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The height zeta function method for rational points on projective space.

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Abstract

We look at n-dimensional rational projective spaces. On these spaces we define a height function and we count the number $N(B)$ of points with height smaller than or equal to a bound B. In particular we are interested in the growth rate of $N(B)$ for large B. We give two different methods to solve this problem. We will generalize the second method to specifically chosen subsets of n-dimensional rational projective spaces.

Contents

1 Introduction

The interest in rational points stems from Elliptical Curve Cryptography, ECC for short. The safety of these ECC algorithms depends on the number of rational points of an elliptic curve on a projective space[1]. This is why we are interested in finding algebraic varieties and other subsets of $\mathbb{RP}^n$ with many rational points[2][3].

In this thesis we look at rational projective spaces of dimension n, denoted $\mathbb{QP}^n$. We define a height function $H : \mathbb{QP}^n \to \mathbb{Z}_{>0}$, which assigns a positive integer to each point on such a space and our aim is to count the number $N(B)$ of points with height smaller than or equal to an arbitrary value called the bound B. Specifically, we are interested in the growth rate of $N(B)$ for large bounds B. The number of points with height at most B will only depend on the space we work with and on the value of B. In this thesis we want to find the exponent E such that $\lim_{B\to\infty} B^{-E}N(B) = C \neq 0$ for $\mathbb{QP}^n$ and specific subsets of $\mathbb{QP}^n$. We will not calculate the constant C explicitly as this constant also depends on the space $\mathbb{QP}^n$ or, if applicable, its subset.

The first goal of this thesis is to find the exponent E for $\mathbb{QP}^n$. The first main theorem we aim to prove states that $E = n + 1$ for $\mathbb{QP}^n$.

Theorem 1. *Let $n \in \mathbb{Z}_{>0}$. The number $N(B)$ of points in $\mathbb{QP}^n$ with height at most B satisfies*

$$\lim_{B\to\infty} B^{-(n+1)}N(B) = C \neq 0$$

for a non-zero constant C.

The second goal of this thesis is to expand Theorem 1 to subsets of $\mathbb{QP}^n$. These subsets X_S are constructed using sets $S \subset \mathbb{Z}_{\geq 0}$ with the requirement that $0 \in S$. The exact definition of X_S can be found in Section 5. The second main theorem of this thesis states that $E = n+1/v$ for $X_S \subset \mathbb{QP}^n$, where $v = \min\{x \in S \setminus \{0\}\}$.

Theorem 2. *Let $S \subset \mathbb{Z}_{\geq 0}$ with $0 \in S$, let $v = \min\{x \in S \setminus \{0\}\}$. The number $N(B)$ of points in $X_S \subset \mathbb{QP}^n$ with height at most B satisfies*

$$\lim_{B\to\infty} B^{-(n+1/v)}N(B) = C \neq 0$$

for a non-zero constant C.

In the Section 2, we describe the main problem more rigorously. We explain what $\mathbb{QP}^n$ is, and we define the height function. In the Section 3, we give the first method of solving the problem for $\mathbb{QP}^n$. This first method uses a different way of thinking about the problem which allows us to solve it using functions from analytic number theory. This method works for $\mathbb{QP}^n$ and allows us to prove Theorem 1. This method even works for some subsets of $\mathbb{QP}^n$, but we do not show this in this thesis. For more general subset of $\mathbb{QP}^n$ this method becomes too complicated and we need a different method. In the Section 4, we develop a second method to prove Theorem 1. For this second method, we first need to introduce p-adic number and fields, which we do in Section 4.1. Next, in Section 4.2 we explain how integration works on p-adic fields and we choose a measure. In Section 4.3, we introduce and prove a Tauberian theorem, which in the end will give the exponent we are looking for. In Section 4.4, we introduce and prove the Poisson Summation Formula. Finally, in Section 4.5, we use both of these theorems along with our knowledge of p-adic fields and integration to prove Theorem 1. This second method can relatively easily be expanded to certain subsets of $\mathbb{QP}^n$. In Section 5, we introduce the aforementioned subsets X_S of $\mathbb{QP}^n$. We start, in Section 5.1, by proving Theorem 2 for subsets $X_S \subset \mathbb{QP}^n$ with $S = \{0, v\}$ for any $v \in \mathbb{Z}_{>0}$. In the next subsection, Section 5.2, we prove Theorem 2 for $X_S \subset \mathbb{QP}^n$ with $S = \{0\} \cup \mathbb{Z}_{\geq v}$ for any $v \in \mathbb{Z}_{>0}$. In Section 5.3, we prove Theorem 2 for sets $X_S \subset \mathbb{QP}^n$ for arbitrary $S \subset \mathbb{Z}_{\geq 0}$ with $0 \in S$. Finally, we give a short outlook on further research that could be done in this field.

2 Problem description

As mentioned in the introduction we will be working with a rational projective space. The rational projective space of dimension n, denoted $\mathbb{QP}^n$, is a subset of $\mathbb{RP}^n$ where we only allow points which have some representation consisting of $n+1$ rational numbers. This can also be thought of as lines through the origin passing through at least one point of $\mathbb{Q}^{n+1} \subset \mathbb{R}^{n+1}$. The height function mentioned in the introduction, denoted $H(x)$ for $x \in \mathbb{QP}^n$, is defined as follows:

Let us use some representation for x by $n+1$ rational numbers $(x_0 : x_1 : \ldots : x_n)$. There exist integers p_i, q_i such that $x_i = p_i/q_i$ for every $0 \leq i \leq n$. Then $(Px_0 : Px_1 : \ldots : Px_n)$ where $P = \prod_{j=0}^{n} q_j$ is a representation of x consisting of integers. Then $(y_0 : y_1 : \ldots : y_n)$ with $y_i = Px_i/\gcd(Px_0, Px_1, \ldots Px_n)$ is a representation of x of coprime integers. Now we define $H(x) = \max\{|y_i| : 0 \leq i \leq n\}$. Note that this procedure always leads to a representation of x consisting of coprime integers. Furthermore, note that any $x \in \mathbb{QP}^n$ has only two representations consisting of coprime integers, these will differ by a factor of -1. This means that the height function always gives the same value independent of the chosen representation of x by coprime integers. This means that the height function is well-defined. Now the main problem we want to solve in this thesis is to find the number of $x \in \mathbb{QP}^n$ for which $H(x) \leq B$ for some bound B. In particular, we want to find its dependence on B for large B.

3 The first method

The first solution to this main problem comes from noticing that any ordered set of $n+1$ coprime integers $(y_0 : y_1 : \ldots : y_n)$ is a representation for one unique $x \in \mathbb{QP}^n$, additionally every x corresponds to two of these ordered sets of coprime integers. This means that the problem reduces to counting ordered sets of coprime integers $(y_0, y_1, \ldots y_n)$ such that $y_i \leq B$ for all i and dividing this by 2. To start the proof, we will look at the case when $n = 1$ which means we need to count how many pairs of coprime integers (y_0, y_1) exist such that $|y_0|, |y_1| \leq B$ for some bound B, for this we will introduce a few functions from analytic number theory.

Definition 1 (Möbius function)**.** The Möbius function of an integer m, denoted $\mu(m)$, is either 0 or ± 1 depending on the prime factors of m in the following way:

$$\mu(m) = \begin{cases} (-1)^{\#\text{prime factors of } m} & \text{if } m \text{ is square-free} \\ 0 & \text{otherwise.} \end{cases}$$

Here m being square-free means that m is not divisible by any square apart from 1. In other words, all of its prime factors only appear once in its prime decomposition. We will also introduce something we will creatively call the χ-function given as follows:

Definition 2 (χ-function)**.** The χ-function of an integer is given by $\chi(m) = \sum_{d>0} \mu(d)\chi_d(m)$ where the functions $\chi_d(m)$ for integers d are given by

$$\chi_d(m) = \begin{cases} 1 & \text{if } d|m \text{ and } m \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We need two results about these functions which we will put in the following lemmas:

Lemma 1. *The Möbius function is multiplicative, meaning that for coprime integers a, b, we have that $\mu(ab) = \mu(a)\mu(b)$.*

Proof. Given two coprime integers a and b, assume that one of them is not square free and let it be a without loss of generality. This means that a is divisible by some square greater than 1, but this means that ab is divisible by the same square. Hence, we have that $\mu(a) = \mu(ab) = 0$, so clearly $\mu(a)\mu(b) = \mu(ab)$.

Now assume both a and b are square-free, since they are coprime, they do not share any prime factors, so the product ab will still be square free. In fact the primes dividing ab will be exactly those dividing a and those dividing b with no overlap, so the number of prime factors of ab is the sum of the number of prime factors of a and the number of prime factors of b, hence we get:

$$\begin{aligned} \mu(ab) &= (-1)^{\#\text{prime factors of } ab} \\ &= (-1)^{\#\text{prime factors of } a + \#\text{prime factors of } b} \\ &= (-1)^{\#\text{prime factors of } a}(-1)^{\#\text{prime factors of } b} \\ &= \mu(a)\mu(b). \end{aligned}$$

This shows that μ is indeed a multiplicative function. □

Lemma 2. *The χ-function has values $\chi(1) = 1$ and $\chi(m) = 0$ for $m \neq 1$.*

Proof. Note that we can rewrite the definition for $\chi(m)$ to be $\chi(m) = \sum_{d|m} \mu(d)$. We use induction on the number of prime factors of m. First of all, we can quickly see that $\mu(1) = 1$, so $\chi(1) = 1$ and $\chi_d(0) = 0$ for all d, so $\chi(0) = 0$. Next, we see that for any prime p, we have $\chi(p) = \mu(1) + \mu(p) = 1 - 1 = 0$. Now we will show that for any $m > 1$ and prime p, $\chi(pm) = 0$ whenever $\chi(m) = 0$. To achieve this, we need to look at the prime factors of m and distinguish two different cases, the first being the case where $p|m$, the second being the case where $p \nmid m$.

Case 1: $p|m$:
In this case notice how every divisor d of pm also divides m or is divisible by p^2, this means that $\mu(d) = 0$ for every divisor d of pm that is not a divisor of m as d is not square-free. Hence,

$$\chi(pm) = \sum_{d|pm} \mu(d) = \sum_{d|m} \mu(d) + \sum_{d|pm, d\nmid m} \mu(d) = \sum_{d|m} \mu(d) = \chi(m).$$

Hence we have shown that in this case $\chi(pm) = \chi(m)$, and by the assumption that $\chi(m) = 0$, we get $\chi(pm) = 0$.

Case 2: $p \nmid m$:
In this case we can split the divisors of pm into two sets, those divisible by p and those not divisible by p. The latter set is the set of divisors of m. Since p and m are coprime, these two sets have the same number of elements. Indeed, if we consider some divisor d of pm with $p \nmid d$, then pd will be a divisor of pm with $p|d$ and vice versa. This means that we can write the following:

$$\chi(pm) = \sum_{d|pm} \mu(d) = \sum_{d|m} \mu(d) + \sum_{d|m} \mu(pd) = \sum_{d|m} (\mu(d) + \mu(p)\mu(d)) = \sum_{d|m} (\mu(d) - \mu(d)) = 0,$$

where we used the fact that for the prime p, $\mu(p) = -1$. This shows that in this case we also have $\chi(pm) = 0$.

We have shown that for $m > 1$, if $\chi(m) = 0$ then $\chi(pm) = 0$ for every prime p. Now we can complete the proof by induction. Assume that we have some number $k > 1$ and some prime $p|k$ assume that for all $1 < k' < k$ we have $\chi(k') = 0$, then we have that either $k/p = 1$ or $\chi(k/p) = 0$ since $k/p < k$. In the first case, we have that $k = p$ for which we already showed that $\chi(p) = 0$. For the second case we use the fact that $k = pk/p$ to see that $\chi(k) = 0$ because $\chi(k/p) = 0$. We conclude that $\chi(1) = 1$ and $\chi(m) = 0$ for $m > 1$. □

These two lemmas are all we need to solve our main problem for the case $n = 1$. From the way we rephrased this problem, it is easy to see that as B goes to infinity, so does the number of points x with $H(x) \leq B$. This leads us to the following proposition.

Proposition 1. *The number $N(B)$ of points $x \in \mathbb{QP}^1$ with $H(x) \leq B$ for some integer B satisfies*

$$\lim_{B\to\infty} N(B)/(2B^2) = \sum_{n>0} \mu(n)/(n^2).$$

Proof. As mentioned before, counting these points will be the same as counting pairs of coprime integers $(y_0 : y_1)$ such that $|y_0|, |y_1| \leq B$ and dividing the total by 2. We will call such a pair $(y_0 : y_1)$ with $|y_0|, |y_1| \leq B$ a valid pair. Now we only have to consider positive integers y_0 and y_1, since if (y_0, y_1) is a valid pair, then so are $(-y_0, -y_1)$, $(-y_0, y_1)$ and $(y_0, -y_1)$. However, this breaks when y_0 or y_1 is 0, but since this only gives four pairs, $(0, \pm 1)$ and $(\pm 1, 0)$, all of which have height 1, we can count them separately. Writing this down in terms of the χ-function, with the notation that $\chi(\gcd(0,0)) := 0$, gives

$$\sum_{i=-B}^{B}\sum_{j=-B}^{B}\chi(\gcd(i,j)) = 4 + 4\sum_{i=1}^{B}\sum_{j=1}^{B}\chi(\gcd(i,j)).$$

As we mentioned earlier, what we have found here is not exactly the number $N(B)$ we are looking for, but it is precisely $2N(B)$, so we have to divide it by 2. This can be done in the following way:

$$\begin{aligned}
N(B) &= \frac{1}{2}\sum_{i=-B}^{B}\sum_{j=-B}^{B}\chi(\gcd(i,j)) \\
&= 2 + 2\sum_{i=1}^{B}\sum_{j=1}^{B}\chi(\gcd(i,j)) \\
&= 2 + 2\sum_{i=1}^{B}\sum_{j=1}^{B}\sum_{d>0}\mu(d)\chi_d(\gcd(i,j)) \\
&= 2 + 2\sum_{d>0}\mu(d)\sum_{i=1}^{B}\sum_{j=1}^{B}\chi_d(\gcd(i,j)) \\
&= 2 + 2\sum_{d>0}\mu(d)\sum_{i=1}^{B}\chi_d(i)\sum_{j=1}^{B}\chi_d(j) \\
&= 2 + 2\sum_{d>0}\mu(d)\left(\sum_{i=1}^{B}\chi_d(i)\right)^2,
\end{aligned}$$

where in the second to last step, we used that $d|\gcd(i,j)$ if and only if $d|i$ and $d|j$. We would like to analyse the behaviour of this function for large B. Since $\sum_{i=1}^{B}\chi_d(i)$ is the number of integers smaller than B which are divisible by d, it always lies between $B/d - 1$ and B/d. By dividing this by B, we can see that the proportion of positive integers below B divisible by d lies between $1/d$ and $1/d - 1/B$. So in the limit as $B \to \infty$, this fraction becomes $1/d$. This means that, if we divide both sides by $2B^2$ and take the limit $B \to \infty$, we get

$$\lim_{B\to\infty}\frac{N(B)}{2B^2} = \lim_{B\to\infty}\frac{1}{2B^2}\left(2 + 2\sum_{d>0}\mu(d)\left(\sum_{i=1}^{B}\chi_d(i)\right)^2\right) = \sum_{d>0}\mu(d)/d^2.$$

This is what we needed to prove. □

Now that we have done the one-dimensional case, we can quite easily see how it generalizes to n dimensions. The only thing that changes is the number of sums we have in the proof as we show in the following theorem.

Theorem 3. *For an n-dimensional rational projective space $\mathbb{QP}^n$, we have that the number $N(B)$ of points x such that $H(x) \leq B$ satisfies*

$$\lim_{B\to\infty} N(B)/(2B)^{n+1} = \sum_{d>0}\mu(d)/(2d^{n+1}).$$

Proof. The proof is very similar to the 1-dimensional case, the only difference is that we are now interested in a quantity with $n+1$ sums:

$$\frac{1}{2}\sum_{i_0=-B}^{B}\cdots\sum_{i_n=-B}^{B}\chi(\gcd(i_0,\dots i_n)),$$

where we again define $\gcd(0,0\dots,0)=0$. In this case however it is easier to skip the step where we only consider the positive part of the sum to find:

$$\begin{aligned}
N(B) &= \frac{1}{2}\sum_{i_0=-B}^{B}\cdots\sum_{i_n=-B}^{B}\chi(\gcd(i_0,\dots i_n)) \\
&= \frac{1}{2}\sum_{i_0=-B}^{B}\cdots\sum_{i_n=-B}^{B}\sum_{d>0}\mu(d)\chi_d(\gcd(i_0,\dots i_n)) \\
&= \frac{1}{2}\sum_{d>0}\mu(d)\left(\sum_{i_0=-B}^{B}\chi_d(i_0)\dots\sum_{i_n=-B}^{B}\chi_d(i_n)\right) \\
&= \frac{1}{2}\sum_{d>0}\mu(d)\left(\sum_{i=-B}^{B}\chi_d(i)\right)^{n+1}.
\end{aligned}$$

We can see that, as for large B, the number of integers between $-B$ and B divisible by d lies between $2(B/d-1)$ and $2B/d$. So the proportion of integers between $-B$ and B divisible by d lies between $1/d$ and $1/d-1/B$. Taking the limit as $B\to\infty$ gives us that this proportion becomes $1/d$. Hence, we can divide by $(2B)^{n+1}$ and calculate the limit:

$$\begin{aligned}
\lim_{B\to\infty}\frac{N(B)}{(2B)^{n+1}} &= \lim_{B\to\infty}\frac{1}{2(2B)^{n+1}}\sum_{i_0=-B}^{B}\cdots\sum_{i_n=-B}^{B}\chi(\gcd(i_0,\dots i_n)) \\
&= \lim_{B\to\infty}\frac{1}{2(2B)^{n+1}}\sum_{d>0}\mu(d)\left(\sum_{i=-B}^{B}\chi_d(i)\right)^{n+1} \\
&= \sum_{d>0}\frac{\mu(d)}{2d^{n+1}}.
\end{aligned}$$

□

These constant terms in the form of $\sum_{d>0}\mu(d)/(2d^{n+1})$ can be calculated but we will not show that here, what is important for us however is that they converge. This is quite easy to check as $|\mu(d)|\leq 1$, so $\sum_{d>0}\left|\frac{\mu(d)}{2d^{n+1}}\right|\leq\sum_{d>0}\frac{1}{d^{n+1}}$, which is famously known to converge for $n>0$, the proof of which can be found in theorem 1.59 of [4]. As we can see this method gives us the behavior of $N(B)$ for large B. However, we want to expand on this by taking various subsets of $\mathbb{QP}^n$ and this method relies on the fact that we use all points of $\mathbb{QP}^n$. This is the reason we will introduce another method to get the same answer for $N(B)$ when we allow all $x\in\mathbb{QP}^n$, but this new method will be expandable to certain subsets of $\mathbb{QP}^n$.

4 The second method

The second method that we will discuss hinges on two main theorems, one is called the Poisson Summation Formula and the other one is a special version of the so-called Tauberian Theorems. Before we use these theorems, we need to set up the mathematical framework using p-adic numbers and p-adic fields, both of which we will refer to as p-adic spaces. This will be done in the following section.

4.1 p-adic spaces

We will start with the definition of p-adic spaces. To understand this definition we need to understand what the completion of a metric space M with respect to its norm $||\dots||$ is. The completion of M is given by the following

$$\{(x_n)_{n\in\mathbb{N}} \text{ is a Cauchy sequence}\}/\sim,$$

where for two Cauchy sequences $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$, $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}}$ if and only if $\lim_{n\to\infty} ||x_n - y_n|| = 0$. This allows us to write down the definition.

Definition 3. The ring of p-adic numbers and the p-adic field are the completions of $\mathbb{Z}$ and $\mathbb{Q}$, respectively, with respect to the norm $||\dots||_p$.

This norm $||\cdot||_p : \mathbb{Q} \to \mathbb{R}_>0$ is defined in the following way. For some nonzero integer $n \in \mathbb{Z}$, which can be written as $p^e n'$ for some $e \in \mathbb{N}$ and n' such that $p \nmid n'$, we define $||n||_p = p^{-e}$. Additionally, we have that $||0||_p = 0$. For some rational number $x \in \mathbb{Q}$, which can be written as n/m with n, m integers, which can again be written as $n = p^e n'$ and $m = p^{e'} m'$ with again $p \nmid n'$ and $p \nmid m'$, then the norm of x will be $||x||_p = p^{e'-e}$. Note that this is well-defined since if we take a different representation for the fraction, we multiply both n and m by λ for some integer λ. If $p \nmid \lambda$, it is clear that this will not change the norm. If $p|\lambda$, let ϵ be the largest integer such that $p^\epsilon|\lambda$, then we can see that e turns into $e+\epsilon$ and e' turns into $e'+\epsilon$. So we see that $e-e'$ will remain $e-e'$ so the norm will stay the same. For completeness we will show that $||\cdot||_p$ is actually a norm on $\mathbb{Q}$ which will then immediately imply that it is also a norm on $\mathbb{Z}$. The definition of a norm can be found in [5], where it is definition 1.1.

Lemma 3. *The function $||\cdot||_p$ is actually a norm.*

Proof. For $||\cdot||_p$ to be a norm, we have to show that $||x||_p = 0$ if and only if $x = 0$, which is trivial from the definition since $p^{-e} \neq 0$ for every $e \in \mathbb{Z}$. Next we have to show that $||x||_p \geq 0$, which again we always have since $p^{-e} > 0$ for every $e \in \mathbb{Z}$.

Then we also need to show that $||x+y||_p \leq ||x||_p + ||y||_p$ for all $x, y \in \mathbb{Q}$, but we will show something stronger instead. We will show that $||x+y||_p \leq \max\{||x||_p, ||y||_p\}$. Let us write $x = n_1/m_1$ and $y = n_2/m_2$ with n_i, m_i integers for $i \in \{1,2\}$. We write these as $n_i = p^{e_i} n_i'$ and $m_i = p^{e_i'} m_i'$ with $p \nmid n_i'$ and $p \nmid m_i'$ for again $i \in \{1,2\}$. We can easily see now that $||x||_p + ||y||_p = p^{e_1'-e_1} + p^{e_2'-e_2}$, and

$$x + y = \frac{p^{e_1} n_1'}{p^{e_1'} m_1'} + \frac{p^{e_2} n_2'}{p^{e_2'} m_2'} = \frac{p^{e_2+e_1'} m_1' n_2' + p^{e_1+e_2'} m_2' n_1'}{p^{e_1'+e_2'} m_1' m_2'}.$$

Note that $p \nmid m_1' m_2'$. We can assume without loss of generality that $(e_1 + e_2') - (e_2 + e_1') = \delta_e \geq 0$. Then we can write

$$x + y = \frac{p^{e_2+e_1'}(m_1' n_2' + p^{\delta_e} m_2' n_1')}{p^{e_1'+e_2'} m_1' m_2'}.$$

From this and the fact that $p \nmid m_1' n_2'$, we get that if $\delta_e \neq 0$, then $p \nmid m_1' n_2' + p^{\delta_e} m_2' n_1'$, and hence $||x + y||_p = p^{(e_1'+e_2')-(e_2+e_1')} = ||y||_p$, while if $\delta_e = 0$, we have that $||x + y||_p \leq p^{(e_1'+e_2')-(e_2+e_1')} = ||y||_p$. If $\delta_e < 0$, a similar calculation shows that $||x + y||_p = ||x||_p$. As we can see in all cases we have that $||x + y||_p \leq \max\{||x||_p, ||y||_p\}$.

Lastly, we need to show that $||xy||_p = ||x||_p ||y||_p$ which can be seen as follows using the same notation as for the previous point:

$$||xy||_p = \left|\left|\frac{p^{e_1} n_1' p^{e_2} n_2'}{p^{e_1'} m_1' p^{e_2'} m_2'}\right|\right|_p = p^{e_1'+e_2'-e_1-e_2} = ||x||_p ||y||_p,$$

where we used that since n_1', n_2', m_1' and m_2' are not divisible by p, neither are $n_1' n_2'$ and $m_1' m_2'$. □

Next we want to find better representations for the elements of $\mathbb{Z}_p$ and $\mathbb{Q}_p$, since the definition using Cauchy sequences is not easy to work with. The easiest way to do this is by noticing that any Cauchy sequence in $\mathbb{Q}_p$ and $\mathbb{Z}_p$ can be written as $(x_n)_{n\in\mathbb{N}}$ with $x_n = \sum_{i=0}^{n} a_i p^i$ with $a_i \in \mathbb{Q}$ and $a_i \in \mathbb{Z}$ respectively. This is clearly a Cauchy sequence as $p^n | x_m - x_n$ for any $n, m \in \mathbb{N}$ with $n < m$, so $||x_m - x_n||_p \leq p^{-n}$. Furthermore, in section 4.4 of [6], it is proven that all elements of $\mathbb{Q}_p$ and $\mathbb{Z}_p$ are of the form $\sum_{i=0}^{\infty} a_i p^i$ with $a_i \in \mathbb{Q}$ and $a_i \in \mathbb{Z}$ respectively. But we can do even better than this and give a nicer representation, since there are multiple representations of this form that give the same elements in $\mathbb{Q}_p$ or $\mathbb{Z}_p$, for example in $\mathbb{Z}_3$ we have $5 \cdot 3^0 = 2 \cdot 3^0 + 1 \cdot 3^1$.

Proposition 2. *All elements in $\mathbb{Z}_p$ can be represented by $\sum_{i=0}^{\infty} x_i p^i$ with $0 \leq x_i < p$ for all $i \geq 0$, and every element of $\mathbb{Q}_p$ can be represented by $\sum_{i=-m}^{\infty} x_i' p^i$ for some integer m and $0 \leq x_i' < p$ for all $i \geq -m$.*

Before we prove this proposition, we require one small lemma.

Lemma 4. *Let p, q be two distinct prime numbers, then we have that $1/q \in \mathbb{Z}_p$.*

Proof. All we have to do is to show that $1/q$ will have a representation of the form $\sum_{i=-m}^{\infty} x_i p^i$ for any prime number $q \neq p$. We can use the fact that p and q are coprime to see that q is invertible in $\mathbb{Z}/p^i\mathbb{Z}$ for every $i \geq 1$. Write $a_i \equiv 1/q \mod p^i$ with $0 < a_i < p^i$, so $p^i | a_i q - 1$. This means that $p^i | q(a_i - a_{i+1})$, and, since q and p^i are coprime, $p^i | (a_i - a_{i+1})$. Let us now look at $\sum_{i=0}^{\infty} x_i p^i$ where $x_i = (a_{i+1} - a_i)/p^i$ for $i > 0$ and $x_0 = a_1$. Then $\sum_{i=0}^{n} x_i p^i = a_{n+1}$ for all $n \geq 0$, and hence, $p^{n+1} | \left(1 - q \sum_{i=1}^{n} x_i p^i\right)$. This means that $||1/q - \sum_{i=0}^{n} x_i p^i||_p \leq p^{-(n+1)}$. So taking the limit as $n \to \infty$, we get $||1/q - \sum_{i=0}^{n} x_i p^i||_p \to 0$. This means that the Cauchy sequence $(\sum_{i=0}^{n} x_i p^i)_{n\in\mathbb{N}}$ converges to $1/q$, and implies that $1/q \in \mathbb{Z}_p$ for all $q \neq p$. □

We can now prove Proposition 2.

Proof. For $\mathbb{Z}_p$ the statement is obvious, since every element of $\mathbb{Z}_p$ is a Cauchy sequence, which is of the form $\sum_{i=0}^{\infty} x_i p^i$ with $x_i \in \mathbb{Z}$. For every i such that $x_i \geq p$ or $x_i < 0$, we can write $x_i = r * p + b$ with $0 \leq b < p$, we can adjust our representations to $\sum_{i=0}^{\infty} x_i' p^i$ by setting $x_i' = b$, $x_{i+1}' = x_{i+1} + r$ and $x_j' = x_j$ for all $j \neq i, i+1$. If we first do this for $i = 0$ and keep doing it for increasing i we will find a representation we want. We can do this since if we generate a new representation by doing this starting from $i = 0$ until $i = N$ for some integer N, the difference of the first N terms of both of these representations will be divisible by p^N, so the norm will be smaller or equal to p^{-N}. So if we set $x_i' = 0$ for $i > N$ and let $N \to \infty$, we obtain the representation we want. This limit exists because $\sum_{i=0}^{N}(x_i' p^i - x_i p^i)$ is a Cauchy sequence with limit 0.

For $\mathbb{Q}_p$, we will use the fact that any element of Z_p has a representation of the form we want. Note that for any prime number q, $1/q$ has a representation of the form $\sum_{i=-m}^{\infty} x_i p^i$ for some integer m and $0 \leq x_i < p$ for all $i \geq -m$. For $q \neq p$, this follows from the fact that by Lemma 4, $1/q \in Z_p$ and for $q = p$, we have $x_{-1} = 1$ and $x_i = 0$ for $i \neq -1$. Next we can see that for any a/b with $a, b \in \mathbb{Z}$, we can use the prime factorization of b. We know a has a representation of the form $\sum_{i=-m}^{\infty} x_i p^i$ and so will $1/q$ for any q in the prime factorization of b. Next we notice that the product of two these representations will be again of the same form, which can be shown as follows

$$\sum_{i=-n}^{\infty} x_i p^i \sum_{j=-m}^{\infty} x_j' p^j = \sum_{i=-n}^{\infty} \sum_{j=-m}^{\infty} x_i x_j' p^{i+j} = \sum_{i=-(n+m)}^{\infty} x_i'' p^i$$

with $x_i'' = \sum_{j=-n}^{i+m} x_j x_{i-j}'$. This representation only has the flaw that x_i'' might not be between 0 and $p-1$, but as we saw in the proof for $\mathbb{Z}_p$, we can change the representation to solve this problem. All of this together shows that a/b will also have a representation of the form in the proposition.

□

This representation becomes especially useful when we perform integration which we will introduce in the next subsection. Finally in this section we will calculate the set of unit elements of $\mathbb{Z}_p$, which we will denote $\mathbb{Z}_p^*$.

Proposition 3. *The set of unit elements of $\mathbb{Z}_p$, denoted $\mathbb{Z}_p^* \subset \mathbb{Z}_p$ are exactly those elements with representation $\sum_{i=0}^{\infty} x_i p^i$, with $0 \leq x_i < p$ for all i, where $x_0 \neq 0$.*

Proof. First of all we note that the prime $p \in \mathbb{Z}_p$ is not invertible. Next note that for any element of $\mathbb{Z}_p$ with representation $\sum_{i=0}^{\infty} x_i p^i$, the sum $\sum_{i=1}^{n} x_i p^i$ for some $n \in \mathbb{N}$ is just an integer. Furthermore since $x_0 \neq 0$, $\sum_{i=0}^{n} x_i p^i$ is coprime with p^k for all $1 \leq k \leq n$. This means it has an inverse in $\mathbb{Z}/p^k\mathbb{Z}$, let us define $a_{k-1} \equiv \left(\sum_{i=0}^{n} x_i p^i\right)^{-1} \mod p^k$. We can see that $p^k | (a_k - a_{k-1})$ since $p \nmid \sum_{i=0}^{n} x_i p^i$, but $p^k | (a_{k-1} - a_k) \sum_{i=0}^{n} x_i p^i$ and p is prime. Furthermore since $a_k \leq p^{k+1}$, we have that $0 \leq (a_k - a_{k-1})p^{-k} \leq p$. Let us now define $y_k = (a_k - a_{k-1})p^{-k}$ and $y_0 = a_0$. Then we see that $\sum_{j=1}^{n} y_j p^j \sum_{i=0}^{n} x_i p^i \equiv 1 \mod p^{n+1}$. If we now let n go to ∞, we get that $\sum_{i=0}^n y_i p^i$ is the inverse of $\sum_{i=0}^n x_i p^i$. This shows that if $x_0 \neq 0$, $\sum_{i=0}^n x_i p^i$ is a unit of $\mathbb{Z}_p$. If however $x_0 = 0$, let $r > 0$ be the smallest integer such that $x_r \neq 0$, then we can write $\sum_{i=0}^n x_i p^i = \sum_{i=r}^n x_i p^i = p^r \sum_{i=0}^n x_{i+r} p^i$. This is now a product of a unit and p^r, which is not a unit for $r > 0$, so this will not be a unit. Hence every unit of $\mathbb{Z}_p$ is of the form $\sum_{i=0}^n x_i p^i$ with $x_0 \neq 0$. □

4.2 p-adic integration

We will choose to use the Haar Measure, the definition of which can be found in section 2.2 of [7], normalized such that $\int_{\mathbb{Z}_p} \mathrm{d}x = 1$, which makes sense in both spaces $\mathbb{Z}_p$ and $\mathbb{Q}_p$ as $\mathbb{Z}_p \subset \mathbb{Q}_p$. This leads to the following lemma.

Lemma 5. *The set of unit elements $\mathbb{Z}_p^* \subset \mathbb{Z}_p$ has volume $1 - 1/p$. The volume of $p^i\mathbb{Z}_p^* \subset \mathbb{Q}_p$ is $(1 - 1/p)p^{-i}$ for every $i \in \mathbb{Z}$.*

Proof. The proof follows directly from the fact that $\int_{\mathbb{Z}_p} \mathrm{d}x = 1$. Since $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, the volume of $p\mathbb{Z}_p$ can be calculated in the following way:

$$\int_{x\in p\mathbb{Z}_p} \mathrm{d}x = \int_{y\in\mathbb{Z}_p} \mathrm{d}(py) = \int_{y\in\mathbb{Z}_p} ||p||_p \mathrm{d}y = 1/p \int_{y\in\mathbb{Z}_p} \mathrm{d}y = 1/p.$$

In the second equality we used a property of the Haar measure, which is $\mathrm{d}(ax) = ||a||_p \mathrm{d}x$ and can be found in section 2.2 of [7]. This means that the volume of $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$ is $1 - 1/p$. Next we calculate the volume of $p^i\mathbb{Z}_p^* \subset \mathbb{Q}_p$ using a similar substitution:

$$\int_{x\in p^i\mathbb{Z}_p^*} \mathrm{d}x = \int_{y\in\mathbb{Z}_p^*} \mathrm{d}(p^i y) = \int_{y\in\mathbb{Z}_p^*} ||p^i||_p \mathrm{d}y = p^{-i} \int_{y\in\mathbb{Z}_p^*} \mathrm{d}y = p^{-i}(1 - 1/p).$$

□

4.3 Poisson Summation Formula

To understand what the Poisson Summation Formula is and where we can use it, we will use the integration to define Fourier transformations on p-adic spaces. The space where we will perform these Fourier transformations is not $\mathbb{Q}_p$, instead it will be a restricted product of copies of $\mathbb{Q}_p$ for different primes, which we will call V. For notation we write P for the set of all prime numbers, and we write $\mathbb{Q}_\infty = \mathbb{R}$ and $\mathbb{Z}_\infty = \mathbb{Z}$. Then we define:

$$V = \left\{ (x_p)_{p\in P\cup\infty} \in \prod_{p\in P\cup\infty} \mathbb{Q}_p : x_p \in \mathbb{Z}_p \text{ for all but finitely many } p \in P \cup \infty \right\}.$$

As we mentioned, we want to perform Fourier transformations in this space. Intuitively this should work similar to Fourier transformations over the field $\mathbb{R}$, where we would have:

$$\hat{f}(\xi) = \int f(x) e^{2\pi\iota\xi x} \mathrm{d}x.$$

The theory behind this in $\mathbb{R}$ can be found in [8]. We want to do this for V, so we need to take $x \in V$ and $\xi \in \mathbb{Q}$. We cannot just fill this in since multiplying elements from $\mathbb{Q}$ and V is not yet defined, but we can define it as follows as $\mathbb{Q} \subset \mathbb{Q}_\infty$ and $\mathbb{Q} \subset \mathbb{Q}_p$ for every prime p. For $\xi \in \mathbb{Q}$ and $x = (x_n)_{n \in P \cup \infty} \in V$, $x\xi = (x_n\xi)_{n \in P \cup \infty}$. So we multiply every element of the sequence x by the rational number ξ. This solves most of our problems, but not all of them, since raising e to the power of $x\xi$ is not yet defined. To fix this, we will insert an extra function which will send $x\xi$ to a real number, by taking what we will call the fractional part:

Definition 4. The fractional part of a number $x \in \mathbb{R}$, denoted by $\{x\}_\infty \in [0,1)$, is the unique number on this interval such that $x \equiv \{x\}_\infty \mod 1$. On $\mathbb{Q}_p$ the fractional part $\{x\}_p$ of $x = \sum_{i=-m}^{\infty} x_i p^i$ is the unique rational number $\xi \in \mathbb{Q} \subset \mathbb{Q}_p$ such that ξ has the representation $\sum_{i=-m}^{-1} x_i p^i$. Note that this means that we can see $\{x\}_p$ as an element of $\mathbb{R}$ since it is a finite sum of rational numbers.

We call this the fractional part of a number since $\{\cdot\}_p$ is a function from $\mathbb{Q}_p$ to $\mathbb{R}$ that is periodic with period 1 in the sense that $\{x\}_p = \{x+1\}_p$. We can easily extend this function to V by defining for any $x = (x_n)_{n \in P \cup \infty} \in V$, $\{x\} = \sum_{p \in P \cup \infty} \{x_p\}_p \in \mathbb{R}$. Note that this sum indeed gives an element of $\mathbb{R}$, because by definition $x_p \notin \mathbb{Z}_p$ for only finitely many p, so $\{x_p\}_p \neq 0$ for only finitely many $p \in P$. And since $\{x\} \in \mathbb{R}$, $e^{\{x\}}$ is defined.

The next thing we need to discus is how the integrals work on V. This is just as we would expect, where for $x = (x_n)_{n \in P \cup \infty} \in V$, $\mathrm{d}x = \prod_{n \in P \cup \infty} \mathrm{d}x_n$. Lastly we have to define the domain of integration of the Fourier transform. If we look for inspiration at the Fourier transform over $\mathbb{R}$ again when we have periodic functions, we set the integration bounds to 0 and 1 since the period is 1. We will be interested in periodic functions, where for any $\xi \in \mathbb{Q}$ and $x \in V$, $f(x) = f(x + (\xi)_{p \in P \cup \infty})$. Naturally it would make sense to choose our domain of integration D such that if $x \in D$, $x + \xi \notin D$ for any $\xi \in \mathbb{Q}$. So if we denote the integration region by $D \subset V$, then we require that if $x \in D$, $x + \xi \notin D$ for all $0 \neq \xi \in \mathbb{Q}$. This defines the following set:

$$D = V/\sim \qquad \text{with } x \sim y \iff \exists z \in \mathbb{Q} \text{ such that } (x-y)_n = z \,\forall n \in P \cup \infty.$$

There is an easier way to describe the set D as shown in the next proposition.

Proposition 4. *Every equivalence class in the set D has precise one representation $[x]$ with $x \in D' := [0,1) \times \prod_{p \in P} \mathbb{Z}_p$.*

Proof. Since $D' \subset V$, we have to show that for any two different $x', y' \in D'$, we cannot have that $x' \sim y'$, and that for every $y \in V$, there exists an $x \in D'$ such that $x \sim y$. First of all let us take $x', y' \in D'$. Then $(x'-y')_p \in \mathbb{Z}_p$ for all $p \in P$ as $(x')_p, (y')_p \in \mathbb{Z}_p$, additionally $-1 < (x'-y')_\infty < 1$. Now let us assume that there exists some $z \in \mathbb{Q}$ such that for all $p \in P \cup \infty$, $(x'-y')_p = z$. Write $z = a/b$ with a, b coprime integers and let q be a prime factor of b. Then $(x'-y')_q = \sum_{i=-m}^{\infty} x_i q^i$ with $m > 0$, hence $(x'-y')_q \notin \mathbb{Z}_q$, so we have a contradiction. This means that b cannot have any prime factors, and hence z will have to be an integer. This means that $(x'-y')_\infty = z$ is an integer, and hence $z = 0$. So we get that

$x' = y'$. Hence we cannot have two distinct $x', y' \in D'$ such that $x' \sim y'$.

Now let us take $y \in V$. We can write $y_p = \sum_{i=-m}^{\infty} x_i p^i$. Since $y \in V$, $m > 0$ for only finitely many p, hence, $\{y_p\}_p \neq 0$ for only finitely many p, and so $\sum_{p \in P} \{y_p\}_p$ is finite. Next we can use Lemma 4 to see that if p and q are distinct primes, $1/p \in \mathbb{Z}_q$. This means that $\{y_p\}_p \in \mathbb{Z}_q$ for any prime $q \neq p$, as $\{y_p\}_p = \sum_{i=-m}^{0} x_i p^i$ and $p^i \in \mathbb{Z}_q$ for any $-m \leq i \leq 0$. Additionally if we look at $\{y_p\}_p \in \mathbb{Q}$ as element of $\mathbb{Q}_p$, we see that $y_p - \{y_p\}_p \in \mathbb{Z}_p$, which gives us $y_p - \sum_{p' \in P} \{y_{p'}\}_{p'} \in \mathbb{Z}_p$ for every prime p. If we now take a look at $\sum_{q \in P} \{y_q\}_q$, we can view this sum as element of V using the fact that we can view any element $x \in \mathbb{Q}$ as element of $\mathbb{Q}_p$ for any $p \in P \cup \{\infty\}$. This also means we can view it as element of V under the map $\mathbb{Q} \to V$ given by $x \mapsto (x)_{p \in P \cup \{\infty\}}$. Using this sum we can construct the element $y - \sum_{q \in P} \{y_q\}_q \in V$ which is equivalent to y under the relation $\sim$, however it does not quite lie in D' yet, for this we need to have that $(y - \sum_{q \in P} \{y_q\}_q)_\infty \in [0, 1)$, but this is quite easy to fix, since we can just take:

$$x = y - \sum_{q \in P} \{y_q\}_q - \lfloor (y - \sum_{q \in P} \{y_q\}_q)_\infty \rfloor.$$

Where we again bring both sums from $\mathbb{Q}$ to V using the same map as before. We can see that $x \in V$ as the sum $\sum_{q \in P} \{y_q\}_q$ only has a finite number of non-zero terms. Furthermore, we clearly have that $x \sim y$ and we see that $x \in D'$ as argued above. □

This proposition means we can use the definition D' instead of more abstract definition of D for any calculations. The first thing we can easily compute using this new description for D is its volume.

Corollary 1. *The volume of D is 1, meaning that $\int_D dx = 1$.*

Proof. The proof becomes easy when we write D as D':

$$\int_D \mathrm{d}x = \int_{D'} \mathrm{d}x = \int_0^1 \mathrm{d}x_\infty \prod_{p \in P} \int_{\mathbb{Z}_p} \mathrm{d}x_p = 1.$$

This last step comes from the fact that $\int_{\mathbb{Z}_p} \mathrm{d}x_p = 1$ for all $p \in P$. □

With this we are almost ready to introduce the two lemmas we will need for the Poisson Summation Formula, but first we need to define the Fourier transform of a function f on V. To introduce the Fourier transform, we need to be able to define continuous functions V and D, so we need a topology on V and D. The spaces $\mathbb{Q}_p$ are metric spaces and as such get their topology from the norm. The space V inherits the product topology from $\mathbb{R}$ and the spaces $\mathbb{Q}_p$, and finally the space D inherits the quotient topology from V.

Definition 5. Let $\phi : D \to \mathbb{R}$ be a continuous function, periodic with period ξ for all $\xi \in \mathbb{Q}$. The Fourier transform of ϕ, denoted $\hat{\phi}$ is given by:

$$\hat{\phi}(\xi) = \int_D \phi(x) e^{-2\pi\iota\{\xi x\}} \mathrm{d}x.$$

For non-periodic but still continuous functions $f : V \to \mathbb{R}$, we have that the Fourier transform is given by:

$$\hat{f}(\xi) = \int_V f(x)e^{-2\pi\iota\{\xi x\}}\mathrm{d}x.$$

This definition is well-defined, taking a different representation of $D \subset V$ gives the same result. The way to check this is to check if the integrant is invariant under addition of ξ for all $\xi \in \mathbb{Q}$. This clearly does not change ϕ, since it is periodic, so $\phi(x) = \phi(x + \eta)$, however we have to check if $e^{2\pi\iota\{\xi x\}} = e^{2\pi\iota\{\xi x+\xi\eta\}}$. For this we have the following proposition.

Proposition 5. *For every rational number $\eta \in \mathbb{Q}$, $\{\eta\} \in \mathbb{Z}$.*

Proof. Let us first write $\eta = a/b$ with a, b coprime. Note that we can assume $0 < \eta < 1$. Indeed, any integer $k \in \mathbb{Z}$ has the property that $k \in \mathbb{Z}_p$ for every $p \in P$. This means that if we look at the definition of $\{\eta\}_p$, we can see that $\{\eta + k\}_p = \{\eta\}_p$ for all $p \in P$, additionally $\{k + \eta\}_\infty = \{\eta\}_\infty$, so we have that $\{k + \eta\} = \{\eta\}$ for every integer k. This means that we can assume that $a, b > 0$ and $a < b$. Let the prime factorisation of b be given by $b = \prod_{i=1}^n q_i^{e_i}$ for primes q_i and integers e_i. Note that if $p \neq q_i$ for all i, then $a/b \in \mathbb{Z}_p$, so $\{a/b\}_p = 0$. This means that:

$$\{\eta\} = \left\{a\prod_{i=1}^n q_i^{-e_i}\right\}_\infty + \sum_{j=1}^n \left\{a\prod_{i=1}^n q_i^{-e_i}\right\}_{q_j} = a\prod_{i=1}^n q_i^{-e_i} + \sum_{j=1}^n \left\{a\prod_{i=1}^n q_i^{-e_i}\right\}_{q_j}.$$

If we look back at the proof of Proposition 2, we can see that:

$$\left\{a\prod_{i=1}^n q_i^{-e_i}\right\}_{q_j} = b_j q_j^{-e_j} \qquad \text{where} \qquad b_j \equiv a q_j^{e_j}\prod_{i=1}^n q_i^{-e_i} \mod q_j^{e_j}.$$

With $0 \leq b_j \leq q_j^{e_j}$ for all $0 < j \leq n$. This means that $b_j q_j^{-e_j}\prod_{i=1}^n q_i^{e_i} \equiv a \mod q_j^{e_j}$. Combining this all we get that:

$$\begin{aligned}
\{\eta\} &= -a\prod_{i=1}^n q_i^{-e_i} + \sum_{j=1}^n \left\{a\prod_{i=1}^n q_i^{-e_i}\right\}_{q_j} \\
&= -a\prod_{i=1}^n q_i^{-e_i} + \sum_{j=1}^n b_j q_j^{-e_j} \\
&= \left(-a + \sum_{j=1}^n b_j q_j^{-e_j}\prod_{k=1}^n q_k^{e_k}\right)\prod_{i=1}^n q_i^{-e_i}.
\end{aligned}$$

Note that $q_i^{e_i} | b_j q_j^{-e_j}\prod_{k=1}^n q_k^{e_k}$ when $i \neq j$, and $q_i^{e_i} | b_j q_j^{-e_j}\prod_{k=1}^n q_k^{e_k} - a$ when $i = j$. These two together mean that $q_i^{e_i} | -a + \sum_{j=1}^n b_j q_j^{-e_j}\prod_{k=1}^n q_k^{e_k}$ for all $0 < i \leq n$. This means that $\{\eta\}$ is an integer. ☐

Proposition 5 has shown that for any rational number η, $\{\eta\}$ is an integer, so $e^{2\pi\iota\{\eta\}} = 1$. The next lemma shows the Fourier inversion formula holds and behaves as expected.

Lemma 6. *Given a continuous function $\phi : V \to \mathbb{R}$, periodic with period ξ for all $\xi \in \mathbb{Q}$, assume that $\sum_{\xi \in \mathbb{Q}} |\hat{\phi}(\xi)| < \infty$, then:*

$$\phi(x) = \sum_{\xi \in \mathbb{Q}} \hat{\phi}(\xi) e^{2\pi\iota\xi x}.$$

Proof. The proof requires more measure theory than we would like to go into in this thesis, it can be found in theorem 4.21 of [7] or Lemma 4.2.2 in [9]. □

Lemma 7. *If $f : V \to \mathbb{R}$ is continuous with an existing Fourier transform $\hat{f}$ and $\sum_{\eta \in \mathbb{Q}} f(x+\eta)$ is uniformly convergent for all $x \in D$, then $\phi(x) = \sum_{\eta \in \mathbb{Q}} f(x+\eta)$ is a periodic function from $V \to \mathbb{R}$ and has the property that $\hat{\phi}(\xi) = \hat{f}(\xi)$.*

Proof. The proof is quite straight-forward and involves just calculations and a few small substitutions:

$$\begin{aligned}
\hat{\phi}(\xi) &= \int_D \phi(x) e^{-2\pi\iota\{\xi x\}} \mathrm{d}x \\
&= \int_D \sum_{\eta \in \mathbb{Q}} f(x+\eta) e^{-2\pi\iota\{\xi x\}} \mathrm{d}x \\
&= \sum_{\eta \in \mathbb{Q}} \int_D f(x+\eta) e^{-2\pi\iota\{\xi x\}} \mathrm{d}x \\
&= \sum_{\eta \in \mathbb{Q}} \int_{D+\eta} f(x) e^{-2\pi\iota\{\xi x - \xi\eta\}} \mathrm{d}x \\
&= \sum_{\eta \in \mathbb{Q}} e^{2\pi\iota\{\xi\eta\}} \int_{D+\eta} f(x) e^{-2\pi\iota\{\xi x\}} \mathrm{d}x.
\end{aligned}$$

The exchange of sum and integral is justified because $\sum_{\eta \in \mathbb{Q}} f(x+\eta)$ is uniformly convergent, so the sum inside the integral is also uniformly convergent. This justifies the exchange as for every $\epsilon > 0$, we can find some finite subset $Q(\epsilon)$ of $\mathbb{Q}$ such that $\sum_{\eta \in \mathbb{Q} \setminus Q(\epsilon)} f(x+\eta) \leq \epsilon$, and uniform convergence makes $Q(\epsilon)$ independent of x. This means that

$$\sum_{\eta \in \mathbb{Q} \setminus Q(\epsilon)} \int_D f(x+\eta) e^{-2\pi\iota\{\xi x\}} \mathrm{d}x < \int_D \epsilon \mathrm{d}x = \epsilon.$$

Next we note that $\xi\eta$ is a rational number, so $\{\xi\eta\}$ is an integer, so $e^{-2\pi\iota\{\xi\eta\}} = 1$, this gives us:

$$\hat{\phi}(\xi) = \sum_{\eta \in \mathbb{Q}} \int_{D+\eta} f(x) e^{-2\pi\iota\{\xi x\}} \mathrm{d}x = \int_V f(x) e^{-2\pi\iota\{\xi x\}} \mathrm{d}x = \hat{f}(\xi).$$

□

Now we have all we need to prove the Poisson Summation Formula which says the following.

Theorem 4 (Poisson Summation Formula)**.** *Let $f : V \to \mathbb{R}$ be a continuous function with Fourier transform $\hat{f}$. Let $\sum_{\xi \in \mathbb{Q}} f(x+\xi)$ be uniformly convergent for all $x \in D$, furthermore let $\sum_{\xi \in \mathbb{Q}} |\hat{f}(\xi)|$ be convergent. Then:*

$$\sum_{\xi \in \mathbb{Q}} \hat{f}(\xi) = \sum_{\xi \in \mathbb{Q}} f(\xi).$$

Proof. The main difficulty of this proof lies in the two the previous lemmas which we have already proven. Let $\phi(x) = \sum_{\xi \in \mathbb{Q}} f(x+\xi)$. Since it is absolutely convergent, we can use Lemma 7 to see that $\hat{\phi}(\xi) = \hat{f}(\xi)$, so we have $\sum_{\xi \in \mathbb{Q}} \hat{f}(\xi) = \sum_{\xi \in \mathbb{Q}} \hat{\phi}(\xi)$. We also have $\sum_{\xi \in \mathbb{Q}} |\hat{f}(\xi)| = \sum_{\xi \in \mathbb{Q}} |\hat{\phi}(\xi)|$ and since the left hand side converges, so does the right hands side. Now we can use Lemma 6 on ϕ with $x = 0$ to see that $\phi(0) = \sum_{\xi \in \mathbb{Q}} \hat{\phi}(\xi)$. Combining these with the definition of ϕ we get:

$$\sum_{\xi \in \mathbb{Q}} \hat{f}(\xi) = \sum_{\xi \in \mathbb{Q}} \hat{\phi}(\xi) = \phi(0) = \sum_{\xi \in \mathbb{Q}} f(\xi).$$

This proves the theorem. □

4.4 Tauberian Theorem

There are several slightly different theorems called Tauberian theorems. All of these start with some function $f(z) := \sum_{n \geq 0} a_n z^n$ with z being complex and this sum being convergent on some region of the complex plane. These theorems then tell us some properties of the constants a_n given some restrictions on f. For example the original theorem by Tauber, found in chapter $II.7$ of [10], states that $\sum_{n \geq 0} a_n = l$ if $f(z)$ has radius of convergence 1, $\lim_{z \to 1} f(z) = l$, f is real valued on the interval $[0, 1)$, and $\sum_{n \leq x} n a_n \to 0$ as $x \to \infty$. While it is interesting, this theorem will not be useful for us. Instead we will look at a slightly modified version of the so-called Ikehara-Ingham-Delange theorem, which is Theorem 7.13 in [10].

Theorem 5. *Let $A(t)$ be a non-decreasing function from $\mathbb{R}$ to $\mathbb{R}$ such that the function $F(z) := z \int_0^\infty A(t) e^{-zt} dt$ converges for $Re(z) > a > 0$. Let there be real numbers $c \geq 0$ and $b > 0$ such that the function*

$$G(z) = \frac{F(z+a)}{z+a} - \frac{c}{z^b}$$

satisfies, for real x,

$$\lim_{x \to 0^+} x^{b-1} \int_{-T}^{T} |G(2x + \imath y) - G(x + \imath y)| dy = 0$$

for every fixed positive real number T. Then

$$\lim_{t \to \infty} A(t) e^{-at} t^{1-b} = \frac{c}{\Gamma(b)}.$$

To prove this theorem we first need two lemmas, their use will become clear when we start proving the theorem.

Lemma 8. *Let $g : \mathbb{R} \to \mathbb{R}$ be an integrable and bounded function on $\mathbb{R}$. Assume there exist real numbers T and K such that*

$$\sup_{x \leq y \leq x+1/T} (g(y) - g(x)) \leq K, \qquad \hat{g}(\tau) = 0,$$

for every $x \in \mathbb{R}$ and $|\tau| \leq T$. By $\hat{g}$ we mean the usual Fourier transform of g, the definition of which can be found in the Appendix. Then

$$||g||_\infty = \sup_{x \in \mathbb{R}} |g(x)| \leq 16K.$$

Proof. Given g, T and K that satisfy the conditions in the statement, if we take the function $h(x) = g(x/T)$, we find that the function h along with $T = 1$ and K also satisfies the conditions, since $\hat{g}(\tau) = \hat{h}(\tau/T)/T$. Hence, we can assume that $T = 1$, as the general case would follow from the inverse of this substitution.

Let us start by looking at the function

$$\beta(t) = \frac{1}{2\pi} \frac{\sin(t/2)^2}{(t/2)^2}.$$

This function is normalised such that $\int_{-\infty}^{\infty} \beta(t)\mathrm{d}t = 1$ and it has the Fourier transform $\hat{\beta}(\tau) = \max(1 - |\tau|, 0)$, which can be found in Appendix 7.1.2. This means that outside the interval $[-1, 1]$, $\hat{\beta}(\tau) = 0$, but this means that $\hat{g}(\tau)\hat{\beta}(\tau) = 0$, since $\hat{g}(\tau) = 0$ on $[-1, 1]$. We can write out exactly what this means:

$$\begin{aligned}
0 &= \hat{g}(\tau)\hat{\beta}(\tau) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\iota\tau x} g(x) e^{-\iota\tau y} \beta(y) \mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\iota\tau(x+y)} g(x)\beta(y) \mathrm{d}x\mathrm{d}y \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\iota\tau z} g(z-y)\beta(y) \mathrm{d}z\mathrm{d}y.
\end{aligned}$$

This is the Fourier transform of the function

$$f(x) = \int_{-\infty}^{\infty} g(x-t)\beta(t)\mathrm{d}t.$$

Since the Fourier transform of f is 0 for every τ and f is continuous, we get that f has to be 0. Next we define σ to be the a function of $||g||_\infty$, with $\sigma \in \{-1, 1\}$ such that $||g||_\infty = \sup_{x\in\mathbb{R}} \sigma g(x)$. In the case where this holds for both $\sigma = 1$ and $\sigma = -1$, we just set

$\sigma = 1$. For every $0 < \epsilon < 1$, there exists some x_0 such that $\sigma g(x_0) \geq (1-\epsilon)||g||_\infty$, then for $x = x_0 - 5\sigma$ we have:

$$\begin{aligned} 0 &= \sigma \int_{-\infty}^{\infty} g(x_0 - 5\sigma - t)\beta(t)\mathrm{d}t \\ &= \sigma \int_{-5}^{5} g(x_0)\beta(t)\mathrm{d}t - \sigma \int_{-5}^{5} (g(x_0) - g(x_0 - 5\sigma - t))\beta(t)\mathrm{d}t + \sigma \int_{|t|>5} g(x_0 - 5\sigma - t)\beta(t)\mathrm{d}t \\ &\geq (1-\epsilon)||g||_\infty \int_{-5}^{5} \beta(t)\mathrm{d}t - \sigma \int_{-5}^{5} (g(x_0) - g(x_0 - 5\sigma - t))\beta(t)\mathrm{d}t - ||g||_\infty \int_{|t|>5} \beta(t)\mathrm{d}t \end{aligned}$$

Next recall that $\sup_{x \leq y \leq x+1}(g(y) - g(x)) \leq K$ for all x from the assumption in the lemma. Hence, for $t \in [-5, 5]$ and $\sigma = 1$ we have:

$$\begin{aligned} g(x_0) - g(x_0 - 5\sigma - t) &\leq \sup_{x_0 - 5\sigma - t \leq y \leq x_0 - 5\sigma - t + 10} (g(y) - g(x_0 - 5\sigma - t)) \\ &\leq \sum_{i=1}^{10} \sup_{x_0 - 5\sigma - t + (i-1) \leq y \leq x_0 - 5\sigma - t + i} (g(y) - g(x_0 - 5\sigma - t)) \leq 10K. \end{aligned}$$

For $t \in [-5, 5]$ and $\sigma = -1$:

$$\begin{aligned} g(x_0 - 5\sigma - t) - g(x_0) &\leq \sup_{x_0 \leq y \leq x_0 + 10} (g(y) - g(x_0)) \\ &\leq \sum_{i=1}^{10} \sup_{x_0 + (i-1) \leq y \leq x_0 + i} (g(y) - g(x_0)) \leq 10K. \end{aligned}$$

So combining these two inequalities we get for $t \in [-5, 5]$:

$$\sigma(g(x_0) - g(x_0 - 5\sigma - t)) \leq 10K.$$

We use this along with the fact that $\int_{-\infty}^{\infty} \beta(t) = 1$, to get:

$$\begin{aligned} 0 &\geq (1-\epsilon)||g||_\infty \int_{-5}^{5} \beta(t)\mathrm{d}t - 10K \int_{-5}^{5} \beta(t)\mathrm{d}t - ||g||_\infty (1 - \int_{-5}^{5} \beta(t))\mathrm{d}t \\ &\geq ((2-\epsilon)||g||_\infty - 10K) \int_{-5}^{5} \beta(t)\mathrm{d}t - ||g||_\infty. \end{aligned}$$

To conclude we use the fact that for $t > 0$, we have $\beta(t) \leq 2/\pi t^{-2}$. Since $\beta(t)$ is symmetric, we get $\int_{|t|>5} \beta(t)\mathrm{d}t \leq 4/\pi \int_5^\infty t^{-2}\mathrm{d}t = 4/(5\pi)$, and hence, $\int_{-5}^{5} \beta(t)\mathrm{d}t \geq 1 - 4/(5\pi)$. We can choose ϵ to be arbitrarily small, which gives us:

$$\begin{aligned} ||g||_\infty &\geq 2||g||_\infty \left(1 - \frac{4}{5\pi}\right) - 10K \left(1 - \frac{4}{5\pi}\right) \\ ||g||_\infty &\leq \frac{10K\left(1 - \frac{4}{5\pi}\right)}{2\left(1 - \frac{4}{5\pi}\right) - 1} \\ &\leq 16K \end{aligned}$$

□

It is quite easy to get a sharper bound for $||g||_\infty$ since we made some rough approximations at times. However, this will not be necessary as all we need in the end is a bound for $||g||_\infty$, it does not matter how sharp this bound is. The next lemma uses Lemma 8, so it will have the same rough bound which could still be sharpened but this again will not be necessary, as we will see later.

Lemma 9. *Let $g : \mathbb{R} \to \mathbb{R}$ be an integrable and bounded function. If we assume that there exist real numbers T and K such that:*

$$\sup_{x \le y \le x+1/T} (g(y) - g(x)) \le K$$

then

$$||g||_\infty \le 16K + 6 \int_{-T}^{T} |\hat{g}(\tau)| d\tau.$$

Proof. We will introduce a new function. Let $\epsilon > 0$, then the function

$$\alpha_\epsilon(t) = \frac{2}{\pi \epsilon t^2} \sin(\epsilon t/2) \sin((2T + \epsilon)t/2)$$

has Fourier transform $\hat{\alpha}_\epsilon(\tau)$ given by $\hat{\alpha}_\epsilon(\tau) = 1$ on $[-T, T]$, $\hat{\alpha}_\epsilon(\tau) = 0$ when $|\tau| > T + \epsilon$ and $\hat{\alpha}_\epsilon(\tau) = (T + \epsilon - \tau)/\epsilon$ for $T < |\tau| < T + \epsilon$. The calculation for this can be found in Appendix 7.1.1. We use this function to define a new function f with the property that $\hat{f} = \hat{g}\hat{\alpha}$. We define f to be the convolution of g and α, defined as $g \star \alpha(x) = \int_{-\infty}^{\infty} g(y)\alpha(x - y)\mathrm{d}y$. Next we note that since g is bounded, $\int_{-(T+\epsilon)}^{T+\epsilon} |\hat{g}(\tau)|\mathrm{d}\tau \le 2(T + \epsilon)||g||_\infty < \infty$. We can use the inverse Fourier transform, defined for example in section 6.2 of [8], to get the following:

$$\begin{aligned}
||f||_\infty &= \sup_{x \in \mathbb{R}} f(x) \\
&= \frac{1}{2\pi} \sup_{x \in \mathbb{R}} \int_{-\infty}^{\infty} \hat{f}(\tau) e^{\iota x \tau} \mathrm{d}\tau \\
&= \frac{1}{2\pi} \sup_{x \in \mathbb{R}} \int_{-(T+\epsilon)}^{T+\epsilon} \hat{f}(\tau) e^{\iota x \tau} \mathrm{d}\tau \\
&\le \frac{1}{2\pi} \int_{-(T+\epsilon)}^{T+\epsilon} |\hat{g}(\tau)\hat{\alpha}(\tau)| \mathrm{d}\tau \\
&\le \frac{1}{2\pi} \int_{-(T+\epsilon)}^{T+\epsilon} |\hat{g}(\tau)| \mathrm{d}\tau.
\end{aligned}$$

Next we look at the function $g - f$. Note that

$$\sup_{x \le y \le x+1/T} ((g - f)(y) - (g - f)(x)) \le \sup_{x \le y \le x+1/T} (g(y) - g(x)) + 2||f||_\infty \le K + 2||f||_\infty,$$

which means we can use Lemma 8 on $g - f$ but instead of the constant K, we have to use $K + 2||f||_\infty$. This gives us $||g - f||_\infty \le 16K + 32||f||_\infty$. The last step is the following:

$$||g||_\infty \leq ||g-f||_\infty + ||f||_\infty \leq 16K + 33||f||_\infty \leq 16K + 6\int_{-(T+\epsilon)}^{T+\epsilon} |\hat{g}(\tau)|\mathrm{d}\tau,$$

where the last step works because $33/(2\pi) < 6$. Now all we need to do is take the limit for $\epsilon \to 0$ to obtain the expression in the statement, which we are allowed to do since all steps hold for all $0 < \epsilon < 1$. □

Combining these two lemmas we can prove Theorem 5.

Proof of Theorem 5. We give this proof in four steps.

First step: We introduce a function $g_x(t)$ and use Lemma 9 to find a bound for $||g_x||_\infty$. First of all, we can assume $A(t) = 0$ when $t \leq 0$, because we are only interested in its behavior as $t \to \infty$. Let us introduce the following function

$$g_x(t) = A(t)e^{-(a+x)t}(1 - e^{-xt})$$

for all positive real x. This has Fourier transform

$$\hat{g}_x(y) = G(x+\iota y) - G(2x+\iota y) + c((x+\iota y)^{-b} - (2x+\iota y)^{-b}).$$

The calculation for this can be found in Appendix 7.1.4. Note that we can write the last two terms on the right hand side as the integral $\int_{2x+\iota y}^{x+\iota y}(-b)\tau^{-(b+1)}\mathrm{d}\tau$ and we can bound this term by

$$\left|\int_{2x+\iota y}^{x+\iota y}(-b)\tau^{-(b+1)}\mathrm{d}\tau\right| = \left|\int_{2x+\iota y}^{x+\iota y} b\tau^{-(b+1)}\mathrm{d}\tau\right| \leq \frac{bx}{|x+\iota y|^{b+1}},$$

where this last inequality bounds the integrand by its supremum. This means that we can write for $x > 0$,

$$\begin{aligned}
\int_{-T}^{T} |c((x+\iota y)^{-b} - (2x+\iota y)^{-b})|\mathrm{d}y &\leq \int_{-T}^{T} \frac{cbx}{|x+\iota y|^{b+1}}\mathrm{d}y \\
&\leq \int_{-T}^{T} \frac{cbx}{\max(x,|y|)^{b+1}}\mathrm{d}y \\
&\leq \int_{-T}^{T} \frac{cbx}{\max(x,|y|)^{b+1}}\mathrm{d}y \\
&\leq \int_{-x}^{x} \frac{cbx}{x^{b+1}}\mathrm{d}y + 2\int_{x}^{T} \frac{cbx}{y^{b+1}}\mathrm{d}y \\
&\leq \frac{2cb}{x^{b-1}} - \frac{2cx}{T^b} + \frac{2c}{x^{b-1}} \leq \frac{2c(b+1)}{x^{b-1}}.
\end{aligned}$$

Now returning to the function $g_x(t)$, we can see that for real $x', y' > 0$, we have that

$$\begin{aligned} g_x(x'+y') - g_x(x') &\geq A(x')e^{-(a+x)(x'+y')}(1-e^{-x(x'+y')}) - A(x')e^{-(a+x)x'}(1-e^{-xx'}) \\ &\geq A(x')e^{-(a+x)(x'+y')}(1-e^{-xx'}) - A(x')e^{-(a+x)x'}(1-e^{-xx'}) \\ &\geq A(x')e^{-(a+x)x'}(1-e^{-xx'})(e^{-(a+x)y'}-1) \\ &\geq g_x(x')(e^{-(a+x)y'}-1) \geq -y'(a+x)||g_x||_\infty, \end{aligned}$$

where the last step holds because $e^{-x} - 1 + x \geq 0$ for all $x > 0$, which can be seen by taking the derivative, which is $1 - e^{-x} \geq 0$ for all $x > 0$ along with the fact that filling in $x = 0$ gives $0 \geq 0$. If we instead have $x' < 0$, we get $A(x') = 0$, so $g_x(x') = 0$ and since $g_x(x'+y') \geq 0$ we clearly have that

$$g_x(x'+y') - g_x(x') \geq -y'(a+x)||g_x||_\infty \tag{1}$$

still holds. This means that we can use Lemma 9 on the function $-g_x(t)$ with $K = (a+x)||g_x||_\infty/T$ which gives us

$$\begin{aligned} ||g_x||_\infty &\leq \frac{16(a+x)||g_x||_\infty}{T} + 6\int_{-T}^{T} |\hat{g}_x(y)|\mathrm{d}y, \\ \left(1 - \frac{16(a+x)}{T}\right)||g_x||_\infty &\leq 6\int_{-T}^{T} |\hat{g}_x(y)|\mathrm{d}y, \\ ||g_x||_\infty &\leq \frac{6T}{T-16(a+x)}\int_{-T}^{T} |\hat{g}_x(y)|\mathrm{d}y. \end{aligned}$$

In last third line we divide by $T - 16(a+x)$, however, since we are allowed to choose T, we will just choose it such that $T > 16(a+x)$ so this stays positive. Later in the proof we will choose T arbitrarily large so this will not be a problem. If we now use the Fourier transform of g_x we obtain the final bound for $||g_x||_\infty$:

$$||g_x||_\infty \leq \frac{6T}{T-16(a+x)}\left(\int_{-T}^{T} |G(x+\iota y) - G(2x+\iota y)|\mathrm{d}y + \frac{2c(b+1)}{x^{b-1}}\right). \tag{2}$$

Second step: We introduce a second function $B_x(t)$, and find a bound for $B_x(x'+y') - B_x(x')$.
We define the function $B_x(t)$ to be 0 for $t \leq 0$ and for $t > 0$ it is defined as $B_x(t) = (c/\Gamma(b))e^{-xt}(1-e^{-xt})t^{b-1}$, where Γ is the usual gamma function. This function has the Fourier transform $\hat{B_x}(y) = c((x+\iota y)^{-b} - (2x+\iota y)^{-b})$ as can be seen in Appendix 7.1.3. We want to use Lemma 9, so we look at $B_x(x'+y') - B_x(x')$ for $y' > 0$. To do this we notice that B_x is continuous and its derivative is, for $t > 0$

$$\begin{aligned} B_x'(t) &= \frac{c(b-1)}{\Gamma(b)}e^{-xt}(1-e^{-xt})t^{b-2} - \frac{cx}{\Gamma(b)}e^{-xt}(1-2e^{-xt})t^{b-1} \\ &= \frac{c}{\Gamma(b)}e^{-xt}t^{b-2}\left((b-1)(1-e^{-xt}) - xt + 2txe^{-xt}\right). \end{aligned}$$

We can use the fact that $1 - e^{-xt} \geq 2xte^{-xt} - xt$ when $xt > 0$. To see why this is true consider the function $f(x) = 1 - e^{-x} - 2xe^{-x} + x$, we can see that $f(0) = 0$. Now take the derivative of f, $f'(x) = 1 - e^{-x} + 2xe^{-x}$ which is clearly bigger than 0 when $x > 0$, so the function is increasing for positive x, and hence, $f(x) \geq 0$ for $x > 0$. Lastly consider $f(xt)$ and the inequality comes out. Using this fact we write:

$$B'_x(t) \leq \frac{cb}{\Gamma(b)} e^{-xt} t^{b-2}(1 - e^{-xt}) \leq \frac{cb}{\Gamma(b)} e^{-xt} t^{b-1}.$$

Next we consider the case $t \leq 0$. In this case the derivative of $B_x(t)$ is just 0. We can use this to write $B_x(x' + y') - B_x(x') = \int_{x'}^{x'+y'} B'_x(t)\mathrm{d}t$. Now we can distinguish three cases: $x' > 0$, $x' \leq 0$ with $x' + y' > 0$ and $x' \leq 0$ with $x' + y' \leq 0$. This last case is straightforward, since then $B_x(x') = B_x(x' + y') = 0$, so we can assume that $x' + y' > 0$. Let us first look at the case where $x' < 0$. We have

$$\begin{aligned} B_x(x' + y') - B_x(x') &= \int_{x'}^{x'+y'} B'_x(t)\mathrm{d}t = \int_{0}^{x'+y'} B'_x(t)\mathrm{d}t \\ &\leq \int_{0}^{x'+y'} \frac{cb}{\Gamma(b)} e^{-xt} t^{b-1}\mathrm{d}t \\ &\leq \int_{0}^{x'+y'} \frac{cb}{\Gamma(b)} t^{b-1}\mathrm{d}t = \frac{c}{\Gamma(b)}(x' + y')^b \leq \frac{c}{\Gamma(b)}(y')^b. \end{aligned}$$

In the case where $x' > 0$, we get

$$\begin{aligned} B_x(x' + y') - B_x(x') &= \int_{x'}^{x'+y'} B'_x(t)\mathrm{d}t \\ &\leq \int_{x'}^{x'+y'} \frac{cb}{\Gamma(b)} e^{-xt} t^{b-1}\mathrm{d}t \\ &\leq e^{-xx'} \frac{cb}{\Gamma(b)} \int_{x'}^{x'+y'} t^{b-1}\mathrm{d}t \\ &\leq e^{-xx'} \frac{c}{\Gamma(b)}((x' + y')^b - (x')^b). \end{aligned}$$

We can improve this bound for $b \geq 1$ by considering the function

$$f_{x'}(y') = (x' + y')^b - (x')^b - b(x' + y')^{b-1}y',$$

which has derivative

$$\frac{\mathrm{d}}{\mathrm{d}y'} f_{x'}(y') = -(b-1)b(x' + y')^{b-2}y'.$$

This derivative is negative when $y' > 0$, so we get that for every $y' > 0$, $f_{x'}(y') \leq 0$. Next consider $b < 1$, in this case we just use the fact that $(x' + y')^b \leq x'^b + y'^b$ since $b \in [0, 1]$ and $x', y' \geq 0$. This means that in general we have:

$$B_x(x' + y') - B_x(x') \leq e^{-xx'} \frac{c}{\Gamma(b)}(\Theta(b-1)b(x' + y')^{b-1}y' + (1 - \Theta(b-1))y'^b).$$

where $\Theta(t)=1$ if $t>0$ and $\Theta(t)=0$ if $t\le 0$. We want to improve this bound by making it independent of x' and y' for $0<x\le 1$ and $0<y'\le 1/T\le 1$. Note that this bound is an increasing function in y', so if we set $y'=1/T$ in the bound, the inequality still holds. This means that for $b<1$, we have

$$e^{-xx'}y'^b \le y'^b \le 1/T^b \le b^b(1+e^{2-b}b^{b+1})\left(\frac{1}{x^{b-1}T}+\frac{1}{T^b}\right)$$

We have, for $y'\le 1/T\le 1$ and $b\ge 1$,

$$b(x'+y')^{b-1}y' \le b(x'+1/T)^{b-1}/T \le b^b\left(\frac{x'^{b-1}}{T}+\frac{1}{T^b}\right).$$

We used that for real $a_1,a_2>0$ and $a_3\ge 1$, $(a_1+a_2)^{a_3}\le(1+a_3)^{a_3}(a_1^{a_3}+a_2^{a_3})$. Now we want to use the fact

$$1+e^{2-b}b^{b+1}\ge \sup_{t\ge 0} e^{-t}t^{b-1}\ge e^{-xx'}(xx')^{b-1}.$$

To see this holds, we first find the value of $t>0$ for which $e^{-t}t^{b-1}$ attains its maximum. Setting the derivative equal to 0 gives $t=b-1$ or $t=0$. So

$$\sup_{t\ge 0} e^{-t}t^{b-1}=e^{-(b-1)}(b-1)^{b-1}\le 1+\left(e^{-(b-1)}(b-1)^{b-1}\right)\left(\frac{b}{b-1}\right)^{b-1}b^2e=1+e^{1-(b-1)}b^{b+1}$$

as $t=0$ gives a minimum. We use this fact to see that for $b\ge 1$,

$$e^{-xx'}b(x'+y')^{b-1}y'\le b^b(1+e^{2-b}b^{b+1})\left(\frac{1}{x^{b-1}T}+\frac{1}{T^b}\right).$$

This means that we can write

$$B_x(x'+y')-B_x(x')\le \frac{c}{\Gamma(b)}b^b(1+e^{2-b}b^{b+1})\left(\frac{1}{x^{b-1}T}+\frac{1}{T^b}\right). \tag{3}$$

Third step: We want to use the equalities found in the previous steps to use Lemma 9 on the function $B_x(t)-g_x(t)$.
Using the bounds for $||g_x||_\infty$ and $B_x(x'+y')-B_x(x')$ from equations (1), (2) and (3), we obtain for $0<y'\le 1/T\le 1$

$$\begin{aligned}(B_x-g_x)(x'+y')-(B_x-g_x)(x')\le&\frac{c}{\Gamma(b)}(1+e^{2-b}b^{b+1})b^b\left(\frac{1}{Tx^{b-1}}+\frac{1}{T^b}\right)+y'(a+x)||g_x||_\infty\\ \le&\frac{c}{\Gamma(b)}(1+e^{2-b}b^{b+1})b^b\left(\frac{1}{Tx^{b-1}}+\frac{1}{T^b}\right)+\frac{6Ty'(a+x)}{T+16(a+x)}\\ &\left(\int_{-T}^{T}|G(x+\iota y)-G(2x+\iota y)|\mathrm{d}y+\frac{2c(b+1)}{x^{b-1}}\right)\\ \le&\frac{c}{\Gamma(b)}(1+e^{2-b}b^{b+1})b^b\left(\frac{1}{Tx^{b-1}}+\frac{1}{T^b}\right)+\frac{6(a+x)}{T+16(a+x)}\\ &\left(\int_{-T}^{T}|G(x+\iota y)-G(2x+\iota y)|\mathrm{d}y+\frac{2c(b+1)}{x^{b-1}}\right)=:K.\end{aligned}$$

Additionally, this new function has Fourier transform

$$(\hat{B}_x - \hat{g}_x)(y) = G(2x + \iota y) - G(x + \iota y).$$

This means that Lemma 9 tells us that

$$||B_x - g_x||_\infty \leq 16K + 6 \int_{-T}^{T} |G(x + \iota y) - G(2x + \iota y)| \mathrm{d}y. \tag{4}$$

Final step: To finish the prove we use the inequalities from the previous steps an take limits to acquire the right result.
We look at the function $x^{b-1}|(B_x - g_x)(1/x)|$ for $x \to 0^+$. By equation (4), we have that

$$x^{b-1}|(B_x - g_x)(1/x)| \leq 16Kx^{b-1} + 6x^{b-1} \int_{-T}^{T} |G(x + \iota y) - G(2x + \iota y)| \mathrm{d}y.$$

The first thing we notice is that by assumption,

$$\lim_{x \to 0^+} x^{b-1} \int_{-T}^{T} |G(x + \iota y) - G(2x + \iota y)| \mathrm{d}y = 0,$$

which means this terms vanishes in the limit $x \to 0^+$. We now want to show that in this limit the $x^{b-1}K$ term also vanishes. To achieve this, we look at the following:

$$\begin{aligned}\lim_{x \to 0^+} 16x^{b-1}K &= \lim_{x \to 0^+} 16\frac{c}{\Gamma(b)}(1 + e^{2-b}b^{b+1})b^b \left(\frac{1}{T} + \frac{x^{b-1}}{T^b}\right) + \frac{6(a+x)}{T + 16(a+x)} 32c(b+1) \\ &= 16\frac{c}{T\Gamma(b)}(1 + e^{2-b}b^{b+1})b^b + \frac{6a}{T + 16a} 32c(b+1).\end{aligned}$$

Let us call this constant $M(T)$, so we have that

$$\lim_{x \to 0^+} x^{b-1}|(B_x - g_x)(1/x)| \leq M(T).$$

This can be rewritten using

$$(B_x - g_x)(1/x) = \left(-A(1/x)e^{-a/x} + \frac{c}{\Gamma(b)x^{b-1}}\right)\frac{e-1}{e^2},$$

and by substituting $z = 1/x$, we see that

$$\lim_{z \to \infty} \left|\frac{A(z)}{e^{az}z^{b-1}c/\Gamma(b)} - 1\right| < \frac{\Gamma(b)e^2}{(e-1)c} M(T).$$

The last thing to note is that we can choose T arbitrarily large, and since $\lim_{T \to \infty} M(T) = 0$, we can choose T such that $M(T)$ becomes arbitrarily small. From this we see that

$$\lim_{z \to \infty} \frac{A(z)}{e^{az}z^{b-1}c/\Gamma b} = 1.$$

This is exactly what we needed to prove.

□

Corollary 2. *Let us have a convergent sum* $\sum_{i=1}^{\infty} a_i/\lambda_i^z$ *for positive integers* a_i, $\lambda_i \in \mathbb{R}$ *for all* $i \in \mathbb{N}$ *and* $z \in \mathbb{C}$. *Assume there exist real numbers* $a, c > 0$ *and a function* $h(z)$ *which is holomorphic for* $Re(z) > a - \epsilon$ *for some* $\epsilon > 0$, *with* $h(a)/a = c$ *and* $\sum_{i=1}^{\infty} a_i/\lambda_i^z = h(z)/(z-a)^b$ *for some integer* $b \geq 0$. *Then* $\lim_{B\to\infty} B^{-a}\log(B)^{1-b}\sum_{\lambda_i \leq B} a_i = c/(a\Gamma(b))$.

Proof. If we choose $A(t) = \sum_{\lambda_i \leq e^t} a_i$, which is clearly an increasing function, we get that

$$F(s) = s\int_0^{\infty} A(t)e^{-st}\mathrm{d}t = s\sum_{i=1}^{\infty}\int_{\ln(\lambda_i)}^{\infty} a_i e^{-st}\mathrm{d}t = \sum_{i=1}^{\infty} a_i/\lambda_i^s = h(s)/(s-a)^b.$$

Now all that is left to show is that

$$\lim_{x\to 0^+} x^{b-1}\int_{-T}^{T}\left|\frac{\frac{h(2x+\iota y+a)}{2x+\iota y+a}-\frac{h(a)}{a}}{(2x+\iota y)^b}-\frac{\frac{h(x+\iota y+a)}{x+\iota y+a}-\frac{h(a)}{a}}{(x+\iota y)^b}\right|\mathrm{d}y = 0.$$

To achieve this, we notice that since $h(z+a)$ is holomorphic for $\mathrm{Re}(z) \geq 0$, the function $h(z+a)/(z+a) - h(a)/a$ is also holomorphic on this same region, as $a > 0$. This means that $(h(z+a)/(z+a) - h(a)/a)z^{-b}$ is holomorphic for $\mathrm{Re}(z) \geq 0$ except in the point where $z = 0$. We will now decompose the integral into a sum of three integrals, let $\epsilon > 0$, then we have that

$$x^{b-1}\int_{-T}^{T}\left|\frac{\frac{h(2x+\iota y+a)}{2x+\iota y+a}-\frac{h(a)}{a}}{(2x+\iota y)^b}-\frac{\frac{h(x+\iota y+a)}{x+\iota y+a}-\frac{h(a)}{a}}{(x+\iota y)^b}\right|\mathrm{d}y = I_1 + I_2 + I_3,$$

with

$$I_1 = x^{b-1}\int_{-\epsilon}^{\epsilon}\left|\frac{\frac{h(2x+\iota y+a)}{2x+\iota y+a}-\frac{h(a)}{a}}{(2x+\iota y)^b}-\frac{\frac{h(x+\iota y+a)}{x+\iota y+a}-\frac{h(a)}{a}}{(x+\iota y)^b}\right|\mathrm{d}y,$$

$$I_2 = x^{b-1}\int_{\epsilon}^{T}\left|\frac{\frac{h(2x+\iota y+a)}{2x+\iota y+a}-\frac{h(a)}{a}}{(2x+\iota y)^b}-\frac{\frac{h(x+\iota y+a)}{x+\iota y+a}-\frac{h(a)}{a}}{(x+\iota y)^b}\right|\mathrm{d}y,$$

$$I_3 = x^{b-1}\int_{-T}^{-\epsilon}\left|\frac{\frac{h(2x+\iota y+a)}{2x+\iota y+a}-\frac{h(a)}{a}}{(2x+\iota y)^b}-\frac{\frac{h(x+\iota y+a)}{x+\iota y+a}-\frac{h(a)}{a}}{(x+\iota y)^b}\right|\mathrm{d}y.$$

Let us start with I_1, define $f(z) := (h(z+a)/(z+a) - h(a)/a)z^b$, this is holomorphic for $\mathrm{Re}(z) \geq 0$ except for $z = 0$, so on this region its derivative exists. Notice that this means that as x approaches 0^+,

$$\lim_{x\to 0^+} x^{-1}\left|\frac{f(2x+\iota y)}{(2x+\iota y)^b}-\frac{f(x+\iota y)}{(x+\iota y)^b}\right| = |f'(\iota y)| \tag{5}$$

for $y \neq 0$. But for $y = 0$, we have that the limit as $x \to 0^+$ diverges. We can therefor conclude that the following holds:

$$\lim_{x\to 0^+}\frac{\left|\frac{f(2x+\iota y)}{(2x+\iota y)^b}-\frac{f(x+\iota y)}{(x+\iota y)^b}\right|}{\left|\frac{f(2x)}{(2x)^b}-\frac{f(x)}{x^b}\right|} \leq 1$$

for any $y \in \mathbb{R}$. This is obvious for $y = 0$ as it will be 1, and for $y \neq 0$ we use equation (5) to see the limit is 0. This means that for $y \neq 0$, there exists a δ such that for $x < \delta$,

$$\frac{\left|\frac{f(2x+\iota y)}{(2x+\iota y)^b} - \frac{f(x+\iota y)}{(x+\iota y)^b}\right|}{\left|\frac{f(2x)}{(2x)^b} - \frac{f(x)}{x^b}\right|} \leq 1.$$

This clearly also holds for $y = 0$. we can take the integral over y to find that for all $x < \delta$,

$$\frac{\int_{-\epsilon}^{\epsilon}\left|\frac{f(2x+\iota y)}{(2x+\iota y)^b} - \frac{f(x+\iota y)}{(x+\iota y)^b}\right| \mathrm{d}y}{2\epsilon\left|\frac{f(2x)}{(2x)^b} - \frac{f(x)}{x^b}\right|} \leq 1.$$

This now allows us to write:

$$\lim_{x\to 0^+}\frac{\int_{-\epsilon}^{\epsilon}\left|\frac{f(2x+\iota y)}{(2x+\iota y)^b} - \frac{f(x+\iota y)}{(x+\iota y)^b}\right| \mathrm{d}y}{2\epsilon\left|\frac{f(2x)}{(2x)^b} - \frac{f(x)}{x^b}\right|} \leq 1.$$

Next we look at the limit

$$\lim_{x\to 0^+} x^{b-1}\left|\frac{f(2x+\iota y)}{2^b x^b} - \frac{f(x+\iota y)}{x^b}\right| = \lim_{x\to 0^+}\left|\frac{f(2x+\iota y)}{2^b x} - \frac{f(x+\iota y)}{x}\right|. \tag{6}$$

Since $h(2x+a)$ and $h(x+a)$ are holomorphic we can use a series expansion in x[11], this will have constant term $h(a)$. Since we can write $f(x) = \frac{1}{x+a}(h(x+a) - h(a) - xh(a)/a)$, $f(x)$ will have a zero in $x = 0$ with a degree of at least 1. This means the limit the limit in equation (6) converges to some limit L. This means that we can write

$$\frac{\lim_{x\to 0^+} x^{b-1}\int_{-\epsilon}^{\epsilon}\left|\frac{f(2x+\iota y)}{(2x+\iota y)^b} - \frac{f(x+\iota y)}{(x+\iota y)^b}\right| \mathrm{d}y}{\lim_{x\to 0^+} x^{b-1}2\epsilon\left|\frac{f(2x)}{(2x)^b} - \frac{f(x)}{x^b}\right|} \leq 1,$$

where the limit in the numerator converges because the limit in the denominator converges. This leads to

$$\lim_{x\to 0^+} x^{b-1}\int_{-\epsilon}^{\epsilon}\left|\frac{f(2x+\iota y)}{(2x+\iota y)^b} - \frac{f(x+\iota y)}{(x+\iota y)^b}\right| \mathrm{d}y \leq 2\epsilon L. \tag{7}$$

For the other two integrals, I_2 and I_3 we will use the fact that $f(z)$ is differentiable on the region of integration, so for every $\epsilon' > 0$, there exists some $\delta > 0$ such that $0 < x < \delta$ implies that

$$\left|\frac{|f(2x+z) - f(x+z)|}{x} - |f'(z)|\right| < \epsilon',$$

for all $z \in [-T, -\epsilon] \cup [\epsilon, T]$. Using the product and chain rules we find that

$$f'(z) = \frac{\frac{h'(z+a)}{z+a} - \frac{h(z+a)}{(z+a)^2}}{z^b} - b\frac{\frac{h(z+a)}{z+a} - \frac{h(a)}{a}}{z^{b+1}}.$$

Since $h(z)$ is holomorphic for $\mathrm{Re}(z) \geq 0$, both it and its derivative are bounded. Hence, $|f'(z)|$ is bounded by $M/|z|^{b+1}$ for some constant M on the intervals $[-T, -\epsilon] \cup [\epsilon, T]$. This means that in this region, $|f'(z)|$ is bounded by M/ϵ^{b+1}. We also have that for $0 < x < \delta$,

$$\begin{aligned}
&x^b \left| \int_\epsilon^T \frac{|f(2x + \iota y) - f(x + \iota y)|}{x} - |f'(\iota y)| \mathrm{d}y \right| \\
&\leq x^b \int_\epsilon^T \left| \frac{|f(2x + \iota y) - f(x + \iota y)|}{x} - |f'(\iota y)| \right| \mathrm{d}y \\
&\leq x^b (T - \epsilon)\epsilon'.
\end{aligned}$$

Since

$$\int_\epsilon^T |f'(\iota y)| \mathrm{d}y \leq (T - \epsilon) M/\epsilon^{b+1},$$

for $x < \delta$, we get by the triangle inequality that

$$I_2 \leq x^b (T - \epsilon) \left(M/\epsilon^{b+1} + \epsilon' \right).$$

And because of symmetry, we also get

$$I_3 \leq x^b (T - \epsilon) \left(M/\epsilon^{b+1} + \epsilon' \right).$$

Note that we can choose ϵ' arbitrarily small. By these two inequalities and equation (7), we see that if we let $\epsilon = \sqrt{x}$, in the limit as $x \to 0^+$ all three integrals, I_1, I_2 and I_3 converge to 0. Hence we conclude that

$$\lim_{x \to 0^+} x^{b-1} \int_{-T}^T \left| \frac{\frac{h(2x+\iota y+a)}{2x+\iota y+a} - \frac{h(a)}{a}}{(2x + \iota y)^b} - \frac{\frac{h(x+\iota y+a)}{x+\iota y+a} - \frac{h(a)}{a}}{(x + \iota y)^b} \right| \mathrm{d}y = 0.$$

This means we have all the requirements for Theorem 5 and we get that as $B \to \infty$

$$\sum_{\lambda_i \leq B} a_i = A(\ln(B)) = \frac{c}{a\Gamma(b)} B^a \ln(B)^{b-1}.$$

□

4.5 Solving the problem

Now that we have all the framework we need, we can take a second look at the problem we want to solve. We will follow the same step as the interlude in [3], but we will do it in general n-dimensions. We need to find the number of points $x \in \mathbb{QP}^n$ with height $H(x) \leq B$ for large B. To do this we view $\mathbb{QP}^n$ as $\mathbb{Q}^n \cup \mathbb{QP}^{n-1}$. Here elements of $\mathbb{Q}^n$ are of the form $(1 : x_1 : \ldots : x_n)$ for rational numbers $x_1 \ldots x_n$. The elements of $\mathbb{QP}^{n-1}$ are of the form $(0 : x_0 : \ldots : x_{n-1})$. We will find for how many $x \in \mathbb{Q}^k \subset \mathbb{QP}^k$ we have $H(x) \leq B$, if we do this for every $k \leq n$, we can use induction to find for how many $x \in \mathbb{QP}^n$ we have $H(x) \leq B$. We can calculate the height of an element of the form $(1 : x_1 : \ldots : x_n)$ using the following lemma.

Lemma 10. *Let $x = (1 : x_1 : \ldots : x_n)$, then the height of x, introduced in Section 2, is given by*

$$H(x) = \max\{1, |x_i| \; |0 < i \leq n\} \prod_{p \in P} \max\{1, ||x_i||_p | 0 < i \leq n\}.$$

Proof. We write $N = \prod_{p \in P} \max\{1, ||x_i||_p | 0 < i \leq n\}$. It is clear since $N \neq 0$, that we can represent x as

$$x = (N : Nx_1 : \ldots : Nx_n)\,.$$

In fact, all of the coordinates in this representation are integers, since for any $p \in P$, we have that $||Nx_i||_p = ||N||_p ||x_i||_p \leq 1$ since if $||x_i||_p \geq 1$, it divides N. Furthermore, these coordinates are a set of coprime integers. To prove this assume that a prime q divides all of these coordinates, let $e > 0$ be the largest integer such that $q^e | N$, then there has to be an index i such that $||x_i||_q = q^e$, but then $q \nmid ||Nx_i||_q$ which is a contradiction. So this representation is the representation of coprime integers we have to use to calculate the height. The height of x is now given by the maximum of all of the coordinates of this representation of x, however since $N > 0$ we have that

$$\max\{N, N||x_i||_p | 0 < i \leq n\} = N \max\{1, ||x_i||_p | 0 < i \leq n\}.$$

This leads immediately to

$$H(x) = \max\{N, |Nx_i| \; |0 < i \leq n\} = \max\{1, |x_i| \; |0 < i \leq n\} \prod_{p \in P} \max\{1, ||x_i||_p | 0 < i \leq n\}.$$

□

This allows us to extend the definition of the height function to a function over V in the following way.

Definition 6. The height function defined on V^n is given by

$$H(x) = \max\{1, |x_{\infty,i}| \; |0 < i \leq n\} \prod_{p \in P} \max\{1, ||x_{p,i}||_p | 0 < i \leq n\}$$

Note that Lemma 10 means we can split the height function into a product over primes in the following way:

$$H(x) = \prod_{p \in P \cup \infty} \max\{1, ||x_{p,i}||_p | 0 < i \leq n\} =: \prod_{p \in P \cup \infty} H_p(x_p).$$

We want to end up using Corollary 2 to find the number of points x with height $H(x) \leq B$. To achieve this, we will look at the function $H(x)^{-s}$ for $s \in \mathbb{C}$ and the following sum

$$\sum_{m \in \mathbb{Q}^n} H(m)^{-s} = \sum_{m \in \mathbb{Q}^n} \hat{H}(m, s),$$

for $\mathrm{Re}(s) > 0$, where $\hat{H}(m,s)$ is the Fourier transform of $H(x)^{-s}$ with respect to x. This equality follows from Theorem 4, which we are allowed to use as

$$\sum_{\xi\in\mathbb{Q}} H((x_i + \delta_{i,j}\xi)_{1\le i\le n})$$

converges uniformly for all $x \in D^n$ and all indices $1 \le j \le n$. This is however still a sum over $\mathbb{Q}^n$, which is not easy to deal with, but using the following lemma, we can write it as a sum over $\mathbb{Z}^n$ instead.

Lemma 11. *Let $f(x) = \prod_{p\in P\cup\infty} f_p(x_p) : [0,1)\times\prod_{p\in P}\mathbb{Q}_p \to \mathbb{R}$ where $f_p(x_p)$ has the property that $f_p(x_p+1) = f_p(x_p)$ for all $x_p \in \mathbb{Q}_p$ and all $p \in P\cup\infty$. Assume that $f(x) = 0$ for all $x \in [0,1)\times\prod_{p\in P}\mathbb{Q}_p \setminus V$, then $\hat{f}(k) = 0$ for all $k \in \mathbb{Q}\setminus\mathbb{Z}$.*

Proof. Let us assume that $k \in \mathbb{Q}\setminus\mathbb{Z}$, we write $k = a/b$ with a, b coprime. Let q be a prime factor of b which has to exist since $k \notin \mathbb{Z}$. We get that

$$\begin{aligned}\hat{f}(k) &= \int_V f(x)e^{-2\pi\iota\{kx\}}\mathrm{d}x = \int_{[0,1]} f_\infty(x)e^{-2\pi\iota kx}\mathrm{d}x \prod_{p\in P}\int_{\mathbb{Q}_p} f_p(x_p)e^{-2\pi\iota\{kx_p\}_p}\mathrm{d}x_p \\ &= \int_{[0,1]} f_\infty(x)e^{-2\pi\iota kx}\mathrm{d}x \prod_{p\in P}\int_{\mathbb{Q}_p} f_p(x_p+\delta_{p,q})e^{-2\pi\iota\{k((x_p)_i+\delta_{p,q})\}_p}\mathrm{d}x_p,\end{aligned}$$

where by $\delta_{p,q}$ we denote the usual delta function which has value 1 if $p = q$ and 0 otherwise. The second equality holds even though V is not equal to $[0,1]\times\prod_{p\in P}\mathbb{Q}_p$, because the function f is zero on $[0,1)\times\prod_{p\in P}\mathbb{Q}_p\setminus V$. Now since $\{c+d\}_q = \{c\}_q + \{d\}_q + \gamma$ for some integer γ combined with the facts that $f_q(x_q+1) = f_q(x_q)$ and $e^{2\pi\iota\gamma} = 1$ we can write this as

$$\begin{aligned}\hat{f}(k) &= \int_{[0,1]} f(x)e^{-2\pi\iota kx}\mathrm{d}x \prod_{p\in P}\int_{\mathbb{Q}_p} f_p(x_p)e^{-2\pi\iota\{kx_p+k\delta_{p,q}\}_p}\mathrm{d}x_p \\ &= \int_{[0,1]} f(x)e^{-2\pi\iota kx}\mathrm{d}x \prod_{p\in P}\int_{\mathbb{Q}_p} f_p(x_p)e^{-2\pi\iota\{k\delta_{p,q}\}_p}e^{-2\pi\iota\{kx_p\}_p}\mathrm{d}x_p \\ &= e^{-2\pi\iota\{k\}_q}\int_{[0,1]} f(x)e^{-2\pi\iota kx}\mathrm{d}x \prod_{p\in P}\int_{\mathbb{Q}_p} f_p(x_p)e^{-2\pi\iota\{kx_p\}_p}\mathrm{d}x_p \\ &= e^{-2\pi\iota\{k\}_q}\hat{f}(k)\end{aligned}$$

Now this means that $e^{-2\pi\iota\{k\}_q} = 1$ or $\hat{f}(k) = 0$. Since $k = a/b$ with $q|b$ and $q \nmid a$, we have that $\{k\}_q \neq 0$, and since $0 \le \{k\}_q < 1$, we get that $e^{-2\pi\iota\{k\}_q} \neq 1$, so $\hat{f}(k) = 0$. □

This lemma also works in n dimensions as we can see in the following corollary.

Corollary 3. *Let $f(x) = \prod_{p\in P\cup\infty} f_p(x) : [0,1)^n \times \prod_{p\in P}\mathbb{Q}_p^n \to \mathbb{R}$ be a function with the property that $f_p(x_p) = f_p(y_p)$ when $x_p - y_p \in \mathbb{Z}^n$ for all $p \in P\cup\infty$. Assume that $f(x) = 0$ for $x \in [0,1)^n \times \prod_{p\in P}\mathbb{Q}_p^n \setminus V^n$. Then we have that $\hat{f}(m) = 0$ if $m \in \mathbb{Q}^n\setminus\mathbb{Z}^n$.*

Proof. This proof follows straight from Lemma 11. Since $m \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ would mean that there exists an index i such that $m_i \in \mathbb{Q} \setminus \mathbb{Z}$ and we can use Lemma 11 on the function $f(m_1 \dots m_{i-1}, x, m_{i+1}, \dots, m_n)$. □

We now show that the function $H^{-s}(x)$ has the required properties to use Corollary 3. Let us write $x_i = p^e a/b$ for some $a, b, e \in \mathbb{Z}$ such that $p \nmid ab$. We have that $||x_i||_p = -e$, while we have $x_i + 1 = (p^e a + b)/b = p^e(a + bp^{-e})/b$. From this we see that if $e > 0$, $||x_i + 1||_p = 0$,if $e < 0$, $||x_i + 1||_p = -e = ||x_i||_p$ and if $e = 0$, $||x_i + 1||_p \leq 0$. Note that for $e = 0$ we do not get $||x_i + 1||_p = 0$ when $p|(a + b)$. This means that

$$H_p(x) = \max\{1, ||x_i||_p | 0 < i \leq n\} = \max\{1, ||x_i + 1||_p | 0 < i \leq n\} = H_p(x + 1).$$

This shows that if $x_p - y_p \in \mathbb{Z}^n$, we have $H_p(x_p)^{-s} = H_p(y_p)^{-s}$ for every $s \in \mathbb{C}$ and $p \in P \cup \infty$. Additionally, if for some i, $x_i \in [0, 1) \times \prod_{p \in P} \mathbb{Q}_p \setminus V$, there are infinitely many $p \in P$ such that $x_i \in \mathbb{Q}_p \setminus \mathbb{Z}_p$. Let us call Q the set of these p. We have $v_p(x_i) \geq 1$ for every $p \in Q$, and as such $H_p(x) \geq p$. This gives $|H_p(x)^{-s}| \leq |p^{-s}|$ for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. From this follows that $\prod_{p \in Q} |H_p(x)^{-s}| \leq |(\prod_{p \in Q} p)^{-s}| = 0$ as there are infinitely many primes p in Q. This means that we can use Corollary 3 to see that $\hat{H}(m, s) = 0$ for $m \in \mathbb{Q}^n \setminus \mathbb{Z}^n$. This means that we can write

$$\sum_{m \in \mathbb{Q}^n} \hat{H}(m, s) = \sum_{m \in \mathbb{Z}^n} \hat{H}(m, s)$$

for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$. In this sum, we can calculate every term separately. We start with the special case $m = 0$. By using that $H(x)^{-s} = 0$ for $x \in [0, 1)^n \times \prod_{p \in P} \mathbb{Q}_p^n \setminus V^n$, we can write

$$\begin{aligned}
\hat{H}(0, s) &= \int_{V^n} H(x)^{-s} \mathrm{d}x \\
&= \int_{[0,1]^n} \max\{1, |x_{i'}| \, | 0 < i' \leq n\}^{-s} \prod_{i=1}^{n} \mathrm{d}x_i \\
&\quad \prod_{p \in P} \int_{\mathbb{Q}_p^n} \max\{1, ||(x_p)_{i'}||_p | 0 < i' \leq n\}^{-s} \prod_{i=1}^{n} \mathrm{d}(x_p)_i \\
&= \prod_{p \in P} \int_{\mathbb{Q}_p^n} \max\{1, ||(x_p)_{i'}||_p | 0 < i' \leq n\}^{-s} \prod_{i=1}^{n} \mathrm{d}(x_p)_i .
\end{aligned}$$

Let us now focus on these integrals for one prime p at a time and define

$$\hat{H}_p(0, s) = \int_{\mathbb{Q}_p^n} \max\{1, ||(x_p)_{i'}||_p | 0 < i' \leq n\}^{-s} \prod_{i=1}^{n} \mathrm{d}(x_p)_i .$$

Note that this is not the Fourier transform of $H_p(x)^{-s}$. Additionally, we can write the integral as

$$\begin{aligned}\hat{H}_p(0,s) &= \int_{\mathbb{Q}_p^n} \max\{1, ||(x_p)_{i'}||_p | 0 < i' \le n\}^{-s} \prod_{i=1}^{n} \mathrm{d}(x_p)_i \\ &= \sum_{j_1,\dots,j_n=-\infty}^{\infty} \int_{(\mathbb{Z}_p^*)^n} \max\{1, ||p^{j_{i'}} x_{i'}||_p | 0 < i' \le n\}^{-s} \prod_{i=1}^{n} p^{-j_i} \mathrm{d}x_i \\ &= \sum_{j_1,\dots,j_n=-\infty}^{\infty} \int_{(\mathbb{Z}_p^*)^n} p^{s \min\{0, j_{i'} | 0 < i' \le n\}} \prod_{i=1}^{n} p^{-j_i} \mathrm{d}x_i \\ &= \sum_{j_1,\dots,j_n=-\infty}^{\infty} p^{s \min\{0, j_{i'} | 0 < i' \le n\}} (1-p^{-1})^n \prod_{i=1}^{n} p^{-j_i}.\end{aligned}$$

The way we evaluate this expression is by considering $k = \min\{0, j_{i'} | 0 < i' \le n\}$, if $k = 0$, we have that $j_i \ge 0$ for all $0 < 1 \le n$, if $k < 0$ and $(j_i)_{0<i\le n} \in [k,\infty)^n \setminus [k+1,\infty)^n \subset \mathbb{Z}^n$. Now we can sum over all possible values of k to get:

$$\hat{H}_p(0,s) = \sum_{k=-\infty}^{-1} \sum_{\substack{(j_i)_{0<i\le n} \in \\ [k,\infty)^n \setminus [k+1,\infty)^n}} p^{sk}(1-p^{-1})^n \prod_{i=1}^{n} p^{-j_i} + \sum_{j_1 \dots j_n = 0}^{\infty} (1-p^{-1})^n \prod_{i=1}^{n} p^{-j_i}.$$

Using the fact that $\sum_{i=0}^{\infty} p^{-i} = 1/(1-p^{-1})$, we get

$$\begin{aligned}\hat{H}_p(0,s) &= \sum_{k=-\infty}^{-1} \sum_{\substack{(j_i)_{0<i\le n} \in \\ [k,\infty)^n \setminus [k+1,\infty)^n}} p^{sk}(1-p^{-1})^n \prod_{i=1}^{n} p^{-j_i} + (1-p^{-1})^n \frac{1}{(1-p^{-1})^n} \\ &= \sum_{k=-\infty}^{-1} p^{sk}(1-p^{-1})^n \sum_{\substack{(j_i)_{0<i\le n} \in \\ [k,\infty)^n \setminus [k+1,\infty)^n}} \prod_{i=1}^{n} p^{-j_i} + 1.\end{aligned}$$

Similarly, for any $k \in \mathbb{Z}$, we have $\sum_{i=k}^{\infty} p^{-i} = p^{-k}/(1-p^{-1})$, which we can use when we split the sum over $[k,\infty)^n \setminus [k+1,\infty)^n$ into a sum over $[k,\infty)^n$ minus a sum over $[k+1,\infty)^n$ to get:

$$
\begin{aligned}
\hat{H}_p(0,s) &= \sum_{k=-\infty}^{-1} p^{sk}(1-p^{-1})^n \sum_{\substack{(j_i)_{0<i\le n}\in \\ [k,\infty)^n\setminus[k+1,\infty)^n}} \prod_{i=1}^{n} p^{-j_i} + 1 \\
&= \sum_{k=-\infty}^{-1} p^{sk}(1-p^{-1})^n \left(\sum_{(j_i)_{0<i\le n}\in[k,\infty)^n} \prod_{i=1}^{n} p^{-j_i} - \sum_{(j_i)_{0<i\le n}\in[k+1,\infty)^n} \prod_{i=1}^{n} p^{-j_i} \right) + 1 \\
&= \sum_{k=-\infty}^{-1} p^{sk}(1-p^{-1})^n \left(\frac{p^{-nk} - p^{-n(k+1)}}{(1-p^{-1})^n} \right) + 1 \\
&= \sum_{k=-\infty}^{-1} p^{(s-n)k}(1-p^{-n}) + 1 \\
&= \sum_{k=0}^{\infty} p^{-k(s-n)}(1-p^{-n}) - (1-p^{-n}) + 1 \\
&= \frac{1-p^{-n}}{1-p^{-(s-n)}} + p^{-n},
\end{aligned}
$$

where this last equality only holds when $\mathrm{Re}(s-n) > 1$, otherwise this sum may diverge. Now what we really want to know is for what $s \in \mathbb{C}$ the function $\hat{H}(0,s)$ converges. More specifically we want to know the largest real number $a > 0$ such that for all $s \in \mathbb{C}$ with $\mathrm{Re}(s) > a$, $\hat{H}(0,s)$ converges. Clearly it diverges for real s with $s \le n$ since then $\hat{H}_p(0,s)$ diverges. To see that it converges for real $s > n+1$, we will write it in the following way:

$$\hat{H}(0,s) = \hat{H}_\infty(0,s) \prod_{p\in P} \hat{H}_p(0,s) \tag{8}$$

$$= \prod_{p\in P} \left(p^{-n} + \frac{1-p^{-n}}{1-p^{-(s-n)}} \right) \tag{9}$$

$$= \prod_{p\in P} \left(1 - (1-p^{-n}) + \frac{1-p^{-n}}{1-p^{-(s-n)}} \right) \tag{10}$$

$$= \prod_{p\in P} \left(1 + (1-p^{-n}) \frac{p^{-(s-n)}}{1-p^{-(s-n)}} \right). \tag{11}$$

Note that if this product converges if it converges absolutely. And because the term $\frac{1-p^{-n}}{1-p^{-(s-n)}}$ is positive for real $s > n$, convergence of the product and absolute convergence of the product are the same for real $s > n$. To figure out how to deal with this product we will need one more theorem from chapter $II.1$ of [10] and one more lemma.

Lemma 12. *Let $f : \mathbb{Z}_{>0} \to R$, then the product $\prod_{p\in P}(1+|f(p)|)$ converges if and only if the sum $\sum_{p\in P} |f(p)|$ converges.*

Proof. Let us first define the set P_n for $n \in \mathbb{Z}_{>0}$ as the set of the n smallest primes. We will first show that if the product converges, the sum also converges.

Let us take a look at the product $\prod_{p\in P_n}(1+|f(p)|)$ for some $n>0$. We get

$$\prod_{p\in P_n}(1+|f(p)|) = 1+\sum_{j=1}^{n}\sum_{\substack{k\in P_n^j:\\ k_i=k_{i'}\implies i=i'}}\frac{1}{j!}\prod_{i=1}^{j}|f(k_i)|.$$

The $\frac{1}{j}$ term is there to avoid double counting. This equation follows from expanding the product on the right had side.

Note that all terms in this sum are positive since we take absolute values, which means that the following holds:

$$1+\sum_{k\in P_n}|f(k)| \leq 1+\sum_{j=1}^{n}\sum_{\substack{k\in P_n^j:\\ k_i=k_{i'}\implies i=i'}}\frac{1}{j!}\prod_{i=1}^{j}|f(k_i)|,$$

as the left side is just the right side for $j=1$. This then means that for every $n\geq 1$

$$0\leq\sum_{k\in P_n}|f(k)| < \prod_{p\in P_n}(1+|f(p)|).$$

And since $\lim_{n\to\infty}\prod_{p\in P_n}(1+|f(p)|)$ converges, so does $\lim_{n\to\infty}\sum_{k\in P_n}|f(k)| = \sum_{k\in P}|f(k)|$.

Let us now assume that $\sum_{k\in P}|f(k)|$ converges, and let $M_n=\sum_{k\in P_n}|f(k)|$. We again use

$$\prod_{p\in P_n}(1+|f(p)|) = 1+\sum_{j=1}^{n}\sum_{\substack{k\in P_n^j:\\ k_i=k_{i'}\implies i=i'}}\frac{1}{j!}\prod_{i=1}^{j}|f(k_i)|,$$

but now we want to use

$$\sum_{\substack{k\in P_n^j:\\ k_i=k_{i'}\implies i=i'}}\frac{1}{j!}\prod_{i=1}^{j}|f(k_i)| \leq \frac{1}{j!}(\sum_{k\in P_n}|f(k)|)^j.$$

This is clearly true for $j=1$ since both sides will be equal to $\sum_{k\in P_n}|f(k)|$. We use induction to show that it is true for every j. Assume that it is true for $j=m\geq 1$, we can multiply both sides of the inequality by $\sum_{k\in P_n}|f(k)|$ to get:

$$\sum_{k'\in P_n}\sum_{\substack{k\in P_n^m:\\ k_i=k_{i'}\implies i=i'}}\frac{1}{m!}|f(k')|\prod_{i=1}^{m}|f(k_i)| \leq \frac{1}{m!}(\sum_{k\in P_n}|f(k)|)^{m+1}.$$

If we now take a closer look at the left hand side, we can see we have all terms in $\sum_{\substack{k\in P_n^{(m+1)}:\\ k_i=k_{i'}\implies i=i'}}\frac{1}{(m+1)!}\prod_{i=1}^{m}|f(k_i)|$ exactly $m+1$ times and some additional positive terms. This means that we can write

$$\sum_{\substack{k\in P_n^{(m+1)}:\\ k_i=k_{i'}\implies i=i'}} \frac{1}{(m+1)!}\prod_{i=1}^{m}|f(k_i)| \leq \frac{1}{m+1}\sum_{k'\in P_n}\sum_{\substack{k\in P_n^{m}:\\ k_i=k_{i'}\implies i=i'}} \frac{1}{m!}|f(k')|\prod_{i=1}^{m}|f(k_i)|$$
$$\leq \frac{1}{m+1}\frac{1}{m!}(\sum_{k\in P_n}|f(k)|)^{m+1}$$
$$\leq \frac{1}{(m+1)!}(\sum_{k\in P_n}|f(k)|)^{m+1}.$$

And hence we have shown by induction that

$$\sum_{\substack{k\in P_n^{j}:\\ k_i=k_{i'}\implies i=i'}} \frac{1}{j!}\prod_{i=1}^{j}|f(k_i)| \leq \frac{1}{j!}(\sum_{k\in P_n}|f(k)|)^{j}.$$

Taking into account that $M_n = \sum_{k\in P_n}|f(k)| \leq \sum_{k\in P}|f(k)| = M$, we can write:

$$\prod_{p\in P_n}(1+|f(p)|) \leq 1+\sum_{j=1}^{n}\frac{1}{j!}M_n^j$$
$$\leq 1+\sum_{j=1}^{n}\frac{1}{j!}M^j.$$

Taking now the limit as $n\to\infty$, we get

$$\prod_{p\in P}(1+|f(p)|) = \lim_{n\to\infty}\prod_{p\in P_n}(1+|f(p)|) \leq \lim_{n\to\infty} 1+\sum_{j=1}^{n}\frac{1}{j!}M^j = e^M.$$

Lastly since $\prod_{p\in P}(1+|f(p)|) > 0$, this means it converges when $\sum_{p\in P}|f(p)|$ converges.

With this we have shown that $\prod_{p\in P}(1+|f(p)|)$ converges if and only if $\sum_{p\in P}|f(p)|$ converges. □

Theorem 6. *Let $f:\mathbb{Z}_{>0}\to\mathbb{R}$ be a function with the property that for coprime integers a,b, $f(ab)=f(a)f(b)$. Then*

$$\prod_{p\in P}\left(1+\sum_{u\geq 1}\frac{f(p^u)}{p^{us}}\right)$$

converges absolutely if and only if $\sum_{m\in\mathbb{Z}_{>0}}\frac{f(m)}{m^s}$ converges absolutely. Furthermore if they converge absolutely, both expressions converge to the same function.

Proof. The proof can be found in chapter $II.1$ of [10] where it is theorem 1.3, however, in this proof the author assumes the result from Lemma 12 but does not prove it. □

By choosing $f(p^u) = (1-p^{-n})p^{nu}$, we can immediately use Theorem 6 to see the product in equation (11) converges absolutely if and only if

$$\sum_{k\in\mathbb{Z}_{>0}} \left| k^{-(s-n)} \prod_{p\in P,\, p|k} (1-p^{-n}) \right|$$

converges. Note that we have $\prod_{p\in P,\, p|k}(1-p^{-n}) < 1$, so we get that this sum converges if and only if

$$\sum_{k\in\mathbb{Z}_{>0}} \left| k^{-(s-n)} \right|$$

converges. We know this converges when $\mathrm{Re}(s) > n+1$, but it diverges for $s = n+1$. This means that if the product in equation (11) converges absolutely for $\mathrm{Re}(s) > n+1$, which means it converges for $\mathrm{Re}(s) > n+1$. However, since for $s = n+1$, all terms in the product in equation (11) are positive and it diverges absolutely, the sum itself diverges.

Next we look at the case $m \neq 0$ with $m \in \mathbb{Q}^n$. This means there is an index $0 < k \leq n$ such that $m_k \neq 0$, we will do the same integral but this time we have to consider the extra term $e^{2\pi\iota\{mx\}}$.

$$\begin{aligned}
\hat{H}(m,s) &= \int_{V^n} H(x)^{-s} e^{-2\pi\iota\{mx\}} \mathrm{d}x \\
&= \int_{[0,1]^n} \max\{1, |x_{i'}||0 < i' \leq n\}^{-s} \prod_{i=1}^{n} e^{-2\pi\iota\{m_i x_i\}_\infty} \mathrm{d}x_i \\
&\prod_{p\in P} \int_{\mathbb{Q}_p^n} \max\{1, ||(x_p)_{i'}||_p | 0 < i' \leq n\}^{-s} \prod_{i=1}^{n} e^{-2\pi\iota\{m_i x_i\}_p} \mathrm{d}(x_p)_i \\
&= \prod_{i=1}^{n} \int_0^1 e^{2\pi\iota\{m_i x\}_\infty} \mathrm{d}x \prod_{p\in P} \int_{\mathbb{Q}_p^n} \max\{1, ||(x_p)_{i'}||_p | 0 < i' \leq n\}^{-s} \prod_{i=1}^{n} e^{-2\pi\iota\{m x_i\}_p} \mathrm{d}(x_p)_i \\
&= 0.
\end{aligned}$$

This last equality holds because $\int_0^1 e^{2\pi\iota\{m_k x\}_\infty} \mathrm{d}x = 0$. From this we gather that our original expression $\sum_{n\in\mathbb{Z}} \hat{H}(n,s)$ has a pole at $s = n+1$ and no poles when $\mathrm{Re}(s) > n+1$. Furthermore this pole at $s = n+1$ has degree 1 as can be seen by Theorem 6 along with the fact that

$$\sum_{k\in\mathbb{Z}_{>0}} k^{-(s-n)} \prod_{p\in P,\, p|k} (1-p^{-n})$$

has a pole of order 1 at $s = n+1$. Furthermore because

$$(s-(n+1)) \sum_{k\in\mathbb{Z}_{>0}} k^{-(s-n)} \prod_{p\in P,\, p|k} (1-p^{-n})$$

is holomorphic for $\mathrm{Re}(s) \geq n+1$, the function $(-(n+1))\hat{H}(0,s)$ is also holomorphic for $\mathrm{Re}(s) \geq n+1$.

This now means we can use the Corollary 2 on $\sum_{m\in\mathbb{Q}} H(m)^{-s} = \hat{H}(0,s)$. More specifically since $\mathbb{Q}^n$ is bijective to $\mathbb{N}$, we can choose a bijective function $\phi : \mathbb{N} \to \mathbb{Q}^n$ such that $\sum_{k\in\mathbb{N}} H(\phi(k))^{-s} = \sum_{m\in\mathbb{Q}^n} H(m)^{-s}$, for which Corollary 2 then gives us that

$$\lim_{B\to\infty} B^{-(n+1)} \sum_{H(\phi(k))\leq B} 1 = C$$

for a nonzero constant C, this is done by setting $\lambda_k = H(\phi(k))$ and $a_k = 1$. Alternatively we can sum over $\mathbb{Q}^n$ instead to get

$$\lim_{B\to\infty} B^{-(n+1)} \sum_{m\in\mathbb{Q}^n : H(m)\leq B} 1 = C. \tag{12}$$

If we now return to the observations made at the start of this section, we have viewed $\mathbb{QP}^n$ as $\mathbb{Q}^n \cup \mathbb{QP}^{n-1}$, we discovered that the number of points $x \in \mathbb{Q}^n$ satisfies equation (12). We can now write $\mathbb{QP}^n = \mathbb{Q}^n \cup \mathbb{Q}^{n-1} \cup \ldots \cup \mathbb{Q}^0$ to see that

$$\lim_{B\to\infty} B^{-(n+1)} N(B) = C.$$

This proves Theorem 1.

5 Expanding to subsets

Now that we have this new method for calculating the number of points of $\mathbb{QP}^n$ with height below some bound B, we can expand it to certain subsets of $\mathbb{QP}^n$. We will denote these subset X_S and they will depend on a set S. The way we will choose these subsets is by considering some $x \in \mathbb{QP}^n$ represented as a sequence of $n+1$ coprime integers $(x_0 : \ldots : x_n)$ and imposing restrictions on x_0. The restrictions we will put on x_0 is that we limit its p-adic valuation $v_p(x_0)$. More specifically we take the set $S \subset \mathbb{N}$ with $0 \in S$ and only allow points $x \in \mathbb{QP}^n$ with $v_p(x_0) \in S$ for all p. In the calculations we will not want to use this representation of coprime integers. Instead, similarly to in the previous section, we will look at the elements of the form $(1 : x'_1 : \ldots : x'_n)$ with $x'_i \in \mathbb{Q}$ for all $1 \leq i \leq n$. On these elements, the restriction is equivalent to stating that $\max(0, -v_p(x_i)|1 \leq i \leq n) \in S$ for all $p \in P$.

We will start with two specific cases, $S = \{0, v\}$ and $S = \{0\} \cup \{n \in \mathbb{Z} : n \geq v\}$ for some $v > 0$. For both of these cases we will prove Theorem 2 and the general case will follow straight from these two cases.

5.1 The first case.

The first case will be the easiest, we will pick $S = \{0, v\}$ for some $0 < v \in \mathbb{N}$. If we go back to the sum $\sum_{m\in\mathbb{Q}^n} H(m)^{-s}$, we now only want to consider $m \in \mathbb{Q}^n$ such that

$\max(0, -v_p(m_i)|1 \leq i \leq n) = 0$ or $\max(0, -v_p(m_i)|1 \leq i \leq n) = v$ for all $p \in P$. However, if we no longer sum over $\mathbb{Q}^n$ we can no longer use Theorem 5, so instead we will sum the function $H(m)^{-s}\delta_v(m)$ with $\delta_v(m) = 0$ when $\max(0, -v_p(m_i)|1 \leq i \leq n) \notin S$ for all $p \in P$ and 1 otherwise. Note that we can write $\delta_v(m) = \prod_{p \in P} \delta_{v,p}(m)$ where $\delta_{v,p}(m) = 0$ when $\max(0, -v_p(m_i)|1 \leq i \leq n) \notin S$ and 1 otherwise. The next steps are similar to what we already did in the previous chapter.

First we use Theorem 4 to write:

$$\sum_{m \in \mathbb{Q}^n} H(m)^{-s}\delta_v(m) = \sum_{m \in \mathbb{Q}^n} \int_V H(x)^{-s}\delta_v(x)e^{2\pi\iota\{mx\}}\mathrm{d}x.$$

We can again use Lemma 11 to see that the part of the sum where $m \in \mathbb{Q}^n \setminus \mathbb{Z}^n$ vanishes. To use this lemma, we have to show that $H(m)^{-s}\delta_v(m)$ is periodic over the integers, we already know from the last chapter that $H(m)^{-s}$ is periodic over $\mathbb{Z}^n$, so all we have to show is that for $x \in \mathbb{Z}^n$ and $m \in \mathbb{Q}^n$, $\delta_v(m + x) = \delta_v(m)$. To do this, notice that if $v_p(m_i) \geq 0$, then $v_p(m_i + x_i) \geq 0$ for any i, since if $v_p(m_i) \geq 0$, m_i is an integer, so $m_i + x_i$ is an integer and we get $v_p(m_i + x_i) \geq 0$. If $v_p(m_i) < 0$, then we can write $m_i = p^{-v_p(m_i)}a/b$, with a, b integers coprime to p, this means we can write $m_i + x_i = p^{v_p(m_i)}(a + bp^{-v_p(m_i)}x_i)/b$, and since x_i is an integer and $v_p(m_i) < 0$, $a + bx_ip^{-v_p(m_i)}$ is an integer coprime to p. This means we get that $v_p(m_i + x_i) = v_p(m_i)$. Both of these facts together mean that $\delta_v(m)$ is periodic over $\mathbb{Z}^n$, which means we have $H(m)^{-s}\delta_v(m) = H(m + x)^{-s}\delta_v(m + x)$ for all $x \in \mathbb{Z}^n$.

This means that by Lemma 11, the sum over $\mathbb{Q}^n$ becomes a sum over $\mathbb{Z}^n$ as all other terms are 0. This means we are left with the following:

$$\sum_{m \in \mathbb{Q}^n} H(m)^{-s}\delta_v(m) = \sum_{m \in \mathbb{Z}^n} \int_V H(x)^{-s}\delta_v(x)e^{2\pi\iota\{mx\}}\mathrm{d}x.$$

Now we can explicitly calculate the integrals in the same way as in the previous chapter. Note that similarly to last chapter, for $m \neq 0$, the integral is 0, as there is some i such that $m_i \neq 0$, so $\int_0^1 \max\{1, |x_\infty|\}^{-s}e^{2\pi\iota x_\infty m_i}\mathrm{d}x_\infty = \int_0^1 e^{2\pi\iota x_\infty m_i}\mathrm{d}x_\infty = 0$. In the case where $m = 0$, the integral becomes

$$\begin{aligned}\int_V H(x)^{-s}\delta_v(x)\mathrm{d}x &= \int_{[0,1]^n} \max\{1, |x_i| : 0 < i \leq n\}^{-s} \prod_{i=1}^n \mathrm{d}x_i \\ &\prod_{p \in P} \int_{\mathbb{Q}_p^n} \max\{1, ||x_i||_p : 0 < i \leq n\}^{-s}\delta_{v,p}(x_1, \ldots, x_n) \prod_{i=1}^n \mathrm{d}x_i \\ &= \prod_{p \in P} \int_{\mathbb{Q}_p^n} \max\{1, ||x_i||_p : 0 < i \leq n\}^{-s}\delta_{v,p}(x_1, \ldots, x_n) \prod_{i=1}^n \mathrm{d}x_i.\end{aligned}$$

We will again calculate this for one specific prime p and take the product later. This will be done in a similar way to the previous chapter:

$$\begin{aligned}
&\int_{\mathbb{Q}_p^n} \max\{1, ||(x_p)_{i'}||_p | 0 < i' \leq n\}^{-s} \delta_{v,p}(x_1, \ldots, x_n) \prod_{i=1}^{n} \mathrm{d}(x_p)_i \\
&= \sum_{j_1,\ldots,j_n=-\infty}^{\infty} \int_{(\mathbb{Z}_p^*)^n} \max\{1, ||p^{j_{i'}} x_{i'}||_p | 0 < i' \leq n\}^{-s} \delta_{v,p}(p^{j_1} x_1, \ldots, p^{j_n} x_n) \prod_{i=1}^{n} p^{-j_i} \mathrm{d}x_i \\
&= \sum_{j_1,\ldots,j_n=-\infty}^{\infty} \int_{(\mathbb{Z}_p^*)^n} p^{s \min\{0, j_{i'} | 0 < i' \leq n\}} \delta_{v,p}(p^{j_1}, \ldots, p^{j_n}) \prod_{i=1}^{n} p^{-j_i} \mathrm{d}x_i \\
&= \sum_{j_1,\ldots,j_n=-\infty}^{\infty} p^{s \min\{0, j_{i'} | 0 < i' \leq n\}} (1 - p^{-1})^n \delta_{v,p}(p^{j_1}, \ldots, p^{j_n}) \prod_{i=1}^{n} p^{-j_i}
\end{aligned}$$

So we get the same except for this extra term $\delta_{v,p}$, which is 0 for $\min(j_i | 1 \leq i \leq n) < 0$ and $\min(j_i | 1 \leq i \leq n) \neq -v$ and 1 otherwise. This allows us to rewrite the sum into three different sum and remove the $\delta_{v,p}$ in this way:

$$\begin{aligned}
&\sum_{j_1,\ldots,j_n \geq 0} (1 - p^{-1})^n \prod_{i=1}^{n} p^{-j_i} + \\
&\sum_{j_1,\ldots,j_n \geq -v} p^{-sv} (1 - p^{-1})^n \prod_{i=1}^{n} p^{-j_i} - \sum_{j_1,\ldots,j_n \geq 1-v} p^{-sv} (1 - p^{-1})^n \prod_{i=1}^{n} p^{-j_i},
\end{aligned}$$

where the first sum deals with the case where $\min(j_i | 1 \leq i \leq n) \geq 0$ and the second two sums together deal with the case where $\min(j_i | 1 \leq i \leq n) = -v$. Using the fact that for $\mathrm{Re}(s) > 1$, we have that $\sum_{i=0}^{\infty} p^{-s} = 1/(1 - p^{-1})$, we get the following expression:

$$\begin{aligned}
&(1 - p^{-1})^n \prod_{i=1}^{n} \frac{1}{1 - p^{-1}} + p^{-sv} (1 - p^{-1})^n \prod_{i=1}^{n} \frac{p^v}{1 - p^{-1}} - p^{-sv} (1 - p^{-1})^n \prod_{i=1}^{n} \frac{p^{v-1}}{1 - p^{-1}} \\
=&1 + p^{-sv} \left(p^{nv} - p^{nv-n} \right) \\
=&1 + p^{(n-s)v} \left(1 - p^{-n} \right).
\end{aligned}$$

So we find that for $\mathrm{Re}(s) > 1$, the final product will be:

$$\int_V H(x)^{-s} \delta_v(x) \mathrm{d}x = \prod_{p \in P} \left(1 + p^{(n-s)v} \left(1 - p^{-n} \right) \right).$$

We proceed in the same way as in the Section 4.5, this time we choose $f(p^u) = 0$ for $u \neq v$ and $f(p^v) = (1 - p^{-n}) p^{nv}$. This allows us to use Theorem 6 which tells us this product converges absolutely if and only if

$$\sum_{x \in \mathbb{Z}_{>0}} f(k)/k^s = \sum_{x \in \mathbb{Z}_{>0}} f(k^v)/k^{vs} = \sum_{x \in \mathbb{Z}_{>0}} k^{(n-s)v} \prod_{p \in P,\, p | k} (1 - p^{-n})$$

converges absolutely. Similarly in Section 4.5, this converges absolutely if $\mathrm{Re}(s) > n + 1/v$, but it diverges absolutely for $s = n + 1/v$. This means the product diverges absolutely for $s = n + 1/v$, but for this value of s, all terms in the product are positive, so the product diverges for $s = n + 1/v$. Furthermore because the function

$$\sum_{x\in\mathbb{Z}_{>0}} k^{(n-s)v} \prod_{p\in P,\, p|k} (1 - p^{-n})$$

has a pole of order 1 in $s = n + 1/v$, so does $\sum_{m\in\mathbb{Q}^n} H(x)^{-s}\delta_v(x)$. Additionally, since

$$(s - (n + 1/v)) \sum_{x\in\mathbb{Z}_{>0}} k^{(n-s)v} \prod_{p\in P,\, p|k} (1 - p^{-n})$$

is holomorphic for $\mathrm{Re}(s) \geq n + 1/v$, the function $(s - (n + 1/v)) \sum_{m\in\mathbb{Q}^n} H(x)^{-s}\delta_v(x)$ is also holomorphic for $\mathrm{Re}(s) \geq n + 1/v$. This means we can use Corollary 2 on $\sum_{m\in\mathbb{Q}^n} H(m)^{-s} = h(s)/(1 + v(n - s))$ and again use a bijection ϕ between $\mathbb{Q}^n$ and $\mathbb{N}$ to get that

$$\lim_{B\to\infty} B^{-(n+1/v)} \sum_{m\in\mathbb{Q}^n : H(m)\leq B} \delta_v(m) = C$$

for a nonzero constant C. And once again this sum is the number of points in our subset of $\mathbb{QP}^n \setminus \mathbb{QP}^{n-1} \cap X_S$, this time however, we can use Theorem 1 that

$$\lim_{B\to\infty} B^{-(n+1/v)} \sum_{x\in\mathbb{QP}^{n-1} : H(x)\leq B} 1 = 0.$$

This means we have

$$\lim_{B\to\infty} B^{-(n+1/v)} \sum_{x\in X_S\subset\mathbb{QP}^n : H(x)\leq B} \delta_v(m) = C \neq 0.$$

This proves Theorem 2 for $S = \{0, v\}$.

5.2 The second case

For the second case we will pick $S = \{0\} \cup \{v + k : k \in \mathbb{Z}_{\geq 0}\}$. The approach is very similar to the first case in the sense that we still want to calculate $\sum_{m\in\mathbb{Q}^n} H(m)^{-s}$ but we again add a function δ_v defined by $\delta_v(m) = 0$ if for some $p \in P$, $-\min(0, v_p(m_i) : 1 \leq i \leq n) \notin S$ and 1 otherwise. We want to calculate

$$\sum_{m\in\mathbb{Q}^m} H(m)^{-s}\delta_v(m) = \sum_{m\in\mathbb{Q}^n} \int_V H(x)^{-s}\delta_v(x)e^{-2\pi\iota\{mx\}}\mathrm{d}x. \tag{13}$$

This equality follows from Theorem 4.

By the same argument as in the previous case, we can again see that if $v_p(m_i) \geq 0$, then $v_p(m_i+x) \geq 0$ for all primes p, $x \in \mathbb{Z}$ and $m_i \in \mathbb{Q}$ and if $v_p(m_i) < 0$ then $v_p(m_i+x) = v_p(m_i)$. Hence we can again conclude that $\delta_v(m) = \delta_v(m+x)$ for all $x \in \mathbb{Z}^n$ and $m \in \mathbb{Q}^n$. We already know that the height function has the property that $H(m) = H(m + x)$ for all $m \in \mathbb{Q}^n$ and

$x \in \mathbb{Z}^n$, as we showed this in Section 4.5. So we get that $H(m)^{-s}\delta_v(m)$ also has this property. This means that we can use Lemma 11 to see that, on the right hand side of equation (13), the sum over $\mathbb{Q}^n$ becomes a sum over $\mathbb{Z}^n$ as the other terms are 0. Hence, we get:

$$\sum_{m\in\mathbb{Q}^m} H(m)^{-s}\delta_v(m) = \sum_{m\in\mathbb{Z}^n}\int_V H(x)^{-s}\delta_v(x)e^{-2\pi\iota\{mx\}}\mathrm{d}x.$$

We can evaluate these integrals, we do this for general m but we quickly see that they are 0 for all nonzero $m \in \mathbb{Q}^n$. We get:

$$\begin{aligned}\int_V \frac{\delta_v(x)}{H(x)^s}e^{-2\pi\{mx\}}\mathrm{d}x &= \int_{[0,1]^n} \max\{1,|x_i| : 0 < i \leq n\}^{-s}\prod_{i=1}^n e^{-2\pi\iota m_i x_i}\mathrm{d}x_i\\ &\prod_{p\in P}\int_{\mathbb{Q}_p^n}\max\{1,||x_i||_p : 0 < i \leq n\}^{-s}\delta_{v,p}((x_i)_{1\leq i\leq n})\prod_{i=1}^n e^{-2\pi\iota\{m_i x_i\}_p}\mathrm{d}x_i\\ &= \prod_{i=1}^n\int_0^1 e^{-2\pi\iota m_i x_i}\mathrm{d}x_i\\ &\prod_{p\in P}\int_{\mathbb{Q}_p^n}\max\{1,||x_i||_p : 0 < i \leq n\}^{-s}\delta_{v,p}((x_i)_{1\leq i\leq n})\prod_{i=1}^n e^{-2\pi\iota\{m_i x_i\}_p}\mathrm{d}x_i,\end{aligned}$$

If there exists an index i such that $m_i \neq 0$, then $\int_0^1 e^{2\pi\iota m_i x_i}\mathrm{d}x_i = 0$. Hence, the whole expression is 0. This means that we only have to consider the case $m = 0$. This gives us:

$$\sum_{m\in\mathbb{Q}^m} H(m)^{-s}\delta_v(m) = \prod_{p\in P}\int_{\mathbb{Q}_p^n}\max\{1,||x_i||_p : 0 < i \leq n\}^{-s}\delta_{v,p}((x_i)_{1\leq i\leq n})\prod_{i=1}^n \mathrm{d}x_i.$$

We can compute all of these integrals for an arbitrary prime p and we take the product afterwards. Expanding the integral gives:

$$\begin{aligned}&\int_{\mathbb{Q}_p^n}\max\{1,||x_i||_p : 0 < i \leq n\}^{-s}\delta_{v,p}((x_i)_{1\leq i\leq n})\prod_{i=1}^n \mathrm{d}x_i\\ &= \sum_{j_1,\ldots,j_n=-\infty}^{\infty}\int_{(\mathbb{Z}_p^*)^n}\max\{1,||p^{j_{i'}}x_{i'}||_p|0 < i' \leq n\}^{-s}\delta_{v,p}((p^{j_i}x_i)_{1\leq i\leq n})\prod_{i=1}^n p^{-j_i}\mathrm{d}x_i\\ &= \sum_{j_1,\ldots,j_n=-\infty}^{\infty}\int_{(\mathbb{Z}_p^*)^n} p^{s\min\{0,j_{i'}|0<i'\leq n\}}\delta_{v,p}((p^{j_i})_{1\leq i\leq n})\prod_{i=1}^n p^{-j_i}\mathrm{d}x_i\\ &= \sum_{j_1,\ldots,j_n=-\infty}^{\infty} p^{s\min\{0,j_{i'}|0<i'\leq n\}}(1-p^{-1})^n\delta_{v,p}((p^{j_i})_{1\leq i\leq n})\prod_{i=1}^n p^{-j_i}\end{aligned}$$

The way we solve this integral is by defining $I_k \subset \mathbb{Z}^n$ such that for $(j_i)_{0<i\leq n}$, we have $\min\{0, j_{i'}|0 < i' \leq n\} = -k$. Note that $I_k = \emptyset$ for $k < 0$. This allows us to rewrite the sum to

$$\sum_{k=0}^{\infty} \sum_{(j_i)_{0<i\leq n}\in I_k} p^{s\min\{0,j_{i'}|0<i'\leq n\}}(1-p^{-1})^n \delta_{v,p}((p^{j_i})_{1\leq i\leq n}) \prod_{i=1}^{n} p^{-j_i}. \tag{14}$$

Note that the $\delta_{v,p}((p^{j_i})_{1\leq i\leq n})$ term is 0 if $\max(0, v_p(p^{j_i}) : 1 \leq i \leq n) \notin S$ and 1 otherwise. This means that it is 1 if and only if there exists an index i such that $j_i \geq v$ or for all indices i, $j_i \leq 0$. We can write this in terms of k, we have that $\delta_{v,p}((p^{j_i})_{1\leq i\leq n}) = 0$ if $k \notin S$ and $\delta_{v,p}((p^{j_i})_{1\leq i\leq n}) = 1$ if $k \in S$. Next, we can see that for $k > 0$, we have $(j_i)_{1\leq i\leq n} \in ([-k,\infty)^n \setminus [1-k,\infty)^n) \cap \mathbb{Z}^n$, while for $k = 0$, we have $(j_i)_{1\leq i\leq n} \in [0,\infty)^n \cap \mathbb{Z}^n$. We will now calculate the sum in equation (14) by calculating the summand for all values of k. For $k = 0$, the summand becomes:

$$\begin{aligned}
\sum_{j_1,\dots,j_n=0}^{\infty} p^{s\min\{0,j_{i'}|0<i'\leq n\}}(1-p^{-1})^n \prod_{i=1}^{n} p^{-j_i} &= \sum_{j_1,\dots,j_n=0}^{\infty} (1-p^{-1})^n \prod_{i=1}^{n} p^{-j_i} \\
&= (1-p^{-1})^n \left(\sum_{j=1}^{\infty} p^{-j}\right)^n \\
&= 1.
\end{aligned}$$

While for $0 < k < v$, we have $-k \notin S$, so the summand is 0. Finally for $k \geq v$, we get:

$$\begin{aligned}
&\sum_{(j_i)_{0<i\leq n}\in I_k} p^{s\min\{0,j_{i'}|0<i'\leq n\}}(1-p^{-1})^n \delta_{v,p}((p^{j_i})_{1\leq i\leq n}) \prod_{i=1}^{n} p^{-j_i} \\
&= \sum_{(j_i)_{0<i\leq n}\in I_k} p^{-sk}(1-p^{-1})^n \delta_{v,p}((p^{j_i})_{1\leq i\leq n}) \prod_{i=1}^{n} p^{-j_i} \\
&= p^{-sk}(1-p^{-1})^n \left(\sum_{j_1,\dots,j_n=-k}^{\infty} \prod_{i=1}^{n} p^{-j_i} - \sum_{j_1,\dots,j_n=1-k}^{\infty} \prod_{i=1}^{n} p^{-j_i} \right) \\
&= p^{-sk}(1-p^{-1})^n \left(\left(\sum_{j=-k}^{\infty} p^{-j_i}\right)^n - \left(\sum_{j=1-k}^{\infty} p^{-j_i}\right)^n \right) \\
&= p^{-sk}(1-p^{-1})^n \left(p^{nk}\left(\sum_{j=0}^{\infty} p^{-j_i}\right)^n - p^{nk-n}\left(\sum_{j=0}^{\infty} p^{-j_i}\right)^n \right) \\
&= p^{-sk}(1-p^{-1})^n \left(p^{nk} - p^{nk-n}\right) \left(\sum_{j=0}^{\infty} p^{-j_i}\right)^n \\
&= p^{(n-s)k}(1-p^{-n}).
\end{aligned}$$

Now that we have calculated the summand of equation (14) for every k, we can calculate the whole sum:

$$\begin{aligned}
&\sum_{k=0}^{\infty} \sum_{(j_i)_{0<i\leq n}\in I_k} p^{s\min\{0,j_{i'}|0<i'\leq n\}}(1-p^{-1})^n \delta_{v,p}((p^{j_i})_{1\leq i\leq n}) \prod_{i=1}^{n} p^{-j_i} \\
&= 1 + \sum_{k=v}^{\infty} p^{(n-s)k}(1-p^{-n}) \\
&= 1 + (1-p^{-n})p^{(n-s)v} \sum_{k=0}^{\infty} p^{(n-s)k} \\
&= 1 + \frac{(1-p^{-n})p^{(n-s)v}}{1-p^{-(s-n)}}.
\end{aligned}$$

If we now return to our product over all primes, we get our expression for $\sum_{m\in\mathbb{Q}^n} H(m)^{-s}$:

$$\sum_{m\in\mathbb{Q}^n} H(m)^{-s} = \prod_{p\in P}\left(1 + \frac{1-p^{-n}}{1-p^{-(s-n)}}p^{(n-s)v}\right).$$

We now want to check for which values of s this product diverges. We use Theorem 6 with $f(p^u) = 0$ for $u < v$ and $f(p^u) = (1-p^{-n})p^{nk}$ for $u \geq v$, to see that it converges absolutely if and only if

$$\sum_{k\in\mathbb{Z}_{>0}} f(k)/k^{-s}$$

converges. Note that for real s we have

$$\sum_{k\in\mathbb{Z}_{>0}} f(k^v)k^{-vs} \leq \sum_{k\in\mathbb{Z}_{>0}} f(k)k^{-s} \leq M(v) \sum_{k\in\mathbb{Z}_{>0}} f(k^v)k^{-vs}$$

for a bound $M(v)$, this means $\sum_{k\in\mathbb{Z}_{>0}} f(k)/k^{-s}$ converges if and only if

$$\sum_{k\in\mathbb{Z}_{>0}} f(k^v)k^{-vs} = \sum_{k\in\mathbb{Z}_{>0}} k^{-v(s-n)} \prod_{p\in P,\, p|k} (1-p^{-n}).$$

This sum diverges for $s = n + 1/v$, but it converges absolutely for $\mathrm{Re}(s) > n + 1/v$. Furthermore it has a pole of order 1 in $s = n+1/v$ and the function $(s-(n+1/v))\sum_{k\in\mathbb{Z}_{>0}} f(k)/k^{-s}$ is holomorphic for $\mathrm{Re}(s) > n + 1/v$. From this we find that $\sum_{m\in\mathbb{Q}^n} H(m)^{-s}\delta_v(m)$ has a pole of order 1 in $s = n + 1/v$ and the function

$$(s-(n+1/v)) \sum_{m\in\mathbb{Q}^n} H(m)^{-s}\delta_v(m)$$

is holomorphic for $\mathrm{Re}(s) > n + 1/v$. This means that we can use the same trick we did in the previous chapter with a bijective function $\phi : \mathbb{N} \to \mathbb{Q}^n$ and use Corollary 2 on $\sum_{k\in\mathbb{N}} H(\phi(k))^{-s}\delta_v(\phi(k))$ to see that

$$\lim_{B\to\infty} B^{-(n+1/v)} \sum_{m\in\mathbb{Q}^n : H(m)<B} \delta_v(n) = C$$

for some constant C. Once again this sum is the number of points in our subset of $\mathbb{QP}^n \setminus \mathbb{QP}^{n-1} \cap X_S$, this time however, we can use Theorem 1 that

$$\lim_{B\to\infty} B^{-(n+1/v)} \sum_{x\in\mathbb{QP}^{n-1} : H(x)\leq B} 1 = 0.$$

This means we have

$$\lim_{B\to\infty} B^{-(n+1/v)} \sum_{x\in X_S \subset \mathbb{QP}^n : H(x)\leq B} \delta_v(m) = C \neq 0.$$

This proves Theorem 2 for $S = \{0\} \cup \mathbb{Z}_{\geq v}$.

5.3 The case for general sets

Lastly, we study the case for general sets S. For any set $S \subset \mathbb{Z}_{\geq 0}$ with $0 \in S$, let v be the smallest nonzero integer such that $v \in S$. Let now $S' = \{0, v\}$ and $S'' = \{0\} \cup \{n \in \mathbb{Z} : n \geq v\}$. Note that since $S' \subset S$, if we let $x \in \mathbb{QP}^n$ be represented by a sequence of $n+1$ coprime integers $x_0, \ldots, x_n$, if $v_p(x_0) \in S'$ then $v_p(x_0) \in S$. And since $S \subset S''$, if $v_p(x_0) \in S$ then $v_p(x_0) \in S''$. From this it follows by definition that

$$X_{S'} \subset X_S \subset X_{S''}.$$

This immediately leads to the following inequalities

$$|\{x \in X_{S'} : H(x) \leq B\}| \leq |\{x \in X_S : H(x) \leq B\}| \leq |\{x \in X_{S''} : H(x) \leq B\}|.$$

Inserting the limits and the power $B^{-(n+1/v)}$ gives:

$$\begin{aligned} C_{S'} &= \lim_{B\to\infty} B^{-(n+1/v)} |\{x \in X_{S'} : H(x) \leq B\}| \\ &\leq \lim_{B\to\infty} B^{-(n+1/v)} |\{x \in X_S : H(x) \leq B\}| \\ &\leq \lim_{B\to\infty} B^{-(n+1/v)} |\{x \in X_{S''} : H(x) \leq B\}| = C_{S''}. \end{aligned}$$

So we have $\lim_{B\to\infty} B^{-(n+1/v)} |\{x \in X_S : H(x) \leq B\}| = C_S$ for some constant C_S and general S where v is the smallest nonzero integer in S.

6 Outlook

We have proven Theorem 1 and Theorem 2. The logical next step is to expand Theorem 2 to more subsets of $\mathbb{QP}^n$. The subsets $X_S \subset \mathbb{QP}^n$ restrict the first coordinate of $(x_0 : x_1 : \ldots : x_n) \in \mathbb{QP}^n$, we could instead restrict any other coordinate. Define $X_{S,i} \subset \mathbb{QP}^n$, for $0 \leq i \leq n$, to be the subset of $\mathbb{QP}^n$ that restricts the ith coordinate of $(x_0 : x_1 : \ldots : x_n) \in \mathbb{QP}^n$ in the same way X_S restricts the first. By symmetry, Theorem 2 should still hold if we consider $X_{S,i}$ instead of X_S. If we try to apply the method of section 5 to $X_{S,i}$ for $i \neq 0$ we run into a problem. First we have to change the δ-function, if we do this however, we can no longer use Lemma 11 on $\sum_{m \in \mathbb{Q}^n} \hat{H}(x, s)\delta(x)$. This means we are stuck with a sum over $\mathbb{Q}^n$ which we are unable to solve.

This becomes a problem when we try to restrict multiple coordinates of $\mathbb{QP}^n$ at the same time, as we can no longer use symmetry to get a result. Let $X_{S_0,\ldots,S_n} \subset \mathbb{QP}^n$ be the subset that restricts the ith coordinate of $(x_0 : x_1 : \ldots : x_n) \in \mathbb{QP}^n$ using S_i for all $0 \leq i \leq n$. So if $(x_0 : x_1 : \ldots : x_n) \in \mathbb{QP}^n$ is a representation of coprime integers, $(x_0 : x_1 : \ldots : x_n) \in X_{S_0,\ldots,S_n}$ if and only if $v_p(x_i) \in S_i$ for all $p \in P$ and all $0 \leq i \leq n$. We can no longer use the method from section 5 to calculate the growth rate of $N(B)$ belonging to $X_{S_0,\ldots,S_n}$, however, it may be possible to combine the method from section 3 with the results of Theorem 2 to find this growth rate.

7 Appendix

7.1 Useful Fourier transforms

The Fourier transform of a function $f : \mathbb{R} \to \mathbb{R}$ is given by

$$\hat{f}(\tau) = \int_{-\infty}^{\infty} e^{-\iota x\tau} f(x)\mathrm{d}x,$$

which can be found in section 6.2 of [8]. The Fourier transform of a continuous function f has the property

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{\iota x\tau}\hat{f}(\tau)\mathrm{d}\tau$$

In the following subsections, we will calculate the Fourier transforms of functions used in the proofs in this thesis.

We will use the fact that for two functions f and g, the Fourier transforms of the product $h = fg$ satisfy

$$\hat{h}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(y)\hat{g}(x-y)\mathrm{d}y. \tag{15}$$

This can be easily shown by substituting $z = x - y$ in the following way:

$$h(t) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(y)\hat{g}(x-y)\mathrm{d}y e^{\iota xt}\mathrm{d}x = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \hat{f}(y)\hat{g}(z)e^{\iota(z+y)t}\mathrm{d}y\mathrm{d}z = f(t)g(t).$$

These are all the tools we need to calculate the Fourier transforms.

7.1.1 Fourier transform of $\alpha_\epsilon(t)$

The function $\alpha_\epsilon(t)$ is defined as

$$\alpha_\epsilon(t) = \frac{2}{\pi\epsilon t^2}\sin(\epsilon t/2)\sin((2T+\epsilon)t/2)$$

Note that we can write this as

$$\alpha_\epsilon(t) = \frac{1}{2\pi\epsilon}\frac{2\sin(\epsilon t/2)}{t}\frac{2\sin((2T+\epsilon)t/2)}{t},$$

so we can use equation (15) together with the Fourier transform of $\sin(at)/t$ to get the Fourier transform of $\alpha_\epsilon(t)$. Let us now define $f_a(t) = \sin(ax)/x$, this function has Fourier transform

$$\begin{aligned}\hat{f}_a(\tau) &= \int_{-\infty}^{\infty} \frac{\sin(ax)}{x} e^{-\iota x\tau}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\sin(ax)}{x}\cos(-x\tau)\mathrm{d}x + \iota\int_{-\infty}^{\infty} \frac{\sin(ax)}{x}\sin(-x\tau)\mathrm{d}x.\end{aligned}$$

The second of these two integrals is 0 since it is the convergent integral of an odd function over a domain that is symmetric around 0. Using the identity $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$ and the fact that $\sin(bx)/x$ is an even function for all b, we can write this as:

$$\begin{aligned}\hat{f}_a(\tau) &= \int_{-\infty}^{\infty} \frac{\sin((a+\tau)x) + \sin((a-\tau)x)}{2x}\mathrm{d}x \\ &= \int_{-\infty}^{\infty} \frac{\sin((a+\tau)x)}{2x}\mathrm{d}x + \int_{-\infty}^{\infty} \frac{\sin((a-\tau)x)}{2x}\mathrm{d}x \\ &= \int_{0}^{\infty} \frac{\sin((a+\tau)x)}{x}\mathrm{d}x + \int_{0}^{\infty} \frac{\sin((a-\tau)x)}{x}\mathrm{d}x \\ &= \int_{0}^{\sigma(a+\tau)\infty} \frac{\sin(x)}{x}\mathrm{d}x + \int_{0}^{\sigma(a-\tau)\infty} \frac{\sin(x)}{x}\mathrm{d}x.\end{aligned}$$

Here σ is the sign function. This immediately gives us that this integral is 0 when $\sigma(a-\tau) = -\sigma(a+\tau)$. This means that we need to compute the integral $\int_0^\infty \sin(x)/x\mathrm{d}x$. We proceed as follows:

$$\begin{aligned}\int_0^\infty \sin(x)/x\mathrm{d}x &= \int_0^\infty \int_0^\infty e^{-xy}\sin(x)\mathrm{d}y\mathrm{d}x \\ &= \int_0^\infty \int_0^\infty e^{-xy}\sin(x)\mathrm{d}x\mathrm{d}y \\ &= \int_0^\infty \left[\frac{-e^{-xy}(y\sin(x)+\cos(x))}{y^2+1}\right]_{x=0}^{\infty}\mathrm{d}y \\ &= \int_0^\infty \frac{1}{y^2+1}\mathrm{d}y \\ &= \arctan(y)|_{y=0}^{\infty} = \pi/2.\end{aligned}$$

Combining everything we get that $\hat{f}_a(\tau) = \pi$ when $-|a| < \tau < |a|$ and $\hat{f}(\tau) = 0$ everywhere else. For the Fourier transform of $\alpha_\epsilon(t)$, we need $\hat{f}_{\epsilon/2}(x)\hat{f}_{T+\epsilon/2}(\tau-x)$, which is 0 for $|x| > |\epsilon/2|$ or $|\tau-x| > T+\epsilon/2$ and π^2 otherwise. This means it is π^2 on the interval $x \in [-\epsilon/2, \epsilon/2] \cap [\tau-(T+\epsilon/2), \tau+(T+\epsilon/2)]$ and 0 outside this interval. This interval has length 0 if $|\tau| > T+\epsilon$, it has length ϵ if $|\tau| < T$ and it has length $T+\epsilon-|\tau|$ if $T \leq |\tau| \leq T+\epsilon$.

Using this we find

$$\hat{\alpha}_\epsilon(\tau) = \frac{1}{\pi^2\epsilon}\int_{-\infty}^{\infty} \hat{f}_{\epsilon/2}(x)\hat{f}_{T+\epsilon/2}(\tau-x)\mathrm{d}x,$$

which is 0 for $|\tau| > T+\epsilon$, it is 1 for $|\tau| < T$ and it is $(T+\epsilon-|\tau|)/\epsilon$ for $T \leq |\tau| \leq T+\epsilon$.

7.1.2 Fourier transform of $\beta(t)$

The function $\beta(t)$ is defined as

$$\beta(t) = \frac{1}{2\pi}\frac{\sin(t/2)^2}{(t/2)^2}.$$

This can we written as

$$\beta(t) = \frac{2}{\pi}\left(\frac{\sin(t/2)}{t}\right)^2.$$

So by equation (15), we can write

$$\hat{\beta}(\tau) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \hat{f}_{1/2}(x) \hat{f}_{1/2}(\tau - x) \mathrm{d}x.$$

In the previous section we calculated $\hat{f}_a(\tau)$, which is π for $|\tau| < a$ and 0 otherwise. This means the function $\hat{f}_{1/2}(x)\hat{f}_{1/2}(\tau - x)$ is 0 for $|x| \geq 1/2$ or $|\tau - x| \geq 1/2$ and π^2 otherwise. It is π^2 on the interval $x \in [-1/2, 1/2] \cap [\tau - 1/2, \tau + 1/2]$, which has length $\max(0, 1 - |\tau|)$, and it is 0 everywhere else.

This means we get

$$\hat{\beta}(\tau) = \max(0, 1 - |\tau|).$$

7.1.3 Fourier transform of $B_x(t)$

The function $B_x(t)$ is defined as

$$B_x(t) = \begin{cases} 0 & \text{if } t \leq 0 \\ \frac{c}{\Gamma(b)} e^{-xt}(1 - e^{-xt}) t^{b-1} & \text{if } t > 0. \end{cases}$$

We can calculate the Fourier transform straight from the definition. We get

$$\begin{aligned}
\hat{B}_x(\tau) &= \int_{-\infty}^{\infty} e^{-\iota t \tau} B_x(t) \mathrm{d}t \\
&= \int_0^{\infty} e^{-\iota t \tau} (c/\Gamma(b)) e^{-xt}(1 - e^{-xt}) t^{b-1} \mathrm{d}t \\
&= \frac{c}{\Gamma(b)} \int_0^{\infty} (e^{-t(\iota\tau + x)} - e^{-t(\iota\tau + 2x)}) t^{b-1} \mathrm{d}t \\
&= \frac{c}{\Gamma(b)} \left(\int_0^{\infty} e^{-t(\iota\tau + x)} t^{b-1} \mathrm{d}t - \int_0^{\infty} e^{-t(\iota\tau + 2x)} t^{b-1} \mathrm{d}t \right) \\
&= \frac{c}{\Gamma(b)} \left(\int_0^{\infty} e^{-u} \frac{u^{b-1}}{(\iota\tau + x)^{b-1}} \frac{\mathrm{d}u}{\iota\tau + x} - \int_0^{\infty} e^{-v} \frac{v^{b-1}}{(\iota\tau + 2x)^{b-1}} \frac{\mathrm{d}v}{\iota\tau + 2x} \right) \\
&= \frac{c}{\Gamma(b)} \left(\frac{\Gamma(b)}{(\iota\tau + x)^b} - \frac{\Gamma(b)}{(\iota\tau + 2x)^b} \right) \\
&= c((\iota\tau + x)^{-b} - (\iota\tau + 2x)^{-b}).
\end{aligned}$$

We used substitutions $u = t(\iota\tau + x)$ and $v = t(\iota\tau + x)$.

7.1.4 Fourier transform of $g_x(t)$

The function $g_x(t)$ is defined as

$$g_x(t) = A(t) e^{-(a+x)t}(1 - e^{-xt})$$

with $a + x > 0$ and $A(t)$ a non-decreasing function which is 0 for $t \leq 0$. We will calculate the Fourier transform in terms of the function $G(z) = \int_0^{\infty} A(t) e^{-(a+x)t} \mathrm{d}t - c/z^b$. We get

$$\begin{aligned}\hat{g}_x(y) &= \int_0^\infty A(t)e^{-(a+x)t}(1-e^{-xt})e^{-\iota ty}\mathrm{d}t \\ &= \int_0^\infty A(t)e^{-(a+x+\iota y)t}\mathrm{d}t - \int_0^\infty A(t)e^{-(a+2x+\iota y)t}\mathrm{d}t \\ &= G(x+\iota y) + c/(x+\iota y)^b - G(2x+\iota y) - c/(2x+\iota y)^b.\end{aligned}$$

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