Asymptotics of Integral Points of Bounded Height on a log Fano Variety

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Abstract

In this thesis we study integral points of bounded height on three log Fano threefolds, following the paper *Integral Points of Bounded Height on a log Fano Threefold* by Florian Wilsch. We parametrize the integral points on the log Fano threefolds using the universal torsor method and obtain lattice points satisfying certain (coprimality) conditions. With the height function induced by log-anticanonical bundles on the threefolds, we bound the integral points, leading to three counting functions. To obtain asymptotic formulae for two of the counting functions, we apply Möbius inversion and we replace sums by integrals. We show that this method cannot be extended in a straightforward way to the third counting function and instead we determine an upper bound.

Contents

1 Introduction

For a Fano variety X over a number field k such that the set of k-rational-points is Zarisky dense in X, and an anticanonical height function $H : X(k) \to \mathbb{R}_{\geq 0}$, Manin's conjecture predicts the asymptotic behaviour of the number of rational points of bounded height that lie in an open subvariety V of X [10]. To be precise, it predicts the asymptotic formula

$$\#\{x \in V(k) \;:\; H(x) \leq B\} \sim cB\log(B)^{r-1},$$

where r is the Picard number of X and c is a positive constant. This conjecture has been proven for specific classes of Fano varieties (see for example [2] for toric varieties), but remains open for many Fano threefolds.

An important tool in studying k-rational points on Fano varieties is the universal torsor method; for example, Salberger used this method to reprove Manin's conjecture for toric varieties [19]. This method introduces a second variety, the torsor, to parametrize the k-rational points of the Fano variety. The $\mathcal{O}_k$-rational points on this torsor are lattice-points satisfying certain coprimality conditions. The height function can be lifted to this torsor and one obtains an expression for $\mathcal{O}_k$-rational points on the torsor in terms of $\mathcal{O}_k$-tuples that satisfy certain equations and an inequality coming from the height function.

In this thesis, we study integral points as opposed to k-rational points for k a number field, and we use the universal torsor method. There is no conjecture like Manin's conjecture for such points, so we study specific examples, hoping to build a foundation for an analogous conjecture.

In particular, we study the paper *Integral Points of Bounded Height on a log Fano Threefold* by Florian Wilsch [24]. In this paper, the author considers the blow-up of $\mathbb{P}^3_{\mathbb{Q}} = \operatorname{Proj}(\mathbb{Q}[a,b,c,d])$ along the smooth conic $C' = V(a^2 + bc, d)$, denoted

$$\pi' : X' \to \mathbb{P}^3_{\mathbb{Q}},$$

and the two open subsets of X' given by $U_1' = X' \setminus \pi'^{-1}(V(b))$ and $U_2' = X' \setminus \pi'^{-1}(V(a))$. Florian Wilsch constructs integral models of X', U_1' and U_2', denoted $\mathcal{X}', \mathcal{U}_1'$ and $\mathcal{U}_2'$, respectively, and a log-anticanonical height function $H' : X(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$.

Theorem 1.1. [24, Theorem 1.1] Let X', U_1', U_2', $\mathcal{X}', \mathcal{U}_1', \mathcal{U}_2'$ and H' be as above. Then there exist open subvarieties $V_1, V_2 \subset X'$ such that the counting functions $N_1(B) = \#\{x \in \mathcal{U}_1'(\mathbb{Z}) \cap V_1(\mathbb{Q}) \;:\; H'(x) \leq B\}$ and $N_2(B) = \#\{x \in \mathcal{U}_2'(\mathbb{Z}) \cap V_2(\mathbb{Q}) \;:\; H'(x) \leq B\}$ satisfy the asymptotic formulae

$$N_1(B) = \frac{20}{3\zeta(2)} B\log(B) + O(B),$$

$$N_2(B) = \frac{20}{3}\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) B\log(B) + O(B\log\log(B)^2).$$

We work through the proofs given in [24] and add many details to the proof of the counting function $N_1(B)$, and summarize the steps taken for the counting function $N_2(B)$.

Additionally, we consider integral points of bounded height on the blow-up of $\mathbb{P}^3_{\mathbb{Q}}$ along the smooth conic $C = V(a^2 + b^2 + c^2, d)$, denoted

$$\pi : X \to \mathbb{P}^3_{\mathbb{Q}},$$

and see if the method Florian Wilsch applies in Section 5 of [24] extends in a straightforward way to this threefold. We remark that the conics C and C' are not isomorphic over $\mathbb{Q}$ as $C(\mathbb{Q}) = \varnothing$ while $(1:1:-1:0) \in C'(\mathbb{Q})$, so any results obtained are new.

Analogous to Florian Wilsch in [24], we consider the open subset of X given by $U = X \setminus \pi^{-1}(V(a))$ and construct integral models of X and U, denoted $\mathcal{X}$ and $\mathcal{U}$, respectively. We also construct a log-anticanonical height function $H : X(\mathbb{Q}) \to \mathbb{R}_{\geq 0}$, which gives rise to the counting function

$$N(B) = \#\{x \in \mathcal{U}(\mathbb{Z}) \cap V(\mathbb{Q}) \; : \; H(x) \leq B\}$$

for an open subvariety $V \subset X$. This counting function gives rise to the following two results:

(1) The method applied in section 5 of [24] does not extend to the counting problem $N(B)$ in a straightforward way;

(2) We have the upper bound $N(B) \ll_\epsilon B^{7/6+\epsilon}$.

Outline of the thesis

In the first chapter, we introduce all the theory needed to apply the universal torsor method to the specific schemes we consider in this thesis. Each subsection outlines a specific topic and we introduce examples whenever possible.

The second chapter concerns the parametrization of the integral points on both blow-ups X and X'. We first determine the Cox rings of both schemes over $\overline{\mathbb{Q}}$ and then determine an explicit expression of integral points on the blow-ups as lattice points satisfying certain conditions, using torsors and their properties. We also define explicit height functions for the integral points on both blow-ups, giving rise to the three counting functions $N_1(B)$, $N_2(B)$ and $N(B)$.

We study these counting functions in Chapter 3. We first work through the proofs Florian Wilsch gives for the asymptotic formula of $N_1(B)$, adding details in many places, and we then summarize the steps taken for the asymptotic formula of $N_2(B)$ in [24]. We show that $N(B)$ can be written in terms of an arithmetic function $\theta_1(b, x, z)$, which is not multiplicative, proving that the method from Florian Wilsch in Section 5 of [24] does not extend in a straightforward way to this counting function. We end the chapter with an upper bound for $N(B)$.

Notation

Throughout this thesis, a k-variety is an integral scheme of finite type over the base field k. For any integer $n \in \mathbb{Z}_{>0}$ and any prime number p, denote by $v_p(n)$ the p-adic order of n; it is defined as $v_p(n) = \max\{k \in \mathbb{Z}_{\geq 0} \ : \ p^k \mid n\}$. We write $f(x) = g(x) + O(e(x))$ as $x \to \infty$ if there are constants $x_0 > 0$, $C > 0$ such that $|f(x) - g(x)| \leq Ce(x)$ for all $x \geq x_0$. We write $f(x) \ll g(x)$ as $x \to \infty$ if $f(x) = O(g(x))$ as $x \to \infty$. For any $\epsilon > 0$, we write $f(x) = g(x) + O_\epsilon(e(x,\epsilon))$ and $f(x) \ll_\epsilon g(x,\epsilon)$ if the implicit constants depend on ϵ. We denote by i the imaginary unit in $\overline{\mathbb{Q}}$.

2 Prerequisites

In this chapter, we introduce all the necessary notions to study integral points using the universal torsor method. Each section introduces a new notion, with examples where possible and proofs of statements when they add clarity. We first introduce the schemes that we are studying and then move towards Cox rings, torsors and integral models. We end this part by introducing the Weil height function and some notions from analytic number theory.

2.1 The Proj-construction

An important construction in scheme theory is the Proj-construction. We follow the exposition in Section II.2 of [13] to introduce the construction for graded rings and then extend this construction to sheaves of graded algebras, following Section II.7 of [13]. We also state a few relevant properties.

Throughout, let S be a graded ring with respect to $\mathbb{Z}_{\geq 0}$ and denote by S_d the d-graded part of S for any $d \in \mathbb{Z}_{\geq 0}$. Then denote by S_+ the ideal $\bigoplus_{d>0} S_d \subset S$. To define the scheme $\mathrm{Proj}(S)$, we first introduce the set $\mathrm{Proj}(S)$ and then impose a topology on this set. We also define the sheaf of rings on $\mathrm{Proj}(S)$.

Definition 2.1. Define the **set** $\mathrm{Proj}(S)$ to be the set of all homogeneous prime ideals $\mathfrak{p}$ that do not contain all of S_+.

For homogeneous ideals $\mathfrak{a} \subset S$, define the subset

$$V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Proj}(S) \; : \; \mathfrak{a} \subseteq \mathfrak{p}\}.$$

It is a quick verification that these subsets satisfy the necessary conditions for being the closed subsets of a topology ([13, Lemma II.2.4]), so we impose the topology on $\mathrm{Proj}(S)$ given by taking such subsets $V(\mathfrak{a})$ as closed subsets.

One must also define the sheaf of rings on $\mathrm{Proj}(S)$. For this, consider an element $\mathfrak{p} \in \mathrm{Proj}(S)$ and let T be the set of all homogeneous elements of S that are not in $\mathfrak{p}$. This set T is multiplicative, so consider the localization $T^{-1}S$. Let $S_{(\mathfrak{p})}$ be the ring of elements of degree zero in the ring $T^{-1}S$. Then for any open subset $U \subseteq \mathrm{Proj}(S)$, define $\mathcal{O}(U)$ to be the set of functions $s : U \to \bigsqcup S_{(\mathfrak{p})}$ such that for each $\mathfrak{p} \in U$, $s(\mathfrak{p}) \in S_{(\mathfrak{p})}$ and such that s is locally a quotient of elements of S. The verification that $\mathcal{O}$ is a presheaf of rings with the natural restrictions is trivial. It follows from the local nature of the construction of $\mathcal{O}$ that it is indeed a sheaf.

Definition 2.2. For any graded ring S, define $(\mathrm{Proj}(S), \mathcal{O})$ to be the topological space together with the sheaf of rings as constructed above.

Proposition 2.3. Let S be a graded ring. The pair $(\mathrm{Proj}(S), \mathcal{O})$ is a scheme.

Proof. See Proposition II.2.5 in [13]. □

Example 2.4. For any ring A, the polynomial ring $A[x_0, \dots, x_n]$ has the natural grading

$$A = \bigoplus_{m \in \mathbb{Z}} A_m, \quad A_m = \{ \sum_{i_0, \dots, i_n \in \mathbb{Z}_{\geq 0}} a x_0^{i_0} \cdots x_n^{i_n} \ : \ a \in A,\ i_0 + \dots + i_n = m \}.$$

With this grading, we find that $\mathrm{Proj}(A[x_0, \dots, x_n]) = \mathbb{P}^n_A$.

Let us now assume that $\mathcal{J}$ is a sheaf of graded algebras over a scheme X. In the following, we assume that X is a noetherian scheme and that $\mathcal{J}$ is a quasi-coherent sheaf of $\mathcal{O}_X$-modules which has a structure of a sheaf of graded $\mathcal{O}_X$-algebras. Thus $\mathcal{J} \cong \bigoplus_{d \geq 0} \mathcal{J}_d$, where $\mathcal{J}_d$ is the homogeneous part of degree d. Assume also that $\mathcal{J}_0 = \mathcal{O}_X$, that $\mathcal{J}_1$ is a coherent $\mathcal{O}_X$-module and that $\mathcal{J}$ is locally generated by $\mathcal{J}_1$ as an $\mathcal{O}_X$-algebra.

Construction 2.5. [13, Page 160] For each open affine subset $U = \mathrm{Spec}(A)$ of X, let $\mathcal{J}(U)$ be the graded A-algebra $\Gamma(U, \mathcal{J}_{|U})$. Consider $\mathrm{Proj}(\mathcal{J}(U))$ and the natural morphism $\pi_U : \mathrm{Proj}(\mathcal{J}(U)) \to U$. As $\mathcal{J}$ is quasi-coherent, for any $f \in A$ with $U_f = \mathrm{Spec}(A_f)$, $\mathrm{Proj}(\mathcal{J}(U_f)) \cong \pi_U^{-1}(U_f)$. It follows that for any two open affine subsets U and V of X, $\pi_U^{-1}(U \cap V)$ is naturally isomorphic to $\pi_V^{-1}(U \cap V)$. With these isomorphisms, one can glue the schemes $\mathrm{Proj}(\mathcal{J}(U))$ to obtain the scheme $\mathrm{Proj}(\mathcal{J})$ together with the morphism $\pi : \mathrm{Proj}(\mathcal{J}) \to X$ such that for any open affine $U \subseteq X$, $\pi^{-1}(U) \cong \mathrm{Proj}(\mathcal{J}(U))$. Remark also that the sheaves $\mathcal{O}$ on each of the $\mathrm{Proj}(\mathcal{J}(U))$ are compatible with this glueing, and they give rise to the sheaf $\mathcal{O}$ on $\mathrm{Proj}(\mathcal{J})$.

Proposition 2.6. Let $X, \mathcal{J}$ be as assumed and $\pi : \mathrm{Proj}(\mathcal{J}) \to X$ as in Construction 2.5 . Then π is a proper morphism.

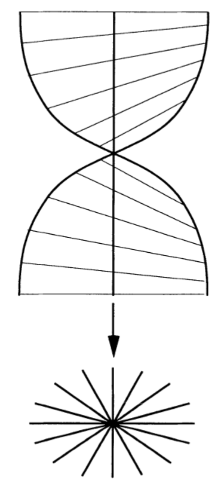
Proof. See the proof of Proposition 7.10.(a) in [13]. □

2.2 Blow-ups

In this section we introduce the blow-up construction on schemes. This construction is a fundamental tool in algebraic geometry as saying that one variety is the blow-up of another along a given subvariety expresses a relationship that is, on the one hand, close enough to relate the structures on the two varieties, and on the other hand, is very flexible. We only introduce the theory on blow-ups needed for this thesis; for a more extensive introduction, see Chapter 1 of [13], Chapter 7 of [12] and Section IV.2 of [8].

We first consider an important example of a blow-up: the blow-up of $\mathbb{A}^2$ at the origin $O = (0,0)$. In the following, let x_1, x_2 be the affine coordinates of $\mathbb{A}^2$ and let y_0, y_1 be the homogeneous coordinates of $\mathbb{P}^1$. With this, closed subsets of $\mathbb{A}^2 \times \mathbb{P}^1$ are defined by polynomials in x_1, x_2, y_0, y_1 that are homogeneous with respect to y_0, y_1.

Define **the blow-up of $\mathbb{A}^2$ at the point** O to be the closed subset X of $\mathbb{A}^2 \times \mathbb{P}^1$ defined by the equation $\{x_1 y_1 = x_2 y_0\}$ ([13, Page 28]). There is a natural morphism $\pi : X \to \mathbb{A}^2$ obtained by restricting the projection morphism from the fiber product $\mathbb{A}^2 \times \mathbb{P}^1 \to \mathbb{A}^2$ to the closed subset. It is not hard to draw this morphism π, see the figure on the right (taken from Example 7.17 of [12]). It looks like a spiral staircase with the steps extending in both directions.



One can show, following page 28 of [13], that for $P \in \mathbb{A}^2$, $P \neq O$, $\pi^{-1}(P)$ consists of one point, and moreover, that there is an isomorphism $X \setminus \pi^{-1}(O) \to \mathbb{A}^2 \setminus O$. One can also show that $\pi^{-1}(O) \cong \mathbb{P}^1$ and that the points of $\pi^{-1}(O)$ are in one-to-one correspondence with the set of lines through O in $\mathbb{A}^2$.

In this thesis, we only consider blow-ups of schemes along closed subschemes of codimension two. These blow-ups can be defined using the Proj-construction that was introduced in the previous section.

Theorem 2.7. [8, Thm IV-23] Let X be a scheme and $Y \subset X$ a closed subscheme. Let $\mathcal{J} = \mathcal{J}_{Y,X} \subset \mathcal{O}_X$ be the ideal sheaf of Y in X. If $\mathcal{A}$ is the sheaf of graded $\mathcal{O}_X$-algebras

$$\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{J}^n = \mathcal{O}_X \oplus \mathcal{J} \oplus \mathcal{J}^2 \oplus \cdots$$

(where the k-th summand is taken to be the k-th graded piece of $\mathcal{A}$), then the scheme $\mathrm{Proj}(\mathcal{A}) \to X$ is the blow-up of X along Y.

The sheaf of graded $\mathcal{O}_X$-algebras $\mathcal{A}$ is sometimes called a Rees algebra. We call the inverse image $E = \pi^{-1}(Y)$ of Y under π the **exceptional divisor** of the blow-up, and Y the **center** of the blow-up. The blow-up of a scheme X along a closed subscheme Y is also denoted $\mathrm{Bl}_Y(X) \to X$.

With Theorem 2.7, one would expect that giving an explicit expression for the blow-up of a scheme X along a closed subscheme Y is easy. However, even when the schemes X and Y are known explicitly, it is not always easy to write down the sheaf of graded $\mathcal{O}_X$-algebras $\mathcal{A}$. The following, however, gives us a specific situation in which $\mathcal{A}$ can be written down explicitly easily.

Recall that a sequence $x, y \in A$ is regular if and only if $x \in A$ is not a zero divisor and $y \in A/(x)$ is not a zero divisor ([22, Tag 00LF]).

Definition 2.8. [8, Definition IV-15] Let X be any scheme, $Y \subset X$ a subscheme. We say that Y is a regular subscheme if it is locally the zero locus of a regular sequence of functions on X.

Assume that $Y \subset X$ is a regular subscheme of codimension two.

Proposition 2.9. [8, Proposition IV-25] Let A be a noetherian ring and $x, y \in A$; let $\mathcal{A}$ be the Rees algebra

$$\mathcal{A} = A[xT, yT] \subset A[T].$$

If $x, y \in A$ is a regular sequence, then

$$\mathcal{A} \cong A[X, Y]/(yX - xY)$$

via the map $X \mapsto xT$, $Y \mapsto yT$.

Example 2.10. Consider the scheme $X = \mathbb{P}^3_{\mathbb{Q}} = \text{Proj}(\mathbb{Q}[a, b, c, d])$ and the smooth closed subscheme $Y = V(a^2 + b^2 + c^2, d)$. Denote by U_0, U_1, U_2, U_3 the standard affine opens of $\mathbb{P}^3_{\mathbb{Q}}$, then we consider the blow-up of X along Y by determining the blow-ups of U_i along $Y_{|U_i}$ for $i = 0, \ldots, 3$. First, observe that $U_3 \cap Y = \varnothing$ so the blow-up of U_3 along $Y_{|U_3}$ is again U_3. In the following, denote by $\mathcal{J}_Y(U_i)$ the ideal sheaf corresponding to $Y_{|U_i}$ for $i = 0, 1, 2$. We work out the following only for $i = 0$, the other computations follow analogously.
We see that $\mathcal{J}_Y(U_0) = (1 + \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2, \frac{d}{a})$ and fix the notation $g_1 = 1 + \left(\frac{b}{a}\right)^2 + \left(\frac{c}{a}\right)^2$, $g_2 = \frac{d}{a}$. Set

$$\mathcal{A}_0 = \bigoplus_{n=0}^{\infty} \mathcal{J}_Y(U_0)^n = \bigoplus_{n=0}^{\infty} (g_1, g_2)^n T^n.$$

For $a \in (g_1, g_2)^n$ and $b \in (g_1, g_2)^m$, we have that $aT^n \cdot bT^m = abT^{mn} \subset \oplus_{n=0}^{\infty} (g_1, g_2)^n T^n = \mathcal{A}_0$, so we can write

$$\mathcal{A}_0 = \mathbb{Q}\left[\frac{b}{a}, \frac{c}{a}, \frac{d}{a}\right][g_1 T, g_2 T],$$

where the grading is given by $\deg(T) = 1$, $\deg(b/a) = \deg(c/a) = \deg(d/a) = 0$.

Through Theorem 2.7, the blow up of U_0 along $Y_{|U_0}$ is given by $\text{Proj}(\mathcal{A}_0) \to U_0$. Following Corollary 3.2 of [3], the scheme $\text{Proj}(\mathcal{A}_0)$ can be obtained by glueing the schemes $\text{Spec}((\mathcal{A}_0)_{(f_i)})$, with $(\mathcal{A}_0)_{(f_i)}$ the degree-zero-part of the localization of $\mathcal{A}_0$ at f_i, where the set $\{f_i\}$ is a collection of homogeneous elements in $(\mathcal{A}_0)_+$ such that for any element of $(\mathcal{A}_0)_+$, some power of this element is in the ideal generated by the f_i's. As $f_1 = g_1 T$ and $f_2 = g_2 T$ satisfy these conditions, $\text{Proj}(\mathcal{A}_0)$ is obtained by glueing

$$\text{Spec}((\mathcal{A}_0)_{(g_1)}),\ \text{Spec}((\mathcal{A}_0)_{(g_2)}).$$

For $i = 1, 2$, we obtain similar results, so the blow up of X along Y can be obtained by glueing the spectra of seven rings: the six rings $(\mathcal{A}_j)_{(g_i)}$ for $j = 0, 1, 2$ and $i = 1, 2$ that arise as described, and the ring $\mathbb{Q}\left[\frac{a}{d}, \frac{b}{d}, \frac{c}{d}\right]$ coming from U_3.

We remark that taking the degree-zero-part of $\mathbb{Q}[a,b,c,d]$ localized at a yields exactly $\mathcal{O}_{\mathbb{P}^3_{\mathbb{Q}}}(U_0)$, and analogously for localizing at b, c, d. Hence, set $\mathcal{A} = \mathbb{Q}[a,b,c,d][(a^2+b^2+c^2)\gamma, d\gamma]$. Then the blow up of X along Y can be obtained by glueing the spectra of the seven rings $\mathcal{A}_{s,t}$ with $s \in \{a,b,c,d\}$ and $t \in \{(a^2+b^2+c^2)\gamma/s^2, d\gamma/s\}$, with $\mathcal{A}_{s.t}$ defined as follows: localize $\mathcal{A}$ at s and take the degree-zero-part with respect to the usual polynomial ring grading of the ring $\mathbb{Q}[a,b,c,d]$. Then localize at t, and take the degree zero part with respect to the grading from the Rees algebra.

Lemma 2.11. Let X be the blow up of $\mathbb{P}^3_{\mathbb{Q}}$ along a smooth closed subscheme Y of codimension two, and let X' be the blow up of $\mathbb{P}^3_{\mathbb{Q}}$ along a smooth closed subscheme Y' of codimension two. Assume that over $\overline{\mathbb{Q}}$, Y and Y' are isomorphic schemes. Then the schemes $\mathrm{Bl}_Y(X)$ and $\mathrm{Bl}_{Y'}(X')$ are isomorphic over $\overline{\mathbb{Q}}$.

Proof. Let $\mathcal{J}$ be the ideal sheaf corresponding to Y, and $\mathcal{J}'$ be the ideal sheaf corresponding to Y'. By assumption, $\mathcal{J}$ and $\mathcal{J}'$ are isomorphic sheaves over $\overline{\mathbb{Q}}$. Hence, the corresponding Rees algebras $\mathcal{A} = \bigoplus_{n=0}^{\infty} \mathcal{J}^n$ and $\mathcal{A}' = \bigoplus_{n=0}^{\infty} \mathcal{J}'^n$ are isomorphic over $\overline{\mathbb{Q}}$ and by construction, the schemes $\mathrm{Bl}_Y(X) = \mathrm{Proj}(\mathcal{A})$ and $\mathrm{Bl}_{Y'}(X') = \mathrm{Proj}(\mathcal{A}')$ are isomorphic over $\overline{\mathbb{Q}}$. □

2.3 Fano varieties

In this section we introduce Cartier divisors, state when they are ample and then what it means to be a log Fano variety. Throughout this section, let X be an integral scheme.

Definition 2.12. [4, Definition 1.2] A **Cartier divisor** on an integral scheme X is a global section of the sheaf $\underline{K(X)}^\times/\mathcal{O}_X^\times$, where $\underline{K(X)}$ is the constant sheaf of rational functions on X. In other words, a Cartier divisor is given by a collection of pairs (U_i, f_i), where $(U_i)_i$ is an affine open cover of X and f_i is a non-zero rational function on U_i such that f_i/f_j is a regular function on $U_i \cap U_j$ that does not vanish at any point of $U_i \cap U_j$.

Intuitively, these Cartier divisors are divisors such that, locally, they can be written as the divisor of a non-zero rational function.

Remark 2.13. [4, Section 1.1] For X a locally factorial scheme, any divisor is locally the divisor of a rational function. There is no distinction between Cartier and Weil divisors.

Example 2.14. Consider the blow up of $\mathbb{P}^3_{\mathbb{Q}} = \mathrm{Proj}(\mathbb{Q}[a,b,c,d])$ along the smooth conic $C = V(a^2+b^2+c^2, d)$ and denote it by $\pi : X \to \mathbb{P}^3_{\mathbb{Q}}$. It is known that $\mathbb{P}^3_{\mathbb{Q}}$ is an integral scheme and as C is non-empty and closed, its corresponding ideal sheaf is non-zero and quasicoherent. Hence, the blow up X is an integral scheme ([22, Tag 02ND]).

From Theorem 1 of III.7 in [18] it follows that nonsingular schemes are locally factorial. As C is a nonsingular subvariety of the nonsingular scheme $\mathbb{P}^3_{\mathbb{Q}}$, Theorem II.8.24 of [13] gives that the blow up X is also nonsingular, and then also locally factorial. We conclude that on the blow up X there is no distinction between Cartier divisors and Weil divisors.

A crucial notion to define Fano varieties is the notion of being *ample*. In the following, a line bundle is an invertible sheaf.

Definition 2.15. [16, Definition 1.2.1] Let X be a proper scheme over a base field k and L a line bundle on X.

(i) L is **very ample** if there exists a closed embedding $X \subseteq \mathbb{P}^n$ of X into some projective space $\mathbb{P}^n$ such that

$$L = \mathcal{O}_X(1) = \mathcal{O}_{\mathbb{P}^n}(1)_{|X}$$

(ii) L is **ample** if $L^{\otimes m}$ is very ample for some $m > 0$

Definition 2.16. A Cartier divisor D on X is ample or very ample if the corresponding line bundle $\mathcal{O}_X(D)$ is.

There are many different definitions for log Fano varieties, but for the purpose of this thesis, we use the following definition.

Definition 2.17. Let X be a nonsingular projective variety. We say that X is a **log Fano** variety if there exists a nonsingular integral closed subscheme D of codimension one such that $-(K_X + D)$ is ample.

2.4 Cox rings

This section gives a short introduction to Cox rings and a simple example of one. Unless stated otherwise, the following is taken from Chapter 1.4 of [1].

We first need to define the notion of normal schemes.

Definition 2.18. [22, Tag 033I] A domain is a normal domain if it is a domain that is integrally closed in its field of fractions. A scheme X is **normal** if and only if for all $x \in X$, the local ring $\mathcal{O}_{X,x}$ is a normal domain.

Let k be an algebraically closed field of characteristic zero and let X be a normal integral scheme of finite type over k with free finitely generated divisor class group $\mathrm{Cl}(X)$. Fix a subgroup $K \subseteq \mathrm{WDiv}(X)$ such that the canonical map $c : K \to \mathrm{Cl}(X)$ sending $D \in K$ to its class $[D] \in \mathrm{Cl}(X)$ is an isomorphism.

Definition 2.19. Define the **Cox sheaf** associated to K to be

$$\mathcal{R} \coloneqq \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}, \qquad \mathcal{R}_{[D]} \coloneqq \mathcal{O}_X(D),$$

where $D \in K$ represents $[D] \in \mathrm{Cl}(X)$ and the multiplication in $\mathcal{R}$ is defined by multiplying homogeneous sections in the field of rational functions $k(X)$.

Remark 2.20. The sheaf $\mathcal{R}$ defined as such is a quasicoherent sheaf and up to isomorphism, it does not depend on the choice of the subgroup $K \subseteq \mathrm{WDiv}(X)$.

Definition 2.21. Define the **Cox ring** of X as the algebra of global sections

$$\mathcal{R}(X) \coloneqq \bigoplus_{[D]\in \mathrm{Cl}(X)} \mathcal{R}_{[D]}(X), \qquad \mathcal{R}_{[D]}(X) \coloneqq \Gamma(X, \mathcal{O}_X(D)).$$

Remark that $\mathcal{R}(X)$ is a graded ring and its grading comes from $\mathrm{Cl}(X)$.

Example 2.22. Consider $\mathbb{P}^n_k$, of which it is known that any hyperplane $H \subseteq \mathbb{P}^n_k$ generates its class group. Let $K \subseteq \mathrm{WDiv}(\mathbb{P}^n_k)$ be the subgroup generated by H. Then the Cox ring of $\mathbb{P}^n_k$ is given by

$$\begin{aligned}\mathcal{R}(\mathbb{P}^n_k) &= \bigoplus_{a\in\mathbb{Z}} \Gamma(\mathbb{P}^n_k, \mathcal{O}_{\mathbb{P}^n_k}(1)^{\otimes a}) \\ &= \bigoplus_{a\in\mathbb{Z}} (k[x_0, \dots, x_n])_a \\ &= k[x_0, \dots, x_n]\end{aligned}$$

and its grading is the standard grading on a polynomial ring.

2.5 Group schemes

In this section we introduce group schemes, for we will consider torsors under group schemes in Section 2.7. We also discuss an important example of a group scheme.

Definition 2.23. [22, Tag 022S] Let S be a scheme. A **group scheme over** S is a pair (G, m) where G is a scheme over S and $m : G \times_S G \to G$ is a morphism of schemes over S with the following property: for every scheme T over S the pair $(G(T), m)$ is a group.

Definition 2.24. [22, Tag 022S] A **morphism** $\varphi : (G, m) \to (G', m')$ **of group schemes over** S is a morphism $\varphi : G \to G'$ of schemes over S such that for every scheme T over S the induced map $\varphi : G(T) \to G'(T)$ is a homomorphism of groups.

Example 2.25. We take $S = \mathrm{Spec}(\mathbb{Z})$ as base scheme and consider $G = \mathbb{G}_{m,\mathbb{Z}} = \mathrm{Spec}(\mathbb{Z}[x, x^{-1}])$. Let $m : G \times_{\mathbb{Z}} G \to G$ be the morphism corresponding to the ring homomorphism given by $x \mapsto x \otimes x$. It follows that with this choice, (G, m) is a group scheme: for any scheme T, $G(T) = \mathrm{Mor}(T, G) = \mathcal{O}_T(T)^\times$, which is the multiplicative group of $\mathcal{O}_T(T)$. Hence, $\mathbb{G}_{m,\mathbb{Z}}$ is a group scheme over $\mathrm{Spec}(\mathbb{Z})$.

Lemma 2.26. Let S_1, S_2 be two group schemes over the base scheme S, then $S_1 \times_S S_2$ is a group scheme over S.

Proof. Let $f : S_1 \to S$ and $g : S_2 \to S$ be the morphisms giving rise to the fiber product and denote by m_1 and m_2, respectively, the morphisms of schemes over S such that $(S_1(T), m_1)$ and $(S_2(T), m_2)$ are groups for any scheme T. Consider the scheme $S_1 \times_S S_2$ with the morphism $m : S_1 \times_S S_2 \times_S S_1 \times_S S_2 \to S_1 \times_S S_2$ defined by $m((s_1, s_2), (s_1', s_2')) = (m_1(s_1, s_1'), m_2(s_2, s_2'))$. Canonically,

$$(S_1 \times_S S_2)(T) = \{(\alpha_1, \alpha_2) \in S_1(T) \times S_2(T) \;:\; f(\alpha_1) = g(\alpha_2)\}$$

which is a subgroup of the Cartesian product $S_1(T) \times S_2(T)$ with the morphism m. □

Example 2.27. The fiber product $\mathbb{G}_{m,\mathbb{Z}}^2 = \mathbb{G}_{m,\mathbb{Z}} \times_{\mathrm{Spec}(\mathbb{Z})} \mathbb{G}_{m,\mathbb{Z}}$ is a group scheme over $\mathrm{Spec}(\mathbb{Z})$.

Remark 2.28. Let G be a group scheme over the scheme S and let P be a property of morphisms. Then we say that G has property P if the structure morphism $f : G \to S$ has property P. If P is a property of schemes, we say that G has property P if G as a scheme has property P.

For example, a group scheme G over S is flat and locally of finite type if the structure morphism $f : G \to S$ is flat and locally of finite type.

Example 2.29. Consider the group scheme $G = \mathbb{G}_{m,\mathbb{Z}}^2$ over $S = \mathrm{Spec}(\mathbb{Z})$ and denote by $f : G \to S$ the structure morphism. The morphism f is locally of finite type if for every $x \in \mathbb{G}_{m,\mathbb{Z}}^2$, there exist affine open neighbourhoods $U = \mathrm{Spec}(A) \subseteq G$, $V = \mathrm{Spec}(R) \subseteq S$ of x and $f(x)$, respectively, such that $f(U) \subseteq V$ and the induced ring homomorphism $R \to A$ is of finite type. Take $A = \mathbb{Z}[x, x^{-1}, y, y^{-1}]$, $R = \mathbb{Z}$ and observe that $A \cong \mathbb{Z}[x, \widetilde{x}, y, \widetilde{y}]/(x\widetilde{x} - 1, y\widetilde{y} - 1)$ as a $\mathbb{Z}$-algebra. Hence, G is indeed locally of finite type.
The morphism $f : \mathbb{G}_{m,\mathbb{Z}} \to S$ is affine, so it is flat if and only if the ring homomorphism $\mathbb{Z} \to \mathbb{Z}[x, x^{-1}, y, y^{-1}]$ is flat, i.e., if $\mathbb{Z}[x, x^{-1}, y, y^{-1}] = \mathbb{Z}[x, \widetilde{x}, y, \widetilde{y}]/(x\widetilde{x} - 1, y\widetilde{y} - 1)$ is a flat $\mathbb{Z}$-module. As $\mathbb{Z}[x, \widetilde{x}, y, \widetilde{y}]$ generate a prime ideal, $\mathbb{Z}[x, x^{-1}, y, y^{-1}]$ is an integral domain and thus torsion-free. As $\mathbb{Z}$ is a Dedekind domain, f is flat.

2.6 The fppf topology

In this section we introduce the main ideas behind the fppf topology. For this we first define when morphisms of schemes are flat and fppf. The goal of this section is not to work out the details, but to give an idea of the construction.

Definition 2.30. [17, Page 7] A homomorphism $f : A \to B$ of rings is **flat** if B is flat as an A-module via f.

Definition 2.31. [17, Page 8] A morphism $f : X \to Y$ of schemes is **flat** if, for all points $x \in X$, the induced map $\mathcal{O}_{Y,f(x)} \to \mathcal{O}_{X,x}$ is flat. Equivalently, f is flat if for any pair V and U of open affines of X and Y, respectively, such that $f(V) \subseteq U$, the map $\mathcal{O}_Y(U) \to \mathcal{O}_X(V)$ is flat.

Definition 2.32. [17, Page 11] A morphism $f : X \to Y$ of schemes is **faithfully flat** if it is flat and surjective.

Definition 2.33. A morphism $f : X \to Y$ of schemes is **fppf** if it is faithfully flat and locally of finite type.

Using these fppf-morphisms, one can construct a Grothendieck topology on schemes: the fppf topology. The idea of the fppf topology on a scheme X is that instead of using open subsets to define a topology, one takes an (fppf)-covering of X instead. Such an (fppf)-covering is a family $\{U_i \to X\}_{i \in I}$ of fppf morphisms of schemes for an index set I such that $X = \bigcup_{i \in I} f_i(U_i)$. For the exact details on how this gives a Grothendieck topology, see Section II.1 of [17], with the note that the author denotes the class of fppf-morphisms by (fl). One can also construct such a covering for other classes of morphisms; for example with open immersions or flat morphisms. Constructing the Grothendieck topology with respect to open immersions returns the Zarisky topology.

To see how one can construct cohomology groups based on the fppf-topology analogous to the cohomology groups based on the Zarisky topology, see Chapter 3 of [17]. For the remainder of this thesis, we fix the following notation: let $H^n(X_{\text{fppf}}, G)$ be the n-th cohomology group of the group scheme G with respect to the fppf-topology.

2.7 Torsors

In this section we introduce torsors under group schemes over arbitrary schemes. We follow this introduction by stating some important properties of torsors. Throughout this section, we consider only schemes endowed with the fppf-topology.

Definition 2.34. [21, Definition 2.2.1] Let X be a scheme. An X-**torsor** under an X-group scheme G is defined as a scheme Y/X equipped with an action of G compatible with the projection to X and satisfying the following equivalent properties:

(i) the morphism $Y \to X$ is *fppf*, and the map $Y \times_X G \to Y \times_X Y$ given by $(y, s) \mapsto (y, ys)$ is an isomorphism;

(ii) there exists a covering $\mathcal{U} = (U_i \to X)$ in the flat topology such that for any i the pair $(Y_{U_i} = Y \times_X U_i, \text{the action of } G_{U_i} = G \times_X U_i)$ is isomorphic to the pair $(G_{U_i}, \text{the right action of } G_{U_i} \text{ on itself})$.

The equivalence of the conditions (i) and (ii) is proven on page 14 of [21].

Example 2.35. The Cox ring of $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ is given by $\mathcal{R}(\mathbb{P}^1_{\overline{\mathbb{Q}}}) = \overline{\mathbb{Q}}[x_0, x_1]$, which is a finitely generated $\overline{\mathbb{Q}}$-algebra. By Construction I.6.3.1 of [1], we have the torsor under $\mathbb{G}^1_{m,\mathbb{Z}}$

$$\operatorname{Spec}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(\widetilde{\overline{\mathbb{Q}}[x_0, x_1]}) \to \mathbb{P}^1_{\overline{\mathbb{Q}}},$$

where $\operatorname{Spec}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(\widetilde{\overline{\mathbb{Q}}[x_0, x_1]})$ is the global spectrum of the sheafification of $\overline{\mathbb{Q}}[x_0, x_1]$ with respect to $\mathbb{P}^1_{\overline{\mathbb{Q}}}$ (see [8, Section I.3.3]). We remark that we can write the above example in a more familiar way: using Section I.6.3 of [1], one can show that

$$\operatorname{Spec}_{\mathbb{P}^1_{\overline{\mathbb{Q}}}}(\widetilde{\overline{\mathbb{Q}}[x_0, x_1]}) = \operatorname{Spec}(\overline{\mathbb{Q}}[x_0, x_1]) \setminus V(x_0, x_1) = \mathbb{A}^2_{\overline{\mathbb{Q}}} \setminus \{(0,0)\},$$

and the action of $\mathbb{G}^1_{\mathrm{m},\mathbb{Z}}$ on $\mathbb{A}^2_{\overline{\mathbb{Q}}} \setminus \{(0,0)\}$ is the diagonal action.

We state some properties of torsors that are important in the next chapter.

Lemma 2.36. Let $f : X \to Y$ be a Y-torsor under $\mathbb{G}^2_{m,Y}$. For any $P \in Y(\mathbb{Z})$, the fiber $f^{-1}(P)$ is a $\mathbb{G}^2_{m,\mathbb{Z}}$-torsor.

Proof. Consider the following cartesian diagram.

$$\begin{array}{ccc} f^{-1}(P) & \longrightarrow & X \\ {\scriptstyle h}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ \operatorname{Spec}(\mathbb{Z}) & \xrightarrow[P]{} & Y \end{array}$$

The morphism f is fppf, i.e., faithfully flat and of finite presentation, by assumption. We know that being flat and being surjective are preserved under base change ([22, Tag 01U9], [18, Prop II.2.4]). Hence, h is a faithfully flat morphism. As being of finite presentation is preserved under base change ([22, Tag 01TS]), the morphism h is fppf.

By assumption, we have an isomorphism of schemes

$$\varphi : X \times_Y \mathbb{G}^2_{m,Y} \to X \times_Y X$$

given by $(x, s) \mapsto (x, xs)$. Observe that per definition,

$$\begin{aligned} X \times_Y \mathbb{G}^2_{m,Y} &= X \times_Y \mathbb{G}_{m,\mathbb{Z}} \times_{\mathbb{Z}} Y \times_Y \mathbb{G}_{m,\mathbb{Z}} \times_{\mathbb{Z}} Y \\ &= X \times_Y Y \times_{\mathbb{Z}} \mathbb{G}^2_{m,\mathbb{Z}} \\ &= X \times_{\mathbb{Z}} \mathbb{G}^2_{m,\mathbb{Z}} \end{aligned}$$

With this, consider the diagram

$$\begin{array}{ccc} X \times_{\mathbb{Z}} \mathbb{G}^2_{m,\mathbb{Z}} & \xrightarrow{\quad\varphi\quad} & X \times_Y X \\ & {\scriptstyle g}\searrow \qquad \swarrow{\scriptstyle g'} & \\ & Y & \end{array}$$

Define the morphisms

$$g(x,s) = f \circ p_1 \circ \varphi(x,s), \quad g'(x,x') = f \circ p_1(x,x'),$$

with $p_1 : X \times_Y X \to X$ the first projection morphism. As g and g' are compositions of morphisms of schemes, they themselves are morphisms of schemes, too. By construction, the above diagram commutes. For any morphism $P : \mathrm{Spec}(\mathbb{Z}) \to Y$, base changing the above diagram with respect to P gives that $g^{-1}(P) \cong (g')^{-1}(P)$.

By definition, $g^{-1}(P) = \mathrm{Spec}(\mathbb{Z}) \times_Y X \times_{\mathbb{Z}} \mathbb{G}^2_{m,\mathbb{Z}} = f^{-1}(P) \times_{\mathbb{Z}} \mathbb{G}^2_{m,\mathbb{Z}}$, and

$$\begin{aligned}(g')^{-1}(P) &= \mathrm{Spec}(\mathbb{Z}) \times_Y X \times_Y X = f^{-1}(P) \times_Y X \\ &= f^{-1}(P) \times_{\mathbb{Z}} \mathrm{Spec}(\mathbb{Z}) \times_Y X = f^{-1}(P) \times_{\mathbb{Z}} f^{-1}(P).\end{aligned}$$

As the isomorphism $g^{-1}(P) \to (g')^{-1}(P)$ is induced by φ, it is the desired isomorphism. Indeed, $f^{-1}(P)$ is a $\mathbb{G}^2_{m,\mathbb{Z}}$-torsor. □

An important property of torsors is that they can be parametrized. Assume that G is a group scheme that is flat and locally of finite-type.

Lemma 2.37. [21, Page 19] There is a one-to-one correspondence

$$\{X\text{-torsors under } G \text{ up to isomorphism}\} \leftrightarrow \text{the group } H^1(X_{\mathrm{fppf}}, G)$$

The cohomology group that parametrizes X-torsors under $\mathbb{G}_m$ can be determined more explicitly with Hilbert's Theorem 90. Remark that in [17], the subscript (fl) refers to what we defined as (fppf) in Section 2.6.

Proposition 2.38. [17, Proposition III.4.9] We have $H^1(X_{\mathrm{fppf}}, \mathbb{G}_m) \cong \mathrm{Pic}(X)$.

2.8 Integral models

In this section we introduce integral models and discuss some of their properties.

Recall that the generic fiber of a scheme with respect to a point is defined as follows.

Definition 2.39. Let $s \in S$ a point of a scheme, and let $\iota_s : \mathrm{Spec}(\kappa(s)) \to S$ be the canonical associated morphism of schemes. Let X be a scheme over S and let X_s be defined by the cartesian diagram

$$\begin{array}{ccc} X_s & \longrightarrow & \mathrm{Spec}(\kappa(s)) \\ \downarrow & & \downarrow \iota_s \\ X & \longrightarrow & S \end{array}$$

We call X_s **the fiber above** s.

Definition 2.40. Let X be a scheme over $\operatorname{Spec}(\mathbb{Q})$. An **integral model**, or simply a model, of X is a scheme $\mathcal{X}$ over $\operatorname{Spec}(\mathbb{Z})$ with generic fiber $\mathcal{X}_\eta = X$.

We say that a model $\mathcal{X}$ is proper if the structure morphism $\mathcal{X} \to \operatorname{Spec}(\mathbb{Z})$ is proper.

Lemma 2.41. Let X be a scheme over $\operatorname{Spec}(\mathbb{Q})$ and let $\mathcal{X}$ be an integral model that is proper over $\mathbb{Z}$. Then $\mathcal{X}(\mathbb{Q}) = \mathcal{X}(\mathbb{Z})$.

Proof. The integral model fits in the following cartesian diagram.

$$\begin{array}{ccc} X & \xrightarrow{\ \gamma\ } & \mathcal{X} \\ \big\downarrow g & & \big\downarrow f \\ \operatorname{Spec}(\mathbb{Q}) & \xrightarrow{\ \varphi\ } & \operatorname{Spec}(\mathbb{Z}) \end{array}$$

Let $\beta \in \mathcal{X}(\mathbb{Z})$. Then $\beta \circ \varphi \in \mathcal{X}(\mathbb{Q})$ as it is the composition of morphisms of schemes and hence also a morphism of schemes. This shows one direction.

Let $\beta \in \mathcal{X}(\mathbb{Q})$, and let us consider the valuation ring $R = \mathbb{Z}_{(p)}$ for a prime p, i.e., the ring of p-adic integers intersected with $\mathbb{Q}$. Its fraction field is $K = \mathbb{Q}$. We know that $\operatorname{Spec}(R) = \{[(0)], [(p)]\}$, where $[(p)]$ is the unique closed point. As the ring homomorphism $\mathbb{Z} \to \mathbb{Z}_{(p)}$ maps the zero ideal of the first ring to the zero ideal of the second ring, we obtain the morphism of schemes $\varphi' : \operatorname{Spec}(\mathbb{Z}_{(p)}) \to \operatorname{Spec}(\mathbb{Z})$ which maps the point $[(0)]$ to $[(0)]$.

Let $p \in \mathbb{Z}$ be an arbitrary prime number and consider the morphism $i : \operatorname{Spec}(\mathbb{Q}) \to \operatorname{Spec}(\mathbb{Z}_{(p)})$ induced by the inclusion morphism $i^\# : \mathbb{Z}_{(p)} \to \mathbb{Q}$ of $\mathbb{Z}_{(p)}$ into its fraction field. This gives the diagram

$$\begin{array}{ccc} \operatorname{Spec}(\mathbb{Q}) & \xrightarrow{\ \beta\ } & \mathcal{X} \\ \big\downarrow i & & \big\downarrow f \\ \operatorname{Spec}(\mathbb{Z}_{(p)}) & \xrightarrow{\ \varphi'\ } & \operatorname{Spec}(\mathbb{Z}) \end{array}$$

As the morphism i is induced by the inclusion of $\mathbb{Z}_{(p)}$ into its fraction field, the unique point in $\operatorname{Spec}(\mathbb{Q})$ is mapped to the point $[(0)]$ in $\operatorname{Spec}(\mathbb{Z}_{(p)})$ and by the above discussion to the point $[(0)]$ in $\operatorname{Spec}(\mathbb{Z})$. Now as both $\operatorname{Spec}(\mathbb{Q})$ and $\mathcal{X}$ are schemes over $\mathbb{Z}$, we know that $\varphi = f \circ \beta$. As $\varphi' \circ i = \varphi$, this shows that the diagram commutes. Following the valuative criterion for properness, we conclude that for all $p \in \mathbb{Z}$, we have a unique morphism of schemes

$$\alpha_p : \operatorname{Spec}(\mathbb{Z}_{(p)}) \to \mathcal{X}$$

such that the resulting triangles in the above diagram commute.

Fix $U_{(p)} = \varphi'(\mathrm{Spec}(\mathbb{Z}_{(p)}) = \{[(0)],[(p)]\} \subset \mathrm{Spec}(\mathbb{Z})$, then for p and q distinct prime numbers, $U_{(pq)} = U_{(p)} \cap U_{(q)} = \{[(0)]\}$. For any p,q distinct prime numbers, set $\varphi'_{pq} = \varphi'_{|U_{(pq)}}$. With the above, we see that $\varphi'_{pq} = \varphi'_{qp} = \mathrm{id}$, so we can glue the sets $U_{(p)}$ to obtain $\mathrm{Spec}(\mathbb{Z})$.

If for $p \neq q$ primes in $\mathbb{Z}$, α_p and α_q agree on the overlap $U_p \cap U_q$, then we can glue the α_p to obtain a morphism α. From the above commuting triangle, it follows that for all $p \in \mathbb{Z}$, $\alpha_p([(0)]) = f([(0)])$. Hence, the morphisms α_p agree on overlaps and glue to a morphism $\alpha : \mathrm{Spec}(\mathbb{Z}) \to \mathcal{X}$. As all morphisms α_p were unique, also α is unique, and we obtain the second direction. This shows that we indeed have the natural bijection $\mathcal{X}(\mathbb{Z}) = \mathcal{X}(\mathbb{Q})$. □

Example 2.42. Let $\mathcal{X} = \mathbb{P}^1_{\mathbb{Z}}$ and $\mathcal{U} = \mathbb{P}^1_{\mathbb{Z}} \setminus \{x_0 = 0\}$. We compute

$$\mathcal{X}(\mathbb{Q}),\ \mathcal{X}(\mathbb{Z}),\ \mathcal{U}(\mathbb{Z}),\ \mathcal{U}(\mathbb{Q}),$$

and determine why the latter two are not equal.

First observe that $\mathcal{U} \cong \mathbb{A}^1_{\mathbb{Z}} \cong \mathrm{Spec}(\mathbb{Z}[T])$, so for any ring R,

$$\begin{aligned}\mathcal{U}(R) &\cong \mathrm{Mor}(\mathrm{Spec}(R),\mathcal{U})\\ &\cong \mathrm{Hom}(\mathbb{Z}[T], R)\\ &\cong R\end{aligned}$$

Hence, $\mathcal{U}(\mathbb{Z}) \cong \mathbb{Z}$, and $\mathcal{U}(\mathbb{Q}) \cong \mathbb{Q}$.

We know that

$$\mathcal{X}(\mathbb{Q}) = \mathbb{P}^1_{\mathbb{Z}}(\mathbb{Q}) \cong (\mathbb{Q}^2 \setminus \{(0,0)\})/\sim,$$

where $(a,b) \sim (c,d)$ if and only if there exists an element $\lambda \in \mathbb{Q}^\times$ such that

$$a = \lambda c,\ b = \lambda d.$$

To determine $\mathcal{X}(\mathbb{Z})$, recall from Example 9.5.1 of [9] that there is a bijection

$$\{\mathcal{X}(\mathbb{Z})\} \leftrightarrow \left\{\begin{smallmatrix}\text{locally free rank-1 modules } L \text{ over } \mathbb{Z}\\ \text{with 2-tuples } (a,b) \in \mathbb{Z}^2 \text{ such that } L = a\mathbb{Z} + b\mathbb{Z}\end{smallmatrix}\right\}$$

Recall that $\mathrm{Pic}(\mathrm{Spec}(\mathbb{Z})) \cong \mathrm{Cl}(\mathbb{Z}) = 0$ and that for any ring R,

$$\mathrm{Cl}(R) = \left\{\begin{smallmatrix}\text{locally free rank-1}\\ \text{modules over } R\end{smallmatrix}\right\} / \cong .$$

Then up to isomorphism, there is a unique $\mathbb{Z}$-module that is locally free of rank one. Hence, we have a bijection

$$\{\mathcal{X}(\mathbb{Z})\} \leftrightarrow \left\{(a,b) \in \mathbb{Z}^2 \text{ such that } \mathbb{Z} = a\mathbb{Z} + b\mathbb{Z}\right\} / \cong$$

It follows that

$$\mathcal{X}(\mathbb{Z}) = \{(a,b) \in \mathbb{Z}^2 \setminus \{(0,0\} \ :\ \gcd(a,b) = 1\}/\sim,$$

for $(a,b) \sim (c,d)$ if and only if there exists an element $\lambda \in \mathbb{Z}^\times$ such that $a = \lambda c$ and $b = \lambda d$.

2.9 Height functions

This part introduces height functions by first looking at a height function on projective space. We then outline how one can use this height function to obtain a height function for any other variety with a rational map to projective space.

Recall that any rational point $P \in \mathbb{P}^n(\mathbb{Q})$ can be written in the form

$$P = (x_0 : \ldots : x_n)$$

with $x_0, \ldots, x_n \in \mathbb{Z}$ and $\gcd(x_0, \ldots, x_n) = 1$.

Definition 2.43. [14, Section B.2] For any $P \in \mathbb{P}^n(\mathbb{Q})$ written as above, define the **Weil height function** (or the **multiplicative height function**) on $\mathbb{P}^n(\mathbb{Q})$ as

$$H(P) = \max\{|x_0|, \ldots, |x_n|\}.$$

When considering a scheme Z with a rational map to $\mathbb{P}^n(\mathbb{Q})$, one can use the Weil height function to determine a height function on the scheme Z. We discuss two examples.

Example 2.44. Consider the blow-up

$$\pi : X' \to \mathbb{P}^3_{\mathbb{Q}}$$

of $\mathbb{P}^3_{\mathbb{Q}} = \operatorname{Proj}(\mathbb{Q}[a,b,c,d])$ along the smooth conic $C' = V(a^2 + bc, d)$ and consider the rational map $f_1 : X' \to \mathbb{P}^9_{\mathbb{Q}}$ induced by the log-anticanonical sheaf $\omega_{X'}(\pi^{-1}(V(a)))^{-1}$. It follows from Exercise II.8.5 of [13] that

$$\omega_{X'}(\pi^{-1}(V(a)))^{-1} = (\omega_{X'} \otimes \mathcal{L}(D))^{-1} = (\pi^*(\mathcal{O}_{\mathbb{P}^3_{\mathbb{Q}}}(1)) \otimes \mathcal{L}(E) \otimes \mathcal{L}(D))^{-1}$$

with $D = \pi^{-1}(V(a))$ and $E = \pi^{-1}(V(a^2 + bc, d))$ the exceptional divisor of the blow-up. Exercise II.8.5 of [13] also gives an isomorphism $\operatorname{Pic}(X') \cong \mathbb{Z}^2$ defined by sending the pullback of any hyperplane in $\mathbb{P}^3_{\mathbb{Q}}$ under π' to $[1,0] \in \mathbb{Z}^2$ and the exceptional divisor to $[0,1] \in \mathbb{Z}^2$. With this isomorphism, the log-anticanonical sheaf corresponds to the element $[3,-1] \in \mathbb{Z}^2$:

$$\begin{aligned} -[\omega_{X'}(D)] &= -\Big([\pi^*(\mathcal{O}_{\mathbb{P}^3_{\mathbb{Q}}}(1))] + [\mathcal{L}(E)] + [\mathcal{L}(D)]\Big) \\ &= [4,0] - [0,1] - [1,0] = [3,-1]. \end{aligned}$$

Hence, the rational map f_1 is defined by global sections on X' of degree $[3,-1]$ that generate the set of all global sections of degree $[3,-1]$ over the global sheaf of X'.

With this rational map f_1, the height of an element $P \in X'(\mathbb{Q})$ is given by $H(f_1(P))$, with H the Weil height function.

We remark that the log-anticanonical sheaf $\omega_{X'}(\pi^{-1}(V(b)))^{-1}$ induces the same rational map $f_1 : X' \to \mathbb{P}^9_{\mathbb{Q}}$: as $V(a)$ and $V(b)$ are both hyperplanes in $\mathbb{P}^3_{\mathbb{Q}}$, their pull-backs under π are in the same class of $\mathrm{Pic}(X')$. Hence, the above computations follow through identically for this choice of log-anticanonical sheaf and the height function one obtains on X' is the same height function as above.

2.10 Arithmetic functions, Möbius inversion and the Legendre symbol

In this section, we introduce a few notions from analytic number theory. We first recall four arithmetic functions and then state Möbius' inversion theorem. We end this part with the Legendre symbol.

We recall the following arithmetic functions.

Definition 2.45. For $n \in \mathbb{Z}_{>0}$,

$$\chi(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{else} \end{cases}$$

Definition 2.46. For $n \in \mathbb{Z}_{>0}$,

$$\omega(n) = \sum_{p|n} 1.$$

Definition 2.47. For $n \in \mathbb{Z}_{>0}$, define the **Möbius function** as

$$\mu(n) = \begin{cases} (-1)^t & \text{if } n = p_1 \cdots p_t \text{ with } p_1, \dots, p_t \text{ distinct primes} \\ 0 & \text{else} \end{cases}$$

Definition 2.48. For $n \in \mathbb{Z}_{>0}$, define the **divisor function** as $d(n) = \sum_{n'|n} 1$.

We state a few properties of the divisor function.

Lemma 2.49. [23, Section I.5.3] For any $n \in \mathbb{Z}_{>0}$, $2^{\omega(n)} \leq d(n)$.

Lemma 2.50. [23, Exercise 166] We have the identity

$$\sum_{n=1}^{\infty} \frac{d(n)}{n^2} = \zeta(2)^2.$$

Lemma 2.51. [23, Corollary I.5.3] For any $\epsilon > 0$, $d(z) \ll_\epsilon z^\epsilon$.

We state an important theorem and we outline how this theorem gives the identities for Möbius inversion as we will apply it later.

Theorem 2.52. [15, Theorem 2 of Chapter 2.2] Let f be an arithmetic function. Define $F(n) \coloneqq \sum_{d|n} f(d)$ for $n \in \mathbb{Z}_{>0}$. Then

$$f(n) = \sum_{d|n} \mu(n/d) F(d)$$

for $n \in \mathbb{Z}_{>0}$.

In Chapter 4, we apply this theorem with F the constant function giving 1, and $f = \chi$: set $x' = x/\alpha$ and $y' = y/\alpha$ and let $g : \mathbb{Z}^2 \to \mathbb{R}$ be any function. Then Theorem 2.52 gives

$$\sum_{\substack{x,y \in \mathbb{Z}, \\ x \neq 0}} \chi(\gcd(x,y)) g(x,y) = \sum_{\substack{x,y' \in \mathbb{Z}, \\ x \neq 0}} \sum_{\alpha | x} \mu(\alpha) g(x, \alpha y') = \sum_{\alpha > 0} \sum_{\substack{x,y' \in \mathbb{Z}, \\ x \neq 0}} \mu(\alpha) g(\alpha x', \alpha y')$$

Recall that an integer a is a quadratic residue modulo a prime p if $x^2 \equiv a \mod p$ is solvable in $x \in \mathbb{Z}$ and $p \nmid a$, and it is not a quadratic residue if the congruence relation is not solvable, but $p \nmid a$ still. With this, the **Legendre symbol** is defined as follows.

Definition 2.53. For $p > 2$ a prime number and $a \in \mathbb{Z}$,

$$\left(\frac{a}{p}\right) = \begin{cases} 1 & \text{if } a \text{ is a quadratic residue modulo } p \\ -1 & \text{if } a \text{ is not a quadratic residue modulo } p \\ 0 & \text{if } p \mid a \end{cases}$$

3 Parametrizing Integral Points

In this chapter we consider the two blow-ups of $\mathbb{P}^3_{\mathbb{Q}} = \text{Proj}(\mathbb{Q}[a,b,c,d])$ along the smooth conics $C = V(a^2+b^2+c^2,d)$ and $C' = V(a^2+bc,d)$, denoted respectively

$$\pi : X \to \mathbb{P}^3_{\mathbb{Q}}, \quad \pi' : X' \to \mathbb{P}^3_{\mathbb{Q}}.$$

We follow the approach taken in [24] to determine for both X and X' a Cox ring over $\overline{\mathbb{Q}}$ and a torsor over $\mathbb{Q}$. After constructing integral models of both the varieties X, X' and their torsors T, T' (denoted by $\mathcal{X}, \mathcal{X}', \mathcal{T}, \mathcal{T}'$, respectively), we apply properties of torsors to determine a 4-to-1 correspondence between integral points on $\mathcal{T}$ and X, and between integral points on $\mathcal{T}'$ and X'.

3.1 Isomorphic conics over $\overline{\mathbb{Q}}$

Lemma 3.1. There is an automorphism

$$\mathbb{P}^3_{\overline{\mathbb{Q}}} \to \mathbb{P}^3_{\overline{\mathbb{Q}}}$$

such that $V(a^2+b^2+c^2,d)$ is mapped to $V(a^2+bc,d)$, i.e., the two conics C and C' are isomorphic over $\overline{\mathbb{Q}}$.

Proof. Observe that over $\overline{\mathbb{Q}}$,

$$a^2 + b^2 + c^2 = a^2 + (b+ic)(b-ic),$$

and consider the morphism

$$\varphi : \mathbb{P}^3_{\overline{\mathbb{Q}}} \to \mathbb{P}^3_{\overline{\mathbb{Q}}}$$
$$(a:b:c:d) \mapsto (a : b+ic : b-ic : d).$$

A point $P = (a:b:c:d) \in V(a^2+b^2+c^2,d) \subset \mathbb{P}^3_{\overline{\mathbb{Q}}}$ is mapped to $\varphi(P) = (a : b+ic : b-ic : d) \in V(a^2+bc,d) \subset \mathbb{P}^3_{\overline{\mathbb{Q}}}$ as

$$a^2 + (b+ic)(b-ic) = a^2 + b^2 + c^2 = 0,$$

so C is mapped to C'.

Observe that φ is a linear projective transformation: it sends an element $\mathbf{x} \in \mathbb{P}^3_{\overline{\mathbb{Q}}}$ to $A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & i & 0 \\ 0 & 1 & -i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \in \text{Gl}_4(\overline{\mathbb{Q}}).$$

Hence, it is immediate that φ is a morphism.

We also observe that $\det(A) = -2i$, so A is an invertible matrix. Hence, φ is bijective for it has an inverse: sending $\mathbf{x}$ to $A^{-1}\mathbf{x}$. So φ is indeed an automorphism. □

Let us now determine explicitly the ring homomorphism $\varphi^{\#} : \mathbb{Q}[a,b,c,d] \to \mathbb{Q}[a,b,c,d]$ corresponding to the morphism φ defined above. Through the Proj-construction, we know that any point

$$(\alpha : \beta : \gamma : \delta) \in \mathbb{P}^3(\overline{\mathbb{Q}})$$

corresponds to the ideal

$$\langle \alpha b - \beta a, \alpha c - \gamma a, \alpha d - \delta a, \beta c - \gamma b, \beta d - \delta b, \gamma d - \delta c \rangle.$$

For $\varphi^{\#}$ to correspond to the morphism φ as defined above, it is required that for all $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{Q}}$

$$(\varphi^{\#})^{-1}(\langle \alpha b - \beta a, \alpha c - \gamma a, \alpha d - \delta a, \beta c - \gamma b, \beta d - \delta b, \gamma d - \delta c \rangle) =$$
$$\langle \alpha(b+ic) - \beta a, \alpha(b-ic) - \gamma a, \alpha d - \delta a, \beta(b-ic) - \gamma(b+ic), \beta d - \delta(b+ic), \gamma d - \delta(b-ic) \rangle.$$

As $(\varphi^{\#})^{-1}$ is a ring homomorphism, this is equivalent to asking that for all $\alpha, \beta, \gamma, \delta \in \overline{\mathbb{Q}}$

$$\begin{aligned}
(\varphi^{\#})^{-1}(\langle \alpha b - \beta a \rangle) &= \langle \alpha(b+ic) - \beta a \rangle \\
(\varphi^{\#})^{-1}(\langle \alpha c - \gamma a \rangle) &= \langle \alpha(b-ic) - \gamma a \rangle \\
(\varphi^{\#})^{-1}(\langle \alpha d - \delta a \rangle) &= \langle \alpha d - \delta a \rangle \\
(\varphi^{\#})^{-1}(\langle \beta c - \gamma b \rangle) &= \langle \beta(b-ic) - \gamma(b+ic) \rangle \\
(\varphi^{\#})^{-1}(\langle \beta d - \delta b \rangle) &= \langle \beta d - \delta(b+ic) \rangle \\
(\varphi^{\#})^{-1}(\langle \gamma d - \delta c \rangle) &= \langle \gamma d - \delta(b-ic) \rangle
\end{aligned}$$

Without loss of generality, we can then require that the inverse image under $\varphi^{\#}$ of the generator on the left equals the generator on the right. Applying $\varphi^{\#}$ to both sides and using that $\varphi^{\#}$ is a ring homomorphism, it is immediate that we require

$$\varphi^{\#}(a) = a, \quad \varphi^{\#}(b) = b - ic, \quad \varphi^{\#}(c) = b + ic, \quad \varphi^{\#}(d) = d.$$

As the morphism $\varphi^{\#}$ is a ring isomorphism over $\overline{\mathbb{Q}}$ and it maps the ideal sheaf corresponding to C' to the ideal sheaf corresponding to C, the pair of morphisms $(\varphi, \varphi^{\#})$ is an isomorphism of schemes over $\overline{\mathbb{Q}}$.

Remark 3.2. As C and C' are isomorphic smooth closed subschemes of codimension two of $\mathbb{P}^3_{\overline{\mathbb{Q}}}$, by Lemma 2.11 the schemes X and X' are isomorphic over $\overline{\mathbb{Q}}$. With the nonsingular integral subscheme $D' = \pi'^{-1}V(a) \subset X'$ of codimension one, the blow-up X' is a log Fano variety (below Remark 2.5 in [24]), i.e., $\omega_{X'}(D')^{-1}$ is ample. As φ leaves the coordinate a unchanged, $[D'] \in \mathrm{Pic}(X')$ is sent to $[D] = [\pi^{-1}(V(a))] \in \mathrm{Pic}(X)$. Hence, the isomorphism φ induces an isomorphism of line bundles $\omega_{X'}(D')^{-1} \cong \omega_X(D)^{-1}$. It is then immediate that also $\omega_X(D)^{-1}$ is ample over $\overline{\mathbb{Q}}$. This implies that X has the ample line bundle $\omega_X(D)^{-1}$ ([22, Tag 0BDC]), such that X together with the divisor $D = \pi^{-1}(V(a))$ is a log Fano variety.

3.2 The Cox rings over $\overline{\mathbb{Q}}$

We determine the Cox rings of X and of X' over $\overline{\mathbb{Q}}$ and we then lift the ring homomorphism from the previous section to obtain an isomorphism between the Cox rings of X and X' over $\overline{\mathbb{Q}}$.

The grading on a Cox ring of a scheme Z is induced by the Picard group of Z. For the blow-ups X and X', it follows from Exercise II.8.5 of [13] that their Picard groups are isomorphic to $\mathbb{Z}^2$. For X, the isomorphism outlined in the exercise is defined by sending the pullback of any plane in $\mathbb{P}^3_{\mathbb{Q}}$ under the morphism π to the element $[1,0] \in \mathbb{Z}^2$, and the exceptional divisor of the blow-up to to the element $[0,1] \in \mathbb{Z}^2$. For the blow-up X', the isomorphism is defined by sending the pullback of any plane in $\mathbb{P}^3_{\mathbb{Q}}$ under the morphism π' to the element $[1,0] \in \mathbb{Z}^2$, and the exceptional divisor of the blow-up to to the element $[0,1] \in \mathbb{Z}^2$.

Lemma 3.3. The Cox ring of X' over $\overline{\mathbb{Q}}$ is

$$R(X'_{\overline{\mathbb{Q}}}) = \overline{\mathbb{Q}}[a,b,c,x,y,z]/(a^2 + bc - yz)$$

and its grading by $\mathrm{Pic}(X'_{\overline{\mathbb{Q}}}) = \mathrm{Pic}(X') = \mathbb{Z}^2$ is given by

a	b	c	x	y	z
1	1	1	1	2	0
0	0	0	-1	-1	1

Proof. As X' is the blow up of $\mathbb{P}^3_{\mathbb{Q}}$ along a smooth conic, X' is in Case 30 of the classification of Fano varieties as in Table 12.3 of [20]. The above Cox ring then follows directly from Theorem 4.5, Case 30 of [7], where we note that there is a typo in the degrees of y and z. □

Lemma 3.4. The Cox ring of X over $\overline{\mathbb{Q}}$ is

$$R(X_{\overline{\mathbb{Q}}}) = \overline{\mathbb{Q}}[a,b,c,x,y,z]/(a^2 + b^2 + c^2 - yz)$$

and its grading by $\mathrm{Pic}(X_{\overline{\mathbb{Q}}}) = \mathrm{Pic}(X) = \mathbb{Z}^2$ is given by

a	b	c	x	y	z
1	1	1	1	2	0
0	0	0	-1	-1	1

Proof. As X is the blow up of $\mathbb{P}^3_{\mathbb{Q}}$ along a smooth conic, X is in Case 30 of the classification of Fano varieties as in Table 12.3 of [20]. The above Cox ring then follows directly from Theorem 4.5, Case 30 of [7], where we note that there is a typo in the degrees of y and z. □

Remark 3.5. As X and X' are isomorphic schemes over $\overline{\mathbb{Q}}$, their Cox rings over $\overline{\mathbb{Q}}$ are isomorphic, too. We construct an isomorphism $\widetilde{\varphi} : \mathcal{R}(X'_{\overline{\mathbb{Q}}}) \to \mathcal{R}(X_{\overline{\mathbb{Q}}})$ as follows. For a, b, c, remark that the isomorphism $\varphi^{\#}$ from the previous section lifts to the Cox rings, i.e., we set $\widetilde{\varphi}(a) = a$, $\widetilde{\varphi}(b) = b - ic$, $\widetilde{\varphi}(c) = b + ic$. As an isomorphism of Cox rings must respect the grading on the respective Cox rings, the following choices are sensible: $\widetilde{\varphi}(x) = x$, $\widetilde{\varphi}(y) = y$, $\widetilde{\varphi}(z) = z$. It is a straightforward verification that this choice of ring homomorphism $\widetilde{\varphi}$ indeed gives an isomorphism of rings over $\overline{\mathbb{Q}}$.

3.3 Torsors for the blow-ups

In this section we determine a torsor $\mathcal{T}$ for an integral model $\mathcal{X}$ of X, and a torsor $\mathcal{T}'$ for an integral model $\mathcal{X}'$ of X'. We will determine a 4-to-1 correspondence between integral points on $\mathcal{T}$ and integral points on X, and between integral points on $\mathcal{T}'$ and integral points on X'. Hence, one can study the integral points on these threefolds by studying the integral points on the corresponding torsors, and as the latter are lattice-points, this simplifies the counting problem.

Let us first determine torsors of X and X' over $\overline{\mathbb{Q}}$.

Lemma 3.6. The variety

$$T'_{\overline{\mathbb{Q}}} = \operatorname{Spec}(\mathcal{R}(X'_{\overline{\mathbb{Q}}})) - V(I'_{\text{irr}}),$$

where $I'_{\text{irr}} = (a, b, c, z)(x, y)$, is a torsor over $X'_{\overline{\mathbb{Q}}}$, and the variety

$$T_{\overline{\mathbb{Q}}} = \operatorname{Spec}(\mathcal{R}(X_{\overline{\mathbb{Q}}})) - V(I_{\text{irr}}),$$

where $I_{\text{irr}} = (a, b, c, z)(x, y)$, is a torsor over $X_{\overline{\mathbb{Q}}}$.

Proof. For the first part, see Lemma 2.1 of [24]. As the blow-ups X and X' are isomorphic over $\overline{\mathbb{Q}}$, the proof of Lemma 2.1 in [24] holds for the second part, too, and we only need to determine an appropriate irrelevant ideal using the isomorphism of Cox rings $\widetilde{\varphi}$ as determined in Remark 3.5. Observing that over $\overline{\mathbb{Q}}$,

$$\widetilde{\varphi}(I'_{\text{irr}}) = (a, b, c, z)(x, y)$$

gives the desired result. □

Consider the rings

$$\mathcal{R}_{\mathbb{Z}}(X) = \mathbb{Z}[a, b, c, x, y, z]/(a^2 + b^2 + c^2 - yz),$$

$$\mathcal{R}_{\mathbb{Z}}(X') = \mathbb{Z}[a, b, c, x, y, z]/(a^2 + bc - yz)$$

and the ideals

$$I_{\text{irr},\mathbb{Z}} = (a, b, c, z)(x, y) \subset \mathcal{R}_{\mathbb{Z}}(X),$$

$$I'_{\mathrm{irr},\mathbb{Z}} = (a,b,c,z)(x,y) \subset \mathcal{R}_{\mathbb{Z}}(X'),$$

Let

$$\pi : \mathcal{X} \to \mathbb{P}^3_{\mathbb{Z}}, \quad \pi' : \mathcal{X}' \to \mathbb{P}^3_{\mathbb{Z}}$$

be the two blow-ups of $\mathbb{P}^3_{\mathbb{Z}}$ along the smooth conics $C = V(a^2+b^2+c^2, d)$ and $C' = V(a^2+bc, d)$, respectively.

Lemma 3.7. The scheme $\mathcal{T} = \mathrm{Spec}(\mathcal{R}_{\mathbb{Z}}(X)) - V(I_{\mathrm{irr},\mathbb{Z}})$ is a $\mathbb{G}^2_{m,\mathcal{X}}$-torsor over $\mathcal{X}$ with the morphism

$$p : \mathcal{T} \to \mathcal{X},$$

and the scheme $\mathcal{T}' = \mathrm{Spec}(\mathcal{R}_{\mathbb{Z}}(X')) - V(I'_{\mathrm{irr},\mathbb{Z}})$ is a $\mathbb{G}^2_{m,\mathcal{X}'}$-torsor over $\mathcal{X}'$ with the morphism

$$p' : \mathcal{T}' \to \mathcal{X}'.$$

Proof. We prove the first statement of the lemma. Observe that

$$V(I_{\mathrm{irr}}) = V(\sqrt{I_{\mathrm{irr}}}) = V(\sqrt{(ax,bx,cx,zx,ay,by,cy)})$$

for yz is a redundant generator for the radical of I_{irr}:

$$(yz)^2 = (a^2+b^2+c^2)yz = ay\cdot az + by\cdot bz + cy\cdot cz \in \sqrt{(ax,bx,cx,zx,ay,by,cy)}.$$

Denote the remaining generators of $\sqrt{I_{\mathrm{irr}}}$ by $f_1,\dots,f_7$. Also observe, using the tables in Lemma 3.4, that the degrees of the two factors of any of the f_i form a basis of $\mathrm{Pic}(X_{\overline{\mathbb{Q}}})$.

Apply Construction 3.1 of [11] with $\overline{X} = X$, $\overline{K} = \overline{\mathbb{Q}}$, $A = \mathbb{Z}$ and $\overline{Y} = T_{\overline{\mathbb{Q}}}$, then indeed $\overline{Y} = \mathrm{Spec}(\mathcal{R}(X_{\overline{\mathbb{Q}}})) - V(f_1,\dots,f_7)$. As the degrees of the two factors of each generator f_i form a basis for the Picard group, conditions (3.1) and (3.3) of [11] are immediately satisfied. Hence, it follows from Theorem 3.3 of [11] that

$$p : \mathrm{Spec}(\mathcal{R}_{\mathbb{Z}}(X)) - V(I_{\mathrm{irr},\mathbb{Z}}) \to \widetilde{X}$$

is a $\mathbb{G}^2_{m,\widetilde{X}}$-torsor over $\widetilde{X}$, with $\widetilde{X}$ an integral model of X that is constructed in Construction 3.1 of [11]. The construction is as follows: for $i \in \{1,\dots,7\}$, let R_i be the degree-zero part of the ring $\mathcal{R}_{\mathbb{Z}}(X)[f_i^{-1}]$, then gluing the $\mathrm{Spec}(R_i)$ gives the scheme $\widetilde{X}$. This integral model $\widetilde{X}$ coincides with the blow-up $\mathcal{X}$, as we show below.

Consider the Rees algebra for $J = (a^2+b^2+c^2, d)$ given by

$$\mathcal{A} = \bigoplus_{n\geq 0} J^n = \mathbb{Z}[a,b,c,d][(a^2+b^2+c^2)\gamma, d\gamma].$$

Observe that we can trivially embed $\mathcal{A}$ into $\mathrm{Frac}(\mathcal{A}) = \mathbb{Q}(a,b,c,d,\gamma)$, and that the map given by $x \mapsto d\gamma$, $y \mapsto (a^2+b^2+c^2)\gamma$ and $z \mapsto \gamma^{-1}$ gives an embedding of $\mathcal{R}_{\mathbb{Z}}(X)$ into $\mathrm{Frac}(\mathcal{A})$.

As seen in Example 2.10, the blow-up $\mathcal{X}$ is obtained by gluing the spectra of the seven rings $\mathcal{A}_{s,t} \subset \mathrm{Frac}(\mathcal{A})$ that are defined as follows: Localize $\mathcal{A}$ at $s \in \{a,b,c,d\}$ and take the degree-0-part with respect to the usual grading on the polynomial ring $\mathbb{Z}[a,b,c,d]$. Then localize the obtained ring in $t \in \left\{\frac{a^2+b^2+c^2}{s^2}\gamma, \frac{d}{s}\gamma\right\}$ and again take the degree-0 part, now with respect to the grading coming from the Rees algebra $\mathcal{A}$.

Observe that the ring $\mathcal{A}_{a,\gamma d/a}$ obtained this way is equal to the ring $(\mathcal{R}_{\mathbb{Z}}(X)[f_1^{-1}])^{(0)}$: by the definition of localization and taking the degree-zero-part,

$$\left(\mathcal{R}_{\mathbb{Z}}(X)[f_1^{-1}]\right)^{(0)} = \left\{\frac{g}{a^n x^n} \ : \ n \in \mathbb{Z}_{\geq 0},\ g \in \mathcal{R}_{\mathbb{Z}}(X),\ \deg(g) = n\right\}$$

with the degree of g the degree of the Cox ring grading. Similarly,

$$\mathcal{A}_a^{(0)} = \left\{\frac{h}{a^m} \ : \ m \in \mathbb{Z}_{\geq 0},\ h \in \mathcal{A},\ \deg(h) = m\right\}$$

where the degree of h is the standard degree of an element of a polynomial ring. Localizing $\mathcal{A}_a^{(0)}$ with respect to $\gamma d/a$ gives

$$\left(\mathcal{A}_a^{(0)}\right)_{\gamma d/a} = \left\{\frac{h' a^{\widetilde{m}}}{(\gamma d)^{\widetilde{m}}} \ : \ h' \in \mathcal{A}_a^{(0)},\ \widetilde{m} \in \mathbb{Z}_{\geq 0}\right\}.$$

Taking the degree-0-part of this with respect to the grading coming from the Rees algebra gives

$$\mathcal{A}_{a,\gamma d/a} = \left\{\frac{h a^{\widetilde{m}}}{a^m (\gamma d)^{\widetilde{m}}} \ : \ m, \widetilde{m} \in \mathbb{Z}_{\geq 0},\ h \in \mathcal{A},\ \deg(h) = m,\ \deg_r(h) = \widetilde{m}\right\}$$

with $\deg_r$ the degree function coming from the Rees algebra. We know that $\deg_r(\gamma) = 1$ and all elements of $\mathbb{Z}[a,b,c,d]$ are considered of degree zero. It is now a straightforward verification that under the embedding of $\mathcal{R}_{\mathbb{Z}}(X)$ into $\mathrm{Frac}(\mathcal{A})$, the two rings $(\mathcal{R}_{\mathbb{Z}}(X)[f_i^{-1}])^{(0)}$ and $A_{a,\gamma d/a}$ are isomorphic.

Analogous to the above, one can show that the seven rings $\mathcal{A}_{s,t}$ are equal to the seven rings $(\mathcal{R}_{\mathbb{Z}}(X)[f_i^{-1}])^{(0)}$ with $i = 1, \ldots, 7$ for (s,t) given by

$$(a, \gamma d/a), (b, \gamma d/b), (c, \gamma d/c), (d, \gamma), (a, (a^2+b^2+c^2)a^{-2}\gamma),$$

$$(b, (a^2+b^2+c^2)b^{-2}\gamma), (c, (a^2+b^2+c^2)c^{-2}\gamma).$$

As the glueing morphisms commute with this isomorphism of rings, the schemes $\widetilde{X}$ and $\mathcal{X}$ coincide.

The proof for the second statement follows entirely analogously. □

Throughout the remainder of this thesis, let

$$\widetilde{\pi} : \mathcal{X} \to \mathbb{P}^3_{\mathbb{Z}}, \quad \widetilde{\pi}' : \mathcal{X}' \to \mathbb{P}^3_{\mathbb{Z}}$$

be the blow-ups of $\mathbb{P}^3_{\mathbb{Z}} = \mathrm{Proj}(\mathbb{Z}[a,b,c,d])$ along $V(a^2+b^2+c^2,d)$ and $V(a^2+bc,d)$, respectively. We also fix the following notation for the remainder of this thesis.

$$D = V(a) \subset \mathbb{P}^3_{\mathbb{Q}}, \quad \overline{D} = V(a) \subset \mathbb{P}^3_{\mathbb{Z}}, \quad U_1 = X - \pi^{-1}(D), \quad \mathcal{U}_1 = \mathcal{X} - \widetilde{\pi}^{-1}(\overline{D})$$
$$D_1' = V(b) \subset \mathbb{P}^3_{\mathbb{Q}}, \quad \overline{D_1'} = V(b) \subset \mathbb{P}^3_{\mathbb{Z}}, \quad U_1' = X' - \pi'^{-1}(D_1'), \quad \mathcal{U}_1' = \mathcal{X}' - \widetilde{\pi}'^{-1}(\overline{D_1'})$$
$$D_2' = V(a) \subset \mathbb{P}^3_{\mathbb{Q}}, \quad \overline{D_2'} = V(a) \subset \mathbb{P}^3_{\mathbb{Z}}, \quad U_2' = X' - \pi'^{-1}(D_2'), \quad \mathcal{U}_2' = \mathcal{X}' - \widetilde{\pi}'^{-1}(\overline{D_2'})$$

Observe that $\mathcal{U}_1$ is an integral model of U_1, $\mathcal{U}_1'$ is an integral model of U_1' and $\mathcal{U}_2'$ is an integral model of U_2'. As torsors are stable under base change, we have the following torsors over $\mathcal{U}_1$, $\mathcal{U}_1'$ and $\mathcal{U}_2'$, respectively.

$$\mathcal{T}_1 = \mathcal{T} - p^{-1}(\widetilde{\pi}^{-1}(V(a)), \quad \mathcal{T}_1' = \mathcal{T}' - p'^{-1}(\widetilde{\pi}'^{-1}(V(b)), \quad \mathcal{T}_2' = \mathcal{T}' - p'^{-1}(\widetilde{\pi}'^{-1}(V(a))$$

Lemma 3.8. The morphism p induces a 4-to-1 correspondence

$$\mathcal{T}(\mathbb{Z}) = \left\{(a,b,c,x,y,z) \in \mathbb{Z}^6 \;:\; {\scriptstyle a^2+b^2+c^2-yz=0, \atop \gcd(a,b,c,z)=\gcd(x,y)=1}\right\} \to \mathcal{X}(\mathbb{Z})$$

and a 4-to-1 correspondence

$$\mathcal{T}_1(\mathbb{Z}) = \left\{(a,b,c,x,y,z) \in \mathbb{Z}^6 \;:\; {\scriptstyle a^2+b^2+c^2-yz=0, \atop a=\pm 1,\ \gcd(x,y)=1}\right\} \to \mathcal{U}(\mathbb{Z}).$$

The morphism p' induces a 4-to-1 correspondence

$$\mathcal{T}'(\mathbb{Z}) = \left\{(a,b,c,x,y,z) \in \mathbb{Z}^6 \;:\; {\scriptstyle a^2+bc-yz=0, \atop \gcd(a,b,c,z)=\gcd(x,y)=1}\right\} \to \mathcal{X}'(\mathbb{Z}),$$

a 4-to-1 correspondence

$$\mathcal{T}_1'(\mathbb{Z}) = \left\{(a,b,c,x,y,z) \in \mathbb{Z}^6 \;:\; {\scriptstyle a^2+bc-yz=0, \atop b=\pm 1,\ \gcd(x,y)=1}\right\} \to \mathcal{U}_1'(\mathbb{Z})$$

and a 4-to-1 correspondence

$$\mathcal{T}_2'(\mathbb{Z}) = \left\{(a,b,c,x,y,z) \in \mathbb{Z}^6 \;:\; {\scriptstyle a^2+bc-yz=0, \atop a=\pm 1,\ \gcd(x,y)=1}\right\} \to \mathcal{U}_2'(\mathbb{Z}).$$

Proof. For any point $P \in \mathcal{X}(\mathbb{Z})$, its fiber $p^{-1}(P)$ is a $\mathbb{G}^2_{\mathrm{m},\mathbb{Z}}$-torsor. From [21, Section 2.2] we have a bijection

$$\left\{\mathbb{G}^2_{\mathrm{m},\mathbb{Z}}\text{-torsors up to isomorphism}\right\} \leftrightarrow H^1_{\mathrm{fppf}}(\mathrm{Spec}(\mathbb{Z}), \mathbb{G}^2_{m,\mathbb{Z}}).$$

From Proposition III.4.9 of [17] we know that $H^1_{\mathrm{fppf}}(\mathrm{Spec}(\mathbb{Z}), \mathbb{G}^2_{m,\mathbb{Z}}) = \mathrm{Pic}(\mathrm{Spec}(\mathbb{Z})) = 0$. Combining this, we find that all fibers $p^{-1}(P)$ are isomorphic to $\mathbb{G}^2_{m,\mathbb{Z}}$. As $\mathbb{G}^2_{\mathrm{m},\mathbb{Z}}(\mathbb{Z}) =$

$\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[x,x^{-1},y,y^{-1}],\mathbb{Z})$ and as such a ring homomorphism must send x and y to an invertible element of $\mathbb{Z}$, there are exactly four ring homomorphisms $\mathbb{Z}[x,x^{-1},y,y^{-1}] \to \mathbb{Z}$. Hence,

$$|\mathbb{G}^2_{m,\mathbb{Z}}(\mathbb{Z})| = 4$$

and we have a 4-to-1 correspondence between $\mathcal{X}(\mathbb{Z})$ and $\mathcal{T}(\mathbb{Z})$.

The proof for the 4-to-1 correspondence between $\mathcal{X}'(\mathbb{Z})$ and $\mathcal{T}'(\mathbb{Z})$ follows entirely analogously.

As $\mathcal{T}$ is a quasi-affine variety, we get a one-to-one correspondence between integral points on $\mathcal{T}$ and ring homomorphisms

$$\varphi^{\#} : \mathbb{Z}[a,b,c,x,y,z]/(a^2+b^2+c^2-yz) \to \mathbb{Z}$$

such that $\varphi^{\#}((a,b,c,z)(x,y)) = \mathbb{Z}$. It is immediate that this is equivalent to requiring that $\gcd(a,b,c,z) = \gcd(x,y) = 1$, which gives the expression for $\mathcal{T}(\mathbb{Z})$ as above.

For the integral points on $\mathcal{U}$, there is an added condition that $\varphi^{\#}((a)) = \mathbb{Z}$, which corresponds to requiring $(a) = 1$, or equivalently, $a = \pm 1$. This gives the expression for $\mathcal{T}_1(\mathbb{Z})$ as above.

The expressions for integral points on $\mathcal{T}_1'$ and $\mathcal{U}_1'$, and $\mathcal{T}_2'$ and $\mathcal{U}_2'$ follow entirely analogously. □

3.4 The height functions

In this section, we construct a height function on $\mathcal{T}$, and on $\mathcal{T}_1$ and $\mathcal{T}_2$, using the Weil height function as defined in Section 2.9.

Let us consider the rational map $\mathcal{T} \to \mathbb{P}^9_{\mathbb{Q}}$ induced by the log-anticanonical bundle $\omega_X(\pi^{-1}(D))^{-1}$ of X. Consider the isomorphism $\mathrm{Pic}(X) \cong \mathbb{Z}^2$ from Exercise II.8.5 of [13], which sends the pullback of any hyperplane in $\mathbb{P}^3_{\mathbb{Q}}$ under π to $[1,0]$ and the exceptional divisor to $[0,1]$. We have seen in Example 2.44 that through this isomorphism, we have the correspondence $-[\omega_X(\pi^{-1}(D))] = [3,-1]$. Hence, the rational map $\mathcal{T} \to \mathbb{P}^9_{\mathbb{Q}}$ is induced by the set of sections of $\mathcal{R}(X_{\overline{\mathbb{Q}}})$ of degree $[3,-1]$ that generate all sections of $\mathcal{R}(X_{\overline{\mathbb{Q}}})$ of degree $[3,-1]$. Observe that the following sections are all the monomials of degree $[3,-1]$ in the Cox ring:

$$a^2x,\ b^2x,\ c^2x,\ ay,\ by,\ cy,\ x^3z^2,\ xyz,\ ax^2z,\ bx^2z,\ cx^2z.$$

As $xyz = (a^2+b^2+c^2)x = a^2x+b^2x+c^2x$ and there are no relations between the remaining sections, we find the generating set

$$\{a^2x,\ b^2x,\ c^2x,\ ay,\ by,\ cy,\ x^3z^2, ax^2z,\ bx^2z,\ cx^2z\}.$$

With this, we obtain for any point $(a,b,c,x,y,z) \in \mathcal{T}(\mathbb{Z})$ the height function

$$H(a,b,c,x,y,z) = \max\{|a^2x|,|b^2x|,|c^2x|,|ay|,|by|,|cy|,|x^3z^2|,|ax^2z|,|bx^2z|,|cx^2z|\}.$$

To study integral points of bounded height on the blow-up X, we actually need a height function on $\mathcal{X}$ instead of on $\mathcal{T}$. For this, we consider the rational map $\mathcal{X} \to \mathbb{P}^9_{\mathbb{Q}}$ using the log-anticanonical line bundle $\omega_{\mathcal{X}}(\pi^{-1}(D))^{-1}$. As this is the same line bundle we used to construct the height function on $\mathcal{T}$, we get a commuting diagram

$$\mathcal{T} \longrightarrow \mathcal{X} \longrightarrow \mathbb{P}^9_{\mathbb{Q}}$$

As X is a proper scheme, every element of $X(\mathbb{Q})$ lifts to a unique point in $\mathcal{X}(\mathbb{Q}) = \mathcal{X}(\mathbb{Z})$, and we saw in the previous section that any element of $\mathcal{X}(\mathbb{Z})$ corresponds to four points $(a,b,c,x,y,z) \in \mathcal{T}(\mathbb{Z})$. By the above commuting diagram, all four points map to the same point in $\mathbb{P}^3_{\mathbb{Q}}(\mathbb{Q})$, so we obtain a well defined height function on X that can be given explicitly as

$$H(\mathrm{x}) = \max\{|a^2x|,|b^2x|,|c^2x|,|ay|,|by|,|cy|,|x^3z^2|,|ax^2z|,|bx^2z|,|cx^2z|\},$$

where x is the image of $(a,b,c,x,y,z) \in \mathcal{T}(\mathbb{Z})$ in $\mathcal{X}(\mathbb{Z})$.

On $\mathcal{T}_1'$ and $\mathcal{T}_2'$ one can construct height functions $\widetilde{H}(a,b,c,x,y,z)$ with respect to D_1' and D_2' entirely analogously. As $[D_1'] = [D_2']$ in $\mathrm{Pic}(X')$, both log-anticanonical bundles $\omega_{X'}(D_1')^{-1}$ and $\omega_{X'}(D_2')^{-1}$ correspond to degree $[3,-1]$ through the isomorphism $\mathrm{Pic}(X') \cong \mathbb{Z}^2$. Going through the above computations again, one finds that on $\mathcal{T}_1'$ and $\mathcal{T}_2'$, the height function is defined as

$$\widetilde{H}(a,b,c,x,y,z) = \max\{|a^2x|,|b^2x|,|c^2x|,|ay|,|by|,|cy|,|x^3z^2|,|ax^2z|,|bx^2z|,|cx^2z|\}.$$

To construct a height function on $\mathcal{X}'$, one obtains an analogous commuting diagram, giving rise to a well-defined height function on X'. Explicitly, it can be given by

$$H(\mathrm{x}) = \max\{|a^2x|,|b^2x|,|c^2x|,|ay|,|by|,|cy|,|x^3z^2|,|ax^2z|,|bx^2z|,|cx^2z|\}$$

where x is the image of $(a,b,c,x,y,z) \in \mathcal{T}'(\mathbb{Z})$ in $\mathcal{X}'(\mathbb{Z})$.

3.5 The counting problems

In this section we combine the parametrizations of integral points on $\mathcal{U}$, $\mathcal{U}_1'$ and $\mathcal{U}_2'$ from section 3.3 with the height functions determined in section 3.4 to obtain explicit counting functions.

First consider $U \subseteq X$. Let $N(B)$ be the counting function

$$N(B) = \#\{\mathrm{x} \in \mathcal{U}(\mathbb{Z}) \cap V(\mathbb{Q}) \;:\; H(\mathrm{x}) \le B\}$$

that counts the integral points of bounded height on $\mathcal{U} = \mathcal{X} - \widetilde{\pi}^{-1}(\overline{V(a)})$ that, as rational points, are in the complement V of $V(abcxz)$. Using the four-to-one correspondence obtained earlier and observing the symmetry in $a = \pm 1$, this counting function becomes

$$N(B) = \frac{1}{2}\#\left\{(b,c,x,y,z) \in \mathbb{Z}^5 \ : \ \substack{1+b^2+c^2-yz=0,\ \gcd(x,y)=1,\\ H(1,b,c,x,y,z)\leq B,\\ b,c,x,z\neq 0}\right\}.$$

Consider $U_1' \subset X'$. The counting function

$$N_1(B) = \#\{\mathrm{x} \in \mathcal{U}_1'(\mathbb{Z}) \cap V(\mathbb{Q}) \ : \ H(\mathrm{x}) \leq B\}$$

counts the integral points of bounded height on $\mathcal{U}_1' = \mathcal{X}' - \widetilde{\pi}'^{-1}(\overline{V(b)})$ that, as rational points, are in the complement V of $V(abxz)$. With the four-to-one correspondence above and because of the symmetry in $b = \pm 1$, this counting function becomes

$$N_1(B) = \frac{1}{2}\#\left\{(a,c,x,y,z) \in \mathbb{Z}^5 \ : \ \substack{a^2+c-yz=0,\ \gcd(x,y)=1,\\ H(a,1,c,x,y,z)\leq B,\\ a,x,z\neq 0}\right\}.$$

Now consider $U_2' \subset X'$. The counting function

$$N_2(B) = \#\{\mathrm{x} \in \mathcal{U}_2'(\mathbb{Z}) \cap V(\mathbb{Q}) \ : \ H(\mathrm{x}) \leq B\}$$

counts the integral points of bounded height on $\mathcal{U}_2' = \mathcal{X}' - \widetilde{\pi}'^{-1}(\overline{V(a)})$ that, as rational points, are in the complement V of $V(abcxz)$. With the four-to-one correspondence above and because of the symmetry in $a = \pm 1$, this counting function becomes

$$N_2(B) = \frac{1}{2}\#\left\{(b,c,x,y,z) \in \mathbb{Z}^5 \ : \ \substack{1+bc-yz=0,\ \gcd(x,y)=1,\\ H(1,b,c,x,y,z)\leq B,\\ b,c,x,z\neq 0}\right\}.$$

In all three counting functions, the height function is defined as

$$H(a,b,c,x,y,z) = \max\{|a^2x|, |b^2x|, |c^2x|, |z^2x^3|, |ay|, |by|, |cy|, |ax^2z|, |bx^2z|, |cx^2z|\}.$$

However, as $(ax^2z)^2 = a^2x \cdot x^3z^2$, the condition $|ax^2z| \leq B$ is redundant in the counting problems. Analogously, the conditions $|bx^2z| \leq B$ and $|cx^2z| \leq B$ are redundant. Hence, without loss of generality, we can use the height function

$$H(a,b,c,x,y,z) = \max\{|a^2x|, |b^2x|, |c^2x|, |z^2x^3|, |ay|, |by|, |cy|\}$$

for all three counting functions.

4 Counting Integral Points

In this chapter we study the counting functions $N(B)$, $N_1(B)$ and $N_2(B)$ defined in the previous chapter. The first section outlines the steps Florian Wilsch takes in Section 4 of [24] to determine an asymptotic expression for $N_1(B)$. In the second section we study the counting function $N_2(B)$ and summarize the steps taken in Section 5 of [24]. In the third section we determine the obstruction to applying the techniques from section 5 in [24] to the counting function $N(B)$, and we determine an upper bound for $N(B)$ in section 4.

4.1 The counting function $N_1(B)$

In this section, we consider the counting function

$$N_1(B) = \frac{1}{2}\#\{(a,c,x,y,z) \in \mathbb{Z}^5 \;:\; {\scriptstyle a^2+c-yz=0,\ \gcd(x,y)=1, \atop H(a,1,c,x,y,z)\le B,\ a,x,z\neq 0}\},$$

with $H(a,1,c,x,y,z) = \max\{|a^2x|,|x|,|c^2x|,|z^2x^3|,|ay|,|y|,|cy|\}$.

Setting $c = yz - a^2$ simplifies the counting function to

$$N_1(B) = \frac{1}{2}\#\{(a,x,y,z) \in \mathbb{Z}^4 \;:\; {\scriptstyle \gcd(x,y)=1, \widetilde{H}_1(a,x,y,z)\le B, \atop a,x,z\neq 0}\},$$

with $\widetilde{H}_1(a,x,y,z) = H(a,1,yz-a^2,x,y,z)$.

Lemma 4.1. We have

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha}\sum_{x',z\in\mathbb{Z}_{\neq 0}}\int_{\substack{|a^2\alpha x'|,|c^2\alpha x'|,\\ |a(a^2+c)z^{-1}|,|c(a^2+c)z^{-1}|,\\ |\alpha^2x'^3z^2|\le B,|a|\ge 1}}\frac{1}{|z|}\mathrm{d}a\mathrm{d}c + O(B).$$

Proof. First apply Möbius inversion as seen in Section 2.10 to get rid of the gcd-condition and obtain

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\mu(\alpha)\sum_{a,x',z\in\mathbb{Z}_{\neq 0}}\#\{y'\in\mathbb{Z} \;:\; \widetilde{H_1}(a,\alpha x',\alpha y',z)\le B\},$$

with $x' = x/\alpha$ and $y' = y/\alpha$. Observe that

$$\#\{y'\in\mathbb{Z} \;:\; \widetilde{H_1}(a,\alpha x',\alpha y',z)\le B\} = \int_{\widetilde{H_1}(a,\alpha x',\alpha y',z)\le B} 1\mathrm{d}y' + O(1),$$

and set

$$V_1(\alpha,a,x',z;B) = \int_{\widetilde{H_1}(a,\alpha x',\alpha y',z)\le B} 1\mathrm{d}y'.$$

Taking the error term $O(1)$ out of the sums over α, a, x', z and forgetting all but two conditions coming from the height condition gives an error term bounded by

$$\sum_{\substack{\alpha>0,\ a,x',z\in\mathbb{Z}_{\neq 0},\\ |a^2\alpha x'|,|\alpha^3 z^2 x'^3|\leq B}} 1. \tag{1}$$

In Equation (1), the sum over a is bounded by

$$\ll \int_{1\leq|a|\leq\lceil B^{1/2}|\alpha x'|^{-1/2}\rceil} 1\mathrm{d}a \ll B^{1/2}\alpha^{-1/2}|x'|^{-1/2}$$

and the sum over z is bounded by

$$\ll \int_{1\leq|z|\leq\lceil B^{1/2}|\alpha x'|^{-3/2}\rceil} 1\mathrm{d}z \ll B^{1/2}\alpha^{-3/2}|x'|^{-3/2}$$

giving the following upper bound for the error term in equation (1)

$$\ll B \sum_{\alpha>0,\ x'\in\mathbb{Z}_{\neq 0}} \frac{1}{|\alpha x'|^2}.$$

Here, the sum over x' is bounded by

$$\ll 1 + \int_{x'=1}^{\infty} x'^{-2}\mathrm{d}x' = 2$$

and the sum over α is bounded by

$$\ll 1 + \int_{\alpha=1}^{\infty} \alpha^{-2}\mathrm{d}\alpha = 2$$

such that the error term in Equation (1) is bounded by $O(B)$.

The counting function is then given by

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\mu(\alpha)\sum_{a,x',z\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B) + O(B).$$

To change the sum over a into an integral over a, we apply Lemma 3.6 of [6] with $t = a,\ \mathbf{x} = y',\ \mathbf{y} = (\alpha, x', z), M = \{(\alpha, x', z, a, y') \in \mathbb{R}^5 : \widetilde{H}_1(a, \alpha x', \alpha y', z) \leq B\}$ and $f : M \to \mathbb{R},\ (\alpha, x', z, a, y') \mapsto 1$. As M is defined by a finite number of inequalities, it is a semi-algebraic set. The graph of the function f is given by $\{(\alpha, x', z, a, y', 1) \in \mathbb{R}^6 : \widetilde{H}_1(a, \alpha x', \alpha y', z) \leq B\}$, which is again a semi-algebraic set as it is defined by a finite number of inequalities. As $f(\mathbf{y}, t, \cdot)$ is a constant function on $M_{\mathbf{y},t} = \{\mathbf{x} \in \mathbb{R} : (\mathbf{y}, t, \mathbf{x}) \in M\}$, it is integrable. Then Lemma 3.6 of [6] gives that there exists a constant $C \in \mathbb{Z}_{>0}$ such that for all $\mathbf{y} \in \mathbb{R}^3$, there exists a partition of $\mathbb{R}$ into at most C intervals I on whose interior $V_{\mathbf{y}}(t) = \int_{\mathbf{x}\in M_{\mathbf{y},t}} f\mathrm{d}x$ is continuously differentiable and monotonic.

Remark that with our choices, $V_{\mathbf{y}}(t) = V_1(\alpha, a, x', z; B)$. On each interval I, the difference between $\sum_{a\in I} V_1(\alpha, a, x', z; B)$ and $\int_{a\in I} V_1(\alpha, a, x', z; B)\mathrm{d}a$ can be bounded by $2\sup_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B)$. Hence,

$$\left|\sum_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B) - \int_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B)\mathrm{d}a\right| \leq C \sup_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B),$$

where C is the constant from Lemma 3.6 of [6]. Hence, we have

$$\sum_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B) = \int_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B)\mathrm{d}a + O(\sup_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B)).$$

We take the error term $O(\sup_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B))$ out of the sum over α, x', z and see that it can be bounded by

$$\ll \sum_{\substack{\alpha>0, x', z\in\mathbb{Z}_{\neq 0}, \\ |\alpha^3 x'^3 z^2|\leq B}} \sup_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B). \tag{2}$$

To determine the supremum of $V_1(\alpha, a, x', z; B)$ with respect to a, we only consider the condition $|(\alpha y' z - a^2)^2 \alpha x'| \leq B$ coming from the height condition. For $z > 0$, this condition is equivalent to

$$\left(\frac{-B^{1/2}}{\alpha^{1/2}|x'|^{1/2}} + a^2\right) z^{-1}\alpha^{-1} \leq y' \leq \left(\frac{B^{1/2}}{\alpha^{1/2}|x'|^{1/2}} + a^2\right) z^{-1}\alpha^{-1},$$

with which $V_1(\alpha, a, x', z; B)$ can be bounded by

$$\ll \frac{2B^{1/2}}{\alpha^{1/2}|x'|^{1/2}} \alpha^{-1}|z|^{-1}.$$

For $z < 0$, one obtains analogously the equivalent condition

$$\left(\frac{-B^{1/2}}{\alpha^{1/2}|x'|^{1/2}} + a^2\right) z^{-1}\alpha^{-1} \geq y' \geq \left(\frac{B^{1/2}}{\alpha^{1/2}|x'|^{1/2}} + a^2\right) z^{-1}\alpha^{-1},$$

with which $V_1(\alpha, a, x', z; B)$ can be bounded by

$$\ll \frac{-2B^{1/2}}{\alpha^{1/2}|x'|^{1/2}} \alpha^{-1} z^{-1} = \frac{2B^{1/2}}{\alpha^{1/2}|x'|^{1/2}} \alpha^{-1}|z|^{-1}.$$

Hence,

$$V_1(\alpha, a, x', z; B) \ll \frac{2B^{1/2}}{\alpha^{1/2}|x'|^{1/2}} \alpha^{-1}|z|^{-1}.$$

As this bound is independent of the variable a,

$$\sup_{a\in\mathbb{Z}_{\neq 0}} V_1(\alpha, a, x', z; B) \ll \frac{B^{1/2}}{\alpha^{3/2}|x'|^{1/2}|z|^1} \ll \frac{B^{1/2}}{\alpha^{1/2}|x'|^{1/2}|z|}$$

where the last inequality is justified as $\alpha \geq 1$. Hence, the error term in Equation (2) can be bounded by

$$\ll \sum_{\substack{\alpha>0, x', z \in \mathbb{Z}_{\neq 0}, \\ |\alpha^3 x'^3 z^2| \leq B}} \frac{B^{1/2}}{\alpha^{1/2} |x'|^{1/2} |z|}. \tag{3}$$

These sums in Equation (3) can be bounded analogously to the sums following Equation (1) and we obtain that the sum over x' can be bounded by

$$\ll \frac{B^{1/2}}{\alpha^{1/2}|z|} B^{1/6} |z|^{-1/3} \alpha^{-1/2}.$$

The sum over z can be bounded by

$$\ll \sum_{z \in \mathbb{Z}_{\neq 0},\, |\alpha^3 z^2| \leq B} |z|^{-4/3} \ll 1 + \int_1^{B^{1/2}\alpha^{-3/2}} z^{-4/3} \mathrm{d}z \ll 4 - 3B^{-1/6} \ll 4,$$

with which the sum over α can be bounded by

$$\ll \sum_{\alpha>0} \alpha^{-3/2} \ll 1 + \int_1^{B^{1/3}} \alpha^{-3/2} \mathrm{d}\alpha \ll 2 - 2B^{-1/6} \ll 2.$$

Combining this, the error term in Equation (2) can be bounded by $O(B^{2/3})$, with which we obtain

$$N_1(B) = \frac{1}{2} \sum_{\alpha>0} \mu(\alpha) \sum_{x', z \in \mathbb{Z}_{\neq 0}} \int_{|a| \geq 1} V_1(\alpha, a, x', z; B) \mathrm{d}a + O(B).$$

Set $V_2(\alpha, x', z; B) = \int_{|a|\geq 1} V_1(\alpha, a, x', z; B) \mathrm{d}a$. Setting $c = \alpha y' |z| - a^2$ and applying the chain rule gives the expression

$$V_2(\alpha, x', z; B) = \int_{\substack{|a^2 \alpha x'|, |\alpha x'|, |c^2 \alpha x'|, \\ |a(a^2+c) z^{-1}|, |(a^2+c) z^{-1}|, |c(a^2+c) z^{-1}|, \\ |\alpha^2 x'^3 z^2| \leq B, |a| \geq 1}} \frac{1}{\alpha |z|} \mathrm{d}a \mathrm{d}c.$$

Observing that the conditions $|\alpha x'| \leq B$ and $|(a^2+c) z^{-1}| \leq B$ are redundant gives the desired expression. $\square$

Lemma 4.2. We have

$$N_1(B) = \frac{1}{2} \sum_{\alpha>0} \frac{\mu(\alpha)}{\alpha^2} \int_{\substack{|a^2 x|, |a^3 z^{-1}|, |c^2 x|, \\ |a^2 c z^{-1}|, |x^3 z^2| \leq B \\ |a|, |z| \geq 1, |x| \geq \alpha}} \frac{1}{|z|} \mathrm{d}a \mathrm{d}c \mathrm{d}x \mathrm{d}z + O(B).$$

Proof. We first move the factor α^{-1} out of $V_2(\alpha, x', z; B)$. We replace the conditions $|a(a^2+c) z^{-1}| \leq B$ and $|c(a^2+c) z^{-1}| \leq B$ in the integral by $|a^3 z^{-1}| \leq B$ and $|c a^2 z^{-1}| \leq B$, i.e., we replace $a^2 + c$ by a^2, which gives the new integral

$$V_2'(\alpha, x', z; B) = \int_{\substack{|a^2 \alpha x'|, |c^2 \alpha x'|, \\ |a^3 z^{-1}|, |c a^2 z^{-1}|, \\ |\alpha^2 x'^3 z^2| \leq B, |a| \geq 1}} \frac{1}{|z|} \mathrm{d}a \mathrm{d}c.$$

However, both of these replacements introduce an error term. Consider first the error term introduced by replacing $|a(a^2+c)z^{-1}| \le B$ by $|a^3z^{-1}| \le B$. This error term can be bounded by

$$\ll \int_{\substack{|a^2\alpha x'|,|c^2\alpha x'|,\\ |a^3z^{-1}|,|c(a^2+c)z^{-1}|,\\ |\alpha^2x'^3z^2|\le B,|a|\ge 1,\\ |a^3z^{-1}+caz^{-1}|>B,\ |a^3z^{-1}|\le B}} \frac{1}{|z|}\mathrm{d}a\mathrm{d}c. \tag{4}$$

The inequalities $|a^3z^{-1}+caz^{-1}| > B,\ |a^3z^{-1}| \le B$ are equivalent to $B - \left|\frac{ac}{z}\right| \le \left|\frac{a^3}{z}\right| \le B + \left|\frac{ac}{z}\right|$, which in turn is equivalent to $\left|a^2 - \frac{B|z|}{|a|}\right| \le |c|$. By setting $a' = a^2 - B|z||a|^{-1}$, this last inequality reduces to $|a'| \le |c|$ and we have that $\left|\frac{\mathrm{d}a'}{\mathrm{d}a}\right| = 2|a| + \frac{B|z|}{|a|^2} \le \sqrt{|a'|}$. Hence, we can bound the error term in Equation (4) by

$$\ll \int_{\substack{|a'|\le|c|,\\ |a^2\alpha x'|,|c^2\alpha x'|,|c(a^2+c)z^{-1}|,\\ |\alpha^3x'^3z^2|\le B,|a|\ge 1}} \frac{1}{\sqrt{|a'|}|z|}\mathrm{d}a'\mathrm{d}c. \tag{5}$$

To write $N_1(B)$ with $V_2'(\alpha, x', z; B)$ instead of $V_2(\alpha, x', z; B)$, we take the bound in Equation (5) out of the sums over α, x', z to obtain an error term bounded by

$$\ll \sum_{\alpha>0} \frac{1}{\alpha} \sum_{x',z\in\mathbb{Z}_{\neq 0}} \int_{\substack{|a'|\le|c|,\\ |\alpha x'c^2|,|\alpha^3x'^3z^2|\le B}} \frac{1}{\sqrt{|a'|}|z|}\mathrm{d}a'\mathrm{d}c. \tag{6}$$

As the integral over a' can be bonded by $\ll 4\sqrt{|c|}$, and the integral over c can be bounded by $\ll B^{\frac{3}{4}}|x'|^{-\frac{3}{4}}\alpha^{-\frac{3}{4}}$, the expression in Equation (6) can be bounded by

$$\ll B^{\frac{3}{4}} \sum_{\alpha>0} \frac{1}{\alpha^{\frac{7}{4}}} \sum_{\substack{x',z\in\mathbb{Z}_{\neq 0},\\ |\alpha^3x'^3z^2|\le B}} |x'|^{-\frac{3}{4}}|z|^{-1}. \tag{7}$$

Analogous to the computations following Equation (1), we can bound the sum over x' in Equation (7) by

$$\ll 1 + \int_1^{\lceil B^{\frac{1}{3}}\alpha^{-1}|z|^{-\frac{2}{3}}\rceil} x'^{-\frac{3}{4}}\mathrm{d}x' \ll B^{\frac{1}{12}}\alpha^{-\frac{1}{4}}|z|^{-\frac{1}{6}}$$

and the sum over z by

$$\ll 1 + \int_0^{\lceil B^{1/2}\alpha^{-3/2}\rceil} z^{-7/6}\mathrm{d}z \ll 14 - 12B^{-1/12}\alpha^{1/4} \ll 14.$$

As $\alpha \ge 1$, $\alpha^{-1/4} \le 1$ and can be forgotten to obtain a larger bound. Hence, the expression in Equation (7) can be bounded by

$$\ll B^{\frac{5}{6}} \sum_{\alpha>0} \alpha^{-\frac{7}{4}}.$$

Here, the sum over α can be bounded by

$$\ll 1 + \int_1^\infty \alpha^{-\frac{7}{4}} \mathrm{d}\alpha \ll \frac{7}{4},$$

with which we obtain that the error term in Equation (6) can be bounded by $O(B^{5/6})$.

The computations for replacing $|c(a^2+c)z^{-1}| \le B$ by $|ca^2 z^{-1}| \le B$ can be done analogously to show that the error term introduced by this replacement is also bounded by $O(B^{5/6})$. Combining this, we obtain

$$N_1(B) = \frac{1}{2} \sum_{\alpha>0} \mu(\alpha) \sum_{x', z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B) + O(B). \tag{8}$$

We now change the sum over z into an integral over z, for which we apply Lemma 3.6 of [6] with $t = z$, $\mathbf{x} = (a, c)$, $\mathbf{y} = (\alpha, x')$, $M = \left\{ (\alpha, x', z, a, c) \in \mathbb{Z}^5 \ : \ \substack{|a^2\alpha x'|, |a^3 z^{-1}|, \\ |c^2 \alpha x'|, |ca^2 z^{-1}|, \\ |\alpha^3 x'^3 z^2| \le B,\ |a| \ge 1} \right\}$ and $f : M \to \mathbb{R}$, $(\alpha, x', z, a, c) \mapsto \frac{1}{|z|}$. As M is defined by a finite number of inequalities, it is a semi-algebraic set. The graph of the function f is given by $\{(\alpha, x', z, a, c, \frac{1}{|z|}) \in \mathbb{R}^6 \ : \ \widetilde{H}_1(a, \alpha x', \alpha y', z) \le B\}$, which is again a semi-algebraic set as it is defined by a finite number of inequalities. As $f(\mathbf{y}, t, \cdot)$ is a constant function on $M_{\mathbf{y},t} = \{\mathbf{x} \in \mathbb{R} \ : \ (\mathbf{y}, t, \mathbf{x}) \in M\}$, it is integrable. Hence, Lemma 3.6 of [6] gives that there exists a constant $C \in \mathbb{Z}_{>0}$ such that for all $\mathbf{y} \in \mathbb{R}^2$, there exists a partition of $\mathbb{R}$ into at most C intervals I on whose interior $V_{\mathbf{y}}(t) = \int_{\mathbf{x} \in M_{\mathbf{y},t}} f \mathrm{d}x$ is continuously differentiable and monotonic.

Remark that $V_{\mathbf{y}}(t) = V_2'(\alpha, x', z; B)$. On each such interval I, the difference between $\sum_{z \in I} V_2'(\alpha, x', z; B)$ and $\int_{z \in I} V_2'(\alpha, x', z; B)$ can be bounded by $2 \sup_{z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B)$. Hence,

$$\left| \sum_{z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B) - \int_{z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B) \mathrm{d}z \right| \le C \sup_{z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B),$$

where C is the constant from Lemma 3.6 of [6]. Hence, we have

$$\sum_{z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B) = \int_{z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B) \mathrm{d}z + O(\sup_{z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B)).$$

Observe that the function $V_2'(\alpha, x', z; B)$ can be bounded by

$$\ll \int_{\substack{|a^3 z^{-1}|, |c^2 \alpha x'| \le B, \\ |a| \ge 1}} \frac{1}{|z|} \mathrm{d}a \mathrm{d}c \ll B^{5/6} \alpha^{-1/2} |x'|^{-1/2} |z|^{-2/3}.$$

It is clear that this bound is largest when $|z|$ is smallest, i.e., when $|z| = 1$, with which

$$\sup_{z \in \mathbb{Z}_{\neq 0}} V_2'(\alpha, x', z; B) \ll B^{5/6} \alpha^{-1/2} |x'|^{-1/2}.$$

Taking the error term $\sup_{z\in\mathbb{Z}_{\neq 0}} V_2'(\alpha,x',z;B)$ out of the sums over α and x' in Equation (8), the total error term can be bounded by

$$\ll \sum_{\alpha>0}\alpha^{-3/2}\sum_{1\le|x'|\le B^{1/3}}\frac{B^{5/6}}{|x'|^{1/2}}. \tag{9}$$

Analogous to the computations following Equation (1), the sum over x' can be bounded by

$$\ll 1+\int_1^{\lceil B^{1/3}\rceil} x'^{-1/2}\mathrm{d}x' \ll B^{1/6}$$

and the sum over α can be bounded by

$$\ll 1+\int_1^{\infty}\alpha^{-3/2}\mathrm{d}\alpha \ll 1,$$

with which we can bound the expression in Equation (9) by $O(B)$. We obtain the expression

$$N_1(B)=\frac{1}{2}\sum_{\alpha>0}\mu(\alpha)\sum_{x'\in\mathbb{Z}_{\neq 0}}\int_{|z|\ge 1} V_2'(\alpha,x',z;B)\mathrm{d}z+O(B). \tag{10}$$

Let us now fix the notation

$$V_3(\alpha,x';B)=\int_{|z|\ge 1} V_2'(\alpha,x',z;B)\mathrm{d}z.$$

To now replace the sum over x' by an integral over x', we apply Lemma 3.6 of [6] with $t=x'$, $\mathbf{x}=(a,c,z)$, $\mathbf{y}=\alpha, M=\left\{(\alpha,x',a,c,z)\in\mathbb{Z}^5 : \begin{smallmatrix}|a^2\alpha x'|,|a^3z^{-1}|,\\ |c^2\alpha x'|,|ca^2z^{-1}|,\\ |\alpha^3x'^3z^2|\le B,\ |a|,|z|\ge 1\end{smallmatrix}\right\}$ and $f: M\to\mathbb{R},\ (\alpha,x',a,c,z)\mapsto\frac{1}{|z|}$. As M is defined by a finite number of inequalities, it is a semi-algebraic set. The graph of the function f is given by $\{(\alpha,x',a,c,z,\frac{1}{|z|})\in\mathbb{R}^6 : \widetilde{H}_1(a,\alpha x',\alpha y',z)\le B\}$, which is again a semi-algebraic set as it is defined by a finite number of inequalities. As $f(\mathbf{y},t,\cdot)$ is a constant function on $M_{\mathbf{y},t}=\{\mathbf{x}\in\mathbb{R} : (\mathbf{y},t,\mathbf{x})\in M\}$, it is integrable, so we know that there exists a constant $C\in\mathbb{Z}_{>0}$ such that for all $\mathbf{y}\in\mathbb{R}$, there exists a partition of $\mathbb{R}$ into at most C intervals I on whose interior $V_{\mathbf{y}}(t)=\int_{\mathbf{x}\in M_{\mathbf{y},t}} f\mathrm{d}x$ is continuously differentiable and monotonic.

Remark that $V_{\mathbf{y}}(t)=V_3(\alpha,x';B)$. On each such interval I, the difference between $\sum_{x'\in I}V_3(\alpha,x';B)$ and $\int_{x'\in I}V_3(\alpha,x';B)$ can be bounded by $2\sup_{x'\in\mathbb{Z}_{\neq 0}}V_3(\alpha,x';B)$. Hence,

$$\left|\sum_{x'\in\mathbb{Z}_{\neq 0}}V_3(\alpha,x';B)-\int_{x'\in\mathbb{Z}_{\neq 0}}V_3(\alpha,x';B)\mathrm{d}x'\right|\le C\sup_{x'\in\mathbb{Z}_{\neq 0}}V_3(\alpha,x';B),$$

where C is the constant from Lemma 3.6 of [6]. Hence, we have

$$\sum_{x'\in\mathbb{Z}_{\neq 0}}V_3(\alpha,x';B)=\int_{x'\in\mathbb{Z}_{\neq 0}}V_3(\alpha,x';B)\mathrm{d}x'+O(\sup_{x'\in\mathbb{Z}_{\neq 0}}V_3(\alpha,x';B)).$$

Using the upper bound we found earlier for $V_2'(\alpha, x', z; B)$, we see that

$$V_3(\alpha, x' : B) \ll \int_{|\alpha^3 x'^3 z^2| \le B} \frac{B^{5/6}}{\alpha^{1/2}|x'|^{1/2}|z|^{2/3}} \mathrm{d}z \ll B\alpha^{-1}|x'|^{-1}.$$

This bound is largest when $|x'|$ is smallest, i.e., when $|x'| = 1$, which gives

$$\sup_{x' \in \mathbb{Z}_{\neq 0}} V_3(\alpha, x' : B) \ll B\alpha^{-1}.$$

Taking the error term $\mathcal{O}(\sup_{x' \in \mathbb{Z}_{\neq 0}} V_3(\alpha, x' : B))$ out of the sum over α in Equation (10) gives that we can bound the total error term by

$$B \sum_{\alpha > 0} \mu(\alpha)\alpha^{-2} \ll B\left(1 + \int_1^\infty \alpha^{-2} \mathrm{d}\alpha\right) \ll B.$$

This combines to the expression

$$N_1(B) = \frac{1}{2} \sum_{\alpha > 0} \mu(\alpha)\alpha^{-1} V_3(\alpha, x'; B) + O(B).$$

Changing the variable $\alpha x'$ to x and using the chain rule (i.e., $dx = \alpha dx'$) then gives the desired expression:

$$N_1(B) = \frac{1}{2} \sum_{\alpha > 0} \frac{\mu(\alpha)}{\alpha^2} \int_{\substack{|a^2 x|, |a^3 z^{-1}|, |c^2 x|, \\ |a^2 c z^{-1}|, |x^3 z^2| \le B \\ |a|, |z| \ge 1, |x| \ge \alpha}} \frac{1}{|z|} \mathrm{d}a \mathrm{d}c \mathrm{d}x \mathrm{d}z + O(B). \qquad \square$$

Proposition 4.3. The number of integral points of bounded height on $\mathcal{U}_1$ satisfies the asymptotic formula

$$N_1(B) = \frac{20}{3\zeta(2)} B \log(B) + O(B).$$

Proof. First, we omit the condition that $|a| \ge 1$, which only adds the case $a = 0$. This introduces an error term bounded by

$$\ll \sum_{\alpha > 0} \frac{\mu(\alpha)}{\alpha^2} \int_{\substack{|c^2 x|, |x^3 z^2| \le B, \\ |z| \ge 1, |x| \ge \alpha}} \frac{1}{|z|} \mathrm{d}c \mathrm{d}x \mathrm{d}z, \tag{11}$$

which we obtained by setting $a = 0$ in the integral from the previous lemma. As $|z| \ge 1$, we can bound the above integral by the integral of 1. Computing the integrals over c, then z and then x gives that the error term in Equation (11) can be bounded by

$$\ll \sum_{\alpha > 0} B\alpha^{-3}.$$

With

$$\sum_{\alpha > 0} \alpha^{-3} \ll 1 + \int_1^\infty \alpha^{-3} \mathrm{d}\alpha \ll \frac{3}{2}$$

the error term in Equation (11) can be bounded by $O(B)$.

We now change variables: set $a' = az^{-1/3}B^{-1/2}$, $c' = cz^{-1/3}B^{-1/3}$ and $x' = xz^{2/3}B^{-1/3}$. With this,

$$\mathrm{d}a' = \mathrm{d}az^{-1/3}B^{-1/2},\ \mathrm{d}c' = \mathrm{d}cz^{-1/3}B^{-1/3},\ \mathrm{d}x' = \mathrm{d}xz^{2/3}B^{-1/3}$$

and hence, $\mathrm{d}a'\mathrm{d}c'\mathrm{d}x' = B^{-1}\mathrm{d}a\mathrm{d}c\mathrm{d}x$.

With this, we find the expression

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2}\int_{\substack{|a'^2x'|,|c'^2x'|,|a'|,\\ |a'^2c'|,|x'|\le 1,\\ 1\le|z|\le|x'|^{3/2}B^{1/2}\alpha^{-3/2}}}\frac{B}{|z|}\mathrm{d}a'\mathrm{d}c'\mathrm{d}x'\mathrm{d}z + O(B).$$

For the remainder of this section, we fix the notation $a = a'$, $c = c'$ and $x = x'$, with which the expression for $N_1(B)$ becomes

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2}\int_{\substack{|a^2x|,|c^2x|,|a|,\\ |a^2c|,|x|\le 1,\\ 1\le|z|\le|x|^{3/2}B^{1/2}\alpha^{-3/2}}}\frac{B}{|z|}\mathrm{d}a\mathrm{d}c\mathrm{d}x\mathrm{d}z + O(B).$$

Computing the integral with respect to z gives

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2}\int_{\substack{|a^2x|,|c^2x|,|a|,\\ |a^2c|,|x|\le 1,\\ 1\le|x|^{3/2}B^{1/2}\alpha^{-3/2}}} B\log(|x|^3B\alpha^{-3})\mathrm{d}a\mathrm{d}c\mathrm{d}x + O(B).$$

We then want to omit the condition $1 \le |x|^{3/2}B^{1/2}\alpha^{-3/2}$, which introduces an error term that in absolute value is bounded by

$$\begin{aligned}
&\ll \left|\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2}\int_{\substack{|a^2x|,|c^2x|,|a|,\\ |a^2c|,|x|\le 1,\\ 1\ge|x|^{3/2}B^{1/2}\alpha^{-3/2}}} B\log(|x|^3B\alpha^{-3})\mathrm{d}a\mathrm{d}c\mathrm{d}x\right| \\
&\ll \sum_{\alpha>0}\frac{1}{\alpha^2}\int_{\substack{|c^2x|,|a|\le 1,\\ |x|^{3/2}B^{1/2}\alpha^{-3/2}\le 1}} |B\log(|x|^3B\alpha^{-3})|\mathrm{d}a\mathrm{d}c\mathrm{d}x. \qquad (12)
\end{aligned}$$

Computing the integral over a and c in Equation (12) gives the upper bound

$$\ll \sum_{\alpha>0}\frac{1}{\alpha^2}\int_{|x|^{3/2}B^{1/2}\alpha^{-3/2}\le 1}|B\log(|x|^3B\alpha^{-3})|\cdot|x|^{-1/2}\mathrm{d}x$$

Here, we compute the integral over x and we find that Equation (12) can be bounded by

$$\ll \sum_{\alpha>0}\frac{1}{\alpha^2}\alpha^{1/2}B^{5/6},$$

which in turn can be bounded by

$$\ll B^{5/6}\left(1+\int_1^\infty \alpha^{-3/2}\mathrm{d}\alpha\right) \ll B^{5/6}.$$

We conclude that

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2}\int_{\substack{|a^2x|,|c^2x|,|a|,\\|a^2c|,|x|\le 1}}\left(B\log(|x|^3B\alpha^{-3})\right)\mathrm{d}a\mathrm{d}c\mathrm{d}x + O(B).$$

Writing $\log(B\alpha^{-3}|x|^3) = \log(B) + \log(\alpha^{-3}|x|^3)$ and splitting the expression for $N_1(B)$ accordingly gives

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2}\int_{\substack{|a^2x|,|c^2x|,|a|,\\|a^2c|,|x|\le 1}}(B\log(B))\,\mathrm{d}a\mathrm{d}c\mathrm{d}x + R_2(B) + O(B)$$

with

$$R_2(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2}\int_{\substack{|a^2x|,|c^2x|,|a|,\\|a^2c|,|x|\le 1}}\left(B\log(|x|^3\alpha^{-3})\right)\mathrm{d}a\mathrm{d}c\mathrm{d}x.$$

In absolute values, $R_2(B)$ can be bounded by

$$\sum_{\alpha>0}\frac{B}{\alpha^2}\int_{|c^2x|,|a|,|x|\le 1}\left(\log(|x|\alpha^{-1})\right)\mathrm{d}a\mathrm{d}c\mathrm{d}x. \tag{13}$$

Computing the integrals over a and then c in Equation (13) gives the upper bound of Equation (13)

$$\ll \sum_{\alpha>0}\frac{B}{\alpha^2}\int_{|x|\le 1}\log(|x|\alpha^{-1})|x|^{-1/2}\mathrm{d}x.$$

We compute the integral over x in the above expression as

$$\begin{aligned}\int_{|x|\le 1}\log(|x|\alpha^{-1})|x|^{-1/2}\mathrm{d}x &= \int_{|x|\le 1}\log(|x|)|x|^{-1/2}\mathrm{d}x - \int_{|x|\le 1}\log(\alpha)|x|^{-1/2}\mathrm{d}x\\ &\ll 2+\log(\alpha),\end{aligned}$$

with which we can bound $|R_2(B)|$ by

$$\ll \sum_{\alpha>0}\frac{B}{\alpha^2}(2+\log(\alpha)) \ll B\sum_{\alpha>0}\frac{1}{\alpha^2} + B\left(1+\int_1^\infty \log(\alpha)\alpha^{-2}\mathrm{d}\alpha\right) \ll B.$$

Combining it all gives the expression

$$N_1(B) = \frac{1}{2}\sum_{\alpha>0}\frac{\mu(\alpha)}{\alpha^2}\int_{\substack{|a^2x|,|c^2x|,|a|,\\|a^2c|,|x|\le 1}}(B\log(B))\,\mathrm{d}a\mathrm{d}c\mathrm{d}x + O(B). \tag{14}$$

We first consider only the integral of the above expression. Computing the integral over x gives

$$\frac{1}{2}\int_{\substack{|a^2x|,|c^2x|,|a|,\\|a^2c|,|x|\le 1}} 1\mathrm{d}a\mathrm{d}c\mathrm{d}x = \int_{\substack{|a|,\\|a^2c|\le 1}}\frac{1}{\max\{1,|a|^2,|c|^2\}}\mathrm{d}a\mathrm{d}c.$$

As $|a| \leq 1$, $\max\{1, |a|^2, |c|^2\} = \max\{1, |c|^2\}$. Again as $|a| \leq 1$, $|a^2c| \leq 1$ imposes no restriction on c, so we distinguish two cases: $|c| > 1$ and $|c| \leq 1$. With this,

$$\int_{\substack{|a|,\\ |a^2c|\leq 1}} \frac{1}{\max\{1, |a|^2, |c|^2\}} \mathrm{d}a\mathrm{d}c = \int_{|c|\leq 1} \int_{|a|\leq 1} 1 \mathrm{d}a\mathrm{d}c + \int_{|c|>1} \int_{|a^2c|\leq 1} \frac{1}{|c|^2} \mathrm{d}a\mathrm{d}c.$$

Computing these integral separately gives

$$\begin{aligned}\int_{|c|\leq 1} \int_{|a|\leq 1} 1 \mathrm{d}a\mathrm{d}c + \int_{|c|>1} \int_{|a^2c|\leq 1} \frac{1}{|c|^2} \mathrm{d}a\mathrm{d}c &= 4 + 4 \int_1^\infty c^{-5/2} \mathrm{d}c \\ &= \frac{20}{3}.\end{aligned}$$

Combining these computations with Equation (14) gives

$$N_1(B) = \frac{20}{3} B \log(B) \sum_{\alpha>0} \frac{\mu(\alpha)}{\alpha^2}.$$

As $\sum_{\alpha>0} \frac{\mu(\alpha)}{\alpha^2} = \zeta(2)^{-1}$ (Equation (3.9) of Chapter 1 in [23]), we obtain the desired asymptotic for $N_1(B)$. □

4.2 The counting function $N_2(B)$

In this section we consider the counting function

$$N_2(B) = \frac{1}{2} \# \left\{ (b, c, x, y, z) \in \mathbb{Z}^5 \ : \ \substack{1+bc-yz=0,\ \gcd(x,y)=1,\\ H(1,b,c,x,y,z)\leq B,\\ b,c,x,z\neq 0} \right\}$$

with

$$H(1, b, c, x, y, z) = \max\{|x|, |b^2x|, |c^2x|, |z^2x^3|, |y|, |by|, |cy|\}.$$

Lemma 4.4. We have

$$N_2(B) = \sum_{b,x,z\in\mathbb{Z}_{\neq 0}} \theta_1(b, x, z) V_1(b, x, z; B) + O(B),$$

where

$$V_1(b, x, z; B) = \frac{1}{2} \int_{\substack{\widetilde{H}_2(b,c,x,z)\leq B,\\ |b|,|c|,|x|,|z|\geq 1}} \frac{1}{|z|} \mathrm{d}c$$

with

$$\widetilde{H}_2(b, c, x, z) = H(1, b, c, x, (1 + bc)z^{-1}, z)$$

and $\theta_1(b, x, z) = \prod_p \theta_1^{(p)}(b, x, z)$ with

$$\theta_1^{(p)}(b, x, z) = \begin{cases} 0 & \text{if } p \mid b, p \mid z, \\ 1 - \frac{1}{p} & \text{if } p \nmid b, p \mid x, \\ 1 & \text{else.} \end{cases}$$

Proof. Applying Möbius inversion as seen in Section 2.10 and setting $y' = y/\alpha$ gives

$$N_2(B) = \frac{1}{2} \sum_{b,x,z \in \mathbb{Z}_{\neq 0}} \sum_{\alpha | x} \mu(\alpha) \widetilde{N}_2(B) \tag{15}$$

with

$$\begin{aligned}\widetilde{N}_2(B) &= \# \left\{ (c, y') \in \mathbb{Z}^2 \ : \ {\scriptstyle c \neq 0,\, 1+bc-\alpha y' z = 0, \atop H(1,b,c,x,\alpha y', z) \leq B} \right\} \\ &= \# \left\{ c \in \mathbb{Z}_{\neq 0} \ : \ {\scriptstyle bc \equiv -1 \mod \alpha |z|, \atop H(1,b,c,x,(1+bc)z^{-1}, z) \leq B} \right\}.\end{aligned}$$

For given b, α, z, the congruence relation $bc \equiv -1 \mod \alpha|z|$ has exactly one solution if $\gcd(b, \alpha z) = 1$ and no solutions otherwise. Hence, in the latter case, $\widetilde{N}_2(B) = 0$. In the first case, we can estimate $\widetilde{N}_2(B)$ as follows. The solutions to the congruence relation are of the form $c = \beta + k|\alpha z|$, where β denotes the unique solution in $\{1, \ldots, \alpha|z| - 1\}$ and $k \in \mathbb{Z}$. We remark that $\beta \in \mathbb{Z}_{\neq 0}$ for we have the condition $c \in \mathbb{Z}_{\neq 0}$. With this, we obtain

$$\begin{aligned}\widetilde{N}_2(B) &= \sum_{\substack{c \in Z_{\neq 0}, \\ \widetilde{H}_2(b,c,x,z) \leq B, \\ bc \equiv -1 \mod \alpha z}} 1 \\ &= \sum_{\substack{k \in \mathbb{Z}, \\ \widetilde{H}_2(b, \beta + k|\alpha z|, x, z) \leq B}} 1 \\ &= \int_{\substack{k \in \mathbb{Z}, \\ \widetilde{H}_2(b, \beta + k|\alpha z|, x, z) \leq B}} 1 \mathrm{d}k + O(1) \\ &= \int_{\substack{\widetilde{H}_2(b,c,x,z) \leq B, \\ |c| \geq 1}} \frac{1}{\alpha |z|} \mathrm{d}c + O(1).\end{aligned}$$

By taking the error term $O(1)$ out of the sums over b, x, z, α in Equation (15) we can bound the error term by

$$\ll \sum_{\substack{b,x,z \in \mathbb{Z}_{\neq 0}, \\ |b^2 x|, |z^2 x^3| \leq B}} \sum_{\alpha | x} |\mu(\alpha)|. \tag{16}$$

As $\sum_{\alpha | x} |\mu(\alpha)| = 2^{\omega(|x|)}$, we bound Equation (16) by

$$\ll \sum_{x \in \mathbb{Z}_{\neq 0}} 2^{\omega(x)} \sum_{\substack{b \in \mathbb{Z}_{\neq 0}, \\ |b^2 x| \leq B}} \sum_{\substack{z \in \mathbb{Z}_{\neq 0}, \\ |z^2 x^3| \leq B}} 1.$$

Analogous to the computations following Equation (1), the sum over z can be bounded by $\ll B^{1/2}|x|^{-3/2}$ and the sum over b can be bounded by $\ll B^{1/2}|x|^{-1/2}$. Using that for all $x \in \mathbb{Z}_{>0}$, $2^{\omega(x)} \leq d(x)$ (Lemma 2.49) and that $\sum_{x=1}^{\infty} d(x) x^{-2} = \zeta(2)^2$ (Lemma 2.50), the sum over x can be bounded by

$$\ll B \sum_{x=1}^{\infty} d(x) x^{-2} \ll B \zeta(2)^2 \ll B.$$

Combining all the above gives

$$N_2(B) = \frac{1}{2} \sum_{b,x,z \in \mathbb{Z}_{\neq 0}} \sum_{\substack{\alpha | x, \\ \gcd(b, \alpha z)=1}} \frac{\mu(\alpha)}{\alpha} \int_{\substack{\widetilde{H}_2(b,c,x,z) \leq B, \\ |b|,|c|,|x|,|z| \geq 1}} \frac{1}{|z|} \mathrm{d}c + O(B).$$

As $\gcd(b, \alpha z)$ and $\mu(\alpha)$ are multiplicative in α, we have

$$\sum_{\substack{\alpha | x, \\ \gcd(b, \alpha z)=1}} \frac{\mu(\alpha)}{\alpha} = \prod_p \begin{cases} 0 & \text{if } p \mid b,\ p \mid z, \\ 1 - \frac{1}{p} & \text{if } p \nmid b,\ p \mid x, \\ 1 & \text{else.} \end{cases}$$

This gives the desired expression for $N_2(B)$. □

Lemma 4.5. We have

$$N_2(B) = \sum_{b,x \in \mathbb{Z}_{\neq 0}} \theta_2(x,z) V_2(x,z;B) + O(B(\log\log(B))^2),$$

where

$$V_2(x,z;B) = \frac{1}{2} \int_{\substack{\widetilde{H}_2(b,c,x,z) \leq B, \\ |b|,|c|,|x|,|z| \geq 1}} \frac{1}{|z|} \mathrm{d}b \mathrm{d}c$$

and $\theta_2(x,z) = \prod_p \theta_2(p)(x,z)$ with

$$\theta_2^{(p)}(x,z) = \begin{cases} \left(1 - \frac{1}{p}\right)^2 & \text{if } p \mid x, z, \\ 1 - \frac{1}{p} + \frac{1}{p^2} & \text{if } p \mid x, p \nmid z, \\ 1 - \frac{1}{p} & \text{if } p \nmid x, p \mid z, \\ 1 & \text{if } p \nmid x, z \end{cases}$$

Idea of proof. Using the conditions $|c^2 x| \leq B$ and $|b(1+bc)z^{-1}| \leq B$ coming from the height condition, the integral $V_1(b,x,z;B)$ can be bounded by

$$\ll \frac{1}{|z|} \min\{B^{1/2}|x|^{-1/2}, B|z||b|^{-2}\} \ll \frac{1}{|z|} \left(\frac{B^{1/2}}{|x|^{1/2}}\right)^{2/3} \left(\frac{B|z|}{|b|^2}\right)^{1/3}. \tag{17}$$

To apply Proposition 3.9 of [5], we first check that $V_1(b,x,z;B)$ satisfies the conditions above Lemma 3.6 in [5]. Setting

$$r = 0,\ s = 2,\ \eta_0 = b,\ \eta_1 = x,\ \eta_2 = z,\ a_1 = a_2 = 1/6$$

$$k_{01} = 2,\ k_{02} = 0,\ k_{11} = 1,\ k_{12} = 3,\ k_{22} = 2,$$

the above bound for $V_1(b,x,z;B)$ implies that $V_1(b,x,z;B)$ indeed satisfies the conditions above Lemma 3.6 in [5].

To apply Proposition 3.9 of [5], we must have $\theta_1(b,x,z) \in \Theta_{1,3}(C,x)$ for some constant $C \in \mathbb{R}_{\geq 0}$. Using Corollary 7.9 of [5], it is enough to show that $\theta_1(b,x,z) \in \Theta'_{4,3}(C,x)$. It is a straightforward verification that indeed $\theta_1(b,x,z) \in \Theta'_{4,3}(C,x)$, using Definition 7.8 of [5]. Then applying Proposition 3.9 of [5] gives the desired result. We remark that it is crucial for the application of [5] that $\theta_1(b,x,z)$ is multiplicative.

Lemma 4.6. We have

$$N_2(B) = \frac{1}{2}\prod_p\left(1-\frac{2}{p^2}+\frac{1}{p^3}\right)\int_{\substack{\widetilde{H}_2(b,c,x,z)\leq B,\\ |b|,|c|,|x|,|z|\geq 1}}\frac{1}{|z|}\mathrm{d}b\mathrm{d}c\mathrm{d}x\mathrm{d}z + O(B(\log\log B)^2).$$

Idea of proof. Using the bound in Equation (17), and using the height condition $|b^2x|\leq B$, we can bound $V_2(x,z;B)$ by

$$\ll B^{2/3}|x|^{-1/3}|z|^{-2/3}\int_{1\leq|b|\leq B^{1/2}|x|^{-1/2}}|b|^{-2/3}\mathrm{d}b.$$

We can bound the integral over b by $\ll B^{1/6}|x|^{-1/6}$, with which we obtain the upper bound for $V_2(x,z;B)$ given by

$$\ll \frac{B}{|xz|}\left(\frac{B}{|x|^3|z|^2}\right)^{-1/6}. \tag{18}$$

To apply Proposition 4.3 of [5], we first check that $V_2(x,z;B)$ satisfies the conditions above Lemma 4.1 in [5]. Setting $r=s=1$, $a_1=\frac{1}{6}$, $k_{1,1}=3$ and $k_{2,1}=2$, the above bound for $V_2(x,z;B)$ implies that $V_2(x,z;B)$ indeed satisfies the conditions above Lemma 4.1 in [5].

To apply Proposition 4.3 of [5], we must have $\theta_2(x,z)\in\Theta_{2,2}(C)$ for some constant $C\in\mathbb{R}_{\geq 0}$. Using Definition 4.2 of [5], this is a straightforward computation. Then applying Proposition 4.3 of [5] gives the desired result.

Proposition 4.7. We have

$$N_2(B) = cB\log(B) + O(B\log\log(B)^2),$$

where

$$c = \frac{20}{3}\prod_p\left(1-\frac{2}{p^2}+\frac{1}{p^3}\right).$$

Proof. Let us fix the notation

$$V_3(B) = \int_{\substack{\widetilde{H}_2(b,c,x,z)\leq B,\\ |b|,|c|,|x|,|z|\geq 1}}\frac{1}{|z|}\mathrm{d}b\mathrm{d}c\mathrm{d}x\mathrm{d}z.$$

Remark that the condition $|(1+bc)z^{-1}|\leq B$ coming from the height condition is redundant, so we can omit it. We replace the conditions $|(1+bc)bz^{-1}|\leq B$ and $|(1+bc)cz^{-1}|\leq B$

in the integral $V_3(B)$ by $|b^2cz^{-1}| \le B$ and $|bc^2z^{-1}| \le B$, i.e., we replace $1+bc$ by bc, which gives the new integral

$$V_3'(B) = \int_{\substack{|b^2x|,|c^2x|,|x^3z^2|,\\ |b^2cz^{-1}|,|bc^2z^{-1}|\le B,\\ |b|,|c|,|x|,|z|\ge 1}} |z|^{-1}\mathrm{d}b\mathrm{d}c\mathrm{d}x\mathrm{d}z.$$

However, both of those replacements introduce an error term. Consider first the error term introduced by replacing $|(1+bc)bz^{-1}| \le B$ by $|b^2cz^{-1}| \le B$. Analogous to the proof of Lemma 4.2, this error term can be bounded by

$$\ll \int_{\substack{|c|,|b|,|x|,|z|\ge 1,\\ \left|\frac{Bz}{b^2}\right|-\frac{1}{|b|}\le |c| \le \left|\frac{Bz}{b^2}\right|+\frac{1}{|b|},\\ |b^2x|,|x^3z^2|\le B}} \frac{1}{|z|}\mathrm{d}c\mathrm{d}b\mathrm{d}x\mathrm{d}z \tag{19}$$

In this equation, the integral over c can be bounded by $\ll |b|^{-1}$, and the integral over x can be bounded by $\ll \min\{B|b|^{-2}, B^{1/3}|z|^{-2/3}\} \ll B^{1/3}|z|^{-2/3}$, with which we can bound Equation (19) by

$$\ll \int_{\substack{|b|,|z|\ge 1,\\ 1\le \left|\frac{Bz}{b^2}\right|+\frac{1}{|b|}}} \frac{1}{|bz|}B^{1/3}|z|^{-2/3}\mathrm{d}b\mathrm{d}z. \tag{20}$$

Remark that $1 \le \left|\frac{Bz}{b^2}\right| + \frac{1}{|b|} \le 2\max\left\{\left|\frac{Bz}{b^2}\right|, \frac{1}{|b|}\right\}$. As $|b^2x| \le B$, we know that $|b| \le B$, with which

$$\left|\frac{Bz}{b^2}\right| = \left|\frac{B}{b}\right| \cdot \left|\frac{z}{b}\right| \ge \left|\frac{z}{b}\right| \ge \frac{1}{|b|}.$$

Hence, we obtain the inequality $1 \le 2\left|\frac{Bz}{b^2}\right|$, which is equivalent to

$$|z|^{-1} \ll B|b|^{-2}.$$

Then also $|z|^{-1/3} \ll B^{1/3}|b|^{-2/3}$, with which the integral in Equation (20) can be bounded by

$$\ll \int_{|b|,|z|\ge 1} B^{2/3}|z|^{-4/3}|b|^{-5/3}\mathrm{d}b\mathrm{d}z. \tag{21}$$

Computing the integrals over b and z gives

$$\int_{|b|\ge 1} |b|^{-5/3}\mathrm{d}b \ll \frac{3}{2},$$

$$\int |z| \ge 1|z|^{-4/3}\mathrm{d}z \ll 3,$$

with which we can bound the error term in Equation (19) by $O(B^{2/3})$.

The computations for replacing $|(1+bc)cz^{-1}| \le B$ by $|bc^2z^{-1}| \le B$ can be done analogously to show that the error term introduced by this replacement is also bounded by $O(B^{2/3})$. Combining this, we obtain

$$N_2(B) = \frac{1}{2}\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right)V_3'(B) + O(B(\log\log B)^2).$$

Analogous to the first step in the proof of Lemma 4.3, we now forget the condition $|b| \geq 1$, which only adds the case $b = 0$. We introduce an error term bounded by

$$\ll \int_{\substack{|c^2x|,|z^2x^3|\leq B,\\ |c|,|x|,|z|\geq 1}} \frac{1}{|z|} \mathrm{d}c\mathrm{d}x\mathrm{d}z \tag{22}$$

In here, the integral over c can be bounded by $\ll \int_1^{B^{1/2}|x|^{-1/2}} 1\mathrm{d}c \ll B^{1/2}|x|^{-1/2}$. The integral over x can then be bounded by $\ll 1 + \int_1^{B^{1/3}|z|^{-2/3}} x^{-1/2}\mathrm{d}x \ll B^{1/6}|z|^{-1/3}$, with which we can bound the integral over z by $\ll \int_{|z|\geq 1} |z|^{-4/3}\mathrm{d}z \ll 3$. Hence, the error term in Equation (22) is bounded by $O(B^{2/3})$.

Omitting the condition $|c| \geq 1$ gives analogous computations and the error term introduced can again be bounded by $\ll O(B^{2/3})$. Hence, we find

$$N_2(B) = \frac{1}{2}\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) V_3''(B) + O(B(\log\log B)^2)$$

with

$$V_3''(B) = \int_{\substack{|b^2x|,|c^2x|,|x^3z^2|,\\ |b^2cz^{-1}|,|bc^2z^{-1}|\leq B,\\ |x|,|z|\geq 1}} |z|^{-1}\mathrm{d}b\mathrm{d}c\mathrm{d}x\mathrm{d}z.$$

Analogous to the proof of Proposition 4.3, we now change variables: set $b' = B^{-1/3}bz^{-1/3}$, $c' = B^{-1/3}cz^{-1/3}$ and $x' = B^{-1/3}xz^{2/3}$. With this,

$$N_2(B) = \frac{1}{2}\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) \int_{\substack{|b'^2x'|,|c'^2x'|,|x'|,\\ |b'^2c'|,|b'c'^2|\leq 1,\\ 1\leq|z|\leq B^{1/2}|x'|^{3/2}}} \frac{B}{|z|}\mathrm{d}b'\mathrm{d}c'\mathrm{d}x'\mathrm{d}z + O(B(\log\log B)^2).$$

For the remainder of this section, we fix the notation $b = b'$, $c = c'$ and $x = x'$, with which the expression for $N_2(B)$ becomes

$$N_2(B) = \frac{1}{2}\prod_p \left(1 - \frac{2}{p^2} + \frac{1}{p^3}\right) V_3''(B) + O(B(\log\log B)^2),$$

with

$$V_3''(B) = \int_{\substack{|b^2x|,|c^2x|,|x|,\\ |b^2c|,|bc^2|\leq 1,\\ 1\leq|z|\leq B^{1/2}|x|^{3/2}}} \frac{B}{|z|}\mathrm{d}b\mathrm{d}c\mathrm{d}x\mathrm{d}z + O(B(\log\log B)^2).$$

We compute the integral over z in $V_3''(B)$ and obtain

$$V_3''(B) = 2B\int_{\substack{|b^2x|,|c^2x|,|x|,\\ |b^2c|,|bc^2|\leq 1,\\ 1\leq B^{1/2}|x|^{3/2}}} \log(B^{1/2}|x|^{3/2})\mathrm{d}b\mathrm{d}c\mathrm{d}x.$$

The remainder of this proof is analogous to the proof of Proposition 4.3. We omit the condition $1 \le B^{1/2}|x|^{3/2}$, which introduces an error term that in absolute value is bounded by

$$\ll B \int_{\substack{|x|\le B^{-1/3},\\ |b^2c|,|bc^2|\le 1}} \left|\log(B^{1/2}|x|^{3/2})\right| \mathrm{d}b\mathrm{d}c\mathrm{d}x. \tag{23}$$

In here, we can bound the integral over x by $\ll B^{-1/3}$, with which Equation (23) can be bounded by $\ll B\int_{|b^2c|,|bc^2|\le 1} B^{-1/3}\mathrm{d}b\mathrm{d}c$. We compute the integral over b to obtain the bound for Equation (23)

$$\ll B^{2/3}\int_{|b|\le|c|^{-2},\ |b|\le|c|^{-1/2}} 1\mathrm{d}b\mathrm{d}c \ll B^{2/3}\left(\int_{|c|\le 1}|c|^{-1/2}\mathrm{d}c + \int_{|c|>1}|c|^{-2}\mathrm{d}c\right) \tag{24}$$

We compute both of the integrals over c and obtain their sum equals $2+1$, such that Equation (23) is bounded by $O(B^{2/3})$. Hence,

$$V_3''(B) = 2B\int_{\substack{|b^2x|,|c^2x|,|x|,\\ |b^2c|,|bc^2|\le 1}} \log(B^{1/2}|x|^{3/2})\mathrm{d}b\mathrm{d}c\mathrm{d}x + O(B^{2/3}).$$

Writing $\log(B^{1/2}|x|^{3/2}) = \frac{1}{2}\log(B) + \log(|x|^{3/2})$ and splitting the integrals over b, c, x accordingly, we obtain

$$V_3''(B) = B\log(B)\int_{\substack{|b^2x|,|c^2x|,|x|,\\ |b^2c|,|bc^2|\le 1}} 1\mathrm{d}b\mathrm{d}c\mathrm{d}x + O(B^{2/3}) + R_4(B),$$

where

$$|R_4(B)| \ll B\int_{\substack{|b^2x|,|c^2x|,|x|,\\ |b^2c|,|bc^2|\le 1}} \log(|x|^{3/2})\mathrm{d}b\mathrm{d}c\mathrm{d}x. \tag{25}$$

In Equation (25), we bound the integral over x by

$$\int_{-1}^{1}\log(|x|^{3/2})\mathrm{d}x \ll \frac{3}{2}(2 - i\pi),$$

which is a constant. We repeat the computations for Equation (24) to conclude that $|R_4(B)| \ll B$, such that

$$V_3''(B) = B\log(B)\int_{\substack{|b^2x|,|c^2x|,|x|,\\ |b^2c|,|bc^2|\le 1}} 1\mathrm{d}b\mathrm{d}c\mathrm{d}x + O(B).$$

Repeating the computations following Equation (14), we see that

$$V_3''(B) = B\log(B)\frac{20}{3} + O(B),$$

as desired. □

4.3 Obstruction to an asymptotic for $N(B)$

In this section, we follow the first steps taken in Section 5 of [24] and we highlight where this method breaks down for the counting function $N(B)$.

Recall the counting function $N(B)$, which can be expressed explicitly as

$$N(B) = \frac{1}{2}\#\left\{(b,c,x,y,z) \in \mathbb{Z}^5 \ : \ \substack{1+b^2+c^2-yz=0,\\ H(1,b,c,x,y,z)\leq B,\\ \gcd(x,y)=1,\ b,c,x,z\neq 0}\right\}.$$

We apply Möbius inversion as seen in Section 2.10 and set $y' = \frac{y}{\alpha}$ to obtain

$$N(B) = \frac{1}{2}\sum_{b,x,z\in\mathbb{Z}_{\neq 0}}\sum_{\alpha|x}\mu(\alpha)\widetilde{N}(B)$$

with

$$\widetilde{N}(B) = \#\left\{(c,y')\in\mathbb{Z}^2 \ : \ \substack{c\neq 0, 1+b^2+c^2-\alpha y' z=0,\\ H(1,b,c,x,\alpha y',z)\leq B}\right\}.$$

Set $g = \gcd(\alpha z, -1-b^2)$, let g' be the unique positive integer such that $v_p(g') = \lceil v_p(g)/2\rceil$ for all primes p. It is immediate that $g' \mid g$ and $g \mid (g')^2$, which justifies setting $g'' = \frac{(g')^2}{g}$.

Lemma 4.8. We have

$$N(B) = \sum_{b,x,z\in\mathbb{Z}_{\neq 0}}\theta_1(b,x,z)V_1(b,x,z;B) + O_\epsilon(B^{5/6+\epsilon})$$

where

$$V_1(b,x,z;B) = \frac{1}{2}\int_{\substack{\widetilde{H}(b,c,x,z)\leq B,\\ |b|,|c|,|x|,|z|\geq 1}}\frac{1}{|z|}\mathrm{d}c$$

with

$$\begin{aligned}\widetilde{H}(b,c,x,z) &= H(1,b,c,x,(1+b^2+c^2)z^{-1},z)\\ &= \max\left\{|x|,|b^2x|,|c^2x|,|z^2x^3|,|(1+b^2+c^2)z^{-1}|,|(1+b^2+c^2)bz^{-1}|,|(1+b^2+c^2)cz^{-1}|\right\}\end{aligned}$$

and

$$\theta_1(b,x,z) = \sum_{\alpha|x}\mu(\alpha)\alpha^{-1}\frac{g}{g'}\eta(q;b'g''),$$

with $\eta(q;b'g'')$ multiplicative in q and for every prime p and $k\in\mathbb{Z}_{>0}$

$$\eta(p^k;b'g'') = \begin{cases} 1+\left(\frac{b'g''}{p}\right) & \text{if } p \nmid 2b'g'' \\ 1 & \text{if } p=2,\ k=1,\ b'g''\equiv 1 \mod 2 \\ 2 & \text{if } p=2,\ k=2,\ b'g''\equiv 1 \mod 4 \\ 4 & \text{if } p=2,\ k\geq 3,\ b'g''\equiv 1 \mod 8 \\ 0 & \text{else}\end{cases}$$

Proof. We observe that the expression for $\widetilde{N}(B)$ above is equal to

$$\widetilde{N}(B) = \#\left\{c \in \mathbb{Z}_{\neq 0} \ : \ \substack{c^2 \equiv -1-b^2 \mod \alpha|z|,\\ H(1,b,c,x,(1+b^2+c^2)z^{-1},z)\leq B}\right\}.$$

We want to study the number of solutions to the congruence relation $c^2 \equiv -1-b^2 \mod \alpha|z|$. As $\gcd(-1-b^2, \alpha z)$ need not be equal to one, we first modify the congruence relation. Observe that any solution $c \in \mathbb{Z}_{\neq 0}$ to $c^2 \equiv -1-b^2 \mod \alpha|z|$ must be divisible by g'. Hence, setting $c' = \frac{c}{g'}$ is sensible. Also set $b' = \frac{-1-b^2}{g}$ and $q = \frac{\alpha|z|}{g}$. With this,

$$\widetilde{N}(B) = \#\left\{c' \in \mathbb{Z}_{\neq 0} \ : \ \substack{g''(c')^2 \equiv b' \mod q,\\ H(1,b,g'c',x,(1+b^2+(g'c')^2)z^{-1},z)\leq B}\right\}.$$

We claim that $\gcd(q, g'') = 1$. Indeed, let p be a prime that divides both q and g''. Then it follows from the congruence relation that p must also divide b', which cannot be for $\gcd(b', q) = 1$ by construction.

As $\gcd(q, g'') = 1$, there exists $\overline{g''} \in \mathbb{Z}/q\mathbb{Z}$ such that $g''\overline{g''} \equiv 1 \mod q$. Hence,

$$\begin{aligned}\widetilde{N}(B) &= \#\left\{c' \in \mathbb{Z}_{\neq 0} \ : \ \substack{(c')^2 \equiv b'g''(\overline{g''})^2 \mod q,\\ H(1,b,g'c',x,(1+b^2+(g'c')^2)z^{-1},z)\leq B}\right\}\\ &= \sum_{\substack{1\leq \rho \leq q,\\ \gcd(\rho,q)=1,\\ \rho^2 \equiv b'g'' \mod q}} \#\left\{c' \in \mathbb{Z}_{\neq 0} \ : \ \substack{c' \equiv \rho\overline{g''} \mod q,\\ H(1,b,g'c',x,(1+b^2+(g'c')^2)z^{-1},z)\leq B}\right\}.\end{aligned}$$

It follows that

$$\begin{aligned}&\#\left\{c' \in \mathbb{Z}_{\neq 0} \ : \ \substack{c' \equiv \rho\overline{g''} \mod q,\\ H(1,b,g'c',x,(1+b^2+(g'c')^2)z^{-1},z)\leq B}\right\}\\ &= \sum_{\substack{k\in\mathbb{Z},\\ H(1,b,g'(\rho\overline{g''}+kq),x,(1+b^2+(g')^2(\rho\overline{g''}+kq)^2)z^{-1},z)\leq B}} 1\\ &= \int_{\substack{k\in\mathbb{R},\\ H(1,b,g'(\rho\overline{g''}+kq),x,(1+b^2+(g')^2(\rho\overline{g''}+kq)^2)z^{-1},z)\leq B}} 1\mathrm{d}k + O(1)\\ &= \int_{\substack{|c'|\geq 1,\\ H(1,b,g'c',x,(1+b^2+(g'c')^2)z^{-1},z)\leq B}} \frac{1}{q}\mathrm{d}c' + O(1)\\ &= \int_{\substack{|c|\geq 1,\\ H(1,b,c,x,(1+b^2+c^2)z^{-1},z)\leq B}} \frac{1}{g'q}\mathrm{d}c + O(1)\\ &= \frac{g}{g'\alpha}\int_{\substack{|c|\geq 1,\\ H(1,b,c,x,(1+b^2+c^2)z^{-1},z)\leq B}} \frac{1}{|z|}\mathrm{d}c + O(1)\end{aligned}$$

Set $V_1(b,x,z;B) = \int_{\substack{|c|\geq 1,\\ H(1,b,c,x,1+b^2+c^2)z^{-1},z)\leq B}} \frac{1}{|z|}\mathrm{d}c$. Then combining all of the above,

$$N(B) = \frac{1}{2}\sum_{\substack{b,x,z\in\mathbb{Z}_{\neq 0},\\ |x|,|b^2x|,|x^3z^2|,\\ |(b^2+2)z^{-1}|,|(b^2+2)bz^{-1}|\leq B}} \sum_{\alpha | x}\mu(\alpha)\alpha^{-1}\frac{g}{g'}\sum_{\substack{1\leq\rho\leq q,\\ \gcd(\rho,q)=1,\\ \rho^2\equiv b'g'' \mod q}} \left(V_1(b,x,z;B) + O(1)\right).$$

We first take the error term $O(1)$ out of the sums over b, x, z, α, ρ and obtain that the error is bounded by

$$\ll \sum_{\substack{b,x,z\in\mathbb{Z}_{\neq 0},\\ |x^3z^2|,|(b^2+2)bz^{-1}|\leq B}} \sum_{\alpha|x}\mu(\alpha) \sum_{\substack{1\leq\rho\leq q,\\ \gcd(\rho,q)=1,\\ \rho^2\equiv b'g'' \mod q}} 1 \tag{26}$$

Set $\eta(q; b'g'') = \sum_{\substack{1\leq\rho\leq q,\\ \gcd(\rho,q)=1,\\ \rho^2\equiv b'g'' \mod q}} 1$. By the Chinese Remainder theorem, this function is multiplicative in q and we know for every prime p and for every $k \in \mathbb{Z}_{>0}$

$$\eta(p^k; b'g'') = \begin{cases} 1 + \left(\frac{b'g''}{p}\right) & \text{if } p \not| \, 2b'g'' \\ 1 & \text{if } p = 2,\ k = 1,\ b'g'' \equiv 1 \mod 2 \\ 2 & \text{if } p = 2,\ k = 2,\ b'g'' \equiv 1 \mod 4 \\ 4 & \text{if } p = 2,\ k \geq 3,\ b'g'' \equiv 1 \mod 8 \\ 0 & \text{else} \end{cases}$$

following from Hensel's lemma and using the Legendre symbol.

It then follows that $\eta(q; b'g'') \leq 2^{\omega(\alpha|z|)+1} \leq 2^{\omega(\alpha)+\omega(|z|)+1}$. As $\mu(\alpha)2^{\omega(\alpha)}$ is multiplicative in α, we obtain

$$\sum_{\alpha|x}\mu(\alpha)\alpha^{-1}2^{\omega(\alpha)} = \prod_p \begin{cases} 1 - \frac{2}{p} & \text{if } p \mid x \\ 1 & \text{else} \end{cases} \leq 1.$$

This combines to the upper bound for Equation (26)

$$\ll \sum_{\substack{b,x,z\in\mathbb{Z}_{\neq 0},\\ |x^3z^2|,|(b^2+2)bz^{-1}|\leq B}} 2^{\omega(|z|)}. \tag{27}$$

In here, the sum over x can be bounded as follows.

$$\begin{aligned} \sum_{\substack{x\in\mathbb{Z}_{\neq 0},\\ |x|\leq B^{1/3}|z|^{-2/3}}} 1 &\ll \int_1^{\lceil B^{1/3}|z|^{-2/3}\rceil} 1 \mathrm{d}x \\ &\ll B^{1/3}|z|^{-2/3} \end{aligned}$$

With this and using Lemma 2.51, the sum over z in Equation (27) can be bounded by

$$\begin{aligned} \sum_{\substack{z\in\mathbb{Z}_{\neq 0},\\ |z|\leq B^{1/2}}} 2^{\omega(|z|)}|z|^{-2/3} &\ll_\epsilon \sum_{\substack{z\in\mathbb{Z}_{\neq 0},\\ |z|\leq B^{1/2}}} |z|^{-2/3+\epsilon} \\ &\ll_\epsilon B^{1/6+\epsilon/2}. \end{aligned}$$

The sum over b in Equation (27) can be bounded by $\ll B^{1/3}$, with which the expression in Equation (27) can be bounded by

$$\ll_\epsilon B^{5/6+\epsilon/2}.$$

Hence, we obtain

$$N(B) = \frac{1}{2} \sum_{b,x,z \in \mathbb{Z}_{\neq 0}} \sum_{\alpha | x} \mu(\alpha)\alpha^{-1} \frac{g}{g'} \sum_{\substack{1 \le \rho \le q, \\ \gcd(\rho, q) = 1, \\ \rho^2 \equiv b' g'' \mod q}} V_1(b, x, z; B) + O_\epsilon(B^{5/6+\epsilon})$$

It is now immediate that

$$\theta_1(b, x, z) = \sum_{\alpha | x} \mu(\alpha)\alpha^{-1} \frac{g}{g'} \sum_{\substack{1 \le \rho \le q, \\ \gcd(\rho, q) = 1, \\ \rho^2 \equiv b' g'' \mod q}} 1 = \sum_{\alpha | x} \mu(\alpha)\alpha^{-1} \frac{g}{g'} \eta(q; b' g''),$$

which is indeed as required. □

In the previous section, we have seen that $\theta_1(b, x, z)$ being multiplicative is crucial in order to apply the paper [5] as Florian Wilsch does in Section 5 of [24]. The following lemma shows that $\theta_1(b, x, z)$ is not multiplicative, which is why the method of Section 5 in [24] does not extend in a straightforward way to the counting function $N(B)$.

Lemma 4.9. The function $\theta_1(b, x, z)$ is not multiplicative in x.

Proof. It suffices to show that for distinct prime numbers p_1, p_2,

$$\theta_1(b, p_1, z)\theta_1(b, p_2, z) \neq \theta_1(b, p_1 p_2, z).$$

As multiplicativity in x must hold for all choices of $b, z \in \mathbb{Z} \neq 0$, we can fix them such that

- for all p such that $v_p(z) \neq 0$, $v_p(-1 - b^2) \geq v_p(z) + 1$
- for all p such that $v_p(z) \neq 0$, $v_p(z)$ is even

With these assumptions on $b, z \in \mathbb{Z}_{\neq 0}$,

$$q = \frac{\alpha |z|}{g} = \prod_p p^{v_p(\alpha z) - \min\{v_p(\alpha z), v_p(-1-b^2)\}} = 1$$

for $v_p(\alpha) \leq 1$ as α can be assumed to be square-free. Then independent of the variable x, $\eta(q; b' g'') = 1$. Observe that

$$\frac{g}{g'} = \prod_p p^{\min\{v_p(\alpha z), v_p(-1-b^2)\} - \lceil \min\{v_p(\alpha z), v_p(-1-b^2)\}/2 \rceil} = \prod_p p^{v_p(\alpha z)/2},$$

then it follows directly that

$$\theta_1(b,p_1,z) = \prod_p p^{v_p(z)/2} - \prod_{p\neq p_1} p^{v_p(z)/2}\cdot p_1^{v_{p_1}(z)/2-1}.$$

and that

$$\theta_1(b,p_1p_2,z) = \prod_p p^{v_p(z)/2} - \prod_{p\neq p_1} p^{v_p(z)/2}\cdot p_1^{v_{p_1}(z)/2-1} - \prod_{p\neq p_2} p^{v_p(z)/2}\cdot p_2^{v_{p_2}(z)/2-1}$$
$$+ \prod_{p\neq p_1,p_2} p^{v_p(z)/2}\cdot p_1^{v_{p_1}(z)/2-1}\cdot p_2^{v_{p_2}(z)/2-1}.$$

It is easy to see then that

$$\theta_1(b,p_1,z)\theta_1(b,p_2,z) = \prod_p p^{v_p(z)/2}\theta_1(b,p_1p_2,z),$$

showing that $\theta_1(b,x,z)$ indeed is not multiplicative in x. □

4.4 An upper bound for $N(B)$

As the method of Section 5 in [24] does not extend in a straightforward way to the counting function $N(B)$, we determine an upper bound for $N(B)$.

Proposition 4.10. Let $N(B)$ be the counting function

$$N(B) = \frac{1}{2}\sum_{b,x,z\in\mathbb{Z}_{\neq 0}} \#\{(c,y)\in\mathbb{Z}^2 \ : \ {\scriptstyle c\neq 0, 1+b^2+c^2-yz=0, (x,y)=1, \atop H(1,b,c,x,y,z)\leq B}\}.$$

Then for all $B\in\mathbb{R}_{>0}$,

$$N(B) \ll_\epsilon B^{7/6+\epsilon}.$$

Proof. As forgetting conditions enlarges the set of points that $N(B)$ counts, one can bound $N(B)$ by

$$\ll \sum_{b,x,z\in\mathbb{Z}_{\neq 0}} \#\{(c,y)\in\mathbb{Z}^2 \ : \ {\scriptstyle c\neq 0, 1+b^2+c^2-yz=0, \atop H(1,b,c,x,y,z)\leq B}\}$$

As in the proof of Lemma 4.8, set $g = \gcd(-1-b^2, z)$, $q = \frac{|z|}{g}$, $b' = \frac{-1-b^2}{g}$, g' to be the unique integer such that $v_p(g') = \lceil v_p(g)/2\rceil$ and $g'' = \frac{g'}{g}$. Then we can write

$$\#\{(c,y)\in\mathbb{Z}^2 \ : \ {\scriptstyle c\neq 0, 1+b^2+c^2-yz=0, \atop H(1,b,c,x,y,z)\leq B}\} = \frac{g}{g'}\eta(q;b'g'')V_1(b,x,z;B),$$

with

$$V_1(b,x,z;B) = \int_{\substack{\widetilde{H}(b,c,x,z)\leq B, \\ |b|,|c|,|x|,|z|\geq 1}} \frac{1}{|z|}\mathrm{d}c$$

with

$$\begin{aligned}\widetilde{H}(b,c,x,z) &= H(1,b,c,x,(1+b^2+c^2)z^{-1},z)\\ &= \max\left\{|x|,|b^2x|,|c^2x|,|z^2x^3|,|(1+b^2+c^2)z^{-1}|,|(1+b^2+c^2)bz^{-1}|,|(1+b^2+c^2)cz^{-1}|\right\}\end{aligned}$$

and with $\eta(q;b'g'')$ multiplicative in q and for every prime p and $k\in\mathbb{Z}_{>0}$, $\eta(p^k;b'g'')$ as given in Lemma 4.8.

It follows from the expression for $\eta(p^k;b'g'')$ that $\eta(q;b'g'')\leq 2^{\omega(q)+1}\leq 2^{\omega(|z|)+1}$. As $\left(\frac{g}{g'}\right)^2\leq g\leq\min\{|z|,1+b^2\}$, and as $1+b^2\leq 2b^2$ for $b\neq 0$, we obtain

$$\frac{g}{g'}\ll\min\{\sqrt{|z|},|b|\}\ll\sqrt{|z|}.$$

Lastly, using the height conditions $|c^2x|\leq B$ and $|(1+b^2+c^2)bz^{-1}|\leq B$,

$$\begin{aligned}V_1(b,x,z;B) &\ll |z|^{-1}\int_{1\leq|c|\leq\min\{B^{1/2}|x|^{-1/2},B^{1/2}|z|^{1/2}|b|^{-1/2}\}}1\mathrm{d}c\\ &\ll |z|^{-1}\min\{B^{1/2}|x|^{-1/2},B^{1/2}|z|^{1/2}|b|^{-1/2}\}\\ &\ll |z|^{-1}B^{1/2}|z|^{1/2}|b|^{-1/2}\end{aligned}$$

Combining all the above, and forgetting all height conditions but $|x^3z^2|\leq B$ and $|(1+b^2+c^2)bz^{-1}|\leq B$, we can bound $N(B)$ by

$$\ll B^{1/2}\sum_{1\leq|b|\leq B^{1/3}}|b|^{-1/2}\sum_{1\leq|z|\leq B^{1/2}}2^{\omega(|z|)}\sum_{1\leq|x|\leq B^{1/3}|z|^{-2/3}}1. \tag{28}$$

Here, the sum over x can be bounded by

$$\ll\lceil B^{1/3}|z|^{-2/3}\rceil-1\ll B^{1/3}|z|^{-2/3}.$$

With this, we find the upper bound for the sum over z in Equation (28)

$$\ll\sum_{\substack{z\in\mathbb{Z}_{\neq0},\\|z|\leq B^{1/2}}}2^{\omega(|z|)}|z|^{-2/3}\ll\sum_{1\leq|z|\leq B^{1/2}}d(z)z^{-2/3}\ll_\epsilon\sum_{1\leq|z|\leq B^{1/2}}z^{-2/3+\epsilon}$$

$$\ll_\epsilon 1+\int_1^{\lceil B^{1/2}\rceil}z^{-2/3+\epsilon}\mathrm{d}z\ll_\epsilon\lceil B^{1/6+\epsilon/2}\rceil\ll_\epsilon B^{1/6+\epsilon/2}$$

where we used Lemma 2.51 and Lemma 2.49. Lastly, the sum over b is bounded by

$$\sum_{\substack{b\in\mathbb{Z}_{\neq0},\\|b|\leq B^{1/3}}}|b|^{-1/2}\ll\sum_{1\leq|b|\leq B^{1/3}}b^{-1/2}\ll B^{1/6}.$$

Combining all these bounds gives indeed

$$N(B)\ll_\epsilon B^{7/6+\epsilon/2}.$$ □

We note that at every step, we threw away information. This makes for a sub-optimal upper bound. It is possible that playing around with the order of summation or taking different upper bounds for g/g' and $V_1(b,x,z)$ gives a tighter upper bound.

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