Morse Theory and The Morse inequalities

by

Pieter Berben

supervised by

Dr. G.S.K.S. Heuts

A Bachelor thesis presented to Utrecht University
for the degree of Bachelor in Mathematics.

Department of Mathematics
Utrecht University
January 26, 2024

Abstract

This bachelor thesis will give a introduction to Morse theory and the Morse inequalities. Morse theory is a method that can be used to determine the cells in the cell structure of a differentiable manifold. Most of the theory discussed in this thesis will be from *Morse theory*, written by Milnor [1]. We will be giving a overview of the theory involved and discuss a few examples. In particular will we be looking at the examples of $\mathbb{CP}^n$, $\mathbb{RP}^2$ and $\mathbb{HP}^n$. We will also discuss the Morse inequalities. In particular, we will discuss the method of proving these inequalities discovered by Witten.

Contents

1 Introduction

Morse theory is a useful method of finding the cell structure of a differentiable manifold M. The method works broadly as followes

If you have a Morse function $f : M \to \mathbb{R}$. Then for every critical point of index λ, the cell structure of M contains a λ cell.

In chapter 2 we will more formally look at this statement as a theorem and look at the proof of this theorem. In paragraph 2.1 we will first look at a example of how you use Morse theory to find the cell structure of a Torus to motivate the intuition behind Morse theory. Then in paragraph 2.1 we will look at how a Morse function and the index are defined. In paragraph 2.3 and 2.4 we will be discussing Morse's lemma an the 1-parameter group of diffeomorphism. These are both important concepts to the proofs of the theorems surrounding Morse theory that we will be discussing in paragraph 2.5.

In chapter 3 we will be discussing more examples of how Morse theory can be used. In paragraph 3.1, 3.2 and 3.3 we will be discussing how we can find the cell structure of $\mathbb{CP}^n$, $\mathbb{RP}^n$ and $\mathbb{HP}^n$. In paragraph 3.4 we will be discussing how we can prove a version of the Reeb sphere theorem using Morse theory.

In chapter 4 we will be discussing the Morse inequalities. This section will be about the proof of the Morse inequalities of Witten, which is not the standerd way of proving these inequalities. These inequalities have something to do with betti numbers, so in paragraph 4.1 we will be discussing what betti numbers are. A important concept to Witten's proof in Witten's conjugation, so in paragraph 4.2 we will be discussing this concept and in paragraph 4.3 we will prove the Morse inequalities.

2 Morse theory

2.1 The Torus

Before we formally state the theorems involved in Morse theory and try to prove then, we will be discussing how you can use Morse theory to find the cell structure of the Torus T. This way we build intuition in the process of using Morse theory before we prove that this process indeed works.

Let T be a torus embedded in $\mathbb{R}^3$ as oriented in figure 1 such that the bottom of the torus has $z = -1$ and the top of the torus has $z = 1$. We let $h : T \to \mathbb{R}$ be the height function on T. This means that h is defined as $h(x, y, z) = z$. We notice that h has four critical point, the top and bottom of the torus and the two saddle point of h. We will denote these critical points as p, q, r, s.

We then define M^a as $M^a = \{x \in T : f(x) \leq a\}$. We will now look at how the cell strucure of M^a changes as a gets larger. We won't do this very precisely, we will rather do this by determining with pictures what M^a is homeomorphic to. What we notice is that

(1) if $a < -1$, the $M^a = \emptyset$

(2) if $-1 < a < f(q)$, then M^a is homotopy equivalent to a disc. This means that M^a is homeomorphic to a 2-cell. We don't have a figure for this, but we notice that M^a looks like a curved disc, so we can tranform it homeomorphically to a disc.

(3) if $f(q) \leq a < f(r)$, then we can see in figure 2 that M^a homotopy equivalent is to a 1-cell attached to a disc as we can transform M^a to such an object. This means that M^a has the cell structure of a 1-cell attached to a 2-cell

(4) if $f(r) \leq a < f(s)$ then we see that M^a is homeomorphic to a 1-cell attached to M^b, with $f(q) \leq b < f(r)$, as we see in figure 3. What this means is that M^a has the cell structure of 2 1-cells attached to a 2-cell.

(5) if $a \geq f(s)$, then we notice that M^a homeomorphic is to the torus. We also see that M^a is homeomorphic to a 0-cell attached to M^b, with $f(r) \leq b < f(s)$.

What we can conclude is that every time a passes a critical point, you add a k-cell to the cell structure to M^a. This way you build the cell structure of T and see that the cell structure is determined by the critical points of h. It is still a bit mysterious what k-cell we need to add once a passes a critical point. This is where the index comes in to play. The index of a critical point is k, then you add a k-cell once a passes this critical point. The definition of this index and the Morse function will be discussed in the next section.

2.2 Morse functions and the index

In this section we will define the Morse function and the index of a critical point of a Morse function. These ideas are fundamental for Morse theory, as we have seen in the example of the Torus. We will be using the definitions as given by Milnor in *Morse Theory* [1].

Let M be a differentiable manifold and let $f : M \to \mathbb{R}$ be a smooth function. A point $p \in M$ is a critical point of f if the differential of f at p $(df)_p : TM_p \to T\mathbb{R}_{f(p)}$ is zero. Because M is differentiable, we can choose a local coordinate system $(x^1, \cdots, x^n)$ in a neighborhood U of p. The Hessian $H(p)$ of f is a function on TM_p that in these local coordinates can be represented by the matrix

$$H(p) = \left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p) \right).$$

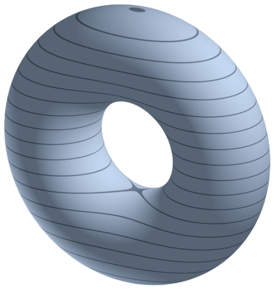

Figure 1: The Torus T. The source of this image can be found here.

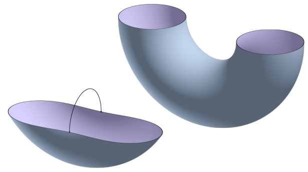

Figure 2: M^a with $f(q) \leq a < f(p)$. The source of this image can be found here
.

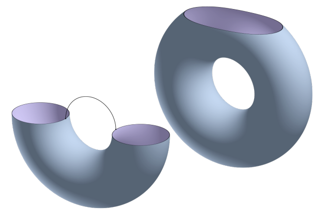

Figure 3: M^a with $f(r) \leq a < f(s)$. The souce if this image can be found here
.

When this matrix is non-singular in p, we call p a non-degenerate critical point. We can now define the Morse function as followes.

Definition 2.1 (Morse function)**.** If $f : M \to \mathbb{R}$ is a smooth map such that every critical point is non-degenerate, then f is a Morse function.

Now that we properly defined what a morse function is, we can prove this more rigorously in the example of the Torus.

Example 2.1 (Height function on the Torus)**.** In the section where we discussed the Torus, we said that the Height function $f : T \to \mathbb{R}$ defined as $f(x, y, z) = z$ is a Morse function.

First we check that f has indeed four critical points as described in the last section. To do this we need to first look at a parametrization of T. This parametrization looks like

$$\begin{aligned} x(\theta, \varphi) &= r \sin\theta \\ y(\theta, \varphi) &= (R + r\cos\theta)\sin\varphi \\ z(\theta, \varphi) &= (R + r\cos\theta)\cos\varphi, \end{aligned}$$

with $\theta, \varphi \in [0, 2\pi)$ and $R, r > 0$ are constants such that $R > r$. In this case (θ, φ) will be our coordinate system. We notice that for any point $p \in T$ the map $(df)_p : TT_p \to T\mathbb{R}_{f(p)}$ can be represented by the matrix

$$\begin{pmatrix} -r\sin\theta\cos\varphi & -(R + r\cos\theta)\sin\varphi \end{pmatrix}.$$

This means that if $(df)_p$ is zero, then $-r\sin\theta\cos\varphi = 0$ and $-(R + r\cos\theta)\sin\varphi = 0$. Because $r < R$ and $-1 \leq \cos\theta \leq 1$, the equation $-(R + r\cos\theta)\sin\varphi = 0$ gives us that $\sin\varphi = 0$. If this is the case, we know that $\cos\varphi \neq 0$, so the equation $-r\sin\theta\cos\varphi = 0$ gives us that $\sin\theta = 0$. This means that the only critical points of T are $\{(0, 0), (0, \pi), (\pi, 0), (\pi, \pi)\}$. These are indeed the four critical points as described earlier. Now we take $p = (0, 0), q = (\pi, 0), s = (\pi, \pi)$ and $t = (0, \pi)$.

Now we have to show that these four points are indeed non-degenerate. To do this we have to look at the Hessian of f. This function is represented by the matrix

$$\begin{pmatrix} -r\cos\theta\cos\varphi & r\sin\theta\sin\varphi \\ r\sin\theta\sin\varphi & -(R + r\cos\theta)\cos\varphi \end{pmatrix}.$$

We notice that

$$\text{if } (\theta,\varphi) = (0,0), \text{ then } H(p) = \begin{pmatrix} -1 & 0 \\ 0 & -(R+r) \end{pmatrix},$$
$$\text{if } (\theta,\varphi) = (\pi,0), \text{ then } H(q) = \begin{pmatrix} 1 & 0 \\ 0 & -(R-r) \end{pmatrix},$$
$$\text{if } (\theta,\varphi) = (0,\pi), \text{ then } H(t) = \begin{pmatrix} 1 & 0 \\ 0 & (R+r) \end{pmatrix}$$
$$\text{and if } (\theta,\varphi) = (\pi,\pi), \text{ then } H(s) = \begin{pmatrix} -1 & 0 \\ 0 & (R-r) \end{pmatrix}.$$

We see that all of these matrices are non-singular, so all the critical points are non-degenerate and f is indeed a Morse function.

We will now be discussing the index of a critical point. This is an important definition as the index determines what n-cell gets attached to the cell structure of M^a when a passes a critical point. We will be using the definition given by Milnor in *Morse Theory* [1].

Definition 2.2 (Index of a critical point)**.** If p is a critical point of the Morse function f, then the index of p is the dimension of the largest subspace of TM_p such that the Hessian H is negative definite.

Example 2.2 (Height function on the Torus)**.** In the last example we have calculated the Hessian of f in every critical point of f. We can use this to calculate the index of every critical point.

Let p be the critical point of h such that $h(p) = -1$. We notice that in $H(p)$ only has negative eigenvalues. This means that $H(p)$ is negative definite over all of TT_p, wich has a dimension of 2, so p has a index of 2.

If we look at the critical point $(0, \pi)$, then we see that the Hessian $H(t)$ has two positive eigenvalues. This means that H is positive definite over TT_t. As a consequence the largest subspace of TT_t where H is negative definite is empty, so $(0, \pi)$ has index 0.

The other two critical points have index 1. This is because $R - r > 0$, so they both have one negative eigenvalue and one positive eigenvalue. For q this means that there is a 1-dimensional subspace of TT_q such that H is negative definite and a 1-dimensional subspace of TT_q such that H is positive definite. As TT_q has dimension 2, this means that if there can't exist a larger subspace of TT_q such that H is negative definite. In the same way we can show that s has index 1.

We notice that these indices match with the cells added to the cell structure of M^a when we pass the critical points in the example of the Torus discussed earlier.

2.3 Morse's lemma

Before we prove the theorems that form the foundation of Morse theory, we need to prove the Lemma of Morse. This lemma is instrumental in proving these theorems. To prove the lemma of Morse, we need a different lemma. We will not give the proof of this Lemma. For those interested, the proof is given by Milnor [1, p. 5].

Lemma 1. Let $f : M \to \mathbb{R}$ be a C^∞ function such that $f(0) = 0$ and the image of f is a convex neighborhood V of 0. Then

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_n),$$

where g_i are C^∞ functions with it's image in V and the property that $g_i(0) = \frac{\partial f}{\partial x_i}(0)$.

We will now use this lemma to prove the lemma of Morse. We will follow the proof given by Milnor in *Morse Theory.*

Lemma 2 (Lemma of Morse)**.** Let p be a non-degenerate critical point of f. There exists a neighborhood U of p with a coordinate system $(y^1, \cdots y^n)$ such that $y^i(p) = 0$ for all $i \in \{1, \cdots, n\}$ and where the identity

$$f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots (y^n)^2 \tag{1}$$

holds throughout. In this identity, λ is the index of p.

Proof. We assume that p the origin is of $\mathbb{R}^n$ and that $f(p) = f(0) = 0$. We can utilize lemma 1 to see that

$$f(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i g_i(x_1, \cdots, x_n).$$

From Lemma 1, we know that $g_j(0) = \frac{\partial f}{\partial x_j}(0)$, so because 0 is a critical point of f, we get that $g_j(0) = 0$, for all $1 \leq j \leq n$. We can now apply Lemma 1 on g_j to get

$$g_j(x_1, \cdots, x_n) = \sum_{i=1}^{n} x_i h_{ij}(x_1, \cdots, x_n).$$

This gives us that

$$f(x_1, \cdots, x_n) = \sum_{i,j=1}^{n} x_i x_j h_{ij}(x_1, \ldots, x_n). \tag{2}$$

We will now use induction to prove that a coordinate system exists such that 1 holds. We suppose that there exists a neighborhood U_1 of 0 with a coordinate system $(u_1, \cdots, u_n)$ such that

$$f = \pm(u_1)^2 + \cdots \pm (u_r)^2 + \sum_{i,j \geq r} u_i u_j H_{ij}(u_1, \cdots, u_n),$$

where the matrix $(H_{ij}(u_1, \cdots, u_n)$ is symmetric. We notice that 2 proves this statement in the case of $r = 0$ as $h_{ij} = h_{ji}$. We can see that this is true in the way we constructed g_i in the proof of lemma 1. As we are using the method of inducuction, we will now we assume that this statement is true for a $0 \leq r \leq n$.

We notice that we can use a linear change of coordinates is the last $n - r + 1$ coordinates to get that $H_{rr}(0) = 0$. Milnor thinks this linear change of coordinates is obvious, but in this proof we will see how this linear change of coordinates looks like. We know that there exists a pair $i, j \geq r$ such that $H_{ij}(0) \neq 0$, as else $(H_{ij}(0))$ is a degenerate matrix. If $i = j$, we can switch u_r with u_i. This will be our linear transformation. If $i \neq j$, so all the diagonal elements are 0, then we use the transformation $x_i = u_i + u_j$, $x_j = u_i - u_j$. Because $\frac{1}{4}u_i u_j = x_i^2 x_j^2$, we have that after this transformation H_{ii} and H_{jj} are non-zero.

We now denote $g(u_1, \cdots, u_n) = \sqrt{|H_{rr}(u_1, \cdots, u_n)|}$. This function is a smooth non-zero function on a smaller neighborhood $U_2 \subset U_1$ of 0. We introduce a new coordinate system as

$$v_i = u_i \text{ if } i \neq r$$

$$v_r(u_1, \cdots, u_n) = g(u_1, \cdots, u_n) \left(u_r + \sum_{i > r} u_i H_{ir}(u_1, \cdots, u_n)/H_{rr}(u_1, \cdots, u_n) \right).$$

The inverse function theorem gives us that this is indeed a coordinate system for a small enough neighborhood U_3 of 0.

We notice that

$$(v_r)^2 = H_{rr}(u_1, \cdots, u_n)u_r^2 + u_r \sum u_i H_{ir}(u_1, \cdots, u_n) + \left(\sum_{i>r} H_{ir}(u_1, \cdots, u_n) \right)^2 / H_{rr}(u_1, \cdots, u_n),$$

this means that if we can define H'_{ij} such that

$$f = \pm v_1 \cdots \pm v_r + \sum_{i,j>r} v_i v_j H'_{ij}(v_1, \cdots, v_n).$$

This proves the induction step and proves that a coordinate system exist such that f can be written as 1.

We assumed that $f(p) = 0$. In the case that $f(p) \neq 0$, we can use the same arguments on the function $h(x_1, \cdots, x_n) = f(x_1, \cdots, x_n) - f(p)$.

We are left to prove that λ is the index of f. We notice that

$$\frac{\partial^2 f}{\partial y_i \partial y_j}(0) = \begin{cases} -2 & \text{if } i = j \leq \lambda \\ 2 & \text{if } i = j > \lambda \\ 0 & \text{else} \end{cases}.$$

This gives us that there exists a subspace of TM_p with dimension λ such that the Hessian is negative definite and there exists a subspace V of TM_p with dimension $n-\lambda$ such that the Hessian is positive definite. If there exists a subspace of TM_p with dimension larger then λ such that the Hessian is negative definite then this space intersects with V, wich is impossible. This gives us that the index of p is given as λ. This proves the lemma. □

Remark 1. Milnor [1] makes the remark that this Lemma shows us that all the critical points of a Morse function are isolated. This is very confinient as it would be hard to use Morse theory as we did in the example of the torus if the critical points are not isolated.

2.4 The 1-parameter group of diffeomorphisms

Another concept which is important to the prove of the theorems related to Morse theory is the 1-parameter group of diffeomorphisms. In this section we will be discussing the definition of this concept and give a lemma related to this concept. We will first give the definition of this 1-parameter group of diffeomorphism.

Definition 2.3 (1-parameter group of diffeomorphisms)**.** Let M be a manifold and

$$\varphi : \mathbb{R} \times M \to M$$

be a C^∞ map such that

(1) for each $t \in \mathbb{R}$ the map $\varphi_t : M \to M$ given by $\varphi_t(x) = \varphi(t, x)$ is a diffeomorphism.

(2) for all $t, s \in \mathbb{R}$ we have that $\varphi_t \circ \varphi_s = \varphi_{t+s}$,

then φ is a 1-parameter group of diffeomorphism of M.

We can use a 1-parameter group of diffeomorphism of M to define a vector field X on M. This vector field is defined as follows. Let $f : M \to \mathbb{R}$ be a smooth function, then

$$X_q(f) = \lim_{h \to 0} \frac{f(\varphi_h(q)) - f(q)}{h}.$$

We say that the vector field X generates φ. We will be using the following Lemma about the 1-parameter group of diffeomorphism in the proofs of the theorems important to Morse theory. We will not give a proof to this lemma, but a proof can be found in *Morse theory* by Milnor [1, p 10].

Lemma 3. Let X be a smooth vector field on M that vanishes outside of a compact subspace $K \subset M$. Then X generates a unique 1-parameter group of diffeomorphism of M.

2.5 Proof of Morse theory

In this section we will be discussing the theorems fundamental to Morse theory. We will first prove two theorems that show how the cell structure of M^a changes as a increases. With these two theorems we can prove how the cell structure is determined by the critical points of a Morse function.

Let $f : M \to \mathbb{R}$ be a Morse function. As in the example of the torus, we will be denoting $M^a = \{x \in M : f(x) \leq a\}$. In the first theorem we will prove that if $f^{-1}[a, b]$ doens't contain critical points, M^a is diffeomorphic to M^b. The proof of this theorem is taken from *Morse theory* by Milnor [1].

Theorem 1. Let $f : M \to \mathbb{R}$ be smooth. Let $a < b$, such that the set $f^{-1}[a, b]$ is compact and contains no critical points of f. Then M^a is diffeomorphic to M^b. We also have that M^a a deformation retract is of M^b, so that the inclusion map $M^a \to M^b$ is a homotopy equivalence.

Proof. First we have to equip the manifold M with a Riemannian metric. This metric determines an inner product on two tangent vectors of M. The gradient of f, denoted as ∇f, is a vector field characterized with the identity

$$\langle \nabla f, v\rangle = D_v f,$$

where v is a tangent vector of M.

We define the following smooth function $p : M \to \mathbb{R}$ as equal to $\frac{1}{\langle \nabla f, \nabla f\rangle}$, through the compact set $f^{-1}[a, b]$ and vanishes outside of a compact neighborhood of $f^{-1}[a, b]$. Let this neighborhood be U, the we have that in $U \setminus f^{-1}[a, b]$, p looks like a smooth function from the values of p in $f^{-1}[a, b]$ to 0. We define the following vector field

$$X_q = p(q)(\nabla f)_q.$$

We notice that this smooth vector field vanishes outside of the compact subspace $f^{-1}[a, b]$, so lemma 3 gives us that X a unique 1-parameter group of diffeomorphism of M generates. We denote this 1-parameter group of diffeomorphism as

$$\varphi_t : M \to M.$$

We fix $q \in M$ and consider the function $t \mapsto f(\varphi_t(q))$. We notice that if $\varphi_t(q) \in f^{-1}(M)$, then

$$df(\varphi_t(q)) = \langle d\varphi_t(q), \nabla f\rangle = \langle X_q, \nabla f\rangle.$$

Substituting the definitions of X an p, we get that

$$\begin{aligned}\langle X_q, \nabla f\rangle &= \langle p(q)(\nabla f)_q, \nabla f\rangle \\ &= \langle \frac{1}{\langle \nabla f, \nabla f\rangle}\nabla f, \nabla f\rangle \\ &= \frac{\langle \nabla f, \nabla f\rangle}{\langle \nabla f, \nabla f\rangle} = 1.\end{aligned}$$

This gives us that the function $t \mapsto f(\varphi_t(q))$ is a linear function with derivative 1. So for a chosen $q \in M$ and $t \in \mathbb{R}$, we have that $f(\varphi_t(q)) = t + f(q)$. We know that $\varphi_{b-a} : M \to M$ is a diffeomorphism. Let $q \in f^{-1}(a)$. Then we notice that $f(\varphi_{b-a}(q)) = b - a + f(q) = b$. This gives us that $\varphi_{b-a}(q) \in f^{-1}(b)$. This means that the function $\varphi_{b-a} : M^a \to M^b$ is well defined. This function is injective, as $\varphi_{b-a} : M \to M$ is injective. By the same argumentation, we can notice that $\varphi_{a-b} : M^b \to M^a$ is also a well defined injective function. This function acts as the inverse to $\varphi_{b-a} : M^a \to M^b$. This gives us that this function is a bijection, and thus a diffeomorphism. This proves the first part of the theorem.

To prove the second part, we define for $r \in [0, 1]$ and $q \in M^b$ the the following maps

$$r_t : M^b \to M^b,$$

by

$$r_t(q) = \begin{cases} q & \text{if } f(q) \leq a \\ \varphi_{t(a-f(q))}(q) & \text{if } a \leq f(q) \leq f(b) \end{cases}.$$

We notice that r_0 is the identity, as $\varphi_0 : M \to M$ is the identity map, and r_1 is a retraction, as the image of $\varphi_{a-f(q)}$ is M^a for all $q \in M^b$. Hence M^a is a deformation retract of M^b. □

The next theorem will say something about when $f^{-1}[a, b]$ does contain a critical point. We show that when this is true, the homotopy type does change. The proof of this theorem is taken from *Morse theory* by Milnor [1].

Theorem 2. Let $f : M \to \mathbb{R}$ be a Morse function and let $p \in M$ be a critical point with index λ. Let $f(p) = c$. Suppose $f^{-1}[c - \varepsilon, c + \varepsilon]$ is compact and contains no critical points of f other then p, for some $\varepsilon > 0$. Then for all $\varepsilon > 0$ small enough, the set $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon}$ with a λ cell attached.

Proof. First we see that the lemma of Morse (lemma 1) gives us that there exists a neighborhood U of p with a coordinate system $(u^1, \cdots, u^n)$ such that

$$f = c - (u^1)^2 - \cdots - (u^\lambda)^2 + (u^{\lambda+1})^2 + \cdots + (u^n)^2.$$

It also gives us that $u^i(p) = 0$ for all $1 \leq i \leq n$. Now we choose $\varepsilon > 0$ small enough such that

(1) The set $f^{-1}[c - \varepsilon, c + \varepsilon]$ is compact and contains no other critical points but p.

(2) The image of U under the diffeomorphic imbedding

$$(u^1, \cdots, u^n) : U \to \mathbb{R}^n$$

contains the closed ball $\{(u^1, \cdots, u^n) : \sum_{i=1}^n (u^i)^2 \leq 2\varepsilon\}$

We know that (1) is possible as we have seen that the critical points of a Morse function are isolated.

Milnor uses the image at the next page to illustrate the situation.

We have here that the coordinate lines represent the planes $u^{\lambda+1} = \cdots = u^n = 0$ and $u^1 = \cdots = u^\lambda = 0$. The circle is the ball of radius $\sqrt{2\varepsilon}$. The regions $M^{c-\varepsilon}$, $f^{-1}[c - \varepsilon, c]$ and $f^{-1}[c, c + \varepsilon]$ are heavily shaded, heavily dotted and lightly dotted respectivly. We define $e^\lambda = \{q \in U : (u^1(q))^2 + \cdots + (u^\lambda(q))^2 \leq \varepsilon, u^{\lambda+1}(q) = \cdots = u^n(q) = 0\}$. In the image the the horizontal line through p represents e^λ.

In the image we notice that $M^{c-\varepsilon} \cap e^\lambda$ is the boundary $\dot{e}^\lambda$ such that e^λ gets attached to $M^{c-\varepsilon}$. We notice that e^λ is a λ-cell, so our goal becomes to show that $e^\lambda \cup M^{c-\varepsilon}$ is a deformation retract of $M^{c+\varepsilon}$.

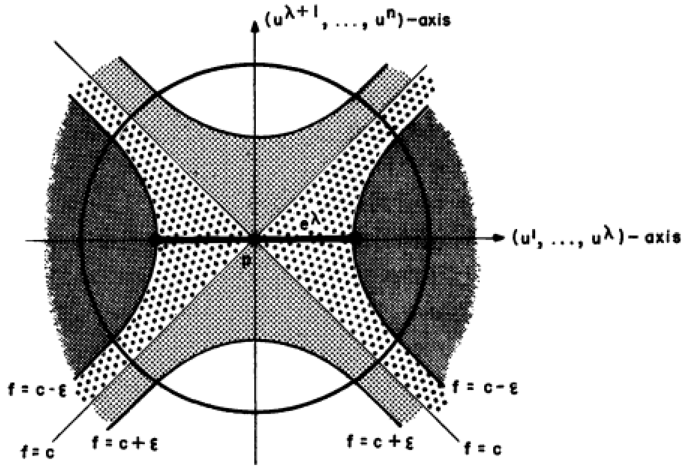

Figure 4: A visual representation used by Milnor to illustrate the situation. This illutration can be found in *Morse theory* by Milnor [1].

To do this we will construct a new function $F : M \to \mathbb{R}$. To do this we first define $\mu : \mathbb{R} \to \mathbb{R}$ as a C^∞ function such that

$$\begin{aligned} &\mu(0) > \varepsilon \\ &\mu(r) = 0 \text{ for all } r \leq 2\varepsilon \\ &-1 < \mu'(r) \leq 0 \text{ for all } r, \end{aligned}$$

where $\mu' = \frac{d\mu}{dr}$. We will asume that such a function exists and define F as

$$F(q) = \begin{cases} f(q) & \text{if } q \notin U \\ f(q) - \mu((u^1(q))^2 + \cdots + (u^\lambda(q))^2 + 2(u^{\lambda+1}(q))^2 + \cdots + 2(u^n(q))^2) & \text{if } q \in U \end{cases}.$$

We notice that this is indeed a smooth function as the composition of two function and the sum of two functions are both smooth. This makes is so F is smooth in U and as F is also smooth outside of U we have that F is smooth.

For the sake of convenience we will define the following functions $\xi, \eta : U \to [0, \infty)$, as

$$\xi = (u^1)^2 + \cdots + (u^\lambda)^2 \tag{3}$$
$$\eta = (u^{\lambda+1} + \cdots + (u^n)^2. \tag{4}$$

This gives us that $f = c - \xi + \eta$ and that

$$F(q) = c - \xi(q) + \eta(q) + \mu(\xi(q) + 2\eta(q)),$$

for all $q \in U$. We will now prove a few of assertions about F.

Assertion 1. The region $F^{-1}(-\infty, c+\varepsilon]$, coincides with the region $M^{c+\varepsilon}$.

proof. Outside of the region $\xi + 2\eta \leq 2\varepsilon$ the functions f and F coincide. Inside the region $\xi + 2\eta \leq 2\varepsilon$ we have that

$$F \leq f = c - \xi + \eta \leq c + \frac{1}{2}\xi + \eta \leq c + \varepsilon.$$

It is true that $F < f$, as $\mu(r) \geq 0$ for all $r \in \mathbb{R}$. This is because for all r, $\mu'(r) \leq 0$ and $\mu(r) = 0$ for all $r \geq 2\varepsilon$, so if there exists a $r \in \mathbb{R}$ such that $\mu(r) < 0$, then we can't have that $\mu'(r) \leq 0$ for all $r \in \mathbb{R}$. □

Assertion 2. The critical points of F are the same as those of f.
proof. We notice that

$$\frac{\partial F}{\partial \xi} = -1 - \mu'(\xi + 2\eta) < 0,$$
$$\frac{\partial F}{\partial \eta} = 1 - 2\mu'(\xi + 2\eta) > 1.$$

The first inequality is true because $\mu'(r) > -1$ for all $r \in \mathbb{R}$ and the second inequality is true because $\mu'(r) \leq 0$ for all $r \in \mathbb{R}$. We also notice that

$$dF = \frac{\partial F}{\partial \xi} d\xi + \frac{\partial F}{\partial \eta} d\eta.$$

In this situation the covectors $d\xi$ and $d\eta$ are only zero at the origin. This gives us that F has no other critical point than the origin. □

Assertion 3. The region $F^{-1}(-\infty, c - \varepsilon]$ is a deformation retract of $M^{c+\varepsilon}$.
proof. Because of assertion 1 and the fact that $F \leq f$, we notice that

$$F^{-1}[c - \varepsilon, c + \varepsilon] \subset f^{-1}[c - \varepsilon, c + \varepsilon].$$

This has as consequence that $F^{-1}[c - \varepsilon, c + \varepsilon]$ is compact. Because of assertion 2 we know that it can contain no critical point except p. But we notice that

$$F(p) = c - \mu(0) < c - \varepsilon.$$

This means that $F^{-1}[c - \varepsilon, c + \varepsilon]$ contains no critical points. Theorem 1 gives us the desired result. □

Before we move on to the last assertion, we want define some things for the sake of convenience. We denote $F^{-1}(-\infty, c + \varepsilon] = M^{c-\varepsilon} \cup H$, where H is the closure of $F^{-1}(-\infty, c + \varepsilon] \setminus M^{c-\varepsilon}$. It is usual that the region $M^{c-\varepsilon} \cup H$ is referred to as $M^{c-\varepsilon}$ with a handle attached. Assertion 3 gives us that $M^{c-\varepsilon} \cup H$ is a deformation retract of $M^{c+\varepsilon}$.

We now denote e^λ as the cell $e^\lambda = \{q \in U : \xi(q) < \varepsilon, \eta(q) = 0\}$. We see that e^λ is contained in H. This is so because $\frac{\partial F}{\partial \xi} < 0$, so for all $q \in e^\lambda$ we have that

$$F(q) \leq F(p) < c - \varepsilon.$$

This gives us that $q \in F^{-1}(\infty, c - \varepsilon]$. But as $f(q) \leq c - \varepsilon$, is is true that $q \notin M^{c-\varepsilon}$. This gives us that $q \in H$.

Assertion 4. $M^{c-\varepsilon}$ is a deformation retract of $M^{c-\varepsilon} \cup H$.
Proof. We denote $r_t : M^{c-\varepsilon} \cup H \to M^{c-\varepsilon} \cup H$ as this deformation retract. We first define r_t as the identity outside of U. Inside of U, r_t will be defined in three cases, as shown in the image.

Case 1. We first look at the region $\xi \leq \varepsilon$. Here r_t corresponds to the transformtation

$$(u^1, \cdots, u^n) \mapsto (u^1, \cdots, u^\lambda, tu^{\lambda+1}, \cdots, tu^n).$$

In this case, we have that r_1 is the identity, and r_0 maps this region to e^λ.

Case 2. Now we look at the region $\varepsilon \leq \xi \leq \eta + \varepsilon$. In this region we let r_t correspond to the transformation

$$(u^1, \cdots, u^n) \mapsto (u^1, \cdots, u^\lambda, s_t u^{\lambda+1}, \cdots, s_t u^n),$$

where s_t is defined as $s_t = t + (1-t)((\xi - \varepsilon)/\eta)^{1/2}$. We notice that r_1 is again the identity and that r_0 the entire region maps to $f^{-1}(c - \varepsilon)$. This is so because $s_0 = ((\xi - \varepsilon)/\eta)^{1/2}$. Let q be in this region. Then we notice that

$$\begin{aligned} f(r_0(q)) &= c - (u^1(q))^2 - \cdots - (u^\lambda(q))^2 + (\xi(q) - \varepsilon)/\eta(q))(u^{\lambda+1}(q))^2 + \cdots + (\xi(q) - \varepsilon)/\eta(q))(u^n(q))^2 \\ &= c - \xi(q) + \frac{\xi(q) + \varepsilon}{\eta(q)} \cdot \eta(q) = c - \varepsilon. \end{aligned}$$

Case 3. Within the region $\eta + \varepsilon \leq \xi$ we denote r_t as the idenity.

We now have defined the deformation retract r_t properly. With this we have proven Assertion 4 and thereby we have proven the theorem. □

Remark 2. In theorem 2 we have assumed that there is one critical point p such that $f(p) = c$. It can happen that we have a manifold with a Morse function f such that there are multiple critical points p such that $f(p) = c$. In this case Milnor [1] makes the following remark. Let $p_1, \cdots, p_k$ be the non-degenerate critical points of f with indices $\lambda_1, \cdots, \lambda_k$ in $f^{-1}(c)$. The we can show in a similar way that $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon} \cup e^{\lambda_1} \cup \cdots \cup e^{\lambda_k}$. We can do this in a similar way as the proof of theorem 2. We follow the same proof, but with a different neighborhood around each critical point and we have to make sure that these neighborhoods don't overlap.

These two theorems give us the foundation of the technique we have used in the example of the torus, but we can't use this yet to say that the cell structure of a manifold is determined by the critical points of the Morse function. To do this, we have to prove one more theorem. Before we will look at this theorem, we will look at a couple of lemma's that will be used in the proof of this theorem. The first lemma is the lemma of Whitehead and we will use a combination of the proof given by Milnor in *Morse theory* [1] and by Whitehead in *On Simply Connected 4-Dimensional Polyhedra*.

Lemma 4 (Whitehead)**.** Let φ_0 and φ_1 be homotopic maps from $\dot{e}^\lambda$ to X. Then the identity map of X extends to a homotopy equivalence

$$k : X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_1} e^\lambda.$$

Proof. We define k as

$$\begin{aligned} &k(x) = x && \text{for } x \in X \\ &k(tu) = 2tu && \text{for } 0 \leq t \leq \frac{1}{2}, u \in \dot{e}^\lambda \\ &k(tu) = \varphi_{2-2t}(u) && \text{for } \frac{1}{2} \leq t \leq 1, u \in \dot{e}^\lambda. \end{aligned}$$

Here we let φ_t denote the homotopy between φ_0 and φ_1. Milnor now assumes that is's trivial to define a corresponding map

$$l : X \cup_{\varphi_1} e^\lambda \to X \cup_{\varphi_0} e^\lambda$$

with similar formula's and that the compositions kl and lk are homotopic to the identity. To find a proof of this claim we look at the proof of lemma 5 from *On Simply Connected 4-Dimensional*

Polyhedra, written by Whitehead [3]. We define l as

$$\begin{aligned} l(x) &= x && \text{for } x \in X \\ l(tu) &= 2tu && \text{for } 0 \le t \le \frac{1}{2}, u \in \dot{e}^\lambda \\ l(tu) &= \varphi_{2t-1}(u) && \text{for } \frac{1}{2} \le t \le 1, u \in \dot{e}^\lambda. \end{aligned}$$

It follows that

$$\begin{aligned} lk(x) &= x && \text{for } x \in X \\ lk(tu) &= 4tu && \text{for } 0 \le t \le \frac{1}{4}, u \in \dot{e}^\lambda \\ lk(tu) &= \varphi_{4t-1} && \text{for } \frac{1}{4} \le t \le \frac{1}{2}, u \in \dot{e}^\lambda \\ lk(tu) &= \varphi_{2-2t} && \text{for } \frac{1}{2} \le t \le 1, u \in \dot{e}^\lambda. \end{aligned}$$

We now define the function $\xi_t : X \cup_{\varphi_0} e^\lambda \to X \cup_{\varphi_0} e^\lambda$ as

$$\begin{aligned} \xi_t(x) &= x && \text{for } x \in X \\ \xi_t(ru) &= (4-3t)ru && \text{for } 0 \le r \le \frac{1}{4-3t}, u \in \dot{e}^\lambda \\ \xi_t(ru) &= \varphi_{(4-3t)r-1}(u) && \text{for } \frac{1}{4-3t} \le r \le \frac{2-t}{4-3t}, u \in \dot{e}^\lambda \\ \xi_t(ru) &= \varphi_{\frac{1}{2}(4-3t)(1-r)}(u) && \text{for } \frac{2-t}{4-3t} \le r \le 1, u \in \dot{e}^\lambda. \end{aligned}$$

Because of the wat we constructed ξ_t, it is obvious that for any $t \in [0,1]$, ξ_t continuous is. It is easy to see that $\xi_1 = 1$ and that $\xi_0 = lk$. This gives us lk homotopic to the identity is. In a very similar way we can define a map $\eta_t : X \cup_{\varphi_1} e^\lambda \to X \cup_{\varphi_1} e^\lambda$ such that $\eta_1 = 1$ and $\eta_0 = kl$. This proves the lemma. □

The next lemma is also important for the proof of this last theorem. For the next lemma we will not give a proof. The proof of this lemma can be found in *Morse theory* by Milnor [1].

Lemma 5. Let $\varphi : \dot{e}^\lambda \to X$ be a attaching map. Any homotopy equivalence $f : X \to Y$ extends to a homotopy equivalence

$$F : X \cup_\varphi e^\lambda \to Y \cup_{f \circ \varphi} e^\lambda.$$

This next theorem is also used in the proof of the last theorem given by Milnor. Milnor uses this theorem without proof and so will we. This theorem is the first theorem from *Combinatorial homotopy I* by Whitehead [4].

Theorem 3. The map $f : X \to Y$ is a homotopy equivalence if, and only if $f_n : \pi_n(x) \to \pi_n(Y)$ is a isomorphism for every n such that $1 \le n \le N+1$, where $N = \max(\Delta X, \Delta Y)$.

In this theorem, ΔX is the minimum dimension of all the cell structures belonging to the manifold X.

With these two lemmas we can now prove the last theorem of this section. This theorem will show us that the cell structure of M is indeed determined by the critical points of the Morse fucntion f. We will follow the proof given by Milnor in *Morse theory* [1].

Theorem 4. If $f : M \to \mathbb{R}$ is a differentiable Morse function and if M^a is compact for all $a \in \mathbb{R}$, then M has the homotopy type of a CW-complex with one cell of dimension λ for each critical point with index λ.

Proof. Let $c_1 < c_2 < \cdots$ be the critical points of the Morse function $f : M \to \mathbb{R}$. Since everty M^a is compact, this sequence has no cluster points. We notice that for $a < c_1$, $M^a = \emptyset$. We will prove this theorem using induction. We notice that for $a < c_1$, M^a has the homotopy type of a CW-complex, as $M^a = \emptyset$, so this proves the base case. We will now assume that M^a has the homotopy type of a CW-complex and that $a \neq c_1, c_2, \cdots$. We let c be the smallest c_i such that $c_i > a$. Then Theorem 1,2 and remarkt 2 give us that $M^{c+\varepsilon}$ has the homotopy type of $M^{c-\varepsilon} \cup_{\varphi_1} e^{\lambda_1} \cup \cdots \cup_{\varphi_j} e^{\lambda_j}$, for certain attaching maps $\varphi_1, \cdots, \varphi_j$, when ε is small enough. Theorem 1 giveus us that there is a homotopy equivalence $h : M^{c-\varepsilon} \to M^a$. We have assumed there $h' : M^a \to K$ where K is a CW-complex. This shows that $h' \circ h \circ \varphi_j$ homotopic is by cellular approximation to a map

$$\psi_j : \dot{e}_j^{\lambda} \to (\lambda_j - 1) \text{ skeleton of } K.$$

This gives us that the cell complex $K \cup_{\psi_1} e^{\lambda_1} \cup \cdots \cup_{\psi_j} e^{\lambda_j}$ is homotopic to $M^{c+\varepsilon}$. This proves the induction step for each $M^{a'}$, and the proof is complete if M is compact. If M is not compact, we can show that M^a is a deformation retract of M is a similar way as the proof of theorem 1.

But this proof only works if there are a finite amount of critical points. If there are a infinite amount of critical points, then we have a sequence of homotopy equivalences $M^{a_i} \to K_i$. We can now look at the union of all the K_i's in the direct limit topology and denote this union as K. If we look at the limit map $g : M \to K$, we notice that this induces an isomorphism of homotopy groups in all dimensions. We can now apply theorem 3 to see that g is indeed a homotopy equivalence. This proves the theorem. □

3 Examples

In this chapter we will discuss a few examples of how to use Morse theory to find the cell structure of a manifold. We will also see that we can use the theorems discussed in the last chapter to prove a version of the Reeb sphere theorem.

3.1 $\mathbb{CP}^n$

In this example we will use Morse theory to determine the cell structure of $\mathbb{CP}^n$. This is as example given by Milnor in *Morse theory* [1].

In this example we will always use the representation of $(z_0 : \cdots : z_n)$ such that $\sum_{i=0}^{n} |z_i|^2 = 1$. We will define the function $f : \mathbb{CP}^n \to \mathbb{R}$ as

$$f(z_0 : z_1 : \cdots : z_n) = \sum_{i=0}^{n} c_i |z_i|^2,$$

where $c_0, c_1, \cdots, c_n$ are distinct real numbers.

We now want to determine the critical points of f. To do this we will look at the following coordinate system. Let U_j be the set of $(z_0 : \cdots : z_n)$ such that $z_j \neq 0$. We now set

$$|z_j| \frac{z_k}{z_j} = x_k + iy_k.$$

We notice that $x_0, y_0, \cdots, x_{j-1}, y_{j-1}, x_{j+1}, y_{j+1}, \cdots, x_n, y_n : U_j \to \mathbb{R}$ are coordinate functions mapping U_j diffeomorphically onto the open unit ball in $\mathbb{R}^n$. We notice that for $k \neq j$,

$$|z_k|^2 = x_k^2 + y_k^2,$$

and that

$$|z_j|^2 = 1 - \sum_{0 \leq k \leq n, k \neq j} (x_k^2 + y_k^2).$$

This gives us that

$$f = c_j + \sum_{0 \leq k \leq n, k \neq j} (c_k - c_j)(x_k^2 + y_k^2),$$

throughout U_j. This gives us that in U_j the only critical point of f is $p_j = (0 : 0 : \cdots : 1 : \cdots : 0)$, where 1 is put on the j'th coordinate. This is true for every $0 \leq j \leq n$, so we notice that f has $n+1$ critical points.

For every critical point in it's respective coordinate system U_j, the Hessian matrix looks like

$$\begin{pmatrix} 2(c_0 - c_j) & 0 & \dots & & \dots & 0 \\ 0 & 2(c_0 - c_j) & \dots & & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ 0 & \dots & 2(c_{j-1} - c_j) & & \dots & 0 \\ 0 & \dots & \dots & 2(c_{j+1} - c_j) & \dots & 0 \\ \vdots & & & & \ddots & \vdots \\ 0 & & \dots & & & 2(c_n - c_j) \end{pmatrix}.$$

We know that $2(c_j - c_k) \neq 0$ for all $0 \leq j, k \leq n$ as $c_1, c_2, \dots c_n$ are all distinct. This gives us that the Hessian is non-singular in every critical point. This means that f is indeed a Morse function.

Without loss of generality, we assume that $c_0 < c_1 < \cdots < c_n$. This gives us, per definition, that the index of p_j is $2j$. Theorem 4 gives us now that $\mathbb{CP}^n$ has the homotopy type of

$$e^0 \cup e^2 \cup \cdots \cup e^{2n}.$$

3.2 $\mathbb{RP}^n$

Inspired by the method used by Milnor to determine the cell structure of $\mathbb{CP}^n$ using Morse theory [1], we can determine the cell structure of $\mathbb{RP}^2$ using Morse theory. In this example we will again always use the representation of $(x_0 : \cdots : x_n) \in \mathbb{RP}^n$ such that $\sum_{i=0}^n x_i^2 = 1$.

To do this, we define a function $f : \mathbb{RP}^n \to \mathbb{R}$ similar to the function we defined for $\mathbb{CP}^n$. We define f as

$$f(x_0 : x_1 : \cdots : x_n) = \sum_{i=0}^n c_i x_i^2,$$

where $c_0, c_1, \cdots, c_n \in \mathbb{R}$ are distinct.

We now need to determine the critical points of f. To do this we look at the set $U_i = \{(x_0 : \cdots : x_n) | x_i \neq 0\}$. We notice that $x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n : U_i \to \mathbb{R}$ are coordinate functions. By the definition of $\mathbb{RP}^n$, we have that $x_i^2 = 1 - \sum_{0 \leq j \leq n, j \neq i} x_j^2$. This gives us that $f(x_0 : \cdots : x_n) = c_i + \sum_{0 \leq j \leq n, j \neq i} (c_j - c_i) x_j^2$ in U_i. This gives us that in U_j the only critical point of f is $p_j = (0 : 0 : \cdots : 1 : \cdots : 0)$, where 1 is put on the j'th coordinate. This gives us that f has $n+1$ critical points. Now we want to show that f is indeed a Morse function. To do this we look at the Hessian matrix for f in each critical point. Let p_i be the critical point of f in U_i, then the Hessian looks like

$$\begin{pmatrix} 2(c_0 - c_i) & 0 & \dots & & & & 0 \\ 0 & 2(c_1 - c_i) & \dots & & & & 0 \\ \vdots & & \ddots & & & & \vdots \\ 0 & & \dots & 2(c_{i-1} - c_i) & \dots & & 0 \\ 0 & & \dots & & 2(c_{i+1} - c_i) & \dots & 0 \\ \vdots & & & & & \ddots & \vdots \\ 0 & & & \dots & & & 2(c_n - c_i) \end{pmatrix}.$$

We notice that $2(c_i - c_j) \neq 0$ for all $1 \leq i, j \leq n$, as $c_1, \cdots, c_n$ are distinct real numbers. This gives us that for all critical points of f, the Hessian is non-singular, so f is indeed a Morse function. If we, without loss of generality, assume that $c_0 < c_1 < \cdots < c_n$, then we see that p_i an index has of i. Theorem 4 gives us now that $\mathbb{RP}^n$ the homotopy type has of the cell structure

$$e^0 \cup e^1 \cup \cdots \cup e^n.$$

3.3 $\mathbb{HP}^n$

We can also use the method of Morse theory to find the cell structure of the quaternionic projective space $\mathbb{HP}^n$. We will do this in a very similar way as Milnor [1] did with $\mathbb{CP}^n$ and we dit with $\mathbb{RP}^n$.

Let $q \in \mathbb{H}$, then $q = a + bi + cj + dk$, with $a, b, c, d \in \mathbb{R}$. The norm on $\mathbb{H}$ is given by $|q|^2 = a^2 + b^2 + c^2 + d^2$. In this example we will always again use the representation of $(q_0 : \cdots : q_n)$ such that $\sum_{i=0}^n |q_i|^2 = 1$.

We will now define a function $f : \mathbb{HP}^n \to \mathbb{R}$ as

$$f(q_0 : \cdots : q_n) = \sum_{i=0}^n c_i |q_i|^2,$$

similar to the case of $\mathbb{CP}^2$ and $\mathbb{RP}^n$. We take a look at the set $U_l = \{(q_0 : \cdots : q_n) : q_i \neq 0\}$. We will now denote

$$|q_l| \frac{q_s}{q_l} = a_s + b_s i + c_s j + d_s k.$$

We notice that the functions $a_s, b_s, c_s, d_s : \mathbb{HP}^n \to \mathbb{R}$ are coordinate functions for all $0 \leq s \leq n$, such that $s \neq l$. We also notice that

$$|q_s|^2 = a_s^2 + b_s^2 + c_s^2 + d_s^2,$$

for $s \neq l$. This means, because of the way $\mathbb{HP}^n$ is defined, that

$$|q_l| = 1 - \sum_{0 \leq r \leq n, r \neq l} (a_r^2 + b_r^2 + c_r^2 + d_r^2).$$

This gives us that in U_l, f looks like

$$f = c_l + \sum_{0 \leq r \leq n, r \neq l} (c_r - c_l)(a_l^2 + b_l^2 + c_l^2 + d_l^2).$$

As a consequence, we see that the only critical point of f in U_l is the origin of U_l. This means that $p_l = (0 : \cdots : 0 : 1 : 0 \cdots : 0)$ is a critical point of f for every position of the 1. Moreover, these are the only critical points of f.

We can now construct the matrix representation of the Hessian of f in U_l in the critical point located in U_l in a similar way as we have done for $\mathbb{CP}^2$ and $\mathbb{RP}^2$. This matrix looks like

$$\begin{pmatrix}
2(c_0 - c_l) & 0 & 0 & 0 & \dots & & & \dots & 0 \\
0 & 2(c_0 - c_l) & 0 & 0 & \dots & & & \dots & 0 \\
0 & 0 & 2(c_0 - c_l) & 0 & \dots & & & \dots & 0 \\
0 & 0 & 0 & 2(c_0 - c_l) & \dots & & & \dots & 0 \\
\vdots & & & & \ddots & & & & \vdots \\
0 & \dots & & & & 2(c_{l-1} - c_l) & & \dots & 0 \\
0 & \dots & & & & \dots & 2(c_{l+1} - c_l) & \dots & 0 \\
\vdots & & & & & & & \ddots & \vdots \\
0 & \dots & & & & \dots & & & 2(c_n - c_l)
\end{pmatrix}.$$

We again see that because $c_1, \cdots, c_n$ are seperate $c_i - c_l \neq 0$. This gives us that the Hessian is non-singular in every critical point, so f is indeed a Morse function.

Without loss of generatity, we can assume that $c_0 < c_1 < \cdots < c_n$. This gives us that p_l has a index of $4l$. Theorem 4 gives us now that $\mathbb{HP}^n$ has the homotopy type of the cell structure

$$e^0 \cup e^4 \cup \cdots \cup e^{4n}.$$

3.4 Reeb Sphere Theorem

In this section we will prove the Reeb sphere theorem as stated in *"Morse theory"* by Milnor [1] and we will be following the proof given by Milnor while giving some more details.

Theorem 5 (Reeb sphere theorem)**.** If M is a compact manifold and $f : M \to \mathbb{R}$ is a Morse function with two critical points, then M is homeomorphic to a sphere.

Proof. The two critical points must be the maximum and minimum of f. We assume that $f(p) = 0$ is the minimun and $f(q) = 1$ is the maximum. We notice that if we take $\varepsilon > 0$ small enough, then $M^\varepsilon = f^{-1}[0, \varepsilon]$ and $f^{-1}[1 - \varepsilon, 1]$ are homeomorphic to n-cells by the theorems discussed in paragraph 2.5. Theorem 1 gives us that M^ε homeomorphic is to $M^{1-\varepsilon}$. This means that M is the union of 2 n-cells matched along their common boundary.

Milnor now assumes it to be easy to contruct a homeomorphism between M and S^n. In this proof we will more explicitly show how this homeomorphism looks like. We will denote $M = D_1 \cup_\varphi D_2$, where $D_1 \cong M^{1-\varepsilon}$, $D_2 \cong f^{-1}[1-\varepsilon, 1]$ and φ is the attaching map between D_1 and D_2. We notice that $S^n = D_1 \cup_{\mathrm{Id}} D_2$. We can now construct the homeomorphism

$$g : S^n \to M$$

as

$$g(x) \begin{cases} x & \text{if } x \in D_1 \\ ||x||\varphi\left(\frac{x}{||x||}\right) & \text{if } x \in D_2 \end{cases}.$$

This is a variation of a trick known as Alexander's trick, named after J. W. Alexander. □

Remark 3. One might believe that it the situation of theorem 5, M might be diffeomorphic to a sphere. This does not have to be true. A differentiable manifold homeomorphic but not diffeomorphic to a n-sphere is called a exotic sphere. One such sphere is discussed in *On manifolds homeomorphic to the 7-sphere* by Milnor [2].

4 The Morse inequalities

When Morse was developing Morse Theory, his original goal was to prove what we now call the Morse inequalities. These inequalities give a relationship between the critical points of a manifold and the Betti numbers of this manifold. In a later section we will go in to more detail about what Betti numbers are and how they are defined.

We will now formulate these inequalities in the forms of theorems. We start with the weak Morse inequalities.

Theorem 6 (Weak Morse inequalities)**.** If $h : M \to \mathbb{R}$ is a Morse function, we let M_p be the amount of critical point of index p. We define B_p to be the pth Betti number. We then have that

$$M_p \geq B_p, \tag{5}$$

for all $p \in \{0, 1, \cdots, \dim M\}$.

The strong Morse inequalities are formulated in the following theorem.

Theorem 7 (Strong Morse inequalities)**.** If $h : M \to \mathbb{R}$ is a Morse function, we let M_p be the amount of critical point of index p. We define B_p to be the pth Betti number. We then have that

$$M_p - M_{p-1} + M_{p-2} \pm \cdots + (-1)^p M_0 \geq B_p - B_{p-1} + B_{p-2} \pm \cdots + (-1)^p B_0. \tag{6}$$

for all $p \in \{0, 1, \cdots, \dim M\}$. For $p = \dim M$ the inequality becomes a equality.

The proofs to these inequalities will be discussed later. The usual way of proving these inequalities is by using combination of the definition of Betti numbers formulated by the homology group of M and the Morse theory discussed earlier. Edward Witten developed in 1982 a different way of proving these inequalities that does not involve the homology group in his paper *Supersymmetry and Morse Theory* [5]. His proof of the Morse inequalities is what we will be discusssing further.

4.1 Betti numbers and cohomology

In this section we will be discussing the definition of the Betti numbers as Witten defined them in his paper *Supersymmetry and Morse Theory* [5]. Witten's method of proving the Morse inequalities also involves some Hodge-theory, which we will also discuss in this section.

The Betti numbers are a sequence of numbers that are usually defined in terms of the homology group of a manifold M. The intuition behind the Betti numbers are that the kth Betti number describes the amount of k-dimensional "holes" in the manifold M. So for instance the Betti numbers of a torus are $b_0 = 1, b_1 = 2, b_2 = 1$. Notice that if $n > \dim M$, then $b_n = 0$. We will be using a different definition of the Betti numbers, a definition that is used in Witten's paper [5, p. 665]. We recall that we have a exterior derivatie $d : \Omega^k(M) \to \Omega^{k+1}(M)$. The definition goes as followes.

Definition 4.1 (Betti numbers)**.** The pth Betti number B_p is the amount of linear independent p-forms ψ such that $d\psi = 0$ and such that there exist no $(p-1)$-form χ such that $d\chi = \psi$.

The first thing we notice is that this definition is equivalent to the dimension of the kth DeRham cohomology space of M. This space is defined as the quotient vector space

$$H^k(M) = Z^k / B^k, \tag{7}$$

where $Z^k = \{\omega \in \Omega^k(M) : d\omega = 0\}$ and $B^k = \{\omega \in \Omega^k(M) : \exists \eta \in \Omega^{k-1}(M), \omega = d\eta\}$.

We will now develop a way of calculating the kth Betti number using Hodge theory. This result will prove useful later. What we want to do is define a Laplacian operator $\Delta : \Omega^k(M) \to \Omega^k(M)$. For this, we need a adjoint operator d^* such that, for $v \in \Omega^k(M)$ and $u \in \Omega^{k+1}(M)$,

$$\langle du, v\rangle = \langle u, d^* v\rangle.$$

To define this adjoint operator, we first need to define a inner product on $\Omega^k(M)$. These are defined in *Extra materiaal: Hodge - De Rham Theory* [7] as followes.

For every $p \in M$, we have a inner product g_p op T_pM. This inner product induces g_p on $\bigwedge^k T_p^* M$. As a consequence, we can define the define for two k-forms $u, v \in \Omega^k(M)$ the smooth function $g(u, v) : M \to \mathbb{R}$ as $g(u, v)(p) = g_p(u_p, v_p)$. Then we define the bilinear form on $\Omega^k(M)$ as

$$\langle u, v\rangle = \int_M g(u, v)\mu_M.$$

Lemma 2.1 from *Extra materiaal: Hodge - De Rham Theory* [7] gives us that this bilinear form is indeed a inner product on $\Omega^k(M)$. We can also rewrite this inner product using the Hodge star operator

$$* : \Omega^k(M) \to \Omega^{n-k},$$

where $n = \dim M$. Then the inner product for $u, v \in \Omega^k$ looks like

$$\langle u, v\rangle = \int_M u \wedge *v.$$

With this notation, we can see in lemma 3.1 from *Extra materiaal: Hodge - De Rham Theory* [7], that

$$d^* = (-1)^{(k-1)n+1} * d * .$$

With this adjoint operator d^*, we can simply define the the Laplacian $\Delta : \Omega^k(M) \to \Omega^k(M)$ as

$$\Delta = dd^* + d^* d.$$

A important result from Hodge theory is a theorem known as the Hodge - De Rham theorem. We will give this theorem without proof. The proof can be found in *Extra materiaal: Hodge-De Rham theorem* by Erik van den Ban [7, Theorem 2.3].

Theorem 8 (Hodge-De Rham)**.** Let M be a compact oriented Riemannian manifold. The space $\mathcal{H}^k$ of harmonic k-forms is finite dimensional. Moreover, the inclusion map $\mathcal{H}^k \to \ker d$ induces a linear isomorphism

$$\mathcal{H}^k \xrightarrow{\simeq} H^k(M).$$

The space $\mathcal{H}^k$ is defined as $\ker \Delta|_{\Omega^k(M)}$. This theorem gives us that

$$\ker \Delta|_{\Omega^k(M)} \cong H^k(M).$$

This means that we can calculate the kth Betti number as $\dim \ker \Delta|_{\Omega^k(M)}$.

4.2 Witten's conjugation

In this section we will be discussing Witten's conjugation. This conjugation is fundamental to the proof of Witten.

Let M be an manifold, let $h : M \to \mathbb{R}$ be a smooth Morse function and let t be a real number, then Witten defines the following conjugation

$$d_t = e^{-ht} d e^{ht}, d_t^* = e^{ht} d^* e^{-ht}. \tag{8}$$

As we are multiplying with a smooth function, we have that $d_t : \Omega^k(M) \to \Omega^{k+1}(M)$. We also see that d_t^* is indeed the adjoint of d_t, as

$$\langle d_t v, u \rangle = \langle e^{-ht} d e^{ht} v, u \rangle = \langle d e^{ht} v, e^{-th} u \rangle = \langle v, e^{th} d^* e^{-th} u \rangle.$$

If we define $H_t^k(M) = Z_t^k / B_t^k$, with $Z^k = \{\omega \in \Omega^k(M) : d_t \omega = 0\}$ and $B^k = \{\omega \in \Omega^k(M) : \exists \eta \in \Omega^{k-1}(M), \omega = d_t \eta\}$, we can use the same arguments using Hodge theory as the $t = 0$ case discussed earlier, to see that

$$\ker \Delta_t|_{\Omega^k(M)} \cong H_t^k(M),$$

where $\Delta_t = d_t d_t^* + d_t^* d_t$.

The following lemma is important to the idea of Witten. In his paper he says it's trivial to prove this lemma, but we can find a more formal proof to this theorem in *Witten Deformation and Morse theory* by Antunes [6]. We will discuss a outline of the proof given by Antunes

Lemma 6. For any $0 \leq k \leq n$,

$$\dim H_t^k(M) = \dim H^k(M).$$

Idea of the proof. The idea is to prove that $H_t^k(M)$ and $H^k(M)$ are isomorphic. This will be a consequence of the fact that $d_t e^{-ht} = e^{-ht} d$ as we can prove that the linear map $\Phi : \Omega^k(M) \to \Omega^k(M)$ given by

$$\alpha \mapsto e^{-th} \alpha,$$

induces a linear map $H^k(M) \to H_t^k(M)$. In the same way we can prove that the linear map $\Psi : \Omega^k(M) \to \Omega^k(M)$ given by

$$\alpha \mapsto \alpha e^{th},$$

induces a linear map $H_t^k(M) \to H^k(M)$ which will be the inverse of the map induced by Φ. This gives us that $H_t^k(M)$ and $H^k(M)$ are isomorphic and proves the lemma. $\square$

As a consequence, we have that

$$\dim \ker \Delta_t|_{\Omega^k(M)} = \dim H_t^k(M) = \dim H^k(M). \tag{9}$$

This means that we can calculate the kth Betti number by looking at the kernel of Δ_t acting on $\Omega^k(M)$ for any t. Witten's idea is now to look at $\dim \ker \Delta_t|_{\Omega^t(k)}$ as $t \to \infty$ and to find the inequalities there.

4.3 The Morse inequalities

In this section we will disuss how we can use lemma 6 to prove the Morse inequalities. The rest of the proof of the Morse inequalities hinges on one more Lemma. We will use the lemma as it is formulated in *Witten Deformation and Morse theory* by Antunes [6].

Lemma 7. For any $c > 0$, there is a $T_0 > 0$ such that for every $t \geq T_0$ the number of eigenvalues of $\Delta_t|_{\Omega^i(M)}$, in $[0, c]$ equals to m_i, for $0 \leq i \leq n$.

The proof of this lemma still requires a lot of calculations that we will not be discussing furtur. A formal proof to this lemma can be found in *Witten Deformation and Morse theory* written by Antunes [6].

Before we can use this lemma to prove the Mores inequalities, we need to define a couple of things. For any $1 \leq i \leq n$ we will denote $F_{t,i}^{[0,c]} \subset \Omega(M)$ as the vector space generated by the eigenspaces of $\Delta_t|_{\Omega^i(M)}$. We will assume that t is large enough that lemma 7 will give us that $\dim F_{t,i}^{[0,c]} = m_i$.

What Antunes also notices in *Witten deformation and Morse theory* [6] is that

$$\begin{aligned} d_t\Delta_t &= d_t(d_t d_t^* + d_t^* d_t) \\ &= d_t^2 d_t^* + d_t d_t^* d_t \\ &= d_t d_t^* d_t = \Delta_t d_t. \end{aligned}$$

This means that im $d_t|_{F_{tf,i}^{[0,c]}} \subset F_{tf,i+1}^{[0,c]}$. This means that we have a finite-dimensional subcomplex of $(\Omega^\bullet(M), d_t)$, given by

$$0 \to F_{t,0}^{[0,c]} \xrightarrow{d_t} F_{t,1}^{[0,c]} \xrightarrow{d_t} \cdots \xrightarrow{d_t} F_{t,n}^{[0,c]} \xrightarrow{d_t} 0.$$

In the same way that we have done for d and d_t, we can look at the cohomology space

$$Z_t^k / B_t^k,$$

with $Z_t^k = \{\omega \in F_{t,i}^{[0,c]} : d_t\omega = 0\}$ and $B_t^k = \{\omega \in F_{t,i}^{[0,c]} : \exists \eta \in F_{t,i-1}^{[0,c]}, \omega = d_t\eta\}$. In the same way we have done this earlier, we can show using Hodge theory that the dimension of this vecor space equals the dimension of the vector space $\ker \Delta_t|_{F_{t,i}^{[0,c]}}$.

Before we prove the Morse inequalities there is one more thing we can show. By the way we have defined $F_{t,i}^{[0,c]}$, we see that $\ker \Delta_t|_{\Omega^i(M)} \subset F_{t,i}^{[0,c]}$. This gives us that $\ker \Delta_t|_{\Omega^i(M)} \subset \ker \Delta_t|F_{t,i}^{[0,c]}$. As $F_{t,i}^{[0,c]} \subset \Omega^i(M)$ we also know that $\ker \Delta_t|_{F_{t,i}^{[0,c]}} \subset \ker \Delta_t|_{\Omega^i(M)}$. This has as consequence that $\ker \Delta_t|_{\Omega^i(M)} = \ker \Delta_t|_{F_{t,i}^{[0,c]}}$.

We can now use this to prove the Morse inequalities.

Proof of the weak Morse inequalities (Theorem 1). We notice that

$$\beta_i = \dim \ker \Delta_t|_{\Omega^i(M)} = \dim \ker \Delta_t|_{F_{tf,i}^{[0,c]}} \leq \dim F_{tf,i}^{[0,c]} = m_i.$$

□

Proof of the strong Morse inequalities (Theorem 2) If we use some finite dimensional linear algebra, we see that

$$\begin{aligned} m_i = \dim F_{tf,i}^{[0,c]} = \dim \ker F_{tf,i}^{[0,c]} + \dim \mathrm{im}\ F_{tf,i}^{[0,c]} = \dim \frac{\ker F_{tf,i}^{[0,c]}}{\ker F_{tf,i-1}^{[0,c]}} + \dim \mathrm{im}\ F_{tf,i-1}^{[0,c]} + \dim \mathrm{im}\ F_{tf,i}^{[0,c]} \\ = \beta_i + \dim \mathrm{im}\ F_{tf,i-1}^{[0,c]} + \dim \mathrm{im}\ F_{tf,i}^{[0,c]}. \end{aligned}$$

Now if we add all the m_i with alternating signs, we get that

$$\sum_{j=0}^{i} (-1)^j m_{i-j} = \sum_{j=0}^{i} (-1)^j \beta_{j-i} + \dim \mathrm{im}\ F_{tf,i}^{[0,c]}.$$

This gives us that for $1 \leq i \leq n$,

$$\sum_{j=0}^{i}(-1)^j m_{i-j} \leq \sum_{j=0}^{i}(-1)^j \beta_{j-i}.$$

Because $\dim \operatorname{im} F_{tf,n}^{[0,c]} = 0$, we have in particular that

$$\sum_{j=0}^{n}(-1)^j m_{i-j} = \sum_{j=0}^{n}(-1)^j \beta_{j-i}.$$

This proves the strong Morse inequalities. □

References

[1] J. Milnor, *Morse theory*, Princeton University Press, Princeton, 1963.

[2] J. Milnor, *On manifolds homeomorphic to the 7-sphere*, Annals of Mathematics, Vol. 64 (1956), pp, 399-405.

[3] J. H. C. Whitehead, *On Simply connected, 4-dimensional polyhedra*, Magdalen College, Oxford, 1948.

[4] J. H. C. Whitehead, *Combinatorial homotopy I*, Magdalen College, Oxford, 1949

[5] E. Witten, Supersymmetry and Morse Theory, *J. Differential Geometry*, (661-692), Princeton University, 1982.

[6] J. A. Antunes, *Witten deformation and Morse theory*, New College of Florida, 2016.

[7] E. van den Ban & E. Looijenga, *Extra materiaal: Hodge - de Rham theorem*, University Utrecht, 2008.