Master's thesis

The hypermultiplet moduli space
of compactified type IIA string theory

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Abstract

We have looked at the hypermultiplet moduli space of the effective supergravity theory obtained from type IIA superstring theory compactified on a Calabi-Yau manifold. It is possible to give this space hypermultiplet moduli space a more intrinsic description as a fibre bundle over the moduli space of deformations of the Calabi-Yau manifold used in the compactification procedure. The fibres may can be interpreted as the symplectised spaces for a compact quotient of the Heisenberg group constructed from the (Weil) intermediate Jacobian of the Calabi-Yau manifold and viewed as a contact manifold. We use the complex structure on the Weil intermediate Jacobian to define a Sasakian (contact metric) structure on this Heisenberg group that can be extended to a metric on the fibres.

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Introduction

One of the most important problems in modern physics is the apparent incompatibility between quantum field theory and general relativity. General relativity provides a very elegant description of gravity that is not only consistent with both Newton's law of gravitation and special relativity in the appropriate limits, but also explains numerous observations of the universe at large scales. Meanwhile, physics at the subatomic level is governed by quantum mechanics. Quantum field theory, and in particular the standard model succeeds at describing the known elementary particles and fundamental forces except gravity (electromagnetism and the weak and strong nuclear forces). While it is not as elegant as general relativity, the standard model is consistent with nearly all experimental observations in particle physics and often describes measurements with great accuracy. Although both theories have been hugely successful in their own domains, attempts to combine the two into a unified quantum theory of gravity have generally proven unsuccessful.

One of the more promising candidates for such a theory is string theory. The principal idea behind string is that the fundamental objects in physics are one dimensional objects, which we call strings, rather than point particles. This simple idea is used to construct an enormously complicated quantum mechanical string theory in which elementary particles correspond to the modes of oscillation of a single type of fundamental string. Amazingly enough, gravity appears naturally in string theory and in a way that is consistent with general relativity at low energies.

An interesting property of string theory is that it puts a restriction on the number of dimensions of space-time, as it can only be formulated consistently for a very specific critical dimension. Although this property could in itself be considered an advantage over theories that say nothing about the dimension of space-time, the fact that this critical dimension equals ten does conflict with the prevailing idea that space-time should in fact be four-dimensional (three spatial dimensions and one time dimension).

One way to get around this discrepancy is through compactification. Instead of considering space-times that extend to infinity in all directions, one considers space-times that are small in six dimensions. This is done by wrapping these unwanted dimensions around a (compact) internal space that is assumed to be so small that we are unable to observe it with current techniques. The resulting effective theory in four dimensions is largely dependent on the geometry of the internal space used in the compactification procedure, which introduces a great amount of freedom to the theory that it did not have before. Calabi-Yau manifolds are of particular interest as the internal spaces used for the compactification because in the low-energy limit these produce four dimensional theories that not only contain gravity, but are also supersymmetric, which is considered desirable. For type IIA, with which this thesis is concerned, this effective low-energy theory is called type IIA supergavity.

Apart from a sector corresponding to gravity, this effective four-dimensional theory contains a number of other fields that split into vector multiplet and hypermultiplets. Because vector multiplets and hypermultiplets do not interact with each other except through gravity, the vector multiplet and hypermultiplet sector may be studied separately. The hypermultiplet fields are described by a non-linear sigma model and parametrise a space, called the hypermultiplet moduli space, that comes with a metric that is known to define a quaternion-Kähler structure. Although we have an explicit description of this metric and this quaternion-Kähler structure, we would like to be able to understand it in terms of the geometry of the Calabi-Yau manifold used in the compactification procedure.

This thesis is the results of the research I have done under the supervision of dr. S.J.G. Vandoren and dr. J. Stienstra to conclude my Masters in Theoretical Physics and Mathematical Sciences. The starting point for the research presented in this document was the point where T.A.F. van der Aalst [1] left off, which is with a description of the hypermultiplet moduli space as a fibre bundle over the complex structure moduli space of the Calabi-Yau manifold with fibres that should be interpreted as the total space of a $\mathbb{C}^*$-bundle over the Weil intermediate Jacobian of the Calabi-Yau. By making use of isometries of the metric, these fibres were given an interpretation as a coset space of an extended version of the Heisenberg group, which was appropriately named the *dilated* Heisenberg group.

We have taken one one step back from this approach and have used the (unextended) Heisenberg group instead of the dilated version to describe these fibres. By equipping this Heisenberg group with a natural contact structure and using the structure of the Weil intermediate Jacobian to extend it to a strictly pseudoconvex Cauchy-Riemann structure we have found a nearly complete description of the fibre metric through a 1-dimensional family of (Sasakian) contact metric structures. The complete expression is obtained by symplectising this structure and extending these contact metric structures to a Kähler structure on the fibres. Finally, we have made a start with a description of the quaternion-Kähler structure on the total space in terms of this construction.

The first chapter deals with some standard definitions and results from mathematics, knowledge of which will be required to be able to understand the chapters that follow it. Chapter 2 discusses the definition of a Calabi-Yau manifolds, the structures on their moduli space and their intermediate Jacobians. In the third chapter an overview of the physical context in which the obtained results should be viewed is provided and the hypermultiplet moduli space is introduced. In chapter 4 contact geometry and CR geometry are discussed and applied to the Heisenberg group that we have used to construct the hypermultiplet moduli space in chapter 5. In the final chapter, we try to give a more intrinsic interpretation to the explicit description of the quaternion-Kähler structure on the hypermultiplet moduli space that was found by Ferrara and Sabharwal [2].

1. Some preliminary mathematics

In this chapter we will introduce some of the mathematical concepts that are used throughout this document. We will discuss some of the most important definitions and results for complex and Kähler manifolds and introduce cohomology groups for these spaces. Some important definitions and results for non-degenerate Complex tori will furthermore be discussed. Everything in this chapter can be found in a variety of standard textbooks, such as [3,4], [5,6] and [7,8,9]. The reader is assumed to know some of the basics of differential and Riemannian geometry.

1.1 Summation convention

We will very often make use of ***Einstein summation convention***, or index notation. This notational convention saves space and is very common in theoretical physics, but not as common among mathematicians. Einstein summation convention states that any unspecified index that appears twice in an expression should be summed over (contracted). For instance, if the indices $i, j, k, \ldots$ are understood to run from 1 to n, then an equation such as

$$\mathrm{d}y_i + \mathcal{Z}_{ij}\mathrm{d}x^j = \mathrm{d}y_i + \left(F_{ij} - i\frac{N_{ik}\bar{X}^k\bar{X}^\ell N_{\ell j}}{\bar{X}^m N_{mn}\bar{X}^n}\right)\mathrm{d}x^j \tag{1.1}$$

should be read as

$$\mathrm{d}y_i + \sum_{j=1}^{n}\mathcal{Z}_{ij}\mathrm{d}x^j = y_i + \sum_{j=1}^{n}\left(F_{ij} - i\frac{\sum_{k,\ell=1}^{n}N_{ik}\bar{X}^k\bar{X}^\ell N_{\ell j}}{\sum_{m,n=1}^{n}\bar{X}^m N_{mn}\bar{X}^n}\right)\mathrm{d}x^j. \tag{1.2}$$

Moreover, because the index i was also left unspecified, this equation is understood to hold for any $i \in \{1, \ldots, n\}$. When there is a distinction between upper indices and lower indices, an expression with contracted indices will in general only be meaningful if all repeated indices come in pairs of one upper index and one lower index.

1.2 Homology and cohomology

1.2.1 Singular (co)homology

On any topological space X we can define singular homology and cohomology groups using formal sums of so-called singular simplices.

The standard k-simplex Δ^k is defined as the convex set in $\mathbb{R}^{n+1}$ generated by the basis vectors $e_0, e_1, \ldots, e_n$, where $e_0 = (1, 0, \ldots, 0)$, $e_1 = (0, 1, 0, \ldots, 0)$ etc. A ***singular k-simplex*** on X is a continuous map $\sigma\colon \Delta^k \to X$ and we call a (finite) formal sum $\sum_\sigma n_\sigma\,\sigma$ of k-simplices with $n_\sigma \in \mathbb{Z}$ a ***singular k-chain***. The set of all such k-chains is a free group, generated by the k-simplices, that we denote by $C_k(X, \mathbb{Z})$. Instead of using integer coefficients, we can write down chains with coefficients in any Abelian ring R and write

$$C_k(X,R) = \left\{ \sum_\sigma r_\sigma\, \sigma \,\middle|\, \sigma\colon \Delta^k \to X,\ r_\sigma \in R \right\} \simeq C_k(X,\mathbb{Z}) \otimes R. \tag{1.3}$$

On this group we have a boundary map $\partial\colon C_k(X,R) \to C_{k-1}(X,R)$, which sends a singular k-simplex to the sum of its faces with appropriate signs. More explicitly, for a generator $\sigma\colon \Delta^k \to X$ we get $\partial\sigma = \sum_{i=0}^k (-1)^i \partial^i \sigma$ with

$$\partial^i \sigma(t_0, \ldots, t_{k-1}) = \sigma(t_0, \ldots, t_{i-1}, 0, t_i, \ldots, t_{k-1}) \tag{1.4}$$

A k-chain $\sigma \in C_k(X,R)$ for which $\partial\sigma = 0$ is called a (singular) k-***cycle*** and it is called a ***boundary*** if $\sigma = \partial\tau$ for some chain $\tau \in C_{k+1}(X,R)$, $\ker\big(\partial\colon C_k(X) \to C_{k-1}(X)\big)$ and $\mathrm{im}\big(\partial\colon C_{k+1}(X) \to C_k(X)\big)$ are the spaces of k-cycles and k-boundaries respectively. Because the boundary operator ∂ satisfies $\partial^2 = 0$ any boundary is also a cycle, i.e.

$$\mathrm{im}\big(\partial\colon C_{k+1}(X) \to C_k(X)\big) < \ker\big(\partial\colon C_k(X) \to C_{k-1}(X)\big), \tag{1.5}$$

which enables us to define singular homology groups.

Definition 1.2.1 **(Singular homology).** *Let X be a topological space, then its k-th* **singular homology group** *with coefficients in the Abelian ring R (e.g. $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$) is defined as*

$$H_k(M,R) = \frac{\ker\big(\partial\colon C_k(X,R) \to C_{k-1}(X,R)\big)}{\mathrm{im}\big(\partial\colon C_{k+1}(X,R) \to C_k(X,R)\big)} \tag{1.6}$$

The chain groups $C_k(X,R)$ can be dualised to obtain the cochain groups $C^k(X,R) = \mathrm{Hom}(C_k(X,\mathbb{Z}),R)$, which come with a coboundary map $\delta = \partial^*\colon C^k(X,R) \to C^{k+1}(X,R)$ defined by $\delta\alpha = \alpha \circ \partial$. Since this coboundary map also satisfies $\delta^2 = 0$ we obtain a definition for singular *co*homology groups that is analogous to that of the singular homology groups.

Definition 1.2.2 **(Singular cohomology).** *Let X be a topological space, then its k-th* **singular cohomology group** *with coefficients in the Abelian ring R (e.g. $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$) is defined as*

$$H^k(M,R) = \frac{\ker\big(\delta\colon C^k(X,R) \to C^{k+1}(X,R)\big)}{\mathrm{im}\big(\partial\colon C^{k-1}(X,R) \to C^k(X,R)\big)} \tag{1.7}$$

The universal coefficient theorems for homology and cohomology tell us that the homology groups and cohomology groups with coefficients in any ring R are completely determined by the groups $H_k(X,\mathbb{Z})$ and $H^k(X,\mathbb{Z})$. A particular consequence of this theorem is the following result.

Proposition 1.2.3. *Let X be a topological space and let K be either $\mathbb{R}$ or $\mathbb{C}$, then we have natural identifications*

$$H_k(X,K) \simeq H_k(X,\mathbb{Z}) \otimes K \qquad \textit{and} \qquad H^k(X,K) \simeq H^k(X,\mathbb{Z}) \otimes K \tag{1.8}$$

for any $k \in \mathbb{N}_0$.

This identification gives us canonical maps $i\colon H^k(X,\mathbb{Z}) \to H^k(X,K)$, which has the torsion subgroup $\mathrm{Tor} = \{\alpha \in H^k(X,\mathbb{Z}) \mid \exists n\colon n\,\alpha = 0\}$ as its kernel. Instead of using the full integer cohomology groups $H^k(X,\mathbb{Z})_{\mathrm{f}}$ we will generally work with the torsion free integer cohomology groups $H^k(X,\mathbb{Z})_{\mathrm{f}} := H^k(X,\mathbb{Z})/\mathrm{Tor} \simeq i(H^k(X,\mathbb{Z})) < H^k(X,\mathbb{R})$ (and analogously for homology groups).

1.2.2 De Rham cohomology

When we talk about the cohomology groups of a smooth manifold, we will mainly be interested in the De Rham cohomology groups. The real and complex De Rham cohomology groups are defined using the complexes

$$0 \to \Omega^0_K(M) \xrightarrow{\mathrm{d}} \Omega^1_K(M) \xrightarrow{\mathrm{d}} \Omega^2_K(M) \xrightarrow{\mathrm{d}} \Omega^3_K(M) \xrightarrow{\mathrm{d}} \dots \tag{1.9}$$

for $K = \mathbb{R}, \mathbb{C}$, where $\Omega^k_K(M)$ is the set of K-valued k-forms on M and d denotes the exterior derivative.

***Definition 1.2.4* (De Rham cohomology).** *Let M be a (smooth) manifold and let K be either $\mathbb{R}$ or $\mathbb{C}$, then the k-th De Rham cohomology group is defined as*

$$H^k_{\text{DR}}(M, K) = \frac{\ker\big(\mathrm{d}\colon \Omega^k_K(M) \to \Omega^{k+1}_K(M)\big)}{\operatorname{im}\big(\mathrm{d}\colon \Omega^{k-1}_K(M) \to \Omega^k_K(M)\big)}. \tag{1.10}$$

In the setting of (smooth) manifolds, any singular k-simplex $\sigma\colon \Delta^k \to M$ is homologous to a smooth k-simplex. For a smooth k-chain $\sigma = \sum_i a_i \sigma_i$ and a k-form α on M we can define the integral

$$\int_\sigma \alpha := \sum_i a_i \int_{\Delta^k} \sigma^* \alpha. \tag{1.11}$$

Stokes' theorem can subsequently be used to show that $\int_{\partial\sigma} \alpha = 0$ and $\int_\tau \mathrm{d}\beta = 0$ if α is closed and τ is a k-cycle, which means that this defines a pairing

$$\textstyle\int\colon H_k(M, \mathbb{R}) \times H^k_{\text{DR}}(M, \mathbb{R}) \to \mathbb{R}, \qquad \langle \sigma, \alpha \rangle([\sigma], [\alpha]) \mapsto \displaystyle\int_\sigma \alpha. \tag{1.12}$$

De Rham's theorem gives us an identification between the De Rham cohomology groups $H^k_{\text{DR}}(M, \mathbb{R})$ and the singular cohomology groups $H^k(M, \mathbb{R})$.

***Theorem 1.2.5* (De Rham's theorem).** *For an n-dimensional manifold, $k \in \mathbb{N}_0$ the map*

$$\Psi\colon H^k_{\text{DR}}(M, \mathbb{R}) \to H^k(M, \mathbb{R}), \qquad \alpha \mapsto \langle \bullet, \alpha \rangle \tag{1.13}$$

is an isomorphism of groups.

Proof: See [7], [3] or [6]. □

Since De Rham's theorem basically tells us that we may view the De Rham cohomology groups and the singular cohomology groups as the same spaces, we will from now on no longer use the label DR to emphasise that we are working with De Rham cohomology groups. It is important to note that De Rham's theorem in particular enables us to naturally view the torsion free integer cohomology groups $H^k(M, \mathbb{Z})_{\text{f}}$ as a subgroup of the real cohomology group $H^3(M, \mathbb{Z})$.

Another important result for the cohomology groups of manifolds is Poincaré duality, which in its most general form gives us an isomorphism between the homology groups $H^k(M, \mathbb{Z})$ and the cohomology groups $H_k(M, \mathbb{Z})$ for compact oriented manifolds. In terms of De Rham cohomology, it can be written as follows:

***Theorem 1.2.6* (Poincaré duality).** *Let M be an n-dimensional compact, oriented manifold and let $K = \mathbb{R}, \mathbb{C}$. The* **intersection pairing**

$$Q\colon H^k(M, R) \times H^{n-k}(M, R) \to \mathbb{R}, \qquad (\alpha, \beta) \mapsto \int_M \alpha \wedge \beta \tag{1.14}$$

defines an isomorphism between $H^k(M, R)$ and $\big(H^{n-k}(M, R)\big)^* \simeq H_{n-k}(M, R)$.

We end our discussion of the cohomology groups of manifolds by stating a result that will allow us to choose a symplectic basis for the middle cohomology groups of Calabi-Yau 3-folds (cf. section 2.1.1).

Proposition 1.2.7. *Let M be a compact oriented n-dimensional manifold and let $\alpha \in H^k(M,\mathbb{Z})_{\mathfrak{f}}$ be a cochain that cannot be written as $\alpha = n\,\beta$ for some $n > 1$ and $\beta \in H^k(M,\mathbb{Z})_{\mathfrak{f}}$. There exists an element $\beta \in H^{n-k}(M,\mathbb{Z})_{\mathfrak{f}}$ such that $Q(\alpha,\beta) = 1$.*

On a manifold of even dimension $2\,n$ with $n \in \mathbb{N}$ odd, the intersection form

$$Q\colon H^n(M,\mathbb{R}) \times H^n(M,\mathbb{R}) \to \mathbb{R}, \qquad (\alpha,\beta) \mapsto \int_M \alpha \wedge \beta \tag{1.15}$$

is a symplectic form on $H^n(M,\mathbb{R})$ and the proposition above can be used to obtain the following result.

Corollary 1.2.8. *Let M be a compact oriented manifold of even dimension $2\,n$ with $n \in \mathbb{N}$ an odd number. There exists a set $\alpha_1,\dots,\alpha_d,\beta^1,\dots,\beta^d$ of generators for $H^n(M,\mathbb{Z})_{\mathfrak{f}} < H^n(M,\mathbb{R})$ such that*

$$Q(\alpha_i,\alpha_j) = Q(\beta^i,\beta^j) = 0 \qquad \textit{and} \qquad Q(\alpha_i,\beta^j) = \delta_i^j. \tag{1.16}$$

These generators are said to form an (integral) **symplectic basis** *for $H^3(M,\mathbb{R})$.*

By Poincaré duality we can also find set of generators $\gamma^1,\dots,\gamma^d,\eta_1,\dots,\eta_d$ for the integer homology groups $H_3(M,\mathbb{Z})_{\mathfrak{f}}$ that is dual to the symplectic basis $(\alpha_i,\beta^i)_i$ in the sense that

$$\int_{\gamma^i} \alpha_j = -\int_{\eta_i} \beta^j = \delta_i^j \qquad \text{and} \qquad \int_{\eta_i} \alpha_j = \int_{\gamma^i} \beta^j = 0. \tag{1.17}$$

These cycles form a set of generators for $H_n(M,\mathbb{Z})_{\mathfrak{f}}$ that satisfy

$$\gamma^i \cap \gamma^j = \eta_i \cap \eta^j = 0 \qquad \text{and} \qquad \gamma^i \cap \eta_j = -\eta_j \cap \gamma^i = \delta_j^i, \tag{1.18}$$

where $\cap$ is the intersection product on $H_n(M,\mathbb{Z})$. Conversely, for any set of generators for $H_n(M,\mathbb{Z})_{\mathfrak{f}}$ a unique dual symplectic basis for $H^n(M,\mathbb{Z})_{\mathfrak{f}}$ may be found.

Any element $\alpha \in H^n(M,\mathbb{R})$ can now be written as

$$\alpha = \sum_i A^i \eta_i - B_i \beta^i = \sum_i \left(\int_{\gamma^i} \alpha\right) \eta_i - \left(\int_{\eta_i} \alpha\right) \beta^i \tag{1.19}$$

where $A^i = \int_{\gamma^i} \alpha$ and $B_i = \int_{\eta_i} \alpha$ are called the ***periods*** of α with respect to the basis $(\gamma^i,\eta_i)_i$

1.3 Complex and Kähler geometry

1.3.1 Complex geometry

Many of the spaces we will be working with will be complex manifolds, which may be viewed as an analogue of real manifolds, but with holomorphic charts. They may also be viewed as real manifolds that admit an integrable almost complex structure.

Definition 1.3.1 **(Almost complex structure).** *An endomorphism $J\colon \mathrm{T}M \to \mathrm{T}M$ on the tangent space of an even dimensional manifold is called an* **almost complex structure** *if $J^2 = -\,\mathrm{id}_{\mathrm{T}M}$. A manifold (M,J) equipped with an almost complex structure is called an almost complex manifold.*

The fact that almost complex manifolds have an even dimension is not actually part of the definition, but rather a consequence of the existence of the endomorphism J. Another consequence of this definition is that any almost complex manifold is automatically orientable. Since an almost complex structure satisfies $J^2 = -\mathrm{id}$ its complex linear extension to the complexified tangent space $\mathrm{T}_{\mathbb{C}}M = \mathrm{T}M \otimes \mathbb{C}$ splits into two bundles of eigenspaces for J that correspond to the eigenvalues $+i$ and $-i$. We have $\mathrm{T}_{\mathbb{C}}M = \mathrm{T}^{1,0}M \oplus \mathrm{T}^{0,1}M$, with

$$\mathrm{T}^{1,0} = \{X \in \mathrm{T}_{\mathbb{C}}M \mid J\,X = i\,X\} \qquad \text{and} \qquad \mathrm{T}^{0,1} = \{X \in \mathrm{T}_{\mathbb{C}}M \mid J\,X = -i\,X\}. \tag{1.20}$$

A section of $\mathrm{T}^{1,0}M$ (resp. $\mathrm{T}^{0,1}M$) is said to be a vector field of type $(1,0)$ (resp. $(0,1)$).

Similarly, a complex-valued 1-form α on M is respectively called a $(1,0)$-form or a $(0,1)$-form if $J^*\alpha := \alpha \circ J = +i\alpha$ or $J^*\alpha - i\alpha$. If we denote the set of complex-valued $(1,0)$ and $(0,1)$-forms by on M by $\Omega^{1,0}(M)$ and $\Omega^{0,1}(M)$ respectively, then the set of all complex-valued 1-forms $\Omega^1_{\mathbb{C}}(M)$ can be written as the direct sum $\Omega^1(M) = \Omega^{1,0}(M) \oplus \Omega^{0,1}(M)$. More generally, we can define the set of (p,q)-forms as

$$\Omega^{p,q}(M) = \textstyle\bigwedge^p \Omega^{1,0}(M) \wedge \bigwedge^q \Omega^{0,1}(M) \tag{1.21}$$

and we have $\Omega^k(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$

We may define a complex manifold as a manifold that admits holomorphic charts.

***Definition 1.3.2* (Complex manifold).** *A* **complex manifold** *of complex dimension n M is a real manifold of even dimension $2\,n$ that can be covered by charts $(U_i, \kappa_i)_i$ that embed $U_i \subseteq M$ in $\mathbb{C}^n$ in such a way that the transition functions $\kappa_i \circ \kappa_j^{-1}\colon \kappa_j(U_i \cap U_j) \to \mathbb{C}^n$ are holomorphic.*

The complex vector space $\mathbb{C}^n$ comes with a natural almost complex structure $J\colon \mathbb{C}^n \to \mathbb{C}^n$ defined by $J\,(z_1, \ldots, z_n) = (i\,z_1, \ldots, i\,z_n)$. If (U, κ) is a chart for a complex manifold M of (complex) dimension n, then this almost complex structure on $\mathbb{C}^n$ can be pulled back along $\kappa\colon U \to \mathbb{C}^n$ to $U \subseteq M$. For any point $x \in M$ the endomorphism $J_x\colon \mathrm{T}_xM \to \mathrm{T}_xM$ does not depend on what chart is chosen to define it exactly because the transition functions are holomorphic, so we see that any complex manifold comes with a unique induced almost complex structure. An almost complex structure that is defined in such a way is called a ***complex structure*** and it can be verified to be integrable.

***Definition 1.3.3* (Integrability).** *An almost complex structure $J\colon \mathrm{T}M \to \mathrm{T}M$ is said to be* **integrable** *if for any two vector fields X and Y of type $(1,0)$ the commutator $[X,Y]$ is again a vector field of type $(1,0)$.*

A very important theorem when dealing with complex and almost complex structures is the Newlander-Nirenberg theorem, which states that integrability is not only a necessary for an almost complex structure to be a complex structure, but also a sufficient.

***Theorem 1.3.4* (Newlander-Nirenberg).** *Let M be an even-dimensional manifold and let $J\colon \mathrm{T}M \to \mathrm{T}M$ be an almost complex structure on M, then J is a complex structure if and only if it is integrable.*

An equivalent condition for the integrability of the almost complex structure J is the vanishing of the **Nijenhuis tensor***,*

$$N(X,Y) = [X,Y] + J[J\,X,Y] + J[X,J\,Y] + [J\,X,J\,Y]. \tag{1.22}$$

Proof: The proof of the first statement is very complex and a little beyond the scope of this text. A partial proof can be found in [6]. The second statement can easily be verified. □

***Proposition/Definition 1.3.5* (Dolbeault operators).** *Let J be a complex structure on the manifold M and let $\pi^{p,q}$ denote the projection $\Omega^{p+q}_{\mathbb{C}}(M) \to \Omega^{p,q}(M)$. For any (p,q)-form $\omega^{(p,q)}$ we have $\mathrm{d}\omega^{(p,q)} = (\pi^{p+1,q} \circ \mathrm{d})\omega^{(p,q)} + (\pi^{p,q+1} \circ \mathrm{d})\omega^{(p,q)}$.*

The operators $\partial = \pi^{p+1,q} \circ \mathrm{d}$ and $\bar{\partial} = \pi^{p,q+1} \circ \mathrm{d}$ on $\Omega^{p,q}(M)$ are called the **Dolbeault operators** *and can naturally be extended to $\Omega^k_{\mathbb{C}}(M) = \bigoplus_{p+q=k} \Omega^{p,q}(M)$. We have $\partial^2 = \bar{\partial}^2 = 0$ and $\partial\bar{\partial} = -\bar{\partial}\partial$.*

The Dolbeault operators ∂ and $\bar{\partial}$ can be defined for any almost complex manifold, but only if the almost complex structure is integrable do they satisfy $\mathrm{d} = \partial + \bar{\partial}$ and $\bar{\partial}^2 = 0$ (both are in fact equivalent conditions).

A function $f\colon M \to \mathbb{C}$ is said to be holomorphic if it is holomorphic on each of the charts of M. This is the case exactly when $\mathrm{d}f$ is of type $(1,0)$, which in turn is true exactly when $\bar{\partial}f = 0$. The Dolbeault operator $\bar{\partial}$ can be used to generalise this notion to differential forms of arbitrary ranks.

Definition 1.3.6 **(Holomorphicity).** *A k-form $\omega^{(k)}$ on a complex manifold (M,J) of (complex) dimension n is said to be* **holomorphic** *if it is of (pure) type $(k,0)$ and $\bar{\partial}\omega^{(k)} = 0$.*

We denote the set of holomorphic p-forms on M by $\Omega^p(M) = \{\alpha \in \Omega^{p,0}(M) \mid \bar{\partial}\alpha = 0\}$.

A chart $\kappa\colon U \hookrightarrow \mathbb{C}^n$ for a complex manifold M provides local complex coordinates $z_1, \ldots, z_n$. The differentials $\mathrm{d}z^i$ form a local basis of $(1,0)$-forms, while their complex conjugates $\mathrm{d}\bar{z}_i$ form a basis of of $(0,1)$-forms. In general, The forms

$$\mathrm{d}z^{i(1)} \wedge \ldots \wedge \mathrm{d}z^{i(p)} \wedge \mathrm{d}\bar{z}^{j(1)} \wedge \ldots \wedge \mathrm{d}\bar{z}^{j(q)} \tag{1.23}$$

with $0 \leq i(1) < \ldots < i(p) \leq n$ and $1 \leq j(1) < \ldots < j(q) \leq n$ describe a basis for all (p,q)-forms on U_i.

We can define Dolbeault-cohomology groups $H^q(M, \Omega^p(M))$ for a complex manifold in a way that is similar to how we had defined the De Rham cohomology groups on a (real) manifold.

Definition 1.3.7 **(Dolbeault cohomology).** *For $p, q \in \mathbb{N}_0$ the Dolbeault cohomology group $H^q(M, \Omega^p(M))$ for a complex manifold M may be defined as the quotient*

$$H^q(M, \Omega^p(M)) = \frac{\ker\big(\bar{\partial}\colon \Omega^{p,q}(M) \to \Omega^{p,q+1}(M)\big)}{\mathrm{im}\big(\bar{\partial}\colon \Omega^{p,q-1}(M) \to \Omega^{p,q}(M)\big)}. \tag{1.24}$$

1.3.2 Kähler geometry

A very important notion that we will use at several stages in the next chapters is the notion of a Kähler structure. A Kähler structure combines the notion of a symplectic structure with a Hermitian metric.

Definition 1.3.8 **(Hermitian metric).** *Let (M,J) be a complex manifold. A Riemannian metric g on M is said to be* **Hermitian** *if it is compatible with J in the sense that*

$$g(JX, JY) = g(X,Y) \tag{1.25}$$

for any two vector fields X and Y on M.

Given a Hermitian metric g, the bilinear form $\omega = g(J\,\bullet, \bullet)$ is antisymmetric ($g(JX,Y) = g(Y,JX) = -g(JY,X)$) and can thus be viewed as a differential form. This form is called the ***fundamental form*** for g and it can furthermore be verified to be of type $(1,1)$. The complex-valued sesquilinear form $h = g - i\,\omega$ is often called the Hermitian metric instead its real part g.

Definition 1.3.9 **(Kähler structure).** *A* **Kähler manifold** *(M,J,g) is a complex manifold (M,J), together with a Hermitian metric g whose fundamental form $\omega = g(J\,\bullet, \bullet)$ is closed.*

The metric g is called the **Kähler metric** *and ω is said to be the* **Kähler form** *for the Kähler structure (J, g).*

Any Kähler manifold (M, J, g) locally admits a so-called ***Kähler potential***. A Kähler potential is a real function K on M such that the Kähler form for g is given by $\omega = i\,\partial\bar{\partial}K$. Any two Kähler potentials are related by a Kähler transformation, which are transformations of the form $K \mapsto K + f + \bar{f}$ for a (locally defined) holomorphic function f on M.

An equivalent way to characterise Kähler metrics uses the Levi-Civita connection. Although we will not use this characterisation for most of this text, it is useful because it shows the link with the definition for quaternion-Kähler structures, which is discussed in section 1.5.

Corollary 1.3.10. *Let M be an even dimensional manifold, g a Riemannian metric and J an almost complex structure on M such that $g(JX, JY) = g(X, Y)$ for all vector fields X and Y on M. The pair (g, J) defines a Kähler structure on M if and only if $\nabla J = 0$, where ∇ denotes the Levi-Civita connection for the metric g.*

Proof: Suppose that $\nabla J = 0$, then $J(\nabla_X Y) = \nabla_X(JY)$ for all vector fields X and Y. Flatness of the Levi-Civita connection then tells us that $[X, Y] = \nabla_X Y - \nabla_Y X$ and hence the Nijenhuis tensor becomes

$$\begin{aligned} N(X, Y) &= [X, Y] + J[JX, Y] + J[X, JY] - [JX, JY] \\ &= \nabla_X Y - \nabla_Y X + J(\nabla_{JX} Y - \nabla_Y(JX)) \\ &\qquad + J(\nabla_X(JY) - \nabla_{JY} X) - \nabla_{JX}(JY) + \nabla_{JY}(JX) \\ &= \nabla_X Y - \nabla_Y X + \nabla_{JX}(JY) + \nabla_Y X \\ &\qquad - \nabla_X Y - \nabla_{JY}(JX) - \nabla_{JX}(JY) + \nabla_{JY}(JX) = 0. \end{aligned} \tag{1.26}$$

Theorem 1.3.4 thus tells us that J is integrable.

Since $Z(g(X, Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$ by definition of the Levi-Civita connection, $\nabla_Z(JX) = J(\nabla_Z X)$ and $\omega = g \circ (J \times \mathrm{id})$ we have that

$$Z(\omega(X, Y)) = g(J\,\nabla_Z X, Y) + g(JX, \nabla_Z Y) = \omega(\nabla_Z X, Y) + \omega(X, \nabla_Z Y) \tag{1.27}$$

for any three vector fields X, Y and Z on M. The exterior derivative of the fundamental form, $\mathrm{d}\omega$, can be expressed through

$$\begin{aligned} (\mathrm{d}\omega)(X, Y, Z) &= X(\omega(Y, Z)) + Y(\omega(Z, X)) + Z(\omega(X, Y)) \\ &\qquad - \omega([X, Y], Z) - \omega([Z, X], Y) - \omega([Y, Z], X) \end{aligned} \tag{1.28}$$

and we see that this vanishes if we plug in $[X, Y] = \nabla_X Y - \nabla_Y X$ and use equation (1.27). Kählerity of (M, g, J) follows. The converse statement can also be proven [6]. □

A property of Kähler manifolds that we will repeatedly make use of is the fact that their Dolbeault cohomology groups can be used to define a decomposition of its complex cohomology groups. We just present this result now without much explanation, but we will say more about it in section 1.3.3

Theorem 1.3.11 **(Hodge decomposition).** *Let (M, J, g) be a compact Kähler manifold, then there exists a decomposition of the cohomology group $H^k(M, \mathbb{C})$ as*

$$H^k(M, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}(M), \tag{1.29}$$

where the spaces $H^{p,q}(M)$ are complex vector spaces that are canonically isomorphic to the Dolbeault cohomology groups $H^q(M, \Omega^p)$. This decomposition depends on the complex structure J on M, but not on its Kähler metric g.

1.3.3 Harmonic forms

Harmonic forms play a very important role in the compactification process described in section 3.2, as well as in the proof of theorem 1.3.11. To define what we mean by a harmonic form on a Riemannian manifold we should first extend the metric on this space to a metric on k-forms for arbitrary k.

Let (M, g) be an oriented Riemannian manifold and let $e_1, \ldots, e_n$ be an oriented local orthonormal basis of vector fields with respect to g and let $e^1, \ldots, e^n$ be the dual basis of 1-forms. We can define a metric on the bundle $\bigwedge^k \mathrm{T}^*M$ by demanding that the basis

$$\left\{ e^{i(1)} \wedge \ldots \wedge e^{i(k)} \;\middle|\; \{i(1) < \ldots < i(k)\} \subseteq \{1, \ldots, n\} \right\} \tag{1.30}$$

is orthonormal with respect to it. This does not depend on the choice made for the original basis $e_1, \ldots, e_n$. The standard volume form μ_M on M is given by $\mu_M = e^1 \wedge \ldots \wedge e^n$.

If M is compact, then we can use this to define the L^2***-metric*** on the space $\Omega^k_{\mathbb{R}}(M)$ of k-forms on a Riemannian manifold (M, g) by setting

$$\langle \alpha, \beta \rangle_{L^2} = \int_M g(\alpha, \beta)\, \mu_M \tag{1.31}$$

for any $\alpha, \beta \in \Omega^k_{\mathbb{R}}(M)$. Directly related to this metric are the notions of the Hodge star operator and the formal adjoint of the exterior derivative d.

Definition 1.3.12 **(Hodge star operator).** *Let (M, g) be a compact n-dimensional Riemannian manifold. The* **Hodge star operator** *is the unique linear map $*\colon \Omega^k_{\mathbb{R}}(M) \to \Omega^{n-k}_{\mathbb{R}}(M)$ for which*

$$\alpha \wedge *\beta = g(\alpha, \beta)\mu_M \qquad \text{and thus} \qquad \langle \alpha, \beta \rangle_{L^2} = \int_M \alpha \wedge *\beta \tag{1.32}$$

for any two k-forms $\alpha, \beta \in \Omega^k(M)$.

On a complex manifold the Hodge star operator can be extended to an operator on $\Omega^k_{\mathbb{C}}(M) \simeq \Omega^k_{\mathbb{R}}(M) \otimes \mathbb{C}$ through complex linear extension. This complex Hodge star operator is characterised by the equation

$$\langle \alpha, \beta \rangle_{L^2} L = \int_M \alpha \wedge \overline{*\beta} = \int_M \mu_M h(\alpha, \beta) \tag{1.33}$$

for $\alpha, \beta \in \Omega^k_{\mathbb{C}}(M)$, where h is the sesquilinear extension of g.

The volume form μ_M on an n-dimensional complex manifold is an (n, n)-form, which tells us that for two pure forms $\alpha^{(p,q)} \in \Omega^{p,q}(M)$ and $\beta^{(p',q')} \in \Omega^{p',q'}(M)$ with $p + q + p' + q' = 2n$, but $p + p' \neq n$ (and thus $q + q' \neq n$), $\alpha \wedge \beta = 0$. As a consequence of this, $\overline{*\omega^{(p,q)}}$ will necessarily be a pure $(n - p, n - q)$-form for $\omega^{(p,q)} \in \Omega^{p,q}(M)$ and thus $*\,\Omega^{p,q}(M) = \Omega^{n-q,n-p}(M)$.

Definition 1.3.13 **(Formal adjoint of d).** *The formal adjoint $\mathrm{d}^\dagger$ of the exterior derivative d on a Riemannian manifold (M, g) is the operator $\mathrm{d}^\dagger\colon \Omega^k_{\mathbb{R}}(M) \to \Omega^{k-1}_{\mathbb{R}}(M)$ that is determined by the relation*

$$\langle \mathrm{d}\alpha, \beta \rangle_{L^2} = \langle \alpha, \mathrm{d}^\dagger \beta \rangle_{L^2} \tag{1.34}$$

for any $\alpha \in \Omega^{k-1}_{\mathbb{R}}(M)$ and any $\beta \in \Omega^k_{\mathbb{R}}(M)$. If M is a complex manifold we can similarly define adjoint operators $\partial^\dagger$ and $\bar{\partial}^\dagger$ for its Dolbeault operators.

It can be shown that the Hodge star operator satisfies $*^2 = (-1)^{k(n-k)}$ and that moreover $\mathrm{d}^\dagger = (-1)^k *^{-1} \mathrm{d}*$, $\partial^\dagger = -*\bar{\partial}*$ and $\bar{\partial}^\dagger = -*\partial*$. We can now define the Laplace operator, or Laplacian, on a Riemannian manifold as a generalisation of the standard Laplace operator $\sum_i \partial_i^2$ on $\mathbb{R}^n$ (equipped with the standard inner product).

***Definition 1.3.14* (Laplace operator).** *The* **Laplace operator** *on a Riemannian manifold (M, g) is defined as the operator $\Delta_{\mathrm{d}} = \mathrm{d}\,\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d}$. A k-form α on (M, g) is said to be* **harmonic** *if $\Delta\alpha = 0$ and we denote the (linear) space of real Harmonic k-forms by $\mathcal{H}^k(M, \mathbb{R})$.*

It can be shown that a form $\omega \in \Omega^k_{\mathbb{R}}(M, \mathbb{R})$ is harmonic if and only if $\mathrm{d}\omega = \mathrm{d}^\dagger\omega = 0$, which means first of all that any harmonic form $\omega \in \mathcal{H}^k(M, \mathbb{R})$ represents a class $[\omega] \in H^k(M, \mathbb{R})$ and secondly that also the Hodge dual $*\omega \in \Omega^{\dim M - k}_{\mathbb{R}}(M)$ is harmonic. One of the main reasons why we are interested in harmonic forms is the following result.

Proposition 1.3.15. *Let $\mathcal{H}^k(M, \mathbb{R})$ be the space of harmonic k-forms on the compact Riemannian manifold (M, g), then the map $\mathcal{H}^k(M, \mathbb{R}) \to H^k(M, \mathbb{R}), \omega \mapsto [\omega]$ is an isomorphism of vector spaces.*

If M has real dimension n, then the Hodge star operator $*\colon \mathcal{H}^k(M, \mathbb{R}) \to \mathcal{H}^{n-k}(M, \mathbb{R})$ induces an isomorphism $H^k(M, \mathbb{R}) \xrightarrow{\sim} H^{n-k}(M, \mathbb{R})$.

For a complex manifold with a Hermitian metric g one can also define Laplacians $\Delta_\partial = \partial\partial^\dagger + \partial^\dagger\partial$ and $\Delta_{\bar\partial} = \bar\partial\bar\partial^\dagger + \bar\partial^\dagger\bar\partial$. Similar to the situation for general Riemannian manifolds, any (p, q)-form that is harmonic with respect to $\Delta_{\bar\partial}$ represents a cohomology class in $H^q(M, \Omega^p(M))$ and we have the following result.

Proposition 1.3.16. *Let $\mathcal{H}^{p,q}(M)$ be the space of harmonic (p, q)-forms on the compact Hermitian manifold (M, J, g), then the map $\mathcal{H}^{p,q}(M) \to H^q(M, \Omega^p(M)), \omega \mapsto [\omega]$ is an isomorphism of complex vector spaces.*

A special (highly non-trivial) property of Kähler manifolds is that $\Delta = 2\,\Delta_\partial = 2\,\Delta_{\bar\partial}$. This can be used to show that the (p, q)-component of any harmonic k-form (with $k = p + q$) is itself harmonic and thus gives us a decomposition

$$\mathcal{H}^k(M, \mathbb{C}) \simeq \mathcal{H}^k(M, \mathbb{R}) \otimes \mathbb{C} = \bigoplus_{p+q=k} \mathcal{H}^{p,q}(M). \tag{1.35}$$

The isomorphism from proposition 1.3.15 induces a corresponding decomposition

$$H^k(M, \mathbb{C}) \simeq H^k(M, \mathbb{R}) \otimes \mathbb{C} = \bigoplus H^{p,q}(M), \tag{1.36}$$

where the spaces $H^{p,q}(M)$ are canonically isomorphic to the Dolbeault cohomology groups $H^q(M, \Omega^p(M))$ by proposition 1.3.16. This makes the decomposition from theorem 1.3.11 explicit.

Definition 1.3.17* (Hodge diamond).** *The* **Hodge numbers** *$h^{p,q}$ of a Kähler manifold (M, J, g) are the (complex) dimensions $h^{p,q} = \dim_{\mathbb{C}}\big(H^{p,q}(M)\big)$ of the Dolbeault cohomology groups of M. These numbers are often presented in the form of a* **Hodge diamond, as the in equation (1.37) (which is for a 3-dimensional complex manifold).*

$$\begin{array}{ccccccc}
 & & & h^{3,3} & & & \\
 & & h^{3,2} & & h^{2,3} & & \\
 & h^{3,1} & & h^{2,2} & & h^{1,3} & \\
h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\
 & h^{2,0} & & h^{1,1} & & h^{0,2} & \\
 & & h^{1,0} & & h^{0,1} & & \\
 & & & h^{0,0} & & &
\end{array} \tag{1.37}$$

These diagram are symmetric under both horizontal and vertical reflections because of the isometries between the spaces $H^{p,q}(M)$ induced by complex conjugation and the Hodge star operator, which tell us that $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$.

If (M, J, g) is a Kähler manifold with Kähler form ω, then an element $\alpha \in H^{p,q}(M, \mathbb{C})$ is said to be ***primitive*** if it cannot be written as $\alpha = \omega \wedge \beta$ for some $\beta \in H^{p-1,q-1}$. On the space $H^{p,q}_{\text{prim}}(M)$ of primitive (p,q)-forms modulo exact forms the complex-linear extension of the Hodge star operator takes a particularly simple form.

Lemma 1.3.18. *Let (M, J, g) be a Kähler manifold with Kähler form ω. If $\alpha \in H^{p,q}(M, \mathbb{C})$ is primitive, then $*\alpha = (-1)^{k(k+1)/2} i^{p-q} \alpha$ for $k = p + q$.*

Another reasons why the Laplace operator is interesting is the fact that it allows us to find a discrete basis of forms on a compact Riemannian manifold with a number of very nice properties.

Theorem 1.3.19. *Let (M, g) be a compact Riemannian manifold, let $\Delta = \mathrm{dd}^\dagger + \mathrm{d}^\dagger\mathrm{d}$ be its Laplace operator and let $k \geq 0$, then the spectrum*

$$\Lambda = \left\{\lambda \mid \exists \alpha \in \Omega^k(M) \colon \Delta\alpha = \lambda\,\omega\right\} \subseteq \mathbb{R} \tag{1.38}$$

is discrete and consists of only non-negative numbers. Moreover, each of the eigenspace $\ker(\Delta - \lambda)$ is finite dimensional and orthogonal with respect to $\langle \bullet, \bullet \rangle_{L^2}$ and any form $\alpha \in \Omega^k(M)$ can be written as a uniformly convergent sum $\alpha = \sum_{\lambda \in \Lambda} \alpha_\lambda$ [10].

By uniform convergence we mean convergence with respect to the L^∞-norm $\|\bullet\|_\infty$ on $\Omega^k(M)$, which is defined by $\|\alpha\|_\infty = \sup_{x \in M} \sqrt{g(\alpha, \alpha)}$. This in particular implies convergence with respect to the L^2-norm.

1.3.4 Holomorphic line bundles

We will briefly discuss some general properties of the first Chern class of a line bundle, which we will later apply to line bundles on complex tori.

***Definition 1.3.20* (Holomorphic line bundle).** *A* **holomorphic vector bundle** *of rank n on a complex manifold M is a real $2n$-dimensional vector bundle $\pi \colon V \to M$ that admits trivialisations $\tau_i \colon \pi^{-1}(U) \to U \times \mathbb{C}^n$ such that the transition functions $\tau_i \circ \tau_j^{-1}$ are holomorphic. A* **holomorphic line bundle** *is a holomorphic vector bundle of rank 1.*

The tensor product $L \otimes L' \to M$ of two holomorphic line bundles $L \to M$ and $L' \to M$ on the compact complex manifold M is again a holomorphic line bundle and for any line bundle $L \to M$ we have $L_0 \otimes L \simeq L$ and $L \otimes L^* \simeq L_0$, where L_0 denotes the trivial line bundle and L^* is the dual bundle for L. The set of isomorphism classes of holomorphic line bundles, equipped with the tensor product, defines an Abelian group that is called the ***Picard group*** and isdenoted by $\mathrm{Pic}(M)$.

Using Čech cohomology it can be shown that this group is naturally isomorphic to the cohomology group $H^1(M, \mathcal{O}_M^*)$, where $\mathcal{O}_M^*$ is the sheaf of nowhere vanishing holomorphic functions on M [7, 6]. If we furthermore let $\mathcal{O}_M$ denote the sheaf of all holomorphic functions on M and $\mathbb{Z}_M$ the constant sheaf of integers on M, then we have a short exact sequence

$$0 \longrightarrow \mathbb{Z}_M \xrightarrow{2\pi i} \mathcal{O}_M \xrightarrow{\exp} \mathcal{O}_M^* \longrightarrow 0, \tag{1.39}$$

that induces a long exact sequence on the cohomology groups,

$$\ldots \to H^1(M, \mathcal{O}_M) \to H^1(M, \mathcal{O}_M^*) \xrightarrow{c_1} H^2(M, \mathbb{Z}) \to H^2(M, \mathcal{O}_M) \to \ldots \tag{1.40}$$

***Definition 1.3.21* (First Chern class).** *Let $L \in H^1(M, \mathcal{O}_M^*)$ be a line bundle on a compact complex manifold M, then the* **first Chern class** *of L is the image $c_1(L) \in H^2(M, \mathbb{Z})$, where c_1 is the connecting homomorphism from equation* (1.40).

Instead of considering the (first) Chern class $c_1(L)$ of a line bundle L as an element of $H^2(M,\mathbb{Z})$, we will often view $c_1(L)$ as an element of the real (De Rham) cohomology group $H^2_{\mathrm{DR}}(M,\mathbb{R})$. Although the (first) Chern class $c_1(L)$ does not completely fix the isometry class of L, it does completely determine L as a smooth vector bundle [7]. There is another, more practical, definition for the first Chern class of a line bundle that can in fact be applied to any smooth vector bundle.

Lemma 1.3.22. *Let $L \to M$ be a holomorphic line bundle on the compact complex manifold M and let Θ be the curvature 2-form for some connection on L, then*

$$c_1(L) = \left[\tfrac{-1}{2\pi i}\Theta\right] \in H^2(M,\mathbb{R}). \tag{1.41}$$

Proof: The equivalence of these definitions is discussed in [7]. □

Note that this lemma in particular states that the cohomology class of the curvature form Θ is independent of the connection chosen. A convenient choice is the Chern connection for a Hermitian metric since the lemma below provides us with a simple way to determine its curvature.

***Lemma/Definition 1.3.23* (Chern connection).** *Let $E \to M$ be a holomorphic vector bundle on the complex manifold M and let h be a Hermitian metric on E. There exists a unique complex connection ∇ on E such that for any two smooth sections σ and τ of L,*

1. *$\nabla(i\,\sigma) = i\,(\nabla\sigma)$ (∇ is a complex connection)*
2. *$\mathrm{d}\big(h(\sigma,\tau)\big) = h(\nabla\sigma,\tau) + h(\sigma,\nabla\tau)$ (∇ is Hermitian)*
3. *$\nabla\sigma - \bar{\partial}_L\sigma \in \Omega^{1,0}(M,L)$.*

This connection is called the **Chern connection** *for h.*

Proof: A proof for this statement may be found in [6] or [11]. □

Lemma 1.3.24. *Let $L \to M$ be a holomorphic line bundle, let h be a Hermitian metric on L and let ∇ be its Chern connection. If we choose a local holomorphic section Φ of L that does not vanish anywhere on its domain then the connection and curvature forms of ∇ are given by*

$$\theta = \partial \log h \qquad \text{and} \qquad \Theta = -\partial\bar{\partial}\log h, \tag{1.42}$$

respectively on the domain of Φ, where $h = h(\Phi,\Phi)$.

Proof: The reader is referred to [11] for a proof of this claim. □

1.3.5 Special Kähler geometry

An interesting class of Kähler manifolds are the so-called special Kähler manifolds, which were first observed to appear in $N = 2$ supergravity theories, such as the one we will introduce in section 3.2 [12]. There are two types of special Kähler manifolds: rigid special Kähler manifolds and local special Kähler manifolds, which appear in (rigid) supersymmetry theories [13] and $N = 2$ supergravity theories [12] respectively.

In the mathematics literature rigid and local special Kähler manifolds are generally referred to as affine special Kähler manifolds and projective special Kähler manifolds respectively. As we will see in section 2.2, the moduli space of Calabi-Yau 3-folds comes with a projective special Kähler structure [14, 15, 16]. A very elegant intrinsic definition for affine special Kähler structures is exists [17].

***Definition 1.3.25* (Affine special Kähler structure).** *An* **affine special Kähler manifold** *is a Kähler manifold (M,J,g) that admits a flat torsion-free connection ∇ such that $\mathrm{d}_\nabla J = 0$.*

It is however the projective special Kähler manifolds that we are mostly interested in. These manifolds can be defined as the orbit spaces for local $\mathbb{C}^*$-action that is defined on affine special Kähler manifold of a specific type [18], but this is not the definition we will use. One of the most explicit of these uses local projective coordinates and a homogeneous prepotential for the Kähler potential [14, 12].

***Definition 1.3.26* (Projective special Kähler structure).** *A* **projective special Kähler manifold** *is an n-dimensional Kähler manifold (M, J, g) that can be covered by open patches U_a that come with a set of complex projective coordinates $X_a^0, \ldots, X_a^n$ and a* **prepotential** *F, which is a holomorphic function $F_a \colon \mathbb{C}^{n+1} \to \mathbb{C}$ such that*

a. *F is homogeneous of degree 2: $F_a(\lambda X_a^0, \ldots, \lambda X_a^n) = \lambda^2 F_a(X_a^0, \ldots, X_m^n)$.*

b. *The function*

$$K_a(X_a, \bar{X}_a) = -\log\left(i \sum_i (\bar{X}_a^i F_{a,i} - X_a^i \bar{F}_{a,i})\right) \tag{1.43}$$

is a Kähler potential for (M, J, g), where $F_{a,i}(X) := \frac{\partial F_a(X)}{\partial X_a^i}$.

c. *On the intersection $U_a \cap U_b$ of two patches, the vectors $(X, \partial F)_a = (X_a^i, F_{a,i}(X))_{i=0}^n$ and $(X, \partial F)_b = (X_b^i, F_{b,i}(X))_{i=0}^n$ are related by a transformation*

$$\begin{pmatrix} \partial F \\ X \end{pmatrix}_a = f_{ab}\, M_{ab} \begin{pmatrix} \partial F \\ X \end{pmatrix}_b \tag{1.44}$$

for some nowhere-vanishing holomorphic function f_{ab} and a constant matrix $M_{ab} \in \mathrm{Sp}(2n+2, \mathbb{R})$. On the overlap $U_a \cap U_b \cap U_c$ of three charts these should satisfy

$$f_{ab}\, f_{bc}\, f_{ca} = 1 \qquad \text{and} \qquad M_{ab} M_{bc} M_{ca} = \mathbf{1}_{2n+2}. \tag{1.45}$$

The homogeneity of a prepotential F_a could alternatively have been expressed through the equation $F_a(X) = \sum_i \frac{1}{2} X_a^i F_{a,i}(X)$, with $F_{a,i}(X) = \frac{\partial F_a(X)}{\partial X_a^i}$. As a consequence of the homogeneity of $F_a(X)$, the derivatives $F_{a,i}(X)$ are homogeneous of degree 1, while the Hessian $F_{ij} = \frac{\partial^2 F(X)}{\partial X^i \partial X^j}$ is homogeneous of degree 0.

Equation (1.44) and (1.44) may not appear to make sense because the coordinates X_a^i on the chart U_a are projective, which is why they should really be interpreted as conditions that hold for any holomorphic functions $X_a^0, \ldots, X_a^n \colon U_a \to \mathbb{C}$ such that $[X_a^0(x), \ldots, X_a^n(x)] = x$ for $x \in U_a$ and the corresponding functions functions $F_{a,i} \colon U_a \to \mathbb{C}, x \mapsto F_{a,i}(X_a(x))$. It can be shown that if condition **(b)** and condition **(c)** hold for one section, they are automatically satisfied by any other section on the same domain.

A closely related definition that does not make use of local coordinates can be found in [14, 19]. We will finish with yet another definition for projective special Kähler manifolds that may be found in [17] and is modelled after variations of the Hodge structure of Calabi-Yau 3-folds as described by [15]. We will encounter special Kähler geometry in this form in section 2.2, where deformations of Calabi-Yau manifolds are discussed.

***Definition 1.3.27* (Projective special Kähler structure).** *Let (M, J, g) be a Kälher manifold of (complex) dimension n. A projective special Kähler structure consists of*

1. *A holomorphic vector bundle $V \simeq (V_{\mathbb{R}}, J)$ of rank $n+1$. We identify V with the $+i$ eigenspace in $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ of J,*
2. *A line sub-bundle $L < V$ whose first Chern class $c_1(L) = [\omega]$ is represented by the Kähler form ω of (M, J, g).*
3. *A flat complex linear connection ∇ on the complexification $V_{\mathbb{C}} = V_{\mathbb{R}} \otimes \mathbb{C}$ of the real vector bundle underlying V such that $\nabla(L) < V$,*

4. *A symplectic form Q on $V_{\mathbb{R}}$ of type $(1,1)$ that is flat with respect to ∇ such that $i\,Q(\bullet,\bar{\bullet})$ defines a Hermitian metric on $L < V < V_{\mathbb{C}}$.*

On any contractible neighbourhood $U \subseteq M$, we can use the flat connection ∇ to identify the fibres of $V_{\mathbb{C}}$ with a single the fibre $V_{\mathbb{C},x} = V_{\mathbb{R},x} \otimes \mathbb{C}$ for $x \in M$. We require that for any such neighbourhood U the map $U \to P(V_{\mathbb{C},x}), x \mapsto L_x$ is an immersion, where $P(V_{\mathbb{C},x})$ is the complex projective space $\big(V_{\mathbb{C},x} \setminus \{0\}\big)/\mathbb{C}^*$.

1.4 Complex tori

We will use the following definition for a complex torus [8].

***Definition 1.4.1* (Complex torus).** *Let V be a complex linear space of dimension n, then by a* **discrete lattice** *in V we mean a discrete subgroup of V (equipped with the additive structure) that is of maximal rank, so it is a free abelian subgroup of V of rank $2\,n$.*

A **complex torus** *of dimension n is a quotient $X = V/\Lambda$, of a complex linear space V of dimension n by a lattice Λ of maximal rank. The complex torus X is topologically simply an $2n$-dimensional torus and it inherits the structure of a complex Lie group from V, so it is in particular an n-dimensional connected compact complex manifold.*

The translations on the vector space V induce diffeomorphisms on the torus X which we also call translation. These coincide with the group translations on X as an Abelian group.

***Definition 1.4.2* (Translation).** *A* **translation** *on a complex torus $X = V/\Lambda$ by $y \in X$ is a map $\tau_x \colon X \to X, x \mapsto x + y$. The translation of a tensor T on X by $y \in X$ is the tensor $\tau_{y*}T$ and T is called translation-invariant when $\tau_{y*}T = T$ for all $y \in X$.*

It can be shown that any compact connected complex Lie group G is a complex torus. The description of such a group as a torus as we have defined it is obtained by taking for v the Lie algebra $V = \mathrm{T}_0 G = \mathfrak{g}$ and for $\Lambda \subseteq V$ the kernel of the exponential map $\exp\colon \mathfrak{g} \to G$ [9].

An effective way to describe a complex torus is through a so-called period matrix. Throughout this section the indices $i, j, k, \ldots$ run from 1 to $n = \dim_{\mathbb{C}} V$ and the indices $s, t, u, \ldots$ run from 1 to $2\,n = \operatorname{rank}\Lambda$.

***Definition 1.4.3* (Period matrix).** *Let $X = V/\Lambda$ be a complex torus of dimension n, let $e_1, \ldots, e_n$ be a (complex) basis for V be and let $\lambda_1, \ldots, \lambda_{2n}$ be an integral basis for the lattice Λ. The* **period matrix** *associated with these bases is the (complex) $n{\times}2n$-matrix $\Omega = \Omega_{is}$ such that*

$$\lambda_s = \sum_i \Omega_{is} e_i. \tag{1.46}$$

The basis $\lambda_1, \ldots, \lambda_{2n}$ is also a real basis for V, so we can define dual coordinates $x_1, \ldots, x_{2n}$ and complex coordinates $z_1, \ldots, z_n$ dual to $e_1, \ldots, e_n$. Since $\mathrm{d}z_i(\lambda_s) = \mathrm{d}z_i(\sum_j \Omega_{js} e_j) = \Omega_{is}$, the corresponding differentials on $X = V/\Lambda$ satisfy

$$\mathrm{d}z_i = \sum_s \Omega_{is} \mathrm{d}x_s \qquad \textit{and} \qquad \mathrm{d}\bar{z}_i = \sum_s \bar{\Omega}_{is} \mathrm{d}x_s, \tag{1.47}$$

so we see that the matrix $\tilde{\Omega} = \binom{\Omega}{\bar{\Omega}}$ defines a change of basis on the complexified cotangent bundle $\mathrm{T}^*_{\mathbb{C}} X$ from $(\mathrm{d}x_s)_s$ to $(\mathrm{d}z_i, \mathrm{d}\bar{z}_i)_i$ [7].

***Remark 1.4.4*.** *Let $\Pi = \Pi_{si}$ be the $2n{\times}n$-matrix for which*

$$\mathrm{d}x_s = \sum_i \Pi_{si} \mathrm{d}z_i + \overline{\Pi}_{si} \bar{\mathrm{d}} z_i, \tag{1.48}$$

then $(\Pi, \overline{\Pi})$ *describes the change of basis in the opposite direction, from* $(\mathrm{d}z_i, \mathrm{d}\bar{z}_i)_i$ *to* $(\mathrm{d}x_s)_s$*, so* $\tilde{\Omega} \cdot \tilde{\Pi} = I_{2n}$*. This matrix is determined by the equations* $\Omega \cdot \Pi = I_n$ *and* $\Omega \cdot \overline{\Pi} = 0$*. By combining this with equations (1.46) it follows that* Π *is characterised by the equations*

$$\sum_s \Pi_{si}\lambda_s = \sum_{s,j} \Omega_{js}\Pi_{si}e_j = e_i \qquad \text{and} \qquad \sum_s \overline{\Pi}_{si}\lambda_s = \sum_{s,j} \Omega_{js}\overline{\Pi}_{si}e_j = 0. \tag{1.49}$$

The matrix Π *is also often referred to as the* period matrix*.*

If we fix a complex basis $(e_i)_i$ for V and an integral basis $(\lambda_s)_s$ for the lattice Λ, then the associated period matrix Ω can be seen as an embedding of the lattice $\Lambda \simeq \mathbb{Z}^{2n}$ into the complex linear space $V \simeq \mathbb{C}^n$. Thus $X = V/\Lambda \simeq \mathbb{C}^n/(\Omega\,\mathbb{Z}^{2n})$ and the complex torus is completely determined by its period matrix. Conversely, any period matrix Ω for which $\tilde{\Omega} = \binom{\Omega}{\overline{\Omega}}$ is invertible describes such an embedding and is hence the period matrix of a complex torus [8].

1.4.1 Non-degenerate complex tori

In section 2.3 we will introduce the Griffiths and Weil intermediate Jacobian of a Calabi-Yau 3-fold. These spaces are complex tori, but they have additional structure in the form of a polarisation, which makes them non-degenerate complex tori. In this section these notions will be introduced and explained.

***Remark 1.4.5* (Polarised manifold).** *A* **polarised manifold** *is a pair* $(M, [\omega])$*, where* M *is a compact complex manifold and* $[\omega] \in H^2(M, \mathbb{Z})_{\mathrm{f}} \subseteq H^2(M, \mathbb{R})$ *is an integral Kähler class [6]. By this we mean that* $[\omega] \in H^2(M, \mathbb{Z})_{\mathrm{f}} \subseteq H^2(M, \mathbb{R})$ *that can be represented by a Kähler form* ω *and is thus in particular also an element of* $H^{1,1}(M)$*. To allow a little more generality we will drop the positivity condition on the Kähler form* ω *and call any cohomology class* $[\omega] \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})_{\mathrm{f}}$ *a (general) polarisation [8].*

Any inner product on the linear space V *naturally induces a translation-invariant metric on the torus* $X = V/\Lambda$ *and it can be shown that with respect to such a metric the harmonic forms on* X *are exactly the translation invariant forms. More explicitly, in terms of the coordinates from definition 1.4.3, the space of harmonic* (p, q)*-forms is given by*

$$\mathcal{H}^{p,q}(X) = \sum_{|I|=p} \sum_{|J|=q} \mathbb{C}\,\mathrm{d}z_I \wedge \mathrm{d}\bar{z}_J, \tag{1.50}$$

and the space of harmonic k*-forms representing integral cohomology classes is given by [7]*

$$\mathcal{H}^k(X, \mathbb{Z}) = \sum_{|I|=k} \mathbb{Z}\,\mathrm{d}x_K. \tag{1.51}$$

Since these forms are translation invariant they define an anti-symmetric bilinear form Q *on* $V \simeq \mathrm{T}_y X$ *(the same one for every* $y \in X$*) that can be shown to take integer values on the lattice* $\Lambda < V$ *and are compatible with the complex structure on* V *in the sense that* $Q(i\,\bullet, i\,\bullet) = Q$*. Conversely, any such bilinear form* Q *uniquely determines a translation invariant form (which we also denote by* Q*) on* $X = V/\Lambda$ *and thus a polarisation* $[Q] \in H^{1,1}(X) \cap H^2(X, \mathbb{Z})_{\mathrm{f}}$*.*

With this in mind, we give the following definition of a polarisation on a complex torus.

***Definition 1.4.6* (Polarisation).** *Let* $X \simeq V/\Lambda$ *be a complex torus and identify* $V \simeq \mathrm{T}_0 X$*. A symplectic form* $Q \colon V \times V \to \mathbb{R}$ *is said to define a* **polarisation** *on* X *if it takes integer values on the lattice* Λ *and is compatible with the complex structure on* X *in the sense that* $Q(i\,\bullet, i\,\bullet) = Q$*.*

It is easily checked that a symplectic form Q on a complex linear space V is compatible with the complex structure if and only if it is the (negative) imaginary part of a Hermitian form h on V, which is completely determined by equation (1.52) below. This allows us to give an alternative definition for a polarisation as a Hermitian form and gives us a canonical way of defining a metric on a non-degenerate torus (cf. 1.4.8).

$$h = g + i\,Q, \qquad g = \mathrm{Re}(h) = Q(i\,\bullet, \bullet), \qquad Q = \mathrm{Im}(h) = g(\bullet, i\,\bullet). \tag{1.52}$$

There is an alternative approach to polarisations on complex tori using the first Chern classes of holomorphic line bundles on these tori. A polarisation is then defined as the first Chern class of a holomorphic line bundle on this torus. The following proposition shows that this approach is equivalent

Proposition 1.4.7. *Let $X = V/\Lambda$ be a complex torus and let $c_1(L)$ be the first Chern class of a line bundle $L \to X$, then there exists a polarisation $Q\colon V \times V \to \mathbb{R}$ on X such that $c_1(L) = [Q]$. Conversely, given a polarisation Q on X, there exists a line bundle $L \to X$ such that $c_1(L) = [Q]$. Any two such line bundles L and L' are related by a translation on X, i.e. $L' = \tau_{y*}L$ for some $y \in X$.*

Proof: The first Chern class $c_1(L)$ is by definition an integral cohomology class and it is of type $(1, 1)$ by lemma 1.3.24. This tells us that it defines a (general) polarisation in the sense of lemma 1.4.5. Conversely, we can construct a line bundle L_ω for any element $[\omega] \in H^{1,1}(M) \cap H^2(M, \mathbb{Z})_{\mathrm{f}}$ such that $c_1(L) = [\omega]$. A method of constructing such line bundles, as well as a proof of their uniqueness up to translations can be found in [7]. □

Since the complex structure and the polarisation on a complex torus are both translation invariant, applying a translation τ_y to $X = V/\Lambda$ does not essentially change anything about the torus. In this sense the notion of a polarisation and that of a line bundle are interchangeable, which is why the line bundle itself is sometimes called a polarisation.

Definition 1.4.8* (Canonical metric).** *A polarisation on a complex torus $X = V/\Lambda$ is a Hermitian form $h\colon V \times V \to \mathbb{C}$ whose imaginary part $Q = \mathrm{Im}(h)$ is a polarisation in the sense of definition 1.4.6. The real part $g = \mathrm{Re}(h)$ of h is called the* ***canonical metric *associated with the polarisation h. Given a complex structure, each of the three forms h and g and Q determine the other two completely since they are linked by the equations*

$$g = \mathrm{Re}(h) = Q(i\,\bullet, \bullet), \qquad Q = \mathrm{Im}(h) = -g(i\,\bullet, \bullet), \qquad \text{and} \qquad h = g + i\,Q. \tag{1.53}$$

The symmetric form g is non-degenerate, but not necessarily positive definite. We say that a polarisation h has **index** *k if the Hermitian form h has index k, i.e. if it has exactly k negative eigenvalues. A complex torus admitting a polarisation of index 0 is called an* **Abelian variety** [8, 9, 7].

Lemma 1.4.9. *Let $Q\colon V \times V \to \mathbb{R}$ be an anti-symmetric bilinear real form on a complex linear space V taking integer values on some lattice $\Lambda < V$ (of maximal rank). There exists an integral basis $\alpha_1, \ldots, \alpha_n, \beta^1, \ldots, \beta^n$ for Λ and integers $d_1, \ldots, d_n$ such that for all $1 \le i, j \le n$*

$$Q(\alpha_i, \alpha_j) = Q(\beta^i, \beta^j) = 0 \qquad \text{and} \qquad Q(\alpha_i, \beta^j) = d_i \delta_i^j. \tag{1.54}$$

We can moreover arrange it such that each of the integers d_i is non-negative and that d_i is a divisor of the next, i.e. $d_i \mid d_{i+1}$ for $i = 1, \ldots, n-1$. With this extra condition, the integers $d_1, \ldots, d_n$ are uniquely determined by Q and are independent of the choice of special basis $(\alpha_i, \beta^i)_i$.

Proof: See [7], section 2.6. □

***Definition 1.4.10* (Principal polarisation).** *Let $Q\colon V \times V \to \mathbb{R}$ define a polarisation on a complex torus $X = V/\Lambda$. By lemma 1.4.10 we can find a basis $(\alpha_i, \beta^i)_i$ for Λ such that equation (1.54) holds for some uniquely determined integers $d_1 \mid d_2 \mid \ldots \mid d_n$.*

These integers $d_1, \ldots, d_n$ are called the **elementary divisors** *of the polarisation. If $d_i = 1$ for all i we say that Q defines a* **principal polarisation** *and we call the basis $(\alpha_i, \beta^i)_i$ a* **symplectic basis** *since Q is represented by the standard symplectic matrix $\Sigma = \left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right)$ when expressed in such a basis.*

Remark 1.4.11. *Let $Q\colon V \times V \to \mathbb{R}$ define a principal polarisation on a complex torus $X = V/\Lambda$ and let $\alpha_1, \ldots, \alpha_n, \beta^1, \ldots, \beta^n$ be a symplectic basis. We can choose real coordinates $x^1, \ldots, x^n, y_1, \ldots, y_n$ on V with respect to this basis such that $\mathrm{d}x^i(\alpha_j) = \mathrm{d}y_j(\beta^i) = \delta^i_j$ and $\mathrm{d}x^i(\beta^j) = \mathrm{d}y_i(\alpha_j) = 0$. The form Q may then be wrtten in terms of these differentials as*

$$Q = \textstyle\sum_{i,j} \mathrm{d}x^i \wedge \mathrm{d}y_i, \tag{1.55}$$

since we can easily check that this 2-form satisfies equation (1.54).

Since the tori we will encounter all carry a canonical principal polarisation, we will from now on assume that every polarisation is of principal type. In the remainder of this section, $X = V/\Lambda$ will therefore denote a complex torus with a principal polarisation of index k defined by the symplectic form $Q\colon V \times V \to \mathbb{R}$. Moreover, we will fix a symplectic basis $\alpha_1, \ldots, \alpha_n, \beta^1, \ldots, \beta^n$ and corresponding coordinates $x^1, \ldots, x^n, y_1, \ldots, y_n$ for which Q is given by equation (1.55).

***Definition 1.4.12* (Normalised period matrix).** *If the vectors $e^i = \beta^i \in V$ for $i = 1, \ldots, n$ form a complex basis for V, then the period matrix Ω associated with this basis will take the special form $\Omega = (\mathcal{Z}, \mathbf{1}_n)$, where $\mathcal{Z} = \mathcal{Z}_{ij}$ is a (complex) $n\times n$-matrix. A period matrix of this form is called a* **normalised period matrix**.

The matrix $\mathcal{Z}$ from definition 1.4.12 will often be referred to as the period matrix as it describes the relevant part of Ω. It is characterised by the equation $\alpha_i = \sum_j \mathcal{Z}_{ji}\beta^j$.

Once we have found an expression for the normalised period matrix, we can use it to give standard expressions for the symplectic form, the Hermitian form and the canonical metric that are associated with polarisation through the following proposition.

Proposition 1.4.13. *Let the $\Omega = (\mathcal{Z}, \mathbf{1}_n)$ be the normalised period matrix for the symplectised basis $(\alpha_i, \beta^i)_i$ and let $(e^i = \beta^i)_i$ be as in definition 1.4.12. The matrix $\mathcal{Z}$ is symmetric and if we write $\mathcal{Z} = \mathcal{X} + i\,\mathcal{Y}$ for two real $n\times n$-matrices $\mathcal{X} = \mathcal{X}_{ij}$ and $\mathcal{Y} = \mathcal{Y}_{ij}$, then $\mathcal{Y} = \mathrm{Im}(\mathcal{Z})$ is non-degenerate and has the same index k as the polarisation. Additionally, we have that $\mathcal{X}\mathcal{Y}^{-1}\mathcal{X} + \mathcal{Y} = 0$.*

Furthermore, the differentials corresponding to the coordinates z_i associated to the basis $e^1, \ldots, e^n$ are given by $\mathrm{d}z_i = \mathrm{d}y_i + \mathcal{Z}_{ij}\mathrm{d}x^i$ and in terms of these differentials, the Hermitian form h, the symplectic form Q and the canonical metric g describing the polarisation are given by

$$Q = -\textstyle\sum_{i,j} \frac{i}{2}\,\mathcal{Y}^{ij}\mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j, \quad g = \sum_{i,j} \mathcal{Y}^{ij}\mathrm{d}z_i\mathrm{d}\bar{z}_j \qquad \textit{and} \quad h = \sum_{i,j} \mathcal{Y}^{ij}\mathrm{d}z_i \otimes \mathrm{d}\bar{z}_j. \tag{1.56}$$

where $\mathcal{Y}^{ij} = \mathcal{Y}^{-1} = \mathrm{Im}(\mathcal{Z})^{-1}$ denotes inverse matrix of $\mathcal{Y}$.

Proof: The period matrix $\Omega = (\mathcal{Z}, \mathbf{1}_n)$ of course still satisfies equation (1.47) by definition, even though we have renamed the coordinates $x_1, \ldots, x_{2n}$ to $x^1, \ldots, x^n, y_1, \ldots, y_n$. This means that the differentials on the torus corresponding to the complex coordinates $z_1, \ldots, z_n$ are given by $\mathrm{d}z_i = \sum_j \mathcal{Z}_{ij}\mathrm{d}x^j + \mathrm{d}y_i$.

Since Q is compatible with the complex structure on V it is a $(1,1)$-form and can be written as $Q = \sum_{i,j} Q^{ij} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j$ for some (complex) invertible matrix Q^{ij} By expanding this using the expression for $\mathrm{d}z_i$ we just obtained and comparing the result with equation (1.55), we see that

$$\begin{aligned} Q &= Q^{ij} \mathrm{d}z_i \wedge \mathrm{d}\bar{z}_j = Q^{ij} (\mathrm{d}y_i + \mathcal{Z}_{ik} \mathrm{d}x^k) \wedge (\mathrm{d}y_j + \overline{\mathcal{Z}}_{j\ell} \mathrm{d}x^\ell) \\ &= \underbrace{Q^{ij} \mathrm{d}y_i \wedge \mathrm{d}y_j}_{=0} + \underbrace{Q^{ij} \overline{\mathcal{Z}}_{ik} \mathrm{d}x^k \wedge \mathcal{Z}_{j\ell} \mathrm{d}x^\ell}_{=0} + \underbrace{Q^{ij} (\mathrm{d}y_i \wedge \bar{\mathcal{Z}}_{j\ell} \mathrm{d}x^\ell + \mathcal{Z}_{ik} \mathrm{d}x^k \wedge \mathrm{d}y_j)}_{=\mathrm{d}x^i \wedge \mathrm{d}y_i} . \end{aligned} \tag{1.57}$$

The fact that the first term should vanish tells us that Q^{ij} is symmetric. We can use this to write the third term as

$$\begin{aligned} Q^{ij} (\mathrm{d}y_i \wedge \bar{\mathcal{Z}}_{j\ell} \mathrm{d}x^\ell + \mathcal{Z}_{ik} \mathrm{d}x^k \wedge \mathrm{d}y_j) &= Q^{ij} (-\bar{\mathcal{Z}}_{jk} \mathrm{d}x^k \wedge \mathrm{d}y_i + \mathcal{Z}_{jk} \mathrm{d}x^k \wedge \mathrm{d}y_i) \\ &= Q^{ij} (\mathcal{Z}_{jk} - \bar{\mathcal{Z}}_{jk})(\mathrm{d}x^i \wedge \mathrm{d}y_k) = 2\,i\, Q^{ij} \mathcal{Y}_{jk} \mathrm{d}x^i \wedge \mathrm{d}y_k = \mathrm{d}x^i \wedge \mathrm{d}y_i, \end{aligned} \tag{1.58}$$

which furthermore tells us that $2\,i\,Q^{ij} = \mathcal{Y}^{ij}$. We see that $\mathcal{Y}$ is symmetric and that the form Q is given by equation (1.56). The corresponding expressions for the hermitian form and canonical metric follow immediately from equation (1.53).

If we now work out the remaining expression in equation (1.57) we find

$$\begin{aligned} 2\,i\,Q^{ij} \overline{\mathcal{Z}}_{ik} \mathcal{Z}_{j\ell} &= \mathcal{Y}^{ij} (\mathcal{X}_{ik} - i\,\mathcal{Y}_{ik})(\mathcal{X}_{j\ell} + i\,\mathcal{Y}_{k\ell}) \\ &= (\mathcal{X}_{ik} \mathcal{Y}^{ij} \mathcal{X}_{j\ell} + \mathcal{Y}_{k\ell}) + i\,(\mathcal{X}_{\ell k} - \mathcal{X}_{k\ell}), \end{aligned} \tag{1.59}$$

which should be symmetric. This can only happen if $\mathcal{X}_{ij} = \mathcal{X}_{ji}$ is symmetric, so we conclude that the entire matrix $\mathcal{Z} = \mathcal{X} + i\,\mathcal{Y}$ is necessarily symmetric. □

Finally, we can also express the complex structure on the torus in terms of the real basis $(\alpha_i, \beta^i)_i$ and the real and imaginary part of the normalised period matrix.

Proposition 1.4.14. *Let the period matrix $\Omega = (\mathcal{Z}, \mathbf{1}_n)$ with $\mathcal{Z} = \mathcal{X} + i\,\mathcal{Y}$ be as in proposition 1.4.13. The complex structure on V is described by the equations*

$$i\,\alpha_i = \mathcal{X}_{ij} \mathcal{Y}^{jk} \alpha_k - (\mathcal{X}_{ij} \mathcal{Y}^{jk} \mathcal{X}_{k\ell} + \mathcal{Y}_{i\ell}) \beta^\ell \qquad \text{and} \qquad i\,\beta^i = \mathcal{Y}^{ij} (\alpha_j - \mathcal{X}_{jk} \beta^k). \tag{1.60}$$

Proof: We have $\alpha_i = \mathcal{Z}_{ij} e^j = \mathcal{Z}_{ij} \beta^j = \mathcal{X}_{ij} \beta^j + i\,\mathcal{Y}_{ij} \beta^j$ by definition of the normalised period matrix Ω. Since $\mathcal{Y}$ is invertible, it immediately follows that

$$i\,\beta^i = \mathcal{Y}^{ij}\,(i\,\mathcal{Y}_{jk} \beta^k) = \mathcal{Y}^{ij}\,(\alpha_j - \mathcal{X}_{jk} \beta^k). \tag{1.61}$$

If we use this to expand $i\,\mathcal{X}_{ij} \beta^j$, we furthermore see that

$$\begin{aligned} i\,\alpha_i &= i\,\mathcal{X}_{ij} \beta^j - \mathcal{Y}_{ij} \beta^j = \mathcal{X}_{ij} \mathcal{Y}^{jk}\,(\alpha_k - \mathcal{X}_{k\ell} \beta^\ell) - \mathcal{Y}_{jk} \beta^k \\ &= \mathcal{X}_{ij} \mathcal{Y}^{jk} \alpha_k - (\mathcal{X}_{ij} \mathcal{Y}^{jk} \mathcal{X}_{k\ell} + \mathcal{Y}_{i\ell}) \beta^\ell, \end{aligned} \tag{1.62}$$

which completes the proof □

Remark 1.4.15. *Let V/Λ be a real torus of dimension $2\,n$, Q a symplectic form that takes integer values on the lattice and $(\alpha_i, \beta^i)_i$ a symplectic basis for this for V with respect to this form. For any symmetric $n \times n$ matrix $\mathcal{Z}$ such that $\mathcal{Y} = \mathrm{Im}(\mathcal{Z})$ is invertible, equation (1.60) defines a complex structure on V that is compatible with Q. The complex structure, the pseudo-Riemannian metric $g = Q(i\bullet, \bullet)$ and the normalised period matrix $(\mathcal{Z}, \mathbf{1})$ satisfying the above two properties are therefore all equivalent parameters.*

1.4.2 Intermediate Jacobians

A very important class of non-degenerate complex tori are the ***intermediate Jacobians*** of compact Kähler manifolds. These tori are defined in terms of just the odd cohomology groups of the Kähler manifold and can be given a complex structure through the Hodge decomposition [8].

For any Kähler manifold M and any integer $k \in \{1, 2, \ldots, \dim_{\mathbb{C}} M\}$, the torsion-free part of the $(2k-1)$-th integral cohomology group, $H^{2k-1}(M, \mathbb{Z})_{\mathrm{f}}$, defines a lattice inside the real cohomology group $H^{2k-1}(M, \mathbb{R})$ of maximal rank. The space

$$\begin{aligned} \mathcal{J}_k(M) &= \operatorname{coker}\bigl(H^{2k-1}(M, \mathbb{Z}) \hookrightarrow H^{2k-1}(M, \mathbb{R})\bigr) \\ &= H^{2k-1}(M, \mathbb{R}) \big/ H^{2k-1}(M, \mathbb{Z})_{\mathrm{f}} \end{aligned} \tag{1.63}$$

is therefore a (real) torus of (real) dimension $h^{2k-1} = \dim(H^{2k-1}(M, \mathbb{R}))$. There are in general several complex structures we can put on $H^{2k-1}(M, \mathbb{R})$, which define different complex tori.

A complex structure on the (real) linear space $H^{2k-1}(M, \mathbb{R})$ is described by an endomorphism $J\colon H^{2k-1}(M, \mathbb{R}) \to H^{2k-1}(M, \mathbb{R})$ for which $J^2 = -\mathrm{id}$. The extension of such an endomorphism J to the complexified space, $H^{2k-1}(M, \mathbb{R}) \otimes \mathbb{C} \simeq H^{2k-1}(M, \mathbb{C})$ is of course diagonalisable and it has eigenvalues $\pm i$, which means that J is completely determined by the eigenspaces $V_{\pm} = \ker(J \mp i) < H^{2k-1}(M, \mathbb{C})$ corresponding to these eigenvalues. There is only one restriction, which comes from the fact that J restricts to an endomorphism on $H^{2k-1}(M, \mathbb{R})$, and that is that $V_- = \overline{V_+}$.

We know that the complex cohomology groups $H^{2k-1}(M, \mathbb{C})$ of the Kähler manifold M can be decomposed into a direct sum of Dolbeault cohomology groups $H^{p,q}(M)$,

$$H^{2k-1}(M, \mathbb{C}) = \bigoplus_{p+q=2k-1} H^{p,q}(M) \tag{1.64}$$

and that moreover $\overline{H^{p,q}(M)} = H^{q,p}(M)$, which we can use the Hodge decomposition to define a complex structure on the odd cohomology groups of the Kähler manifold M and turn the tori $\mathcal{J}_k(M)$ into complex tori by defining the eigenspaces V_+ and V_- as sums of such spaces $H^{p,q}(M)$.

There are two important classes of intermediate Jacobians, the Weil intermediate Jacobians and the Griffiths intermediate Jacobians.

Definition 1.4.16 **(Griffiths intermediate Jacobian).** *The k-th* **Griffiths intermediate Jacobian** *of a Kähler manifold M is the complex torus*

$$\mathcal{J}_k^{\mathrm{G}}(M) = (H^{2k-1}(M, \mathbb{R}) / H^{2k-1}(M, \mathbb{Z})_{\mathrm{f}}, J^{\mathrm{G}}) \tag{1.65}$$

where the complex structure J^{G} is defined by the eigenspaces $V^{\mathrm{G}} = \bigoplus_{q>p} H^{p,q}(M)$ and $\overline{V^{\mathrm{G}}}$ corresponding to the eigenvalues $+i$ and $-i$ respectively.

and

Definition 1.4.17 **(Weil intermediate Jacobian).** *The k-th* **Weil intermediate Jacobian** *of a Kähler manifold M is the complex torus*

$$\mathcal{J}_k^{\mathrm{W}}(M) = (H^{2k-1}(M, \mathbb{R}) / H^{2k-1}(M, \mathbb{Z})_{\mathrm{f}}, J^{\mathrm{W}}) \tag{1.66}$$

where the complex structure J^{W} is defined by the eigenspaces

$$V^{\mathrm{W}} = \bigoplus_{1+p-q \in 4\mathbb{Z}} H^{p,q}(M) \tag{1.67}$$

and $\overline{V^{\mathrm{W}}}$ corresponding to the eigenvalues $+i$ and $-i$ respectively.

It is easily verified that the extension of J^{G} and J^{W} to $H^{2k-1}(M,\mathbb{C}) \simeq H^{2k-1}(M,\mathbb{R}) \otimes \mathbb{C}$ can also be defined through the equation

$$J^{\text{G}}\omega^{(p,q)} = -i^{\operatorname{sgn}(p-q)}\omega^{(p,q)} \qquad \text{and} \qquad J^{\text{W}}\omega^{(p,q)} = -i^{p-q}\omega^{(p,q)} \tag{1.68}$$

for $\omega^{(p,q)} \in H^{(p,q)}(M) \subseteq H^{2k-1}(M,\mathbb{C})$.

Using the following rather trivial lemma, we can also give a description of these intermediate Jacobians as quotient spaces of a direct su'm of Dolbeault cohomology groups.

Lemma 1.4.18. *Let H be a real linear space with complexification $H_{\mathbb{C}} = H \otimes \mathbb{C}$ and let $V < H_{\mathbb{C}}$ be a linear subspace such that $H_{\mathbb{C}} = V \oplus \overline{V}$. The composition*

$$\varphi = \pi_V \circ i \colon H \to H_{\mathbb{C}} = V \oplus \overline{V} \to V \tag{1.69}$$

of the inclusion map $i\colon H \hookrightarrow H_{\mathbb{C}}$ and the projection $\pi\colon H_{\mathbb{C}} = V \oplus \overline{V} \to V$ is a isomorphism between real linear spaces.

Proof: It is easily shown that the map

$$\varphi^{-1}\colon V \to H^{2k-1}(M,\mathbb{R}), \qquad \alpha \mapsto \alpha_+ + \overline{\alpha}_+. \tag{1.70}$$

is the inverse of $\varphi = \pi_V \circ i$. □

By applying this lemma to the spaces $H^{2k-1}(M,\mathbb{R})$, $H^{2k-1}(M,\mathbb{R})_{\mathbb{C}} \simeq H^{2k-1}(M,\mathbb{C})$ and the eigenspaces $V_\pm$ of a complex structure on $H^{2k-1}(M,\mathbb{R})$, we obtain a real linear isomprhism $\varphi\colon H^{2k-1}(M,\mathbb{R}) \to V$.

***Corollary 1.4.19* (Alternative description).** *Let $(\mathcal{J}_k(M), J)$ either be the k-th Weil or Griffiths intermediate Jacobian and let $V < H^{2k-1}(M,\mathbb{C}) \simeq \mathrm{T}_{\mathbb{C}}\mathcal{J}_k(M)$ be the eigenspace corresponding to the eigenvalue $+i$. If we write $\varphi\colon H^{2k-1}(M,\mathbb{R}) \to V$ for the composition of the inclusion $H^{2k-1}(M,\mathbb{R}) \hookrightarrow H^{2k-1}(M,\mathbb{C})$ and the projection $H^3(M,\mathbb{C}) = V \oplus \overline{V} \to V$, then it induces an isomorphism (of complex Lie groups)*

$$\mathcal{J}_k(M) = \frac{H^{2k-1}(M,\mathbb{R})}{H^{2k-1}(M,\mathbb{Z})_{\mathrm{f}}} \simeq \frac{V}{\varphi(H^{2k-1}(M,\mathbb{Z})_{\mathrm{f}})}. \tag{1.71}$$

Proof: We can write any $\alpha \in H^{2k-1}(M,\mathbb{R})$ as $\alpha = \alpha_+ + \bar{\alpha}_+$ with $\alpha_+ \in V$ and $\alpha_- = \bar{\alpha}_+ \in \overline{V}$ and we have $\varphi(\alpha) = \alpha_+$ by definition of the isomorophism φ. It follows that

$$\varphi(J\,\alpha) = \varphi(i\,\alpha_+ - i\,\alpha_-) = i\,\alpha_+ = i\,\varphi(\alpha), \tag{1.72}$$

so we can view the real linear map $\varphi\colon H^{2k-1}(M,\mathbb{R}) \to V$ as a complex linear isomorphism. This tells us that $\varphi(H^{2k-1}(M,\mathbb{R}))/\varphi(H^{2k-1}(M,\mathbb{Z})_{\mathrm{f}})$ and $(H^{2k-1}(M,\mathbb{R}),J)/H^{2k-1}(M,\mathbb{Z})_{\mathrm{f}}$ define the same complex torus. □

Both the Griffiths intermediate Jacobian and the Weil intermediate Jacobian come with a polarisation, but we will postpone their description until section 2.3, where we will be looking at the intermediate Jacobians on the middle cohomology group of a Calabi-Yau 3-fold. For general k, the the Griffiths intermediate Jacobian $\mathcal{J}_k^{\text{G}}$ and the Weil intermediate Jacobian $\mathcal{J}_k^{\text{W}}$ will not be isomorphic, but for $k = 1$ and $k = \dim_{\mathbb{C}} M$, the two tori coincide. The special cases $\mathcal{J}_1$ and $\mathcal{J}_{2n-1}$ are referred to as the Picard variety of M and the Albanese variety respectively. In general, the Weil intermediate Jacobian are very different spaces.

The Griffiths intermediate is often simply referred to as *the* intermediate Jacobian [7]. Unlike the Weil intermediate Jacobian it varies holomorphically when the Kähler structure used to define it is varied, which is often a great advantage. The Weil intermediate Jacobian on the other hand is an Abelian variety, something that is not generally true for the Griffiths intermediate Jacobian [8].

1.5 Quaternion-Kähler structures

A definition that is commonly used for quaternion-Kähler manifolds is that they are $4n$-dimensional Riemannian manifold (M, g) whose holonomy group is contained in the group $\mathrm{Sp}(1).\,\mathrm{Sp}(n) = \big(\mathrm{Sp}(1)\times\mathrm{Sp}(n)\big)/\{\pm I\}$ [20,5]. With this definition any orientable 4-dimensional manifold would be quaternion-Kähler since $\mathrm{Sp}(1).\,\mathrm{Sp}(1) \simeq \mathrm{SO}(4)$. Many results that do hold for quaternion-Kähler manifolds of dimension greater than 4 do not hold for general orientable 4-dimensional manifolds, which is why a more restrictive definition is sometimes used in this case [5]. It will be assumed that $n > 1$.

A different, but equivalent way of defining quaternion-Kähler manifolds is through a parallel bundle of endomorphisms [5, 21]. It is this definition that we will focus on.

Definition 1.5.1 **(Quaternion-Kähler manifold).** *A quaternion-Kähler manifold (M, g, H) is a Riemannian manifold (M, g) with a 3-dimensional subbundle H of $\mathrm{End}(\mathrm{T}M)$ such that*

a. *The vector bundle H locally admits a basis I, J, K that satisfy the quaternionic algebra,*

$$I^2 = J^2 = K^2 = -\,\mathrm{id}_{\mathrm{T}M}, \tag{1.73a}$$

$$I\,J = -J\,I = K, \qquad J\,K = -K\,J = I, \qquad K\,I = -I\,K = J, \tag{1.73b}$$

where the first equation just says that I, J and K are almost complex structures.

b. *The metric g is Hermitian for any section L of H for which $L^2 = -\,\mathrm{id}_{\mathrm{T}M}$, i.e.*

$$g(L\,X, L\,Y) = g(X, Y) \tag{1.74}$$

for any such section L and any two vector fields X and Y on M.

c. *H is parallel with respect to the Levi-Civita connection ∇ on (M, g).*
In other words: $\nabla_X I$, $\nabla_X J$ and $\nabla_X K$ are linear combinations of I, J and K for any local basis I, J, K for H.

Even though it is not part of this definition, the (real) dimension of a quaternion-Kähler manifold is always divisible by 4. We will briefly mention a few other important properties of quaternion-Kähler manifolds.

Let (g, H) be a quaternion-Kähler structure and let I, J, K be a local basis of almost complex structures in $H < \mathrm{End}(\mathrm{T}M)$ satisfying equation (1.73). For a set of real numbers $x, y, z \in \mathbb{R}$, the endomorphism $L = x\,I + y\,J + z\,K$ is also an almost complex structure exactly when $x^2 + y^2 + z^2 = 1$. This tells us that inside H there is an S^2-bundle of (local) almost complex structures, which one can show is also parallel with respect to the Levi-Civita connection for g and which complete describes the quaternion-Kähler structure.

Remark 1.5.2. *Quaternion-Kähler manifolds are also sometimes called quaternionic Kähler manifolds. Although quaternion-Kähler manifolds are always quaternionic, they will in general* not *be Kähler. In the definition of the quaternion-Kähler structure We do not require the existence of a global basis of almost complex structures, so a quaternion-Kähler manifold need not even be an almost complex manifold.*

One particularly interesting property of quaternion-Kähler manifolds (of dimension greater than 4) is expressed by the following theorem [22, 23].

Theorem 1.5.3 **(Einstein property).** *A quaternion-Kähler manifold (M, g, H) of dimension greater than 4 is Einstein, i.e. its Ricci tensor is given by $\mathrm{Ric}_g = \lambda\,g$ for some constant $\lambda \in \mathbb{R}$.*

The scalar curvature of a quaternion-Kähler manifold (M, g, H) of dimension $4\,n$ with $\mathrm{Ric} = \lambda\,g$ is given by $4\,n\,\lambda$. A special class of quaternion-Kähler manifolds are the hyperkähler manifolds, for which the almost complex structures I, J and K are globally defined and integrable, which means that they define three Kähler structures (I,g), (J,g) and (K,g) such that $I\,J = -J\,I = K$. These manifolds can alternatively be defined as quaternion-Kähler manifolds with vanishing scalar curvature or as Riemannian manifolds with holonomy contained in $\mathrm{Sp}(n)$.

Quaternion-Kähler manifolds are often studied by looking at their twistor spaces, which are complex manifolds, unlike the quaternion-Kähler manifolds themselves [20, 5, 24].

Definition 1.5.4 **(Twistor space).** *The* **twistor space** Z *of a* $4n$*-dimensional quaternion-Kähler manifoxld* (M,g,H) *is defined as the total space of the* S^2*-bundle of almost complex structures inside* H*, i.e.*

$$Z = \left\{ L \in H_x \mid x \in M, L^2 = -\,\mathrm{id}_{\mathrm{T}_x M} \right\}. \tag{1.75}$$

This is a manifold of (real) dimension $4\,n{+}2$ *and it comes with a canonical (integrable) complex structure.*

The twistor space Z of a quaternion-Kähler manifold (M,g,H) is in fact a complex contact manifold, with a contact bundle defined by the horizontal directions in $Z \subseteq H < \mathrm{End}(\mathrm{T}M)$ with respect to the Levi-Civita connection. If the scalar curvature of g is positive, then Z is in fact a Kähler(-Einstein) manifold. A lot can already be said about quaternion-Kähler manifolds with positive scalar curvature through their twistor spaces, but not much is known about those with negative curvature, such as the quaternion-Kähler structure on the hypermultiplet moduli space, which is described in chapter 6.

2. Calabi-Yau 3-folds and their intermediate Jacobians

In this chapter we will introduce the notion of a Calabi-Yau manifold and Calabi-Yau 3-folds in particular and present a number of important properties of these spaces. We will give a description of the moduli space of geometric deformations of Calabi-Yau 3-folds, which we will need in section 3.2. Finally, the Griffiths and Weil intermediate Jacobians of Calabi-Yau 3-folds and their canonical metrics will be discussed.

2.1 Calabi-Yau manifolds

There are many different definitions for Calabi-Yau manifolds in use in both the physics and mathematics literature [5, 25, 26, 27, 28, 29, 30], not all of which are equivalent. The definition on which we will focus is the following [25].

***Definition 2.1.1* (Calabi-Yau manifold).** *A* **Calabi-Yau manifold** $(\mathcal{Y}, J, g)$ *is a compact Kähler manifold whose canonical bundle, $K_{\mathcal{Y}}$, is trivial. We will only consider connected Calabi-Yau manifolds whose fundamental groups is finite, $|\pi_1(\mathcal{Y})| < \infty$.*

The canonical bundle of a complex manifold of dimension n is defined to be the holomorphic vector bundle of $(n, 0)$-forms. Triviality of the canonical bundle of an n-dimensional Calabi-Yau manifold $\mathcal{Y}$ is equivalent to the existence of a (globally defined) nowhere vanishing holomorphic n-form $\Omega \in \Omega^n(\mathcal{Y})$. This n-form Ω is closed since $\mathrm{d}\Omega = \bar{\partial}\Omega = 0$, so it can be used to represents a class $[\Omega] \in H^{3,0}(\mathcal{Y})$. This class is non-trivial since the requirement that Ω (and hence also $\bar{\Omega}$) is nowhere-vanishing implies that $\int_{\mathcal{Y}} \Omega \wedge \bar{\Omega} \neq 0$.

It can easily be shown that any Calabi-Yau manifold $\mathcal{Y}$ has a vanishing (real) first Chern class, but if definition 2.1.1 is used the converse is not generally true. Calabi-Yau manifolds owe their name to Yau's theorem, which was originally conjectured by Calabi [31] and finally proven by Yau [32,33]. The theorem can in particular be used to find Ricci-flat Kähler metrics on any Kähler manifold with vanishing first Chern class and thus in particular on Calabi-Yau manifolds [32, 33, 5, 25, 26].

***Theorem 2.1.2* (Yau).** *Let (M, J, g) be a Kähler manifold with vanishing first Chern class and let $[\omega]$ be the cohomology class of its Kähler form $g(J\bullet, \bullet)$. The (complex) manifold (M, J) admits a unique Ricci-flat Kähler metric whose Kähler form is contained in $[\omega]$.*

Calabi-Yau manifolds are often defined as Kähler manifolds whose first Chern class vanishes and Ricci-flat Kähler metrics on such manifolds are called ***Calabi-Yau metrics*** [5]. Any Ricci-flat Kähler metric on a compact manifold M has a vanishing first (real) Chern class, so the existence of such metrics is equivalent to this definition. Alternatively, this condition can be described using holonomy since a given Riemannian metric is both Ricci-flat and Kähler (for some complex structure) exactly when its *restricted* holonomy group group is contained in $\mathrm{SU}(n)$ [27], where n is the complex dimension of the manifold M. The proposition below allows us to compare this with definition 2.1.1 and in particular tells us that every definition given thus far is equivalent when $\mathcal{Y}$ is simply connected.

Proposition 2.1.3. *Let (M, J, g) be a compact Ricci-flat Kähler manifold of (complex) dimension n, then (M, J) is Calabi-Yau according to definition 2.1.1 if and only if the (global) holonomy group of g is contained in $\mathrm{SU}(n)$.*

We will only be concerned with Calabi-Yau manifolds of (complex) dimension 3, which we will refer to as ***Calabi-Yau* 3*-folds***. By using the Calabi-Yau property, a lot can already be said about the Hodge structure of such manifolds and with an extra condition on the fundamental group, the entire Hodge diamond is determined by two numbers, $h^{1,1}$ and $h^{1,2}$.

***Theorem 2.1.4* (Hodge diamond).** *The* **Hodge diamond** *of a connected Calabi-Yau 3-fold $\mathcal{Y}$ with finite fundamental group is of the form*

$$\begin{array}{ccccccc} & & & h^{3,3} & & & \\ & & h^{3,2} & & h^{2,3} & & \\ & h^{3,1} & & h^{2,2} & & h^{1,3} & \\ h^{3,0} & & h^{2,1} & & h^{1,2} & & h^{0,3} \\ & h^{2,0} & & h^{1,1} & & h^{0,2} & \\ & & h^{1,0} & & h^{0,1} & & \\ & & & h^{0,0} & & & \end{array} = \begin{array}{ccccccc} & & & 1 & & & \\ & & 0 & & 0 & & \\ & 0 & & h^{1,1} & & 0 & \\ 1 & & h^{1,2} & & h^{1,2} & & 1 \\ & 0 & & h^{1,1} & & 0 & \\ & & 0 & & 0 & & \\ & & & 1 & & & \end{array} \tag{2.1}$$

and is thus in particular determined completely by the two Hodge numbers $h^{1,1}$ and $h^{1,2}$.

We had already seen in section 1.3.3 that the Hodge numbers $h^{p,q}$ satisfy $h^{p,q} = h^{q,p} = h^{n-p,n-q} = h^{n-q,n-p}$, but for Calabi-Yau manifolds there is another symmetry. For any q the holomorphic 3-form $\Omega \in \Omega^3(\mathcal{Y})$ induces an isomorphism $H^{0,q}(\mathcal{Y}) \xrightarrow{\sim} H^{n,q}(\mathcal{Y}), \alpha \mapsto \alpha \wedge \Omega$, which tells us that $h^{0,q} = h^{n,q}$.

The assumptions that $\mathcal{Y}$ is connected and has a finite fundamental group finally tell us that $H^0(\mathcal{Y}, \mathbb{C}) \simeq \mathbb{C}$ and $H^1(\mathcal{Y}, \mathbb{C}) = 0$ and hence that $h^{0,0} = 1$ and $h^{1,0} = h^{0,1} = 0$. Together with the symmetries described above this leaves only Hodge diamonds of the form (2.1). This result will prove to be very important because the hypermultiplet moduli space will be described nearly entirely in terms of these cohomology groups.

2.1.1 The middle cohomology group

The cohomology groups of a Calabi-Yau manifold play a very important role for the structure of their moduli spaces and for the fields that are present in the string theory models compactified on such manifolds. If we look at the Hodge diamond in equation (2.1), we see that the interesting real cohomology groups of a Calabi-Yau 3-fold $\mathcal{Y}$ are $H^3(\mathcal{Y}, \mathbb{R})$ and $H^2(\mathcal{Y}, \mathbb{R}) \simeq H^4(\mathcal{Y}, \mathbb{R})$ and that these have dimension $2\,(1 + h^{1,2})$ and $h^{1,1}$ respectively. In this thesis, we will mostly be interested in the ***middle cohomology group***, $H^3(\mathcal{Y}, \mathbb{R})$, of the Calabi-Yau 3-fold $\mathcal{Y}$.

***Definition 2.1.5* (Intersection form).** *Let $\mathcal{Y}$ be an oriented Calabi-Yau 3-fold. For $R = \mathbb{Z}, \mathbb{R}, \mathbb{C}$ we define the* **intersection form** *on $H^3(\mathcal{Y}, R)$ as the anti-symmetric bilinear form*

$$Q\colon H^3(X, R) \times H^3(X, R) \to R, \qquad (\alpha, \beta) \mapsto \int_{\mathcal{Y}} \alpha \wedge \beta \tag{2.2}$$

***Proposition 2.1.6* (Symplectic basis).** *The free part of the integral cohomology group $H^3(\mathcal{Y},\mathbb{Z})_\mathrm{f}$ admits a a basis $(\alpha_i,\beta^i)_{i=0}^{h^{1,2}}$ such that for all $i,j\in\{0,\dots,h^{1,2}\}$*

$$Q(\alpha_i,\alpha_j)=Q(\beta^i,\beta^j)=0 \qquad \text{and} \qquad Q(\alpha_i,\beta^j)=\delta_i^j. \tag{2.3}$$

We call such a basis a **symplectic basis***.*

The intersection form will play a very important role, as will become apparent later, as will the existence of a symplectic basis for $H^3(\mathcal{Y},\mathbb{Z})_\mathrm{f}$ with respect to this form. It follows that we can write any $\alpha\in H^3(\mathcal{Y},R)$ as $\alpha=A^i\alpha_i-B_i\beta^i$ for $A^i,B_i\in R$, where R is either $\mathbb{Z}$, $\mathbb{R}$ or $\mathbb{C}$.

We can use the intersection form to construct a Hermitian form h on $H^3(\mathcal{Y},\mathbb{C})$ that respects the Hodge decomposition.

***Corollary 2.1.7* (Hermitian form).** *The sesquilinear form h on $H^3(\mathcal{Y},\mathbb{C})$, defined by*

$$h\colon H^3(\mathcal{Y},\mathbb{C})\times H^3(\mathcal{Y},\mathbb{C})\to\mathbb{C},\qquad (\alpha,\beta)\mapsto 2\,i\,Q(\alpha,\bar\beta)=2\,i\int_{\mathcal{Y}}\alpha\wedge\bar\beta \tag{2.4}$$

is Hermitian and the Hodge decomposition

$$H^3(\mathcal{Y},\mathbb{C})=H^{3,0}(\mathcal{Y})\oplus H^{2,1}(\mathcal{Y})\oplus H^{1,2}(\mathcal{Y})\oplus H^{0,3}(\mathcal{Y}) \tag{2.5}$$

is orthogonal with respect to it. h is positive definite on $H^{3,0}(\mathcal{Y})\oplus H^{1,2}(\mathcal{Y})$ and negative definite on $H^{2,1}(\mathcal{Y})\oplus H^{0,3}(\mathcal{Y})$.

Proof: That h is sesquilinear follows from its definition and that it is Hermitian is easily verified using the anti-symmetry Q. For any $\alpha\in H^{p,q}$ and $\beta\in H^{p',q'}$ with $p+q=p'+q'=3$, we have that $\bar\beta\in H^{q',p'}$ and therefore that $\int\alpha\wedge\bar\beta=0$ unless $p+q'=p'+q=3$, i.e. $p=p'$ and $q=q'$. Orthogonality of the Hodge decomposition with respect to h follows.

Since $H^1(\mathcal{Y},\mathbb{C})=0$ we know that the entire third cohomology group of $\mathcal{Y}$ is primitive, which can be used to show that $\alpha=i^{q-p}*\alpha=i^{3-2p}*\alpha=-i\,(-1)^p*\alpha$ for any $\alpha\in H^{p,q}$ with $p+q=3$ [6]. Thus

$$\begin{aligned} h(\alpha,\alpha)&=2\,i\int_{\mathcal{Y}}\alpha\wedge\bar\alpha=2\,i\int_{\mathcal{Y}}\alpha\wedge\overline{-i(-1)^p*\alpha}\\ &=-2\,(-1)^p\int_{\mathcal{Y}}\alpha\wedge\overline{*\alpha}=-2(-1)^p\|\alpha\|^2_{L^2}. \end{aligned} \tag{2.6}$$

Since the L^2-norm is positive definite, we can immediately read off that h is positive definite on the spaces $H^{p,q}(\mathcal{Y})$ for p odd and negative definite for p even. □

We can use this Hermitian form to define a projection map P that we will need a few times in the sections that follow.

***Lemma 2.1.8*.** *Let Ω be a holomorphic 3-form on $\mathcal{Y}$, then the complex linear map*

$$P\colon H^3(\mathcal{Y},\mathbb{C})\to H^3(\mathcal{Y},\mathbb{C}),\qquad \alpha\mapsto\frac{h(\alpha,\Omega)\,\Omega-h(\alpha,\bar\Omega)\,\bar\Omega}{h(\Omega,\Omega)} \tag{2.7}$$

projects $H^3(\mathcal{Y},\mathbb{C})$ onto $H^{3,0}(\mathcal{Y})\oplus H^{0,3}(\mathcal{Y})$ along $H^{2,1}(\mathcal{Y})\oplus H^{1,2}(\mathcal{Y})$.

Proof: Note that $h(\bar\Omega,\bar\Omega)=2\,i\,Q(\bar\Omega,\Omega)=-2\,i\,Q(\Omega,\bar\Omega)=-h(\Omega,\Omega)$. Since we know from corollary 2.1.7 that the Hodge decomposition is orthogonal with respect to h, it follows from a simple calculation that $P(\alpha)=\alpha$ for $\alpha=a\,\Omega+b\,\bar\Omega\in H^{3,0}(\mathcal{Y})\oplus H^{0,3}(\mathcal{Y})$ and that $P(\beta)=0$ for $\beta\in H^{1,2}(\mathcal{Y})\oplus H^{2,1}(\mathcal{Y})$. □

2.2 The moduli space of deformations

Yau's theorem tells us that any Calabi-Yau manifold admits a Ricci-flat Kähler metric and that such metrics are completely determined by a complex structure and the cohomology class of its Kähler form. Since this Ricci flat Kähler metric will to correspond to dynamical variables in the effective supergravity theories we will consider, we are very much interested in the possible variations of this Calabi-Yau structure. If we are given a complex structure, then the family of Kähler metrics can be identified with the so-called Kähler cone.

Lemma/Definition 2.2.1 **(Kähler cone).** *The* **Kähler cone** *of a complex manifold* (M, J) *is the space*

$$\mathcal{K}_J = \left\{[\omega] \in H^2(\mathcal{Y}, \mathbb{R}) \mid \omega \text{ is a Kähler form}\right\}. \tag{2.8}$$

This space $\mathcal{K}_J$ *is a convex cone and an open subset of* $H^{1,1}(M) \cap H^2(M, \mathbb{R})$.

The statement that $\mathcal{K}_J$ is a convex cone, which means that $\lambda\,[\omega] + [\omega'] =\in \mathcal{K}_J$ for any $\lambda > 0$ and any $[\omega], [\omega'] \in \mathcal{K}_J$, can easily be verified since $\lambda\,\omega + \omega'$ is a Kähler form if ω and ω' are. The same is true for $\omega + \lambda\,\eta$ for any Kähler form ω, $\eta \in H^{1,1}(M) \cap H^2(M, \mathbb{R})$ a closed form and $\lambda > 0$ sufficiently small, which tells us that $\mathcal{K}_J \subseteq H^{1,1}(M) \cap H^2(M, \mathbb{R})$ is open.

Yau's theorem gives us the following corollary, which tells us about the space of Kähler structures on a Calabi-Yau manifolds [27].

Corollary 2.2.2. *Let* $(\mathcal{Y}, J)$ *be a complex manifold admitting Kähler structures with vanishing first Chern class, then the space of Ricci-flat Kähler metrics on* $(\mathcal{Y}, J)$ *is a smooth manifold of dimension* $h^{1,1}(\mathcal{Y})$ *and can be identified with the Kähler cone* $\mathcal{K}_J$.

The other type of deformations of the Calabi-Yau structure on $\mathcal{Y}$ are the deformations of the complex structure. The word deformation has a well-defined meaning in this context.

Definition 2.2.3 **(Universal family).** *A* **family of complex manifolds** *is a proper holomorphic submersion* $\phi\colon \mathscr{X} \to B$ *for two complex manifolds* $\mathscr{X}$ *and* B [6, 34]. *If* B *is connected and we fix a point* $0 \in B$*, then we say that* $\pi\colon \mathscr{X} \to B$ *is a family of deformations of the complex manifold* $X_0 = \pi^{-1}(0)$ *and we call any other fibre* $\pi^{-1}(b)$ *a* **deformation** *of* X_0.

This family of deformations for a complex manifold X_0 *is said to be* **complete** *if for any other family of complex manifolds* $\rho\colon \mathscr{Y} \to S$ *such that there exists a holomorphic isomorphism* $f_0\colon \rho^{-1}(s) \to X_0$ *for some* $s \in S$*, there exists a holomorphic map* $g\colon U \to B$ *defined on some neighbourhood* $U \subseteq S$ *of* s *with* $g(s) = 0$ *for which there exists another holomorphic function* $f\colon \rho^{-1}(U) \to \mathscr{X}$ *such that* $f|_{\rho^{-1}(s)} = f_0$ *and* $\pi \circ f = g \circ \rho$*. We call the family* **universal** *if the map* g *is unique for any such family.*

A general complex manifolds will not have a universal family of deformations, but fortunately for us Calabi-Yau manifolds do. The Hodge numbers, the triviality of the canonical bundle and the existence of Kähler structures are all invariant under small deformations, which means that we can easily restrict ourselves to those deformations that are themselves Calabi-Yau manifolds.

The ***(complex structure) moduli space*** of the Calabi-Yau manifold $\mathcal{Y}$ is the quotient of the space of all (integrable) complex structures on $\mathcal{Y}$ that come from a Calabi-Yau structure by the group of diffeomorphisms [26]. It can locally be identified with the base space of a universal family of deformations of $\mathcal{Y}$ [27], for which we have the following result.

Theorem/Definition 2.2.4. *Let* $(\mathcal{Y}, J)$ *be a Calabi-Yau manifold of dimension* n*, then the base space of a universal family of deformations of* $(\mathcal{Y}, J)$ *is a complex manifold of dimension* $h^{n-1,1}$.

This result, which is due to Tian [35] and Todorov [36], is far from trivial. By combining corollary 2.2.2 and theorem 2.2.4 a moduli space of general deformations of the Calabi-Yau structure (J, g) on $\mathcal{Y}$ is obtained, which is a real $(h^{1,1} + 2\,h^{1,2})$-dimensional manifold.

2.2.1 Deformations of the metric

One of the ways to look at the space of deformations of Calabi-Yau structures is by considering deformations of Ricci-flat Kähler metrics instead of looking at Kähler classes and complex structures. This approach is particularly interesting for us because it connects with the way these deformations appear through the compactification procedure in supergravity theory.

Let $\mathcal{Y}$ be a Calabi-Yau 3-fold with complex structure J and let g be a compactible Ricci flat Kähler metric on $\mathcal{Y}$. If we choose a set of local complex coordinates z^i on $\mathcal{Y}$ for $i = 1, 2, 3$, we can write the metric g as $g_{i\bar{\jmath}}\mathrm{d}z^i\mathrm{d}\bar{z}^j$ for some real coefficients $g_{i\bar{\jmath}}$. A general metric $g' = g + h$ can then (locally) be written as

$$g' = (g_{i\bar{\jmath}} + 2\,h_{i\bar{\jmath}})\mathrm{d}z^i\mathrm{d}\bar{z}^j + (h_{ij}\mathrm{d}z^i\mathrm{d}z^j + h_{\bar{\imath}\bar{\jmath}}\mathrm{d}\bar{z}^i\mathrm{d}\bar{z}^j), \tag{2.9}$$

with $h_{\bar{\imath}\bar{\jmath}} = \overline{h_{ij}}$ symmetric and $h_{i\bar{\jmath}} = \overline{h_{\bar{\imath}j}}$. We would like the deformed metric to satisfy the Calabi-Yau condition and therefore in particular be Ricci flat. This will put strong restrictions on the coefficients $h_{i\bar{\jmath}}$ and h_{ij}.

It is simpler to consider *infinitesimal* deformation of the metric instead of finite ones, so we write down $g' = g + \delta g$, where the infinitesimal deformation δg is given by

$$\delta g = 2\,\delta g_{i\bar{\jmath}}\mathrm{d}z^i\mathrm{d}\bar{z}^j + (\delta g_{ij}\mathrm{d}z^i\mathrm{d}z^j + \overline{\delta g_{ij}}\mathrm{d}\bar{z}^i\mathrm{d}\bar{z}^j), \tag{2.10}$$

for some infinitesimal parameters $\delta g_{i\bar{\jmath}} = \overline{\delta g_{i\bar{\jmath}}}$ and $g_{ij} = g_{ji}$. It may be shown that also the deformed metric $g' = g + \delta g$ is Ricci flat, up to first order in the deformation, if and only if the real $(1,1)$-form $k = i\,\delta g_{i\bar{\jmath}}\mathrm{d}z^i \wedge \mathrm{d}\bar{z}^j = \delta g(J\,\bullet, \bullet)$ and the $T^{1,0}\mathcal{Y}$-valued $(0,1)$-form $\ell = g^{i\bar{\jmath}}\delta g_{\bar{\jmath}\bar{k}}\mathrm{d}\bar{z}^k \otimes \frac{\partial}{\partial z^i}$ are harmonic [37,16]. The holomorphic 3-form Ω can subsequently be used to show that the latter of these is harmonic exactly when the $(2,1)$-form $\imath_\ell\Omega = \Omega(\ell(\bullet), \bullet, \bullet)$ is [36]. This gives us respective identifications between the infinitesimal deformations of the type $\delta g_{i\bar{\jmath}}$ and δg_{ij} of the Calabi-Yau metric g that preserve the Ricci-flatness of the metric and the spaces $H^{1,1}(\mathcal{Y}) \cap H^2(\mathcal{Y}, \mathbb{R})$ and $H^{2,1}(\mathcal{Y})$.

It can be shown that all such infinitesimal deformations can be extended to finite deformations to different Ricci-flat metrics. The deformations of the type $\delta g = \delta g_{i\bar{\jmath}}\mathrm{d}z^i\mathrm{d}\bar{z}^j$ with $\delta g(J, \bullet) \in H^{1,1}(\mathcal{Y}) \cap H^2(\mathcal{Y}, \mathbb{R})$ correspond exactly to the deformations of the metric parametrised by the Kähler cone. The other type of deformations, for which $\delta g = \delta g_{ij}\mathrm{d}z^i\mathrm{d}z^j + \overline{\delta g_{ij}}\mathrm{d}\bar{z}^i\mathrm{d}\bar{z}^j$ corresponds to an element of $H^{2,1}(\mathcal{Y})$, transform the metric g into a metric that is no longer Hermitian with respect to the complex structure J. The metric will however be Hermitian and even Kähler with respect to a *different* complex structure [30]. This gives us a complex $h^{1,2}$-dimensional family of complex structure deformations that locally corresponds exactly to the complex structure moduli.

2.2.2 Complex structure moduli

We consider the complex structure moduli space $\mathcal{M}_{\mathbb{C}}$ of a Calabi-Yau manifold $\mathcal{Y}$, which is locally the base space for a universal family of deformations $(\mathcal{Y}_t)_t$ and describe the projective special Kähler structure on it.

For a universal family of deformations $\mathscr{Y} \to U$ of the Calabi-Yau manifold $\mathcal{Y}$, the cohomology groups $H^3(X_b, \mathbb{C})$ for $b \in U$ define a vector bundle over the base space U and these can be combined to obtain a vector space $\mathcal{H}^3$ over $\mathcal{M}_{\mathbb{C}}$ called the ***Hodge bundle***.[1] Since the cohomology groups $H^3(\mathcal{Y}_t, \mathbb{C}) \simeq H^3(\mathcal{Y}_t, \mathbb{Z}) \otimes \mathbb{C}$ can be defined in terms of just

[1] This can actually be done for any family of complex manifolds [6].

the topology of $\mathcal{Y}_t$ and without any reference to the complex structure, nearby fibres of $\mathcal{H}^3$ can be identified, making $\mathcal{H}^3$ flat. This flatness is expressed through the (flat) ***Gauss-Manin connection*** ∇, which can be defined by requiring that (local) sections that take values in $H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}} < H^3(\mathcal{Y},\mathbb{C})$ are flat with respect to it.

The Hodge decomposition $H^3(\mathcal{Y}_t,\mathbb{C}) = \bigoplus_{p+q=3} H^{p,q}(\mathcal{Y}_t)$ does depend on the complex structure on $\mathcal{Y}_t$, which in particular means that the position of $H^{3,0}(\mathcal{Y}_t)$ inside $H^3(\mathcal{Y},\mathbb{C})$ changes as $t \in \mathcal{M}_{\mathbb{C}}$ is varied. The following theorem tells us that the complex structure moduli space $\mathcal{M}_{\mathbb{C}}$ can locally be viewed as a submanifold of the complex projective space $P\big(H^3(\mathcal{Y},\mathbb{C})\big) = \big\{\mathbb{C}\,\alpha \mid \alpha \in H^3(\mathcal{Y},\mathbb{C}) \setminus \{0\}\big\}$ [6, 25, 27].

Theorem/Definition 2.2.5 **(Period map).** *Let $(\mathcal{Y}, J)$ be a Calabi-Yau manifold, let $U \subseteq \mathcal{M}_{\mathbb{C}}$ be a contractible open subset in the complex structure moduli space for $\mathcal{Y}$ and let $0 \in U$. If we use the Gauss-Manin connection to identify the fibres $\mathcal{H}^3_t = H^3(\mathcal{Y}_t,\mathbb{C})$ of the Hodge bundle with $H^3(\mathcal{Y}_0,\mathbb{C})$ for $t \in U$, then any element $t \in \mathcal{M}_{\mathbb{C}}$ uniquely determines a line $H^{3,0}(\mathcal{Y}_t) < H^3(\mathcal{Y},\mathbb{C})$ and the map*

$$\Phi\colon U \to P(H^3(\mathcal{Y},\mathbb{C}), \qquad t \mapsto \big[H^{3,0}(\mathcal{Y}_t)\big], \tag{2.11}$$

is a holomorphic immersion. This map is called a **period map** *for the Calabi-Yau manifold $\mathcal{Y}$.*

This tells us that the cohomology groups $H^{3,0}(\mathcal{Y}_t) < H^3(\mathcal{Y}_t, \mathbb{C}$ define a holomorphic line bundle $\mathcal{H}^{3,0}$ inside $\mathcal{H}^3$. Another very important property of the period map is ***Griffiths transversality***, which basically says that under infinitesimal deformations of the complex structure on $\mathcal{Y}$, $H^{3,0}(\mathcal{Y}_t)$ can mix with $H^{2,1}(\mathcal{Y}_t)$, but not with $H^{0,3}(\mathcal{Y}_t)$ or $H^{1,2}(\mathcal{Y}_t)$.

Proposition 2.2.6 **(Griffiths transversality).** *Let Φ be a period map from definition 2.2.5 and let $t \in \mathcal{M}_{\mathbb{C}}$ be in the domain of Φ, then the image of $\mathrm{d}\Phi\colon \mathrm{T}\mathcal{M} \to \mathrm{T}_{\Phi(t)}P(H^3(\mathcal{Y},\mathbb{C}))$ is $P_*(H^{3,0}(\mathcal{Y}_t) \oplus H^{2,1}(\mathcal{Y}_t))$.*

The projection map $P\colon H^3(\mathcal{Y},\mathbb{C}) \setminus \{0\} \to P\big(H^3(\mathcal{Y},\mathbb{C})\big)$ gives $H^3(\mathcal{Y},\mathbb{C})$ the interpretation of a holomorphic line bundle. Because the period map $\Phi\colon \mathcal{M}_{\mathbb{C}} \to P\big(H^3(\mathcal{Y},\mathbb{C})\big)$ is an immersion this line bundle can be pulled back to a holomorphic line bundle on the complex structure moduli space $\mathcal{M}_{\mathbb{C}}$ whose fibre at $t \in \mathcal{M}_{\mathbb{C}}$ can be identified with the Dolbeault cohomology group $H^{3,0}(\mathcal{Y}_t)$.

If we choose a contractible open subset $U \subseteq \mathcal{M}_{\mathbb{C}}$, fix a point $0 \in U$ and write $\mathcal{Y} = \mathcal{Y}_0$, then a symplectic basis $(\alpha_i, \beta^i)_i$ for the intersection form on Q on $H^3(\mathcal{Y},\mathbb{Z})$ defines a flat basis of sections of $\mathcal{H}^3$ on U. If we subsequently let $\Omega\colon t \mapsto \Omega_t \in H^{3,0}(\mathcal{Y}_t)$ be a local holomorphic section of the holomorphic line bundle $\mathcal{H}^{3,0} < \mathcal{H}$, then we can write $\Omega_t = X^i(t)\alpha_i - F_i(t)\beta^i$, where X^i and F_i are the periods

$$X^i(t) = \int_{\gamma^i} \Omega_t \qquad \text{and} \qquad F_i(t) = \int_{\eta_i} \Omega_t \tag{2.12}$$

of Ω with respect to the basis $(\gamma^i, \eta_i)_i$ for $H_3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$ dual to $(\alpha_i, \beta_i)_i$. These periods are (local) holomorphic functions on $\mathcal{M}_{\mathbb{C}}$ and locally determine the point $t \in \mathcal{M}_{\mathbb{C}}$, but there is a large redundancy because $H^3(\mathcal{Y},\mathbb{C})$ has (complex) dimension $2 + 2\,h^{1,2}$ and $\dim_{\mathbb{C}} \mathcal{M}_{\mathbb{C}} = h^{1,2}$.

Theorem 2.2.7. *For some choice of symplectic basis $(\alpha_i, \beta^i)_i$ the periods $X^i = \int_{\gamma^i} \Omega$ locally define a set of complex projective coordinates on $\mathcal{M}_{\mathbb{C}}$. There exists a holomorphic function $F\colon \mathbb{C}^{1+h^{1,2}} \to \mathbb{C}$ that depends on the coordinates X^i and is homogeneous of degree 2 such that the periods $F_i = \int_{\eta_i} \Omega_t$ are are given by $F_i = \frac{\partial F(X)}{\partial X^i}$.*

Proof: Let Ω be a local holomorphic section of $\mathcal{H}^{3,0}$ on a contractible open subset $U \subseteq \mathcal{M}_{\mathbb{C}}$ as before. It is always possible to choose an integral symplectic basis $(\alpha_i, \beta^i)_i$ for $H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$

such that (locally) the complex structure on $\mathcal{Y}$ is determined entirely by the coordinates X^i and we can thus write $F_i(t) = F_i(X(t))$ [15]. This leaves only a redundancy of one complex coordinate, which corresponds to the rescaling of the holomorphic form Ω and hence of the coordinates X^i. Since the section Ω is only determined up to multiplication by a nowhere-vanishing holomorphic function by the complex structure on the Calabi-Yau manifold, we see that the coordinates X^i define a set of complex projective local coordinates for $\mathcal{M}_{\mathbb{C}}$.

It is not hard to show that the derivatives of the holomorphic $(3,0)$-form Ω,

$$\Omega_i(X) := \frac{\partial\Omega(X)}{\partial X^i} = \alpha_i - \frac{\partial F_j(X)}{\partial X^i}\beta^j \qquad \text{and} \qquad \frac{\partial\Omega(X)}{\partial \bar{X}^i} = -\frac{\partial F_j(X)}{\partial \bar{X}^i}\beta^j \tag{2.13}$$

are contained in $H^{3,0}(\mathcal{Y}) \oplus H^{2,1}(\mathcal{Y})$ as a consequence of Griffiths transversality.

Because $Q\big(H^{3,0}(\mathcal{Y}) \oplus H^{2,1}(\mathcal{Y}), H^{3,0}(\mathcal{Y}) \oplus H^{2,1}(\mathcal{Y})\big) = 0$ we see that

$$0 = Q\big(\Omega_k, \tfrac{\partial\Omega}{\partial \bar{X}^i}\big) = \int_{\mathcal{Y}} \Big(\alpha_k - \tfrac{\partial F_j}{\partial X^k}\beta^j\Big) \wedge \Big(-\tfrac{\partial F_j}{\partial \bar{X}^i}\beta^j\Big) = -\frac{\partial F_k}{\partial \bar{X}^i}, \tag{2.14}$$

from which we can conclude that $F_i(X)$ are holomorphic functions. A similar calculation furthermore shows that

$$\begin{aligned} 0 = Q(\Omega, \Omega_i) &= \int_{\mathcal{Y}} \big(X^j\alpha_j - F_j\beta^j\big) \wedge \Big(\alpha_i - \tfrac{\partial F_j}{\partial X^i}\beta^j\Big) \\ &= -X^j\tfrac{\partial F_j}{\partial X^i} + F_i = -X^j\tfrac{\partial F_j}{\partial X^i} - \tfrac{\partial X^j}{\partial X^i}F_j + 2\,F_i = -\tfrac{\partial(X^jF_j)}{\partial X^i} + 2\,F_i. \end{aligned} \tag{2.15}$$

In other words: The periods F_i can be expressed as the derivatives $\frac{\partial F(X)}{\partial X^i}$, where $F(X)$ is defined as the holomorphic function $F(X) = \frac{1}{2}X^iF_i(X)$, which is now homogenous of degree 2 by definition. □

An almost direct consequence of this is that the complex structure moduli space can be given the structure of a ***projective special Kähler manifold*** [12, 14, 15].

Corollary 2.2.8. *The complex structure moduli space $\mathcal{M}_{\mathbb{C}}$ is projective special Kähler. For a local holomorphic section Ω of the bundle $H^{3,0}(\mathcal{Y})$ and the projective coordinates $X^i = \int_{\gamma^i}\Omega$ from theorem 2.2.7, the function $F = \frac{1}{2}X^iF_i$ defines a prepotential for $\mathcal{M}_{\mathbb{C}}$ and the Kähler potential on $\mathcal{M}_{\mathbb{C}}$ is given by*

$$\begin{aligned} K_{\mathbb{C}} &= -\log\Big(i\textstyle\int_{\mathcal{Y}}\Omega_t \wedge \bar{\Omega}_t\Big) = -\log\big(\tfrac{1}{2}h(\Omega,\Omega)\big) \\ &= -\log\big(-i\,X^i\bar{F}_i + i\,\bar{X}^iF_i\big) = -\log\big(-X^iN_{ij}\bar{X}^j\big)\,, \end{aligned} \tag{2.16}$$

where $N_{ij} = 2\,\mathrm{Im}(F_{ij}) = 2\,\mathrm{Im}\big(\frac{\partial^2 F}{\partial X^i \partial X^j}\big)$ and $h = 2\,i\,Q(\bullet, \bar{\bullet})$ is the Hermitian form from corollary 2.1.7.

Proof: We can use the homogeneity of F to show that that $-i\,X^i\bar{F}_i + i\,\bar{X}^iF_i = -X^iN_{ij}X^j$ and an explicit calculation furthermore shows that

$$\int_{\mathcal{Y}}\Omega_t \wedge \bar{\Omega}_t = \int_{\mathcal{Y}}(X^i\alpha_i - F_i\beta^i) \wedge (\bar{X}^i\alpha_i - \bar{F}_i\beta^i) = -X^i\bar{F}_i + \bar{X}^iF_i, \tag{2.17}$$

which tells us that equation (2.16) is consistent. We will compare the situation we have with the situation in definition 1.3.26.

We have seen in theorem 2.2.7 that the periods $X^i = \int_{\eta^i}\Omega$ locally define complex projective coordinates on $\mathcal{M}_{\mathbb{C}}$ and that there exists a prepotential F that is homogeneous of degree 2 and satisfies $F_i = \int_{\gamma_i}\Omega = \frac{\partial F}{\partial X^i}$. On a different chart for $\mathcal{M}_{\mathbb{C}}$ two things can be different: The symplectic basis[2] $(\alpha_i, \beta^i)_i$ for Q and the choice for the holomorphic section Ω for $\mathcal{H}^{3,0}$. Any two symplectic bases $(\alpha_i, \beta^i)_i$ and $(\tilde{\alpha}_i, \tilde{\beta}^i)_i$ are related by a symplectic matrix $M \in \mathrm{Sp}(2\,h^{1,2}+2, \mathbb{R})$ and any other holomorphic section is given by $\tilde{\Omega} = f\,\Omega$ for some nowhere-vanishing holomorphic function f. This tells us that the corresponding functions $\tilde{X} = (\tilde{X}^i)_i$ and $\partial\tilde{F} = (\tilde{F}_i)_i$ such that $\tilde{\Omega} = \tilde{X}^i\tilde{\alpha}_i - \tilde{F}_i\tilde{\beta}^i$ are related to the original coordinates X and ∂F by

$$\begin{pmatrix} \partial\tilde{F} \\ \tilde{X} \end{pmatrix} = f\,M \begin{pmatrix} \partial F \\ X \end{pmatrix}. \tag{2.18}$$

Here we have used that $\Omega = (\alpha, \beta) \cdot \Sigma \cdot \left(\begin{smallmatrix} \partial F \\ X \end{smallmatrix}\right)$, where $\Sigma = \left(\begin{smallmatrix} 0 & \mathbf{1}_n \\ -\mathbf{1}_n & 0 \end{smallmatrix}\right)$ is the standard $2(1+h^{1,2}) \times 2(1+h^{1,2})$ symplectic matrix and that for any symplectic matrix $M \in \mathrm{Sp}(2h^{1,2}+2, \mathbb{R})$ this is equal to $(\alpha, \beta)\,M^{\mathrm{t}} \cdot \Sigma \cdot M \left(\begin{smallmatrix} \partial F \\ X \end{smallmatrix}\right)$. The additional consistency condition from equation (1.45) is automatically satisfied by such functions f and matrices M because of the nature of their definition in terms of holomorphic sections and bases.

The only thing that we should still check is that the function $K_{\mathbb{C}}$ is a Kähler potential. By working out the derivatives explicitly, we can show that the matrix $K_{i\bar{j}}$ is given by

$$\begin{aligned} g_{\mathbb{C}} = i\,\partial\bar{\partial}K(\bullet, J\,\bullet) &= \frac{\partial^2 K(X, \bar{X})}{\partial X^i\,\partial\bar{X}^j}\mathrm{d}X^i\mathrm{d}\bar{X}^j \\ &= \frac{-1}{h(\Omega,\Omega)}\left(h(\Omega_i,\Omega_j) - \frac{h(\Omega_i,\Omega)\,h(\Omega,\Omega_j)}{h(\Omega,\Omega)}\right)\mathrm{d}X^i\mathrm{d}\bar{X}^j \\ &= \frac{-1}{XN\bar{X}}\left(N_{ij} - \frac{N_{ik}\bar{X}^k X^\ell N_{\ell j}}{XN\bar{X}}\right)\mathrm{d}X^i\mathrm{d}\bar{X}^j, \end{aligned} \tag{2.19}$$

where we have written $\Omega_i = \frac{\partial\Omega}{\partial X^i} = \alpha_i - F_{ij}\beta^j$.

By writing $h(\Omega_i, \Omega_j) = h(P\,\Omega_i, \Omega_j) + h((1-P)\Omega_i, \Omega_j)$ for the projection map P from lemma 2.1.8 and out the resulting expression we can rewrite the second line from equation (2.19) as

$$g_{\mathbb{C}} = \frac{-1}{h(\Omega,\Omega)}h\big((1-P)\Omega_i, \Omega_j\big)\mathrm{d}X^i\mathrm{d}\bar{X}^j \tag{2.20}$$

Corollary 2.1.7 tells us that for any i, $-h\big((1-P)\Omega_i, \Omega_i\big)$ is non-negative and only vanishes if $\Omega_i = \lambda\,\Omega = \lambda\,X^i\Omega_i$ (the last step follows from the homogeneity of F) for some $\lambda \in \mathbb{C}$. By the same corollary, $h(\Omega, \Omega) > 0$.

The tangent space to $\mathcal{M}_{\mathbb{C}}$ at t corresponds to $P_*\big(H^{3,0}(\mathcal{Y}_t) \oplus H^{2,1}(\mathcal{Y}_t)\big)$, where it the direction $H^{3,0}(\mathcal{Y}_t)$ that is projected out. We have $\mathrm{d}X^i(\Omega_j) = \frac{\partial}{\partial\varphi}|_{\varphi=0}\int_{\eta^i}(\Omega + \varphi\Omega_j) = \delta^i_j$ and $\mathrm{d}X^i(\Omega) = X^i$, so we see that the direction for which $-h\big((1-P)\Omega_i, \Omega_i\big)$ vanishes corresponds exactly to the direction that is projected out. We find that $g_{\mathbb{C}}$ is positive definite as a metric on $\mathcal{M}_{\mathbb{C}}$ and because it was obtained from a potential it is automatically Kähler. □

This Kähler metric $g_{\mathbb{C}}$ on the complex structure moduli space is often referred to as the ***Weil-Petersson metric*** [36].

2.3 The intermediate Jacobians

For the remainder of this chapter, $\mathcal{Y}$ will denote a Calabi-Yau 3-fold with a fixed Ricci-flat Kähler structure and we will fix an integral basis $(\alpha_i, \beta^i)_i$ for $H^3(\mathcal{Y}, \mathbb{Z})_{\mathrm{f}} \hookrightarrow H^3(\mathcal{Y}, \mathbb{C})$ that is symplectic with respect to the intersection form $Q = \int_{\mathcal{Y}} \bullet \wedge \bullet$. Since the Calabi-Yau 3-fold $\mathcal{Y}$

[2] N.B. The intersection form Q is defined on the Hodge bundle $\mathcal{H}^3$ because its definition is topological in nature.

has trivial first and fifth cohomology groups, its only non-trivial intermediate Jacobians use the middle cohomology group, $H^3(\mathcal{Y},\mathbb{R})$. These Jacobians use the space

$$\mathcal{J}_2(\mathcal{Y}) = H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}, \tag{2.21}$$

which has (real) dimension $b^3 = h^{3,0} + h^{2,1} + h^{1,2} + h^{0,3} = 2(1 + h^{1,2})$. As we have seen in section 1.4.2, there are different ways to define the complex structure on this space using its Hodge decomposition. This gives us the ***Griffiths intermediate Jacobian*** $J_2^{\mathrm{G}}(\mathcal{Y})$ and the ***Weil intermediate Jacobian*** $J_2^{\mathrm{W}}(\mathcal{Y})$, which we will simply write as $J^{\mathrm{G}} = J_2^{\mathrm{G}}(\mathcal{Y})$ and $J^{\mathrm{W}} = J_2^{\mathrm{W}}(\mathcal{Y})$.

Especially the Weil intermediate Jacobian will become important for the description of the hypermultiplet moduli space, as we will see in chapter 5. By using a number of properties for the intersection form $Q\colon H^3(\mathcal{Y},\mathbb{R}) \times H^3(\mathcal{Y},\mathbb{R}) \to \mathbb{R}$, we can show that it defines a polarisation on both tori.

***Proposition 2.3.1* (Polarisation).** *For any Calabi-Yau 3-fold $\mathcal{Y}$, the intersection form*

$$Q\colon H^3(\mathcal{Y},\mathbb{R}) \times H^3(\mathcal{Y},\mathbb{R}) \to \mathbb{R}, \qquad (\alpha,\beta) \mapsto \int_{\mathcal{Y}} \alpha \wedge \beta \tag{2.22}$$

defines a principal polarisation on both the Weil and the Griffiths intermediate Jacobian.

Proof: We already know that the intersection form is anti-symmetric annd non-degenerate and that it takes integer values on the lattice $H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$, so the only thing that remains to be checked is compatibility with the complex structure.

We can write any $\alpha \in H^3(\mathcal{Y},\mathbb{R})$ as $\alpha = \sum_{p+q=3} \alpha^{p,q}$ with $\alpha^{p,q} \in H^{p,q}(\mathcal{Y})$. If we denote (the complex extension of) the complex structure on intermediate Jacobian (either Weil or Griffiths) by J, then $J\,\alpha^{p,q} = \lambda^{p,q}\alpha^{p,q}$ for $\lambda^{p,q} \in \{i,-i\}$. The fact that J is a real endomorphism implies that $\lambda^{p,q} = \bar{\lambda}^{q,p}$ and we know that $Q(H^{p,q},H^{p',q'}) = 0$ unless $p+p' = q+q' = 3$, so

$$\begin{aligned} Q(J\,\alpha, J\,\beta) &= \sum_{p+q=3}\sum_{p'+q'=3} Q(J\,\alpha^{p,q}, J\,\beta^{p',q'}) = \sum_{p+q=3} Q(\lambda^{p,q}\,\alpha^{p,q}, \lambda^{q,p}\,\beta^{q,p}) \\ &= \sum_{p+q=3} \lambda^{p,q}\bar{\lambda}^{p,q} Q(\alpha^{p,q},\beta^{q,p}) = \sum_{p+q=3}\sum_{p'+q'=3} Q(\alpha^{p,q},\beta^{p',q'}) = Q(\alpha,\beta), \end{aligned} \tag{2.23}$$

from which we conclude that Q defines a polarisation on both $\mathcal{J}^{\mathrm{W}}$ and $\mathcal{J}^{\mathrm{G}}$. This polarisation is furthermore principal because we had already seen that there exists a symplectic basis for the lattics $H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$ with respect to Q. $\square$

2.3.1 The Griffiths intermediate Jacobian

The complex structure on the Griffiths intermediate Jacobian for a Calabi-Yau 3-fold $\mathcal{Y}$ is defined by the complex structure J on $H^3(\mathcal{Y},\mathbb{R})$ with eigenspaces V and $\overline{V}$ corresponding to the eigenvalues $+i$ and $-i$, where V and $\overline{V}$ come from the decomposition

$$H^3(\mathcal{Y},\mathbb{C}) = \underbrace{H^{3,0}(\mathcal{Y}) \oplus H^{2,1}(\mathcal{Y})}_{\overline{V}} \oplus \underbrace{H^{1,2}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})}_{V}. \tag{2.24}$$

We can also use the alternative description from corollary 1.4.19 for the intermediate Jacobian as a quotient of the complex linear space $V = H^{1,2}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})$. This description makes it easy to calculate the normalised period matrix of $\mathcal{J}^{\mathrm{G}}(\mathcal{Y})$.

***Proposition 2.3.2* (Period matrix).** *Let $(\alpha_i,\beta^i)_i$ be a symplectic basis for $H^3(\mathcal{Y},\mathbb{Z})$ with respect to the intersection form Q and let $\Omega = X^i\alpha_i - F_i\beta^i \in H^{3,0}(\mathcal{Y})$ be a holomorphic 3-form* *with periods X^i and F_i. The normalised period matrix for the Griffiths intermediate Jacobian of $\mathcal{Y}$ is given by the matrix $\Omega^{\mathrm{G}} = (\mathcal{Z}^{\mathrm{G}}, \mathbf{1})$ with*

$$\mathcal{Z}^{\mathrm{G}}_{ij} = F_{ij}, \tag{2.25}$$

where $F_{ij} = \frac{\partial F_i}{\partial Z^j} = \frac{\partial^2 F}{\partial X^i \partial X^j}$ is the Hessian of the prepotential $F = \frac{1}{2} X^i F_i$.

Proof: Let φ denote the canonical complex linear map $\varphi\colon H^3(\mathcal{Y},\mathbb{R}) \to H^{0,3}(\mathcal{Y}) \oplus H^{1,2}(\mathcal{Y})$ from corollary 1.4.19 and let π be the projection map $\pi\colon H^3(\mathcal{Y},\mathbb{C}) \to H^{0,3}(\mathcal{Y}) \oplus H^{1,2}(\mathcal{Y})$, then φ is simply the restriction of π to $H^3(\mathcal{Y},\mathbb{R})$. The derivative $\Omega_i = \frac{\partial \Omega}{\partial X^i} = \alpha_i - F_{ij}\beta^j$ is an element of $\overline{V}^{\mathrm{G}} = H^{3,0}(\mathcal{Y}) \oplus H^{2,1}(\mathcal{Y}) = \ker \pi$ by Griffiths transversality, which tells us that

$$\varphi(\alpha_i) - F_{ij}\varphi(\beta^j) = \pi^{\mathrm{G}}(\alpha_i) - F_{ij}\pi^{\mathrm{G}}(\beta^j) = \pi^{\mathrm{G}}(\alpha_i - F_{ij}\beta^j) = \pi^{\mathrm{G}}(\Omega_i) = 0, \tag{2.26}$$

so $\varphi(\alpha_i) = F_{ij}\varphi(\beta^j)$. This tells us that F_{ij} is the normalised period matrix for $V^{\mathrm{G}}/\varphi(H^3(\mathcal{Y},\mathbb{Z}))$ with respect to the symplectic basis $(\varphi(\alpha_i), \varphi(\beta_i))_i$. This torus can be identified with the Griffiths intermediate Jacobian by corollary 1.4.19. □

Let $(x^i, y_i)_i$ be the real coordinates on $H^3(\mathcal{Y},\mathbb{R})$ that correspond to the symplectic basis $(\alpha_i, \beta^i)_i$ and write the normalised period matrix from proposition 2.3.2 as $\mathcal{Z}^{\mathrm{G}}_{ij} = F_{ij} = \mathcal{X}^{\mathrm{G}}_{ij} + i\,\mathcal{Y}^{\mathrm{G}}_{ij}$, with $\mathcal{X}^{\mathrm{G}} = \operatorname{Re} \mathcal{Z}^{\mathrm{G}}$ and $\mathcal{Y}^{\mathrm{G}} = \operatorname{Im} \mathcal{Z}^{\mathrm{G}}$.

Lemma 2.3.3. *Let $h^{\mathrm{G}} = g^{\mathrm{G}} + i\,Q$ be the Hermitian form associated with Q, with $g^{\mathrm{G}} = Q(J^{\mathrm{G}}\bullet, \bullet)$, and let $\varphi\colon H^3(\mathcal{Y},\mathbb{R}) \to V = H^{1,2}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})$ be the complex linear isomorphism introduced in corollary 1.4.19. We have for any $\alpha, \beta \in H^3(\mathcal{Y},\mathbb{R})$ that*

$$h^{\mathrm{G}}(\alpha,\beta) = h(\varphi\alpha, \varphi\beta), \tag{2.27}$$

where $h = 2\,i\,Q(\bullet, \bar{\bullet})$ is the Hermitian form from corollary 2.1.7.

The polarisation on the Griffiths intermediate Jacobian defined by Q has index 1.

Proof: A simple calculation shows that

$$\begin{aligned} \operatorname{Im}[h(\varphi(\alpha), \varphi(\beta))] &= \operatorname{Im}[2\,i\,Q(\varphi(\alpha), \overline{\varphi(\beta)})] = Q(\varphi(\alpha), \overline{\varphi(\beta)}) + Q(\overline{\varphi(\alpha)}, \varphi(\beta)) \\ &= Q(\varphi(\alpha) + \overline{\varphi(\alpha)}, \varphi(\beta) + \overline{\varphi(\beta)}) = Q(\alpha, \beta) \end{aligned} \tag{2.28}$$

since $Q(V,V) = Q(\overline{V}, \overline{V}) = 0$ for $V = H^{1,2}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})$ and $\alpha = \varphi(\alpha) + \overline{\varphi(\alpha)}$ for α real. Because h is Hermitian it immediately follows that

$$\operatorname{Re}[h(\varphi(\alpha), \varphi(\beta))] = \operatorname{Im}[h(i\,\varphi(\alpha), \varphi(\beta))] = \operatorname{Im}[h(\varphi(J^{\mathrm{G}}\alpha), \varphi(\beta))] = Q(J^{\mathrm{G}}\alpha, \beta). \tag{2.29}$$

The signature of h and hence of h^{G} follows from corollary 2.1.7. □

***Corollary 2.3.4* (Canonical metric).** *The canonical metric $g^{\mathrm{G}} = Q(J^{\mathrm{G}}\bullet, \bullet)$ on the Griffiths intermediate Jacobian is given by*

$$g^{\mathrm{G}} = \mathcal{Y}_{\mathrm{G}}^{ij} \mathrm{d}z_i\, \mathrm{d}\bar{z}_j = 2\,N^{ij}(\mathrm{d}x_i + F_{ik}\mathrm{d}y^k)(\mathrm{d}x_j + \bar{F}_{j\ell}\mathrm{d}y^\ell), \tag{2.30}$$

where $z_i = x_i + \mathcal{Z}^{\mathrm{G}}_{ij} y^j$ are the standard complex coordinates on $\mathcal{J}^{\mathrm{G}}(\mathcal{Y})$ and $\mathcal{Y}_{\mathrm{G}}^{ij} = 2\,N^{ij}$ denotes the inverse matrix for $\mathcal{Y}^{\mathrm{G}}_{ij} = \operatorname{Im} \mathcal{Z}^{\mathrm{G}}_{ij} = \frac{1}{2} N_{ij}$.

This metric is positive definite on $\big(H^{2,1}(\mathcal{Y}) \oplus H^{1,2}(\mathcal{Y})\big) \cap H^3(\mathcal{Y},\mathbb{R})$ and negative definite on $\big(H^{3,0}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})\big) \cap H^3(\mathcal{Y},\mathbb{R})$ and these two subspaces are perpendicular for g^{G}.

Proof: Equation (2.30) follows directly from proposition 1.4.13 and the second result follows lemma 2.3.3 and corollary 2.1.7 □

2.3.2 The Weil intermediate Jacobian

As the complex structure on the Griffiths intermediate Jacobian, the complex structure on the Weil intermediate Jacobian is defined in terms of the Hodge decomposition for $H^3(\mathcal{Y},\mathbb{C})$. The complex structure J^{W} is defined by the eigenspaces V and $\overline{V}$ corresponding to the eigenvalues $+i$ and $-i$,

$$H^3(\mathcal{Y},\mathbb{C}) = \underbrace{H^{0,3}(\mathcal{Y}) \oplus H^{2,1}(\mathcal{Y})}_{\overline{V}} \oplus \underbrace{H^{1,2}(\mathcal{Y}) \oplus H^{3,0}(\mathcal{Y})}_{V}. \tag{2.31}$$

This choice enables us to give a very simple characterisation for the canonical metric on the Weil intermediate Jacobian.

Lemma 2.3.5. *The canonical metric g^{W} for the Weil intermediate Jacobian J^{W} coincides with the L^2-metric, so it is in particular positive definite.*

Additionally, the spaces $(H^{1,2}(\mathcal{Y})\oplus H^{2,1}(\mathcal{Y}))\cap H^3(\mathcal{Y},\mathbb{R})$ and $(H^{3,0}(\mathcal{Y})\oplus H^{0,3}(\mathcal{Y}))\cap H^3(\mathcal{Y},\mathbb{R})$ are mutually orthogonal with respect to g^{W}.

Proof: The extension of the J to the complex cohomology group $H^3(\mathcal{Y},\mathbb{C})$ is characterised by $J^{\mathrm{W}}\omega^{(p,q)} = -i^{p-q}\omega^{(p,q)}$ for $\omega^{(p,q)} \in H^{p,q}(\mathcal{Y})$ and we know from lemma 1.3.18 that $*\omega^{(p,q)} = i^{p-q}\omega^{p,q}$ for this a form. This tells us that $J^{\mathrm{W}}\omega = -*\omega$ for all $\omega \in H^3(\mathcal{Y},\mathbb{C})$ and hence that

$$g^{\mathrm{W}}(\alpha,\beta) = -Q(\alpha, J^{\mathrm{W}}\beta) = Q(\alpha, *\beta) = \int_{\mathcal{Y}} \alpha \wedge *\beta =: \langle \alpha,\beta\rangle_{L^2} \tag{2.32}$$

for all $\alpha,\beta \in H^3(\mathcal{Y},\mathbb{R})$. Because $g^{\mathrm{W}} = \langle \bullet,\bullet\rangle_{L^2}$, the Dolbeault cohomology groups are orthogonal with respect to it [6]. . $\square$

Computations with the Griffiths intermediate Jacobian in terms of the Hessian F_{ij} were relatively simple because the derivatives of the holomorphic 3-form Ω were contained in a single eigenspace of J^{G}. We can use these expressions for the Griffiths intermediate to find similar expressions for the Weil intermediate Jacobian, but things become significantly more messy.

	$H^{3,0}(\mathcal{Y})$	$H^{2,1}(\mathcal{Y})$	$H^{1,2}(\mathcal{Y})$	$H^{0,3}(\mathcal{Y})$
J^{W}	$+i$	$-i$	$+i$	$-i$
J^{G}	$-i$	$-i$	$+i$	$+i$
$J^{\mathrm{W}}J^{\mathrm{G}}$	$+1$	-1	-1	$+1$
P	$+1$	0	0	$+1$

Table 2.1: The eigenspaces and eigenvalues of J^{W}, J^{G}, $J^{\mathrm{W}}J^{\mathrm{G}}$ and $P = \frac{1}{2}(1 + J^{\mathrm{W}}J^{\mathrm{G}})$.

Lemma 2.3.6. *The complex structures J^{W} and J^{G} for the Weil and the Griffiths intermediate Jacobian commute. The map $P = \frac{1}{2}(1 + J^{\mathrm{W}}J^{\mathrm{G}})$ projects $H^3(\mathcal{Y},\mathbb{R})$ onto $\big(H^{3,0}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})\big) \cap H^3(\mathcal{Y},\mathbb{R})$ along $\big(H^{2,1}(\mathcal{Y}) \oplus H^{1,2}(\mathcal{Y})\big) \cap H^3(\mathcal{Y},\mathbb{R})$ and coincides the projection P from lemma 2.1.8.*

Proof: Both complex structures were defined using the Hodge decomposition and their eigenvalues and eigenspaces have been illustrated in table 2.1. We immediately see that J^{W} and J^{G} commute and that their product equals the identity on $\big(H^{3,0}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})\big) \cap H^3(\mathcal{Y},\mathbb{R})$ and minus the identity on $\big(H^{2,1}(\mathcal{Y}) \oplus H^{1,2}(\mathcal{Y})\big) \cap H^3(\mathcal{Y},\mathbb{R})$. It follows that P is the desired projection map and thus corresponds with the projection P we had defined before. $\square$

We can give an alternative description for the projection map P by using the Hermitian form $h = 2\,i\,Q(\bullet, \bar{\bullet})$ from corollary 2.1.7.

We can use the projection map P to relate the metrics on the Weil and the Griffiths intermediate Jacobian.

Proposition 2.3.7 **(Canonical metric).** *The spaces* $\ker P = \big(H^{2,1}(\mathcal{Y}) \oplus H^{1,2}(\mathcal{Y})\big) \cap H^3(\mathcal{Y}, \mathbb{R})$ *and* $\operatorname{im} P = \big(H^{3,0}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})\big) \cap H^3(\mathcal{Y}, \mathbb{R})$ *are orthogonal with respect to both* g^{W} *and* g^{G}*. We have* $g^{\mathrm{W}}|_{\ker P} = g^{\mathrm{G}}|_{\ker P}$ *and* $g^{\mathrm{W}}|_{\operatorname{im} P} = -g^{\mathrm{G}}|_{\operatorname{im} P}$ *and we can write*

$$g^{\mathrm{W}} = g^{\mathrm{G}} - 2\,g^{\mathrm{G}}(P\bullet, \bullet) \qquad \text{and} \qquad g^{\mathrm{G}} = g^{\mathrm{W}} - 2\,g^{\mathrm{W}}(P\bullet, \bullet), \tag{2.33}$$

where P is the projection map from lemma 2.3.6 and lemma 2.1.8.

Proof: These properties can be derived directly from the behaviour of the intersection form Q with respect to the Hodge decomposition and table 2.1, but we can also take a different approach. Since we can define P as $P = \frac{1}{2}(1 + J^{\mathrm{W}} J^{\mathrm{G}})$, we have $2\,J^{\mathrm{G}} P = J^{\mathrm{G}} - J^{\mathrm{W}}$ and thus

$$g^{\mathrm{G}} - 2\,g^{\mathrm{G}}(P\bullet, \bullet) = Q(J^{\mathrm{G}}(1 - 2\,P)\bullet, \bullet) = Q(J^{\mathrm{W}}\bullet, \bullet) = g^{\mathrm{W}}. \tag{2.34}$$

This derivation can be repeated with the labels W and G interchanged to obtain the second relation and the equality $g^{\mathrm{W}}(P\bullet, \bullet) = -g^{\mathrm{G}}(P\bullet, \bullet)$.

We already know from lemma 2.3.5 that the decomposition $H^3(\mathcal{Y}, \mathbb{R}) = \operatorname{im} P \oplus \ker P$ is orthogonal with respect g^{W} and because $P^2 = P$ we have that $P\,\alpha' = \alpha'$ for $\alpha' \in \operatorname{im} P$. Equation (2.33) now tells us that for $\alpha = \alpha' + \alpha''$ and $\beta = \beta' + \beta''$ with $\alpha', \beta' \in \operatorname{im} P$ and $\alpha'', \beta'' \in \ker P$,

$$\begin{aligned} g^{\mathrm{G}}(\alpha, \beta) &= g^{\mathrm{W}}(\alpha, \beta) - 2\,g^{\mathrm{W}}(P\,\alpha, \beta) \\ &= g^{\mathrm{W}}(\alpha', \beta') + g^{\mathrm{W}}(\alpha'', \beta'') - 2\,g^{\mathrm{W}}(\alpha'', \beta' + \beta'') \\ &= g^{\mathrm{W}}(\alpha', \beta') - g^{\mathrm{W}}(\alpha'', \beta''), \end{aligned} \tag{2.35}$$

which tells us that $g^{\mathrm{W}} = g^{\mathrm{G}}$ on $\ker P$ and $g^{\mathrm{W}} = -g^{\mathrm{G}}$ on $\operatorname{im} P$. □

Lemma 2.3.8. *The restriction* $g^{\mathrm{W}}|_{\operatorname{im} P} = -g^{\mathrm{G}}|_{\operatorname{im} P}$ *of the canonical metrics* g^{W} *and* g^{G} *to* $\operatorname{im} P$ *is given by the equation*

$$g^{\mathrm{W}}(\alpha, \beta) = \frac{1}{2\,h(\Omega, \Omega)} \Big(h(\alpha, \Omega)\,\overline{h(\beta, \Omega)} + h(\beta, \Omega)\,\overline{h(\alpha, \Omega)} \Big) \tag{2.36}$$

for $\alpha, \beta \in \operatorname{im} P = (H^{3,0}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})) \cap H^3(\mathcal{Y}, \mathbb{R})$*. Here* $\Omega \in H^{3,0}(\mathcal{Y})$ *denotes a holomorphic 3-form on* $\mathcal{Y}$ *and* $h = 2\,i\,Q(\bullet, \bar{\bullet})$*.*

Proof: Since for $\alpha, \beta \in H^3(\mathcal{Y}, \mathbb{R})$ we obviously have $\overline{\beta} = \beta$, lemma 2.1.8 tells us that

$$\begin{aligned} 2\,i\,g^{\mathrm{W}}(\alpha, \beta) &= 2\,i\,g^{\mathrm{W}}(P\,\alpha, \beta) = 2\,i\,Q(J^{\mathrm{W}}(P\,\alpha), \overline{\beta}) \\ &= h\Big(\tfrac{1}{h(\Omega,\Omega)} J^{\mathrm{W}}\big(h(\alpha, \Omega)\Omega - h(\alpha, \bar{\Omega})\bar{\Omega}\big), \beta \Big) \\ &= \tfrac{1}{h(\Omega,\Omega)} \Big(h(\alpha, \Omega) h(i\,\Omega, \beta) - h(\alpha, \bar{\Omega}) h(-i\,\bar{\Omega}, \beta) \Big) \\ &= \tfrac{i}{h(\Omega,\Omega)} \Big(h(\alpha, \Omega)\overline{h(\beta, \Omega)} + \overline{h(\alpha, \Omega)} h(\beta, \Omega) \Big), \end{aligned} \tag{2.37}$$

where we have used that h is a Hermitian form and that $h(\alpha, \bar{\Omega}) = 2\,i\,Q(\alpha, \Omega) = -h(\Omega, \alpha)$ for $\alpha \in H^3(\mathcal{Y}, \mathbb{R})$ and similarly for $h(\bar{\Omega}, \beta)$. □

The normalised period matrix for the Weil intermediate Jacobian is significantly more complicated than that for the Griffiths intermediate Jacobian.

Proposition 2.3.9. *Let $(\alpha_i, \beta^i)_i$ be a symplectic basis for $H^3(\mathcal{Y}, \mathbb{Z})$ with respect to the intersection form Q and let $\Omega = X^i\alpha_i - F_i\beta^i \in H^{3,0}(\mathcal{Y})$ be a holomorphic 3-form with periods X^i and F_i. The normalised period matrix for the Weil intermediate Jacobian corresponding to the basis $(\alpha_i, \beta^i)_i$ is given by $\Omega^{\mathrm{W}} = (\mathcal{Z}^{\mathrm{W}}, \mathbf{1})$ with*

$$\mathcal{Z}^{\mathrm{W}}_{ij} = F_{ij} - i\frac{N_{ik}\bar{X}^k\bar{X}^\ell N_{\ell j}}{\bar{X}N\bar{X}}, \tag{2.38}$$

where $F_{ij} = \frac{\partial F_i}{\partial X^j}$ and $N_{ij} = 2\,\mathrm{Im}(F_{ij}) = -i\,F_{ij} + i\,\bar{F}_{ij}$.

Proof: Let ρ_i denote the complex-valued 1-form $\rho_i = \mathrm{d}y_i + \mathcal{Z}^{\mathrm{W}}_{ij}\mathrm{d}x^j$ on $\mathcal{J}^{\mathrm{W}}_2(\mathcal{Y})$, where $\mathcal{Z}^{\mathrm{W}}$ is the matrix from equation (2.38). If we apply the 1-form ρ_i to $\bar{\Omega} \in H^{0,3}(\mathcal{Y}) < \big(\mathrm{T}\mathcal{J}^{\mathrm{W}}_2(\mathcal{Y})\big) \otimes \mathbb{C}$ we obtain

$$\begin{aligned}\rho_i(\bar{\Omega}) &= \Big(\mathrm{d}y_i + \big(F_{ij} - i\tfrac{N_{ik}\bar{X}^k\bar{X}^\ell N_{\ell j}}{\bar{X}N\bar{X}}\big)\mathrm{d}x^j\Big)(\bar{X}^m\alpha_m - \bar{F}_m\beta^m) \\ &= -\bar{F}_i + F_{ij}\bar{X}^j - i\tfrac{\bar{X}N\bar{X}}{\bar{X}N\bar{X}}N_{ik}\bar{X}^k = -\bar{F}_{ij}\bar{X}^j + F_{ij}\bar{X}^j - iN_{ik}\bar{X}^k \\ &= 2\,i\,\mathrm{Im}(F_{ij})\bar{X}^j - 2\,i\,\mathrm{Im}(F_{ij})\bar{X}^j = 0\end{aligned} \tag{2.39}$$

For $\Omega_k = \frac{\partial\Omega}{\partial X^j} = \alpha_j - F_{jk}\beta^k \in H^{2,1}(\mathcal{Y}) \oplus H^{3,0}(\mathcal{Y})$ we also have that

$$\rho_i\big((1-P)\Omega_j\big) = \rho_i\left(\Omega_j - \frac{h(\Omega_j,\Omega)\Omega - h(\Omega_j,\bar{\Omega})\bar{\Omega}}{h(\Omega,\Omega)}\right) = 0, \tag{2.40}$$

as can be shown by explicitly working out this expression using the formula for ρ_i and that $h(\Omega_i, \Omega) = 2N_{ij}\bar{X}^j$ and $h(\Omega_i, \bar{\Omega}) = 0$.

The fact that $\rho_i(\bar{\Omega}) = 0$ tells us that ρ_i vanishes on $H^{0,3}(\mathcal{Y})$ and because $\rho_i((1-P)\Omega_j) = 0$ we can also conclude that $\rho_i|_{H^{2,1}(\mathcal{Y})} = 0$. This leaves $H^{3,0}(\mathcal{Y}) \oplus H^{1,2}(\mathcal{Y})$, which is the $+i$ eigenspace of J^{W}, from which we learn that $J^{\mathrm{W}*}\rho_i = i\,\rho_i$.

Because moreover $\rho_i(\beta^j) = \delta^j_i$, we can conclude that $\rho_0, \ldots, \rho_{h^{1,2}}$ is the basis of complex linear 1-forms dual to $e^0, \ldots, e^{h^{1,2}}$ (defined by $e^i = \beta^i$). We already know from proposition 1.4.13 that these forms are given by $\mathrm{d}z_i = \mathrm{d}y_i + \mathcal{Z}_{ij}\mathrm{d}x^i$, where $\mathcal{Z}_{ij}$ is the normalised period matrix for the torus with respect to the symplectic basis $(\alpha_i, \beta^i)_i$, so it necessarily follows that the matrix $\mathcal{Z}^{\mathrm{W}}_{ij}$ is the period matrix for the Weil intermediate Jacobian. □

Corollary 2.3.10. *The canonical metric on the Weil intermediate Jacobian is positive definite and it is given by*

$$g^{\mathrm{W}} = Q \circ (J^{\mathrm{W}} \times \mathrm{id}) = \mathcal{Y}^{ij}_{\mathrm{W}}\mathrm{d}z_i\mathrm{d}\bar{z}_j = \mathcal{Y}^{ij}_{\mathrm{W}}(\mathrm{d}y_i + \mathcal{Z}^{\mathrm{W}}_{ik}\mathrm{d}x_k)(\mathrm{d}y_j + \bar{\mathcal{Z}}^{\mathrm{W}}_{j\ell}\mathrm{d}x^\ell) \tag{2.41}$$

where $z_i = y_i + \mathcal{Z}^{\mathrm{W}}_{ij}x^i$ (for $\mathcal{Z}^{\mathrm{W}}$ the period matrix from equation (2.38)) and $\mathcal{Y}^{ij}_{\mathrm{W}}$ denotes the inverse matrix for $\mathcal{Y}^{\mathrm{W}} = \mathrm{Im}(\mathcal{Z}^{\mathrm{W}})$.

Proof: Equation (2.41) follows immediately from proposition 2.3.9 and proposition 1.4.13, but it can also be obtained by using equation (2.30) and proposition 2.3.7. □

3. String theory

The context in which the results presented in this thesis is provided by the effective supergravity theory that describes compactified type IIA superstring theory in the low-energy limit. In this chapter we will give a short overview of what string theory is and in particular what this low-energy effective theory looks like, without going into too many details. We will explain the ideas behind Calabi-Yau compactification a bit more thoroughly and will finally introduce the hypermultiplet moduli space of the four-dimensional theory that is obtained. For a more complete review a standard textbooks, such as [38], [39] or [40], may be consulted.

3.1 (Super)string theory

The principal idea behind string theory is that nature should not be described in terms of point particles, but instead in terms of extended 1-dimensional objects called ***strings***. These can either be open (lines with two endpoints) or closed (loops). Fundamental strings come with a discrete set of oscillatory modes, each of which will correspond to a different type of elementary particle in the quantised version of string theory.

One of the reason why string theory is popular is the fact that gravity appears naturally from it, making it a candidate theory for reconciling the standard model and Einstein's theory of general relativity. If string theory does describe gravity, the string scale should be related to the Planck scale, which corresponds to energies of roughly 10^{19} GeV. This is well beyond the reach of modern particle accelerators and explains why strings, if they exist, have not yet been observed.

In the classical situation, we can say that a fundamental (bosonic) string moving through d-dimensional space-time spans a 2-dimensional surface called the ***worldsheet*** of the string. This is described by an embedding of a 2-dimensional surface Σ into the d-dimensional ***target space*** X. An interaction between two particles (strings) is then simply represented by a worldsheet with splitting strings (cf. figure 3.1).

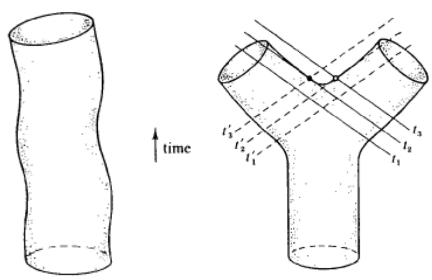

Figure 3.1: The worldsheet of a single string and a splitting string.
This figure originally appeared in [38].

By choosing local coordinates τ and σ on Σ one can parametrise this embedding of the worldsheet by as $x^\mu(\tau, \sigma)$ (for $\mu = 0, \ldots, d-1$). The dynamics of these strings is described by extremisation of the surface area of the worldsheet and can therefore described by Nambu-Goto action,

$$S_{\text{NG}} = \frac{1}{2\,\pi\,\alpha'}\text{Area}(\Sigma) = -\frac{1}{2\,\pi\,\alpha'}\int \mathrm{d}^2\sigma\sqrt{-\det\left(\frac{\partial x^\mu}{\partial\sigma^{(\alpha)}}\frac{\partial x^\nu}{\partial\sigma^{(\beta)}}\eta_{\mu\nu}\right)}. \tag{3.1}$$

The only free parameter in string theory is the so-called ***Regge slope***, or the ***string scale***. Contraction of indices is done using (background) metric $\eta_{\mu\nu}$ on the target space, which we have for now assumed is just the d-dimensional Minkowski metric.

A point particle is characterised by its position and its momentum, but this no longer suffices for these extended objects. To describe a fundamental string we require, in addition to its centre of mass and its centre of momentum, the amplitudes of its modes of oscillation.

A quantum theory of strings is obtained from the classical theory by not only introducing operators to describe the position and the momentum of the string, but also for these modes of oscillation, which will act as creation operators for oscillations on the string. This quantisation procedure can only be carried out consistently if the target space is 26-dimensional, so we say that 26 is the ***critical dimension*** of (bosonic) string theory. An interesting property of this quantised string theory is that its ***spectrum*** includes a massless spin 2 particle that takes the role of the graviton. The spectrum of the theory however does not include any fermions, but does include a tachyon, which is a particle with a negative squared mass, which is undesirable because it leads to instabilities.

These problems can be solved by complementing every bosonic degree of freedom with an additional fermionic degree of freedom and extending the original action (or in fact a different, but equivalent action called the Polyakov action) to a supersymmetric action. The words ***bosonic*** and ***fermionic*** refer to the way fields behave under Lorentz transformations (on the worldsheet in this case), which can either be through a representation of the Lorentz group itself (bosonic) or of its spin group (fermionic). The resulting theory is called a ***superstring theory***. ***Supersymmetry*** is a very important concept and a number of results that we will use depend on it. We will not discuss this in detail here and just state that it is a continuous symmetry of the action that transforms bosonic fields into fermionic fields and vice versa. Similar to bosonic string theory, such a superstring theory can only be formulated consistently on a target space of some critical dimension, which is $d = 10$ in this case.

There are a couple of different superstring theories that can all be related to each other through a number of dualities. We will only be considering one of these, namely type IIA superstring theory. The II tells us that we are only considering closed strings and the A tells us which physical states are allowed to appear, as some had to be projected out to get rid of the tachyon.

Quantised string theory can be described in the path integral formalism, where basically every possible worldsheet gets a weight $e^{-S_{\text{P}}}$ determined by the (supersymmetric) action S_{P}. The scattering amplitude of a process, which is basically the probability of this process occurring, is then determined by adding up the weights of all worldsheets that describe this process. This "sum" is in fact a functional integral. For instance for an interaction with two incoming strings and two outgoing strings corresponding to certain real particles all worldsheets with four external lines (cylinders) corresponding to these particle should be considered, even those with loops (cf. figure 3.2).

String theory could have been formulated using any Ricci-flat (i.e. satisfying Einstein's field equations in vacuum) metric instead of the standard 10-dimensional Minkowski metric. What is amazing is that in the path integral formulation the theory does not change by changing this background metric, because the new situation corresponds to the original situation with an added background of gravitons. Similarly, ***background fields*** can be added for each of the other massless states in the spectrum.

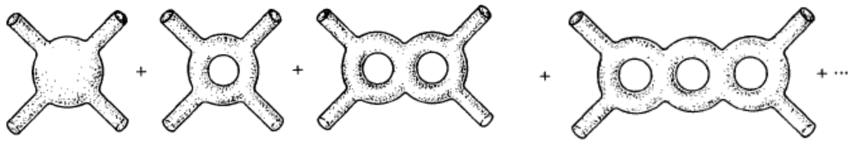

Figure 3.2: Quantum corrections correspond to worldsheets with loops. This figure originally appeared in [38].

3.1.1 The low-energy limit

The string scale α' was the only free parameter and determines all energy scales, so taking the limit for $\alpha' \to 0$, corresponds to the low energy limit [40, 41]. In this limit it makes sense to consider just the massless states of the theory, which can to good approximation be described by a classical ***effective theory*** consisting of fluctuations of the background fields corresponding to particles from the massless spectrum and an effective action that reproduces field equations and scattering amplitudes. The effective low-energy theory for type IIA superstring theory is called ***type IIA supergravity***. It is a classical supersymmetric theory that contains the Einstein-Hilbert action from general relativity.

Table 3.1: Type IIA bosonic massless spectrum

ϕ	scalar	dilaton	NS-NS
$B^{(2)}$	2-form	Kalb-Ramond field	NS-NS
δg	traceless symmetric $(0,2)$-tensor	graviton	NS-NS
$A^{(1)}$	1-form		R-R
$A^{(3)}$	3-form		R-R

The massless spectrum of type IIA string theory consists of four sectors: the Ramond-Ramond, the R-NS, the NS-R and the NS-NS sectors, where R stands for Ramond and NS for Neveu-Schwarz and these refer to the type of boundary conditions that are put on the left-moving and the right-moving modes of the string. The NS-R and the R-NS sectors of the massless spectrum only contain fermionic fields and the NS-NS and the R-R sector make up the bosonic part of the spectrum, which is the part we will be interested in in this thesis. Apart from the graviton, all of the fields that make up the bosonic part of the massless spectrum are completely anti-symmetric $(0, k)$ tensors for some $k \in \mathbb{N}_0$ and can thus be interpreted as differential forms on the 10-dimensional target space. The particles in the bosonic massless spectrum have been presented in table 3.1 and all come with a fermionic superpartner.

By only considering the original string theory at the ***tree level***, i.e. by only considering worldsheet diagrams without loops, an action S_{eff} can be obtained for the fields from table 3.1. This action can be written as $S_{\text{eff}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}}$, where S_{NS}, S_{R} and S_{CS} are given by equation (3.2). The first two contain the fields from the NS-NS-sector and the R-R-sector respectively and the third is called the ***Chern-Simons term***.

$$S_{\text{NS}} = -\frac{1}{(2\pi)^7\alpha'^4}\int \mu_X e^{-2\phi}\left(-R_g + 4|\mathrm{d}\phi|^2 + \tfrac{1}{2}|H^{(3)}|^2\right), \tag{3.2a}$$

$$S_{\text{R}} = -\frac{1}{2(2\pi)^7\alpha'^4}\int \mu_X \left(|F^{(2)}|^2 + |F^{(4)}|^2\right) \tag{3.2b}$$

and

$$S_{\text{cs}} = -\frac{1}{2(2\pi)^7\alpha'^4}\int B^{(2)} \wedge F^{(4)} \wedge F^{(4)}. \tag{3.2c}$$

Here $\mu_X = \sqrt{|g|}\mathrm{d}^{10}x$ denotes the standard volume form on X and R_g is the scalar curvature for the metric g. The 3-form $H^{(3)} = \mathrm{d}B^{(2)}$ is the field strength for the Kalb-Ramond field and $F^{(2)} = \mathrm{d}A^{(1)}$ and $F^{(4)} = \mathrm{d}A^{(3)}$ are the field strengths for $A^{(1)}$ and $A^{(3)}$ respectively. The squared norm $|\alpha|^2$ of a differential form α is taken with respect to the metric induced by g, which means that $\alpha \wedge *\alpha = \mu_X|\alpha|^2$ by definition of the Hodge star operator.

The complete effective action is obtained by adding ***quantum corrections***, which come in two flavours: (perturbative) ***loop corrections*** and (non-perturbative) ***instanton corrections*** that correspond to non-trivial embeddings of higher dimensional objects called ***branes*** into the target space. The exact nature of branes and their appearance in string theory is not important for understanding this text and the interested reader is referred to [42].

The contribution of a constant background field ϕ_0 for the dilaton gives a contribution to the original (Polyakov) action for the worldsheet, which is proportional to $-\phi_0$ times the Euler characteristic $\chi_\Sigma = 2 - 2\,g$ of the worldsheet considered. This means that in the limit for $\phi_0 \to \infty$, the contribution from loop diagrams will be small and the approach taken, by viewing contributions from loop diagrams as perturbations, is justified. For smaller values of ϕ the action (3.2) will not be valid and both perturbative and non-perturbative correction should really be considered as well.

3.2 Compactified string theory

The world we live in is manifestly 4-dimensional, but as we have noted above, superstring theory can only be consistently formulated for a 10-dimensional target space. This serious discrepancy between string theory and the physics it hopes to describe can be resolved by ***compactification*** of six of the ten dimensions of the target space [39, 40, 43].

By the compactification of string theory on a compact ***internal manifold*** K we basically mean that instead of using 10-dimensional Minkowski as the target space for the theory, a manifold of the form $M \times K$ is used instead, where $M \cong \mathbb{R}^4$ is the standard uncompactified 4-dimensional space-time. The background metrics that are considered are of the form $g = \eta + g_K$, where η is the standard 4-dimensional Minkowski metric on $\mathbb{R}^4$ and g_K is some (Ricci-flat) metric on K with respect to which the size of the internal manifold ($\mathrm{Vol}_{g_K}(K)$) is small. The idea is that while we can see the 4-dimensional ***external*** space-time M, the internal manifold K will be so small that we do not observe it.

An additional assumption that is made is that there should be an unbroken supersymmetry in four dimensions. Preservation of supersymmetry may seem like an odd condition since we know that supersymmetry should eventually be broken at some energy scale, but there are strong arguments that suggest that some supersymmetry should survive in the 4-dimensional theory at high energies [39, 40, 43]. This extra condition for the background requires that the background metric on the internal manifold is not only Ricci flat, but also K"ahler.

Since one of the definitions for Calabi-Yau manifolds that is in use is that a Calabi-Yau manifold is a compact, Ricci-flat Kähler manifold, we refer to this type of compactification as ***Calabi-Yau compactification***. We will put a slightly stronger restriction on the internal manifold K, namely that it is a Calabi-Yau 3-fold in the sense of definition 2.1.1, i.e. that it has a trivial canonical bundle, and that its fundamental group is finite. One may recall that triviality of the fundamental group of a Ricci-flat Kähler manifold is equivalent to the condition that its *global* holonomy group is contained in $\mathrm{SU}(3)$. To emphasise that we are dealing with Calabi-Yau compactifications, the internal manifold will from now on be denoted by the letter $\mathcal{Y}$ instead of K.

3.2.1 Switching to four dimensions

From now on the coordinate x will be used to denote points in M and y for points in $\mathcal{Y}$, which means that (x, y) is a general point on the product $X \cong M \times \mathcal{Y}$. The first thing we should do to express the original 10-dimensional theory in terms of 4-dimensional fields is split each of the forms $B^{(2)}$, $A^{(1)}$ and $A^{(3)}$, as well as the graviton δg, into the parts that live on M and parts that live on $\mathcal{Y}$. For a k-form $\omega^{(k)}$ this means that we should write $\omega^{(k)}(\hat{x}) = \sum_{p+q=k} \omega^{(p,q)}(\hat{x})$, where $\omega^{(p)(q)}$ is a p-form on M and a q-form on $\mathcal{Y}$.

As an operator acting on differential forms, the ***Laplace operator*** $\Delta = \Delta_{g_\mathcal{Y}} = \mathrm{d}\mathrm{d}^\dagger + \mathrm{d}^\dagger \mathrm{d}$ on the internal manifold $(\mathcal{Y}, g_\mathcal{Y})$ acting on q-forms is known to have some very nice properties. Because $\mathcal{Y}$ is compact $\Delta_\mathcal{Y}$ has a discrete spectrum Λ containing only non-negative eigenvalues, its eigenspaces are finite-dimensional and mutually orthogonal and any q-form can be written as a sum of eigenforms (cf. theorem 1.3.19). This allows us to fix a discrete basis $\left(\alpha^{(q)}_{\mathcal{Y},\lambda,s}\right)_{\lambda,s}$ (with $\lambda \in \Lambda$ and $s = 1, \ldots, \dim\ker(\Delta - \lambda)$) of eigenforms for Δ with eigenvalues $\lambda \geq 0$ for any$q \geq 0$.

Consequently, we can write the any k-form $\omega^{(k)}$ on $\mathbb{R}^4 \times \mathcal{Y}$ as

$$\omega^{(k)}(x, y) = \sum_{p+q=k} \omega^{(p)(q)}(x, y) = \sum_{p+q=k} \sum_{\lambda,s} \omega^{(p)}_{M,\lambda,s}(x) \wedge \alpha^{(q)}_{\mathcal{Y},\lambda,s}(y), \tag{3.3}$$

where $\alpha^{(q)}_{\mathcal{Y},\lambda,s}$ on $\mathcal{Y}$ form the aforementioned discrete basis of q-forms on $\mathcal{Y}$ and the coefficients $\omega^{(p)}_{M,\lambda,s}$ are p-forms on the 4-dimensional external space M [39]. These 4-dimensional fields, of which there are infinitely many, now carry all the information that the original 10-dimensional field $\omega^{(k)}$ did. Something similar can be done for the graviton field δg. As we will see, all but a finite number of these will come with a mass term from the 4-dimensional point of view.

The original 10-dimensional theory was described by the action $S_{\text{eff}} = S_{\text{NS}} + S_{\text{R}} + S_{\text{CS}}$ equation (3.2) and it is of the form $S_{\text{eff}} = \int_M \mathrm{d}^4x \int_\mathcal{Y} \mathrm{d}^6y \mathcal{L}_{\text{eff}}$ for some Lagrangian density $\mathcal{L}_{\text{eff}}$ that depends on the fields ϕ, δg, $B^{(2)}$, $A^{(1)}$ and $A^{(3)}$ and their derivatives. An action for the 4-dimensional fields is obtained by "simply" integrating out the internal Calabi-Yau manifold. Performing this integral yields the action $S_{4\text{D}} = \int \mathrm{d}^4x \mathcal{L}_{4\text{D}}$, with a 4-dimensional Lagrangian density given by [39]

$$\mathcal{L}_{4\text{D}} = \int_\mathcal{Y} \mathrm{d}^6y \mathcal{L}_{\text{eff}}. \tag{3.4}$$

The dynamics of the 4-dimensional fields can be described completely in terms of this action.

For small variations of the fields, the equations of motion imposed by the action (3.2) for the fields $\omega^{(0)} = \phi$, $\omega^{(2)} = B^{(2)}$, $\omega^{(1)} = A^{(1)}$ and $\omega^{(3)} = A^{(3)}$ read $\Delta_X \omega^{(k)} = 0$ (plus small higher order terms). Here $\Delta_X = \Box_M + \Delta_\mathcal{Y}$ is the Laplace operator on $M \times \mathcal{Y}$, which in our case can be written as the sum of the ***D'Alembert (wave) operator*** $\Box_M = \Delta_M = \partial^\mu \partial_\mu$ on 4-dimensional space-time and the Laplacian on the internal Calabi-Yau manifold $\mathcal{Y}$. For the field $\omega^{(p)}_{M,\lambda,s} \wedge \alpha^{(q)}_{\mathcal{Y},\lambda,s}$ this becomes

$$\begin{aligned} \Delta_X \left(\omega^{(p)}_{M,\lambda,s} \wedge \alpha^{(q)}_{\mathcal{Y},\lambda,s}\right) &= \left(\Box_M \omega^{(p)}_{M,\lambda,s}\right) \wedge \alpha^{(q)}_{\mathcal{Y},\lambda,s} + \omega^{(p)}_{M,\lambda,s} \wedge \left(\Delta_\mathcal{Y} \alpha^{(q)}_{\mathcal{Y},\lambda,s}\right) \\ &= \left((\Box_M + \lambda)\omega^{(p)}_{M,\lambda,s}\right) \wedge \alpha^{(q)}_{\mathcal{Y},\lambda,s} = 0, \end{aligned} \tag{3.5}$$

i.e. $(\Delta_M + \lambda)\omega^{(p)}_{M,\lambda,s} = 0$ for each of the 4-dimensional (coefficient) fields $\omega^{(p)}_{M,\lambda,s}$. and all λ, s that contribute to the sum in equation (3.3). We see that the field $\omega^{(p)}_{M,\lambda,s}$ gets an effective mass $\sqrt{\lambda}$ when viewed as a 4-dimensional field.

The Laplace operator on $\mathcal{Y}$ and its eigenvalues are inversely proportional to the metric on $\mathcal{Y}$ and thus proportional to $\mathrm{Vol}(\mathcal{Y})^{-1/3}$, so the assumption that the internal manifold is small tell us that the fields $\omega^{(p)}_{M,\lambda,s}$ for $\lambda > 0$ will be very massive. Because we are working in the low-energy limit we choose to only consider the massless fields (the ***zero modes***), which are exactly those four-dimensional fields that correspond to ***harmonic forms*** on $\mathcal{Y}$, i.e. those for which $\lambda = 0$.

Although these harmonic forms $\alpha^{(q)}_{\mathcal{Y},0,s}$ depend on the (Kähler) metric that is put on $\mathcal{Y}$ in a particular point $x \in M$, the (De Rham) cohomology classes in $H^q(\mathcal{Y}, \mathbb{R})$ they represent do not. This enables us to naturally view the combined fields $\omega^{(p)}_{\mathcal{Y},0,s}$ for $s = 1, \ldots, b^q$ into a single $H^q(\mathcal{Y}, \mathbb{R})$-valued p-form $\sum_s \omega^{(p)}_{\mathcal{Y},0,s} \otimes [\alpha^{(q)}_{\mathcal{Y},0,s}]$.

Because of the assumption we had made back in section 2.1 that the Calabi-Yau 3-fold $\mathcal{Y}$ is connected and has a finite fundamental group, its Hodge diamond is given by equation (2.1) for some integers $h^{1,1}$ and $h^{1,2}$. This tells us exactly what 4-dimensional fields we can expect to obtain. An overview of these fields has been given in table 3.2, though we will say a little bit about each of them. For more details on this, see for instance [44].

The dilaton ϕ is just a scalar on X, and the only harmonic 0-forms on $\mathcal{Y}$ are constant functions, so ϕ becomes a scalar on M and similarly $A^{(1)}$ only contributes a 1-form on M. The 3-form $A^{(3)}$ from table 3.1 gives us three contributions to the massless spectrum of the 4-dimensional theory, namely the fields $A^{(3)}$, $A^{(1)(2)}$ and $a^{(3)}$ from table 3.2, which are 0-forms, 2-forms and 3-forms on $\mathcal{Y}$ respectively (and a 3-form, a 1-form and a scalar on M). Of these only the last, which we call the ***Ramond-Ramond** 3**-form*** (to remind us that it originally came from the Ramond-Ramond sector), will be important for us.

The Kalb-Ramond 2-form provides two fields, namely the field b, which is a harmonic 2-form on $\mathcal{Y}$ and a scalar on M, and another field $B^{(2)}$ that is constant on $\mathcal{Y}$ and a 2-form on M. By either using the field equations for $B^{(2)}$ or by introducing a Lagrangian multiplier, $B^{(2)}$ may be dualised to a scalar field σ called the ***Kalb-Ramond axion*** [44, 2], which satisfies $H^{(3)} := \mathrm{d}B^{(2)} = *\mathrm{d}\sigma$, where $*$ now denotes the 4-dimensional Hodge star operator.

Table 3.2: Massless bosonic spectrum in 4 dimensions

Field		Description	Multiplet
ϕ	$\in C^\infty(M)$	dilaton	hyper-multiplets
σ	$\in C^\infty(M)$	(Kalb-Ramond) axion	
$a^{(3)}$	$\in C^\infty(M, H^3(\mathcal{Y}, \mathbb{R}))$	(Ramond-Ramond) 3-form	
t	$\in C^\infty(M, \mathcal{M}_{\mathbb{C}})$	complex structure moduli	
ω	$\in C^\infty(M, \mathcal{K}_{\mathcal{Y}})$	Kähler moduli	vector multiplets
b	$\in C^\infty(M, H^2(\mathcal{Y}, \mathbb{R}))$		
$A^{(1)(2)}$	$\in \Omega^1(M, H^2(\mathcal{Y}, \mathbb{R}))$		
δg	traceless symmetric $(0,2)$-tensor	graviton	Gravity multiplet
$A^{(1)}$	$\in \Omega^1(M)$		
$A^{(3)}$	$\in \Omega^3(M)$		

For the graviton δg the situation is similar, is different but nevertheless similar. Once again, only zero-modes are considered. The part of δg that is a $(0,2)$-tensor on M and a scalar on $\mathcal{Y}$ only has one zero-mode, which is constant on $\mathcal{Y}$, and thus contributes a 4-dimensional graviton field. The part that is a 1-form on both M and on $\mathcal{Y}$ does not have any zero-modes as a consequence of the fact that $b^1 = h^{1,0} + h^{0,1} = 0$ [44, 39], which still leaves us with a scalar field on M that corresponds to a deformation of the metric on $\mathcal{Y}$ [39, 43].

Of all possible deformations of the metric on the internal Calabi-Yau manifold, only those that preserve Ricci-flatness will contribute. These deformations have already been discussed in section 2.2.1, where we had seen that they split up into two types: deformations of the complex structure on $\mathcal{Y}$ and deformations of the Kähler class. Since these together determine a deformed metric on $\mathcal{Y}$, we see that we should view the complex structure moduli $t \in \mathcal{M}_{\mathbb{C}}$ and the Kähler moduli $d \in \mathcal{K}_{\mathcal{Y}}$ as fields in the 4-dimensional theory, so these have been included in table 3.2.

The fields in table 3.2 have been split into a ***gravity multiplet***, ***vector multiplets*** and ***hypermultiplets***, which are sets of bosonic fields (and their fermionic superpartners) that transform into each other under supersymmetry transformations. A hypermultiplet for instance is a set of four scalar fields (and their fermionic superpartners) that may together be viewed as a quaternion and transform accordingly [45]. We will see that the vector multiplets fields and the hypermultiplets decouple in the effective action (3.7) and therefore only interact via gravity, which is in fact a general result and does not only apply to the present situation.

3.2.2 The four-dimensional action

As we have seen, the effective 4-dimensional theory comes with an action $S_{\text{4D}} = \int_M \mathrm{d}^6 y \mathcal{L}_{\text{eff}}$ that is obtained by integrating out the internal Calabi-Yau manifold from the original 10-dimensional action. Although computing this action is not a simple task, the Lagrangian density $\mathcal{L}_{\text{4D}}$ has already been found [44, 40]. This effective action can be written as the sum $S_{\text{4D}} = S_{\text{g}} + S_{\text{hm}} + S_{\text{vm}}$ of three separate contributions corresponding to gravity, the hypermultiplet fields and the vector multiplet fields.

To be able to present this action we should first discuss how to introduce coordinates for the fields in table 3.2. Recall from proposition 2.1.6 that $H^3(\mathcal{Y}, \mathbb{R})_{\text{f}}$ admits a basis $\alpha_0, \ldots, \alpha_{h^{1,2}}, \beta^1, \ldots, \beta^{h^{1,2}}$ of generators for $H^3(\mathcal{Y}, \mathbb{Z})_{\text{f}}$ that is symplectic with respect to the intersection form Q. This basis can be used to write the field $a^{(3)}$ as

$$a^{(3)}(x) = \sum_{i=0}^{h^{1,2}} A^i(x)\alpha_i - B_i(x)\beta^i, \tag{3.6}$$

for the (real) scalar fields $A^0, \ldots, A^h, B_0, \ldots, B_h$ on M. Unless stated otherwise, the indices $i, j, k, \ldots$ will from now on always run from 0 to $h^{1,2}$.

We have also seen that the complex structure on the Calabi-Yau manifold $\mathcal{Y}$ is completely determined by a holomorphic 3-form $\Omega = X^i\alpha_i - F_i\beta^i$ and that for the right symplectic basis the periods X^i (locally) form a set of complex projective coordinates for the complex structure moduli space. This made it possible to view the complex structure moduli moreover as a projective special Kähler manifold with a prepotential $F = \frac{1}{2}X^iF_i$ and Kähler potential $K_{\mathbb{C}} = -\log\bigl(i\int_{\mathcal{Y}} \Omega \wedge \overline{\Omega}\bigr)$.

The space of Kähler moduli, which is the Kähler cone $\mathcal{K}_{\mathcal{Y}}$, could be viewed as an open subset of $H^2(\mathcal{Y}, \mathbb{R})$, which enables us to combine the Kähler moduli ω and the field $b \in H^2(\mathcal{Y}, \mathbb{R})$ into a single complex field $W = b + i\,\omega$ that takes values in the ***complexified Kähler moduli space*** $\mathcal{M}_k = H^2(\mathcal{Y}, \mathbb{R}) + i\,\mathcal{K}_{\mathcal{Y}}$. By subsequently choosing a basis $v_1, \ldots, v_{h^{1,1}}$ for $H^2(\mathcal{Y}, \mathbb{Z})$ it is possible to expand this field as $W = \sum_\gamma W^\gamma v_\gamma$. Also the complexified Kähler is projective special Kähler with Kähler potential $K_{\text{K}} = -\log\bigl(\int_{\mathcal{Y}} \omega \wedge \omega \wedge \omega\bigr) = -\log \mathrm{Vol}(\mathcal{Y})$. Since $A^{(1)(2)}$ is Harmonic it is in particular closed and $\mathcal{F} = \mathrm{d}A^{(1)(2)}$ is a 2-form on both M and on $\mathcal{Y}$, which enables us to write $\mathcal{F} = \sum_\gamma \mathcal{F}^\gamma v_\gamma$.

We can now write down the effective 4-dimensional action $S_{\text{eff}} = S_{\text{g}} + S_{\text{vm}} + S_{\text{hm}}$. In its first contribution we recognise the Einstein-Hilbert action from general relativity

$$S_{\text{g}} = \int_M \mu_g R_g = \int_M \mathrm{d}^4x\sqrt{|g|}R_g, \tag{3.7a}$$

where $\mu = \mathrm{d}^4x\sqrt{|g|}$ is the volume form on $M \cong \mathbb{R}^4$ for the metric g and R_g is its scalar curvature. The vector multiplet contribution is

$$S_{\text{vm}} = -\int \left[\mu_M \frac{\partial^2 K_{\text{K}}(W, \overline{W})}{\partial W^\lambda\, \partial \overline{W}^\gamma} \partial_\mu W^\lambda \partial^\mu \overline{W}^\gamma - \tfrac{1}{2}\,\mathrm{Im}(\mathcal{N}_{\lambda\gamma}\mathcal{F}^{+\lambda} \wedge *\mathcal{F}^{+\gamma})\right], \tag{3.7b}$$

where K_K is a Kähler potential on the complexified Kähler moduli space, $\mathcal{N}$ is some (complex) invertible matrix that only depends on W and $\mathcal{F}^{\gamma,\pm} = \mathcal{F}^{\gamma} \mp i * \mathcal{F}^{\gamma}$. Finally, there is a contribution for the fields in the hypermultiplet sector, which is given by

$$\begin{aligned} S_{\text{hm}} = -\int \mu_M \Bigg[&\frac{\partial^2 K_{\mathbb{C}}(X,\overline{X})}{\partial X^i \, \partial \overline{X}^j} \partial_\mu X^i \partial^\mu \overline{X}^j + \frac{1}{(2\,\psi)^2}(\partial_\mu \psi)(\partial^\mu \psi) \\ &+ \frac{1}{(2\,\psi)^2}(\partial_\mu \sigma - A^i \overleftrightarrow{\partial}_\mu B_i)(\partial^\mu \sigma - A^i \overleftrightarrow{\partial}^\mu B_i) \\ &- \frac{1}{2\,\psi} \operatorname{Im}(\mathcal{N})^{ij} (\partial_\mu B_i - \mathcal{N}_{ik} \partial_\mu A^k)(\partial^\mu B_i - \overline{\mathcal{N}}_{ik} \partial^\mu A^k) \Bigg]. \end{aligned} \tag{3.7c}$$

Here $\psi = e^{\phi}$ describes the dilaton field, the matrix $\mathcal{N}$ is given by

$$\mathcal{N}_{ij} = \bar{F}_{ij} + i \frac{N_{ik} X^k X^\ell N_{\ell j}}{X\,N\,X}, \tag{3.8}$$

with $N_{ij} = 2\,\operatorname{Im}(F_{ij})$, and $\operatorname{Im}(\mathcal{N})^{ij}$ is the inverse of its imaginary part. The Kähler potential $K_{\mathbb{C}}(X,\overline{X})$ can be shown to correspond to the canonical Kähler structure on the complex structure moduli space and is given by [43, 40]

$$K_{\mathbb{C}}(X,\bar{X}) = -\log\left(i \int_{\mathcal{Y}} \Omega \wedge \overline{\Omega} \right) = -\log(i\,\overline{X}^i F_i - i\,X^i \overline{F}_i). \tag{3.9}$$

We should not forget that the action $S_{\text{4D}} = S_{\text{g}} + S_{\text{vm}} + S_{\text{hm}}$ described above was obtained in the limit for $\phi \to \infty$ and is an approximation of a quantum corrected expression. We can observe that this effective action has no contributions containing both vector multiplet and hypermultiplets, which is no coincidence [46] and will therefore still hold once these correction terms have been added.

3.3 The scalar moduli space

At the moment we are only be interested in the fields from the 4-dimensional theory from the previous section that are scalars on the external manifold $M \cong \mathbb{R}^4$ (but not necessarily on $\mathcal{Y}$). This includes most of the fields from the vector multiplet and hypermultiplet sectors, but it excludes $A^{(1,2)}$, and the entire gravity multiplet. Although the appearance of the graviton δg is quite essential, we will also ignore it for now and fix the standard 4-dimensional Minkowski metric on M.

The moduli space $\mathcal{M}$ of the classical 4-dimensional theory described by these scalar fields in these sectors of table 3.2 and the scalar part of the action $S_{\text{vm}} + S_{\text{hm}}$ from equation (3.7) is the space of all vacuum states, by which we mean all states that minimise the energy. This action only contains kinetic terms as consequence of the fact that we had put all massive fields to zero, which means that the vacuum states are exactly those states for which all fields are constant. The moduli space can thus be identified with the space of all values the fields can take at any single given point. On this space, the symbols ϕ, σ, $a^{(3)}$ etc. will denote coordinates rather than fields and we will denote the (moduli) space they parametrise by $\mathcal{M}$.

For a mathematician, the notion of a moduli space has a slighty different meaning that also happens to apply in this case. Choosing a field configuration consists of choosing a specific Calabi-Yau 3-fold, which comes with a Ricci-flat metric, and a number of harmonic forms on this Calabi-Yau manifold at every point $x \in M$. We can view the combination of a Calabi-Yau manifold and these forms as a Calabi-Yau manifold with some extra structure and these objects are exactly parametrised by the scalar fields in the hypermultiplet and vector multiplet sectors from table 3.2. The moduli space $\mathcal{M}$ can therefore be viewed as an extension of the geometric moduli space described in section 2.2 to include this additional structure.

The scalar part of the action $S_{\text{vm}} + S_{\text{hm}}$ describes a ***non-linear sigma model***, by which we mean that it is of the form

$$S[\psi] = -\int_M \mu_M \sum_{\sigma,\tau=1}^{N} G_{\sigma\tau}(\psi)\,(\partial_\mu\psi^\sigma)(\partial^\mu\psi^\tau) \tag{3.10}$$

for some field $\psi \in C^\infty(M, \mathcal{M})$ that we can be expressed in terms of local coordinates on $\mathcal{M}$ as $\psi = (\psi^1, \ldots, \psi^N)$. The matrix $G_{\sigma\tau}(\psi)$ is symmetric and is allowed to depend on the point ψ on the moduli space $\mathcal{M}$ (otherwise we say the sigma model is linear). In such situations the action naturally induces a (pseudo-)metric on the (moduli) space $\mathcal{M}$, namely

$$G = \sum_{\sigma,\tau=1}^{N} G_{\sigma\tau}\mathrm{d}\psi^\sigma\mathrm{d}\psi^\tau \tag{3.11}$$

where the symbols $\psi^1, \ldots, \psi^N$ have now been used as coordinates on $\mathcal{M}$ rather than fields on M.

In our case we obtain the metric $g_{\mathcal{M}} = g_{\text{vm}} + g_{\text{hm}}$ on the scalar part of the moduli space with respect to which the vector multiplet fields and the hypermultiplet fields correspond to orthogonal directions. More explicitly, we have

$$g_{\text{vm}} = \frac{\partial^2 K_{\mathrm{K}}(W, \overline{W})}{\partial W^\lambda\,\partial \overline{W}^\gamma}\mathrm{d}W^\lambda\mathrm{d}\overline{W}^\gamma \tag{3.12}$$

and

$$\begin{aligned} g_{\text{hm}} = &\frac{\partial^2 K_{\mathbb{C}}(X, \bar{X})}{\partial X^i\,\partial \bar{X}^j}\mathrm{d}X^i\mathrm{d}\bar{X}^j + \frac{1}{(2\,\psi)^2}\mathrm{d}\psi^2 + \frac{1}{(2\,\psi)^2}(\mathrm{d}\sigma - A^i\overleftrightarrow{\mathrm{d}}B_i)^2 \\ &- \frac{1}{2\,\psi}\,\mathrm{Im}(\mathcal{N})^{ij}(\mathrm{d}B_i - \mathcal{N}_{ik}\mathrm{d}A^k)(\mathrm{d}B_i - \bar{\mathcal{N}}_{ik}\mathrm{d}A^k). \end{aligned} \tag{3.13}$$

We can view the scalar moduli space as the product $\mathcal{M} = \mathcal{M}_{\text{vm}} \times \mathcal{M}_{\text{hm}}$ of two Riemannian manifolds, the ***vector multiplet moduli space*** $\mathcal{M}_{\text{vm}}$ and the ***hypermultiplet moduli space*** $\mathcal{M}_{\text{hm}}$, which are parametrised by the fields in the vector multiplet sector and the hypermultiplet sector respectively. Since these two spaces are orthogonal with respect to the induced metric described above and the vector multiplet part of this metric does not depend on the hypermultiplet fields and vice versa, we can study the metrics on both spaces separately. Since the vector multiplet moduli space $\mathcal{M}_{\text{vm}}$ is just the complexified Kähler cone $\mathcal{K}_{\mathcal{Y}}$ for the Calabi-Yau manifold $\mathcal{Y}$, it is a projective special Kähler manifold and as such it is understood quite well [47, 39].

The geometry of the hypermultiplet moduli has also been studied intensively and it has been found to be a ***quaternion-Kähler manifold*** [48, 49] with negative scalar curvature. This is quite a general statement that follows from supersymmetry, so it is even true when quantum corrections are taken into account. By using the ***local c-map***, S. Ferrara and S. Sabharwal were able to give an explicit description of the metric on the hypermultiplet moduli space and its quaternion-Kähler structure [2] and these results have later been verified through a more direct method [44, 50]. Nevertheless, the precise nature of the quaternion-Kähler structure on $\mathcal{M}_{\text{hm}}$ is not yet fully understood and requires further study. By finding a more intrinsic description for the quaternion-Kähler structure on the hypermultiplet moduli space without quantum corrections we may also gain a better understanding of this space and it may even help us to find the corrected action.

3.3.1 Peccei-Quinn symmetries

The hypermultiplet action S_{hm} has a number of continuous symmetries that are parametrised by $2\,h^{1,2}+3$ continuous parameters and are known collectively as the (continuous) ***Peccei-Quinn symmetries*** [51, 52] and correspond to the transformations

$$\begin{aligned} A^i &\mapsto A^i + a^i, & \sigma &\mapsto \sigma + s + a^i B_i - b_i A^i, \\ B_i &\mapsto B_i + b_i, & \psi &\mapsto \psi \end{aligned} \tag{3.14}$$

for $a^i, b_i, s \in \mathbb{R}$. Additionally, there is also a continuous ***scaling symmetry***,

$$\begin{aligned} A^i &\mapsto \lambda\, A^i, & \sigma &\mapsto \lambda^2 \sigma \\ B_i &\mapsto \lambda\, B_i, & \psi &\mapsto \lambda^2 \psi, \end{aligned} \tag{3.15}$$

parametrised by $\lambda > 0$. These symmetries correspond to isometries for the induced metric g_{hm} from equation (3.13) on the hypermultiplet moduli space $\mathcal{M}_{\text{hm}}$. Together the Peccei-Quinn symmetries from equation (3.14) describe the group action of a $(2\,h^{1,2}+3)$-dimensional ***Heisenberg group*** [53, 2], as we will see in section 5.1.

While the scaling symmetry (3.15) is broken when perturbative corrections are taken into account, the Peccei-Quinn symmetries will be preserved to any orders in perturbation theory [54, 55]. Explicit descriptions for this quantum-corrected quaternion-Kähler metric, which does not yet include non-perturbative corrections, can be found in [56] or [55].

Once also instanton contributions are added, the continuous Peccei-Quinn symmetries do break. These leave only a discrete subgroup of the group of isometries unbroken [54, 57]. There are two important types of instanton contributions to the hypermultiplet action that correspond to so-called D2-brane and NS5-branes and these can be considered separately. The first break the Peccei-Quinn isometries by restricting $(a^i, b_i) \in \mathbb{R}^{2+2h^{1,2}}$ from equation (3.14) to a lattice, $\mathbb{Z}^{2+2h^{1,2}}$, while contributions from the latter break the shift isometry in σ, which is parametrised by s in equation (3.14), to a discrete subgroup [54]. Although this cannot be said with absolute certainty, it seems most likely that when the two are combined, the remaining symmetries correspond to the transformations (3.14) for $a^i, b_i \in \mathbb{Z}$ and $s \in \mathbb{Z}$ (or possibly $s \in n^{-1}\mathbb{Z}$ for some $n \in \mathbb{N}$).

States that are related through these unbroken symmetries are completely indistinguishable in the full effective theory and should therefore be identified. This makes each of the fields A^i, B_i and thus means that $a^{(3)} = A^i\alpha_i - B_i\beta^i$ becomes an element of the ***intermediate Jacobian*** $H^3(\mathcal{Y}, \mathbb{R})/H^3(\mathcal{Y}, \mathbb{Z})$, while σ should now be viewed as element of the circle $\mathbb{R}/\mathbb{Z}$. No identifications are made for the dilaton field ϕ.

4. Contact and Cauchy-Riemann geometry

In this chapter contact metric manifold and Cauchy-Riemann (CR) manifolds are introduced, some important properties of these spaces will be presented and it will be explained how Sasakian structures are related to both of these concepts. We will finally apply all this to the Heisenberg group and explain how a the specific Kähler structure we will encounter in section 5.2 can be described in terms of contact and CR geometry.

4.1 Hyperplane fields

The central object in contact geometry is a hyperplane field [58, 59, 60], so the first thing we should do is define what we mean by that and discuss a few other general notions.

Definition 4.1.1 **(Hyperplane field).** *A* **hyperplane field** *F on a manifold M is a codimension 1 subbundle of the tangent bundle $\mathrm{T}M$. A (locally)* **defining form** *for F is a (locally defined) 1-form on M for which $F = \ker \eta$.*

For any hyperplane field $F < \mathrm{T}M$ we can also consider the (real) line bundle $\mathrm{T}M/F$. From now on the manifold M will be assumed to be connected and co-orientable, as defined below.

Definition 4.1.2 **(Co-orientability).** *The hyperplane field F is said to be* **co-orientable** *when $\mathrm{T}M/F$ is orientable (and hence trivial) and a* **co-orientation** *of F is a choice of one of the two connected components of $\mathrm{T}M/F \setminus (M \times \{0\})$, which is denoted by $(\mathrm{T}M/F)^+$.*

A section X of $\mathrm{T}M/F$ is said to be **positive** *if $X(x) \in (\mathrm{T}M/F)^+$ for all $x \in M$ and* **negative** *if $X(x) \in (\mathrm{T}M/F)^- = -(\mathrm{T}M/F)^+$ and we call a defining form η for F positive (resp. negative) if $\eta(X) > 0$ for all positive (resp. negative) sections of $\mathrm{T}M/F$.*

On any co-orientable hyperplane field we can define the Frobenius form by using the commutator $[\bullet, \bullet]$ for vector fields.

Definition 4.1.3 **(Frobenius form).** *The* **Frobenius form** *of a distribution $F < \mathrm{T}M$ is the anti-symmetric bilinear map $\omega \colon F \times F \to \mathrm{T}M/F$ given by*

$$\omega(X, Y) = \pi_{\mathrm{T}M/F}([X, Y]) \tag{4.1}$$

for any two sections X and Y of F. Here $\pi_{\mathrm{T}M/F}$ is the canonical projection map from $\mathrm{T}M$ onto $\mathrm{T}M/F$.

That ω is linear over $C^\infty(M)$ is easily verified since for any function f on M and any two sections X and Y of F we have

$$\omega(X, f\,Y) = \pi_{\mathrm{T}M/F}([X, f\,Y]) = \pi_{\mathrm{T}M/F}(f\,[X,Y] + X(f)\,Y) = f\,\pi_{\mathrm{T}M/F}([X,Y]) \tag{4.2}$$

because $\pi_{\mathrm{T}M/F}$ is $C^\infty(M)$-linear and $\ker \pi_{\mathrm{T}M/F} = F$. This means that the value of $\omega(X,Y)$ in the point $x \in M$ only depends on the value of X and Y in $x \in M$ and hence that, unlike the commutator bracket, the Frobenius form can be defined on the fibres of F rather than on sections.

Any defining form for the (co-orientable) hyperplane field $F < \mathrm{T}M$ vanishes on F by definition and thus defines a nowhere-vanishing homomorphism $\eta\colon \mathrm{T}M/F \to M \times \mathbb{R}$, i.e. a trivialisation for $\mathrm{T}M/F$. Conversely, any trivialisation $\varphi\colon \mathrm{T}M/F \to \mathbb{R}$ for $\mathrm{T}M/F$ comes from a unique defining form $\eta_\varphi = \varphi \circ \pi_{\mathrm{T}M/F}$ for $F < \mathrm{T}M$. This not only gives us a one-one correspondence between trivialisations of $\mathrm{T}M/F$ and defining forms for F, but also enables us to relate the Frobenius form of a hyperplane field to the external derivative of some defining form [61].

Proposition 4.1.4. *Let $F < \mathrm{T}M$ be a (co-orientable) hyperplane field and let $\eta = \varphi \circ \pi_{\mathrm{T}M/F}$ be a defining form for F, then $\mathrm{d}\eta|_F = -\varphi \circ \omega$, where ω is the Frobenius form for F.*

Proof: By using the intrinsic formula for the exterior derivative, we see that for any two sections X and Y of the bundle $F = \ker \eta < \mathrm{T}M$ we have that

$$\begin{aligned} \mathrm{d}\eta(X,Y) &:= X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = -\eta([X,Y]) \\ &= -\varphi\big(\pi_{\mathrm{T}M/F}([X,Y])\big) = -\varphi(\omega(X,Y)) \end{aligned} \tag{4.3}$$

by definition of η and ω. □

For the remainder of this chapter, the manifold M will be assumed to be of dimension $2\,n+1$ for some integer n.

4.2 Contact geometry

Contact geometry is often considered to be the odd-dimensional counterpart of symplectic geometry, which can only be defined for even-dimensional manifolds. Contact structures are often defined in terms of a contact form, which is a special type of defining form [24,60,58].

***Definition 4.2.1* (Contact form).** *A* **contact form** *η on M is a nowhere-vanishing 1-form for which one of the following equivalent conditions holds:*

1. *The $(2n+1)$-form $\eta \wedge (\mathrm{d}\eta)^n$ is nowhere-vanishing (it is a volume form),*
2. *The 2-form $\mathrm{d}\eta$ is non-degenerate on $F = \ker \eta$ (it is a symplectic form on F).*

Equivalence of these definitions is easily verified, as is the existence of a nowhere vanishing vector field ξ on M for which $\mathrm{d}\eta(\xi, \bullet) = 0$, which gives rise to the following definition.

***Definition 4.2.2* (Reeb vector field).** *The* **Reeb vector field** *associated with a contact form η is the unique vector field ξ on M for which $\eta(\xi) = 1$ and $\imath_\xi \mathrm{d}\eta = \mathrm{d}\eta(\xi, \bullet) = 0$.*

Although some authors [24] call the contact form itself the contact structure, it is more common to call the hyperplane field defined by such a form the contact structure [58, 60] and the contact condition can alternatively also be formulated in terms of the Frobenius tensor [61]

Proposition/Definition 4.2.3 **(Contact structure).** *A* **contact structure** *on M is a (co-orientable) hyperplane field $F < \mathrm{T}M$ for which one of the following equivalent conditions holds:*

1. *The Frobenius form ω of F is non-degenerate,*
2. *There exists a contact form η on M such that $F = \ker \eta$,*
3. *Any defining form for F is a contact form.*

A hyperplane field F satisfying these conditions is also called a **contact bundle** *and a manifold (M, F) equipped with a contact structure is called a* **contact manifold**.

Proof: Let η be some defining form for the hyperplane field $F < \mathrm{T}M$, then $\eta\colon \mathrm{T}M/F \to M \times \mathbb{R}$ defines a trivialisation and $\mathrm{d}\eta|_{F\times F} = -\eta \circ \omega$, as we had seen in proposition 4.1.4. This tells us that $\mathrm{d}\eta$ is non-degenerate on F exactly when ω is non-degenerate and since the defining form η was arbitrary it proves that each of the three conditions above are equivalent. □

It is not hard to show that for any contact form η one can locally choose coordinates $x^1, \dots, x^n, y_1, \dots, y_n, z$ with respect to which η is given by

$$\eta = \mathrm{d}z - \sum_i y_i \,\mathrm{d}x^i. \tag{4.4}$$

Such a coordinate system is called a ***Darboux coordinate system*** [24]. By instead choosing the coordinates $z' = z - \frac{1}{2}x^i y_i$, $x'^i = x^i/\sqrt{2}$ and $y'^i = y^i/\sqrt{2}$ we obtain

$$\eta = \mathrm{d}z' - \sum_i y'_i \overleftrightarrow{\mathrm{d}} x'^i, \tag{4.5}$$

where $y_i \overleftrightarrow{\mathrm{d}} x^i$ is short for $y_i \mathrm{d}x^i - x^i \mathrm{d}y_i$.

Remark 4.2.4. *A subbundle $F \subseteq \mathrm{T}M$ of rank k on an n-dimensional manifold is said to be* **integrable** *if M locally admits a folation into k-dimensional integral submanifolds (submanifolds that are tangent to F). By the Frobenius theorem, a hyperplane field $F < \mathrm{T}M$ is integrable exactly when its Frobenius form vanishes, which means that a contact bundle will never be integrable. The contact condition is in fact a much a stronger condition than non-integrability, as it guarantees that no submanifold of M of dimension greater than $\lfloor n \rfloor$ will be an integral submanifold with respect to a contact bundle $F < \mathrm{T}M$, which is why a contact distribution is sometimes said to be* **maximally non-integrable** [60].

4.2.1 Symplectisation

Let (M, F) be a co-orientable contact manifold. The vector bundle $(\mathrm{T}M/F)^*$ dual to $\mathrm{T}M/F$ can be naturally identified with the bundle

$$\mathcal{D} = \{\alpha \in \mathrm{T}^*_x M \mid x \in M, F_x \subseteq \ker(\alpha)\} \subseteq \mathrm{T}^*M, \tag{4.6}$$

which is generated by the defining forms for $F \subseteq \mathrm{T}M$. Co-orientability of (M, F) tells us that a globally defined nowhere-vanishing section η of $\mathcal{D}$ exists and hence that $\mathcal{D}$ is trivial. This bundle comes with a canonical 1-form θ defined as the restriction of the Liouville form on T^*M to the submanifold $\mathcal{D}$.

Definition 4.2.5 **(Liouville form).** *The cotangent bundle T^*M carries a canonical 1-form, called the* **Liouville form**, *which is given by*

$$\theta_{\eta_x} = \pi^* \eta \tag{4.7}$$

at the point $\eta_x \in \mathrm{T}^*M$, where $\pi\colon \mathrm{T}^*M \to M$ is the standard projection of T^*M onto M.

It is a well-known result that the exterior derivative $\mathrm{d}\theta$ of the Liouville form is non-degenerate as a 2-form on T^*M and thus defines a canonical symplectic structure on $\mathrm{T}^*M \setminus (M \times \{0\})$. Because of the fact that $F < \mathrm{T}M$ is a contact bundle, it turns out that the restricion of this symplectic structure to one of the connected components of $\mathcal{D} \setminus (M \times \{0\})$ is still non-degenerate.

***Theorem/Definition 4.2.6* (Symplectisation).** *Let η be a positive contact form on the contact manifold (M, F) and let ξ be the associated Reeb vector field and $\tilde{M}$ the bundle of positive defining forms,*

$$\tilde{M} = \{\alpha \in \mathcal{D} \mid \alpha(\xi) > 0\} = \{\lambda\, \eta_x \mid x \in M, \lambda > 0\} \subseteq \mathrm{T}^*M. \tag{4.8}$$

The restriction of the canonical symplectic form $\mathrm{d}\theta$ on T^*M, where θ is the Liouvile form from 4.2.5, to $\tilde{M}$ defines a symplectic structure $\tilde{\omega}$ on $\tilde{M}$. The symplectic manifold $(\tilde{M}, \tilde{\omega})$ we thus obtain is called the **symplectisation** [62, 60] *of the contact manifold (M, F).*

We will postpone the proof of the statement that the 2-form $\tilde{\omega}$ defines a symplectic structure for now and first discuss an alternative description of the symplectisation $\tilde{M}$.

Remark 4.2.7. *The canonical projection map $\pi\colon \tilde{M} \to M$ allows us to transfer any exterior k-form σ on M to a k-form $\pi^*\sigma$ on $\tilde{M}$, which we will usually also denote by σ This in particular means that we can extend a preferred defining form $\eta = \pi^*\eta$ to a 1-form on $\tilde{M}$. At any point $\eta_x \in \tilde{M}$, the Liouville form is given by $\theta_{\eta_x} = \pi^*\eta_x = \eta_x$.*

Although intrinsic, the definition above is not very practical to work with. We will therefore mainly work with a more explicit description of the symplectisation, which can be introduced by fixing a defining form and defining a coordinate on the fibres of $\tilde{M}$ [62, 60, 59].

***Proposition 4.2.8* (Explicit description).** *Let (M, F) be a co-oriented contact manifold and let η be a positive defining form for F. The map $\psi_\eta\colon M \times \mathbb{R}_{>0} \to \tilde{M} \subseteq \mathrm{T}^*M, (x, r) \mapsto r\, \eta_x$ is a diffeomorphism such that $\psi_\eta^*\theta = r\, \eta$ and*

$$\psi_\eta^*\tilde{\omega} = \mathrm{d}(r\, \eta) = \mathrm{d}r \wedge \eta + r\, \mathrm{d}\eta. \tag{4.9}$$

This 2-form on $M \times \mathbb{R}$ is non-degenerate and closed, so it defines a symplectic form on $M \times \mathbb{R}_{>0}$.

For this reason we will often identify $(\tilde{M}, \tilde{\omega})$ with $(M \times \mathbb{R}_{>0}, \mathrm{d}r \wedge \eta + r\, \mathrm{d}\eta)$ or just refer to this space as the symplectisation of M.

Proof: The defining form η defines a trivialisation $\tau\colon \mathcal{D} \to M \times \mathbb{R}, r\eta_x \mapsto (x, r)$, so its restriction to the symplectisation $\tilde{M}$ is a diffeomorphism $\tilde{M} \cong M \times \mathbb{R}_{>0}$. Since $\theta_{r\eta_x} = r\eta_x$ for all $x \in M$ and all $r > 0$, we immediately see that for $(x, r) \in M \times \mathbb{R}_{>0}$,

$$(\psi_\eta^*\theta)_{(x,r)} = \psi_\eta^*(\theta_{r\,\eta}) = \psi_\eta^*(\pi^* r\, \eta) = (\pi \circ \psi_\eta)^*(r\, \eta) = r\, \eta \tag{4.10}$$

since $\pi \circ \psi_\eta = \mathrm{id}_M$. It subsequently follows that

$$\psi_\eta^*\tilde{\omega} = \psi_\eta^*\mathrm{d}\theta = \mathrm{d}\psi_\eta^*\theta = \mathrm{d}(r\, \eta) = \mathrm{d}r \wedge \eta + r\, \mathrm{d}\eta. \tag{4.11}$$

Since η is a defining form for the contact bundle F, $\mathrm{d}\eta$ is non-degenerate on F, but $\mathrm{d}\eta(\xi) = 0$ by definition of ξ, so $\pi^*\mathrm{d}\eta$ is non-degenerate on $F_x \times \{0\} \subseteq \mathrm{T}_x(M \times \mathbb{R})$ and vanishes on the subspace of $\mathrm{T}_x(M \times \mathbb{R})$ spanned by $(\xi, 0)$ and ∂_r. Meanwhile, $\mathrm{d}r \wedge \eta$ is non-degenrate on exactly this subspace and vanishes on F_x. The form $\tilde{\omega} = \mathrm{d}(r\, \eta) = \mathrm{d}r \wedge \eta + r\, \mathrm{d}\eta$ is therefore non-degenrate on the whole of $\mathrm{T}_x\tilde{M}$ for all $x \in M$. It is exact and therefore closed, which means that it defines a symplectic structure on $M \times \mathbb{R}$ □

Proof of theorem 4.2.6: This is a direct consequence of proposition 4.2.8 since we can use the diffeomorphism ψ_η to transform the symplectic form $\psi_\eta^*\tilde{\omega}$ on $M \times \mathbb{R}$ back to $\tilde{M}$. □

It is possible to define the space $\tilde{M}$ using any hyperplane field $F \subseteq \mathrm{T}M$, but the 2-form $\tilde{\omega} = \mathrm{d}\theta$ will only be non-degenerate if (M, F) is a contact manifold. A contact structure can therefore equivalently be defined as a hyperplane field $F < \mathrm{T}M$ for which the symplectic structure on T^*M induces a symplectic structure on the space $(\mathrm{T}M/F)^*$ [60]. From now on we will mostly work with the explicit description of the symplectisation as $M \times \mathbb{R}_{>0}$ introduced in proposition 4.2.8 for some fixed defining form η and write $\tilde{\omega}$ for the induced symplectic form $\psi_\eta^* \tilde{\omega} = \mathrm{d}r \wedge \eta + r\, \mathrm{d}\eta$.

4.2.2 Contact metric structures

There exists a special class of metrics on contact manifolds, namely those that are part of a contact metric structure. The definition of a contact metric structure makes use of an almost contact structure, which is strongly related to the notion of a contact structure and that of an almost complex structure.

***Definition 4.2.9* (Almost contact structure).** *An* **almost contact structure** (ϕ, ξ, η) *on* M *is an endomorphism* ϕ *on* $\mathrm{T}M$ *such that*

$$\phi^2 = -\operatorname{id}_{\mathrm{T}M} + \xi \otimes \eta, \tag{4.12}$$

for the vector field ξ *and the* 1*-form* η *on* M *such that* $\eta(\xi) = 1$*. A manifold admitting such a structure is called an* **almost contact manifold***.*

As will be made more explicit in proposition 4.2.13, any contact manifold (M, F) with $F = \ker \eta$ and corresponding Reeb vector field ξ admits an almost contact stucture (ϕ, ξ, η), but the converse is generally not true [24, 58]. It is however possible to prove that any non-compact connected manifold with an almost contact structure also admits a contact structure and that the two can be related through homotopy [60], but this is not what we are interested in.

Lemma 4.2.10. *Let* (ϕ, ξ, η) *be an almost contact structure on* M*, then* $\ker \phi$ *is a (real) line bundle spanned by* ξ *and* $\operatorname{im} \phi = \ker \eta$*.*

Proof: Since $\eta(\xi) = 1$ we see that $\phi^2(\xi) = -\xi + \eta(\xi)\, \xi = 0$ and hence

$$0 = \phi^3(\xi) = \phi^2(\phi(\xi)) = -\phi(\xi) + \eta(\phi(\xi))\, \xi, \tag{4.13}$$

so $\phi(\xi) = \eta(\phi(\xi))\, \xi$. It subsequently follows that $\phi(\xi) = 0$ since

$$0 = \phi^2(\xi) = \phi(\eta(\phi(\xi)\, \xi)) = \eta(\phi(\xi))\, \phi(\xi) = \eta(\phi(\xi))^2\, \xi, \tag{4.14}$$

which is only possibly if $\eta(\phi(\xi)) = 0$ and thus $\phi(\xi) = \eta(\phi(\xi))\, \xi = 0$.

This means that for any vector field X on M,

$$\begin{aligned} \phi(X) &= -\phi(-X + \eta(X)\, \xi) = -\phi(\phi^2(X)) \\ &= -\phi^2(\phi(X)) = \phi(X) - \eta(\phi(X))\, \xi, \end{aligned} \tag{4.15}$$

which tells us that $\eta(\phi(X)) = 0$ and hence that $\eta \circ \phi = 0$, i.e. $\operatorname{im} \phi < \ker \eta$.

If $\phi(X) = 0$ for some $X \in \mathrm{T}_x M$ then we have $\phi^2(X) = -X + \eta(X)\, \xi_x = 0$, so we necessarily have $X \in \mathbb{R}\, \xi_x$. Since we had already seen that $\phi(\xi) = 0$ it follows that for any $x \in M$, $\ker \phi = \mathbb{R}\, \xi_x$, which in particular means that $\operatorname{rank} \phi = \dim M - 1 = 2\, n$. We had also shown that $\operatorname{im} \phi < \ker \eta$, but for $x \in M$ both $\operatorname{im} \phi_x$ and $\ker \eta_x$ are subspaces of $\mathrm{T}_x M$ dimension $2\, n$, so they are in fact equal. $\square$

By using the definition of an almost contact structure we can now subsequently introduce the notion of an almost contact metric structure and a contact metric structure.

***Definition 4.2.11* (Almost contact metric structure).** *A metric g on a almost contact manifold (M, ϕ, ξ, η) is said to be be compatible with the almost contact structure if*

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{4.16}$$

for any two vector fields X and Y on M. Together with the almost contact structure, g forms an **almost contact metric structure** *or* **almost contact Riemannian structure** *(ϕ, ξ, η, g).*

***Definition 4.2.12* (Contact metric structure).** *An almost contact metric structure (ϕ, ξ, η, g) on M is said to be a* **contact metric structure** *if*

$$g(X, \phi Y) = \tfrac{1}{2}\mathrm{d}\eta(X, Y) \tag{4.17}$$

for any two vector fields X and Y on M and the manifold (M, ϕ, ξ, η, g) is then called a **contact metric manifold** *or a* **contact Riemannian manifold**.

Note that this definition in particular implies that $\mathrm{d}\eta|_F$ is non-degenerate and hence that $F = \ker\eta$ defines a contact structure on M. If a metric g exists such that (ϕ, ξ, η, g) is a contact metric structure for a given almost contact structure (ϕ, ξ, η), this metric is in fact completely fixed by equation (4.16) and (4.17), since the first tells us that $g(\xi, \bullet) = \eta$ and the latter fixes g on $F = \operatorname{im}\phi$. The metric g is given by

$$g = \tfrac{1}{2}\mathrm{d}\eta(\phi\bullet, \bullet) + \eta \otimes \eta \tag{4.18}$$

Proposition 4.2.13. *Let (M, F) be a contact structure such that $F = \ker\eta$ for the contact form η whose Reeb vector field is ξ. There exist an endomorphism $\phi\colon \mathrm{T}M \to \mathrm{T}M$ and a metric g on M such that (ϕ, ξ, η, g) is a contact metric structure.*

Proof: See [24] for a proof of this proposition using the polar decomposition of matrices. □

Definition 4.2.14. *A contact metric structure (ϕ, ξ, η, g) is said to be* **K-contact** *if ξ is a Killing vector field for g, i.e. $\mathcal{L}_\xi g = 0$, or equivalently if $\mathcal{L}_\xi\phi = 0$.*

If (ϕ, ξ, η, g) is a contact metric structure, then g is given by equation (4.18) and thus

$$\mathcal{L}_\xi g = \tfrac{1}{2}\mathrm{d}(\mathcal{L}_\xi\eta)(\phi\,\bullet, \bullet) + \tfrac{1}{2}\mathrm{d}\eta((\mathcal{L}_\xi\phi)\,\bullet, \bullet) + (\mathcal{L}_\xi\eta) \otimes \eta + \eta \otimes (\mathcal{L}_\xi\eta) \tag{4.19}$$

A direct consequence of the definition of the Reeb vector field ξ is that $\mathcal{L}_\xi\eta = 0$, which also means that $\mathcal{L}_\xi\mathrm{d}\eta = 0$. When we plug this into equation (4.19) we are left with $\mathcal{L}_\xi g = \frac{1}{2}\mathrm{d}\eta((\mathcal{L}_\xi\phi)\bullet, \bullet)$, which vanishes exactly when $\mathcal{L}_\xi\phi = 0$ (N.B. $(\mathcal{L}_\xi\phi)(\xi) = 0$ always holds).

4.3 Cauchy-Riemann structures

Another interesting type of structure that we can define on a hyperplane field are the so-called CR structures.

***Definition 4.3.1* (CR structure).** *A* **CR structure** *(Cauchy-Riemann structure) on a manifold M is an involutive subbundle $H < \mathrm{T}_\mathbb{C}M$ of the complexified tangent bundle for which $H \cap \overline{H} = 0$. Involutivity of H means that $[X, Y] \in H$ for any two sections of H. If $H < \mathrm{T}_\mathbb{C}$ is a CR structure then the pair (M, H) is said to be a CR manifold.*

A complex structure is the same thing as a CR structure of maximal rank on an even-dimensional manifold M and satisfies $\mathrm{T}_\mathbb{C}M = H \oplus \overline{H}$. Although the definition of a CR structure given above makes no reference to the rank of H or the dimension of M, we will from now on always assume that the dimension of M is an odd number $2n+1$ and that the rank of H is maximal ($\dim_{\mathbb{C}} H = n$). This assumption is usually made when dealing with CR manifolds in relation to contact geometry since then $H_{\mathbb{R}} = (H \oplus \overline{H}) \cap \mathrm{T}M$ is a hyperplane field. In analogy to the corresponding definition on a complex manifold, we have a notion of holomorphicity for functions between CR manifolds.

Definition 4.3.2 **(CR-holomorphicity).** *A complex function $f\colon M \to \mathbb{C}$ on the CR manifold (M,H) is said to be* **CR-holomorphic** *if $\mathrm{d}f(X) = X(f) = 0$ for all $X \in H$. A map φ from one CR manifold (M,H) to a CR manifold (M',H') is called* **CR-holomorphic** *if $\varphi_*(H) \subseteq H'$.*

A CR structure induces an endomorphism J_H on $H_{\mathbb{R}} = (H \oplus \overline{H}) \cap \mathrm{T}M$ satisfying $J_H^2 = -\operatorname{id}_{H_{\mathbb{R}}}$ whose complex eigenspace are H and $\overline{H}$ and correspond to the eigenvalues $+i$ and $-i$ respectively. Since the hyperplane field $H_{\mathbb{R}}$ and the endomorphism J_H completely determine the CR structure, we will generally denote the CR structure $H < \mathrm{T}_{\mathbb{C}}M$ by the pair $(H_{\mathbb{R}}, J_H)$.

Definition 4.3.3 **(Levi form).** *The Frobenius form $\omega_H\colon H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathrm{T}M/H_{\mathbb{R}}$ for the hyperplane field associated with a CR structure H on M is called the* **Levi form** *of the CR structure.*

It can easily be verified that (the complex extension of) the Levi form ω_H vanishes on $H \times H$ and $\bar{H} \times \bar{H}$ because $[H,H] \subseteq H$ by the CR condition. This tells us that the bilinear form $\omega_H(J_H\bullet,\bullet)\colon H_{\mathbb{R}} \times H_{\mathbb{R}} \to \mathrm{T}M/H_{\mathbb{R}}$ is symmetric.

Definition 4.3.4 **(Strictly pseudoconvex CR structure).** *Let $(H_{\mathbb{R}}, J_H)$ define a CR structure and fix a co-orientation for $H_{\mathbb{R}}$, then we say that a 2-form ω is* **positive** *if $\omega(J_H X, X) \in (\mathrm{T}M/H_{\mathbb{R}})^+$ for any $X \in H_{\mathbb{R}} \setminus (M \times \{0\})$. A CR structure is called* **Levi non-degenerate** *if the Levi form ω_H is non-degenerate and* **strictly pseudoconvex** *if $\omega(J_H\bullet,\bullet)$ is positive for a co-orientation of $H_{\mathbb{R}}$.*

Since the Levi-form is just the Frobenius form of the hyperplane field $H_{\mathbb{R}}$, the hyperplane field $H_{\mathbb{R}}$ will define a contact structure exactly when the CR structure is Levi non-degenerate. A strictly pseudoconvex CR manifold can furthermore be equipped with a contact metric structure, but this structure is not unique since a defining form for $H_{\mathbb{R}}$ should first be specified.

Definition 4.3.5 **(Underlying CR structure).** *Let (ϕ, ξ, η, g) be a contact metric structure on M such that $(H_{\mathbb{R}}, J_H) = (\ker \eta, \phi|_{H_{\mathbb{R}}})$ be a CR structure, then this CR structure is said to be the* **underlying CR structure** *for the contact metric structure.*

Proposition 4.3.6. *If $(H_{\mathbb{R}}, J_H)$ is a strictly pseudoconvex CR structure (with corresponding orientation) and η is a positive defining form for $H_{\mathbb{R}}$, then (ϕ, η, ξ, g) is a contact metric structure if ξ is the corresponding Reeb vector field, ϕ is given by $\phi(\xi) = 0$ and $\phi_{H_{\mathbb{R}}} = J_H$ and $g = \frac{1}{2}\mathrm{d}\eta(\phi\bullet,\bullet) + \eta \otimes \eta$.*

A CR structure $(H_{\mathbb{R}}, J_H)$ on M is the underlying CR structure for a contact metric structure if and only if it is strictly pseudoconvex.

Definition 4.3.7 **(CR automorphism).** *a* **CR automorphism** *on a CR manifold $(M, H_{\mathbb{R}}, J_H)$ is a diffeomorphism φ from M to itself such that $\varphi_*(H_{\mathbb{R}}) \subseteq H_{\mathbb{R}}$ and $\varphi_* \circ J_H = J_H \circ \varphi_*$.*

Proposition 4.3.8. *The group $\mathfrak{CR}(M, H_{\mathbb{R}}, J_H)$ of CR automorphisms for the CR manifold $(M, H_{\mathbb{R}}, J_H)$ is a Lie group. Its Lie algebra can be describes through the infinitesimal CR automorphisms and is given by*

$$\mathfrak{cr}(M, H_{\mathbb{R}}, J_H) = \{\xi \in \mathfrak{X}(M) \mid [\xi, H_{\mathbb{R}}] \subseteq H_{\mathbb{R}} \text{ and } \mathcal{L}_\xi J = 0\}. \tag{4.20}$$

4.3.1 Sasakian structure

A Sasakian structure is a special kind of contact metric structure that is of particular interest to us.

***Definition 4.3.9* (Sasakian structure).** *A* **Sasakian structure** *on M is a contact metric structure (ϕ, ξ, η, g) on M that is K-contact and for which $(F, J) = (\ker \eta, \phi|_F)$ is a CR structure. A CR structure that underlies a Sasakian structure is said to be of* **Sasaki type**.

Note that all elements of the Sasakian structure (ϕ, ξ, η, g) are completely determined by the CR structure $(F, J) = (\ker \eta, \phi|_F)$ and the Reeb (Killing) vector field ξ. The complete tuple (ϕ, ξ, η, g) is determined by the equations $\eta(\xi) = 1$, $\eta|_F = 0$, $\phi(\xi) = 0$, $\phi|_F = J$ and $g = \frac{1}{2}\mathrm{d}\eta(\phi\bullet, \bullet) + \eta \otimes \eta$.

***Definition 4.3.10* (Normal almost contact structure).** *Let (ϕ, ξ, η) describe an almost contact structure on M. We can define an almost complex structure $\tilde{J}$ on the space $\tilde{M} = M \times \mathbb{R}$ by*

$$\tilde{J}(X + \kappa\, \partial_r) = \phi\, X - \kappa\, \xi + \eta(X)\, \partial_r, \tag{4.21}$$

were r is the standard coordinate on $\mathbb{R}$ and $\xi \sim (\xi, 0)$ is the Reeb vector field associated with preferred defining form η. The almost contact structure (ϕ, ξ, η) is said to be **normal** *when $\tilde{J}$ is integrable.*

We know that an almost complex structure is integrable if and only if its ***Nijenhuis tensor*** vanishes, which we can use to obtain a similar result for for almost contact structures.

Theorem 4.3.11. *An almost contact structure (ϕ, ξ, η) is normal if and only if the $(1, 2)$-tensor defined by*

$$N^{(1)}(X, Y) = -\phi^2[X, Y] + \phi[\phi\, X, Y] + \phi[X, \phi\, Y] - [\phi(X), \phi(Y)] - \mathrm{d}\eta(X, Y)\, \xi, \tag{4.22}$$

vanishes.

Sketch of the proof: The tangent space of $\tilde{M}$ is spanned by $\mathrm{T}(M \times \{0\}) \simeq \mathrm{T}M$ together with the vector ∂_t. Since the Nijenhuis tensor is anti-symmetric, $N(\partial_t, \partial_t) = 0$ and we only need to check whether $N(X, Y)$ and $N(X, \partial_t)$ vanish for vector fields X and Y on M. A lengthy calculation shows that

$$N_{\tilde{J}}(X, Y) = N^{(1)}(X, Y) + N^{(2)}(X, Y)\, \partial_t \qquad \text{and} \qquad N(X, \partial_t) = N^{(3)}(X), \tag{4.23}$$

where $N^{(1)}$ is the object described in equation (4.22), $N^{(2)}(X, Y) = \eta(N^{(1)}(\phi\, X, Y))$ and $N^{(3)}(X) = -\phi(N^{(1)}(X, \xi))$. We immediately see that the vanishing of $N^{(1)}$ is a necessary and sufficient condition for the $\tilde{J}$ to be integrable and thus for (ϕ, ξ, η) to be normal.

The tensorial nature of $N^{(1)}$ can be explicitly verified. □

Instead of using the characterisation for Sasakian structure given in definition 4.3.9, a Sasakian structure is often defined as a normal contact metric structure or as a contact metric structure whose (real) cone admits a particular kind of Kähler metric. All of these definitions are equivalent. There are also other characterisations for Sasakian structures that we will not go into here [63, 64]

***Theorem/Definition 4.3.12* (Alternative definitions).** *Let g be a Riemannian metric on M, let ξ and η be a vector field and a 1-form on M respectively and let $\phi\colon \mathrm{T}M \to \mathrm{T}M$ be an endomorphism on the tangent bundle $\mathrm{T}M$. The following statements are all equivalent*

1. *(ϕ, ξ, η, g) is a Sasakian structure.*

2. (ϕ, ξ, η, g) *is a normal contact metric structure on* M.
3. *The vector* ξ *is a Killing vector field on* (M, g) *of unit length, the endomorphism* ϕ *on* M *is given by* $\phi(X) = -\nabla_X \xi$ *satisfying*

$$(\nabla_X \phi)(Y) = g(X, Y)\, \xi - \eta(Y)\, X, \tag{4.24}$$

4. *There exists a dilation invariant complex structure* $\tilde{J}$ *(i.e.* $\mathcal{L}_{r\partial_r}\tilde{J} = 0$*) on the cone* $M \times \mathbb{R}_{>0}$ *with respect to which the metric* $\mathrm{d}r^2 + r^2\, g$ *on* $M \times \mathbb{R}_{>0}$ *is Kähler.*
   If we denote the canonical projection from $M \times \mathbb{R}_{>0}$ *to* M *by* $\tilde{\pi}$*, then* $\xi = -\tilde{\pi}_* \tilde{J}(r\partial_r)$*,* $\eta = g(\xi, \bullet)$ *and* ϕ *is determined by* $\phi(\xi) = 0$ *and* $g(\bullet, \phi\bullet) = \frac{1}{2}\mathrm{d}\eta$.

Definition 4.3.13 **(Sasakian η-Einstein structure).** *A* **Sasaki-Einstein structure** *is a Sasakian structure* (ϕ, ξ, η, g) *which is also Einstein, i.e. one for which the Ricci* Ric_g *is proportional to* g*. A Sasakian structure is said to be* **Sasakian η-Einstein** *if the Ricci tensor for* g *is given by*

$$\mathrm{Ric}_g = \lambda\, g + \nu\, \eta \otimes \eta \tag{4.25}$$

for some constants $\lambda, \nu \in \mathbb{R}$.

Definition 4.3.14 **(Sasakian cone).** *Let* $(M, H_{\mathbb{R}}, J_H)$ *be a strictly pseudoconvex CR manifold and let* η *be a positive defining form for* $H_{\mathbb{R}}$*, then the (unrestricted)* **Sasakian cone** *is defined as the subset*

$$\mathfrak{cr}^+(M, H_{\mathbb{R}}, J_H) = \{\xi \in \mathfrak{cr}(M, H_{\mathbb{R}}, J_H) \mid \eta(\xi) > 0\}. \tag{4.26}$$

The Sasakian cone is a convex cone since we can easily check that $\lambda\,\xi + \xi' \in \mathfrak{cr}^+(M, H_{\mathbb{R}}, J_H)$ for all $\xi, \xi' \in \mathfrak{cr}^+(M, H_{\mathbb{R}}, J_H)$ and all $\lambda > 0$.

Theorem 4.3.15. *Let* $(H_{\mathbb{R}}, J_H)$ *be a strictly pseudoconvex CR structure on* M*, then* $\xi \in \mathfrak{cr}^+(M, H_{\mathbb{R}}, J_H)$ *for any Sasakian structure* (ϕ, ξ, η, g) *on* M *with* $(H_{\mathbb{R}}, J_H)$ *as its underlying CR structure* $(H_{\mathbb{R}}, J_H)$*. Moreover, if we let* $\mathcal{S}(M, H_{\mathbb{R}}, J_H)$ *denote the set of all Sasakian structures on* M *which have* $(H_{\mathbb{R}}, J_H)$ *as their underlying CR structure, then the map*

$$\mathcal{S}(M, H_{\mathbb{R}}, J_H) \to \mathfrak{cr}^+(M, H_{\mathbb{R}}, J_H) \qquad (\phi, \xi, \eta, g) \mapsto \xi \tag{4.27}$$

is a bijection.

4.4 The Heisenberg group

The most simple example of a contact structure is the one defined on the odd-dimensional Euclidean space $\mathbb{R}^{2n+1}$ by using equation (4.4) to define a contact structure on the entire space. A more interesting way to construct a contact structure on odd-dimensional vector spaces is by using a symplectic vector space to construct a Heisenberg group.

Definition 4.4.1 **(Heisenberg (Lie) group).** *Let* (V, ω) *be a symplectic vector space, then we define the* **Heisenberg group** *associated with this space to be* $\mathcal{H}(V, \omega) = (V \times \mathbb{R}, \cdot)$*. Group multiplication on* $\mathcal{H}(V, \omega)$ *is defined by*

$$(v, s) \cdot (w, t) = (v + w, s + t - \omega(v, w)), \tag{4.28}$$

for $v, w \in V$ *and* $s, t \in \mathbb{R}$.

It can easily be verified that the product defined in equation (4.28) defines a group structure on $V \times \mathbb{R}$ with neutral element $(0, 0)$ such that the inverse of an element $(v, s) \in \mathcal{H}(V, \omega)$ is simply $(-v, -s)$. Since $V \times \mathbb{R}$ is a linear space, it carries a natural smooth structure with respect to which both group multiplication and inversion are smooth The Heisenberg group therefore carries the structure of a (connected) Lie group. For the remainder of this section, (V, ω) will denote a $2n$-dimensional real symplectic vector space and $\mathcal{H} = \mathcal{H}(V, \omega)$ will be its associated Heisenberg group.

Remark 4.4.2. *It is always possible to choose a basis on V with respect to which ω looks like the standard symplectic form. If we then write $v = (x^1, \dots, x^n, y_1, \dots, y_n)$ and $w = (x'^1, \dots, x'^n, y'_1, \dots, y'_n)$ in terms of the corresponding coordinates we have $\omega(v, w) = x^i y'_i - y_i x'^i$. If we add an additional coordinate s on $\mathbb{R}$, then group multiplication on $\mathcal{H}(V, \omega)$ will be described by*

$$(v, s) \cdot (w, t) = (x, y, s) \cdot (x', y', t) = (x + x', y + y', s + t - x^i y'_i + y_i x'^i). \tag{4.29}$$

For this reason, the $2n + 1$-dimensional Heisenberg group is sometimes defined as the space $\mathbb{R}^{2\,n+1}$ with the multiplication described by equation (4.29) [65].

Because we will require the commutator of left-invariant vector fields on $\mathcal{H}$ on a number of occasions it will be useful to know the Lie algebra of $\mathcal{H}$.

***Lemma 4.4.3* (Heisenberg algebra).** *The Lie algebra, $\mathfrak{H}(V, \omega)$, of $\mathcal{H}(V, W)$ is the linear space $V \times \mathbb{R}$, equipped with the Lie bracket defined by*

$$[(v, s), (w, t)] = (0, -2\,\omega(v, w)) \tag{4.30}$$

Proof: A simple calculation shows that the commutator of two group elements (v, s) and (w, t) is given by

$$\begin{aligned} [(v, s), (w, t)] &= (v, s) \cdot (w, t) \cdot (v, s)^{-1} \cdot (w, t)^{-1} \\ &= (v + w, s + t - \omega(v, w) \cdot (-v - w, -s - t - \omega(v, w)) \\ &= (0, -2\,\omega(v, w)). \end{aligned} \tag{4.31}$$

Since $(0, 0)$ is the unit element, this proves the lemma. □

Recall that a vector field X on $\mathcal{H}$ is called ***left-invariant*** if $\ell_{(v,s)*} X = X$ for all $(v, s) \in \mathcal{H}$ and that such a vector field is completely determined by its value at the unit element through $X(v, s) = \ell_{(v,s)*} X(0, 0)$. Here $\ell_{(v,s)} \colon \mathcal{H} \to \mathcal{H}, (w, t) \mapsto (v, s) \cdot (w, t)$ denotes left translation by $(v, s) \in \mathcal{H}$ [66].

***Lemma 4.4.4* (Invariant vector fields).** *The left-invariant vector field $X^{(w,t)}$ on $\mathcal{H}$ for which $X^{(w,t)}(0, 0) = (w, t)$ are given by*

$$X^{(w,t)}(v, s) = (w, t - \omega(v, w)). \tag{4.32}$$

Proof: The value of the left-invariant vector field $X^{(w,t)}$ for which $X^{(w,t)}(0, 0) = (w, t)$ at $(v, s) \in \mathcal{H}(V, \omega)$ follows by taking the derivative of $\ell_{(v,s)}$

$$\begin{aligned} X^{(w,t)}(v, s) = \ell_{(v,s)*}(w, t) &= \tfrac{\mathrm{d}}{\mathrm{d}\varphi}\big|_{\varphi=0} \ell_{(v,s)}(\varphi\, w, \varphi\, t) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}\varphi}\big|_{\varphi=0} (v + \varphi\, w, s + \varphi\, t - \omega(v, \varphi\, w)) \\ &= (w, t - \omega(v, w)). \end{aligned} \tag{4.33}$$

This corresponds exactly with equation (4.32). □

Note that this in particular means that the vector field $\xi = X^{(0,1)} = (0, 1) = \partial_s$ is invariant. The fact that we already know the Lie algebra of $\mathcal{H}$ enables us to write the commutator of two invariant vector fields in terms of symplectic structure and this vector field ξ.

Corollary 4.4.5 **(Commutator bracket).** *The vector field commutator of two left-invariant vector fields $X^{(v,s)}$ and $X^{(w,t)}$ is given by*

$$[X^{(v,s)}, X^{(w,t)}] = X^{([(v,s),(w,t)])} = -2\,\omega(v,w)\,\xi, \tag{4.34}$$

where $\xi = X^{(0,1)} = \partial_s$.

Proof: This is a direct consequence of lemma 4.4.3 and a general result for Lie groups relating the commutator bracket of invariant vector fields to the Lie algebra. □

4.4.1 Contact and CR structures on the Heisenberg group

The left-invariant differential 1-forms on $\mathcal{H}$ are exactly those forms that evaluate to constant functions when applied to an invariant vector field. This means that a 1-form α is invariant if and only if for any $(v,s) \in \mathrm{T}_{(0,0)}\mathcal{H} \simeq V \times \mathbb{R}$ the function $\alpha(X^{(v,s)})\colon \mathcal{H} \to \mathbb{R}$ is constant . We will be particularly interested in the one 1-form η defined by

$$\eta(X^{(v,s)}) = s \tag{4.35}$$

for $(v,s) \in \mathrm{T}\mathcal{H} \simeq V \times \mathbb{R}$.

Lemma 4.4.6. *Let η be the left-invariant 1-form on $\mathcal{H}(V,\omega)$ defined above, then for any two left-invariant vector fields $X^{(v,s)}$ and $X^{(w,t)}$*

$$\mathrm{d}\eta(X^{(v,s)}, X^{(w,t)}) = 2\,\omega(v,w) \tag{4.36}$$

Proof: This can easily be verified through an explicit computation, as this shows that

$$\begin{aligned} \mathrm{d}\eta(X^{(v,s)}, X^{(w,t)}) &= X^{(v,s)}\eta(X^{(w,t)}) - X^{(w,t)}\eta(X^{(v,s)}) - \eta([X^{(v,s)}, X^{(w,t)}]) \\ &= X^{(v,s)}(t) - X^{(w,t)}(s) - \eta(-2\,\omega(v,w)\,\xi) = 2\,\omega(v,w), \end{aligned} \tag{4.37}$$

where $\xi = X^{(0,1)}$ as before. □

There is a natural way to define a hyperplane field on the Heisenberg group that is invariant with respect to the group structure [67] by taking the subspace $V < \mathrm{T}_{(0,0)}\mathcal{H}$ and extending it to other points in the group through left-translations. Using the lemma above, it becomes straightforward to show that this hyperplane field defines a contact structure on $\mathcal{H}(V,\omega)$.

Corollary 4.4.7 **(Contact structure on $\mathcal{H}$).** *The hyperplane field $F < \mathrm{T}\mathcal{H}$ generated by the vector fields $X^{(w,0)}$ for $w \in \mathrm{T}_{(0,0)}V \simeq V$ defines a contact structure. The invariant form η from equation (4.35) is a defining form for F and $\xi = \partial_s$ is the corresponding Reeb vector field.*

Proof: The 1-form η was defined to satisfy $\eta(X^{(v,s)}) = s$, so $\eta(\xi) = 1$ and $\eta(X^{(v,0)}) = 0$. This shows that η is nowhere vanishing and that moreover $\ker\eta = F$. We know that ω is non-degenerate on V, which tells us that $\mathrm{d}\eta$ is non-degenerate on F since we had seen that $\mathrm{d}\eta(X^{(v,0)}, X^{(w,0)}) = \omega(v,w)$. Lemma 4.4.6 furthermore tells us that $\mathrm{d}\eta(\xi,\bullet) = 0$ and we had already seen that $\eta(\xi) = 1$, so ξ is the Reeb vector field corresponding to η. □

The imaginary part of the Hermitian form on this space is a symplectic form, so instead of constructing the Heisenberg group from a (real) symplectic vector space, it is also possible to start with a Hermitian vector space. If we construct the Heisenberg group in this way we can use the complex structure to obtain a CR structure.

Proposition 4.4.8. *Let (V, h) be a Hermitian vector space of (complex) dimension n with Hermitian form h and let $\mathcal{H} = \mathcal{H}(V, \omega)$ be the Heisenberg group for the symplectic vector space (V, ω), with $\omega = \mathrm{Im}(h)$. The bundle F from corollary 4.4.7, generated by the fields $X^{(v,0)}$ for $v \in V$, together with the endomorphism $J\colon F \to F$ defined through*

$$J(X^{(v,0)}) = X^{(i\,v,0)}, \tag{4.38}$$

defines a Levi-nondegenerate CR structure $(H_{\mathbb{R}}, J_H) = (F, J)$ on $\mathcal{H}$. If the Hermitian form h is a Hermitian metric on V, then (F, J) is strictly pseudoconvex.

Proof: It is easily verified that equation (4.38) completely defines J as a linear endomorphism and that moreover $J^2 = -\,\mathrm{id}_F$. This tells us that J has eigenvalues $+i$ and $-i$ with eigenspaces H and $\bar{H}$ respectively, such that $F_{\mathbb{C}} = F \otimes \mathbb{C} = H \oplus \bar{H}$ and $H \cap \bar{H} = 0$.

Compatibility of the Hermitian form h and the complex structure on V tells us that $\omega(i\,v, i\,w) = \omega(v, w)$ for all $v, w \in V$ and thus, by corollary 4.4.5,

$$[X^{(v,0)}, X^{(w,0)}] = -2\,\omega(v, w) = -2\,\omega(i\,v, i\,w) = [J\,X^{(v,0)}, J\,X^{(w,0)}]. \tag{4.39}$$

A basis for the sections of H is given by the left-invariant sections $(1 - i\,J)X^{(v,0)}$ for $v \in V$. The commutator of two such sections is given by

$$\begin{aligned}
[(1 - i\,J)X^{(v,0)}, (1 - i\,J)X^{(w,0)}] &= [X^{(v,0)}, X^{(w,0)}] + i^2[J\,X^{(v,0)}, J\,X^{(w,0)}] \\
&\quad - i[J\,X^{(v,0)}, X^{(w,0)}] - i[X^{(v,0)}, J\,X^{(w,0)}] \\
&= [X^{(v,0)}, X^{(w,0)}] - [X^{(v,0)}, X^{(w,0)}] \\
&\quad - i[J^2 X^{(v,0)}, J\,X^{(w,0)}] - i[X^{(v,0)}, J\,X^{(w,0)}] = 0
\end{aligned} \tag{4.40}$$

which tells us that the commutator of any two sections X and Y of H is again a section of H, i.e. that H is involutive.

Non-degeneracy of the Levi-form means exactly that F is a contact bundle, which we had already shown in corollary 4.4.7. If moreover, the form h is positive definite then, since $h = \omega(i\bullet, \bullet) + i\,\omega$,

$$\mathrm{d}\eta(J\,X^{(v,0)}, X^{(v,0)}) = 2\,\omega(i\,v, v) = 2\,h(v, v) > 0. \tag{4.41}$$

From this we learn that the CR structure (F, J) is strictly pseudoconvex. □

4.4.2 Sasakian structure on the Heisenberg group

Let h be a Hermitian metric on the complex vector space V and let $\omega = \mathrm{Im}(h)$, let $\mathcal{H} = \mathcal{H}(V, \omega)$ be the associated Heisenberg group and let (F, J) be the strictly pseudoconvex CR structure defined in proposition 4.4.8. By proposition 4.3.6 this means that any positive defining form η for F gives us a contact metric structure on $\mathcal{H}$ with (F, J) as its underlying CR structure.

We would like to equip the Heisenberg group with a metric, but while proposition 4.3.6 gives us a prescription to find one by fixing a positive defining form for $F < \mathrm{T}M$, it does not tell us which contact form to use. Both the contact structure and the CR structure have been defined in such a way that they are invariant under group translations, we would also like to require left-invariance for the contact metric structure.

The positive left-invariant defining forms for $F < \mathrm{T}M$ are the forms $\eta_\lambda = \lambda\,\eta$ for some $\lambda > 0$, which correspond to the Reeb vector fields $\xi_\lambda = \lambda^{-1}\xi$, where η is still the contact form from equation (4.35) and $\xi = \partial_s$. This gives us a 1-dimensional family of left-invariant contact metric structures $(\phi_\lambda, \xi_\lambda, \eta_\lambda, g_\lambda)$ for which $\phi|_F = J$. The endomorphism $\phi = \phi_\lambda$ does not depend on λ and is given by

$$\phi(X^{(v,s)}) = J(X^{(v,0)}) = X^{(i\,v,0)}, \tag{4.42}$$

as we can easily verify. The almost contact structure $(\phi, \xi_\lambda, \eta_\lambda)$ fixes the metric g_λ, which is given by

$$g_\lambda = \tfrac{1}{2}\mathrm{d}\eta_\lambda(\phi\,\bullet, \bullet) + \eta_\lambda \otimes \eta_\lambda = \tfrac{1}{2}\lambda\,\mathrm{d}\eta(\phi\,\bullet, \bullet) + \lambda^2 \eta \otimes \eta. \tag{4.43}$$

Apart from being invariant these contact metric structures have another interesting property, namely that they are in fact Sasakian structures.

Proposition 4.4.9. *The contact metric structure $(\phi, \xi_\lambda, \eta_\lambda, g_\lambda)$ on $\mathcal{H}$ is a Sasakian structure for all $\lambda > 0$.*

Proof: Since $(\phi, \xi_\lambda, \eta_\lambda, g_\lambda)$ comes with an underlying CR structure by construction, the only thing we need to verify is that this contact metric structure is K-contact, i.e. that $\mathcal{L}_{\xi_\lambda}\phi = 0$. That this holds is a simple consequence of the fact that the field ξ_λ is central in the algebra of left-invariant vector fields and the fact that ϕ has been defined to be left-invariant. These tell us that for any left-invariant vector field $X^{(v,s)}$ on $\mathcal{H}$

$$\begin{aligned}(\mathcal{L}_{\xi_\lambda}\phi)(X^{(v,s)}) &= \lambda^{-1}[\xi, \phi\,X^{(v,s)}] - \lambda^{-1}\phi\,[\xi, X^{(v,s)}] \\ &= \lambda^{-1}[\xi, X^{(iv,s)}] - \lambda^{-1}\phi(0) = 0\end{aligned} \tag{4.44}$$

and that therefore $\mathcal{L}_\xi\phi = 0$. □

Remark 4.4.10. *Although not every compatible contact metric structure is Sasakian, there are many Sasakian structures on the Heisenberg group* [68] *with this same underlying (left-invariant) CR structure. By also requiring left-invariance of the Sasakian structure we have reduced the freedom we have to choose the Sasakian structure on the Heisenberg group to a single parameter $\lambda > 0$.*

4.5 Kähler structures

Let $(\phi, \xi_\lambda, \eta_\lambda, g_\lambda)_{\lambda>0}$ be a family of Sasakian structures on a manifold M, with $\xi_\lambda = \lambda^{-1}\xi$, $\eta_\lambda = \lambda\,\eta$ and $g_\lambda = \eta_\lambda^2 + \frac{1}{2}\mathrm{d}\eta_\lambda(\phi\,\bullet, \bullet)$, and let (F, J) be their underlying CR structure. Since we will not make any other assumptions about these Sasakian structures or the manifold M, eveything in this section will in particular apply to the Sasakian structures on the Heisenberg groups defined in the previous section.

We can apply definition 4.2.6 and proposition 4.2.8 to this space to equip $\tilde{M} \subseteq \mathrm{T}^*M$ with a canonical symplectic structure $\tilde{\omega}$. The symplectised space could be described more explicitly as the space $M \times \mathbb{R}$ through the diffeomorphism

$$\psi_{\eta_\lambda} : M \times \mathbb{R}_{>0} \to \tilde{M}, \qquad (x, t) \mapsto t\,\eta_{\lambda,x} = \eta_{\lambda t,x}. \tag{4.45}$$

for $\lambda > 0$. On this manifold the canonical symplectic structure becomes $\mathrm{d}(t\,\eta_\lambda) = \mathrm{d}t \wedge \eta_\lambda + t\,\mathrm{d}\eta_\lambda$. Even though it is not directly manifest in this description, the symplectic structure $\tilde{\omega}$ is completely independent of the value of λ, as it is completely determined by the contact bundle $F = \ker\eta_\lambda = \ker\eta$.

Because (ϕ, ξ, η, g) is Sasakian, the almost contact structure (ϕ, ξ, η) is normal, which means that the complex structure $\tilde{J}$ on $\tilde{M} \cong M \times \mathbb{R}_{>0}$ defined by

$$\tilde{J}(X + \kappa\,\partial_r) = \phi\,X - \kappa\,\xi_\lambda + \eta_\lambda(X)\,\partial_r, \tag{4.21}$$

is integrable, where $r\colon \mathbb{R}_{>0} \to \mathbb{R}, t \mapsto r(t)$ is some coordinate on the positive real line $\mathbb{R}_{>0}$. Equation (4.21) completely determines the complex structure $\tilde{J}$ up to a reparametrisation for $\mathbb{R}_{>0}$, but such a reparametrisation changes the form of the canonical symplectic structure $\tilde{\omega} = \mathrm{d}(t\,\eta)$. For any choice for the coordinate $r(t)$ with $r'(t) > 0$ we have the following result.

Theorem 4.5.1. *The almost complex structure $\tilde{J}$ on $\tilde{M}$ defined by equation (4.21) for the coordinate $r(t)$, is compatible with the symplectic structure $\tilde{\omega}$. If $r'(t) > 0$ then $\tilde{J}$ and $\tilde{\omega}$ together define a Kähler structure with Kähler metric $\tilde{g} = \tilde{\omega}(\tilde{J}\,\bullet, \bullet)$.*

Proof: Since $\mathrm{d}t(X + \kappa\,\partial_r) = \kappa\,\mathrm{d}t(\partial_r)$ for all $X \in \mathrm{T}M$ and $\kappa \in \mathbb{R}$, we have for any two vector fields X and Y on M and for $\kappa, \chi \in \mathbb{R}$ that

$$\begin{aligned}
&\tilde{\omega}(\tilde{J}(X + \kappa\,\partial_r), \tilde{J}(Y + \chi\,\partial_r)) \\
&\qquad = \big(\mathrm{d}t \wedge \eta_\lambda + t\,\mathrm{d}\eta_\lambda\big)\big(\phi\,X - \kappa\,\xi_\lambda + \eta_\lambda(X)\,\partial_r, \phi\,Y - \chi\,\xi_\lambda + \eta_\lambda(Y)\,\partial_r\big) \\
&\qquad = \big(\mathrm{d}t(\eta_\lambda(X)\,\partial_r)\,\eta_\lambda(-\chi\,\xi_\lambda) - \mathrm{d}t(\eta_\lambda(Y)\,\partial_r)\,\eta_\lambda(-\kappa\,\xi_\lambda) + t\,\mathrm{d}\eta_\lambda(\phi\,X, \phi\,Y)\big) \\
&\qquad = -\mathrm{d}t(\chi\,\partial_r)\,\eta_\lambda(X) + \mathrm{d}t(\kappa\,\partial_r)\,\eta_\lambda(Y) + t\,\mathrm{d}\eta_\lambda(X, Y) \\
&\qquad = \big(\mathrm{d}t \wedge \eta_\lambda + t\,\mathrm{d}\eta_\lambda\big)\big(X + \kappa\,\partial_r, Y + \chi\,\partial_r\big) = \tilde{\omega}(X + \kappa\,\partial_r, Y + \chi\,\partial_r).
\end{aligned} \tag{4.46}$$

This tells us that the almost complex structure $\tilde{J}$ and the symplectic structure $\tilde{\omega}$ are compatible.

By definition of the Sasakian structure on M, the almost complex structure $\tilde{J}$ from equation (4.21) is integrable. Since it is moreover compatible with the symplectic structure $\tilde{\omega}$ we can conclude that $g = \tilde{\omega}(\tilde{J}\,\bullet, \bullet)$ is a pseudo-Kähler metric.

If we write $t = t(r)$, then the symmetric form $\tilde{g}$ is given by

$$\begin{aligned}
\tilde{g} &= \tilde{\omega} \circ (\tilde{J} \times \mathrm{id}) = \mathrm{d}(t(r)\,\eta_\lambda) \circ (\tilde{J} \times \mathrm{id}) \\
&= t'(r)(\tilde{J}^*\mathrm{d}r \otimes \eta_\lambda - \tilde{J}^*\eta_\lambda \otimes \mathrm{d}r) + t(r)\mathrm{d}\eta_\lambda \circ (\phi \times \mathrm{id}) \\
&= t'(r)\,(\eta_\lambda \otimes \eta_\lambda + \mathrm{d}r \otimes \mathrm{d}r) + t(r)\,\mathrm{d}\eta_\lambda \circ (\phi \times \mathrm{id}).
\end{aligned} \tag{4.47}$$

We see that the pseudo-Kähler metric $\tilde{g}$ is positive definite (and hence Kähler) if and only if the derivative $t'(r) = r'(t)^{-1}$ only takes positive values (and $t > 0$). □

If $r(t) = \frac{1}{2}\log(2\,t)$, and thus $t(r) = \frac{1}{2}\exp(2\,r)$, then the metric $\tilde{g} = \tilde{\omega}(\tilde{J}\,\bullet, \bullet)$ from equation (4.47) becomes the cone metric from definition 4.3.12,

$$\begin{aligned}
\tilde{g} &= e^{2r}(\eta_\lambda \otimes \eta_\lambda + \mathrm{d}r \otimes \mathrm{d}r) + \tfrac{1}{2}e^{2r}\mathrm{d}\eta_\lambda(\phi\,\bullet, \bullet) \\
&= \mathrm{d}\rho^2 + \rho^2\big(\eta_\lambda^2 + \tfrac{1}{2}\mathrm{d}\eta_\lambda(\phi\,\bullet, \bullet)\big) = \mathrm{d}\rho^2 + \rho^2 g_\lambda
\end{aligned} \tag{4.48}$$

with $\rho = e^r = \sqrt{2\,t}$. This is however not the Kähler structure we are looking for, so we will instead make a different choice for the relation between the coordinates t and r that relate the symplectic and the complex structure.

Corollary 4.5.2. *For all $\lambda > 0$ the canonical symplectic structure $\tilde{\omega} = \mathrm{d}(t\,\eta_\lambda)$ on $\tilde{M} \cong M \times \mathbb{R}_{>0}$, together with the metric*

$$\tilde{g}_\lambda = \tfrac{1}{4}\mathrm{d}(\log t)^2 + g_{2\lambda t} = \tfrac{1}{4}\mathrm{d}(\log t)^2 + \eta_{2\lambda t}^2 + \tfrac{1}{2}\mathrm{d}\eta_{2\lambda t}(\phi\,\bullet, \bullet) \tag{4.49}$$

describes a Kähler structure. The corresponding complex structure is given by

$$\tilde{J}|_{F \times \{0\}} = \phi_F, \qquad \tilde{J}(\partial_{\log t}) = \tfrac{1}{4}\,\xi_{\lambda t} \qquad \text{and} \qquad \tilde{J}(\xi_{\lambda t}) = -4\partial_{\log t} \tag{4.50}$$

and the Kähler potential for g_λ is $K = \frac{1}{2}\log t$.

Proof: We obtain this Kähler structure through theorem 4.5.1 by taking $r(t) = -(4\,t)^{-1} < 0$ and thus $t(r) = -(4\,r)^{-1}$. The complex structure is obtained by rewriting equation (4.21) in terms of the coordinates t and equation (4.47) gives us the metric

$$\tilde{g}_\lambda = \tfrac{1}{4r^2}(\eta_\lambda^2 + \mathrm{d}r^2) - \tfrac{1}{4r}\mathrm{d}\eta_\lambda \circ (\phi \times \mathrm{id}), \tag{4.51}$$

which we can rewrite in terms of the coordinate t, using that $\eta_\kappa = \kappa\,\eta$, to obtain equation (4.49)

Equation (4.50) tells us that $J^*\mathrm{d}\log t = -4\,\eta_{\lambda t} = -4\,t\,\eta_\lambda$, which enables us to verify that

$$i\partial\bar{\partial}K = \tfrac{i}{2}\mathrm{d}(1 + i\,J^*)\mathrm{d}(\tfrac{1}{2}\log t) = -\tfrac{1}{4}\mathrm{d}(J^*\mathrm{d}\log t) = \mathrm{d}(t\,\eta_\lambda) = \tilde{\omega} \tag{4.52}$$

where we have used that the integrability of $\tilde{J}$ tells us that $\partial\bar{\partial}K = \mathrm{d}\bar{\partial}K$ and that $\bar{\partial}K = (1 + i\,J^*)\mathrm{d}K$. □

Although the metric g_λ in equation (4.49) does explicitly depend on the parameter λ any scaling of λ can be compensated by rescaling the coordinate $t \in \mathbb{R}_{>0}$. This is possible because λ only appears as $\lambda\,t$ and $\mathrm{d}(\log t) = \mathrm{d}(\log(\lambda t))$. We had originally introduced the symplectised space $\tilde{M}$ in theorem 4.2.6 as a subspace of the cotangent bundle consisting exactly of the elements $\eta_{\lambda,x}$ for $\lambda > 0$ and $x \in M$ and the coordinate t was introduced through the diffeomorphism ψ_{η_λ} from equation (4.45).

Because λ and t again only appear in the combination $\lambda\,t$, the metric $\psi_{\eta_\lambda *}\tilde{g}_\lambda$ on $\tilde{M} \subseteq \mathrm{T}^*M$ does not depend on λ. We can make this a little more explicit through the following proposition.

Proposition 4.5.3. *The metric $\psi_{\eta_\lambda *}\tilde{g}_\lambda$ on $\tilde{M} \subseteq \mathrm{T}^*M$ does not depend on λ*

Proof: For any $(x,t) \in M \times \mathbb{R}_{>0}$ and $\kappa, \lambda > 0$ we have

$$\psi_{\eta_\kappa}^{-1}(\psi_{\eta_\lambda}(x,t)) = \psi_{\eta_\kappa}^{-1}(t\,\eta_{\lambda,x}) = \psi_{\eta_\kappa}^{-1}(t\,\lambda\,\kappa^{-1}\eta_{\kappa,x}) = (x, \lambda\,\kappa^{-1}t), \tag{4.53}$$

which we can use to show that

$$\begin{aligned}(\psi_{\eta_\kappa}^{-1} \circ \psi_{\eta_\lambda})_*\tilde{g}_\lambda &= (\psi_{\eta_\kappa}^{-1} \circ \psi_{\eta_\lambda})_*\big(\tfrac{1}{4}\mathrm{d}(\log t)^2 + \eta_{2\lambda t}^2 + \tfrac{1}{2}\mathrm{d}\eta_{2\lambda t}(\phi\,\bullet,\bullet)\big)\\ &= \tfrac{1}{4}\mathrm{d}(\log(\kappa\,\lambda^{-1}t))^2 + \eta_{2\lambda(\kappa\,\lambda^{-1}t)}^2 + \tfrac{1}{2}\mathrm{d}\eta_{2\lambda(\kappa\,\lambda^{-1}t)}(\phi\,\bullet,\bullet)\\ &= \tfrac{1}{4}\mathrm{d}(\log t)^2 + \eta_{2\kappa t}^2 + \tfrac{1}{2}\mathrm{d}\eta_{2\kappa t}(\phi\,\bullet,\bullet) = \tilde{g}_\kappa.\end{aligned} \tag{4.54}$$

We see that $\psi_{\eta_\lambda *}\tilde{g}_\lambda = \psi_{\eta_\kappa *}\tilde{g}_\kappa$ for all $\kappa, \lambda > 0$, so $\psi_{\eta_\lambda *}\tilde{g}_\lambda$ does not depend on λ. □

Note that instead of taking $r(t) = -(4\,t)^{-1}$ we could have used $r(t) = -(4\,\alpha\,t)^{-1}$ for any constant $\alpha > 0$. This would result in the metric $\frac{1}{4\alpha}(\mathrm{d}\log t)^2 + \frac{4}{\alpha}g_{2\alpha\lambda t}$, which is related to the metric $\tilde{g}_\lambda$ from equations (4.49) by a constant factor and a rescaling of the coordinate t.

5. Fibration of the hypermultiplet moduli space

In section 3.3 we had introduced the hypermultiplet moduli space, $\mathcal{M}_{\text{hm}}$, of type IIA string theory compactified on a family of Calabi-Yau manifolds $(\mathcal{Y}_t)_{t\in\mathcal{M}_{\mathbb{C}}}$. We had seen that the effective action (3.7c) could be used to equip this space with a canonical metric.

Because this metric is induced by the hypermultiplet part of the action of a supergravity theory, we know that it is a quaternion-Kähler structure [48], something which has later been explicitly verified [2]. We will describe a construction for the hypermultiplet moduli space that includes the quaternion-Kähler metric g_{hm} in terms of the complex structure moduli space and the Weil intermediate Jacobian of the internal Calabi-Yau manifold. By trying to understand this construction we hope to gain a better understanding of the quaternion-Kähler structure it produces.

5.1 Fibrations

The hypermultiplet sector of the effective type IIA compactified theory consists of the complex structure moduli, a harmonic 3-form $a^{(3)} \in H^3(\mathcal{Y}, \mathbb{R})$ and two scalars ϕ and σ that describe the dilaton and the Kalb-Ramond axion respectively. We can view the hypermultiplet moduli space $\mathcal{M}_{\text{hm}}$ as a fibre bundle over the complex moduli space $\mathcal{M}_{\mathbb{C}}$ with fibres $\mathcal{M}_t$ parametrised by $a^{(3)} \in H^3(\mathcal{Y}, \mathbb{R})$, and $\phi, \sigma \in \mathbb{R}$ for $t \in \mathcal{M}_{\mathbb{C}}$ [1]. With respect to such a fibration, the hypermultiplet metric in equation (3.13) becomes an orthogonal sum of the canonical metric on the base manifold $\mathcal{M}_{\mathbb{C}}$ and a metric on each of the fibres. As we have seen in section 2.2 the complex structure moduli space can (locally) be described through the periods $X^i = \int_{\gamma^i} \Omega$ of the holomorphic 3-form Ω with respect to some symplectic basis of cycles $\gamma^0, \ldots, \gamma^{h^{1,2}}, \eta_1, \ldots, \eta_{h^{1,2}} \in H_3(\mathcal{Y}, \mathbb{Z})_{\text{f}}$. These locally form a set of complex projective coordinates on $\mathcal{M}_{\mathbb{C}}$. The base space admits a projective special Kähler structure with a metric given by

$$g_{\text{base}} = \frac{\partial^2 K(X, \bar{X})}{\partial X^i \, \partial \bar{X}^j} \mathrm{d}X^i \mathrm{d}\bar{X}^j, \tag{5.1}$$

for the Kähler potential K expressed in terms of the periods $F_i = \int_{\eta_i} \Omega$ as

$$K = -\log\left(i \int_{\mathcal{Y}_t} \Omega \wedge \bar{\Omega}\right) = -\log\big(i\,\bar{X}^i F_i - i\,X^i \bar{F}_i\big)\,. \tag{5.2}$$

The periods F_i could alternatively be written as the derivatives $F_i = \frac{\partial F(X)}{\partial X^i}$ for the prepotential $F(X) = \frac{1}{2} F_i X^i$, which was holomorphic and homogenous of degree 2. We will often encounter the second derivative, or Hessian, $F_{ij} = \frac{\partial^2 F}{\partial X^i X^j} = \frac{\partial F_i}{\partial X^j}$.

The fields $A^i = \int_{\gamma^i} a^{(3)}$ and $B_i = \int_{\eta_i} a^{(3)}$ are the periods of $a^{(3)} \in H^3(\mathcal{Y}, \mathbb{R})$ with respect to the same basis for $H_3(\mathcal{Y}, \mathbb{Z})_{\text{f}}$. In terms of these periods and the coordinates $\sigma \in \mathbb{R}$ and

$\psi = e^{\phi} > 0$ the (quaternion-Kähler) metric on the total hypermultiplet moduli space is given by $g_{\text{hm}} = g_{\text{base}} + g_t$ over the point $t \in \mathcal{M}_{\mathbb{C}}$ in the base space, where g_{base} is the metric from equation (5.1). The metric on the fibre $\mathcal{M}_t$ is

$$\begin{aligned} g_t =& \frac{1}{(2\,\psi)^2}\mathrm{d}\psi^2 + \frac{1}{(2\,\psi)^2}(\mathrm{d}\sigma - A^i\overleftrightarrow{\mathrm{d}}B_i)^2 \\ &- \frac{1}{2\,\psi}\,\mathrm{Im}(\mathcal{N})^{ij}(\mathrm{d}B_i - \mathcal{N}_{ik}\mathrm{d}A^k)(\mathrm{d}B_i - \bar{\mathcal{N}}_{ik}\mathrm{d}A^k), \end{aligned} \tag{5.3}$$

with $\mathrm{Im}(\mathcal{N})^{ij}$ the inverse of the imaginary part of the $(1+h^{1,2}) \times (1+h^{1,2})$-matrix

$$\mathcal{N}_{ij} = \bar{F}_{ij} + i\,\frac{N_{ik}X^kX^\ell N_{\ell j}}{XNX}, \tag{5.4}$$

where $N_{ij} = 2\,\mathrm{Im}(F_{ij})$. The matrix $\mathcal{N}_{ij}$ will be referred to as the ***period matrix*** for reasons that will become clear in section 5.2.

5.1.1 Further fibration

The fibres $\mathcal{M}_t$ themselves can also be interpreted as a fibre bundle, but before we can do this we should recall the Peccei-Quinn symmetries

$$\begin{aligned} &A^i \mapsto A^i + a^i, \qquad && \sigma \mapsto \sigma + s + a^iB_i - b_iA^i, \\ &B_i \mapsto B_i + b_i, && \psi \mapsto \psi. \end{aligned} \tag{3.14}$$

Since these symmetries do not affect the base space they are symmetries of the metric g_t on the fibres $\mathcal{M}_t$ for $t \in \mathcal{M}_{\mathbb{C}}$. For A^i, B_i and s integers these symmetries are expected to be preserved in the full non-perturbative theory, as described in section 3.3. These unbroken symmetries require us to make some identifications on the fibres $\mathcal{M}_t$, namely

$$(A^i, B_i, \sigma, \psi) \sim (A^i + a^i, B_i, \sigma + a^iB_i, \psi), \tag{5.5a}$$
$$(A^i, B_i, \sigma, \psi) \sim (A^i, B_i + b_i, \sigma - b_iA^i, \psi) \tag{5.5b}$$
$$(A^i, B_i, \sigma, \psi) \sim (A^i, B_i, \sigma + s, \psi). \tag{5.5c}$$

for $a^0, \ldots, a^{h^{1,2}}, b_0, \ldots, b_{h^{1,2}}, s \in \mathbb{Z}$. Equation (5.5) can be concisely summarised by

$$(a^{(3)}, \sigma, \psi) \sim (a^{(3)} + \alpha, \sigma - Q(\alpha, a^{(3)}), \psi), \tag{5.6}$$

for $a^{(3)} = A^i\alpha_i - B_i\beta^i \in H^3(\mathcal{Y}, \mathbb{R})$ and $\alpha = a^i\alpha_i - b_i\beta^i \in H^3(\mathcal{Y}, \mathbb{Z})_{\mathrm{f}}$.

The dilaton $\psi = e^{\phi}$ is unaffected by these symmetries, so the fibre $\mathcal{M}_t$ can be written as $\mathcal{M}_t = \mathcal{M}_t' \times \mathbb{R}_{>0}$, with $\psi > 0$ as the standard coordinate on $\mathbb{R}_{>0}$ and $\mathcal{M}_t' \cong \big(H^3(\mathcal{Y}, \mathbb{R}) \times \mathbb{R}\big)/\sim$ the space parametrised by just the $a^{(3)}$ and σ and . The identification (5.5c) tells us that $\sigma \in \mathbb{R}/\mathbb{Z}$ lives on a circle, rather than on the real line. Although we cannot do the same for the Ramond-Ramond 3-form $a^{(3)} = A^i\alpha_i - B_i\beta^i$, equations (5.5a) and (5.5b) do tell us that the projection map

$$\pi_{S^1} : \mathcal{M}_t' \cong \big(H^3(\mathcal{Y}, \mathbb{R}) \times \mathbb{R}\big)/\sim\; \to H^3(\mathcal{Y}, \mathbb{R})/H^3(\mathcal{Y}, \mathbb{Z})_{\mathrm{f}} \qquad (a^{(3)}, \sigma) \mapsto a^{(3)} \tag{5.7}$$

is well-defined. It is easily verified that this defines a ***circle bundle*** over the (real) torus $H^3(\mathcal{Y}, \mathbb{R})/H^3(\mathcal{Y}, \mathbb{Z})_{\mathrm{f}}$.

Proposition 5.1.1. *The space $\mathcal{M}_t' \cong \big(H^3(\mathcal{Y}, \mathbb{R}) \times \mathbb{R}\big)/\sim$ is the total space of a principal* $\mathrm{U}(1)$*-bundle (circle bundle) over the torus $H^3(\mathcal{Y}, \mathbb{R})/H^3(\mathcal{Y}, \mathbb{Z})$ with the projection map π_{S^1} from equation* (5.7).

Proof: The fibre of this bundle at $a^{(3)} \in H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$ is parametrised by $\sigma \in \mathbb{R}/\mathbb{Z}$ or, alternatatively, by $z = e^{2\pi i\sigma} \in S^1 \simeq \mathrm{U}(1) \subseteq \mathbb{C}$. The only thing that could spoil our attempt to interpret $\mathcal{M}'_t$ as a principal $\mathrm{U}(1)$ bundle is the possibility that this group action on its fibres depends on the element $a^{(3)} \in H^3(\mathcal{Y},\mathbb{R})$ chosen to represent an element of the torus $H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$. The point $(a^{(3)},\sigma)$ is identified with $(a^{(3)}+\alpha,\sigma-Q(\alpha,a^{(3)}))$ for $a^{(3)} \in H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$, so $(a^{(3)},z) \sim (a^{(3)}+\alpha, e^{2\pi i Q(\alpha,a^{(3)})}z)$. Because this identification corresponds to multiplication by a constant element of $\mathrm{U}(1) \subseteq \mathbb{C}$ on the fibre we can conclude that $\mathcal{M}'_t$ is indeed a principal $\mathrm{U}(1)$ bundle. $\square$

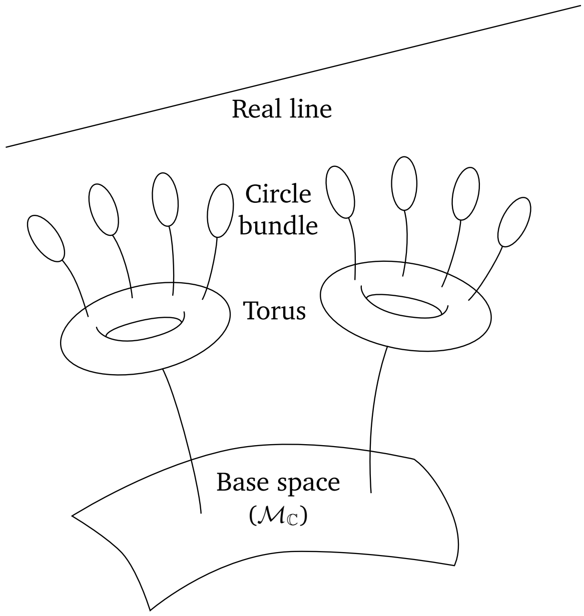

Figure 5.1: A diagrammatic representation of the structure of the hypermultiplet moduli space.

All in all, we have seen that the hypermultiplet moduli space is a fibre bundle over the complex moduli space $\mathcal{M}_{\mathbb{C}}$ with fibres $\mathcal{M}_t = \mathcal{M}'_t \times \mathbb{R}$ that are the product of the real line parametrised by the dilaton, and a circle bundle over the torus $H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$.

This structure is concisely summarised by figure 5.1, where we note that it is not at all obvious where the real line should be put. It can be argued that $\mathcal{M}_t = \mathcal{M}'_t \times \mathbb{R}$ should be viewed as a bundle over $\mathbb{R}$ [1], which would require us to place the line between the base space and the torus, while it is on the other hand tempting to combine the dilaton and the axion to form a $\mathbb{C}^*$-bundle on the torus. We will see that it is in fact possible to combine the dilaton and the axion to obtain *part* of a holomorphic line bundle.

5.1.2 The Heisenberg group

We may recall that the ***intersection form*** Q, which we had originally introduced in definition 2.1.5 as the bilinear form

$$Q\colon H^3(\mathcal{Y},\mathbb{R}) \times H^3(\mathcal{Y},\mathbb{R}), \qquad (\alpha,\beta) \mapsto \int_{\mathcal{Y}} \alpha \wedge \beta, \tag{5.8}$$

defines a symplectic structure on the third cohomology group $H^3(\mathcal{Y},\mathbb{R})$ of the Calabi-Yau manifold $\mathcal{Y}$. With respect to this form the basis $(\alpha_i,\beta^i)_{i=0}^{h^{1,2}}$ we have used before to write

$a^{(3)} = A^i\alpha_i - B_i\beta^i$ is a symplectic basis. We can use the intersection form to define a Heisenberg group (cf. definition 4.4.1).

***Definition 5.1.2* (Heisenberg group).** *The* **Heisenberg group** $\mathcal{H}_{\mathcal{Y}} = \mathcal{H}(H^3(\mathcal{Y},\mathbb{R}),Q)$ *is the space* $H^3(\mathcal{Y},\mathbb{R}) \times \mathbb{R}$*, with a multiplicative structure defined by*

$$(\alpha,s)\cdot(\beta,t) = (\alpha+\beta, s+t+Q(\alpha,\beta)). \tag{5.9}$$

for all $\alpha,\beta \in H^3(\mathcal{Y},\mathbb{R})$ *and* $s,t \in \mathbb{R}$*.*

Since the Heisenberg group $\mathcal{H}_{\mathcal{Y}}$ is essentially the space $H^3(\mathcal{Y},\mathbb{R}) \times \mathbb{R}$ it defines a group action on this space through group multiplication. The reason why we are interested in the group structure on $\mathcal{H}_{\mathcal{Y}}$ is that this group action turns out to describe the Peccei-Quinn isometries by defining a group action on the fibres $\mathcal{M}_t$.

***Proposition 5.1.3* (Group action).** *The map* $\mathcal{H}_{\mathcal{Y}} \times (H^3(\mathcal{Y},\mathbb{R})\times\mathbb{R}) \to (H^3(\mathcal{Y},\mathbb{R})\times\mathbb{R})$*, defined by*

$$\big((\alpha,s),(a^{(3)},\sigma)\big) \mapsto (\alpha,s)\cdot(a^{(3)},\sigma) = (a^{(3)}+\alpha, \sigma+s-Q(\alpha,a^{(3)})) \tag{5.10}$$

defines a group action of the Heisenberg group $\mathcal{H}(H^3(\mathcal{Y},\mathbb{R}),Q)$ *on* $\mathcal{M}_t$*.*

For $\alpha = a^i\alpha_i - b_i\beta^i \in H^3(\mathcal{Y},\mathbb{R})$ *and* $s \in \mathbb{R}$*, this group action corresponds to one of the Peccei-Quinn isometries from equation* (3.14)*.*

Proof: Equation (5.10) obviously describes a group action since it is defined through group multiplication on $\mathcal{H}_{\mathcal{Y}} = (H^3(\mathcal{Y},\mathbb{R})\times\mathbb{R},\cdot)$. If we write $\alpha = a^i\alpha_i - b_i\beta^i$ and $a^{(3)} = A^i\alpha_i - B_i\beta^i$, then $a^{(3)}+\alpha = (A^i+a^i)\alpha_i - (B_i+b_i)\beta^i$ and $-Q(\alpha,a^{(3)}) = a^iB_i - b_iA^i$, so in terms of the coordinates A^i, B_i, σ, ψ on $\mathcal{M}_t$, equation (5.10) reads

$$(a^i,b_i,s)\cdot(A^i,B_i,\sigma,\psi) = (A^i+a^i, B_i+b_i, \sigma+s+a^iB_i-b_iA^i), \tag{5.11}$$

which coincides with the transformation (3.14). □

Through the discrete cohomology group $H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}} < H^3(\mathcal{Y},\mathbb{R})$ we can define a discrete subgroup of the Heisenberg group by virtue of the fact $Q\big(H^3(\mathcal{Y},\mathbb{Z}),H^3(\mathcal{Y},\mathbb{Z})\big) \subseteq \mathbb{Z}$.

***Definition 5.1.4* (Discrete Heisenberg group).** *The* **discrete Heisenberg group** *for the Calabi-Yau manifold* $\mathcal{Y}$ *is the group* $\mathcal{H}_{\mathcal{Y},\mathbb{Z}} = (H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}} \times \mathbb{Z},\cdot)$*, where group multiplication is defined by equation* (5.9)*.*

Since $\mathcal{H}_{\mathcal{Y},\mathbb{Z}}$ is a subgroup of $\mathcal{H}_{\mathcal{Y}}$ it also inherits the group action from proposition 5.1.3. We can use this to identify the total space $\mathcal{M}'_t$ of the circle bundle $\pi_{S^1}: \mathcal{M}'_t \to H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$ with the coset space $\mathcal{H}_{\mathcal{Y},\mathbb{Z}}\backslash\mathcal{H}_{\mathcal{Y}}$.

Theorem 5.1.5. *The space* $\mathcal{M}'_t$ *is diffeomorphic to the coset space* $\mathcal{H}_{\mathcal{Y},\mathbb{Z}}\backslash\mathcal{H}_{\mathcal{Y}}$*.*

Proof: A general element of $\mathcal{H}_{\mathcal{Y},\mathbb{Z}}$ can be written as (a,s) for $a = a^i\alpha_i - b_i\beta^i$ with $a^i, b_i \in \mathbb{Z}$ and for $s \in \mathbb{Z}$. When we compare the identifications (5.5) with the group action in equation (5.11) we see that two points $(a^{(3)},\sigma)$ and $(b^{(3)},\tau)$ are identified exactly when $(b^{(3)},\tau) = (a,s)\cdot(a^{(3)},\sigma)$ for some $a,s \in \mathcal{H}_{\mathcal{Y},\mathbb{Z}}$. Because the group action of $\mathcal{H}_{\mathcal{Y},\mathbb{Z}}$ on $H^3(\mathcal{Y},\mathbb{R})\times\mathbb{R} \cong \mathcal{H}_{\mathcal{Y}}$ corresponds to multiplication within $\mathcal{H}_{\mathcal{Y}}$, we conclude that

$$\mathcal{M}'_t = \big(H^3(\mathcal{Y},\mathbb{R})\times\mathbb{R}\big)/\mathcal{H}_{\mathcal{Y},\mathbb{Z}} \cong \mathcal{H}_{\mathcal{Y},\mathbb{Z}}\backslash\mathcal{H}_{\mathcal{Y}}. \tag{5.12}$$

□

Remark 5.1.6. *It is possible to extend the Heisenberg group to include the scaling symmetry from equation* (3.15)*. The resulting group is the semidirect product* $\mathcal{H}_{\mathcal{Y}} \rtimes \mathbb{R}$ *of the original*

*Heisenberg group with the real line and the appropriate name "**dilated Heisenberg group**" has been suggested for it [1]. This Lie group can be identified with the space $\big(H^3(\mathcal{Y},\mathbb{R})\times\mathbb{R}\big)\times\mathbb{R}$ through its transitive group action, which tells us that*

$$\mathcal{M}_t = \mathcal{M}_t' \times \mathbb{R} \cong \mathcal{H}_{\mathcal{Y},\mathbb{Z}}\backslash\big(\mathcal{H}_{\mathcal{Y}} \rtimes \mathbb{R}\big) \tag{5.13}$$

since $\mathcal{M}_t$ is obtained from $\big(H^3(\mathcal{Y},\mathbb{R})\times\mathbb{R}\big)\times\mathbb{R}$ by identifying points related through the discrete Peccei-Quinn symmetries.

Although the advantage of the dilated Heisenberg group is that it describes the entire fibre $\mathcal{M}$ instead of just part of it, the fact that it includes the scaling symmetry means that it will probably be less useful when perturbative corrections are added (cf. section 3.3.1). Instead of looking at the extended group we will therefore focus on the original Heisenberg group $\mathcal{H}_{\mathcal{Y}}$, which we know from section 4.4.1 comes with a contact structure, and use the symplectisation process to extend this space to include the dilaton.

5.2 The fibre metric

For the moment we will fix the complex structure on the Calabi-Yau manifold $\mathcal{Y}=\mathcal{Y}_t$ by fixing a point $t\in\mathcal{M}_{\mathbb{C}}$. We have argued that this fibre consists of the product of the real line parametrised by the dilaton and a circle bundle on a $2(1+h^{1,2})$-dimensional torus. The metric g_t on this space from equation (5.3) consists of multiple parts, we will start by first discussing the contribution to this metric from the torus and then showing how the rest of the metric corresponds to expressions from chapter 4.

5.2.1 The Weil intermediate Jacobian

We have seen the intermediate Jacobian $\mathcal{J}_2(\mathcal{Y}) := H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$ appear in the hypermultiplet moduli space as the base space for the circle bundle $\pi_{S^1}:\mathcal{M}_t'\to\mathcal{J}_2$. For fixed $\psi>0$, the contribution to the metric g_t from equation (5.3) that comes from this torus is the bilinear form

$$\frac{1}{2\psi}g_{\mathrm{torus}} = -\frac{1}{2\,\psi}\,\mathrm{Im}(\mathcal{N})^{ij}(\mathrm{d}B_i-\mathcal{N}_{ik}\mathrm{d}A^k)(\mathrm{d}B_i-\bar{\mathcal{N}}_{ik}\mathrm{d}A^k), \tag{5.14}$$

which restricts to a metric on $\mathcal{J}_2(\mathcal{Y})$.

Up to the factor $2\,\psi$ in front, the metric from (5.14) has the form of a canonical metric on a non-degenerate complex torus with normalised period matrix $\bar{\mathcal{N}}$ (cf. equation (1.56)). It turns out that the torus this period matrix belongs to is the ***Weil intermediate Jacobian*** of the Calabi-Yau manifold $\mathcal{Y}_t$ [1].

Proposition 5.2.1. *The metric g_{torus} from equation (5.14) is the canonical metric g^{W} for the Weil intermediate Jacobian of $\mathcal{Y}_t$.*

Proof: The normalised period matrix for the Weil intermediate Jacobian with respect to the symplectic basis $(\alpha_i,\beta^i)_i$ has been described in proposition 2.3.9. It was given by the $n\times 2n$-matrix $\Omega=(\mathcal{Z}^{\mathrm{W}},\mathbf{1})$ with

$$\mathcal{Z}^{\mathrm{W}}_{ij} = F_{ij} - i\frac{N_{ik}\bar{X}^k\bar{X}^\ell N_{\ell j}}{\bar{X}N\bar{X}} = \bar{\mathcal{N}}_{ij}. \tag{5.15}$$

We can write $a^{(3)} = x^i\alpha_i + y_i\beta^i$, with $x^i=A^i$ and $y_i=-B_i$ and combine these in a new (complex) coordinate $z_i = y_i + \mathcal{Z}_{ij}x^j = -(B_i-\bar{\mathcal{N}}_{ij}A^j)$. The metric in equation (5.14) then becomes

$$\mathcal{Y}^{ij}_{\mathrm{W}}(B_i-\mathcal{Z}_{ik}A^k)(B_j-\bar{\mathcal{Z}}_{j\ell}A^\ell) = \mathcal{Y}^{ij}_{\mathrm{W}}\mathrm{d}z_i\mathrm{d}\bar{z}_j, \tag{5.16}$$

where $\mathcal{Y}_{\mathrm{W}}^{ij} = (\mathcal{Y}^{\mathrm{W}})^{-1}$ is the inverse of $\mathrm{Im}(\mathcal{Z}^{\mathrm{W}}) = -\mathrm{Im}(\mathcal{N})$. This is exactly the canonical metric g^{W} for the Weil intermediate Jacobian from corollary 2.3.10. □

This tells us that the tori that are fibred over each point $t \in \mathcal{M}_{\mathbb{C}}$ should be interpreted as the Weil intermediate Jacobians of the Calabi-Yau manifold $\mathcal{Y}_t$. Note that the complex structure on the Weil intermediate Jacobian is given by $J^{\mathrm{W}} = -*$ and that the intersection form Q and the canonical metric $g^{\mathrm{W}} = g_{\text{torus}}$ are related through $g^{\mathrm{W}} = Q(J^{\mathrm{W}}\bullet, \bullet)$.

5.2.2 Contact metric structure

On top of these intermediate Jacobians $\mathcal{J}_2^{\mathrm{W}}(\mathcal{Y}) = H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$, we had a circle bundle parametrised by the axion field $\sigma \in \mathbb{R}/\mathbb{Z}$. Because this circle is the result of a partial breaking of the Peccei-Quinn symmetries to a discrete symmetry group corresponding to the discrete Heisenberg group we could view the total as the coset space

$$\mathcal{M}'_t \cong \mathcal{H}_{\mathcal{Y},\mathbb{Z}} \backslash \mathcal{H}_{\mathcal{Y}}. \tag{5.17}$$

By fixing the coordinate $\psi \in \mathbb{R}_{>0}$ we can view $\mathcal{M}'_t$ as a submanifold of the fibre $\mathcal{M}_t = \mathcal{M}'_t \times \mathbb{R}_{>0}$ and consider the restriction of the metric g_t from equation (5.3) to this submanifold. The resulting metric is the expression

$$g'_{t,\psi} = \frac{1}{2\psi} \underbrace{\mathrm{Im}(\bar{\mathcal{N}})^{ij}(\mathrm{d}B_i - \mathcal{N}_{ik}\mathrm{d}A^k)(\mathrm{d}B_i - \bar{\mathcal{N}}_{ik}\mathrm{d}A^k)}_{\pi_{S^1}^* g^{\mathrm{W}}} + \frac{1}{(2\psi)^2} \underbrace{(\mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i)^2}_{\eta^2}, \tag{5.18}$$

which is a combination of the pullback of the canonical metric g^{W} on the Weil intermediate Jacobian along $\pi_{S^1} : \mathcal{M}'_t \to \mathcal{J}_2$ and η^2 with $\eta = \mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i$. The dilaton, parametrised by $\psi > 0$, introduces a grading on the tangent spaces of $\mathcal{M}'_t$.

Although it is the compact quotient $\mathcal{M}'_t \cong \mathcal{H}_{\mathcal{Y},\mathbb{Z}} \backslash \mathcal{H}_{\mathcal{Y}}$ we are most interested in, it is more practical to work with the Heisenberg group $\mathcal{H}_{\mathcal{Y}} \cong \mathbb{R}^{2h^{1,2}+3}$ instead. We will therefore consider the metric $g'_{t,\psi}$ from equation (5.18), which is left-invariant because we had introduced this group to describe its isometries, as a metric on $\mathcal{H}_{\mathcal{Y}}$. In general, any left-invariant object on $\mathcal{H}_{\mathcal{Y}}$ can be transferred to the quotient $\mathcal{H}_{\mathcal{Y},\mathbb{Z}} \backslash \mathcal{H}_{\mathcal{Y}}$. In the form $\mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i = \mathrm{d}(\sigma + A^i B_i)$ we recognise the standard contact form on $\mathbb{R}^{2h^{1,2}+3}$ from equation (4.5).

Lemma 5.2.2. *The 1-form $\eta = \mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i$ on $H^3(\mathcal{Y},\mathbb{R}) \times \mathbb{R}$ coincides with the invariant contact form η from section 4.4.1, which is characterised by*

$$\eta(\xi) = 1 \qquad \text{and} \qquad \eta|_F = 0 \tag{5.19}$$

where $\xi = \partial_\sigma$ is the (Reeb) vector field pointing along the circle bundle directions and $F < \mathrm{T}\mathcal{H}_{\mathcal{Y}}$ is the left-invariant hyperplane field for which $F_{(0,0)} = H^3(\mathcal{Y},\mathbb{R}) \times \{0\} < \mathrm{T}_{(0,0)}\mathcal{H}_{\mathcal{Y}}$.

Proof: Left-invariance of the form $\mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i$ on $\mathcal{H}_{\mathcal{Y}}$ is a direct consequence of its invariance under the Peccei-Quinn isometries, but it can also easily be verified explicitly by performing a translation by $a = a^i \alpha_i - b_i \beta^i$,

$$\begin{aligned}
\ell_{(a,s)_*}(\mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i) &= \mathrm{d}(\sigma - Q(a, a^{(3)})) - (A^i + a^i)\overleftrightarrow{\mathrm{d}}(B_i + b_i) \\
&= \mathrm{d}(\sigma + a^i B_i - b_i A^i) - (A^i + a^i)\overleftrightarrow{\mathrm{d}}(B_i + b_i) \\
&= (\mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i)
\end{aligned} \tag{5.20}$$

and using that the basis $(\alpha_i, \beta^i)_i$ in which $a^{(3)} = A^i \alpha_i - B_i \beta^i$ has been expressed is symplectic with respect to Q.

Since $\eta = \mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i$ equals $\mathrm{d}\sigma$ at $(0,0) \in \mathcal{H}_{\mathcal{Y}}$ it vanishes on $H^3(\mathcal{Y},\mathbb{R}) \times \{0\} < \mathrm{T}\mathcal{H}_{\mathcal{Y},(0,0)}$ and thus on the whole of $F < \mathrm{T}\mathcal{H}_{\mathcal{Y}}$ and η moreover obviously satisfies

$$\eta(\xi) = (\mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i)(\partial_\sigma) = 1. \tag{5.21}$$

□

If we let $X^{(v,s)}$ denote the left-invariant vector field on $\mathcal{H}_{\mathcal{Y}}$ for which $X^{(v,s)}(0,0) = (v,s) \in \mathrm{T}_{(0,0)}\mathcal{H}_{\mathcal{Y}} \simeq H^3(\mathcal{Y},\mathbb{R}) \times \mathbb{R}$, then the contact form η can alternatively be defined by setting

$$\eta(X^{(v,s)}) = s \tag{5.22}$$

for all $v \in H^3(\mathcal{Y},\mathbb{R})$ and $s \in \mathbb{R}$, which is how we defined it in section 4.4.1. The contact bundle $F = \ker\eta < \mathrm{T}\mathcal{H}_{\mathcal{Y}}$ is therefore generated by the invariant vector fields $X^{(v,0)}$ for $v \in H^3(\mathcal{Y},\mathbb{R})$.

We had argued that the torus $\mathcal{J}_2 \cong H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$ used to construct the circle bundle $\pi_{S^1} \colon \mathcal{M}'_t \to \mathcal{J}_2$ should be interpreted as a Weil intermediate Jacobian, which means that the cohomology group $H^3(\mathcal{Y},\mathbb{R})$ comes with the complex structure $J^{\mathrm{W}} = -*$. Because the intersection form Q defines a polarisation on the Weil intermediate Jacobian (see proposition 2.3.1) it is the imaginary part of a Hermitian structure $h^{\mathrm{W}} = g^{\mathrm{W}} + i\,Q$ and we can use proposition 4.4.8 to define a Levi non-degenerate CR structure on the Heisenberg group $\mathcal{H}_{\mathcal{Y}} \cong H^3(\mathcal{Y},\mathbb{R}) \times \mathbb{R}$.

Proposition 5.2.3. *Let $F = \ker\eta$ be the contact bundle defined by the contact form η from lemma 5.2.2 and let J be the left-invariant endomorphism $J \colon F \to F$ on F defined by*

$$J\big(X^{(v,0)}\big) = X^{(J^{\mathrm{W}}v,0)}, \tag{5.23}$$

for $v \in H^3(\mathcal{Y},\mathbb{R}) \simeq F_{(0,0)}$ and $X^{(v,0)}$ the left-invariant vector field for which $(X^{(v,0)})(0,0) = (v,0)$. This endomorphism defines a strictly pseudoconvex CR structure (F,J) on $\mathcal{H}_{\mathcal{Y}}$.

Proof: It follows from a simple application of proposition 4.4.8 that (F,J) is a CR structure. We already know that the metric $g^{\mathrm{W}} = Q(J^{\mathrm{W}}\bullet,\bullet)$ is positive definite, which tells us that $h^{\mathrm{W}} = g^{\mathrm{W}} + i\,Q$ is a Hermitian inner product, so we can conclude that this CR structure is strictly pseudoconvex. □

Note that the group structure on $\mathcal{H}_{\mathcal{Y}}$ was defined in terms of only the intersection form Q, which only depends on the topology of $\mathcal{Y}_t$. This means that the contact structure on $\mathcal{H}_{\mathcal{Y}}$ is independent of the complex structure on $\mathcal{Y}_t$ and is thus the same for every point $t \in \mathcal{M}_{\mathbb{C}}$. The same is not true for the CR structure (F,J) as it uses the complex structure J^{W} and thus depends on the Hodge structure of $H^3(\mathcal{Y},\mathbb{R})$.

Since the CR structure defined in proposition 5.2.3 is strictly pseudoconvex it is the underlying CR structure of a contact metric structure by proposition 4.3.6. By following the steps described in section 4.4.2 we can obtain a 1-dimensional family of left-invariant contact metric structures that turned out to be Sasakian.

Corollary 5.2.4. *Let $\phi \colon \mathrm{T}\mathcal{H}_{\mathcal{Y}} \to \mathrm{T}\mathcal{H}_{\mathcal{Y}}$ be the endomorphism on the tangent bundle of the Heisenberg group defined by $\phi(X^{(v,s)}) = J(X^{(v,0)}) = X^{(J^{\mathrm{W}}v,0)}$, let $\eta_\lambda = \lambda\,\eta = \lambda(\mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i)$, let $\xi_\lambda = \lambda^{-1}\partial_\sigma$ and let g_λ denote the metric*

$$g_\lambda = \tfrac{1}{2}\mathrm{d}\eta_\lambda(\phi\bullet,\bullet) + \eta_\lambda \otimes \eta_\lambda = \tfrac{1}{2}\lambda\,\mathrm{d}\eta(\phi\bullet,\bullet) + \lambda^2\eta\otimes\eta \tag{5.24}$$

for $\lambda > 0$. Together these make up a left-invariant Sasakian structure $(\phi,\xi_\lambda,\eta_\lambda,g_\lambda)$ with (F,J) as its underlying CR structure and every such Sasakian structure is of this form.

Proof: This follows directly from proposition 5.2.3 and proposition 4.4.9. □

The relevance of these Sasakian structures becomes clear if we compare it with the metrics $g'_{t,\psi}$ we already had on the Heisenberg group $\mathcal{H}_\mathcal{Y}$. Not only do we see the expression $\eta^2 = (\mathrm{d}\sigma - A^i \overleftrightarrow{\mathrm{d}} B_i)^2$ appear, but in addition to this we have a contribution of the form $\frac{1}{2}\mathrm{d}\eta(\phi\bullet,\bullet)$, which corresponds to the pullback of the metric on the Weil intermediate Jacobian since $\frac{1}{2}\mathrm{d}\eta = \mathrm{d}(A^i \overleftrightarrow{\mathrm{d}} B_i) = \mathrm{d}A^i \wedge \mathrm{d}B_i$, which restricts to the intersection form Q on $H^3(\mathcal{Y},\mathbb{R}) < \mathrm{T}\mathcal{H}_\mathcal{Y}$, and ϕ restricts to the complex structure J^{W}.

Proposition 5.2.5. *The metric g_λ from equation (5.24) coincides with the metric $g'_{t,\psi}$ from equation (5.18) for $\lambda = \frac{1}{2\psi}$, i.e.*

$$g'_{t,\psi} = g_{(2\psi)^{-1}} = \frac{1}{4\psi}\mathrm{d}\eta(\phi\bullet,\bullet) + \frac{1}{(2\psi)^2}\eta\otimes\eta. \tag{5.25}$$

Proof: We know that both metrics, $g'_{t,\psi}$ and g_λ, are left-invariant with respect to the group structure on $\mathcal{H}_\mathcal{Y}$, which means that we only need to compare the two at the unit element $(0,0)\in\mathcal{H}_\mathcal{Y}$. For the invariant vector fields $X^{(v,s)}$ and $X^{(w,t)}$ we have

$$\begin{aligned} g_\lambda(X^{(v,s)},X^{(w,t)}) &= \tfrac{1}{2}\lambda\,\mathrm{d}\eta(\phi\,X^{(v,s)},X^{(w,t)}) + \lambda^2\eta(X^{(v,s)})\eta(X^{(w,t)}) \\ &= \lambda\,Q(J^{\mathrm{W}}v,w) + \lambda^2 s\,t = \lambda\,g^{\mathrm{W}}(v,w) + \lambda^2 s\,t \end{aligned} \tag{5.26}$$

since $\phi\,X^{(v,s)} = X^{(J^{\mathrm{W}}v,0)}$ by definition of ϕ and $\mathrm{d}\eta(X^{(v,s)},X^{(w,t)}) = 2\,Q(v,w)$ by lemma 4.4.6. For these vector fields we moreover have

$$\begin{aligned} g'_{t,\psi}(X^{(v,s)},X^{(w,t)}) &= \tfrac{1}{2\psi}g^{\mathrm{W}}(\pi_{S^1}(v,s),\pi_{S^1}(w,t)) + \tfrac{1}{(2\psi)^2}\eta(X^{(v,s)})\eta(X^{(w,t)}) \\ &= \tfrac{1}{2\psi}g^{\mathrm{W}}(v,w) + \tfrac{1}{(2\psi)^2}s\,t = g_{(2\psi)^{-1}}(X^{(v,s)},X^{(w,t)}), \end{aligned} \tag{5.27}$$

which completes the proof. □

Because all of these structures are left-invariant they can be transferred to the quotient $\mathcal{M}'_t \cong \mathcal{H}_{\mathcal{Y},\mathbb{Z}}\backslash\mathcal{H}_\mathcal{Y}$. Now that we have found a description for the metric on the submanifolds $\mathcal{M}'_t\times\{\psi\}$ of $\mathcal{M}_t = \mathcal{M}'_t\times\mathbb{R}_{>0}$ as the left-invariant Sasakian metric $g_{(2\psi)^{-1}}$ on the Heisenberg group, we are just one step away from a description of the entire metric g_t from equation (5.3).

5.2.3 Kähler structure

We can use what we already know about the invariant Sasakian structures on $\mathcal{M}'_t$ to say something about the metric g_t on the entire fibre $\mathcal{M}_t = \mathcal{M}'_t\times\mathbb{R}_{>0}$. By using what we have learnt in section 5.2.2 we can express the metric on $\mathcal{M}_t$ from equation (5.3) as

$$g_t = \tfrac{1}{(2\psi)^2}\mathrm{d}\psi^2 + \tfrac{1}{(2\psi)^2}\eta\otimes\eta + \tfrac{1}{2\psi}\pi^*_{S^1}g^{\mathrm{W}} = \tfrac{1}{(2\psi)^2}\mathrm{d}\psi^2 + g'_{(2\psi)^{-1}}, \tag{5.28}$$

where $g_{t,\psi}$ is the Sasakian metric on $\mathcal{M}'_t$ from equation (5.25).

***Proposition 5.2.6* (Symplectisation).** *Let η be the defining form from lemma 5.2.2 for the contact structure $F < \mathrm{T}\mathcal{M}'_t$ on $\mathcal{M}'_t$ and let ψ be the standard coordinate on $\mathbb{R}_{>0}$, then the 2-form*

$$\tilde{\omega} = \tfrac{1}{4}\mathrm{d}\big(\psi^{-1}\eta\big) = \tfrac{1}{4\psi}\mathrm{d}\eta - \tfrac{1}{(2\psi)^2}\mathrm{d}\psi\wedge\eta \tag{5.29}$$

defines a symplectic structure on $\mathcal{M}_t = \mathcal{M}'_t\times\mathbb{R}_{>0}$.

Proof: By defining $t = (4\,\psi)^{-1} > 0$ we can write $\tilde{\omega} = \mathrm{d}(t\,\eta)$, which we recognise from proposition 4.2.8 as the canonical symplectic structure on the symplectised manifold $\tilde{\mathcal{M}}'_t \cong \mathcal{M}'_t\times\mathbb{R}$. □

Proposition 5.2.6 gives us an identification between the fibre $\mathcal{M}_t = \mathcal{M}'_t \times \mathbb{R}$ and the symplectisation $\tilde{\mathcal{M}}'_t \subseteq \mathrm{T}^*M$ of the contact manifold $(\mathcal{M}'_t, F)$. Since the symplectic form $\tilde{\omega}$ can be defined in terms of just the contact structure $F < \mathrm{T}\mathcal{M}'_t$ it is in particular independent of the complex structure and therefore does not depend on $t \in \mathcal{M}_{\mathbb{C}}$. The reason for choosing this particular identification between the coordinates t and ψ becomes clear when we use corollary 4.5.2 to equip $\mathcal{M}'_t$ with a Kähler structure.

Theorem 5.2.7. *Let (F, J) be the (strictly pseudoconvex) CR structure on $\mathcal{M}'_t = \mathcal{H}_{\mathcal{Y},\mathbb{Z}} \backslash \mathcal{H}_{\mathcal{Y}}$ defined in proposition 5.2.3, let $\xi = \partial_\sigma$ and let η be the invariant defining form for F for which $\eta(\xi) = 1$. The almost complex structure $\tilde{J}$ on $\mathrm{T}\mathcal{M}_t$ defined by*

$$\tilde{J}|_{F\times\{0\}} = J, \qquad \tilde{J}(\partial_\psi) = \xi \qquad \text{and} \qquad \tilde{J}(\xi) = -\partial_\psi \tag{5.30}$$

is integrable. The metric $\tilde{g} = g_t$ from equation (5.28) on $\mathcal{M}_t$ is Kähler, its Kähler form is $\tilde{\omega} = \frac{1}{4}\mathrm{d}\big(\psi^{-1}\eta\big)$ and the function $K\colon \mathcal{M}_t \to \mathbb{R}_{>0}, (a^{(3)}, \sigma, \psi) \mapsto -\frac{1}{2}\log\psi$ is a Kähler potential for it.

Proof: In proposition 5.2.6 we had identified $\mathcal{M}_t$ with the symplectised space $\tilde{\mathcal{M}}'_t$ as described in proposition 4.2.8 by setting $t = (4\,\psi)^{-1}$. By subsequently applying corollary 4.5.2 we obtain the integrable complex structure $\tilde{J}$ and the Kähler metric

$$\tilde{g}_\lambda = \tfrac{1}{4}\mathrm{d}(\log t)^2 + g_{2\lambda t} = \tfrac{1}{4}\mathrm{d}(\log\psi)^2 + g_{(2\psi)^{-1}\lambda}, \tag{5.31}$$

which equals g_t for $\lambda = 1$. We have $-\frac{1}{2}\log\psi = \frac{1}{2}\log t + \log 2$, which differs from the Kähler potential from corollary 4.5.2 by the constant $\log 2$ and is therefore a Kähler potential as well. $\square$

5.2.4 Complex coordinates

We have just seen that $\tilde{J}$ defines a complex structure and we have managed to show that $K = -\frac{1}{2}\log\psi$ is a Kähler potential on $\mathcal{M}_t$ with respect to it. We would however also like to have a set of complex coordinates to be able to describe this structure more explicitly. Such coordinates are not hard to find, but before we do this we should show that the projection $\mathcal{M}_t \to \mathcal{J}_2^{\mathrm{W}}$ is a holomorphic map and introduce some notation for forms on $\mathcal{M}_t$.

Lemma 5.2.8. *Let π denote the projection map from $\mathcal{M}_t \cong \mathcal{H}_{\mathcal{Y},\mathbb{Z}}\backslash\big(H^3(\mathcal{Y},\mathbb{R}) \times \mathbb{R} \times \mathbb{R}_{>0}\big)$ to the torus $\mathcal{J}_2 \cong H^3(\mathcal{Y},\mathbb{R})/H^3(\mathcal{Y},\mathbb{Z})_{\mathrm{f}}$. The projection map π is holomorphic and $\tilde{J}^*(\pi^*\alpha) = \pi^*(J^{\mathrm{W}*}\alpha)$ for any 1-form α on $\mathcal{J}_2$.*

Proof: This is a consequence of that fact that the invariant vector field $X^{(v,s)}$ is given by $X^{(v,s)}(a^{(3)}, \sigma) = (v, s - Q(a^{(3)}, v))$ and thus satisfies $\pi_* X^{(v,s)} = v$. A general tangent vector at $(a^{(3)}, \sigma, \psi) \in \mathcal{M}_t = \mathcal{M}'_t \times \mathbb{R}_{>0}$ can be written as $X = X^{(v,s)} + \kappa\,\partial_\psi = X^{(v,0)} + s\,\xi + \kappa\partial_\psi$ for some $v \in H^3(\mathcal{Y},\mathbb{R})$, $s \in \mathbb{R}$, $\kappa \in \mathbb{R}$ and $\xi = X^{(0,1)} = \partial_\sigma$ and we can use (5.30) to show that

$$\begin{aligned} \tilde{J}^*(\pi^*\alpha)(X) &= \tilde{J}^*(\pi^*\alpha)(\kappa\,\partial_\psi + s\,\xi + X^{(v,0)}) \\ &= (\pi^*\alpha)(\kappa\,\xi - s\,\partial_\psi + X^{(J^{\mathrm{W}}v,0)}) \\ &= \alpha(\pi_*(\kappa\,\xi - s\,\partial_\psi + X^{(J^{\mathrm{W}}v,0)})) \\ &= \alpha(J^{\mathrm{W}}v) = J^{\mathrm{W}*}\alpha(\pi_* X) = \pi^*(J^{\mathrm{W}*}\alpha)(X). \end{aligned} \tag{5.32}$$

This confirms the second claim. The first claim is a direct consequence of this since the fact that $\tilde{J}^*\pi^* = \pi^* J^{\mathrm{W}*}$ can be rephrased as $\pi_* \circ \tilde{J} = J^{\mathrm{W}} \circ \pi_*$ for $\pi_* = \mathrm{d}\pi$. $\square$

The Weil intermediate Jacobian had been discussed in some detail in section 2.3 in terms of the symplectic basis $(\alpha_i, \beta^i)_i$ and the normalised period matrix

$$\mathcal{Z}^{\mathrm{W}}_{ij} = \bar{\mathcal{N}}_{ij} = F_{ij} - i\frac{N_{ik}\bar{X}^k\bar{X}^\ell N_{\ell j}}{\bar{X}N\bar{X}}. \tag{5.33}$$

The complex structure on $\mathcal{J}_2^{\mathrm{W}}(\mathcal{Y})$ could be expressed by specifying the complex coordintes $z_i = y_i + \mathcal{Z}_{ij}x^j = -B_i + \mathcal{Z}_{ij}A^j$, which tells us that $\mathrm{d}z_i$ are holomorphic $(1,0)$-forms on $\mathcal{J}_2^{\mathrm{W}}(\mathcal{Y})$. Since not only $\mathrm{d}z_i$, but also $g^{\mathrm{W}}(a^{(3)}, \bullet)$ and $Q(a^{(3)}, \bullet) = g^{\mathrm{W}}(a^{(3)}, J^{\mathrm{W}}\bullet)$ define differential forms on the torus, we can consider the 1-forms $\pi^*\mathrm{d}z_i$, $\pi^* g^{\mathrm{W}}(a^{(3)}, \bullet)$ and $\pi^*Q(a^{(3)}, \bullet)$ on $\mathcal{M}_t$. We will from now on simply denote these forms on $\mathcal{M}_t$ by $\mathrm{d}z_i$, $g^{\mathrm{W}}(a^{(3)}, \bullet)$ and $Q(a^{(3)}, \bullet)$ and leave out the pullback π^* to keep things clean. Because $X^{(v,s)}(a^{(3)}, \sigma, \psi) = (v, s - Q(a^{(3)}, v))$ and $\pi_* X^{(v,s)} = v$ we have that

$$\big(\mathrm{d}\sigma + \pi^*Q(a^{(3)}, \bullet)\big)(X^{(v,s)}) = s - Q(a^{(3)}, v) + Q(a^{(3)}, \pi_*X^{(v,s)}) = s. \tag{5.34}$$

This property completely characterised the invariant contact form η, so we can conclude that $\eta = \mathrm{d}\sigma + Q(a^{(3)}, \bullet)$. It is now not hard to find the complex coordinates we were looking for.

Proposition 5.2.9. *The complex coordinates $z_i = -B_i + \mathcal{Z}^{\mathrm{W}}_{ij}A^j$ on $(H^3(\mathcal{Y}, \mathbb{R}), J^{\mathrm{W}})$ on the torus (cf. corollary 2.3.10) and the additional coordinate*

$$\tau = -\sigma + i\,\psi + \tfrac{i}{2} g^{\mathrm{W}}(a^{(3)}, a^{(3)}) = -\sigma + i\,\psi + \tfrac{i}{2} z_i \mathcal{Y}_{\mathrm{W}}^{ij}\bar{z}_j \tag{5.35}$$

together (locally) describe a set of complex coordinates on $\mathcal{M}_t \cong \mathcal{H}_{\mathcal{Y},\mathbb{Z}}\backslash\big(H^3(\mathcal{Y}, \mathbb{R}) \times \mathbb{R} \times \mathbb{R}_{>0}\big)$ with respect to the complex structure $\tilde{J}$ described in theorem 5.2.7.

Proof: Since $\mathrm{d}z_i = \pi^*\mathrm{d}z_i$ is the pullback of a 1-form on $\mathcal{J}_2$ and $J^{\mathrm{W}*}\mathrm{d}z_i = +i\,\mathrm{d}z_i$, lemma 5.2.8 tells us that

$$\tilde{J}^*\mathrm{d}z_i = \tilde{J}^{\mathrm{W}*}\mathrm{d}z_i = +i\,\mathrm{d}z_i, \tag{5.36}$$

so the coordinates z_i locally define a set of holomorphic functions on $\mathcal{M}_t$.

The differential of τ is given by

$$\mathrm{d}\tau = -\mathrm{d}\sigma + i\,\mathrm{d}\psi + \mathrm{d}\big(\tfrac{i}{2} g^{\mathrm{W}}(a^{(3)}, a^{(3)})\big) = -\mathrm{d}\sigma + i\,\mathrm{d}\psi + i\, g^{\mathrm{W}}(a^{(3)}, \bullet), \tag{5.37}$$

since we can check that $\mathrm{d}g(a^{(3)}, a^{(3)})(X) = \frac{\partial}{\partial\varphi}\big|_{\varphi=0} g^{\mathrm{W}}(a^{(3)} + \varphi\, X, a^{(3)} + \varphi\, X) = 2\, g^{\mathrm{W}}(a^{(3)}, X)$.

We can use this to work out that

$$\begin{aligned}(\tilde{J}^* - i)\mathrm{d}\tau &= i\,\tilde{J}^*(i\,\mathrm{d}\sigma + \mathrm{d}\psi + g^{\mathrm{W}}(a^{(3)}, \bullet)) + i\,\mathrm{d}\sigma + \mathrm{d}\psi + g^{\mathrm{W}}(a^{(3)}, \bullet)\\ &= \mathrm{d}\psi - \tilde{J}^*\mathrm{d}\sigma + g^{\mathrm{W}}(a^{(3)}, \bullet) + i\,\big(\mathrm{d}\sigma + g^{\mathrm{W}}(a^{(3)}, J^{\mathrm{W}}\bullet) + \tilde{J}^*\mathrm{d}\psi\big)\\ &= \mathrm{d}\psi - \tilde{J}^*\underbrace{\big(\mathrm{d}\sigma + Q(a^{(3)}, \bullet)\big)}_{\eta} + i\,\big(\underbrace{\mathrm{d}\sigma + Q(a^{(3)}, \bullet)}_{\eta} + \tilde{J}\mathrm{d}\psi\big),\end{aligned} \tag{5.38}$$

where we have used that $g^{\mathrm{W}}(a^{(3)}, \bullet) = -Q(a^{(3)}, J^{\mathrm{W}}\bullet) = -\tilde{J}^*Q(a^{(3)}, \bullet)$. We can derive from equation (5.30) that $\tilde{J}^*\eta = \mathrm{d}\psi$ and hence also that $\tilde{J}^*\mathrm{d}\psi = -\eta$, which tells us that the right-hand side of equation (5.38) vanishes. This can only happen when $\tilde{J}^*\mathrm{d}\tau = +i\mathrm{d}\tau$, so τ is a holomorphic function on $\mathcal{M}_t$.

Because each of the coordinates z_i and τ is holomorphic and their differentials are linearly independent they locally describe a set of complex coordinates on $\mathcal{M}_t$. □

Although it may seem unimportant, the appearance of the metric g^{W} in the coordinate τ turns out to be very significant.

We note that the form of the coordinate τ corresponds to the coordinate S that is often introduced on the hypermultiplet moduli space of rigid Calabi-Yau manifolds (those for which $h^{1,2} = 0$) [57, 69, 70], but we have chosen to denote it by a different letter to stress the fact that it does not correspond to the coordinate S from [2].

5.2.5 Interpretation in terms of a line bundle

The fibres of the bundle $\mathcal{M}_t \to \mathcal{J}_2(\mathcal{Y}_t)$ are parametrised by the dilaton $\psi \in \mathbb{R}_{>0}$ and the Kalb-Ramond axion $\sigma \in \mathbb{R}/\mathbb{Z}$, which can also be viewed as an element $e^{2\pi i\sigma}$ of the unit circle in $\mathbb{C}$. Since this axion defines a $\mathrm{U}(1)$-bundle and the coordinate ψ is globally defined, they can be combined into a $\mathbb{C}^*$-***bundle***. We can do slightly better than this however.

The total space $(\mathcal{M}_t, \tilde{J})$ is a complex manifold and the projection map : $\mathcal{M}_t \to \mathcal{J}_2^{\mathrm{W}}(\mathcal{Y}_t)$ onto the Weil intermediate Jacobian is holomorphic by lemma 5.2.8. The complex structure on the fibres is induced by the complex structure $\tilde{J}$ from theorem 5.2.7 and is thus given by $\tilde{J}(\partial_\psi) = \partial_\sigma$ and $\tilde{J}(\partial_\sigma) = -\partial_\psi$, which means that $-\sigma + i\,\psi$ is a complex coordinate on a given fibre. This coordinate lives on the upper half plane, on which some identifications need to be made due to the fact that $\sigma \in \mathbb{R}/\mathbb{Z}$ is periodic. These identifications are neatly captured by switching to a new coordinate $z = e^{2\pi i(-\sigma+i\psi)} = e^{-2\pi i\sigma}e^{-2\pi\psi}$, which lies in a punctured disk $D_1 \setminus \{0\} = \{z \in \mathbb{C} \in 0 < |z| < 1\} \subseteq \mathbb{C}$.

This gives the space $\mathcal{M}_t$, which we may recall was itself a fibre over $t \in \mathcal{M}_{\mathbb{C}}$, an interpretation as the total space of a bundle of complex disks (see figure 5.2). The extension of this bundle to a bundle of complex planes, which is achieved by allowing $z = 0$ ($\psi = +\infty$) and $z \geq 1$ ($\psi \leq 0$), can naturally be viewed as a smooth complex line bundle, but interpretating it as a holomorphic line bundle takes another step. The first problem is that the coordinate $z = e^{-2\pi i(\sigma - i\psi)}$ on the fibres is not a complex coordinate on the total space $\mathcal{M}_t$, but a more important (related) issue is the fact that transition functions for the trivialisations they define are not holomorphic.

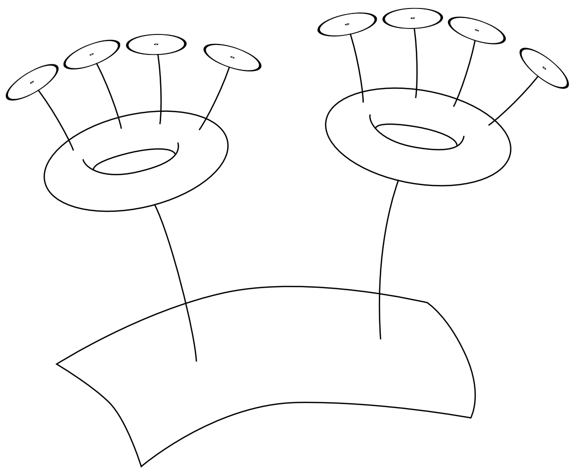

Figure 5.2: The hypermultiplet moduli with the circle and line bundles combined into a bundle of disks.

We had already seen that the fibres of $\mathcal{M}_t \to \mathcal{J}_2(\mathcal{Y}_t)$ can alternatively be parametrised by $\tau = -\sigma + i\,\psi + \frac{i}{2} g^{\mathrm{W}}(a^{(3)}, a^{(3)})$, which lies in the half plane defined by $\mathrm{Im}\,\tau > \frac{1}{2} g^{\mathrm{W}}(a^{(3)}, a^{(3)})$. Unlike z however, it can be combined with the canonical coordinates on $\mathcal{J}_2(\mathcal{Y}_t)$ to form a set of complex coordinates on the total space. As z, τ covers each fibre multiple times due to the periodicity of $\sigma = \mathrm{Re}\,\tau$, which we can again remedy by exponentiating it and switching to the coordinate

$$q = e^{2\pi i\tau} = z\,e^{-\pi g^{\mathrm{W}}(a^{(3)},a^{(3)})} = e^{-2\pi i\sigma}e^{-2\pi\psi - \pi g^{\mathrm{W}}(a^{(3)},a^{(3)}))} \tag{5.39}$$

With respect to this coordinate the radius of the disks varies as $\psi > 0$ restricts q to the punctured disk with $0 < |q| < \exp(-\pi\, g^{\mathrm{W}}(a^{(3)}, a^{(3)}))$. Anticipating what lies ahead, we will denote the bundle of complex planes obtained by allowing $q = 0$ and $|q| \geq \exp(-\pi\, g^{\mathrm{W}}(a^{(3)}, a^{(3)}))$ by $L \to \mathcal{J}_2$. The proposition below shows that this bundle can be interpreted as a holomorphic line bundle and hence that $\mathcal{M}_t$ can be interpreted as a (smooth) bundle of complex disks inside L.

Proposition 5.2.10. *The bundle : $L \to \mathcal{J}_2(\mathcal{Y}_t)$ of complex planes is a holomorphic line bundle. It can locally be trivialised on a contractible open subset $U \subseteq \mathcal{J}_2(\mathcal{Y}_t)$ by*

$$\varphi_U : L_U \to U \times \mathbb{C}, \qquad (a^{(3)}, \sigma, \psi) \mapsto (a^{(3)}, q) \tag{5.40}$$

where $q = e^{-2\pi i\sigma - 2\pi\psi - \pi g^{\mathrm{W}}(a^{(3)}, a^{(3)})}$ as in equation (5.39) and we have identified U with a subset of $H^3(\mathcal{Y}, \mathbb{R})$.

Proof: We already know that the coordinate q and the projection $\mathcal{M}_t \to \mathcal{J}_2^{\mathrm{W}}(\mathcal{Y}_t)$ are holomorphic, so it is easily verified that their extension to L by allowing $q = 0$ and $q > \exp(-\pi g^{\mathrm{W}}(a^{(3)}, a^{(3)}))$ are holomorphic as well. As a consequence, each of the trivialisations φ_U are holomorphic as well and we only need to check that the transition functions for L are linear in the fibres.

Let $a^{(3)}$ and $a^{(3)\prime} = a^{(3)} + \alpha$, with $\alpha \in H^3(\mathcal{Y}, \mathbb{Z})$ represent the same point in $\mathcal{J}_2(\mathcal{Y}_t)$, then $(a^{(3)}, \sigma, \psi) \sim (a^{(3)\prime}, \sigma - Q(\alpha, a^{(3)}), \psi)$ (cf. equation (5.6)). If we fix a point $p = [a^{(3)}, \sigma, \psi] = [a^{(3)} + \alpha, \sigma - Q(\alpha, a^{(3)}), \psi] \in L$ and two trivialisations φ and φ' that identify the basepoint for p with $a^{(3)}$ and $a^{(3)} + \alpha$ respectively, then

$$\begin{aligned} \varphi'_x(p) &= e^{-2\pi i(\sigma - Q(\alpha, a^{(3)})) - 2\pi\psi - \pi g^{\mathrm{W}}(a^{(3)} + \alpha, a^{(3)} + \alpha)} \\ &= e^{2\pi i Q(\alpha, a^{(3)}) - 2\pi g^{\mathrm{W}}(\alpha, a^{(3)}) - \pi g^{\mathrm{W}}(\alpha,\alpha)} e^{-2\pi i\sigma - 2\pi\psi - 2\pi g^{\mathrm{W}}(a^{(3)}, a^{(3)})} \\ &= e^{2\pi i Q(\alpha, a^{(3)}) - 2\pi g^{\mathrm{W}}(\alpha, a^{(3)}) - \pi g^{\mathrm{W}}(\alpha,\alpha)} \varphi_x(p). \end{aligned} \tag{5.41}$$

Since the factor $e^{2\pi i Q(\alpha, a^{(3)}) - 2\pi g^{\mathrm{W}}(\alpha, a^{(3)}) - \pi g^{\mathrm{W}}(\alpha,\alpha)}$ only depends on the base point and not on any coordinates on the fibre, it is constant on the fibres and the transition function from the trivialisation φ to φ' is a linear isomorphism. That this transition function is holomorphic can be read off explicitly from this factor since $g^{\mathrm{W}} = Q(J^{\mathrm{W}}\bullet, \bullet)$. We conclude that L is a holomorphic line bundle. $\square$

Proposition 5.2.11. *The Chern class $c_1(L)$ equals the polarisation Q on the Weil intermediate Jacobian and this completely fixes the line bundle L up to translations on the torus.*

Proof: We can locally define a Hermitian inner product h on the line bundle L by setting

$$h(q, q) = e^{2\pi g^{\mathrm{W}}(a^{(3)}, a^{(3)})}|q|^2 = e^{2\pi g^{\mathrm{W}}(a^{(3)}, a^{(3)})}|e^{-2\pi i\sigma - 2\pi\psi - \pi g^{\mathrm{W}}(a^{(3)}, a^{(3)})}|^2 = e^{-4\pi\psi}. \tag{5.42}$$

This in fact globally defines an inner product on L since the coordinate ψ is globally defined. Locally, we can specify a holomorphic section s of $L \to \mathcal{J}_2(\mathcal{Y}_t)$ by fixing $q = 1$ and for this section we have $h = h(s, s) = e^{2\pi g^{\mathrm{W}}(a^{(3)}, a^{(3)})}$. Lemma 1.3.24 tells us that the curvature form for this connection is given by

$$\begin{aligned} \Theta &= -\partial\bar{\partial}\log h = -\tfrac{i}{2}\mathrm{d}J^{\mathrm{W}*}\mathrm{d}\log h = -\pi\, i\, \mathrm{d}J^{\mathrm{W}*}\mathrm{d}g^{\mathrm{W}}(a^{(3)}, a^{(3)}) \\ &= -2\,\pi\, i\, \mathrm{d}g^{\mathrm{W}}(a^{(3)}, J^{\mathrm{W}}\bullet) = -2\,\pi\, i\, \mathrm{d}Q(a^{(3)}, \bullet) = -2\,\pi\, i\, Q \end{aligned} \tag{5.43}$$

and by lemma 1.3.22 the first Chern class of L is given by

$$c_1(L) = \left[\tfrac{-1}{2\pi i}\Theta\right] = [Q]. \tag{5.44}$$

Proposition 1.4.7 finally tells us that this property fixes L up to a translation on the torus, which finishes the proof. $\square$

6. The Quaternion-Kähler structure

The fact that the hypermultiplet moduli space comes with a quaternion-Kähler structure can be derived from just its behaviour under supersymmetry transformations [48] and we already have an explicit description of this structure by Ferrara and Sabharwal [2]. In this chapter we will express this structure in terms of the construction described in chapter 5 and try to give it an interpretation in terms of the complex structure moduli space and the disk bundle on the Weil intermediate Jacobian.

6.1 Local frames

By considering different parts from the fibre bundle described in section 5.1 separately we had found a construction for the metric on the entire hypermultiplet moduli space. This metric was given by

$$\begin{aligned} g_{\mathrm{hm}} = K_{i\bar{\jmath}}\mathrm{d}X^i\mathrm{d}\bar{X}^j &+ \frac{1}{(2\,\psi)^2}\mathrm{d}\psi^2 + \frac{1}{(2\,\psi)^2}(\mathrm{d}\sigma - A^i\overleftrightarrow{\mathrm{d}}B_i)^2 \\ &+ \frac{1}{2\,\psi}\mathcal{Y}_{\mathrm{W}}^{ij}(\mathrm{d}B_i - \overline{\mathcal{Z}}^{\mathrm{W}}_{ik}\mathrm{d}A^k)(\mathrm{d}B_i - \mathcal{Z}^{\mathrm{W}}_{ik}\mathrm{d}A^k) \end{aligned} \tag{6.1}$$

for $\mathcal{Z}^{\mathrm{W}}_{ij} = F_{ij} - i\frac{N_{ik}\bar{X}^k\bar{X}^\ell N_{\ell j}}{\bar{X}N\bar{X}}$ and $\mathcal{Y}_{\mathrm{W}}^{ij}$ the inverse of $\mathcal{Y}^{\mathrm{W}} = \mathrm{Im}(\mathcal{Z}^{\mathrm{W}})$.

Here $K(X,\bar{X}) = -\log(-X^iN_{ij}\bar{X}^j)$ is a Kähler potential described in section 2.2 and $K_{i\bar{\jmath}}$ is its second derivative,

$$K_{i\bar{\jmath}} = \frac{\partial^2 K(X,\bar{X})}{\partial X^i\,\partial\bar{X}^j} = \frac{-1}{XN\bar{X}}\left(N_{ij} - \frac{N_{ik}\bar{X}^kX^\ell N_{\ell j}}{XN\bar{X}}\right). \tag{6.2}$$

Note that $K_{i\bar{\jmath}}\bar{X}^j = K_{\bar{\imath}j}X^i = 0$.

The description Ferrara and Sabharwal have given for the quaternion-Kähler structure on the hypermultiplet moduli space was expressed in terms of local orthonormal frames [2]. We will write these frames in terms of the coordinates we have been working with and use the construction for the hypermultiplet moduli space from the previous chapter to give it a more intrinsic interpretation.

Although working with homogenous coordinates $X^0,\ldots,X^{h^{1,2}}$ have their advantages, it will now be convenient to use the inhomogeneous coordinates $Z^1,\ldots,Z^{h^{1,2}}$, given by $Z^a = X^a/X^0$, instead and to write $Z^0 = X^0/X^0 = 1$. We can always locally do this because at least one of the projective coordinates X^i is always non-zero and we can just rearrange the indices in the symplectic basis $(\alpha_i,\beta^i)_i$. For the remainder of this chapter the letters $a,b,\ldots$ will denote inhomogenous indices and will run from 1 to n, while the indices $i,j,\ldots$ are still homogenous and run from 0 to n.

The inhomogeneous expression for the metric on the complex structure moduli space is given by

$$g_{\mathbb{C}} = K^{[0]}_{a\bar{b}}\mathrm{d}Z^a\mathrm{d}\bar{Z}^b := \frac{\partial^2 K^{[0]}(Z,\bar{Z})}{\partial Z^a\,\partial\bar{Z}^b}\mathrm{d}Z^a\mathrm{d}\bar{Z}^b, \tag{6.3}$$

where $K^{[0]}$ denotes the inhomogeneous Kähler potential $K^{[0]} = -\log(-Z^a N_{ab}\bar{Z}^b)$.

Lemma 6.1.1. *We can locally find a basis $e^{(1)},\ldots,e^{(h^{1,2})}$ of $(1,0)$-forms such that the metric $g_{\mathbb{C}}$ from equation (6.3) can be written as*

$$g_{\mathbb{C}} = e^{(a)}\bar{e}^{(a)} = P^{(a)}_b\bar{P}^{(a)}_c\mathrm{d}Z^b\mathrm{d}\bar{Z}^c = K^{[0]}_{b\bar{c}}\mathrm{d}Z^b\mathrm{d}\bar{Z}^c. \tag{6.4}$$

By using the $n\times n$-matrix $P^{(a)}_b$ for which $e^{(a)} = P^{(a)}_b\mathrm{d}Z^b$ we can write $K^{[0]}_{b\bar{c}} = P^{(a)}_b\bar{P}^{(a)}_c$.

Proof: The tangent spaces $\mathrm{T}_t\mathcal{M}_{\mathbb{C}} \simeq \mathrm{T}^{1,0}\mathcal{M}_{\mathbb{C}}$ can be viewed as complex linear spaces on which the metric $g_{\mathbb{C}}$ extends to a Hermitian metric $h_{\mathbb{C}} = g_{\mathbb{C}} - i\,g(J_{\mathbb{C}}\,\bullet,\bullet)$. Through Gram-Schmidt orthonormalisation it is possible to locally find an orthonormal (complex) basis $e_{(1)},\ldots,e_{(h^{1,2})}$ for this Hermitian metric. Now, let $e^{(1)},\ldots,e^{(h^{1,2})}$ denote the dual basis of $(1,0)$-forms defined by $e^{(a)}(e_{(b)}) = \delta^{(a)}_{(b)}$ (and $e^{(a)}(J_{\mathbb{C}}\,e_{(b)}) = i\,\delta^{(a)}_{(b)}$). It can now easily be verified that $h_{\mathbb{C}} = e^{(a)}\otimes\bar{e}^{(a)}$ and hence that the real metric $g_{\mathbb{C}} = \mathrm{Re}\,h_{\mathbb{C}} = \frac{1}{2}\big(h_{\mathbb{C}} + \overline{h_{\mathbb{C}}}\big)$ is given by $g_{\mathbb{C}} = e^{(a)}\bar{e}^{(b)}$. $\square$

We will often refer to this orthonormal basis of $(1,0)$-forms $e^{(a)} = P^{(a)}_b\mathrm{d}Z^b$ as the ***vielbein*** for the complex structure moduli space.

If we extend P to a $(n+1)\times n$ matrix by writing $P^{(a)}_0 := -P^{(a)}_b Z^b$, then we have for $i,j = 0,\ldots,h^{1,2}$ that [2]

$$P^{(a)}_i\bar{P}^{(a)}_j = \frac{-1}{ZN\bar{Z}}\left(N_{ij} - \frac{N_{ik}\bar{Z}^k Z^\ell N_{\ell j}}{ZN\bar{Z}}\right) = (X^0)^2 K_{i\bar{j}}. \tag{6.5}$$

This is clearly true for $i,j>0$, but if either $i=0$ or $j=0$ then we need to use that

$$Z^a K^{[0]}_{a\bar{b}} = (X^0)^2 Z^i K_{i\bar{b}} - (X^0)^2 Z^0 K_{0\bar{b}} = -(X^0)^2 K_{0\bar{b}}. \tag{6.6}$$

Using the $(n+1)\times n$ matrix $P^{(a)}_i$, Ferrara and Sabharwal managed to find a set of vielbein 1-forms that also include the fibres of the hypermultiplet moduli space [2]. Expressed in the by now familiar coordinates σ, ψ, A^i and B_i, in addition to the inhomogeneous coordinates $X^a = Z^a/Z^0$, these are given by

$$\begin{aligned}
e^{(a)} &= P^{(a)}_b\mathrm{d}Z^b = e^{(a)}_b\mathrm{d}Z^b, && (6.7\mathrm{a})\\
E^{(a)} &= i\sqrt{2}\,e^{(\widetilde{K}-K^{[0]})/2}P^{(a)}_i N^{ij}(\mathrm{d}B_j - \overline{\mathcal{Z}}^{\mathrm{W}}_{jk}\mathrm{d}A^k), && (6.7\mathrm{b})\\
u &= i\sqrt{2}\,e^{(\widetilde{K}+K^{[0]})/2}Z^i(\mathrm{d}B_i - \overline{\mathcal{Z}}^{\mathrm{W}}_{ij}\mathrm{d}A^j), && (6.7\mathrm{c})\\
v &= \tfrac{1}{2\,\psi}(\mathrm{d}\psi - i\,(\mathrm{d}\sigma - A^i\overleftrightarrow{\mathrm{d}}B_i)), && (6.7\mathrm{d})
\end{aligned}$$

where $\mathcal{Z}^{\mathrm{W}}_{ij} = F_{ij} - i\frac{N_{ik}\bar{Z}^k\bar{Z}^\ell N_{\ell j}}{\bar{Z}N\bar{Z}} = \bar{\mathcal{N}}$ is the period matrix for the Weil intermediate Jacobian and $\mathcal{Y}^{ij}_{\mathrm{W}}$ and N^{ij} denote the inverse of $\mathcal{Y}^{\mathrm{W}} = \mathrm{Im}(\mathcal{Z}^{\mathrm{W}})$ and $N_{ij} = 2\,\mathrm{Im}(F_{ij})$. Furthermore, $K = -\log(-Z^i N_{ij}\bar{Z}^j)$ and $\widetilde{K} = -\log(2\,\psi)$ are still the Kähler potentials on the base space and the fibres respectively.

Proposition 6.1.2. *Together, the forms $E^{(a)}$ and u form a vielbein for the torus part of the metric g_{hm} in the sense that the metric $\frac{1}{2\psi}g_{\mathrm{torus}}$ from equation (5.14) can be written as*

$$\tfrac{1}{2\psi}g_{\mathrm{torus}} = \bar{E}^{(a)}E^{(a)} + \bar{u}\,u. \tag{6.8}$$

This sum above is orthogonal and $\bar{u}\,u$ and $\bar{E}^{(a)}E^{(a)}$ vanish on $(H^{3,0}(\mathcal{Y})\oplus H^{0,3}(\mathcal{Y}))\cap H^3(\mathcal{Y},\mathbb{R})$ and on $(H^{2,1}(\mathcal{Y})\oplus H^{1,2}(\mathcal{Y}))\cap H^3(\mathcal{Y},\mathbb{R})$ respectively.

Proof: The first statement can be directly verified through an explicit computation,

$$\begin{aligned}
\bar{E}^{(a)}E^{(a)}+\bar{u}\,u &= 2\,e^{\widetilde{K}+K}\big(e^{-2K}N^{ik}P^{(a)}_i\bar{P}^{(a)}_j N^{j\ell}+Z^k\bar{Z}^\ell\big)\times\\
&\qquad\times(\mathrm{d}B-\overline{\mathcal{Z}}^{\mathrm{W}}\mathrm{d}A)_k(\mathrm{d}B-\mathcal{Z}^{\mathrm{W}}\mathrm{d}A)_\ell\\
&=\frac{-1}{\psi\,ZN\bar{Z}}\left(-\bar{Z}NZ\,N^{ik}\left(N_{ij}-\frac{N_{im}\bar{Z}^mZ^nN_{jn}}{\bar{Z}NZ}\right)N^{j\ell}+Z^k\bar{Z}^\ell\right)\times\\
&\qquad\times(\mathrm{d}B-\overline{\mathcal{Z}}^{\mathrm{W}}\mathrm{d}A)_k(\mathrm{d}B-\mathcal{Z}^{\mathrm{W}}\mathrm{d}A)_\ell\\
&=\frac{-1}{\psi}\left(-N^{k\ell}+\frac{\bar{Z}^kZ^\ell+Z^k\bar{Z}^\ell}{\bar{Z}NZ}\right)(\mathrm{d}B-\overline{\mathcal{Z}}^{\mathrm{W}}\mathrm{d}A)_k(\mathrm{d}B-\mathcal{Z}^{\mathrm{W}}\mathrm{d}A)_\ell\\
&=\frac{-1}{\psi}\left(-N^{k\ell}+\frac{\bar{Z}^kZ^\ell+Z^k\bar{Z}^\ell}{\bar{Z}NZ}\right)(\mathrm{d}B-\overline{\mathcal{Z}}^{\mathrm{W}}\,A)_k(\mathrm{d}B-\mathcal{Z}^{\mathrm{W}}\,A)_\ell\\
&=\frac{-1}{2\,\psi}\mathcal{Y}_{\mathrm{W}}^{ij}(\mathrm{d}B_i-\overline{\mathcal{Z}}^{\mathrm{W}}_{ik}\mathrm{d}A^k)_k(\mathrm{d}B_j-\mathcal{Z}^{\mathrm{W}}_{j\ell}\mathrm{d}A^\ell)_\ell.
\end{aligned}\tag{6.9}$$

Here we have used that the $\mathcal{Y}^{\mathrm{W}}_{ij}=\mathrm{Im}(\mathcal{Z}^{\mathrm{W}}_{ij})$ and its inverse $\mathcal{Y}_{\mathrm{W}}^{ij}$ are given by

$$2\,\mathcal{Y}^{\mathrm{W}}_{ij}=N^{k\ell}-\tfrac{N_{ik}\bar{Z}^k\bar{Z}^\ell N_{j\ell}}{\bar{Z}N\bar{Z}}-\tfrac{N_{ik}Z^kZ^\ell N_{j\ell}}{ZNZ},\qquad \mathcal{Y}_{\mathrm{W}}^{ij}=-N^{ij}+\tfrac{\bar{Z}^iZ^j+Z^i\bar{Z}^j}{\bar{Z}NZ},\tag{6.10}$$

which can be explicitly verified to be the case.

In addition to this, we have that (N.B. $N_{ij}=2\,\mathrm{Im}(F_{ij})$),

$$\begin{aligned}
Z^i\overline{\mathcal{Z}}^{\mathrm{W}}_{ij}&=Z^i\left(\bar{F}_{ij}+i\frac{N_{ik}Z^kZ^\ell N_{j\ell}}{Z\,N\,Z}\right)=Z^i\bar{F}_{ij}+i\frac{(Z^iN_{ik}Z^k)Z^\ell N_{j\ell}}{Z\,N\,Z}\\
&=Z^i\bar{F}_{ij}+Z^i(F_{ij}-\bar{F}_{ij})=Z^iF_{ij}=(X^0)^{-1}F_j,
\end{aligned}\tag{6.11}$$

from which it follows that for $a^{(3)}=A^i\alpha_i-B_i\beta^i\in H^3(\mathcal{Y},\mathbb{R})$,

$$\begin{aligned}
u(a^{(3)})&=i\,\sqrt{2}\,e^{(\widetilde{K}+K^{[0]})/2}Z^i(\mathrm{d}B_i-\overline{\mathcal{Z}}^{\mathrm{W}}_{ij}\mathrm{d}A^j)(a^{(3)})\\
&=\frac{i\,\sqrt{2}\,e^{(\widetilde{K}+K^{[0]})/2}}{X^0}(X^i\mathrm{d}B_i-\bar{F}_i\mathrm{d}A^i)(A^j\alpha_j-B_j\beta^j)\\
&=\frac{i\,\sqrt{2}\,e^{(\widetilde{K}+K^{[0]})/2}}{X^0}(X^iB_i-F_iA^i)\\
&=-\frac{i\,\sqrt{2}\,e^{(\widetilde{K}+K^{[0]})/2}}{X^0}Q(\Omega,a^{(3)})=-\frac{e^{(\widetilde{K}+K^{[0]})/2}}{\sqrt{2}\,X^0}h(\Omega,a^{(3)})
\end{aligned}\tag{6.12}$$

where $\Omega=X^i\alpha_i-F_i\beta^i\in H^{3,0}(\mathcal{Y})$ and $h=2\,i\,Q(\bullet,\bar{\bullet})$ is the Hermitian form from corollary 2.1.7. This not only tells us that that u and $\bar{u}$ vanish on $(H^{1,2}(\mathcal{Y})\oplus H^{2,1}(\mathcal{Y}))\cap H^3(\mathcal{Y},\mathbb{R})$, but also that for any $a^{(3)}\in\big(H^{3,0}(\mathcal{Y})\oplus H^{0,3}(\mathcal{Y})\big)\cap H^3(\mathcal{Y},\mathbb{R})$,

$$\begin{aligned}
\bar{u}(a^{(3)})\,u(a^{(3)})&=\tfrac{1}{2}\,|X^0|^{-2}e^{K^{[0]}+\widetilde{K}}h(\Omega,a^{(3)})\,\overline{h(\Omega a^{(3)})}\\
&=\frac{h(\Omega,a^{(3)})\,\overline{h(\Omega,a^{(3)})}}{2\,\psi\,h(\Omega,\Omega)}=\frac{\overline{h(a^{(3)},\Omega)}\,h(a^{(3)},\Omega)}{2\,\psi\,h(\Omega,\Omega)},
\end{aligned}\tag{6.13}$$

where we have used that h is Hermitian and have expanded the exponentials of $\widetilde{K}=-\log(2\psi)$ and $K^{[0]}=-\log(-Z^iN_{ij}\bar{Z}^j)=-\log\big(\tfrac{1}{2}|X^0|^{-2}h(\Omega,\Omega)\big)$. A quick comparison with equation (2.36) from lemma 2.3.8 show that $\bar{u}\,u=(2\,\psi)^{-1}g^{\mathrm{W}}$ on $a^{(3)}\in\big(H^{3,0}(\mathcal{Y})\oplus H^{0,3}(\mathcal{Y})\big)\cap H^3(\mathcal{Y},\mathbb{R})$.

Since $\bar{u}\,u + \bar{E}^{(a)}E^{(a)} = \frac{1}{2\psi}g^{\mathrm{W}}$ and the Hodge decomposition is perpendicular with respect to g^{W}, we conclude that the sum $\bar{u}\,u + \bar{E}^{(a)}E^{(a)}$ is orthogonal and that $\bar{u}\,u$ and $\bar{E}^{(a)}E^{(a)}$ vanish on $(H^{3,0}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})) \cap H^3(\mathcal{Y},\mathbb{R})$ and $(H^{3,0}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})) \cap H^3(\mathcal{Y},\mathbb{R})$ respectively. □

Corollary 6.1.3. *The hypermultiplet metric can be written as*

$$g_{\mathrm{hm}} = \underbrace{e^{(a)}\bar{e}^{(a)}}_{\textit{complex moduli}} + \underbrace{E^{(a)}\bar{E}^{(a)} + u\,\bar{u}}_{\textit{the torus}} + \underbrace{v\,\bar{v}}_{\textit{“line bundle”}}. \tag{6.14}$$

Proof: The forms $e^{(a)}$ had been defined at the beginning of this section and satisfy $g_{\mathbb{C}} = e^{(a)}\bar{e}^{(a)}$ by definition. We have just shown in proposition 6.1.2 that the torus part of the metric is given by $E^{(a)}\bar{E}^{(a)} + u\,\bar{u}$ and a quick calculation shows that

$$v\,\bar{v} = (2\,\psi)^{-2}|\mathrm{d}\psi - i\,(\mathrm{d}\sigma - A^i\overleftrightarrow{\mathrm{d}}B_i)|^2 = (2\,\psi)^{-2}\mathrm{d}\psi^2 + (2\,\psi)^{-2}(\mathrm{d}\sigma - A^i\overleftrightarrow{\mathrm{d}}B_i)^2. \tag{6.15}$$

By adding all of these we obtain the metric g_{hm} from equation (6.1). □

To describe the quaternion-Kähler structure on the hypermultiplet moduli space, it will be convenient to write $e^{+(0)} = u$, $e^{+(a)} = e^{(a)}$, $e^{-(0)} = v$ and $e^{-(a)} = E^{(a)}$ to combine $e^{(a)}$ with u and $E^{(a)}$ with v into new vielbein 1-forms $e^{\alpha(i)} = (e^{+(i)}, e^{-(i)})$, given by Equation (6.14) then reads

$$g_{\mathrm{hm}} = e^{\alpha(i)}\bar{e}^{\alpha(i)} := \sum_{\alpha=\pm}\sum_{i=0}^{n} e^{\alpha(i)}\bar{e}^{\alpha(i)}. \tag{6.16}$$

Note that instead of combining $E^{(a)}$ with v, which would make sense since these forms together describe the torus, we have $e^{+(i)} = (u, e^{(a)})$ and $e^{-(i)} = (v, E^{(a)})$.This will become later in section 6.2.1

6.1.1 The connection

The Levi-Civita connection ∇ for g_{hm} induces a natural connection on the cotangent bundle that we will also denote by ∇ and is given by

$$(\nabla_X\alpha)(Y) = X(\alpha(Y)) - \alpha(\nabla_X Y). \tag{6.17}$$

for any 1-form α and any two vector fields X and Y on $\mathcal{M}_{\mathrm{hm}}$. This connection can be extended to a connection on the complexified cotangent bundle through complex linear extension. Since we have a (complex) basis $(e^{\alpha(i)})_{\alpha(i)}$ for the complexified cotangent bundle we can give a more explicit description through the connection 1-forms φ and χ, which are matrices $\varphi^{\alpha(i)}{}_{\beta(j)}$ and $\chi^{\alpha(i)}{}_{\beta(j)}$ such that

$$\nabla_X e^{\alpha(i)} = \varphi^{\alpha(i)}{}_{\beta(j)}(X)e^{\beta(j)} + \chi^{\alpha(i)}{}_{\beta(j)}(X)\bar{e}^{\beta(j)}, \tag{6.18a}$$

$$\nabla_X \bar{e}^{\alpha(i)} = \bar{\varphi}^{\alpha(i)}{}_{\beta(j)}(X)\bar{e}^{\beta(j)} + \bar{\chi}^{\alpha(i)}{}_{\beta(j)}(X)e^{\beta(j)} \tag{6.18b}$$

for any vector field $X \in \mathrm{T}\mathcal{M}_{\mathrm{hm}}$ on $\mathcal{M}$.

Proposition 6.1.4. *The Levi-Citiva connection is completely characterised by the equations*

$$\varphi^{\alpha(i)}{}_{\beta(j)} + \bar{\varphi}^{\beta(j)}{}_{\alpha(i)} = \chi^{\alpha(i)}{}_{\beta(j)} + \chi^{\beta(j)}{}_{\alpha(i)} = 0, \tag{6.19a}$$

$$\varphi^{\alpha(i)}{}_{\beta(j)} \wedge e^{\beta(j)} + \chi^{\alpha(i)}{}_{\beta(j)} \wedge \bar{e}^{\beta(j)} = \mathrm{d}e^{\alpha(i)}. \tag{6.19b}$$

Proof: The Levi-Civita connection of a metric is characterised by flatness of this metric, which translates to

$$\begin{aligned}\nabla_X g = \nabla_X\big(e^{\alpha(i)}\bar{e}^{\alpha(i)}\big) &= \big(\nabla_X e^{\alpha(i)}\big)\bar{e}^{\alpha(i)} + e^{\alpha(i)}\big(\nabla_X \bar{e}^{\alpha(i)}\big)\\ &= \big(\varphi^{\alpha(i)}{}_{\beta(j)}(X)e^{\beta(j)} + \chi^{\alpha(i)}{}_{\beta(j)}(X)\bar{e}^{\beta(j)}\big)\bar{e}^{\alpha(i)}\\ &\quad + e^{\alpha(i)}\big(\bar{\varphi}^{\alpha(i)}{}_{\beta(j)}(X)\bar{e}^{\beta(j)} + \bar{\chi}^{\alpha(i)}{}_{\beta(j)}(X)e^{\beta(j)}\big)\\ &= \big(\varphi^{\alpha(i)}{}_{\beta(j)} + \bar{\varphi}^{\beta(j)}{}_{\alpha(i)}\big)e^{\beta(j)}\bar{e}^{\alpha(i)}\\ &\quad + \chi^{\alpha(i)}{}_{\beta(j)}\bar{e}^{\alpha(i)}\bar{e}^{\beta(j)} + \bar{\chi}^{\alpha(i)}{}_{\beta(j)}e^{\alpha(i)}e^{\beta(j)} = 0\end{aligned} \tag{6.20}$$

for all $X \in \mathrm{T}\mathcal{M}_{\text{hm}}$. This equivalent to equation (6.19a). In addition to this, the Levi-Civita connection is required to be torsion free, or equivalently,

$$\begin{aligned}&\mathrm{d}e^{\alpha(i)}(X,Y) - \big((\nabla_X e^{\alpha(i)})(Y) - (\nabla_Y e^{\alpha(i)})(X)\big)\\ &\quad = \big(X(e^{\alpha(i)}(Y)) - Y(e^{\alpha(i)}(X)) - e^{\alpha(i)}([X,Y])\big)\\ &\qquad - \big(X(e^{\alpha(i)}(Y)) - e^{\alpha(i)}(\nabla_X Y) - Y(e^{\alpha(i)}(X)) + e^{\alpha(i)}(\nabla_Y X)\big)\\ &\quad = -e^{\alpha(i)}([X,Y]) + e^{\alpha(i)}(\nabla_X Y - \nabla_Y X) = e^{\alpha(i)}(T_\nabla(X,Y)) = 0,\end{aligned} \tag{6.21}$$

which gives us equation (6.19b). □

Theorem 6.1.5. *The matrices $\varphi^{\alpha(i)}{}_{\beta(j)}$ and $\chi^{\alpha(i)}{}_{\beta(j)}$ from equation (6.18) are given by*

$$\varphi^{\alpha(i)}{}_{\beta(j)} = -p^\alpha{}_\beta \delta^{(i)}_{(j)} - q^{(i)}{}_{(j)}\delta_{\alpha\beta} \qquad \text{and} \qquad \chi^{\alpha(i)}{}_{\beta(j)} = -t^{(i)}{}_{(j)}\epsilon_{\alpha\beta}. \tag{6.22}$$

Here $p^\alpha{}_\beta$ is a traceless anti-Hermitian $2{\times}2$-matrix of 1-forms given by $p^+{}_+ = -p^-{}_- = \tilde{v}$ and $p^+{}_- = -\bar{p}^-{}_+ = -u$, with $\tilde{v} = \frac{i}{2}\,\mathrm{Im}(v - (\bar{Z}NZ)^{-1}\bar{Z}^i N_{ij}\mathrm{d}Z^j)$. Of the two 1-form valued $(1+n){\times}(1+n)$ matrices, $q^{(i)}{}_{(j)}$ and $t^{(i)}{}_{(j)}$, q is anti-Hermitian and t is symmetric.

Proof: Through a lengthy calculation it can be shown that the exterior derivatives of the vielbein 1-forms $e^{\alpha(i)}$ are given by [2]

$$\mathrm{d}e^{\alpha(i)} = -p^\alpha{}_\beta \wedge e^{\beta(i)} - q^{(i)}{}_{(j)} \wedge e^{\alpha(j)} - t^{(i)}{}_{(j)}\epsilon_{\alpha\beta} \wedge \bar{e}^{\beta(j)} \tag{6.23}$$

for matrices p, q and t of the form described above. Since the matrices p^α_β and $q^{(i)}{}_{(j)}$ are both anti-Hermitian, and $t^{(i)}{}_{(j)}\epsilon_{\alpha\beta}$ is anti-symmetric, it is easily verified that equation (6.19) is satisfied. □

6.2 The quaternion-Kähler structure

Using the orthonormal frame $e^{\alpha(i)}$ we can define three almost complex structures, $J^{(1)}$, $J^{(2)}$ and $J^{(3)}$ on $\mathcal{M}_{\text{hm}}$. These can be defined by specifying $J^{(u)*}e^{\alpha(i)}$ for $\alpha = \pm$ and $i = 1, \ldots, h^{1,2}$ and demanding that $J^{(u)*}\bar{e}^{\alpha(i)} = \overline{J^{(u)*}e^{\alpha(i)}}$ (because J is real).

In this section we will often use the Pauli-matrices, which are the anti-Hermitian traceless matrices $\sigma^{(u)}$ for $u = 1, 2, 3$, given by

$$\sigma^{(1)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^{(2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^{(3)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{6.24}$$

The product of two Pauli-matrices is given by $\sigma^{(u)}\sigma^{(v)} = \delta_{(u)(v)} + i\,\epsilon_{(u)(v)(w)}\sigma^{(w)}$.

Lemma 6.2.1. *The endomorphisms $J^{(u)}\colon \mathrm{T}M \to \mathrm{T}M$ on the tangent bundle of M defined through*

$$J^{(u)*}e^{\alpha(i)} = i\,\sigma^{(u)}_{\alpha\beta}e^{\beta(i)} \tag{6.25}$$

define three almost complex structures for $u = 1, 2, 3$ that satisfy the quaternionic algebra, $J^{(u)}J^{(v)} = -\delta_{(u)(v)} + \epsilon_{(u)(v)(w)}J^{(w)}$. Each of these almost complex structures is compatible with the metric g_{hm}.

Proof: Using that $\sigma^{(u)}\sigma^{(v)} = \delta_{(u)(v)} + i\,\epsilon_{(u)(v)(w)}\sigma^{(w)}$, we find that

$$\begin{aligned}(J^{(u)}J^{(v)})^*e^{\alpha(i)} &= J^{(v)*}(J^{(u)*}e^{\alpha(i)}) = -\sigma^{(u)}_{\alpha\beta}\sigma^{(v)}_{\beta\gamma}e^{\gamma(i)}\\ &= -\delta_{(u)(v)}e^{\alpha(i)} - i\,\epsilon_{(v)(u)(w)}\sigma^{(w)}_{\alpha\beta}e^{\beta(i)}\\ &= \big(-\delta_{(u)(v)} + \epsilon_{(u)(v)(w)}J^{(w)*}\big)e^{\alpha(i)},\end{aligned} \tag{6.26}$$

so the three endomorphisms $J^{(u)}$ are almost complex structures and they satisfy the quaternionic algebra.

Together with the metric $g = e^{\alpha(i)}\bar{e}^{\alpha(i)}$, $J^{(u)}$ moreover defines an almost Hermitian structure for $u = 1, 2, 3$ since for any two vector fields X and Y on $\mathcal{M}_{\text{hm}}$,

$$\begin{aligned}g(J^{(u)}X, J^{(u)}Y) &= \big(e^{\alpha(i)}\bar{e}^{\alpha(i)}\big)(J^{(u)}X, J^{(u)}Y) = \big(J^{(u)*}e^{\alpha(i)}J^{(u)*}\bar{e}^{\alpha(i)}\big)(X,Y)\\ &= (i\,\sigma^{(u)}_{\alpha\beta}e^{\beta(i)}\overline{i\,\sigma^{(u)}_{\alpha\gamma}}\bar{e}^{\gamma(i)})(X,Y) = \big(\sigma^{(u)}_{\gamma\alpha}\sigma^{(u)}_{\alpha\beta}e^{\beta(i)}\bar{e}^{\gamma(i)}\big)(X,Y)\\ &= \big(e^{\alpha(i)}\bar{e}^{\alpha(i)}\big)(X,Y) = g(X,Y)\end{aligned} \tag{6.27}$$

where we have used that the Pauli-matrices are Hermitian, i.e. that $\bar{\sigma}^{(u)}_{\alpha\beta} = \sigma^{(u)}_{\beta\alpha}$. □

Since the almost complex structures $J^{(u)}$ are compatible with the metric g_{hm} on the hypermultiplet moduli space, we can define a set of fundamental forms $\omega^{(u)} = g_{\text{hm}} \circ (J^{(u)} \times \mathrm{id})$. These are given by

$$\begin{aligned}\omega^{(u)} &= g_{\text{hm}} \circ (J^{(u)} \times \mathrm{id}) = (e^{\alpha(i)}\bar{e}^{\alpha(i)}) \circ (J^{(u)} \times \mathrm{id})\\ &= \tfrac{1}{2}\big(J^{(u)*}e^{\alpha(i)}\big) \otimes \bar{e}^{\alpha(i)} + \tfrac{1}{2}\big(J^{(u)*}\bar{e}^{\alpha(i)}\big) \otimes e^{\alpha(i)}\\ &= \tfrac{1}{2}\Big(i\,\sigma^{(u)}_{\alpha\beta}e^{\beta(i)}\Big) \otimes \bar{e}^{\alpha(i)} + \tfrac{1}{2}\Big(\overline{i\,\sigma^{(u)}_{\alpha\beta}}\bar{e}^{\beta(i)}\Big) \otimes e^{\alpha(i)}\\ &= \tfrac{i}{2}\sigma^{(u)}_{\alpha\beta}\Big(e^{\beta(i)} \otimes \bar{e}^{\alpha(i)} - \bar{e}^{\alpha(i)} \otimes e^{\beta(i)}\Big) = -\tfrac{i}{2}\,\sigma^{(u)}_{\alpha\beta}\,\bar{e}^{\alpha(i)} \wedge e^{\beta(i)}.\end{aligned} \tag{6.28}$$

Proposition 6.2.2. *The fundamental forms $\omega^{(u)}$ satisfy $\nabla_Z\omega^{(u)} = \frac{i}{2}[\sigma^{(u)}, p(Z)]_{\alpha\beta}\bar{e}^{\alpha(i)} \wedge e^{\beta(i)}$ for $u = 1, 2, 3$, where p is the matrix from theorem 6.1.5.*

Proof: Equation (6.18a) and theorem 6.1.5 allow us to explicitly work out the covariant derivatives $\nabla_Z\omega^{(u)}$

$$\begin{aligned}\nabla_Z\omega^{(u)} &= -\tfrac{i}{2}\sigma^{(u)}_{\alpha\beta}\big(\nabla_Z(\bar{e}^{\alpha(i)} \wedge e^{\beta(i)})\big)\\ &= -\tfrac{i}{2}\sigma^{(u)}_{\alpha\beta}\big((\nabla_Z\bar{e}^{\alpha(i)}) \wedge e^{\beta(i)} + \bar{e}^{\alpha(i)} \wedge (\nabla_Z e^{\beta(i)})\big)\\ &= -\tfrac{i}{2}\sigma^{(u)}_{\alpha\beta}\big((\bar{\varphi}^{\alpha(i)}{}_{\gamma(j)}(Z)\,\bar{e}^{\gamma(j)} + \bar{\chi}^{\alpha(i)}{}_{\gamma(j)}(Z)\,e^{\gamma(j)}) \wedge e^{\beta(i)}\\ &\qquad + \bar{e}^{\alpha(i)} \wedge (\varphi^{\beta(i)}{}_{\gamma(j)}(Z)\,e^{\gamma(j)} + \chi^{\beta(i)}{}_{\gamma(j)}(Z)\,\bar{e}^{\gamma(j)})\big)\\ &= \tfrac{i}{2}\sigma^{(u)}_{\alpha\beta}\Big(\bar{p}^{\alpha}{}_{\gamma}(Z)\,\bar{e}^{\gamma(i)} \wedge e^{\beta(i)} + p^{\beta}{}_{\gamma}(Z)\,\bar{e}^{\alpha(i)} \wedge e^{\gamma(i)}\\ &\qquad + \bar{q}^{(i)}{}_{(j)}(Z)\,\bar{e}^{\alpha(j)} \wedge e^{\beta(i)} + q^{(i)}{}_{(j)}(Z)\,\bar{e}^{\alpha(i)} \wedge e^{\beta(j)}\\ &\qquad + \bar{t}^{(i)}{}_{(j)}(Z)\,\epsilon_{\alpha\gamma}e^{\gamma(j)} \wedge e^{\beta(i)} + t^{(i)}{}_{(j)}(Z)\,\epsilon_{\beta\gamma}\bar{e}^{\alpha(i)} \wedge \bar{e}^{\gamma(j)}\Big)\end{aligned} \tag{6.29}$$

We can use the fact that the matrix $t^{(i)}_{\ (j)}$ to rewrite

$$\begin{aligned}\sigma^{(u)}_{\alpha\beta}t^{(i)}_{\ (j)}(Z)\,\epsilon_{\beta\gamma}\bar{e}^{\alpha(i)}\wedge\bar{e}^{\gamma(j)} &= \tfrac{1}{2}\sigma^{(u)}_{\alpha\beta}t^{(i)}_{\ (j)}(Z)\,\epsilon_{\beta\gamma}\big(\bar{e}^{\alpha(i)}\wedge\bar{e}^{\gamma(j)}+\bar{e}^{\alpha(j)}\wedge\bar{e}^{\gamma(i)}\big)\\ &= \tfrac{1}{2}t^{(i)}_{\ (j)}(Z)\big(\sigma^{(u)}_{\alpha\beta}\,\epsilon_{\beta\gamma}-\sigma^{(u)}_{\gamma\beta}\,\epsilon_{\beta\alpha}\big)\bar{e}^{\alpha(i)}\wedge\bar{e}^{\gamma(j)},\end{aligned} \tag{6.30}$$

which vanishes because we can explicitly that $\sigma^{(u)}\epsilon$ is symmetric for $u=1,2,3$. This tells us that the last line from equation (6.29) vanishes. Similarly, it follows from the fact that $q^{(i)}_{\ (j)}$ is anti-hermitian that

$$\bar{q}^{(i)}_{\ (j)}\bar{e}^{\alpha(j)}\wedge e^{\beta(i)}+q^{(i)}_{\ (j)}\bar{e}^{\alpha(i)}\wedge e^{\beta(j)}=\big(\bar{q}^{(i)}_{\ (j)}+q^{(j)}_{\ (i)}\big)\,\bar{e}^{\alpha(j)}\wedge e^{\beta(i)}=0. \tag{6.31}$$

This leaves us with

$$\begin{aligned}\nabla_Z\omega^{(u)} &= \tfrac{i}{2}(\sigma^{(u)}_{\gamma\beta}\bar{p}^{\gamma}_{\ \alpha}(Z)+\sigma^{(u)}_{\alpha\gamma}p^{\gamma}_{\ \beta}(Z))\,\bar{e}^{\alpha(i)}\wedge e^{\beta(i)}\\ &= \tfrac{i}{2}(-p^{\alpha}_{\ \gamma}(Z)\sigma^{(u)}_{\gamma\beta}+\sigma^{(u)}_{\alpha\gamma}p^{\gamma}_{\ \beta}(Z))\,\bar{e}^{\alpha(i)}\wedge e^{\beta(i)}\\ &= \tfrac{i}{2}[\sigma^{(u)},p(Z)]_{\alpha\beta}\bar{e}^{\alpha(i)}\wedge e^{\beta(i)},\end{aligned} \tag{6.32}$$

which proves the theorem □

Lemma 6.2.3. *We have that $(\nabla_Z\omega^{(u)})(X,Y)=g_{\mathrm{hm}}((\nabla_Z J^{(u)})X,Y)$ for $u=1,2,3$ and any three vector fields X, Y and Z on $\mathcal{M}_{\mathrm{hm}}$.*

Proof: The covariant derivatives of these fundamental forms are given by

$$\begin{aligned}(\nabla_Z\omega)(X,Y) &= Z(\omega(X,Y))-\omega(\nabla_Z X,Y)-\omega(X,\nabla_Z Y)\\ &= Z(g(J\,X,Y))-g(J\,\nabla_Z X,Y)-\omega(J\,X,\nabla_Z Y)\\ &= g(\nabla_Z(J\,X),Y))-g(J\,(\nabla_Z X),Y)=g((\nabla_Z J)\,X),Y)),\end{aligned} \tag{6.33}$$

where we have used flatness of the metric with respect to its Levi-Civita connection. □

Before we can conclude that $J^{(u)}$ defines a quaternion-Kähler structure on the domain where the basis $e^{\alpha(i)}$ is defined, we still need to verify that the bundle generated by these almost complex structures is preserved by the connection. This turns out to be the case.

Corollary 6.2.4. *The almost complex structures $J^{(u)}$ for $u=1,2,3$ from lemma 6.2.1, together with the metric g_{hm}, define a quaternion-Kähler structure on $\mathcal{M}_{\mathrm{hm}}$.*

Proof: Since $\sigma^{(u)}$ is Hermitian and $p(Z)$ is anti-Hermitian for $u=1,2,3$ and any (real) vector field Z, we have that

$$\begin{aligned}[\sigma^{(u)},p(Z)]_{\alpha\beta} &= \sigma^{(u)}_{\alpha\gamma}p(Z)_{\gamma\beta}-p(Z)_{\alpha\gamma}\sigma^{(u)}_{\gamma\beta}\\ &= -\bar{\sigma}^{(u)}_{\gamma\alpha}\bar{p}(Z)_{\beta\gamma}+\bar{p}(Z)_{\gamma\alpha}\bar{\sigma}^{(u)}_{\beta\gamma}=[\bar{\sigma}^{(u)},\bar{p}(Z)]_{\beta\alpha},\end{aligned} \tag{6.34}$$

so $[\sigma^{(u)},p(Z)]$ is a Hermitian matrix. Since it is a commutator it is furthermore traceless and will therefore be a linear combination of Pauli matrices, $[\sigma^{(u)},p(Z)]=A^{(u)}_{\ (v)}\sigma^{(v)}$ for some coefficients $A^{(u)}_{\ (v)}$.

Proposition 6.2.2 now tells us that $\nabla_Z\omega^{(u)}=\frac{i}{2}A^{(v)}_{\ (u)}\sigma^{(v)}\bar{e}^{\alpha(i)}e^{\beta(i)}=A^{(v)}_{\ (u)}\omega^{(v)}$, so if we finally apply lemma 6.2.3 we see that $\nabla_Z J^{(u)}=A^{(v)}_{\ (u)}J^{(v)}$. We already know from lemma 6.2.1 that the almost complex structures $J^{(u)}$ satisfy the quaternionic algebra and that they are compatible with the metric g_{hm}, so it follows that $\mathcal{M}_{\mathrm{hm}}$ is quaternion-Kähler. □

Remark 6.2.5. *We can provide a more explicit description of the Quaternion-Kähler structure by working out the expressions in equation (6.25) for $u = 1, 2, 3$. This gives us*

$$J^{(1)*}e^{\pm(i)} = i\, e^{\mp(i)} \qquad J^{(2)*}e^{\pm(i)} = \mp\, e^{\mp(i)} \qquad J^{(3)*}e^{\pm(i)} = \pm i\, e^{\pm(i)} \tag{6.35}$$

and if we use theorem 6.1.5 to work out $\nabla_Z J^{(u)}$ we obtain

$$\nabla_Z J^{(1)} = \tilde{v}(Z)\, J^{(2)} + 2\,\mathrm{Re}(u)(Z)\, J^{(3)} \tag{6.36a}$$
$$\nabla_Z J^{(2)} = -\,2\,\mathrm{Im}(u)(Z)\, J^{(3)} - \tilde{v}(Z)\, J^{(1)} \tag{6.36b}$$
$$\nabla_Z J^{(3)} = -\,2\,\mathrm{Re}(u)(Z)\, J^{(1)} + 2\,\mathrm{Im}(u)(Z)\, J^{(2)}, \tag{6.36c}$$

where $\tilde{v} = \frac{1}{2}\,\mathrm{Im}\big(v - \frac{\bar{Z}^i N_{ij} \mathrm{d}Z^j}{\bar{Z}NZ}\big)$.

6.2.1 Intrinsic interpretation

The hypermultiplet moduli space metric consists of a number of orthogonal pieces that we have studied separately in sections 5.1 and 5.2.

$$g_{\mathrm{hm}} = \underbrace{\tfrac{\partial^2 K(X,\bar{X})}{\partial X^i \partial \bar{X}^j} \mathrm{d}X^i \mathrm{d}\bar{X}^j}_{\text{complex structure moduli}} + \underbrace{\tfrac{1}{2\psi} g^{\mathrm{W}}}_{\text{torus}} + \underbrace{\tfrac{1}{(2\psi)^2}\big(\mathrm{d}\psi^2 + \eta^2\big)}_{\text{circle+line}}. \tag{6.37}$$

To this decomposition of g_{hm} corresponds a decomposition of the tangent bundle into orthogonal pieces

$$\mathrm{T}_{(t,a^{(3)},\psi,\sigma)}\mathcal{M}_{\mathrm{hm}} \simeq \overbrace{\underbrace{\mathrm{T}_t\mathcal{M}_{\mathbb{C}}}_{\simeq H^{2,1}(\mathcal{Y})}}^{\text{base space}} \oplus \overbrace{\underbrace{\mathrm{T}_{a^{(3)}}\big(H^3(\mathcal{Y},\mathbb{R})\big)}_{\text{torus (twisted)}} \oplus \underbrace{\mathbb{R}\,\partial_\sigma \oplus \mathbb{R}\,\partial_\psi}_{\text{circle+line}}}^{\text{fibre}}, \tag{6.38}$$

where we say that the torus directions are "twisted" because they are determined by the contact bundle on the Heisenberg group and there is no way to locally be embedded the torus in $\mathcal{M}_{\mathrm{hm}}$ such a way that it is tangent to this bundle since contact bundle are non-integrable (cf. remark 4.2.4), . We had seen that the circle and the real line parametrised by σ and ψ combine into a bundle of complex disks over the Weil intermediate Jacobian.

By using the Dolbeault cohomology groups, the tangent space $H^3(\mathcal{Y},\mathbb{R})$ to the torus can be further decomposed into the pieces $\big(H^{3,0}(\mathcal{Y}) \oplus H^{0,3}(\mathcal{Y})\big) \cap H^3(\mathcal{Y},\mathbb{R})$ and $\big(H^{2,1}(\mathcal{Y}) \oplus H^{1,2}(\mathcal{Y})\big) \cap H^3(\mathcal{Y},\mathbb{R})$, which are also orthogonal (cf. lemma 2.3.5). All-in-all, there are four even-dimensional pieces, each of which comes with its own complex structure: The complex structure moduli space, these two orthogonal parts of the Weil intermediate Jacobian and the aforementioned punctured disks.

We know that both the base space and the fibres are Kähler manifolds and hence that both come with a complex structure that we denote by $J_{\mathbb{C}}$ and $\tilde{J}$ respectively (cf. corollary 2.2.8 and theorem 5.2.7).

The vielbein 1-forms $e^{(a)}$, $E^{(a)}$, u and v describe exactly these same pieces, as we had seen in corollary 6.1.3 and proposition 6.1.2. The forms $e^{(a)} = P^{(a)}_b \mathrm{d}Z^b$ are clearly $(1,0)$-forms on the base space with respect to its standard complex structure and each of the forms $E^{(a)}$, u and v are $(0,1)$-forms for the complex structure $\tilde{J}$ on the fibres. This is true for v because $v = \frac{1}{2\psi}(\mathrm{d}\psi - i\,\eta)$ and $\tilde{J}^*\eta = \mathrm{d}\psi$ and for $E^{(a)}$ and u because they are linear combinations of the forms $(\mathrm{d}B_i - \overline{Z}^{\mathrm{W}}_{ij}\mathrm{d}A^j)$ (recall that $z_i = -B_i + Z^{\mathrm{W}}_{ij}A^j$ were complex coordinates on the Weil intermediate Jacobian and hence on $\mathcal{M}_t$, cf. proposition 5.2.9).

Lemma 6.2.6. *Let $J^{(0)}\colon \mathrm{T}\mathcal{M}_{\mathrm{hm}} \to \mathrm{T}\mathcal{M}_{\mathrm{hm}}$ be the almost complex structure on $\mathrm{T}\mathcal{M}_{\mathrm{hm}}$ that equals $J_{\mathbb{C}}$ on the base space $\mathcal{M}_{\mathbb{C}}$ and $-\tilde{J}$ on the fibres $\mathcal{M}_t$. The vielbein 1-forms $e^{+(i)} = (u, e^{(a)})$ and $e^{-(i)} = (v, E^{(a)})$ from equation (6.7) satisfy*

$$J^{(0)*} e^{\alpha(i)} = +i\, e^{\alpha(i)}. \tag{6.39}$$

for $\alpha = \pm$ and $i = 0, \ldots, h^{1,2}$

Proof: The validity of this statement has been discussed above this lemma. □

We also have an explicit descriptions for the almost complex structures $J^{(u)}$ $(u = 1, 2, 3)$ that make up the quaternion-Kähler structure on the total space in terms of these vielbeins. By combining the decomposition (6.38) we can now give a slightly more intrinsic description of the quaternion-Kähler structure on $\mathcal{M}_{\mathrm{hm}}$.

Theorem 6.2.7. *The almost complex structure $J^{(u)}$ described in lemma 6.2.1 and remark 6.2.5 satisfy the following properties.*

- *$J^{(3)}$ respects the decomposition from equation (6.38). It acts on $\mathrm{T}_t\mathcal{M}_{\mathbb{C}}$ through the canonical complex structures on the complex structure moduli space and on $\mathbb{R}\,\partial_\sigma \oplus \mathbb{R}\,\partial_\psi$ as $\tilde{J}$. On the torus directions it acts as the complex structure corresponding to the* Griffiths *intermediate Jacobian instead of the Weil intermediate Jacobian.*
- *$J^{(1)}$ and $J^{(2)}$ interchange vectors along the $H^{1,2} \oplus H^{2,1}$-directions of the torus with tangent vectors for $\mathcal{M}_{\mathbb{C}}$ and vectors along the $H^{3,0} \oplus H^{0,3}$-directions with elements of $\mathbb{R}\,\partial_\sigma \oplus \mathbb{R}\,\partial_\psi$.*

Proof: The almost complex structures $J^{(u)}$ were defined in terms of the forms $e^{\alpha(i)}$. The third almost complex structure was given by (cf. remark 6.2.5)

$$J^{(3)*}u = i\,u, \qquad J^{(3)*}e^{(a)} = i\,e^{(a)}, \qquad J^{(3)*}v = -i\,v, \qquad J^{(3)*}E^{(a)} = -i\,E^{(a)}. \tag{6.40}$$

We see that it equals $J^{(0)}$ on $\mathrm{T}\mathcal{M}_{\mathbb{C}}$ and on the torus directions corresponding to the $H^{3,0} \oplus H^{0,3}$-part of the torus, while it describes $-J^{(0)}$ on the remaining parts of the decomposition (6.38). We see that the $H^{2,1} \oplus H^{1,2}$-part and the $H^{3,0} \oplus H^{0,3}$-part of the torus directions are treated differently: $J^{(3)}$ corresponds to $J^{\mathrm{W}} = J^{\mathrm{G}}$ on the first and to $-J^{\mathrm{W}} = J^{\mathrm{G}}$ on the latter. It appears that on the torus $J^{(3)}$ should be interpreted as the almost complex structure corresponding to the Griffiths intermediate Jacobian rather than the Weil intermediate Jacobian.

The other two almost complex structures are given by

$$J^{(1)*}u = i\,v, \qquad J^{(1)*}e^{(a)} = i\,E^{(a)}, \qquad J^{(1)*}v = i\,u, \qquad J^{(1)*}E^{(a)} = i\,e^{(a)}, \tag{6.41}$$

$$J^{(2)*}u = -v, \qquad J^{(2)*}e^{(a)} = -E^{(a)}, \qquad J^{(2)*}v = +u, \qquad J^{(2)*}E^{(a)} = +e^{(a)}, \tag{6.42}$$

from which we can immediately read off that they both interchange the $H^{3,0} \oplus H^{0,3}$-part and the $H^{2,1} \oplus H^{1,2}$-part of the torus with $\mathrm{T}\mathcal{M}_{\mathbb{C}}$ and $\mathbb{R}\,\partial_\sigma \oplus \mathbb{R}\,\partial_\psi$ respectively. □

DISCUSSION

We were interested in the (known) quaternion-Kähler metric on the hypermultiplet moduli space of the effective supergravity theory that arises from type IIA superstring theory in the low energy limit after compactification on a Calabi-Yau 3-fold. By finding a more intrinsic description for this space and its metric we ultimately hope to gain a better understanding of the quaternionic nature of this space.

The hypermultiplet moduli space can be described as a fibre bundle over the moduli space of the Calabi-Yau manifold used in the compactification procedure, with fibres consisting of a torus with a circle and a line bundle corresponding to the (Kalb-Ramond) axion and the dilaton respectively. This fibre bundle is flat and for the metric we wish to describe base space and fibres are orthogonal. Since we already have a description of the metric on the complex structure moduli space (cf. section 2.2.2), we have initially mostly focused on the fibres. A previous study [1] has shown that the torus in this construction should be viewed as the Weil intermediate Jacobian of the Calabi-Yau manifold and that the entire fibre can be viewed as a coset space of an extended version of a Heisenberg group (c.f. remark 5.1.6).

Instead of considering this extended version of the Heisenberg group, we have looked at these fibres as a direct product of a compact quotient of an (unextended) Heisenberg group and a real line that corresponds to the dilaton field. This compact quotient can naturally be interpreted as a contact manifold because it inherits a contact structure from the Heisenberg group. By combining this contact structure with the complex structure on the Weil intermediate Jacobian, we can view this space as a strictly pseudoconvex Cauchy-Riemann manifold. This Cauchy-Riemann structure underlies a 1-dimension family of (Sasakian) contact metric structures, which describe the metric on the fibre metric for fixed values of the dilaton field. The entire fibre metric is obtained as a Kähler metric on the symplectisation of this contact manifold obtained by choosing a specific extension of the Cauchy-Riemann structure to a complex structure (cf. section 5.2).

Since the dilaton and the axion combine into a $\mathbb{C}^*$-bundle over the Weil intermediate Jacobian, we were interested in the possibility that they may in fact form (part of) a holomorphic line bundle and in particular one that corresponds to the canonical polarisation of this torus. We discovered that the dilaton and the axion can together be interpreted as a bundle of punctured disks inside exactly such a bundle (cf. section 5.2.5).

The tangent bundle of the complete hypermultiplet moduli space splits into directions corresponding to the complex structure moduli space, the torus and these complex disks. With this in mind we have examined the explicit quaternion-Kähler structure found by Ferrara and Sabharwal [2] and found that it exhibits some interesting behaviour (cf. section 6.2). One of the almost complex structures that make up the quaternion-Kähler structure acts separately on each of the components from the above construction. Interestingly enough, it seems to correspond to the complex structure of the Griffiths intermediate Jacobian on the torus, while it we had just seen that it is the Weil intermediate Jacobian that plays a big role in the construction of the quaternion-Kähler metric.

Since quantum corrections still have to be taken into account and we know that also the corrected expressions will be quaternion-Kähler, an important question that is what freedom we have to deform this structure. It may be promising to first look at the one-loop corrected version of the quaternion-Kähler metric that is presented in [56], in which we also recognise a contact form that can be used in combination with a well-chosen complex structure to express large part of the metric quaternion-Kähler metric.

Although we have made a start, a fully intrinsic description of this quaternion-Kähler structure in terms of the construction described above has not yet been obtained. The complex structure moduli space and the fibres over it remain for a large part two separate and unrelated pieces, so we would like to learn more about how the structures on each can be related and why the combination of the two results in a quaternion-Kähler manifold. The interpretation we have given to the hypermultiplet moduli space was completely in terms of spaces defined using a Calabi-Yau 3-fold $\mathcal{Y}$, but the entire quaternion-Kähler metric could be expressed in terms of just the projective special Kähler structure of the complex structure moduli space. By finding a natural way to combine the Heisenberg group and this moduli space we may also learn something about the more general situation.

The fibres over the complex structure moduli space are complex manifolds, which have a strictly pseudoconvex boundary at infinity ($\psi = e^{\phi} \to \infty$) that corresponds to the (quotient of the) Heisenberg group. The expression we have worked with is only valid in the limit towards this boundary, so quantum corrections should only have an effect outside this boundary. It may be interesting to note that this situation is very similar to that of so-called asymptotically complex hyperbolic manifolds, which are complex manifolds that have exactly the same asymptotic behaviour towards their boundary [71,72]. Moreover, the boundary of such spaces comes with a strictly pseudoconvex CR structure and can has been linked with the Heisenberg group [73]. To which extent this may be relevant or helpful remains to be seen.

Although it is possibly just a coincidence, the fact that the twistor space for the quaternion-Kähler structure on the hypermultiplet moduli space is a complex contact manifold of exactly twice the dimension of the Heisenberg group, which we had interpreted as a real contact manifold, can at least be said to be interesting.

In short, there are still many unanswered questions and unsolved problems and thus a lot of opportunities for future research.

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Bibliography

[1] T. van der Aalst, *A geometric interpretation of the c-map*, Master's thesis, Utrecht University, 2008, Available from: http://www1.phys.uu.nl/wwwitf/teaching/2008/TvdAalst.pdf.

[2] S. Ferrara and S. Sabharwal, *Quaternionic manifolds for type II superstring vacua of Calabi-Yau spaces*, Nuclear Physics B **332**, 317 (1990).

[3] G. E. Bredon, *Topology and Geometry*, Springer-Verslag, 1993.

[4] A. Hatcher, *Algebraic Topology*, Cambridge University Press, 2002, Available from: http://www.math.cornell.edu/~hatcher/AT/AT.pdf.

[5] A. L. Besse, *Einstein manifolds*, Springer-Verslag, 1987, Available from: http://books.google.com/books?id=6I_XgRJaBL0C.

[6] C. Voisin, *Hodge theory and complex algebraic geometry I*, Cambridge University Press, 2002.

[7] P. Griffiths and J. Harris, *Principles of algebraic geometry*, John Wiley & Sons, Inc., 1978.

[8] C. Birkenhake and H. Lange, *Complex tori*, Birkhäuser, Boston, 1999, Available from: http://books.google.com/books?id=tKdbFhbEpuAC.

[9] C. Birkenhake and H. Lange, *Complex Abelian Varieties*, Springer-Verslag, 2004, Available from: http://books.google.com/books?id=MOW2gEP7HIkC.

[10] F. W. Warner, *Foundations of differentiable manifolds and Lie groups*, Springer-Verslag, 1983, Available from: http://books.google.com/books?id=iaeUqc2yQVQC.

[11] W. Ballmann, *Lectures on K̈hler Manifolds*, ESI lectures in mathematics and physics, European Mathematical Society, 2006, Available from: http://books.google.com/books?id=e9mp7QXSx08C.

[12] B. de Wit and A. van Proeven, *Potentials and Symmetries of General Gauged $N = 2$ Supergravity*, Nuclear Physics B **245**, 89 (1984).

[13] G. Sierra and P. K. Townsend, *An Introduction to $N = 2$ Rigid Supersymmetry*, p. 396, World Scientific, 1983, in: Supersymmetry and Supergravity.

[14] A. Strominger, *Special Geometry*, Commun. Math. Phys. **133**, 163 (1990).

[15] R. L. Bryant and P. A. Griffiths, *Some Observations on the Infinitesimal Period Relations for Regular Threefolds with Trivial Canonical Bundle*, Arithmetic and Geometry, Progress in Mathematics **2**, 77 (1983).

[16] P. Candelas and X. C. de la Ossa, *Moduli Space of Calabi-Yau Manifolds*, Nuclear Physics B **355**, 455 (1990).

[17] D. S. Freed, *Special Kähler Manifolds*, Communications in Mathematical Physics **203**, 31 (1999).

[18] D. Alekseevsky, V. Cortés, and C. Devchand, *Special complex manifolds*, Journal of Geometry and Physics **42**, 85 (2001).

[19] P. FRÉ, *Lectures on special Kähler geometry and electric-magnetic duality rotations*, Nuclear Physics B -Proceedings Supplements **45**, 59 (1996).

[20] S. SALAMON, *Quaternionic Kähler Manifolds*, Inventiones Mathematicae **67**, 143 (1982).

[21] S. ISHIHARA, *Quaternion Kählerian Manifolds and Fibred Riemannian Spaces with Sasakian 3-structure*, Kodai Math. Sem. Rep. **25**, 321 (1973), Available from: http://projecteuclid.org/euclid.kmj/1138846820.

[22] D. ALEKSEEVSKII, *Riemannian Spaces with Exceptional Holonomy Groups*, Funktional Anal. i Priložen **2**, 1 (1968).

[23] M. BERGER, *Remarques sur le Groupe d'Holonomie des Variétés Riemanniennes*, C. R. Acad. Sci. Paris **262**, 1316 (1966).

[24] D. E. BLAIR, *Riemannian Geometry of Contact and Symplectic Manifolds*, Progress in Mathematics, Birkhäuser, 2002, Available from: http://books.google.com/books?id=TkXiK2Ea2XQC.

[25] C. VOISIN, *Mirror Symmetry*, AMS Bookstore, 1999, Available from: http://books.google.com/books?id=NyYE4C_dT4cC.

[26] M. W. GROSS, D. HUYBRECHTS, and D. D. JOYCE, *Calabi-Yau Manifolds and Related Geometries*, Springer-Verslag, 2003, Available from: http://books.google.com/books?id=7mtvbcxUqFEC.

[27] D. D. JOYCE, *Compact Manifolds with Special Holonomy*, Oxford University Press, 2000, Available from: http://books.google.com/books?id=c3P-YUD8GZQC.

[28] T. HÜBSCH, *Calabi-Yau Manifolds*, World Scientific, 1994, Available from: http://books.google.com/books?id=Z5zSbFktn1EC.

[29] K. HORI, *Mirror Symmetry*, AMS Bookstore, 2003, Available from: http://books.google.com/books?id=uGYRaAhFGx0C.

[30] B. GREENE, *String theory on Calabi-Yau manifolds*, 1997, arXiv:hep-th/9702155v1.

[31] E. CALABI, *The Space of Kähler Metrics*, Proc. Int. Congr. Math. Amsterdam **2**, 206 (1954).

[32] S.-T. YAU, *Calabi's Conjecture and some New Results in Algebraic Geometry*, Proceedings of the National Academy of Science **74**, 1798 (1977).

[33] S.-T. YAU, *On Ricci Curvature of a Compact Kähler Manifold and Complex Monge-Ampère Equation*, Comm. Pure and App. Math. **31** (1979).

[34] K. UENO and Y. SHIMIZU, *Advances in Moduli Theory*, AMS Bookstore, 2002, Available from: http://books.google.com/books?id=N7zc8cLP_aoC.

[35] G. TIAN, Smoothness of the Universal Deformation Space of Compact Calabi-Yau Manifolds and its Petersson-Weyl Metric, in *Mathematical Aspects of String Theory*, edited by S.-T. YAU, pp. 629–646, World Scientific, 1997.

[36] A. N. TODOROV, *The Weil-Petersson Geometry of the Moduli Space of* $\mathrm{SU}(n \geq 3)$ *(Calabi-Yau) Manifolds I*, Commun. Math. Phys **126**, 325 (1989).

[37] P. CANDELAS, *Yukawa Coupling between* $(2,1)$*-forms*, Nuclear Physics B **298**, 458 (1988).

[38] M. B. GREEN, J. H. SCHWARZ, and E. WITTEN, *Superstring theory*, volume 1 & 2, Cambridge University Press, 1987.

[39] J. POLCHINSKI, *String Theory*, volume 1-2, Cambridge University Press, 1998.

[40] K. BECKER, M. BECKER, and J. H. SCHWARZ, *String Theory and M-theory*, Cambridge University Press, 2007.

[41] G. 'T HOOFT, *Introduction to String Theory*, Lecture notes, 2004, Available from: http://www.phys.uu.nl/thooft/lectures/string.html.

[42] C. V. JOHNSON, *D-Branes*, Cambridge Monographs on Mathematical Physics, Cambridge University Press, 2006, Available from: http://www.google.com/books?id=pnClL_tua_wC.

[43] P. CANDELAS, G. T. HOROWITZ, A. STROMINGER, and E. WITTEN, *Vacuum Configurations for Superstrings*, Nuclear Physics B **258**, 46 (1985).

[44] M. BODNER, A. C. CADAVID, and S. FERRARA, $(2, 2)$ *Vacuum Configurations for Type IIA Superstrings: $N = 2$ Supergravity Lagrangians and Algebraic Geometry*, Class. Quantum Grav. **8**, 789 (1991).

[45] P. FAYET, *Fermi-Bose Hypersymmetry*, Nuclear Physics B **1**, 135 (1976).

[46] B. DE WIT, P. G. LAUWERS, and A. VAN PROEVEN, *Lagrangians of $N = 2$ Supergravity - Matter Systems*, Nuclear Physics B **255**, 569 (1985).

[47] P. S. ASPINWALL, *Compactification, Geometry and Duality: $N = 2$*, 2000, arXiv:hep-th/0001001v2.

[48] J. BAGGER and E. WITTEN, *Matter Couplings in $N = 2$ Supergravity*, Nuclear Physics B **222**, 1 (1983).

[49] S. CECOTTI, S. FERRARA, and L. GIRARDELLO, *Geometry of Type II Superstrings and the Moduli of Superconformal Field Theories*, International Journal of Modern Physics A **4**, 2475 (1989).

[50] R. BÖHM, H. GÜNTHER, C. HERRMANN, and J. LOUIS, *Compactifications of Type IIB Theory on Calabi-Yau Threefolds*, Nuclear Physics B **569**, 229 (2000).

[51] R. D. PECCEI and H. R. QUINN, *Constraints Imposed by CP Conservation in the Presence of Instanton*, Phys. Rev. D **16** (1977).

[52] R. D. PECCEI and H. R. QUINN, *CP Conservation in the Presence of Instantons*, Phys. Rev. Lett. **38**, 1440 (1977).

[53] B. DE WIT, F. VANDERSEYPEN, and A. VAN PROEVEN, *Symmetry structure of special geometries*, Nuclear Physics B **400**, 463 (1993), arXiv:hep-th/9210068.

[54] S. ALEXANDROV, B. PIOLINE, F. SAUERESSIG, and S. VANDOREN, *D-instantons and Twistors*, JHEP **03**, 044 (2009), arXiv:0812.4219.

[55] D. ROBLES-LLANA, F. SAUERESSIG, and S. VANDOREN, *String Loop Corrected Hypermultiplet Moduli Spaces*, JHEP **03**, 081 (2006), arXiv:hep-th/0602164.

[56] S. ALEXANDROV, *Quantum Covariant c-map*, JHEP **05**, 093 (2007), arXiv:hep-th/0702203.

[57] K. BECKER and M. BECKER, *Instanton Action for Type II Hypermultiplets*, Nuclear Physics B **551**, 102 (1999).

[58] H. GEIGES, *A Brief History of Contact Geometry and Topology*, Expositiones Mathematicae **19**, 25 (2001).

[59] F. BOURGEOIS, *Introduction to Contact Homology*, Lecture notes, 2003, Available from: http://homepages.ulb.ac.be/~fbourgeo/pspdf/Berder.pdf.

[60] J. ETNYRE, *Contact Manifolds*, article for the Encyclopedia of Mathematical Physics, 2002, Available from: http://people.math.gatech.edu/~etnyre/preprints/phys.html.

[61] L. ORNEA and M. VERBITSKY, *Sasakian Structures on CR Manifolds*, Geom Dedicata **125**, 150 (2007).

[62] A. C. DE SILVA, *Lectures on symplectic geometry*, Springer-Verslag, 2001, Available from: http://books.google.com/books?id=r9i6pXc7GEQC.

[63] V. APOSTOLOV, *Perspectives in Riemannian Geometry*, AMS Bookstore, 2006, Available from: http://books.google.nl/books?id=Yy5WKQcQYpsC.

[64] G. MARINESCU and N. YEGANEFAR, *Embeddability of some strongly pseudoconvex CR manifolds*, Trans. Amer. Math. Soc. **359**, 4757 (2007), arXiv:math/0403044v1.

[65] R. S. PONGE, *Heisenberg Calculus and Spectral Theory of Hypoelliptic Operators on Heisenberg Manifolds*, Number 906 in Memoirs of the American Mathematical Society, American Mathematical Society, 2008.

[66] E. P. VAN DEN BAN, *Lie Groups*, Lecture notes, 2003, Available from: http://www.math.uu.nl/people/ban/lecnotes/lie2005.pdf.

[67] W. M. GOLDMAN, *Complex Hyperbolic Geometry*, Oxford University Press, 1999, Available from: http://books.google.com/books?id=wilE1U0Fh9IC.

[68] C. P. BOYER, *The Sasakian Geometry of the Heisenberg Group*, Bull. Math. Soc. Sci. Math. Roumanie **Tome 52 (100)**, 251 (2009), arXiv:0904.1406.

[69] S. V. KETOV, *Universal Hypermultiplet Metrics*, Nuclear Physics B **604**, 256 (2001).

[70] L. ANGUELOVA, M. ROČEK, and S. VANDOREN, *Quantum Corrections to the Universal Hypermultiplet and Superspace*, Phys. Rev. D **70**, 066001 (2004), arXiv:hep-th/0402132.

[71] V. MINERBE and D. MAERTEN, *A Mass for Asymptotically Complex Hyperbolic Manifolds*, To be published, 2009, arXiv:0911.0306.

[72] O. BIQUARD and M. HERZLICH, *A Burns-Epstein Invariant for ACHE 4-manifolds*, Duke Math. J. **126**, 53 (2005).

[73] L. CAPOGNA, D. DANIELLI, S. D. PAULS, and J. T. TYSON, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, volume 259 of *Progress in Mathematics*, Springer-Verslag, 2007, Available from: http://www.google.com/books?id=RUqqvNiCBHIC.