: $(\mathbf{x} - \theta_i \mathbb{1}_{\{0<x_i\}}\mathbf{e}_i;\ \boldsymbol{\theta})$	*at rate* $p_i\mu_i(x_i)$	*departure:* $(\mathbf{x}' - \theta'_i \mathbb{1}_{\{0<x'_i\}}\mathbf{e}_i;\ \boldsymbol{\theta}')$	*at rate* $p_i\mu_i(x'_i)$
i-j jump: $(\mathbf{x} - \theta_i \mathbb{1}_{\{0<x_j<N_j\}}\mathbf{e}_i + \theta_i \mathbb{1}_{\{0<x_j<N_j\}}\mathbf{e}_j;\ \boldsymbol{\theta})$	*at rate* $p_{ij}\mu_i(x_i)$	*i-j jump:* $(\mathbf{x}' - \theta'_i \mathbb{1}_{\{0<x'_j<N_j\}}\mathbf{e}_i + \theta'_i \mathbb{1}_{\{0<x'_j<N_j\}}\mathbf{e}_j;\ \boldsymbol{\theta}')$	*at rate* $p_{ij}\mu_i(x'_i)$
breakdown: $(\mathbf{x};\ \boldsymbol{\theta} - \theta_i\mathbf{e}_i)$	*at rate* δ_i	*breakdown:* $(\mathbf{x}';\ \boldsymbol{\theta}' - \theta'_i\mathbf{e}_i)$	*at rate* δ_i
repair: $(\mathbf{x};\ \boldsymbol{\theta} + (1-\theta_i)\mathbf{e}_i)$	*at rate* β_i	*repair:* $(\mathbf{x}';\ \boldsymbol{\theta}' + (1-\theta'_i)\mathbf{e}_i)$	*at rate* β_i
	for all $i, j \in \{1, ..., M\}, j \neq i$		for all $i, j \in \{1, ..., M\}, j \neq i$

Each station has a capacity constraint N_i on the total number of jobs at each service station i, thus upon an arrival instant the number of customers x_i at station i determines if the arriving job is accepted to join the queue or not. The indicator $\mathbb{1}_{\{x_i<N_i\}} = 1$ if there is place for a new arrival and $\mathbb{1}_{\{x_i<N_i\}} = 0$ if the arrival is rejected. The differences of the processes X and X' occur only in the first line of Table 5.1. In the original system the i-th unit vector is added to the number of customers at station i if $x_i < N_i$. When we consider the modified system we also need to have that station i is working (that is if $\theta_i = 1$) for an arrival to enter the system. Departures and i-j jumps only occur from station i if that station is working. Breakdowns occur at rate δ_i only if station i is working (that is exactly as $\theta_i = 1$). Repairs take place at rate β, only if the system is down, that is when $\theta_i = 1$ or, equivalently if $(1 - \theta_i) = 0$.

The goal is to compare the original system X with the (easier) second system X' which is a modification of the original system. Common sense says that the modified system gives a lower bound on the number of customers in the system, simply because less customers are accepted to the network. Or, in process X' the *loss rate* is higher.

In the next section we consider the model described above for the single-server case and show that by choosing the right state space description we can use coupling arguments to compare the two processes. In Section 5.1.3 we generalize this coupling to the case with an arbitrary number of servers.

5.2 Single-server queue

To get more intuition on what is happening in both systems, and how to construct a coupling, we consider both Jackson networks with just one service station ($M = 1$). For simplicity we assume the service rate to be constant whenever there are customers at the server; $\mu(n) = \mu$ for $1 \leq n \leq N$. A graphical representation of processes X and X' is given in Figures 5.1 and 5.2, where the maximum number of customers in the system is $N = 4$.

$$\begin{tikzcd}
\theta = 1 \ \textit{(up)} & (0,1) \arrow[r, shift left, "\lambda"] \arrow[d, shift left, "\delta"] & (1,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] \arrow[d, shift left, "\delta"] & (2,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] \arrow[d, shift left, "\delta"] & (3,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] \arrow[d, shift left, "\delta"] & (4,1) \arrow[l, shift left, "\mu"] \arrow[d, shift left, "\delta"] & \text{Network } \mathbf{X} \\
\theta = 0 \ \textit{(down)} & (0,0) \arrow[r, "\lambda"] \arrow[u, shift left, "\beta"] & (1,0) \arrow[r, "\lambda"] \arrow[u, shift left, "\beta"] & (2,0) \arrow[r, "\lambda"] \arrow[u, shift left, "\beta"] & (3,0) \arrow[r, "\lambda"] \arrow[u, shift left, "\beta"] & (4,0) \arrow[u, shift left, "\beta"] &
\end{tikzcd}$$

Figure 5.1: Original Jackson network for $M = 1$ and $N = 4$.

$$\begin{tikzcd}
\theta = 1 \ \textit{(up)} & (0,1) \arrow[r, shift left, "\lambda"] \arrow[d, shift left, "\delta"] & (1,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] \arrow[d, shift left, "\delta"] & (2,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] \arrow[d, shift left, "\delta"] & (3,1) \arrow[l, shift left, "\mu"] \arrow[r, shift left, "\lambda"] \arrow[d, shift left, "\delta"] & (4,1) \arrow[l, shift left, "\mu"] \arrow[d, shift left, "\delta"] & \text{Network } \mathbf{X'} \\
\theta = 0 \ \textit{(down)} & (0,0) \arrow[u, shift left, "\beta"] & (1,0) \arrow[u, shift left, "\beta"] & (2,0) \arrow[u, shift left, "\beta"] & (3,0) \arrow[u, shift left, "\beta"] & (4,0) \arrow[u, shift left, "\beta"] &
\end{tikzcd}$$

Figure 5.2: Modified Jackson network for $M = 1$ and $N = 4$.

Studying these figures gives us the idea that coupling techniques might work to prove not only that the adapted model gives a lower bound for the steady-state mean loss rate (as is done in [van Dijk, 1998]), but we can even get a stronger result, namely that in steady state the number of customers in X stochastically dominates the number of customers in system X':

$$\mathbb{P}(X \leq n) \leq \mathbb{P}(X' \leq n) \text{ for all } n = 0, 1, \ldots, M.$$

Explicit rates for a coupling of X and X' are given in Table 5.2. Considering the Tables 5.1 and 5.2, it is not difficult to check that this is indeed a coupling of X and X'.

Table 5.2: A coupling of the single-server Jackson networks

Coupling $(\hat{X}, \hat{X}')$

$((x,\ \theta),(x',\ \theta'))$	*arrival:* $\longmapsto ((x+\mathbb{1}_{\{x<N\}},\ \theta),(x'+\theta'\mathbb{1}_{\{x'<N\}},\ \theta'))$	*at rate* λ
$((x,\ \theta),(x',\ \theta'))$	*departure:* $\longmapsto ((x-\theta\mathbb{1}_{\{0<x\}},\ \theta),(x'-\theta'\mathbb{1}_{\{0<x'\}},\ \theta'))$	*at rate* μ
$((x,\ \theta),(x',\ \theta'))$	*breakdown:* $\longmapsto ((x,0),(x',0))$	*at rate* δ
$((x,\ \theta),(x',\ \theta'))$	*repair:* $\longmapsto ((x,1),(x',1))$	*at rate* β

What we want to achieve is that in steady-state the number of customers in system X is bigger than the number of customers in the modified system X', hence, at start we are interested in the relation

$$R^{coord} = \{((x,\ \theta),(x',\ \theta')) : x \geq x'\}. \tag{5.1}$$

The coupling given in Table 5.2 however does not preserve this relation. This can easily be shown by considering again the Figures 5.1 and 5.2. Namely, let X be in state $(3,1)$ and X' in state $(3,0)$ at some point in time. Then $((3,1),(3,0)) \in R^{coord}$, but with a positive rate of $\mu > 0$ the process X goes to state $(2,1)$ while process X' remains in state $(3,0)$. But $((2,1),(3,0)) \notin R^{coord}$, and this shows that R^{coord} is not invariant for the coupling .

For this reason, the state space was chosen such that also θ is included and the coupling in Table 5.2 preserves actually relation $R^{breakdowns}$, which is a *subrelation* of R^{coord}. The subrelation $R^{breakdowns}$ is invariant for the coupling in Table 5.2, where

$$R^{breakdowns} = \{((x,\ \theta),(x',\ \theta')) : x \geq x' \text{ and } \theta = \theta'\}. \tag{5.2}$$

Or in other words, in steady-state, the processes X and X' are stochastically related with respect to the relation $R^{breakdowns}$. We have to remark here that for the elements $(x,\ \theta)$ and $(x',\ \theta')$ to be related, the equality of the breakdown-indicator ($\theta = \theta'$) is crucial.

Again, this is not difficult to check. By looking at Table 5.2 we see that there is no positive rate for which an element in $R^{breakdown}$ can jump to an element not in $R^{breakdown}$.

Theorem 5.1. *Let X and X' be single-server Jackson networks which are subject to breakdowns, as described in the beginning of this section. Then, in steady-state:*

$$\mathbb{P}(X \leq n) \leq \mathbb{P}(X' \leq n) \text{ for all } n = 0,1,\ldots,M.$$

Proof. In Table 5.2, a coupling $(\hat{X}, \hat{X}')$ of X and X' is given for which $R^{breakdown}$ is invariant. Hence,

$$\mathbb{P}((\hat{X}, \hat{X}') \in R^{breakdown}) = 1,$$

and because $R^{breakdowns} \subseteq R^{coord}$,

$$\mathbb{P}((\hat{X}, \hat{X}') \in R^{coord}) = 1.$$

Hence, $\mathbb{P}(\hat{X}$ has less customers than $\hat{X}'$ at every station$) = 1$. With this coupling, we showed that the steady-state distributions of X and X' are stochastically related with respect to R^{coord}, and because the coupling couples the distributions of X and X', we conclude that indeed

$$\mathbb{P}(X \leq n) \leq \mathbb{P}(X' \leq n) \text{ for all } n = 0, 1, \ldots, M.$$

□

A consequence of Theorem 5.1 is that the mean loss rate in the original model is less than or equal to the mean loss rate in the modified system, which is also shown in [van Dijk, 1998].

5.3 Coupling of the Jackson network example

We extend the coupling of the single-server networks of the previous section to the case where X and X' have a general number of M servers. The coupling is a Markov process living in the product space $S \times S \subseteq (\mathbb{N}^M \times \{0,1\}^M) \times (\mathbb{N}^M \times \{0,1\}^M)$. We denote an element of the product set by $((\mathbf{x}, \boldsymbol{\theta}), (\mathbf{x}', \boldsymbol{\theta}'))$. First, we show that the coupling given in Table 5.3 coupled indeed the processes X and X'. Second, we show that this coupling preserves a subrelation of the coordinate-wise relation. This gives us the result which states that the steady-state distributions of X and X' are stochastically related with respect to the coordinate-wise relation.

Lemma 5.2. *The process $(\hat{X}, \hat{X}')$ in Table 5.3 couples the processes X and X'.*

Proof. To show that $(\hat{X}, \hat{X}')$ is indeed a coupling of X and X', we have to verify that the marginal transition rates of $(\hat{X}, \hat{X}')$ match with the transition rates of X and X'. This is easily verifiable. Furthermore all the transition rates are non-negative. The transitions map $S \times S$ into $S \times S$. Theorem 2.10 implies that $(\hat{X}, \hat{X}')$ is a coupling of X and X'.

□

Note that in this coupling the initial states of $\boldsymbol{\theta}$ can be different. But if at some moment in time $\theta_i = \theta_i'$, then from that moment on they will remain equal. This holds for each $i = 1, ..., M$. For each i the set $\{((\mathbf{x}, \boldsymbol{\theta}), (\mathbf{x}', \boldsymbol{\theta}')) : \theta_i = \theta_i'\}$ is invariant for the coupling of Table 5.3.

Define the following relation on $S \times S$:

$$R^{breakdowns} := \{((\mathbf{x},\ \boldsymbol{\theta}), (\mathbf{x}',\ \boldsymbol{\theta}')) : x_i \geq x_i' \text{ and } \theta_i = \theta_i' \text{ for all } i = 1, \ldots, M.\} \tag{5.3}$$

For all $(\mathbf{x}, \boldsymbol{\theta})$ and $(\mathbf{x}', \boldsymbol{\theta}')$ with $(\mathbf{x}, \boldsymbol{\theta}) \sim (\mathbf{x}', \boldsymbol{\theta}')$, the coupling preserves relation $R^{breakdowns}$.

Lemma 5.3. *Relation $R^{breakdowns}$ is invariant for the coupling of Table 5.3.*

Table 5.3: A coupling of the multiple-server Jackson networks.

$$\frac{\text{Coupling } (\hat{X}, \hat{X}') \text{ of } X \text{ and } X'}{\Big((\mathbf{x},\boldsymbol{\theta}),(\mathbf{x}',\boldsymbol{\theta}')\Big) \longmapsto \ \ldots}$$

arrival:

$$\Big((\mathbf{x} + \mathbb{1}_{\{x_i<N_i\}}\mathbf{e}_i, \boldsymbol{\theta})\ ,\ (\mathbf{x}' + \theta_i'\mathbb{1}_{\{x_i'<N_i\}}\mathbf{e}_i, \boldsymbol{\theta}')\Big) \qquad \textit{at rate} \quad \lambda_i$$

departure:

$$\Big((\mathbf{x} - \theta_i\mathbb{1}_{\{0<x_1\}}\mathbf{e}_i, \boldsymbol{\theta})\ ,\ (\mathbf{x}' - \theta_i'\mathbb{1}_{\{0<x_1'\}}\mathbf{e}_i, \boldsymbol{\theta}')\Big) \qquad \textit{at rate} \quad p_i \cdot \min\{\mu_i(x_i), \mu_i(x_i')\}$$

$$\Big((\mathbf{x} - \theta_i\mathbb{1}_{\{0<x_1\}}\mathbf{e}_i, \boldsymbol{\theta})\ ,\ (\mathbf{x}', \boldsymbol{\theta}')\Big) \qquad \textit{at rate} \quad p_i \cdot (\mu_i(x_i) - \mu_i(x_i'))^+$$

$$\Big((\mathbf{x}, \boldsymbol{\theta})\ ,\ (\mathbf{n}' - \theta_i'\mathbb{1}_{\{0<x_1'\}}\mathbf{e}_i, \boldsymbol{\theta}')\Big) \qquad \textit{at rate} \quad p_i \cdot (\mu_i(x_i') - \mu_i(x_i))^+$$

i-j jump:

$$\Big((\mathbf{x} - \theta_i\mathbb{1}_{\{0<x_j<N_j\}}\mathbf{e}_i + \theta_i\mathbb{1}_{\{0<x_j<N_j\}}\mathbf{e}_j;\ \boldsymbol{\theta})\ ,\ (\mathbf{x}' - \theta_i'\mathbb{1}_{\{0<x_j'<N_j\}}\mathbf{e}_i + \theta_i'\mathbb{1}_{\{0<x_j'<N_j\}}\mathbf{e}_j;\ \boldsymbol{\theta}')\Big)$$
$$\textit{at rate} \quad p_{ij} \cdot \min\{\mu_i(x_i), \mu_i(x_i')\}$$

$$\Big((\mathbf{x} - \theta_i\mathbb{1}_{\{0<x_j<N_j\}}\mathbf{e}_i + \theta_i\mathbb{1}_{\{0<x_j<N_j\}}\mathbf{e}_j;\ \boldsymbol{\theta})\ ,\ (\mathbf{x}', \boldsymbol{\theta}')\Big) \qquad \textit{at rate} \quad p_{ij} \cdot (\mu_i(x_i) - \mu_i(x_i'))^+$$

$$\Big((\mathbf{x}, \boldsymbol{\theta})\ ,\ (\mathbf{x}' - \theta_i'\mathbb{1}_{\{0<x_j'<N_j\}}\mathbf{e}_i + \theta_i'\mathbb{1}_{\{0<x_j'<N_j\}}\mathbf{e}_j;\ \boldsymbol{\theta}') \qquad \textit{at rate} \quad p_{ij} \cdot (\mu_i(x_i') - \mu_i(x_i))^+$$

breakdown:

$$\Big((\mathbf{x}, \boldsymbol{\theta} - \theta_i\mathbf{e}_i)\ ,\ (\mathbf{x}', \boldsymbol{\theta}' - \theta_i'\mathbf{e}_i)\Big) \qquad \textit{at rate} \quad \delta_i$$

repair:

$$\Big((\mathbf{x}, \boldsymbol{\theta} + (1-\theta_i)\mathbf{e}_i)\ ,\ (\mathbf{x}', \boldsymbol{\theta}' + (1-\theta_i')\mathbf{e}_i)\Big) \qquad \textit{at rate} \quad \beta_i$$

for all $i, j \in \{1, ..., M\}, j \neq i$

Where we use the notation $(\mu)^+ := \max\{\mu, 0\}$ *for* $\mu \in \mathbb{R}$ *to describe the departure and jump rates.*

Proof. When $((\mathbf{x},\ \boldsymbol{\theta}), (\mathbf{x}',\ \boldsymbol{\theta}')) \in R^{breakdowns}$ we write

$$(\mathbf{x}, \boldsymbol{\theta}) \sim (\mathbf{x}', \boldsymbol{\theta}') \Leftrightarrow \begin{cases} x_i \geq x_i' & \text{for all } i \\ \theta_i = \theta_i' & \text{for all } i. \end{cases} \tag{5.4}$$

Suppose that $R^{breakdowns}$ is *not* invariant for the coupling. Then there must exist two states A and B in the product space such that A is in the relation but B is not, and there is a positive rate to jump from state A to B. Write $A = ((\mathbf{x}_A, \boldsymbol{\theta}_A), (\mathbf{x}_A', \boldsymbol{\theta}_A'))$ and $B = ((\mathbf{x}_B, \boldsymbol{\theta}_B), (\mathbf{x}_B', \boldsymbol{\theta}_B'))$. State $A \in R^{breakdowns}$, hence $x_{Ai} \geq x_{Ai}'$ and $\theta_{Ai} = \theta_{Ai}'$ for all $i = 1, \ldots, M$. But, for state B there is an i such that (1) $\theta_{Bi} \neq \theta_{Bi}'$ or (2) $x_{Bi} < x_{Bi}'$.

$$((\mathbf{x}_A, \boldsymbol{\theta}_A), (\mathbf{x}_A', \boldsymbol{\theta}_A')) \longmapsto ((\mathbf{x}_B, \boldsymbol{\theta}_B), (\mathbf{x}_B', \boldsymbol{\theta}_B')).$$

(1). Suppose there is an i such that $\theta_{Bi} \neq \theta_{Bi}'$. Then the jump out of the relation must be caused either by a breakdown or by a repair of station i. When a breakdown of station i occurs and the stations were down ($\theta_{Ai} = \theta_{Ai}' = 0$), in both systems nothing happens. When $\theta_{Ai} = \theta_{Ai}' = 1$, both systems go down at rate δ_i. Thus then $\theta_{Bi} = \theta_{Bi}' = 0$. Similarly, if a repair occurs when the servers were down in both systems this will cause a repair, and if they were already up nothing happens. Therefore, θ_{Bi} remains always equal to θ'_{Bi} with probability one, and $\theta_i := \theta_{Ai} = \theta'_{Ai} = \theta_{Bi} = \theta'_{Bi}$ is true.

(2). Suppose $x_{Bi} < x'_{Bi}$ for a certain i. If $x_{Ai} \geq x'_{Ai}$ and in one jump we go to a state where $x_{Bi} < x'_{Bi}$, we must have $x_{Ai} = x'_{Ai}$. There are two possibilities. Either (i) a station-i-departure occurs in X but not in X',

$$x_{Ai} \longmapsto x_{Bi} = x_{Ai} - 1 \text{ and } x'_{Ai} \longmapsto x'_{Bi} = x'_{Ai},$$

or (ii) a station-i-arrival occurs in X' but not in X

$$x_{Ai} \longmapsto x_{Bi} = x_{Ai} \text{ and } x'_{Ai} \longmapsto x'_{Bi} = x'_{Ai} + 1.$$

(i) The event $x_{Ai} \longmapsto x_{Bi} = x_{Ai} - 1$ and $x'_{Ai} \longmapsto x'_{Bi} = x'_{Ai}$ can happen by a departure or by a jump of a customer from a server j to i.

Departure
Observe that $\theta_{Ai} = \theta'_{Ai} = \theta_i$, therefore, a departure can only cause this jump if $p_i \cdot (\mu_i(x_{Ai}) - \mu_i(x'_{Ai}))^+$ is positive. But $x_{Ai} = x'_{Ai}$, thus this rate is zero.

i-j jump
A jump from i to j occurs only if $\theta_{Ai} = \theta'_{Ai} = \theta_i = 1$ and if $x_{Ai} = x'_{Ai} < N_i$. An i-j jump happens in X and not in X' only if $p_{ij} \cdot (\mu_i(x_{Ai}) - \mu_i(x'_{Ai}))^+$ is positive. This is not the case because $\mu_i(x_{Ai}) = \mu_i(x'_{Ai})$. We conclude that this possibility can not happen.

(ii) Suppose $x_{Ai} \longmapsto x_{Bi} = x_{Ai}$ and $x'_{Ai} \longmapsto x'_{Bi} = x'_{Ai} + 1$. This jump is possible when an arrival occurs (at rate λ_i) or by a jump from a station j to i.

Arrival
We know that $x_{Ai} = x'_{Ai}$ so $\mathbb{1}_{\{x_{Ai} < N_i\}} = \mathbb{1}_{\{x'_{Ai} < N_i\}}$. For an arrival this means that an arrival in X' always goes together with an arrival in X.

j-i jump
For an j-i jump, $\theta_{Aj} = \theta'_{Aj} = 1$ (otherwise the customer can never leave station j). Such a jump from j to i in X' and not in X can only happen if the rate $p_{ji} \cdot (\mu_j(x_{Aj}) - \mu_j(x'_{Aj}))^+$ is positive. We know that $x_{Aj} \geq x'_{Aj}$, hence, if $\mu_j(x_j)$ is non-decreasing in x_j for every j this rate is zero.

We conclude that a jump out of the relation can never happen and thus that the relation is absorbing in the coupling of Table 5.3.

□

Combining Lemma's 5.2 and 5.3, we can state the following theorem.

Theorem 5.4. *Let X and X' be two M-station Jackson networks where the departure channels are subject to breakdowns, as described in Section 5.1. In the steady-state, process X is coordinate-wise stochastically bigger than X':*

$$\mathbb{P}(X_i \geq s_i, \textit{ for all } i) \geq \mathbb{P}(X_i \geq s_i, \textit{ for all } i) \textit{ for all } \mathbf{s} \in \mathbb{R}^M.$$

Proof. By Lemma's 5.2 and 5.3, there exists a coupling $(\hat{X}, \hat{X}')$ of X and X' such that, the relation $R^{breakdowns}$ is invariant for this coupling. Theorem 3.19 ensures that $X \sim_{st} X'$ with respect to R^{coord}. In steady state

$$\mathbb{P}((\hat{X}, \hat{X}') \in R^{breakdowns}) = 1.$$

Since $R^{breakdowns}$ is a subrelation if R^{coord},

$$\mathbb{P}((\hat{X}, \hat{X}') \in R^{coord}) = 1.$$

Hence, in the steady-state the processes X and X' are stochastically related with respect to the relation R^{coord}:

$$\mathbb{P}(X_i \geq s_i, \text{ for all } i) \geq \mathbb{P}(X_i \geq s_i, \text{ for all } i) \text{ for all } \mathbf{s} \in \mathbb{R}^M.$$

$\square$

Chapter 6

Conclusion

Stochastic comparison is a strong comparison method which does not only compare means and expectations of random elements, but compares the distributions of the stochastic elements under consideration. The recently introduced stochastic comparison with respect to relations is more generally applicable than stochastic ordering. It allows us to compare processes on different state spaces for non-trivial relations on the product space. With the help of coupling arguments, strong results on the comparison of processes can be obtained.

In this thesis, we have studied the stochastic comparison of Markov queueing networks. An introduction on couplings of random elements and stochastic processes was given in Chapter 2. Definitions of the coupling of transition (rate) matrices are presented and we proved that a coupling of transition (rate) matrices of Markov processes is equal to the transition (rate) matrix of a coupling of the processes, in both discrete and continuous-time. Chapter 3 gave an introduction on stochastic comparability. We have shown that comparability of processes in ordered spaces is naturally extended to stochastic comparability with respect to relations. In Chapter 4 we studied a basic Markov queueing network and presented a new necessary and sufficient conditions for the stochastic comparison of two of those Markovian queueing networks, for the coordinate wise order relation and the summation relation. Theorems 4.4 and 4.8 summarize these results, and we have proved that the conditions of 4.8 are indeed equivalent to an alternative characterization presented in [Delgado et al., 2004]. Although the theorem in [Delgado et al., 2004] would be more useful for numerical computation, Theorem 4.4 gives analytically simpler conditions. In Chapter 5, we have studied an example of two Jackson networks, where the servers are subject to breakdowns, and we have illustrated how strong comparison results follow from a coupling which preserves some subrelation.

Altogether, we have presented a thorough overview of the theory of stochastic comparison with respect to relations. We have shown how to derive conditions, on the transition rates of Markov queueing networks, which ensure that these networks are stochastically comparable with respect to a certain relation. Furthermore, we have shown how constructions of explicit couplings can be used to prove strong comparison results in an intuitive way.

Discussion and further research

I studied several examples which are not discussed in this thesis. These are examples where it is not (yet) known whether or not stochastic comparison can be proved. In [van Dijk, 1998], queueing networks are presented for which mean value analysis can be used to derive bounds on certain system quantities, and it is claimed that stochastic comparison using order-relations can not be proved in these examples. However, no proof is given and thus, this remains an open question. It may be possible that there exists a choice of state space and subrelation for which the coupling method works. Although I tried to construct these, I did not manage to find such couplings. There is not yet a characterization of where the coupling method fails. Further investigation can be made to explore the boundaries of stochastic comparison.

There are two important directions for extending the theory of stochastic comparison. First, it would be useful to study more complex queueing networks. Secondly, stochastic comparison using relations gives the possibility of comparing networks that are highly different of each other. It is interesting to further explore this.

There are many extensions of queueing networks that could be considered. For example, we can add batch arrivals, priority classes of customers or server-sharing models. We can also think of adding more dependencies in the network models, such as state-dependent transition probabilities. In this work, we assumed that the service requirements and inter-arrival times are independent and exponentially distributed. We can think of relaxed constraints or simply other distributions. All of these extensions are useful for modeling real networks in various applications.

When working with relations, instead of orders, we are not restricted to have equal state spaces in both models. This invites us to think in a much broader way of comparable models to provide bounds on the original network. This seems especially promising for high-dimensional networks. A bounding model with simplified state space can be constructed, for example, by grouping sets of servers. Furthermore, infinite-dimensional networks can be bounded by finite-dimensional models. It is, however, not directly obvious which merging and bounding operations can be used while ensuring stochastic comparability.

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Index