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Geodesics in the Carroll limit

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Abstract

The focus of this thesis is the Carroll limit, the limit of vanishing speed of light. This can be thought of as the opposite of the Galilean (or Newtonian) limit, in which we take the limit of infinite speed of light. We look at taking this limit in the context of finding geodesics in general relativity, because there is not a lot of literature about this subject yet. Given a spacetime, we can take the Carroll limit in different stages of the process of finding geodesics, and it turns out that this will give us different, but similar, results for the existence of certain geodesics. All will be illustrated with examples in Minkowski, Schwarzschild, and de Sitter spacetime. The main result of this thesis is the Carroll limit of the geodesic equations being written down. We also found a non-trivial Carroll geodesic in the Schwarzschild spacetime, which shows that there can be moving particles in the Carroll limit in a non-flat spacetime.

Contents

Notation & Conventions

For the sake of clarity, we will use some consistency in notation throughout this thesis.

We indicate the end of definitions, remarks, consequences, and cases with a $\triangleleft$, the end of solutions with a $\circ$, and the end of proofs with a $\square$. We underline any definitions we use or introduce.

Throughout this text, we explicitly write down all the powers of the speed of light c because they play a major role. If we say that a quantity X "scales like c^y," we mean that $\lim_{c\to 0} Xc^{-y}$ is finite and non-zero. If we say that a quantity X "does not scale with c," we mean that X scales like c^0. We may write the symbol $\sim$, by which we mean "scales like".

We also expand quantities in terms of powers of c, the speed of light. It is useful to note that the dimensions of the coefficients are not the same as those of the original quantity. To illustrate what we mean, let us look at an example in which we expand the radius R:

$$R = \sum_{i=0}^{\infty} R_i c^i. \tag{1}$$

The coefficient R_0 has dimensions of a radius, but for all $i \geq 1$, R_i will not have dimensions of a radius.

We will encounter tensors, for which we use the standard tensor notation. This includes the Einstein summation convention, and the use of free indices: e.g. for the $(1,0)$ tensor T and the number A, the statement $T^\mu = A$ means that for all μ, the statement $T^\mu = A$ is true. We may also refer to the tensor T as T^μ. Furthermore, Greek indices run over all spacetime coordinates ($\mu \in \{0,\ldots,D-1\}$), while Latin indices run over the non-timelike coordinates ($i \in \{1,\ldots,D-1\}$). When there is a time coordinate, say t, we set $x^0 := t$, so x^0 has different dimensions than x^i, and as a consequence, the elements of the metric do not all have the same dimensions.

In writing down metrics, we use the standard notation of the line element. For example, the 3-dimensional Euclidean metric with coordinates (x, y, z) defined by $g_{ij} := \delta_{ij}$ is written down as $ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$. In general, we introduce a metric $g_{\mu\nu}$ by writing down $ds^2 := g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu$. Metrics will always have the signature $(-1, 1, 1, 1)$.

Chapter 1

Introduction

In this thesis, we will examine the Carroll limit of vanishing speed of light in the context of geodesics.

Contrary to what one might expect, this limit is not named after the physicist who thought of it. Instead, it is named after the mathematician and writer Lewis Carroll, who played with mathematical concepts in his novels. An example of interest is one in his novel *Through the Looking-Glass, and What Alice Found There*, the sequel to *Alice's Adventures in Wonderland*. In this story, the *Red Queen's race* takes place, during which the Queen says the following ([1]):

> "Now, here, you see, it takes all the running you can do, to keep in the same place. If you want to get somewhere else, you must run at least twice as fast as that!"

In special relativity, the speed of light c is regarded as the universal speed limit. Thus, "all the running you can do" can be interpreted as travelling with the speed of light. In the Carroll limit, we find that light stays in the same place, because we send the speed of light to 0. This is probably one of the main reasons why the limit of $c \to 0$ was associated with Lewis Carroll, and consequently named the Carroll limit by Lévy-Leblond ([5]) in 1965.

The Carroll limit can be seen as the counterpart of the more famous Newtonian limit $c \to \infty$. The Newtonian limit of general relativity is called Newton-Cartan geometry, introduced by Élie Cartan in [2] and [3]. Analogously, the Carroll limit of general relativity is also a new kind of geometry, named Carrollian geometry. A good mathematical treatment of both geometries and their structures can be found in [23]. In [18], an action for Newtonian gravity is constructed, in the process using Newton-Cartan geometry, and in the end they find equations of motion that "generalize Newtonian gravity by allowing for the effect of gravitational time dilation."

In the case of particle motion, we take the Newtonian limit if we are interested in particles that move with a tiny fraction of the speed of light. Analogously, we take the Carroll limit if we are interested in particles that move with a speed of which the speed of light is a tiny fraction. This may sound strange, so why are we interested in this limit at all?

First of all, general relativity allows the solution $x = 100ct$ in 2-dimensional Minkowski space just as well as it allows $x = 0.01ct$: these superluminal particles are called tachyons. Touching on a cosmological topic, we can also be interested in recessional speeds in a de Sitter spacetime that far exceed the speed of light, as we consider particles that are lots of Hubble radii apart ([24]). In the Carroll limit, spatially separated points are also causally separated, in the sense that they cannot send light signals to each other. This sparks interest in systems that have subsystems that are completely causally disconnected from each other. This would be a system exhibiting characteristics of a strict Carroll limit, i.e. setting $c = 0$. An example of a system exhibiting characteristics of an approximate Carroll limit, i.e. looking at characteric speeds v for which $v \gg c$, is a condensed matter system at a temperature lower than the Fermi temperature, with a relatively low Fermi velocity ([24]). An example of a system that can exhibit low Fermi velocities in practice is so-called "twisted bilayer graphene" with a twist angle close to the "magic angle" ([29]). Another example is the interface between the metal palladium and a material called WTe_2 ([28]). In these systems, there can be subluminal particles that have speeds exceeding the Fermi velocity. That would be reminiscent of a moving Carroll-like particle in this context.

Furthermore, in certain spacetimes there are hypersurfaces that have Carrollian symmetries: "(...) the geometry of a black hole horizon can be described as a Carrollian geometry emerging from an ultra-relativistic limit where the near-horizon radial coordinate plays the role of a virtual velocity of light tending to zero" ([21]).

One can also look at taking the Carroll limit of (perfect) fluids, which is done in [24]. They show that the Carroll limit of a perfect fluid with non-zero energy density gives us an equation of state that models dark energy.

Now we know why it is interesting to study the Carroll limit, let us introduce the area of study in this thesis: geodesics.

In general relativity, we start with the Einstein equations. From there, we determine a metric that is a solution to these equations. Given such a metric, we determine the geodesic equations, and a solution to the geodesic equations describes the motion of particles. This is a multi-step process, and it is not clear yet if we can take the Carroll limit anywhere in this process and end up with the same particle motions. Therefore, we can only speak of *a* Carroll limit until we completely specify what we mean.

We have to specify two important things to specify a Carroll limit, and one of them is exactly *when* we take the limit of $c \to 0$. The other thing we need to specify is how we let the integration constants (that we get from solving the differential equations we encounter in the process mentioned above) scale with c. For example, an integration constant we get from solving the Einstein equations could be the Schwarzschild radius in the Schwarzschild solution. It is given by $R_S := \frac{2GM}{c^2}$, so one could think that it scales with c^{-2}. However, we are now promoting c, which is normally a constant, to a variable which we intend to make very small. This means that constants of nature could depend on c in a way that we do not know about. We will have to specify how GM will scale with c to specify taking the Carroll limit in a Schwarzschild spacetime: depending on the scaling of GM, the Carroll limit of expressions in Schwarzschild spacetime may give different results.

In this thesis, we will consider different Carroll limits of geodesics, guided by the examples of Minkowski, Schwarzschild, and de Sitter spacetimes.

In the next chapter, we give a recapitulation of the essential knowledge of general relativity that we will use in this thesis.

In Chapter 3, we give a short overview of what has been done regarding the Carroll limit, and where this thesis fits into the literature.

In Chapter 4, we introduce our sample spacetimes of Minkowski, Schwarzschild, and de Sitter, and we take a brief look at which scalings of the integration constants in the metrics we have to consider.

In Chapter 5, we take Carroll limits of geodesics in the three sample spacetimes.

In Chapter 6, we consider the small-c expansion of the geodesic equations. Consequently, we find solutions to the leading (i.e. non-vanishing) orders of these equations.

In Chapter 7, we take the Carroll limit within the action for geodesics, and look at its Euler-Lagrange equations, which we regard as another limit of the geodesic equations.

We wrap up by summarizing our results and giving suggestions for future research in the Conclusion and outlook.

Chapter 2

The essentials of general relativity

This chapter does not serve as an introduction to the tools of general relativity that will be necessary to follow this thesis. Instead, it serves as a quick reminder for those who are already somewhat familiar with the subject.

As described in the introduction, we will deal with a multi-step process to find geodesics. We start with Einstein's vacuum field equations in D dimensions[1]:

$$R_{\mu\nu} = \frac{\Lambda}{\frac{D}{2} - 1} g_{\mu\nu}, \tag{2.1}$$

where $R_{\mu\nu}$ is the Ricci tensor, $g_{\mu\nu}$ is the metric, and Λ is the cosmological constant. Solutions of the Einstein equations are metrics: they determine the characteristics of the D-dimensional spacetime. From the metric, we determine the Christoffel symbols $\Gamma^\rho_{\mu\nu}$ of the Levi-Civita connection,

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu} \right) \tag{2.2}$$

with which we can construct the geodesic equations:

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2} + \Gamma^\rho_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0. \tag{2.3}$$

A solution $x^\mu(s)$ of these geodesic equations is called a <u>geodesic</u>, and s serves only as a parameter, which we will take to have dimensions of length in this thesis. In this thesis, we will call the image of the function $x^i : s \to M$ the <u>trajectory</u> of the particle on the geodesic: every particle has a trajectory, which will be a 0- or 1-dimensional submanifold of the spacetime manifold.

Definition 2.1. Given a geodesic, we define

$$\varepsilon := -g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s}. \tag{2.4}$$

This will be a number that is constant along the geodesic (see Lemma A.1 in Appendix A for a proof of this). If $\varepsilon < 0$, we call the geodesic <u>spacelike</u>. If $\varepsilon = 0$, the geodesic is <u>lightlike</u>. Lastly, if $\varepsilon > 0$, the geodesic is <u>timelike</u>. We can always reparameterize our geodesic, so without loss of generality, we can assume that $\varepsilon \in \{-1, 0, 1\}$. ◁

[1] Here we eliminated the Ricci scalar from the standard form of the Einstein field equations to make the equations look as short and simple as possible.

As massless particles will always travel at the speed of light, they will not move in the Carroll limit. Therefore, the only interesting cases will appear for particles with non-zero mass, which is why we can include the mass in quantities below.

We get the equivalent of the geodesic equations if we consider the action for a relativistic particle

$$S = \int \mathcal{L}\,\mathrm{d}s, \tag{2.5}$$

where $\mathcal{L}$ is the following Lagrangian:

$$\mathcal{L} := \pm|m|c\sqrt{\pm g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}}, \tag{2.6}$$

where we pick the $+$-signs if we want to find spacelike geodesics, and the $-$-signs if we want to find timelike geodesics. Then the Euler-Lagrange equations associated with this action will be the geodesic equations. We can define the canonical momentum p^μ as follows:

$$p^\mu := \frac{\partial\mathcal{L}}{\partial\dot{x}_\mu} = |m|c\frac{\mathrm{d}x^\mu}{\mathrm{d}s}, \tag{2.7}$$

where we wrote $\dot{x}^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$.

Remark 2.2. From the definition of ε in equation (2.4) and of the momentum in equation (2.7), we get the following dispersion relation:

$$p_\mu p^\mu + |m|^2c^2\varepsilon = 0. \tag{2.8}$$

We see that we get the familiar dispersion relation

$$p_\mu p^\mu + m^2c^2 = 0 \tag{2.9}$$

if we let m be imaginary for particles on spacelike geodesics. Note that this dispersion relation is general: it holds in all spacetimes. $\triangleleft$

Lastly, we want to explicitly mention that we do not accept "solutions" to the geodesic equations for which there is a μ such that $\left|\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right| \to \infty$. We always want particles on geodesics to be described by a set of finite coordinates. Another way of looking at it is to consider the following Minkowskian worldline action ([24], section 3.1)

$$S = -\int \left(E\dot{t} - \vec{p}\cdot\dot{\vec{x}}\right)\,\mathrm{d}s. \tag{2.10}$$

The associated equations of motion are equivalent to conservation of energy and momentum, which are equivalent to the geodesic equations in Minkowski spacetime. We want this action to have a finite integrand, also for non-zero energy and/or momentum, so we want $\left|\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right| < \infty$. Note that we can always reparametrize our geodesic: in particular, we can always rescale s with powers of c. This will give us a parameter, let us call it $\widetilde{s}$ wherever we introduce it, with dimensions different from distance. This reparameterization is an important consideration for later chapters, in which the Carroll limit of some finite coordinate can become infinite.

Conserved quantities

Given a metric, we can sometimes find quantities of particles that are conserved along geodesics, which can aid in describing these geodesics. The process of finding these conserved quantities involves Lie derivatives. The following coordinate expression for the Lie derivative will come in handy later.

Remark 2.3. The Lie derivative $\mathcal{L}_X A$ of a $(0,2)$-tensor A along a vector X is the following in tensor notation:

$$(\mathcal{L}_X A)_{\mu\nu} = X^\sigma \partial_\sigma A_{\mu\nu} + A_{\sigma\nu}\partial_\mu X^\sigma + A_{\mu\sigma}\partial_\nu X^\sigma. \tag{2.11}$$

◁

Definition 2.4. Let a metric be given. A vector X^μ is called a Killing vector if

$$(\mathcal{L}_X g)_{\mu\nu} = 0. \tag{2.12}$$

In particular, if the metric g does not depend on a coordinate x^α for some specific α, then $(\partial_\alpha)^\mu$ is a Killing vector. ◁

Remark 2.5. Given a Killing vector X^μ, the expression

$$X_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \tag{2.13}$$

defines a conserved quantity. See Lemma A.2 in Appendix A for a proof. ◁

Definition 2.6. If $(\partial_t)^\mu$ is a Killing vector, we define the energy E (which is conserved) as

$$E := -|m|c g_{t\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s}. \tag{2.14}$$

This energy relates to the time component of the momentum as

$$E = -g_{t\mu} p^\mu = -p_t. \tag{2.15}$$

If $(\partial_\phi)^\mu$ is a Killing vector, we define the angular momentum L (which is conserved) as

$$L := |m|c g_{\phi\mu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s}. \tag{2.16}$$

It is important to note that both of these quantities, and conserved quantities in general, depend on reparameterizations (and thus rescalings) of s. ◁

For metrics that have $(\partial_t)^\mu$ as Killing vector and for which $g_{ti} = 0$, the dispersion relation becomes

$$E^2 g^{tt} + p_i p^i + m^2 c^2 = 0. \tag{2.17}$$

Chapter 3

An overview of the Carroll limit

Let us start by defining the focus of this thesis, the Carroll limit, followed by some general remarks.

Definition 3.1. The Carroll limit, or Carrollian limit, of an expression A is defined as

$$\lim_{c \to 0} A. \tag{3.1}$$

We say that we take the Carroll limit of A if we apply the operator $\lim\limits_{c \to 0}$ to A. ◁

As the speed of light is a dimensionful quantity, we should explain that the way we take the limit of $c \to 0$ is to multiply every c by a number ϵ, and then taking the limit of $\epsilon \to 0$. In short,

$$\lim_{c \to 0} A(c) := \lim_{\epsilon \to 0} A(c\epsilon). \tag{3.2}$$

Some call the Carroll limit an ultra-relativistic or non-relativistic limit. However, just like in Galilean relativity, there is still a concept of relativity in the Carroll limit, and ultra-relativity is often used in the context of speeds approaching the speed of light, not far exceeding it. Accordingly, we will not refer to the Carroll limit in this manner.

The correct way of thinking about "taking the Carroll limit" is to look at expansions in powers of c, and keeping only the leading order terms. Just like a true Galilean limit is unphysical because light particles would then travel at infinite speeds, a true Carroll limit is unphysical because then there would be no concept of causality.

3.1 Carroll algebra

The first mention of the limit of sending the speed of light to zero in the context of general relativity is from Jean-Marc Levy-Leblond in [5], in 1965. He named the corresponding symmetry group (the analogue of the Poincaré group) the Carroll group, after Lewis Carroll. He defined the Carroll algebra from the Poincaré algebra analogously to how the Galilei algebra is defined from the Poincaré algebra. If we only consider boosts (which make up the interesting part of the coordinate transformations in this context), we get that the boost parameter $\vec{b}$ describes the Carroll boosts:

$$\text{Carroll:} \qquad t' = t - \vec{b} \cdot \vec{x}, \qquad \vec{x}' = \vec{x}. \tag{3.3}$$

Compare this to the Galilei boosts described by the boost parameter $\vec{v}$:

$$\text{Galilei:} \qquad t' = t, \qquad \vec{x}' = \vec{x} - \vec{v}t, \tag{3.4}$$

and one sees that the roles of time and space are switched to some degree. The boost transformations of velocities become

$$\vec{v}'(t') = \frac{\vec{v}(t)}{1 - \vec{b} \cdot \vec{v}}, \tag{3.5}$$

which are rescalings. It means that, by simply performing boosts, one can never put a moving particle to rest, and one can never make a non-moving particle move. These are two separate kinds of particles, unable to transition between eachother. A good exposition about this is found in section 2 of [24]. There, it is also explained that the Carroll limit of the energy-momentum tensor indicates that there cannot be energy flux in a Carrollian spacetime. This implies the following statement.

Remark 3.2. All moving Carroll particles have zero energy, and only non-moving Carroll particles have a non-zero energy, which is determined by the dispersion relation. In the latter case, we let $|m|$ scale like c^{-2} (so we keep $|m|c^2$ fixed) such that the energy becomes non-zero, in order to have a non-trivial dispersion relation, which then yields the rest energy

$$E = mc^2. \tag{3.6}$$

In the first case, we let $|m|$ scale like c^{-1} (so we keep $|m|c$ fixed) such that the momentum is finite and non-zero: in this case, the dispersion relation becomes

$$p_i p^i = -m^2 c^2. \tag{3.7}$$

Note that moving particles are on spacelike geodesics, so $-m^2c^2 > 0$.
It is unclear if there is an underlying motivation for manually choosing these scaling behaviours for $|m|$ for zero and non-zero energies, other than the fact that these are the only scalings that yield finite and non-trivial equations of motion.

A remark on notation: if $|m|c$ is being held constant in the Carroll limit, we will write $|m|c$, also after the limit is taken. If $|m|c^2$ is being held constant, then we write $|m|c^2$ after taking the limit. We do this to avoid issues in notation between the two different scalings. $\triangleleft$

Levy-Leblond also wrote down the Lie algebra corresponding to the Carroll group, which describes the geometry that remains after one takes the Carroll limit of the Poincaré group. See Figure 3.1 for a comparison of the Lie algebras of the Carroll, Poincaré, and Galilei groups. We see that in Carroll geometry, all boosts commute, which we explain by noting that boosts are only rescalings of the velocity. We also see that boosts and time translations commute: for non-zero energy, we have zero velocity, so trivial boosts, and for non-trivial boosts, we have non-zero velocity, so zero energy. In general, we see that P_0 commutes with all other symmetry generators, which means that P_0 is now a central charge. We get two inequivalent algebras from $P_0 = 0$ and $P_0 \neq 0$. This corresponds to the inequivalence of the zero and non-zero energy particles.

We see that Levy-Leblond focused on the infinitesimal transformations, i.e. the local behaviour of a spacetime after taking the Carroll limit. A logical next step is to zoom out and look at what happens at the level of the whole spacetime manifold.

3.2 Carroll manifolds

Regarding the movements of particles (on geodesics), Bergshoeff and others state in [10] that “a single free such Carroll particle has no non-trivial dynamics (‘the Carroll particle does not move’)”, but that “the single Carroll particle in a non-trivial background (...) has non-trivial dynamics.” This

Carroll	Poincaré	Galilée
$[\mathrm{J}_i, \mathrm{J}_j] = i\varepsilon_{ijk}\mathrm{J}_k$	$[\mathrm{J}_i, \mathrm{J}_j] = i\varepsilon_{ijk}\mathrm{J}_k$	$[\mathrm{J}_i, \mathrm{J}_j] = i\varepsilon_{ijk}\mathrm{J}_k$
$[\mathrm{J}_i, \mathrm{K}_j] = i\varepsilon_{ijk}\mathrm{K}_k$	$[\mathrm{J}_i, \mathrm{K}_j] = i\varepsilon_{ijk}\mathrm{K}_k$	$[\mathrm{J}_i, \mathrm{K}_j] = i\varepsilon_{ijk}\mathrm{K}_k$
$[\mathrm{K}_i, \mathrm{K}_j] = 0$	$[\mathrm{K}_i, \mathrm{K}_j] = -\, i\varepsilon_{ijk}\mathrm{J}_k$	$[\mathrm{K}_i, \mathrm{K}_j] = 0$
$[\mathrm{J}_i, \mathrm{P}_j] = i\varepsilon_{ijk}\mathrm{P}_k$	$[\mathrm{J}_i, \mathrm{P}_j] = i\varepsilon_{ijk}\mathrm{P}_k$	$[\mathrm{J}_i, \mathrm{P}_j] = i\varepsilon_{ijk}\mathrm{P}_k$
$[\mathrm{K}_i, \mathrm{P}_j] = i\delta_{ij}\mathrm{P}_0$	$[\mathrm{K}_i, \mathrm{P}_j] = i\delta_{ij}\mathrm{P}_0$	$[\mathrm{K}_i, \mathrm{P}_j] = 0$
$[\mathrm{J}_i, \mathrm{P}_0] = 0$	$[\mathrm{J}_i, \mathrm{P}_0] = 0$	$[\mathrm{J}_i, \mathrm{P}_0] = 0$
$[\mathrm{K}_i, \mathrm{P}_0] = 0$	$[\mathrm{K}_i, \mathrm{P}_0] = i\mathrm{P}_i$	$[\mathrm{K}_i, \mathrm{P}_0] = i\mathrm{P}_i$
$[\mathrm{P}_i, \mathrm{P}_j] = 0$	$[\mathrm{P}_i, \mathrm{P}_j] = 0$	$[\mathrm{P}_i, \mathrm{P}_j] = 0$
$[\mathrm{P}_i, \mathrm{P}_0] = 0$	$[\mathrm{P}_i, \mathrm{P}_0] = 0$	$[\mathrm{P}_i, \mathrm{P}_0] = 0$

Figure 3.1: The commutation relations of the infinitesimal generators that define the Lie algebras of the Carroll, Poincaré, and Galilei groups. Rotations are generated by J_i, boosts by K_i, and translations by P_μ. Source: [5], page 6.

is true for particles on timelike or lightlike geodesics, but in this thesis we will allow for free particles on spacelike geodesics to exist, and these have non-trivial dynamics.

Gibbons and others looked at spacetime manifolds endowed with metrics in the context of the Carroll limit in [11] and [12]. The latter in particular is a good source to get more insight into the mathematical structure of Carroll spacetimes. They conclude that in the Carroll limit, the manifold itself does not change, and in particular keeps the same dimensionality. However, the metric $g_{\mu\nu}$ becomes degenerate, i.e. not invertible, which makes it a pseudometric. We will follow Gibbons et al. in only accepting pseudometrics of rank 3 as true Carrollian metrics.[1] From the definition of the Christoffel symbols, we see that the Levi-Civita connection will not be defined anymore. They described a logical way to get around this problem to construct some (but not all) of the Christoffel symbols, which we explain in Appendix B for the interested reader. However, Hartong and others have constructed a formalism in which we do not have this problem at all, and the resulting Christoffel symbols agree with those that Gibbons et al. found. The idea is to expand the metric in powers of c using the formalism of vielbeins, which is seen in [22] and [13]. We will introduce this formalism here, using similar notation to that used in an unpublished work by Hartong and others, and we will use this formalism in Chapter 6 to find geodesics.

The first step is to diagonalize the matrix that describes the metric g, which we do by a one-form transformation. We want to construct a basis of one-forms $\{\sigma^A\}$ ($A \in \{0, 1, 2, 3\}$) as functions of $\{\mathrm{d}x^\mu\}$ such that $g = \eta_{AB}\sigma^A \otimes \sigma^B$, where η is the Minkowski metric. This basis is called an orthogonal coframe, and it is not unique. We define the vielbeins

$$T_\mu := \sigma^0_\mu, \qquad E^a_\mu := \sigma^a_\mu, \tag{3.8}$$

where $a \in \{1, 2, 3\}$. With σ^A_μ we mean $\sigma^A(\partial_\mu)$, so "the $\mathrm{d}x^\mu$-part of σ^A". Thus, we now have

$$g_{\mu\nu} = -c^2 T_\mu T_\nu + \sum_a E^a_\mu E^a_\nu. \tag{3.9}$$

[1]This will become important when the time comes to choose scalings for integration constants: we will not accept any scalings that lead to metrics collapsing to having rank less than 3.

Next, we want to write the inverse metric in a diagonal way. Having the coframe, this is now simple, because we can define the unique frame $\{f_A\}$ as functions of $\{\partial_\mu\}$ that corresponds to the coframe by demanding that $\sigma^A f_B = \delta^A_B$. Then $g^{-1} = \eta^{AB} f_A \otimes f_B$. We define

$$\widetilde{T}^\mu := -f_0^\mu, \qquad \widetilde{E}_a^\mu := f_a^\mu. \tag{3.10}$$

We now have

$$g^{\mu\nu} = -\frac{1}{c^2}\widetilde{T}^\mu \widetilde{T}^\nu + \sum_a \widetilde{E}_a^\mu \widetilde{E}_a^\nu. \tag{3.11}$$

We want to expand the metric and its inverse, so we want to expand the vielbeins that are introduced above. We therefore make the following assumption.

Assumption 3.3. *We assume that T_μ, E_μ^a, $\widetilde{T}^\mu$, and $\widetilde{E}_a^\mu$ have expansions in c^2, for which we use the following notation:*

$$T_\mu = \tau_\mu + O(c^2), \qquad E_\mu^a = e_\mu^a + \pi_\mu^a c^2 + O(c^4), \tag{3.12}$$
$$\widetilde{T}^\mu = v^\mu + M^\mu c^2 + O(c^4), \qquad \widetilde{E}_a^\mu = \tilde{e}_a^\mu + O(c^2). \tag{3.13}$$

Note that this is an expansion in c^2, not in c as the most general choice of expansion. A footnote on page 4 of [17] states the following about this: "One can argue [7] that odd powers of c will only appear at higher order than we are interested in. This implies that one does not lose any generality by restricting to even powers here. Still this will be one of the few assumptions we put into the formalism from the start. It might be interesting to allow odd terms in the expansion from the beginning and see directly from the equations of motion that they can be consistently put to zero."

Definition 3.4. We now define

$$h_{\mu\nu} := \sum_a e_\mu^a e_\nu^a, \qquad h^{\mu\nu} := \sum_a \tilde{e}_a^\mu \tilde{e}_a^\nu, \tag{3.14}$$
$$\Phi_{\mu\nu} := \sum_a 2 e^a_{(\mu} \pi^a_{\nu)}, \tag{3.15}$$
$$\overline{h}^{\mu\nu} := h^{\mu\nu} - 2v^{(\mu} M^{\nu)}, \tag{3.16}$$
$$\overline{\Phi}_{\mu\nu} := \Phi_{\mu\nu} - \tau_\mu \tau_\nu, \tag{3.17}$$
$$\hat{\tau}_\mu := \tau_\mu - h_{\mu\sigma} M^\sigma, \tag{3.18}$$
$$K_{\mu\nu} := -\frac{1}{2}\left(\mathcal{L}_v h\right)_{\mu\nu}. \tag{3.19}$$

We call $K_{\mu\nu}$ the <u>extrinsic curvature</u> of the spacetime. Note that it is symmetric, because $h_{\mu\nu}$ is symmetric and the Lie derivative conserves this symmetry. ◁

With these definitions, we can write the expansion of the metric and its inverse as follows:

$$g_{\mu\nu} = h_{\mu\nu} + \overline{\Phi}_{\mu\nu} c^2 + O(c^4), \tag{3.20}$$
$$g^{\mu\nu} = -\frac{1}{c^2} v^\mu v^\nu + \overline{h}^{\mu\nu} + O(c^2). \tag{3.21}$$

Remark 3.5. Intuitively, $h_{\mu\nu}$ and $h^{\mu\nu}$ are the leading order of the spatial part of the metric and the inverse metric, respectively. In Lemma A.4 in Appendix A, it is shown that the tensor rank of $h_{\mu\nu}$ is 3 and that the tensor rank of $K_{\mu\nu}$ is at most 3. In the sample spacetimes we discuss in this thesis, the extrinsic curvature is either 0 or it is positive definite on its support. This may give a false sense of non-negative definiteness of the extrinsic curvature. This is not always the case, a counterexample being the extrinsic curvature of a Kasner metric. ◁

There are a lot of identities concerning the newly-introduced quantities, and we will list the important ones below. However, we start out with a few identities of the sections themselves:

$$T_\lambda \widetilde{T}^\lambda = -\sigma^0 f_0 = -1, \qquad E^a_\lambda \widetilde{E}^\lambda_b = \sigma^a \sigma_b = \delta^a_b, \tag{3.22}$$
$$T_\lambda \widetilde{E}^\lambda_a = \sigma^0 f_a = 0, \qquad E^a_\lambda \widetilde{T}^\lambda = \sigma^a \sigma_0 = 0. \tag{3.23}$$

Definition 3.6. For every $n \in \mathbb{Z}$, introduce the operator $\cdot_{(n)}$ that takes an expression, expands it in powers of c, and gives the coefficient for c^n. So,

$$\left[\sum_{i=-\infty}^{\infty} a_i c^i\right]_{(n)} = a_n. \tag{3.24}$$

◁

Note that for quantities A and B that both have a leading order of c^0, we have

$$A_{(0)} B_{(0)} = (AB)_{(0)}. \tag{3.25}$$

We now calculate the following identities:

$$\tau_\sigma v^\sigma = [T_\sigma]_{(0)} \left[\tilde{T}^\sigma\right]_{(0)} \overset{(3.12)\ \&\ (3.13)\ \&\ (3.25)}{=} \left[T_\sigma \tilde{T}^\sigma\right]_{(0)} = -1, \tag{3.26}$$
$$h^{\mu\sigma} h_{\sigma\nu} = \delta^\mu_\nu + v^\mu \tau_\nu, \tag{3.27}$$
$$v^\sigma \Phi_{\sigma\mu} = -h_{\mu\sigma} M^\sigma, \tag{3.28}$$
$$v^\sigma h_{\sigma\mu} = \left[\tilde{T}^\sigma \sum_a E^a_\sigma E^a_\mu\right]_{(0)} \overset{(3.23)}{=} 0, \tag{3.29}$$
$$\tau_\sigma h^{\sigma\mu} = \left[T_\sigma \sum_a \tilde{E}^\sigma_a \tilde{E}^\mu_a\right]_{(0)} \overset{(3.23)}{=} 0, \tag{3.30}$$
$$K_{\mu\nu} = -\frac{1}{2}\left(v^\lambda \partial_\lambda h_{\mu\nu} + h_{\lambda\nu}\partial_\mu v^\lambda + h_{\mu\lambda}\partial_\nu v^\lambda\right) \overset{(3.29)}{=} \frac{1}{2} v^\lambda \left(\partial_\mu h_{\lambda\nu} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right), \tag{3.31}$$
$$v^\sigma K_{\mu\sigma} = -\frac{1}{2}\left(v^\lambda v^\sigma \partial_\lambda h_{\mu\sigma} + v^\lambda h_{\mu\sigma}\partial_\lambda v^\sigma + h_{\lambda\sigma} v^\sigma \partial_\mu v^\lambda\right) \overset{(3.29)}{=} 0, \tag{3.32}$$
$$v^\sigma \hat{\tau}_\sigma \overset{(3.18)}{=} v^\sigma \tau_\sigma - v^\sigma h_{\sigma\rho} M^\rho \overset{(3.26)\ \&\ (3.29)}{=} -1, \tag{3.33}$$
$$h_{\mu\sigma} \overline{h}^{\sigma\nu} \overset{(3.16)\ \&\ (3.27)\ \&\ (3.29)}{=} \delta^\nu_\mu + v^\nu \tau_\mu - h_{\mu\sigma} M^\sigma v^\nu \overset{(3.18)}{=} \delta^\nu_\mu + v^\nu \hat{\tau}_\mu, \tag{3.34}$$
$$v^\sigma \overline{\Phi}_{\sigma\mu} \overset{(3.17)\ \&\ (3.28)}{=} -h_{\mu\sigma} M^\sigma - v^\sigma \tau_\sigma \tau_\mu \overset{(3.26)\ \&\ (3.18)}{=} \hat{\tau}_\mu. \tag{3.35}$$

Definition 3.7. We define the <u>Carroll compatible connection</u> $C^\rho_{\mu\nu}$ as ([13][2], [31])

$$C^\rho_{\mu\nu} := -v^\rho \partial_\mu \hat{\tau}_\nu + \frac{1}{2}\overline{h}^{\rho\sigma}\left(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu} - 2K_{\mu\sigma}\hat{\tau}_\nu\right). \tag{3.36}$$

◁

[2] We use equation (2.63), calculate $\overline{h}^{\nu\lambda}\hat{\tau}_\nu = 2v^\lambda \overline{\Phi}$ (with $\overline{\Phi}$ defined differently than in this thesis, namely in equation (2.66) in [13]), and use the fact that the connection is independent of $\overline{\Phi}$, so we set it to 0 to get our definition in equation (3.36).

Remark 3.8. Note that this connection is not symmetric, while the Levi-Civita connection is. This is a torsionful connection. The connection $C^{\rho}_{\mu\nu}$ is called Carroll compatible because $h_{\mu\nu}$ and v^{ρ} - the components that the leading order of $g_{\mu\nu}$ and $g^{\mu\nu}$ are constructed with, respectively - are in the kernel of the covariant derivative $\overset{(C)}{\nabla}$ that is based on the connection $C^{\rho}_{\mu\nu}$:

$$\overset{(C)}{\nabla}_{\alpha} h_{\mu\nu} = 0, \qquad \overset{(C)}{\nabla}_{\alpha} v^{\rho} = 0. \tag{3.37}$$

A proof of this can be found in Lemma A.5 in Appendix A. Please note that this connection is not the only Carroll compatible connection that can be constructed (see [13], page 12), but this is the Carroll compatible connection that we will be using throughout this work. ◁

We have now introduced the formalism and its notation, and we have provided some useful identities of the vielbeins. We will put this framework to work in Chapter 6, where we find an expression for the Christoffel symbols of the Levi-Civita connection and use it to write down an expansion of the geodesic equations.

This thesis will build upon what is introduced in this section, but one can also consider taking the Carroll limit of the gravity action and altering the theory of gravity as a whole. We briefly touch on this in the next section.

3.3 Carroll theories

We could also look at existing theories, i.e. actions, and take the Carroll limit at the level of the action. We could say that the limiting action produces a Carroll theory. We look at the Einstein-Hilbert action (which produces general relativity), but we also touch on general Lorentz invariant field theories and string theories.

3.3.1 Einstein-Hilbert action

We could also choose to look at the Einstein equations or the Einstein-Hilbert action in the Carroll limit. This is done in chapter 3.2 in [16]. They start with the vacuum Einstein-Hilbert action[3]

$$S_{\text{EH}} = -\frac{c^4}{16\pi G_N} \int \det\left(T_\mu, E^a_\mu\right) \left[\widetilde{E}^\mu_a \widetilde{E}^\nu_b R_{\mu\nu}{}^{ab}(J) - \frac{1}{c^2} \widetilde{T}^\mu \widetilde{T}^\nu R_{\mu\nu}{}^{00}(J)\right] d^D x, \tag{3.38}$$

where G_N is Newton's gravitational constant, and $R_{\mu\nu}{}^{AB}(J)$ is the "curvature associated to the Lorentz transformations" (see [16] for specifics). The Carroll limit of this action is given by

$$S_{\text{Car}} = \frac{1}{16\pi G_C} \int \det\left(\tau_\mu, e^a_\mu\right) h^{\mu\nu} \left(R_{\mu\nu} - v^\rho \tau_\sigma R_{\mu\rho\nu}{}^{\sigma}\right) d^D x, \tag{3.39}$$

where $G_C := \frac{G_N}{c^4}$, which is kept fixed.

In chapter 3 of [26], the Einstein-Hilbert action is also studied, but there the curvature tensors are already expanded in terms of the vielbeins we introduced in the previous section. They start with the familiar form of the Einstein-Hilbert action:

$$S_{\text{EH}} = \frac{c^3}{16\pi G_N} \int R\sqrt{-g}\; d^D x. \tag{3.40}$$

[3] This action is rewritten from the original in [16] to match the notation introduced in the previous section. We also introduce the standard notation of $d^D x$ for $\mathrm{d}x^0\,\mathrm{d}x^1 \ldots \mathrm{d}x^{D-1}$.

They find that the leading order ("LO") is of order c^2, and it is given by[4]

$$S_{\text{LO}} = \frac{1}{16\pi G_N} \int \left(K_{\mu\nu} h^{\mu\rho} h^{\nu\sigma} K_{\rho\sigma} - K_{\mu\nu} h^{\mu\nu} K_{\rho\sigma} h^{\rho\sigma} \right) \det \left(\tau_\mu, e^a_\mu \right) d^D x \tag{3.41}$$

This gives equations that restrict $K_{\mu\nu}$ and $h^{\mu\nu}$. The next-to-leading order (order c^4) of the Einstein-Hilbert action is also given in this article, but it is more involved, and the study of the associated equations of motion is left to future work.

The gravity theory resulting from such a Carroll action (either as stated in (3.39) or as an action for every order of c following the work in [26]) is known as Carroll gravity. An interesting aspect about the Carroll action that the original Einstein-Hilbert action does not have is that it carries a Weyl symmetry. We will not go further into this, but see [25] for the Weyl symmetry in the action, and see [20] for the consequent Weyl covariance in the equations of motion.

3.3.2 Field theories

A good introduction of Carroll field theories can be found in chapter 4 of [24], where they consider a simple scalar field theory and Maxwell theory. An intriguing point is that there are two different Carroll limits ("contractions"), depending on how the canonical variables are scaled with c. Just like Le Bellac and Lévy-Leblond proved in the case of Galilean limits of electromagnetism in [6], it was proven in [12] that electromagnetism has two inequivalent Carroll contractions, which are named the "electric" and "magnetic" contractions. In essence, these two Carroll contractions differ only in a different choice of scaling the canonical field ϕ and the canonical momentum π_ϕ.

Recently, in [27], it was shown that *any* Lorentz invariant theory has two contractions, which are named electric and magnetic in the general case as well, even though there might not be any electromagnetic duality involved in the theory considered. It is suspected that the electric contraction of general relativity (defined by the Einstein-Hilbert action) corresponds to the theory defined by the leading order of the Einstein-Hilbert action seen in equation (3.41).

3.3.3 String theory

One could also look at relativistic strings and take the Carroll limit: a "'stringy' Carrollian limit", as Cardona et al. call it in [14]. In this article, they conclude that "the free Carroll string does not move". We suspect that, by starting with the Nambu-Goto action

$$S_{\text{NG}} = -T \int \mathrm{d}\tau \, \mathrm{d}\sigma \tag{3.42}$$

with tension T instead of $|T|$, they implicitly make the assumption of timelike or lightlike strings, therefore not finding the non-trivial dynamics of spacelike strings.[5] Just like Bergshoeff et al. found in [10] for particles, Cardona et al. find the following for strings: "If we consider Carroll strings coupled to Carroll gravity the strings will have a non-trivial dynamics like in the case of the Carroll particle coupled to Carroll gauge fields."

We will not go into Carroll theories in this thesis.

[4] This action is also rewritten from the original in [26] to match our notation.

[5] Just like we did not want to assume anything about the real or imaginary nature of the mass in writing down the action in equation (2.5) by writing $|m|$ instead of m, it seems expected that one would want to write $|T|$ instead of T in the Nambu-Goto action to allow for imaginary tension, because tension is the string analogue to mass.

3.4 Carroll quantum equations

A separate field of study would be to try to get statements about quantum field theory in a Carrollian limit. Marsot gives this a push in the right direction in [30] by introducing the Carroll limit of the Klein-Gordon equation,

$$(\partial_t)^2 \psi(\vec{x}, t) = -\frac{m^2 c^4}{\hbar^2} \psi(\vec{x}, t). \tag{3.43}$$

He mentions that the Klein-Gordon equation results from the Poincaré group and that the Schrödinger equation results from the Galilei group, after which he derives the equation resulting from the Carroll group, which is

$$\partial_t \psi = \frac{\mathrm{i} m c^2}{\hbar} \psi. \tag{3.44}$$

We thought it important to briefly mention this work, but we will not say more about quantum field theory in the context of the Carroll limit in this thesis.

Chapter 4

Carroll limits of metrics

In this chapter, we look at the sample spacetimes of Minkowski, Schwarzschild, and de Sitter, and the scalings of their parameters. We will only accept scalings that result in a (true) Carroll metric, i.e. a metric of rank 3. It will become evident that it is convenient to consider two different coordinate systems for de Sitter spacetime: comoving and static coordinates. We will introduce them both. We will also state the quantities $\hat{\tau}_\mu$, v^μ, $h_{\mu\nu}$, the extrinsic curvature $K_{\mu\nu}$, and the Carroll compatible connection $C^\rho_{\mu\nu}$ for every metric, as these are quantities that we will use later.

4.1 Minkowski

The Minkowski spacetime is given by introducing the coordinates (t, x, y, z) and the line element of the Minkowski metric

$$ds^2 = -c^2 \mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{4.1}$$

For this metric, we have the following quantities[1]:

$$\hat{\tau}_t = 1, \qquad v^t = -1, \qquad h_{xx} = h_{yy} = h_{zz} = 1, \qquad K_{\mu\nu} = 0. \tag{4.2}$$

The Carroll compatible connection $C^\rho_{\mu\nu}$ is given by

$$C^\rho_{\mu\nu} = 0. \tag{4.3}$$

There are no parameters, so the only Carroll limit of the Minkowski metric is the 3-dimensional Euclidean metric

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{4.4}$$

4.2 Schwarzschild

We introduce the coordinates (t, r, θ, ϕ) and the Schwarzschild radius

$$R_S := \frac{2GM}{c^2}, \tag{4.5}$$

[1] The components of $\hat{\tau}_\mu$, v^μ, $h_{\mu\nu}$, and $K_{\mu\nu}$ that are not written down are zero. The analogous list of quantities for the other metrics should be read in the same way.

where G is Newton's gravitational constant and M is an integration constant with units of mass. The Schwarzschild metric is then

$$ds^2 = -c^2\left(1 - \frac{R_S}{r}\right) \mathrm{d}t^2 + \frac{1}{1-\frac{R_S}{r}} \mathrm{d}r^2 + r^2 d\Omega^2, \tag{4.6}$$

where

$$d\Omega^2 := \mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2. \tag{4.7}$$

For this metric, we have the following quantities:

$$\hat{\tau}_t = \sqrt{\frac{r - R_S}{r}}_{(0)}, \qquad v^t = -\sqrt{\frac{r}{r-R_S}}_{(0)}, \tag{4.8}$$

$$h_{rr} = \left(\frac{r}{r-R_S}\right)_{(0)}, \qquad h_{\theta\theta} = \left(r^2\right)_{(0)}, \qquad h_{\phi\phi} = \left(r^2 \sin^2\theta\right)_{(0)}, \tag{4.9}$$

$$K_{\mu\nu} = 0. \tag{4.10}$$

The non-zero components of the Carroll compatible connection $C^{\rho}_{\mu\nu}$ are given by

$$C^t_{rt} = \left(\frac{R_S}{2r(r-R_S)}\right)_{(0)}, \tag{4.11}$$

$$C^r_{rr} = -\left(\frac{R_S}{2r(r-R_S)}\right)_{(0)}, \qquad C^r_{\theta\theta} = -(r-R_S)_{(0)}, \qquad C^r_{\phi\phi} = -\left[(r-R_S)\sin^2\theta\right]_{(0)}, \tag{4.12}$$

$$C^\theta_{r\theta} = C^\theta_{\theta r} = \left(\frac{1}{r}\right)_{(0)}, \qquad C^\theta_{\phi\phi} = -\left(\sin\theta\cos\theta\right)_{(0)}, \tag{4.13}$$

$$C^\phi_{r\phi} = C^\phi_{\phi r} = \left(\frac{1}{r}\right)_{(0)}, \qquad C^\phi_{\theta\phi} = C^\phi_{\phi\theta} = \left(\frac{1}{\tan\theta}\right)_{(0)}. \tag{4.14}$$

If we let the Schwarzschild radius R_S not scale with c, then the resulting Carroll metric is

$$ds^2 = \frac{1}{1-\frac{R_S}{r}}\mathrm{d}r^2 + r^2 d\Omega^2. \tag{4.15}$$

We accept this scaling behaviour, because the Carroll metric has rank 3.
If we let $R_S \sim c^{<0}$, then the patch of validity of these coordinates becomes non-existent, so we do not accept this scaling behaviour.
If we let $R_S \sim c^{>0}$, then the Carroll metric is

$$ds^2 = \mathrm{d}r^2 + r^2 d\Omega^2. \tag{4.16}$$

We accept this scaling behaviour, because the Carroll metric has rank 3. We note that this is the same 3-dimensional Euclidean metric we get when we perform the Carroll limit on the Minkowski metric. This is what we expect, because this can be interpreted as a weak gravity limit or a small black hole mass limit, both of which are expected to be Minkowskian.

4.2.1 Isotropic coordinates

We can transform the Schwarzschild metric to isotropic coordinates, defined by

$$\rho := \frac{1 + \sqrt{1 - \frac{R_S}{r}}}{1 - \sqrt{1 - \frac{R_S}{r}}} \tag{4.17}$$

This defines an extension of the Schwarzschild spacetime. We will only look at this metric in the context of non-scaling Schwarzschild radius. Then the resulting Carroll metric becomes

$$ds^2 = \left(\frac{(\rho+1)^2 R_S}{4\rho^2} \right)^2 \left(\mathrm{d}\rho^2 + \rho^2 d\Omega^2 \right). \tag{4.18}$$

The inverse transformation is

$$r = \frac{R_S}{4} \left(\sqrt{\rho} + \frac{1}{\sqrt{\rho}} \right)^2, \tag{4.19}$$

from which we see that $r(\rho) = r(1/\rho)$. This is the two-fold $\mathbb{Z}_2$-symmetry that gives us the familiar diagram of the Einstein-Rosen bridge ([4]), in which r is plotted against $\log(\rho)$. In the following chapter, we will encounter a non-trivial geodesic, which we will examine in these isotropic coordinates.

4.3 de Sitter

The de Sitter spacetime is one that has multiple widely used coordinate systems. We will consider the so-called comoving and static coordinates. The reason that we consider both coordinate systems is because the comoving coordinates make use of the Hubble constant H, while the static coordinates include the Hubble radius

$$R_H = \frac{c}{H}. \tag{4.20}$$

As one can see, these constants have different scaling behaviour. Therefore, if we want the Hubble constant not to scale with c, the comoving coordinates are more insightful. If, on the other hand, we want the Hubble constant to scale with c^1, we prefer to look at the static coordinates. As we will see, the coordinate transformation between these two coordinate systems depends on c, which results in problems if we want to fix a certain scaling of H and compare the Carroll limit of the de Sitter metric in these coordinates; they will not necessarily be the same.

4.3.1 de Sitter in comoving coordinates

We introduce the coordinates (t, x, y, z) and the Hubble constant H, with units of inverse time. The de Sitter metric in comoving coordinates is

$$ds^2 = -c^2\, \mathrm{d}t^2 + \mathrm{e}^{2Ht} \left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \right]. \tag{4.21}$$

Note that there is no timelike Killing vector, so we cannot define a conserved energy.

For this metric, we have the following quantities:

$$\hat{\tau}_t = 1, \qquad v^t = -1, \tag{4.22}$$

$$h_{xx} = h_{yy} = h_{zz} = \left(\mathrm{e}^{2Ht}\right)_{(0)}, \tag{4.23}$$

$$K_{xx} = K_{yy} = K_{zz} = \left(H\mathrm{e}^{2Ht}\right)_{(0)}. \tag{4.24}$$

Note that the extrinsic curvature is non-zero. The non-zero components of the Carroll compatible connection $C^{\rho}_{\mu\nu}$ are given by

$$C^x_{tx} = C^y_{ty} = C^z_{tz} = H_{(0)}. \tag{4.25}$$

If we let H not scale with c, then the Carroll metric becomes a Euclidean metric with a scaling factor:

$$ds^2 = \mathrm{e}^{2Ht}\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right], \tag{4.26}$$

which we accept as it has rank 3.

If we let H scale with $c^{<0}$, then the metric becomes infinite in the Carroll limit, which we do not accept.

If we let H scale with $c^{>0}$, then the Carroll metric is the, by now familiar, 3-dimensional Euclidean metric

$$ds^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2, \tag{4.27}$$

which we accept. We expect this limiting metric, because we can see the $H \to 0$ limit as the limit of vanishing expansion coefficient in the de Sitter spacetime, which is expected to be Minkowskian.

4.3.2 de Sitter in static coordinates

To get the de Sitter metric in static coordinates, we start with the metric in comoving coordinates. The first step is to convert to spherical coordinates on the spatial part of the metric, and substitute $H = \frac{c}{R_H}$:

$$ds^2_{\mathrm{dS}} = -c^2\,\mathrm{d}t^2 + \mathrm{e}^{2ct/R_H}(\mathrm{d}r^2 + r^2 d\Omega^2). \tag{4.28}$$

Now we perform the coordinate transformation:

$$\rho := r\mathrm{e}^{ct/R_H}, \qquad \tau := t - \frac{R_H}{2c}\log\left(-1 + \frac{r^2}{R_H^2}\mathrm{e}^{2ct/R_H}\right) \tag{4.29}$$

to get the <u>de Sitter metric in static coordinates</u>:

$$ds^2 = -\left(1 - \frac{\rho^2}{R_H^2}\right)c^2\,\mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{1 - \frac{\rho^2}{R_H^2}} + \rho^2 d\Omega^2. \tag{4.30}$$

We note that these coordinates are only valid for $0 \leq \rho < R_H$. Note also that the expressions of the static coordinates in terms of the comoving coordinates depend on c, which hints at the difference

in scaling behaviour of the coordinates.
For this metric, we have the following quantities:

$$\hat{\tau}_t = \sqrt{1 - \frac{\rho^2}{R_H^2}}_{(0)}, \qquad v^t = -\sqrt{\frac{R_H^2}{R_H^2 - \rho^2}}_{(0)}, \tag{4.31}$$

$$h_{\rho\rho} = \left(\frac{R_H^2}{R_H^2 - \rho^2}\right)_{(0)}, \qquad h_{\theta\theta} = \left(\rho^2\right)_{(0)}, \qquad h_{\phi\phi} = \left(\rho^2 \sin^2\theta\right)_{(0)}, \tag{4.32}$$

$$K_{\mu\nu} = 0. \tag{4.33}$$

Note that now the extrinsic curvature of de Sitter spacetime is suddenly equal to 0. This is caused by the fact that this is the extrinsic curvature of the spatial part of the metric, and with the coordinate transformation in equation (4.29), we have mixed the time and space coordinates: what was once a purely time coordinate is now pointing in a partly time and partly spatial direction. This alters the extrinsic curvature as we have defined it.
The non-zero components of the Carroll compatible connection $C^{\rho}_{\mu\nu}$ are given by

$$C^{\tau}_{\rho\tau} = -\left(\frac{\rho}{R_H^2 - \rho^2}\right)_{(0)}, \tag{4.34}$$

$$C^{\rho}_{\rho\rho} = \left(\frac{\rho}{R_H^2 - \rho^2}\right)_{(0)}, \qquad C^{r}_{\theta\theta} = -\left(\rho - \frac{\rho^3}{R_H^2}\right)_{(0)}, \qquad C^{r}_{\phi\phi} = -\left[\left(\rho - \frac{\rho^3}{R_H^2}\right)\sin^2\theta\right]_{(0)}, \tag{4.35}$$

$$C^{\theta}_{r\theta} = C^{\theta}_{\theta r} = \left(\frac{1}{r}\right)_{(0)}, \qquad C^{\theta}_{\phi\phi} = -\left(\sin\theta\cos\theta\right)_{(0)}, \tag{4.36}$$

$$C^{\phi}_{r\phi} = C^{\phi}_{\phi r} = \left(\frac{1}{r}\right)_{(0)}, \qquad C^{\phi}_{\theta\phi} = C^{\phi}_{\phi\theta} = \left(\frac{1}{\tan\theta}\right)_{(0)}. \tag{4.37}$$

If we let R_H not scale with c, then the Carroll metric is

$$ds^2 = \frac{\mathrm{d}\rho^2}{1 - \frac{\rho^2}{R_H^2}} + \rho^2 d\Omega^2, \tag{4.38}$$

which we accept, because it has rank 3.

If we let R_H scale with $c^{<0}$, then the Carroll metric is the 3-dimensional Euclidean metric

$$\mathrm{d}r^2 + r^2 d\Omega^2, \tag{4.39}$$

which we accept.

If we let R_H scale with $c^{>0}$, then the patch of validity of these coordinates becomes non-existent, so we do not accept this scaling behaviour.

Remark 4.1. To come back to the statement of the Carroll limits of de Sitter space not necessarily being the same for some choice of scaling for H, we have now seen an example. If we keep the Hubble radius R_H fixed in the Carroll limit, then $H \to 0$ and the limit of the de Sitter metric in comoving coordinates will be the 3-dimensional Euclidean metric, but the limit of the de Sitter metric in static coordinates will be the metric given in equation (4.38), which is not the 3-dimensional Euclidean metric. $\triangleleft$

Chapter 5

Carroll limits of geodesics

In this chapter, we are given geodesics, of which we take a Carroll limit. We cannot treat this in general, because we have to start with a geodesic being given to us. For concreteness, we will only treat the sample spacetimes of Minkowski, Schwarzschild, and de Sitter. Let us start with the basics by looking at Minkowski spacetime.

5.1 Minkowski

The Minkowski metric is given by

$$ds^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{5.1}$$

Because of the complete spatial translation and rotation symmetry, for sake of finding geodesics (which is our only goal), we can set $y = z = 0$ without loss of generality. The resulting metric is

$$ds^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2. \tag{5.2}$$

This metric is constant, so the Christoffel symbols are $\Gamma^\rho_{\mu\nu} = 0$, which means that the geodesic equations are

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}\widetilde{s}^2} = 0. \tag{5.3}$$

We named the parameter $\widetilde{s}$ in anticipation of rescaling the parameter with a dimensionless constant later.

To solve this equation, we introduce an integration constant u with units of speed, and a dimensionless integration constant k_x. Then the solution to the geodesic equations is

$$t = \frac{1}{u}\widetilde{s} + \text{constant}, \qquad x = k_x\widetilde{s} + \text{constant}. \tag{5.4}$$

Because of the complete translation symmetry of Minkowski spacetime, we can set all the terms named 'constant' to zero, so we are left with

$$t = \frac{1}{u}\widetilde{s}, \qquad x = k_x\widetilde{s}. \tag{5.5}$$

This implies

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k_x u \tag{5.6}$$

The number ε is:

$$\varepsilon = -g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\tilde{s}}\frac{\mathrm{d}x^\nu}{\mathrm{d}\tilde{s}} = c^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}}\right)^2 - \left(\frac{\mathrm{d}x}{\mathrm{d}\tilde{s}}\right)^2 = \frac{c^2}{u^2} - k_x^2. \tag{5.7}$$

The momentum and energy of the particles on this geodesic according to this parameterization are

$$E = \frac{|m|c^3}{u}, \qquad p^x = |m|ck_x. \tag{5.8}$$

Note that these quantities do not satisfy the dispersion relation in (2.17). This has to do with the fact that we did not reparameterize our geodesic such that $\varepsilon \in \{-1, 0, 1\}$.

If we are looking at a lightlike geodesic, $\varepsilon = 0$, then $|k_x u| = c$ and $m = 0$, so the dispersion relation is satisfied.

If we are looking at a timelike or spacelike geodesic, then we want to normalize ε, which we can do with the following reparameterization:

$$s := \sqrt{\left|\frac{c^2}{u^2} - k_x^2\right|}\tilde{s}. \tag{5.9}$$

We check:

$$\varepsilon = -g_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}s}\frac{\mathrm{d}x^\nu}{\mathrm{d}s} = \frac{c^2 - k_x^2u^2}{|c^2 - k_x^2u^2|} = \pm 1. \tag{5.10}$$

The energy and momentum of the particles on this geodesic are then

$$E := |m|c^3\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{|m|c^3}{\sqrt{|c^2 - k_x^2u^2|}}, \qquad p^x = \frac{|m|ck_xu}{\sqrt{|c^2 - k_x^2u^2|}}. \tag{5.11}$$

It is now easy to check that the dispersion relation (2.17) is satisfied.

Thus, the geodesic in the right parameterization (for $|k_xu| \neq c$) is

$$t = \frac{1}{\sqrt{|c^2 - k_x^2u^2|}}s, \qquad x = \frac{k_xu}{\sqrt{|c^2 - k_x^2u^2|}}s. \tag{5.12}$$

Remark 5.1. We see that the geodesic is spacelike if and only if $|k_xu| > c$, that the geodesic is lightlike if and only if $|k_xu| = c$, and that the geodesic is timelike if and only if $|k_xu| < c$. $\triangleleft$

To specify exactly what Carroll limit we are taking, we have already determined that we will take the limit after the geodesics are written down, but we are yet to determine the scaling behaviour of the integration constants. There are multiple ways to scale these parameters, which we will treat in a case-based manner. Note however that u cannot scale like $c^{>0}$, because then the coordinate t becomes infinite. As we only want to consider scaling speeds either like c^1 or not with c, we conclude that u does not scale with c.

Case 1. If k_x does not scale with c, then the Carroll limit of the geodesic becomes

$$t = \frac{1}{u|k_x|}s, \qquad x = \frac{k_x}{|k_x|}s. \tag{5.13}$$

For the sake of simplicity, let us assume that $k_x \neq 0$, otherwise we would have a non-moving lightlike particle in the Carroll limit. The coordinate velocity of the particle is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = k_xu, \tag{5.14}$$

which is finite and non-zero. The ε is given by

$$\varepsilon = -1, \tag{5.15}$$

so the geodesic is spacelike. We can state the energy and momentum of the particle by choosing to keep $|m|c$ fixed in the Carroll limit (because this is a moving particle):

$$E = 0, \qquad p^x = |m|c. \tag{5.16}$$

We see that, indeed, the energy is zero for these moving particles, and that the momentum is non-zero, as predicted in Remark 3.2. ◁

Case 2. If k_x scales like $c^{>0}$, then the reparameterization we performed is not valid, so we look at the first parameterization. The Carroll limit of the geodesic becomes

$$t = \frac{1}{u}\widetilde{s}, \qquad x = 0. \tag{5.17}$$

We see that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 0, \tag{5.18}$$

and we see from

$$\varepsilon = 0 \tag{5.19}$$

that this geodesic becomes lightlike in the Carroll limit. In this case, we cannot rescale $|m|$ such that we get a valid non-trivial dispersion relation, so this non-moving particle has zero energy. An intuitive way of seeing this might be to consider this lightlike particle to have zero mass, so the fact that its momentum is zero must imply zero energy because of the dispersion relation. ◁

5.2 Schwarzschild

The Schwarzschild metric is given by

$$ds^2 = -c^2\left(1 - \frac{R_S}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{R_S}{r}}\mathrm{d}r^2 + r^2 d\Omega^2. \tag{5.20}$$

By the complete spatial rotational symmetry of this metric, we can set $\theta = \frac{\pi}{2}$ without loss of generality for finding geodesics. The resulting metric is

$$ds^2 = -c^2\left(1 - \frac{R_S}{r}\right)\mathrm{d}t^2 + \frac{1}{1-\frac{R_S}{r}}\mathrm{d}r^2 + r^2 \mathrm{d}\phi^2. \tag{5.21}$$

We note that these coordinates are only valid for $r > R_S$. For this metric, we have

$$\varepsilon = c^2\left(1 - \frac{R_S}{r}\right)\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{1}{1-\frac{R_S}{r}}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2. \tag{5.22}$$

Remark 5.2. If we take the Carroll limit in the equation above, we see that no timelike geodesics are possible, that particles on lightlike geodesics are non-moving, and that particles on spacelike geodesics are always moving. This is the very familiar dichotomy that we have seen before in Minkowski spacetime. ◁

The metric is independent of t and ϕ, so we have energy and angular momentum conservation:

$$E := |m|c^3\left(1-\frac{R_S}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}s}, \qquad L := |m|cr^2\frac{\mathrm{d}\phi}{\mathrm{d}s}. \tag{5.23}$$

We recall that $p^\mu = |m|c\frac{\mathrm{d}x^\mu}{\mathrm{d}s}$. We can then write down the dispersion relation from equation (2.17):

$$\frac{E^2}{c^2\left(1-\frac{R_S}{r}\right)} = p_i p^i + |m|^2c^2\varepsilon. \tag{5.24}$$

The non-zero Christoffel symbols of this metric are

$$\Gamma^t_{tr} = \Gamma^t_{rt} = \frac{R_S}{2r(r-R_S)}, \tag{5.25}$$

$$\Gamma^r_{tt} = \frac{R_S(r-R_S)c^2}{2r^3}, \qquad \Gamma^r_{rr} = -\frac{R_S}{2r(r-R_S)}, \qquad \Gamma^r_{\phi\phi} = R_S - r \tag{5.26}$$

$$\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = \frac{1}{r}. \tag{5.27}$$

Thus, the geodesic equations are

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \frac{R_S}{r(r-R_S)}\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}r}{\mathrm{d}s} = 0, \tag{5.28}$$

$$\frac{\mathrm{d}^2r}{\mathrm{d}s^2} + \frac{R_S(r-R_S)c^2}{2r^3}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \frac{R_S}{2r(r-R_S)}\left(\frac{\mathrm{d}r}{\mathrm{d}s}\right)^2 - (r-R_S)\left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0, \tag{5.29}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{2}{r}\frac{\mathrm{d}r}{\mathrm{d}s}\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0. \tag{5.30}$$

It is not easy to see solutions to these equations, so we will first consider particles with a radial trajectory, then those with a circular trajectory. General solutions involve Jacobi elliptic functions, and it is not worth the hassle of big calculations to discuss these general geodesics.

5.2.1 Radial trajectories

Solution 5.3. What follows is the geodesic that is the result of putting a particle at a radius r_0 at $s = 0$ with zero speed and letting it fall towards $r = R_S$.

Given a radius $r_0 > R_S$, a solution ([15], page 9, equation (32)) can be construed numerically from the following statements about the derivatives:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{1}{c}\frac{\sqrt{1-\frac{R_S}{r_0}}}{1-\frac{R_S}{r}}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = -\sqrt{\frac{R_S}{r}-\frac{R_S}{r_0}}, \qquad r(s=0) = r_0, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}s} = 0. \tag{5.31}$$

For this geodesic the following is true:

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}r}{\mathrm{d}s}}{\frac{\mathrm{d}t}{\mathrm{d}s}} = -c\frac{\sqrt{\frac{R_S}{r}-\frac{R_S}{r_0}}\left(1-\frac{R_S}{r}\right)}{\sqrt{1-\frac{R_S}{r_0}}}. \tag{5.32}$$

From the differential equation for r and the condition $r(s=0) = r_0$, we find

$$s = r_0\sqrt{\frac{r_0}{R_S}}\arctan\left(\sqrt{\frac{r_0}{r}-1}\right) + \frac{r_0 - r}{\sqrt{\frac{R_S}{r}-\frac{R_S}{r_0}}}. \tag{5.33}$$

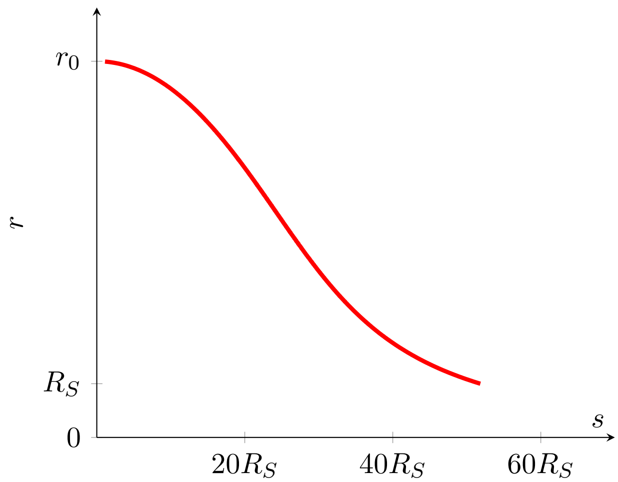

Figure 5.1: Solution 5.3, with $r_0 = 7R_S$.

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We see from $\frac{\mathrm{d}r}{\mathrm{d}s} < 0$ that this expression is invertible. However, it cannot be turned into an analytical expression for $r(s)$, so we show a graph in Figure 5.1.

We calculate:

$$\varepsilon = c^2 \left(1 - \frac{R_S}{r}\right) \left(\frac{1}{c} \frac{\sqrt{1 - \frac{R_S}{r_0}}}{1 - \frac{R_S}{r}}\right)^2 - \frac{1}{1 - \frac{R_S}{r}} \left(\frac{R_S}{r} - \frac{R_S}{r_0}\right) = 1, \tag{5.34}$$

so this is a timelike geodesic (as expected, because our particle had speed 0 at $s = 0$). The energy that particles on this geodesic have is

$$E = |m| c^2 \sqrt{1 - \frac{R_S}{r_0}}, \tag{5.35}$$

so, instead of r_0 parameterizing the geodesic, we can also look at it as if the energy of the particle parameterizes the geodesic. The momentum is given by

$$p^r = -|m| c \sqrt{\frac{R_S}{r} - \frac{R_S}{r_0}}. \tag{5.36}$$

The proper "parametric distance" Δs that our particle takes to fall from $r = r_0$ to $r = R_S$ is ([15], page 9, equation (34))

$$\Delta s = r_0 \left(\sqrt{\frac{r_0}{R_S}} \frac{\pi}{2} + 1\right). \tag{5.37}$$

To take a Carroll limit of this geodesic, we have to specify how the Schwarzschild radius scales with c. If r_0 scales with $c^{>0}$, then our particle starts at a point that is outside the validity of these Schwarzschild coordinates. If r_0 scales with $c^{<0}$ then we drop the particle at $r = \infty$ in the Carroll limit, which does not make sense. Therefore, we let r_0 be independent of c, and the only parameter we can choose the scaling of will be R_S. There is a problem, though. From the expression for $\frac{\mathrm{d}t}{\mathrm{d}s}$, we see that no scaling of R_S will make it finite in the Carroll limit. We conclude that we must rescale s with c as follows:

$$\widetilde{s} := \frac{s}{c}. \tag{5.38}$$

Note that the new parameter $\widetilde{s}$ has dimensions of time. The geodesic is then described by the following statements:

$$\frac{\mathrm{d}t}{\mathrm{d}\widetilde{s}} = \frac{\sqrt{1-\frac{R_S}{r_0}}}{1-\frac{R_S}{r}}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\widetilde{s}} = -c\sqrt{\frac{R_S}{r}-\frac{R_S}{r_0}}, \qquad r(s=0)=r_0, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\widetilde{s}} = 0. \tag{5.39}$$

From this, it is easily seen that all Carroll limits will yield non-moving particles, because $\frac{\mathrm{d}r}{\mathrm{d}\widetilde{s}}$ goes to zero for all valid scalings of R_S. We see that $\varepsilon = 0$, which is consistent with the fact that non-moving particles are lightlike. Again, we conclude that both the energy and the momentum of this particle become zero in the Carroll limit.

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Solution 5.4. For all constants $A \in \mathbb{R}$, a geodesic can be construed numerically from the following statements:

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{A}{c\left(1-\frac{R_S}{r}\right)}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{A^2+1-\frac{R_S}{r}}, \qquad \phi = \text{constant}. \tag{5.40}$$

The energy associated with this geodesic is

$$E := |m|c^3\left(1-\frac{R_S}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}s} = A|m|c^2, \tag{5.41}$$

so A can be seen as the dimensionless energy of the particle on the geodesic.

The equation for $\frac{\mathrm{d}r}{\mathrm{d}s}$ implies

$$s = \frac{r}{A^2+1}\sqrt{A^2+1-\frac{R_S}{r}} + \frac{R_S}{(A^2+1)\sqrt{A^2+1}}\operatorname{arctanh}\left(\sqrt{\frac{A^2+1-\frac{R_S}{r}}{A^2+1}}\right) + \text{constant}. \tag{5.42}$$

Note that s only depends on A^2, so the sign of A is irrelevant to $r(s)$. The sign of A comes into play in $t(s)$, so the sign of $r(t)$ depends on the sign of A. Again, the expression for $s(r)$ is invertible but it is not possible to give an analytical expression $r(s)$, so we graph it in Figure 5.3. The graph demonstrates that this is a particle that starts at the event horizon and goes out radially to $r = \infty$. For $A < 0$, the particle goes in radially to $r = R_S$, towards the singularity. This can be nicely illustrated in a Penrose diagram, seen in Figure 5.2.

We calculate

$$\varepsilon = \frac{A^2}{\left(1-\frac{R_S}{r}\right)} - \frac{A^2+1-\frac{R_S}{r}}{1-\frac{R_S}{r}} = -1, \tag{5.43}$$

so this is a spacelike geodesic. This is what we expect, because only particles on spacelike geodesics can start at the event horizon and have a velocity away from the black hole. We also calculate

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{\frac{\mathrm{d}r}{\mathrm{d}s}}{\frac{\mathrm{d}t}{\mathrm{d}s}} = c\frac{\sqrt{A^2+1-\frac{R_S}{r}}}{A}\left(1-\frac{R_S}{r}\right). \tag{5.44}$$

From $\frac{\mathrm{d}t}{\mathrm{d}s}$ we see that A must scale like $c^{\geq 1}$.

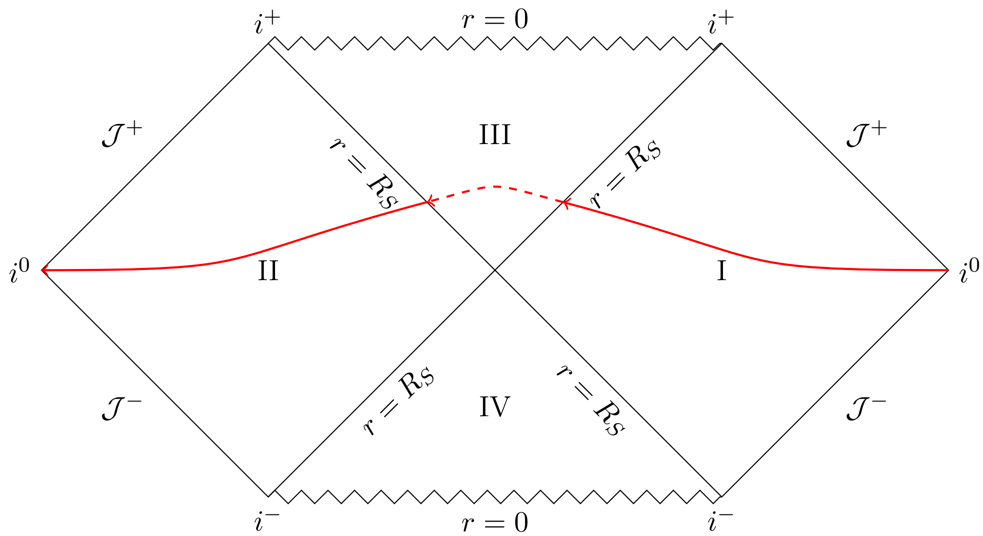

Figure 5.2: The Penrose diagram of a Schwarzschild spacetime, in which the solid red curves indicate the trajectories of the ingoing and outgoing particles. The dotted line indicates the possible completion of a trajectory of a single geodesic if it were extended into Kruskal coordinates, but we did not check this.

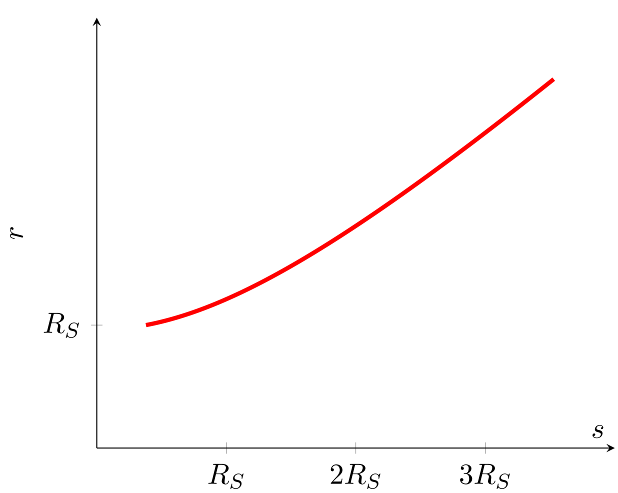

Figure 5.3: Solution 5.4 with $A = 0.2$ and constant $= 0$.

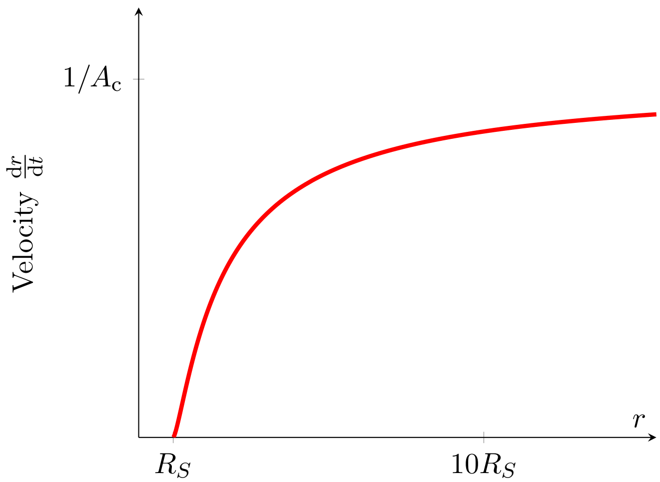

Figure 5.4: Velocity $\frac{\mathrm{d}r}{\mathrm{d}t}$ in solution 5.4, where $A = A_c c + O(c^2)$ and R_S does not scale with c.

Case 1. If R_S does not scale with c and A scales like c^1, we define $A_c := \lim_{c\to 0} \frac{A}{c}$, which has units of inverse speed, and we get the following:

$$\lim_{c\to 0} s = r\sqrt{1-\frac{R_S}{r}} + R_S \operatorname{arctanh}\left(\sqrt{1-\frac{R_S}{r}}\right) + \text{constant}, \tag{5.45}$$

$$\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{A_c}{1-\frac{R_S}{r}}, \tag{5.46}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{A_c}\left(1-\frac{R_S}{r}\right)^{3/2}, \tag{5.47}$$

$$\frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1-\frac{R_S}{r}}. \tag{5.48}$$

This choice of scaling behaviour of R_S and A is interesting, because all the quantities above remain finite and non-zero, which is a behaviour that we have not seen before. We plot $\frac{\mathrm{d}r}{\mathrm{d}t}$ as a function of r in Figure 5.4. Its radial velocity increases as its radius (and thus time) increases. We can explain this by the gravitational redshift that an outside observer sees: the particle will seem to stand still with respect to coordinate time when near the event horizon due to the redshift of the information sent from the particle to the outside observer.

If we keep $|m|c$ fixed in this Carroll limit, then we get

$$E = 0, \qquad E^2 g^{tt} = 0, \qquad p^r = |m|c\sqrt{1-\frac{R_S}{r}}, \qquad |\vec{p}|^2 = |m|^2c^2. \tag{5.49}$$

The expected zero energy with a non-zero momentum is seen again, and we see that the value of p^r makes sure that the dispersion relation (2.17) is satisfied. ◁

Case 2. If R_S does not scale with c and A scales like $c^{>1}$, then we get the following limit of this geodesic:

$$t = \text{constant}, \qquad \frac{\mathrm{d}r}{\mathrm{d}s} = \sqrt{1-\frac{R_S}{r}}, \qquad \phi = \text{constant}. \tag{5.50}$$

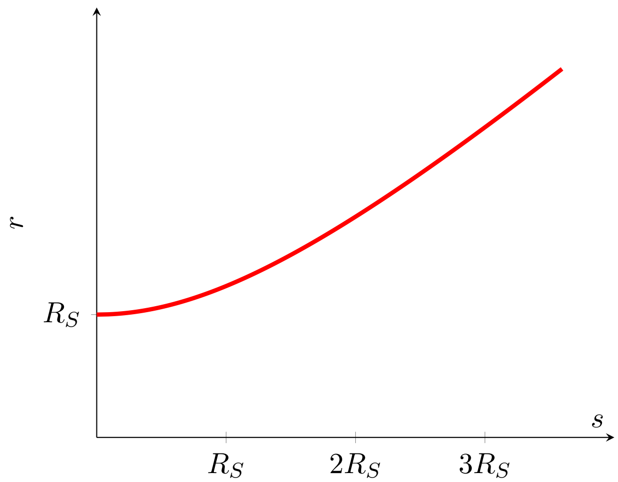

Figure 5.5: Solution 5.4 with $A = 0$ and constant $= 0$.

The energy of this particle is 0 for every scaling of $|m|$, and the momentum is still given by

$$p^r = |m|c\sqrt{1 - \frac{R_S}{r}}. \tag{5.51}$$

So, we choose to keep $|m|c$ constant in the Carroll limit and we end up in the familiar situation of having zero energy and non-zero momentum. Just like in the previous case, we see that the dispersion relation is satisfied.

This scaling yields the same result as just taking $A = 0$ in the original geodesic. We plot $r(s)$ in Figure 5.5. ◁

Case 3. If R_S scales like $c^{>0}$, then the geodesic becomes in the Carroll limit:

$$t = \lim_{c\to 0} \frac{As}{c} + \text{constant}, \qquad r = \lim_{c\to 0} \sqrt{A^2+1}|s|. \tag{5.52}$$

This is a subset of the geodesics we encounter in Minkowski spacetime, namely for $u = \frac{c}{A}$ and $k_x = \sqrt{A^2+1}$. This is to be expected, because $R_S \to 0$ with this scaling, from which we expect that our geodesics behave like in Minkowski spacetime, because we saw that with this scaling, the Schwarzschild metric becomes the 3-dimensional Euclidean metric. ◁

Remark 5.5. The trajectory of the geodesic in Solution 5.4 is given in isotropic coordinates - up to translations of s with a constant - by

$$s = \frac{R_S\left(\sqrt{\rho}+\frac{1}{\sqrt{\rho}}\right)^2}{4(A^2+1)}\sqrt{A^2+1-\frac{4}{\left(\sqrt{\rho}+\frac{1}{\sqrt{\rho}}\right)^2}} + \frac{R_S}{(A^2+1)^{3/2}}\operatorname{arctanh}\left(\sqrt{\frac{A^2+1-\frac{4}{\left(\sqrt{\rho}+\frac{1}{\sqrt{\rho}}\right)^2}}{A^2+1}}\right), \tag{5.53}$$

which we plot in Figure 5.6.

We see that the coordinate ρ describes the geodesic until the Scharzschild radius R_S at $\log(\rho) = 0$, but also probes further into the domain of $\log(\rho) \ll 0$. In this domain, the trajectory of the particle

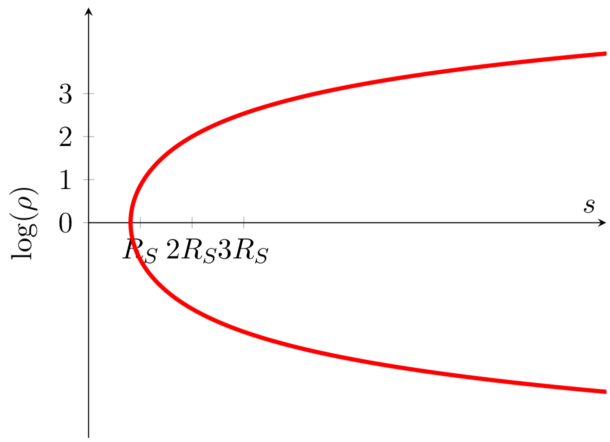

Figure 5.6: Solution 5.4 with $A = 1$ in isotropic coordinates.

on this geodesic is defined by $s(\rho) = s(1/\rho)$. Furthermore, we saw in Solution 5.4 that the sign of A determines the radial direction of the particle: $A > 0$ is outwards and $A < 0$ is inwards. We can consider a geodesic 1 with $A_1 < 0$ and a geodesic 2 with $A_2 > 0$, where $|A_1| = |A_2|$ and $A_1, A_2 \sim c^1$. The scaling with c^1 makes sure that $\frac{\mathrm{d}t}{\mathrm{d}s}$ and $\frac{\mathrm{d}r}{\mathrm{d}t}$ are finite, as we saw in Case 1 of Solution 5.4. We can now glue geodesics 1 and 2 together at the Schwarzschild radius $\log(\rho) = 0$, and we plot the trajectory of a particle on these glued geodesics in Figure 5.7. We see that the particle comes in from infinity, goes radially into the event horizon, and at that moment (at $\log(\rho) = 0$) it switches from geodesic 1 to geodesic 2, and it appears at the other sector of the Schwarzschild spacetime and goes out radially to infinity. In summary, the particle comes from infinity, goes through the bottleneck at $r = R_S$ and goes off to infinity again. This all happens in a static spacetime of the form

$$f(\rho)\left(\mathrm{d}\rho^2 + \rho^2 d\Omega^2\right), \tag{5.54}$$

where f is any function (see the metric in equation (4.18)), with the $\mathbb{Z}_2$ symmetry of $r(\rho) = r(1/\rho)$. From the properties of wormhole spacetimes mentioned in ([9], equation (48)), we could say that the particle went through the Schwarzschild wormhole, also called the Einstein-Rosen bridge ([4]). Note that the Carroll limit is vital in this process, because it ensures that the metric is of the form of that of a wormhole. The scaling of A with c is vital, because it ensures that we can pick different signs for $\frac{\mathrm{d}t}{\mathrm{d}s}$, which makes the gluing possible.

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5.2.2 Circular trajectories

For these solutions, we have $\frac{\mathrm{d}r}{\mathrm{d}s} = 0$, so the geodesic equations become

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2} = 0, \tag{5.55}$$

$$\frac{R_S c^2}{2r^3}\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 = \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2, \tag{5.56}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} = 0. \tag{5.57}$$

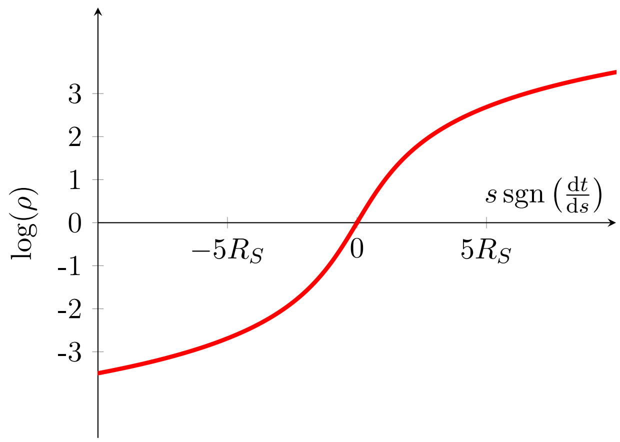

Figure 5.7: The result of gluing two geodesics of Case 1 of Solution 5.4 together to get a particle that goes from one sector to the other sector of Schwarzschild spacetime. The trajectory seen here corresponds to the trajectory of the solid lines in the Penrose diagram in Figure 5.2.

From the top and bottom equations, we get that t and ϕ are both linear in s. The middle equation gives a relation between the two integration constants we get. Thus, we only have one solution.

Solution 5.6. Let r_0 be a distance. The following is the only geodesic that has a circular trajectory:

$$t = \sqrt{\frac{2r}{R_S}}\frac{rs}{r_0 c} + \text{constant}, \qquad r = \text{constant}, \qquad \phi = \frac{s}{r_0} + \text{constant}. \tag{5.58}$$

We calculate:

$$\varepsilon = c^2\left(1 - \frac{R_S}{r}\right)\frac{2r^3}{R_S r_0^2 c^2} - r^2\frac{1}{r_0^2} = \left[\frac{2r}{R_S} - 3\right]\left(\frac{r}{r_0}\right)^2. \tag{5.59}$$

We see that for circular orbits at $r > \frac{3}{2}R_S$ we have timelike particles, at $r = \frac{3}{2}R_S$ we have lightlike particles, and at $R_S < r < \frac{3}{2}R_S$ we have spacelike particles. We calculate

$$r\frac{\mathrm{d}\phi}{\mathrm{d}t} = c\sqrt{\frac{R_S}{2r}}. \tag{5.60}$$

Lightlike particles can only have circular orbits at $r = \frac{3}{2}R_S$.

For timelike and spacelike particles, one can construct the potential ([19])

$$V(r) = \frac{1}{2}\varepsilon - \varepsilon\frac{R_S}{2r} + \frac{A^2}{2r^2} - \frac{R_S A^2}{2r^3}, \tag{5.61}$$

where A is defined as

$$A := r^2\frac{\mathrm{d}\phi}{\mathrm{d}s}. \tag{5.62}$$

From this potential, it follows that particles on timelike geodesics have an innermost stable circular orbit radius of $3R_S$ (discussed in [19]), and that particles on spacelike geodesics on a circular orbit of any radius (i.e. any $R_S < r < \frac{3}{2}R_S$) are on a stable circular orbit. Said another way: every spacelike geodesic with a circular trajectory is stable.

To determine the energy and angular momentum of a timelike or lightlike particle in a circular orbit with radius r, we need to reparameterize in order to normalize ε, so we define

$$\widetilde{s} := \sqrt{\left|\frac{2r}{R_S} - 3\right|}\frac{r}{r_0}s. \tag{5.63}$$

Then we can calculate the energy and angular momentum:

$$E := |m|c^3\left(1 - \frac{R_S}{r}\right)\frac{\mathrm{d}t}{\mathrm{d}\widetilde{s}} = |m|c^2\frac{1 - \frac{R_S}{r}}{\sqrt{\left|1 - \frac{3R_S}{2r}\right|}}, \qquad L := |m|cr^2\frac{\mathrm{d}\phi}{\mathrm{d}\widetilde{s}} = |m|cr\frac{1}{\sqrt{\left|\frac{2r}{R_S} - 3\right|}}. \tag{5.64}$$

We see that both the energy and angular momentum become large when the radius tends to the radius of lightlike geodesics, $r = \frac{3}{2}R_S$. We also see that the energy and angular momentum are related by

$$L = \frac{r^2}{r - R_S}\sqrt{\frac{R_S}{2r}}\frac{E}{c}. \tag{5.65}$$

As was the case with Solution 5.1, to get valid geodesics in the Carroll limit, we have to perform a rescaling of s by c. In this case, we get non-moving particles, and $\varepsilon = 0$. The dispersion relation then implies that the energy is also zero.

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5.3 de Sitter in comoving coordinates

The de Sitter metric in comoving coordinates is

$$ds^2 = -c^2\,\mathrm{d}t^2 + \mathrm{e}^{2Ht}\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right]. \tag{5.66}$$

By the complete spatial rotational and translational symmetry of this metric, we can set $y = z = 0$ without loss of generality for finding geodesics. The resulting metric is

$$ds^2 = -c^2\,\mathrm{d}t^2 + \mathrm{e}^{2Ht}\mathrm{d}x^2. \tag{5.67}$$

For this metric, we have

$$\varepsilon = c^2\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \mathrm{e}^{2Ht}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2. \tag{5.68}$$

The Christoffel symbols are given by

$$\Gamma^t_{\gamma\lambda} = \frac{1}{c^2}H\mathrm{e}^{2Ht}\delta^x_\gamma\delta_{\lambda x}, \qquad \Gamma^x_{\gamma\lambda} = H(\delta^t_\gamma\delta^x_\lambda + \delta^t_\lambda\delta^x_\gamma). \tag{5.69}$$

Thus, the geodesic equations are

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2} + \frac{1}{c^2}H\mathrm{e}^{2Ht}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 = 0, \tag{5.70}$$

$$\frac{\mathrm{d}^2x}{\mathrm{d}s^2} + 2H\frac{\mathrm{d}t}{\mathrm{d}s}\frac{\mathrm{d}x}{\mathrm{d}s} = 0. \tag{5.71}$$

Note that there is no energy conservation, because there is no timelike Killing vector.

Solution 5.7. The following is a geodesic for any distance x_∞:

$$t = \frac{1}{H}\log\left(\frac{Hs}{c}\right), \qquad x = \frac{c^2}{H^2 s} + x_\infty. \tag{5.72}$$

We see that the notation is justified, because $x_\infty = \lim_{s\to\infty} x$.

We can express x in terms of t by

$$x = \frac{c}{H}\mathrm{e}^{-Ht} + x_\infty, \tag{5.73}$$

which is the solution commonly found in the literature. We can now calculate:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -c\mathrm{e}^{-Ht}. \tag{5.74}$$

We calculate ε for this geodesic:

$$\varepsilon = c^2\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2 - \mathrm{e}^{2Ht}\left(\frac{\mathrm{d}x}{\mathrm{d}s}\right)^2 = 0. \tag{5.75}$$

This geodesic describes throwing a massless particle: at $t = 0$ there is a speed of c, and the light moves away from the position $x = \frac{c}{H} + x_\infty$. The light will then travel towards the spatial point $(x_\infty, 0, 0)$, which is exactly one Hubble radius $\frac{c}{H}$ away from its starting point. Thus, the light will converge to $x = x_\infty$, which explains the decreasing speed we found in equation (5.74): space expands in front of the massless particle, which makes the lightlike geodesic converge to a constant point in space.

From the expression for t, we see that we have to keep H constant in the Carroll limit. Then we need to rescale s as follows:

$$\widetilde{s} := \frac{s}{c}. \tag{5.76}$$

Note that $\widetilde{s}$ has dimensions of time. The geodesic is given by

$$t = \frac{1}{H}\log\left(H\widetilde{s}\right), \qquad x = \frac{c}{H^2\widetilde{s}} + x_\infty. \tag{5.77}$$

Case 1. We already determined that the only scaling of H is to keep it fixed in the Carroll limit. Therefore, this geodesic has only one Carroll limit, and it is given by

$$t = \frac{1}{H}\log\left(H\widetilde{s}\right), \qquad x = x_\infty. \tag{5.78}$$

We see that the particle is non-moving, which we expect, because the particle was lightlike. ◁

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Solution 5.8. Let u be a non-zero (constant) speed and let $A \in \mathbb{R}_{>0}$ be a positive constant. Then (inspiration: [8], page 22, equation (4.34))

$$t = \frac{1}{H}\log\left(\frac{uA}{c}\sinh\left(\frac{Hs}{u}\right)\right), \qquad x = -\frac{c^2}{uAH\tanh\left(\frac{Hs}{u}\right)} + \text{constant}. \tag{5.79}$$

is a geodesic. We can use

$$\sinh\left(\frac{Hs}{u}\right) = \frac{c}{uA}\mathrm{e}^{Ht} \tag{5.80}$$

to calculate $\frac{\mathrm{d}x}{\mathrm{d}t}$:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c\mathrm{e}^{-Ht}\frac{1}{\sqrt{1+\left(\frac{c}{uA}\mathrm{e}^{Ht}\right)^2}}. \tag{5.81}$$

The ε of this geodesic is

$$\varepsilon = \frac{c^2}{u^2}, \tag{5.82}$$

which makes it timelike.

We see from the expression for t that there is only one way to scale the three parameters such that t remains finite in the Carroll limit.

Case 1. If H and u do not scale with c and A scales like c^1, then we define $A_\mathrm{c} := \lim_{c\to 0} \frac{A}{c}$, and the geodesic in the Carroll limit will be

$$t = \frac{1}{H}\log\left(uA_\mathrm{c}\sinh\left(\frac{Hs}{u}\right)\right), \qquad x = \text{constant}. \tag{5.83}$$

We see that in this limit, $\varepsilon = 0$, which makes this non-moving particle lightlike, as expected. There is no energy conservation, so it does not make sense to see if this particle has zero or non-zero energy. However, we can write down the four-momentum:

$$p^0 = \frac{|m|c}{u\tanh\left(\frac{Hs}{u}\right)}, \qquad p^i = 0. \tag{5.84}$$

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Solution 5.9. In the previous solution, we can take both u and A to be complex. We see from ε, which has to be real, that u has to be either real or imaginary. In the case of imaginary u, we have to take A real to keep the coordinates real. We conclude that A must always be real. This solution will consider the imaginary-u-version of the previous solution. We write $u = \mathrm{i}\tilde{u}$, where $\tilde{u} \in \mathbb{R}$, so the geodesic becomes

$$t = \frac{1}{H}\log\left(\frac{\tilde{u}A}{c}\sin\left(\frac{Hs}{\tilde{u}}\right)\right), \qquad x = -\frac{c^2}{\tilde{u}AH\tan\left(\frac{Hs}{\tilde{u}}\right)} + \text{constant}. \tag{5.85}$$

We see from the expression for t that s can only take values $0 < s < \frac{\tilde{u}\pi}{H}$. The geodesic's ε now becomes

$$\varepsilon = -\frac{c^2}{\tilde{u}^2}, \tag{5.86}$$

which makes it spacelike, and the expression $\frac{\mathrm{d}x}{\mathrm{d}t}$ is

$$\frac{\mathrm{d}x}{\mathrm{d}t} = c\mathrm{e}^{-Ht}\frac{1}{\sqrt{1-\left(\frac{c}{\tilde{u}A}\mathrm{e}^{Ht}\right)^2}}. \tag{5.87}$$

We see from the expression for t that H must not scale with c.

Remark 5.10. If $\tilde{u}$ scales like c^1, then we encouter a peculiarity. First of all, if we require our coordinates to be real, s can only take values in $\left(0, \frac{\tilde{u}\pi}{H}\right)$, so in the Carroll limit, s cannot take on any value. If, for some reason, we decide that complex values for our coordinates are allowed, then we encounter the fact that both x and t contain periodic functions of $\tilde{u}$ that have zero in their image. This is problematic, because then the coordinates do not have a well-defined Carroll limit (exactly in the same manner that $\lim\limits_{x\to\infty} \sin(x)$ does not exist). Thus, we do not consider scaling $\tilde{u}$ like c^1 as an option. ◁

From the fact that $\tilde{u}$ does not scale with c, we see from the expression for t that A must scale like c^1.

Case 1. With the scaling behaviour mentioned above, the geodesic in the Carroll limit will be

$$t = \frac{1}{H}\log\left(\tilde{u}A_{\mathrm{c}}\sin\left(\frac{Hs}{\tilde{u}}\right)\right), \qquad x = \text{constant}. \tag{5.88}$$

Just like in the previous solution, the particle on this geodesic is lightlike. The four-momentum is given by

$$p^0 = \frac{|m|c}{\tilde{u}\tan\left(\frac{Hs}{\tilde{u}}\right)}, \qquad p^i = 0. \tag{5.89}$$

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Solution 5.11. A stationary particle is also on a geodesic. For any speed u, a geodesic is given by

$$t = \frac{1}{u}s + \text{constant}, \qquad x = \text{constant}. \tag{5.90}$$

We have discussed the possible Carroll scalings and their accompanying Carroll limits in the Minkowski section, so we will not repeat them here.

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Remark 5.12. We see that the coordinate time in the solution above is linear in s. However, s is just a parameter that lets us 'walk along the geodesic': we can apply diffeomorphisms to s and the above solution will still be a solution. This yields

$$t = \frac{1}{u}f(s), \qquad x = \text{constant}, \tag{5.91}$$

with f any diffeomorphism of s that returns a value with the correct dimensions, namely distance. In particular, if the spatial coordinates are constant along the geodesic, we can freely reparameterize s without altering the constantness of the spatial coordinates. Thus, for non-moving geodesics, the time coordinate has freedom. ◁

Remark 5.13. It is interesting to point out that all (valid) Carroll limits of the geodesics we discussed in de Sitter spacetime in comoving coordinates are non-moving. ◁

This is a good time to comment on the role of recessional velocities in de Sitter spacetime.

5.3.1 Remarks about recessional velocities

In de Sitter spacetime, we can pick two points a and b in the manifold with time coordinate t' and spatial coordinates (x_a, y_a, z_a) and (x_b, y_b, z_b) respectively, and we can then pick our coordinates such that $y_a = y_b$ and $z_a = z_b$. For each time t', define the distance $D(t')$ as

$$D(t') := \int_a^b ds\Big|_{t=t'}, \tag{5.92}$$

where the integral is defined over a geodesic of the spatial part of the metric that goes through a and b. We have

$$D(t') = \int_a^b \mathrm{e}^{Ht'}|\mathrm{d}x| = \mathrm{e}^{Ht'}\int_a^b |\mathrm{d}x| = \mathrm{e}^{Ht'}\int_a^b ds\Big|_{t=0} = \mathrm{e}^{Ht'}D(0). \tag{5.93}$$

Note that the derivation of this equation did not depend on the precise form of the geodesic, but only on the metric. If we now define the recessional velocity $v(D)$ as

$$v(D) := \frac{\mathrm{d}D}{\mathrm{d}t}, \tag{5.94}$$

then we get to Hubble's law

$$v(D)(t) = \frac{\mathrm{d}D}{\mathrm{d}t}(t) = \frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{Ht}D(0)\right] = HD(t). \tag{5.95}$$

As the distance D can be arbitrarily big (because we can pick any two points a and b in the de Sitter manifold), there is no upper limit to what values the recessional velocity can take on. This leads to recessional velocities being able to be much larger than the speed of light, which sparks interest in coupling it to the Carroll limit.

Remark 5.14. The derivation of Hubble's law above does not contain the speed of light anywhere. This is because we measure D in a snapshot of constant time, and the spatial part of the de Sitter metric does not contain any factors of c. This means that in any Carroll limit, Hubble's law still stands. The scaling of H does of course influence the precise form of Hubble's law. $\triangleleft$

In particular, we can look at two particles that are on two different stationary geodesics, and we find that their recessional velocity is non-zero.

Remark 5.15. This argument supports the idea that Hubble's law does not have anything to do with the geodesic equations. On one hand, we can have two particles on different stationary geodesics that have non-zero and increasing recessional velocity with respect to each other. On the other hand, we can have particles on non-stationary (be it timelike, lightlike, or spacelike) geodesics that cross paths, and at the time they cross paths they will have zero recessional velocity, because their spatial distance measured at that time will be zero. Conclusion: a large recessional velocity ($\gg c$) does not imply that the geodesic velocity $\frac{\mathrm{d}x}{\mathrm{d}s}$ will be large (or even non-zero) in the Carroll limit. $\triangleleft$

However, the velocity v_{tot} of a particle measured by an observer (for example by measuring its redshift or blueshift) is given by

$$v_{\mathrm{total}} = v_{\mathrm{geod}} + v_{\mathrm{rec}}, \tag{5.96}$$

where we wrote the geodesic (coordinate) velocity as v_{geod} and the recessional velocity as v_{rec}. As we have seen (Remark 5.13), the geodesic velocity of particles on the geodesics discussed will be zero in the Carroll limit. Nevertheless, an observer can still measure a non-zero velocity of a particle, which will be equal to the recessional velocity. So, even if all coordinate velocities are zero, there is still a sense of non-zero velocity in the Carroll limit of de Sitter spacetime, generated by the Hubble constant H.

5.4 de Sitter in static coordinates

The de Sitter metric in static coordinates is

$$ds^2 = -\left(1 - \frac{\rho^2}{R_H^2}\right) c^2 \, \mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{1 - \frac{\rho^2}{R_H^2}} + \rho^2 d\Omega^2. \tag{5.97}$$

By the spatial rotational symmetry of this metric, we can set $\theta = \frac{\pi}{2}$, and we are left with

$$ds^2 = -\left(1 - \frac{\rho^2}{R_H^2}\right) c^2 \, \mathrm{d}\tau^2 + \frac{\mathrm{d}\rho^2}{1 - \frac{\rho^2}{R_H^2}} + \rho^2 \mathrm{d}\phi^2. \tag{5.98}$$

For this metric, we have

$$\varepsilon = \left(1 - \frac{\rho^2}{R_H^2}\right) c^2 \left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2 - \frac{1}{1 - \frac{\rho^2}{R_H^2}} \left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 - \rho^2 \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2. \tag{5.99}$$

The Christoffel symbols are given by

$$\Gamma^{\tau}_{\tau\rho} = \Gamma^{\tau}_{\rho\tau} = -\frac{\rho}{R_H^2 - \rho^2}, \tag{5.100}$$

$$\Gamma^{\rho}_{\tau\tau} = -\left(1 - \frac{\rho^2}{R_H^2}\right) \frac{c^2 \rho}{R_H^2}, \qquad \Gamma^{\rho}_{\rho\rho} = \frac{\rho}{R_H^2 - \rho^2}, \qquad \Gamma^{\rho}_{\phi\phi} = -\left(1 - \frac{\rho^2}{R_H^2}\right) \rho, \tag{5.101}$$

$$\Gamma^{\phi}_{\rho\phi} = \Gamma^{\phi}_{\phi\rho} = \frac{1}{\rho}. \tag{5.102}$$

This yields the geodesic equations:

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s^2} - \frac{2\rho}{R_H^2 - \rho^2} \frac{\mathrm{d}\tau}{\mathrm{d}s} \frac{\mathrm{d}\rho}{\mathrm{d}s} = 0, \tag{5.103}$$

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}s^2} - \left(1 - \frac{\rho^2}{R_H^2}\right) \frac{c^2 \rho}{R_H^2} \left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2 + \frac{\rho}{R_H^2 - \rho^2} \left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 - \left(1 - \frac{\rho^2}{R_H^2}\right) \rho \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}\right)^2 = 0, \tag{5.104}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} + \frac{2}{\rho} \frac{\mathrm{d}\rho}{\mathrm{d}s} \frac{\mathrm{d}\phi}{\mathrm{d}s} \tag{5.105}$$

As the metric does not depend on τ or ϕ, we have conservation of energy and angular momentum:

$$E = |m| c^3 \left(1 - \frac{\rho^2}{R_H^2}\right) \frac{\mathrm{d}\tau}{\mathrm{d}s}, \qquad L = |m| c \rho^2 \frac{\mathrm{d}\phi}{\mathrm{d}s}. \tag{5.106}$$

Unfortunately, if we transform Solution 5.7 and Solution 5.8 to static coordinates, we end up outside of the range of validity of the static coordinates ($\tau \to \infty$ and $\rho \geq R_H$, respectively). However, we are able to take inspiration from Solution 5.9, out of which the following geodesic is born.

Solution 5.16. The following is a geodesic:

$$\tau = \text{constant}, \qquad \rho = \left| R_H \sin\left(\frac{s + \text{constant}}{R_H}\right) \right|, \qquad \phi = \text{constant}. \tag{5.107}$$

Note that we get a Minkowski-like geodesic in the limit of $R_H \to \infty$, which we expect.
Also note that although ρ has points at which it is not a differentiable function of s, this is a

coordinate manifestation. We can go to Euclidean coordinates for the spatial part of the metric and it will read $x = R_H \sin\left(\frac{s+\text{constant}}{R_H}\right)$, which is a smooth function of s.
We calculate:

$$\varepsilon = -\frac{1}{1-\frac{\rho^2}{R_H^2}}\left(\frac{\mathrm{d}\rho}{\mathrm{d}s}\right)^2 = -1, \tag{5.108}$$

so this is a spacelike geodesic. We remind the reader that we only accept R_H scaling like $c^{\leq 0}$.

Remark 5.17. If we keep R_H constant in the Carroll limit, the Hubble radius remains finite while the speed of light becomes 0. We have $H = \frac{c}{R_H} \to 0$, so all recessional velocities $v = HD$ (see equation (5.94)) become 0. As the speed of light itself is 0, this does not mean that every two points can be connected by a timelike geodesic like in 'original' Minkowski space: timelike geodesics do not exist in the Carroll limit of this metric when R_H is kept fixed (see equation (5.99)). Thus, there is no such thing as the Hubble sphere being a causal patch. Only particles on spacelike geodesics move, and in these coordinates we only find solutions in which they only move within their Hubble sphere. This gives a local character to spacelike geodesics, much like we are used to from the timelike geodesics in the 'original' de Sitter spacetime. ◁

Case 1. If we let R_H not scale with c, then the geodesic remains unchanged. It is a spacelike geodesic, and the energy and momentum are given by

$$E = 0, \qquad p^r = \frac{|m|c}{R_H}\cos\left(\frac{s+\text{constant}}{R_H}\right)\operatorname{sgn}\left(\sin\left(\frac{s+\text{constant}}{R_H}\right)\right), \tag{5.109}$$

where sgn is the signum function. We see the familiar zero energy and non-zero momentum for this spacelike geodesic. ◁

Case 2. If we let R_H scale like $c^{<0}$, then the geodesic in the Carroll limit becomes

$$\tau = \text{constant}, \qquad \rho = |s+\text{constant}|, \qquad \phi = \text{constant}. \tag{5.110}$$

This is a subset of the geodesics we encounter in Minkowski spacetime, namely for $u \to \infty$ and $k_x = 1$. This is to be expected, because for $R_H \to \infty$, we saw that we get the 3-dimensional Euclidean metric. ◁

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Solution 5.18. Non-moving particles at $\rho = 0$ are also on a geodesic:

$$\tau = \frac{s}{c} + \text{constant}, \qquad \rho = 0, \qquad \phi = f(s), \tag{5.111}$$

f is an arbitrary function of s. We calculate:

$$\varepsilon = 1, \tag{5.112}$$

so this is a timelike geodesic. The energy and momentum are given by

$$E = |m|c^2, \qquad p^i = 0. \tag{5.113}$$

As with all stationary geodesics, we choose the scaling of $|m|$ such that the energy is non-zero when we can, so this is a particle with non-zero energy and zero momentum, which is what we expected from this non-moving particle.

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Chapter 6

Carroll limit of the geodesic equations

We can also take the Carroll limit at the level of the geodesic equations. Here, we are given a metric and we seek to expand the Christoffel symbols in powers of c to get an expansion of the geodesic equations.

6.1 The expansions

Building upon the formalism introduced in Chapter 3, we are now ready to construct the expansion of the Christoffel symbols. We do this by applying the principle of least action to the action

$$S = |m|c \int \sqrt{g_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s}} \, \mathrm{d}s, \tag{6.1}$$

in which we use the expansion we have constructed for the metric $g_{\mu\nu}$ (equation (3.20)). The equation of motion we get from the action

$$S = |m|c \int \sqrt{\left(h_{\mu\nu} + c^2 \overline{\Phi}_{\mu\nu}\right) \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s}} \, \mathrm{d}s \tag{6.2}$$

is

$$\left(h_{\alpha\nu} + c^2 \overline{\Phi}_{\alpha\nu}\right) \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}s^2} + \frac{1}{2}\Big[\partial_\mu \left(h_{\alpha\nu} + c^2 \overline{\Phi}_{\alpha\nu}\right) + \partial_\nu \left(h_{\alpha\mu} + c^2 \overline{\Phi}_{\alpha\mu}\right) - \partial_\alpha \left(h_{\mu\nu} + c^2 \overline{\Phi}_{\mu\nu}\right)\Big] \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0. \tag{6.3}$$

If we now multiply this equation by the expansion for the inverse metric (equation (3.21)), then the equation above implies[1]:

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2} + \left[-\frac{1}{c^2} v^\rho K_{\mu\nu} + C^\rho_{\mu\nu} + \frac{1}{2} v^\rho \left(\partial_\mu \hat{\tau}_\nu - \partial_\nu \hat{\tau}_\mu + \mathcal{L}_v \overline{\Phi}_{\mu\nu}\right) + \overline{h}^{\rho\sigma} K_{\mu\sigma} \hat{\tau}_\nu + O(c^2)\right] \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0, \tag{6.4}$$

where we used equations (3.29), (3.34), (3.35), (3.31), and (3.36). If we compare this to the geodesic equation

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2} + \Gamma^\rho_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \frac{\mathrm{d}x^\nu}{\mathrm{d}s} = 0, \tag{6.5}$$

[1] At this part in the process, normally we multiply the equation above by the inverse metric $g^{\rho\alpha}$ (which is invertible), and then we get an equivalent statement. Here, we multiply by the leading and sub-leading order of the inverse metric, of which we do not know if it is invertible. This means that now we get only an implication instead of an equivalence.

we see that the expansion of the Christoffel symbols is

$$\Gamma^\rho_{\mu\nu} = -\frac{1}{c^2} v^\rho K_{\mu\nu} + C^\rho_{\mu\nu} + \frac{1}{2} v^\rho \left(\partial_\mu \hat{\tau}_\nu - \partial_\nu \hat{\tau}_\mu + \mathcal{L}_v \overline{\Phi}_{\mu\nu}\right) + \overline{h}^{\rho\sigma} K_{\mu\sigma} \hat{\tau}_\nu + O(c^2). \tag{6.6}$$

We write

$$\left(\Gamma^\rho_{\mu\nu}\right)_{(-2)} := -v^\rho K_{\mu\nu}, \tag{6.7}$$

$$\left(\Gamma^\rho_{\mu\nu}\right)_{(0)} := C^\rho_{\mu\nu} + \frac{1}{2} v^\rho \left(\partial_\mu \hat{\tau}_\nu - \partial_\nu \hat{\tau}_\mu + \mathcal{L}_v \overline{\Phi}_{\mu\nu}\right) + \overline{h}^{\rho\sigma} K_{\mu\sigma} \hat{\tau}_\nu. \tag{6.8}$$

We must now also make the following assumptions to continue with this method.

Assumption 6.1. *To be able to write down the geodesic equations as a series in c, we must assume, given a solution $x^\rho(s)$, that $\frac{\mathrm{d}x^\rho}{\mathrm{d}s}$ has a series expansion in c. However, we want to take the $c \to 0$ limit, so we want the expansion to have a leading order. As the geodesic equation is invariant under diffeomorphisms of s, we can choose this order, and we choose c^0 to be the leading order.*[2] *In short, we assume*

$$\frac{\mathrm{d}x^\rho}{\mathrm{d}s} = \sum_{i=0}^{\infty} c^i \frac{\mathrm{d}x^\rho}{\mathrm{d}s}_{(i)}, \tag{6.9}$$

where for every $n \geq 0$ we have that $\frac{\mathrm{d}x^\rho}{\mathrm{d}s}_{(n)}$ is independent of c.

We see immediately that

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2} \overset{(6.9)}{=} \frac{\mathrm{d}}{\mathrm{d}s}\left[\sum_{i=0}^{\infty} c^i \frac{\mathrm{d}x^\rho}{\mathrm{d}s}_{(i)}\right] = \sum_{i=0}^{\infty} c^i \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x^\rho}{\mathrm{d}s}_{(i)}\right), \tag{6.10}$$

so for every $n \geq 0$ we define

$$\frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2}_{(n)} := \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathrm{d}x^\rho}{\mathrm{d}s}_{(n)}\right). \tag{6.11}$$

With this notation, we can write the geodesic equation in the expanded form as follows:

$$\left[\sum_{i=0}^{\infty} c^i \frac{\mathrm{d}^2 x^\rho}{\mathrm{d}s^2}_{(i)}\right] + \left[\frac{1}{c^2}\left(\Gamma^\rho_{\mu\nu}\right)_{(-2)} + \left(\Gamma^\rho_{\mu\nu}\right)_{(0)} + O(c^2)\right]\left[\sum_{i=0}^{\infty} c^i \frac{\mathrm{d}x^\mu}{\mathrm{d}s}_{(i)}\right]\left[\sum_{j=0}^{\infty} c^j \frac{\mathrm{d}x^\nu}{\mathrm{d}s}_{(j)}\right] = 0. \tag{6.12}$$

6.2 Geodesic equations by order

From the geodesic equation in expanded form, we get one equation for every order of c, starting with the leading order of c^{-2}. We will discuss these equations below. We are only interested in solving for $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}_{(0)}$, because the subleading orders of $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}$ will go to 0 in the Carroll limit.

[2] This choice is logical in anticipation of looking at the Carroll limit: a leading order of $c^{<0}$ would give infinities in the Carroll limit for $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}$, and a leading order of $c^{>0}$ would only give trivial dynamics (i.e. non-moving particles) in the Carroll limit. In general, when looking at geodesics, we are only interested in the order that "survives" the Carroll limit, and that is the order of c^0.

6.2.1 Order c^{-2}

We start by writing down the leading order of the geodesic equation:

$$\left(\Gamma^{\rho}_{\mu\nu}\right)_{(-2)} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(0)} = 0. \tag{6.13}$$

We know that $v \neq 0$, so an equivalent statement is

$$K_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(0)} = 0. \tag{6.14}$$

This does not completely determine $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)}$, because this is 1 equation for the 4 components of $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)}$.

6.2.2 Order c^{-1}

This order of the geodesic equation gives

$$K_{\mu\nu} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(1)} = 0. \tag{6.15}$$

This adds no knowledge about $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)}$.

6.2.3 Order c^{0}

This order of the geodesic equation gives

$$\frac{\mathrm{d}^2x^{\rho}}{\mathrm{d}s^2}_{(0)} + 2\left(\Gamma^{\rho}_{\mu\nu}\right)_{(-2)} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(2)} + \left(\Gamma^{\rho}_{\mu\nu}\right)_{(-2)} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(1)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(1)} + \left(\Gamma^{\rho}_{\mu\nu}\right)_{(0)} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(0)} = 0. \tag{6.16}$$

This looks like an unmanageable equation, but there is a clever contraction we can perform.

Consequence 6.2. We see from equations (3.29) and (6.7) that

$$h_{\sigma\rho}\left(\Gamma^{\rho}_{\mu\nu}\right)_{(-2)} = 0. \tag{6.17}$$

Therefore, it is a good idea to contract the geodesic equation with $h_{\sigma\rho}$, and we get the following:

$$h_{\sigma\rho}\left[\frac{\mathrm{d}^2x^{\rho}}{\mathrm{d}s^2}_{(0)} + 2\left(\Gamma^{\rho}_{\mu\nu}\right)_{(-2)} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(2)} + \left(\Gamma^{\rho}_{\mu\nu}\right)_{(-2)} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(1)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(1)} + \left(\Gamma^{\rho}_{\mu\nu}\right)_{(0)} \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(0)}\right] \tag{6.18}$$

$$\overset{(3.29)}{=} h_{\sigma\rho}\left[\frac{\mathrm{d}^2x^{\rho}}{\mathrm{d}s^2}_{(0)} + \left(C^{\rho}_{\mu\nu} + \overline{h}^{\rho\lambda}K_{\mu\lambda}\hat{\tau}_{\nu}\right) \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(0)}\right] \tag{6.19}$$

$$\overset{(3.34)\ \&\ (3.32)}{=} h_{\sigma\rho}\frac{\mathrm{d}^2x^{\rho}}{\mathrm{d}s^2}_{(0)} + \left(h_{\sigma\rho}C^{\rho}_{\mu\nu} + K_{\mu\sigma}\hat{\tau}_{\nu}\right) \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(0)}. \tag{6.20}$$

We get the following equation:

$$h_{\sigma\rho}\frac{\mathrm{d}^2x^{\rho}}{\mathrm{d}s^2}_{(0)} + \left(h_{\sigma\rho}C^{\rho}_{\mu\nu} + K_{\mu\sigma}\hat{\tau}_{\nu}\right) \frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^{\nu}}{\mathrm{d}s}_{(0)} = 0. \tag{6.21}$$

In a spacetime of general dimension D, these are D equations for the D-dimensional vector $\frac{\mathrm{d}x^{\mu}}{\mathrm{d}s}_{(0)}$. Given that the tensor ranks of $h_{\sigma\rho}$ is $D-1$, we need at least one more equation to completely determine $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}_{(0)}$. We do have one more equation, namely equation (6.14). However, this does not completely determine $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}_{(0)}$: we will see in Remark 7.1 in Chapter 7 that the set of equations (6.14) and (6.21) comprise $D-1$ independent equations. Thus, we cannot find geodesics by testing them only for satisfaction of equations (6.14) and (6.21); we have to test them for satisfaction of equation (6.16). ◁

Remark 6.3. We can also contract equation (6.16) with τ_ρ, but $\tau_\rho \left(\Gamma^\rho_{\mu\nu}\right)_{(-2)}$ is not necessarily equal to 0. This means that the contracted equation will contain $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}_{(2)}$, which we know nothing about yet. We conclude that this contraction does not give us a known equation for $\frac{\mathrm{d}x^\mu}{\mathrm{d}s}_{(0)}$. ◁

Other equations derived from the geodesic equation will have to be derived from the part of the geodesic equation that is of higher order than c^0. In this chapter, we seek to take the limit of the geodesic equations, so the goal is to expand the geodesic equations in powers of c (which we have done) and disregard the equations that are of order $c^{\geq 1}$. So, this is it: the set of equations (6.14) and (6.16) define the Carroll limit of the geodesic equations, which we will call the leading order geodesic equations, referring to the fact that they are the equations corresponding to the two leading orders of c that do not vanish in the Carroll limit.

Remark 6.4. Note that equation (6.21) is merely a consequence of one of the leading order geodesic equations (6.16). However, the facts that $\operatorname{rank}(h_{\mu\nu}) = 3$ and that in our example spacetimes $h_{t\mu} = 0$ imply that equation (6.21) is equivalent to the three equations we get when we fill in $\rho = 1, 2, 3$ into equation (6.16). We will use this in finding geodesics. ◁

It is now time to see what this entails for our sample spacetimes.

6.3 Minkowski

Recall that for the Minkowski metric, we have

$$K_{\mu\nu} = 0, \qquad h_{ii} = 1, \qquad \hat{\tau}_t = 1, \qquad C^\rho_{\mu\nu} = 0, \quad \mathcal{L}_v \overline{\Phi}_{\mu\nu} = 0. \tag{6.22}$$

We see now that equation (6.14) becomes $0 = 0$, and that equation (6.16) becomes

$$\frac{\mathrm{d}^2 x^\mu}{\mathrm{d}s^2}_{(0)} = 0. \tag{6.23}$$

These are the original geodesic equations for Minkowski spacetimes, except we take the leading order of the second derivative. Thus, in particular, the solutions we found in the previous chapter are also solutions of these leading order geodesic equations.

Solution 6.5. The general solution to the leading order geodesic equations of Minkowski spacetime is given by

$$\frac{\mathrm{d}t}{\mathrm{d}s}_{(0)} = \frac{1}{u}, \qquad \frac{\mathrm{d}x}{\mathrm{d}s}_{(0)} = k_x, \tag{6.24}$$

where k_x is a dimensionless constant and u is a constant with dimensions of speed. This is completely analogous to the solution found in the previous chapter, but now there is no such thing as a choice of scaling for u and k_x anymore: we have already performed the Carroll limit. ◦

6.4 Schwarzschild

For the Schwarzschild metric, we have (for $\theta = \frac{\pi}{2}$)

$$K_{\mu\nu} = 0, \tag{6.25}$$

$$h_{rr} = \left(\frac{r}{r - R_S}\right)_{(0)}, \qquad h_{\phi\phi} = (r^2)_{(0)}, \tag{6.26}$$

$$\hat{\tau}_t = \sqrt{\frac{r - R_S}{r}}_{(0)}, \tag{6.27}$$

$$C^t_{rt} = \left(\frac{R_S}{2r(r - R_S)}\right)_{(0)}, \tag{6.28}$$

$$C^r_{rr} = -\left(\frac{R_S}{2r(r - R_S)}\right)_{(0)}, \qquad C^r_{\phi\phi} = -(r - R_S)_{(0)}, \tag{6.29}$$

$$C^\phi_{r\phi} = C^\phi_{\phi r} = \left(\frac{1}{r}\right)_{(0)}, \tag{6.30}$$

$$\mathcal{L}_v \overline{\Phi}_{rt} = \mathcal{L}_v \overline{\Phi}_{tr} = \frac{R_S}{2r^2\sqrt{1 - \frac{R_S}{r}}}_{(0)}. \tag{6.31}$$

We see that equation (6.14) becomes $0 = 0$, and that equation (6.21) becomes the following two equations:

$$\left(\frac{r}{r - R_S}\right)_{(0)} \left[\frac{\mathrm{d}^2 r}{\mathrm{d}s^2}_{(0)} - \left(\frac{R_S}{2r(r - R_S)}\right)_{(0)} \left(\frac{\mathrm{d}r}{\mathrm{d}s}_{(0)}\right)^2 - (r - R_S)_{(0)} \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}_{(0)}\right)^2\right] = 0, \tag{6.32}$$

$$(r^2)_{(0)} \left[\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}_{(0)} + \left(\frac{2}{r}\right)_{(0)} \frac{\mathrm{d}r}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}\phi}{\mathrm{d}s}_{(0)}\right] = 0. \tag{6.33}$$

Equation (6.16) for $\rho = t$ becomes

$$\frac{\mathrm{d}^2 t}{\mathrm{d}s^2}_{(0)} + \frac{R_S}{r(r - R_S)}_{(0)} \frac{\mathrm{d}r}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}t}{\mathrm{d}s}_{(0)} = 0. \tag{6.34}$$

This last equation and the equations between the square brackets are identical to the original geodesic equations, except for the fact that we are missing the $\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2$ term, because this term scales like c^2, and except for the fact that we take the leading order of every term. The absence of the $\left(\frac{\mathrm{d}t}{\mathrm{d}s}\right)^2$-term has big consequence: if we put a particle at some point in the spacetime with spatial velocity 0, so $\frac{\mathrm{d}r}{\mathrm{d}s} = r\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0$, then it will never start moving. A non-moving particle will remain non-moving. Similarly, a moving particle will experience an acceleration that is proportional to some function of its radial position, but also proportional to its spatial speed. A moving particle will therefore remain moving. This is the behaviour that is described in section 3.3 of [24] in the case of Minkowski spacetime: a dichotomy between moving and non-moving particles.

Let us first see if the solutions we found in the previous chapter are also solutions of the leading order geodesic equations.

Remark 6.6. Solution 5.3 does not meet the expansion requirements, because $\frac{\mathrm{d}t}{\mathrm{d}s}$ is of order c^{-1}. Therefore, we reparameterize s to

$$\tilde{s} := \frac{s}{c} \tag{6.35}$$

such that the new solution becomes

$$\frac{\mathrm{d}t}{\mathrm{d}\tilde{s}} = \frac{\sqrt{1-\frac{R_S}{r_0}}}{1-\frac{R_S}{r}}, \qquad \frac{\mathrm{d}r}{\mathrm{d}\tilde{s}} = -c\sqrt{\frac{R_S}{r}-\frac{R_S}{r_0}}, \quad r(\tilde{s}=0)=r_0, \qquad \frac{\mathrm{d}\phi}{\mathrm{d}\tilde{s}} = 0. \tag{6.36}$$

It follows that

$$\left.\frac{\mathrm{d}x^i}{\mathrm{d}\tilde{s}}\right._{(0)} = 0, \tag{6.37}$$

and thus

$$\left.\frac{\mathrm{d}^2 t}{\mathrm{d}\tilde{s}^2}\right._{(0)} = 0. \tag{6.38}$$

Consequently, we introduce an integration constant u with dimensions of speed and we can write down the geodesic

$$t_{(0)} = \frac{1}{u}s + \text{constant}, \qquad r_{(0)} = r_0, \qquad \phi_{(0)} = \text{constant}. \tag{6.39}$$

We see that this is still a solution of the leading order geodesic equations, but it is an uninteresting solution because the particles on this geodesic do not move. ◁

Remark 6.7. Solution 5.4 will be the same story as above, except in the interesting case of $A = A_c c + O(c^2)$ and $R_S \sim c^0$. The solution becomes

$$\left.\frac{\mathrm{d}t}{\mathrm{d}s}\right._{(0)} = \left(\frac{A_c}{1-\frac{R_S}{r}}\right)_{(0)}, \qquad \left.\frac{\mathrm{d}r}{\mathrm{d}s}\right._{(0)} = \sqrt{1-\frac{R_S}{r}}_{(0)}, \qquad \left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right._{(0)} = 0, \tag{6.40}$$

which is also a solution of the leading order geodesic equations. ◁

Remark 6.8. Solution 5.6 describing particles with circular trajectories does not meet the expansion requirements, but after rescaling of s and taking the Carroll limit, the solution becomes

$$\left.\frac{\mathrm{d}t}{\mathrm{d}s}\right._{(0)} = \sqrt{\frac{2r}{R_S}\frac{r}{r_0}}, \qquad \left.\frac{\mathrm{d}r}{\mathrm{d}s}\right._{(0)} = 0, \qquad \left.\frac{\mathrm{d}\phi}{\mathrm{d}s}\right._{(0)} = 0, \tag{6.41}$$

which describes the non-moving particle again, so this becomes the same solution as in Remark 6.6. ◁

These are the two solutions to the leading order geodesic equations that we can find. We see that they match the Carroll limits of the geodesics we found in regular general relativity in the previous chapter.

6.5 de Sitter in comoving coordinates

Recall that for the de Sitter metric in comoving coordinates, we have

$$K_{ii} = \left(H\mathrm{e}^{2Ht}\right)_{(0)}, \qquad h_{ii} = \left(\mathrm{e}^{2Ht}\right)_{(0)}, \qquad \hat{\tau}_t = 1, \qquad C^{\rho}_{\mu\nu} = H_{(0)}\delta^{\rho}_i\delta^t_{\mu}\delta^i_{\nu}. \tag{6.42}$$

We see that equation (6.14) becomes

$$\left(H\mathrm{e}^{2Ht}\right)_{(0)}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}s}_{(0)}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}s}_{(0)}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}s}_{(0)}\right)^2\right]=0, \tag{6.43}$$

and that equation (6.21) becomes the following:

$$\left(\mathrm{e}^{2Ht}\right)_{(0)}\left[\frac{\mathrm{d}^2x^i}{\mathrm{d}s^2}_{(0)}+2H_{(0)}\frac{\mathrm{d}x^i}{\mathrm{d}s}_{(0)}\frac{\mathrm{d}t}{\mathrm{d}s}_{(0)}\right]=0. \tag{6.44}$$

Just like in the case of Schwarzschild, we see that the equations between the square brackets are identical to the original equations, except that we take the leading order of every term, and that now we do not have the $\frac{\mathrm{d}^2t}{\mathrm{d}s^2}$ term.

Solution 6.9. If $H_{(0)}\neq 0$, then the first equation becomes

$$\frac{\mathrm{d}x^i}{\mathrm{d}s}_{(0)}=0, \tag{6.45}$$

so we only find geodesics belonging to non-moving particles. Then equation (6.16) with $\rho=t$ becomes

$$\frac{\mathrm{d}^2t}{\mathrm{d}s^2}_{(0)}+\left(H\mathrm{e}^{2Ht}\right)_{(0)}\left[\left(\frac{\mathrm{d}x}{\mathrm{d}s}_{(1)}\right)^2+\left(\frac{\mathrm{d}y}{\mathrm{d}s}_{(1)}\right)^2+\left(\frac{\mathrm{d}z}{\mathrm{d}s}_{(1)}\right)^2\right]=0, \tag{6.46}$$

from which we see that $\frac{\mathrm{d}t}{\mathrm{d}s}_{(0)}$ depends on $\frac{\mathrm{d}\vec{x}}{\mathrm{d}s}_{(1)}$. We also saw this in the original geodesic equations in the reparameterized geodesic in Solution 5.7: x is of order c^1 and it influences the equation of motion for t, which is of order c^0. We do not have any equations specifying $\frac{\mathrm{d}\vec{x}}{\mathrm{d}s}_{(1)}$, so we cannot determine t. ∘

Even though we could not determine t using the leading order geodesic equations, we did determine that this geodesic accommodates non-moving particles. This agrees with the fact that we could not find any Carroll limits in the previous chapter such that the geodesics in comoving coordinates of de Sitter spacetime would yield moving particles.

6.6 de Sitter in static coordinates

Recall that for the de Sitter metric in static coordinates, we have (for $\theta=\frac{\pi}{2}$)

$$K_{\mu\nu}=0, \tag{6.47}$$

$$h_{\rho\rho}=\left(\frac{1}{1-\frac{\rho^2}{R_H^2}}\right)_{(0)}, \qquad h_{\phi\phi}=\left(\rho^2\right)_{(0)}, \tag{6.48}$$

$$\hat{\tau}_t=\sqrt{1-\frac{\rho^2}{R_H^2}}_{(0)}, \tag{6.49}$$

$$C^{\tau}_{\rho\tau}=-\left(\frac{\rho}{R_H^2-\rho^2}\right)_{(0)}, \tag{6.50}$$

$$C^{\rho}_{\rho\rho}=\left(\frac{\rho}{R_H^2-\rho^2}\right)_{(0)}, \qquad C^{\rho}_{\phi\phi}=-\left(\rho-\frac{\rho^3}{R_H^2}\right)_{(0)}, \tag{6.51}$$

$$C^{\phi}_{\rho\phi}=C^{\phi}_{\phi\rho}=\left(\frac{1}{\rho}\right)_{(0)}. \tag{6.52}$$

We see that equation (6.14) becomes $0 = 0$, and that equation (6.21) becomes the following:

$$\left(\frac{1}{1-\frac{\rho^2}{R_H^2}}\right)_{(0)} \left[\frac{\mathrm{d}^2\rho}{\mathrm{d}s^2}_{(0)} + \left(\frac{\rho}{R_H^2-\rho^2}\right)_{(0)} \left(\frac{\mathrm{d}\rho}{\mathrm{d}s}_{(0)}\right)^2 - \left(\rho - \frac{\rho^3}{R_H^2}\right)_{(0)} \left(\frac{\mathrm{d}\phi}{\mathrm{d}s}_{(0)}\right)^2\right] = 0, \tag{6.53}$$

$$\left(\rho^2\right)_{(0)} \left[\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2}_{(0)} + \left(\frac{2}{\rho}\right)_{(0)} \frac{\mathrm{d}\rho}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}\phi}{\mathrm{d}s}_{(0)}\right] = 0. \tag{6.54}$$

Furthermore, equation 6.16 with $\rho = \tau$ becomes

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s^2}_{(0)} - \left(\frac{2\rho}{R_H^2-\rho^2}\right)_{(0)} \frac{\mathrm{d}\tau}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}\rho}{\mathrm{d}s}_{(0)} = 0. \tag{6.55}$$

Again, we see that the equation above and the equations between the square brackets are identical to the original equations, except for the fact that we are missing the $\left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2$ term, and that we take the leading order of every term.

6.6.1 Radial trajectories

For radial trajectories in the interesting case of $R_H \sim c^0$, we have $\frac{\mathrm{d}\phi}{\mathrm{d}s} = 0$ and the leading order geodesic equations become

$$\frac{\mathrm{d}^2\tau}{\mathrm{d}s^2}_{(0)} - \left(\frac{2\rho}{R_H^2-\rho^2}\right)_{(0)} \frac{\mathrm{d}\tau}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}\rho}{\mathrm{d}s}_{(0)} = 0, \tag{6.56}$$

$$\frac{\mathrm{d}^2\rho}{\mathrm{d}s^2}_{(0)} + \left(\frac{\rho}{R_H^2-\rho^2}\right)_{(0)} \left(\frac{\mathrm{d}\rho}{\mathrm{d}s}_{(0)}\right)^2 = 0. \tag{6.57}$$

Solution 6.10. A radial solution to the leading order geodesic equations of de Sitter in static coordinates is given by

$$\tau_{(0)} = T\tan\left(\frac{s+\text{constant}}{R_H}\right) + \text{constant}, \qquad \rho_{(0)} = R_H\left|\sin\left(\frac{s+\text{constant}}{R_H}\right)\right|, \qquad \phi_{(0)} = \text{constant}, \tag{6.58}$$

where T is an integration constant with dimensions of time. This is Case 1 of Solution 5.16, but then with a non-trivial time coordinate. This change is due to the disappearance of the $\left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2$ term in the leading order geodesic equations. ∘

Solution 6.11. Again, non-moving particles are solutions of the leading order geodesic equations, for which we introduce an integration constant u with dimensions of speed:

$$\tau_{(0)} = \frac{1}{u}s, \qquad \rho_{(0)} = \text{constant}, \qquad \phi_{(0)} = \text{constant}. \tag{6.59}$$

Note that $\rho_{(0)}$ is not necessarily equal to 0, and we get this extra freedom compared to the original non-moving geodesic in Solution 5.18 because of the disappearance of the $\left(\frac{\mathrm{d}\tau}{\mathrm{d}s}\right)^2$ term in the leading order geodesic equations. ∘

These are also the only solutions for which $\frac{\mathrm{d}r}{\mathrm{d}s}_{(0)} = 0$: there are no moving particles on circular trajectories.

Chapter 7

Carroll limit of the action

Another point at which we can take the Carroll limit is at the level of the action. Here, we are given a metric and we seek to expand the action in powers of c. We already did this in equation (6.2), but taking the Carroll limit in this action leaves us with the following Carroll action:

$$S_{\text{Carroll}} = |m|u \int \sqrt{h_{\mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^\nu}{\mathrm{d}s}_{(0)}}\, \mathrm{d}s, \tag{7.1}$$

where u is a non-scaling constant with dimensions of speed, such that the dimensions of the action are correct. The equations of motion following from this action are

$$h_{\alpha\nu} \frac{\mathrm{d}^2 x^\nu}{\mathrm{d}s^2}_{(0)} + \frac{1}{2} \left(\partial_\mu h_{\alpha\nu} + \partial_\nu h_{\alpha\mu} - \partial_\alpha h_{\mu\nu} \right) \frac{\mathrm{d}x^\mu}{\mathrm{d}s}_{(0)} \frac{\mathrm{d}x^\nu}{\mathrm{d}s}_{(0)} = 0. \tag{7.2}$$

We recall that $h_{\mu\nu}$ has tensor rank $D-1$, so we will get $D-1$ independent equations of motion from this action. The tensor rank of $h_{\mu\nu}$ also implies that there is a coordinate system $\{x^A\}$ in which $h_{0A} = 0$ for all A. This means that $\frac{\mathrm{d}x^0}{\mathrm{d}s}$ is not present in this action, so it will also be absent in the associated equations of motion. In our sample spacetimes, this is always the time coordinate (t or τ). This means that the time coordinate of geodesics is not determined by the equations of motion in (7.2).

Remark 7.1. The set of equations (6.14) & (6.21) is equivalent to equation (7.2). The equivalence is demonstrated using the following relations: (3.36), (3.29), (3.34), (3.31), and (3.32). ◁

From Remark 7.1, it follows that all solutions to the leading order geodesic equations we found in the previous chapter are also solutions to the equations of motion presented in (7.2). We also know that these equations do not determine the time coordinate of geodesics. Therefore, the solutions to the equations of motion associated with the "Carroll action" given in 7.1 are exactly those solutions we found in the previous chapter, but with the extra freedom in the time coordinate. We see that taking the Carroll limit earlier in the process, i.e. in the action rather than in the geodesic equations, leaves more freedom for solutions. This either indicates that this is just an incomplete theory of geodesics (because geodesics are not completely determined by the action), or this indicates that the action given in equation (7.1) is incorrect, and there is work to be done finding the correct c-independent action that has the leading order geodesic equations as its Euler-Lagrange equations.

Chapter 8

Conclusion and outlook

The goal of this thesis was to look into Carrollian geodesics in curved spacetimes. We found that there exist particles with a non-zero coordinate velocity $\frac{\mathrm{d}\vec{x}}{\mathrm{d}t}$ not only in the Minkowski but also in the non-flat Schwarzschild spacetime. When we expanded the geodesic equations and looked only at the leading orders c^{-2}, c^{-1}, and c^{0}, we found what are in essence the same geodesics, with some slightly different from the Carroll limit of the geodesics we found in regular general relativity, which is explained by some terms missing in the leading order geodesic equations that are present in the regular geodesic equations. Finally, we performed the Carroll limit within the action, which led to equations of motion that do not determine the coordinate x^0. We conclude that this is probably not the correct Carroll action.

We also found in our sample geodesics that the moving (spacelike) particles all have zero energy, in agreement with [24]. The (lightlike) non-moving particles have zero momentum, but not all non-moving particles have non-zero energy. We even found such a non-moving particle with zero energy in the flat Minkowski case.

For future research, we think that an interesting problem is to find the correct Carroll action, by which we mean an action that is independent of the speed of light such that its Euler-Lagrange equations constitute the leading order geodesic equations we found in Chapter 6.

It would also be interesting to see if the theory of Carroll gravity,[1] governed by its own "Carrollian Einstein field equations", produces geodesics. If so, it would be interesting to compare the resulting geodesic equations to the leading order geodesic equations presented in this thesis. It might also be interesting to compare specific geodesics in Minkowksi, Schwarzschild, de Sitter, or other spacetimes to the different sets of geodesics presented here. This will give an expanded sense of the extent to which taking the Carroll limit and going through the process of finding geodesics commute.

[1] See Appendix C for some statements about the leading orders of the curvature tensors.

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Appendices

Appendix A

Proofs

In some of these proofs, we use that the Levi-Civita connection is metric compatible, i.e.

$$\nabla_\rho g_{\mu\nu} = 0. \tag{A.1}$$

Lemma A.1. *Given a geodesic, the number ε is constant along the geodesic.*

Proof. Define $V^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$. Then $\varepsilon = -V_\mu V^\mu$, and the geodesic equation can be written as $V^\nu \nabla_\nu V^\mu = 0$. We can now calculate how ε changes along the geodesic:

$$\frac{\mathrm{D}}{\mathrm{d}s}\varepsilon = V^\nu \nabla_\nu \varepsilon = -V^\nu \nabla_\nu (V^\mu V_\mu) = -V^\nu V^\mu \nabla_\nu V_\mu - V^\nu V_\mu \nabla_\nu V^\mu \tag{A.2}$$

$$\overset{\text{metric compatibility}}{=} -2V_\mu V^\nu \nabla_\nu V^\mu \overset{\text{geodesic equation}}{=} -2V_\mu \cdot 0 = 0. \tag{A.3}$$

□

Lemma A.2. *Given a Killing vector K^μ, the quantity*

$$K_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s} \tag{A.4}$$

is conserved along a geodesic.

Proof. Let K^μ be a Killing vector. We first prove the Killing equation $\nabla_{(\mu} K_{\mu)} = 0$:

$$\nabla_{(\mu} K_{\mu)} = \frac{1}{2}\left(\nabla_\mu K_\nu + \nabla_\nu K_\mu\right) \overset{\text{metric compatibility}}{=} \frac{1}{2}\left[g_{\rho\nu}\nabla_\mu K^\rho + g_{\mu\rho}\nabla_\nu K^\rho + K^\sigma \nabla_\sigma g_{\mu\nu}\right] \tag{A.5}$$

$$= \frac{1}{2}\left[K^\sigma\left(\nabla_\sigma g_{\mu\nu} + \Gamma^\rho_{\mu\sigma} g_{\rho\nu} + \Gamma^\rho_{\nu\sigma} g_{\sigma\mu}\right) + g_{\rho\nu}\left(\nabla_\mu K^\rho - \Gamma^\rho_{\mu\lambda} K^\lambda\right) + g_{\mu\rho}\left(\nabla_\nu K^\rho - \Gamma^\rho_{\nu\lambda} K^\lambda\right)\right] \tag{A.6}$$

$$= \frac{1}{2}\left[K^\sigma \partial_\sigma g_{\mu\nu} + g_{\sigma\nu}\partial_\mu K^\sigma + g_{\mu\sigma}\partial_\nu K^\sigma\right] = \frac{1}{2}\left(\mathcal{L}_K g\right)_{\mu\nu} = 0, \tag{A.7}$$

where the last equality comes from the definition of K^μ being a Killing vector.

Define again $V^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$. Then the rate of change of $K_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$ along a geodesic is

$$\frac{\mathrm{D}}{\mathrm{d}s}\left[K_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}\right] = V^\nu \nabla_\nu \left[K_\mu V^\mu\right] = V^\nu V^\mu \nabla_\nu K_\mu + V^\nu K_\mu \nabla_\nu V^\mu = V^\nu V^\mu \nabla_{(\nu} K_{\mu)} + K_\mu V^\nu \nabla_\nu V^\mu = 0, \tag{A.8}$$

where the first term is zero because K^μ is a Killing vector, and the second term is zero because of the geodesic equation. We conclude that the quantity $K_\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}s}$ does not change along a geodesic, so it is a conserved quantity. □

Definition A.3. The rank of a tensor is the minimum number of simple tensors that sum to that tensor. In particular, the rank of a $(0,2)$-tensor $A_{\mu\nu}$ is the same as the matrix rank of the corresponding matrix $\delta^{\mu\rho}A_{\rho\nu}$. ◁

Lemma A.4. *Let us work in a D-dimensional spacetime, with the quantities that we introduced related to the expansion of the geodesic equations. The following statements are true:*

$$v \neq 0, \tag{A.9}$$

$$\operatorname{rank}(K_{\mu\nu}) \leq D-1, \tag{A.10}$$

$$\operatorname{rank}(h_{\mu\nu}) = D-1. \tag{A.11}$$

Proof. We know that $v^\sigma\tau_\sigma = -1$ from equation (3.26), so $v \neq 0$.

We also know that $v^\nu K_{\sigma\nu} = 0$, so $\delta^{\mu\sigma}K_{\sigma\nu}v^\nu = 0$. So the non-zero vector v is in the kernel of the matrix $\delta^{\mu\sigma}K_{\sigma\nu}$, so $\delta^{\mu\sigma}K_{\sigma\nu}$ has matrix rank smaller than or equal to $D-1$. We conclude that $\operatorname{rank}(K_{\mu\nu}) \leq D-1$.

Similarly, we conclude from $v^\nu h_{\sigma\nu} = 0$ that $\operatorname{rank}(h_{\mu\nu}) \leq D-1$. We also know from equation (3.27) that $-v^\mu\tau_\nu + h^{\mu\rho}h_{\rho\nu} = \delta^\mu_\nu$, so by subadditivity of the rank, we have

$$\operatorname{rank}(-v^\mu\tau_\nu) + \operatorname{rank}(h^{\mu\rho}h_{\rho\nu}) \geq \operatorname{rank}(-v^\mu\tau_\nu + h^{\mu\rho}h_{\rho\nu}) = \operatorname{rank}(\delta^\mu_\nu) = D, \tag{A.12}$$

$$\iff \operatorname{rank}(h^{\mu\rho}h_{\rho\nu}) \geq D - \operatorname{rank}(v^\mu\tau_\nu). \tag{A.13}$$

But $v^\mu\tau_\nu$ is a non-zero (by equation (3.26)) simple tensor, so it has rank 1. Thus,

$$\operatorname{rank}(h^{\mu\rho}h_{\rho\nu}) \geq D-1. \tag{A.14}$$

For all matrices A and B, we know that $\operatorname{rank}(AB) \leq \operatorname{rank}(B)$, so

$$\operatorname{rank}(h_{\mu\nu}) = \operatorname{rank}(\delta^{\mu\sigma}h_{\sigma\nu}) \geq \operatorname{rank}\left(\delta_{\rho\lambda}h^{\lambda\mu}\delta^{\rho\sigma}h_{\sigma\nu}\right) = \operatorname{rank}(h^{\sigma\mu}h_{\sigma\nu}) = \operatorname{rank}(h^{\mu\rho}h_{\rho\nu}) \geq D-1. \tag{A.15}$$

We conclude that $\operatorname{rank}(h_{\mu\nu}) = D-1$. □

Lemma A.5. *The vector v^ρ and the tensor $h_{\mu\nu}$ are in the kernel of $\overset{(C)}{\nabla}$:*

$$\overset{(C)}{\nabla}_\alpha h_{\mu\nu} = 0, \qquad \overset{(C)}{\nabla}_\alpha v^\rho = 0. \tag{A.16}$$

Proof. We calculate

$$\overset{(C)}{\nabla}_\alpha v^\rho := \partial_\alpha v^\rho + C^\rho_{\alpha\lambda}v^\lambda \overset{(3.36)\ \&\ (3.31)}{=} \partial_\alpha v^\rho - v^\lambda v^\rho \partial_\alpha \hat{\tau}_\lambda + v^\lambda \overline{h}^{\rho\sigma}\partial_\alpha h_{\lambda\sigma} \tag{A.17}$$

$$\overset{(3.34)\ \&\ (3.29)}{=} \partial_\alpha v^\rho - v^\lambda v^\rho \partial_\alpha \hat{\tau}_\lambda + v^\lambda \partial_\alpha\left(\delta^\rho_\lambda + v^\rho\hat{\tau}_\lambda\right) \overset{(3.33)}{=} 0. \tag{A.18}$$

We also calculate

$$\overset{(C)}{\nabla}_\alpha h_{\mu\nu} := \partial_\alpha h_{\mu\nu} - C^\lambda_{\alpha\mu}h_{\lambda\nu} - C^\lambda_{\alpha\nu}h_{\lambda\mu} \tag{A.19}$$

$$\overset{(3.36)\ \&\ (3.34)}{=} \partial_\alpha h_{\mu\nu} - \frac{1}{2}\left(\delta^\sigma_\nu + v^\sigma\hat{\tau}_\nu\right)\left(\partial_\alpha h_{\mu\sigma} + \partial_\mu h_{\alpha\sigma} - \partial_\sigma h_{\alpha\mu} - 2K_{\alpha\sigma}\hat{\tau}_\mu\right) \tag{A.20}$$

$$- \frac{1}{2}\left(\delta^\sigma_\mu + v^\sigma\hat{\tau}_\mu\right)\left(\partial_\alpha h_{\nu\sigma} + \partial_\nu h_{\alpha\sigma} - \partial_\sigma h_{\alpha\nu} - 2K_{\alpha\sigma}\hat{\tau}_\nu\right) \tag{A.21}$$

$$\overset{(3.31)}{=} \frac{1}{2}\partial_\alpha\left(h_{\mu\nu} - h_{\nu\mu}\right) = 0, \tag{A.22}$$

where the last equality comes from $h_{\mu\nu}$ being symmetric. □

Appendix B

Mathematical derivation of the Christoffel symbols of pseudometrics

We show how Gibbons et al. constructed some, but not all, of the Christoffel symbols of a pseudometric in [11] and [12]. It is not necessary to know this in order to understand the rest of the thesis, but along the way it gives a reminder of how to think about manifolds, connections, and metrics.

Let L be a Lorentzian manifold with a Lorentzian metric g. We will look at metric-compatible connections, so we will look at $\nabla_X g$, where X is a vector field. This is only possible if we see the metric g as a section. This means that we see g as a tensor field that gives an element in $T^*L \otimes T^*L$ for every point $l \in L$:

$$g \ : \ L \to T^*L \otimes T^*L. \tag{B.1}$$

In a coordinate basis x^μ, this means

$$g(l) = g_{\mu\nu}(l)\, \mathrm{d}x^\mu \otimes \ \mathrm{d}x^\nu. \tag{B.2}$$

Thus, to be clear, for every $l \in L$, $g(l)$ is a function $T_l L \otimes T_l L \to \mathbb{R}$. We are now about to take the Carroll limit, and we will call our Lorentzian manifold C after taking this limit to indicate that c is now set to 0 and we are in the Carroll regime. In the Carroll limit, the metric g becomes degenerate, in the sense that there is exactly one nowhere-zero tangent vector field ξ_u in the kernel of g. This means that for all tangent vector fields $\chi \ : \ C \to TC$ and all $x \in C$, we have:

$$g(x)\left(\xi_u(x), \chi(x)\right) = 0. \tag{B.3}$$

In every point $x \in C$, we can choose a basis of $T_x M$ containing $\xi_u(x)$. This way, u becomes a coordinate, and $\xi_u = \partial_u$. In a basis containing u, the equation above reads

$$0 = g_{\mu\nu}(x)\left(\mathrm{d}x^\mu \xi_u(x)\right) \otimes \left(\mathrm{d}x^\nu \chi(x)\right) = g_{u\nu}(x) 1 \otimes \left(\mathrm{d}x^\nu \chi(x)\right), \tag{B.4}$$

$$\iff g_{u\nu} = 0. \tag{B.5}$$

Now, we note that there are $|\mathbb{R}|$ hypersurfaces that are orthogonal to the vector field ξ_u, so we can define a hypersurface Σ_u for every $u \in \mathbb{R}$. This allows us to write

$$C = \mathbb{R} \times \Sigma := \bigcup_{u \in \mathbb{R}} \Sigma_u. \tag{B.6}$$

For every hypersurface Σ_u, we can consider the bundle $T\Sigma_u \otimes T\Sigma_u$ as a sub-bundle of $TC \otimes TC$. By construction, for every $l \in L$, $g(l)$ is non-zero on $T_l\Sigma_u \otimes T_l\Sigma_u$, because $\xi_u(l) \notin T_l\Sigma_u$. Together with the fact that g was a (true) metric before taking the Carroll limit - which amounted to manually turning on a nowhere-zero tangent vector field in the kernel of the metric - this means that

$$\widehat{g}_u := g\Big|_{\Sigma_u} \tag{B.7}$$

is a (true) metric on Σ_u. This means that the Levi-Civita connection is the unique torsion-free metric-compatible connection on Σ_u, which we can define through the Christoffel symbols

$$\widehat{\Gamma}^C_{AB} = \frac{1}{2}\widehat{g}^{CZ}\left(\partial_A \widehat{g}_{BZ} + \partial_B \widehat{g}_{AZ} - \partial_Z \widehat{g}_{AB}\right), \tag{B.8}$$

where we omitted the subscript u for clarity purposes, and the capitalized Latin indices run over all coordinates except u.

Definition B.1. Let u be fixed. Define the projection $P_{T\Sigma_u} : TC \to T\Sigma_u$ according to the usual linear algebraic definition of a projection on a vector space. Then, for every vector field $\chi : C \to TC$, define its projection $\widehat{\chi} : \Sigma_u \to T\Sigma_u$ by defining for every $x \in \Sigma_u$:

$$\widehat{\chi}(x) := P_{T\Sigma_u}\left(\chi(x)\right). \tag{B.9}$$

$\triangleleft$

We note that for all points $x \in C$, there is a unique $u \in \mathbb{R}$ such that $x \in \Sigma_u$. Take a point $x \in C$, determine the u such that $x \in \Sigma_u$, and write $x = x_u$. For all vector fields $\chi, \psi : C \to TC$, we have

$$g\left(x_u\right)\left(\chi\left(x_u\right), \psi\left(x_u\right)\right) = \widehat{g}_u\left(x_u\right)\left(\widehat{\chi}\left(x_u\right), \widehat{\psi}\left(x_u\right)\right). \tag{B.10}$$

This is a direct consequence of the way we have constructed $\widehat{g}$. From this, it is a logical step to define part of the Christoffel symbols of the pseudometric g as

$$\Gamma^C_{AB} = \widehat{\Gamma}^C_{AB}. \tag{B.11}$$

This is consistent with the expression for the Christoffel symbols we get in equation (6.6). To make the connection pseudometric-compatible, we require

$$\Gamma^A_{u\mu} = \Gamma^A_{\mu u} = 0. \tag{B.12}$$

This leaves freedom for the other components:

$$\Gamma^u_{\mu\nu} = \text{arbitrary}. \tag{B.13}$$

We could be inclined to argue that we want our connection to be torsion-free, thus requiring $\Gamma^u_{\mu\nu} = \Gamma^u_{\nu\mu}$, but we have seen that torsion is naturally introduced into Carrollian spacetime connections: the Carroll compatible connection has non-zero torsion. Therefore, we do not want to impose zero torsion.

Appendix C

Some results about curvature tensors

We have an expression for the Christoffel symbols, so we can now construct the Riemann tensor:

$$R_{\mu\nu\sigma}{}^{\rho} := -\partial_\mu \Gamma^{\rho}_{\nu\sigma} + \partial_\nu \Gamma^{\rho}_{\mu\sigma} - \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} + \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}. \tag{C.1}$$

We see immediately that $\mathcal{C}_{<-4}\left[R_{\mu\nu\sigma}{}^{\rho}\right] = 0$, and we can calculate

$$\mathcal{C}_{-4}\left[R_{\mu\nu\sigma}{}^{\rho}\right] = -\left(\Gamma^{\rho}_{\mu\lambda}\right)_{(-2)}\left(\Gamma^{\lambda}_{\nu\sigma}\right)_{(-2)} + \left(\Gamma^{\rho}_{\nu\lambda}\right)_{(-2)}\left(\Gamma^{\lambda}_{\mu\sigma}\right)_{(-2)} \tag{C.2}$$

$$\overset{(6.7)}{=} -v^{\rho}K_{\mu\lambda}v^{\lambda}K_{\nu\sigma} + v^{\rho}K_{\nu\lambda}v^{\lambda}K_{\mu\sigma}, \overset{(3.32)}{=} 0, \tag{C.3}$$

so the leading order of the Riemann tensor is c^{-2}. Define the torsion of the Carroll compatible connection as

$$T^{\lambda}{}_{\mu\nu} := 2C^{\lambda}_{[\mu\nu]}. \tag{C.4}$$

This is a tensor, as opposed to $C^{\lambda}_{\mu\nu}$. Also write the covariant derivative with respect to the Carroll compatible connection as

$$\nabla := \overset{(C)}{\nabla}, \tag{C.5}$$

as a shorthand. We have:

$$\mathcal{C}_{-2}\left[R_{\mu\nu\sigma}{}^{\rho}\right] := -\partial_\mu\left(\Gamma^{\rho}_{\nu\sigma}\right)_{(-2)} + \partial_\nu\left(\Gamma^{\rho}_{\mu\sigma}\right)_{(-2)} - \left(\Gamma^{\rho}_{\mu\lambda}\right)_{(-2)}\left(\Gamma^{\lambda}_{\nu\sigma}\right)_{(0)} - \left(\Gamma^{\rho}_{\mu\lambda}\right)_{(0)}\left(\Gamma^{\lambda}_{\nu\sigma}\right)_{(-2)} \tag{C.6}$$

$$+ \left(\Gamma^{\rho}_{\nu\lambda}\right)_{(-2)}\left(\Gamma^{\lambda}_{\mu\sigma}\right)_{(0)} + \left(\Gamma^{\rho}_{\nu\lambda}\right)_{(0)}\left(\Gamma^{\lambda}_{\mu\sigma}\right)_{(-2)} \tag{C.7}$$

$$= 2v^{\rho}\nabla_{[\mu}K_{\nu]\sigma} + v^{\rho}K_{\lambda\sigma}T^{\lambda}{}_{\mu\nu} \tag{C.8}$$

$$+ v^{\lambda}K_{\nu\sigma}\left(\frac{1}{2}v^{\rho}\left(\partial_\mu\hat{\tau}_\lambda - \partial_\lambda\hat{\tau}_\mu + \mathcal{L}_v\overline{\Phi}_{\mu\lambda}\right) + \overline{h}^{\rho\gamma}K_{\mu\gamma}\hat{\tau}_\lambda\right) \tag{C.9}$$

$$- v^{\lambda}K_{\mu\sigma}\left(\frac{1}{2}v^{\rho}\left(\partial_\nu\hat{\tau}_\lambda - \partial_\lambda\hat{\tau}_\nu + \mathcal{L}_v\overline{\Phi}_{\nu\lambda}\right) + \overline{h}^{\rho\gamma}K_{\nu\gamma}\hat{\tau}_\lambda\right). \tag{C.10}$$

This gives the following leading order of the Ricci tensor:

$$\mathcal{C}_{-2}\left[R_{\mu\sigma}\right] = 2v^{\nu}\nabla_{[\mu}K_{\nu]\sigma} + v^{\nu}K_{\lambda\sigma}T^{\lambda}{}_{\mu\nu} \tag{C.11}$$

$$+ v^{\lambda}K_{\nu\sigma}\left(\frac{1}{2}v^{\nu}\left(\partial_\mu\hat{\tau}_\lambda - \partial_\lambda\hat{\tau}_\mu + \mathcal{L}_v\overline{\Phi}_{\mu\lambda}\right) + \overline{h}^{\nu\gamma}K_{\mu\gamma}\hat{\tau}_\lambda\right) \tag{C.12}$$

$$- v^{\lambda}K_{\mu\sigma}\left(\frac{1}{2}v^{\nu}\left(\partial_\nu\hat{\tau}_\lambda - \partial_\lambda\hat{\tau}_\nu + \mathcal{L}_v\overline{\Phi}_{\nu\lambda}\right) + \overline{h}^{\nu\gamma}K_{\nu\gamma}\hat{\tau}_\lambda\right). \tag{C.13}$$

Considering the Ricci scalar, we know from the above that $\mathcal{C}_{<-4}[R] = 0$, and we can calculate

$$\mathcal{C}_{-4}[R] = \mathcal{C}_{-2}[g^{\mu\sigma}]\,\mathcal{C}_{-2}[R_{\mu\sigma}] = -v^{\mu}v^{\sigma}v^{\nu}\left(\nabla_{\mu}K_{\nu\sigma} - \nabla_{\nu}K_{\mu\sigma}\right) = 0. \tag{C.14}$$

This means that the Ricci scalar is of order c^{-2} or higher. Unfortunately, the expression for the order of -2 is unknown, because the second term in the expression below is not known:

$$\mathcal{C}_{-2}[R] = \mathcal{C}_{0}[g^{\mu\sigma}]\,\mathcal{C}_{-2}[R_{\mu\sigma}] + \mathcal{C}_{-2}[g^{\mu\sigma}]\,\mathcal{C}_{0}[R_{\mu\sigma}]\,. \tag{C.15}$$

This makes the construction of the Einstein-Hilbert Lagrangian

$$\mathcal{L} := R\sqrt{\det\left(g_{\mu\lambda}\delta^{\lambda\nu}\right)} \tag{C.16}$$

impractical following the presented method.

For more about the action, see e.g. [16] or [26].

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