Utrecht University

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Semigroups and Energy Estimates

Bachelor Thesis

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Mathematics

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January, 2025

Abstract

In this thesis we will study a certain type of initial value problem, known as the Cauchy problem, through two different methods. The first one is the theory of semigroups and the second is an estimate known as energy estimate. We will develop both the necessary results of semigroup theory and the energy estimate and apply them to the Cauchy problem. We conclude the thesis by applying both methods to the wave equation with initial conditions.

Contents

1 Introduction

Some of the most important equations in physics are differential equations like the wave equation or the Schrödinger equation. As such, many tools have been developed over the years to study these equations.

In this thesis we will examine two methods for studying the *Cauchy problem*. This problem consists of finding $u : [0, T] \to X$ such that

$$\begin{cases} \frac{du(t)}{dt} = A(t)u(t) + f(t) & \text{if } t \in (0, T), \\ u(0) = x. \end{cases} \tag{1.1}$$

We first take a look at the theory of semigroups, a generalization of the exponential function, and apply this to the special case of (1.1) where $A(t) = A$ is constant with respect to t and $f = 0$. With the help of the Hille-Yosida theorem, we will see that under suitable conditions on A and x, there exists a unique solution.

The second method we will look at is an energy estimate similar to the one found in [Hör07]. Under suitable conditions on $A(t)$, we will be able to provide an upper bound on u in terms of $u(0), A(t)u(t)$ and $du(t)/dt$. This energy estimate can be used to then find unique solutions to (4). In [Hör07], the space X in consideration are the Sobolev spaces $H_s(\mathbb{R}^n)$, which can be generated by the operator $(I - \Delta)^{1/2}$. In this thesis, we will generalize the results in [Hör07] by generalizing the Sobolev spaces. We will achieve this by replacing $(I - \Delta)^{1/2}$ by a more general operator B satisfying certain conditions.

For this thesis, we assume the reader is familiar with the concepts of Banach and Hilbert spaces and functional analysis of bounded operators, as found in [RY08].

In Chapter 2 we develop the tools necessary to study these methods. We will first introduce some necessary concepts from (unbounded) operator theory found in [Sch12]. We then develop the more generalized Sobolev spaces mentioned earlier. These generalized Sobolev spaces are an original construction developed for this thesis. We finish the section off with some important results we will need from distribution theory, which will be based off of the theory developed in [DK10].

In Chapter 3 we develop semigroup theory. We cover some important properties of semigroups and work towards the Hille-Yosida theorem. We will then show how the theory is applied to the Cauchy problem, with $A(t) = A$ constant with respect to t and $f = 0$. This section will consist of a selection of results found in [Paz83].

In Chapter 4, we will develop our more general energy estimate. We will state conditions on $A(t)$ under which the energy estimate will hold and prove the solvability of the Cauchy problem under these conditions. As stated before, the content of this section will be a generalization of results found in [Hör07].

In Chapter 5, we apply both methods to the wave equation with initial conditions. We will reduce this second order problem to a Cauchy problem and apply the semigroup theory and energy estimate. The semigroup treatment of the wave equation is inspired by [Paz83]. The energy estimate treatment is inspired by discussions with thesis advisor Michał Wrochna.

2 Preliminaries

In this chapter we deal with some preliminaries needed to develop the following chapters. We will give a brief overview of linear operators, which is based on Chapter 1 of [Sch12]. Using this theory, we will define Sobolev spaces, which we will use when we develop the energy estimate. We end the chapter with a section on distribution theory, which we will need when we discuss the energy estimate. This section is based on [DK10].

2.1 Linear Operators

In this section we assume $(X, ||.||_X)$, $(Y, ||.||_Y)$ and $(Z, \|.\|_Z)$ are Banach spaces. Throughout this thesis we will assume that all Banach and Hilbert spaces are over $\mathbb{C}$.

Definition 2.1.1. Let $A \subset X$ be a linear subspace. We call a linear map $T : A \to Y$ a *linear operator* from X to Y and denote $D(T) := A$ the *domain* of T. When referring to a bounded linear operator $T : X \to Y$, we assume this to mean $D(T) = X$. We call T *densely defined* if $D(T)$ is a dense subspace of X.

Definition 2.1.2. Let S, T be two linear operators from X to Y. Then $S \subset T$ whenever $D(S) \subset D(T)$ and $Sx = Tx$ for all $x \in D(S)$. In particular, $S = T$ whenever $S \subset T$ and $T \subset S$.

When we get to Sobolev spaces, we will be interested in the composition of linear operators.

Definition 2.1.3. Let $S : X \to Y$ and $T : Y \to Z$ be linear operators with domains $D(S)$ and $D(T)$ respectively. Then we define the linear operator $TS : X \to Z$ with domain $D(TS) := \{x \in D(S) : Sx \in D(T)\}$ by $TSx = T(Sx)$ for all $x \in D(TS)$.

Whenever X and Y are Banach spaces, we can turn $X \oplus Y$ into a Banach space by endowing $X \oplus Y$ with the norm

$$\|(x, y)\|_{X\oplus Y} = \|x\|_X + \|y\|_Y\,, \quad \forall (x, y) \in X \oplus Y.$$

For any linear operator $T : X \to Y$, the graph $\text{gr}(T) := \{(x, Tx) : x \in D(T)\}$ is a subspace of $X \oplus Y$.

Definition 2.1.4. Let $T : X \to Y$ be a linear operator. We say T is *closed* if $\text{gr}(T)$ is a closed subspace of $X \oplus Y$.

The following result is a useful characterization of closed operators that we will also use in this thesis.

Theorem 2.1.5. *Let $T : X \to Y$ be a linear operator. Then T is closed if and only if for every sequence $(x_n)_{n\in\mathbb{N}} \subset D(T)$ such that $x_n \to x$ in X and $Tx_n \to y$ in Y as $n \to \infty$, we find $x \in D(T)$ and $Tx = y$.*

just like bounded operators, it makes sense it talk about invertibility of linear operators.

Definition 2.1.6. Let $T : X \to Y$ be a linear operator with domain $D(T)$. Then we say T is invertible if T is a bijection from $D(T)$ to Y and if the map T^{-1} is in $B(Y, X)$, i.e. is a bounded operator from Y to X.

Definition 2.1.7. Let $T : X \to Y$ be a linear operator. We define the *resolvent set of* T as $\rho(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is invertible}\}$. For $\lambda \in \rho(T)$ we denote the resolvent

$$R(\lambda : T) = (\lambda I - T)^{-1}. \tag{2.1}$$

Remark 2.1.8. Sometimes the resolvent of T is defined as $(T-\lambda I)^{-1}$ for $\lambda \in \rho(T)$, which is just $-(\lambda I-T)^{-1}$. Throughout the thesis we will consistently be working with the resolvent as defined in Definition 2.1.7.

Now let $(H_1, \langle ., . \rangle_1)$ and $(H_2, \langle ., . \rangle_2)$ be Hilbert spaces. In this thesis, we will assume the convention that inner products are linear in the first argument and anti-linear in the second argument. Furthermore, let $T : H_1 \to H_2$ be a densely defined linear operator. We define

$$D(T^*) := \{y \in H_2 : \exists z \in H_1 \text{ such that } \forall x \in H_1 \ \langle Tx, y\rangle_2 = \langle x, z\rangle_1\}.$$

Let $y \in D(T^*)$. Then there exists $z \in H_1$ such that $\langle Tx, y\rangle_2 = \langle x, z\rangle_1$. As $D(T) \subset H_1$ is dense, it follows that z is unique. Thus we can define the following linear operator.

Definition 2.1.9. Let $T : H_1 \to H_2$ be a densely defined linear operator. Then we define the linear operator $T^* : H_2 \to H_1$ with domain $D(T^*)$ by $T^*y = z$, with $z \in H_2$ the unique element such that

$$\langle Tx, y\rangle_2 = \langle x, z\rangle_1 = \langle x, T^*y\rangle_1 \quad \forall x \in D(T). \tag{2.2}$$

We call T^* the adjoint of T. Furthermore, we say that T is self-adjoint if $T = T^*$.

Lemma 2.1.10. *Let T be a densely defined operator from H_1 to H_2, then T is closed if and only if $T = (T^*)^*$.*

We will finish this section off with a definition that is needed in defining arbitrary powers of a linear operator, which we will need for defining Sobolev spaces.

Definition 2.1.11. We call a linear operator $T : H \to H$ *(strictly) positive* or positive definite if

$$\langle Tx, x\rangle > 0 \tag{2.3}$$

for all $x \in D(T)$ such that $x \neq 0$. We also denote this by $T > 0$.

2.2 Sobolev Spaces

In this section we construct the (generalized) Sobolev spaces. For this, we will work with a Hilbert space $(H, \langle.,.\rangle)$. Let B be a positive, self-adjoint and invertible operator on H. Then as can be seen in [Sch12], for any $s \in \mathbb{R}$ we can define the self-adjoint operators B^s on H. This family of operators, known as the fractional powers of B, has the following properties.

(i) for all $s \in \mathbb{R}$, $D(B^s) \subset H$ is dense.

(ii) $B^0 = I_H$ and $B^1 = B$.

(iii) For all $s \geq 0$, $B^{-s}x \in D(B^s)$ and $B^sB^{-s}x = x$ for all $x \in H$. If in particular $x \in D(B^s)$, then $B^{-s}B^sx = B^sB^{-s}x = x$.

(iv) Let $s, r \in \mathbb{R}$. If either $x \in D(B^rB^s)$ or $x \in D(B^sB^r)$, then both inclusions are true and in addition, $x \in D(B^{r+s})$ and $B^{r+s}x = B^rB^sx = B^sB^rx$.

In particular, the following two properties can be derived from property (iii) and (iv).

(v) Let $s, r \leq 0$, then $B^{r+s}x = B^rB^sx = B^sB^rx$ for all $x \in H$.

(vi) Let $r, s > 0$. Then $D(B^{r+s}) = D(B^rB^s) = D(B^sB^r)$ and $B^{r+s}x = B^rB^sx = B^sB^rx$.

These operators can be constructed using spectral theory. For more information on this topic, we refer the reader to [Sch12]. We can now define *(generalized) Sobolev spaces of order s.*

Definition 2.2.1. Let $s \geq 0$ and B a positive, self-adjoint and invertible operator on H. We define $H_s := D(B^s)$ and the bilinear map $\langle x, y\rangle_s := \langle B^sx, B^sy\rangle$. We also define the map $\|x\|_s := \|B^sx\|$.

Remark 2.2.2. From the definition it should be clear that $H_s \subset H_0 = H$ for all $s \geq 0$. In fact, for any $s \geq r \geq 0$, it is the case that $H_s \subset H_r$. Let $x \in H_s$. Then writing $s = (s - r) + r$, we see $s - r, r \geq 0$, so by property (vi) we find $x \in H_r$.

The notations in this definition are quite suggestive and for a good reason: $\langle.,.\rangle_s$ is an inner-product on H_s with associated norm $\|.\|_s$ and the space $(H_s, \langle.,.\rangle_s)$ is a Hilbert space for all $s \in \mathbb{R}$.

Lemma 2.2.3. *Let $s \geq 0$ and let $(H_s, \langle.,.\rangle_s)$ as in Definition 2.2.1. Then $(H_s, \langle.,.\rangle_s)$ is a Hilbert space.*

Proof. Let $(x_n)_{n\in\mathbb{N}} \subset H_s$ be a Cauchy sequence. Let $\varepsilon > 0$. Then there exists an $N \in \mathbb{N}$ such that for all $n, m \geq N$ we find $\|B^sx_n - B^sx_m\| = \|x_n - x_m\|_s < \varepsilon$. Thus $(B^sx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H, so there exists an $x \in H$ such that $\lim_{n\to\infty} B^sx_n = x$ in H. Now we find that $x = B^sB^{-s}x$ by property (iii). Thus $B^{-s}x \in H_s$ and for $\varepsilon > 0$ we find there exists an $N \in \mathbb{N}$ such that for all $n \geq N$ we find

$$\left\|x_n - B^{-s}x\right\|_s = \|B^sx_n - x\| < \varepsilon,$$

Thus $(x_n)_{n\in\mathbb{N}}$ converges in H_s. □

Now we wish to extend our family of Sobolev spaces to the negative reals.

Definition 2.2.4. Let $s > 0$. Then we define $(H_{-s}, \|.\|_{-s}) := (H_s', \|.\|_{H_s \to \mathbb{C}})$, with $\|.\|_{H_s \to \mathbb{C}}$ the usual operator norm on H_s'.

We can now prove that there exists a very nice dense embedding from H into H_{-s} for $s \in \mathbb{R}$. Using this embedding, we will be able to view our family of Sobolev spaces $(H_s)_{s\in\mathbb{R}}$ as a chain of Hilbert spaces, where $H_s \subset H_r$ for all $s \geq r$.

Lemma 2.2.5. *Let $s > 0$. We define $\iota : H \to H_{-s}$ by*

$$\iota(y)(x) = \left\langle B^s x, B^{-s} y \right\rangle . \tag{2.4}$$

Then ι is injective, $\iota(H) \subset H_{-s}$ is dense and $\|\iota(y)\|_{-s} = \|B^{-s}y\|$.

Proof. Say $\iota(y) = \iota(z)$. Then for all $x \in H_s$, we find $\langle B^s x, B^{-s}y\rangle = \langle B^s x, B^{-s}z\rangle$, so $\langle B^s x, B^{-s}(y-z)\rangle = 0$. Now by property (iii) and (v) we find $B^{-2s}(y-z) = B^{-s}B^{-s}(y-z) \in H_s$ and $B^s B^{-2s}(y-z) = B^{-s}(y-z)$. Thus

$$\left\|B^{-s}y - B^{-s}z\right\|^2 = \left\langle B^s B^{-2s}(y-z), B^{-s}(y-z)\right\rangle = 0,$$

so $y = z$ and ι is injective.

Now we show $\iota(H) \subset H_{-s}$ is dense. Let $f \in H_{-s}$ and let $\varepsilon > 0$. As H_s is a Hilbert space, by the Riesz representation theorem there exists a $z \in H_s$ such that $f(x) = \langle x, z\rangle_s = \langle B^s x, B^s z\rangle$ for all $x \in H_s$. Now $B^s z \in H$, thus as $H_s \subset H$ is dense, there exists $y \in H_s$ such that $\|y - B^s z\| < \varepsilon$. Now as $y \in H_s$, we know $B^s y \in H$. Now let $g = \iota(B^s y)$. Then

$$|f(x) - g(x)| = |\langle B^s x, B^s z - y\rangle| \leq \|x\|_s \|B^s z - y\| .$$

Thus $\|f - g\|_{-s} \leq \|B^s z - y\| < \varepsilon$. We can conclude that $\iota(H) \subset H_{-s}$ is dense.

Now we show the final claim. By the Cauchy-Schwarz inequality, we find that

$$|\iota(y)(x)| = \left|\left\langle B^s x, B^{-s}y\right\rangle\right| \leq \|B^s x\| \left\|B^{-s}y\right\| = \|x\|_s \left\|B^{-s}y\right\| .$$

It follows that $\|\iota(y)\| \leq \|B^{-s}y\|$.
Now for $y \in H$ with $y \neq 0$, it follows that $B^{-2s}y \in H_s$, thus for $x = B^{-2s}y / \|B^{-s}y\|$ we find $\|x\|_s = 1$, so

$$\|\iota(y)\| \geq |\iota(y)(x)| = \frac{1}{\|B^{-s}y\|}\left\langle B^s B^{-2s}y, B^{-s}y\right\rangle = \left\|B^{-s}y\right\| .$$

As the claim is clearly true for $y = 0$ as well, we find $\|\iota(y)\|_{-s} = \|B^{-s}y\|$ for all $y \in H$. ☐

This lemma implies that we can effectively view H as a dense subspace of H_{-s} for all $s > 0$. If we then have $x \in H_{-s}$, we know that there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset H$ such that $\lim_{n\to\infty} \iota(x_n) = x$ in H_{-s}. Then $(\iota(x_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in H_{-s}, so by Lemma 2.2.5 we know $(B^{-s}x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in H and thus converges. Thus for we can define

$$B^{-s}x = \lim_{n\to\infty} B^{-s}x_n, \tag{2.5}$$

where the limit is taken in H. The uniqueness of limits and the injectivity of ι guarantee that this limit is unique and thus that $B^{-s}x$ is well-defined for all $x \in H_{-s}$. Now as B^s is a self-adjoint operator on H_s, it follows that it is closed. Thus one can easily derive that B^{-s} is still injective on H_{-s}. Of course, when $x \in H$, our new definition and original definition for B^{-s} coincide as H is embedded in H_{-s}. We can now turn H_{-s} into a Hilbert space by introducing the inner product $\langle x, y\rangle_{-s} := \langle B^{-s}x, B^{-s}y\rangle$. Notice how this is indeed an inner product, as B^{-s} is linear and injective on H_{-s}. Now let $x \in H_{-s}$ with $\iota(x_n) \to x$ in H_{-s} as $n \to \infty$. Then

$$\left\langle B^{-s}x, B^{-s}x\right\rangle = \lim_{n\to\infty}\left\langle B^{-s}x_n, B^{-s}x_n\right\rangle = \lim_{n\to\infty}\left\|B^{-s}x_n\right\|^2 = \lim_{n\to\infty}\|\iota(x_n)\|_{-s}^2 = \|x\|_{-s}^2 .$$

Thus the norm induced by $\langle .,. \rangle_{-s}$ is the same as the dual space norm on H_{-s}, thus we find that $(H_{-s}, \langle .,. \rangle_{-s})$ is a Hilbert space.

Also notice that we can naturally view H_{-r} as a subspace of H_{-s} for any $s \geq r \geq 0$ using the map $\iota : H_{-r} \to H_{-s}$ given by

$$\iota(f)(x) = \langle B^s x, B^r y_f \rangle , \tag{2.6}$$

where y_f is uniquely associated to f by the Riesz representation theorem. Similarly to the proof of 2.2.5 it follows that this map is an injection. Using these inclusions, we can now view $(H_s)_{s\in\mathbb{R}}$ as a family of Hilbert spaces such that $H_s \subset H_r$ for all $s \geq r \in \mathbb{R}$.

Now let $s > 0$ and $x \in H$. With this new framework, we can now define $B^s x$. As $H_s \subset H$ is dense, we can find a sequence $(x_n)_{n\in\mathbb{N}} \subset H_s \subset H$ such that $x_n \to x$ in H as $n \to \infty$, so in particular, $(x_n)_{n\in\mathbb{N}}$ is Cauchy in H. Now we cannot necessarily say $(B^s x_n)_{n\in\mathbb{N}}$ is Cauchy in H, but it is Cauchy in H_{-s}, as $\|B^s x_n - B^s x_m\|_{-s} = \|B^{-s} B^s x_n - B^{-s} B^s x_m\| = \|x_n - x_m\|$. Thus it follows that this sequence actually converges in H_{-s}, so we can define

$$B^s x := \lim_{n\to\infty} B^s x_n \tag{2.7}$$

with the limit taken in H_{-s}. Here also, B^s remains injective on H, as B^{-s} is self-adjoint on H and hence closed.
Now if $x \in H_s$ and $r > s$, we can define

$$B^r x := B^{r-s} B^s x. \tag{2.8}$$

As $x \in H_s$, we know $B^s x \in H$, thus with our new definition, we find $B^r x \in H_{s-r}$. Combining this with our initial assumptions (iv), we find that

$$B^r x = B^{r-s} B^s x \quad \forall r, s \in \mathbb{R}. \tag{2.9}$$

Now for any $s > 0$ we know $H_s \subset H$ is dense and $H \subset H_{-s}$ is dense. But in fact, a much stronger result is true.

Theorem 2.2.6. *Let $s \geq r$. Then $H_s \subset H_r$ is dense.*

Proof. Let $x \in H_r$ and $\varepsilon > 0$. Then $B^r x \in H$. Now as $s - r \geq 0$, we find $H_{s-r} \subset H$ is dense. Thus there exists $y \in H_{s-r}$ such that $\|B^r x - y\| < \varepsilon$. Now as $y \in H_{s-r}$, it follows that $B^{s-r} y \in H$, so $B^{-r} y = B^{-s} B^{s-r} y \in H_s$. We then see that

$$\left\|x - B^{-r} y\right\|_r = \left\|B^r x - B^r B^{-r} y\right\| = \|B^r x - y\| < \varepsilon.$$

□

Using a similar map as in Lemma 2.2.5, we can now show that H_{-s} is isometrically isomorphic to H_s' for any $s \in \mathbb{R}$. The proof is mostly similar to the proof of Lemma 2.2.5

Theorem 2.2.7. *Let $s \in \mathbb{R}$. Then $\Psi_s : H_{-s} \to H_s'$ given by*

$$\Psi_s(y)(x) = \left\langle B^s x, B^{-s} y \right\rangle \tag{2.10}$$

is an isometric and anti-linear isomoprhism.

Proof. The proof is mostly similar to the proof of Lemma 2.2.5. We will thus only show that Ψ_s is surjective. Let $f \in H_s'$. Then there exists $z \in H_s$ such that for all $x \in H_s$ $f(x) = \langle x, z \rangle_s = \langle B^s x, B^s z \rangle$. Now let $y = B^{2s} z \in H_{-s}$. Then $f(x) = \langle B^s x, B^s z \rangle = \langle B^s x, B^{-s} y \rangle = \Psi_s(y)(x)$ for all $x \in H_s$, thus $\Psi_s(y) = f$. □

Example 2.2.8 (Standard Sobolev spaces)**.** It is known that $(I-\Delta)u = \mathcal{F}^{-1}\left[(1+\|.\|^2)\mathcal{F}u\right]$ for $u \in \mathcal{S}'(\mathbb{R}^n)$. Here $\mathcal{F}$ is the Fourier transform, $\mathcal{S}'(\mathbb{R}^n)$ is the space of tempered distributions on $\mathbb{R}^n$ and

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} \tag{2.11}$$

is known as the Laplacian, where $\partial/\partial x_i$ is the i^{th} derivative in a distributional sense. If we now define $D(I-\Delta) = \{f \in L^2(\mathbb{R}^n) : (I-\Delta)f \in L^2(\mathbb{R}^n)\}$, it is clear that $D(I-\Delta)$ is dense in $L^2(\mathbb{R}^n)$ as $C_0^\infty(\mathbb{R}^n) \subset D(I-\Delta) \subset L^2(\mathbb{R}^n)$. Using the properties of the Fourier transformation, it can be shown that $(I-\Delta)$ is self-adjoint, positive and invertible. It follows that there exists a unique positive, self-adjoint and invertible operator $(I-\Delta)^{1/2}$ such that $(I-\Delta)^{1/2}(I-\Delta)^{1/2} = (I-\Delta)$.

Now letting $H = L^2(\mathbb{R}^n), B = (I-\Delta)^{1/2}$ and $D(B) = D((I-\Delta)^{1/2})$, we obtain the standard Sobolev spaces which we will denote by $H_s(\mathbb{R}^n)$. For integers $k \geq 0$, we also obtain the following characterization of $H_k(\mathbb{R}^n)$:

$$H_k(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \partial^\alpha u \in L^2(\mathbb{R}^n) \text{ for all } |\alpha| \leq k\}. \tag{2.12}$$

Here $\alpha \in \mathbb{N}^n$ and $\partial^\alpha = \partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}$ with derivatives taken in the distributional sense. For more details on the construction and properties of these spaces, we refer the reader to [DK10]. △

Definition 2.2.9. We define the sesquilinear map $(.,.)_s : H_s \times H_{-s} \to \mathbb{C}$ by $(x,y)_s = \langle B^s x, B^{-s} y\rangle$.

Now assume we have a linear map $A : \cup_{s\in\mathbb{R}} H_s \to \cup_{s\in\mathbb{R}} H_s$ such that for all $s \in \mathbb{R}$ the restriction $A|_{H_s} : H_s \to H_{s-1}$ is bounded and for all $s, r \in \mathbb{R}$, we get $A|_{H_s} = A|_{H_r}$ on $H_s \cap H_r$. We will denote $A_s = A|_{H_s}$ and $A_s' : H_{s-1}' \to H_s'$ as the dual operator.

Definition 2.2.10. We define the H_s-adjoint of A as the map $A_s^* : H_{-s+1} \to H_{-s}$ defined by

$$A_s^* = \Psi_s^{-1} A_s' \Psi_{s-1}. \tag{2.13}$$

The term adjoint is then justified by the following theorem.

Theorem 2.2.11. *Let $s \in \mathbb{R}$. Then for all $x \in H_s$ and $y \in H_{-s+1}$,*

$$(Ax, y)_{s-1} = (x, A_s^* y)_s. \tag{2.14}$$

Proof. Let $x \in H_s$ and $y \in H_{-s+1}$. The result then follows straight from the definitions as

$$(Ax,y)_{s-1} = \Psi_{s-1}(y)(A_s x) = A_s' \Psi_{s-1}(y)(x)$$

and

$$(x, A_s^* y)_s = \Psi_s(A_s^* y)(x) = \Psi_s(\Psi_s^{-1} A_s' \Psi_{s-1} y)(x) = A_s' \Psi_{s-1}(y)(x)$$

□

2.3 Derivatives and distributions

Now that we have our Hilbert spaces H_s, we can study function spaces of functions $(0,T) \to H_s$. We start with $C^1((0,T); H_s)$. Throughout this thesis, we write $d_s u/dt$ for the derivative of $u : (0,T) \to H_s$. if $s = 0$, we shall write $d/dt := d_0/dt$. For this section, assume $T > 0$.

We start of with a variation of partial integration, which is compatible with the map $(.,.)_s$. This result relies on the following key lemma.

Lemma 2.3.1. *Let $u \in C^1((0,T); H_s)$. Then*

$$B^s \frac{d_s u}{dt} = \frac{d}{dt}(B^s u). \tag{2.15}$$

Proof. The proof simply relies on filling in the definitions. We see

$$\left\| \frac{B^s u(t+h) - B^s u(t)}{h} - B^s \frac{d_s u(t)}{dt} \right\| = \left\| \frac{u(t+h)-u(t)}{h} - \frac{d_s u(t)}{dt} \right\|_s .$$

As $h \to 0$, the right hand side goes to 0, thus the result follows. □

To avoid boundary terms, we introduce to space $C_0^\infty(I, H_s)$, which is the space of smooth functions from an open $I \subset \mathbb{R}$ to H_s with compact support.

Lemma 2.3.2. *Let $u \in C^1((0,T); H_s)$ and $v \in C_0^\infty((0,T); H_{-s})$. Then*

$$\int_0^T \left(u(t), \frac{d_{-s} v(t)}{dt} \right)_s dt = -\int_0^T \left(\frac{d_s u(t)}{dt}, v(t) \right)_s dt \tag{2.16}$$

Proof. Using the previous lemma we find

$$\begin{aligned}
\int_0^T \left(u(t), \frac{d_{-s} v(t)}{dt} \right)_s dt &= \int_0^T \left\langle B^s u(t), B^{-s} \frac{d_{-s} v(t)}{dt} \right\rangle dt \\
&= \int_0^T \left\langle B^s u(t), \frac{d}{dt} B^{-s} v(t) \right\rangle dt \\
&= -\int_0^T \left\langle \frac{d}{dt} B^s u(t), B^{-s} v(t) \right\rangle dt \\
&= -\int_0^T \left\langle B^s \frac{d_s u(t)}{dt}, B^{-s} v(t) \right\rangle dt \\
&= \int_0^T \left(\frac{d_s u(t)}{dt}, v(t) \right)_s dt
\end{aligned}$$

□

For our discussion of energy estimates, functions are not enough, we will need Banach space-valued distributions. For standard distribution theory (of $\mathbb{C}$-valued distributions), we refer the reader to [DK10].

Definition 2.3.3. For $(\phi_j)_{j \in \mathbb{N}} \subset C_0^\infty(I; H_s)$ we say $\phi_j \to \phi$ as $j \to \infty$ if

(i) there exists a compact set K such that supp$\phi_j \subset K$ for all $j \in \mathbb{N}$.

(ii) For every $k \in \mathbb{N}$, the sequence $(d^k \phi_j / dt^k)_{j \in \mathbb{N}}$ converges uniformly on I to $d\phi/dt$.

Definition 2.3.4. Let $u : C_0^\infty(I; H_{-s}) \to \mathbb{C}$ be a linear map. Then we call u a *distribution* if

$$(\phi_j)_{j \in \mathbb{N}}, \quad \lim_{n \to \infty} \phi_j = \phi \Longrightarrow \lim_{j \to \infty} u(\phi_j) = u(\phi). \tag{2.17}$$

We denote $\mathcal{D}'(I; H_s)$ as the space of all such distributions.

Example 2.3.5. Let $u \in L^1_{\text{loc}}(I; H_s)$. Then the map $u_{\text{dis}} : C_0^\infty(I; H_{-s}) \to \mathbb{C}$ defined by

$$u_{\text{dis}}(\phi) = \int_I (u(t), \phi(t))_s dt \tag{2.18}$$

is a distribution in $\mathcal{D}'(I; H_s)$. This follows from the fact that there is some $K \subset \mathbb{R}$ compact such that supp$\phi_j \subset K$ for all $j \in \mathbb{N}$ and

$$|u(\phi)| \le \sup_{t \in I} \|\phi(t)\|_{-s} \int_{I \cap K} \|u(t)\|_s \, dt.$$

△

Lemma 2.3.6. *Let $I \subset \mathbb{R}$ be bounded, $f, g \in L^\infty(I; H_s)$ and $f_{dis} = g_{dis}$, then $f = g$ a.e.*

Proof. Let $h = f - g$. It is known that $C_0^\infty(I; H_s) \subset L^1(I, H_s)$ is dense. Thus there exists $(\phi_j)_{j\in\mathbb{N}} \subset C_0^\infty(I; H_s)$ such that $\phi_j \to h$ as $j \to \infty$. Now it should be clear that $B^{2s}\phi_j(t) \in H_{-s}$ for all t and that

$$\frac{d_{-s}B^{2s}\phi(t)}{dt} = B^{2s}\frac{d_s\phi}{dt}.$$

Thus by an inductive argument, we conclude $(B^{2s}\phi_j)_{j\in\mathbb{N}} \subset C_0^\infty(I; H_{-s})$. Now as $f_{\text{dis}} = g_{\text{dis}}$, it easily follows that $h_{\text{dis}} = 0$, so for all $j \in \mathbb{N}$ we find

$$\int_I \langle h(t), \phi_j(t)\rangle_s \, dt = \int_I (h(t), B^{2s}\phi(t))_s dt = 0.$$

Thus it follows that

$$\int_I \|h(t)\|_s^2 \, dt = \int_I \langle h(t), h(t) - \phi_j(t)\rangle_s \, dt \leq \text{ess sup}_{t\in I} \|h(t)\|_s \cdot \int_I \|h(t) - \phi_j(t)\|_s \, dt$$

Now as $\phi_j \to h$ in $L^\infty(I, H_s)$ and I is bounded, it follows that the right-hand side tends to 0, thus we can conclude that $h = 0$ a.e., thus $f = g$ a.e. □

Inspired by Lemma 2.3.2, we can now define what it means for these distributions to be differentiable.

Definition 2.3.7. Let $u \in \mathcal{D}'((0, T); H_s)$. Then we define the derivative $d_{s,\text{dis}}u/dt$ by

$$\frac{d_{s,\text{dis}}u}{dt}(\phi) = -u\left(\frac{d_{-s}\phi}{dt}\right), \tag{2.19}$$

for all $\phi \in C_0^\infty((0, T); H_{-s})$.

It should be clear that $d_{s,\text{dis}}u/dt$ is also a distribution. if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and if there exists $v \in L^1_{\text{loc}}(\mathbb{R}^n)$ such that $d_{s,\text{dis}}u/dt = v_{\text{dis}}$, then we call u *weakly derivative* with *weak derivative* $d_{s,\text{w}}u/dt = v$.

3 Semigroups of Bounded Linear Operators

In this chapter we will give an overview of semigroups. This chapter is based on Chapter 1 and 4 of [Paz83]. Throughout this chapter, let $(X, ||.||)$ be a Banach space.

3.1 Continuous Semigroups

Definition 3.1.1. Let $T(t)$ be a family of bounded linear operators on X for $t \geq 0$. We call $T(t)$ a *semigroup on X* if

(i) $T(0) = I$ (the identity operator on X),

(ii) $T(t+s) = T(t)T(s)$ for all $t, s \in [0, \infty)$ (The semigroup property).

Furthermore, we define the linear operator A on X with domain

$$D(A) := \left\{ x \in X : \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \text{ exists.} \right\} \tag{3.1}$$

by

$$Ax = \lim_{t \downarrow 0} \frac{T(t)x - x}{t} \tag{3.2}$$

for $x \in D(A)$. We call A the *(infinitesimal) generator* of the semigroup $T(t)$.

In this thesis, we will primarily be interested in continuous semigroups. It turns out that the following definition will be enough to ensure (uniform) continuity.

Definition 3.1.2. Let $T(t)$ be a semigroup. We call the semigroup *strongly continuous* or C_0 if

$$\lim_{t \downarrow 0} T(t)x = x \tag{3.3}$$

for all $x \in X$. We call the semigroup *uniformly continuous* if

$$\lim_{t \downarrow 0} ||T(t) - I|| = 0. \tag{3.4}$$

Notice how any uniformly continuous semigroup is also a C_0 semigroup as

$$\|T(t)x - x\| \leq \|T(t) - I\| \, \|x\| \to 0$$

as $t \downarrow 0$. A very useful property of C_0 is that they are exponentially bounded.

Theorem 3.1.3. *Let $T(t)$ be a C_0 semigroup. Then there exist $M \geq 1$ and $\omega \geq 0$ such that*

$$\|T(t)\| \leq Me^{\omega t}. \tag{3.5}$$

Proof. Assume that there exists no $\eta > 0$ such that $\|T(t)\|$ is bounded for all $t \in [0, \eta]$. Then for every $n \in \mathbb{N}$, there exists a $t_n \in [0, 1/n]$ such that $\|T(t_n)\| \geq n$. Then clearly $\lim_{n\to\infty} t_n = 0$. As $T(t)$ is a C_0 semigroup, it follows that $\lim_{n\to\infty} T(t_n)x = x$, thus $(\|T(t_n)\|)_{n\in\mathbb{N}}$ is a bounded sequence. Furthermore, by the uniform boundedness principle it follows that there exists an $x \in X$ such that $(\|T(t_n)x\|)_{n\in\mathbb{N}}$ is unbounded, a contradiction. Thus there exists an $\eta > 0$ and $M \in \mathbb{R}$ such that $\|T(t)\| \leq M$ for all $t \in [0, \eta]$. Thus $M \geq \|T(0)\| = \|I\| = 1$. Now let $t \geq 0$. Then there exists an $n \in \mathbb{N}$ and $0 \leq \delta < \eta$ such that $t = n\eta + \delta$. Thus by the semigroup property

$$\|T(t)\| = \|T(n\eta + \delta)\| = \|T(\eta)^n T(\delta)\| \leq \|T(\eta)\|^n \|T(\delta)\| \leq M^{n+1}.$$

As $t = n\eta + \delta$, we find

$$n = \frac{t - \delta}{\eta} \leq \frac{t}{\eta}.$$

Furthermore, as $M \geq 1$, we can define $\omega = \eta^{-1} \log M \geq 0$. Then

$$e^{\omega t} = M^{\frac{t}{\eta}},$$

so we can conclude that

$$\|T(t)\| \leq M^{n+1} \leq M M^{t/\eta} = M e^{\omega t}.$$

□

Corollary 3.1.4. *Let $T(t)$ be a C_0 semigroup. Then for every $x \in X$ the map $t \mapsto T(t)x$ is a continuous map from $[0, \infty)$ to X. Furthermore, if $T(t)$ is uniformly continuous, then the map $t \mapsto T(t)$ is continuous from $[0, \infty)$ to $B(X, X)$.*

Proof. Let $s, t \geq 0$, $x \in X$ and first assume $t \geq s$. Then

$$\|T(t)x - T(s)x\| \leq \|T(s)\| \, \|T(t-s)x - x\| \xrightarrow{(t \downarrow s)} 0. \tag{3.6}$$

Now assume $t \leq s$. Then by Theorem 3.1.3 there exist $M \geq 1$ and $\omega \geq 0$ such that

$$\|T(t)x - T(s)x\| \leq \|T(t)\| \, \|x - T(s-t)x\| \leq M e^{\omega t} \|T(s-t)x - x\| \xrightarrow{(t \uparrow s)} 0. \tag{3.7}$$

Thus

$$\lim_{t \to s} \|T(t)x - T(s)x\| = 0,$$

so the claim follows. Now say $T(t)$ is uniformly continuous. Then from (3.6) and (3.7) we find

$$\|T(t)x - T(s)x\| \leq \|T(s)\| \, \|T(t-s)x - x\| \leq \|T(s)\| \, \|T(t-s) - I\| \, \|x\|,$$

$$\|T(t)x - T(s)x\| \leq M e^{\omega t} \|T(s-t)x - x\| \leq M e^{\omega t} \|T(s-t) - I\| \, \|x\|.$$

Thus

$$\|T(t) - T(s)\| \leq \|T(s)\| \, \|T(t-s) - I\| \xrightarrow{(t \downarrow s)} 0,$$

$$\|T(t) - T(s)\| \leq M e^{\omega t} \|T(s-t) - I\| \xrightarrow{(t \uparrow s)} 0.$$

□

The nice thing about about C_0 semigroups is that they have some nice properties related to differentiability and integration. In particular, we will begin to see the resemblance between the exponential function and C_0 semigroups.

Before we get that, we need to introduce the concept of integration over Banach space valued functions. Such integrals are called *Bochner integrals*. If the Banach space is $\mathbb{C}$, then the Bochner integral is just the Lebesgue integral. The construction of these integrals is not particularly important here, all we need to know is that the integral behaves as we expect it to. The integral is linear and continuous functions are integrable. Another important result is the following.

Lemma 3.1.5. *Let $A : X \to Y$ be a closed operator on where X, Y are Banach spaces and let $f : I \to X$ be Bochner integrable for $I \subset \mathbb{R}$. If $f(t) \in D(A)$ for all $t \in I$ and $f, A \circ f$ are both Bochner integrable, then*

$$\int_I A(f(t)) dt = A \left(\int_I f(t) dt \right). \tag{3.8}$$

If T is bounded, then the integrability of f implies the integrability of $A \circ f$ and the same result holds true. For more information on the Bochner integral, we refer the reader to [Are+11].

Now we can continue with semigroups.

Theorem 3.1.6. *Let $T(t)$ be a C_0 semigroup and A be its generator. Then* *(i) For all $x \in X$,*

$$\lim_{h \downarrow 0} \frac{1}{h} \int_t^{t+h} T(s)x ds = T(t)x. \tag{3.9}$$

(ii) For all $x \in X$ and $t \geq 0$, $\int_0^t T(s)x ds \in D(A)$ and

$$A\left(\int_0^t T(s)x ds\right) = T(t)x - x. \tag{3.10}$$

(iii) For $x \in D(A)$, $T(t)x \in D(A)$ and

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax. \tag{3.11}$$

(iv) For $x \in D(A)$ and $t, s \geq 0$,

$$T(t)x - T(s)x = \int_s^t T(\tau)Ax d\tau = \int_s^t AT(\tau)x d\tau. \tag{3.12}$$

Proof. Part (i) follows from the continuity of $t \to T(t)x$ and the fundamental theorem of calculus. For part (ii), notice how for $0 \leq h \leq t$

$$\begin{aligned}
(T(h) - I)\left(\int_0^t T(s)x ds\right) &= \int_0^t T(s+h)x ds - \int_0^t T(s)x ds \\
&= \int_h^{t+h} T(s)x ds - \int_0^t T(s)x ds \\
&= \int_t^{t+h} T(s)x ds - \int_0^h T(s)x ds.
\end{aligned}$$

By part (i) we find

$$\lim_{h \downarrow 0} \frac{T(h) - I}{h}\left(\int_0^t T(s)x ds\right) = T(t)x - T(0)x = T(t)x - x.$$

Thus claim (ii) follows. Now let $x \in D(A)$. Then

$$\frac{T(h) - I}{h} T(t)x = T(t)\frac{T(h)x - x}{h}.$$

As $T(t)$ is a bounded operator, it is continuous from X to X, thus

$$\lim_{h \downarrow 0} \frac{T(h) - I}{h} T(t)x = \lim_{h \downarrow 0} T(t)\frac{T(h)x - x}{h} = T(t)Ax.$$

It follows that $T(t)x \in D(A)$ and that $AT(t)x = T(t)Ax$. This also implies

$$\frac{d^+}{dt} T(t)x = \lim_{h \downarrow 0} \frac{T(t+h)x - T(t)x}{h} = T(t) \lim_{h \downarrow 0} \frac{T(h)x - x}{h} = T(t)Ax.$$

For $t > h \geq 0$, Notice how

$$\begin{aligned}
\left\| T(t-h)\frac{T(h)x - x}{h} - T(t)Ax \right\| &= \left\| T(t-h)\frac{T(h)x - x}{h} + T(t-h)Ax - T(t-h)Ax - T(t)Ax \right\| \\
&\leq \|T(t-h)\| \left\| \frac{T(h)x - x}{h} \right\| + \|T(t-h)Ax - T(t)Ax\| \\
&\leq Me^{\omega(t-h)} \left\| \frac{T(h)x - x}{h} \right\| + \|T(t-h)Ax - T(t)Ax\|.
\end{aligned}$$

Using the continuity of $t \mapsto T(t)x$, we find that the right hand side goes to 0 as $h \downarrow 0$. Thus for the left derivative we find

$$\begin{aligned}\frac{d^-}{dt}T(t)x &= \lim_{h\uparrow 0}\frac{T(t+h)x - T(t)x}{h}\\ &= \lim_{h\downarrow 0}\frac{T(t-h)x - T(t)x}{-h}\\ &= \lim_{h\downarrow 0}\frac{T(t)x - T(t-h)x}{h}\\ &= \lim_{h\downarrow 0}T(t-h)\frac{T(h)x - x}{h}\\ &= T(t)Ax.\end{aligned}$$

Thus part (iii) follows. Part (iv) then follows from integrating integrating (3.11) from s to t. □

The following corollary follows quite easily from the proofs of Theorem 3.1.6(i) and (ii) and the continuity of $t \mapsto T(t)$ for uniformly bounded semigroups $T(t)$.

Corollary 3.1.7. *Let $T(t)$ be a uniformly continuous semigroup and A be its generator. Then for all $t \geq 0$,*

(i)

$$\lim_{h\downarrow 0}\frac{1}{h}\int_t^{t+h} T(s)ds = T(t), \tag{3.13}$$

(ii)

$$A\left(\int_0^t T(s)ds\right) = T(t) - I. \tag{3.14}$$

Corollary 3.1.8. *Let $T(t)$ be a C_0 semigroup and A its generator. Then $D(A)$ is dense in X and A is a closed operator. If moreover, $T(t)$ is uniformly continuous, then A is a bounded operator.*

Proof. For $x \in X$, define

$$x_t = t^{-1}\int_0^t T(s)xds.$$

Then by Theorem 3.1.6(i) we find $x_t \in D(A)$ and $\lim_{t\downarrow 0} x_t = x$, thus $x \in \overline{D(A)}$. Thus $D(A)$ is dense in X. Now let $(x_n)_{n\in\mathbb{N}} \subset D(A)$ such that $x_n \to x$ and $Ax_n \to y$ as $n \to \infty$. By Theorem 3.1.6(iv) we know for all $n \in \mathbb{N}$,

$$T(t)x_n - x_n = \int_0^t T(s)Ax_n ds.$$

As $T(t)$ is a C_0 semigroup, there exist constants M, ω such that

$$\|T(s)Ax_n\| \leq \|T(s)\|\,\|Ax_n\| \leq Me^{\omega s}\|Ax_n\| \leq Me^{\omega t}\|Ax_n\|$$

for all $0 \leq s \leq t$. As $T(s)$ is a continuous map on X for all $s \geq 0$, it follows that $\lim_{n\to\infty} T(s)Ax_n = T(s)y$ for all $s \geq 0$. Thus by the Dominated Convergence Theorem, we find that

$$T(t)x - x = \lim_{n\to\infty}(T(t)x_n - x_n) = \lim_{n\to\infty}\int_0^t T(s)Ax_n ds = \int_0^t T(s)yds.$$

Thus we find by Theorem 3.1.6(i) that

$$\lim_{t\downarrow 0}\frac{T(t)x - x}{t} = \lim_{t\downarrow 0} t^{-1}\int_0^t T(s)yds = y.$$

Thus we find that $x \in D(A)$ and that $Ax = y$, so it follows that A is a closed operator.

Now let $T(t)$ be a uniformly continuous semigroup and A its generator. Then by Corollary 3.1.7 (i) we find

$$\lim_{h\downarrow 0} h^{-1}\int_t^{t+h} T(s)ds = T(t)$$

for all $t \geq 0$. Thus in particular there exists an $h > 0$ such that

$$\left\| I - h^{-1}\int_0^h T(s)ds \right\| < 1.$$

It follows that $I - \left(I - h^{-1}\int_0^t T(s)ds\right) = h^{-1}\int_0^t T(s)ds$ is invertible, thus $\int_0^t T(s)ds$ is invertible. Now by Corollary 3.1.7 (ii) we find

$$A\left(\int_0^t T(s)ds\right) = T(t) - I,$$

Thus we find

$$A = (T(t) - I)\left(\int_0^t T(s)ds\right)^{-1}.$$

As the product of bounded operators, we see that A is also a bounded operator. □

Corollary 3.1.9. *Let $T(t)$ be a uniformly continuous semigroup with generator A. Then $t \mapsto T(t)$ is a differentiable map and*

$$\frac{d}{dt}T(t) = AT(t) = T(t)A. \tag{3.15}$$

Proof. From Corollary 3.1.7(ii) it follows that

$$A\left(\frac{1}{t}\int_0^t T(s)ds\right) = \frac{1}{t}A\left(\int_0^t T(s)ds\right) = \frac{T(t)-I}{t}.$$

As A is continuous, it follows by Corollary 3.1.7(i) that

$$\lim_{t\downarrow 0}\frac{T(t)-I}{t} = A.$$

This rest of the proof is essentially the same as the proof of Theorem 3.1.6(iii). □

We will finish this section with a uniqueness property that will be very useful in the next section and when we will be studying the Cauchy problem.

Corollary 3.1.10. *Let $T(t)$ and $S(t)$ be C_0 semigroups with generators A and B respectively. Then $A = B$ if and only if $T(t) = S(t)$.*

Proof. If $T(t) = S(t)$, then clearly $A = B$ by the uniqueness of limits. Thus assume $A = B$ and let $x \in D := D(A) = D(B)$ and $t \geq 0$. We define the map $\varphi : [0,t] \to X$ by $s \mapsto T(t-s)S(s)x$. Then by Theorem 3.1.6 and a result similar to the product rule for differentiation, we find that this map is differentiable with

$$\begin{aligned}\frac{d}{ds}T(t-s)S(s)x &= -AT(t-s)S(s)x + T(t-s)BS(s)x\\ &= -AT(t-s)S(s)x + T(t-s)AS(s)x\\ &= -AT(t-s)S(s)x + AT(t-s)S(s)x\\ &= 0.\end{aligned}$$

Thus φ is constant, so $T(t)x = \varphi(0) = \varphi(t) = S(t)x$. Thus $T(t) = S(t)$ on $D(A)$. As $D(A)$ is dense in X and $T(t), S(t)$ are bounded, it follows that $T(t) = S(t)$ on X. □

3.2 Generators of Continuous Semigroups

In this section we will be characterizing the generators of C_0 semigroups $T(t)$ such that $\|T(t)\| \leq e^{\omega t}$, where $\omega \geq 0$. We will achieve this in steps. Before we can characterize the generators of such C_0 semigroups, we need to characterize the generators of uniformly continuous semigroups. After this we look at generators of *semigroups of contractions*; these are semigroups $T(t)$ such that $\|T(t)\| \leq 1$. This characterization is known as the *Hille-Yosida Theorem.* Using the result result of the Hille-Yosida theorem, we will finish this section with the classification of the desired class of C_0 semigroups.

Theorem 3.2.1. *Let A be a linear operator on X. Then A generates a uniformly continuous semigroup if and only if A is a bounded operator.*

Proof. If A is the generator of a uniformly continuous semigroup, then A is bounded by Corollary 3.1.8. Thus let A be a bounded operator. For $t \geq 0$, define

$$T(t) = e^{tA} := \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}. \tag{3.16}$$

Notice how this series converges, as X is a Banach space and

$$\sum_{n=0}^{\infty} \left\| \frac{(tA)^n}{n!} \right\| = \sum_{n=0}^{\infty} \frac{(t\|A\|)^n}{n!} = e^{t\|A\|} < \infty.$$

and $B(X)$ is a Banach space. This implies that $T(t)$ is bounded for all $t \geq 0$. Furthermore, it is clear from the definition that $T(0) = I$. As $T(t)$ and $T(s)$ are both absolutely convergent for all $t, s \geq 0$, it follows that their product is the Cauchy product, from which we obtain

$$\begin{aligned} T(t)T(s) &= \left(\sum_{n=0}^{\infty} \frac{(tA)^n}{n!} \right) \left(\sum_{n=0}^{\infty} \frac{(sA)^n}{n!} \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(tA)^k}{k!} \frac{(sA)^{n-k}}{(n-k)!} \\ &= \sum_{n=0}^{\infty} \frac{A^n}{n!} \sum_{k=0}^{n} \binom{n}{k} t^k s^{n-k} \\ &= \sum_{n=0}^{\infty} \frac{A^n}{n!} (t+s)^n \\ &= T(t+s). \end{aligned}$$

It follows that $T(t)$ is a semigroup. Now let B be the generator of $T(t)$. Notice how for all $x \in X$ we find

$$\begin{aligned} \left\| \frac{T(t)x - x}{t} - Ax \right\| &\leq \left\| \frac{T(t) - I}{t} - A \right\| \|x\| \\ &= \left\| t^{-1} \left(\sum_{n=0}^{\infty} \frac{(tA)^n}{n!} - I \right) - A \right\| \|x\| \\ &= \left\| \sum_{n=1}^{\infty} \frac{t^{n-1} A^n}{n!} - A \right\| \|x\| \\ &= \left\| \sum_{n=2}^{\infty} \frac{t^{n-1} A^n}{n!} \right\| \|x\| \\ &\leq \sum_{n=2}^{\infty} \frac{t^{n-1} \|A\|^n}{(n-2)!} \|x\| \\ &= t \|A\|^2 e^{t\|A\|} \|x\| \xrightarrow{(t \downarrow 0)} 0. \end{aligned}$$

Thus it follows that $D(B) = X$ and

$$Bx = \lim_{t\downarrow 0} \frac{T(t)x - x}{t} = Ax.$$

Thus A is the generator of $T(t)$. Now we also find

$$\|T(t) - I\| = \left\| \sum_{n=0}^{\infty} \frac{(tA)^n}{n!} - I \right\| \leq \sum_{n=1}^{\infty} \frac{(t\|A\|)^n}{(n-1)!} = te^{t\|A\|} \xrightarrow{(t\downarrow 0)} 0.$$

Thus $T(t)$ is uniformly continuous, so A generates a uniformly continuous semigroup.

□

Corollary 3.2.2. *Let $T(t)$ be a uniformly continuous semigroup $T(t)$. Then:*

(i) There exists a unique bounded linear operator A such that $T(t) = e^{tA}$ and A is the generator of $T(t)$.

(ii) For A as in (i), $\|T(t)\| \leq e^{t\|A\|}$

(iii) the map $t \mapsto T(t)$ is differentiable and

$$\frac{d}{dt}T(t) = AT(t) = T(t)A, \tag{3.17}$$

where A as in (i).

Proof. By Theorem 3.2.1 we know that the generator A of $T(t)$ is a bounded operator. Then again by Theorem 3.2.1 we know that A generates the uniformly continuous semigroup e^{tA}. By Corollary 3.1.10 we know then that $T(t) = e^{tA}$. Furthermore, we saw

$$\|T(t)\| = \left\|e^{tA}\right\| = \left\| \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} \right\| \leq e^{t\|A\|}.$$

Now in the proof of Theorem 3.2.1 we have seen for $h > 0$ that $\lim_{h\downarrow 0} h^{-1}(T(h) - I) = A$, thus following the proof of Theorem 3.1.6 (iii), we can conclude that $\frac{d}{dt}T(t) = AT(t) = T(t)A$. □

Now for any C_0 semigroup $T(t)$, we have seen that there exist constants $M \geq 1, \omega \geq 0$ such that $\|T(t)\| \leq Me^{\omega t}$. When $\omega = 0$, we call $T(t)$ *uniformly bounded* and if $M = 1$ as well, we call $T(t)$ a *semigroup of contractions.*

Theorem 3.2.3 (Hille-Yosida theorem)**.** *Let A be a linear operator. Then A is the generator of a semigroup of contractions if and only if*

(i) $\overline{D(A)} = X$ and A is closed.

(ii) $(0, \infty) \subset \rho(A)$ and for every $\lambda > 0$

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda}. \tag{3.18}$$

We will prove this theorem in two parts, as we will need some key lemmas for the sufficiency proof. First we deal with necessity.

Proof of Theorem 3.2.3 (Necessity). Assume A is the generator of a C_0 semigroup $T(t)$. Then (i) follows by Corollary 3.1.8. Now let $\lambda > 0$ and $x \in X$. As $t \mapsto T(t)x$ is continuous and $\|T(t)x\| \leq \|x\|$ for all $t \geq 0$, it follows that we can define the following integral:

$$R(\lambda)x = \int_0^{\infty} e^{-\lambda t}T(t)xdt. \tag{3.19}$$

Notice that $\|T(t)x\|$ is bounded uniformly in t, so the integral exists and $R(\lambda)$ is a linear map. Furthermore,

$$\|R(\lambda)x\| = \left\| \int_0^{\infty} e^{-\lambda t}T(t)xdt \right\| \leq \int_0^{\infty} e^{-\lambda t} \|T(t)x\| dt \leq \|x\| \int_0^{\infty} e^{-\lambda t}dt = \frac{\|x\|}{\lambda}. \tag{3.20}$$

Now we show $R(\lambda : A) = R(\lambda)$. Let $x \in X$. Then using the continuity of $x \mapsto T(t)x$ for $t \geq 0$, we find

$$\begin{aligned} \frac{T(h)-I}{h}R(\lambda)x &= h^{-1}T(h)\int_0^\infty e^{-\lambda t}T(t)xdt - h^{-1}\int_0^\infty e^{-\lambda t}T(t)xdt \\ &= h^{-1}\int_0^\infty e^{-\lambda t}T(t+h)xdt - h^{-1}\int_0^\infty e^{-\lambda t}T(t)xdt \\ &= h^{-1}\int_h^\infty e^{-\lambda(t-h)}T(t)xdt - h^{-1}\int_0^\infty e^{-\lambda t}T(t)xdt \\ &= \frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt - e^{\lambda h}\cdot\frac{1}{h}\int_0^h e^{-\lambda t}T(t)xdt \end{aligned}$$

As $T(t)$ is a semigroup, clearly $e^{-\lambda t}T(t)$ is as well, thus by Theorem 3.1.6 we find

$$\lim_{h\downarrow 0}\left(\frac{e^{\lambda h}-1}{h}\int_0^\infty e^{-\lambda t}T(t)xdt - e^{\lambda h}\cdot\frac{1}{h}\int_0^h e^{-\lambda t}T(t)xdt\right) = \lambda R(\lambda)x - x.$$

Thus $AR(\lambda)x = \lambda R(\lambda)x - x$ for all $x \in X$ so

$$(\lambda I - A)R(\lambda) = I_X.$$

Now let $x \in D(A)$. Then by Theorem 3.1.6(iii) we find

$$R(\lambda)Ax = \int_0^\infty e^{-\lambda t}T(t)Axdt = \int_0^\infty e^{-\lambda t}AT(t)xdt = \int_0^\infty A\left(e^{-\lambda t}T(t)x\right)dt$$

As A is a closed operator, we know that

$$\int_0^\infty A\left(e^{-\lambda t}T(t)x\right)dt = A\left(\int_0^\infty e^{-\lambda t}T(t)xdt\right) = AR(\lambda)x.$$

Thus $R(\lambda)Ax = \lambda R(\lambda)x - x$ for all $x \in D(A)$, so

$$R(\lambda)(\lambda I - A) = I_{D(A)}.$$

It follows that $\lambda I - A$ is invertible for all $\lambda > 0$ and $R(\lambda : A) = R(\lambda)$. By (3.20) we find

$$\|R(\lambda : A)\| = \|R(\lambda)\| \leq \frac{1}{\lambda}.$$

□

For the other direction, we will approximate the operator A by a family of bounded operators, which we now know are the generators of uniformly continuous semigroups. Assume A is a linear operators such that conditions (i) and (ii) in Theorem 3.2.3 hold. Then clearly $AR(\lambda : A) = \lambda R(\lambda : A) - I$ for all $\lambda > 0$. We define the *Yosida approximation* by

$$A_\lambda = \lambda AR(\lambda : A) = \lambda^2 R(\lambda : A) - \lambda I. \tag{3.21}$$

Clearly A_λ is a bounded linear operator on X. The term approximation is justified by the following lemma.

Lemma 3.2.4. *Let A satisfy conditions (i) and (ii) of Theorem 3.2.3 and A_λ be its Yosida approximation. Then for all $x \in D(A)$,*

$$\lim_{\lambda\to\infty} A_\lambda x = Ax. \tag{3.22}$$

Proof. We will first prove $\lim_{\lambda\to\infty}\lambda R(\lambda : A)x = x$ for all $x \in X$. Let $x \in D(A)$. Then we see that

$$\begin{aligned} \|\lambda R(\lambda : A)x - x\| &= \|(\lambda R(\lambda : A) - I)x\| \\ &= \|R(\lambda : A)Ax\| \\ &\leq \frac{1}{\lambda}\|Ax\| \xrightarrow{(\lambda\to\infty)} 0. \end{aligned}$$

Now let $x \in X$ and $\varepsilon > 0$. Then as $D(A) \subset X$ is dense, there exists $y \in D(A)$ such that $\|x - y\| < \varepsilon/3$. Thus there exists $R > 0$ such that for all $\lambda \geq R$ $\|\lambda R(\lambda : A)y - y\| < \varepsilon/3$, and we find

$$\begin{aligned}\|\lambda R(\lambda : A)x - x\| &\leq \|(\lambda R(\lambda : A)(x - y)\| + \|\lambda R(\lambda : A)y - y\| + \|y - x\| \\ &\leq \|\lambda R(\lambda : A)y - y\| + (\lambda \|R(\lambda : A)\| + 1) \|x - y\| \\ &\leq \|\lambda R(\lambda : A)y - y\| + 2\varepsilon/3 \\ &< \varepsilon.\end{aligned}$$

Thus it follows that $\lim_{\lambda\to\infty} \lambda R(\lambda : A)x = x$ for all $x \in X$. In particular, for $x \in D(A)$ we find

$$\lim_{\lambda\to\infty} A_\lambda x = \lim_{n\to\infty} \lambda AR(\lambda : A)x = \lim_{\lambda\to\infty} \lambda R(\lambda : A)(Ax) = Ax.$$

□

Now as A_λ is a bounded operator for $\lambda > 0$, we know that A_λ generates the uniformly continuous semigroup e^{tA_λ} by Corollary 3.2.2.

Lemma 3.2.5. *Let A satisfy conditions (i) and (ii) of Theorem 3.2.3 and A_λ be its Yosida approximation. Then for all $x \in X$ and $\lambda, \mu > 0$,*

$$\left\|e^{tA_\lambda}x - e^{tA_\mu}x\right\| \leq t \|A_\lambda x - A_\mu x\| . \tag{3.23}$$

Proof. For any $t \geq 0$, we find that

$$\left\|e^{tA_\lambda}\right\| \leq \left\|e^{t\lambda^2 R(\lambda:A)}\right\| \left\|e^{-t\lambda I}\right\| \leq e^{t\lambda^2\|R(\lambda:A)\|}e^{-t\lambda} \leq 1.$$

Examining the definitions, it is not too difficult to see that $A_\lambda, A_\mu, e^{tA_\lambda}$ and e^{tA_μ} all commute. Now let $t \geq 0$. Then by Theorem 3.1.6 (iii) we find that the map $s \mapsto e^{tsA_\lambda}e^{t(1-s)A_\mu}$ is differentiable, so we find that

$$\begin{aligned}\left\|e^{tA_\lambda}x - e^{tA_\mu}x\right\| &= \left\|\int_0^1 \frac{d}{ds} e^{tsA_\lambda}e^{t(1-s)A_\mu}xds\right\| \\ &\leq \int_0^1 \left\|te^{tsA_\lambda}e^{t(1-s)A_\mu}(A_\lambda x - A_\mu x)\right\| ds \\ &\leq \int_0^1 te^{\|tsA_\lambda\|}e^{\|t(1-s)A_\mu\|} \|A_\lambda x - A_\mu x\| ds \\ &\leq t \|A_\lambda x - A_\mu x\| .\end{aligned}$$

□

We are now ready to prove the other direction of the Hille-Yosida theorem.

Proof of Theorem 3.2.3 (sufficiency). Let $x \in D(A)$ and $t \geq 0$, then we know by Lemma 3.2.5 that

$$\left\|e^{tA_\lambda}x - e^{tA_\mu}x\right\| \leq \|A_\lambda x - A_\mu x\| \leq t \|A_\lambda x - Ax\| + t \|Ax - A_\mu x\| . \tag{3.24}$$

By Lemma 3.2.4, we can thus conclude that $(e^{A_n}x)_{\lambda\in\mathbb{N}}$ is a Cauchy sequence and thus it converges to some value $L \in X$. Now let $\varepsilon > 0$, then there exists an $N \in \mathbb{N}$ and $R > 0$ such that for all $n \geq N$ and all $\lambda, \mu \geq R$,

$$\begin{aligned}\left\|e^{tA_\lambda}x - e^{tA_\mu}x\right\| &< \varepsilon/2, \\ \left\|e^{tA_n}x - L\right\| &< \varepsilon/2.\end{aligned}$$

Thus let $\lambda \geq R$ and $n \geq \max(N, R)$, then

$$\left\|e^{tA_\lambda}x - L\right\| \leq \left\|e^{tA_\lambda}x - e^{tA_n}x\right\| + \left\|e^{tA_n}x - L\right\| < \varepsilon.$$

Thus $(e^{tA_\lambda}x)_{\lambda>0}$ converges for all $x \in D(A)$, and as $D(A) \subset X$ is dense, the limit exists for all $x \in X$. Thus we can define

$$T(t)x := \lim_{\lambda\to\infty} e^{tA_\lambda}x. \tag{3.25}$$

Now as e^{tA_λ} is a semigroup for all $\lambda > 0$, it follows by the properties of limits that $T(t)$ is also a semigroup. Also notice how (3.24) implies that this convergence is uniform in t on bounded intervals. Now we know for every $\lambda > 0$, e^{tA_λ} is a uniformly continuous semigroup, so $t \mapsto e^{tA_\lambda}x$ is continuous by Corollary 3.1.4, thus $t \mapsto T(t)x$ is also continuous. It follows that $T(t)$ is a C_0 semigroup.

Now we have seen in Lemma 3.2.5 that $\left\|e^{tA_\lambda}\right\| \leq 1$, thus

$$\|T(t)x\| = \lim_{\lambda\to\infty} \left\|e^{tA_\lambda}x\right\| \leq \lim_{\lambda\to\infty} \left\|e^{tA_\lambda}\right\| \|x\| = \|x\|.$$

Thus $T(t)$ is a semigroup of contractions. It remains to show that A is its generator. Let $x \in D(A)$. Then by Theorem 3.1.6 (iv) we know that

$$T(h)x - x = \lim_{\lambda\to\infty} \left(e^{hA_\lambda}x - x\right) = \lim_{\lambda\to\infty} \int_0^h e^{tA_\lambda} A_\lambda x dt = \int_0^h T(t)Axdt, \tag{3.26}$$

with the last inequality following by the uniform convergence on bounded intervals. Let B be the generator of $T(t)$. Dividing both sides of (3.26) by h and taking the limit down to 0, we find by Theorem 3.1.6 (i) that $x \in D(B)$ and $Bx = Ax$, thus $A \subset B$. Now let $x \in D(B)$. Now by assumption (ii), we know $1 \in \rho(A)$, so $I - A$ is invertible. Furthermore, as B is the generator of a C_0 semigroup of contractions, we know it must by true that $1 \in \rho(B)$, as we have shown that assumption (ii) holds in this case. Thus $I - B$ is also invertible. Now for any $x \in X$, we know $(I - A)^{-1}x \in D(A) \subset D(B)$, thus we have seen that $(I - B)(I - A)^{-1}x = (I - A)(I - A)^{-1}x = x$, so $(I - A)^{-1}x = (I - B)^{-1}x$. Now let $x \in D(B)$. Then $(I - B)x \in X$, so we know $(I - A)^{-1}(I - B)x \in D(A)$. Now we know that $(I - A)^{-1} = (I - B)^{-1}$, so $(I - A)^{-1}(I - B)x = (I - B)^{-1}(I - B)x = x \in D(A)$. Thus $D(A) = D(B)$, so it follows that $A = B$. □

We can generalize this theorem easily to C_0 semigroups $T(t)$ such that $\|T(t)\| \leq e^{\omega t}$ for some $\omega \in \mathbb{R}$. Let $T(t)$ be such a semigroup. Then we can define $S(t) = e^{-\omega t}T(t)$, which is clearly a semigroup of contractions. Now

$$\begin{aligned}\frac{S(t)x - x}{t} &= \frac{e^{-\omega t}S(t)x - x}{t} \\ &= e^{-\omega t}\frac{S(t)x - x}{t} + \frac{e^{-\omega t}x - x}{t}.\end{aligned}$$

It is then clear that $\lim_{t\downarrow 0} t^{-1}(T(t)x - x)$ exists if and only if $\lim_{t\downarrow 0} t^{-1}(S(t)x - x)$ exists. Let A and B be the generator of $T(t)$ and $S(t)$ respectively. Then we find that $D(A) = D(B)$ and $B = A - \omega I$.

We can also reverse the above process. For any semigroup of contractions $S(t)$ with generator B, we can define $T(t) = e^{\omega t}S(t)$. Then $T(t)$ is a C_0 semigroup satisfying $\|T(t)\| \leq e^{\omega t}$. Similarly as before, we can deduce that $A + \omega I$ is the generator of $T(t)$. This gives us the following corollary.

Corollary 3.2.6. *Let A be a linear operator. Then A is the generator of a C_0 semigroup satisfying $\|T(t)\| \leq e^{\omega t}$ if and only if*

(i) $\overline{D(A)} = X$ *and A is closed.*

(ii) $(\omega, \infty) \subset \rho(A)$ *and for every $\lambda > \omega$*

$$\|R(\lambda : A)\| \leq \frac{1}{\lambda - \omega}. \tag{3.27}$$

Proof. Let $T(t)$ be a C_0 semigroup satisfying $\|T(t)\| \leq e^{\omega t}$ with generator A. Then $S(t) = e^{-\omega t}T(t)$ is a semigroup of contractions, so by the Hille-Yosida theorem we know $D(A - \omega I)$ is dense in X and $A - \omega I$ is closed. Thus $D(A) = D(A - \omega I)$ is dense and by the linearity of limits it follows easily that A is closed.

Now we also know that $(0,\infty) \subset \rho(A-\omega I)$ and for $\lambda > 0$ $\|R(\lambda : A-\omega I)\| \leq \frac{1}{\lambda}$. Now let $\lambda > \omega$. Then $\lambda I - A = (\lambda-\omega)I - (A-\omega I)$. As $\lambda - \omega > 0$, it follows that $\lambda I - A$ is invertible, so $\lambda \in \rho(A)$. Furthermore,

$$\|R(\lambda : A)\| = \|R(\lambda-\omega : A-\omega I)\| \leq \frac{1}{\lambda-\omega}.$$

Now assume A satisfies conditions (i) and (ii) and let $B = A - \omega I$. Then again clearly $D(B)$ is dense in X and B is closed. Now let $\lambda > 0$. That $\lambda + \omega > \omega$. Thus $\lambda + \omega \in \rho(A)$, so $\|R(\lambda+\omega : A)\| \leq \frac{1}{\lambda}$. Notice how $(\lambda+\omega)I - A = \lambda I - (A-\omega I) = \lambda I - B$. Thus $\lambda \in \rho(B)$ and $\|R(\lambda : B)\| = \|R(\lambda+\omega : A)\| \leq \frac{1}{\lambda}$. By the Hille-Yosida theorem it follows that B is the generator of a semigroup of contractions $S(t)$. Thus $B + \omega I = A - \omega I + \omega I = A$ is the generator of the semigroup $T(t) = e^{\omega t}S(t)$ satsifying $\|T(t)\| \leq e^{\omega t}$. □

3.3 The Cauchy Problem

We will now turn to studying the Cauchy problem using the tools of semigroup theory. Let A be a linear operator on a Banach space X and let $x \in X$. wish to find functions $u \in C^1((0,\infty);X) \cap C^0([0,\infty);X)$ such that

$$\begin{cases} \frac{du(t)}{dt} = Au(t), & t > 0, \\ u(0) = x. \end{cases} \tag{3.28}$$

Say A is the generator of a C_0 semigroup. Then we have seen for all $x \in D(A)$ that

$$\frac{d}{dt}T(t)x = AT(t)x$$

and of course $T(0)x = x$. Thus $u(t) := T(t)x$ is clearly a solution to (3.28). It turns out that solution will also be unique. To prove this, we first need a lemma.

Lemma 3.3.1. *Let $u : [0,T] \to X$ be continuous. If*

$$\left\| \int_0^T e^{ns} u(s)ds \right\| \leq M \tag{3.29}$$

for all $n \in \mathbb{N}$, then $u = 0$.

Proof. Let $f \in X'$ and let $\phi : [0,T] \to \mathbb{C}$ be given by $\phi(t) = f(u(t))$. Then ϕ is continuous and

$$\left| \int_0^T e^{ns}\phi(s)ds \right| = \left| f\left(\int_0^T e^{ns}u(s)ds \right) \right| \leq \|f\| \left\| \int_0^T e^{ns}u(s)ds \right\| \leq M\|f\|.$$

Now consider the series

$$\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kn\tau} = 1 - \exp(-e^{n\tau}).$$

Then clearly the series converges uniformly in τ on bounded intervals. It then follows for $0 \leq t < T$ that

$$\begin{aligned} \left| \int_0^T \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k!} e^{kn(t-T+s)}\phi(s)ds \right| &= \left| \sum_{k=1}^{\infty} \int_0^T \frac{(-1)^{k-1}}{k!} e^{kn(t-T+s)}\phi(s)ds \right| \\ &\leq \sum_{k=1}^{\infty} \frac{1}{k!} e^{kn(t-T)} \left| \int_0^T e^{kns}\phi(s)ds \right| \\ &\leq M\|f\| \left(\exp\left(e^{n(t-T)}\right) - 1 \right). \end{aligned}$$

Taking the limit we find that the integral converges to 0 as $n \to \infty$. Furthermore, we also know

$$\int_0^T \sum_{k=1}^\infty \frac{(-1)^{k-1}}{k!} e^{kn(t-T+s)} \phi(s)ds = \int_0^T \left(1 - \exp\left(-e^{n(t-T+s)}\right)\right) \phi(s)ds$$
$$= \int_0^{T-t} \left(1 - \exp\left(-e^{n(t-T+s)}\right)\right) \phi(s)ds + \int_{T-t}^T \left(1 - \exp\left(-e^{n(t-T+s)}\right)\right) \phi(s)ds.$$

By Lebesgue's dominated convergence theorem, it follows that

$$\int_0^{T-t} \left(1 - \exp\left(-e^{n(t-T+s)}\right)\right) \phi(s)ds + \int_{T-t}^T \left(1 - \exp\left(-e^{n(t-T+s)}\right)\right) \phi(s)ds$$
$$\xrightarrow{(n\to\infty)} \int_0^{T-t} (1-1)\,\phi(s)ds + \int_{T-t}^T (1-0)\,\phi(s)ds$$
$$= \int_{T-t}^T \phi(s)ds.$$

We can conclude for all $0 \leq t < T$ that

$$\int_{T-t}^T \phi(s)ds = 0.$$

Now say ϕ is not 0 on $(0, T]$. Then there exists a $0 < t \leq T$ such that $|\phi(T-t)| \neq 0$. Now for every $0 < \delta < t$ we find

$$0 = \int_{T-t}^T \phi(s)ds = \int_{T-t}^{T-(t-\delta)} \phi(s)ds + \int_{T-(t-\delta)}^T \phi(s)ds = \int_{T-t}^{T-t+\delta} \phi(s)ds = \int_0^\delta \phi(T-t+s)ds.$$

Dividing both sides by δ and taking the limit $\delta \downarrow 0$ we find $\phi(T-t) = 0$, a contradiction. Thus $\phi = 0$ on $(0, T]$ and thus by continuity on $[0, T]$. As $f \in X'$ was arbitrary, it follows from the Hahn-Banach theorem that $u(t) = 0$ for all $t \in [0, T]$. □

Theorem 3.3.2. *Let $x \in D(A)$ and A be the generator of a C_0 semigroup satisfying $\|T(t)\| \leq e^{\omega t}$. Then there exists a unique solution u of* (3.28)*.*

Proof. We have already established existence. Say v, w are both solutions. Then it is clear that $u = v - w$ satisfies

$$\begin{cases} \frac{du(t)}{dt} = Au(t), & t > 0, \\ u(0) = 0. \end{cases} \tag{3.30}$$

For now assume $\omega = 0$ and let $\lambda > 0$. By the Hille-Yosida theorem $R(\lambda : A)$ exists and is a continuous operator on X, so it follows quite easily that for $s > 0$,

$$\frac{d}{ds} R(\lambda : A)u(s) = R(\lambda : A)Au(s) = \lambda R(\lambda : A)u(s) - u(s).$$

Then for $s > 0$ it follows that

$$\frac{d}{ds} e^{\lambda(t-s)} R(\lambda : A)u(s) = -\lambda e^{\lambda(t-s)} R(\lambda : A)u(s) + \lambda e^{\lambda(t-s)} R(\lambda : A)u(s) - e^{\lambda(t-s)} u(s) = -e^{\lambda(t-s)} u(s).$$

Integrating both sides from 0 to $t > 0$ and noting $u(0) = 0$, we find

$$R(\lambda : A)u(t) = -\int_0^t e^{\lambda(t-s)} u(s)ds. \tag{3.31}$$

Now for every $0 < \sigma < t$ it follows by the Hille-Yosida theorem that

$$e^{-\sigma\lambda} \|R(\lambda : A)\| \leq \frac{e^{-\sigma\lambda}}{\lambda} \xrightarrow{(\lambda\to\infty)} 0.$$

using (3.31) we find

$$\left\| \int_0^{t-\sigma} e^{\lambda(t-\sigma-s)} u(s) ds \right\| = \| R(\lambda : A) u(t) \| \leq \| R(\lambda : A) \| \, \| u(t) \| \xrightarrow{(\lambda \to \infty)} 0.$$

Thus it follows that there exists an $M > 0$ such that

$$\left\| \int_0^{t-\sigma} e^{\lambda s} u(t-\sigma-s) ds \right\| = \left\| \int_0^{t-\sigma} e^{\lambda(t-\sigma-s)} u(s) ds \right\| \leq M$$

for all $\lambda > 0$. Thus in particular, it follows from Lemma 3.3.1 that $u(t-\sigma-s) = 0$ for all $s \in [0, t-\sigma]$, so $u = 0$ on $[0, t-\sigma]$. As $t > 0$ and $0 < \sigma < t$ were arbitrary, it follows that $u(t) = 0$ for all $t \geq 0$, thus it follows that $v = w$.

Now say $\omega \in \mathbb{R}$ arbitrary. As u is a solution of (3.30), it follows that $e^{-\omega t} u(t)$ is a solution of

$$\begin{cases} \frac{du(t)}{dt} = (A - \omega I) u(t), \quad t > 0, \\ u(0) = 0. \end{cases} \tag{3.32}$$

Now as A is the generator of $T(t)$, it follows that $A - \omega I$ is the generator of $e^{-\omega t} T(t)$. As $e^{-\omega t} T(t)$ is a semigroup of contractions, it follows from the $\omega = 0$ case that $e^{-\omega t} u(t) = 0$ for all $t \geq 0$ and thus again $v = w$. □

4 Energy Estimates

In this chapter, we give a generalization of Lemma 23.1.1 and Theorem 23.1.2 found in [Hör07]. Where [Hör07] uses the standard Sobolev space $H_s(\mathbb{R}^n)$ and a framework particular to partial differential equations on $\mathbb{R}^n$, we will show these results hold in the more general Sobolev spaces we have developed in Chapter 2.2, under assumptions formulated in a more abstract way. Our goal is to prove that there exists a unique solution to the Cauchy problem under certain conditions by developing what is often referred to as an *energy estimate*. In nice cases like the wave equation, the idea is that one can define an appropriate energy function and show that conservation of energy holds. However, this is not always possible, which is why we will develop an energy estimate which generalizes the property of energy conservation. Now we will be working with a specific energy estimate developed in [Hör07], but there are multiple takes on energy estimates. For a different take, we refer the reader to [Tay23].

4.1 Energy Estimates

In this section we study the well-posedness of the Cauchy problem

$$\begin{cases} \frac{du}{dt} + A(t) u(t) = f(t) \text{ for } t \in (0, T), \\ u(0) = x, \end{cases} \tag{4.1}$$

for given Cauchy data x. We will examine this problem in generalized Sobolev spaces. To this end, we make the following assumptions. Let $(H, \langle ., . \rangle)$ be a Hilbert space and let $(H_s)_{s \in \mathbb{R}}$ be the Sobolev space generated by the self-adjoint positive and invertible linear operator B. We assume that for all $t \in [0, T]$ the operator $A(t) : \cup_{s \in \mathbb{R}} H_s \to \cup_{s \in \mathbb{R}} H_s$ satisfies the following conditions for all $s \in \mathbb{R}$:

(i) $A(t)$ is a bounded operator from H_s to H_{s-1}.

(ii) The map $t \mapsto A(t)$ from $[0, T]$ to $B(H_s, H_{s-1})$ is continuous.

(iii) There exists a $c \in \mathbb{R}$ such that $\operatorname{Re} \langle A(t) u, u \rangle \geq -c \langle u, u \rangle$ for all $u \in H_1$.

(iv) There exists a $D_s > 0$ such that $\| (A(t) - B^s A(t) B^{-s}) v \| \leq D_s \| v \|$ for all $v \in H_1$.

Remark 4.1.1. In [Hör07], the author works in the setting of pseudo-differential operators. As such, Hörmander was able to impose more natural assumptions on $A(t)$ than we have done here, to reach the same conclusions as ours in the setting of pseudo-differential calculus. As such, it might prove more difficult to prove condition (iv) for a given family of operators $A(t)$ in this generalized framework.

We will now develop an important estimate that will help us in characterizing solutions of our Cauchy problem. Recall that throughout this chapter, $d/dt := d_0/dt$.

Lemma 4.1.2. *Let $s \in \mathbb{R}$ and let $\lambda \in \mathbb{R}$ be greater than some number dependent on s. Then for every $u \in C^1([0,T]; H_s) \cap ([0,T]; H_{s+1})$ and $p \in [0,\infty]$*

$$\left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u(t)\right\|_s^p dt\right)^{1/p} \leq \|u(0)\|_s + 2\int_0^T e^{-\lambda t/2}\left\|\frac{d_s u}{dt} + A(t)u(t)\right\|_s dt, \tag{4.2}$$

where we interpret the left-hand side as the supremum norm when $p = \infty$.

Proof. We first assume that $s = 0$. Then by assumption (iii),

$$\mathrm{Re}\,\langle A(t)v, v\rangle \geq -c\,\langle v, v\rangle$$

for all $v \in H_1$. Let $f(t) = \frac{d}{dt}u(t) + A(t)u(t) \in H$ for all $t \in [0,T]$. Then for any t we find

$$\begin{aligned}
2\mathrm{Re}\,\langle f(t), u(t)\rangle\, e^{-2ct} &= 2\mathrm{Re}\left\langle \frac{d}{dt}u(t), u(t)\right\rangle e^{-2ct} + 2\mathrm{Re}\,\langle A(t)u(t), u(t)\rangle\, e^{-2ct} \\
&= \frac{d}{dt}\left(\|u(t)\|^2 e^{-2ct}\right) + 2c\,\langle u(t), u(t)\rangle\, e^{-2ct} + 2\mathrm{Re}\,\langle A(t)u(t), u(t)\rangle\, e^{-2ct} \\
&\geq \frac{d}{dt}\left(\|u(t)\|^2 e^{-2ct}\right).
\end{aligned}$$

Integrating both sides from 0 to $t \leq T$ we find

$$\|u(t)\|^2 e^{-2ct} \leq \|u(0)\|^2 + 2\int_0^t \mathrm{Re}\,\langle f(s), u(s)\rangle\, e^{-2cs} ds.$$

Now by Cauchy-Schwarz we find $\mathrm{Re}\,\langle f(s), u(s)\rangle \leq |\langle f(s), u(s)\rangle| \leq \|f(s)\|\,\|u(s)\|$, thus

$$\left(\|u(t)\|\, e^{-ct}\right)^2 \leq \|u(0)\|^2 + 2\int_0^t \|u(s)\|\, e^{-cs}\,\|f(s)\|\, e^{-cs} ds.$$

Now we define $M(t) := \sup_{0\leq\tau\leq t}\|u(\tau)\|\, e^{-c\tau}$. Then for all $0 \leq \tau \leq t \leq T$, it follows that

$$\begin{aligned}
\left(\|u(\tau)\|\, e^{-c\tau}\right)^2 &\leq \|u(0)\|^2 + 2\int_0^\tau \|u(s)\|\, e^{-cs}\,\|f(s)\|\, e^{-cs} ds \\
&\leq \|u(0)\|^2 + 2\int_0^t \|u(s)\|\, e^{-cs}\,\|f(s)\|\, e^{-cs} ds \\
&\leq \|u(0)\|^2 + 2M(t)\int_0^t \|f(s)\|\, e^{-cs} ds.
\end{aligned}$$

where the second inequality follows from the positivity of the integrand. Taking the supremum over τ on both sides yields

$$M(t)^2 \leq \|u(0)\|^2 + 2M(t)\int_0^t \|f(s)\|\, e^{-cs} ds.$$

From this, it follows that

$$\begin{aligned}
\left(M(t) - \int_0^t e^{-cs}\,\|f(s)\|\, ds\right)^2 &= M(t)^2 - 2M(t)\int_0^t e^{-cs}\,\|f(s)\|\, ds + \left(\int_0^t e^{-cs}\,\|f(s)\|\, ds\right)^2 \\
&\leq \|u(0)\|^2 + \left(\int_0^t e^{-cs}\,\|f(s)\|\, ds\right)^2 \\
&\leq \left(\|u(0)\| + \int_0^t e^{-cs}\,\|f(s)\|\, ds\right)^2.
\end{aligned}$$

Thus it follows that

$$e^{-ct}\|u(t)\| \le M(t) \le \|u(0)\| + 2\int_0^t e^{-cs}\|f(s)\|\,ds,$$

Multiplying both sides by $e^{(c-\lambda)t}$ yields

$$e^{-\lambda t}\|u(t)\| \le e^{(c-\lambda)t}\|u(0)\| + 2\int_0^t e^{-\lambda s}\|f(s)\|\,e^{(c-\lambda)(t-s)}ds.$$

Now let $\lambda > 2c$. Then $c-\lambda < -\frac{\lambda}{2}$, so

$$\begin{aligned} e^{-\lambda t}\|u(t)\| &\le e^{-\lambda t/2}\|u(0)\| + 2\int_0^t e^{-\lambda s}\|f(s)\|\,e^{-\lambda(t-s)/2}ds \\ &= e^{-\lambda t/2}\left(\|u(0)\| + 2\int_0^t e^{-\lambda s}\|f(s)\|\,e^{\lambda s/2}ds\right) \\ &\le e^{-\lambda t/2}\left(\|u(0)\| + 2\int_0^T e^{-\lambda s/2}\|f(s)\|\,ds\right). \end{aligned}$$

Taking the $L^p([0,T])$-norm on both sides we find

$$\left(\int_0^T e^{-p\lambda t}\|u(t)\|^p\,dt\right)^{1/p} \le \left(\|u(0)\| + 2\int_0^T e^{-\lambda t/2}\|f(t)\|\,dt\right)\left(\int_0^T e^{-p\lambda t/2}dt\right)^{1/p}.$$

If we now assume that $\lambda > 0$, then

$$\int_0^T e^{-p\lambda t/2}dt = -\frac{2}{p\lambda}e^{-p\lambda t/2}\Big|_0^T \le \frac{2}{p\lambda} \le \frac{2}{\lambda}.$$

Thus

$$\left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u(t)\right\|^p dt\right)^{1/p} \le \|u(0)\| + 2\int_0^T e^{-\lambda t/2}\|f(t)\|\,ds.$$

Now let $s \in \mathbb{R}$ and let $t \in [0,T]$. Then by assumption (iv) we find that

$$\begin{aligned} \operatorname{Re}\left\langle B^s A(t) B^{-s} v, v\right\rangle &= \operatorname{Re}\langle A(t)v, v\rangle - \operatorname{Re}\left\langle (A(t) - B^s A(t) B^{-s})v, v\right\rangle \\ &\ge -c\langle v, v\rangle - \left\|(A(t) - B^s A(t) B^{-s})v\right\|\|v\| \\ &\ge (-c - D_s)\langle v, v\rangle \end{aligned}$$

for all $v \in H_1$. Now let $u \in C^1([0,T];H_s) \cap C^0([0,T];H_{s+1})$. Then we know that $(t \mapsto B^s u(t)) \in C^1([0,T];H_0) \cap C^0([0,T];H_1)$ and that $\frac{d}{dt}B^s u(t) = B^s \frac{d_s}{dt}u(t)$. As $B^s A(t) B^{-s}$ satisfies assumption (iii), it follows from the $s=0$ case that for all λ large enough

$$\begin{aligned} \left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u(t)\right\|_s^p dt\right)^{1/p} &= \left(\frac{\lambda}{2}\int_0^T \left\|B^s(e^{-\lambda t}u(t))\right\|^p dt\right)^{1/p} \\ &= \left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}B^s u(t)\right\|^p dt\right)^{1/p} \\ &\le \|B^s u(0)\| + 2\int_0^T e^{-\lambda t/2}\left\|\frac{d}{dt}B^s u(t) + B^s A(t) B^{-s} B^s u(t)\right\| ds \\ &= \|u(0)\|_s + 2\int_0^T e^{-\lambda t/2}\left\|B^s\frac{d_s}{dt}u(t) + B^s A(t) u(t)\right\| ds \\ &= \|u(0)\|_s + 2\int_0^T e^{-\lambda t/2}\left\|\frac{d_s}{dt}u(t) + A(t)u(t)\right\|_s ds \end{aligned}$$

□

This energy estimate can now be used to to derive the existence of unique solutions to the Cauchy problem, which satisfy the energy estimate.

Theorem 4.1.3. *Let $A(t)$ satisfy assumptions (i)-(iv) and let $s \in \mathbb{R}$. Then for every $x \in H_s$ and $f \in L^1((0,T);H_s)$, there exists $u \in C([0,T];H_s)$ such that u is a solution of*

$$\begin{cases} \frac{d_{s-1}u}{dt} + A(t)u(t) = f(t) \text{ for } t \in (0,T), \\ u(0) = x, \end{cases} \tag{4.3}$$

and satisfies the energy estimate (4.2) for $s-1$.

Proof. (Existence) Let $N > 0$ be larger than some number dependent on s, let $v \in C_0^\infty((-\infty,T);H_N)$. Now let us define $h(t) = -\frac{dv}{dt}(t) + B^{-s}A_s^*(t)B^s v(t)$. Notice that $\mathrm{Re}\,\langle B^{-s}A_s^*(t)B^s w, w\rangle \geq (-c-D)\|w\|^2$ for all $w \in H_s \cap H_1$. Then we find that as $v(t) \in H_N \subset H_s \cap H_1$,

$$\begin{aligned}
&2\mathrm{Re}\,\langle h(T-t), v(T-t)\rangle e^{2c(T-t)} = \\
&2\mathrm{Re}\left\langle -\frac{dv}{dt}(T-t), v(T-t)\right\rangle e^{2c(T-t)} + 2\mathrm{Re}\left\langle B^{-s}A_s^*(T-t)B^s v(T-t), v(T-t)\right\rangle e^{2c(T-t)} = \\
&\frac{d}{dt}\left(\|v(T-t)\|^2 e^{2c(T-t)}\right) + 2c\|v(T-t)\|^2 e^{2c(T-t)} + 2\mathrm{Re}\left\langle B^{-s}A^*(T-t)B^s v(T-t), v(T-t)\right\rangle e^{2c(T-t)} \geq \\
&\frac{d}{dt}\left(\|v(T-t)\|^2 e^{2c(T-t)}\right).
\end{aligned}$$

Integrating both sides from 0 to t we find

$$\|v(T-t)\|^2 e^{2c(T-t)} - \|v(T)\|^2 e^{2cT} \leq 2\int_0^t \mathrm{Re}\,\langle h(T-s), v(T-s)\rangle e^{2c(T-s)} ds.$$

Now by definition of v, we know $v(T) = 0$. Using Cauchy-Schwarz we find that

$$\begin{aligned}
\int_0^t \mathrm{Re}\,\langle h(T-s), v(T-s)\rangle e^{2c(T-s)} ds &\leq \int_0^t \|h(T-s)\| \|v(T-s)\| e^{2c(T-s)} ds \\
&\leq \int_0^T \|h(T-s)\| \|v(T-s)\| e^{2c(T-s)} ds \\
&\leq \sup_{0\leq t\leq T} \|v(T-t)\| e^{c(T-t)} \int_0^T e^{cs}\|h(s)\| ds.
\end{aligned}$$

Letting $M := \sup_{0\leq t\leq T}\|v(T-t)\| e^{c(T-t)} = \sup_{0\leq t\leq T}\|v(t)\| e^{ct}$, we find

$$M^2 \leq 2M\int_0^T e^{ct}\|h(t)\| dt,$$

Thus we see

$$\sup_{0\leq t\leq T}\|v(t)\| e^{ct} = M \leq 2\int_0^T e^{ct}\|h(t)\| dt.$$

Using the fact that $1 \leq e^{ct} \leq e^{cT}$ for all $t \in [0,T]$, we can easily conclude that there exists a $C > 0$ such that

$$\sup_{0\leq t\leq T}\|v(t)\| \leq C\int_0^T \|h(t)\| dt. \tag{4.4}$$

As $N > 0$, it follows that B^{-N} is continuous on H, thus it follows that $d_N v/dt = dv/dt$. If also $N > -s$, we find $-s-N < 0$, so B^{-s-N} is continuous on H. it follows that

$$\frac{dB^{-s}v}{dt} = B^{-s}\frac{d_N v}{dt} = B^{-s}\frac{dv}{dt}$$

If we also choose N large enough so that $N+s>1,s$, we have $B^{-s}v(t)\in H_s\cap H_1$. It follows that the estimate (4.4) is also valid if we replace v with $B^{-s}v$. If we now define $g(t)=-dv/dt+A_s^*(t)v(t)$, we get

$$\begin{aligned}\sup_{0\le t\le T}\|v(t)\|_{-s} &= \sup_{0\le t\le T}\left\|B^{-s}v(t)\right\| \\ &\le C\int_0^T\left\|-\frac{dB^{-s}v}{dt}(t)+B^{-s}A_s^*(t)B^sB^{-s}v(t)\right\|dt \\ &= C\int_0^T\left\|B^{-s}\left(-\frac{dv}{dt}(t)+A_s^*(t)v(t)\right)\right\|dt \\ &= C\int_0^T\|g(t)\|_{-s}\,dt\end{aligned}$$

From this we obtain that

$$\begin{aligned}\left|\int_0^T(f(t),v(t))_sdt+(x,v(0))_s\right| &\le \int_0^T|(f(t),v(t))_s|\,dt+|(x,v(0))_s| \\ &\le \int_0^T\|f(t)\|_s\|v(t)\|_{-s}\,dt+\|x\|_s\|v(t)\|_{-s} \\ &\le \tilde{C}\sup_{0\le t\le T}\|v(t)\|_{-s} \\ &\le C\tilde{C}\int_0^T\|g(t)\|_{-s}\,dt.\end{aligned}$$

Now we define

$$S:=\left\{g\in L^1((0,T);H_{-s}):\exists v\in C_0^\infty((-\infty,T);H_N)\text{ such that }g(t)=-\frac{dv}{dt}(t)+A_s^*(t)v(t)\quad\forall t\in(0,T)\right\}$$

and $\Psi_S:S\to\mathbb{C}$ by

$$\Psi_S(g)=\int_0^T(f(t),v(t))_s\,dt+(x,v(0))_s\,.$$

Then we have shown that Ψ_S is an anti-linear functional on S. Using the Hahn-Banach theorem, we can extend this functional to obtain a functional $\Psi\in(L^1((0,T);H_{-s}))'$. Thus there exists a $\varphi\in L^\infty((0,T);H'_{-s})$ such that

$$\Psi(g)=\int_0^T\varphi(t)(g(t))dt.$$

Now we have seen that $H'_{-s}\cong H_s$ and thus that for each $t\in(0,T)$ there exists a $u(t)\in H_s$ such that $\varphi(t)(g(t))=(u(t),g(t))_s$. Now as $\|u(t)\|_s=\|\varphi\|_{H_s\to\mathbb{C}}$, it follows that $u\in L^\infty((0,T);H_s)$. Thus we find

$$\int_0^T\left(u(t),-\frac{dv}{dt}(t)+A_s^*(t)v(t)\right)_s dt=\int_0^T(u(t),g(t))_s\,dt=\Psi(g)=\int_0^T(f(t),v(t))_s+(x,v(0))_s\,. \tag{4.5}$$

In the case that $v\in C_0^\infty((0,T);H_N)$, we find

$$\int_0^T\left(u(t),-\frac{dv}{dt}(t)+A_s^*(t)v(t)\right)_s dt=\int_0^T(f(t),v(t))_s\,. \tag{4.6}$$

Rewriting (4.6) in terms of distributions, we find

$$-u_{\text{dis}}\left(\frac{d_sv}{dt}\right)+u_{\text{dis}}(A_s^*(t)v(t))=f_{\text{dis}}(v),$$

thus

$$\frac{d_{s,\mathrm{dis}}u}{dt}(v) + (Au)_{\mathrm{dis}}(v) = f_{\mathrm{dis}}(v)$$

for all $v \in C_0^\infty((0,T);H_N)$. Now we know that $H_N \subset H_{-s}$ is dense. It follows that $C_0^\infty((0,T);H_N) \subset C_0^\infty((0,T);H_{-s})$[1]. Thus we can conclude that

$$\frac{d_{s,\mathrm{dis}}u}{dt} + (Au)_{\mathrm{dis}} = f_{\mathrm{dis}}.$$

Now assume that $f \in C_0^\infty((0,T);H_N)$. Then in particular $f \in L^\infty((0,T);H_{s-1})$, and as $t \to A(t)$ is continuous on $[0,T]$, we find $f - A(.)u \in L^\infty((0,T);H_{s-1})$. Thus it follows from equation (4.6) that $du_{s-1,\mathrm{w}}/dt := f - A(.)u$ is the weak derivative of u and $d_{s,\mathrm{dis}}u/dt(v) = (d_{s-1,\mathrm{w}}u/dt)_{\mathrm{dis}}(v)$ for all $v \in C_0^\infty((0,T);H_s)$. Now following an analogue of the Lebesgue differentiation theorem, we can conclude u is differentiable a.e. from $(0,T)$ to H_{s-1} and thus that u is continuous a.e. from $[0,T]$ to H_{s-1}, so essentially $u \in C([0,T];H_{s-1})$. As u is differentiable a.e., it follows that $du_{s-1,\mathrm{w}}/dt = d_{s-1}u/dt$ a.e. Now just as before, we find that $d_{s-2}u/dt = d_{s-1}u/dt = f - A(.)u \in C([0,T];H_{s-2})$, so $u \in C^1([0,T];H_{s-2})$.
Now if $x \in H_N$, we can replace s by $s+2$ and find that we have found $u \in C([0,T];H_{s+1}) \cap C^1([0,T];H_s)$, provided again that $N > s+2$, so the estimate from Lemma 4.1.2 applies to u and

$$\frac{d_s u(t)}{dt} + A(t)u(t) = f(t).$$

Now if we go back to (4.5), we find by partial integration that

$$(u(0), v(0))_s = (x, v(0))_s$$

for all $v \in C_0^\infty((-\infty,T);H_N)$. Then it follows that for all $w \in H_N$ we have $(u(0),w)_s = (x,w)_s$. Now let $w \in H$ and $\varepsilon > 0$. Then $B^s w \in H_{-s}$, so as $H_N \subset H_{-s}$, it follows that there is a $y \in H_N$ such that $\|B^s w - y\|_{-s} < \varepsilon$. Thus

$$\begin{aligned} |\langle B^s(u(0)-x), w\rangle| &= |(u(0)-x, B^s w)_s| \\ &= |(u(0)-x, y)_s + (u(0)-x, B^s w - y)_s| \\ &\leq \|u(0)-x\|_s \|B^s w - y\|_{-s} \\ &\leq \varepsilon \|u(0)-x\|_s . \end{aligned}$$

As $\varepsilon \downarrow 0$, we find that $\langle B^s(u(0)-x), v\rangle$ is 0 for all $v \in H$, and thus it follows that $u(0) = x$.

Now we assume $f \in L^1((0,T);H_s)$ and $x \in H_s$ are arbitrary. As $H_N \subset H_s$ and $C_0^\infty((0,T);H_N) \subset L^1((0,T);H_s)$ are dense inclusions (the last one again by tensor argument), there exist $(f_n)_{n\in\mathbb{N}} \subset C_0^\infty((0,T);H_N)$ and $(x_n)_{n\in\mathbb{N}} \subset H_N$ such that

$$\|x - x_n\|_s \to 0, \qquad \int_0^T \|f(t) - f_n(t)\|_s \, dt \xrightarrow{(n\to\infty)} 0.$$

Thus for each $n \in \mathbb{N}$ there exists $u_n \in C([0,T];H_{s+1}) \cap C^1([0,T];H_s)$ such that $u_n(0) = x_n$ and

$$\frac{d_s u_n}{dt}(t) + A(t)u_n(t) = f_n(t).$$

Then of course, for all $n,m \in \mathbb{N}$ we find $u_n - u_m \in C([0,T];H_{s+1}) \cap C^1([0,T];H_s)$, so by Lemma 4.1.2 we find

[1] For more details on how the truth of this statement, we refer the reader to Chapter 43 and 44 of [Tre67], in particular Theorem 44.1.

$$\frac{\lambda}{2}e^{-\lambda T}\sup_{0\le t\le T}\|u_n(t)-u_m(t)\|_s \le$$
$$\|u_n(0)-u_m(0)\|_s + 2\int_0^T e^{-\lambda t/2}\left\|\frac{d_s(u_n-u_m)}{dt}(t)+A(t)(u_n(t)-u_m(t))\right\|_s dt \le$$
$$\|x_n-x_m\|_s + 2\int_0^T \|f_n(t)-f_m(t)\|_s\,dt.$$

It follows that $(u_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $C([0,T];H_s)$, and thus has a limit $u\in C([0,T];H_s)$. Note that clearly $u(0)=x$. Now as B^{-1} is bounded on H and $A(t)$ is bounded from H_s to H_{s-1}, it follows that $f_n\to f$ in $L^1((0,T);H_{s-1})$ and $Au_n\to Au$ in $C([0,T];H_{s-1})$. Thus we can conclude that $d_{s-1}u_n/dt = d_s u_n/dt \to f(t)-A(t)u(t)$ in $L^1((0,T);H_{s-1})$. Now also note that

$$u_n(t)=\int_0^t \frac{d_{s-1}u_n(s)}{ds}ds,$$

so again by Lebesgue's Differentiation Theorem, it follows that u is differentiable a.e. from $(0,T)$ to H_{s-1} and $d_{s-1}/dt=\lim_{n\to\infty} d_{s-1}u_n/dt$ in $L^1((0,T);H_{s-1})$. Thus it follows that

$$\frac{d_{s-1}u(t)}{dt}+A(t)u(t)=f(t).$$

Now for $n\in\mathbb{N}$ we know that

$$\left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u_n(t)\right\|_{s-1}^p dt\right)^{1/p} \le \|u_n(0)\|_{s-1} + 2\int_0^T e^{-\lambda t/2}\left\|\frac{d_{s-1}u_n}{dt}+A(t)u_n(t)\right\|_{s-1} dt.$$

Thus taking the limit $n\to\infty$ in H_{s-1} we find that

$$\left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u(t)\right\|_{s-1}^p dt\right)^{1/p} \le \|u(0)\|_s + 2\int_0^T e^{-\lambda t/2}\left\|\frac{d_{s-1}u}{dt}+A(t)u(t)\right\|_{s-1} dt.$$

□

Let us reflect a bit on the general idea behind this proof. If there exists a solution u, then we can apply integration by parts and get 4.5. In the proof, we work backwards and try to construct this u so that it satisfies this property. We were able to achieve this by choosing a subspace S consisting of exactly those g that are of the desired form $-dv(t)/dt + A_s^*(t)v(t)$ and a functional that acts on S in a way that mimics the integration by parts property we want. The Hahn-Banach theorem then produces a function u defined on the whole interval with this property.

At this point we only have a solution in a distributional sense. We show the result holds under the stronger assumptions that f is smooth and that $f(t)$ and x have as high a regularity as we want. We can then approximate the f and x given in the theorem by sequences $(x_n)_{n\in\mathbb{N}}$ and $(f_n)_{n\in\mathbb{N}}$ for which solutions $(u_n)_{n\in\mathbb{N}}$ exist and use the energy estimate with $p=\infty$ to prove these solutions converge to a solution $u\in C([0,T];H_s)$. In doing so, we give up a degree of regularity, because $A(t)$ reduces the regularity of u.

5 The wave equation

In this chapter, we will apply the theory we have developed for semigroups and the energy estimate of the previous two section to the equation

$$\begin{cases} \frac{d^2u}{dt^2} = \Delta u - mu, \\ u(0)=x_1, \\ \frac{du(0)}{dt} = x_2, \end{cases} \tag{5.1}$$

where x_1 and x_2 are given. This second-order differential equation is known as the wave equation, which is of particular importance to many areas of physics. However, the theory we have developed thus far is for particular first-order problems. Thus, before applying our theories, we will have to reformulate this problem. The semigroup treatment of the wave equation is based on Chapter 7.4 of [Paz83].

5.1 Semigroup Solution

To apply semigroup theory to the wave equation, we need the equation to be of the same form as the Cauchy problem. It is not too difficult to see that (5.1) is equivalent to

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta - mI & 0 \end{pmatrix}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{cases} \tag{5.2}$$

Indeed if u is a solution of (5.1), then $\begin{pmatrix} u \\ \frac{du}{dt} \end{pmatrix}$ is a solution of (5.2) and conversely, if $\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a solution of (5.2), then u_1 is a solution of (5.1). We will define this new operator properly. Take the space $H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ with the norm

$$\|u\| = \|(u_1, u_2)\| = \left(\|u_1\|_{H_1}^2 + \|u_2\|_{L^2}^2\right)^{1/2} \tag{5.3}$$

for all $u = (u_1, u_2) \in H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. This clearly turns $H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ into a Banach space.

Definition 5.1.1. Let $D(A) = H^2(\mathbb{R}^n) \oplus H^1(\mathbb{R}^n)$ and for $u = (u_1, u_2) \in D(A)$ define

$$Au = (u_2, \Delta u_1 - mu_1). \tag{5.4}$$

By construction we see that A is densely defined. Furthermore, we know Δ with $D(\Delta) = H_2(\mathbb{R}^n)$ is self-adjoint, so it is a closed operator. It follows easily that A is then a closed operator on $H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. With these, we already have half of the requirements to apply the Hille-Yosida theorem. To get the other half, we are going to need two preliminary lemmas.

Lemma 5.1.2. *If $\alpha > 0$, $s \geq 0$ and $f \in H_s(\mathbb{R}^n)$, then there exists a unique $u \in H_{s+2}(\mathbb{R}^n)$ satisfying*

$$u + \alpha(mI - \Delta)u = f. \tag{5.5}$$

Proof. Let $\mathcal{F}f$ be the Fourier transform of f and let $v = (1 + \alpha m + \alpha \|.\|^2)^{-1}\mathcal{F}f$. As $f \in H_s(\mathbb{R}^n)$, we know $(1 + \|.\|^2)^{s/2}\mathcal{F}f \in L^2(\mathbb{R}^n)$. Now for $\xi \in \mathbb{R}^n$ we find

$$\int_{\mathbb{R}^n} \left|(1 + \|\xi\|^2)^{(s+2)/2} v(\xi)\right|^2 d\xi = \int_{\mathbb{R}^n} (1 + \alpha m + \alpha \|\xi\|^2)^{-1}(1 + \|\xi\|^2) \left|(1 + \|\xi\|^2)^{s/2}\mathcal{F}f(\xi)\right|^2 d\xi.$$

Now notice how $(1 + \alpha m + \alpha \|\xi\|^2)^{-1}(1 + \|\xi\|^2) \leq M := \max(1, \alpha^{-1})$. Thus it follows that

$$\int_{\mathbb{R}^n} (1 + m + \alpha \|\xi\|^2)^{-1}(1 + \|\xi\|^2) \left|(1 + \|\xi\|^2)^{s/2}\mathcal{F}f(\xi)\right|^2 d\xi \leq M \int_{\mathbb{R}^n} \left|(1 + \|\xi\|^2)^{s/2}\mathcal{F}f(\xi)\right|^2 d\xi < \infty.$$

Thus we find $(1 + \|\xi\|^2)^{(s+2)/2}v(\xi) \in L^2(\mathbb{R}^n)$. Now clearly $(1 + \|\xi\|^2)^{(s+2)/2}v(\xi) = (1 + \|\xi\|^2)^{(s+2)/2}\mathcal{F}\mathcal{F}^{-1}v$, then it follows that $u := \mathcal{F}^{-1}v \in H_{s+2}(\mathbb{R}^n)$ and

$$u = \mathcal{F}^{-1}v = \mathcal{F}^{-1}(1 + \alpha m + \alpha \|.\|^2)^{-1}\mathcal{F}f) = (I + \alpha(mI - \Delta))^{-1}f,$$

thus existence follows. Now say two solutions u, v exist, then $w = u - v$ is a solution of $w + \alpha m w - \alpha \Delta w = 0$, which implies $\mathcal{F}w + \alpha m \mathcal{F}w + \alpha \|.\|^2 \mathcal{F}w = \mathcal{F}(w + \alpha m w - \alpha \Delta w) = \mathcal{F}(0) = 0$. now fix $\xi \in \mathbb{R}$ then $\mathcal{F}w(\xi)(1 + m + \alpha \|\xi\|^2) = \mathcal{F}w(\xi) + m\mathcal{F}w(\xi) + \alpha \|\xi\|^2 \mathcal{F}w(\xi) = 0$. Thus we can conclude $\mathcal{F}w = 0$ and thus $w = 0$, so $u = v$. □

Lemma 5.1.3. *For every* $f = (f_1, f_2) \in C_0^\infty(\mathbb{R}^n) \oplus C_0^\infty(\mathbb{R}^n)$ *nonzero* $\lambda \in \mathbb{R}$, *then for every* $s \geq 2$ *there exists a unique* $u = (u_1, u_2) \in H_s(\mathbb{R}^n) \oplus H_{s-2}(\mathbb{R}^n)$ *such that*

$$u - \lambda A u = f \tag{5.6}$$

and for all $0 \leq |\lambda(m-1)| < 1$,

$$\|u\| \leq (1 - |\lambda(m-1)|)^{-1} \|f\| . \tag{5.7}$$

Proof. Now we know $C_0^\infty(\mathbb{R}^n) \subset H_s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$. Thus for $i = 1, 2$, by the previous lemma we know there exist $v_i \in H_s(\mathbb{R}^n)$ such that

$$v_i + \lambda^2 m v_i - \lambda^2 \Delta v_i = f_i.$$

Now we know $v_i \in H_s(\mathbb{R}^n) \subset H_{s-2}(\mathbb{R}^n)$. Define $u_1 = v_1 + \lambda v_2$ and $u_2 = v_2 - \lambda m v_1 + \lambda \Delta v_1$. Then clearly $(u_1, u_2) \in H_s(\mathbb{R}^n) \oplus H_{s-2}(\mathbb{R}^n)$ for every $s \geq 2$ and

$$\begin{aligned}
(u_1, u_2) - \lambda A(u_1, u_2) &= (u_1, u_2) - (\lambda u_2, \lambda \Delta u_1 - \lambda m u_1) \\
&= (v_1 + \lambda v_2, v_2 - \lambda m v_1 + \lambda \Delta v_1) - (\lambda v_2 - \lambda^2 m v_1 + \lambda^2 \Delta v_1, \lambda \Delta v_1 + \lambda^2 \Delta v_2 - \lambda m v_1 - \lambda^2 m v_2) \\
&= (v_1 + \lambda^2 m v_1 - \lambda^2 \Delta v_1, v_2 + \lambda^2 m v_2 - \lambda^2 \Delta v_2) \\
&= (f_1, f_2).
\end{aligned}$$

Now for the last claim, we find

$$\begin{aligned}
\|(f_1, f_2)\|^2 &= \|f_1\|_{H_1}^2 + \|f_2\|_{L^2}^2 \\
&= \left\langle (I - \Delta)^{1/2} f_1, (I - \Delta)^{1/2} f_1 \right\rangle + \langle f_2, f_2 \rangle \\
&= \langle f_1 - \Delta f_1, f_1 \rangle + \langle f_2, f_2 \rangle \\
&= \langle u_1 - \lambda u_2 - \Delta u_1 + \lambda \Delta u_2, u_1 - \lambda u_2 \rangle + \langle u_2 + \lambda m u_1 - \lambda \Delta u_1, u_2 + \lambda m u_1 - \lambda \Delta u_1 \rangle \\
&= \langle u_1 - \Delta u_1, u_1 \rangle - \lambda \langle (I - \Delta) u_1, u_2 \rangle - \lambda \langle (I - \Delta) u_2, u_1 \rangle + |\lambda|^2 \langle (I - \Delta) u_2, u_2 \rangle + \|u_2\|_{L^2}^2 - \\
&\qquad \lambda \langle u_2, \Delta u_1 \rangle - \lambda \langle \Delta u_1, u_2 \rangle + |\lambda|^2 \|\Delta u_1\|_{L^2}^2 + \langle \lambda m u_1, u_2 - \lambda \Delta u_1 \rangle + \langle u_2 + \lambda m u_1 - \lambda \Delta u_1, \lambda m u_1 \rangle \\
&\geq \langle u_1 - \Delta u_1, u_1 \rangle - 2\lambda \mathrm{Re} \langle (I - \Delta) u_1, u_2 \rangle + \|u_2\|_{L^2}^2 + 2\lambda m \mathrm{Re} \langle u_1, u_2 \rangle + \\
&\qquad 2\lambda^2 m \mathrm{Re} \langle u_1, -\Delta u_1 \rangle + \lambda^2 m^2 \|u_1\|_{L^2(\mathbb{R}^n)} \\
&\geq \langle u_1 - \Delta u_1, u_1 \rangle + \|u_2\|_{L^2}^2 - 2 |\lambda(m-1)| \mathrm{Re} \langle u_1, u_2 \rangle \\
&\geq \|(u_1, u_2)\|^2 - |\lambda(m-1)| (\|u_1\|_{L^2}^2 + \|u_2\|_{L^2}^2) \\
&\geq (1 - |\lambda(m-1)|) \|(u_1, u_2)\|^2 .
\end{aligned}$$

Then if $0 \leq |\lambda(m-1)| < 1$, we find that $\|(f_1, f_2)\|^2 \geq (1 - |\lambda(m-1)|)^2 \|(u_1, u_2)\|^2$, so we find

$$\|(f_1, f_2)\| \geq (1 - |\lambda(m-1)|) \|(u_1, u_2)\| . \tag{5.8}$$

□

Now we know $C_0^\infty(\mathbb{R}^n) \subset H_1(\mathbb{R}^n), C_0^\infty(\mathbb{R}^n) \subset L^2(\mathbb{R}^n)$ are dense inclusions, so it clearly follows that $C_0^\infty(\mathbb{R}^n) \oplus C_0^\infty(\mathbb{R}^n) \subset H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ is dense. Now for $0 \leq |\lambda(m-1)| < 1$ we have just seen $C_0^\infty(\mathbb{R}^n) \oplus C_0^\infty(\mathbb{R}^n)$ is in the image of $I - \lambda A$. Now A is closed, thus so is $I - \lambda A$ is closed. Using that $C_0^\infty(\mathbb{R}^n) \oplus C_0^\infty(\mathbb{R}^n) \subset H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ is dense, equation (5.8) and that $I - \lambda A$ is closed, we find that the range of $I - \lambda A$ is $H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. As such we obtain the following corollary.

Corollary 5.1.4. *Let* $f = (f_1, f_2) \in H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ *and real* $\lambda \in \mathbb{R}$ *satisfy* $0 \leq |\lambda(m-1)| < 1$, *then*

$$u - \lambda A u = f \tag{5.9}$$

has a unique solution $u = (u_1, u_2) \in H_2(\mathbb{R}^n) \oplus H_1(\mathbb{R}^n)$ *and*

$$\|u\| \leq (1 - |\lambda(m-1)|)^{-1} \|f\| . \tag{5.10}$$

With this corollary, we can now finally solve the wave equation using the Hille-Yosida theorem.

Theorem 5.1.5. *the operator A is the infinitesimal generator of a C_0 semigroup $T(t)$ on $H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ satisfying*

$$\|T(t)\| \leq e^{|m-1|t}. \tag{5.11}$$

Proof. We've already established A is closed and densely defined. Let $\mu > |m-1|$. Then $0 \leq \left|\mu^{-1}(m-1)\right| < 1$, so by Corollary 5.1.4 we know $I - \mu^{-1}A$ bijective, thus so is $\mu I - A$. Now let $f \in H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$. Then there exists $u \in H_2(\mathbb{R}^n) \oplus H_1(\mathbb{R}^n)$ such that $(I - \mu^{-1}A)u = f$, so $(\mu I - A)^{-1} f = \mu^{-1} u$, thus again by Corollary 5.1.4 we find

$$\left\|(\mu I - A)^{-1} f\right\| = \mu^{-1} \|u\| \leq \mu^{-1}(1 - \mu^{-1}|m-1|)^{-1} \|f\| = (\mu - |m-1|)^{-1} \|f\|.$$

Thus $(|m-1|, \infty) \subset \rho(A)$ and for all $\mu > |m-1|$ we find $\|R(\mu : A)\| \leq (\mu - |m-1|)^{-1}$. The result follows from the Hille-Yosida theorem. □

The semigroup obtained by this theorem is the desired solution.

Corollary 5.1.6. *Let $x_1 \in H_2(\mathbb{R}^n), x_2 \in H_1(\mathbb{R}^n)$. Then there exists a unique $u \in C^1([0,\infty); L^2(\mathbb{R}^n))$ satisfying*

$$\begin{cases} \frac{d^2u(t)}{dt^2} = \Delta u(t) - mu(t), \\ u(0) = x_1, \\ \frac{du(0)}{dt} = x_2. \end{cases} \tag{5.12}$$

Proof. Let $T(t)$ be the C_0 semigroup generated by A and let $(u_1(t), u_2(t)) = T(t)(x_1, x_2)$. We then know that

$$\frac{d}{dt}(u_1(t), u_2(t)) = \frac{d}{dt} T(t)(f_1, f_2) = AT(t)(f_1, f_2) = A(u_1(t), u_2(t)) = (u_2(t), \Delta u_1(t) - mu_1(t)).$$

Notice this derivative is with respect to the norm $\|.\|_{H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)}$. It follows easily that $du_1/dt = u_2$ in $L^2(\mathbb{R}^n)$. It also follows that $du_2(t)/dt = \Delta u_1(t)$ in $L^2(\mathbb{R}^n)$. Thus clearly $u_1 \in C^1([0,\infty); L^2(\mathbb{R}^n))$ and satisfies (5.12). □

Using the semigroup estimate, we see for the solution $u := u_1$ that

$$\|u\|_1 \leq \|(u_1, u_2)\| = \|T(t)(x_1, x_2)\| \leq e^{|m-1|t} \|(x_1, x_2)\| = e^{|m-1|t} \left(\|x_1\|_1 + \|x_2\|\right)^{1/2}$$

5.2 Energy Estimate Solution

Let us return to the equation

$$\begin{cases} \frac{d}{dt}\begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} 0 & I \\ \Delta - mI & 0 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \\ \begin{pmatrix} u_1(0) \\ u_2(0) \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \end{cases} \tag{5.13}$$

This problem is first order in t, which is what we need to apply the energy estimate from the previous chapter. However, unlike in the application of semigroup theory, we need our operator to work on Sobolev spaces. As we have a positive mass term, the operator $mI - \Delta$ is positive, which implies that $\pm(mI - \Delta)^{1/2}$ are both invertible. This way, we can reformulate our problem, as

$$\begin{pmatrix} 0 & I \\ \Delta - mI & 0 \end{pmatrix} = \begin{pmatrix} I & I \\ i(mI-\Delta)^{1/2} & -i(mI-\Delta)^{1/2} \end{pmatrix} \begin{pmatrix} i(mI-\Delta)^{1/2} & 0 \\ 0 & -i(mI-\Delta)^{1/2} \end{pmatrix} \begin{pmatrix} \frac{1}{2}I & -\frac{i}{2}(mI-\Delta)^{-1/2} \\ \frac{1}{2}I & \frac{i}{2}(mI-\Delta)^{-1/2} \end{pmatrix}.$$

Notice how we can write this as

$$\begin{pmatrix} 0 & I \\ \Delta - mI & 0 \end{pmatrix} = P \begin{pmatrix} i(mI-\Delta)^{1/2} & 0 \\ 0 & -i(mI-\Delta)^{1/2} \end{pmatrix} P^{-1}. \tag{5.14}$$

This also illustrates why we introduced m, as this allows us to invert the matrix P. Then (5.14) implies that if $u_\pm$ are solutions of

$$\begin{cases} \frac{du_\pm(t)}{dt} = \pm(mI-\Delta)^{1/2}u_\pm(t), \\ u_\pm(0) = x_\pm, \end{cases} \tag{5.15}$$

where $x_+ = P^{-1}x_1$ and $x_- = P^{-1}x_2$, then $(u_1(t), u_2(t)) = (Pu_+(t), Pu_-(t))$ is a solution of (5.13). We will study one of these Cauchy problems, and the other one follows by flipping the signs.

Definition 5.2.1. Let $A : \mathcal{S}' \to \mathcal{S}'$ be given by $Af = i(mI-\Delta)^{1/2}f$.

We will now check the four conditions we need on A to able to apply our results from Chapter 4.

Lemma 5.2.2. *Let $s \in \mathbb{R}$ and let $t \in [0,T]$. Then*

(i) A is a bounded operator from $H_s(\mathbb{R}^n)$ to $H_{s-1}(\mathbb{R}^n)$.

(ii) The map $t \mapsto A$ from $[0,T]$ to $B(H_s, H_{s-1})$ is continuous.

(iii) There exists a $c \in \mathbb{R}$ such that $Re\,\langle Au, u\rangle \geq -c\,\langle u, u\rangle$ for all $u \in H_1(\mathbb{R}^n)$.

(iv) There exists a $D_s > 0$ such that $\|(A - B^sAB^{-s})v\| \leq D_s\,\|v\|$ for all $v \in H_1(\mathbb{R}^n)$.

Proof. For (i) let $s \in \mathbb{R}$ and $f \in H_s(\mathbb{R}^n)$. Then we know $(1+\|.\|^2)^{s/2}f \in L^2(\mathbb{R}^n)$. Now it should be clear that $(1+\|\xi\|^2)^{-1}(m+\|\xi\|^2) \leq \max(m,1)$ for all $\xi \in \mathbb{R}^n$. Thus

$$\begin{aligned} \int_{\mathbb{R}^n} \left|(1+\|\xi\|^2)^{(s-1)/2}(m+\|\xi\|^2)^{1/2}\mathcal{F}f(\xi)\right|^2 d\xi &= \int_{\mathbb{R}^n} (1+\|\xi\|^2)^{-1}(m+\|\xi\|^2)\left|(1+\|\xi\|^2)^{s/2}\mathcal{F}f(\xi)\right|^2 d\xi \\ &\leq \max(1,m)\int_{\mathbb{R}^n}\left|(1+\|\xi\|^2)^{s/2}\mathcal{F}f(\xi)\right|^2 d\xi. \end{aligned}$$

Thus it follows that there is a $C > 0$ such that

$$\left\|(1+\|.\|^2)^{(s-1)/2}(m+\|.\|^2)^{1/2}\mathcal{F}f\right\| \leq C\left\|(1+\|.\|^2)^{s/2}\mathcal{F}f\right\|.$$

Thus it follows that

$$\begin{aligned} \|Af\|_{s-1} &= \left\|(I-\Delta)^{(s-1)/2}i(mI-\Delta)^{1/2}f\right\| \\ &= \left\|\mathcal{F}^{-1}\left[(1+\|.\|^2)^{(s-1)/2}(m+\|.\|^2)^{1/2}\mathcal{F}f\right]\right\| \\ &= (2\pi)^{-n/2}\left\|(1+\|.\|^2)^{(s-1)/2}(m+\|.\|^2)^{1/2}\mathcal{F}f\right\| \\ &\leq (2\pi)^{-n/2}C\left\|(1+\|.\|^2)^{s/2}\mathcal{F}f\right\| \\ &= C\left\|\mathcal{F}^{-1}\left[(1+\|.\|^2)^{s/2}\mathcal{F}f\right]\right\| \\ &= C\left\|(I-\Delta)^{s/2}f\right\| \\ &= C\,\|f\|_s\,. \end{aligned}$$

It follows that A is bounded from $H_s(\mathbb{R}^n)$ to $H_{s-1}(\mathbb{R}^n)$. Now (ii) is immediately obvious, as A is constant from $[0,T]$ to $B(H_s, H_{s-1})$ for all $s \in \mathbb{R}$. Now for (iii) let $f \in H_1(\mathbb{R}^n)$. Then of course, $f, Af \in L^2(\mathbb{R}^n)$, so we get

$$\begin{aligned}\langle Af, f\rangle &= \left\langle \mathcal{F}^{-1}\left[i(m+\|.\|^2)^{1/2}\mathcal{F}f\right], \mathcal{F}^{-1}\mathcal{F}f\right\rangle \\ &= (2\pi)^{-n}\left\langle i(m+\|.\|^2)^{1/2}\mathcal{F}f, \mathcal{F}f\right\rangle \\ &= (2\pi)^{-n}\int_{\mathbb{R}^n} \overline{i(m+\|\xi\|^2)^{1/2}\mathcal{F}f(\xi)}\mathcal{F}f(\xi)d\xi \\ &= -i(2\pi)^{-n}\int_{\mathbb{R}^n}(m+\|\xi\|^2)^{1/2}\left|\mathcal{F}f(\xi)\right|^2 d\xi.\end{aligned}$$

Clearly, then $\operatorname{Re}\langle Af, f\rangle = 0$, so $\operatorname{Re}\langle Af, f\rangle \geq -c\langle f, f\rangle$ for $c = 0$. Finally for (iv), it should be clear that A and B^s commute for all $s \in \mathbb{R}$, so the result follows trivially. □

A direct result of this lemma and Theorem 4.1.3 directly lead to the following result:

Corollary 5.2.3. *For every $s \in \mathbb{R}$ and $x_- \in H_s(\mathbb{R}^n)$, there exists $u \in C([0,T]; H_s(\mathbb{R}^n))$ such that*

$$\begin{cases} \frac{d_{s-1}u}{dt} + Au = 0, \\ u(0) = x_-. \end{cases} \tag{5.16}$$

Furthermore, for all $\lambda \in \mathbb{R}$ large enough and for all $p \in [1, \infty]$,

$$\left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u(t)\right\|_{s-1}^p dt\right)^{1/p} \leq \|x_-\|_{s-1}. \tag{5.17}$$

Theorem 5.2.4. *Let $x_1 \in H_2(\mathbb{R}^n), x_2 \in H_1(\mathbb{R}^n)$. Then there exists a unique $u \in C^1([0,\infty); L^2(\mathbb{R}^n))$ satisfying*

$$\begin{cases} \frac{d^2u(t)}{dt^2} = \Delta u(t) - mu(t), \\ u(0) = x_1, \\ \frac{du(0)}{dt} = x_2. \end{cases} \tag{5.18}$$

Proof. We can prove this in steps. We first define A and P on $H_1(\mathbb{R}^n) \oplus L^2(\mathbb{R}^n)$ as follows. Let $D(A) = H_1(\mathbb{R}^n) \oplus H_1(\mathbb{R}^n)$ and $D(P) = H_1(\mathbb{R}^n) \oplus H_1(\mathbb{R}^n)$ and define

$$\begin{aligned} A(v_1, v_2) &= (i(mI-\Delta)^{1/2}v_1, -i(mI-\Delta)^{1/2}v_2) \\ P(v_1, v_2) &= (v_1+v_2, i(mI-\Delta)^{1/2}(v_1-v_2)). \end{aligned}$$

It can easily be seen that P is invertible and that

$$P^{-1}(v_1, v_2) = \frac{1}{2}(v_1 - i(mI-\Delta)^{-1/2}v_2, v_1 + i(mI-\Delta)^{-1/2}v_2).$$

Now by the definitions, it should be clear that $(x_+, x_-) := P^{-1}(x_1, x_2) \in H_2(\mathbb{R}^n) \oplus H_2(\mathbb{R}^n)$. By Corollary 5.2.3 it then follows that there exist $(u_+, u_-) \in C([0,T]; H_2(\mathbb{R}^n)) \oplus C([0,T]; H_2(\mathbb{R}^n))$ such that

$$\begin{cases} \left(\frac{d_1u_+(t)}{dt}, \frac{d_1u_-(t)}{dt}\right) = A(u_+(t), u_-(t)), \\ (u_+(0), u_-(0)) = (x_+, x_-). \end{cases}$$

Recall that the subscript 1 signifies that the derivative limit is taken in $H_1(\mathbb{R}^n)$. Reducing regularity does not change the derivative, thus we find

$$\frac{d}{dt}(u_+(t), u_-(t)) = \left(\frac{d_1u_+(t)}{dt}, \frac{du_-(t)}{dt}\right) = \left(\frac{d_2u_+(t)}{dt}, \frac{d_2u_-(t)}{dt}\right) = A(u_+(t), u_-(t)).$$

Now we can define

$$(u_1(t), u_2(t)) = P(u_+(t), u_-(t)) = (u_+(t) + u_-(t), i(mI - \Delta)^{1/2}(u_+(t) - u_-(t))).$$

Notice then how $(u_1(0), u_2(0)) = (x_1, x_2)$. Now we we have seen that $i(mI - \Delta)^{1/2}$ is bounded from $H_2(\mathbb{R}^n)$ to $H_1(\mathbb{R}^n)$, so as $u_+, u_- \in C([0,T]; H_2(\mathbb{R}^n))$, it follows that $u_1, u_2 \in C([0,T]; H_1(\mathbb{R}^n)$. Substituting (u_+, u_-) for (u_1, u_2), we find

$$P^{-1}\frac{d}{dt}(u_1(t), u_2(t)) = \frac{d}{dt}P^{-1}(u_1(t), u_2(t)) = AP^{-1}(u_1(t), u_2(t)).$$

Thus

$$\frac{d}{dt}(u_1(t), u_2(t)) = PAP^{-1}(u_1(t), u_2(t)).$$

With some simple algebra, one can deduce that $PAP^{-1}(u_1(t), u_2(t)) = (u_2(t), \Delta u_1(t))$. Now just like in the proof using semigroup theory, we deduce that $u := u_1 \in C^1([0,T]; L^2(\mathbb{R}^n))$ is a solution of (5.18).

□

Now we know that $u := u_1 = u_+ + u_-$. We know $u_\pm$ satisfy the energy estimates

$$\left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u_\pm(t)\right\|_1^p dt\right)^{1/p} \leq \|x_\pm\|_1$$

for all λ large enough. Thus we find

$$\begin{aligned}
\left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u(t)\right\|_1^p dt\right)^{1/p} &\leq \left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u_+(t)\right\|_1^p dt\right)^{1/p} + \left(\frac{\lambda}{2}\int_0^T \left\|e^{-\lambda t}u_-(t)\right\|_1^p dt\right)^{1/p} \\
&\leq \|x_+\|_1 + \|x_-\|_1 \\
&\leq \|x_1\|_1 + \left\|(mI - \Delta)^{-1/2}x_2\right\|_1.
\end{aligned}$$

6 Conclusion

In this thesis, we have developed two different methods for proving the solvability of the Cauchy problem. First we developed semigroup theory, in particular the Hille-Yosida Theorem, which produces a solution in the form of a semigroup under the right conditions. We also developed an abstract energy estimate which can also be used to prove the solvability of the Cauchy problem, again under certain abstract conditions. To compare the two methods, we have applied both to a well-known second order problem, finding a solution to the wave equation for specified initial values. To apply the methods, we first had to reduce the problem to an appropriate first order problem. Through this first order problem, we were able to find solutions that satisfied the wave equation.

Though we we were able to apply both methods to prove the wave equation with initial conditions is solvable, the application of the semigroup method was more technically involved. This comes down to the conditions of the Hille-Yosida theorem. Unlike the (abstract) energy estimate, which was developed specifically for the study of differential equations, the Hille-Yosida theorem is a more abstract result that is not bound to the study of differential equations. As such, the conditions of the energy estimate method are more easily verified when studying a problem like the solvability of the wave equation with initial conditions.

It is possible that there are more complicated problems than the solvability of the wave equation with initial conditions, in which case the Hille-Yosida theorem is too difficult to apply directly. We focused on the problem

$$\begin{cases} \frac{du(t)}{dt} = Au(t) \quad t \in (0,T), \\ u(0) = x, \end{cases}$$

where A is independent of t and the differential equation is homogeneous. As can be seen in Chapter 4 and 5 of [Paz83], it is a lot more involved to generalize to the same setting we used for developing the energy estimate. But when we are able to apply the Hille-Yosida theorem, we find a solution in the form of a semigroup, in which case we gain a lot of information about the solution from the theory of semigroups.

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