Utrecht University

Faculty of Science

Systems of forms in C_i fields

A discussion of S. Lang's and M. Nagata's results

Bachelor Thesis

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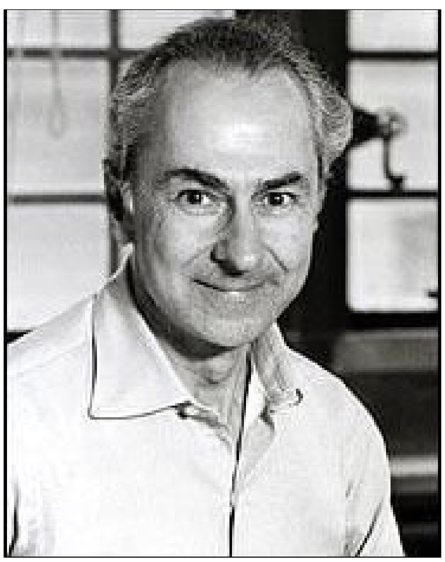
Mathematics



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Abstract

The notion of algebraically closed fields can be generalised by C_i fields. These are fields with the following property. Let f be a homogenous polynomial with coefficients in a field F and $i \in \mathbb{N}$ an integer. Then f has a non trivial zero in F if the number of variables of f is greater than the i-th power of the degree of f. A field is C_0 if and only if it is algebraically closed. The definition of a C_i field can thus be seen as a measurement of how close a field is to being algebraically closed. One can generalise the notion of C_i fields to strongly C_i fields by considering polynomials without constant term instead of homogenous polynomials. In this thesis the algebraic properties of (strongly) C_i fields are investigated. First, it is proven that a field is C_0 if and only if it is algebraically closed. Then it is shown that finite fields are strongly C_1. We investigate under what circumstances the notion of C_i fields can be extended to non trivial common zeros of systems of homogenous polynomials. These results are used to show that algebraic extensions of C_i fields are also C_i. An important open question about C_i fields is whether C_i fields are also strongly C_i. We show that this is the case for C_0 fields and provide a condition under which a C_i field is strongly C_{i+1}.

Contents

1 Introduction

There exist many homogenous polynomials (forms) with coefficients in $\mathbb{R}$ that only have the trivial zero $(0,\ldots,0) \in \mathbb{R}^n$, regardless of the number of variables. Consider for example

$$X_1^2 + X_2^2 \ldots + X_n^2. \tag{1.1}$$

In contrast to this, there exist fields F with the following property. Let $f \in F[X_1,\ldots,X_n]$ be a form of degree d and $i \in \mathbb{N}$ an integer. Then f has a non trivial zero in F if $n > d^i$. Such fields are called C_i fields. A field is C_0 if and only if it is algebraically closed. The definition of a C_i field can thus be seen as a measurement of how close a field is to being algebraically closed. In this thesis the algebraic properties of C_i fields are investigated. Searching for solutions of polynomials is one of the most fundamental questions in algebra. The material in this thesis guarantees the existence of solutions of a wide range of polynomials.

In chapter 2 C_i fields are defined. One can extend the notion of C_i fields to strongly C_i fields by considering polynomials without constant term instead of forms. In chapter 3 it is proven that a field is (strongly) C_0 if and only if it is algebraically closed. Chapter 4 provides a proof of Chevalley's Theorem. This theorem implies that finite fields are strongly C_1. In Chapters 5 and 6 we investigate under what circumstances the notion of a C_i field can be extended to systems of forms. A proof is provided for the following result. Let $f_1,\ldots,f_r$ be forms in n common variables, of degree d and with coefficients in a C_i field F. Then $f_1,\ldots,f_r$ have a non trivial common zero in F if $n > rd^i$. A similar statement can be made if $f_1,\ldots,f_r$ are of different degree. In that case F is required to admit normic forms[1] of order i of any degree. In chapter 7 it is shown that algebraic field extensions of (strongly) C_i fields are also (strongly) C_i. An important open question about C_i fields is whether C_i fields are also strongly C_i. In Chapter 7 a condition is provided under which a C_i field is strongly C_{i+1}.

The content of this thesis is based on a paper by Serge Lang [1] and a paper by Masayoshi Nagata [2]. Chapters 2, 3, 5 and 7 are based on Lang's paper. The proofs from Nagata's paper are used in Chapter 6. Chapter 4 is based on chapter 25 of volume two of the book "Algebra" by Falko Lorenz [3, p. 120]. The results presented by Lang and Nagata are extended in several ways. The substitution ξ is introduced to provide one general method for many of the proofs. Second, results from Lang and Nagata are combined into more general statements. Examples of this are that Theorem 5.2.2 is true for polynomials without constant term of degree at most d or that Corollary 7.1.2 and Theorem 7.2.1 are true for general C_i fields. Third, it is proven that a field F admits normic forms of order i of any degree if F admits normic forms of order i of degree p for every prime number p. We assume that the reader has some knowledge about groups, rings and fields. The course "Rings and Galois theory" as well as a small part of the course "Group theory" are used as prerequisites.

Finally I would like to thank my supervisor M. Pieropan. I am especially grateful that she was willing to supervise me under the special circumstances caused by a concussion. I felt extremely supported throughout this entire process. Her patience, excellent advice and the freedom she gave, made the writing of this thesis very enjoyable.

[1] Normic forms are defined in section 5.1.

2 Defining (strongly) C_i fields

In this chapter the main objects of this thesis are introduced, namely (strongly) C_i fields. We first recall what a homogeneous polynomial is and use it in the definition of a C_0 field. After that the definition of a C_i field is given. Then we look at strongly C_i fields and we finish by discussing some basic properties of (strongly) C_i fields.

2.1 Definition of C_i fields

In order to understand the definition of C_i fields given by Lang, one needs to know what a form is. Recall that the degree of a term of a polynomial is the sum of the powers of each variable. For example, the polynomial $X_1^5 + X_2^3 X_3^6 \in \mathbb{R}[X_1, X_2, X_3]$ has one term of degree five and one term of degree nine. Let us now define a form.

Definition 2.1.1 (form)**.** Let R be a ring. A non constant polynomial $f \in R[X_1, \ldots, X_n]$ is called a form in R if every term has the same degree.

An example of a form in $\mathbb{R}[X_1, X_2, X_3]$ of degree seven is $X_1^7 + X_1^3 X_2^4 + 2X_1 X_2 X_3^5 + 3X_2^6 X_3$. Constant polynomials are excluded from the definition of forms to avoid confusion. One could view a constant polynomial as a polynomial where every term has degree zero. The definition of C_i fields uses the notion of non trivial zeros of polynomials.

Definition 2.1.2 (non trivial zero)**.** Consider a polynomial $f \in R[X_1, \ldots, X_n]$, where R is some ring. A non trivial zero of f in R is given by $(a_1, \ldots, a_n) \in R^n$, where $f(a_1, \ldots, a_n) = 0$ and at least one a_i is not equal to zero for some $i \in \{1, \ldots, n\}$.

If it is not specified in what ring the (non) trivial zero lies, it is implied that its coordinates are elements of the ring that also contains the coefficients of the polynomial. Let us now continue to the definition of a C_0 field.

Definition 2.1.3 (C_0)**.** A field F is called C_0 if every form in F in n variables and of degree d, with $n = d$, has a non trivial zero in F. It is required that $n, d \in \mathbb{N}$ and $n, d > 1$.

The definition above is slightly different than the definition presented by Lang. The condition $n, d > 1$ was added. In the case where $n = d = 1$, a form looks like $aX_1 \in F[X_1]$. Such a form only has the trivial zero, so that case is excluded from the definition. We now make some comments about notation.

Notation 2.1.4. Throughout this thesis the number of variables of a polynomial is denoted by n and the degree of a polynomial by d, thus implying that $n, d \in \mathbb{N}$. When considering a system of polynomials $f_1, \ldots, f_r$, their numbers of variables will be denoted by $n_1, \ldots, n_r$ and their degrees by $d_1, \ldots, d_r$ respectively. If $n_1 = \ldots = n_r$ or $d_1 = \ldots = d_r$, the notation n or d is used instead. The name of the polynomial is sometimes used in the notation of the number of variables and degree. For example, the number of variables a polynomial ϕ will be written as n_ϕ and the degree as d_ϕ. A field F is assumed to be unequal to $\{0\}$.

Theorem 3.2.1 states that a field is C_0 if and only if it is algebraically closed. Considering this, one can determine how close a field is to being algebraically closed by looking at how close it is to being C_0. One could ask how many variables are needed relative to the degree of a form in order to ensure the existence of a non trivial zero. This property is captured in the following definition.

Definition 2.1.5 (C_i)**.** Let $i \in \mathbb{N}$ be some integer. A field F is called C_i if every form in F in n variables and of degree d, with $n > d^i$, has a non trivial zero in F.

In section 3.3 it is shown that using $i = 0$ in this definition, is equivalent to our definition of C_0 fields. The case where $n = 1$ is automatically excluded from the definition of C_i fields, since that would require the degree to be zero and $i \neq 0$. A form is never a constant polynomial, thus there are no forms of degree zero.

2.2 Definition of strongly C_i fields

There are several ways to generalise the notion of a C_i field. One could study common zeros of a system of forms. In chapter 5 and chapter 6 conditions are provided that guarantee the existence of non trivial common zeros of systems of forms in C_i fields. Another way to make a stronger statement is to consider polynomials without constant term instead of forms. We will define strongly C_i fields in a similar way to C_i fields, but using polynomials without constant term instead of forms. From now on a polynomial without constant term with coefficients in a ring R is called a polynomial without constant term in R. Note that we also consider the zero polynomial to be a polynomial without constant term.

Definition 2.2.1 (strongly C_0)**.** A field F is called strongly C_0 if every polynomial without constant term in F in n variables and of degree d, with $n = d$, has a non trivial zero in F. It is required that $n, d \in \mathbb{N}$ and $n, d > 1$.

In his paper, Lang does not define strongly C_0 and strongly C_i fields separately. We do this to highlight the similarities between C_i and strongly C_i fields. Towards the end of section 3.3, it is shown that defining strongly C_0 fields in this way is equivalent to using $i = 0$ in the definition below.

Definition 2.2.2 (strongly C_i)**.** Let $i \in \mathbb{N}$ be some integer. A field F is called strongly C_i if every polynomial without constant term in F in n variables of degree d, with $n > d^i$, has a non trivial zero in F.

Since polynomials without constant term can be the zero polynomial, there is a case where $d = 0$. In that case any number of variables greater than zero satisfies the inequality $n > d^i$. This is not a problem, since the zero polynomial has a non trivial zero in every field.

2.3 Basic properties of (strongly) C_i fields

In this section some basic properties of (strongly) C_i fields are discussed. From the definition of (strongly) C_i fields, it follows that a strongly C_i field is also C_i. This is captured in the following lemma.

Lemma 2.3.1. *Let F be a strongly C_i field. Then F is C_i.*

Proof. A form is a special case of a polynomial without constant term. From this it follows that being strongly C_i guarantees the existence of non trivial zeros of the appropriate forms. □

Whether or not the converse is true is still an open problem in mathematics. Theorem 3.3.1 states that C_0 fields are also strongly C_0 fields. One of the final results of this thesis is a sufficient condition under which a C_i field is strongly C_{i+1}. This is presented in Theorem 7.2.1.

Consider a C_i field F and $j \in \mathbb{N}$ such that $j > i$. Then F is also C_j. The same is true in the case of strongly C_i fields. This is proven in the lemma below. The regular and strong case are covered at the same time. One needs to replace C_i by strongly C_i and form by polynomial without constant term.

Lemma 2.3.2. *Let $i, j \in \mathbb{N}$ be integers such that $j > i$. Let F be a C_i field. Then F is C_j.*

Proof. First consider the case where $i = 0$. Let f be a form in F in n variables and of degree d, with $n > d^j$. Consider the polynomial $f(x_1, \ldots, x_d, x_1, \ldots, x_1)$. This is a form in F in d variables and of degree d. Since F is C_0 it follows that there exist $a_1, \ldots, a_d \in F$ not all equal to zero, such that $f(a_1, \ldots, a_d, a_1, \ldots, a_1) = 0$. Thus $(a_1, \ldots, a_d, a_1, \ldots, a_1)$ is a non trivial zero of f in F. Because of this, F is C_j. Now consider the case where $i > 0$. Let f be a form in F in n variables and of degree d, with $n > d^j$. Since $j > i$, it follows that $n > d^j > d^i$. Thus f has a non trivial zero in F, since F is C_i. □

3 (Strongly) C_0 fields

In this chapter (strongly) C_0 fields are discussed. It turns out that being algebraically closed is equivalent to being C_0. In order to prove this, the result stated in Theorem one in Lang's paper is used. This theorem is proven in section 3.1. After that, we show that the C_0 fields are precisely the algebraically closed fields. This is done in section 3.2. The fact that C_0 fields are algebraically closed is used to show that C_0 fields are strongly C_0 fields. This is discussed in section 3.3, alongside a comment on our definition of (strongly) C_i fields.

3.1 Field norms

In this section we prove Theorem one from Lang's paper: "if a field F admits one extension E of degree n then there exists in F a form of degree n, in n variables and having only the trivial zero" [1, p. 374]. In the proof, the notion of the field norm is used. Field norms are introduced below. Using the field norm one can define a form in F of degree n in n variables with only the trivial zero in F, thus satisfying the criteria of Theorem one from Lang's paper (Theorem 3.1.4). First some notation is introduced.

Notation 3.1.1. Let F be a field and E a finite field extension of degree $n \in \mathbb{N}_{>0}$. Let $\omega_1, \ldots, \omega_n$ be a basis of E considered as F-vector space. Then there exist $a_{i,j,k} \in F$ for all $1 \leq i, j, k \leq n$ such that

$$\omega_i \omega_j = a_{i,j,1}\omega_1 + a_{i,j,2}\omega_2 + \ldots + a_{i,j,n}\omega_n.$$

An element $x = x_1\omega_1 + \ldots + x_n\omega_n \in E$ can also be viewed as the vector $(x_1, \ldots, x_n)^T \in F^n$. We will denote the vector $(a_{i,j,1}, a_{i,j,2}, \ldots, a_{i,j,n})^T \in F^n$ by $\vec{a_{i,j}}$. Let $x \in E$, the map $m_x : E \to E$ is defined as $m_x(y) = xy$.

It follows that m_x is a F-linear map. Given $x, y_1, y_2 \in E$ and $\lambda \in F$, then $m_x(\lambda y_1 + y_2) = x(\lambda y_1 + y_2) = \lambda x y_1 + x y_2 = \lambda m_x(y_1) + m_x(y_2)$. We are now ready to define the field norm.

Definition 3.1.2 (field norm)**.** Using the same notation as above. The field norm $N_{E/F}(x)$ of an element $x \in E$ is defined as the determinant of the matrix corresponding to the F-linear map m_x.

Since the determinant of a matrix corresponding to m_x is not dependent on the choice of basis, the field norm is well-defined. We now give an example of the calculation of the field norm.

Example 3.1.3. Consider the field extension $\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}$ of degree 3. The field $\mathbb{Q}(\sqrt[3]{2})$ can be seen as a $\mathbb{Q}$-linear vector space with $1, \sqrt[3]{2}, \sqrt[3]{2}^2$ as basis. Given $x \in \mathbb{Q}(\sqrt[3]{2})$, then $x = x_1 + x_2\sqrt[3]{2} + x_3\sqrt[3]{2}^2$, where $x_1, x_2, x_3 \in \mathbb{Q}$. Let us compute the matrix associated with the map m_x. In order to do this we consider the image of the basis vectors under the map m_x. First we get

$$m_x(1) = x = x_1 + x_2\sqrt[3]{2} + x_3\sqrt[3]{2}^2.$$

Secondly we obtain

$$m_x(\sqrt[3]{2}) = x\sqrt[3]{2} = x_1\sqrt[3]{2} + x_2\sqrt[3]{2}^2 + x_3\sqrt[3]{2}^3 = 2x_3 + x_1\sqrt[3]{2} + x_2\sqrt[3]{2}^2.$$

Finally we have

$$m_x(\sqrt[3]{2}^2) = x\sqrt[3]{2}^2 = x_1\sqrt[3]{2}^2 + x_2\sqrt[3]{2}^3 + x_3\sqrt[3]{2}^4 = 2x_2 + 2x_3\sqrt[3]{2} + x_1\sqrt[3]{2}^2.$$

Using this, one can find the matrix M_x corresponding to the map m_x under the basis of choice.

$$M_x = \begin{pmatrix} \vec{m_x(1)} & \vec{m_x(\sqrt[3]{2})} & \vec{m_x(\sqrt[3]{2}^2)} \end{pmatrix} = \begin{pmatrix} x_1 & 2x_3 & 2x_2 \\ x_2 & x_1 & 2x_3 \\ x_3 & x_2 & x_1 \end{pmatrix}$$

The only thing left to do is to calculate the determinant of M_x.

$$\begin{aligned} N_{\mathbb{Q}(\sqrt[3]{2})/\mathbb{Q}}(x) = \det(M_x) &= \begin{vmatrix} x_1 & 2x_3 & 2x_2 \\ x_2 & x_1 & 2x_3 \\ x_3 & x_2 & x_1 \end{vmatrix} \\ &= x_1 \begin{vmatrix} x_1 & 2x_3 \\ x_2 & x_1 \end{vmatrix} - x_2 \begin{vmatrix} 2x_3 & 2x_2 \\ x_2 & x_1 \end{vmatrix} + x_3 \begin{vmatrix} 2x_3 & 2x_2 \\ x_1 & 2x_3 \end{vmatrix} \\ &= x_1(x_1^2 - 2x_3x_2) - x_2(2x_3x_1 - 2x_2^2) + x_3(4x_3^2 - 2x_2x_1) \\ &= x_1^3 - 2x_1x_3x_2 - 2x_2x_3x_1 + 2x_2^3 + 4x_3^3 - 2x_3x_2x_1 \\ &= x_1^3 + 2x_2^3 + 4x_3^3 - 6x_1x_2x_3 \end{aligned} \tag{3.1}$$

From (3.1) it follows that the field norm of an element $x = x_1 + x_2\sqrt[3]{2} + x_3\sqrt[3]{2}^2 \in \mathbb{Q}(\sqrt[3]{2})$ is given by $x_1^3 + 2x_2^3 + 4x_3^3 - 6x_1x_2x_3$. Note that this polynomial is indeed a form in $\mathbb{Q}$ of degree three in three variables. △

Let us now prove Theorem one from Lang's paper, which is stated below. We added the condition that the degree of the extension should be bigger than one.

Theorem 3.1.4. *If a field F admits one extension E of degree $n > 1$, then there exists in F a form of degree n, in n variables and having only the trivial zero in F.*

Proof. The strategy of this proof is to show that the polynomial f, defined as the field norm of the general element $x = x_1\omega_1 + \ldots + x_n\omega_n \in E$, is a form in F of degree n, in n variables and with only the trivial zero in F. We first calculate the matrix M_x corresponding to the map m_x under our choice of basis. Consider the image of the basis vector ω_i under the map m_x.

$$\begin{aligned}
m_x(\omega_i) &= x\omega_i \\
&= (x_1\omega_1 + \ldots + x_n\omega_n)\omega_i \\
&= x_1\omega_1\omega_i + \ldots + x_n\omega_n\omega_i \\
&= x_1(a_{1,i,1}\omega_1 + \ldots + a_{1,i,n}\omega_n) + \ldots + x_n(a_{n,i,1}\omega_1 + \ldots + a_{n,i,n}\omega_n) \\
&= (x_1a_{1,i,1} + x_2a_{2,i,1} + \ldots + x_na_{n,i,1})\omega_1 + \ldots + (x_1a_{1,i,n} + x_2a_{2,i,n} + \ldots + x_na_{n,i,n})\omega_n
\end{aligned} \tag{3.2}$$

Using (3.2), one can compute the matrix M_x, corresponding to the map m_x under our choice of basis.

$$\begin{aligned}
M_x &= \begin{pmatrix} \vec{m_x(\omega_1)} & \vec{m_x(\omega_2)} & \ldots & \vec{m_x(\omega_n)} \end{pmatrix} \\
&= \begin{pmatrix} x_1a_{1,1,1} + x_2a_{2,1,1} + \ldots + x_na_{n,1,1} & \ldots & x_1a_{1,n,1} + x_2a_{2,n,1} + \ldots + x_na_{n,n,1} \\ \vdots & \vdots & \vdots \\ x_1a_{1,1,n} + x_2a_{2,1,n} + \ldots + x_na_{n,1,n} & \ldots & x_1a_{1,n,n} + x_2a_{2,n,n} + \ldots + x_na_{n,n,n} \end{pmatrix} \\
&= \begin{pmatrix} x_1\vec{a_{1,1}} + x_2\vec{a_{2,1}} + \ldots + x_n\vec{a_{n,1}} & \ldots & x_1\vec{a_{1,n}} + x_2\vec{a_{2,n}} + \ldots + x_n\vec{a_{n,n}} \end{pmatrix}
\end{aligned}$$

Let us now calculate the determinant of M_x. This is done by using the fact that the determinant of a matrix is linear in each column.

$$\begin{aligned}
\det(M_x) &= \begin{vmatrix} x_1\vec{a_{1,1}} + x_2\vec{a_{2,1}} + \ldots + x_n\vec{a_{n,1}} & \ldots & x_1\vec{a_{1,n}} + x_2\vec{a_{2,n}} + \ldots + x_n\vec{a_{n,n}} \end{vmatrix} \\
&= x_1 \begin{vmatrix} \vec{a_{1,1}} & x_1\vec{a_{1,2}} + x_2\vec{a_{2,2}} + \ldots + x_n\vec{a_{n,2}} & \ldots & x_1\vec{a_{1,n}} + x_2\vec{a_{2,n}} + \ldots + x_n\vec{a_{n,n}} \end{vmatrix} \\
&+ x_2 \begin{vmatrix} \vec{a_{2,1}} & x_1\vec{a_{1,2}} + x_2\vec{a_{2,2}} + \ldots + x_n\vec{a_{n,2}} & \ldots & x_1\vec{a_{1,n}} + x_2\vec{a_{2,n}} + \ldots + x_n\vec{a_{n,n}} \end{vmatrix} \\
&+ \ldots \\
&+ x_n \begin{vmatrix} \vec{a_{n,1}} & x_1\vec{a_{1,2}} + x_2\vec{a_{2,2}} + \ldots + x_n\vec{a_{n,2}} & \ldots & x_1\vec{a_{1,n}} + x_2\vec{a_{2,n}} + \ldots + x_n\vec{a_{n,n}} \end{vmatrix} \\
&= \sum_{k_1,\ldots,k_n \in \{1,\ldots,n\}} x_{k_1}x_{k_2}\cdots x_{k_n} \begin{vmatrix} \vec{a_{k_1,1}} & \vec{a_{k_2,2}} & \ldots & \vec{a_{k_n,n}} \end{vmatrix}
\end{aligned} \tag{3.3}$$

The coefficients of the polynomial in (3.3) are determinants of matrices with only elements of F as entries, since all $a_{i,j,k} \in F$. Thus all these coefficients are elements of F. Assuming that these coefficients are not all equal to zero, it follows that $f \in F[x_1, \ldots, x_n]$, given by $f(x_1, \ldots, x_n) = N_{E/F}(x_1\omega_1 + \ldots + x_n\omega_n)$, is indeed a form in F in n variables of degree n. At the end of this proof we show that it is not possible for all these coefficients to be zero. First it is shown that f only has the trivial zero. From (3.3) it follows that $\det(M_0) = 0$, thus $f(0, \ldots, 0) = N_{E/F}(0) = 0$. Now let $x \in E$ with $x \neq 0$. Since E is a field, there exists x^{-1} such that $xx^{-1} = x^{-1}x = 1$. The F-linear map $m_{x^{-1}}$ is the inverse of m_x. Given $a \in E$, then $m_x(m_{x^{-1}}(a)) = m_x(x^{-1}a) = xx^{-1}a = a$ and similarly $m_{x^{-1}}(m_x(a)) = a$. Thus the matrix $M_{x^{-1}}$ is the inverse of the matrix M_x. The determinant of a matrix is 0 if and only if it is not invertible. Because of this $\det(M_x) \neq 0$ if $x \neq 0$. This is the same as saying that $f(x_1, \ldots, x_n) \neq 0$ if $x_i \neq 0$ for some $1 \leq i \leq n$, since $x = 0$ if and only if $x_1, \ldots, x_n = 0$. From this it also follows that not all coefficients of the polynomial given in (3.3) are zero, otherwise $f(x_1, \ldots, x_n) = 0$ for all $x_i \in F$. □

3.2 C_0 fields are algebraically closed

In this subsection it is shown that a field is C_0 if and only if it is algebraically closed. First it is proven that algebraically closed implies C_0. Second Theorem 3.1.4 is used to show that C_0 implies algebraically closed. This result is stated in a theorem below.

Theorem 3.2.1. *A field is C_0 if and only if it is algebraically closed.*

Proof. Let F be an algebraically closed field. Given a form $f \in F[X_1, \ldots, X_n]$ of degree n in n variables, with $n > 1$. We need to show that f has a non trivial zero in F. Let $a \in F$ be unequal to zero. Such an $a \in F$ exists, since $F \neq \{0\}$. F is algebraically closed, thus the polynomial $f(X_1, a, \ldots, a) \in F[X_1]$ has a zero in F, which we call X_1^*. Since $a \neq 0$ we know that $(X_1^*, a, .., a)$ is a non trivial zero of f in F. Now, let F be a C_0 field. Given a non-constant polynomial $f \in F[X_1]$. Assume that f has no zeros in F. Then there exists a finite extension E/F, such that E contains a zero of f. Let n be the degree of the extension E. Since E contains a zero of f and F does not, it follows that $E \neq F$. Because of this $n > 1$. Theorem 3.1.4 implies that there exists a form in F of degree n, in n variables and with only the trivial zero in F. This is a contradiction with the fact that F is C_0. Our assumption that f has no zeros in F is wrong and F is algebraically closed. □

3.3 C_0 fields are strongly C_0

In this subsection strongly C_0 fields are investigated. First it is shown that being C_0 is equivalent to being strongly C_0. Second our definition of (strongly) C_0 fields is discussed. As explained in section 2.2, Lang does not define strongly C_0 and strongly C_i fields separately. We chose to format Definition 2.2.1 in this way to highlight the similarities between C_i and strongly C_i fields. In this subsection it is proven that our definition of strongly C_i fields is equivalent to Lang's definition.

Theorem 3.3.1. *A field is strongly C_0 if and only if it is C_0.*

Proof. Let F be a strongly C_0 field. From Lemma 2.3.1 it follows that F is C_0. Now let F be a C_0 field. Theorem 3.2.1 implies that F is algebraically closed. We now present an argument that is similar to our proof that algebraically closed implies C_0. Let f be a polynomial without constant term in F in n variables of degree n, where $n > 1$. Consider $f(X_1, a, \ldots, a) \in F[X_1]$, where $a \neq 0$. This polynomial has a zero X_1^* in F, since F is algebraically closed. Thus $(X_1^*, a, \ldots, a)$ is a non trivial zero of f in F and it follows that F is strongly C_0. □

Recall that it is still an open problem whether or not C_i fields are also strongly C_i. As proven above, this statement is true if $i = 0$. Next our definition of strongly C_i fields is discussed. Lang uses a different definition in his paper. He does not use a separate definition for strongly C_0 fields and general strongly C_i fields. Instead Lang uses $i = 0$ in his general definition. Below the definition of strongly C_0 fields given by Lang is presented. After that it is shown that it is equivalent to Definition 2.2.1.

Definition 3.3.2 (strongly C_0 (Lang))**.** A field F is called strongly C_0 if every polynomial without constant term in F in n variables, where $n > 1$, has a non trivial zero in F.

Lang's choice of definition is slightly stronger than Definition 2.2.1. Definition 2.2.1 requires that the polynomial without constant term has degree $d = n$. From this it follows that Lang's definition implies Definition 2.2.1. Now let F be a strongly C_0 field under Definition 2.2.1. We use an argument that is similar to our proof that an algebraically closed field is strongly C_0. F is also a C_0 field and thus algebraically closed. Let f be a polynomial without constant term in F in n variables, with $n > 1$. Given some $a \in F$ such that $a \neq 0$. Such an a exists, since $F \neq \{0\}$. Then $f(X_1, a, \ldots, a) \in F[X_1]$ has a zero in F, since F is algebraically closed. Because of this, f has a non trivial zero in F, thus F is also strongly C_0 under Lang's definition. It follows that Definition 2.2.1 and Definition 3.3.2 are equivalent. In almost the same way, the definition of C_0 fields is equivalent to using $i = 0$ in the definition of C_i fields. The only difference with the proof given for strongly C_i fields is that one has to consider forms instead of polynomials without constant term.

4 Finite fields

The main result of this chapter is that finite fields are strongly C_1. In the first section it is shown that a finite field is not algebraically closed. In order to do this, it is proven that $X^q - X = 0$ for all $X \in F$. Here F denotes a finite field with q elements. This proof uses Lagrange's Theorem from group theory. In section 4.2 everything is prepared for the proof in the final section. Section 4.3 contains a proof of Chevalley's Theorem. This theorem implies that a finite field is strongly C_1. The material in this chapter is not from Lang's or Nagata's paper. Sections 4.2 and 4.3 are heavily based on chapter 25 of volume two of the book "Algebra" by Falko Lorenz [3, p. 120].

4.1 Finite fields are not (strongly) C_0

In this section it is shown that $X^q - X = 0$ for all $X \in F$, where F is a finite field with q elements. This is then used to show that a finite field is not algebraically closed and thus not (strongly) C_0. In the remainder of this chapter F denotes a finite field with q elements. The material in this section is covered in most courses on group theory. This section uses Lagrange's Theorem, which is stated below. This theorem can be found in the book "Groups and Symmetry" by Mark Armstrong [4, Theorem 11.1].

Theorem 4.1.1 (Lagrange)**.** *The order of a subgroup of a finite group is always a divisor of the order of the group.*

Let us now move on to the main result from this section. First consider $X \in F$ such that $X \neq 0$.

Theorem 4.1.2. *Let F be a finite field with q elements. Then $X^{q-1} = 1$ for all $X \in F \setminus \{0\}$.*

Proof. The units of a ring form a group with respect to the multiplication. Because of this, $F^* = F \setminus \{0\}$ is a finite group with respect to the multiplication of F. Consider the subgroup generated by some $X \in F^*$, which we denote by $\langle X \rangle$. From the fact that F^* is a finite group, it follows that $\langle X \rangle$ has a finite number of elements. Thus there exist $l, m \in \mathbb{N}$, such that $l \neq m$ and $X^m = X^l$, otherwise $\langle X \rangle$ would have an infinite number of elements. Assume without loss of generality that $l < m$. By multiplying the left and the right hand side by X^{-l}, we obtain the following

$$X^m X^{-l} = X^l X^{-l}$$
$$X^{m-l} = 1.$$

This shows that there exists a number $m - l \in \mathbb{N}_{>0}$, such that $X^{m-l} = 1$. Call the smallest of such numbers k. Then the subgroup $\langle X \rangle$ has elements $\{1, X, \ldots, X^{k-1}\}$. It follows that $\langle X \rangle$ has order k. By applying Theorem 4.1.1 it follows that k divides the order of F^*, which is $q - 1$. This implies the following for some $r \in \mathbb{N}_{>0}$

$$X^{q-1} = X^{kr} = (X^k)^r = 1^r = 1. \qquad \square$$

From Theorem 4.1.2 one can obtain a more general result, where $X = 0$ is included.

Corollary 4.1.3. *Let F be a finite field with q elements. Then $X^q - X = 0$ for all $X \in F$.*

Proof. Assume that $X = 0$. Then clearly $X^q - X = 0$. Now assume that $X \neq 0$. Theorem 4.1.2 implies the following

$$X^q - X = X(X^{q-1} - 1) = X \cdot 0 = 0. \qquad \square$$

Corollary 4.1.3 can be used to show that a finite field F is not (strongly) C_0.

Corollary 4.1.4. *Let F be a finite field. Then F is not C_0.*

Proof. Assume that F has q elements and consider the polynomial $X^q - X + 1$. Corollary 4.1.3 implies that $X^q - X + 1$ has no zero in F, since $1 \neq 0$. It follows that F is not algebraically closed. Theorem 3.2.1 and Theorem 3.3.1 tell us that being algebraically closed and being (strongly) C_0 are equivalent. Thus F is not (strongly) C_0. $\square$

4.2 Preparation for proof of Chevalley's Theorem

This section lays the groundwork for the proof of Chevalley's Theorem. First some notation is introduced and after that some preliminary results are proven.

Let $\mathfrak{a}$ be the ideal $(X_1^q - X_1, X_2^q - X_2, \ldots, X_n^q - X_n) \subseteq F[X_1, \ldots, X_n]$. Note that all polynomials in $\mathfrak{a}$ vanish on F^n. This follows from Corollary 4.1.3. For the proof of Chevalley's Theorem, the notion of a reduced polynomial is needed.

Definition 4.2.1 (reduced polynomial)**.** A polynomial $f \in F[X_1, \ldots, X_n]$ is called reduced if each variable appears in it only with exponents less than q.

Now some properties of reduced polynomials are discussed. Quite a lot of lemmas are covered, but most build on each other and by using them, the proof of Chevalley's Theorem will be fairly short. The two lemmas stated below might seem trivial, but when considering finite fields they are not. For example, the polynomials in $\mathfrak{a}$ vanish on all points of F^n, but are not necessarily equal to zero. First the case of one variable is covered and then induction is used to extend it to multiple variables.

Lemma 4.2.2. *Let $f \in F[X_1]$ be a reduced polynomial such that f vanishes on all points of F. Then the coefficients of f are all zero, thus $f = 0$.*

Proof. From the fact that f is reduced, it follows that f can be written in the following manner

$$f(X_1) = \sum_{i=0}^{q-1} a_i X_1^i.$$

Here $a_i \in F$ for all $i \in \{0, \ldots, q-1\}$. Since F is a field, one can consider F as a vector space. Note that $f(x) = 0$ for all $x \in F$. This gives us the following system of q F-linear equations.

$$\begin{aligned} a_0 + x_0 a_1 + \ldots + x_0^{q-1} a_{q-1} &= 0 \\ &\vdots \\ a_0 + x_{q-1} a_1 + \ldots + x_{q-1}^{q-1} a_{q-1} &= 0 \end{aligned} \tag{4.1}$$

Here $x_0, \ldots, x_{q-1}$ are all the elements of F. We want to solve this system for $a_0, \ldots, a_{q-1} \in F$. One can prove that (4.1) is F-linearly independent by showing that the determinant of the following matrix is non-zero.

$$M = \begin{pmatrix} 1 & x_0 & \ldots & x_0^{q-1} \\ 1 & x_1 & \ldots & x_1^{q-1} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_{q-1} & \ldots & x_{q-1}^{q-1} \end{pmatrix}$$

The matrix M is a VanderMonde matrix [5, Theorem 6.5.1]. It follows that

$$\det(M) = \prod_{k<j} (x_j - x_k).$$

Here k, j are elements of $\{0, \ldots, q-1\}$. Note that $x_j - x_k \neq 0$ if $j \neq k$. A field has no zero divisors, thus $\det(M) \neq 0$. It follows that (4.1) contains q F-linearly independent equations in q unknowns. Because of this (4.1) has exactly one solution. This is the solution where $a_i = 0$ for all $i \in \{0, \ldots, q-1\}$. From this it follows that $f = 0$. □

One can extend Lemma 4.2.2 to the case of n variables by using induction.

Lemma 4.2.3. *Let $f \in F[X_1, \ldots, X_n]$ be a reduced polynomial such that f vanishes on all points of F^n. It follows that $f = 0$.*

Proof. The strategy of this proof is induction on the number of variables of f. Consider the case where $n = 1$. This case is dealt with by Lemma 4.2.2. Assume that the lemma is true up to and including $n-1$ variables. From the fact that f is reduced, it follows that f can be written in the following manner

$$f(X_1, \ldots, X_n) = \sum_{i=0}^{q-1} f_i(X_1, \ldots, X_{n-1}) X_n^i.$$

Here $f_i \in F[X_1, \ldots, X_{n-1}]$ are also reduced. Consider the polynomial $f(y_1, \ldots, y_{n-1}, X_n) \in F[X_n]$ for any $y_1, \ldots, y_{n-1} \in F$. Then $f(y_1, \ldots, y_{n-1}, X_n)$ vanishes on every point of F. Lemma 4.2.2 implies that $f_i(y_1, \ldots, y_{n-1}) = 0$ for all $i \in \{0, \ldots, q-1\}$. It follows that $f_i(X_1, \ldots, X_{n-1})$ are reduced polynomials that vanish at all points of F^{n-1}. From our induction hypothesis it follows that $f_i(X_1, \ldots, X_{n-1}) = 0$ for all $i \in \{0, \ldots, q-1\}$, thus $f = 0$. □

Lemma 4.2.3 implies that two reduced polynomials that take on the same values in each point of F^n are the same. An example of this is the lemma below.

Lemma 4.2.4. *Let $f \in F[X_1, \ldots, X_n]$ be a reduced polynomial such that f vanishes at every point of $F^n \setminus \{0\}$ and $f(0) = 1$. It follows that*

$$f = (1 - X_1^{q-1})(1 - X_2^{q-1}) \cdots (1 - X_n^{q-1}).$$

Proof. Let $f \in F[X_1, \ldots, X_n]$ be a reduced polynomial such that f vanishes at every point of $F^n \setminus \{0\}$ and $f(0) = 1$. Now let $h \in F[X_1, \ldots, X_n]$ be the following polynomial

$$h = (1 - X_1^{q-1})(1 - X_2^{q-1}) \cdots (1 - X_n^{q-1}).$$

Theorem 4.1.2 tells us that $X^{q-1} = 1$ for all $X \in F \setminus \{0\}$. From this it follows that h is a reduced polynomial that vanishes at every point of $F^n \setminus \{0\}$ and $h(0) = 1$. Now consider the polynomial $g = f - h$. Then g is a reduced polynomial that vanishes on all points of F^n. Lemma 4.2.3 implies that $g = 0$. From this it follows that $f = h$. □

The lemmas below prove that for each polynomial $f \in F[X_1, \ldots, X_n]$, there exists a reduced polynomial f^* that takes on the same values as f in each point of F^n. Lemma 4.2.7 shows that this polynomial f^* is actually unique. In order to prove this the result from the lemma stated below is needed.

Lemma 4.2.5. *Let f be a polynomial in $F[X_1, \ldots, X_n]$ that is not reduced. Let f' be a polynomial in $F[X_1, \ldots, X_n]$ that is equal to f except for the fact that in one term of f one factor X_j^q is replaced by X_j for some $j \in \{1, \ldots, n\}$. Then $f - f' \in \mathfrak{a}$.*

Proof. From the fact that f is not reduced it follows that some term of f contains a factor X_j^q. The polynomials f and f' only differ in the term where the factor X_j^q is replaced by X_j. When subtracting one from the other, all terms that are not changed disappear against its counterpart in the other polynomial. It follows that $f - f'$ is of the following form, where $a_{k_1,\dots,k_n} \in F$ and $k_1, \dots, k_n \in \mathbb{N}$ correspond to the term where X_j^q is replaced by X_j.

$$\begin{aligned} f - f' &= a_{k_1,\dots,k_n} X_1^{k_1} \cdots X_j^{k_j} \cdots X_n^{k_n} - a_{k_1,\dots,k_n} X_1^{k_1} \cdots X_j^{k_j+1-q} \cdots X_n^{k_n} \\ &= a_{k_1,\dots,k_n} X_1^{k_1} \cdots X_j^{k_j-q} \cdots X_n^{k_n} (X_j^q - X_j) \in \mathfrak{a} \end{aligned}$$ □

The result stated below follows from Lemma 4.2.5.

Lemma 4.2.6. *Let $f \in F[X_1, \dots, X_n]$ be a polynomial. Then there exists a reduced polynomial $f^* \in F[X_1, \dots, X_n]$ such that $f \equiv f^* \mod \mathfrak{a}$ and $d_{f^*} \leq d_f$.*

Proof. If f is a reduced polynomial we simply say $f = f^*$. Now assume that f is not reduced. The polynomial f^* is constructed in the following manner. In every term of f, replace each factor X_j^q by X_j as many times as possible for all $j \in \{1, \dots, n\}$. This is a finite procedure, since f has a finite number of terms and each term has finite degree. The result is a reduced polynomial with coefficients in F of degree less than or equal to d_f. In order to finish this proof we need to show that $f \equiv f^* \mod \mathfrak{a}$. Consider $m+1$ polynomials $f_0, \dots, f_m$ constructed in the following way. We start with $f_0 = f$. After that f_{t+1} follows from f_t by changing one factor X_j^q to X_j in one term of f_t, for some $j \in \{1, \dots, n\}$. This is done as many times as possible, thus $f_m = f^*$. Let us introduce the polynomials $g_t = f_t - f_{t+1}$. From Lemma 4.2.5 it follows that $g_t \in \mathfrak{a}$ for all $t \in \{0, \dots, m-1\}$. Now consider $f - f^*$.

$$\begin{aligned} f - f^* &= f_0 - f_m \\ &= f_0 - f_1 + f_1 - f_2 + \dots + f_{m-2} - f_{m-1} + f_{m-1} - f_m \\ &= g_0 + g_1 + \dots + g_{m-2} + g_{m-1} \in \mathfrak{a} \end{aligned} \tag{4.2}$$

From (4.2) it follows that $f - f^* \in \mathfrak{a}$ and thus $f \equiv f^* \mod \mathfrak{a}$. □

Lemma 4.2.3 implies that the reduced polynomial generated in Lemma 4.2.6 is unique. This result is stated in a lemma below. It also follows that the ideal $\mathfrak{a}$ is equal to the subset of $F[X_1, \dots, X_n]$ that consists of polynomials that vanish on F^n. These results are not needed for the proof of Chevalley's Theorem. They are included because of their mathematical beauty.

Lemma 4.2.7. *Let $f \in F[X_1, \dots, X_n]$ be a polynomial. Then there exists a unique reduced polynomial $f^* \in F[X_1, \dots, X_n]$ such that $f \equiv f^* \mod \mathfrak{a}$. Furthermore $d_{f^*} \leq d_f$. Finally it follows that if f vanishes on F^n, then $f \in \mathfrak{a}$.*

Proof. Let $f^* \in F[X_1, \dots, X_n]$ be a reduced polynomial such that $f \equiv f^* \mod \mathfrak{a}$ and $d_{f^*} \leq d_f$. Such a polynomial exists according to Lemma 4.2.6. Let $f_* \in F[X_1, \dots, X_n]$ be a reduced polynomial such that $f \equiv f_* \mod \mathfrak{a}$ as well. Then there exist $g_1, g_2 \in \mathfrak{a}$ such that $f = f^* + g_1$ and $f = f_* + g_2$. Let us now consider $f^* - f_*$.

$$f^* - f_* = f - g_1 - (f - g_2) = g_2 - g_1 \in \mathfrak{a} \tag{4.3}$$

From (4.3) it follows that $f^* - f_* \in \mathfrak{a}$ and thus $f^* - f_*$ vanishes on all point of F^n. Note that $f^* - f_*$ is a reduced polynomial. Lemma 4.2.3 implies that $f^* - f_* = 0$ and thus $f^* = f_*$. Now assume that f vanishes on F^n. The polynomial $g_1 \in \mathfrak{a}$ vanishes on F^n. Thus $f^* = f - g_1$ vanishes on F^n. From Lemma 4.2.3 it follows that $f^* = 0$ and thus $f = g_1 \in \mathfrak{a}$. □

4.3 Finite fields are strongly C_1

In this section it is shown that a finite field is strongly C_1. This is done by proving Chevalley's Theorem. This proof of Chevalley's Theorem builds upon the lemmas from section 4.2. The fact that a finite field is strongly C_1 is a corollary of Chevalley's Theorem. Let us now state the theorem below.

Theorem 4.3.1 (Chevalley). *Let F be a finite field and let $f_1, \dots, f_r \in F[X_1, \dots, X_n]$ be polynomials without constant term. Suppose that the sum of the degrees of $f_1, \dots, f_r$ is less than n. Then $f_1, \dots, f_r$ have a non trivial common zero in F.*

Proof. The strategy of this proof is to argue by contradiction. We assume that $f_1, \dots, f_r$ do not have a common non trivial zero in F. Then the lemmas from section 4.2 are used to construct a polynomial f such that $d_{f^*} > d_f$. This is a contradiction. Consider the following polynomial

$$f = (1 - f_1^{q-1})(1 - f_2^{q-1}) \cdots (1 - f_r^{q-1}).$$

The degree of f is computed below. Here d_j denotes the degree of f_j for some $j \in \{1, \dots, r\}$. From the fact that the sum of the degrees of $f_1, \dots, f_r$ is less than n it follows that

$$d_f = (q-1) \sum_{j=1}^{r} d_j < (q-1)n.$$

The polynomials $f_1, \dots, f_r$ are polynomials without constant term. Because of this $f_1, \dots, f_r$ all vanish at zero, thus

$$f(0, \dots, 0) = (1 - 0^{q-1})(1 - 0^{q-1}) \cdots (1 - 0^{q-1}) = 1.$$

Assume that $f_1, \dots, f_r$ do not have a non trivial common zero in F. Given some $(y_1, \dots, y_n) \in F^n$ where $(y_1, \dots, y_n) \neq 0$. Then there is a f_j such that $f_j(y_1, \dots, y_n) \neq 0$ for some $j \in \{1, \dots, r\}$. Theorem 4.1.2 implies that

$$\begin{aligned} f(y_1, \dots, y_n) &= (1 - f_1(y_1, \dots, y_n)^{q-1}) \cdots (1 - f_j(y_1, \dots, y_n)^{q-1}) \cdots (1 - f_r(y_1, \dots, y_n)^{q-1}) \\ &= (1 - f_1(y_1, \dots, y_n)^{q-1}) \cdots 0 \cdots (1 - f_r(y_1, \dots, y_n)^{q-1}) = 0. \end{aligned}$$

Thus f vanishes on $F^n \setminus \{0\}$ and $f(0) = 1$. From Lemma 4.2.6 it follows that there exists a reduced polynomial $f^* \in F[X_1, \dots, X_n]$ such that $f \equiv f^* \mod \mathfrak{a}$ and $d_{f^*} \leq d_f$. This ensures the existence of a polynomial $g \in \mathfrak{a}$ such that $f^* = f - g$. The polynomial g lies in $\mathfrak{a}$ and thus vanishes on F^n. From this it follows that f^* takes on the same values as f. Thus f^* is a reduced polynomial that vanishes on $F^n \setminus \{0\}$ and $f^*(0) = 1$. Lemma 4.2.4 implies that

$$f^* = (1 - X_1^{q-1})(1 - X_2^{q-1}) \cdots (1 - X_n^{q-1}).$$

It follows that $d_{f^*} = (q-1)n > d_f$. This is a contradiction. □

Chevalley's Theorem implies that a finite field is strongly C_1, as shown below.

Corollary 4.3.2. *A finite field is strongly C_1.*

Proof. Let f be a polynomial without constant term in F such that $n > d$. By applying Theorem 4.3.1 with $r = 1$, it follows that f has a non trivial zero in F. □

From the proof above it becomes clear that Chevalley's Theorem is a more powerful result than the fact that a finite field is strongly C_1. In the following chapters we consider systems of forms in C_i fields and systems of polynomials without constant term in strongly C_i fields. We prove two results concerning general (strongly) C_i fields that are similar to Chevalley's Theorem. Theorem 6.2.2 shows that polynomials without constant term $f_1, \ldots, f_r$ of degree at most d in a strongly C_i field F, have a common non trivial zero in F if $n > rd^i$. The requirement that $n > rd^i$ is more restricting than the requirement that $n > d_1 + \ldots + d_r$. So this result is not exactly a generalisation of Chevalley's Theorem. Second a condition is provided under which polynomials without constant term $f_1, \ldots, f_r$ have a common non trivial zero in a strongly C_i field F, if $n > d_1^i + \ldots + d_r^i$. This is done in Theorem 5.3.1. This condition is based on normic polynomials, which are introduced in Defenition 5.1.16. Theorem 5.3.1 requires strongly C_i fields to admit normic polynomials of order i of any degree. By combining Theorem 3.1.4 with the fact that finite fields admit finite field extensions of any degree, it follows that finite fields admit normic polynomials of order one of any degree. Thus Chevalley's Theorem is a special case of Theorem 5.3.1.

5 Lang's results on systems of forms in C_i fields

In this chapter non trivial solutions of systems of forms, or polynomials without constant term, in (strongly) C_i fields are discussed. A form f, in a C_i field F, has a non trivial zero in F as long as the number of variables of f exceeds the i-th power of the degree of f. This gives rise to the question under what circumstances multiple forms have a common non trivial zero in a C_i field. In this chapter, requirements are presented that guarantee the existence of non trivial common zeros of multiple forms in a C_i field. This chapter covers most of the results from part one of Lang's paper. In Nagata's paper stronger versions of some of the results from this chapter are proven. In chapter 6 the results from this chapter are compared to the results from Nagata's paper. Section 5.1 contains the groundwork for the proofs presented later on. First some lemmas are discussed. We formulated these lemmas ourselves in order to make the proofs presented later on more elegant. After that the notion of a normic form is introduced. Section 5.1 finishes with a lemma from Lang's paper on normic forms. In the beginning of section 5.2, systems of polynomials without constant term in algebraically closed fields are examined. This is Theorem two from Lang's paper. Then Theorem three from Lang's paper is proven. This theorem provides a condition that guarantees the existence of a non trivial solution of a system of forms of equal degree in a C_i field. Something similar can be done for systems of polynomials without constant term in strongly C_i fields. In section 5.3, Theorem four from Lang's paper is proven. This theorem is similar to Theorem three, without the constraint that the forms are of the same degree.

5.1 Normic forms

This section prepares us for the proofs presented in the remainder of this chapter. First the notation of a specific substitution of polynomials is introduced. Many of the results from the remainder of this thesis rely on this substitution. After this, some properties of this substitution are discussed. These results are then combined in three corollaries, which are used throughout the rest of this thesis. After that, normic forms are defined. We show that if a field has a normic form of order i, then it has normic forms of order i in arbitrarily many variables. The lemmas presented below are quite technical, but take care of a lot of the details later on.

Notation 5.1.1. Let ϕ and $f_1, \ldots, f_r$ be polynomials with coefficients in a field F. Here $f_1, \ldots, f_r$ are polynomials in n common variables. We substitute $f_1, \ldots, f_r$ as many times as possible in ϕ. Each set $f_1, \ldots, f_r$ uses n new variables. The polynomials $f_1, \ldots, f_r$ need to fit into ϕ as complete sets, thus the last couple variables of ϕ are replaced by zero. The result is called $\xi(\phi, f_1, \ldots, f_r)$. The polynomial $\xi(\phi, f_1, \ldots, f_r)$ is thus defined in the following manner, where each occurrence of $|$ denotes the use of n new variables.

$$\xi(\phi, f_1, \ldots, f_r) = \phi(f_1, \ldots, f_r|f_1, \ldots, f_r| \ldots |f_1, \ldots, f_r|0, \ldots, 0)$$

It follows that $\xi(\phi, f_1, \ldots, f_r)$ is a polynomial in $\lfloor \frac{n_\phi}{r} \rfloor n$ variables. Throughout the remainder of this thesis, the notation $s = \lfloor \frac{n_\phi}{r} \rfloor$ will be used.

There are cases where the polynomial $\xi(\phi, f_1, \ldots, f_r)$ is equal to the zero polynomial. For example the polynomial $\phi(X_1, X_2) = X_1 - X_2$. It follows that $\xi(\phi, f_1, f_1) = 0$ for all $f_1 \in F[X_1, \ldots, X_n]$. Another example is a polynomial without constant term ϕ such that $n_\phi < r$. In that case every variable is replaced by zero. Since ϕ does not have a constant term the result is the zero polynomial. Below we cover a condition that guarantees that $\xi(\phi, f_1, \ldots, f_r) \neq 0$.

Lemma 5.1.2. *Let $\phi \in F[X_1, \ldots, X_{n_\phi}]$ and $f_1, \ldots, f_r \in F[X_1, \ldots, X_n]$ be polynomials such that $n_\phi \geq r$. Assume that ϕ only has the trivial zero in F and that $f_1, \ldots, f_r$ are not all equal to zero as a polynomial. Then $\xi(\phi, f_1, \ldots, f_r)$ is not equal to zero as a polynomial.*

Proof. The strategy of this proof is to argue by contradiction. Assume that $\xi(\phi, f_1, \ldots, f_r) = 0$. From the fact that $f_1, \ldots, f_r$ are not all equal to zero as a polynomial, it follows that there exist $X^*_{1,1}, \ldots, X^*_{1,n}, \ldots, X^*_{s,1}, \ldots, X^*_{s,n} \in F$ such that the following element $a \in F^{n_\phi}$ is non trivial.

$$a = (f_1(X^*_{1,1}, \ldots, X^*_{1,n}), \ldots, f_r(X^*_{1,1}, \ldots, X^*_{1,n}), \ldots$$
$$\ldots, f_1(X^*_{s,1}, \ldots, X^*_{s,n}), \ldots, f_r(X^*_{s,1}, \ldots, X^*_{s,n}), 0, \ldots, 0)$$

Since $\xi(\phi, f_1, \ldots, f_r) = 0$, it follows that $\phi(a) = 0$. Thus a is a non trivial zero of ϕ in F. This is a contradiction with the fact that ϕ only has the trivial zero in F. Because of this $\xi(\phi, f_1, \ldots, f_r)$ is not the zero polynomial. □

If one not only requires that one $f_1, \ldots, f_r$ are unequal to zero, but instead assumes that $f_1, \ldots, f_r$ only have the trivial common zero in F, a stronger version of Lemma 5.1.2 is true. When $f_1, \ldots, f_r$ only have the trivial common zero in F and ϕ only has the trivial zero in F, then $\xi(\phi, f_1, \ldots, f_r)$ only has the trivial zero in F. This is proven below.

Lemma 5.1.3. *Let $\phi \in F[X_1, \dots, X_{n_\phi}]$ and $f_1, \dots, f_r \in F[X_1, \dots, X_n]$ be polynomials such that $n_\phi \geq r$. Assume that ϕ only has the trivial zero in F and that $f_1, \dots, f_r$ only have the trivial common zero in F. Then $\xi(\phi, f_1, \dots, f_r)$ only has the trivial zero in F.*

Proof. This proof argues by contradiction. Assume that $(X^*_{1,1}, \dots, X^*_{1,n}, \dots, X^*_{s,1} \dots, X^*_{s,n}) \in F^{sn}$ is a non trivial zero of $\xi(\phi, f_1, \dots, f_r)$ in F. From the fact that $f_1, \dots, f_r$ only have the trivial common zero in F, it follows that the following element of F^{sr} is non trivial.

$$(f_1(X^*_{1,1}, \dots, X^*_{1,n}), \dots, f_r(X^*_{1,1}, \dots, X^*_{1,n}), \dots \\ \dots, f_1(X^*_{s,1}, \dots, X^*_{s,n}), \dots, f_r(X^*_{s,1}, \dots, X^*_{s,n}))$$

Since $(X^*_{1,1}, \dots, X^*_{1,n}, \dots, X^*_{s,1} \dots, X^*_{s,n})$ is a zero of $\xi(\phi, f_1, \dots, f_r)$, one can create a non trivial zero of ϕ in the following manner

$$(f_1(X^*_{1,1}, \dots, X^*_{1,n}), \dots, f_r(X^*_{1,1}, \dots, X^*_{1,n}), \dots \\ \dots, f_1(X^*_{s,1}, \dots, X^*_{s,n}), \dots, f_r(X^*_{s,1}, \dots, X^*_{s,n}), 0, \dots, 0).$$

This is a contradiction since ϕ only has the trivial zero. □

The results below concern the degree of $\xi(\phi, f_1, \dots, f_r)$. First it is shown that the product of two forms ϕ, ψ in F is again a form in F of degree $d_\phi d_\psi$. Then this result is extended to non zero polynomials with coefficients in F. Finally it is shown that the degree of $\xi(\phi, f_1, \dots, f_r)$ is at most $d_\phi d$, if $f_1, \dots, f_r$ are of degree at most d. This is done by first considering the case where $r \mid n_\phi$ and then extending this result to the case where $r \nmid n$. Let us now consider the product of two forms.

Lemma 5.1.4. *Let ϕ, ψ be forms in F in n common variables. Then $\phi\psi$ is a form in F of degree $d_\phi d_\psi$.*

Proof. First we show that $\phi\psi \neq 0$. From the fact that F is a field, it follows that F is a domain. This implies that $F[X_1, \dots, X_n]$ is a domain and thus has no zero divisors. Forms are defined as non constant polynomials, thus $\phi, \psi \neq 0$. Because of this $\phi\psi \neq 0$. The polynomials ϕ and ψ can be written as shown below

$$\phi(X_1, \dots, X_n) = \sum_{k_1+\dots+k_n=d_\phi} a_{k_1,\dots,k_n} X_1^{k_1} \cdots X_n^{k_n}$$
$$\psi(X_1, \dots, X_n) = \sum_{j_1+\dots+j_n=d_\psi} b_{j_1,\dots,j_n} X_1^{j_1} \cdots X_n^{j_n}.$$

Here all $k_1, \dots, k_n, j_1, \dots, j_n$ are elements of $\mathbb{N}$ and all $a_{k_1,\dots,k_n}, b_{j_1,\dots,j_n}$ are elements of F. The product $\phi\psi$ is described in the following manner

$$\begin{aligned}
\phi\psi(X_1,\ldots,X_n) &= \phi(X_1,\ldots,X_n)\psi(X_1,\ldots,X_n)\\
&= \Big(\sum_{k_1+\ldots+k_n=d_\phi} a_{k_1,\ldots,k_n} X_1^{k_1}\cdots X_n^{k_n}\Big)\Big(\sum_{j_1+\ldots+j_n=d_\psi} b_{j_1,\ldots,j_n} X_1^{j_1}\cdots X_n^{j_n}\Big)\\
&= \sum_{\substack{k_1+\ldots+k_n=d_\phi,\\ j_1+\ldots+j_n=d_\psi}} a_{k_1,\ldots,k_n} X_1^{k_1}\cdots X_n^{k_n} b_{j_1,\ldots,j_n} X_1^{j_1}\cdots X_n^{j_n} \qquad (5.1)\\
&= \sum_{\substack{k_1+\ldots+k_n=d_\phi,\\ j_1+\ldots+j_n=d_\psi}} a_{k_1,\ldots,k_n} b_{j_1,\ldots,j_n} X_1^{k_1+j_1}\cdots X_n^{k_n+j_n}.
\end{aligned}$$

From (5.1) it follows that $\phi\psi$ is the sum of terms of degree $k_1+j_1+\ldots+k_n+j_n$. The fact that $k_1+\ldots+k_n=d_\phi$ and $j_1+\ldots+j_n=d_\psi$ implies that each non-zero term of $\phi\psi$ has degree $d_\phi+d_\psi$. Since $\phi\psi\neq 0$, it follows that $\phi\psi$ is a form of degree $d_\phi+d_\psi$. □

Lemma 5.1.4 can be extended to the case of non zero polynomials.

Lemma 5.1.5. *Let $\phi,\psi\in F[X_1,\ldots,X_n]$ be non zero polynomials. Then $\phi\psi\in F[X_1,\ldots,X_n]$ is a non zero polynomial of degree $d_\phi+d_\psi$.*

Proof. The polynomials ϕ and ψ can be written in the following manner

$$\phi(X_1,\ldots,X_n) = \sum_{k_1+\ldots+k_n\le d_\phi} a_{k_1,\ldots,k_n} X_1^{k_1}\cdots X_n^{k_n}$$

$$\psi(X_1,\ldots,X_n) = \sum_{j_1+\ldots+j_n\le d_\psi} b_{j_1,\ldots,j_n} X_1^{j_1}\cdots X_n^{j_n}.$$

Here all $k_1,\ldots,k_n,j_1,\ldots,j_n$ are elements of $\mathbb{N}$ and all $a_{k_1,\ldots,k_n}, b_{j_1,\ldots,j_n}$ are elements of F. The product $\phi\psi$ can thus be written as shown below.

$$\begin{aligned}
\phi\psi(X_1,\ldots,X_n) &= \Big(\sum_{k_1+\ldots+k_n\le d_\phi} a_{k_1,\ldots,k_n} X_1^{k_1}\cdots X_n^{k_n}\Big)\Big(\sum_{j_1+\ldots+j_n\le d_\psi} b_{j_1,\ldots,j_n} X_1^{j_1}\cdots X_n^{j_n}\Big)\\
&= \sum_{\substack{k_1+\ldots+k_n\le d_\phi,\\ j_1+\ldots+j_n\le d_\psi}} a_{k_1,\ldots,k_n} X_1^{k_1}\cdots X_n^{k_n} b_{j_1,\ldots,j_n} X_1^{j_1}\cdots X_n^{j_n}\\
&= \sum_{\substack{k_1+\ldots+k_n\le d_\phi,\\ j_1+\ldots+j_n\le d_\psi}} a_{k_1,\ldots,k_n} b_{j_1,\ldots,j_n} X_1^{k_1+j_1}\cdots X_n^{k_n+j_n}
\end{aligned}$$

The terms of $\phi\psi$ that have the highest degree, are those where $k_1+\ldots+k_n=d_\phi$, $j_1+\ldots+j_n=d_\psi$ and $a_{k_1,\ldots,k_n}, b_{j_1,\ldots,j_n}\neq 0$. Note that such terms exist, otherwise ϕ and ψ would not be non zero polynomials of degree d_ϕ and d_ψ. All that is left to do is show that the sum of the terms where $k_1+\ldots+k_n=d_\phi$ and $j_1+\ldots+j_n=d_\psi$, is not zero. Denote the polynomial that consists of all the terms of maximal degree of a polynomial f by f_{d_f}. Observe that $\phi\psi_{d_{\phi\psi}}=\phi_{d_\phi}\psi_{d_\psi}$. Note that f_{d_f} is a form if f_{d_f} is non-constant. The fact that ϕ,ψ are non zero polynomials in combination with Lemma 5.1.4 implies that $\phi\psi_{d_{\phi\psi}}\neq 0$, thus $\phi\psi$ is a non zero polynomial of degree $d_\phi+d_\psi$. □

Let us now examine the degree of $\xi(\phi, f_1, \ldots, f_r)$, where $f_1, \ldots, f_r$ have degree at most d. First the case where $r \mid n_\phi$ is discussed. Then this result is extended to the case where $r \nmid n_\phi$.

Lemma 5.1.6. *Let $\phi \in F[X_1, \ldots, X_{n_\phi}]$ and $f_1, \ldots, f_r \in F[X_1, \ldots, X_n]$ be polynomials. Assume that the degree of $f_1, \ldots, f_r$ is at most d and that $r \mid n_\phi$. Then $\xi(\phi, f_1, \ldots, f_r)$ is a polynomial of degree at most $d_\phi d$.*

Proof. The strategy of this proof is induction on the degree of ϕ. First consider the case where $d_\phi = 0$. Then ϕ is a constant polynomial. Thus $\xi(\phi, f_1, \ldots, f_r)$ is also a constant polynomial. It follows that $\xi(\phi, f_1, \ldots, f_r)$ has degree zero, which is less than or equal to $0 \cdot d$. Now assume that Lemma 5.1.6 is true for all degrees up to d_ϕ. Also assume that $d_\phi > 0$. The polynomial ϕ can be written in the following manner

$$\phi(X_1, \ldots, X_{n_\phi}) = a + \sum_{j=1}^{n_\phi} \phi_j(X_1, \ldots, X_{n_\phi}) X_j. \tag{5.2}$$

Here ϕ_j is a polynomial of degree at most $d_\phi - 1$ for all $j \in \{1, \ldots, n_\phi\}$ and $a \in F$ is some constant. Note that the way ϕ is written in (5.2) is not necessarily unique. This is not a problem for the remainder of the proof. We now consider $\xi(\phi, f_1, \ldots, f_r)$.

$$\begin{aligned}\xi(\phi, f_1, \ldots, f_r) &= a + \sum_{j=1}^{n_\phi} \phi_j(f_1, \ldots, f_r | \ldots | f_1, \ldots, f_r) f_j^* \\ &= a + \sum_{j=1}^{n_\phi} \xi(\phi_j, f_1, \ldots, f_r) f_j^*\end{aligned}$$

Here f_j^* denotes one of the polynomials $f_1, \ldots, f_r$, with the proper variables according to the substitution. From the induction hypothesis it follows that $\xi(\phi_j, f_1, \ldots, f_r)$ is a polynomial of degree at most $(d_\phi - 1)d$. Note that f_j^* is a polynomial of degree at most d. Lemma 5.1.5 implies that $\xi(\phi_j, f_1, \ldots, f_r) f_j^*$ is a polynomial of degree at most $d_\phi d$. Since $\xi(\phi, f_1, \ldots, f_r)$ can be written as the sum of polynomials of this form and one constant term, it follows that $\xi(\phi, f_1, \ldots, f_r)$ is a polynomial of degree at most $d_\phi d$. □

Lemma 5.1.6 is extended to the case where $r \nmid n_\phi$ below.

Lemma 5.1.7. *Let $\phi \in F[X_1, \ldots, X_{n_\phi}]$ and $f_1, \ldots, f_r \in F[X_1, \ldots, X_n]$ be polynomials. Assume that the degree of $f_1, \ldots, f_r$ is at most d. Then $\xi(\phi, f_1, \ldots, f_r)$ is a polynomial of degree at most $d_\phi d$.*

Proof. Consider the polynomial ϕ^* in sr variables

$$\phi^*(X_1, \ldots, X_{sr}) = \phi(X_1, \ldots, X_{sr}, 0, \ldots, 0).$$

It follows that $d_{\phi^*} \leq d_\phi$, depending on whether the terms of ϕ of degree d_ϕ contain any of the variables that are replaced by zero. Note that $\xi(\phi, f_1, \ldots, f_r) = \xi(\phi^*, f_1, \ldots, f_r)$. Lemma 5.1.6 implies that the degree of $\xi(\phi^*, f_1, \ldots, f_r)$ is less than or equal to $d_\phi^* d$. From the fact that $\xi(\phi, f_1, \ldots, f_r) = \xi(\phi^*, f_1, \ldots, f_r)$, it follows that the degree of $\xi(\phi, f_1, \ldots, f_r)$ is less than or equal to $d_\phi^* d$, which is less than or equal to $d_\phi d$. □

One can show that $\xi(\phi, f_1, \dots, f_r)$ is a polynomial without constant term, if $\phi, f_1, \dots, f_r$ are polynomials without constant term.

Lemma 5.1.8. *Let $\phi \in F[X_1, \dots, X_{n_\phi}]$ and $f_1, \dots, f_r \in F[X_1, \dots, X_n]$ be polynomials without constant term. Then $\xi(\phi, f_1, \dots, f_r)$ is a polynomial without constant term.*

Proof. First, note that the sum of two polynomials without constant term is again a polynomial without constant term. Second, observe that the product of two polynomials without constant term is also a polynomial without constant term. Since $\phi, f_1, \dots, f_r$ are all polynomial without constant term, it follows that after the substitution, each term of ϕ becomes the product of finitely many polynomials without constant term. Thus each term of ϕ becomes a polynomial without constant term after the substitution. The polynomial $\xi(\phi, f_1, \dots, f_r)$ is obtained by adding all these terms together. Because of this $\xi(\phi, f_1, \dots, f_r)$ is a polynomial without constant term. □

The results from Lemma 5.1.8, Lemma 5.1.7 and Lemma 5.1.3 are now combined in one corollary. This corollary is used throughout the remainder of this thesis.

Corollary 5.1.9. *Let $f_1, \dots, f_r$ be polynomials without constant term in a field F of degree at most d in n common variables. Let ϕ be a polynomial without constant term in F such that $n_\phi \geq r$. Assume that ϕ only has the trivial zero in F and that $f_1, \dots, f_r$ only have the trivial common zero in F. It follows that $\xi(\phi, f_1, \dots, f_r)$ is a polynomial without constant term of degree at most $d_\phi d$, in sn variables and that only has the trivial zero in F.*

Proof. The fact that $\phi, f_1, \dots, f_r$ are polynomials without constant term in combination with Lemma 5.1.8 implies that $\xi(\phi, f_1, \dots, f_r)$ is a polynomial without constant term. Since $f_1, \dots, f_r$ are of degree at most d, it follows from Lemma 5.1.7 that the degree of $\xi(\phi, f_1, \dots, f_r)$ is less than or equal to $d_\phi d$. Finally, $n_\phi \geq r$, ϕ only has the trivial zero in F and $f_1, \dots, f_r$ only have the trivial common zero in F. This implies that $\xi(\phi, f_1, \dots, f_r)$ only has the trivial zero in F, according to Lemma 5.1.3. □

Now consider the case where only one polynomial ψ is substituted into ϕ. In that case the degree of $\xi(\phi, \psi)$ is exactly $d_\phi d_\psi$. The fact that ϕ only has the trivial zero in F is not needed to ensure that $\xi(\phi, \psi) \neq 0$. There are three reasons why the degree of $\xi(\phi, \psi)$ is exactly $d_\phi d_\psi$. First, $r \mid n_\phi$, since $r = 1$, so there are no zeros substituted into ϕ. Secondly, only polynomials of the same degree are substituted into ϕ, since only one polynomial is substituted into ϕ. Finally, each substitution of ψ uses new variables, so terms of maximal degree can not be subtracted from one another and cancel each other out. For this reason, it is needed that ψ is non constant to ensure that $\xi(\phi, \psi) \neq 0$ in the lemma below. For example, $\phi(X_1, X_2) = X_1 - X_2$ would result in $\xi(\phi, \psi) = 0$ if ψ is constant, since ψ does not contain any variables.

Lemma 5.1.10. *Let ϕ, ψ be polynomials with coefficients in F. Then $\xi(\phi, \psi)$ is a polynomial of degree $d_\phi d_\psi$. Furthermore, if ϕ is non zero and ψ is non constant, then $\xi(\phi, \psi)$ is different than zero as a polynomial.*

Proof. First assume that ψ is constant. Then $\xi(\phi,\psi)$ is a constant polynomial. Thus $\xi(\phi,\psi)$ has degree $0 = d_\phi \cdot 0$. Now assume that ψ is not a constant polynomial. The polynomial ϕ can be written in the following manner

$$\phi(X_1,\ldots,X_{n_\phi}) = \sum_{k_1+\ldots+k_{n_\phi}\leq d_\phi} a_{k_1,\ldots,k_{n_\phi}} X_1^{k_1}\cdots X_{n_\phi}^{k_{n_\phi}}.$$

Here $a_{k_1,\ldots,k_{n_\phi}}$ is an element of F for all $k_1,\ldots,k_{n_\phi} \in \mathbb{N}$. Now consider the substitution $\xi(\phi,\psi)$.

$$\xi(\phi,\psi) = \sum_{k_1+\ldots+k_{n_\phi}\leq d_\phi} a_{k_1,\ldots,k_{n_\phi}} \psi_1^{k_1}\cdots \psi_{n_\phi}^{k_{n_\phi}}$$

Here ψ_j denotes the polynomial ψ, but with the j-th set of n_ψ variables for all $j \in \{1,\ldots,n_\phi\}$. A polynomial of degree d, raised to the power k, results in a polynomial of degree dk. In the case of the zero polynomial this is clearly true. The case of non zero polynomials follows from applying Lemma 5.1.5 $k-1$ times. By combining this with Lemma 5.1.5 we conclude that after the substitution ξ, each non zero term of ϕ becomes a polynomial of degree $k_1 d_\psi + \ldots + k_{n_\phi} d_\psi = d_\psi(k_1+\ldots+k_{n_\phi})$. The terms of $\xi(\phi,\psi)$ of maximal degree are contained in those polynomials where $k_1+\ldots+k_{n_\phi} = d_\phi$ and $a_{k_1,\ldots,k_{n_\phi}} \neq 0$. Note that one can not subtract one of these terms from another, since each ψ uses different variables. It follows that $\xi(\phi,\psi)$ has degree $d_\psi d_\phi$.

Now assume that ϕ is non zero as a polynomial and that ψ is non constant. If ϕ is also non constant, then $d_\phi, d_\psi > 0$. Because of this the degree of $\xi(\phi,\psi)$ is bigger than zero and thus $\xi(\phi,\psi) \neq 0$. Now let ϕ be a constant polynomial, then $\xi(\phi,\psi) = \phi \neq 0$. □

Below a corollary similar to Corollary 5.1.9 is presented. In this case only one polynomial ψ is substituted into ϕ. Note that we do not make any comments about the existence of non trivial zeros. Later the corollary given below is used when it is not clear that ϕ,ψ only have the trivial zero. If ϕ,ψ only have the trivial zero in F and $n_\phi > 0$, then Lemma 5.1.3 is used separately to imply that $\xi(\phi,\psi)$ only has the trivial zero in F.

Corollary 5.1.11. *Let ψ,ϕ be polynomials without constant term in a field F. Then $\xi(\phi,\psi)$ is a polynomial without constant term of degree $d_\phi d$.*

Proof. Lemma 5.1.8 implies that $\xi(\phi,\psi)$ is a polynomial without constant term, since ϕ,ψ are polynomials without constant term. From Lemma 5.1.10 it follows that the degree of $\xi(\phi,\psi)$ is $d_\psi d_\phi$. □

Finally consider the case where $\phi, f_1,\ldots,f_r$ are forms. Since a form is also a polynomial without constant term, some of the earlier results still hold. We show that $\xi(\phi,f_1,\ldots,f_r)$ is a form. It also follows that the degree of $\xi(\phi,f_1,\ldots,f_r)$ is exactly $d_\phi d$ if $f_1,\ldots,f_r$ are all of degree d. First the case where $r \mid n_\phi$ is discussed, then this result is extended to the more general case.

Lemma 5.1.12. *Let $\phi \in F[X_1,\ldots,X_{n_\phi}]$ and $f_1,\ldots,f_r \in F[X_1,\ldots,X_n]$ be forms. Assume that $f_1,\ldots,f_r$ are of degree d and that $r \mid n_\phi$. It follows that either $\xi(\phi,f_1,\ldots,f_r)$ is a form of degree $d_\phi d$ or $\xi(\phi,f_1,\ldots,f_r) = 0$.*

Proof. The strategy of this proof is induction on the degree of ϕ. From the fact that $r \mid n_\phi$ it follows that $n_\phi \geq r$. First consider the case where $d_\phi = 1$. Then ϕ can be written in the following manner

$$\phi(X_1, \ldots, X_{n_\phi}) = \sum_{k=1}^{n_\phi} a_k X_k. \tag{5.3}$$

All a_k are elements of F. From (5.3) it follows that $\xi(\phi, f_1, \ldots, f_r)$ is the sum of forms of degree d and thus a form of degree $d = d_\phi d$, unless all forms cancel each other out. In that case $\xi(\phi, f_1, \ldots, f_r) = 0$. Assume that Lemma 5.1.12 is true for all degrees up to d_ϕ and that $d_\phi > 1$. Then ϕ can be written in the following manner

$$\phi(X_1, \ldots, X_{n_\phi}) = \sum_{k=1}^{n_\phi} \phi_k(X_1, \ldots, X_{n_\phi}) X_k. \tag{5.4}$$

Here ϕ_k is a form of degree $d_\phi - 1$ or zero as a polynomial, for all $k \in \{1, \ldots, n_\phi\}$. From (5.4) it follows that $\xi(\phi, f_1, \ldots, f_r)$ can be written as shown below.

$$\begin{aligned} \xi(\phi, f_1, \ldots, f_r) &= \phi(f_1, \ldots, f_r | \ldots | f_1, \ldots, f_r) \\ &= \sum_{k=1}^{n_\phi} \phi_k(f_1, \ldots, f_r | \ldots | f_1, \ldots, f_r) f_k^* \\ &= \sum_{k=1}^{n_\phi} \xi(\phi_k, f_1, \ldots, f_r) f_k^* \end{aligned}$$

Here f_k^* denotes the appropriate form, in the proper variables, according to the substitution. The induction hypothesis implies that $\xi(\phi_k, f_1, \ldots, f_r)$ is a form of degree $(d_\phi - 1)d$ or that $\xi(\phi_k, f_1, \ldots, f_r) = 0$. By applying Lemma 5.1.4 it follows for each $k \in \{1, \ldots, n_\phi\}$, that $\xi(\phi_k, f_1, \ldots, f_r) f_k^*$ is a form of degree $(d_\phi - 1)d + d = d_\phi d$ or that $\xi(\phi_k, f_1, \ldots, f_r) f_k^* = 0$. Thus $\xi(\phi, f_1, \ldots, f_r)$ is the sum of forms of degree $d_\phi d$ and zero polynomials. It follows that $\xi(\phi, f_1, \ldots, f_r)$ is a form of degree $d_\phi d$ or zero as a polynomial. □

Let us now extend the result from Lemma 5.1.12 to the case where $r \nmid n_\phi$. This proof is similar to that of Lemma 5.1.7.

Lemma 5.1.13. *Let $\phi \in F[X_1, \ldots, X_{n_\phi}]$ and $f_1, \ldots, f_r \in F[X_1, \ldots, X_n]$ be forms. Assume that $f_1, \ldots, f_r$ are of degree d. It follows that either $\xi(\phi, f_1, \ldots, f_r)$ is a form of degree $d_\phi d$ or $\xi(\phi, f_1, \ldots, f_r) = 0$.*

Proof. First consider the case where $n_\phi < r$. Then $\xi(\phi, f_1, \ldots, f_r) = \phi(0, \ldots, 0) = 0$, since ϕ is a form and in particular a polynomial without constant term. Assume that $n_\phi \geq r$. Consider the polynomial ϕ^* defined in the following way

$$\phi^*(X_1, \ldots, X_{sr}) = \phi(X_1, \ldots, X_{sr}, 0, \ldots, 0).$$

From the fact that ϕ is a form it follows that ϕ^* is a form of degree d_ϕ or that $\phi^* = 0$. The latter is the case if each term of ϕ contains one of the variables that is replaced by zero. Note that $r \mid n_{\phi^*}$ and that $\xi(\phi, f_1, \ldots, f_r) = \xi(\phi^*, f_1, \ldots, f_r)$. Assume that $\phi^* = 0$. Then

$\xi(\phi^*, f_1, \ldots, f_r) = 0$ and thus $\xi(\phi, f_1, \ldots, f_r) = 0$. Now assume that $\phi^* \neq 0$. By applying Lemma 5.1.12 to ϕ^* it follows that either $\xi(\phi^*, f_1, \ldots, f_r)$ is a form of degree $d_{\phi^*}d = d_\phi d$ or that $\xi(\phi^*, f_1, \ldots, f_r) = 0$. Thus $\xi(\phi, f_1, \ldots, f_r)$ is a form of degree $d_\phi d$ or $\xi(\phi, f_1, \ldots, f_r) = 0$. □

The results about forms are combined in a corollary similar to Corollary 5.1.9.

Corollary 5.1.14. *Let $f_1, \ldots, f_r$ be forms in a field F of degree d in n common variables. Let ϕ be a form in F, such that $n_\phi \geq r$. Assume that ϕ only has the trivial zero in F and that $f_1, \ldots, f_r$ only have the trivial common zero in F. It follows that $\xi(\phi, f_1, \ldots, f_r)$ is a form of degree $d_\phi d$, in sn variables and that only has the trivial zero in F.*

Proof. Since ϕ only has the trivial zero, $n_\phi \geq r$ and $f_1, \ldots, f_r \neq 0$, Lemma 5.1.2 implies that $\xi(\phi, f_1, \ldots, f_r)$ is not equal to zero as a polynomial. By combining this with Lemma 5.1.13, it follows that $\xi(\phi, f_1, \ldots, f_r)$ is a form of degree $d_\phi d$. Lemma 5.1.3 implies that $\xi(\phi, f_1, \ldots, f_r)$ only has the trivial zero in F. □

These were all the results about the substitution ξ. Let us continue with the definition of a normic form. Lang uses normic forms to provide a sufficient condition for the existence of common non trivial zeros of systems of forms in a C_i field or similarly, systems of polynomials without constant term in strongly C_i fields. This result is presented in Theorem 5.2.2. The definition of a normic form of order $i \in \mathbb{N}$ is given below.

Definition 5.1.15 (normic form)**.** A form f in a field F is called normic of order $i \in \mathbb{N}$ if $n = d^i$ and f only has the trivial zero in F. Here it is required that $i > 0$ and $d > 1$.

The following is an example of a normic form. Let F be a field that admits a finite field extension of degree $n > 1$. Then Theorem 3.1.4 provides a normic form of order one in F. Definition 5.1.15 requires that $i > 0$ and $d > 1$. The case where $d = 1$ results in a form that looks like $aX_1 = 0$, with $a \in F$ and $a \neq 0$. This form only has the trivial zero regardless of what F looks like. This normic form does not work with the theorems presented later on, thus it is excluded from the definition. For a similar reason normic forms of order zero are excluded from the definition. This is not a problem, since normic forms of order i are used in combination with C_i fields. The C_0 case is treated separately. Normic polynomials without constant term are defined similarly to normic forms. We will refer to these as normic polynomials.

Definition 5.1.16 (normic polynomial)**.** A polynomial without constant term f in a field F is called normic of order $i \in \mathbb{N}$ if $n = d^i$ and f only has the trivial zero in F. Here it is required that $i > 0$ and $d > 1$.

Many of the proofs presented after this are similar for C_i fields and normic forms of order i, as well as for strongly C_i fields and normic polynomials of order i. To avoid repetition, only one proof is presented. One needs to replaces C_i field by strongly C_i field and normic form by normic polynomial. Any differences in the proofs will be pointed out as we go along. Consider the following lemma presented by Lang.

Lemma 5.1.17. *Let F be a field with a normic form of order i, then F has a normic form of order i in arbitrarily many variables.*

Proof. Denote by ϕ, ψ two normic forms of order i in F. These exist since it is allowed that $\phi = \psi$. From the definition of normic forms it follows that $n_\phi = d_\phi^i$, where $d_\phi > 1$ and $i > 0$. Because of this $n_\phi > 1$. Similarly it follows that $n_\psi > 1$. Since ϕ, ψ are normic forms, they only have the trivial zero. Corollary 5.1.14 implies that $\xi(\phi, \psi)$ is a form in $n_\phi n_\psi$ variables, of degree $d_\phi d_\psi$ and that only has the trivial zero in F. Note that $n_\phi = d_\phi^i$ and $n_\psi = d_\psi^i$. Because of this $n_\phi n_\psi = (d_\phi d_\psi)^i$. Thus $\xi(\phi, \psi)$ is a normic form of order i. From the fact that $n_\phi, n_\psi > 1$, it follows that normic forms in arbitrarily many variables can be generated by continuing this process. □

Lemma 5.1.17 is also true for strongly C_i fields and normic polynomials. Instead of Corollary 5.1.14 one needs to use Corollary 5.1.11 in combination with Lemma 5.1.3.

5.2 Systems of forms of equal degree in C_i fields

In this section Theorem three from Lang's paper is proven. This theorem states that the existence of a normic form of order i in a C_i field F guarantees the existence of a non trivial common zero of a system of r forms of degree d in F, as long as $n > rd^i$. First the case of C_0 fields is discussed. After that Theorem three is proven by using Corollary 5.1.14 in the case of C_i fields and Corollary 5.1.9 in the case of strongly C_i fields. One can make a slightly stronger statement when considering strongly C_i fields. In that case the polynomials without constant term $f_1, \ldots, f_r$ can be of degree at most d. This section ends with a Corollary about C_1 fields. In the case of C_1 fields, Theorem 3.1.4 implies the existence of a normic form of order 1.

Consider the case where F is a C_0 field. According to Theorem 3.2.1, F is algebraically closed. We will not provide a proof for the theorem given below. It is regarded as a classical result and thus Lang does not provide a proof in his paper. The proof of this theorem uses algebraic geometry and is beyond the scope of this thesis. If you are interested, a proof can be found in chapter 25 of volume two of the book "Algebra" by Falko Lorenz [3, p. 123].

Theorem 5.2.1. *Let $f_1, \ldots, f_r$ be polynomials without constant term in an algebraically closed field F in n common variables. Then $f_1, \ldots, f_r$ have a non trivial common zero in F if $n > r$.*

Note that the degree of the polynomials is not relevant in Theorem 5.2.1. For the (strongly) C_i case the degree does matter.

Theorem 5.2.2. *Let $f_1, \ldots, f_r$ be forms of degree d in a field F in n common variables. Here F is a C_i field that admits a normic form of order i. Then $f_1, \ldots, f_r$ have a non trivial common zero in F if $n > rd^i$.*

Proof. The strategy of this proof is as follows. We assume that $f_1, \ldots, f_r$ only have the trivial common zero in F and obtain a contradiction. Let N be a normic form of order i in F. One can show that $f = \xi(N, f_1, \ldots, f_r)$ is a form in F with only the trivial zero in F. It follows that $n_f > d_f^i$ if N has enough variables. According to Lemma 5.1.17, N can have arbitrarily many variables. Since F is C_i it follows that f has a non trivial zero. This is a contradiction. The actual proof is given below.

From Lemma 5.1.17 it follows that F has a normic form N of order i in arbitrarily many variables. Choose N such that $n_N \geq r$. Later another requirement for n_N is added. We use the notation $f = \xi(N, f_1, \dots, f_r)$. Assume that $f_1, \dots, f_r$ only have the trivial common zero in F. Note that N only has the trivial zero in F, since N is a normic form. From Corollary 5.1.14 it follows that f is a form in ns variables, of degree $d_N d = \sqrt[i]{n_N} d$ and with only the trivial zero in F. We want to take n_N such that $n_f > d_f^i$. Consider a new symbol $t = n_N - sr$, representing the number of zeros that are substituted in N. Note that $t < r$. Choose n_N, and thus s, large enough such that $n - rd^i > \frac{td^i}{s}$. Note that $n - rd^i > 0$, since $n > rd^i$. From this it follows that $n_f > d_f^i$ as shown below.

$$\begin{aligned} n - rd^i &> \frac{td^i}{s} \\ sn - srd^i &> td^i \\ sn &> td^i + srd^i \\ sn &> (t + sr)d^i \\ sn &> (\sqrt[i]{t + sr}d)^i \\ sn &> (\sqrt[i]{n_N}d)^i \\ n_f &> d_f^i \end{aligned}$$

Thus there exists a normic form N of order i in F such that f is a form with only the trivial zero in F that satisfies $n_f > d_f^i$. From the fact that F is a C_i field, it follows that f has a non trivial zero in F. This is a contradiction. Thus $f_1, \dots, f_r$ do have a common non trivial zero in F. □

The proof given above still holds for the strongly C_i case by using Corollary 5.1.9. The difference is that the degree of f is smaller than or equal to $d_N d$. This only makes it easier for the condition $n_f > d_f^i$ to be satisfied. In the case of strongly C_i fields one could actually make a stronger statement than the result from Lang's paper. Corollary 5.1.9 still works if $f_1, \dots, f_r$ are of degree at most d. Thus one can omit the condition that $f_1, \dots, f_r$ are of equal degree. This result is similar to Theorem 1b from Nagata's paper. The requirement that $n > rd^i$ is the least restricting requirement under which Theorem 5.2.2 is true. Let f be a normic form of order i in F. In that case $r = 1$ and $n = d^i$. However f does not have a non trivial zero in F. The requirement that $n > rd^i$ is needed. Something similar is true for Theorems 5.3.1, 6.2.1, 6.2.2. If F is not C_0, the existence of a normic form of order one is ensured by Theorem 3.1.4. This gives rise to the result in the following corollary. Since a normic form of order one is also a normic polynomial of order one, the corollary below also works for the strongly C_1 case, where $f_1, \dots, f_r$ can be of degree at most d.

Corollary 5.2.3. *Given a C_1 field F. Let $f_1, \dots, f_r$ be forms in F, in n common variables and of degree d. Then $f_1, \dots, f_r$ have a common non trivial zero in F if $n > rd$.*

Proof. First consider the case where F is C_0. Then Theorem 5.2.1 ensures the existence of the relevant non trivial zero. Now let F be a C_1 field which is not C_0. Theorem 3.2.1 implies that F is not algebraically closed. Thus there exists a finite extension E of F of degree $k \in \mathbb{N}_{>1}$. From Theorem 3.1.4 it follows that F has a normic form of order one. Now apply Theorem 5.2.2 to obtain the desired result. □

5.3 Systems of forms of different degree in C_i fields

In this subsection Theorem four from Lang's paper is proven. It is similar to Theorem 5.2.2, but without the requirement that $f_1, \ldots, f_r$ are of the same degree. In order to do this, the existence of normic forms of order i of any degree is required. Contrary to this, Theorem 5.2.2 only requires the existence of one normic form of order i. In section 7.2 the results from this section are used to show that a C_i field that admits normic forms of order i of any degree is actually strongly C_{i+1}.

Theorem 5.3.1. *Given a C_i field F that admits normic forms of order i of any degree. Let $f_1, \ldots, f_r$ be forms in F in n common variables and of degrees $d_1, \ldots, d_r$ respectively. Then $f_1, \ldots, f_r$ have a non trivial common zero in F if $n > d_1^i + \ldots + d_r^i$.*

Proof. The strategy of this proof is as follows. The forms $f_1, \ldots, f_r$ are used to construct a particular system of forms of the same degree. Then we assume that $f_1, \ldots, f_r$ only have the trivial common zero in F. This implies that the system of forms only has the trivial common zero in F. Theorem 5.2.2 is used to show that the system of forms does have a non trivial common zero in F. This is a contradiction. The actual proof is given below.

Let $N_1, \ldots, N_r$ be normic forms of order i, each of degree $d_2 \cdots d_r, d_1 d_3 \cdots d_r, \ldots, d_1 \cdots d_{r-1}$ respectively. Consider the system of equations shown below, where each row contains $d_1^i, \ldots, d_r^i$ equations respectively.

$$\begin{gathered} \xi(N_1, f_1) \mid \xi(N_1, f_1) \mid \ldots \mid \xi(N_1, f_1) \\ \xi(N_2, f_2) \mid \xi(N_2, f_2) \mid \ldots \mid \xi(N_2, f_2) \\ \vdots \\ \xi(N_r, f_r) \mid \xi(N_r, f_r) \mid \ldots \mid \xi(N_r, f_r) \end{gathered} \tag{5.5}$$

In the first row each individual substitution of f_1 uses n new variables. Thus each instance of $\mid$ denotes the introduction of n_{N_1} sets of n new variables. In the following rows the same variables are used. The j-th row contains $n_{N_j} n d_j^i = d_1^i \cdots d_{j-1}^i d_{j+1}^i \cdots d_r^i n d_j^i = n(d_1 \cdots d_r)^i$ variables. Thus each row has the same number of variables. From Corollary 5.1.11 it follows that all these individual equations are polynomials without constant term of degree $d_{N_j} d_j = d_1 \cdots d_r$. From the fact that N_j and f_j are forms it follows that $d_j, d_{N_j} > 0$ and thus $\xi(N_j, f_j)$ is not zero as a polynomial. Lemma 5.1.13 implies that all $\xi(N_j, f_j)$ are forms. Thus (5.5) contains a system of $d_1^i + \ldots + d_r^i$ forms. A normic form only has the trivial zero. Because of this, a zero of $\xi(N_j, f_j)$ is a vector such that all f_j substituted into N_j are zero. Assume that $f_1, \ldots, f_r$ only have the trivial common zero in F. Then the system of forms in (5.5) only has the trivial common zero in F. We now apply Theorem 5.2.2. From the fact that $n > d_1^i + \ldots + d_r^i$, it follows that $n(d_1 \cdots d_r)^i > (d_1^i + \ldots + d_r^i)(d_1 \cdots d_r)^i$. The number of total variables is bigger than the number of forms times the i-th power of the degree of these forms. Theorem 5.2.2 implies that the system in (5.5) has a non trivial common zero in F. This is a contradiction. Thus $f_1, \ldots, f_r$ have a non trivial common zero in F. □

The proof given above also works for the case of strongly C_i fields. The main difference is that polynomials without constant term can be zero as a polynomial. If one of the polynomials without constant term f_j is equal to zero as a polynomial, one can remove it and perform the proof with $r-1$ polynomials without constant term, as follows. This can be done until there are no polynomials left that are zero as a polynomial. If $f_1, \ldots, f_r$ are all zero as a polynomial, then every non trivial element of F^n is a common non trivial zero in F. Assume without loss of generality that f_r is zero as a polynomial. From the condition $n > d_1^i + \ldots + d_r^i$, it follows that $n > d_1^i + \ldots + d_{r-1}^i$. Furthermore, a non trivial common zero of $f_1, \ldots, f_{r-1}$ is also a non trivial common zero of $f_1, \ldots, f_r$. The second difference is that one can not use the fact that N_j is a form to imply that $d_{N_j} > 0$. However, the fact that N_j only has the trivial zero in F implies that N_j is not zero as a polynomial. There exists an alternative version of the proof given above in the case of strongly C_i fields. Lemmas 5.1.7 and 5.1.8 can be used instead of Corollary 5.1.11. In that case the stronger version of Theorem 5.2.2, in which the degree of the polynomials without constant term is at most d, is needed. If Lemma 5.1.7 is used, one only knows that all $\xi(N_j, f_j)$ have degree at most $d_1 \cdots d_r$.

The existence of normic forms of order i of degree p for every prime number p is enough to ensure the existence of normic forms of order i of any degree. The same is true for normic polynomials of order i. In the case of normic polynomials one needs to use Corollary 5.1.11 and Lemma 5.1.3 instead of Corollary 5.1.14 in the proof below. This result is not discussed in Lang's or Nagata's paper.

Theorem 5.3.2. *Let F be a field that has normic forms of order i of degree p for every prime number p. Then F admits normic forms of order i of any degree.*

Proof. Let $d \in \mathbb{N}$ be some degree with prime factorisation $d = p_1 p_2 \cdots p_n$. Let $N_1, \ldots, N_n$ be normic forms in F of order i of degree $p_1, \ldots, p_n$ respectively. We inductively define n polynomials ϕ_t, with coefficients in F, where $t \in \{1, \ldots, n\}$. Start with $\phi_1 = N_1$. The polynomial ϕ_t is then defined as follows

$$\phi_t = \xi(\phi_{t-1}, N_t).$$

The polynomial ϕ_t is a normic form of order i in F of degree $p_1 \cdots p_t$, as follows. Since N_1 is a normic form of order i in F of degree p_1, this is true for $t = 1$. Assume that it holds for $t-1$. Corollary 5.1.14 implies that ϕ_t is a form in F, in $n_{\phi_{t-1}} n_{N_t}$ variables and of degree $p_1 \cdots p_{t-1} p_t$. From Corollary 5.1.14 it also follows that ϕ_t only has the trivial zero. The polynomials ϕ_{t-1} and N_t are normic forms. Because of this $n_{\phi_{t-1}} = (p_1 \cdots p_{t-1})^i$ and $n_{N_t} = p_t^i$. Thus $n_{\phi_t} = d_{\phi_t}^i$ and it follows that ϕ_t is a normic form of order i of degree $p_1 \cdots p_{t-1} p_t$. By considering ϕ_n, the existence of a normic form of order i in F of degree $p_1 \cdots p_n = d$ is guaranteed. □

6 Nagata's results on systems of forms in C_i fields

In this chapter the material from Nagata's paper is discussed. A theorem similar to Theorem 5.2.2 is proven, together with a stronger version for strongly C_i fields. The difference between the theorems from this chapter and Theorem 5.2.2 is that Theorems 6.2.1 and 6.2.2 do not require the existence of normic forms. In section 6.1 some notation and three lemmas are introduced. This lays the groundwork for the proof provided in section 6.2. Section 6.3 comments on the differences between the methods of Lang and Nagata.

6.1 Preparation for proof of Nagata's results

This section prepares everything needed for the proof in section 6.2. First some notation is introduced. After that the results from two lemmas presented by Nagata are discussed. In this thesis these lemmas are formulated in a different manner. Three lemmas are used to get the desired results. This makes these results a bit more precise. The case of C_i fields and strongly C_i fields is examined at the same time. The main difference is in the language. One needs to replace form by polynomial without constant term, and C_i field by strongly C_i field. Besides that, one needs to use the versions of lemmas and corollaries about polynomials without constant term instead of forms. Any further differences are highlighted as we go along.

Let $f_1, \dots, f_r$ be forms in n variables of degree d in a field F. In the case of polynomials without constant term, the degree of $f_1, \dots, f_r$ can differ, but is at most d. Let ϕ be a form in F such that $n_\phi \geq r$. We define the polynomials $\phi^{(t)}(f_1, \dots, f_r)$ inductively for $t \in \mathbb{N}$. Start with $\phi^{(0)}(f_1, \dots, f_r) = \phi$. From there, let $\phi^{(t-1)}(f_1, \dots, f_r)$ be a polynomial in $n_{\phi^{(t-1)}}$ variables and of degree $d_{\phi^{(t-1)}}$. We introduce the notation $s_t = \lfloor \frac{n_{\phi^{(t)}}}{r} \rfloor$, where $s_0 = \lfloor \frac{n_\phi}{r} \rfloor$. Then $\phi^{(t)}(f_1, \dots, f_r)$ is the following polynomial in ns_{t-1} variables

$$\phi^{(t)}(f_1, \dots, f_r) = \xi(\phi^{(t-1)}(f_1, \dots, f_r), f_1, \dots, f_r).$$

We now present three lemmas concerning properties of the polynomial $\phi^{(t)}(f_1, \dots, f_r)$.

Lemma 6.1.1. *Let $f_1, \dots, f_r$ be forms in a field F with only the trivial common zero in F. Let ϕ be a form in F with only the trivial zero in F. Then $\phi^{(t)}(f_1, \dots, f_r)$ is a form in F with only the trivial zero in F.*

Proof. The strategy of this proof is induction on t. From the assumption that ϕ is a form in F with only the trivial zero in F, it follows that $\phi^{(0)}(f_1, \dots, f_r) = \phi$ is a form in F with only the trivial zero in F. Now assume that $\phi^{(t-1)}(f_1, \dots, f_r)$ is a form in F with only the trivial zero in F. From the fact that $n_\phi \geq r$, it follows that $n_{\phi^{(t)}} \geq r$ for all $t \in \mathbb{N}$. Corollary 5.1.14 implies that $\phi^{(t)}(f_1, \dots, f_r)$ is a form in F that only has the trivial zero in F. □

In the next lemma it is shown that the number of variables of $\phi^{(t)}(f_1, \dots, f_r)$ becomes arbitrarily large as t goes to infinity, if $n > rd^i$ and $d > 1$ for some $i \in \mathbb{N}_{>0}$.

Lemma 6.1.2. *If $n > rd^i$, then $n_{\phi^{(t)}} \geq n(d^i)^{t-1}$ for all $t > 0$.*

Proof. This proof relies on the fact that $\phi^{(t)}(f_1, \dots, f_r)$ has $n\lfloor \frac{n_{\phi^{(t-1)}}}{r} \rfloor$ variables for $t > 0$. This follows from the number of variables of the substitution ξ. The strategy of this proof is induction on t. Assume that $t = 1$. Note that $n_{\phi^{(0)}} = n_\phi \geq r$. From this it follows that

$$n_{\phi^{(1)}} = n\lfloor \frac{n_{\phi^{(0)}}}{r} \rfloor = n\lfloor \frac{n_\phi}{r} \rfloor \geq n = n(d^i)^0.$$

Assume that this lemma is true for $t-1$ and that $t > 1$. From the fact that $n > rd^i$, it follows that $\frac{n}{r} > d^i$. This implies that

$$\lfloor \frac{n(d^i)^t}{r} \rfloor = \lfloor (d^i)^t \frac{n}{r} \rfloor \geq \lfloor (d^i)^t (d^i) \rfloor = \lfloor (d^i)^{t+1} \rfloor = (d^i)^{t+1}. \tag{6.1}$$

From (6.1) it follows that

$$n_{\phi^{(t)}} = n\lfloor \frac{n_{\phi^{(t-1)}}}{r} \rfloor \geq n\lfloor \frac{n(d^i)^{t-2}}{r} \rfloor \geq n(d^i)^{t-1}. \qquad \square$$

In the following lemma, it is shown that the ratio between the number of variables and the i-th power of the degree of $\phi^{(t)}(f_1, \dots, f_r)$ goes to infinity as t goes to infinity, if $n > rd^i$ for some $i \in \mathbb{N}_{>0}$. Later this is used to show that there exists a t such that $n_{\phi^{(t)}} > d^i_{\phi^{(t)}}$. Lemma 6.1.3 requires that $f_1, \dots, f_r$ only have the trivial common zero and that ϕ only has the trivial zero. This is needed to ensure that $\phi^{(t)}(f_1, \dots, f_r)$ is not zero as a polynomial.

Lemma 6.1.3. *Let $f_1, \dots, f_r$ be forms in a field F with only the trivial common zero in F. Let ϕ be a form in F with only the trivial zero in F. If $n > rd^i$, where $d > 1$ and $i \in \mathbb{N}_{>0}$, then there exists a $t^* \in \mathbb{N}$ such that $\frac{n_{\phi^{(t)}}}{(d_{\phi^{(t)}})^i}$ becomes arbitrarily large as t becomes arbitrarily large, for all $t \geq t^*$.*

Proof. Lemma 6.1.1 shows that $\phi^{(t)}(f_1, \dots, f_r)$ is a form with only the trivial zero in F. It follows that $\phi^{(t)}(f_1, \dots, f_r)$ is not zero as a polynomial. Thus $d_{\phi^{(t)}} > 0$ for all $t \in \mathbb{N}$. From Corollary 5.1.14 it follows that $d_{\phi^{(t)}} = d_{\phi^{(t-1)}}d$. Note that $n_{\phi^{(t)}} = n\lfloor \frac{n_{\phi^{(t-1)}}}{r} \rfloor$. Let us define a new symbol $c_t = \frac{n_{\phi^{(t)}}}{r} - \lfloor \frac{n_{\phi^{(t)}}}{r} \rfloor$. From the fact that $n > rd^i$, it follows that there exists a $g \in \mathbb{R}$ such that $1 < g < \frac{n}{rd^i}$. Note that n, r, d, i, g are all constant and that $c_t \in [0, 1)$. Since $d > 1$ and $i > 0$, Lemma 6.1.2 implies the existence of a $t^* \in \mathbb{N}$, such that the following inequality holds for all $t \geq t^*$.

$$\begin{aligned}
n_{\phi^{(t-1)}} &> \frac{1}{\frac{n}{rd^i} - g} \frac{n}{d^i} c_{t-1} \\
(\frac{n}{rd^i} - g)n_{\phi^{(t-1)}} &> \frac{n}{d^i} c_{t-1} \\
(\frac{n}{rd^i} - g)n_{\phi^{(t-1)}} - \frac{n}{d^i} c_{t-1} &> 0 \\
\frac{n}{rd^i} n_{\phi^{(t-1)}} - g n_{\phi^{(t-1)}} - \frac{n}{d^i} c_{t-1} &> 0 \\
\frac{n}{rd^i} n_{\phi^{(t-1)}} - \frac{n}{d^i} c_{t-1} &> g n_{\phi^{(t-1)}} \\
\frac{n}{rd^i} \frac{n_{\phi^{(t-1)}}}{d^i_{\phi^{(t-1)}}} - \frac{n c_{t-1}}{d^i d^i_{\phi^{(t-1)}}} &> g \frac{n_{\phi^{(t-1)}}}{d^i_{\phi^{(t-1)}}} \\
\frac{\frac{n}{r} n_{\phi^{(t-1)}}}{d^i d^i_{\phi^{(t-1)}}} - \frac{n c_{t-1}}{d^i d^i_{\phi^{(t-1)}}} &> g \frac{n_{\phi^{(t-1)}}}{d^i_{\phi^{(t-1)}}} \\
\frac{n(\frac{n_{\phi^{(t-1)}}}{r} - c_{t-1})}{d^i d^i_{\phi^{(t-1)}}} &> g \frac{n_{\phi^{(t-1)}}}{d^i_{\phi^{(t-1)}}} \\
\frac{n\lfloor \frac{n_{\phi^{(t-1)}}}{r} \rfloor}{d^i d^i_{\phi^{(t-1)}}} &> g \frac{n_{\phi^{(t-1)}}}{d^i_{\phi^{(t-1)}}} \\
\frac{n_{\phi^{(t)}}}{d^i_{\phi^{(t)}}} &> g \frac{n_{\phi^{(t-1)}}}{d^i_{\phi^{(t-1)}}}
\end{aligned}$$

From the fact that $g > 1$, it follows that the ratio $n_{\phi^{(t)}}/d^i_{\phi^{(t)}}$ will grow indefinitely as t increases for all $t \geq t^*$. $\square$

In the proof above the cases where $d = 1$ or $i = 0$ are excluded. In that case one could have that $\lfloor \frac{n}{r} \rfloor = 1$. Then it is not certain that $\frac{n_{\phi(t)}}{(d_{\phi(t)})^i}$ goes to infinity for sufficiently large t. This exception is not treated in Nagata's paper. In the case of polynomials without constant term there are two differences to the proof presented above. First, Corollary 5.1.9 implies that $d_{\phi^{(t)}} \leq d_{\phi^{(t-1)}} d$ instead of $d_{\phi^{(t)}} = d_{\phi^{(t-1)}} d$. The proof presented above still works in that case. Actually the ratio $n_{\phi^{(t)}}/d^i_{\phi^{(t)}}$ might grow faster. Second, polynomials without constant term can be zero. This does not provide any problems, since it is still assumed that $f_1, \dots, f_r$ only have the trivial common zero in F and that ϕ only has the trivial zero in F. The proof above does not rely on the fact that forms are not zero as a polynomial. The version of Lemma 6.1.1 about polynomials without constant term implies that $\phi^{(t)}(f_1, \dots, f_r)$ only has the trivial zero in F for all $t \in \mathbb{N}$. From this it follows that $\phi^{(t)}(f_1, \dots, f_r)$ is not zero as a polynomial. Thus $d_{\phi^{(t)}} > 0$ for all $t \in \mathbb{N}$.

6.2 Systems of forms in C_i fields without normic forms

This section presents a proof of Theorem 1a and Theorem 1b from Nagata's paper. These theorems are similar to Theorem 5.2.2, but they do not require the existence of a normic form. Both theorems are stated below. A proof is only provided for the case of C_i fields. There is one difference between the C_i and strongly C_i case. The degree of the polynomials without constant term $f_1, \dots, f_r$ does not have to be equal, but is at most d. In the case of forms the degree needs to be equal. The main reason for this is that Lemma 5.1.13 requires $f_1, \dots, f_r$ to have the same degree and Lemma 5.1.8 still works if $f_1, \dots, f_r$ are of different degrees. If $f_1, \dots, f_r$ are forms of different degrees, then $\phi^{(t)}(f_1, \dots, f_r)$ is not necessarily a form.

Theorem 6.2.1. *Let $f_1, \dots, f_r$ be forms in a C_i field F in n common variables and each of degree d. If $n > rd^i$, then $f_1, \dots, f_r$ have a common non trivial zero in F.*

Theorem 6.2.2. *Let $f_1, \dots, f_r$ be polynomials without constant term in a strongly C_i field F in n common variables and of degree at most d. If $n > rd^i$, then $f_1, \dots, f_r$ have a common non trivial zero in F.*

Proof. The strategy of this proof is to argue by contradiction. We assume that $f_1, \dots, f_r$ only have the trivial common zero. The lemmas from the previous section guarantee the existence a form in F with only the trivial zero in F, that also satisfies the criterium that the number of variables is bigger than the i-th power of the degree. This is a contradiction with the fact that F is a C_i field. Thus our assumption is not true.

The case where $d = 1$ is first discussed. In that case $f_1, \dots, f_r$ is a system of F-linear equations with $n > r$. It follows that $f_1, \dots, f_r$ have a non trivial common zero in F. Now assume that $d > 1$. Consider the case where F is algebraically closed. In that case Theorem 5.2.1 provides the desired result. Now assume that F is not algebraically closed. Theorem 3.2.1 shows that being algebraically closed is equivalent to being C_0. Thus F is not C_0. Because of this there exists a form ϕ in F, in n_ϕ variables, of degree d_ϕ and that only has the trivial zero in F. Note that $n_\phi, d_\phi > 1$ and that $n_\phi = d_\phi$. Assume without loss of generality that $n_\phi \geq r$. If this is not the case then one can consider $\phi^{(t)}(\phi)$ for large enough t. From

Lemma 6.1.1 it follows that $\phi^{(t)}(\phi)$ is a form that only has the trivial zero in F. Corollary 5.1.14 implies that $n_{\phi^{(t)}} = n_\phi^{t+1}$. Since $n_\phi > 1$, it follows that there exists $T \in \mathbb{N}$ such that $n_{\phi^{(T)}} \geq r$. Now $\phi^{(T)}(\phi)$ is called the new ϕ. Assume that $f_1, \ldots, f_r$ only have the trivial common zero in F. From Lemma 6.1.1 it follows that $\phi^{(t)}(f_1, \ldots, f_r)$ only has the trivial zero in F for all $t \in \mathbb{N}$. Lemma 6.1.3 implies that there exists a $t^* \in \mathbb{N}$ such that $\frac{n_{\phi^{(t^*)}}}{(d_{\phi^{(t^*)}})^i} > 1$. It follows that $n_{\phi^{(t^*)}} > d^i_{\phi^{(t^*)}}$. From the fact that F is C_i it follows that $\phi^{(t^*)}(f_1, \ldots, f_r)$ has a non trivial zero in F. This is a contradiction. Thus $f_1, \ldots, f_r$ do have a common non trivial zero in F. □

Theorems 6.2.1 and 6.2.2 give rise to an alternative definition of (strongly) C_i fields. This definition is not included in Nagata's or Lang's paper. Theorem 6.2.1 shows that if a field F is C_i then each system of forms $f_1, \ldots, f_r$ in F, where $n > rd^i$, has a common non trivial zero in F. The converse is also true. Let F be a field such that each system of forms $f_1, \ldots, f_r$ in F, where $n > rd^i$ for some $i \in \mathbb{N}$, has a common non trivial zero in F. Then F is C_i. This follows by considering the case where $r = 1$. Something similar is true in the case of polynomials without constant term and strongly C_i fields. This gives us the following alternative definition of (strongly) C_i fields. This definition is more general than the original definition but also equivalent to the original definition.

Definition 6.2.3 (alternative C_i)**.** Let $i \in \mathbb{N}$ be an integer. A field F is called C_i if every system of r forms in F, in n variables and of degree d, with $n > rd^i$, has a non trivial common zero in F.

Definition 6.2.4 (alternative strongly C_i)**.** Let $i \in \mathbb{N}$ be an integer. A field F is called strongly C_i if every system of r polynomials without constant term in F, in n variables and of degree at most d, with $n > rd^i$, has a non trivial common zero in F.

6.3 Comparison between Lang's and Nagata's results

In this section the differences between the results from Lang's and Nagata's papers are examined. First, Theorem 5.2.2 is compared to Theorems 6.2.1 and 6.2.2. After that it is explained that Nagata's method does not work for Theorem 5.3.1.

The main difference between Theorem 5.2.2 and Theorems 6.2.1, 6.2.2, is that Theorem 5.2.2 requires the existence of a normic form. As explained below, Lang uses this normic form for two reasons. We use some notation from the proof of Theorem 5.2.2. More specifically, N is used to denote a normic form of order i. First, Lang uses the fact that a normic form only has a trivial zero to ensure that $\xi(N, f_1, \ldots, f_r)$ only has a trivial zero if $f_1, \ldots, f_n$ only have the trivial common zero. Nagata works around this by splitting the problem in one case where F is algebraically closed and one case where F is not algebraically closed. A field is C_0 if and only if it is algebraically closed. If a field is not C_0, the existence of a form with only the trivial zero follows, thus the specific existence of a normic form is not needed. Second, Lang uses the normic form to ensure that $\xi(N, f_1, \ldots, f_r)$ has the proper amount of variables and degree needed. Nagata avoids this by showing that the ratio between the number of variables and i-th power of the degree of $\phi^{(t)}(f_1, \ldots, f_r)$ grows to infinity as t goes to infinity, as long as $n > rd^i$, $i > 0$ and $d > 1$. Using this, Nagata can construct a form where the number of variables is bigger than the i-th power of the degree. Nagata avoids the use of a normic form by repeatedly using the substitution ξ. Another difference between Theorem 5.2.2 and Theorem 6.2.2 is that, in the strongly C_i case, Nagata requires $f_1, \dots, f_r$ to have a degree of at most d. Lang requires them to have a degree of exactly d. As discussed in section 5.2, one can extend the proof presented by Lang to include the case where the degree of the polynomials without constant term is at most d. The proof of Theorem 5.2.2 still works if one allows $f_1, \dots, f_r$ to be of different degree. The reason for this is that $\xi(\phi, f_1, \dots, f_r)$ is still a polynomial without constant term in this case. When considering forms, $f_1, \dots, f_r$ need to be of the same degree to ensure that $\xi(\phi, f_1, \dots, f_r)$ is again a form. A C_i field that is also C_{i-1} does not admit normic forms of order i. An interesting question to investigate is whether there are C_i fields, that are not C_{i-1}, which do not have a normic form of order i. Similarly, one can ask whether there are strongly C_i fields, that are not strongly C_{i-1}, which do not have a normic polynomial of order i. The result that every C_i field that is not C_{i-1} has a normic form of order i would make Theorem 6.2.1 and Theorem 5.2.2 equivalent. This would indicate whether Theorems 6.2.1, 6.2.2 truly are stronger results.

We now discuss why one cannot extend Nagata's method to Theorem 5.3.1. It might be possible to combine the methods from the proofs of Theorem 6.2.1 and Theorem 5.3.1. This would result in a theorem similar to Theorem 5.3.1, but without requiring the existence of normic forms of order i of any degree. In order for the method from the proof of Theorem 5.3.1 to work, each form in the system of forms needs to be of the same degree. This requires the existence of forms $\phi_1, \dots, \phi_r$ in F with only the trivial zero in F, such that the degree of $\phi_1^{(t)}(f_1), \phi_2^{(t)}(f_2), \dots, \phi_r^{(t)}(f_r)$ is the same after t steps. This is a problem since the existence of forms with only the trivial zero of specific degrees is not guaranteed in a general C_i field. Secondly, the value of t is not known from the beginning. It is unclear for what value of t the degrees of $\phi_1^{(t)}(f_1), \phi_2^{(t)}(f_2), \dots, \phi_r^{(t)}(f_r)$ should be equal.

7 Applications of results on systems of forms

In this chapter two final results are proven. In section 7.1 it is shown that an algebraic field extension of a (strongly) C_i field is again a (strongly) C_i field. In section 7.2, Theorem 5.3.1 is used to provide a condition under which a C_i field is a strongly C_{i+1} field.

7.1 Algebraic extensions of (strongly) C_i fields are (strongly) C_i

In this subsection it is proven that an algebraic field extension of a (strongly) C_i field is again a (strongly) C_i field. First, finite field extensions are discussed. Then this result is extended to algebraic field extensions. In order to prove this, Lang uses Theorem 5.2.2 in the case of C_i fields and Theorem 5.3.1 in the case of strongly C_i fields. These theorems require the existence of normic forms of order i. Because of this, the results from Lang's paper only work for (strongly) C_i fields that admit normic forms of order i. The proof presented here uses Theorem 6.2.1 in the case of C_i fields and Theorem 6.2.2 in the case of strongly C_i fields. By doing this, the existence of normic forms of order i is not required. We will cover the C_i case and only highlight the differences with the strongly C_i case. Let us first consider finite field extensions.

Theorem 7.1.1. *Let E be a finite field extension of a C_i field F. Then E is a C_i field.*

Proof. Consider a form f in E in n variables and of degree d, such that $n > d^i$. Denote the degree of E over F by r and let $\omega_1, \ldots, \omega_r$ be a basis of E considered as F-vector space. Then there exist $a_{i,j,k} \in F$ for all $1 \leq i, j, k \leq r$ such that

$$\omega_i \omega_j = a_{i,j,1}\omega_1 + a_{i,j,2}\omega_2 + \ldots + a_{i,j,r}\omega_r. \tag{7.1}$$

Consider the following form X in E of degree one in r variables

$$X(X_1, \ldots, X_r) = X_1\omega_1 + \ldots + X_r\omega_r.$$

Since $\omega_1, \ldots, \omega_r$ form a basis of E considered as F-vector space, it follows that $\omega_1, \ldots, \omega_r$ are F-linearly independent. Thus X only has the trivial zero in F. In particular, X is non constant. Let us define the polynomial $f^* = \xi(f, X)$. From the fact that f is a form, it follows that f is non zero. Lemma 5.1.10 implies that $f^* = \xi(f, X)$ is not zero as a polynomial. Because of this, f^* is a form of degree d_f in nr variables, according to Lemma 5.1.13.
Now replace every $\omega_i\omega_j$ in f^* by the appropriate expression from (7.1). Note that this does not change the number of variables or the degree of terms of f^*. After this operation, f^* is still a form in nr variables of degree d, but can now be written in the following manner

$$\begin{aligned} f^*(X_{1,1}, \ldots, X_{1,r}, \ldots, X_{n,1}, \ldots, X_{n,r}) =& f_1(X_{1,1}, \ldots, X_{1,r}, \ldots, X_{n,1}, \ldots, X_{n,r})\omega_1 \\ &+ \ldots + \\ &f_r(X_{1,1}, \ldots, X_{1,r}, \ldots, X_{n,1}, \ldots, X_{n,r})\omega_r. \end{aligned} \tag{7.2}$$

Here, each $f_1, \ldots, f_r$ is a polynomial with coefficients in F. Since f^* is a form in nr variables of degree d, it follows that $f_1, \ldots, f_r$ are either zero as a polynomial or a form in nr variables of degree d. From the fact that $n > d^i$, it follows that $nr > d^i r$. Theorem 6.2.1 guarantees the existence of a non trivial common zero in F of all $f_1, \ldots, f_r$ that are not zero as a polynomial. This is also a non trivial common zero of all $f_1, \ldots, f_r$. Denote this zero by $(X^*_{1,1}, \ldots, X^*_{1,r}, \ldots, X^*_{n,1}, \ldots, X^*_{n,r})$. It follows from (7.2) that $(X^*_{1,1}, \ldots, X^*_{1,r}, \ldots, X^*_{n,1}, \ldots, X^*_{n,r})$ is a zero of f^*. This zero is used to generate n elements in E. These elements are the coordinates of a non trivial zero of f in E.

$$X^*_v = X^*_{v,1}\omega_1 + \ldots + X^*_{v,r}\omega_r$$

In the equation above $v \in \{1, \ldots, n\}$. As shown below, $(X^*_1, \ldots, X^*_n)$ is a zero of f.

$$\begin{aligned} f(X^*_1, \ldots, X^*_n) &= f(X^*_{1,1}\omega_1 + \ldots + X^*_{1,r}\omega_r, \ldots, X^*_{n,1}\omega_1 + \ldots + X^*_{n,r}\omega_r) \\ &= f^*(X^*_{1,1}, \ldots, X^*_{1,r}, \ldots, X^*_{n,1}, \ldots, X^*_{n,r}) = 0 \end{aligned}$$

Since $(X^*_{1,1}, \ldots, X^*_{1,r}, \ldots, X^*_{n,1}, \ldots, X^*_{n,r})$ is non trivial, there exists a $v \in \{1, \ldots, n\}$ such that $(X^*_{v,1}, \ldots, X^*_{v,r}) \neq (0, \ldots, 0)$. From the fact that $\omega_1, \ldots, \omega_r$ are F-linear independent, it follows that X^*_v is not equal to zero. Because of this $(X^*_1, \ldots, X^*_n)$ is a non trivial zero of f in E. It follows that E is C_i. □

The proof of the strongly C_i case is very similar. One difference is that $f_1, \ldots, f_r$ are not necessarily polynomials without constant term of equal degree, but of degree at most d. From Corollary 5.1.11 it follows that f^* is a polynomial without constant term of exactly degree d. However, it is unclear whether $f_1, \ldots, f_r$ each contain a term of degree d. One can still apply Theorem 6.2.2, thus the proof still works. Theorem 7.1.1 can be extended to algebraic field extensions. This is done by using the fact that a polynomial only has a finite number of coefficients.

Corollary 7.1.2. *Let E be an algebraic field extension of a C_i field F. Then E is a C_i field.*

Proof. Let f be a form in E such that $n > d^i$. Let $a_1, \ldots, a_k \in E$ be the coefficients of f for some $k \in \mathbb{N}$. From the fact that E is an algebraic extension, it follows that $a_1, \ldots, a_k$ are all algebraic over F. Because of this $F(a_1, \ldots, a_k)$ is a finite extension of F. Thus $F(a_1, \ldots, a_k)$ is a C_i field according to Theorem 7.1.1. Since $f \in F(a_1, \ldots, a_k)[X_1, \ldots, X_n]$, f has a non trivial zero in $F(a_1, \ldots, a_k) \subseteq E$. This is also a non trivial zero of f in E. □

7.2 Condition under which C_i fields are strongly C_{i+1}

In this section Theorem 5.3.1 is used to show that a C_i field that admits normic forms of order i of any degree is a strongly C_{i+1} field. This is then applied to C_1 fields in a corollary. This corollary states that a C_1 field that admits a finite field extension of degree p for every prime number p is strongly C_2. Let us now present the main result of this section. Naturally this result does not have a similar statement for strongly C_i fields.

Theorem 7.2.1. *Let F be a C_i field that admits normic forms of order i of any degree, then F is strongly C_{i+1}.*

Proof. Let f be a polynomial without constant term in F such that $n > d^{i+1}$. If f is zero as a polynomial, it has a non trivial zero in F. Now assume that f is not zero as a polynomial. The case where $d = 1$ is treated separately. If $d = 1$, then f looks as follows

$$f(X_1, \ldots, X_n) = \sum_{k=1}^{n} a_k X_k.$$

Here $a_k \in F$ for all $k \in \{1, \ldots, n\}$. Since $n > 1$, f has a non trivial zero in F. Now assume that $d > 1$. The polynomial f can be written in the following manner

$$f(X_1, \ldots, X_n) = f_1(X_1, \ldots, X_n) + \ldots + f_d(X_1, \ldots, X_n).$$

Here f_j is a form in n variables of degree j for all $j \in \{1, \ldots, d\}$. The polynomials $f_1, \ldots, f_d$ are a system of forms in n variables of degrees $d, d-1, \ldots, 1$ respectively. The following inequalities hold

$$d^i + (d-1)^i + \ldots + 1^i < d^i + d^i + \ldots + d^i = dd^i < n.$$

Theorem 5.3.1 implies that $f_1, \ldots, f_d$ have a non trivial common zero in F. This non trivial commmon zero is also a non trivial zero of f in F, thus F is strongly C_{i+1}. □

From Theorem 5.3.2 it follows that Theorem 7.2.1 only requires the existence of normic forms of order i of degree p for all prime numbers p. We finish this section with a corollary about C_1 fields. This result is not found in Lang's or Nagata's paper. The following result is a combination of Theorem 3.1.4, Theorem 7.2.1 and Theorem 5.3.2.

Corollary 7.2.2. *Let F be a C_1 field that has finite field extensions of degree p for every prime number p. Then F is strongly C_2.*

Proof. From Theorem 3.1.4 and the fact that F has finite field extensions of degree p for every prime number p, it follows that F has normic forms of order one of degree p for every prime number p. Theorem 5.3.2 implies that F has normic forms of order one of any degree. By combining this with Theorem 7.2.1, it follows that F is strongly C_2. □

8 Conclusion

In this section we summarise the results from this thesis and describe some interesting future questions. After introducing (strongly) C_i fields, two examples of C_i fields have been discussed. First, it was proven that the algebraically closed fields are exactly the (strongly) C_0 fields. Second, it was shown that finite fields are strongly C_1 and not C_0. Then systems of forms in C_i fields were considered. Nagata showed that a system of r forms in n common variables, each of degree d in a C_i field, has a non trivial common zero in F if $n > rd^i$. In the case of polynomials without constant term, $f_1, \ldots, f_r$ are required to have degree at most d. Lang proved a similar result concerning forms of different degree. In that case F is required to admit normic forms of order i of any degree. Let F be a C_i field that admits normic forms of order i of any degree. A system of r forms in n common variables of degrees $d_1, \ldots, d_r$ in F, has a non trivial common zero in F if $n > d_1^i + \ldots + d_r^i$. This statement also holds for polynomials without constant term and strongly C_i fields. Nagata's result implies that algebraic extensions of (strongly) C_i fields are (strongly) C_i as well. Lang's result implies that a C_i field that admits normic forms of order i of any degree is strongly C_{i+1}.

In this thesis we expand upon some of the results presented by Lang and Nagata. In section 3.3 it is shown that using $i = 0$ in the general definition of (strongly) C_i fields is equivalent to our separate definition of (strongly) C_0 fields. Definitions 2.1.5 and 2.2.2 contain the general definitions of (strongly) C_i fields and Definitions 2.1.3 and 2.2.1 contain our definitions of (strongly) C_0 fields. In section 5.1 the substitution ξ is introduced to provide one general method for many of the proofs. In the proof of Lemma 6.1.3, the case where $d = 1$ is excluded. In that case the lemma is not necessarily true. This is not done in Nagata's paper. Also results from Lang and Nagata are combined into stronger statements. It is shown that Theorem 5.2.2 is true for polynomials without constant term of degree at most d, instead of polynomials without constant term of degree exactly d. It is proven that Corollary 7.1.2 and Theorem 7.2.1 are true for general C_i fields. Not just C_i fields that admit normic forms of order i. Finally we were able to make the requirement that Theorem 5.3.1 places on C_i fields slightly less restricting. Theorem 5.3.2 states that a field F admits normic forms of order i of any degree, if F admits normic forms of order i of degree p for every prime number p.

The material from this thesis gives rise to several interesting questions. As discussed in section 6.3, one could wonder whether there exist C_i fields, that are not C_{i-1}, which do not admit normic forms of order i. This would show whether Theorems 6.2.1, 6.2.2 are actually stronger results than Theorem 5.2.2. Secondly, we would like to extend Theorem 5.3.1 to (strongly) C_i fields that do not necessarily have normic forms of order i of any given degree. The fact that Theorems 6.2.1, 6.2.2 hold for fields without normic forms make it plausible that this is possible. Another way to go about this is to show that each (strongly) C_i field, that is not (strongly) C_{i-1}, has normic forms or polynomials of order i of any given degree. If one would show that Theorem 5.3.1 is true for general (strongly) C_i fields, then this would imply that a C_i field is strongly C_{i+1}. Both Lang and Nagata comment in their paper that they find this likely to be the case. This would get us one step closer to answering the question whether a C_i field is strongly C_i.

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