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Spectral sequences obtained from towers in the ∞-category of spectra

Décalage, exact couples and Massey products

Master Thesis

Mathematial Sciences

Author: *Julie Creemers*

Supervisor:

Dr. Lennart Meier
Mathematical Institute, Utrecht University

Second Reader:

Dr. Gijs Heuts
Mathematical Institute, Utrecht University

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Abstract

There are various ways of constructing spectral sequences from the infinity category of towers of spectra. Classically there is the approach via exact couples; we will also discuss the décalage functor as constructed by Hedenlund. We will show that this construction of spectral sequences from towers and the construction via exact couples give isomorphic spectral sequences. This can be proved by showing that both methods of can be related to another third construction method by Lurie, using recent work by Antieau. Furthermore, the décalage construction yields a functor of infinity operads and provides a way to construct multiplicative spectral sequences. Then we can define Massey products on such a multiplicative spectral sequence. Lastly, we will discuss a possible relation between these Massey products and Toda brackets on homotopy groups of spectra, analogous to Moss' theorem for the Adams spectral sequence.

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Introduction

Spectral sequences are tools used to compute many objects in algebraic topology. Notable historical examples are the Atiyah-Hirzebruch spectral sequence, relating generalized cohomology theories to singular cohomology of spaces and the Adams spectral sequence, which arises from cohomology and is used to determine stable homotopy groups of spectra. There are many more, see for example [19]. In certain cases a spectral sequence has a multiplicative structure. This can be used to determine the differentials of the spectral sequence and to compute multiplicative structures on the objects the sequence converges to. For instance, multiplication in the cohomological Serre spectral sequence is used to determine ring structures of cohomology rings. Furthermore, there are also higher multiplicative structures, such as Massey products and Toda brackets, which we will explained later.

There are several ways of constructing spectral sequences; often they arise from towers or filtrations of spectra. For instance, Lurie provides a way of constructing spectral sequences from filtrations in a stable ∞-category in Higher Algebra [16], which we will also discuss in this thesis. However, the focus will be on Hedenlund's method of constructing spectral sequences from towers in the ∞-category of spectra, since this method yields multiplicative spectral sequences [12, Paper II]. She constructs spectral sequences via the décalage functor. This is an ∞-categorical functor

$$\mathrm{Tow}(\mathrm{Sp}) := \mathrm{Fun}(N(\mathbb{Z})^{op}, \mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$$

and defined using the Beilinson t-structure on $\mathrm{Tow}(\mathrm{Sp})$. Let $X \in \mathrm{Tow}(\mathrm{Sp})$. Then the Beilinson-Whitehead tower of X is the tower of towers arising from the Beilinson t-structure on the ∞-category of towers of spectra:

$$\dots \longrightarrow \tau^{Bei}_{\geq n+1}(X) \longrightarrow \tau^{Bei}_{\geq n}(X) \longrightarrow \tau^{Bei}_{\geq n-1}(X) \longrightarrow \dots$$

For each k, we can take the colimit of the tower $\tau^{Bei}_{\geq k}(X)$, which yields another tower of spectra

$$\dots \longrightarrow \mathrm{colim}_{i\in\mathbb{Z}}\, \tau^{Bei}_{\geq n+1}(X_i) \longrightarrow \mathrm{colim}_{i\in\mathbb{Z}}\, \tau^{Bei}_{\geq n}(X_i) \longrightarrow \mathrm{colim}_{i\in\mathbb{Z}}\, \tau^{Bei}_{\geq n-1} X_i \longrightarrow \dots$$

This resulting tower is called the **décalée** of X, which yields the décalage functor $\mathrm{Déc}\colon \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$. The décalage functor turns out to be a lax symmetric monoidal functor and can be iterated. The functor $E^{*,*}\colon \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{SSeq}$ is then defined by mapping a tower X to the spectral sequence

$$E^{n,s}_r(X) := \pi_n \,\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)).$$

That is, we take the homotopy groups of the $((r-1)n+s)$'th associated graded of the $(r-1)$-fold décalee of X. This construction generalizes several spectral sequence. Among them is the Atiyah-Hirzebruch spectral sequence, see [7].

A classical way of constructing spectral sequences is via exact couples. This method can also be applied to towers of spectra. That is, from a tower of spectra we get exact couples

$$\begin{array}{ccc} \oplus_{n,s}\pi_n(X(s)) & \xrightarrow{\quad i \quad} & \oplus_{n,s}\pi_n(X(s)) \\ & {\scriptstyle \kappa}\nwarrow \quad \swarrow{\scriptstyle j} & \\ & E_1(X) := \oplus_{n,s}\pi_n \,\mathrm{Gr}^s(X) & \end{array}$$

from which we can derive another exact couple. Iterating this precess will yield a spectral sequence, see [13]. Therefore, it makes sense to ask the question whether Lurie's method, décalage and the construction via exact couples yield isomorphic spectral sequences when applied to towers in the ∞-category of spectra. Indeed this is true and part of this thesis will be focussed on showing this. In particular, we will relate Lurie's method and the construction via exact couples and discuss Antieau's recent proof relating the décalage constructing with Lurie's method. This is an important result, as it will allow us to use the language of either method in proofs and discussions on these spectral sequences.

As mentioned, the construction of spectral sequences via décalage yields multiplicative spectral sequences. In particular, we shall see that it is possible to define higher multiplication structures on these spectral sequences as well, called Massey products. Furthermore, we want to relate them to another higher multiplication structure, Toda brackets, on homotopy groups of spectra. Massey products were initially introduced by Massey in 1958. They were later used by him to topologically describe Borromean Rings and their n-fold generalizations, Brunnian links in knot theory [18]. They have also been used to describe elements or differentials in various spectral sequences, such as the Adams spectral sequence, see [20], the Eilenberg-Moore spectral sequence [19, Ch. 8] and the Atiyah-Hirzebruch spectral sequence on twisted K-theory, see [2].

Generally, they are defined as follows, see also [19, p. 302,303]. Let Γ be a differential graded algebra and consider its corresponding cohomology ring $H^*(\Gamma)$. Let $\alpha = [a], \beta = [b], \gamma = [c] \in H^*(\Gamma)$ and write $\bar{a} = (-1)^{|a|+1}a$. Suppose $\alpha\beta = 0$ and $\beta\gamma = 0$. Then the Massey product is defined as

$$\langle \alpha, \beta, \gamma \rangle := \{[\bar{a}y + \bar{x}c] \colon d(x) = \bar{a}b,\, d(y) = \bar{b}c\} \subset H^{|\alpha|+|\beta|+|\gamma|+1}(\Gamma).$$

Furthermore, Toda introduced the Toda bracket originally to compute stable homotopy groups of spheres. They can be defined on general triangulated categories as follows. Let $\mathcal{C}$ be a triangulated category. Suppose we have a sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} W$$

in $\mathcal{C}$ such that $g \circ f = 0 = h \circ g$. Then we get an induced diagram

$$\begin{array}{ccccccc} X & \xrightarrow{f} & Y & \xrightarrow{i} & C_f & \longrightarrow & X[1] \\ \| & & \downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \psi} \\ X & \xrightarrow{f} & Y & \xrightarrow{g} & Z & \xrightarrow{h} & W \end{array}$$

where C_f is the cone of f. Note that neither the factorization of g into $\phi \circ i$ and the factorization of $h \circ \phi$ through C_f are unique. Therefore the Toda bracket $\langle f, g, h \rangle$ is defined to be the set of maps $\psi \colon X[1] \to W$ arising from such diagrams as the one above.

Moss discussed Massey products on the Adams spectral sequence and used them to describe the differentials in this spectral sequence, see [20]. Furthermore, in the same paper he proved that under certain conditions for a convergent Adams spectral sequence one can find an element in the Massey product of permanent cycles converging to an element in the corresponding Toda bracket. This was generalized by Belmont and Kong for spectral sequences arising from towers in an arbitrary symmetric monoidal stable topological model category and with a multiplicative Toda bracket instead of a composition Toda bracket, see [4].

We will use the multiplicative structure of décalage spectral sequences on associative algebra objects in Tow(Sp) to define Massey products on these spectral sequences. The goal then is to discuss a similar statement as the result by Belmont and Kong and give a potential proof strategy. In particular, we will need the result relating spectral sequences from exact couples with the décalage spectral sequence, as the paper by Belmont and Kong is written in the language of exact couples. The statement is as follows.

Pretheorem. Suppose we have a tower $X \in \mathrm{Tow}(\mathrm{Sp})$ with a multiplication $\mu \colon X \otimes X \to X$, associative up to coherent homotopy; and $X(-\infty) := \mathrm{colim}_{\mathbb{Z}} X$. Consider the associated multiplicative décalage spectral sequence $E_r^{n,s}(X) \Rightarrow \pi_n X(-\infty)$, which we assume to be weakly convergent. Let

$$a \in E_r^{n,s}(X),\, a' \in E_r^{n',s'}(X),\, a'' \in E_r^{n'',s''}(X)$$

be permanent cycles converging as in the set-up of Section 6.2 to

$$\alpha \in \pi_n X(-\infty),\, \alpha' \in \pi_{n'} X(-\infty)\,, \alpha'' \in \pi_{n''} X(-\infty)$$

such that aa' and $a'a''$ are d_r-boundaries and $\alpha\alpha'$ and $\alpha'\alpha''$ are null-homotopic. Then under a certain hypothesis, there exists an element in the Massey product $\langle [a], [a'], [a''] \rangle \subset E_{r+1}(X)$ converging to an element in the multiplicative Toda bracket $\langle \alpha, \alpha', \alpha'' \rangle$.

For this thesis, we will assume knowledge of category theory, basic knowledge of ∞-categories as well as a background in algebraic topology. For a first introduction in ∞-categories, we refer to [14]. We will provide an introduction in stable and (symmetric) monoidal ∞-categories as well as a description of the ∞-category of spectra in Chapter 2 and 3. Before this, in Chapter 1, we will briefly discuss spectral sequences and spectra, hence prior knowledge of spectral sequences is not necessary, but it is advisable as spectral sequences can seem rather complicated at first sight. In Chapter 4, we will discuss the décalage functor and Hedenlund's proof of multiplicativity of the resulting décalage spectral sequence. In Chapter 5, we will relate the décalage spectral sequence to one obtained via exact couples. In Chapter 6, we will define Massey products and Toda products and discuss their relation. In particular, we will give a proof strategy for our main statement.

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1 Spectra and Spectral Sequences

We will start with some preliminaries on spectra and spectral sequences. Spectra of spaces are historically used to determine stable behaviour of spaces, which is motivated by Freudenthal's suspension theorem. We can also view spectra as (co)homology theories and vice versa, see [10, §. 4.E].

On the other hand, spectral sequences are a useful tool for computations of homotopy and (co)homology groups. They can be used to relate generalized (co)homology theories with (co)homology of spaces, amongst other applications.

1.1 Spectral Sequences

We will start with a discussion of spectral sequences and the way they can be graded, as well as what it means for spectral sequences to converge.

Definition 1.1. Let R be a ring. Then a **spectral sequence of R-modules** consists of

- R-modules E_r for $r \geq 1$,
- R-module homomorphisms $d_r \colon E_r \to E_r$ with the property that $d_r \circ d_r = 0$,
- isomorphisms $\phi_r \colon E_{r+1} \to H(E_r) := \ker(d_r)/\operatorname{im}(d_r)$.

We refer to a spectral sequence (E_r, d_r) as a **cohomological spectral sequence** if all modules E_r are bigraded and the differential has bidegree $(r, 1-r)$. That is, for each $i, j \in \mathbb{Z}$ we get that

$$d_r \colon E_r^{i,j} \to E_r^{i+r,i-r+1},$$

and $E_{r+1}^{i,j} = \ker(d_r \colon E_r^{i,j} \to E_r^{i+r,i-r+1})/\operatorname{im}(d_r \colon E_r^{i,j} \to E_r^{i-r,i+r-1})$. Similarly, a **homological spectral sequence** has a differential with bidegree $(-r, r+1)$ and a spectral sequence is **Adams** graded if the differential $d_r \colon E_r \to E_r$ has bidegree $(r, r-1)$.

Remark 1.2. The grading conventions are somewhat arbitrary; one can move from one to another via linear transformations. When visualizing a spectral sequence, the first index is usually placed on the x-axis and the second is placed on the y-axis. The exception is the Adams graded spectral sequences. This is often written down with s on the vertical axis, and $t-s$ on the horizontal axis, where $t-s$ is the total degree of an element $x \in E_r^{s,t}$. This changes the bidegree of the differential d_r to $(-1, r)$ relative to the x and y-axis. That is, if E denotes an Adams graded spectral sequence, then by placing s on the x-axis and $t-s$ on the y-axis, we obtain the spectral sequence $\tilde{E}$ which then satisfies $\tilde{E}_r^{n,q} = E_r^{q,n+q}$, with n on the x-axis and q on the y-axis. The total degree then changes to $n+q-q = n$. The latter grading convention will be used in this thesis.

Classically, one can construct spectral sequences using exact couples, see [13] for more details. In short, the method goes as follows. Consider R-modules (E, A) together with maps i, j, k such that the triangle

$$\begin{array}{ccc} A & \xrightarrow{\quad i \quad} & A \\ & {\scriptstyle k}\nwarrow \qquad \swarrow{\scriptstyle j} & \\ & E & \end{array}$$

is exact at every corner. This is also called an **exact couple**. Set $d = j \circ k$. Then $d \circ d = 0$. From this we can derive another exact couple

$$\begin{array}{ccc} A_2 & \xrightarrow{\quad i_2 \quad} & A_2 \\ & {\scriptstyle k_2}\nwarrow \qquad \swarrow{\scriptstyle j_2} & \\ & E_2 & \end{array}$$

where $A_2 := \operatorname{im}(i) \subset A$ and $E_2 := H(E, d) = \ker(d)/\operatorname{im}(d)$ and the maps are defined by

- $i_2 = i\mid_{\operatorname{im}(i)}$;
- $k_2[e] = k(e)$;
- $j_2(a) = j(b)$ for $a = i(b)$.

We leave it to the reader to check that these maps are indeed well-defined and yield an exact couple. Iterating this process on the resulting **derived exact couple**, we get a spectral sequence where $E_{r+1}(X) \cong H(E_r, d_r = j_r\kappa_r)$ by definition. In fact, as a quotient of the E_1-page, the E_r-page of this spectral sequence can be rewritten as

$$E_r \cong \frac{\kappa^{-1}(\operatorname{im}(i^{r-1})}{j(\ker(i^{r-1})}.$$

Remark 1.3. Applying this method to double cochain complexes and its total complex, this construction is what results in the well-known cohomological Serre spectral sequence. Furthermore, if the double complexes are multiplicative, this yields a product structure on the spectral sequence. For a precise construction, we refer to [6]. Other spectral sequences which can be constructed using exact couples include the Atiyah-Hirzebruch spectral sequence, see [13].However, if we want to consider product structures on spectral sequences in general, looking at them from the perspective of exact couples makes it very hard to determine well-defined product structures on spectral sequences.

Next, we remark that we can form a of spectral sequences, SSeq, by defining morphisms as follows.

Definition 1.4. Let (D_r, d_r^D, ϕ_r^D) and (E_r, d_r^E, ϕ_r^E) be two spectral sequences. Then a map of spectral sequences $g\colon D \to E$ consists of morphisms

$$g_r\colon D_r \to E_r$$

such that the maps g_r are compatible with the structure maps of the spectral sequences. That is,

$$g_r \circ d_r^D = d_r^E \circ g_r \text{ and } H(g_r) \circ \phi_r^D = \phi_r^E \circ g_{r+1}.$$

Often one is interested in whether a spectral sequence converges or not and if so, to what. We follow Boardman [5, p. 63] for the definition of convergence.

Definition 1.5 (E_∞-page)**.** Let $(E_r^{n,s}, d_r, \phi_r)$ be a spectral sequence. Then we have subgroups

$$0 = B_1^{n,s} \subset B_2^{n,s} \subset \ldots Z_2^{n,s} \subset Z_1^{n,s} = E_1^{n,s},$$

where $Z_r^{n,s}$ consists of those elements of $E_1^{n,s}$ which are killed by $d_1, \ldots d_r$, i.e. d_r**-cycles**. Similarly, $B_r^{n,s}$ consists of those elements of $E_1^{n,s}$ which lie in the image of $d_1, \ldots d_r$, i.e. d_r**-boundaries**. Then we have

$$E_r^{n,s} \cong \frac{Z_{r-1}^{n,s}}{B_{r-1}^{n,s}}.$$

Furthermore, we define

$$Z_\infty^{n,s} := \bigcap_r Z_r^{n,s},$$

of which the elements are called the **infinite cycles**, and

$$B_\infty^{n,s} := \bigcup_r B_r^{n,s},$$

of which the elements are called the **infinite boundaries**. Next, we define the $E_\infty-$**page** by

$$E_\infty^{n,s} := \frac{Z_\infty^{n,s}}{B_\infty^{n,s}}.$$

Hence on E_∞ all in- and outgoing differentials are zero.

Example 1.6. In the case of the spectral sequence obtained via exact couples we have $Z_r = \kappa^{-1}(\mathrm{im}(i^{r-1}))$ and $B_r = j(\ker(i^{r-1}))$.

Definition 1.7 (Convergence)**.** Let $(E_r^{n,s}, d_r, \phi_r)$ be a spectral sequence. Let M^* be a graded ring. Then we say that E_r **converges weakly** to M, often written $E_2^{n,s} \implies M^n$, if

- there is an exhaustive filtration on M^*, i.e.

$$0 \subseteq \cdots \subseteq F^{-1}M \subseteq F^0M \subseteq F^1M \subseteq \cdots \subseteq M$$

  with $M = \bigcup F^iM$;

- there are isomorphisms

$$E_\infty^{n,s} \cong \frac{F^sM^n}{F^{s+1}M^n}$$

  for all $n, s \in \mathbb{Z}$.

We say that E_r **converges** to M if E_r converges weakly to M and the filtration is Hausdorff as well. That is, $0 = \bigcap F^iM$. Lastly, we say that E_r **converges** strongly to M if E_r converges weakly to M and the filtration is complete Hausdorff. For a precise definition of this, we refer to [5, Part I].

1.2 Spectra

In this section we discuss spectra from several points of view. Firstly we look at them from the perspective of spaces.

Definition 1.8. A **spectrum** consists of an infinite sequence of pointed spaces $\{X_n\}_{n\in\mathbb{Z}}$ together with maps $\Sigma X_n \to X_{n+1}$.

Example 1.9. The simplest example is the sphere spectrum $\mathbb{S}$, which is defined by $\mathbb{S}_n = \Sigma^n(*)$, i.e. the n-sphere. More generally, for any pointed topological space X, we have the **suspension spectrum** $\Sigma^\infty X$, defined by $\Sigma^\infty X_n = \Sigma^n X$ and the obvious maps, with $X_{-n} = X$ for all $n \in \mathbb{N}$.

Example 1.10. Another important example of a spectrum consists of the Eilenberg-MacLane spaces. Recall that for an abelian group A, $n \in \mathbb{N}$, the Eilenberg-MacLane spaces $K(A, n)$ are determined by

$$\pi_k(K(A,n), A) \cong \begin{cases} A & \text{for } k = n \\ 0 & \text{else} \end{cases}$$

and unique up to weak homotopy equivalence. Furthermore, we have weak homotopy equivalences $K(A, n) \to \Omega K(A, n + 1)$, which makes the Eilenberg-Maclane spaces of A into a spectrum by taking the adjoints of these equivalences.

Such a spectrum, where the adjoint maps of the suspension maps into the loop spaces are in fact weak equivalences, is also called a **Ω-spectrum**. A space X_0 admits a delooping if we can write $X_0 \simeq \Omega X_1 \simeq \Omega X_2 \ldots$.

Originally, the motivation for came from studying homotopy groups of spheres and the Freudenthal Suspension Theorem, which states the following, see also [10, Cor. 4.24].

Theorem 1.11. *Let X be an $(n-1)-$connected CW complex. Then $\pi_i(X) \to \pi_{i+1}(\Sigma X)$ induced by taking the suspension functor is an isomorphism for $i < 2n - 1$ and surjective for $i = 2n - 1$.*

As a consequence, for any suitable pointed space X, the sequence

$$\pi_k(X) \to \pi_{k+1}(\Sigma X) \to \pi_{k+2}(\Sigma^2 X) \to \ldots$$

must eventually stabilize.

Definition 1.12. The k**'th stable homotopy group** of a pointed space X is defined as

$$\pi_k^{st}(X) := \operatorname{colim}_n \pi_{k+n}(\Sigma^n X)$$

In other words, we consider the suspension spectrum of X and its homotopy groups and see where they stabilize. For the sphere spectrum $\mathbb{S}$, the k'th homotopy group of $\mathbb{S}$ is given by $\pi_k^{st}(*)$. In general, if we have a spectrum $\{X_n\}_{n \geq \mathbb{N}}$, its k'th homotopy group is given by

$$\pi_k^{st} \operatorname{colim}_n \pi_{k+n}(X_n).$$

Alternatively, we can view spectra as generalized reduced cohomology theories. Indeed, the suspension isomorphisms of a generalized cohomology theory give rise to spectra and in turn any spectrum defines a generalized (co)homology theory by Brown's Representability Theorem, see [10, Thm. 4E.1].

Theorem 1.13. *Every reduced cohomology theory on the category of base-pointed CW complexes and base-point preserving maps has the form $h^n(X) = [X, K_n]_*$, where $\{K_n\}_{n\in\mathbb{Z}}$. is a loop spectrum.*

2 Monoidal ∞-categories

2.1 Monoidal ∞-categories

Since we will look at spectral sequences from an infinity-categorical perspective, the following two chapters will be concerned with generalizing certain 1-categorical concepts to an ∞-categorical setting, which go somewhat further than a first introduction in ∞-categories such as in [14]. In this chapter, we will consider what it means for ∞-categories to have a monoidal structure. Recall that an ordinary monoidal category $\mathcal{C}$ is a category with a pairing $\otimes$ and certain natural transformations which satisfy certain associativity conditions, also called MacLane's pentagram, see [21, App. E.2].

Alternatively, it turns out that we can encode the monoidal structure of a monoidal category in one single map, which is a Grothendieck opfibration, see [9, §. 4.1]. It is via this viewpoint we can define monoidal ∞-categories, as defining associativity conditions via a "pentagram" would result in an infinite amount of conditions to set. In this section we mostly use the survey made by Moritz Groth [9], which in turn relies heavily on multiple works by Lurie. We start with the following analogue of ordinary categorical coCartesian lifts.

Definition 2.1. Let $\mathcal{C}, \mathcal{D}$ be ∞-categories, $p\colon \mathcal{C} \to \mathcal{D}$ a functor. Then a morphism $f\colon c_1 \to c_2$ in $\mathcal{C}$ is **p-coCartesian** or a p-coCartesian lift of $\alpha = p(f)\colon d_1 \to d_2$ if the map

$$\phi\colon \mathcal{C}_{f/} \to \mathcal{C}_{c_1/} \times_{\mathcal{D}_{p(c_1)/}} \mathcal{D}_{p(f)/}$$

is a trivial Kan fibration.

Here the map ϕ is induced by the diagram

$$\begin{array}{ccc} \mathcal{C}_{f/} & \xrightarrow{p} & \mathcal{D}_{p(f)/} \\ \downarrow & & \downarrow \\ \mathcal{C}_{c_1/} & \xrightarrow{p} & \mathcal{D}_{p(c_1)/}, \end{array}$$

where the vertical arrows are given by the forgetful functor. This means that if we have a 2-simplex σ

$$\begin{array}{ccc} & d_2 & \\ {}^{p(f)}\nearrow & & \searrow \\ d_1 & \longrightarrow & d \end{array}$$

and a lift $c_1 \to e$ in $\mathcal{C}$ of $d_1 \to d$ then we have a unique lift of the above 2-simplex in $\mathcal{C}$, i.e.

$$\begin{array}{ccc} & c_2 & \\ {}^{f}\nearrow & & \searrow \\ c_1 & \longrightarrow & e, \end{array}$$

up to equivalence. Next, consider a map $p\colon C \to D$ between ∞-categories. Then we have the following defintion as analogue of Grothendieck opfibrations for ordinary categories.

Definition 2.2. A map $p\colon \mathcal{C} \to \mathcal{D}$ between ∞-categories is a **coCartesian fibration** if p is an inner fibration and for every vertex $c \in \mathcal{C}$ and morphism $\alpha\colon p(c) \to d$ in $\mathcal{D}$ there is a p-coCartesian lift $f\colon c \to c'$ of α in $\mathcal{C}$.

We can then define monoidal ∞-categories as follows.

Definition 2.3. (Monoidal ∞-category) A **monoidal ∞-category** is a coCartesian fibration $p\colon \mathcal{C}^{\otimes} \to N(\Delta^{op}) \cong N(\Delta)^{op}$ such that the **Segal maps**

$$\sigma_n\colon \mathcal{C}^{\otimes}_{[n]} \to (\mathcal{C}^{\otimes}_{[1]})^{\times n}$$

induced by $\langle i-1, i\rangle^{op}\colon [n] \to [1]$ in Δ^{op} are equivalences. The fiber $\mathcal{C} := \mathcal{C}^{\otimes}_{[1]}$ is the underlying monoidal ∞-category.

In particular, this implies that for a monoidal ∞-category the fiber $\mathcal{C}^{\otimes}_{[0]}$ is contractible. Also, the map $s_0\colon [0] \to [1]$ in Δ^{op} induces a functor $s\colon * \simeq \mathcal{C}^{\otimes}_{[0]} \to \mathcal{C}^{\otimes}_{[1]}$ which yields a unit in $\mathcal{C}^{\otimes}_{[1]}$ up to equivalence. Furthermore, by definition the maps $\langle 0,1\rangle\colon [1] \to [2]$ and $\langle 1,2\rangle\colon [1] \to [2]$ yield an equivalence

$$\sigma_2\colon \mathcal{C}^{\otimes}_{[2]} \to \mathcal{C}^{\otimes}_{[1]} \times \mathcal{C}^{\otimes}_{[1]}$$

Write σ_2^{-1} for a homotopy inverse. Then together with the functor induced by $d_1\colon [2] \to [1]$ we get a map

$$\mathcal{C}^{\otimes}_{[1]} \times \mathcal{C}^{\otimes}_{[1]} \xrightarrow{\sigma_2^{-1}} \mathcal{C}^{\otimes}_{[2]} \to \mathcal{C}^{\otimes}_{[1]},$$

defined up to equivalence. This yields an associative and unital structure on $\mathcal{C}^{\otimes}_{[1]}$ up to coherent homotopy. Furthermore, for a monoidal ∞-category $\mathcal{C} = \mathcal{C}^{\otimes}_{[1]}$ the homotopy category $h(\mathcal{C})$ is in fact a monoidal category in the ordinary categorical sense.

Example 2.4. One way of generating monoidal ∞-categories is by taking the nerve of ordinary monoidal categories. That is if $p\colon \mathcal{C}^{ot} \to \Delta^{op}$ is an ordinary monoidal structure, then taking the nerve

$$N(p)\colon N(\mathcal{C}^{\otimes}) \to N(D^{op})$$

yields a monoidal ∞-category. In particular, since $\mathrm{id}\colon \Delta^{op} \to \Delta^{op}$ is isomorphic to the Grothendieck construction on the one-point category and thus a monoidal structure, we have that

$$\mathrm{id} = N(\mathrm{id})\colon N(\Delta^{op}) \to N(\Delta^{op})$$

is a monoidal ∞-category. For more on this, as well as a generalization we refer to [9, Ex. 4.15].

Analogous to the ordinary categorical sense, we can define algebra objects of a monoidal ∞-category. We note that for ordinary categories, these are defined as objects A within a monoidal category $\mathcal{C}$ together with maps $\mu\colon A \otimes A \to A$ and $\eta\colon 1_M \to A$ compatible with the monoidal structure in $\mathcal{C}$. For instance, monoids or algebra objects in $(\mathrm{Ab}, \otimes)$ are rings. Again, such a definition is hard to translate in a convenient way to the ∞-categorical setting. However, the algebra objects can also be described using maps, see [9, Prop. 4.21], which we can take as a definition for ∞-categories.

Definition 2.5. Let $\alpha\colon [m] \to [n]$ be a morphism in Δ. Then α is **convex**, if α is injective and the image is an interval, i.e., $\mathrm{im}(\alpha) = [\alpha[0], \alpha(m)] \subset [n]$.

Definition 2.6. Let $p\colon \mathcal{C}^{\otimes} \to N(\Delta^{op})$ be a monoidal ∞-category. Then an **(associative) algebra object** of $\mathcal{C}$ is a section $s\colon N(\Delta^{op}) \to \mathcal{C}^{\otimes}$ such that arrows in $N(\Delta^{op})$ defined by convex maps α in Δ are sent to p-coCartesian arrows in $\mathcal{C}^{\otimes}$.

This results in a map on $A[1]$ which is associative and unital up to coherent homotopy. Furthermore, algebra objects are in fact a special case of functors between monoidal ∞-categories which preserve the monoidal structure to some extent. To be more precise, algebra objects are lax monoidal functors. These are defined as follows.

Definition 2.7. Let $p\colon \mathcal{C}^{\otimes} \to N(\Delta^{op})$ and $q\colon \mathcal{D}^{\otimes} \to N(\Delta^{op})$ be two monoidal ∞-categories. Then a functor $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $N(\Delta^{op})$, i.e.

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{\quad F \quad} & \mathcal{D}^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & N(\Delta^{op}) & \end{array}$$

is called **lax monoidal** if it sends p-coCartesian lifts of convex maps in $N(\Delta^{op})$ to q-coCartesian maps in $\mathcal{D}^{\otimes}$. Furthermore, such a functor F is called **monoidal** or **strong monoidal** if it sends all p-coCartesian arrows in $\mathcal{C}^{\otimes}$ to q-coCartesian maps in $\mathcal{D}^{\otimes}$.

In particular, we write $\text{Fun}^{lax}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ and $\text{Fun}^{mon}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ for the full subcategories of the ∞-category $\text{Fun}_{N(\Delta^{op})}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ of lax and strong monoidal functors respectively. Also, we get an ∞-category of associative algebra objects of a monoidal ∞-category $p\colon \mathcal{C}^{\otimes} \to N(\Delta^{op})$ via

$$\text{Alg}_{\mathbb{A}_\infty}(\mathcal{C}^{\otimes}) := \text{Fun}^{lax}(N(\Delta^{op}), \mathcal{C}^{\otimes}).$$

Indeed, sections of $p\colon \mathcal{C}^{\otimes} \to N(\Delta^{op})$ lifting convex maps to p-coCartesian arrows are precisely the lax monoidal functors between the identity on $N(\Delta^{op})$ and p.

2.2 Symmetric Monoidal ∞-categories

So far we have only discussed monoidal ∞-categories and functors between them. However, it is desirable to have an ∞-categorical definition of symmetric monoidal categories as well. For this we note that Δ, since it is ordered, cannot give us the notion of symmetry we want. Therefore we consider the following category.

Definition 2.8. Let Fin_* be the category of finite pointed sets, which is spanned by objects $\langle n\rangle = \{0, \dots, n\}$, $n \in \mathbb{N}$, with 0 as base-point and base-point preserving maps.

First, as an analogue of the maps $\langle i, i-1\rangle\colon [1] \to [n]$ in Δ, we consider projection maps $p_i\colon \langle n\rangle \to \langle 1\rangle$ in Fin_* defined by $p_i(j) = 0$ for $j \neq i$ and $p_i(i) = 1$. We then have the following definition.

Definition 2.9. A **symmetric monoidal ∞-category** is a coCartesian fibration $p\colon \mathcal{C}^{\otimes} \to N(\text{Fin}_*)$ such that the Segal maps

$$s_n\colon \mathcal{C}^{\otimes}_{\langle n\rangle} \to \left(\mathcal{C}^{\otimes}_{\langle 1\rangle}\right)^{\times n},$$

induced by projections $p_i\colon \langle n\rangle \to \langle 1\rangle$ are equivalences.

Then for a symmetric monoidal ∞-category, the map $m\colon \langle 2\rangle \to \langle 1\rangle$ by $1, 2 \mapsto 1$ induces a map

$$\mathcal{C}^{\otimes}_{\langle 1\rangle} \times \mathcal{C}^{\otimes}_{\langle 1\rangle} \xrightarrow{s_n^{-1}} \mathcal{C}^{\otimes}_{\langle 2\rangle} \to \mathcal{C}^{\otimes}_{\langle 1\rangle},$$

where s_n^{-1} is an inverse for the equivalence given by the Segal maps. Furthermore, as before, we get a unit induced by the constant map $\langle 1\rangle \to \langle 0\rangle$. Additionally, the twist map $t\colon \langle 2\rangle \to \langle 2\rangle$ will give $\mathcal{C}^{\otimes}_{\langle 1\rangle}$ a commutative structure, up to homotopy.

Example 2.10. As for ordinary monoidal categories, an ordinary symmetric monoidal category can also be defined by a Grothendieck opfibration $p\colon \mathcal{C}^{\otimes} \to \text{Fin}_*$ satisfying the Segal condition and taking the nerve then yields a symmetric monoidal ∞-category. Also, $N(\text{id})\colon N(\text{Fin}_*) \to N(\text{Fin}_*)$ is a symmetric monoidal ∞-category. Conversely, the homotopy category of a symmetric monoidal ∞-category is an ordinary symmetric monoidal category.

As before, we can define commutative algebra objects and symmetric monoidal functors. In particular, as an analogue of convex maps, we have the following definition.

Definition 2.11. A morphism $\alpha\colon \langle m\rangle \to \langle n\rangle$ in Fin_* is called **inert** if $\alpha^{-1}(i)$ is a singleton for every $i \geq 1$.

In particular, the projection maps p_i from before are inert. We can then now define the desired algebra objects and functors.

Definition 2.12. Let $p\colon \mathcal{C}^{\otimes} \to N(\text{Fin}_*)$ and $q\colon \mathcal{D}^{ot} \to N(\text{Fin}_*)$ be two symmetric monoidal ∞-categories. Then a functor $F\colon \mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ over $N(\text{Fin}_*)$, i.e.

$$\begin{array}{ccc} \mathcal{C}^{\otimes} & \xrightarrow{\quad F \quad} & \mathcal{D}^{\otimes} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle q} & \\ & N(\Delta^{op}) & \end{array}$$

is called **lax symmetric monoidal** if it sends p-coCartesian lifts of inert maps in $N(\mathrm{Fin}_*)$ to q-coCartesian maps in $\mathcal{D}^{\otimes}$. Furthermore, such a functor F is called **symmetric monoidal** or **strong symmetric monoidal** if it sends all p-coCartesian arrows in $\mathcal{C}^{\otimes}$ to q-coCartesian maps in $\mathcal{D}^{\otimes}$.

As before, we have ∞-categories $\mathrm{Fun}^{sLax}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}) \subset \mathrm{Fun}^{sMon}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$ spanned by lax and (strong) symmetric monoidal functors in $\mathrm{Fun}_{N(\mathrm{Fin}_*)}(\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes})$. We then define **commutative algebra objects** E of a symmetric monoidal ∞-category $\mathcal{C}^{\otimes}$ to be the lax monoidal functors between the identity $N(\mathrm{Fin}_*) \to N(\mathrm{Fin}_*)$ and $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$. We write

$$\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C}^{\otimes}) = \mathrm{Fun}^{sLax}(N(\mathrm{Fin}_*), \mathcal{C}^{\otimes})$$

for the ∞-category of commutative algebra objects on $\mathcal{C}^{\otimes}$.

Since we want to be able to determine whether functors are monoidal or not, we note that we can construct new monoidal functors from previously existing ones in the following way.

Proposition 2.13. *Let $\mathcal{C}^{\otimes}, \mathcal{D}^{\otimes}, \mathcal{E}^{\otimes}$ be symmetric monoidal ∞-categories.*

- *Then the composition $G \circ F \colon \mathcal{C}^{\otimes} \to \mathcal{E}^{\otimes}$ of two (lax) symmetric monoidal functors F, G is (lax) symmetric monoidal.*
- *There is an inverse equivalence between the ∞-category of lax symmetric monoidal right-adjoints $\mathcal{C}^{\otimes} \to \mathcal{D}^{\otimes}$ and oplax symmetric monoidal left-adjoints $\mathcal{D}^{\otimes} \to \mathcal{C}^{\otimes}$ [11, Prop. A]*

Remark 2.14. As is mentioned on [11, p. 2], a symmetric monoidal ∞-category can be described via a coCartesian fibration $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*)$, but also by a Cartesian fibration $\mathcal{C}^{\otimes} \to N(\mathrm{Fin}_*^{op})$. An oplax symmetric monoidal functor is then a functor between Cartesian fibrations over $N(Fin_*)$ which sends Cartesian lifts of inert maps to Cartesian maps. In particular, any strong symmetric monoidal functor is also oplax symmetric monoidal.

Remark 2.15. Symmetric monoidal ∞-categories are in turn specific cases of ∞-operads. Indeed, ordinary symmetric monoidal categories are specific kinds of coloured operads, which are collections of objects together with some kind of "multilinear" maps. This in turn can be generalized to the ∞-categorical setting. For the precise formulation of this we refer to [16, §. 2.1.1].

3 Stable ∞-categories

3.1 Stable ∞-categories and t-structures

In this section, we will discuss stable ∞-categories, t-structures and the ∞-category of spectra. An interesting property of stable ∞-categories is that their homotopy category is triangulated and fiber and cofiber sequences are the same. We will follow the approach of Groth [9, Section 5] and Lurie [16, Ch. 1] in defining stable infinity categories and further discussion.

Definition 3.1. Let $\mathcal{C}$ be an ∞-category. We say that C is **pointed** if it has a zero object. That is, if it has an object which is both terminal and initial in $\mathcal{C}$.

In particular, let C be a pointed ∞-category, say with zero object 0. Then for all objects $x, y \in C$ the mapping spaces $\mathrm{Map}_{\mathcal{C}}(0, y)$ and $\mathrm{Map}_{\mathcal{C}}(x, 0)$ are contractible, hence we have a map

$$\begin{array}{ccc} x & \xrightarrow{0_{x,y}} & y \\ & \searrow \quad \nearrow & \\ & 0 & \end{array}$$

uniquely defined up to a contractible Kan complex.

Example 3.2. Let C be an ∞-category with a terminal object $*$. Then the coslice ∞-category $C_{*/}$ is a pointed ∞-category, with a zero object $* \to *$. In particular, we write $\mathcal{S}_* := \mathcal{S}_{*/}$ for the ∞-category of pointed spaces. Note that $* := \Delta^0$. The zero object $\Delta^0 \to \Delta^0$ in $\mathcal{S}$ is also referred to as the **0-sphere**, and denoted by S^0.

Definition 3.3. Let $\mathcal{C}$ be a pointed ∞-category. Then a **triangle** in $\mathcal{C}$ is a diagram $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ given by

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & z \end{array}$$

where 0 is a zero object of $\mathcal{C}$. Furthermore, we say that a triangle $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ is **exact** or **a fiber sequence** if it is a pullback square in $\mathcal{C}$, i.e. $q\colon \Delta^1 \times \Delta^1 \cong (\Lambda^2_2)^{\triangleleft} \to \mathcal{C}$ is a limit cone. Similarly, we say that $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ is **coexact** or **a cofiber sequence** if it is a pushout square in $\mathcal{C}$, i.e. $q\colon \Delta^1 \times \Delta^1 \cong (\Lambda^2_0)^{\triangleright} \to \mathcal{C}$ is a colimit cone.

If we have a triangle $q\colon \Delta^1 \times \Delta^1 \to \mathcal{C}$ by

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & & \downarrow g \\ 0 & \longrightarrow & z \end{array}$$

which is a fiber sequence, we refer to x as a **fiber** of g, whereas if q is a cofiber sequence, we refer to z as a **cofiber** of f. In particular, fibers and cofibers are unique up to equivalence.

If enough limits and colimits exist in an ∞-category, we can define the loop and suspension functors. In particular, suppose C is a finitely complete and cocomplete, pointed ∞-category. Then we can consider the full subcategory

$$\mathcal{C}^{\Sigma} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$$

spanned by cofiber sequences in $\mathcal{C}$ with zero objects in both the upper right and the lower left corner, i.e. pushout squares

$$\begin{array}{ccc} x & \longrightarrow & 0' \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & y. \end{array}$$

Note that in a way, these diagrams are determined by the objects $x \in \mathcal{C}$. Similarly, we can consider the full subcategory

$$\mathcal{C}^{\Omega} \subseteq \mathrm{Fun}(\Delta^1 \times \Delta^1, \mathcal{C})$$

spanned by fiber sequences

$$\begin{array}{ccc} x & \longrightarrow & 0' \\ \downarrow & \lrcorner & \downarrow \\ 0 & \longrightarrow & y \end{array}$$

in $\mathcal{C}$. In a way, these diagrams are determined by the objects $x \in \mathcal{C}$, in particular, as a consequence of [15, 4.3.2.15] the evaluation maps

$$\mathrm{ev}_{(0,0)} \colon \mathcal{C}^{\Sigma} \to \mathcal{C}$$

and

$$\mathrm{ev}_{(1,1)} \colon \mathcal{C}^{\Omega} \to \mathcal{C}$$

are trivial Kan fibrations. Hence they admit sections s^{Σ} and s^{Ω} which are unique up to equivalence [17, Cor. 1.5.5.5]. We then define the suspension and loop functors as follows.

Definition 3.4 (Suspension and loop)**.** Let $\mathcal{C}$ be a pointed finitely complete and finitely cocomplete ∞-category. Then we define the **suspension functor** Σ as the composition

$$\Sigma \colon \mathcal{C} \xrightarrow{s_{\Sigma}} \mathcal{C} \xrightarrow{\mathrm{ev}_{(1,1)}} \mathcal{C}.$$

Similarly, the **loop functor** is defined as the composition

$$\Omega \colon \mathcal{C} \xrightarrow{s_{\Omega}} \mathcal{C} \xrightarrow{\mathrm{ev}_{(0,0)}} \mathcal{C}.$$

In particular, for any object $c \in \mathcal{C}$, $\Sigma(c)$ fits into a diagram

$$\begin{array}{ccc} c & \longrightarrow & 0' \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma(c). \end{array}$$

which is unique up to equivalence. We have a pullback diagram instead of a pushout for $\Omega(c)$. Moreover, the following theorem holds.

Theorem 3.5. *Let $\mathcal{C}$ be a finitely complete and finitely cocomplete pointed ∞-category. Then the suspension functor Σ is left adjoint to the loop functor Ω.*

Remark 3.6. Technically, we can define the suspension functor and loop functor if every morphism in $\mathcal{C}$ admits a cofiber and fiber respectively. We do not require the existence of all finite (co)limits for the definition of each to be well-defined. Also, the adjunction above holds if both functors are defined.

Proof sketch. The unit of this adjunction $\eta \colon \mathrm{id}_{\mathcal{C}} \to \Omega \circ \Sigma$ is given by setting, for every $c \in \mathcal{C}$ the morphism η_c to be the map $\eta_c \colon c \to \Omega\Sigma(c)$ induced by the diagram by universal property of the pullback

$$\begin{array}{ccc} c & \longrightarrow & 0' \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & \Sigma(c). \end{array}$$

In turn, we can define a co-unit $\epsilon\colon \Sigma \circ \Omega \to \mathrm{id}_{\mathcal{C}}$ by setting $\epsilon_c\colon \Sigma\Omega(c) \to c$ to be the morphism induced by the diagram

$$\begin{array}{ccc} \Omega(c) & \longrightarrow & 0' \\ \downarrow & \ulcorner & \downarrow \\ 0 & \longrightarrow & c. \end{array}$$

Intuitively, it is not hard to see that these indeed yield an adjunction in the sense of [17, 6.2.1.1]. This can also be proved by using that $\mathrm{Map}_{\mathcal{C}}(-, d)\colon \mathcal{C}^{op} \to \mathcal{S}$ preserves limits. □

For stable ∞-categories, the suspension and loop functor satisfy an even stronger statement. In particular, we have the following definition [16, 1.1.1.9].

Definition 3.7. Let $\mathcal{C}$ a pointed ∞-category. Then $\mathcal{C}$ is stable if

- every morphism in $\mathcal{C}$ admits a fiber and a cofiber
- every triangle is a cofiber sequence if and only if it is a fiber sequence.

Note that if $\mathcal{C}$ is stable, then both suspension and loop functor are well-defined and $\mathcal{C}^{\Sigma} = \mathcal{C}^{\Omega}$. So then Σ and Ω are mutually inverse equivalences. Even stronger, we have the following theorem [16, Cor. 1.4.2.27] [9, Thm. 5.12].

Theorem 3.8. *Let $\mathcal{C}$ be a pointed ∞-category. Then the following are equivalent.*

- *$\mathcal{C}$ is stable*
- *$\mathcal{C}$ has all finite colimits and $\Sigma\colon \mathcal{C} \to \mathcal{C}$ is an equivalence.*
- *$\mathcal{C}$ has all finite limits and $\Omega\colon \mathcal{C} \to \mathcal{C}$ is an equivalence.*

Furthermore, if $\mathcal{C}$ is pointed and finitely complete and cocomplete, then $\mathcal{C}$ is stable if and only if every square is a pullback if and only if it is a pushout.

Another kind of nice ∞-categories are presentable ∞-categories. These are defined as follows, see also [15, p. 312].

Definition 3.9. An ∞-category $\mathcal{C}$ is presentable if it has all (small) colimits and is accessible.

Roughly this means that $\mathcal{C}$ is generated under suitable filtered colimits by a small full subcategory.

Example 3.10. The ∞-category $\mathcal{S}$ of spaces is presentable, in particular it is generated under colimits by one object, $\Delta^0 = *$.

The reason why presentable ∞-categories are relevant is mostly because as with ordinary categories, we have an adjoint functor theorem, see also [15, Cor. 5.5.2.9]. This states the following.

Theorem 3.11. *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between presentable ∞-categories. Then*

- *F has a right adjoint if and only if it preserves small colimits.*
- *F has a left adjoint if and only if it preserves small limits and it is accessible.*

3.2 The homotopy category of a stable ∞-category

In this section, we will describe the structure of the homotopy category of a stable ∞-category. For the notion of an additive category we refer to [16, 1.1.2.1]. We will state the definition of a triangulated category, as defined by Verdier [16, 1.1.2.5].

Definition 3.12. A **triangulated** category is an additive category $\mathcal{C}$ together with an equivalence $\Sigma\colon \mathcal{C} \to \mathcal{C}$, also denoted by $X \mapsto X[1]$, and a collection of **distinguished triangles**

$$X \to Y \to Z \to X[1],$$

satisfying Verdier's four axioms, see [16, 1.1.2.5] for the precise formulation.

Remark 3.13. For $X \in \mathcal{C}$ a triangulated category, we write $X[n]$ for the n-fold interation of Σ, for any $n \in \mathbb{Z}$.

Theorem 3.14. *Let $\mathcal{C}$ be a stable ∞-category. Then $h\mathcal{C}$ has the structure of a triangulated category.*

Proof sketch. While we will not check all axioms, but we will give a description of the triangulated structure on the category $h\mathcal{C}$. Firstly, since $\mathcal{C}$ is stable, it is in particular pointed. Hence $\mathcal{C}$ has a zero object, which also yields a zero object of $h\mathcal{C}$. Furthermore, the suspension functor is an equivalence $\Sigma\colon \mathcal{C} \to \mathcal{C}$, which also yields an equivalence $\Sigma\colon h\mathcal{C} \to h\mathcal{C}$.

For additivity, to see that $\hom_{h\mathcal{C}}(c,d)$ is a group, we note that $\hom_{h\mathcal{C}}(c,d) \cong \pi_0 \operatorname{Map}_{\mathcal{C}}(c,d)$. Furthermore, since pullbacks and pushouts in the ∞-category of (pointed) spaces are precisely (pointed) homotopy pullbacks and pushouts of simplicial sets, we have that the suspension functor Σ is in fact determined by equivalences $\operatorname{Map}_{\mathcal{C}}(\Sigma(c),d) \simeq \Omega \operatorname{Map}_{\mathcal{C}}(c,d)$. Furthermore, as for spaces, we have that $\pi_n\Omega \simeq \pi_{n+1}$, hence we get that

$$\begin{aligned} \hom_{h\mathcal{C}}(\Sigma(c),d) &\cong \pi_0 \operatorname{Map}_{\mathcal{C}}(\Sigma(c),d) \\ &\cong \pi_0\Omega \operatorname{Map}_{\mathcal{C}}(c,d) \\ &\cong \pi_1 \operatorname{map}_{\mathcal{C}}(c,d). \end{aligned}$$

So $\hom_{h\mathcal{C}}(\Sigma(c),d)$ is a group. Since Σ is an equivalence, every object in $h\mathcal{C}$ can be written as $\Sigma(c')$ for some object $c' \in \mathcal{C}$. Therefore for all $c, d \in \mathcal{C}$, we have that $\hom_{h\mathcal{C}}(c,d) isagroup$. Also, $h\mathcal{C}$ has all finite products and coproducts, see [16, 1.1.2.9], so $h\mathcal{C}$ is an additive category.

Next, we can define the distinguished triangles as follows [16, Def. 1.1.2.11] [9, p. 62]. In a stable ∞-category $\mathcal{C}$, we can extend any cofiber sequence

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \downarrow & \ulcorner & \downarrow g \\ 0 & \longrightarrow & z \end{array}$$

to a diagram

$$\begin{array}{ccccc} x & \xrightarrow{f} & y & \longrightarrow & 0' \\ \downarrow & \ulcorner & \downarrow g & \ulcorner & \downarrow \\ 0 & \longrightarrow & z & \xrightarrow{h} & w. \end{array}$$

Since a pasting lemma holds similarly for ∞-categorical pushouts as in the ordinary categorical sense, it follows that the outer diagram is also a cofiber sequence, hence we get a morphism $\phi\colon \Sigma(x) \simeq w$ by definition of the suspension functor. This yields a sequence

$$x \xrightarrow{f} y \xrightarrow{g} z \xrightarrow{\phi\circ h} w[1],$$

and a well-defined triangle in $h\mathcal{C}$ when passing to homotopy classes. We then set a triangle in $h\mathcal{C}$ to be distinguished if it is isomorphic to a triangle

$$x \xrightarrow{[f]} y \xrightarrow{[g]} z \xrightarrow{[\phi\circ h]},$$

as discussed above. In [16, Thm. 1.1.2.14] it is proved that this class of distinguished triangles in fact satisfies Verdier's axioms, hence $h\mathcal{C}$ is a triangulated ∞-category. □

3.3 t-structures

In addition to being triangulated, some categories also come with a t-structure, which where introduced first by Beilinson, Bernstein and Deligne in [3]. However, we will mostly follow Lurie's discussion of this in [16, §. 1.2.1].

Definition 3.15. Let $\mathcal{C}$ be a triangulated ordinary category. Then a t-structure on $\mathcal{C}$ consists of a pair of full subcategories $\mathcal{C}_{\leq 0}$ and $\mathcal{C}_{\geq 0}$ satisfying the following properties:

- for every $X \in \mathcal{C}_{\geq 0}$ and $Y \in \mathcal{C}_{\leq 0}$, we have that $\hom_{\mathcal{C}}(X, Y[-1]) = 0$.
- We have inclusions $\mathcal{C}_{\geq 0}[1] \subseteq C_{\geq 0}$ and $\mathcal{C}_{\leq 0}[-1] \subseteq \mathcal{C}_{\leq 0}$.
- For any $X \in \mathcal{C}$, there exists a distinguished triangle

  $$X' \to X \to X'' \to X'[1]$$

  where $X' \in \mathcal{C}_{\geq 0}$ and $X'' \in C_{\leq 0}[-1]$.

Remark 3.16. For convenience, we write $\mathcal{C}_{\geq n} := \mathcal{C}_{\geq 0}[n]$ for the image of $\mathcal{C}_{\geq 0}$ under the n-fold iteration of the map Σ. Similarly, we write $\mathcal{C}_{\leq n} := \mathcal{C}_{\leq 0}[n]$.

Theorem 3.17. *Let $\mathcal{C}$ be a triangulated category with a t-structure. Then $\forall n \in \mathbb{Z}$, the inclusion $\mathcal{C}_{\leq n} \to \mathcal{C}$ admits a left adjoint. Similarly, the inclusion $\mathcal{C}_{\geq n} \to \mathcal{C}$ admits a right adjoint. We denote these left and right adjoints by $\tau_{\leq n}$ and $\tau_{\geq n}$ respectively [3, Prop. 1.3.3].*

For stable ∞-categories, we saw that $h\mathcal{C}$ is in fact a triangulated ∞-category. Hence we can define t-structures on stable ∞-categories as follows, see also [16, Def. 1.2.1.4].

Definition 3.18. Let $\mathcal{C}$ be a stable ∞-category. Then a t-structure on $\mathcal{C}$ is a t-structure on the homotopy category $h\mathcal{C}$. We write $\mathcal{C}_{\leq n}, \mathcal{C}_{\geq n}$ for the full subcategories spanned by objects of $h\mathcal{C}_{\leq n}$ and $h\mathcal{C}_{\geq n}$ respectively.

As in the ordinary categorical case, we have the following theorem, see also [16, Prop. 1.2.1.5].

Theorem 3.19. *Let $\mathcal{C}$ be a stable ∞-category. Then $\forall n \in \mathbb{Z}$, the inclusion $\mathcal{C}_{\leq n} \to \mathcal{C}$ admits a left adjoint. Similarly, the inclusion $\mathcal{C}_{\geq n} \to \mathcal{C}$ admits a right adjoint. We denote these left and right adjoints by $\tau_{\leq n}$ and $\tau_{\geq n}$ respectively. Furthermore, for every $n \in \mathbb{Z}$, the maps $\tau_{\leq n}, \tau_{\geq n}$ map $\mathcal{C}_{\leq m}$ into itself and for any object $X \in \mathcal{C}$, there is a cofiber sequence in $\mathcal{C}$*

$$\tau_{\geq n}X \to X \to \tau_{\leq n-1}X.$$

As a consequence, all $\mathcal{C}_{\geq m}$ are closed under colimits and all $\mathcal{C}_{\leq m}$ are closed under limits.

Remark 3.20. Note that cofiber sequences $\tau_{\geq n}X \to X \to \tau_{\leq n-1}X$ are also fiber sequences. Also, any cofiber sequence $\tau_{\geq n}X \to X \to \tau_{\leq n-1}X$ in a stable category $\mathcal{C}$ corresponds to a diagram

$$\begin{array}{ccc} \tau_{\geq n}X & \xrightarrow{f} & X \\ \downarrow & \ulcorner & \downarrow g \\ 0 & \longrightarrow & \tau_{\leq n-1}X \end{array}$$

which, since $\mathcal{C}$ has all finite colimits and hence all cofibers, can be extended to a diagram

$$\begin{array}{ccccc} \tau_{\geq n}X & \xrightarrow{f} & X & \longrightarrow & 0' \\ \downarrow & \ulcorner & \downarrow g & \ulcorner & \downarrow \\ 0 & \longrightarrow & \tau_{\leq n-1}X & \longrightarrow & W \end{array}$$

which will precisely yield a distinguished triangle in $h\mathcal{C}$ as defined in the previous section. Indeed, in general distinguished triangles in $h\mathcal{C}$ are given by (co)fiber sequences in $\mathcal{C}$, hence the terms distinguished triangle and cofiber sequence are sometimes used interchangeably.

Definition 3.21. Let $\mathcal{C}$ be a stable ∞-category. We define $\mathcal{C}^{\heartsuit}$, the **heart** of $\mathcal{C}$, to be the full subcategory of $\mathcal{C}$ given by $\mathcal{C}_{\leq 0} \cap \mathcal{C}_{\geq 0} \subseteq \mathcal{C}$.

Remark 3.22. We have that $h(\mathcal{C}^{\heartsuit}) = (h\mathcal{C})^{\heartsuit} := h\mathcal{C}_{\leq 0} \cap h\mathcal{C}_{\geq C}$. Hence as a property of t-structures on ordinary categories, $h(\mathcal{C}^{\heartsuit})$ is an abelian category, see [3, Thm. 1.3.6]. Furthermore, $\mathcal{C}^{\heartsuit}$ is equivalent to the nerve of its homotopy category $h(\mathcal{C}^{\heartsuit})$ [16, Rk. 1.2.1.12].

Lastly, we define a **homotopy group functor** on stable ∞-categories by

$$\pi_n := \tau_{\geq 0} \circ \tau_{\leq 0} \circ [-n] \colon \mathcal{C} \to \mathcal{C}^{\heartsuit}.$$

Remark 3.23. There is an equivalence of functors $\mathcal{C} \to \mathcal{C}_{\leq m} \cap \mathcal{C}_{\geq n}$

$$\theta \colon \tau_{\leq m} \circ \tau_{\geq n} \to \tau_{\geq n} \circ \tau_{\leq m}$$

for every $m, n \in \mathbb{Z}$ [16, Prop. 1.2.1.10], hence it does not matter if we define $\pi_0 = \tau_{\geq 0} \circ \tau_{\leq 0}$ or the other way around.

We conclude this paragraph with a discussion of a possible interaction between t-structures and monoidal structures.

Definition 3.24. Let $F \colon \mathcal{C} \to \mathcal{D}$ be a functor between stable ∞-categories. Then F is **exact** if and only if F preserves 0-objects and maps fiber sequences (or exact triangles) to fiber sequences.

Remark 3.25. This is equivalent to F preserving finite limits or colimits [16, 1.1.4.1].

Definition 3.26. A t-structure on a symmetric monoidal stable ∞-category $\mathcal{C}$ is **compatible with the monoidal structure** [12, App. II.A] if

(1) the tensor product $\otimes \colon \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is exact in both variables,

(2) the monoidal unit is contained in $\mathcal{C}_{\geq 0}$,

(3) $\mathcal{C}_{\geq 0}$ is closed under tensor products.

3.4 The ∞-category of spectra

In the last paragraph of this section we define and discuss the ∞-category of spectra. In particular, we will look at the ∞-category of spectrum objects in $\mathcal{C}$, written as $\mathrm{Sp}(\mathcal{C})$, for a sufficiently nice pointed ∞-category, as well as the ∞-category Sp of spectra of spaces and its monoidal structure and t-structure. For (pre)-spectrum objects, we mostly follow [9, §. 5.2].

Definition 3.27. Let $\mathcal{C}$ be a finitely (co)complete pointed ∞-category. Then we define a **prespectrum object** of $\mathcal{C}$ to be a functor

$$X \colon N(\mathbb{Z} \times \mathbb{Z}) \to \mathcal{C}$$

such that for all $i \neq j$ the object $X(i,j)$ is a zero object in $\mathcal{C}$.

We can visualize a pre-spectrum object via a diagram

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & X_{(n+1,n+1)} & & \dots \\
 & & \uparrow & & \uparrow & & \\
 & 0' & \longrightarrow & X_{(n,n)} & \longrightarrow & 0'' & \\
 & \uparrow & & \uparrow & & & \\
\dots & X_{(n-1,n-1)} & \longrightarrow & 0''' & & &
\end{array}$$

By definition of the suspension and loop functor, we have induced maps

$$\alpha_n \colon \Sigma X_{n+1} \to X_n \text{ and } \beta_n \colon X_n \to \Omega X_{n+1}.$$

Definition 3.28. Let X be a pre-spectrum object of $\mathcal{C}$. Then X is a **spectrum below** n if for every $m < n$ the maps $\beta_m \colon X_m \to \Omega X_{m+1}$ are equivalences. We write $\mathrm{Sp}_n(\mathcal{C})$ for the full subcategory of $\mathrm{Fun}(N(\mathbb{Z} \times \mathbb{Z}), \mathcal{C})$ of spectra below n on $\mathcal{C}$. Furthermore, we say that X is a **spectrum object** if the maps β_m are equivalences for all $m \in \mathbb{Z}$. We write $\mathrm{Sp}(\mathcal{C})$ for the ∞-category of spectrum objects of $\mathcal{C}$.

Note, for an ∞-category $\mathcal{C}$ which is not pointed with suitable limits, we can set $\mathrm{Sp}(\mathcal{C}) := \mathrm{Sp}(\mathcal{C}_*)$. In particular, for a pointed ∞-category, there is an equivalence $\mathrm{Sp}(\mathcal{C}_*) \simeq \mathrm{Sp}(\mathcal{C})$. Lurie actually first defines spectrum objects in ∞-categories before defining them for pointed ∞-categories, see [16, §. 1.4.2], but this is in fact equivalent.

Remark 3.29. The definition of spectrum objects is equivalent to the following notion. For $\mathcal{C}$ an ∞-category with finite limits, the loop functor on the pointed ∞-category $\mathcal{C}_*$ exists. We can define $\mathrm{Sp}(\mathcal{C})$ to be the limit in Cat_∞ of the tower

$$\dots \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_* \xrightarrow{\Omega} \mathcal{C}_*,$$

see also [16, Rk. 1.4.25]. Indeed, the objects of $\mathrm{Sp}(\mathcal{C})$ are then precisely sequences of pointed objects $\{X_n\}_{n\in\mathbb{Z}}$ together with equivalences $X_n \simeq \Omega X_{n+1}$ [17, 7.6.6.12]. This yields a spectrum object by setting $X(n,n) = X_n$ and 0 else.

Example 3.30. Consider $\mathcal{S}$, the ∞-category of spaces. Then we set $\mathrm{Sp} := \mathrm{Sp}(\mathcal{S}_*)$ to be the ∞-category of spectra of spaces, often simply referred to as the **∞-category of spectra**.

Groth refers to the spectrum of an ∞-category $\mathcal{C}$ as the stabilization of $\mathcal{C}$, indeed, the following holds [16, 1.4.2.17].

Theorem 3.31. *Let $\mathcal{C}$ be an ∞-category with finite limits. Then the ∞-category $\mathrm{Sp}(\mathcal{C}) = \mathrm{Sp}(\mathcal{C}_*)$ is stable.*

In particular, Sp is a stable ∞-category, as $\mathcal{S}$ as all small (co)limits. We can define loop and suspension functors on ∞-categories of spectrum objects as follows.

Definition 3.32. Let $\mathcal{C}$ be a pointed ∞-category with finite limits. The **n-th loop spectrum functor** is the evaluation functor $\Omega^{\infty-n} \colon \mathrm{Sp}(\mathcal{C}) \to \mathcal{C}$ by $X \mapsto X(n,n)$. If $\mathcal{C}$ is also a presentable ∞-category and has all finite colimits, this functor has a left-adjoint $\Sigma^{\infty-n} \colon \mathcal{C} \to \mathrm{Sp}(\mathcal{C})$, which we refer to as the **n-th suspension spectrum functor**, see [9, p. 66].

Example 3.33. In particular, the ∞-category of pointed spaces is presentable, hence we have an adjunction

$$\Sigma_+^{n-\infty} \dashv \Omega^{n-\infty}$$

Furthermore, for the ∞-category of spaces, we can in fact give a nice description of $\Sigma_+^{n-\infty}$, see [9, p. 65]. Let $D_n \subseteq \mathrm{Sp}_n$ be the full subcategory of spectra below n such that $\alpha_m \colon \Sigma X_m \to X_{m+1}$ is an equivalence for all $m \geq n$. Then $X \in D_n$ is determined by X_n and the evaluation map $\mathrm{ev}_n\colon D_n \to \mathcal{S}_*$ is a trivial Kan fibration. Hence we have a section of this map, s_n, and we set

$$\Sigma_+^{n-\infty}\colon \mathcal{S}_* \to \mathrm{PSp}$$

to be the composition of s_n with the inclusion functor $D_n \hookrightarrow \mathrm{PSp}$. Composed with a localization functor $L\colon \mathrm{PSp} \to \mathrm{Sp}$, this yields the n'th suspension spectrum functor.

Remark 3.34. We refer to the **sphere spectrum** $\mathbb{S}$ as the image of the zero-sphere $S_0 \in \mathcal{S}_*$ under Σ_+^∞. As Δ^0 does for $\mathcal{S}$, the ∞-category Sp is generated under colimits by the sphere spectrum. Furthermore, colimit preserving functors $F\colon \mathrm{Sp} \to \mathcal{C}$, for $\mathcal{C}$ presentable, are determined by their value on the sphere spectrum, as are colimit preserving functors on $\mathcal{S}$ by their evaluation of the point [9, Prop. 5.3, Cor. 5.27].

For the remainder of this paragraph we will look at a symmetric monoidal structure and a t-structure on the ∞-category Sp. We can look at the ∞-category of stable, presentable ∞-categories, Pr^L_{St}, which can be endowed with a closed symmetric monoidal structure [9, 5.34], for which Sp is the monoidal unit. The following theorem will then yield a way to endow Sp with a monoidal structure.

Theorem 3.35. *For every symmetric monoidal ∞-category $\mathcal{C}^\otimes \to N(\mathrm{Fin})$ the ∞-category $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathcal{C})$ has an initial object. Moreover, a commutative algebra object E is initial if and only if the unit map $\mathbb{S} \to E_{\langle 1\rangle}$ is an equivalence.*

Hence Sp corresponds to an initial object in $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Pr}^L_{St})$. Now, via straightening-unstraightening, we have that commutative monoids in Cat_∞ are precisely symmetric monoidal ∞-categories, see also [9, p. 58-59], hence $\mathrm{Alg}_{\mathbb{E}_\infty}(\mathrm{Pr}^L_{St}) \simeq \mathrm{Pr}^{L,\otimes}_{St}$, the ∞-category of stable, presentable, symmetric monoidal closed ∞-categories and symmetric monoidal, colimit-preserving functors. This means Sp can be endowed with a symmetric monoidal structure $\mathrm{Sp}^\otimes$, of which we call the tensor product the **smash product**.

Theorem 3.36. *The **smash product** on Sp is uniquely determined by the following properties:*

(1) The functor $\otimes\colon \mathrm{Sp} \times \mathrm{Sp} \to \mathrm{Sp}$ preserves colimits in both variables.

(2) The sphere spectrum $\mathbb{S}$ is the monoidal unit of $\mathrm{Sp}^\otimes$.

Next, we define a t-structure on Sp as follows. We set $\mathrm{Sp}_{\leq -1}$ to be the full subcategory of Sp on those objects X for which $\Omega^\infty(X)$ is contractible. This yields a t-structure via [16, Thm. 1.4.3.6.]. The proof of Theorem 1.4.3.6 also yields that $\mathrm{Sp}_{\geq 0}$ is the full subcategory of Sp spanned by

$$\{X \in \mathrm{Sp} \mid \pi_n(X) \simeq 0 \,\forall n < 0\}$$

and $\mathrm{Sp}_{\leq 0}$ is the full subcategory of Sp spanned by

$$\{X \in \mathrm{Sp} \mid \pi_n(X) \simeq 0 \,\forall n > 0\}.$$

Here π_n refers to the usual homotopy group functor of spectra. Furthermore, Theorem 1.4.3.6 also states that the heart of Sp with this t-structure is equivalent to the nerve of the category of abelian groups, i.e. $\mathrm{Sp}^\heartsuit \simeq N(\mathrm{Ab})$ and the homotopy group functor $\pi_n\colon \mathrm{Sp} \to \mathrm{Sp}^\heartsuit$ corresponds to the usual homotopy group functor.

Theorem 3.37. *The t-structure on Sp is compatible with the monoidal structure on Sp given by the smash product [16, Lemma 7.1.1.7].*

4 Décalage

In this section we look at a way of constructing spectral sequences from towers in the ∞-category of spectra. Furthermore, we can endow the ∞-category of towers of spectra with a symmetric monoidal structure and we will see that the construction of spectral sequences is in a way compatible with this structure. This yields a recipe for constructing multiplicative spectral sequences, which are difficult to get via traditional methods. This theory is laid out by Hedenlund in her PhD thesis [12] and we also used De Potter's master thesis [7] as an overview of Hedenlund's theory. For the most part we will not give proofs, as they are already done in detail by Hedenlund. For interested readers, we note that both Hedenlund and De Potter give an application: Hedenlund uses décalage to construct multiplicative Tate spectral sequences and De Potter applies the theory to construct the multiplicative Leray-Serre-Atiyah-Hirzebruch spectral sequence.

4.1 The ∞-category of towers and the associated graded

We start with a discussion of towers of a stable ∞-category and the associated graded functor.

Definition 4.1. Let $\mathcal{C}$ be a stable ∞-category. Then $\mathrm{Tow}(\mathcal{C})$ is the ∞-category of towers given by the functor category $\mathrm{Fun}(N(\mathbb{Z}^{op}), \mathcal{C})$.

Note that for a stable ∞-category, the ∞-category $\mathrm{Tow}(\mathcal{C})$ is stable as well since limits and colimits are computed pointwise. We can visualise a tower as a sequence

$$\cdots \to X(i+1) \to X(i) \to X(i-1) \to \dots$$

of objects in $\mathcal{C}$.

Provided $\mathcal{C}$ is endowed with a symmetric monoidal structure, we get a symmetric monoidal structure on $\mathrm{Tow}(\mathcal{C})$ via Day Convolution. See [12, §. II.1.3] for further references. In particular, for $X, Y \in \mathrm{Tow}(\mathcal{C})$, we have

$$(X \otimes Y)(n) := \mathrm{colim}_{i+j \geq n} X(i) \otimes X(j).$$

In this case, a map $X \otimes Y \to Z$ in $\mathrm{Tow}(\mathcal{C})$ is called a **pairing** of towers. The unit of this monoidal structure is the tower defined by

$$1_{\mathrm{Tow}(\mathcal{C})}(n) = \begin{cases} 1_{\mathcal{C}} & \text{for } n \leq 0 \\ 0 & \text{else} \end{cases},$$

where $1_{\mathcal{C}}$ is the monoidal unit of $\mathcal{C}$. We can visualize this as

$$\cdots * \to * \to * \to 1_{\mathcal{C}} \to 1_{\mathcal{C}} \to \dots.$$

Aside from a monoidal structure, if $\mathcal{C}$ is a stable ∞-category with a t-structure, we can define a t-structure on $\mathrm{Tow}(\mathcal{C})$ in the following way [12, §. II.1.4].

Definition 4.2 (Canonical t-structure on $\mathrm{Tow}(\mathcal{C})$)**.** Suppose $\mathcal{C}$ is a stable ∞-category with a t-structure. Then we define $\tau^{can}_{\geq 0}\colon \mathrm{Tow}(\mathcal{C}) \to \mathrm{Tow}(\mathcal{C})$ where for $X \in \mathrm{Tow}(\mathcal{C})$,

$$\tau^{can}_{\geq 0} X(n) := \tau_{\geq n} X(n).$$

Here $\tau_{\geq n}$ comes from the t-structure on $\mathcal{C}$. The essential image of this functor we denote by $\mathrm{Tow}^{can}_{\geq 0}(\mathcal{C})$, and it is spanned by the objects

$$\{X \in \mathrm{Tow}(\mathcal{C}) \mid X(n) \in \mathcal{C}_{\geq n}\}.$$

Remark 4.3. To show that this is indeed a well-defined t-structure, by the dual of [16, Prop. 1.2.1.16] it is enough to show that $\mathrm{Tow}^{can}_{\geq 0}(\mathcal{C})$ is closed under extensions, which Hedenlund does in [12, Prop. II.1.22].

Also, if $\mathcal{C}$ is endowed with both a symmetric monoidal structure and a t-structure, this t-structure works nicely with Day convolution. In particular, the following holds [12, Prop. II.1.23].

Proposition 4.4. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category, with a t-structure that is compatible with the monoidal structure. Then the canonical t-structure on $\mathrm{Tow}(\mathcal{C})$ is compatible with the monoidal structure via Day Convolution.*

Proof. By Definition 3.26, there are three things to check, namely that the tensor product is exact in both variables, that $\mathrm{Tow}^{can}_{\geq 0}(\mathcal{C})$ contains the monoidal unit and that $\mathrm{Tow}^{can}_{\geq 0}(\mathcal{C})$ is closed under the tensor product. Exactness of Day convolution follows from the fact that Day convolution is defined by taking a colimit over tensor products in $\mathcal{C}$. Since colimits commute with colimits, see the dual of [15, 5.5.2.3], and the t-structure on $\mathcal{C}$ is compatible with the monoidal structure on $\mathcal{C}$, the tensor on $\mathcal{C}$ is exact in both variables, therefore so is the tensor on $\mathrm{Tow}(\mathcal{C})$.

The monoidal unit $1_{\mathrm{Tow}(\mathcal{C})}$ is contained in $\mathrm{Tow}^{can}_{\geq 0}(\mathcal{C})$, since $1_{\mathcal{C}}$ is contained in $\mathcal{C}_{\geq 0}$. Next, suppose $X, Y \in \mathrm{Tow}(\mathcal{C})_{\geq 0}$. For each i, j we have that $X(i) \otimes Y(j)$ lies in $\mathcal{C}_{\geq i+j}$ as $X(i) \in \mathcal{C}_{\geq i}$ and $Y(j) \in \mathcal{C}) \geq j$. Since for $m \geq n$ we have $\mathcal{C}_{\geq m} \subseteq \mathcal{C}_{\geq n}$ and all ∞-categories $\mathcal{C}_{\geq m}$ are closed under colimits, indeed $X \otimes Y(n) := \mathrm{colim}_{i+j \geq n} X(i) \otimes X(j)$ lies in $\mathcal{C}_{\geq n}$. So $X \otimes Y$ is in $\mathrm{Tow}_{\geq 0}(\mathcal{C})$. □

Definition 4.5 (Whitehead Tower)**.** Let $\mathcal{C}$ be a stable ∞-category with a t-structure. Then we define $\tau_{\geq *} \colon \mathcal{C} \to \mathrm{Tow}(\mathcal{C})$ by

$$\tau_{\geq *}(X)(n) := \tau_{\geq n}(X).$$

In other words, $\tau_{\geq *}$ is the composition of the constant functor $\mathcal{C} \to \mathrm{Tow}(\mathcal{C})$ mapping X to the constant tower at X and the functor $\tau^{can}_{\geq 0} \colon \mathrm{Tow}(\mathcal{C}) \to \mathrm{Tow}^{can}_{\geq 0}(\mathcal{C})$.

Proposition 4.6. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category, with a t-strucutre that is compatible with the monoidal structure. Then the Whitehead tower functor is lax symmetric monoidal.*

Proof. One can show that the constant functor $\mathcal{C} \to \mathrm{Tow}(\mathcal{C})$ is strong symmetric monoidal. In turn, the functor $\iota_{\geq 0} \colon \mathrm{Tow}\,\mathcal{C}_{\geq 0} \to \mathrm{Tow}(\mathcal{C})$ is symmetric monoidal, since the canonical t-structure is compatible with Day Convolution, see also the dual of [16, Prop. 2.2.1.9]. The functor $\tau^{can}_{\geq 0}$ is its right adjoint. Since strong symmetric monoidal functors are in particular oplax symmetric monoidal functors, see Proposition 2.13, it follows that $\tau^{can}_{\geq 0}$ is lax symmetric monoidal. The composition of lax symmetric monoidal functors is lax symmetric monoidal, hence indeed $\tau_{\geq *}$ is lax symmetric monoidal. □

We have for now gathered all information needed on the ∞-category of towers. We will continue with the associated graded functor.

Definition 4.7 (Associated Graded)**.** Let $\mathcal{C}$ be a stable ∞-category, $X \in \mathrm{Tow}(\mathcal{C})$. For $q \in \mathbb{Z}$, we write

$$\mathrm{Gr}^q \colon \mathrm{Tow}(\mathcal{C}) \to \mathcal{C}$$

for the functor sending a tower X to $X(q)/X(q+1) := \mathrm{cofib}(X(q+1) \to X(q))$. This is the **q'th associated graded functor**. We can then define a **total associated graded functor** $\mathrm{Gr} \colon \mathrm{Tow}(\mathcal{C}) \to \prod_{\mathbb{Z}} \mathcal{C}$ by

$$X \mapsto (X(q)/X(q+1))_{q \in \mathbb{Z}}.$$

The following useful proposition was proved by De Potter in his thesis [7, Prop. 4.4].

Proposition 4.8. *Let $i \geq j \geq k \geq l$; $X \in \mathrm{Tow}(\mathcal{C})$. Then*

$$\begin{array}{ccc} X(k)/X(i) & \longrightarrow & X(l)/X(i) \\ \downarrow & & \downarrow \\ X(k)/X(j) & \longrightarrow & X(l)/X(j) \end{array}$$

is a pushout square.

Proposition 4.9. *Let $\mathcal{C}$ be a symmetric monoidal stable ∞-category. Then the total associated graded functor* $\mathrm{Gr}\colon \mathrm{Tow}(\mathcal{C}) \to \prod_{\mathbb{Z}} \mathcal{C}$ *is strong symmetric monoidal with respect to Day Convolution. In particular, for all $q \in \mathbb{Z}$, we have*

$$\mathrm{Gr}^q(X \otimes Y) \simeq \bigoplus_{i+j=q} \mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y).$$

For the proof we refer to [12, Prop. II.1.3]. Eventually we wish to construct a chain complex using the associated graded, hence we define a differential as follows.

Definition 4.10 (Differential)**.** Let $X \in \mathrm{Tow}(\mathcal{C})$, for $\mathcal{C}$ stable. Then for each $i \in \mathbb{Z}$, we have a map $\delta^i\colon \mathrm{Gr}^i(X) \to \mathrm{Gr}^{i+1}(X)[1]$ in $\mathcal{C}$ via

$$\begin{array}{ccccc}
\mathrm{Gr}^{i+1}(X) & \longrightarrow & X(i)/X(i+2) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Gr}^i(X) & \xrightarrow{\delta^i} & \mathrm{Gr}^{i+1}(X)[1]
\end{array}$$

Remark 4.11. Note that the left square is a push-out square, as well as the outer diagram, which induces δ^i and means that the right square is also a push-out square. As it turns out, one can prove that δ^i is indeed a differential, i.e. $\delta \circ \delta \simeq 0$, via pasting push-out squares. In particular, for $i \geq j \geq k$, we have commutative diagram

$$\begin{array}{ccccccccc}
X(i) & \longrightarrow & X(j) & \longrightarrow & X(k) & \longrightarrow & 0 & & \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & X(j)/X(i) & \longrightarrow & X(k)/X(i) & \longrightarrow & X(i)[1] & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \longrightarrow & X(j)/X(k) & \xrightarrow{\kappa} & X(j)[1] & \xrightarrow{j} & X(j)/X(k)[1]
\end{array}$$

in which all squares are pushouts. This immediately implies that we can factor δ as $j \circ \kappa$.

Furthermore, if $\mathcal{C}$ is a symmetric monoidal stable ∞-category, the differentials work nicely with the monoidal structure [12, Prop. II.1.21].

Proposition 4.12 (Leibniz Rule)**.** *Let $\phi\colon X \otimes Y \to Z$ be a pairing of towers in* $\mathrm{Tow}(\mathcal{C})$*. Then we have a commuting diagram in $\mathcal{C}$ as follows*

$$\begin{array}{ccc}
\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y) & \xrightarrow{\delta^i_X \otimes 1 + 1 \otimes \delta^j_Y} & \big(\mathrm{Gr}^{i+1}(X) \otimes \mathrm{Gr}^j(Y)\big) \oplus \big(\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^{j+1}(Y)\big) \\
\Big\downarrow {\scriptstyle \mathrm{Gr}^{i,j}(\phi)} & & \Big\downarrow {\scriptstyle \mathrm{Gr}^{i+1,j}(\phi) \oplus \mathrm{Gr}^{i,j+1}(\phi)} \\
\mathrm{Gr}^{i+j}(Z) & \xrightarrow{\delta^{i+j}_Z} & \mathrm{Gr}^{i+j+1}(Z)[1].
\end{array}$$

4.2 The Beilinson t-structure

From now on we will only consider the case where $\mathcal{C} = \mathrm{Sp}$. Recall that Sp is a symmetric monoidal ∞-category, with a compatible t-structure, hence we can use all results from the previous paragraph. In this paragraph we will discuss an additional t-structure on Tow(Sp) besides the canonical one. The heart of this t-structure is particularly important, as it is equivalent to the category of chain complexes of abelian groups. We will follow Hedenlund's approach in Chapter [12, II.2], for original sources on the Beilinson t-structure we refer to the ones listed by her in [12, §. II.2.1].

Definition 4.13 (Beilinson t-structure)**.** For $n \in \mathbb{Z}$ we define the **Beilinson t-structure** by setting $\mathrm{Tow}(\mathrm{Sp})^{Bei}_{\geq n}$ to be the full subcategory of Tow(Sp) spanned by

$$\{X \in \mathrm{Tow}(\mathrm{Sp}) \mid \mathrm{Gr}^q(X) \in \mathrm{Sp}_{\geq n-q}\ \forall q \in \mathbb{Z}\}.$$

Furthermore, we set $\mathrm{Tow}(\mathrm{Sp})^{Bei}_{\leq n}$ to be the full subcategory of Tow(Sp) spanned by

$$\{X \in \mathrm{Tow}(\mathrm{Sp}) \mid X(q) \in \mathrm{Sp}_{\leq n-q}\ \forall q \in \mathbb{Z}\}.$$

Proposition 4.14. *The Beilinson t-structure is well-defined and compatible with Day Convolution.*

Proof. We mostly give an overview of the steps in the proof. For more details, we refer to Hedenlund [12, Prop. II.2.1 & II.2.7]. To show that the t-structure is well-defined, we note that the first condition of a t-structure holds by construction. For the other two conditions, the idea is to construct $\tau_{\leq -1}\colon \mathrm{Tow}(\mathrm{Sp})^{Bei} \to \mathrm{Tow}(\mathrm{Sp})^{Bei}_{\leq -1}$ and show that it indeed maps into $\mathrm{Tow}(\mathrm{Sp})^{Bei}_{\leq -1}$. The construction of $\tau_{\leq -1}$ is as follows. Since Tow(Sp) is presentable and $\mathrm{Tow}(\mathrm{Sp})^{Bei}_{\geq 0}$ is closed under colimits, the inclusion map has a right-adjoint as a consequence of the adjoint functor theorem. This we call $\tau^{Bei}_{\geq 0}$. Then $\tau_{\leq -1}$ is defined by mapping each X to the cofiber of $\tau^{Bei}_{\geq 0}(X) \to X$.

To show that the t-structure is compatible with Day convolution, we remark that we only need to show that the unit is contained in $\mathrm{Tow}(\mathrm{Sp})^{Bei}_{\geq 0}$ and that $\mathrm{Tow}(\mathrm{Sp})^{Bei}_{\geq 0}$ is closed under the tensor product, as we already saw that the tensor product is exact in both variables. Firstly, we note that the q'th associated graded of $1_{\mathrm{Tow(Sp)}}$ is given by the unit of Sp for $q = 0$ and 0 otherwise. The unit, which is the sphere spectrum, is in $\mathrm{Sp}_{\geq 0}$ because the t-structure of Sp is compatible with the smash product. So it follows that indeed $1_{\mathrm{Tow}(spc)}$ is in $\mathrm{Tow}(\mathrm{Sp})^{Bei}_{\geq 0}$.

Furthermore, to show that for $X, Y \in \mathrm{Tow}(\mathrm{Sp})^{Bei}_{\geq 0}$, their tensor $X \otimes Y$ is in $\mathrm{Tow}(\mathrm{Sp})^{Bei}_{\geq 0}$, we note that the total associated graded functor is lax symmetric monoidal. Combined with the fact the t-structure on Sp is compatible with the smash product, this means that all $\mathrm{Gr}^i(X) \otimes \mathrm{Gr}^j(Y)$ lie in $\mathrm{Sp}_{\geq -i-j}$. Using also that $\mathrm{Sp}_{\geq m}$ is also closed under colimits for all $m \in \mathbb{Z}$, the result will follow by a similar reasoning as in Proposition 4.4. □

For discussing the heart of the Beilinson t-structure and its isomorphism to Ch(Ab), we need to understand the Eilenberg-MacLane functor. It is defined as follows.

Definition 4.15. Let Ch(Ab) be the ordinary category of chain complexes. Then the **Eilenberg-MacLane functor** is the canonical functor

$$H\colon \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Sp}\,.$$

Formally, this is the composition of (1) the localization functor L from Ch(Ab) to the derived ∞-category $\mathcal{D}(\mathrm{Ab})$, (2) the ∞-categorical equivalence $E\colon \mathcal{D}(\mathrm{Ab}) \simeq \mathrm{Mod}_{H\mathbb{Z}}$, see [16, Thm. 7.1.2.13] and (3) the forgetful functor $U\colon \mathcal{D}(\mathrm{Ab}) \simeq \mathrm{Mod}_{H\mathbb{Z}} \to \mathrm{Sp}\,$.

Remark 4.16. This functor is lax symmetric monoidal, as the monoidal structure on $D(\mathrm{Ab})$ is such that $N(\mathrm{Ch}(\mathrm{Ab})) \to D(\mathrm{Ab})$ is lax symmetric monoidal, the forgetful functor is lax symmetric monoidal and so is the equivalence $\mathcal{D}(\mathrm{Ab}) \simeq \mathrm{Mod}_{H\mathbb{Z}}$, see [16, Thm. 7.1.2.13]. The model structure on Ch(Ab) has as weak equivalences those chain maps which induce isomorphisms on homology, see [16, 7.1.2.8]. These then become actual isomorphisms in the homotopy category of $\mathcal{D}(\mathrm{Ab})$.

Corollary 4.17. *Let* $C \in \mathrm{Ch}(\mathrm{Ab})$*. Then we have a natural isomorphism*

$$H_n(C) \cong \pi_n H(C)$$

Proof. First, we note that the forgetful functor sits in an adjunction

$$\mathrm{Mod}_{\mathbb{Z}} \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Sp}\ ,$$

where the free functor F is given by $- \otimes H\mathbb{Z}$. Hence we have

$$\begin{aligned} \pi_n H(C) &\cong \mathrm{Ho}(\mathrm{Sp})(\Sigma^n \mathbb{S}, H(C)) \\ &\cong \mathrm{Ho}(\mathrm{Mod}_{\mathbb{Z}})(\Sigma^n H(\mathbb{Z}), EL(C)) \\ &\cong \mathrm{Ho}(\mathcal{D}(\mathrm{Ab}))(\Sigma^n \mathbb{Z}, L(C)) \end{aligned}$$

Here $\Sigma^n \mathbb{Z}$ is the chain complex $\cdots \to 0 \to \mathbb{Z} \to 0 \to \dots$ with $\mathbb{Z}$ in degree n. By definition of the model structure and definition of the derived category, we have that $\mathrm{Ho}(\mathcal{D}(\mathrm{Ab}))(\Sigma^n \mathbb{Z}, L(C))$ is precisely $H_n(C)$. □

We will now state the theorem given by Hedenlund which relates the heart of the Beilinson t-structure with chain complexes of abelian groups.

Theorem 4.18. *There is an equivalence of abelian categories* $\mathrm{Tow}\,(\mathrm{Sp})^{Bei,\heartsuit} \simeq \mathrm{Ch}(\mathrm{Ab})$. *Here the equivalence is given by the functor* $\Phi\colon \mathrm{Tow}\,(\mathrm{Sp})^{Bei,\heartsuit} \to \mathrm{Ch}(\mathrm{Ab})$ *where*

$$X \mapsto \cdots \to \pi_1(\mathrm{Gr}^{-1}(X)) \to \pi_0(\mathrm{Gr}^0(X)) \to \dots$$

and the differentials are induced by the differentials from Definition 4.10.

Its inverse $\Psi\colon \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Tow}\,(\mathrm{Sp})^{Bei,\heartsuit}$ *is defined by*

$$\mathcal{C} \mapsto \Phi\mathcal{C}\colon \mathbb{Z}^{op} \to \mathrm{Sp};\ \textit{where } \Psi\mathcal{C}(n) = H(\sigma_{\leq n}\mathcal{C}),$$

where H *is the Eilenberg-MacLane functor and* $\sigma_{\leq n}\colon \mathrm{Ch}(\mathrm{Ab}) \to \mathrm{Ch}(\mathrm{Ab})$ *is the stupid truncation functor.*

The proof of this consists of showing that Ψ is well-defined and has a right-adjoint, which is shown to be naturally isomorphic to Φ. Lastly, Hedenlund shows that this adjunction is in fact an equivalence of categories. For the details, which require quite some work, we refer to [12, Prop. II.2.10]. This theorem essentially makes the décalage functor and the corresponding construction of spectral sequences work the way that it does.

Note that we can define a symmetric monoidal structure on $\mathrm{Tow}\,(\mathrm{Sp})^{Bei,\heartsuit}$ via

$$X \otimes_{Bei,\heartsuit} Y := \tau^{Bei}_{\leq 0}(X \otimes Y),$$

where $X \otimes Y$ comes from Day Convolution.

Theorem 4.19. *The equivalence in Theorem 4.18 is strong symmetric monoidal.*

We will not prove this in detail, this is done in [12, Prop. II.2.11], but it is worth mentioning that the proof uses the lax symmetric monoidality of the homotopy group functor

$$\pi_*\colon \mathrm{Sp} \to \prod_{\mathbb{Z}} \mathrm{Ab},$$

which is lax symmetric monoidal by compatibility of the t-structure on Sp. The proof also uses the fact that the total associated graded functor is strong symmetric monoidal.

Remark 4.20. Under the isomorphism of Theorem 4.18, we can view the Beilinson homotopy groups $\pi_n^{Bei} X$ as the chain complexes

$$\cdots \to \pi_{n+1}(\mathrm{Gr}^{-1}(X)) \to \pi_n(\mathrm{Gr}^0(X)) \to \cdots \to \pi_{n-1}(\mathrm{Gr}^1(X)) \to \dots,$$

with $\pi_n(\mathrm{Gr}^0(X))$ in degree 0. The resulting functor

$$\pi_*^{Bei}\colon \mathrm{Tow}(\mathrm{Sp}) \to \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab})$$

is lax symmetric monoidal, see also [12, Prop. II.2.12 and II.2.13].

4.3 Décalage and the Construction of Multiplicative Spectral Sequences

We will now define the décalée of a tower of spectra. We start with the definition, see also [12, II.2.15]. Next, we will give Hedenlunds construction of the spectral sequence.

Definition 4.21 (Décalage)**.** Let $X \in \mathrm{Tow}(\mathrm{Sp})$. Then the Beilinson-Whitehead tower of X, similarly to Definition 4.5, is the tower of towers

$$\cdots \to \tau_{\geq n+1}^{Bei}(X) \to \tau_{\geq n}^{Bei}(X) \to \tau_{\geq n-1}^{Bei}(X) \to \ldots$$

Since for each k, we have that $\tau_{\geq k}(X)$ is a tower, we can take the colimit of this tower. This in turn yields a tower of spectra

$$\cdots \to \mathrm{colim}_{i\in\mathbb{Z}}\, \tau_{\geq n+1}^{Bei}(X)(i) \to \mathrm{colim}_{i\in\mathbb{Z}}\, \tau_{\geq n}^{Bei}(X)(i) \to \mathrm{colim}_{i\in\mathbb{Z}}\, \tau_{\geq n-1}^{Bei}(X)(i) \to \ldots$$

This resulting tower is the **décalée** of X.

Proposition 4.22. *Definition 4.21 yields a lax symmetric monoidal functor Déc:* $\mathrm{Tow}(\mathrm{Sp}) \to \mathrm{Tow}(\mathrm{Sp})$.

Proof. The proof is also stated in [12, II.2.19]. Firstly, note that taking the Whitehead tower is lax symmetric monoidal, see Proposition 4.6. Furthermore, we can commute colimits and this is symmetric monoidal, by exactness of Day convolution and the smash product. □

Remark 4.23. This functor is not idempotent, so we can iterate it. Thus we write $\mathrm{Déc}^n$ for the n-fold iteration.

We now state an important theorem that will be necessary for the construction of spectral sequences from towers.

Theorem 4.24. *There is a lax symmetric monoidal equivalence of functors*

$$\mathrm{Gr} \circ \mathit{Déc} \simeq H \circ \Sigma^{tot} \circ \pi_*^{Bei}.$$

Here $H\colon \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \to \prod_{\mathbb{Z}} \mathrm{Sp}$ *is the Eilenberg-MacLane functor and* $\Sigma^{tot}\colon \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab}) \to \prod_{\mathbb{Z}} \mathrm{Ch}(\mathrm{Ab})$ *is the shift-functor defined by*

$$\Sigma^{tot}(\{\mathcal{C}_*^k\}_{k\in\mathbb{Z}}) = \{\mathcal{C}_{*-k}^k\}_{k\in\mathbb{Z}}.$$

Remark 4.25. For the proof of this we refer to [12, Thm. II.2.20, II.2.22]. The fact that the compositions are equivalent can be seen via a degree wise computation and commuting certain colimits, whereas the monoidality of the equivalence follows mostly from the monoidality of the functors involved, and the monoidality of the previous equivalence in Theorem 4.18.

We will now construct spectral sequences using décalage. For the definition of spectral sequences, we refer to Definition 1.1.

Theorem 4.26. *Let* $X \in \mathrm{Tow}(\mathrm{Sp})$ *be a tower of spectra. Then we have a spectral sequence defined by*

$$E_{n,s}^r := \pi_n \,\mathrm{Gr}^{(r-1)n+s}(\mathit{Déc}^{r-1}(X)),$$

for $r \geq 0$ *with differentials* $d^r\colon E_{n,s}^r \to E_{n-1,s+r}^r$ *induced by the push-out*

$$\begin{array}{ccccc}
\mathrm{Gr}^{(r-1)n+s+1}(\mathit{Déc}^{r-1}(X)) & \longrightarrow & \frac{\mathit{Déc}^{r-1}(X)((r-1)n+s)}{\mathit{Déc}^{r-1}(X)((r-1)n+s+2)} & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \mathrm{Gr}^{(r-1)n+s}(X) & \xrightarrow{\delta^r} & \mathrm{Gr}^{(r-1)n+s+1}(\mathit{Déc}^{r-1}(X))[1]
\end{array}$$

as in Definition 4.10.

Proof. We will include the proof as in [12, Thm. II.3.2]. Firstly, note that as in Definition 4.10, $\delta^r \circ \delta^r = 0$, which must then also hold for $d^r = \pi_n(\delta^r)$. It remains to show that for each n, s, r, we have isomorphisms

$$E^{r+1}_{n,s} \cong H(E^r_{n,s}, d^r).$$

First, note that by Theorem 4.18

$$\pi_n(\mathrm{Gr}^q \, \mathrm{Déc}(X)) \cong \pi_n(H(\pi_q^{Bei}(X)[q])).$$

Next, by Corollary 4.17, we have that

$$\begin{aligned}\pi_n(H(\pi_q^{Bei}(X)[q])) &\cong H_n(\pi_q^{Bei}(X)[q]) \\ &= \frac{\ker(d_n \colon \pi_n(\mathrm{Gr}^{q-n}(X)) \to \pi_{n-1}(\mathrm{Gr}^{q-n+1}(X)))}{\mathrm{im}(d_{n+1} \colon \pi_{n+1}(\mathrm{Gr}^{q-n-1}(X)) \to \pi_n(\mathrm{Gr}^{q-n+1}(X)))}.\end{aligned}$$

If we apply this to $E^{r+1}_{n,s}$, we get

$$\begin{aligned}E^{r+1}_{n,s} &= \pi_n \, \mathrm{Gr}^{rn+s}(\mathrm{Déc}^r(X)) \\ &\cong \frac{\ker(d_n \colon \pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)) \to \pi_{n-1}(\mathrm{Gr}^{(r-1)n+s+1}(\mathrm{Déc}^{r-1}(X))))}{\mathrm{im}(d_n \colon \pi_{n+1}(\mathrm{Gr}^{(r-1)n+s-1}(\mathrm{Déc}^{r-1}(X))) \to \pi_n(\mathrm{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X))))} \\ &= H(E^r_{n,s}, d_r).\end{aligned}$$

□

We will now discuss the multiplicativity of these spectral sequences. For this we first go back to the category of spectral sequences SSeq. On this category we can define a structure which is not quite a monoidal structure, but we can talk about multilinear maps. In particular, SSeq is a (coloured) operad, see Remark 2.15. The multilinear maps are built by iterating the construction below.

Definition 4.27. Let $(E, d^r_E), (D, d^r_D)$ and (C, d^r_C) be spectral sequences, for convenience we assume they are Adams graded, i.e. have bidegree $(-1, r)$. A **bilinear map** $\phi \colon (C, D) \to E$ consists of maps

$$\phi^r \colon C^r_{n,s} \otimes D^r_{n',s'} \to E^r_{n+n',s+s'},$$

where

$$d^r \circ \phi^r = \phi^r(d^r \otimes 1 + 1 \otimes d^r) \colon C^r_{n,s} \otimes D^r_{n',s'} \to E^r_{n+n'-1,s+s'+r}$$

and the diagram

$$\begin{array}{ccc} C^r_{n,s} \otimes D^r_{n',s'} & \xrightarrow{\phi^r} & E^r_{n+n',s+s'} \\ \downarrow & & \downarrow \\ H(C^r_{n,s} \otimes D^r_{n',s'}) & \xrightarrow{H(\phi^r)} & H(E^r_{n+n',s+s'}) \end{array}$$

commutes for all n, s, n', s', r.

We now claim that the construction of spectral sequences via décalage preserves the operad structure. In particular, the following theorem holds [12, Thm. II.3.5].

Theorem 4.28. *The functor $E \colon \mathrm{Tow}(\mathrm{Sp}) \to \mathrm{SSeq}$ which sends a tower X to a spectral sequence E^r as defined in Theorem 4.26 is a map of ∞-operads.*

Proof. First note that the statement makes sense, as any ordinary operad can be made into an ∞-categorical operad using a nerve construction and any symmetric monoidal ∞-category is an ∞-operad. We give an overview of the proof, for more details, see [12, II.3.5]. Since SSeq is an ordinary category, all higher coherencies in Tow(Sp) are irrelevant for proving this, i.e. we only have to show that multilinear maps in Tow(Sp) are mapped to multilinear maps in SSeq. For maps of ∞-operads, see [16, §. 2.1.2].

Note that the monoidal structure on Tow(Sp) is symmetric, hence it is enough to only consider bilinear maps, i.e. pairings $X \otimes Y \to Z$ in Tow(Sp). Since both the total associated graded and the décalage functor are lax symmetric monoidal, as well as the homotopy group functor $\pi_*\colon \mathrm{Sp} \to \prod_{\mathbb{Z}} \mathrm{Ab}$, it follows that any pairing $\phi\colon X \otimes Y \to Z$ immediately induces maps

$$\phi^r\colon \pi_n \operatorname{Gr}^{(r-1)n+s}(\mathrm{Déc}^{r-1}(X)) \otimes \pi_n \operatorname{Gr}^{(r-1)n'+s'}(\mathrm{Déc}^{r-1}(Y)) \to \pi_n \operatorname{Gr}^{(r-1)(n+n')+s+s'}(\mathrm{Déc}^{r-1}(Z)).$$

In particular, the functors $E^r\colon \mathrm{Tow(Sp)} \to \prod_{\mathbb{Z}\times\mathbb{Z}} \mathrm{Ab}$ are all lax symmetric monoidal. Hence it remains to check the conditions of the previous definition. The first condition follows from the fact that both the décalage functor and the homotopy group functor are lax symmetric monoidal, as well as the Leibniz rule, see 4.12.

For the second condition, we note that the isomorphisms $E^{r+1}_{n,s} \cong H(E^r_{n,s}, d^r)$ come from the symmetric monoidal equivalence in Theorem 4.18. Combined with the fact that both the homotopy group functor $\pi_*\colon \mathrm{Sp} \to \prod_{\mathbb{Z}} \mathrm{Ab}$ and the Eilenberg-MacLane functor are lax symmetric monoidal, we get a commutative diagram

$$\begin{array}{ccc}
\pi_i(\operatorname{Gr}^p('\mathrm{Déc}(X))) \otimes \pi_j(\operatorname{Gr}^q(\mathrm{Déc}(Y))) & \longrightarrow & \pi_{i+j}(\operatorname{Gr}^{p+q}(\mathrm{Déc}(Z))) \\
\downarrow & & \downarrow \\
H_{i+j-p-q}(\pi^{Bei}_p(X) \otimes \pi^{Bei}_q(Y)) & \longrightarrow & H_{i+j-p-q}(\pi^{Bei}_{p+q}(Z))
\end{array}$$

Applying this with setting i, j, p, q the right indices and also with the iterated décalage functor, we get that indeed, the second condition holds for the spectral sequence constructed via décalage. Hence the functor E sends bilinear maps to bilinear maps. □

Remark 4.29. We end this section with a remark on signs. As is usual for (bi)graded abelian groups, we tensor in accordance with the Koszul sign formula. That is, for maps $f\colon C_{n,s} \to C_{n+k,s+l}$ and $g\colon D_{n',s'} \to D_{n'+k',s'+l'}$, where C, D are bigraded abelian groups, we have

$$f \otimes g(x \otimes y) = (-1)^{|x|\cdot|g|} f(x) \otimes g(y)$$

when evaluating them. Here $|x|, |g|$ denote the total degrees of x and the map g respectively. In the case of Adams spectral sequences C, D with grading $(-1, r)$, the total degrees $|x|$ and $|g|$ are n and k' respectively. That is, the total degree depends only on the first grading index, see also Remark 1.2. In particular, this means that in Definition 4.27, we have

$$(d_r \otimes 1 + 1 \otimes d_r)(x \otimes y) = (-1)^{n\cdot 0} d_r(x) \otimes y + (-1)^{n\cdot -1} x \otimes d_r(y).$$

So the first property in Definition 4.27 translates to

$$d_r(a \cdot a') = d_r(a) \cdot a' + (-1)^n a \cdot d_r(a'),$$

for $a \in C^r_{n,s}$ and $a' \in C^r_{n',s'}$. This is analogous to Property (ii) in [20, Thm. 2.1].

5 Exact couples

In their paper [4], Belmont and Kong use the description of the spectral sequence they consider via exact couples in order to rewrite the E^r-page as a workable subquotient of the E^1-page. Ideally, we wish to do the same for the spectral sequence obtained via décalage. Originally décalage was defined by Deligne [8]. The functor 'turns the page' in the spectral sequence of a filtered chain complex F_*C_*, which can be obtained via exact couples. Therefore, it makes sense to ask whether the spectral sequence obtained via décalage on a tower of spectra is in fact the same as one arising from exact couples, that is, whether the ∞-categorical functor also 'turns the page'. More precisely, let X be a tower of spectra and consider the following exact couple.

$$\begin{array}{ccc} \oplus_{n,s}\pi_n(X)(s) & \xrightarrow{\quad i \quad} & \oplus_{n,s}\pi_n(X)(s) \\ & {}_{\kappa}\nwarrow \quad \swarrow_{j} & \\ & E^1(X) := \oplus_{n,s}\pi_n \operatorname{Gr}^s(X) & \end{array}$$

Here i is induced by the maps $X^{s+1} \to X^s$, the map j is induced by the maps $X^s \to \operatorname{Gr}^s(X)$ and the map κ comes from the map $\operatorname{Gr}^s(X) \to X^{s+1}[1]$ induced by the cofiber sequence. Recall from Chapter 1 that we can derive another exact couple from this by setting $d = j \circ \kappa$ and taking homology with respect to this differential d. In particular, this yields a spectral sequence.

Note that the map d is precisely the map $d^1 : \pi_n \operatorname{Gr}^s(X) \to \pi_{n-1} \operatorname{Gr}^{s+1}(X)$ of the E^1-page of the décalage spectral sequence, see also Remark 4.11. Furthermore, by Theorem 4.26 the E^2-page obtained via décalage is isomorphic to $H(E^1, d^1)$. Also, the E^2-page of the décalage spectral sequence fits in an exact couple

$$\begin{array}{ccc} \oplus_{n,s}\pi_n\text{Déc}(X)(n+s) & \xrightarrow{\quad i \quad} & \oplus_{n,s}\pi_n\text{Déc}(X)(n+s) \\ & {}_{\kappa}\nwarrow \quad \swarrow_{j} & \\ & E^2_{\text{Déc}}(X) := \oplus_{n,s}\pi_n \operatorname{Gr}^{n+s}(\text{Déc}(X)) & \end{array}$$

which defines the differential of the E^2-page of the décalage spectral sequence. Ideally, this differential is compatible with the differential $d^2 = j^2\kappa^2$ of the spectral sequence obtained via exact couples under isomorphism $H(E^1, d^1) \cong \pi_n \operatorname{Gr}^{n+s}(\text{Déc}(X))$. In fact, we want this to hold for all $r \geq 1$. Proving this directly is quite complicated. However, Antieau has proved a link between the décalage spectral sequence and another method of constructing a spectral sequence by Lurie [1, Thm. 4.13]. We will prove that this third method of constructing a spectral sequence results in the same spectral sequence as the one obtained via derived exact couples.

5.1 Spectral sequence: Lurie's method and relation to exact couples

The third method of constructing a spectral sequence from a tower of spectra is one described by Lurie in [16, Section 1.2.2] and by Antieau in [1, Section 4]. Let $X \in \text{Tow}(\text{Sp})$. For convenience, we write $X(i,j) := \text{cofib}(X(j) \to X(i))$. Then for every $n, s \in \mathbb{Z}$ and $r \geq 0$, we have a commutative diagram of cofiber sequences

$$\begin{array}{ccccc} X(s+r) & \longrightarrow & X(s) & \longrightarrow & X(s,s+r) \\ \downarrow & & \downarrow & & \downarrow \\ X(s+1) & \longrightarrow & X(s-r+1) & \longrightarrow & X(s-r+1,s+1) \end{array}$$

This in turn yields a commutative diagram of cofiber sequences

$$\begin{array}{ccccc} X(s+r,s+2r) & \longrightarrow & X(s,s+2r) & \longrightarrow & X(s,s+r) \\ \downarrow & & \downarrow & & \downarrow \\ X(s+1,s+r+1) & \longrightarrow & X(s-r+1,s+r+1) & \longrightarrow & X(s-r+1,s+1), \end{array}$$

see also Proposition 4.8. We then set

$$\tilde{E}^r_{n,s}(X) := \mathrm{im}(e^r_{n,s} : \pi_n X(s,s+r) \to \pi_n X(s-r+1,s+1)),$$

where $e^r_{n,s}$ is induced by the above diagram of cofiber sequences. We define the differential via the following commutative diagram.

$$\begin{array}{ccccc} \pi_n X(s,s+r) & \twoheadrightarrow & \tilde{E}^r_{n,s}(X) & \rightarrowtail & X(s-r+1,s+1) \\ \downarrow & & \Big\downarrow{\scriptstyle d^r} & & \downarrow \\ \pi_{n-1} X(s+r,s+2r) & \twoheadrightarrow & \tilde{E}^r_{n-1,s+r}(X) & \rightarrowtail & X(s+1,s+r+1). \end{array}$$

The left and right-most vertical maps are given by the boundary maps induced by the previous diagram. The desired, dotted arrow then exists as a consequence of functoriality of epi-mono factorizations, as is also stated by Antieau [1, Not. 4.4]. This yields a spectral sequence, see also Lurie [16, Prop. 1.2.2.7]. We will not give the details of the proof as is given there, but we will briefly describe the isomorphisms

$$\tilde{E}^{r+1}(X) \cong \frac{\ker(d^r)}{\mathrm{im}(d^r)},$$

as are detailed in [16, Prop. 1.2.2.7]. Write

$$\tilde{Z}^r_{n,s} \colon = \ker(d^r \colon \tilde{E}^r_{n,s}(X) \to \tilde{E}^r_{n-1,s+r}(X))$$

and

$$\tilde{B}^r_{n,s} \colon = \mathrm{im}(d^r \colon \tilde{E}^r_{n+1,s-r}(X) \to \tilde{E}^r_{n,s}(X)).$$

For each $n, s \in \mathbb{Z}$ we have that

$$\tilde{E}^{r+1}_{n,s}(X) := \mathrm{im}(\pi_n X(s,s+r+1) \to \pi_n X(s-r,s+1)).$$

We can factor the map $e^{r+1}_{n,s}$ that defines $\tilde{E}^{r+1}_{n,s}$ as

$$\pi_n X(s,s+r+1) \to \pi_n X(s,s+r) \xrightarrow{e^r_{n,s}} \pi_n X(s-r+1,s+1) \to \pi_n X(s-r,s+1).$$

The composition

$$\tau^{r+1}_{n,s} \colon \pi_n X(s,s+r+1) \to \pi_n X(s,s+r) \xrightarrow{e^r_{n,s}} \pi_n X(s-r+1,s+1)$$

maps into $\tilde{Z}^r_{n,s}$ and yields an epi-mono factorization

$$\pi_n X(s,s+r+1) \overset{\tau^{r+1}_{n,s}}{\twoheadrightarrow} \tilde{Z}^r_{n,s} \twoheadrightarrow \frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}} \hookrightarrow \frac{\tilde{E}^r_{n,s}}{\tilde{B}^r_{n,s}} \hookrightarrow \pi_n X(s-r,s+1).$$

For details on this, we refer again to [16, Prop. 1.2.2.7]. Important is that $\tau^{r+1}_{n,s}$ is an epimorphism onto $\tilde{Z}^r_{n,s}$. Thenwe have isomorphisms $\tilde{E}^{r+1}_{n,s}(X) \cong \frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}}$ as a consequence of uniqueness of epi-mono factorizations in Ab. In particular, the isomorphisms $\phi^{r+1}_{n,s} \colon \tilde{E}^{r+1}_{n,s}(X) \to \frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}}$ are given by

$$\phi^{r+1}_{n,s}(x) = [\tau^{r+1}_{n,s}(y)], \text{ where } e^{r+1}_{n,s}(y) = x.$$

The inverse of $\phi^{r+1}_{n,s}$ is given by $\psi^{r+1}_{n,s} : \frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}} \to \tilde{E}^{r+1}_{n,s}(X)$ with

$$\psi^{r+1}_{n,s}[x] := e^{r+1}_{n,s}(y), \text{ where } [\tau^{r+1}_{n,s}(y)] = [x].$$

Note that $\tau^{r+1}_{n,s}$ is in fact an epimorphism onto $\tilde{Z}^r_{n,s}$, so we can always just pick $y \in \pi_n X(s, s+r+1)$ such that $\tau^{r+1}_{n,s}(y) = x$. We will now give our own proof that for any tower of spectra X, this spectral sequence is isomorphic to the spectral sequence arising from exact couples.

Theorem 5.1. *Let X be a tower of spectra, $r \geq 1$. Let*

$$\tilde{E}^r_{n,s}(X) := \operatorname{im}(\pi_n X(s, s+r) \to \pi_n X(s-r+1, s+1))$$

be r-page of the spectral sequence discussed above. Write $E^r_{n,s}(X)$ for the spectral sequence obtained via exact couples as in the introduction of this section. Then $\tilde{E}^r_{n,s}(X) \cong E^r_{n,s}(X)$ as subquotients of the E^1-page, in a way that is compatible with the differentials, i.e.

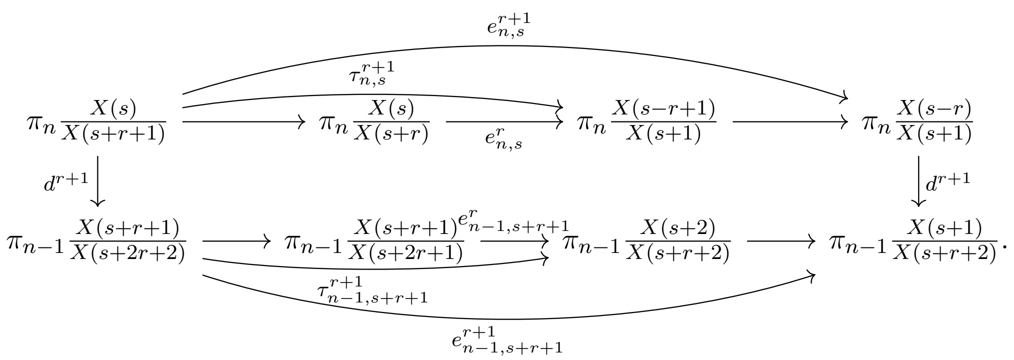
$$\begin{array}{ccc} \tilde{E}^r_{n,s}(X) & \xrightarrow{\cong} & E^r_{n,s}(X) \\ {\scriptstyle d^r}\downarrow & & \downarrow{\scriptstyle d^r} \\ \tilde{E}^r_{n-1,s+r} & \xrightarrow{\cong} & E^r_{n-1,s+r} \end{array}$$

commutes.

Proof. First, for every $n, s \in \mathbb{Z}$ and $r \geq 1$ we define a map $\delta^{r+1} : \frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}} \to \frac{\tilde{Z}^r_{n-1,s+r+1}}{\tilde{B}^r_{n-1,s+r+1}}$ via the following diagram

$$\begin{array}{ccc} \tilde{E}^{r+1}_{n,s}(X) & \underset{\psi^{r+1}_{n,s}}{\overset{\phi^{r+1}_{n,s}}{\rightleftarrows}} & \frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}} \\ {\scriptstyle d^{r+1}}\downarrow & & \downarrow{\scriptstyle \delta^{r+1}} \\ \tilde{E}^{r+1}_{n-s,s+r+1} & \underset{\psi^{r+1}_{n-1,s+r+1}}{\overset{\phi^{r+1}_{n-1,s+r+1}}{\rightleftarrows}} & \frac{\tilde{Z}^r_{n-1,s+r+1}}{\tilde{B}^r_{n-1,s+r+1}}. \end{array}$$

That is, $\delta^{r+1} = \phi^{r+1}_{n-1,s+r+1} d^{r+1} \psi^{r+1}_{n,s}$. We have a commutative diagram



Hence for $[x] \in \frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}}$ we have

$$\begin{aligned} \delta^{r+1}[x] &= \phi^{r+1}_{n-1,s+r+1} d^{r+1} \psi^{r+1}_{n,s}[x] \\ &= \phi^{r+1}_{n-1,s+r+1} d^{r+1} e^{r+1}_{n,s}(y), \text{ where } \tau^{r+1}_{n,s}(y) = x \text{ for some } y \in \pi_n X(s, s+r+1), \\ &= \phi^{r+1}_{n-1,s+r+1} e^{r+1}_{n-1,s+r+1} d^{r+1}(y) \\ &= [\tau_{n-1,s+r+1} d^{r+1}(y)]. \end{aligned}$$

Indeed, clearly $d^{r+1}(y)$ is a lift of $e^{r+1}_{n-1,s+r+1}d^{r+1}(y)$ under $e_{n-1,s+r+1}$.

As mentioned at the start of this section, we obtain the exact couple spectral sequence from the exact couple with maps

$$i : \pi_n X(s+1) \to \pi_n X(s),$$

$$j_{s,s+1} : \pi_n X(s) \to \pi_n X(s, s+1)$$

$$\kappa_{s,s+1} : \pi_n X(s, s+1) \to \pi_{n-1} X(s+1).$$

As in Example 1.6, note that for $r \geq 1$ we can write $E^{r+1}_{n,s}(X) = \frac{Z^r_{n,s}}{B^r_{n,s}}$, with

$$Z^r_{n,s} := \kappa^{-1}_{s,s+1}(\mathrm{im}(i^r : \pi_{n-1}X(s+r+1) \to \pi_{n-1}X(s+1))) \subset \pi_n X(s, s+1)$$

and

$$B^r_{n,s} := j_{s,s+1}(\ker(i^r : \pi_n X(s) \to \pi_n X(s-r))) \subset \pi_n X(s, s+1).$$

Then $E^{r+1}_{n,s}(X)$ becomes a subquotient of the E^1-page. Under this identification, the differential $d^{r+1}_{ex.cpl.}$ becomes the map

$$\frac{\kappa^{-1}(\mathrm{im}(i^r : \pi_{n-1}X(s+r+1) \to \pi_{n-1}X(s+1)))}{j(\ker(i^r : \pi_n X(s) \to X(s-r))} \to \frac{\kappa^{-1}(\mathrm{im}(i^r : \pi_{n-2}X(s+2r+1) \to \pi_{n-2}X(s+r+1)))}{j(\ker(i^r : \pi_{n-1}X(s+r+1) \to \pi_{n-1}X(s+1))}$$

where for a class $[x]$ with $x \in Z^r_{n,s}$ we lift $\kappa_{s,s+1}(x)$ to $\pi_{n-1}X(s+r+1)$ along

$$i^r : \pi_{n-1}X(s+r+1) \to X(s+1)$$

and then map into $\pi_{n-1}X(s+r+1, s+r+2)$ via j. In a diagram:

$$\begin{array}{ccccc} & & \pi_{n-1}X(s+r+1) & \xrightarrow{j} & \pi_{n-1}\frac{X(s+r+1)}{X(s+r+2)} \\ & & \overset{lift}{\nearrow} \big\downarrow i & & \\ \pi_n \frac{X(s)}{X(s+1)} & \xrightarrow{\kappa} & \pi_{n-1}X(s+1). & & \end{array}$$

Note that $\frac{Z^r_{n,s}}{B^r_{n,s}}$ is a subquotient of the E_1-page of the exact couple spectral sequence; but $\frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}}$ is a subquotient of the $\tilde{E}_r$-page of Lurie's spectral sequence in this notation. The first page of both spectral sequences is defined by

$$\tilde{E}^1_{n,s}(X) = E^1_{n,s}(X) = \pi_n X(s, s+1)$$

together with the differential

$$d_1 : \pi_n X(s, s+1) \xrightarrow{\kappa} \pi_{n-1}X(s+1) \xrightarrow{j} \pi_{n-1}X(s+1, s+2).$$

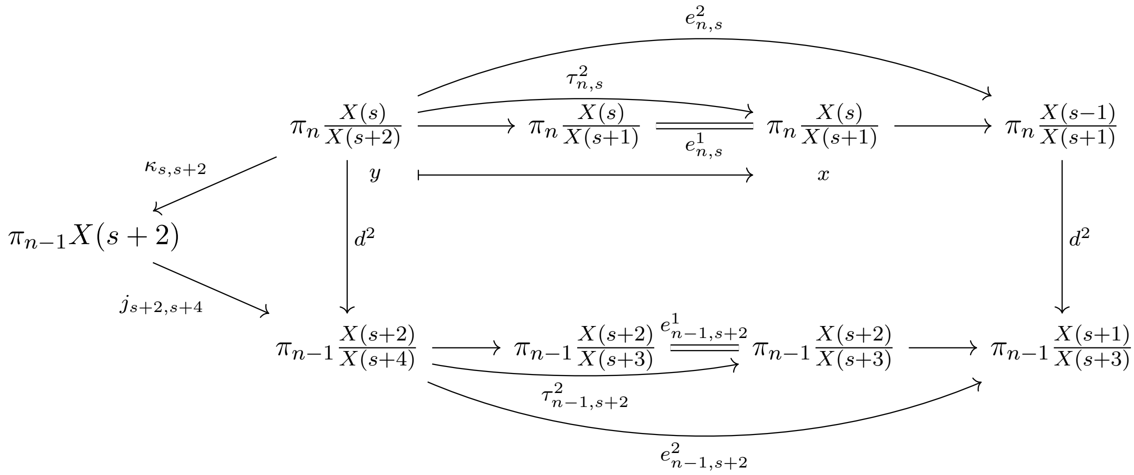
As a consequence

$$\tilde{E}^2_{n,s} \underset{\psi^2_{n,s}}{\overset{\phi^2_{n,s}}{\cong}} \frac{\tilde{Z}^1_{n,s}}{\tilde{B}^1_{n,s}} = \frac{Z^1_{n,s}}{B^1_{n,s}} = E^2_{n,s}(X).$$

We will now show that δ^2 and $d^2_{ex.cpl}$ are the same map. First, we note that

$$e^1_{n,s} : \pi_n X(s, s+1) \to \pi_n X(s, s+1)$$

is simply the identity map. We then keep the diagram



in mind. Note that in general $d^{r+1}: \pi_n X(s, s+r+1) \to \pi_n X(s+r+1, s+r+2)$ from Lurie's spectral sequence factors as

$$\pi_n X(s, s+r+1) \xrightarrow{\kappa_{s,s+r+1}} \pi_{n-1} X(s+r+1) \xrightarrow{j_{s+r+1,s+2r+2}} \pi_n X(s+r+1, s+2r+2)$$

by Remark 4.11, as is also indicated in the diagram. Then, for $[x] \in \frac{\tilde{Z}^1_{n,s}}{\tilde{B}^1_{n,s}}$ we have

$$\delta^2[x] = [\tau^2_{n-1,s+2} d^2(y)]$$

where $\tau^2_{n,s}(y) = x$. Note that the diagram of cofiber sequences

$$\begin{array}{ccccc} X(s+4) & \longrightarrow & X(s+2) & \longrightarrow & X(s+2, s+4) \\ \downarrow & & \downarrow & & \downarrow \\ X(s+3) & \longrightarrow & X(s+2) & \longrightarrow & X(s+2, s+3) \end{array}$$

implies that the map $\tau^2_{n-1,s+2} \circ j_{s+2,s+4}$ is simply $j_{s+2,s+3} : \pi_{n-1} X(s+2) \to \pi_{n-1} X(s+2, s+3)$. Therefore,

$$\begin{aligned} \delta^2[x] &= [\tau^2_{n-1,s+2} d^2(y)] \\ &= [j_{s+2,s+3} \kappa_{s,s+2}(y)]. \end{aligned}$$

Using again a diagram of cofiber sequences as above, we have a commutative diagram

$$\begin{array}{ccc} \pi_n X(s, s+1) & \xleftarrow{\tau^2_{n,s}} & \pi_n X(s, s+2) \\ \kappa_{s,s+1} \downarrow & & \downarrow \kappa_{s,s+2} \\ \pi_{n-1} X(s+1) & \longleftarrow & \pi_{n-1} X(s+2). \end{array}$$

Therefore $\kappa_{s,s+2}(y)$ is in fact a lift of $\kappa_{s,s+1}(\tau^2_{n,s}(y)) = \kappa_{s,s+1}(x)$ under $i : \pi_{n-1} X(s+2) \to \pi_{n-1} X(s+1)$ and therefore

$$\delta^2[x] = [j_{s+2,s+3} \kappa_{s,s+2}(y)] = d^2_{ex.cpl}[x]$$

by definition of $d^2_{ex.cpl.}$, which does not depend on the choice of lifts, as these are quotiented out. So for $r = 1$, the theorem holds.

We will now assume for $1 \leq i \leq r-1$ we have that $\frac{\tilde{Z}^i_{n,s}}{\tilde{B}^i_{n,s}}$ when viewed as a sub-quotient of $\tilde{E}^1_{n,s} = E^1_{n,s}$ via applying the isomorphisms $\phi^i, \ldots, \phi^2, \phi^1$ is the same as the sub-quotient $\frac{Z^i_{n,s}}{B^i_{n,s}}$ of the E_1-page. We also assume that these isomorphisms are compatible with the differentials. Here ϕ^1 is simply the identity $E^1 = \tilde{E}^1$. That is, we assume the diagram

$$\begin{array}{ccc}
\frac{\tilde{Z}^i_{n,s}}{\tilde{B}^i_{n,s}} & \xrightarrow{\cong} & \frac{Z^i_{n,s}}{B^i_{n,s}} \\
\delta_i \downarrow & & \downarrow d^{i+1}_{ex.cpl.} \\
\frac{\tilde{Z}^i_{n-1,s+i+1}}{\tilde{B}^i_{n-1,s+i+1}} & \xrightarrow{\cong} & \frac{Z^i_{n-1,s+i+1}}{B^i_{n-1,s+i+1}}
\end{array}$$

commutes. This in turn yields isomorphisms

$$\frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}} \cong \frac{Z^r_{n,s}}{B^r_{n,s}}$$

via the commutative diagram

$$\begin{array}{ccccc}
\tilde{E}^r_{n,s} & \xrightarrow{\phi^r} & \frac{\tilde{Z}^{r-1}_{n,s}}{\tilde{B}^{r-1}_{n,s}} & \xrightarrow{\cong} & \frac{Z^{r-1}_{n,s}}{B^{r-1}_{n,s}} \\
d^r \downarrow & & \delta^r \downarrow & & d^r_{ex.cpl} \downarrow \\
\tilde{E}^r_{n-1,s+r} & \xrightarrow{\phi^r} & \frac{\tilde{Z}^{r-1}_{n-1,s+r}}{\tilde{B}^{r-1}_{n-1,s+r}} & \xrightarrow{\cong} & \frac{Z^{r-1}_{n-1,s+r}}{B^{r-1}_{n-1,s+r}}.
\end{array}$$

Indeed, this diagram implies that $\tilde{Z}^r_{n,s} \cong Z^r_{n,s}$ and $\tilde{B}^r_{n,s} \cong B^r_{n,s}$. To clarify this, consider $r = 2$. Then the discussion of the base case satisfies the hypothesis and yields isomorphisms

$$\frac{\tilde{Z}^2_{n,s}}{\tilde{B}^2_{n,s}} \cong \frac{\ker(d^2_{ex.cpl.} : \frac{Z_1^{n,s}}{B^1_{n,s}} \to \frac{Z_1^{n-1,s+2}}{B^1_{n-1,s+2}})}{\operatorname{im}(d^2_{ex.cpl.} : \frac{Z_1^{n+1,s-2}}{B^1_{n+1,s-2}} \to \frac{Z_1^{n,s}}{B^1_{n,s}})} \cong \frac{Z^2_{n,s}}{B^2_{n,s}}$$

via

$$[x] \mapsto [\phi^2_{n,s}(x)] = [[\tau^2_{n,s}(y)]] \mapsto [\tau^2_{n,s}(y)], \text{ where } e^2_{n,s}(y) = x.$$

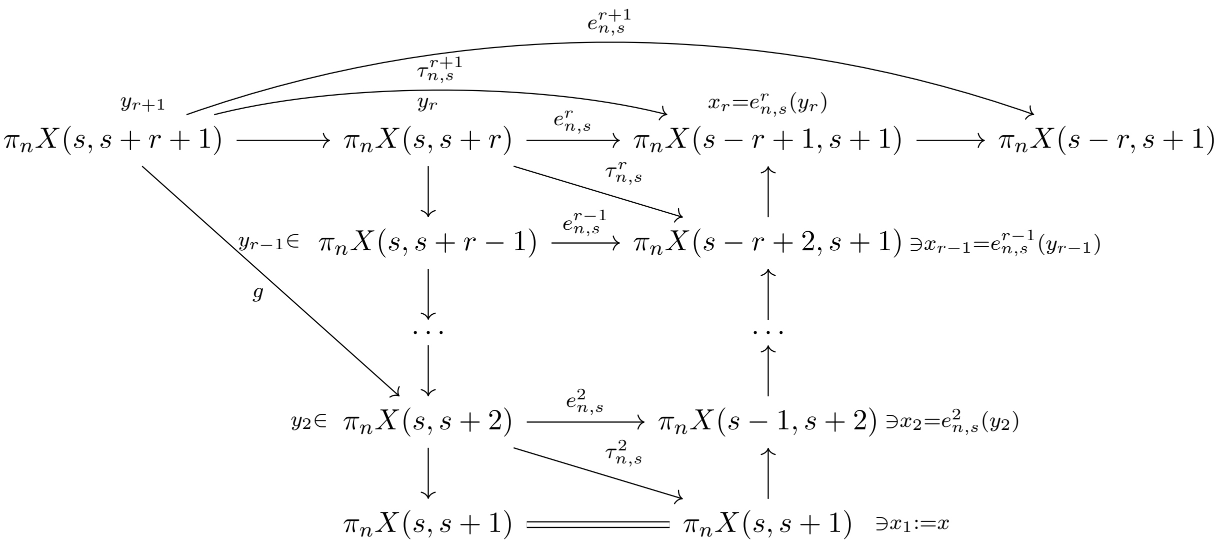
and

$$[\psi^2_{n,s}[x']] = [e^2_{n,s}(y')] \leftarrow\!\shortmid [[x']] \leftarrow\!\shortmid [x], \text{ where } x = \tau^2_{n,s}(y').$$

In general, this means that the isomorphsims

$$\tilde{E}^{r+1}_{n,s} \xleftarrow{\psi^{r+1}_{n,s}} \frac{\tilde{Z}^r_{n,s}}{\tilde{B}^r_{n,s}} \xleftarrow{\psi^{r,\dots,1}} \frac{Z^r_{n,s}}{B^r_{n,s}}$$

are given by the following. Consider the diagram



Then a class $[x] \in \frac{Z^r_{n,s}}{B^r_{n,s}}$ is mapped to $e^{r+1}_{n,s}(y_{r+1}) \in \tilde{E}^{r+1}_{n,s}$, where y^{r+1} is chosen by the process of taking lifts

$$y_2 \text{ such that } \tau^2_{n,s}(y_2) = x,$$

$$y_3 \text{ such that } \tau^3_{n,s}(y_3) = x_2 := e^2_{n,s}(y_2),$$

$$\dots$$

In particular, this implies that we simply choose y^{r+1} as a lift of $\sigma(x)$ with

$$\sigma : \pi_n X(s, s+1) \to \pi_n X(s-r+1, s+1).$$

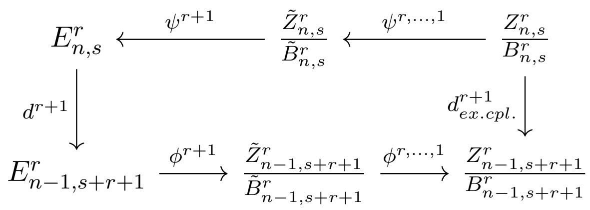
Similarly, we note that

$$\tilde{E}^{r+1}_{n,s} \xrightarrow{\phi^{r+1}_{n,s}} \frac{\tilde{Z}_{n,s}}{\tilde{B}_{n,s}} \xrightarrow{\phi^{r,\dots,1}} \frac{Z^r_{n,s}}{B^r_{n,s}}$$

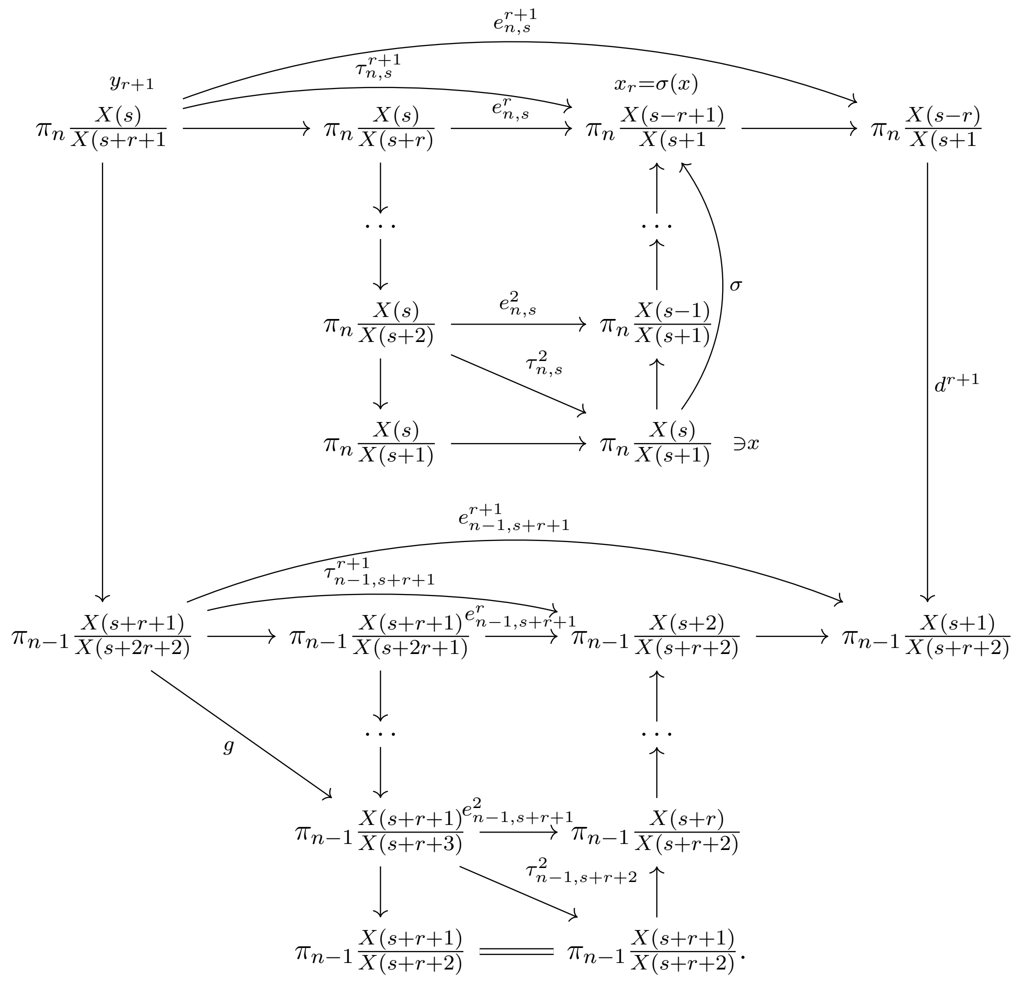
is given by sending

$$x' \in \tilde{E}^{r+1}_{n,s} = \operatorname{im}(\pi_n X(s, s+r+1) \xrightarrow{e^{r+1}_{n,s}} \pi_n X(s+1, s+r+2))$$

to $[\tau^2_{n,s}(y_2')]$, where y_2 is the image of y'_{r+1} under the map $g : \pi_n X(s, s+r+1) \to \pi_n X(s, s+2)$ and $e^{r+1}_{n,s}(y'_{r+1}) = x'$. We will now show that



commutes. We can combine the information above in a commutative diagram



Then for $[x] \in \frac{Z^r_{n,s}}{B^r_{n,s}}$ we have that

$$\begin{aligned}\phi^{r,\dots,1}\delta^{r+1}\psi^{r,\dots,1}[x] &= \phi^{r,\dots,1}\phi^{r+1}d^{r+1}e^{r+1}_{n,s}(y^{r+1})\\ &= \phi^{r,\dots,1}\phi^{r+1}e^{r+1}_{n-1}d^{r+1}(y^{r+1})\\ &= [\tau^2_{n-1,s+r+1}\circ g\circ d^{r+1}(y^{r+1})].\end{aligned}$$

Recall that we can factor $d^{r+1}: \pi_nX(s,s+r+1) \to \pi_{n-1}X(s+r+1,s+2r+2)$ as

$$\pi_nX(s,s+r+1) \xrightarrow{\kappa_{s,s+r+1}} X(s+r+1)$$

As for the base case, note that $j_{s+r+1,s+2r+2}\circ\tau^2_{n-1,s+r+1}\circ g$ is simply the map $j_{s+r+1,s+r+2}$. Therefore

$$\begin{aligned}\phi^{r,\dots,1}\delta^{r+1}\psi^{r,\dots,1}[x] &= [\tau^2_{n-1,s+r+1}\circ g\circ d^{r+1}(y^{r+1})]\\ &= [j_{s+r+1,s+r+2}\kappa_{s,s+r+1}(y_{r+1})].\end{aligned}$$

Furthermore, we have commutative diagrams

$$\begin{array}{ccc} y_{r+1} & \longmapsto & x_r \\ \pi_nX(s,s+r+1) & \xrightarrow{\tau^{r+1}_{n,s}} & \pi_nX(s-r+1,s+1) \\ \downarrow{\scriptstyle \kappa_{s,s+r+1}} & & \downarrow{\scriptstyle \kappa_{s-r+1,s+1}} \\ \pi_{n-1}X(s+r+1) & \longrightarrow & \pi_{n-1}X(s+1) \end{array}$$

and

$$\begin{array}{ccc} x=x_1 & \longmapsto & x_r \\ \pi_nX(s,s+1) & \xrightarrow{\sigma} & \pi_nX(s-r+1,s+1) \\ & {\scriptstyle \kappa_{s,s+1}}\searrow & \downarrow{\scriptstyle \kappa_{s-r+1,s+1}} \\ & & \pi_{n-1}X(s+1) \end{array}$$

This means that $\kappa_{s,s+r+1}(y^{r+1})$ is in fact a lift of $\kappa_{s,s+1}(x)$ along $\pi_{n-1}X(s+r+1) \to \pi_{n-1}X(s+1)$. Therefore, by definition of $d^{r+1}_{ex.cpl.}$ we have that

$$\begin{aligned}\phi^{r,\dots,1}\delta^{r+1}\psi^{r,\dots,1}[x] &= [j_{s+r+1,s+r+2}\kappa_{s,s+r+1}(y_{r+1})]\\ &= d^{r+1}_{ex.cpl.}[x].\end{aligned}$$

This completes the proof. □

5.2 Method III and relation to the Décalage spectral sequence.

We will now discuss Lurie's method of constructing a spectral sequence and the link with the décalage functor. Antieau proves the following result [1, Thm. 4.13]. He proves it for general stable ∞-categories with sequential limits and colimits. We will only discuss it for Sp.

Theorem 5.2. *Let $X \in \mathrm{Tow}(\mathrm{Sp})$; write $\tilde{E}^r_{n,s}(X)$ for the spectral sequence obtained via Lurie's method in Section 5.1. Then for all $r \geq 1$ we have*

$$\tilde{E}^{r+1}_{n,s}(X) \cong \tilde{E}^r_{n,n+s}(D\acute{e}c(X)),$$

compatible with d^r and d^{r+1} respectively.

We will briefly give an overview of Antieau's proof.

Proof sketch. First, we will show that $\tilde{E}^{r+1}_{n,s}(X) \cong \tilde{E}^{r}_{n,n+s}(\mathrm{Déc}(X))$ are isomorphic. For this, we introduce the following notation. Let $i \leq j$ and let $\mathrm{Gr}^{[i,j)}(X)$ be the cofiber of $X(j) \to X(i)$, which can be filtered by the tower

$$\cdots \to 0 \to X(j-1)/X(j) \to \cdots \to X(i)/X(j) \simeq X(i)/X(j) \simeq \ldots .$$

Similarly, we write $\mathrm{Gr}^{[i,\infty)}(X)$ for the cofiber of $X(\infty) \to X(i)$, which we can also filter by

$$\cdots \to X(i+2)/X(\infty) \to X(i+1)/X(\infty) \to X(i)/X(\infty) \simeq X(i)/X(\infty) \simeq \ldots,$$

and we write $\mathrm{Gr}^{(-\infty,j]}(X) = \mathrm{cofib}(X(j) \to X(-\infty))$, which we can filter by

$$\cdots \to 0 \to X(j-1)/X(j) \to X(j-2)/X(j) \to \ldots$$

Note for $i \leq j \leq k$ that we have a fiber sequence

$$\mathrm{Gr}^{[j,k)}(X) \to \mathrm{Gr}^{[i,k)}(X) \to \mathrm{Gr}^{[i,j)}(X).$$

Then we also have a fiber sequence of towers

$$\mathrm{Déc}(\mathrm{Gr}^{[j,k)}(X)) \to \mathrm{Déc}(\mathrm{Gr}^{[i,k)}(X)) \to \mathrm{Déc}(\mathrm{Gr}^{[i,j)}(X)).$$

This is a non-trivial fact, which Antieau shows in [1, Lemma 4.20, 4.21].

For notational convenience we consider only the isomorphisms $E^{r+1}_{0,0}(X) \cong E^{r}_{0,0}(\mathrm{Déc}(X))$ and corresponding differentials. There is a commutative diagram

$$\begin{array}{ccccccccc}
\pi_0 \mathrm{Gr}^{[0,r+1)}(X) & \xleftarrow[\cong]{1} & \pi_0 \mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)) & \xrightarrow{2} & \pi_0 \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)) & \xleftarrow[\cong]{3} & \pi_0 \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^{[0,\infty)}(X)) & \xrightarrow{4} & \pi_0 \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(X)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\pi_0 \mathrm{Gr}^{[-r,1)}(X) & \xleftarrow{5} & \pi_0 \mathrm{Gr}^{[-r+1,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[-r,1)}(X)) & \xrightarrow[\cong]{6} & \pi_0 \mathrm{Gr}^{[-r+1,1)}(\mathrm{Déc}(\mathrm{Gr}^{[-r,1)}(X)) & \xleftarrow{7} & \pi_0 \mathrm{Gr}^{[-r+1,1)}(\mathrm{Déc}(\mathrm{Gr}^{[-r,\infty)}(X)) & \xrightarrow[\cong]{8} & \pi_0 \mathrm{Gr}^{[-r+1,1)}(\mathrm{Déc}(X))
\end{array}$$

Here, when we take the décalée of the associated graded, we mean that we take the décalée of the corresponding tower as discussed above. Furthermore, the vertical maps on the far left and the far right respectively determine $\tilde{E}^{r+1}_{0,0}(X)$ and $\tilde{E}^{r}_{0,0}(\mathrm{Déc}(X))$. So if we can show that the maps in the above diagram have the properties stated, then we indeed have an isomorphism $\tilde{E}^{r+1}_{0,0}(X) \cong \tilde{E}^{r}_{0,0}(\mathrm{Déc}(X))$.

Antieau discusses all maps separately; we will discuss the first three to give an idea of the proof techniques he uses and to demonstrate where the maps actually come from. For (1), note that we have a (co)fiber sequence

$$\tau^{Bei}_{\geq 0}(\mathrm{Gr}^{[0,r+1)}(X)) \to \mathrm{Gr}^{[0,r+1)}(X) \to \tau^{Bei}_{\leq -1}(\mathrm{Gr}^{[0,r+1)}(X)).$$

Here we again view $\mathrm{Gr}^{[0,r+1)}(X)$ as a filtration. Then we can take the colimit, which yields a cofiber sequence

$$\mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)) \to \mathrm{Gr}^{[0,r+1)}(X) \to \mathrm{Gr}^{[-\infty,-1]}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)).$$

Indeed,

$$\begin{aligned}
\mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)) &:= \frac{\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)(0)}{\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)(\infty)} \\
&\simeq \frac{\tau^{Bei}_{\geq 0} \mathrm{Gr}^{[0,r+1)}(X)(-\infty)}{0} \\
&\simeq \tau^{Bei}_{\geq 0} \mathrm{Gr}^{[0,r+1)}(X)(-\infty)
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{Gr}^{(-\infty,-1]}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)) &= \frac{\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))(-\infty)}{\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))(0)} \\
&\simeq \frac{\mathrm{Gr}^{[0,r+1)}(X)}{\tau^{Bei}_{\geq 0} \mathrm{Gr}^{[0,r+1)}(X)(-\infty)} \\
&\simeq \tau^{Bei}_{\leq -1} \mathrm{Gr}^{[0,r+1)}(X)(-\infty).
\end{aligned}$$

Then (1) is the map induced by $\mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))) \to \mathrm{Gr}^{[0,r+1)}(X)$. Furthermore, by the above discussion on filtering the associated graded, we can filter $\mathrm{Gr}^{[-\infty,-1]}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))$ as

$$\cdots \to 0 \to \frac{\mathrm{Gr}^{[r,r+1)}(X)}{\mathrm{Déc}(\mathrm{Gr}^{[r,r+1)}(X))(0)} \to \cdots \to \frac{\mathrm{Gr}^{[0,r+1)}(X)}{\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))(0)} \simeq \frac{\mathrm{Gr}^{[0,r+1)}(X)}{\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))(0)} \simeq \dots,$$

with associated graded pieces

$$\frac{\mathrm{Gr}^{[s,r+1)}(X)}{\mathrm{Déc}(\mathrm{Gr}^{[s,r+1)}(X))(0)} \Big/ \frac{\mathrm{Gr}^{[s+1,r+1)}(X)}{\mathrm{Déc}(\mathrm{Gr}^{[s+1,r+1)}(X))(0)} \cong \frac{\mathrm{Gr}^{s}(X)}{\mathrm{Déc}(\mathrm{Gr}^{s}(X))(0)}.$$

Since $\mathrm{Déc}(\mathrm{Gr}^s(X)(0)) \cong \tau_{\geq -s}(\mathrm{Gr}^s(X))$ by [1, Lemma 4.24], it follows that

$$\mathrm{Gr}^{(-\infty,-1]}\,\mathrm{Déc}(\mathrm{Gr}^s(X)) \simeq \tau_{\leq -s-1}\,\mathrm{Gr}^s(X).$$

The objects in the filtration have $0 \leq s \leq r$, hence it follows that $\mathrm{Gr}^{[-\infty,-1]}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))$ is an iterated extension of objects in $\mathrm{Sp}_{\leq -1}$. Since $\mathrm{Sp}_{\leq -1}$ is closed under extensions, we have that $\mathrm{Gr}^{[-\infty,-1]}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)) \in \mathrm{Sp}_{\leq -1}$. Therefore $\pi_{0,1}(\mathrm{Gr}^{[-\infty,-1]}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))) \cong 0$, so indeed by exactness the map (1) is an isomorphism.

Secondly, the map (2) is induced by the canonical map

$$\mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))) \to \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))).$$

Similarly as in (1); we can filter both the source and the target of this map. The first one as

$$0 \to \mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[r,r+1)}(X))) \to \cdots \to \mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)) \simeq \mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)) \simeq \dots$$

and the second one as

$$0 \to \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^{[r,r+1)}(X))) \to \cdots \to \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))) \simeq \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))) \simeq \dots.$$

Using Lemma 4.24 again, we have for $0 \leq s \leq r$ that the associated graded of these filtrations is given by

$$\mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^s(X))) \cong \tau_{\geq -s}\,\mathrm{Gr}^s(X) \text{ and } \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^s(X))) \cong \tau_{<-s+r}\tau_{\geq -s}\,\mathrm{Gr}^s(X).$$

Hence the fiber of (2) has a filtration with associated graded objects $\tau_{\geq -s+r}\,\mathrm{Gr}^s(X)$. Therefore, the fiber is in $\mathrm{Sp}_{\geq 0}$, by analysing the long exact sequences induced by taking the associated graded. In particular, π_{-1} of the fiber of

$$\mathrm{Gr}^{[0,\infty)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X))) \to \mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^{[0,r+1)}(X)))$$

is 0, so (2) is indeed an epimorphism.

Next, map (3) is the first map where the isomorphism from Theorem 4.18 comes into play. In particular, we can filter the fiber $\mathrm{Gr}^{[0,r)}(\mathrm{Déc}(\mathrm{Gr}^{[r+1,\infty)}(X)))$ of the map that induces (3) by

$$0 \to \cdots \to \frac{\mathrm{Déc}(\mathrm{Gr}^{[r+1,\infty)}(X))(r-1)}{\mathrm{Déc}(\mathrm{Gr}^{[r+1,\infty)}(X))(r)} \to \cdots \to \frac{\mathrm{Déc}(\mathrm{Gr}^{[r+1,\infty)}(X))(0)}{\mathrm{Déc}(\mathrm{Gr}^{[r+1,\infty)}(X))(r)} \simeq \frac{\mathrm{Déc}(\mathrm{Gr}^{[r+1,\infty)}(X))(0)}{\mathrm{Déc}(\mathrm{Gr}^{[r+1,\infty)}(X))(r)},$$

with associated graded pieces $\mathrm{Gr}^s\,\mathrm{Déc}(\mathrm{Gr}^{[r+1,\infty)}(X)) \cong \tau^{Bei}_{\leq s}\tau^{Bei}_{\geq s}(\mathrm{Gr}^{[r+1,\infty)}(X))$, which is precisely $\pi^{Bei}_s\,\mathrm{Gr}^{[r+1,\infty)}(X)[s]$. By Theorem 4.18 we can view $\pi^{Bei}_s\,\mathrm{Gr}^{[r+1,\infty)}(X)$ as the chain complex

$$\dots 0 \to \pi_{-r-1+s}\,\mathrm{Gr}^{r+1}(X) \to \pi_{-r-2+s}\,\mathrm{Gr}^{r+2}(X) \to \dots$$

with $\pi_{-r-1+s}\,\mathrm{Gr}^{r+1}(X)$ in homological degree $(-r-1)$. Note, then $H^{-b}(\pi^{Bei}_s\,\mathrm{Gr}^{[r+1,\infty)}(X)) \cong 0$ for $b < r+1$. It follows that the homotopy groups π_{-b} of $\pi^{Bei}_s\,\mathrm{Gr}^{[r+1,\infty)}(X)$ are zero for $b > r+1$, hence

$\pi_{-b}\pi_s^{Bei}\,\mathrm{Gr}^{[r+1,\infty)}(X)[s] \cong 0$ for $b < r+1-s$. It follows in particular that the associated graded objects of the fiber have zero homotopy groups π_{-b} for $b < 2$. Via several long exact sequences, it then follows that the fiber itself must also have zero homotopy groups π_{-b} for $b < 2$, so (3) is an isomorphism. Similar arguments yield the remaining maps and their corresponding properties. The commutativity of this diagram yields isomorphic epi-mono factorizations of all vertical maps; hence $\tilde{E}^{r+1}_{0,0}(X) \cong \tilde{E}^{r}_{0,0}(\mathrm{D\acute{e}c}(X))$.

Next, he shows that the isomorphisms $\tilde{E}^{r+1}_{0,0}(X) \cong \tilde{E}^{r}_{0,0}(\mathrm{D\acute{e}c}(X))$ induced by the 2x5-diagram above are compatible with the maps d^r and d^{r+1}. This involves more complicated diagrams than before. Firstly, there is a commutative diagram

$$\begin{array}{ccccccccc}
\mathrm{Gr}^{[r+1,2r+2)}(X) & \leftarrow & \mathrm{Gr}^{[r,\infty)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,2r+2)}(X)) & \rightarrow & \mathrm{Gr}^{[r,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,2r+2)}(X)) & \leftarrow & \mathrm{Gr}^{[r,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,\infty)}(X)) & \rightarrow & \mathrm{Gr}^{[r,2r)}(X) \\
\| & & \downarrow & & \downarrow & & \downarrow & & \| \\
\mathrm{Gr}^{[r+1,2r+2)}(X) & \leftarrow & \mathrm{Gr}^{[0,\infty)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,2r+2)}(X)) & \rightarrow & P & \leftarrow & \mathrm{Gr}^{[r,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,\infty)}(X)) & \rightarrow & \mathrm{Gr}^{[r,2r)}(X) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}^{[0,2r+2)}(X) & \leftarrow & \mathrm{Gr}^{[0,\infty)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,2r+2)}(X)) & \rightarrow & \mathrm{Gr}^{[0,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,2r+2)}(X)) & \leftarrow & \mathrm{Gr}^{[0,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,\infty)}(X)) & \rightarrow & \mathrm{Gr}^{[0,2r)}(\mathrm{D\acute{e}c}(X)) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{Gr}^{[0,r+1)}(X) & \leftarrow & \mathrm{Gr}^{[0,\infty)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,r+1)}(X)) & \rightarrow & \mathrm{Gr}^{[0,r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,r+1)}(X)) & \leftarrow & \mathrm{Gr}^{[0,r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,\infty)}(X)) & \rightarrow & \mathrm{Gr}^{[0,r)}(\mathrm{D\acute{e}c}(X))
\end{array}$$

where the bottom 3x5-diagram consists of cofiber sequences in the columns and where P is the cofiber of

$$\mathrm{Gr}^{[0,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,2r+2)}(X)) \to \mathrm{Gr}^{[0,r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,r+1)}(X)).$$

Note that

$$\mathrm{Gr}^{[r,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,2r+2)}(X)) \to \mathrm{Gr}^{[0,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,2r+2)}(X)) \to \mathrm{Gr}^{[0,r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,r+1)}(X))$$

is zero, so the top vertical map into P indeed exists. This in turn induces a diagram

$$\begin{array}{ccccccccccccc}
\tilde{E}^{r+1}_{0,0}(X) & \twoheadleftarrow & \pi_0\,\mathrm{Gr}^{[0,r+1)}(X) & \xleftarrow{\cong} & \pi_0\,\mathrm{Gr}^{[0,\infty)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,r+1)}(X)) & \twoheadrightarrow & \pi_0\,\mathrm{Gr}^{[0,r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,r+1)}(X)) & \xleftarrow{\cong} & \pi_0\,\mathrm{Gr}^{[0,r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,\infty)}(X)) & \twoheadrightarrow & \pi_0\,\mathrm{Gr}^{[0,r)}(\mathrm{D\acute{e}c}(X)) & \twoheadrightarrow & \tilde{E}^{r}_{0,0}(\mathrm{D\acute{e}c}(X)) \\
\downarrow d^{r+1} & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow d^{r} \\
 & & \pi_{-1}\,\mathrm{Gr}^{[r+1,2r+2)}(X) & \leftarrow & \pi_{-1}\,\mathrm{Gr}^{[0,\infty)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,2r+2)}(X)) & \rightarrow & P & \xleftarrow{B} & \pi_{-1}\,\mathrm{Gr}^{[r,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,\infty)}(X)) & \rightarrow & \pi_{-1}\,\mathrm{Gr}^{[r,2r)}(\mathrm{D\acute{e}c}(X)) & & \\
 & & \| & & A \uparrow \cong & & \uparrow & & \uparrow & & \| & & \\
\tilde{E}^{r+1}_{-1,r+1}(X) & \twoheadleftarrow & \pi_{-1}\,\mathrm{Gr}^{[r+1,2r+2)}(X) & \xrightarrow{\cong} & \pi_{-1}\,\mathrm{Gr}^{[r,\infty)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,2r+2)}(X)) & \twoheadrightarrow & \pi_{-1}\,\mathrm{Gr}^{[r,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,2r+2)}(X)) & \xleftarrow{\cong} & \pi_{-1}\,\mathrm{Gr}^{[r,2r)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[r+1,\infty)}(X)) & \twoheadrightarrow & \pi_{-1}\,\mathrm{Gr}^{[r,2r)}(\mathrm{D\acute{e}c}(X)) & \twoheadrightarrow & \tilde{E}^{r+1}_{-1,r}(\mathrm{D\acute{e}c}(X))
\end{array}$$

where it is shown that the maps A and B have the desired properties by similar arguments as for the maps (1)-(8); for details see the full proof. The bottom row and the top row come from the diagram that defines the isomorphisms $\tilde{E}^{r+1}_{0,0}(X) \cong \tilde{E}^{r}_{0,0}(\mathrm{D\acute{e}c}(X))$ and $\tilde{E}^{r+1}_{-1,r+1}(X) \cong \tilde{E}^{r}_{-1,r}(\mathrm{D\acute{e}c}(X))$. Next, by chasing this diagram, it follows that the two maps from $\pi_0\,\mathrm{Gr}^{[0,\infty)}\,\mathrm{D\acute{e}c}(\mathrm{Gr}^{[0,r+1)}(X))$ to $\tilde{E}^{r}_{-1,r}(\mathrm{D\acute{e}c}(X))$ are the same. Here the first map is the top row composed with d^r and the second one comes from following the inverse of A and the bottom row. This then implies that indeed the isomorphisms are compatible with the differentials. □

This has the following immediate consequence [1, Cor. 4.14].

Corollary 5.3. *Let $X \in \mathrm{Tow}(\mathrm{Sp})$. For $r \geq 1$, we have*

$$\tilde{E}^{r}_{n,s}(X) \cong \tilde{E}^{1}_{n,(r-1)n+s}(D\acute{e}c^{r-1}(X))$$

compatible with the d^1 and d^r differentials respectively.

Note that the $\tilde{E}^{1}_{n,(r-1)n+s}(\mathrm{D\acute{e}c}^{r-1}(X))$ is precisely the definition of the E^r-page of the décalage spectral sequence. Hence combined with Theorem 5.1 this yields the following important result.

Corollary 5.4. *Let $X \in \mathrm{Tow}(\mathrm{Sp})$; write $E^{r}_{n,s}(X)$ for the spectral sequence obtained via exact couples. Then for $r \geq 1$, we have*

$$E^{r}_{n,s}(X) \cong E^{1}_{n,(r-1)n+s}(D\acute{e}c^{r-1}(X))$$

as sub-quotients of the abelian group $\pi_n\,\mathrm{Gr}^s(X)$, compatible with the d^1 and d^r differentials respectively.

In particular, this means that the $E^r_{n,s}(X) := \pi_n \operatorname{Gr}^{(r-1)n+s} \operatorname{Déc}^{r-1}(X)$ is isomorphic to the subquotient of $E^1_{n,s}(X) = \pi_n \operatorname{Gr}^s(X)$:

$$\frac{Z^{r-1}_{n,s}}{B^{r-1}_{n,s}} = \frac{\{a \in \pi_n \operatorname{Gr}^s(X) \colon \kappa(a) \text{ lifts to } \pi_{n-1}X(s+r)\}}{\{a \in \pi_n \operatorname{Gr}^s(X) \colon a \text{ lifts to } \ker(\pi_n X(s) \to \pi_n X(s-r+1))\}}.$$

Since the spectral sequence via décalage and the one obtained by Lurie's method are the same, it follows that Luries convergence statement also holds for the décalage spectral sequence, as was conjectured by De Potter in his master thesis [7, Conjecture 4.23]. It is formulated as follows.

Theorem 5.5. *Let $X \in \operatorname{Tow}(\mathrm{Sp})$ such that $X(k) \simeq 0$ for some $k >> 0$. Then the spectral sequence obtained via Lurie's method converges strongly. In particular, we have*

$$\tilde{E}^r_{n,s}(X) \implies \pi_n \operatorname{colim}_{\mathbb{Z}} X,$$

with filtration on the target given by $F^s\pi_n \operatorname{colim}_{\mathbb{Z}} X := \operatorname{im}(\pi_n X(s) \to \pi_n \operatorname{colim}_{\mathbb{Z}} X)$.

A more general convergence statement is discussed by Antieau in [1, Chapter 6].

6 Massey Products and Toda Brackets

In this section, we will discuss Massey products and define them on the resulting spectral sequence from Section 4.3. We will also discuss multiplicative Toda Brackets on homotopy groups of spectra. Moss showed that under certain conditions elements in the Massey product on the Adams spectral sequence converge to elements in the Toda Bracket on homotopy groups [20]. As was also mentioned in the introduction, a generalization of this was given by Belmont and Kong, see [4]. In the following section we will formulate and provide a proof strategy for a similar statement for the spectral sequence obtained via décalage. Before doing so we will define Massey products and discuss the 'crossing differential hypothesis', one of the conditions Moss, Belmont and Kong used in their proofs.

6.1 Massey Products and the Crossing differential hypothesis

We will define Massey products on the spectral sequence analogous to the definition in [19, p. 302,303] and [4, Def. 3.6]. Consider the following setting. Let X be an associative algebra object in Tow(Sp). This yields a pairing $\mu\colon X \otimes X \to X$, which is unital and associative up to coherent homotopy. By Theorem 4.28, this yields a corresponding pairing of spectral sequences $\mu\colon E^*_{*,*}(X) \otimes E^*_{*,*}(X) \to E^*_{*,*}(X)$, with $E^r_{n,s}(X) = \pi_n \operatorname{Gr}^{(r-1)n+s}(\text{Déc}^{r-1}(X))$. This is again associative. For convenience, we will often write simply write xy for $\mu(x \otimes y)$.

Definition 6.1 (Massey Product). Suppose $r \geq 1$. Let $[a] \in E^{r+1}_{n,s}(X), [a'] \in E^{r+1}_{n',s'}(X)$ and $[a''] \in E^{r+1}_{n'',s''}(X)$. Here we view $[a]$ as a class of $H(E^r_{n,s}(X), d_r) \cong E^{r+1}_{n,s}(X)$. Let $[a]\dot{[}a'] = 0$ and $[a']\dot{[}a''] = 0$. Then the **Massey product** is defined as the set

$$\langle [a], [a'], [a''] \rangle := \{[\bar{a} \cdot b' + \bar{b} \cdot a''] \text{ with } d_r(b) = \bar{a}a' \text{ and } d_r(b') = \bar{a}'a''\},$$

which is a subset of $E^{r+1}_{n+n'+n''+1,s+s'+s''-r}(X)$. Here we define $\bar{x} = (-1)^{|x|+1}x$, where $|x|$ is the total degree of x in $E^{r+1}(X)$, see Remark 1.2.

Note that as a consequence of the Leibniz rule

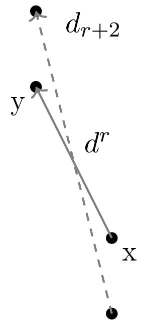
$$\begin{aligned}
d_r(\bar{a} \cdot b' + \bar{b} \cdot a'') &= d_r(\bar{a} \cdot b') + d_r(\bar{b} \cdot a'') \\
&= d_r(\bar{a}) \cdot b' + (-1)^{|\bar{a}|}\bar{a} \cdot d_r(b') + d_r(\bar{b}) \cdot a'' + (-1)^{|\bar{b}|}\bar{b} \cdot d_r(a'') \\
&= 0 + (-1)^{|\bar{a}|}(-1)^{|a|+1}a \cdot (-1)^{|a'|+1}a'a'' + (-1)^{|b|+1}(-1)^{|a|+1}aa' \cdot a''' + 0 \\
&= (-1)^n(-1)^{n+1}(-1)^{n'+1}a \cdot a'a'' + (-1)^{n+n'+2}(-1)^{n+1}aa' \cdot a'' \\
&= (-1)^{n'}a \cdot a'a'' + (-1)^{n'+1}aa' \cdot a'' \\
&= 0
\end{aligned}$$

So elements in the Massey product indeed define homology classes in $E^{r+1}(X)$ as claimed.

As was mentioned before, Belmont and Kong base their proof on a similar idea by Moss. Moss in particular uses the 'crossing differential hypothesis', as the condition is called by Belmont and Kong, to give a restriction on when we can find an element in the Massey product that converges to an element in the Toda bracket. Here, this is defined the same as by Belmont and Kong [4, Def. 2.4].

Definition 6.2. Let $E^*_{*,*}$ be a spectral sequence. Then $E^*_{*,*}$ satisfies the crossing differential hypothesis in degree (r, n, s) if every element in $E^{s-m+1}_{n+1,m}$ for $0 \leq m \leq s - r + 1$ is a permanent cycle.

Remark 6.3. As is also explained by Belmont and Kong, for an element $y \in E^r_{n,s}$ with $d^r(x) = y$, this means that the crossing differential hypothesis is satisfied in degree (r, n, s) if there are no differentials going from degree $n + 1$ to n where the filtration degree of the source is less than $s - r$ and filtration degree of the target degree larger than s. That is, similar situations like the one below cannot happen.



We will now formulate a similar theorem as Belmont and Kong, but then for the décalage spectral sequence, see also [4, Prop. 2.6]. For this, we first need to assume that the spectral sequence $E^r(X)$ is weakly convergent, see Definition 1.7. In particular, we assume that

$$E^1_{n,s}(X) \implies \pi_n(X(-\infty)),$$

where $X(-\infty) = \operatorname{colim}_{i\in\mathbb{Z}} X(i)$ and $\pi_n(Y)$ is endowed with the following filtration:

$$F^k\pi_n(X(-\infty)) = \operatorname{im}(\pi_n X(k) \to \pi_n X(-\infty)).$$

Theorem 6.4. *Let $X \in \operatorname{Tow}\operatorname{Sp}$; and $E^r_{n,s}(X) := \pi_n \operatorname{Gr}^{(r-1)n+s}(Déc^{r-1}(X))$ the corresponding décalage spectral sequence converging weakly to $\pi_n(X(-\infty))$. Let $a \in \pi_n \operatorname{Gr}^s(X) = E^1_{n,s}(X)$ be a d^r-boundary and suppose $\alpha \in \pi_n(X(s))$ is a lift of a such that $\alpha \in \ker(\pi_n X(s) \to \pi_n X(-\infty))$. If the crossing differential hypothesis holds in degree (r, n, s), then α lies in $\ker(\pi_n X(s) \to \pi_n X(s-r))$.*

The proof needs the following lemma.

Lemma 6.5. *Let $E^r_{n,s}(X)$ be weakly convergent. Write*

$$R_s = \bigcap_{r\geq 0} \operatorname{im}(\pi_n X(s+r) \to \pi_n X(s)).$$

Then the map $R_{s+1} \to R_s$ is injective.

Proof. Recall that the filtration was given by $F^s(\pi_n X(-\infty)) := \operatorname{im}\pi_n(X(s) \to \pi_n X(-\infty))$; which is precisely the filtration discussed by Boardman in [5, Lemma 5.6]. By Corollary 5.4, the décalage spectral sequence is in fact the same as the one obtained via derived exact couples. Hence we have an exact sequence

$$0 \to \frac{F^s\pi_n X(-\infty)}{F^{s+1}\pi_n X(-\infty)} \to E^\infty_{n,s} \to R_{s+1} \to R_s.$$

Since the spectral sequence is weakly convergent, it follows that $\frac{F^s\pi_n X(-\infty)}{F^{s+1}\pi_n X(-\infty)} \to E^\infty_{n,s}$ is an isomorphism, hence $R_{s+1} \to R_s$ is injective. □

Remark 6.6. The exact sequence implies that the converse holds as well; i.e. $E^r_{n,s}(X)$ is weakly convergent if for all $s \in \mathbb{Z}$ the maps $R_{s+1} \to R_s$ are injective.

We will now give the proof as done by Belmont and Kong [4, Prop. 2.6], with some additional explanation; this proof is directly applicable to our setting.

Proof of Theorem 6.4. Firstly, we note that by Corollary 5.4, the d^r-boundaries of the spectral sequence $E^r_{n,s}(X)$ viewed as a quotient of the E^1-page are precisely the elements in

$$B^r_{n,s} := \{a \in \pi_n \operatorname{Gr}^s(X) \colon a \text{ lifts to } \ker(\pi_n X(s) \to \pi_n X(s-r))\}.$$

Hence the statement of the theorem makes sense. Indeed, we can always find some lift of a which lies in the kernel of $\pi_n X(s) \to \pi_n X(-\infty)$, since this factors through $\pi_n X(s-r)$ and we know that there is some lift

$$\beta \in \ker(\pi_n X(s) \to \pi_n X(s-r).$$

So if $\alpha = \beta$, we are done. If not, then $\alpha - \beta \neq 0$, say $\gamma \in \pi_n X(s)$. Since

$$\alpha \in \ker(\pi_n X(s) \to \pi_n Y),$$

we can find a minimal k such that

$$\alpha \in \ker(\pi_n X(s) \to \pi_n X(s-k)).$$

We will proceed by contradiction. Suppose $k > r$. Then $\gamma = \alpha - \beta$ maps to $\alpha - 0$ in $\pi_n X(s-r)$, hence to 0 in $\pi_n X(s-k)$. So

$$\gamma \in \ker(\pi_n X(s) \to \pi_n(X(s-k))),$$

but

$$\gamma \notin \ker(\pi_n X(s) \to \pi_n(X(s-k+1)).$$

Write $i_{s-k+1,s}(\gamma)$ for the image of γ in $\pi_n X(s-k+1)$. By Lemma 6.5, the map

$$i_R \colon R_{s-k+1} \to R_{s-k}$$

is injective. Hence we can conclude that $i_{s-k+1,s}(\gamma) \notin R_{s-k+1}$. Indeed, if $i_{s-k+1,s}(\gamma) \in R_{s-k+1}$, then

$$i_R(i_{s-k+1,s}(\gamma)) = i_{s-k+1,s}(\gamma) = 0.$$

This is a contradiction as $i_{s-k+1,s}(\gamma) \neq 0$ and i_R is injective. So $i_{s-k+1,s}(\gamma) \notin R_{s-k+1}$.

But then there must be some m such that $i_{s-k+1,s}(\gamma)$ lifts to some $\gamma' \in \pi_n X(s+m)$ but not to any element in $\pi_n X(s+m+1)$. In particular $m \geq 1$. Indeed,

$$\gamma = \alpha - \beta \in \ker(\pi_n X(s) \to \pi_n \operatorname{Gr}^s(X))$$

and therefore lifts to $\pi_n X(s+1)$. This then means that $i_{s-k+1,s}\gamma)$ lifts to $\pi_n X(s+1)$ and so $m \geq 1$. We also note that since $i_{s-k+1,s} = 0$, we have that

$$i_{s-k+1,s}(\gamma) \in \operatorname{im}(\kappa \colon \pi_{n+1} \operatorname{Gr}^{s-k} X \to \pi_n X(s-k+1)),$$

say

$$i_{s-k+1,s}(\gamma) = \kappa(c).$$

Next we consider $d_{k+m}(c)$. By identification of the décalage spectral sequence with the spectral sequence obtained via exact couples, $d_{k+m}(c) = [j(\gamma')]$ in $E^r_{n,s+m}$ as sub-quotient of the E^1-page. We know that $j(\gamma') \neq 0$, as

$$\gamma' \notin \operatorname{im}(\pi_n X(s+m+1) \to \pi_n X(s+m)).$$

In fact, we have that $[j(\gamma')] \neq 0$. That is, $j(\gamma')$ is not a d_i-boundary for all $i < k+m$. Indeed, suppose it is a d_i-boundary for some $i < k+m$. Then $j(\gamma') \in \pi_n \operatorname{Gr}^{s+m}(X)$ has a lift

$$\tilde{\gamma} \in \ker(\pi_n X(s+m) \to \pi_n X(s+m-i)),$$

which then also lies in $\ker(\pi_n X(s+m) \to \pi_n X(s-k+1))$. In particular, $j(\gamma) = j(\gamma')$, so

$$\gamma' - \gamma'' \in im(\pi_n X(s+m+1) \to \pi_n X(s+m)).$$

Let $\delta \in \pi_n X(s+m+1)$ be a lift of $\gamma - \gamma''$. Then, since

$$\gamma'' \in \ker(\pi_n X(s+m) \to \pi_n X(s-k+1)),$$

it follows that δ is a lift of $i^{k-1}(\gamma)$. But this is a contradiction with the definition of m. So $[j(\gamma')]$ cannot be zero in $E^{k+m}_{n,k+m}$.

Therefore, we have a differential

$$d_{k+m} \colon E^{k+m}_{n+1,s-k} \to E^{k+m}_{n,s+m}$$

which is non-zero. However, since $k > r$, we have that $s-k < s-r$ and $s+m > s$, as $m \geq 1$. This yields a contradiction with the crossing differential hypothesis in degree (r,n,s), so we must have that $k \leq r$, which is a contradiction. Hence indeed

$$\alpha \in \ker(\pi_n X(s) \to \pi_n X(s-k) \to \pi_n X(s-r)).$$

This concludes the proof. □

This theorem has the following consequence, which is an analogue of Lemma 2.12 in [4].

Corollary 6.7. *Let $a \in \pi_n \operatorname{Gr}^s(X)$ be a d^r-boundary; and let $\alpha \in \pi_n X(s)$ be a lift of a such that $\alpha \in \ker(i_{s-r,s} : \pi_n X(s) \to \pi_n X(s-r))$. Then we have a null-homotopy $h\colon i_{s-r,s} \circ \alpha \simeq 0$, i.e. a commutative diagram*

$$
\begin{array}{ccccccc}
\Sigma^n \mathbb{S} & \xrightarrow{\alpha} & X(s) & \xrightarrow{i_{s-r+1,s}} & X(s-r+1) & \longrightarrow & 0 \\
\downarrow & & \downarrow{\scriptstyle i_{s-r,s}} & & \downarrow & & \downarrow \\
0 & \longrightarrow & X(s-r) & \longrightarrow & X(s-r) & \longrightarrow & \operatorname{Gr}^{s-r}(X).
\end{array}
$$

Let $\beta\colon \Sigma^{1+n}\mathbb{S} \to \operatorname{Gr}^{s-r}(X)$ be the induced map. Write $[\beta]^r$ for the class in the sub-quotient $E^r \cong Z^{r-1}/B^{r-1}$ of the E^1-page. Then we have

$$d^r[\beta]^r = [a]$$

in $E^r_{n,s}$.

Proof. First note that for $a \in \pi_n \operatorname{Gr}^s(X)$ a d^r-boundary, the lift $\alpha \in \ker(\pi_n X(s) \to \pi_n X(s-r))$ indeed exists by Theorem 6.4. Recall that d^r is induced by

$$
\begin{array}{ccc}
 & \pi_{n-1} X(s) & \xrightarrow{j_s} & \pi_{n-1} \frac{X(s)}{X(s+1)} \\
 & \nearrow{\scriptstyle \mathit{lift}} \quad \downarrow{\scriptstyle i_{s-r+1,s}} & & \\
\pi_n \frac{X(s-r)}{X(s-r+1)} \xrightarrow{\kappa_{s-r,s-r+1}} & \pi_{n-1} X(s-r+1) & &
\end{array}
$$

Since α is a lift of a it remains to show that $\kappa_{s-r,s-r+1} \circ \beta \simeq i_{s-r+1,s} \circ \alpha$. Note that we can equivalently write β as the third map in the diagram

$$
\begin{array}{ccccc}
\Sigma^n \mathbb{S} & \xrightarrow{\alpha} & X(s) & \xrightarrow{i_{s-r+1,s}} & X(s-r+1) \\
\downarrow & & \downarrow{\scriptstyle i_{s-r,s}} & & \downarrow \\
0 & \longrightarrow & X(s-r) & \longrightarrow & X(s-r) \\
\downarrow & & & & \downarrow \\
\Sigma^{n+1}\mathbb{S} & & \xrightarrow{\beta} & & \operatorname{Gr}^{s-r}(X) \\
\downarrow & & & & \downarrow{\scriptstyle \kappa} \\
\Sigma^{n+1}\mathbb{S} & & \xrightarrow{\Sigma(i_{s-r+1,s}\alpha)} & & \Sigma X(s-r+1)
\end{array}
$$

Hence indeed $\kappa_{s-r,s-r+1} \circ \beta \simeq \Sigma(i_{s-r+1,s} \circ \alpha)$. This means that we can lift the homotopy class $\kappa[\beta] \in \pi_n X(s-r+1)$ to $\pi_n X(s-r+1+k)$ for $1 \le k \le r-1$, so $[\beta]$ is in fact a d^k-cycle for $1 \le k \le r-1$. Also,

$$d^r[\beta]^r = j[\alpha]^r = [a]^r,$$

where we write $[\beta]^r$ for the class in the sub-quotient $E^r \cong Z^{r-1}/B^{r-1}$ of the E^1-page. □

6.2 Set-up

For the remainder of this section we assume that the spectral sequence induced by X is weakly convergent. Furthermore, we note that the pairing $\mu\colon X \otimes X \to X$ induces commutative diagrams

$$\begin{array}{ccccc} X(i+k)\otimes X(j) & \longrightarrow & X(i)\otimes X(j) & \longleftarrow & X(i)\otimes X(j+l) \\ \downarrow & & \downarrow & & \downarrow \\ X(i+j+k) & \longrightarrow & X(i+j) & \longleftarrow & X(i+j+l) \end{array}$$

and

$$\begin{array}{ccccc} W_{i,j} & \longrightarrow & X(i)\otimes X(j) & \longrightarrow & \mathrm{Gr}^i(X)\otimes \mathrm{Gr}^j(X) \\ \downarrow & & \downarrow & & \downarrow \\ X(i+j+1) & \longrightarrow & X(i+j) & \longrightarrow & \mathrm{Gr}^{i+j}(X) \end{array}$$

Here the horizontal rows are cofiber sequences in Sp and $W_{i,j} := X(i)\otimes X(j+1)\cup_{X(i+1)\otimes X(j+1)} X(i+1)\otimes X(j)$. For reference, see [12, p. 198,199]. Furthermore, we assume we have commutative diagrams

$$\begin{array}{ccc} X(i)\otimes X(j) & \xrightarrow{\mu} & X(i+j) \\ \downarrow & & \downarrow \\ X(-\infty)\otimes X(\infty) & \xrightarrow[\mu]{} & X(-\infty) \end{array}$$

which is certainly true in the case we have a tower $\cdots \to X(k) \to \cdots \to X(i) \simeq X(i) \simeq \ldots$.

The following conjecture is an analogue of Assumption 3.14 (5) in [4]. In particular, we claim that we do not have to assume anything in our setting; instead the result can likely be proved with the theory we already have. Indeed both $a\cdot a$ and $\mu(\alpha\otimes\alpha')$ arise from taking the associated graded of a pairing induced by $\mu\colon X\otimes X\to X$. We will assume this conjecture in the remainder of this chapter.

Conjecture 6.8. Let $a \in E^r_{n,s}(X) = \pi_n \mathrm{Gr}^{(r-1)n+s}(\mathrm{D\acute{e}c}^{r-1}(X))$ and $a' \in E^r_{n',s'}(X)$. Let $\alpha\colon \Sigma^n\mathbb{S} \to \mathrm{Gr}^s(X), \alpha'\colon \Sigma^{n'}\mathbb{S} \to \mathrm{Gr}^{s'}(X)$ be representatives on the E^1-page of a, a' respectively. Then

$$\mu(\alpha\cdot\alpha')\colon \Sigma^n\mathbb{S}\otimes\Sigma^{n'}\mathbb{S} \to \mathrm{Gr}^s(X)\otimes\mathrm{Gr}^{s'}(X) \to \mathrm{Gr}^{s+s'}(X)$$

is a representative of $a\cdot a'$ on the E^1-page.

Lastly, we have the following assumption on the E^∞-page of our weakly convergent spectral sequence.

Assumption 6.9. Assume the isomorphisms

$$E^\infty_{n,s}(X) \cong \frac{F^s\pi_n X(-\infty)}{F^{s+1}\pi_n X(-\infty)} := \frac{\mathrm{im}(\pi_n X(s)\to\pi_n X(-\infty))}{\mathrm{im}(\pi_n X(s+1)\to\pi_n X(-\infty))}$$

are essentially given by lifting a representative $\alpha\colon \Sigma^n\mathbb{S} \to \pi_n X(-\infty)$ of a class in $\frac{F^s\pi_n X(-\infty)}{F^{s+1}\pi_n X(-\infty)}$ to $\pi_n X(s)$ and then mapping into $\pi_n \mathrm{Gr}^s(X)$ and taking its class in $E^\infty_{n,s}$.

This is usually the case with spectral sequences obtained via exact couples. For the remainder of this chapter we will work in the following setting. Let

$$\alpha\colon \Sigma^n\mathbb{S}\to X(-\infty),\ \alpha'\colon \Sigma^{n'}\mathbb{S}\to X(-\infty),\ \alpha''\colon \Sigma^{n''}X(-\infty)$$

be maps representing elements in $\pi_n X(-\infty)$, $\pi_{n'}X(-\infty)$ and $\pi_{n''}X(-\infty)$ respectively, such that $\alpha\alpha'\simeq 0$ and $\alpha'\alpha''\simeq 0$. Suppose also that they are detected by permanent cycles $a\in E^r_{n,s}(X), a'\in E^r_{n',s'}(X)$ and $a''\in E^r_{n'',s''}(X)$. That is, these permanent cycles have representatives

$$\alpha_{**}\colon \Sigma^n\mathbb{S}\to\mathrm{Gr}^s(X),\ \alpha'_{**}\colon \Sigma^{n''}\mathbb{S}\to\mathrm{Gr}^{s'}(X),\ \alpha''_{**}\colon \Sigma^{n''}\mathbb{S}\to\mathrm{Gr}^{s''}(X)$$

on the E^1-page, together with lifts

$$\alpha_*\colon \Sigma^n\mathbb{S}\to X(s),\ \alpha'_*\colon \Sigma^{n''}\mathbb{S}\to X(s'),\ \alpha''_*\colon \Sigma^{n''}\mathbb{S}\to X(s'')$$

such that $\iota_s(\alpha_*) = \alpha$, with $\iota_s\colon X(s) \to X(-\infty)$ and $j_s(\alpha_*) = \alpha_{**}$, with $j_s\colon X(s) \to \mathrm{Gr}^s(X)$ and similarly for α', α''. Then by Conjecture 6.8 and Assumption 6.9 we also have that

$$\iota_{s+s'}\mu(\alpha_*, \alpha'_*) \simeq \mu(\alpha, \alpha') \text{ and } j_{s+s'}(\mu(\alpha_*, \alpha'_*)) \simeq \mu(\alpha_{**}, \alpha'_{**}),$$

where $\mu(\alpha_{**}, \alpha'_{**})$ is a representative of $a \cdot a'$. Note that this set-up is similar to the one by Belmont and Kong [4, 3.5.2].

Lemma 6.10. *Suppose that $E^*(X)$ satisfies the crossing differential hypothesis in degree $(r, n+n', s+s')$ and $(r, n'+n'', s'+s'')$. Then the following compositions are null-homotopic*

$$\Sigma^{n+n'}\mathbb{S} \xrightarrow{\alpha_*\alpha'_*} X(s+s') \xrightarrow{i_{s+s'-r,s+s'}} X(s+s'-r)$$

and

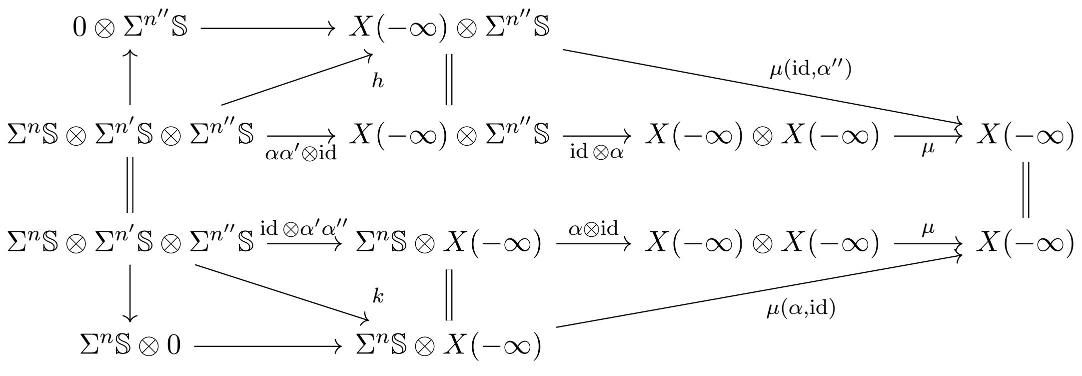
$$\Sigma^{n'+n''}\mathbb{S} \xrightarrow{\alpha'_*\alpha''_*} X(s'+s'') \xrightarrow{i_{s'+s''-r,s'+s''}} X(s'+s''-r)$$

Proof. As a consequence of the assumptions and Conjecture 6.8 note that $\alpha_*\alpha'_*$ and $\alpha'_*\alpha''_*$ are representatives of the permanent cycles aa' and $a'a''$; with lifts $\alpha\alpha' \simeq 0$ and $\alpha'\alpha'' \simeq 0$. Then by Moss' theorem, see Theorem 6.4, it follows directly that both compositions are null-homotopic. □

6.3 Toda brackets

We will now define multiplicative Toda brackets on $\pi_n X(-\infty)$ of a tower $X \in \mathrm{Tow}(\mathrm{Sp})$ analogously to [4, §. 3]. After that, we will state our main conjecture and give an idea of a possible proof. For this, we use the assumptions on the multiplication as stated in the previous section.

Definition 6.11 (Toda bracket). Let $X \in \mathrm{Tow}(\mathrm{Sp})$ and let $\alpha, \alpha', \alpha''$ be representatives of elements in $\pi_n X(-\infty), \pi_{n'} X(-\infty), \pi_{n''} X(-\infty)$ respectively, with the assumptions as in Section 6.2. Suppose $0 \simeq \mu(\alpha, \alpha')$ and $0 \simeq \mu(\alpha', \alpha'')$, via null-homotopies h and k respectively. Then this also induces null-homotopies $h : 0 \simeq \alpha\alpha' \otimes \mathrm{id}$ and $k : 0 \simeq \mathrm{id} \otimes \alpha'\alpha''$. Recall that $\mu : X \to X \to X$ is associative up to coherent homotopy, hence we get a commutative diagram



Since $- \otimes -$ preserves zero objects in both variables, this induces a map $g\colon \Sigma^{1+n+n'+n''}\mathbb{S} \to X(-\infty)$. Then we define **the Toda bracket** of α, α' and α'' as

$$\langle \alpha, \alpha', \alpha'' \rangle := \{[g] \in \pi_{n+n'+n''+1} X(-\infty) \mid g \text{ arises via the above construction}\}.$$

Remark 6.12. The reason why we get a set of maps is because the diagram depends on the choice of null-homotopies h, k. In line with this, recall that for zero objects, we have that $\mathrm{map}_{\mathrm{Sp}}(x, 0)$ and $\mathrm{map}_{\mathrm{Sp}}(0, y)$ are contractible Kan complexes for any $x, y \in \mathrm{Sp}$. Therefore, also zero maps $0_{x,y}\colon x \to y$ are defined up to contractible choice. So the class of $g : \Sigma^{1+n+n'+n''}\mathbb{S} \to X(-\infty)$ does not really depend on the choice of zero maps.

Furthermore, we note that $\mathrm{Fun}(\Delta^2, \mathrm{Sp}) \to \mathrm{Fun}(\Lambda^1_2, \mathrm{Sp})$ is a trivial Kan fibration. Hence if we have maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, they form a fixed diagram $\Lambda^1_2 \to \mathrm{Sp}$; and the possible compositions which are given by the fiber of $X \xrightarrow{f} Y \xrightarrow{g} Z$ along $\mathrm{Fun}(\Delta^2, \mathrm{Sp}) \to \mathrm{Fun}(\Lambda^1_2, \mathrm{Sp})$ form a contractible Kan complex, see also [14, Cor. 1.3.44]. Therefore it also does not matter much which compositions we pick in the above diagram and the associativity of $\mu\colon X \otimes X \to X$ specifies the commuting square in the middle of the diagram, so the homotopies h and k are also really the only objects which yield different maps.

We will now formulate our main statement and give a proof sketch.

Pretheorem 6.13. Let $X \in \mathrm{Tow}(\mathrm{Sp})$ an associative algebra object. Write $\mu\colon X \otimes X \to X$ for the resulting pairing. Consider the associated multiplicative décalage spectral sequence

$$E^r_{n,s}(X) \Rightarrow \pi_n X(-\infty)$$

and assume it is weakly convergent. Assume the spectral sequence satisfies the properties in the set-up of Section 6.2. Let

$$a \in E^r_{n,s}(X),\ a' \in E^r_{n',s'}(X),\ a'' \in E^r_{n'',s''}(X)$$

be permanent cycles converging as in Section 6.2 to α, α', $\alpha'' \in \pi_*(X(-\infty))$ such that aa' and $a'a''$ are d^r-boundaries and $\alpha\alpha'$ and $\alpha'\alpha''$ are null-homotopic. Assume the crossing differential hypothesis holds in degrees $(r, n+n', s+s')$ and $(r, n'+n'', s'+s'')$. Then there exists an element in the Massey product $\langle [a], [a'], [a''] \rangle \subset E^{r+1}_{n-1,s+s'+s''-r}(X)$ converging to an element in the Toda bracket $\langle \alpha, \alpha', \alpha'' \rangle$.

Proof sketch of Pretheorem 6.13. The idea of the proof is to construct a map

$$\gamma : \Sigma^{1+n+n'+n''}\mathbb{S} \to X(s+s'+s''-r)$$

via a similar diagram as in the definition of the Toda bracket and show that its composition with

$$\iota_{s+s'+s''-r} : X(s+s'+s''-r) \to X(-\infty)$$

is an element of the Toda bracket and that it lifts to an element in the Massey-product under

$$j_{s+s'+s''-r} \colon : X(s+s'+s''-r) \to \mathrm{Gr}^{s+s'+s''-r}(X).$$

Indeed, by the set-up of Section 6.2 and in particular the way the spectral sequence converges, the class of $j_{s+s'+s''-r} \circ \gamma$ in the E^{r+1}−page converges to the class of $\iota_{s+s'+s''-r} \circ \gamma$ in $\pi_{n+n'+n''+1}(X(-\infty))$. So if $[j_{s+s'+s''-r} \circ \gamma]_{r+1}$ is an element of the Massey-product $\langle [a], [a'], [a''] \rangle$ and $\iota_{s+s'+s''-r} \circ \gamma$ is a representative of an element in $\langle \alpha, \alpha', \alpha'' \rangle$, the statement holds. We will construct γ as follows. Consider lifts

$$\alpha_* \in \pi_n X(s), \alpha'_* \in \pi_{n'} X(s'), \alpha''_* \pi_{n''} X(s'')$$

of $\alpha, \alpha', \alpha''$ as in Section 6.2. Then by Lemma 6.10 we have null-homotopies $h_1 : 0 \simeq i_{s+s'-r,s+s'} \alpha_* \alpha'_*$ and $h_2 : 0 \simeq i_{s'+s''-r,s'+s''} \alpha'_* \alpha''_*$ and therefore a commutative diagram:

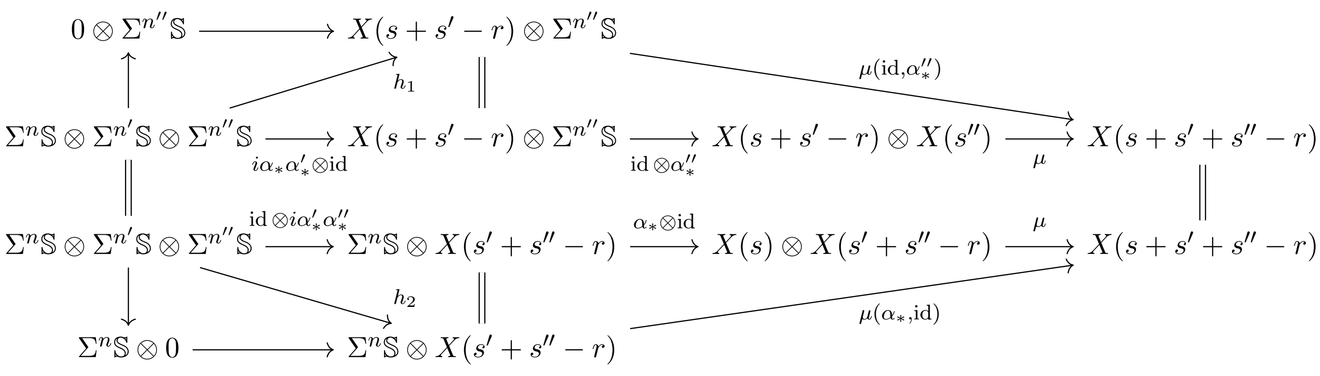


Figure 1

Note that we supressed the indices to make the diagram less complicated. This diagram yields a map $\gamma : \Sigma^{1+n+n'+n''}\mathbb{S} \to X(s+s'+s''-r)$. Then we first want to prove the following.

Conjecture 6.14. Let $\iota_{s+s'+s''-r}\colon X(s+s'+s''-r) \to X(-\infty)$ be the colimit map. Then $\iota_{s+s'+s''-r}\circ\gamma$ represents an element in the Toda bracket $\langle \alpha, \alpha', \alpha''\rangle$.

The idea for proving this is as follows. Firstly,

$$\iota_{s+s'+s''-r} \circ \gamma : \Sigma(\Sigma^n \mathbb{S} \otimes \Sigma^{n'}\mathbb{S} \otimes \Sigma^{n''}\mathbb{S}) \to X(-\infty)$$

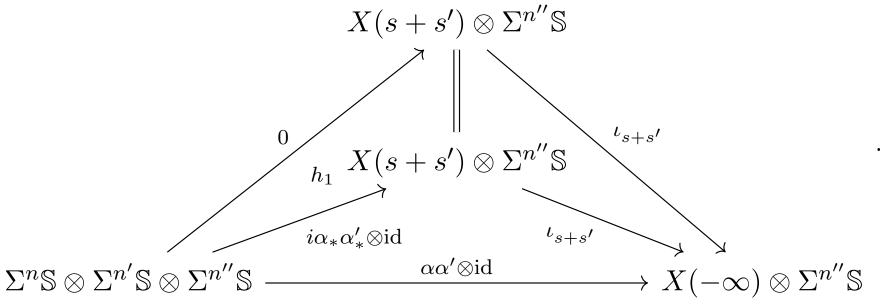
corresponds to a commutative diagram $\Delta^1 \times \Delta^1 \to \mathrm{Sp}$

$$\begin{array}{ccc} \Sigma^n\mathbb{S} \otimes \Sigma^{n'}\mathbb{S} \otimes \Sigma^{n''}\mathbb{S} & \longrightarrow & 0 \\ \downarrow & & \downarrow \\ 0 & \longrightarrow & X(-\infty). \end{array}$$

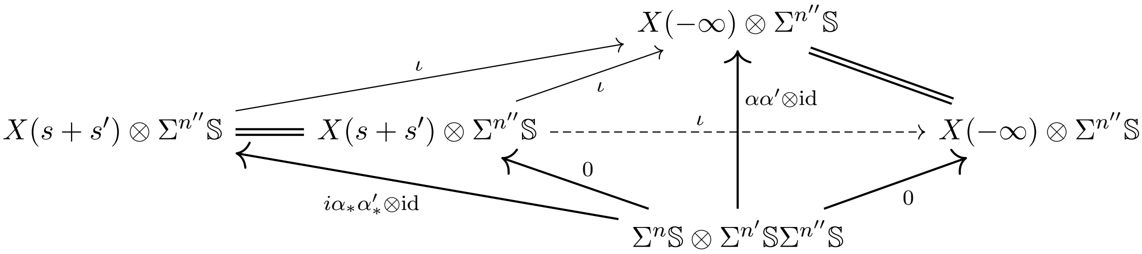
We want to show that we can be more specific. In particular, we want to show that $\iota \circ \gamma$ comes from a diagram as in Definition 6.11. Note that by the assumptions in the set-up of Section 6.2, we have that

$$\iota_{s+s'-r} i_{s+s'-r,s+s'} \alpha_* \alpha'_* \simeq \iota_{s+s'} \alpha_* \alpha'_* \simeq \alpha\alpha'.$$

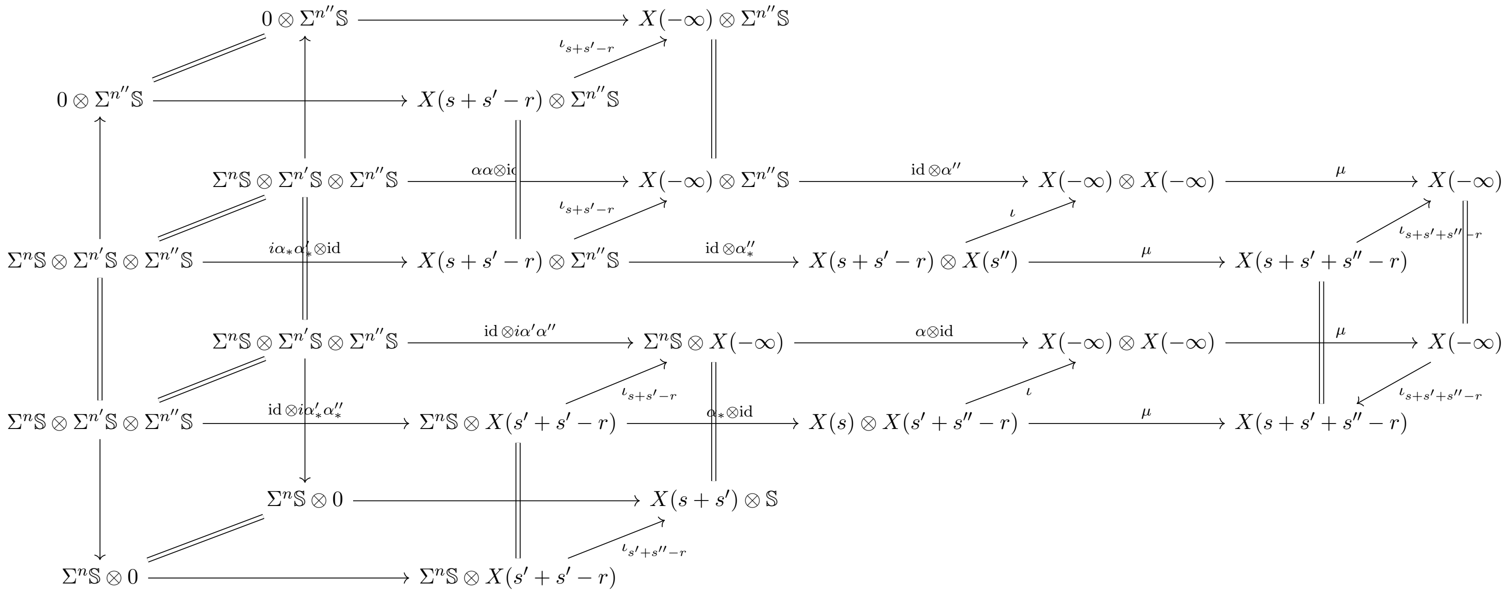
This yields a horn $\Lambda^3_1\colon \to \mathrm{Sp}$



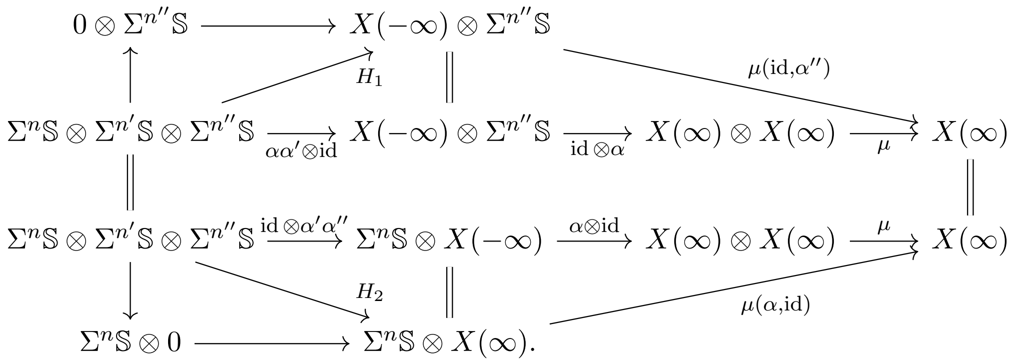
Similarly, we can fill a second horn which results in a filled diamond



Here the left 2-simplex or null-homotopy surrounded by thick arrows is h_1 and we write H_1 for the null-homotopy $0 \simeq \alpha\alpha' \otimes \mathrm{id}$ surrounded by thick arrows on the right. Similarly, h_2 induces a null-homotopy $H_2 : \mathrm{id} \otimes \alpha'\alpha'' \simeq 0$. Furthermore, since $\mu : X \otimes X \to X$ is associative up to coherent homotopy in a way that is compatible with the filtrations, we get a cube $\Delta^1 \times \Delta^1 \times \Delta^1 \to \mathrm{Sp}$ obtained from:



where the back diagram is



This yields a map $\gamma' : \Sigma^{1+n+n'+n''}\mathbb{S} \to X(-\infty)$. By definition, this defines an element in the Toda-bracket. Also, because H_1 and H_2 are constructed by composition with ι from h_1 and h_2, intuitively this means that in fact

$$\gamma' \simeq \iota \circ \gamma.$$

It still needs to be made precise when two diagrams and homotopies induce the same maps on suspension.

Next, we assume $n, n', n'' = 0$ and we will give an idea for showing that γ maps to an element in the Massey product $\langle [a], [a'], [a''] \rangle$, where a, a', a'' are permanent cycles detecting $\alpha, \alpha', \alpha''$ as in Section 6.2. For this, we will first construct an element which is in the Massey-product and we will then show that $j_{s+s'+s''-r} \circ \gamma$ can be written as a representative of this element, up to homotopy. Here $j_{s+s'+s''-r}$ is the map $j_{s+s'+s''-r} : X(s+s'+s''-r) \to \mathrm{Gr}^{s+s'+s''}(X)$.

Conjecture 6.15. Consider $\gamma : \Sigma\mathbb{S} \to X(s+s'+s''-r)$ induced by Figure 1. Then $j_{s+s'+s''-r} \circ \gamma$ is homotopic to a representative of an element in the Massey-product $\langle [a], [a'], [a''] \rangle$.

We will now discuss a proof strategy for this second conjecture. To fill in the details, as well as working it our for higher degrees, is a possible subject for further research.

Proof sketch. First, we will construct an element in the Massey product $\langle [a], [a'], [a''] \rangle$. Consider the following diagram:

$$\begin{array}{ccccccc}
\mathbb{S} \simeq \mathbb{S} \otimes \mathbb{S} & \xrightarrow{\alpha_* \alpha'_*} & X(s+s') & \longrightarrow & X(s+s'-r+1) & \longrightarrow & 0 \\
\downarrow & \searrow^{\tilde{h}_1} & \downarrow{\scriptstyle i_{s+s'-r,s+s'}} & & \downarrow & & \downarrow \\
0 & \longrightarrow & X(s+s'-r) & = & X(s+s'-r) & \longrightarrow & \mathrm{Gr}^{s+s'-r}(X).
\end{array}$$

Write $\beta^- \colon \Sigma\mathbb{S} \to \mathrm{Gr}^{s'+s''-r}(X)$ for the induced map. Then similarly, $\beta^+ \colon \Sigma\mathbb{S} \to \mathrm{Gr}^{s+s'-r}(X)$ is the map induced by

$$\begin{array}{ccccccc}
\mathbb{S} \simeq \mathbb{S} \otimes \mathbb{S} & \xrightarrow{\alpha'_* \alpha''_{**}} & X(s'+s'') & \longrightarrow & X(s'+s''-r+1) & \longrightarrow & 0 \\
\downarrow & \searrow^{\tilde{h}_2} & \downarrow{\scriptstyle i_{s'+s''-r,s'+s''}} & & \downarrow & & \downarrow \\
0 & \longrightarrow & X(s'+s''-r) & = & X(s'+s''-r) & \longrightarrow & \mathrm{Gr}^{s'+s''-r}(X).
\end{array}$$

Here $\tilde{h}_1$ and $\tilde{h}_2$ are homotopies induced by $h_1 : 0 \simeq i\alpha_*\alpha'_*$ and $h_2 : 0 \simeq i\alpha'_*\alpha''_*$ by taking compositions as in the discussion of Conjecture 6.14. We then claim that the class

$$[a \cdot [\beta^+]^r - [\beta^-]^r \cdot a''] \in E^{r+1}_{1,s+s'+s''-r}(X)$$

is an element of the Massey product $\langle a, a', a'' \rangle$. Here $[\beta^+]^r$ refers to the class in $E^r(X)$ determined by the d_r-cycle β^+, see also Corollary 6.7. By that same corollary, we have that $d^r[\beta^+]^r = aa'$ and $d^r[\beta^-]^r = a'a''$. It follows that

$$d^r(-[\beta^+]^r) = -aa' = \bar{a}a'$$

and

$$d^r(-[\beta^-]^r) = -a'a'' = \bar{a}'a''.$$

Then by Definition 6.1,

$$[\bar{a} \cdot -[\beta^+]^r + \overline{-[\beta^-]^r} \cdot a'']$$

is an element of the Massey product $\langle [a], [a'], [a''] \rangle$. The degree of $-[\beta^-]^r$ is $(1, s+s'-r)$, hence

$$\begin{aligned}
[\bar{a} \cdot -[\beta^+]^r + \overline{-[\beta^-]^r} \cdot a''] &= [-a \cdot -[\beta^+]^r - (-1)^2[\beta^-]^r \cdot a''] \\
&= [a \cdot [\beta^+]^r - [\beta^-]^r \cdot a''].
\end{aligned}$$

So indeed $[a \cdot [\beta^+]^r - [\beta^-]^r \cdot a'']$ is in the Massey product.

Now, by the assumptions in Section 6.2 and specifically Conjecture 6.8, we have that a representative of $[a \cdot [\beta^+]^r]$ is given by the map

$$\mu(\alpha_{**}, \beta^+) : \Sigma\mathbb{S} \simeq \mathbb{S} \otimes \Sigma(\mathbb{S}) \xrightarrow{\alpha_{**} \otimes \beta^+} \mathrm{Gr}^s(X) \otimes \mathrm{Gr}^{s'+s''-r}(X) \xrightarrow{\mu} \mathrm{Gr}^{s+s'+s''-r}(X).$$

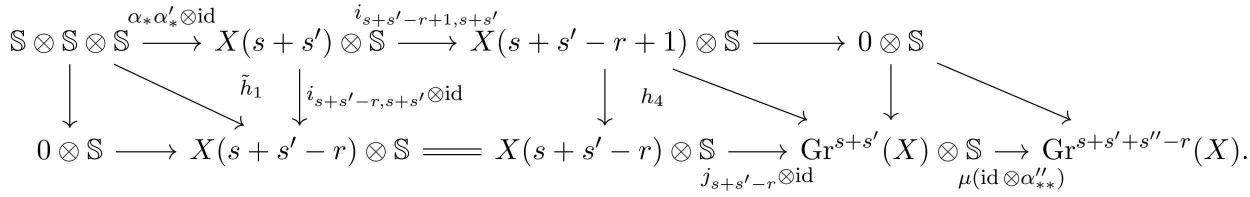
By definition of β^+, this means that $\mu(\alpha_{**}, \beta^+)$ is the map induced by

$$\begin{array}{ccccccccc}
\mathbb{S} \otimes \mathbb{S} \otimes \mathbb{S} & \xrightarrow{\mathrm{id} \otimes \alpha'_* \alpha''_*} & \mathbb{S} \otimes X(s'+s'') & \xrightarrow{\mathrm{id} \otimes i_{s'+s''-r+1,s'+s''}} & \mathbb{S} \otimes X(s'+s''-r+1) & \longrightarrow & \mathbb{S} \otimes 0 & & \\
\downarrow & \searrow^{\tilde{h}_2} & \downarrow{\scriptstyle \mathrm{id} \otimes i_{s'+s''-r,s'+s''}} & & \downarrow & \searrow^{h_3} & \downarrow & \searrow & \\
\mathbb{S} \otimes 0 & \longrightarrow & \mathbb{S} \otimes X(s'+s''-r) & = & \mathbb{S} \otimes X(s'+s''-r) & \xrightarrow[\mathrm{id} \otimes j_{s'+s''-r}]{} & \mathbb{S} \otimes \mathrm{Gr}^{s'+s''-r}(X) & \xrightarrow[\mu(\alpha_{**} \otimes \mathrm{id})]{} & \mathrm{Gr}^{s+s'+s''-r}(X).
\end{array}$$

Note that

$$\mu(\alpha_{**} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes j_{s'+s''-r}) \circ (\mathrm{id} \otimes i_{s'+s''-r}) \circ (\mathrm{id} \otimes \alpha'_* \alpha''_*) \simeq j_{s+s'+s''-r}\mu(\alpha_*, i_{s'+s''-r,s'+s''}\alpha'_*\alpha''_*)$$

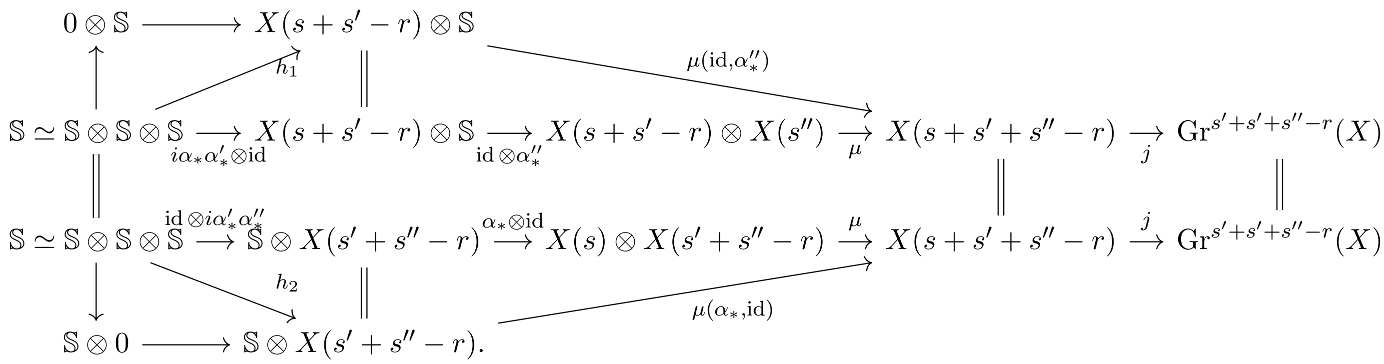
by compatibility of μ with the associated graded and the fact that $\alpha_{**} \simeq j_s\alpha_*$. Similarly, the map $\mu(\beta^-, \alpha''_{**}) : \Sigma\mathbb{S} \simeq \Sigma\mathbb{S} \otimes \mathbb{S} \to \mathrm{Gr}^{s+s'+s''-r}(X)$ is induced by



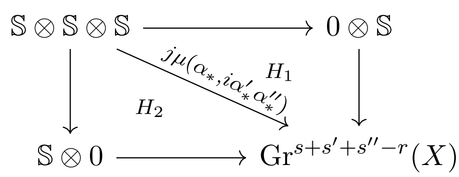
Note that we have

$$\mu(\mathrm{id}\otimes\alpha''_{**})\circ(j_{s+s'-r}\otimes\mathrm{id})\circ(i_{s+s'-r,s+s}\otimes\mathrm{id})\circ(\alpha_*\alpha'_*\otimes\mathrm{id})\simeq j_{s+s'+s''-r}\mu(i'_{s+s'-r,s+s}\alpha_*\alpha'_*,\alpha''_*)$$
$$\simeq j_{s+s'+s''-r}\mu(\alpha_*,i_{s'+s''-r,s'+s''}\alpha'_*\alpha''_*)$$

where the first is obtained from similar reasoning as before and the second via associativity. We will now try to relate this to the map $j_{s+s'+s''-r}\circ\gamma$. Note that $j\circ\gamma:\Sigma\mathbb{S}\to\mathrm{Gr}^{s+s'+s''-r}(X)$ is the map obtained from the diagram



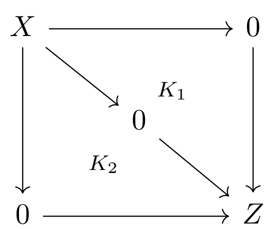
See also Figure 1. We again suppressed indices in the maps to make the diagram more readable. To be more precise, $j\circ\gamma$ is the map induced by the diagram



where H_1 is obtained from h_1 by taking compositions according to the above diagram, together with the associativity relation, and H_2 is induced by h_2 by taking compositions as well.

We now want to write $j_{s+s'+s''-r}\circ\gamma$ as $\mu(\alpha_{**},\beta^+)-\mu(\beta^-,\alpha''_{**})$. The idea is to do this similarly as one would for topological spaces. That is, show that $j\circ\gamma$ is homotopic to a map which is zero in the 'middle' of the suspension and therefore is homotopic to a map on $\Sigma\mathbb{S}\oplus\Sigma\mathbb{S}$. This is the approach Belmont and Kong use as well [4]. We want to translate this idea to the ∞-category of spectra. In particular, we consider the following statement.

Lemma 6.16. *Suppose we have a diagram*

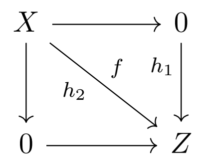


in Sp*, where* K_1, K_2 *are squares* $\Delta_1 \times \Delta_1 \to \mathrm{Sp}$*. Then the outer square yields a map* $k : \Sigma X \to Z$*, which defines an operation on* $[\Sigma X, Z]$*. Furthermore, this operation coincides with the usual addition* $\Sigma X \to Z \oplus Z \to Z$*. That is,* $k \simeq k_1 + k_2$*, where* k_1 *and* k_2 *are the maps* $\Sigma X \to Z$ *corresponding to squares* K_1 *and* K_2 *respectively.*

A proof strategy would first involve showing that the operation is well-defined; and then possibly an Eckmann-Hilton argument to show that the two operations coincide. Note that the maps $\mu(\alpha_{**}, \beta^+), \mu(\beta^-, \alpha''_{**})$ and $j_{s+s'+s''-r} \circ \gamma$ are all obtained from diagrams with essentially the same map on the diagonal. Furthermore, the homotopies $\tilde{h}_1$ and H_1 and $\tilde{h}_2$ and H_2 are nearly the same and both h_3 and h_4 come from taking cofibers. For a rigid proof, one would have to show that we can indeed relate the homotopies in the way we think they can be related, so that we can use the following statement to write $[j_{s+s'+s''-r} \circ \gamma]$ as

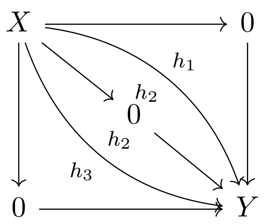
$$[\mu(\alpha_{**}, \beta^+)] - [\mu(\beta^-, \alpha''_{**})].$$

Lemma 6.17. *Let* $h_1, h_2, h_3 : \Delta^2 \to \mathrm{Sp}$ *be three null-homotopies of the same map* $f : X \to Z$*. Write* $[h_1, h_2] : \Sigma X \to Z$ *for class of the maps in* $[\Sigma X, Z]$ *induced by*

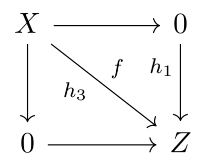


Then $[h_1, h_2] + [h_2, h_1] + [h_3, h_1] = 0$ *in* $[\Sigma X, Z]$*.*

For proving this, the idea is to use Lemma 6.16 to note that we can write $[h_1, h_2] + [h_2, h_3]$ as the map induced by diagram



Then the goal is to show, probably by considering the homotopies as morphisms in mapping spaces, that the middle diagram is essentially the same as taking the identity homotopy from f to itself. That is, this induces the same map as the diagram



and therefore, since swapping diagrams is the same as inverting, see[16, 1.2.2.10], we have indeed that $[h_1, h_2] + [h_2, h_3] + [h_3, h_1] = 0$. □

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