UTRECHT UNIVERSITY
Institute for Theoretical Physics

Theoretical Physics master thesis

Exploration of Momentum-Dependent Power Law in Strange Metals

Author:
LUO D. Boyang

Supervisors:
Prof. dr. ir. H.T.C. Stoof
drs. O. A. Eastman
Second Examiner:
Dr. L. (Lars) Fritz

January 29, 2025

Abstract

High-temperature superconductors, particularly cuprates, have long intrigued both theoretical and experimental researchers. Despite recent progress in phenomenological explanations, such as the application of the AdS/CMT, the exact physical mechanism underlying high-temperature superconductivity remains elusive. This thesis investigates a physical model of the strange metal phase, a hallmark of high-temperature superconducting systems, with the aim of gaining insights into the nature of high-T_c superconductivity. One key feature of the strange metal phase is its momentum-dependent power law behavior, which deviates from the predictions of conventional Fermi-liquid theory, posing a significant theoretical challenge. Inspired by the work of M. Khodas, et al. on one-dimensional Luttinger liquids[1], which demonstrated momentum-dependent power law, this thesis explores an alternative approach for extending these results to higher dimensions. While the original work relies on techniques such as bosonization, which are difficult to generalize to two-dimensional systems like cuprates, we propose a novel approach utilizing path-integral formalism and the Hubbard-Stratonovich transformation, serving as an analog to bosonization in two-dimensions. By incorporating a linearization scheme, we successfully reproduce the momentum-dependent power law behavior in one-dimensional systems, validating the robustness of our method. Building on this success, we extend our framework to two-dimensional systems, laying the groundwork for future studies. We anticipate that this generalization will elucidate the momentum-dependent power law behavior in two-dimensional strange metals, advancing our understanding of the mechanisms driving high-temperature superconductivity.

Contents

Chapter 1

Introduction

Cuprates, a large class of high-T_c superconducting systems, have captured significant research interest for decades. However, their underlying mechanisms remain poorly understood. The exact mechanism of the superconducting phase in these systems remains a mystery, and this enigma extends to two other notable phases—the strange metal phase and the pseudogap phase (as shown in the phase diagram in Fig. 1.1). In this thesis, we focus on exploring the strange metal phase, particularly its hallmark momentum-dependent power law behavior.

The term non-Fermi liquid arises from the inability of Fermi-liquid theory to describe the strange metal phase. In Fermi-liquid theory, interacting particles are treated as quasi-independent quasi-particles, with a finite lifetime $\tau(q,\omega)$ characterized by the nonzero imaginary part of the electron self-energy $\Sigma(q,\omega)$. Here, q and ω represent the momentum and energy of the electron, respectively. The self-energy can be experimentally measured using angle-resolved photoemission spectroscopy (ARPES), which probes the spectral function $A(q,\omega)$. The spectral function is expressed as

$$\begin{aligned} A(q,\omega) &\equiv \frac{1}{\pi}\mathrm{Im}\frac{\hbar}{-\hbar\omega+\epsilon_q+\Sigma(q,\omega)-\mu} \\ &= \frac{1}{\pi}\mathrm{Im}\frac{\hbar}{-\hbar\omega+\epsilon_q+\mathrm{Re}[\Sigma(q,\omega)]-\mu+i/2\tau(q,\omega)}, \end{aligned} \tag{1.1}$$

from which we find that $\tau(q,\omega) = \frac{1}{2\,\mathrm{Im}[\Sigma(q,\omega)]}$. Here, ϵ_q is the dispersion relation and μ is the Fermi energy. The distinction between Fermi liquids and strange metals lies in the behavior of the imaginary part of the self-energy. In strange metals, the imaginary part of the self-energy exhibits a momentum-dependent power-law behavior along the nodal direction (discussed in Chapter 5), while Fermi liquids do not. Specifically, for strange metals along nodal direction, $\mathrm{Im}[\Sigma(q,\omega)] \propto \omega^{a(q)}$, where $a(q)$

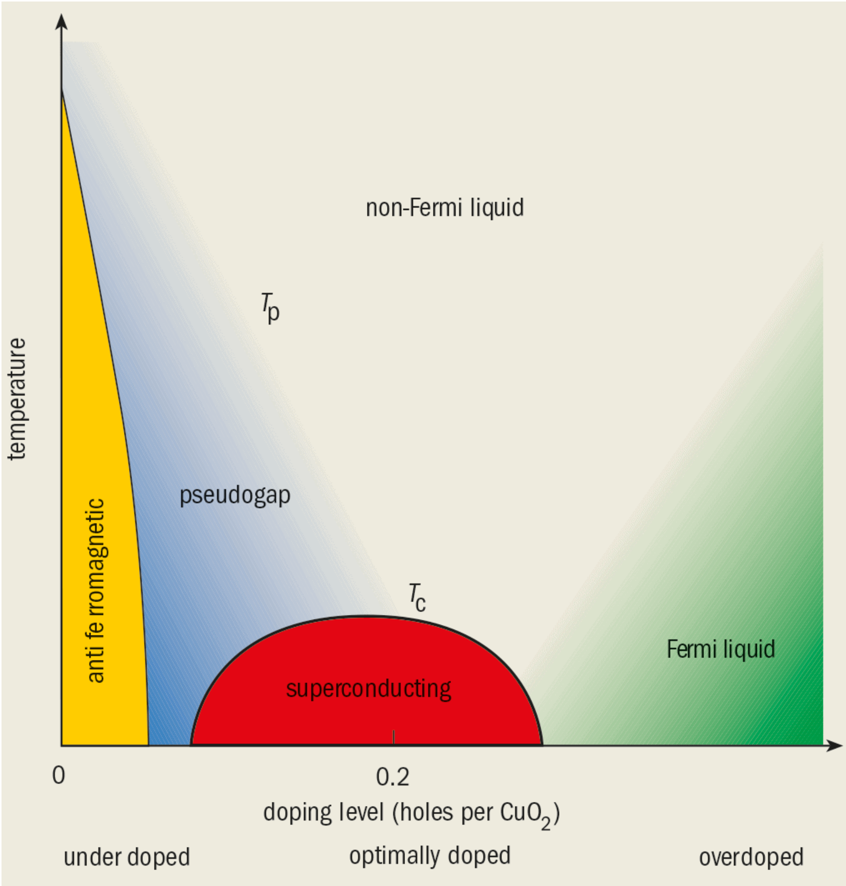


Figure 1.1: Phase diagram of cuprates

encodes the momentum-dependent exponent. While for Fermi liquid, it usually shows $\text{Im}[\Sigma(q,\omega)] \propto \omega^\gamma$, with constant γ.

A semi-holographic theory has been proposed, where the electron interacts with a conformal field, yielding the result [2]

$$\frac{2\text{Im}[\Sigma(k,\omega)]}{v_F} = \lambda \frac{(\hbar\omega)^{2\alpha(k)}}{(\hbar\omega_N)^{2\alpha(k)-1}}, \tag{1.2}$$

where λ is a coupling constant describing the strength of the interaction normalized to an energy scale $\hbar\omega_N = 0.5eV$. While this result provides a promising framework for studying strongly correlated systems, it leaves fundamental questions unanswered —most notably, the origin of the conformal field. The application of techniques from black hole physics to high-T_c superconductivity is intriguing, but theoretical physics aims not only to compute quantities but also to uncover the underlying logic. Key questions arise: Is the conformal field an emergent effective field originating from electromagnetic interactions? If not, does it imply the existence of interactions missing from standard quantum field theory?

This thesis addresses whether strange metals can be described using the standard statistical field theory (SFT) framework. Our findings suggest that this may indeed be possible. By employing novel techniques, such as linearization, we demonstrate that the momentum-dependent power law can be reproduced within the standard SFT framework in one-dimensional systems. Further generalization in two-dimensional systems remains an open question.

Our approach combines statistical field theory—specifically, the path-integral formalism and the Hubbard-Stratonovich transformation—with a dispersion linearization scheme. This work draws inspiration from the research of M. Khodas, et al. [1], who studied the Green's function of a quadratic dispersion in a one-dimensional system using linearization. Their work revealed momentum-dependent power law behavior and Luttinger-liquid phenomena for "deep" fermions. However, their methods cannot, as currently formulated, be generalized to two dimensions, a critical limitation for studying cuprates.

In Chapter 4, we validate our approach by reproducing the one-dimensional results of Khodas, et al. In Chapter 5, we extend this method to two dimensions, laying the groundwork for studying high-T_c superconductors, which are inherently two-dimensional systems.

Before exploring these results, we first review statistical field theory in Chapter 2. In Chapter 3, we present a calculation based on Fock theory that exhibits features similar to those of a Luttinger liquid. This analysis offers valuable physical intuition for understanding our subsequent findings.

Chapter 2

Statistical field theory review

2.1 Basic setting

In condensed matter physics, the thermal grand-canonical partition function

$$Z \equiv \mathrm{Tr}\left[e^{-\beta(H-\mu N)}\right]$$

is a fundamental quantity used to describe systems at equilibrium. It encapsulates the full probability distribution of microscopic states and provides a basis for calculating equilibrium expectation values of physical observables. Here, H denotes the Hamiltonian operator, and N represents the particle number operator. The parameters $\beta = 1/k_B T$ and μ correspond to the Fermi energy (chemical potential). For a given observable O, its expectation value is given by

$$\langle O \rangle = \frac{1}{Z}\mathrm{Tr}\left[Oe^{-\beta(H-\mu N)}\right].$$

This computation can be performed in the operator formalism using many-particle quantum states, as was used by M. Khodas, et al. [1]. Here, we employ the path-integral formalism, which is more adaptable for generalizing to higher dimensions for our calculation later. In the path-integral formalism, the partition function is expressed as [3]

$$Z = \int \mathcal{D}[\phi^*]\mathcal{D}[\phi]\, e^{-S[\phi^*,\phi]/\hbar}, \tag{2.1}$$

where $S[\phi^*, \phi]$ is the Euclidean action (neglecting spin contributions)

$$S[\phi^*,\phi] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x}\, \phi^*(\mathbf{x},\tau) \left\{ \hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} + V^{\mathrm{ex}}(\mathbf{x}) - \mu \right\} \phi(\mathbf{x},\tau)$$
$$+ \frac{1}{2}\int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x} \int \mathrm{d}\mathbf{x}'\, \phi^*(\mathbf{x},\tau)\phi^*(\mathbf{x}',\tau)V(\mathbf{x}-\mathbf{x}')\phi(\mathbf{x}',\tau)\phi(\mathbf{x},\tau). \quad (2.2)$$

Here, all definitions follow precisely as presented in Chapter 7 of U.Q.F. [3]. For instance, $\phi(\mathbf{x},\tau)$ represents the electron field, corresponding to the eigenvalue of the coherent state $|\phi\rangle$, which is introduced to establish the path-integral formalism. Here, τ denotes the time parameter, ranging from 0 to $\hbar\beta$. In deriving this formalism, we assume that the electrons are subject to an external potential $V^{\mathrm{ex}}(\mathbf{x})$ and interact via a two-body potential $V(\mathbf{x}-\mathbf{x}')$.

To compute a physical observable, we first express it in terms of field operators, then transform it into the path-integral formalism. This is achieved by modifying Z to include source terms

$$Z[J,J^*] = \int \mathcal{D}[\phi^*]\mathcal{D}[\phi]\, e^{-S[\phi^*,\phi]/\hbar + (\phi|J) + (J|\phi)}, \quad (2.3)$$

where the notation

$$(f|g) = \int \mathrm{d}\mathbf{x} \int \mathrm{d}\tau\, f^*(\mathbf{x},\tau)g(\mathbf{x},\tau)$$

is used. For instance, the Green's function of an electron is defined as [3]

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') \equiv -\langle T\phi(\mathbf{x},\tau)\phi^\dagger(\mathbf{x}',\tau')\rangle,$$

and can be expressed in the path-integral formalism as

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = -\frac{1}{Z}\int \mathcal{D}[\phi^*]\mathcal{D}[\phi]\, \phi(\mathbf{x},\tau)\phi^*(\mathbf{x}',\tau')e^{-S[\phi^*,\phi]/\hbar}.$$

Introducing the source terms, it can also be written as

$$G(\mathbf{x},\tau;\mathbf{x}',\tau') = -\frac{1}{Z}\frac{\delta}{\delta J^*(\mathbf{x},\tau)}\frac{\delta}{\delta J(\mathbf{x}',\tau')}Z\Big|_{J,J^*=0}. \quad (2.4)$$

2.2 Hubbard-Stratonovich transformation

To compute the Green's function, we need to evaluate (2.3) as a function of J and J^*. However, the interaction term in (2.2) contains a fourth-order term in the field ϕ, which cannot be integrated exactly. A common approach to address this issue is to convert the interaction term into a Gaussian form. One method involves using perturbation theory by assuming V is relatively small and expanding the exponential function in a Taylor series. This reduces the calculation to derivatives of the non-interacting partition function $Z_0[J, J^*]$, which has a Gaussian form. However, this approach restricts the calculation to a finite number of Feynman diagrams and does not capture the full physical behavior.

Instead, we employ the Hubbard-Stratonovich transformation, which reformulates (2.2) into a Gaussian expression by introducing an auxiliary field κ. This technique is advantageous because, in principle, it allows for exact calculations without relying on perturbation theory. In practice, approximations are often necessary since the path integral over the auxiliary field cannot generally be performed exactly. Nevertheless, this method enables summation over infinitely many Feynman diagrams, offering a significant improvement over standard perturbation theory. The transformation uses the identity:

$$1 = \int \mathcal{D}[\kappa] \, \exp\left\{ \frac{1}{2\hbar} (\kappa - V\phi^*\phi | V^{-1} | \kappa - V\phi^*\phi) \right\},$$

where the notation is

$$\begin{aligned}
&(\kappa - V\phi^*\phi | V^{-1} | \kappa - V\phi^*\phi) \\
&= \int_0^{\hbar\beta} \mathrm{d}\tau \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{x}' \left(\kappa(\mathbf{x}, \tau) - \int \mathrm{d}\mathbf{x}'' \phi^*(\mathbf{x}'', \tau)\phi(\mathbf{x}'', \tau) V(\mathbf{x}'' - \mathbf{x}) \right) \\
&\quad \times V^{-1}(\mathbf{x} - \mathbf{x}') \left(\kappa(\mathbf{x}', \tau) - \int \mathrm{d}\mathbf{x}'' V(\mathbf{x}' - \mathbf{x}'') \phi^*(\mathbf{x}'', \tau)\phi(\mathbf{x}'', \tau) \right). \qquad (2.5)
\end{aligned}$$

Using this transformation, the partition function becomes

$$\begin{aligned}
Z[J, J^*] &= \int \mathcal{D}[\phi^*]\mathcal{D}[\phi] \int \mathcal{D}[\kappa] \, \exp\left\{ -\frac{1}{\hbar} S[\phi^*, \phi] + (J|\phi) + (\phi|J) \right\} \\
&\quad \times \exp\left\{ \frac{1}{2\hbar} (\kappa - V\phi^*\phi | V^{-1} | \kappa - V\phi^*\phi) \right\} \\
&= \int \mathcal{D}[\kappa] \exp\left\{ \frac{1}{2\hbar} (\kappa | V^{-1} | \kappa) \right\} \int \mathcal{D}[\phi^*]\mathcal{D}[\phi] \\
&\quad \times \exp\left\{ (\phi | G_0^{-1} - \Sigma | \phi) + (J|\phi) + (\phi|J) \right\}, \qquad (2.6)
\end{aligned}$$

where $\hbar\Sigma = \kappa$ and $G_0^{-1} = -\frac{1}{\hbar}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} + V^{\mathrm{ex}}(\mathbf{x}) - \mu \right)$. We choose κ as $\kappa = \kappa' + \langle\kappa\rangle$, such that on average, $\langle\kappa(\mathbf{x}, \tau)\rangle = \int \mathrm{d}\mathbf{x}' \, V(\mathbf{x} - \mathbf{x}') \langle\phi^*(\mathbf{x}', \tau)\phi(\mathbf{x}', \tau)\rangle$. The physical significance of this choice is that this average corresponds to the mean-field theory. Since $V^{\mathrm{ex}}(\mathbf{x})$ is added to account for the positively charged background in which electrons move, we have

$$V^{\text{ex}}(\mathbf{x}) = -n_e \int d\mathbf{x}' \, V(\mathbf{x} - \mathbf{x}') = -\langle \kappa(\mathbf{x}, \tau) \rangle,$$

which implies that the external potential $V^{\text{ex}}(\mathbf{x})$ cancels out with the average value $\langle \kappa(\mathbf{x}, \tau) \rangle$. Thus, we can use $G_0^{-1} = -\frac{1}{\hbar}\left(\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu\right)$ if κ' is used in place of κ. However, for convenience, we retain the notation κ for κ'.

The integral over ϕ is Gaussian and can be computed using the standard formula, yielding

$$Z[J, J^*] = \int \mathcal{D}[\kappa] \exp\left\{-\frac{1}{\hbar} S^{\text{eff}}[\kappa] - (J|G|J)\right\},$$

where

$$S^{\text{eff}}[\kappa] = -\frac{1}{2}(\kappa|V^{-1}|\kappa) + \hbar \, \text{Tr}\left[\log(-G^{-1})\right],$$

and $G^{-1} = G_0^{-1} - \kappa/\hbar$. The Green's function from (2.4) is then

$$G(\mathbf{x}, \tau; \mathbf{x}', \tau') = \frac{\int \mathcal{D}[\kappa] \, G(\mathbf{x}, \tau; \mathbf{x}', \tau', \kappa) \exp\left\{-\frac{1}{\hbar} S^{\text{eff}}[\kappa]\right\}}{\int \mathcal{D}[\kappa] \exp\left\{-\frac{1}{\hbar} S^{\text{eff}}[\kappa]\right\}}, \tag{2.7}$$

where $G(\mathbf{x}, \tau; \mathbf{x}', \tau', \kappa) = 1/(G_0^{-1} - \kappa/\hbar)$, satisfying the differential equation

$$\left\{\hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2\nabla^2}{2m} - \mu + \kappa(\mathbf{x}, \tau)\right\} G(\mathbf{x}, \tau; \mathbf{x}', \tau', \kappa) = -\hbar\delta(\tau - \tau')\delta(\mathbf{x} - \mathbf{x}'). \tag{2.8}$$

What we need to do next is to find the solution for this differential equation and plug it in (2.7). This will be addressed in Chapter 4. In the next chapter, we will first calculate the Fock theory in one dimension to gain deeper insights into our results.

Chapter 3

Fock theory

3.1 Dyson equation

The simplest way to understand Fock theory is through the Dyson equation. Consider using perturbation theory to compute the interacting Green's function. The result is typically calculated as a sum of Feynman diagrams to a certain order of approximation. Consider the electron-phonon interaction model, this argument is shown diagrammatically in Fig 3.1.

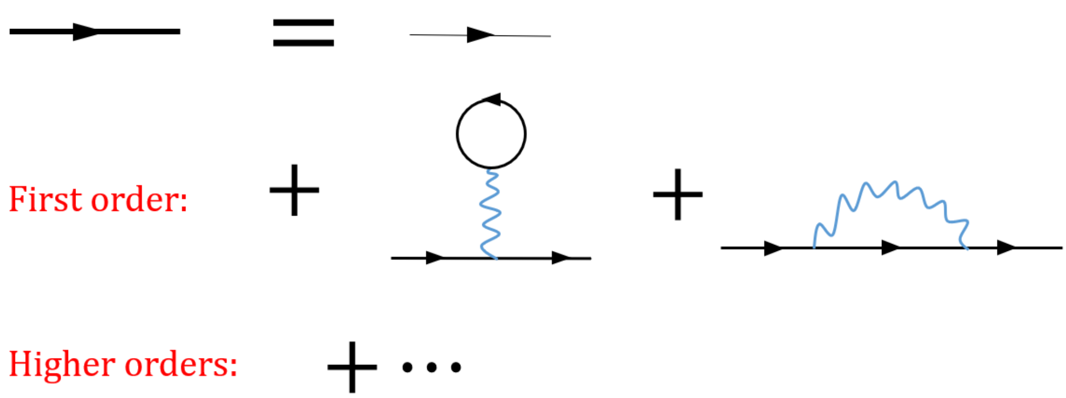


Figure 3.1: Perturbation theory to the first order.

However, these diagrams can be reorganized based on the concept of n-particle irreducibility, which determines whether a diagram can be decomposed into smaller diagrams connected by non-interacting Green's function propagators. A visual representation of this concept is shown in Fig 3.2.

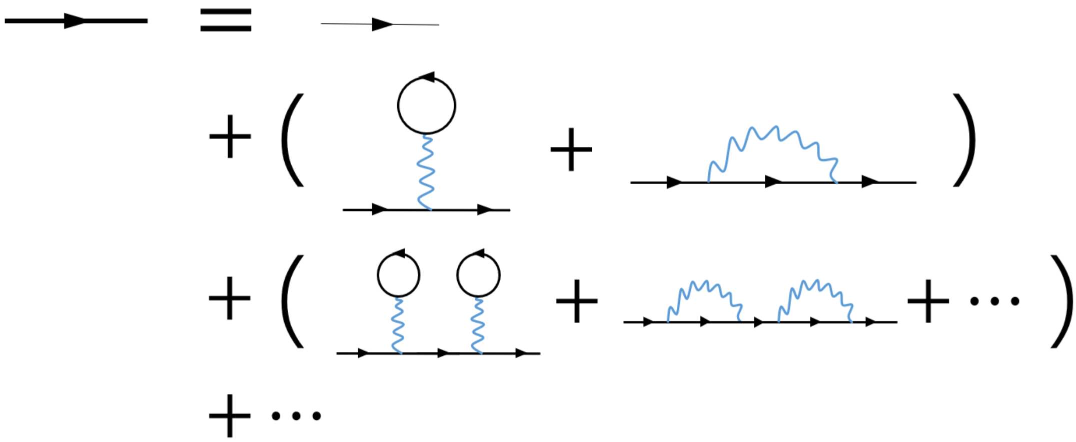


Figure 3.2: Regrouping Feynman diagrams by n-particle irreducibility.

Following this reorganization, the Dyson equation emerges, linking the interacting Green's function $G(\mathbf{k}, i\omega_n)$ with the non-interacting Green's function $G_0(\mathbf{k}, \omega_n)$ in momentum-space. The equation is given by

$$G(\mathbf{k}, i\omega_n) = G_0(\mathbf{k}, i\omega_n) + G_0(\mathbf{k}, i\omega_n)\Sigma(\mathbf{k}, i\omega_n)G(\mathbf{k}, i\omega_n),$$

which simplifies to

$$G(\mathbf{k}, i\omega_n) = \frac{-\hbar}{-i\hbar\omega_n + \epsilon_{\mathbf{k}} + \hbar\Sigma(\mathbf{k}, i\omega_n) - \mu},$$

where $\Sigma(\mathbf{k}, i\omega_n)$ is the self-energy. It is shown diagrammatically in Fig 3.3. For further details, refer to Chapter 8 of *U.Q.F.*[3].

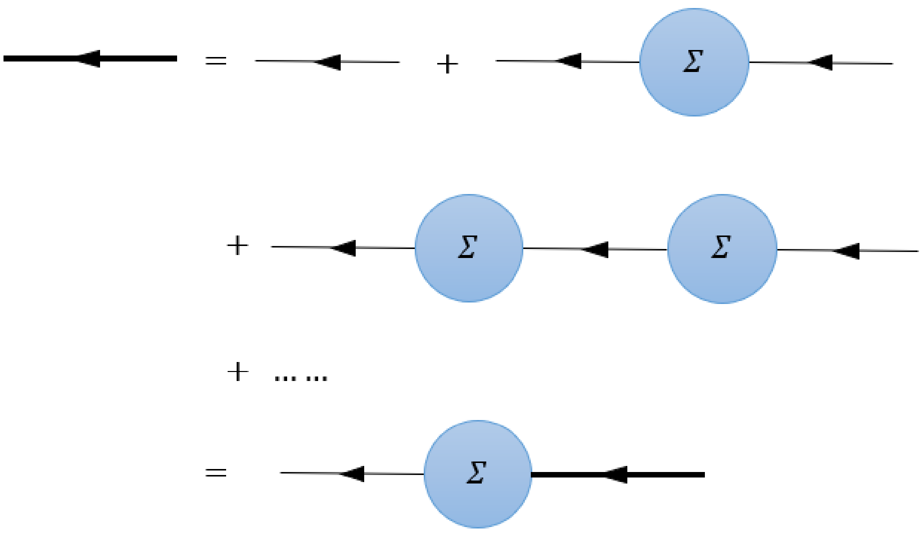


Figure 3.3: Diagrammatic representation of the Dyson equation.

3.2 Self-energy for Fock theory in one-dimension

The Fock theory arises by considering the self-energy Σ as represented by the first-order diagram in Fig. 3.4.

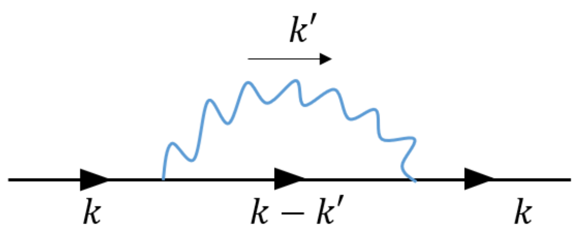


Figure 3.4: Fock self-energy diagram. The solid black line represents the non-interacting electron propagator, and the curly blue line represents the phonon propagator.

To calculate the self-energy, we evaluate the integral

$$\Sigma(k, i\omega_n) = \int \frac{\mathrm{d}k'}{2\pi} \frac{1}{\hbar\beta} \sum_{n'} G_0(k-k', i\omega_n - i\omega_{n'}) G_{\mathrm{ph}}(k', i\omega_{n'}), \tag{3.1}$$

where $G_0(k, i\omega_n)$ is the non-interacting electron Green's function

$$G_0(k, i\omega_n) = \frac{-\hbar}{-i\hbar\omega_n + \frac{\hbar^2 k^2}{2m} - \mu} = \frac{1}{i\omega_n - (\epsilon_k - \mu)/\hbar},$$

and $G_{\mathrm{ph}}(k', i\omega_{n'})$ is the phonon propagator

$$G_{\mathrm{ph}}(k', i\omega_{n'}) = \frac{2v_F k'}{(i\omega_{n'})^2 - (v_F k')^2} = \left[\frac{1}{i\omega_{n'} - v_F|k'|} - \frac{1}{i\omega_{n'} + v_F|k'|}\right].$$

Substituting into (3.1), we have

$$\Sigma(k, i\omega_n) = \int \frac{\mathrm{d}k'}{2\pi} \frac{1}{\hbar\beta} \sum_{n'} \frac{-1}{i\omega_{n'} - i\omega_n + (\epsilon_{k-k'} - \mu)/\hbar} \left[\frac{1}{i\omega_{n'} - v_F|k'|} - \frac{1}{i\omega_{n'} + v_F|k'|}\right]. \tag{3.2}$$

The Matsubara frequency $\omega_{n'} = \frac{2n\pi}{\hbar\beta}$ arises because we are summing over bosonic phonon propagators. After performing the Matsubara sum, (3.2) becomes (Detailed calculation is left in Appendix A)

$$\Sigma(k, i\omega_n) = \int \frac{\mathrm{d}k'}{2\pi} \Bigg[\left[1 - N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|)\right] \frac{1}{i\omega_n - (\epsilon_{k-k'} - \mu)/\hbar - v_F|k'|} + \left[N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|)\right] \frac{1}{i\omega_n - (\epsilon_{k-k'} - \mu)/\hbar + v_F|k'|} \Bigg], \tag{3.3}$$

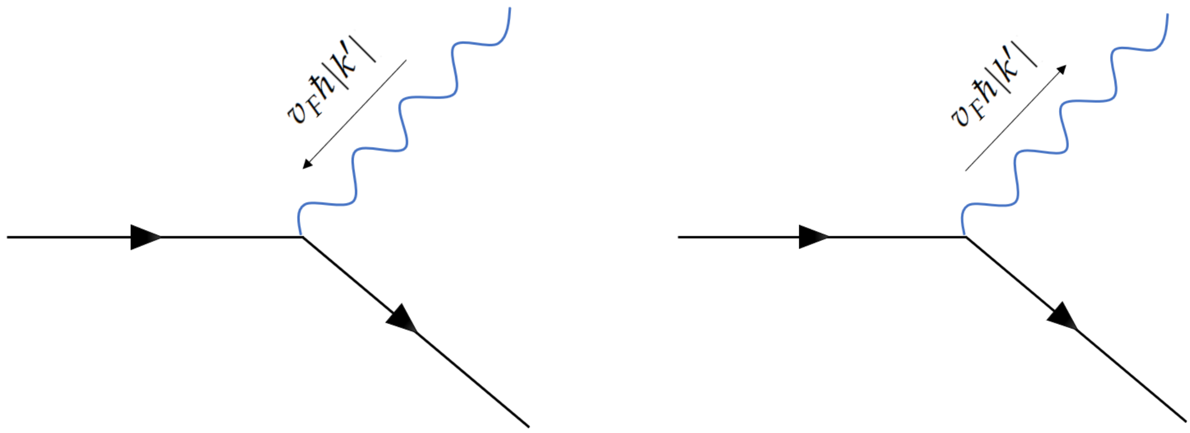
where N_{FD} and N_{BE} are the Fermi-Dirac and Bose-Einstein distributions, respectively.

At $T=0$, $N_{BE}(v_F\hbar|k'|)=0$ and $N_{FD}(\epsilon_{k-k'})=\Theta(-\epsilon_{k-k'}+\mu)$, which turns Eq.(3.3) into

$$\Sigma(k,i\omega_n)=\int\frac{dk'}{2\pi}\Bigg[\Theta(\epsilon_{k-k'}-\mu)\frac{1}{i\omega_n-(\epsilon_{k-k'}-\mu)/\hbar-v_F|k'|} + \Theta(-\epsilon_{k-k'}+\mu)\frac{1}{i\omega_n-(\epsilon_{k-k'}-\mu)/\hbar+v_F|k'|}\Bigg]. \tag{3.4}$$

This implies that the first term contributes only above the Fermi surface, while the second term contributes below it.

The two terms in Eq.(3.3) may be understood as two separate processes, phonon emission and absorption, as indicated by the poles of the two terms (see Fig 3.5). The two poles tell us that there is energy conservation $\epsilon_k=\epsilon_{k-k'}+\hbar v_F|k'|$ for phonon emission and $\epsilon_k=\epsilon_{k-k'}-\hbar v_F|k'|$ for phonon absorption. The prefactors can be understood



(a) phonon absorption **(b)** phonon emmission

Figure 3.5: Two terms contributing to Eq.(3.3)

with the help of second order perturbation theory in quantum mechanics which states that the energy correction for a weak interaction with Hamiltonian $H=H_0+H'$ to the second order equals

$$\Delta E^{(2)}=\sum_I\frac{|\langle I|H'|i\rangle|^2}{E_i-E_I} \tag{3.5}$$

where $|i\rangle$ is the initial state of the system and the summation $|I\rangle$ is over possible intermediate states. Here we should take $H'=\alpha\left(\Phi^\dagger_{p-q}\Phi_p a^\dagger_q+\Phi^\dagger_{p-q}\Phi_p a_{-q}\right)+c.c.$, where α indicate the intensity of the interaction and Φ and a are the field operators for electron and phonon respectively. If we want to calculate the energy level shift for the electron state with energy ϵ_k due to phonon absorption, we see it's proportional to the probability of the electron state with momentum k turns into an electron state with momentum $k-k'$ absorbing a phonon with energy $\hbar v_F\,|k'|$, which is $(1-N_{FD}(\epsilon_{k-k'}))\,N_{BE}(\hbar v_F\,|k'|)$. However, this is not the full story. Since we are assuming that the state with energy

ϵ_k is already occupied in order to calculate the probability of the previous process, we need to be careful. Because when we calculate the energy shift for other states, for example for an electron with energy $\epsilon_{k-k'}$, we must exclude the intermediate state in Eq.(3.5) because it's already occupied hence cannot accommodates another electron by Pauli's exclusion principle. In fact this is true for calculating all energy states. This modifies the energy level by subtracting the reversed process with the probability $N_{FD}(\epsilon_{k-k'})\left(1+N_{BE}(\hbar v_F\left|k'\right|)\right)$. These sum up to

$$\begin{aligned}&\frac{\left(1-N_{FD}(\epsilon_{k-k'})\right)N_{BE}(\hbar v_F\left|k'\right|)}{\epsilon_k-\epsilon_{k-k'}}-\frac{N_{FD}(\epsilon_{k-k'})\left(1+N_{BE}(\hbar v_F\left|k'\right|)\right)}{\epsilon_{k-k'}-\epsilon_k}\\=&\frac{N_{FD}(\epsilon_{k-k'})+N_{BE}(\hbar v_F\left|k'\right|)}{\epsilon_k-\epsilon_{k-k'}}.\end{aligned}\tag{3.6}$$

This gives the prefactor of the first term. Similarly, there are two terms contributing to the phonon emission process, the probability of the electron state with momentum k turns into an electron state with momentum $k-k'$ emitting a phonon with energy $\hbar v_F\left|k'\right|$, which is $\left(1-N_{FD}(\epsilon_{k-k'})\right)\left(1+N_{BE}(\hbar v_F\left|k'\right|)\right)$, and its reversed process with the probability $N_{FD}(\epsilon_{k-k'})N_{BE}(\hbar v_F\left|k'\right|)$. These sum up to

$$\begin{aligned}&\frac{\left(1-N_{FD}(\epsilon_{k-k'})\right)\left(1+N_{BE}(\hbar v_F\left|k'\right|)\right)}{\epsilon_k-\epsilon_{k-k'}}-\frac{N_{FD}(\epsilon_{k-k'})N_{BE}(\hbar v_F\left|k'\right|)}{\epsilon_{k-k'}-\epsilon_k}\\=&\frac{1-N_{FD}(\epsilon_{k-k'})+N_{BE}(\hbar v_F\left|k'\right|)}{\epsilon_k-\epsilon_{k-k'}}.\end{aligned}\tag{3.7}$$

which gives the prefactor of the first term.

3.3 The importance of the imaginary part of self-energy

In principle, the integral (3.3) can be evaluated. However, rather than the complete result, we are primarily interested in the imaginary part of this integral for the following reason. For a complex self-energy, the imaginary part typically represents the lifetime of the state. A simple argument illustrates this: for a state with energy $\omega=\mathrm{Re}[E]-i\mathrm{Im}[E]$, the wave function takes the form

$$\phi(x)e^{-i\mathrm{Re}[E]t}e^{-\mathrm{Im}[E]t},\tag{3.8}$$

which decays if $\mathrm{Im}[E]$ is positive. What's more, the imaginary part not only reflects the state's lifetime but also indicates the possible states involved in the interaction. To see this, consider the non-interacting Green's function

$$G_0(\mathbf{k},\omega)=\frac{-\hbar}{-\hbar\omega+\epsilon_{\mathbf{k}}-\mu}.\tag{3.9}$$

If we introduce an infinitesimal imaginary part, the Green's function becomes

$$G_0(\mathbf{k}, \omega) = \lim_{\epsilon \to 0} \frac{-\hbar}{-\hbar\omega + \epsilon_{\mathbf{k}} - \mu + i\epsilon}. \tag{3.10}$$

The imaginary part of this Green's function is then

$$\mathrm{Im}[G_0(\mathbf{k}, \omega)] = -\pi\delta\left(\omega - \frac{\epsilon_{\mathbf{k}} - \mu}{\hbar}\right). \tag{3.11}$$

This result shows that, for a non-interacting Green's function, its imaginary part is a delta function that lies on the dispersion relation. This aligns with the fact that, for non-interacting electrons, the states strictly follow the on-shell dispersion relation. However, this is generally not true for interacting Green's functions. In such cases, a non-zero broadening of the imaginary part around the dispersion relation indicates that the possible states are no longer restricted to on-shell states.

The imaginary part of the self-energy is sometimes incorporated into the definition of the spectral function as follows

$$A(\mathbf{k}, \omega) = -\frac{1}{\pi}\mathrm{Im}[G(\mathbf{k}, \omega)]. \tag{3.12}$$

Suppose the Green's function can be expressed in terms of the self-energy $\Sigma = \mathrm{Re}[\Sigma] + i\mathrm{Im}[\Sigma]$, i.e.,

$$\begin{aligned} G(\mathbf{k}, \omega) =& \frac{1}{\omega - (\epsilon_{\mathbf{k}} - \mu)/\hbar - \Sigma/\hbar} \\ =& \frac{\omega - (\epsilon_{\mathbf{k}} + \mathrm{Re}[\Sigma] - \mu)/\hbar + i\mathrm{Im}[\Sigma]/\hbar}{(\omega - (\epsilon_{\mathbf{k}} + \mathrm{Re}[\Sigma] - \mu)/\hbar)^2 + (\mathrm{Im}[\Sigma]/\hbar)^2}. \end{aligned} \tag{3.13}$$

Then the spectral function becomes

$$A(\mathbf{k}, \omega) = -\frac{1}{\pi\hbar}\frac{\mathrm{Im}[\Sigma]}{(\omega - (\epsilon_{\mathbf{k}} + \mathrm{Re}[\Sigma] - \mu)/\hbar)^2 + (\mathrm{Im}[\Sigma]/\hbar)^2}. \tag{3.14}$$

As we can see, apart from a non-zero factor $-\frac{1}{(\omega - (\epsilon_{\mathbf{k}} + \mathrm{Re}[\Sigma] - \mu)/\hbar)^2 + (\mathrm{Im}[\Sigma]/\hbar)^2}$, the spectral function is proportional to the imaginary part of the self-energy. Thus, the self-energy serves as a good proxy for the spectral function to extract information about possible states.

3.4 Plots of the imaginary part of self-energy and spectral function

Following the calculation in Appendix A, we can obtain the imaginary part of the Fock self-energy in (3.3). Since the result is lengthy and not particularly insightful, we only present the plot of the result in Fig. 3.6. We only mention here that the result is a self-energy $\Sigma \propto \omega^{-\frac{1}{2}}$, hence not a momentum-dependent power law. This provides a comparison to our later result in Chapter 4.

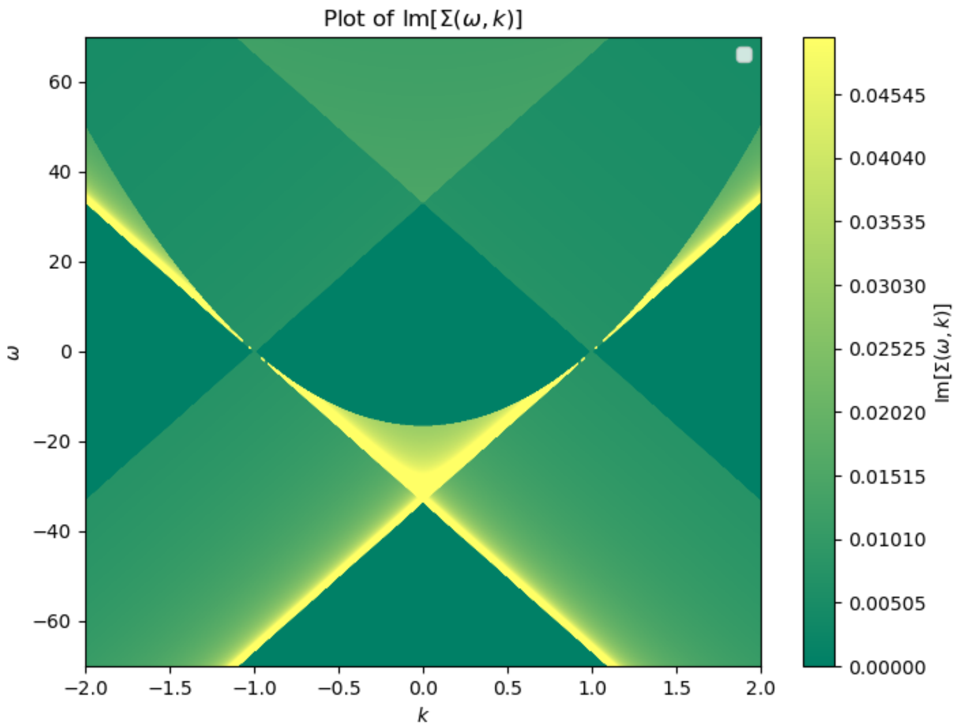


Figure 3.6: Imaginary part of $\Sigma(k, i\omega_n)$ for the Fock diagram in Fig. 3.4.

If we neglect the real part of the self-energy $\text{Re}[\Sigma_{\text{Fock}}]$ for our diagrammatic purposes, the spectral function can be expressed as

$$A_{\text{Fock}}(k,\omega) = -\frac{1}{\pi\hbar}\frac{\text{Im}[\Sigma_{\text{Fock}}]}{(\omega-(\epsilon_k-\mu)/\hbar)^2+(\text{Im}[\Sigma_{\text{Fock}}]/\hbar)^2}. \tag{3.15}$$

This is plotted in Fig. 3.7, with the restriction that the phonon momentum satisfies $|k'| \leq 0.1k_F$, mimicking the long-wavelength approximation of phonons.

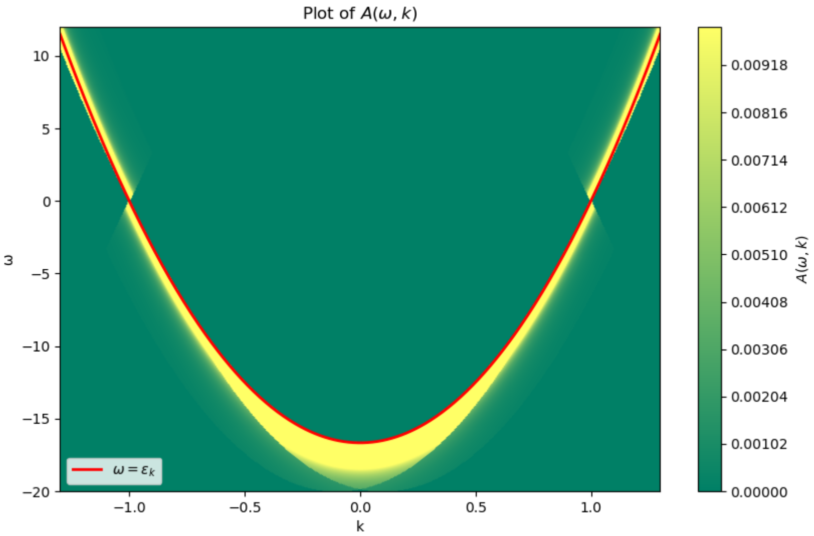


Figure 3.7: Spectral function for Fock theory in one-dimension

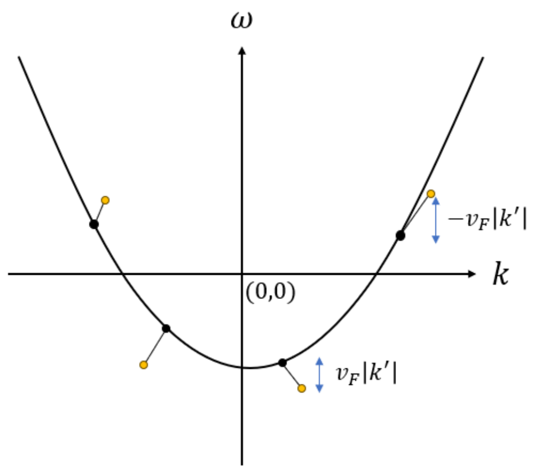


Figure 3.8: Diagrammatic analysis.

As we have seen in Eq.(3.4), the first term which corresponds to phonon emission only happens above the Fermi surface, and the second term which corresponds to phonon absorption only happens below the Fermi surface. This can be seen directly in the plots, for energies above the Fermi surface ($\omega > 0$), only energy-decreasing processes are possible. Conversely, for energies below the Fermi surface ($\omega < 0$), only energy-increasing processes occur. A diagrammatic illustration with some possible processes of this is shown in Fig. 3.8. In the plot, the yellow dots represent the quasi-particle state with energy ω, and they can either emit or absorb an phonon with energy $\hbar v_F|k'|$ and end up to the electron state with energy $\epsilon_{k-k'}$.

This behavior may possibly be understood as follows: below the Fermi surface, the Green's function describes hole propagation, whereas above the Fermi surface, it describes particle propagation. For particles, emitting a phonon lower the energy. For holes, emitting a phonon increase the energy, as it corresponds to a higher-energy electron losing energy, and creating a higher-energy hole state. At zero temperature, there are no phonon because there are no lattice vibration which explains why only emitting process are possible.

Chapter 4

One-dimensional linearization scheme

In this approach, we analyze the wave function of an electron with momentum $\hbar q$. We assume the wave function has the form of a plane wave with momentum $\hbar q$ multiplied by a slowly varying function $\tilde{\phi}(x, \tau)$ w.r.t x, i.e.,

$$\phi(x, \tau) = e^{iqx}\tilde{\phi}(x, \tau). \tag{4.1}$$

This implies that, when decomposed in momentum space, $\tilde{\phi}(k, \omega) \neq 0$ if $k \ll \Lambda$ for a certain maximal value Λ comparably small to q. Referring to (2.6), the action for the ϕ part is given by

$$S[\phi^*, \phi, \kappa] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x}\, \phi^*(\mathbf{x}, \tau) \left\{ \hbar \frac{\partial}{\partial \tau} - \frac{\hbar^2 \nabla^2}{2m} - \mu + \kappa(x, \tau) \right\} \phi(\mathbf{x}, \tau). \tag{4.2}$$

The κ field couples to ϕ through the interaction term $\phi^*\phi\kappa$. As we will demonstrate later, the κ field behaves analogously to a phonon, and this interaction represents electron-phonon scattering. We adopt the long-wavelength approximation for the phonon field, represented by the κ field. Under this approximation, if an electron originally possesses momentum $\hbar q$ (corresponding to a plane-wave solution e^{iqx}), it will most probably be excited to configurations of the form (4.1) due to the interaction. This assumption underpins the choice of (4.1).

After substituting ϕ by (4.1), the action for ϕ becomes (neglecting the term $\frac{\partial^2 \tilde{\phi}(x,\tau)}{\partial x^2} \propto k^2$, as we assume the momentum k of $\tilde{\phi}$ to be small)

$$S[\phi^*, \phi, \kappa] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}x\, \tilde{\phi}^*(x, \tau) \left\{ \hbar \frac{\partial}{\partial \tau} + \frac{\hbar^2 q^2}{2m} - \frac{i\hbar^2 q}{m} \frac{\partial}{\partial x} - \mu + \kappa(x, \tau) \right\} \tilde{\phi}(x, \tau). \tag{4.3}$$

This can be understood as a linearization of the momentum for each plane-wave component of ϕ. Since ϕ is a superposition of plane waves with momentum close to $\hbar q$, their momentum can be written as

$$\frac{\hbar^2(q+k)^2}{2m} = \frac{\hbar^2 q^2}{2m} + \frac{\hbar^2 qk}{m} + \frac{\hbar^2 k^2}{2m} \approx \frac{\hbar^2 q^2}{2m} + \frac{\hbar^2 qk}{m}, \tag{4.4}$$

where we consider k to be relatively small. Thus, we sum over all configurations that deviate from the plane wave with momentum $\hbar q$ by a quantum fluctuation of momentum k. Accordingly, k is replaced by $-i\frac{\partial}{\partial x}$, while q is treated as a classical parameter representing the wave number of the mean field. Typically, in research on one-dimensional systems, the linear dispersion is derived by assuming that, in one dimension, electron motion is purely collective due to the constraints of limited dimensionality and the repulsive forces between electrons. However, in this thesis, we begin with a quadratic dispersion relation, as in M. Khodas et al.[1], and thus do not adopt this perspective.

Using the action in (4.3), equation (2.8) becomes

$$\left\{\hbar\frac{\partial}{\partial\tau} + \frac{\hbar^2 q^2}{2m} - \frac{i\hbar^2 q}{m}\frac{\partial}{\partial x} - \mu + \kappa(x,\tau)\right\}\tilde{G}(x,\tau;x',\tau',\kappa) = -\hbar\delta(\tau-\tau')\delta(x-x'). \tag{4.5}$$

Remark. By the choice of (4.1), the Green's function can be expressed as $G(\mathbf{x},\tau;\mathbf{x}',\tau',\kappa) = \tilde{G}(\mathbf{x},\tau;\mathbf{x}',\tau',\kappa)e^{iq(x-x')}$. Once this equation is solved, the Green's function can be substituted into (2.7) to obtain the desired $G(\mathbf{x},\tau;\mathbf{x}',\tau')$.

4.1 Turning kappa field into a local phase transformation

The advantage of linearizing the momentum is that it allows us to replace κ by a local phase transformation $\phi(x,\tau) \to \phi(x,\tau)e^{i\theta(x,\tau)}$. We can fix the phase by choosing a function $\theta(x,\tau)$ such that the minimal coupling term $\kappa\phi^*\phi$ vanishes in (4.5). This approach effectively separates the path integral over κ and ϕ. Under a phase transformation,

$$\begin{aligned}
S[\phi^*,\phi] &= \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}x\, \tilde{\phi}^*(x,\tau)e^{-i\theta(x,\tau)}\left\{\hbar\frac{\partial}{\partial\tau} + \frac{\hbar^2 q^2}{2m} - \frac{i\hbar^2 q}{m}\frac{\partial}{\partial x} - \mu + \kappa(x,\tau)\right\}\tilde{\phi}(x,\tau)e^{i\theta(x,\tau)} \\
&= \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}x\, \tilde{\phi}^*(x,\tau)\left\{\hbar\frac{\partial}{\partial\tau} + \frac{\hbar^2 q^2}{2m} - \frac{i\hbar^2 q}{m}\frac{\partial}{\partial x} - \mu\right\}\tilde{\phi}(x,\tau) \\
&\quad + \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}x\, \tilde{\phi}^*(x,\tau)\left\{i\hbar\frac{\partial}{\partial\tau}\theta(x,\tau) + \frac{\hbar^2 q}{m}\frac{\partial}{\partial x}\theta(x,\tau) + \kappa(x,\tau)\right\}\tilde{\phi}(x,\tau).
\end{aligned} \tag{4.6}$$

We see that $\theta(x,\tau)$ must satisfy the equation

$$i\hbar\frac{\partial}{\partial\tau}\theta(x,\tau)+\frac{\hbar^2 q}{m}\frac{\partial}{\partial x}\theta(x,\tau)+\kappa(x,\tau)=0. \tag{4.7}$$

By Wick-rotating $\tau \to it$, as explained in Appendix B.1, this equation becomes

$$\hbar\frac{\partial}{\partial t}\theta(x,t)+\frac{\hbar^2 q}{m}\frac{\partial}{\partial x}\theta(x,t)+\kappa(x,t)=0. \tag{4.8}$$

The solution of this differential equation can be obtained via Fourier transformation (see Appendix B.3)

$$\theta(x,t)=\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}\omega}{2\pi}\frac{-i\kappa(k,\omega)}{\hbar(\omega-v_q k)}e^{ikx}e^{-i\omega t}, \tag{4.9}$$

where $v_q=\frac{\hbar}{m}q$. This solution confirms that we can eliminate the minimal coupling term $\kappa\phi^*\phi$ in the action by introducing a local $U(1)$ transformation.

The Green's function for the transformed field $\tilde{\phi}$ satisfies the equation

$$\left\{\hbar\frac{\partial}{\partial\tau}+\frac{\hbar^2q^2}{2m}-\frac{i\hbar^2 q}{m}\frac{\partial}{\partial x}-\mu\right\}\tilde{G}_0(x,\tau;x',\tau')=-\hbar\delta(\tau-\tau')\delta(x-x'), \tag{4.10}$$

indicating it is the non-interacting Green's function with linearized momentum. However, the original Green's function G is given by

$$G(x,\tau;x',\tau',\kappa)=G_0(x,\tau;x',\tau')e^{i(\theta(x,\tau)-\theta(x',\tau'))}, \tag{4.11}$$

where

$$\begin{aligned}G_0(x,\tau;x',\tau')&=\tilde{G}_0(x,\tau;x',\tau')e^{iq(x-x')}\\&=\int\frac{\mathrm{d}k}{2\pi}\sum_n\frac{1}{\hbar\beta}\frac{-\hbar}{-i\hbar\omega_n+\frac{\hbar^2q^2}{2m}+\hbar v_q k-\mu}e^{i(q+k)(x-x')}e^{-i\omega_n(\tau-\tau')}.\end{aligned} \tag{4.12}$$

After Wick-rotation and adopting the zero-temperature approximation (see Appendix B.1), this becomes

$$\begin{aligned}G_0(x,t;x',t')&=\tilde{G}_0(x,t;x',t')e^{iq(x-x')}\\&=\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega-(\frac{\hbar^2q^2}{2m}+\hbar v_q k-\mu)/\hbar}e^{i(q+k)(x-x')}e^{-i\omega(t-t')}.\end{aligned} \tag{4.13}$$

A deeper reason for Wick-rotation under zero-temperature here is that we want to work with the continuous parameter ω instead of $i\omega_n$ because only in this case will our calculation be valid.

The complete Green's function is

$$G(x,t;x',t') = G_0(x,t;x',t') \frac{\int \mathrm{d}[\kappa]\, e^{i(\theta(x,t)-\theta(x',t'))} e^{-\frac{1}{\hbar}S^{\text{eff}}[\kappa]}}{\int \mathrm{d}[\kappa]\, e^{-\frac{1}{\hbar}S^{\text{eff}}[\kappa]}}$$
$$= G_0(x,t;x',t') \left\langle e^{i(\theta(x,t)-\theta(x',t'))} \right\rangle_\kappa$$
$$= G_0(x,t;x',t') e^{-\frac{1}{2}\left\langle [\theta(x,t)-\theta(x',t')]^2 \right\rangle_\kappa}, \tag{4.14}$$

where the average of odd powers of κ vanishes, and

$$\left\langle [\theta(x,t)-\theta(x',t')]^{2n} \right\rangle_\kappa = \binom{2n}{2}\binom{2n-2}{2}\cdots\binom{4}{2}\left\langle [\theta(x,t)-\theta(x',t')]^2 \right\rangle_\kappa^n, \tag{4.15}$$

with $\binom{n}{m}$ being the binomial coefficient.

This method is only applicable in the case of linearized momentum. If the second derivative term $\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$ is present in the action, then

$$S[\phi^*,\phi] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}x\, \phi^*(x,\tau) \left\{ \hbar\frac{\partial}{\partial\tau} - \frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} - \mu \right\} \phi(x,\tau)$$
$$+ \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}x\, \phi^*(x,\tau) \left\{ i\hbar\frac{\partial}{\partial\tau}\theta(x,\tau) - \frac{\hbar^2}{2m}\left(\frac{\partial\theta(x,\tau)}{\partial x}\right)^2 \right.$$
$$\left. + i\frac{\hbar^2}{2m}\frac{\partial^2\theta(x,\tau)}{\partial x^2} + i\frac{\hbar^2}{m}\frac{\partial\theta}{\partial x}\frac{\partial}{\partial x} + \kappa(x,\tau) \right\} \phi(x,\tau). \tag{4.16}$$

The additional derivatives and non-linear terms make this approach inapplicable. This is the precise reason for the linearization introduced at the outset.

4.2 Two-point correlation function of the auxiliary field

The quantity $\left\langle [\theta(x,t)-\theta(x',t')]^2 \right\rangle_\kappa$ is calculated in Appendix B.4. The result is

$$\left\langle [\theta(x,t)-\theta(x',t')]^2 \right\rangle_\kappa = 2\int \frac{\mathrm{d}k}{2\pi}\int \frac{\mathrm{d}\omega}{2\pi} \frac{\langle \kappa(k,\omega)\kappa(-k,-\omega)\rangle_\kappa}{\hbar^2(\omega - v_q k)^2}$$
$$\times \left(1 - \cos(k(x-x') - \omega(t-t'))\right). \tag{4.17}$$

Substituting this result into Eq. (4.14), we obtain

$$
\begin{aligned}
&G(x,t;x',t') \\
=&G_0(x,t;x',t') \\
&\times \exp\left\{ \int \frac{dk}{2\pi} \int \frac{d\omega}{2\pi} \frac{\langle \kappa(k,\omega)\kappa(-k,-\omega)\rangle_\kappa}{\hbar^2(\omega - v_q k)^2} \left[1 - \cos(k(x-x') - \omega(t-t'))\right] \right\}.
\end{aligned} \tag{4.18}
$$

To proceed, we need to evaluate $\langle \kappa(k,\omega)\kappa(-k,-\omega)\rangle_\kappa$. This calculation has been performed in U.Q.F. (8.139) [3]. In their derivation, the main contribution arises from the diagram shown in Fig. 4.1. The bubble diagram describes the annihilation of a phonon and the creation of an electron-hole pair, which occurs only near the Fermi surface at $T = 0$ for low-energy phonons. This aligns with the long-wavelength approximation adopted in this work.

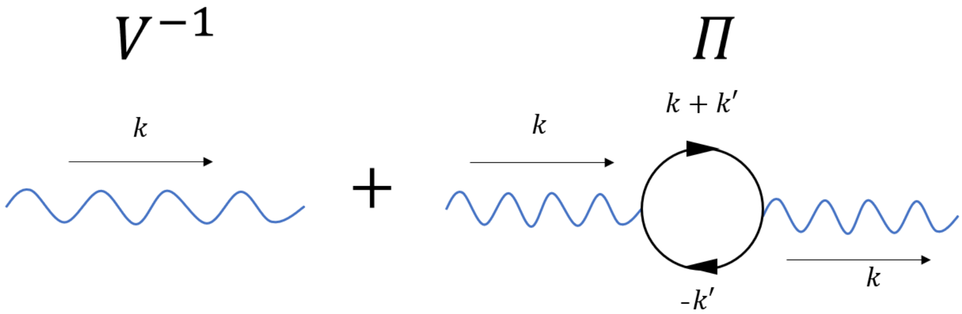


Figure 4.1: Feynman diagram of the two-point correlation function of the auxiliary field

The result is

$$
\langle \kappa(k,\omega)\kappa(-k,-\omega)\rangle_\kappa = -iG_k(k,\omega) = \frac{-i\hbar}{V^{-1}(k) - \Pi(k,\omega)}. \tag{4.19}
$$

Here, $\Pi(k,\omega)$ is given by

$$
\Pi(k,\omega) = 2\int \frac{dk'}{2\pi} \frac{N_{FD}(\epsilon_{k+k'}) - N_{FD}(\epsilon_{k'})}{-\hbar\omega + \epsilon_{k+k'} - \epsilon_{k'}}. \tag{4.20}
$$

At $T = 0$, where $N_{FD}(\epsilon) = \Theta(\mu - \epsilon)$, (4.20) simplifies to

$$
\begin{aligned}
\Pi(k,\omega) =&2\int \frac{dk'}{2\pi} \frac{\Theta(\mu - \epsilon_{k+k'}) - \Theta(\mu - \epsilon_{k'})}{-\hbar\omega + \epsilon_{k+k'} - \epsilon_{k'}} \\
=&\frac{1}{\pi}\int dk' \left[\frac{\Theta(\mu - \epsilon_{k'})}{-\hbar\omega + \epsilon_{k'} - \epsilon_{k'-k}} - \frac{\Theta(\mu - \epsilon_{k'})}{-\hbar\omega + \epsilon_{k+k'} - \epsilon_{k'}} \right].
\end{aligned} \tag{4.21}
$$

The Heaviside step function Θ restricts k' to the range $[-k_F, k_F]$. Substituting $\epsilon_k = \frac{\hbar^2 k^2}{2m}$

and performing some algebraic manipulations, we find

$$\Pi(k,\omega) = \frac{m}{\pi\hbar^2}\int_{-k_F-\frac{m\omega}{\hbar k}}^{k_F-\frac{m\omega}{\hbar k}} \frac{\mathrm{d}k'}{k'^2-\left(\frac{\hbar^2 k}{2m}\right)^2}. \tag{4.22}$$

Using the standard formula

$$\int \frac{\mathrm{d}x}{x^2-a^2} = \frac{\ln|x-a|-\ln|x+a|}{2a} \approx -\frac{1}{x}, \quad \text{if} \quad a \ll 1, \tag{4.23}$$

we obtain(assuming k small due to long-wave approximation)

$$\Pi(k,\omega) = \Pi_L(k,\omega)+\Pi_R(k,\omega) = \frac{k}{\pi\hbar}\frac{2v_F k}{\omega^2-v_F^2k^2}. \tag{4.24}$$

Here

$$\Pi_L(k,\omega) = \frac{1}{\pi}\frac{-k}{\hbar\omega+\hbar v_F k} \tag{4.25}$$

is the contribution from left-moving electrons, and

$$\Pi_R(k,\omega) = \frac{1}{\pi}\frac{k}{\hbar\omega-\hbar v_F k} \tag{4.26}$$

is the contribution from right-moving electrons.

Substituting (4.19) and (4.24) into (4.18), we obtain

$$\begin{aligned} &G(x,t;x',t') \\ =&G_0(x,t;x',t') \\ &\times \exp\left\{\frac{-iV}{\hbar}\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-v_F^2k^2}{\omega^2-v^2k^2}\frac{1-\cos(k(x-x')-\omega(t-t'))}{(\omega-v_qk)^2}\right\}. \end{aligned} \tag{4.27}$$

Here, $v = v_F\sqrt{\frac{2V}{\pi\hbar v_F}+1}$, and $V(k)$ is approximated as a constant $V(0)$ due to the long-wavelength approximation, which assumes k is small.

4.3 Green's function of one-dimensional model

Before performing the calculation in (4.27), we first address some subtleties. The most physical definition of the electron's Green's function is given by[4]

$$\begin{aligned} G(k,t-t') =& -i\left\langle T\psi_k(t)\psi_k^\dagger(t')\right\rangle \\ =& -i\Theta(t-t')\left\langle \psi_k(t)\psi_k^\dagger(t')\right\rangle + i\Theta(t'-t)\left\langle \psi_k^\dagger(t')\psi_k(t)\right\rangle, \end{aligned} \tag{4.28}$$

where $\psi_k^\dagger(t)$ and $\psi_k(t)$ are the creation and annihilation operators for an electron state with momentum k at time t. This function measures the particle states in the vacuum. For instance, when $t > t'$, it measures the probability of creating an electron with momentum k at t' and annihilating it at a later time t. In the zero-temperature limit without interaction, this probability is non-zero only if $|k| > k_F$, meaning the electron state lies above the Fermi surface and is unoccupied in the vacuum ground state. Conversely, when $t < t'$, it measures the probability of annihilating an electron with momentum k at t and recreating it at time t'. In this case, the non-interacting Green's function is non-zero only if $|k| < k_F$, indicating the state resides within the Fermi sea and is already occupied at zero temperature.

In the interacting case, while the above statements are not strictly correct, they remain approximately valid under certain conditions. Specifically, for electron states "deep" within the Fermi sea or "high above" it, the long-wavelength approximation ensures that interactions are insufficient to excite states deeply within the Fermi sea or to create very high-energy electrons far above it. Based on this reasoning, our calculation will separately consider two cases: one for "deep" fermions (associated with $t < t'$) and another for "high above" fermions (associated with $t > t'$).

We now return to the task of integrating (4.27) to obtain the real-space Green's function. First, we integrate over ω using the method outlined in Section B.1. The integral separates into two parts (the reason for this separation will become clear later). The first part is

$$\int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - v_F^2 k^2}{\omega^2 - v^2 k^2} \frac{1}{(\omega - v_q k)^2}, \tag{4.29}$$

which can be evaluated using contour integration (assuming $q > 0$), yielding (see Appendix B.5)

$$\int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - v_F^2 k^2}{\omega^2 - v^2 k^2} \frac{1}{(\omega - v_q k)^2} = \frac{i}{k} c_1(q) \left[\Theta(-k) - \Theta(k)\right], \tag{4.30}$$

where $c_1(q)$ is defined as

$$c_1(q) = -\frac{v^2 - v_F^2}{2v(v_q + v)^2}. \tag{4.31}$$

The second part of the integral is

$$-\int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - v_F^2 k^2}{\omega^2 - v^2 k^2} \frac{\cos(k(x - x') - \omega(t - t'))}{(\omega - v_q k)^2}. \tag{4.32}$$

To simplify, we use partial fraction decomposition to separate the poles

$$\begin{aligned}
&\int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - v_F^2 k^2}{\omega^2 - v^2 k^2} \frac{\cos(k(x-x') - \omega(t-t'))}{(\omega - v_q k)^2} \\
=&\int \frac{\mathrm{d}\omega}{2\pi} \left[\frac{1}{k} \left(\frac{c_1(q)}{\omega + vk} + \frac{c_2(q)}{\omega - vk} + \frac{c_3(q)}{\omega - v_q k} \right) + \frac{c_4(q)}{(\omega - v_q k)^2} \right] \times \cos(k\bar{x} - \omega\bar{t}),
\end{aligned} \tag{4.33}$$

where $\bar{x} = x - x'$ and $\bar{t} = t - t'$. The coefficients $c_i(q)$ are

$$\begin{aligned}
c_1(q) &= -\frac{v^2 - v_F^2}{2v(v_q + v)^2} & c_2(q) &= \frac{v^2 - v_F^2}{2v(v_q - v)^2} \\
c_3(q) &= -\frac{2v_q(v^2 - v_F^2)}{(v_q^2 - v^2)^2} & c_4(q) &= \frac{v_q^2(v^2 - v_F^2)}{v^2(v_q^2 - v^2)}.
\end{aligned} \tag{4.34}$$

The separation into two parts is necessary because the separated poles are not all convergent as $|\omega| \to \infty$. Consequently, contour integration cannot be applied directly to the first term if its poles are separated.

Using contour integration (see Appendix B.5), we find the results for both terms. Combining them, we summarize the results as follows.

For "deep" fermions (hole propagation, $t < t'$ and $q > 0$), the result is

$$\begin{aligned}
&\frac{-iV}{\hbar} \int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - v_F^2 k^2}{\omega^2 - v^2 k^2} \frac{1 - \cos(k(x-x') - \omega(t-t'))}{(\omega - v_q k)^2} \\
&= \Theta(-\bar{t}) \frac{V}{2\hbar} \int \frac{\mathrm{d}k}{2\pi} \Big\{ \Theta(-k) \Big[k^{-1} \Big(c_1(q)(e^{-ik(\bar{x}+v\bar{t})} - 1) - c_2(q)(e^{ik(\bar{x}-v\bar{t})} - 1) \\
&-c_3(q)(e^{ik(\bar{x}-v_q\bar{t})} - 1) \Big) - ic_4(q)\bar{t}e^{ik(\bar{x}-v_q\bar{t})} \Big] \\
&+ \Theta(k) \Big[-k^{-1} \Big(c_1(q)(e^{ik(\bar{x}+v\bar{t})} - 1) - c_2(q)(e^{-ik(\bar{x}-v\bar{t})} - 1) \\
&-c_3(q)(e^{-ik(\bar{x}-v_q\bar{t})} - 1) \Big) - ic_4(q)\bar{t}e^{-ik(\bar{x}-v_q\bar{t})} \Big] \Big\},
\end{aligned} \tag{4.35}$$

where we have used $c_1 + c_2 + c_3 = 0$.

For "higher above" fermions (particle propagation, $t > t'$ and $q > 0$), the result is

$$\begin{aligned}
&\frac{-iV}{\hbar} \int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - v_F^2 k^2}{\omega^2 - v^2 k^2} \frac{1 - \cos(k(x-x') - \omega(t-t'))}{(\omega - v_q k)^2} \\
=&\Theta(\bar{t}) \frac{V}{2\hbar} \int \frac{\mathrm{d}k}{2\pi} \Big\{ \Theta(-k) \Big[k^{-1} \Big(c_1(q)(e^{ik(\bar{x}+v\bar{t})} - 1) - c_2(q)(e^{-ik(\bar{x}-v\bar{t})} - 1) \\
&-c_3(q)(e^{-ik(\bar{x}-v_q\bar{t})} - 1) \Big) - ic_4(q)\bar{t}e^{-ik(\bar{x}-v_q\bar{t})} \Big] \\
&+ \Theta(k) \Big[-k^{-1} \Big(c_1(q)(e^{-ik(\bar{x}+v\bar{t})} - 1) - c_2(q)(e^{ik(\bar{x}-v\bar{t})} - 1) \\
&-c_3(q)(e^{ik(\bar{x}-v_q\bar{t})} - 1) \Big) - ic_4(q)\bar{t}e^{ik(\bar{x}-v_q\bar{t})} \Big] \Big\}.
\end{aligned} \tag{4.36}$$

If we assume $q < 0$, the result can be obtained using the same method and is as follows:

For "deep" fermions (hole propagation, $t < t'$ and $q < 0$),

$$
\begin{aligned}
&\frac{-iV}{\hbar}\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-v_F^2k^2}{\omega^2-v^2k^2}\frac{1-\cos(k(x-x')-\omega(t-t'))}{(\omega-v_qk)^2}\\
=&\Theta(-\bar{t})\frac{V}{2\hbar}\int\frac{\mathrm{d}k}{2\pi}\Big\{\Theta(-k)\Big[k^{-1}\Big(c_1(q)(e^{-ik(\bar{x}+v\bar{t})}-1)-c_2(q)(e^{ik(\bar{x}-v\bar{t})}-1)\\
&+c_3(q)(e^{ik(\bar{x}-v_q\bar{t})}-1)\Big)-ic_4(q)\bar{t}e^{ik(\bar{x}-v_q\bar{t})}\Big]\\
&+\Theta(k)\Big[-k^{-1}\Big(c_1(q)(e^{ik(\bar{x}+v\bar{t})}-1)-c_2(q)(e^{-ik(\bar{x}-v\bar{t})}-1)\\
&+c_3(q)(e^{-ik(\bar{x}-v_q\bar{t})}-1)\Big)-ic_4(q)\bar{t}e^{-ik(\bar{x}-v_q\bar{t})}\Big]\Big\}. \qquad (4.37)
\end{aligned}
$$

For "higher above" fermions (particle propagation, $t > t'$ and $q < 0$),

$$
\begin{aligned}
&\frac{-iV}{\hbar}\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-v_F^2k^2}{\omega^2-v^2k^2}\frac{1-\cos(k(x-x')-\omega(t-t'))}{(\omega-v_qk)^2}\\
=&\Theta(\bar{t})\frac{V}{2\hbar}\int\frac{\mathrm{d}k}{2\pi}\Big\{\Theta(-k)\Big[k^{-1}\Big(c_1(q)(e^{ik(\bar{x}+v\bar{t})}-1)-c_2(q)(e^{-ik(\bar{x}-v\bar{t})}-1)\\
&+c_3(q)(e^{-ik(\bar{x}-v_q\bar{t})}-1)\Big)-ic_4(q)\bar{t}e^{-ik(\bar{x}-v_q\bar{t})}\Big]\\
&+\Theta(k)\Big[-k^{-1}\Big(c_1(q)(e^{-ik(\bar{x}+v\bar{t})}-1)-c_2(q)(e^{ik(\bar{x}-v\bar{t})}-1)\\
&+c_3(q)(e^{ik(\bar{x}-v_q\bar{t})}-1)\Big)-ic_4(q)\bar{t}e^{ik(\bar{x}-v_q\bar{t})}\Big]\Big\}. \qquad (4.38)
\end{aligned}
$$

4.3.1 Finishing the k integral

We now address the remaining k integral in the equations above. Since the integrand is an even function of k, the integration can be simplified by calculating from 0 to ∞ and doubling the result. Specifically, we evaluate

$$
\begin{aligned}
\Theta(-\bar{t})\frac{V}{\hbar}\int_0^\infty\frac{\mathrm{d}k}{2\pi}\Bigg[&-k^{-1}\big(c_1(q)(e^{ik(\bar{x}+v\bar{t})}-1)-c_2(q)(e^{-ik(\bar{x}-v\bar{t})}-1)\\
&-\mathrm{sgn}(q)c_3(q)(e^{-ik(\bar{x}-v_q\bar{t})}-1)\big)-ic_4(q)\bar{t}e^{-ik(\bar{x}-v_q\bar{t})}\Bigg], \qquad (4.39)
\end{aligned}
$$

for hole propagation (deep fermion), and

$$
\begin{aligned}
\Theta(\bar{t})\frac{V}{\hbar}\int_0^\infty\frac{\mathrm{d}k}{2\pi}\Bigg[&-k^{-1}\big(c_1(q)(e^{-ik(\bar{x}+v\bar{t})}-1)-c_2(q)(e^{ik(\bar{x}-v\bar{t})}-1)\\
&-\mathrm{sgn}(q)c_3(q)(e^{ik(\bar{x}-v_q\bar{t})}-1)\big)-ic_4(q)\bar{t}e^{ik(\bar{x}-v_q\bar{t})}\Bigg], \qquad (4.40)
\end{aligned}
$$

for particle propagation.

To compute these integrals, we introduce a cutoff $e^{-\Lambda^{-1}k}$ for k, as the long-wave approximation for phonons is employed. This leads to two distinct types of integrals.

Type 1:

$$f(X,\Lambda) \equiv -\int_0^\infty \frac{\mathrm{d}k}{2\pi} e^{-k\Lambda^{-1}} \frac{e^{ikX}-1}{k}, \tag{4.41}$$

which can be evaluated using the following trick:

$$\begin{aligned}\frac{\partial f(X,\Lambda)}{\partial \Lambda^{-1}} &= \int_0^\infty \frac{\mathrm{d}k}{2\pi} e^{-k\Lambda^{-1}} (e^{ikX}-1) \\ &= \frac{1}{2\pi(-\Lambda^{-1}+iX)} \left[e^{k(-\Lambda^{-1}+iX)} \right]_0^\infty + \frac{1}{2\pi\Lambda^{-1}} \left[e^{-k\Lambda^{-1}} \right]_0^\infty \\ &= \frac{1}{2\pi} \left(\frac{1}{\Lambda^{-1}-iX} - \frac{1}{\Lambda^{-1}} \right) \\ &= \frac{1}{2\pi} \frac{i\Lambda^2 X}{1-i\Lambda X} = \frac{1}{2\pi} \frac{\partial \log(1-i\Lambda X)}{\partial \Lambda^{-1}}. \end{aligned} \tag{4.42}$$

Integrating (4.42) yields

$$f(X,\Lambda) = \frac{1}{2\pi} \log(1-i\Lambda X) + f_0(X). \tag{4.43}$$

Since $f(X,\Lambda) \to 0$ as $\Lambda \to 0$, the constant $f_0(X)$ is determined to be 0. Thus,

$$f(X,\Lambda) = \frac{1}{2\pi} \log(1-i\Lambda X). \tag{4.44}$$

Type 2:

$$g(X,\Lambda) \equiv \int_0^\infty \frac{\mathrm{d}k}{2\pi} e^{-\frac{k}{\Lambda}} e^{ikX} = \frac{1}{2\pi(\Lambda^{-1}-iX)}. \tag{4.45}$$

Using these results, equations (4.39) and (4.40) can be evaluated directly. For hole propagation (deep fermion),

$$\begin{aligned} G_{\text{hole}}(x,t;x',t') = \Theta(-\bar{t}) G_0(x,t;x',t') \exp\left(-\frac{V}{2\pi\hbar} \frac{ic_4(q)\bar{t}}{\Lambda^{-1}+i(\bar{x}-v_q\bar{t})} \right) \\ \times \left(1-i\Lambda(\bar{x}+v\bar{t})\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(1+i\Lambda(\bar{x}-v\bar{t})\right)^{-\frac{c_2(q)V}{2\pi\hbar}} \\ \times \left(1+i\Lambda(\bar{x}-v_q\bar{t})\right)^{-\text{sgn}(q)\frac{c_3(q)V}{2\pi\hbar}}, \end{aligned} \tag{4.46}$$

where $\bar{x} = x - x'$, $\bar{t} = t - t'$, $v = v_F\sqrt{\frac{2V}{\pi\hbar v_F}+1}$, and V and Λ (the cutoff for k) are constants. For particle propagation,

$$
\begin{aligned}
G_{\text{part}}(x,t;x',t') = \Theta(\bar{t}) G_0(x,t;x',t') \exp\left(-\frac{V}{2\pi\hbar}\frac{ic_4(q)\bar{t}}{\Lambda^{-1} - i(\bar{x} - v_q\bar{t})}\right) \\
\times \left(1 + i\Lambda(\bar{x} + v\bar{t})\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(1 - i\Lambda(\bar{x} - v\bar{t})\right)^{-\frac{c_2(q)V}{2\pi\hbar}} \\
\times \left(1 - i\Lambda(\bar{x} - v_q\bar{t})\right)^{-\text{sgn}(q)\frac{c_3(q)V}{2\pi\hbar}} .
\end{aligned}
\tag{4.47}
$$

The non-interacting Green's function is essentially a delta function (the proof is provided in Appendix B.7),

$$
G_0(x,t;x',t') = ie^{iq\bar{x}}e^{-i\left(\frac{\hbar q^2}{2m} - \frac{\mu}{\hbar}\right)\bar{t}}\delta(\bar{x} - v_q\bar{t})\left[\Theta(-\bar{t}) - \Theta(\bar{t})\right]. \tag{4.48}
$$

This simplifies the real-space Green's functions to

For hole propagation:

$$
\begin{aligned}
G_{\text{hole}}(x,t;x',t') = i\Theta(-\bar{t})e^{iq\bar{x}}e^{-i\left(\frac{\hbar q^2}{2m} - \frac{\mu}{\hbar} + \frac{\Lambda c_4(q)V}{2\pi\hbar}\right)\bar{t}}\delta(\bar{x} - v_q\bar{t}) \\
\times \left(1 - i\Lambda(\bar{x} + v\bar{t})\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(1 + i\Lambda(\bar{x} - v\bar{t})\right)^{-\frac{c_2(q)V}{2\pi\hbar}} .
\end{aligned}
\tag{4.49}
$$

For particle propagation:

$$
\begin{aligned}
G_{\text{part}}(x,t;x',t') = -i\Theta(\bar{t})e^{iq\bar{x}}e^{-i\left(\frac{\hbar q^2}{2m} - \frac{\mu}{\hbar} + \frac{\Lambda c_4(q)V}{2\pi\hbar}\right)\bar{t}}\delta(\bar{x} - v_q\bar{t}) \\
\times \left(1 + i\Lambda(\bar{x} + v\bar{t})\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(1 - i\Lambda(\bar{x} - v\bar{t})\right)^{-\frac{c_2(q)V}{2\pi\hbar}} .
\end{aligned}
\tag{4.50}
$$

As observed, the second-order pole contributes only a small shift to the energy dispersion. The two Green's functions (excluding the delta function) along the direction $\bar{x} = v_q\bar{t}$ are plotted in Fig. 4.2 and Fig. 4.3. Both Green's functions exhibit a power-law decay at large $\bar{t}$, with a correlation length $\left(\Lambda\sqrt{|v_q^2 - v^2|}\right)^{-1}$, if we define the correlation length ξ for the power law as $(\xi/\bar{t})^{\alpha}$ for positive α.

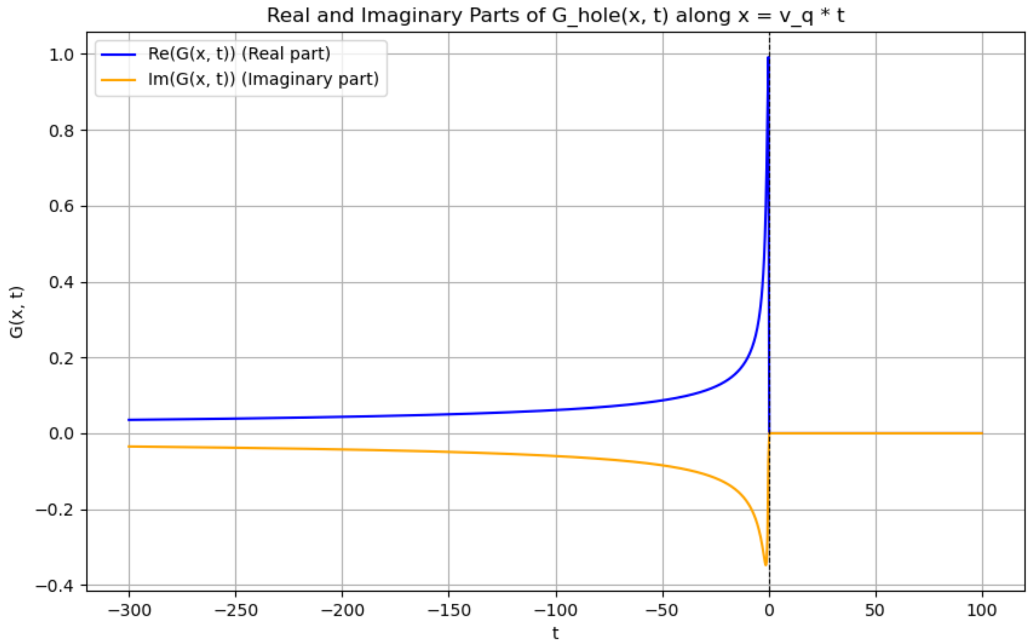


Figure 4.2: Plot of Green's function for hole propagation with $v_q = 0.5v_F$

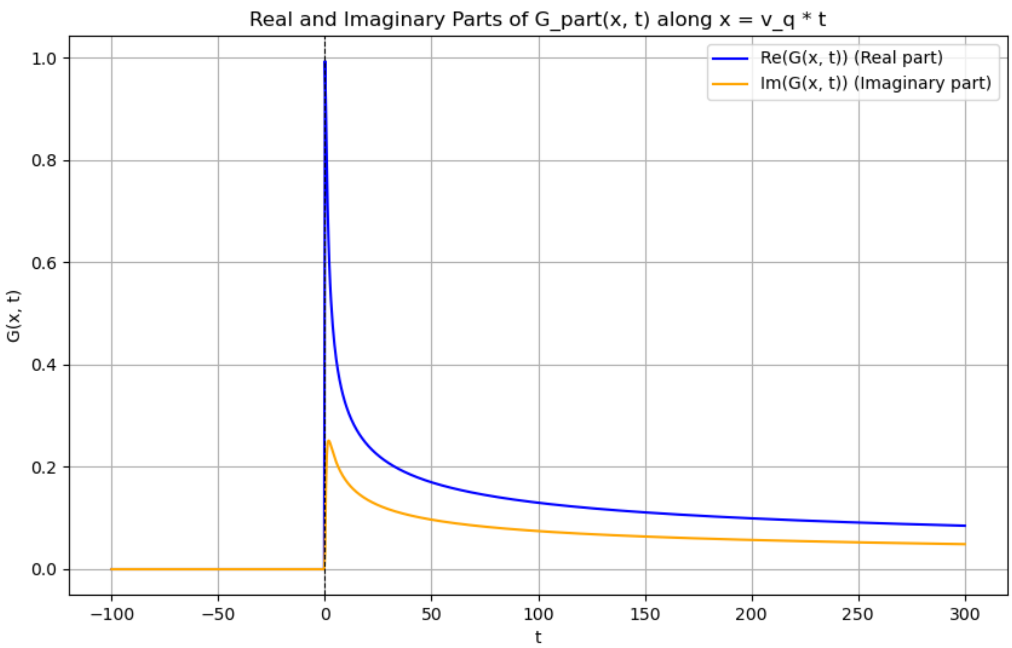


Figure 4.3: Plot of Green's function for particle propagation with $v_q = 3v_F$

4.4 Momentum-space Green's function

To compare our results with experiments, we calculate the Green's function in momentum space. This is obtained by Fourier transforming equations (4.49) and (4.50)

$$G(\omega, q) = \int \mathrm{d}\bar{t} \int \mathrm{d}\bar{x}\, G(\bar{x}, \bar{t}) e^{i\omega\bar{t}} e^{-iq\bar{x}}, \tag{4.51}$$

where the integral over $\bar{x}$ in (4.51) can be evaluated directly. For deep fermions, we have

$$\begin{aligned} G_{\text{hole}}(q, \omega) &= i \int \mathrm{d}\bar{t}\, \Theta(-\bar{t}) e^{i\left(\omega - \frac{\epsilon_q - \mu}{\hbar} - \frac{\Lambda V c_4(q)}{2\pi\hbar}\right)\bar{t}} \\ &\quad \times \left(1 - i\Lambda(v_q + v)\bar{t}\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(1 + i\Lambda(v_q - v)\bar{t}\right)^{-\frac{c_2(q)V}{2\pi\hbar}} \\ &= i \int \mathrm{d}\bar{t}\, \Theta(-\bar{t}) e^{i\tilde{\omega}\bar{t}} \\ &\quad \times \left(1 - i\Lambda(v_q + v)\bar{t}\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(1 + i\Lambda(v_q - v)\bar{t}\right)^{-\frac{c_2(q)V}{2\pi\hbar}}, \end{aligned} \tag{4.52}$$

where the shifted energy is defined as $\tilde{\omega} = \omega - \frac{\epsilon_q - \mu}{\hbar} - \frac{\Lambda V c_4(q)}{2\pi\hbar}$. For particle propagation,

$$\begin{aligned} G_{\text{part}}(q, \omega) &= -i \int \mathrm{d}\bar{t}\, \Theta(\bar{t}) e^{i\tilde{\omega}\bar{t}} \\ &\quad \times \left(1 + i\Lambda(v_q + v)\bar{t}\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(1 - i\Lambda(v_q - v)\bar{t}\right)^{-\frac{c_2(q)V}{2\pi\hbar}}. \end{aligned} \tag{4.53}$$

Using the convolution theorem, $\mathcal{F}(u \cdot v) = \mathcal{F}(u) * \mathcal{F}(v)$, and the associativity of convolution, we can compute the integrals over $\bar{t}$ as

$$\begin{aligned} G_{\text{hole}}(q, \omega) &= i\mathcal{F}_{\bar{t}}\left\{\Theta(-\bar{t})\right\} * \mathcal{F}_{\bar{t}}\left\{\left(1 - i\Lambda(v_q + v)\bar{t}\right)^{\frac{c_1(q)V}{2\pi\hbar}}\right\} \\ &\quad * \mathcal{F}_{\bar{t}}\left\{\left(1 + i\Lambda(v_q - v)\bar{t}\right)^{-\frac{c_2(q)V}{2\pi\hbar}}\right\}, \end{aligned} \tag{4.54}$$

and

$$\begin{aligned} G_{\text{part}}(q, \omega) &= -i\mathcal{F}_{\bar{t}}\left\{\Theta(\bar{t})\right\} * \mathcal{F}_{\bar{t}}\left\{\left(1 + i\Lambda(v_q + v)\bar{t}\right)^{\frac{c_1(q)V}{2\pi\hbar}}\right\} \\ &\quad * \mathcal{F}_{\bar{t}}\left\{\left(1 - i\Lambda(v_q - v)\bar{t}\right)^{-\frac{c_2(q)V}{2\pi\hbar}}\right\}. \end{aligned} \tag{4.55}$$

The Fourier transform $\mathcal{F}_{\bar{t}}$ is defined as

$$\mathcal{F}_{\bar{t}}\left\{f(\bar{t})\right\} = \int \mathrm{d}\bar{t}\, f(\bar{t}) e^{i\tilde{\omega}\bar{t}}. \tag{4.56}$$

Applying the identity derived in Appendix B.8,

$$\mathcal{F}_{\bar{t}}\left[(1+iW\bar{t})^{-\alpha}\right] = \frac{1}{|W|}\frac{2\pi}{\Gamma(\alpha)}\Theta\left(\frac{\tilde{\omega}}{W}\right)\left(\frac{\tilde{\omega}}{W}\right)^{\alpha-1} e^{-\frac{\tilde{\omega}}{W}}, \tag{4.57}$$

and the Fourier transforms of the Heaviside functions,

$$\mathcal{F}_{\bar{t}}\left\{\Theta(\bar{t})\right\} = \frac{1}{i\tilde{\omega}} + \pi\delta(\tilde{\omega}), \quad \mathcal{F}_{\bar{t}}\left\{\Theta(-\bar{t})\right\} = -\frac{1}{i\tilde{\omega}} + \pi\delta(\tilde{\omega}), \tag{4.58}$$

to compute

$$\begin{aligned} &G_{\text{hole}}(q,\omega) \\ &= i\left\{-\frac{1}{i\tilde{\omega}} + \pi\delta(\tilde{\omega})\right\} \\ &\quad * \left\{\frac{1}{|\Lambda(v_q+v)|}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{-\Lambda(v_q+v)}\right)\left(\frac{\tilde{\omega}}{-\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1} e^{\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\} \\ &\quad * \left\{\frac{1}{|\Lambda(v_q-v)|}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{\Lambda(v_q-v)}\right)\left(\frac{\tilde{\omega}}{\Lambda(v_q-v)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1} e^{-\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}, \end{aligned} \tag{4.59}$$

and

$$\begin{aligned} &G_{\text{part}}(q,\omega) \\ &= -i\left\{\frac{1}{i\tilde{\omega}} + \pi\delta(\tilde{\omega})\right\} \\ &\quad * \left\{\frac{1}{|\Lambda(v_q+v)|}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1} e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\} \\ &\quad * \left\{\frac{1}{|\Lambda(v_q-v)|}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1} e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}. \end{aligned} \tag{4.60}$$

Since convolution is associative, we can compute it in any order. For example, we may first calculate the convolution of the last two terms for both hole and particle propagation, and then convolve the result with the first term. For deep fermions, where $\epsilon_q \ll \mu$, it follows that $|v_q| < v_F < v$. This implies $v - v_q > 0$ and $v + v_q > 0$. Then, the convolution of the last two terms simplifies to

$$
\begin{aligned}
&\left\{\frac{1}{|\Lambda(v_q+v)|}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(-\tilde{\omega}\right)\left(\frac{-\tilde{\omega}}{\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}e^{\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\}\\
&*\left\{\frac{1}{|\Lambda(v_q-v)|}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(-\tilde{\omega}\right)\left(\frac{-\tilde{\omega}}{\Lambda(v-v_q)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{-\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}\\
=&\frac{4\pi^2}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\left(\Lambda(v+v_q)\right)^{\frac{c_1(q)V}{2\pi\hbar}}\left(\Lambda(v-v_q)\right)^{\frac{-c_2(q)V}{2\pi\hbar}}\\
&\times\int \mathrm{d}\omega'\Theta\left(-\omega'\right)\left(-\omega'\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}e^{\frac{\omega'}{\Lambda(v+v_q)}}\Theta\left(\omega'-\tilde{\omega}\right)\left(\omega'-\tilde{\omega}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{\frac{\tilde{\omega}-\omega'}{\Lambda(v-v_q)}}\\
=&\frac{4\pi^2}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\left(\Lambda(v+v_q)\right)^{\frac{c_1(q)V}{2\pi\hbar}}\left(\Lambda(v-v_q)\right)^{\frac{-c_2(q)V}{2\pi\hbar}}e^{\frac{\tilde{\omega}}{\Lambda(v-v_q)}}\\
&\times\Theta(-\tilde{\omega})(-\tilde{\omega})^{-\frac{\left(c_1(q)-c_2(q)\right)V}{2\pi\hbar}-1}\int_0^1 \mathrm{d}u\,(u)^{-\frac{c_1(q)V}{2\pi\hbar}-1}\left(1-u\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{\frac{2v_q\tilde{\omega}}{\Lambda(v_q^2-v^2)}u}\\
=&\frac{4\pi^2}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}+\frac{c_2(q)V}{2\pi\hbar}\right)}\left(\Lambda(v+v_q)\right)^{\frac{c_1(q)V}{2\pi\hbar}}\left(\Lambda(v-v_q)\right)^{\frac{-c_2(q)V}{2\pi\hbar}}e^{\frac{\tilde{\omega}}{\Lambda(v-v_q)}}\\
&\times\Theta(-\tilde{\omega})(-\tilde{\omega})^{-\frac{\left(c_1(q)-c_2(q)\right)V}{2\pi\hbar}-1}M\left(-\frac{c_1(q)V}{2\pi\hbar},-\frac{c_1(q)V}{2\pi\hbar}+\frac{c_2(q)V}{2\pi\hbar},\frac{2v_q\tilde{\omega}}{\Lambda(v_q^2-v^2)}\right),
\end{aligned}
\tag{4.61}
$$

where $M(a,b,z)$ is Kummer's confluent hypergeometric function

$$
M(a,b,z)=\frac{\Gamma(b)}{\Gamma(a)\Gamma(b-a)}\int_0^1 e^{zu}u^{a-1}(1-u)^{b-a-1}\,\mathrm{d}u. \tag{4.62}
$$

For fermions well above the Fermi sea, i.e., $|q|\gg k_F$, it follows that $|v_q|>v$. In this regime, either $v-v_q<0$ or $v+v_q<0$, leading to two sub-cases.

Case 1: Right-moving electron ($v_q>v$, i.e., $v-v_q<0$ and $v+v_q>0$) In this case, we have

$$
\begin{aligned}
&\left\{\frac{1}{|\Lambda(v_q+v)|}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\}\\
&*\left\{\frac{1}{|\Lambda(v_q-v)|}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}\\
=&\left\{\frac{1}{\Lambda(v_q+v)}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(\tilde{\omega}\right)\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\}\\
&*\left\{\frac{1}{\Lambda(v_q-v)}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(-\tilde{\omega}\right)\left(\frac{-\tilde{\omega}}{\Lambda(v_q-v)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}
\end{aligned}
$$

$$
\begin{aligned}
=&\frac{4\pi^2}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\left(\Lambda(v_q+v)\right)^{\frac{c_1(q)V}{2\pi\hbar}}\left(\Lambda(v_q-v)\right)^{\frac{-c_2(q)V}{2\pi\hbar}}e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\\
&\times\int \mathrm{d}\omega'\Theta\left(\omega'\right)\Theta\left(\omega'-\tilde{\omega}\right)\left(\omega'\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}\left(\omega'-\tilde{\omega}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{-\frac{2v_q}{\Lambda(v_q^2-v^2)}\omega'}\\
=&4\pi^2\left(\Lambda(v_q+v)\right)^{\frac{c_1(q)V}{2\pi\hbar}}\left(\Lambda(v_q-v)\right)^{\frac{-c_2(q)V}{2\pi\hbar}}e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\\
&\times\left[\frac{\Theta(\tilde{\omega})}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}e^{-\frac{2v_q}{\Lambda(v_q^2-v^2)}\tilde{\omega}}(\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}U\left(\frac{c_2(q)V}{2\pi\hbar},-\frac{c_1(q)V}{2\pi\hbar}+\frac{c_2(q)V}{2\pi\hbar},\frac{2v_q\tilde{\omega}}{\Lambda(v_q^2-v^2)}\right)\right.\\
&\left.+\frac{\Theta(-\tilde{\omega})}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}(-\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}U\left(-\frac{c_1(q)V}{2\pi\hbar},-\frac{c_1(q)V}{2\pi\hbar}+\frac{c_2(q)V}{2\pi\hbar},\frac{-2v_q\tilde{\omega}}{\Lambda(v_q^2-v^2)}\right)\right].
\end{aligned}
\tag{4.63}
$$

Here, $U(a,b,z)$ is the Tricomi confluent hypergeometric function, defined as

$$
U(a,b,z)=\frac{1}{\Gamma(a)}\int_0^\infty e^{-zt}t^{a-1}(1+t)^{b-a-1}\,\mathrm{d}t. \tag{4.64}
$$

Case 2: Left-moving electron ($v_q<-v$, i.e., $v-v_q>0$ and $v+v_q<0$) In this scenario, we have

$$
\begin{aligned}
&\left\{\frac{1}{|\Lambda(v_q+v)|}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\}\\
&*\left\{\frac{1}{|\Lambda(v_q-v)|}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}\\
=&\left\{\frac{1}{-\Lambda(v_q+v)}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(-\tilde{\omega}\right)\left(\frac{-\tilde{\omega}}{-\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\}\\
&*\left\{\frac{1}{-\Lambda(v_q-v)}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(\tilde{\omega}\right)\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}\\
=&\frac{4\pi^2}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\left(-\Lambda(v_q+v)\right)^{\frac{c_1(q)V}{2\pi\hbar}}\left(-\Lambda(v_q-v)\right)^{\frac{-c_2(q)V}{2\pi\hbar}}e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\\
&\times\int \mathrm{d}\omega'\Theta\left(\omega'-\tilde{\omega}\right)\Theta\left(\omega'\right)\left(\omega'-\tilde{\omega}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}\left(\omega'\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{\frac{2v_q}{\Lambda(v_q^2-v^2)}\omega'}
\end{aligned}
$$

$$
\begin{aligned}
=&4\pi^2\big(-\Lambda(v_q+v)\big)^{\frac{c_1(q)V}{2\pi\hbar}}\big(-\Lambda(v_q-v)\big)^{\frac{-c_2(q)V}{2\pi\hbar}}e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\\
\times&\Bigg[\frac{\Theta(\tilde{\omega})}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}e^{\frac{2v_q}{\Lambda(v_q^2-v^2)}\tilde{\omega}}(\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}U\left(-\frac{c_1(q)V}{2\pi\hbar},-\frac{c_1(q)V}{2\pi\hbar}+\frac{c_2(q)V}{2\pi\hbar},\frac{-2v_q\tilde{\omega}}{\Lambda(v_q^2-v^2)}\right)\\
+&\frac{\Theta(-\tilde{\omega})}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}(-\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}U\left(\frac{c_2(q)V}{2\pi\hbar},-\frac{c_1(q)V}{2\pi\hbar}+\frac{c_2(q)V}{2\pi\hbar},\frac{2v_q\tilde{\omega}}{\Lambda(v_q^2-v^2)}\right)\Bigg].
\end{aligned}
\tag{4.65}
$$

These terms, when convolved with the Fourier transform of $\Theta(-\bar{t})$ and $\Theta(\bar{t})$ respectively, yield the corresponding momentum-space Green's function. However, since the spectral function is derived from the imaginary part of the Green's function, we retain only the convolution with the delta function component.

4.5 The spectral function

As defined in Section 3.2, the spectral function, which provides information about the possible states, is given by

$$
A(k,\omega)=-\frac{1}{\pi}\mathrm{Im}G(k,\omega). \tag{4.66}
$$

From previous results, the spectral function for a deep fermion is expressed as

$$
\begin{aligned}
A_{\text{hole}}(q,\omega)=&-\{\delta(\tilde{\omega})\}\\
*&\left\{\frac{1}{|\Lambda(v_q+v)|}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{-\Lambda(v_q+v)}\right)\left(\frac{\tilde{\omega}}{-\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}e^{\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\}\\
*&\left\{\frac{1}{|\Lambda(v_q-v)|}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{\Lambda(v_q-v)}\right)\left(\frac{\tilde{\omega}}{\Lambda(v_q-v)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{-\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}.
\end{aligned}
\tag{4.67}
$$

For a fermion far above the Fermi level, the spectral function is

$$
\begin{aligned}
A_{\text{part}}(q,\omega)=&\{\delta(\tilde{\omega})\}\\
*&\left\{\frac{1}{|\Lambda(v_q+v)|}\frac{2\pi}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)\left(\frac{\tilde{\omega}}{\Lambda(v_q+v)}\right)^{-\frac{c_1(q)V}{2\pi\hbar}-1}e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}}\right\}\\
*&\left\{\frac{1}{|\Lambda(v_q-v)|}\frac{2\pi}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)\left(\frac{\tilde{\omega}}{-\Lambda(v_q-v)}\right)^{\frac{c_2(q)V}{2\pi\hbar}-1}e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}}\right\}.
\end{aligned}
\tag{4.68}
$$

The convolution of a function with the Dirac delta function is

$$f * \delta(\tilde{\omega}) = \int \mathrm{d}\omega' \, f(\omega')\delta(\tilde{\omega} - \omega') = f(\tilde{\omega}). \tag{4.69}$$

Thus, the expressions become

$$\begin{aligned}
&A_{\text{hole}}(q,\omega) \\
&= -\frac{4\pi^2}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar} + \frac{c_2(q)V}{2\pi\hbar}\right)} \left(\Lambda(v+v_q)\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(\Lambda(v-v_q)\right)^{\frac{-c_2(q)V}{2\pi\hbar}} e^{\frac{\tilde{\omega}}{\Lambda(v-v_q)}} \\
&\quad \times \Theta(-\tilde{\omega})(-\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1} M\left(-\frac{c_1(q)V}{2\pi\hbar}, -\frac{c_1(q)V}{2\pi\hbar} + \frac{c_2(q)V}{2\pi\hbar}, \frac{2v_q\tilde{\omega}}{\Lambda(v_q^2 - v^2)}\right),
\end{aligned} \tag{4.70}$$

$$\begin{aligned}
&A^R_{\text{part}}(q,\omega) \\
=&4\pi^2\left(\Lambda(v_q+v)\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(\Lambda(v_q-v)\right)^{\frac{-c_2(q)V}{2\pi\hbar}} e^{\frac{\tilde{\omega}}{\Lambda(v_q-v)}} \\
\times&\left[\frac{\Theta(\tilde{\omega})}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)} e^{-\frac{2v_q}{\Lambda(v_q^2-v^2)}\tilde{\omega}} (\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1} U\left(\frac{c_2(q)V}{2\pi\hbar}, -\frac{c_1(q)V}{2\pi\hbar} + \frac{c_2(q)V}{2\pi\hbar}, \frac{2v_q\tilde{\omega}}{\Lambda(v_q^2 - v^2)}\right)\right. \\
&\left. + \frac{\Theta(-\tilde{\omega})}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)} (-\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1} U\left(-\frac{c_1(q)V}{2\pi\hbar}, -\frac{c_1(q)V}{2\pi\hbar} + \frac{c_2(q)V}{2\pi\hbar}, \frac{-2v_q\tilde{\omega}}{\Lambda(v_q^2 - v^2)}\right)\right],
\end{aligned} \tag{4.71}$$

and

$$\begin{aligned}
&A^L_{\text{part}}(q,\omega) \\
=&4\pi^2\left(-\Lambda(v_q+v)\right)^{\frac{c_1(q)V}{2\pi\hbar}} \left(-\Lambda(v_q-v)\right)^{\frac{-c_2(q)V}{2\pi\hbar}} e^{-\frac{\tilde{\omega}}{\Lambda(v_q+v)}} \\
\times&\left[\frac{\Theta(\tilde{\omega})}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)} e^{\frac{2v_q}{\Lambda(v_q^2-v^2)}\tilde{\omega}} (\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1} U\left(-\frac{c_1(q)V}{2\pi\hbar}, -\frac{c_1(q)V}{2\pi\hbar} + \frac{c_2(q)V}{2\pi\hbar}, \frac{-2v_q\tilde{\omega}}{\Lambda(v_q^2 - v^2)}\right)\right. \\
&\left. + \frac{\Theta(-\tilde{\omega})}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)} (-\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1} U\left(\frac{c_2(q)V}{2\pi\hbar}, -\frac{c_1(q)V}{2\pi\hbar} + \frac{c_2(q)V}{2\pi\hbar}, \frac{2v_q\tilde{\omega}}{\Lambda(v_q^2 - v^2)}\right)\right].
\end{aligned} \tag{4.72}$$

Here, R and L denote right-moving (positive momentum) and left-moving (negative momentum) electrons, respectively. The term $\tilde{\omega} = \left(\omega - \frac{\epsilon_q - \mu}{\hbar} - \frac{\Lambda V c_4(q)}{2\pi\hbar}\right)$ represents the modified dispersion relation.

4.6 Momentum-dependent power law

When ω approaches the dispersion curve, i.e., $\tilde{\omega} \to 0$, the result for a deep fermion can be approximated using the property of Kummer's function ($M \to 1$) as

$$\begin{aligned} &A_{\text{hole}}(q,\omega) \\ &= -\frac{4\pi^2}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}+\frac{c_2(q)V}{2\pi\hbar}\right)}\left(\Lambda(v+v_q)\right)^{\frac{c_1(q)V}{2\pi\hbar}}\left(\Lambda(v-v_q)\right)^{\frac{-c_2(q)V}{2\pi\hbar}} \\ &\times\Theta\left(-\tilde{\omega}\right)\left(-\tilde{\omega}\right)^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}. \end{aligned} \tag{4.73}$$

This describes the momentum-dependent power law in the one-dimensional case. On the lower side of the dispersion curve, the spectral function exhibits a momentum-dependent power law, while on the upper side, it vanishes. Similarly, using the property of the Tricomi function

$$U(a,b,z) \to \frac{\Gamma(1-b)}{\Gamma(a-b+1)}+\frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}, \quad z\to 0, \quad b\notin\mathbb{Z}, \tag{4.74}$$

we can approximate $A^L_{\text{part}}(q,\omega)$ and $A^R_{\text{part}}(q,\omega)$ when ω is close to the dispersion curve. Here, we assume that $-\frac{c_1(q)V}{2\pi\hbar}+\frac{c_2(q)V}{2\pi\hbar}$ is small for a small coupling constant V. We get

$$\begin{aligned} A^R_{\text{part}}(q,\omega) =& 4\pi^2\big(\Lambda(v_q+v)\big)^{\frac{c_1(q)V}{2\pi\hbar}}\big(\Lambda(v_q-v)\big)^{\frac{-c_2(q)V}{2\pi\hbar}} \\ &\times\left[\frac{\Gamma\left(1+\frac{c_1(q)V}{2\pi\hbar}-\frac{c_2(q)V}{2\pi\hbar}\right)}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)\Gamma\left(1+\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta(\tilde{\omega})(\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}\right. \\ &\left.+\frac{\Gamma\left(1+\frac{c_1(q)V}{2\pi\hbar}-\frac{c_2(q)V}{2\pi\hbar}\right)}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)\Gamma\left(1-\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta(-\tilde{\omega})(-\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}\right], \end{aligned} \tag{4.75}$$

and

$$\begin{aligned} A^L_{\text{part}}(q,\omega) =& 4\pi^2\big(-\Lambda(v_q+v)\big)^{\frac{c_1(q)V}{2\pi\hbar}}\big(-\Lambda(v_q-v)\big)^{\frac{-c_2(q)V}{2\pi\hbar}} \\ &\times\left[\frac{\Gamma\left(1+\frac{c_1(q)V}{2\pi\hbar}-\frac{c_2(q)V}{2\pi\hbar}\right)}{\Gamma\left(\frac{c_2(q)V}{2\pi\hbar}\right)\Gamma\left(1-\frac{c_2(q)V}{2\pi\hbar}\right)}\Theta(\tilde{\omega})(\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}\right. \\ &\left.+\frac{\Gamma\left(1+\frac{c_1(q)V}{2\pi\hbar}-\frac{c_2(q)V}{2\pi\hbar}\right)}{\Gamma\left(-\frac{c_1(q)V}{2\pi\hbar}\right)\Gamma\left(1+\frac{c_1(q)V}{2\pi\hbar}\right)}\Theta(-\tilde{\omega})(-\tilde{\omega})^{-\frac{(c_1(q)-c_2(q))V}{2\pi\hbar}-1}\right]. \end{aligned} \tag{4.76}$$

This demonstrates a momentum-dependent power law on both sides of the dispersion curve.

4.7 Plot of the spectral function

The result above is plotted in Fig. 4.4, where "deep" fermions are shown in the center, and "higher above" fermions are on the two sides. As seen in the figure, for the "deep" fermion, the spectral function exhibits Luttinger liquid behavior, meaning it vanishes above the dispersion relation for hole propagation. For fermions far above the Fermi surface, the spectral function shows Fermi-liquid-like behavior, with a non-zero spectral function on both sides of the dispersion. This is in good agreement with the predictions made in [1], as shown in Fig. 4.5. However, in both cases, the spectral function displays a momentum-dependent power law as it approaches the dispersion relation. This suggests that, for fermions far above the Fermi level, the system may not behave as a true quasi-particle as predicted by Fermi-liquid theory, which typically predicts a spectral function $A_q(\omega) \sim \omega^\gamma$, with constant γ.

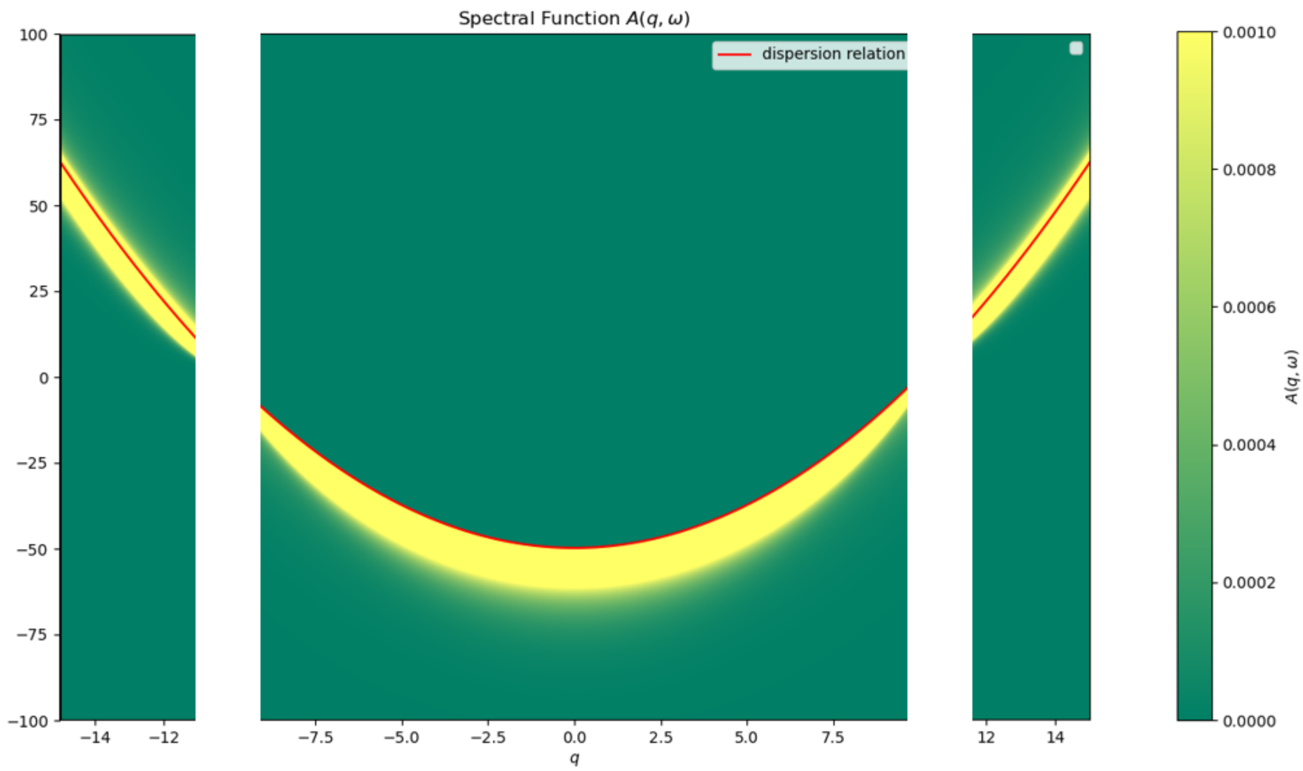


Figure 4.4: Spectral function of One-dimensional Luttinger model

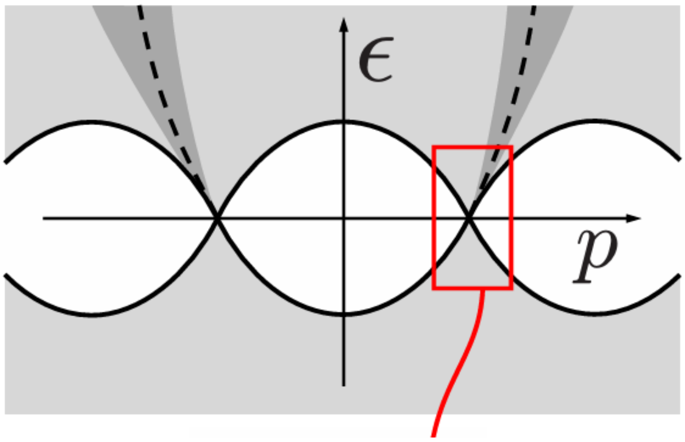
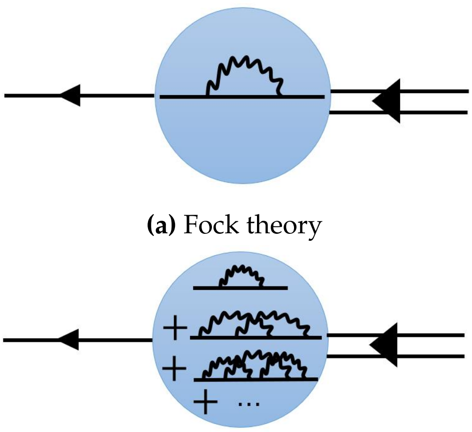


Figure 4.5: Spectral function predicted by M. Khodas et al.[1]

For comparison, we refer to the spectral function of the Fock diagram calculated in section 3.2. As seen in Fig.3.7, both the spectral function from our results and the Fock diagram share some common features. For instance, for the "deep" fermion, there are no possible states above the dispersion curve. Similarly, for fermions above the Fermi surface, the spectral function is non-zero on both sides of the dispersion. The most interesting feature is that, for the Fock diagram, the spectral function becomes zero if (ω, k) is too far from the dispersion, due to the fact that the interaction with low-energy phonons does not propagate that far. This suggests that one possible interpretation of our one-dimensional calculation is that we have summed over all types of diagrams for the self-energy, not just the Fock diagram, as illustrated in Fig. 4.6. This interpretation is reasonable, as we have not made any further approximations or expansions of diagrams, apart from the linearization scheme and the long-wavelength approximation.

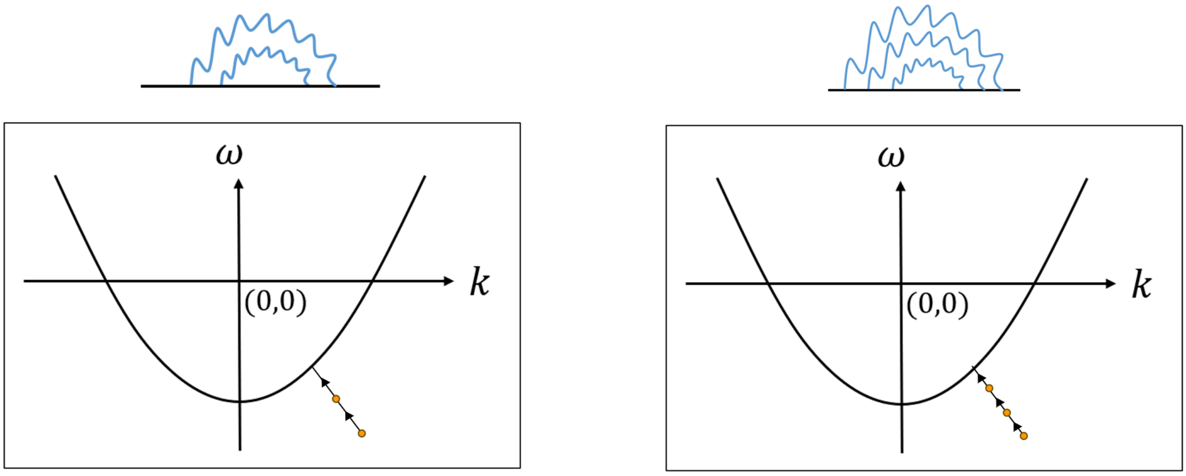


(a) Fock theory

(b) Linearization scheme

Figure 4.6: Possible diagrammatic interpretation of our results

In the linearized model, for a quasi-particle state far from the dispersion, many diagrams can contribute to the non-zero imaginary part of the spectral function, as illustrated in Fig. 4.7.



(a) Non-zero spectral function contribution from the two-loop diagram

(b) Non-zero spectral function contribution from the three-loop diagram

Figure 4.7: Different possible processes contributing to the spectral function

Chapter 5

Exploration in two-dimension

To generalize our result from one dimension to two dimensions, we essentially follow the logic in Chapter 3, replacing 1D momentum scalars with 2D vectors and the spatial derivative $\frac{\partial}{\partial x}$ with the gradient operator ∇, among other adjustments.

5.1 Generalization(Isotropic model)

In the two-dimensional case, the action is given by

$$S[\phi^*, \phi] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x}\, \phi_q^*(\mathbf{x}, \tau) \left\{ \hbar \frac{\partial}{\partial \tau} + \frac{\hbar^2 q^2}{2m} - \frac{i\hbar^2 \mathbf{q} \cdot \nabla}{m} - \mu + \kappa(\mathbf{x}, \tau) \right\} \phi_q(\mathbf{x}, \tau). \tag{5.1}$$

We now replace κ by a local phase transformation $\phi_q(\mathbf{x}) \to \tilde{\phi}_q(\mathbf{x}) e^{i\theta(\mathbf{x},\tau)}$, which modifies the action to

$$\begin{aligned} S[\phi^*, \phi] = &\int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x}\, \phi_q^*(\mathbf{x}, \tau) \left\{ \hbar \frac{\partial}{\partial \tau} + \frac{\hbar^2 q^2}{2m} - \frac{i\hbar^2 \mathbf{q} \cdot \nabla}{m} - \mu \right\} \phi_q(\mathbf{x}, \tau) \\ &+ \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x}\, \phi_q^*(\mathbf{x}, \tau) \left\{ i\hbar \frac{\partial}{\partial \tau} \theta(\mathbf{x}, \tau) + \frac{\hbar^2}{m} \mathbf{q} \cdot \nabla \theta(\mathbf{x}, \tau) + \kappa(\mathbf{x}, \tau) \right\} \phi_q(\mathbf{x}, \tau). \end{aligned} \tag{5.2}$$

We choose $\theta(\mathbf{x}, \tau)$ such that the second term vanishes. This leads to the differential equation

$$i\hbar \frac{\partial}{\partial \tau} \theta(\mathbf{x}, \tau) + \frac{\hbar^2}{m} \mathbf{q} \cdot \nabla \theta(\mathbf{x}, \tau) + \kappa(\mathbf{x}, \tau) = 0. \tag{5.3}$$

As before, we solve this equation by first Wick-rotating and then performing a Fourier transform. The result is

$$\theta(\mathbf{x},t) = \int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \int \frac{\mathrm{d}\omega}{2\pi} \frac{-i\kappa(\mathbf{k},\omega)}{\hbar(\omega - \mathbf{v_q}\cdot\mathbf{k})} e^{i\mathbf{k}\cdot\mathbf{x}} e^{-i\omega t}, \tag{5.4}$$

where $\mathbf{v_q} = \frac{\hbar}{m}\mathbf{q}$.

The Green's function we are calculating becomes

$$\begin{aligned} &G_{\mathbf{q}}(\mathbf{x},t;\mathbf{x}',t') \\ =&G_{\mathbf{q}0}(\mathbf{x},t;\mathbf{x}',t') e^{-\frac{1}{2}\left\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\right\rangle_\kappa} \\ =&\tilde{G}_{\mathbf{q}0}(\mathbf{x},t;\mathbf{x}',t') e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')} e^{-\frac{1}{2}\left\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\right\rangle_\kappa}, \end{aligned} \tag{5.5}$$

where $G_{\mathbf{q}0}(\mathbf{x},t;\mathbf{x}',t')$ is the 2D non-interacting Green's function. The term $\left\langle[\theta(\mathbf{x},t) - \theta(\mathbf{x}',t')]^2\right\rangle_\kappa$ can be computed using the same method as before, yielding

$$\begin{aligned} \left\langle[\theta(\mathbf{x},t) - \theta(\mathbf{x}',t')]^2\right\rangle_\kappa = -2\int \frac{\mathrm{d}\mathbf{k}}{(2\pi)^2} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\langle\kappa(\mathbf{k},\omega)\kappa(-\mathbf{k},-\omega)\rangle_\kappa}{\hbar^2(\omega - \mathbf{v_q}\cdot\mathbf{k})^2} \\ \times\left[1 - \cos(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}') - \omega(t-t'))\right]. \end{aligned} \tag{5.6}$$

5.1.1 Kappa propagator

Here we calculate $\langle\kappa(\mathbf{k},\omega)\kappa(-\mathbf{k},-\omega)\rangle_\kappa$ in (5.6). We already know (see (4.19))

$$\langle\kappa(\mathbf{k},\omega)\kappa(-\mathbf{k},-\omega)\rangle_\kappa = -iG_k(\mathbf{k},\omega) = \frac{-i\hbar}{V^{-1}(\mathbf{k}) - \Pi(\mathbf{k},\omega)}, \tag{5.7}$$

where $\Pi(\mathbf{k},\omega)$ is given by

$$\Pi(\mathbf{k},\omega) = 2\int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2} \frac{N_{FD}(\epsilon_{\mathbf{k}+\mathbf{k}'}) - N_{FD}(\epsilon_{\mathbf{k}'})}{-\hbar\omega + \epsilon_{\mathbf{k}+\mathbf{k}'} - \epsilon_{\mathbf{k}'}}. \tag{5.8}$$

At $T = 0$, this simplifies to

$$\begin{aligned} \Pi(\mathbf{k},\omega) &= 2\int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2} \frac{\Theta(\mu - \epsilon_{\mathbf{k}+\mathbf{k}'}) - \Theta(\mu - \epsilon_{\mathbf{k}'})}{-\hbar\omega + \epsilon_{\mathbf{k}+\mathbf{k}'} - \epsilon_{\mathbf{k}'}} \\ &= 2\int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2} \frac{\Theta(\mu - \epsilon_{\mathbf{k}'})}{-\hbar\omega + \epsilon_{\mathbf{k}'} - \epsilon_{\mathbf{k}'-\mathbf{k}}} - 2\int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2} \frac{\Theta(\mu - \epsilon_{\mathbf{k}'})}{-\hbar\omega + \epsilon_{\mathbf{k}+\mathbf{k}'} - \epsilon_{\mathbf{k}'}}. \end{aligned} \tag{5.9}$$

Using $\epsilon_{\mathbf{k}} = \frac{\hbar^2\mathbf{k}^2}{2m}$, we rewrite

$$\Pi(\mathbf{k},\omega) =2\int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2} \frac{\Theta(\mathbf{k_F}^2 - \mathbf{k}'^2)}{-\hbar\omega - \frac{\hbar^2k^2}{2m} + \frac{\hbar^2\mathbf{k}\cdot\mathbf{k}'}{m}} - 2\int \frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2} \frac{\Theta(\mathbf{k_F}^2 - \mathbf{k}'^2)}{-\hbar\omega + \frac{\hbar^2k^2}{2m} + \frac{\hbar^2\mathbf{k}\cdot\mathbf{k}'}{m}}$$

$$
\begin{aligned}
=&\frac{1}{2\pi^2}\int_0^{k_F} k'\mathrm{d}k'\left[\int \mathrm{d}\theta\frac{1}{-\hbar\omega-\frac{\hbar^2k^2}{2m}+\frac{\hbar^2kk'}{m}\cos\theta}-\int \mathrm{d}\theta\frac{1}{-\hbar\omega+\frac{\hbar^2k^2}{2m}+\frac{\hbar^2kk'}{m}\cos\theta}\right]\\
=&\frac{1}{2\pi^2}\frac{m}{\hbar^2k}\int_0^{k_F}\mathrm{d}k'\left[\int \mathrm{d}\theta\frac{1}{\cos\theta-\frac{m}{\hbar^2kk'}(\hbar\omega+\frac{\hbar^2k^2}{2m})}-\mathrm{d}k'\int \mathrm{d}\theta\frac{1}{\cos\theta-\frac{m}{\hbar^2kk'}(\hbar\omega-\frac{\hbar^2k^2}{2m})}\right],
\end{aligned}
\tag{5.10}
$$

where We have aligned the z-axis with the direction of $\mathbf{k}$, leveraging the isotropic nature of the dispersion relation. The θ integral can be evaluated using the formula (valid for $a>1$, which is ensured by the long-wavelength approximation $k\to 0$)

$$
\int_{-\pi}^{\pi}\mathrm{d}\theta\frac{1}{\cos\theta-a}=-\frac{2\pi}{\sqrt{a^2-1}}. \tag{5.11}
$$

This can be proven by making the substitution $t=\tan\frac{\theta}{2}$, which yields

$$
\cos\theta=\frac{1-t^2}{1+t^2}, \qquad \& \qquad \mathrm{d}\theta=\frac{2}{1+t^2}\mathrm{d}t. \tag{5.12}
$$

We find, in the long-wavelength approximation ($k\to 0$)

$$
\begin{aligned}
\Pi(\mathbf{k},\omega)=&\frac{1}{\pi}\frac{m}{\hbar^2k}\int_0^{k_F}\mathrm{d}k'\left[\frac{1}{\sqrt{\left(\frac{m}{\hbar^2kk'}\right)^2\left(\hbar\omega-\frac{\hbar^2k^2}{2m}\right)^2-1}}-\frac{1}{\sqrt{\left(\frac{m}{\hbar^2kk'}\right)^2\left(\hbar\omega+\frac{\hbar^2k^2}{2m}\right)^2-1}}\right]\\
\approx&\frac{1}{\pi}\frac{m}{\hbar^2k}\int_0^{k_F}\mathrm{d}k'\left[\frac{1}{\sqrt{\left(\frac{m\omega}{\hbar kk'}\right)^2-1-\frac{m\omega}{\hbar k'^2}}}-\frac{1}{\sqrt{\left(\frac{m\omega}{\hbar kk'}\right)^2-1+\frac{m\omega}{\hbar k'^2}}}\right]\\
=&\frac{1}{\pi}\frac{m}{\hbar^2k}\int_0^{k_F}\mathrm{d}k'\frac{\sqrt{\left(\frac{m\omega}{\hbar kk'}\right)^2-1+\frac{m\omega}{\hbar k'^2}}-\sqrt{\left(\frac{m\omega}{\hbar kk'}\right)^2-1-\frac{m\omega}{\hbar k'^2}}}{\sqrt{\left(\left(\frac{m\omega}{\hbar kk'}\right)^2-1\right)^2-\left(\frac{m\omega}{\hbar k'^2}\right)^2}}\\
\approx&\frac{1}{\pi}\frac{m}{\hbar^2k}\int_0^{k_F}\mathrm{d}k'\frac{\frac{m\omega}{\hbar kk'}\left(1-\frac{1}{2}(\frac{\hbar kk'}{m\omega})^2(1-\frac{m\omega}{\hbar k'^2})\right)-\frac{m\omega}{\hbar kk'}\left(1-\frac{1}{2}(\frac{\hbar kk'}{m\omega})^2(1+\frac{m\omega}{\hbar k'^2})\right)}{\left(\frac{m\omega}{\hbar kk'}\right)^2}\\
=&\frac{1}{\pi}\frac{m}{\hbar^2k}\int_0^{k_F}\mathrm{d}k'\frac{\frac{m\omega}{\hbar k'^2}}{\left(\frac{m\omega}{\hbar kk'}\right)^3}=\frac{1}{\pi}\frac{k^2}{m\omega^2}\int_0^{k_F}k'\mathrm{d}k'\\
=&\frac{k_F^2k^2}{2\pi m\omega^2}=\frac{\mu k^2}{\pi\hbar^2\omega^2}
\end{aligned}
\tag{5.13}
$$

Substituting (5.13) into (5.7) yields

$$
\langle\kappa(\mathbf{k},\omega)\kappa(-\mathbf{k},-\omega)\rangle_\kappa=\frac{-i\hbar}{V^{-1}-\Pi(\mathbf{k},\omega)}=-i\hbar V\frac{\omega^2}{\omega^2-v^2k^2}, \tag{5.14}
$$

where $v = \frac{\mu V}{\pi\hbar^2}$.

5.1.2 Green's function

As in the 1D case, the problem reduces to solving the following integral

$$
\begin{aligned}
&-\frac{1}{2}\left\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\right\rangle_\kappa \\
=&-\frac{iV}{\hbar}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2}{(\omega^2-v^2\mathbf{k}\cdot\mathbf{k})(\omega-\mathbf{v_q}\cdot\mathbf{k})^2}\left[1-\cos(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t'))\right].
\end{aligned}
\tag{5.15}
$$

Similar to the 1D case, the integrand can be decomposed as

$$
\begin{aligned}
&\frac{\omega^2}{(\omega^2-v^2\mathbf{k}\cdot\mathbf{k})(\omega-\mathbf{v_q}\cdot\mathbf{k})^2} \\
=&\frac{1}{k}\left[\frac{c_1(\mathbf{q})}{\omega-vk}-\frac{c_2(\mathbf{q})}{\omega+vk}-\frac{c_3(\mathbf{q})}{\omega-\mathbf{v_q}\cdot\mathbf{k}}+\frac{c_4(\mathbf{q})}{(\omega-\mathbf{v_q}\cdot\mathbf{k})^2}\right],
\end{aligned}
\tag{5.16}
$$

where the coefficients $c_i(\mathbf{q})$ are given by

$$
\begin{aligned}
c_1(\mathbf{q}) &= \frac{1}{2v\left(\frac{\mathbf{v_q}\cdot\mathbf{k}}{vk}-1\right)^2}, & c_2(\mathbf{q}) &= \frac{1}{2v\left(\frac{\mathbf{v_q}\cdot\mathbf{k}}{vk}+1\right)^2}, \\
c_3(\mathbf{q}) &= \frac{2\frac{\mathbf{v_q}\cdot\mathbf{k}}{vk}}{v\left[\left(\frac{\mathbf{v_q}\cdot\mathbf{k}}{vk}\right)^2-1\right]^2}, & c_4(\mathbf{q}) &= \frac{\left(\frac{\mathbf{v_q}\cdot\mathbf{k}}{vk}\right)^2}{\left(\frac{\mathbf{v_q}\cdot\mathbf{k}}{vk}\right)^2-1},
\end{aligned}
\tag{5.17}
$$

satisfying $c_1(\mathbf{q}) - c_2(\mathbf{q}) - c_3(\mathbf{q}) = 0$.

Next, we calculate the integral

$$
\begin{aligned}
&-\frac{1}{2}\left\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\right\rangle_\kappa \\
=&-\frac{iV}{\hbar}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2}{(\omega^2-v^2\mathbf{k}\cdot\mathbf{k})(\omega-\mathbf{v_q}\cdot\mathbf{k})^2} \\
&+\frac{iV}{\hbar}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\frac{1}{k}\int\frac{\mathrm{d}\omega}{2\pi}\left[\frac{c_1(\mathbf{q})}{\omega-vk}-\frac{c_2(\mathbf{q})}{\omega+vk}-\frac{c_3(\mathbf{q})}{\omega-\mathbf{v_q}\cdot\mathbf{k}}+\frac{c_4(\mathbf{q})}{(\omega-\mathbf{v_q}\cdot\mathbf{k})^2}\right] \\
&\times\cos(\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')-\omega(t-t')).
\end{aligned}
\tag{5.18}
$$

The detailed steps for evaluating these integrals follow the method used in the 1D case. For clarity, we simplify the notation by defining $\bar{\mathbf{x}} = \mathbf{x} - \mathbf{x}'$ and $\bar{t} = t - t'$. The result is summarized as

For "deep" fermions (hole propagation, $t < t'$),

$$
\begin{aligned}
&-\frac{1}{2}\left\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\right\rangle_\kappa \\
&+\frac{V}{\hbar}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\frac{1}{k}\Bigg\{\Theta(-\mathbf{v_q}\cdot\mathbf{k})\left[\frac{c_1(\mathbf{q})}{2}\left[e^{-i(\mathbf{k}\cdot\bar{\mathbf{x}}-vk\bar{t})}-1\right]+\frac{c_2(\mathbf{q})}{2}\left[e^{i(\mathbf{k}\cdot\bar{\mathbf{x}}+vk\bar{t})}-1\right]\right. \\
&\qquad\left.+\frac{c_3(\mathbf{q})}{2}\left[e^{i\mathbf{k}\cdot(\bar{\mathbf{x}}-\mathbf{v_q}\bar{t})}-1\right]+\frac{i\bar{t}c_4(\mathbf{q})}{2}e^{i\mathbf{k}\cdot(\bar{\mathbf{x}}-\mathbf{v_q}\bar{t})}\right] \\
&\quad+\Theta(\mathbf{v_q}\cdot\mathbf{k})\left[\frac{c_1(\mathbf{q})}{2}\left[e^{-i(\mathbf{k}\cdot\bar{\mathbf{x}}-vk\bar{t})}-1\right]+\frac{c_2(\mathbf{q})}{2}\left[e^{i(\mathbf{k}\cdot\bar{\mathbf{x}}+vk\bar{t})}-1\right]\right. \\
&\qquad\left.-\frac{c_3(\mathbf{q})}{2}\left[e^{-i\mathbf{k}\cdot(\bar{\mathbf{x}}-\mathbf{v_q}\bar{t})}-1\right]+\frac{i\bar{t}c_4(\mathbf{q})}{2}e^{-i\mathbf{k}\cdot(\bar{\mathbf{x}}-\mathbf{v_q}\bar{t})}\right]\Bigg\}
\end{aligned}
\tag{5.19}
$$

For "higher above" fermions (particle propagation, $t > t'$),

$$
\begin{aligned}
&-\frac{1}{2}\left\langle[\theta(\mathbf{x},t)-\theta(\mathbf{x}',t')]^2\right\rangle_\kappa \\
=&\frac{V}{\hbar}\int\frac{\mathrm{d}\mathbf{k}}{(2\pi)^2}\frac{1}{k}\Bigg\{\Theta(-\mathbf{v_q}\cdot\mathbf{k})\left[\frac{c_1(\mathbf{q})}{2}\left[e^{i(\mathbf{k}\cdot\bar{\mathbf{x}}-vk\bar{t})}-1\right]+\frac{c_2(\mathbf{q})}{2}\left[e^{-i(\mathbf{k}\cdot\bar{\mathbf{x}}+vk\bar{t})}-1\right]\right. \\
&\qquad\left.+\frac{c_3(\mathbf{q})}{2}\left[e^{-i\mathbf{k}\cdot(\bar{\mathbf{x}}-\mathbf{v_q}\bar{t})}-1\right]-\frac{i\bar{t}c_4(\mathbf{q})}{2}e^{-i\mathbf{k}\cdot(\bar{\mathbf{x}}-\mathbf{v_q}\bar{t})}\right] \\
&\quad+\Theta(\mathbf{v_q}\cdot\mathbf{k})\left[\frac{c_1(\mathbf{q})}{2}\left[e^{i(\mathbf{k}\cdot\bar{\mathbf{x}}-vk\bar{t})}-1\right]+\frac{c_2(\mathbf{q})}{2}\left[e^{-i(\mathbf{k}\cdot\bar{\mathbf{x}}+vk\bar{t})}-1\right]\right. \\
&\qquad\left.-\frac{c_3(\mathbf{q})}{2}\left[e^{i\mathbf{k}\cdot(\bar{\mathbf{x}}-\mathbf{v_q}\bar{t})}-1\right]-\frac{i\bar{t}c_4(\mathbf{q})}{2}e^{i\mathbf{k}\cdot(\bar{\mathbf{x}}-\mathbf{v_q}\bar{t})}\right]\Bigg\}
\end{aligned}
\tag{5.20}
$$

which resembles the 1D result in section 4.3.

In the 2D case, extra care is needed because the coefficients $c_i(\mathbf{q})$ depend on the angle between $\mathbf{k}$ and $\mathbf{v_q}$, which complicates the $\mathbf{k}$-integration. This final calculation has not been completed by the end of this thesis.

5.1.3 Non-interacting Green's Function

We can extend our 1D calculation to the 2D case to determine the non-interacting Green's function. The result is

$$
\begin{aligned}
G_0(\mathbf{x},t;\mathbf{x}',t') &= \int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega-\frac{\hbar q^2}{2m}-\mathbf{v_q}\cdot\mathbf{k}'+\frac{\mu}{\hbar}}e^{i\mathbf{k}'\cdot(\mathbf{x}-\mathbf{x}')}e^{-i\omega(t-t')} \\
&= -\frac{i}{2\pi}\int\frac{\mathrm{d}\mathbf{k}'}{(2\pi)^2}e^{i\mathbf{k}'\cdot\bar{\mathbf{x}}}\Bigg\{\Theta(\bar{t})\Theta\left(\frac{\hbar q^2}{2m}+\mathbf{v_q}\cdot\mathbf{k}'-\frac{\mu}{\hbar}\right)e^{-i\left[\frac{\hbar q^2}{2m}+\mathbf{v_q}\cdot\mathbf{k}'-\frac{\mu}{\hbar}\right]\bar{t}} \\
&\qquad-\Theta(-\bar{t})\Theta\left(-\frac{\hbar q^2}{2m}-\mathbf{v_q}\cdot\mathbf{k}'+\frac{\mu}{\hbar}\right)e^{-i\left[\frac{\hbar q^2}{2m}+\mathbf{v_q}\cdot\mathbf{k}'-\frac{\mu}{\hbar}\right]\bar{t}}\Bigg\}.
\end{aligned}
\tag{5.21}
$$

However, a difficulty arises at this point. In the 1D case, we have

$$\int \mathrm{d}k\, \Theta(ak+b)e^{ikx} = \pi e^{-i\frac{b}{a}x}\delta(x) = \pi\delta(x). \tag{5.22}$$

In the 2D case, this relation no longer holds. Instead, what we obtain is

$$\int \mathrm{d}\mathbf{k}'\, e^{i\mathbf{k}'\cdot(\bar{\mathbf{x}}-\mathbf{v}_\mathbf{q}\bar{t})}\, \Theta\left(\frac{\hbar q^2}{2m} + \mathbf{v}_\mathbf{q}\cdot\mathbf{k}' - \frac{\mu}{\hbar}\right) = \frac{2\pi\delta((\bar{\mathbf{x}}-\mathbf{v}_\mathbf{q}\bar{t})\cdot\hat{\mathbf{v}}_{\mathbf{q}\perp})}{(\bar{\mathbf{x}}-\mathbf{v}_\mathbf{q}\bar{t})\cdot\mathbf{v}_\mathbf{q}} e^{i\left(\frac{\hbar q^2}{2m}-\frac{\mu}{\hbar}\right)\frac{(\bar{\mathbf{x}}-\mathbf{v}_\mathbf{q}\bar{t})\cdot\mathbf{v}_\mathbf{q}}{|\mathbf{v}_\mathbf{q}|}}.$$

The absence of a true delta function in this result complicates the calculation for momentum space Green's function. The complete solution for the isotropic case remains unresolved due to time constraints.

5.2 Anisotropic model

Due to time constraints, this thesis does not address the anisotropic case. However, for completeness, we outline the foundation for future study.

When generalizing our 1D results to the 2D isotropic case, we have simplistically chosen the dispersion relation as $\epsilon_\mathbf{k} = \frac{\hbar^2\mathbf{k}^2}{2m}$. However, real cuprates exhibit an anisotropic dispersion, and the Fermi surface has been experimentally observed, as shown in Fig. 5.1. This suggests that the actual dispersion is more complex than the simple quadratic form. It is anticipated that, through an appropriate choice of a model for the dispersion relation, we can linearize the momentum along the nodal direction (see the schematic in Fig. 5.2) and thereby reveal momentum-dependent power law behaviors.

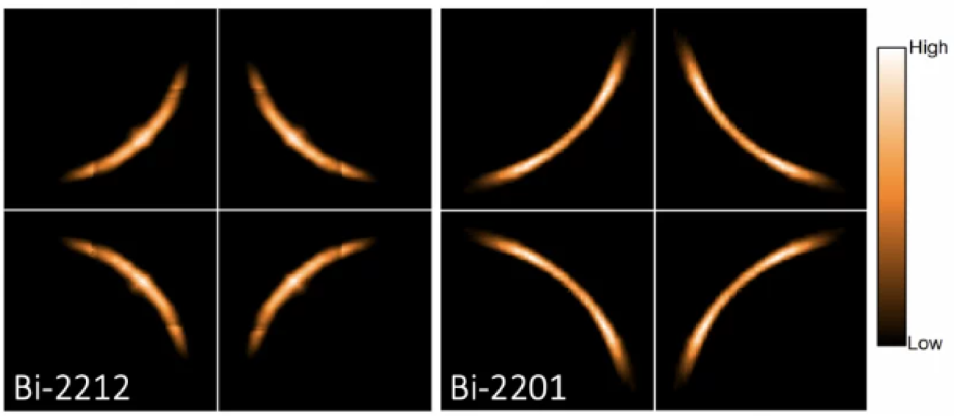


Figure 5.1: ARPES intensity mappings at the Fermi energy, showing the Fermi arcs in Bi2212 (left), and Bi2Sr2CuO6+δ (Bi2201, right)[5].

As an example, consider the tight-binding energy dispersion

$$\epsilon_\mathbf{k} = -2t\left(\cos(k_x a) + \cos(k_y a)\right). \tag{5.23}$$

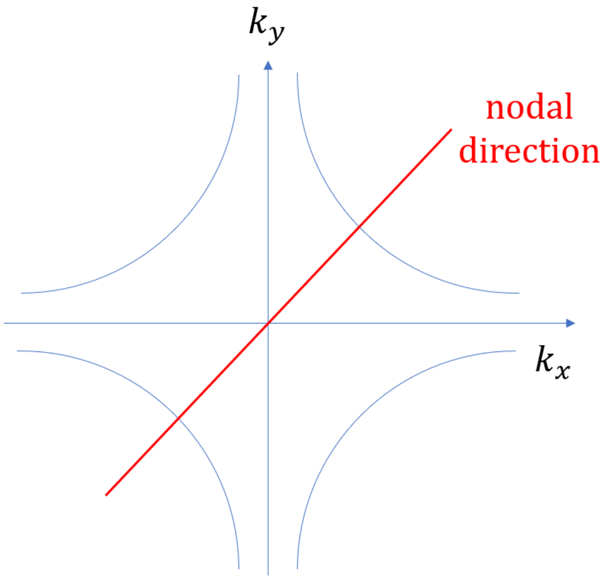
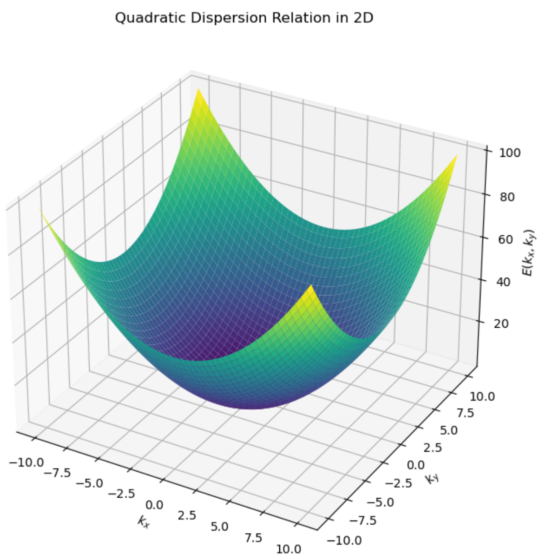
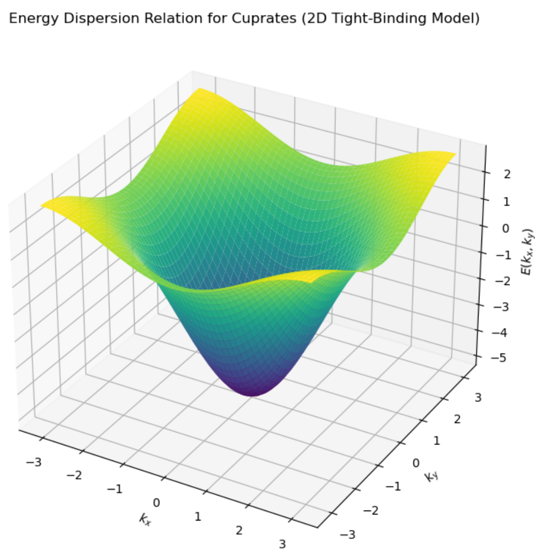


Figure 5.2: Schematic drawing for nodal direction of the dispersion in cuprates.

A comparison between the quadratic dispersion and tight-binding dispersion is shown in Fig.5.3.



(a) Quadratic dispersion



(b) Tight-binding dispersion

Figure 5.3: Comparison of two energy dispersion relations.

Assuming $\mathbf{k} = \mathbf{q} + \mathbf{k}'$, we can expand this dispersion around $\mathbf{q}$ to obtain a linearized form

$$\begin{aligned}\epsilon_{\mathbf{k}} &= -2t\left(\cos(q_x a + k'_x a) + \cos(q_y a + k'_y a)\right) \\ &= -2t\big(\cos(q_x a)\cos(k'_x a) - \sin(q_x a)\sin(k'_x a) \\ &\quad + \cos(q_y a)\cos(k'_y a) - \sin(q_y a)\sin(k'_y a)\big) \\ &\approx -2t\left(\cos(q_x a) + \cos(q_y a) - a\left(\sin(q_x a)k'_x + \sin(q_y a)k'_y\right)\right). \end{aligned} \tag{5.24}$$

After quantization, this leads to the action

$$S[\phi^*, \phi] = \int_0^{\hbar\beta} \mathrm{d}\tau \int \mathrm{d}\mathbf{x}\, \phi_q^*(\mathbf{x}, \tau) \Bigg\{ \hbar \frac{\partial}{\partial \tau} - 2t \left(\cos(q_x a) + \cos(q_y a)\right)$$
$$+ 2i\hbar a t \left(\sin(q_x a) \frac{\partial}{\partial x} + \sin(q_y a) \frac{\partial}{\partial y} \right) - \mu + \kappa(\mathbf{x}, \tau) \Bigg\} \phi_q(\mathbf{x}, \tau). \tag{5.25}$$

By introducing a $U(1)$ phase transformation, the κ field can be removed from the action. The κ propagator can then be computed using the new dispersion, with all calculations proceeding similarly to the isotropic case.

Although the results are not yet obtained, it remains uncertain whether this approach leads to momentum-dependent power law. However, there are many potential outcomes depending on the specific dispersion model chosen. With a well-designed dispersion model, it may be possible to achieve momentum-dependent power law behavior.

Chapter 6

Conclusion

So far, we have primarily focused on one-dimensional (1D) calculations and developed a linearization scheme for computing the Green's function under the long-wave approximation. We have successfully verified the validity of this approach in a 1D system by reproducing the characteristic results of M. Khodas *et al.* [1]. This progress paves the way for further investigations into the momentum-dependent power law behavior. As demonstrated, the Hubbard-Stratonovich transformation can be seen as a generalization of 1D bosonization. Utilizing this tool, we aim to extend its application to different models. As a summary, the steps of our method are illustrated in Fig. 6.1.

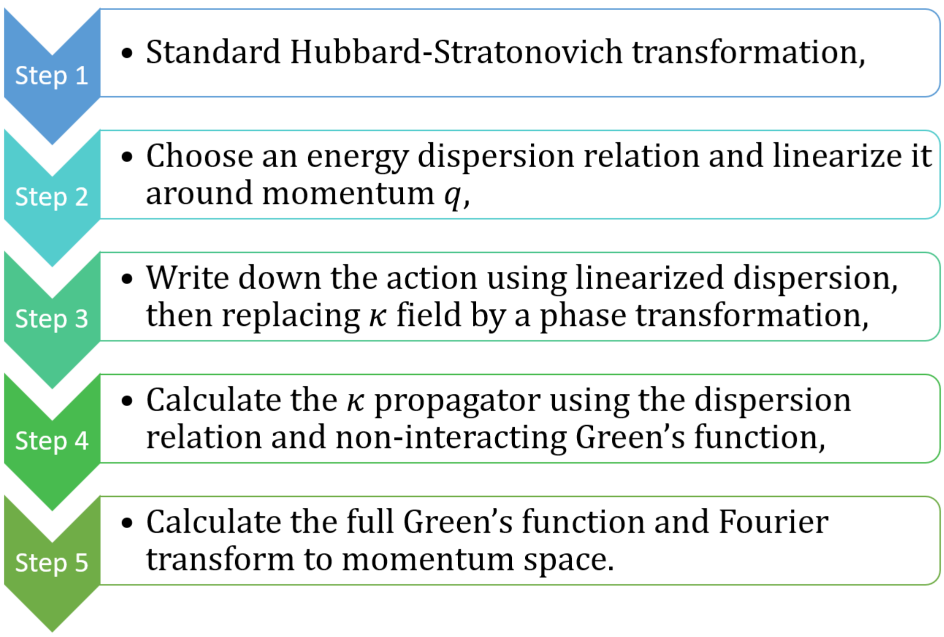


Figure 6.1: Flowchart outlining the steps for calculating the Green's function in our linearization scheme.

We have attempted to apply our method to a two-dimensional (2D) isotropic model, achieving partial results. However, the incompleteness of the calculation means we do not yet have conclusive results. Nevertheless, our partial findings demonstrate how generalization to 2D can be approached, providing a valuable example for future research.

Additionally, we have presented an example of applying our method to a model with anisotropic dispersion, offering insights into potential future work. Although we have not completed the generalization of our method to 2D systems, the framework we have established provides a solid foundation for further research and exploration of various models.

Finally, during our investigation of 1D systems, we find that the auxiliary κ field only produces a phase changing to the field ϕ. In some sense, the κ field resembles the scalar potential ϕ in electrodynamics, and our approach effectively involved choosing a gauge in which κ vanishes. This observation leads us to consider a deeper question: whether it would be advantageous to start from relativistic quantum electrodynamics and construct our model based on Dirac fermions. For Dirac fermions, the Hamiltonian is given by

$$H = \int d^3x\, \psi^\dagger(x) \left(-i\vec{\alpha} \cdot \vec{\nabla} + \beta m \right) \psi(x).$$

The corresponding action in the path-integral formalism can be written as

$$S_E[\psi, \bar{\psi}] = \int_0^\beta d\tau \int d^3x\, \bar{\psi} \left(\gamma^0 \partial_\tau - i\vec{\gamma} \cdot \vec{\nabla} + m \right) \psi,$$

or, in the interacting case,

$$S_E[\psi, \bar{\psi}] = \int_0^\beta d\tau \int d^3x\, \bar{\psi} \left(\gamma^0 D_\tau - i\vec{\gamma} \cdot \vec{D} + m \right) \psi,$$

where $D_\mu = \partial_\mu + ieA_\mu$. If we assume that A_i vanishes for $i = 1, 2, 3$ and only ϕ is non-zero, we could choose a gauge to cancel ϕ in the action. This could be an intriguing direction for further study.

Appendices

Chapter A

Continuation of the calculation for the Fock diagram in Chapter 3

We provide a detailed calculation for (3.3).

$$
\begin{aligned}
&\Sigma(k, i\omega_n) \\
&= \int \frac{\mathrm{d}k'}{2\pi} \frac{1}{\hbar\beta} \sum_{n'} \frac{-1}{i\omega_{n'} - i\omega_n + \frac{\epsilon_{k-k'} - \mu}{\hbar}} \left[\frac{1}{i\omega_{n'} - v_F|k'|} - \frac{1}{i\omega_{n'} + v_F|k'|} \right] \\
&= -\int \frac{\mathrm{d}k'}{2\pi} \frac{1}{2\pi i} \int_C \mathrm{d}z \frac{1}{z - i\omega_n + \frac{\epsilon_{k-k'} - \mu}{\hbar}} \left[\frac{1}{z - v_F|k'|} - \frac{1}{z + v_F|k'|} \right] \frac{1}{e^{\hbar\beta z} - 1} \\
&= -\int \frac{\mathrm{d}k'}{2\pi} \left[\frac{1}{e^{i\beta\hbar\omega_n} e^{-\beta(\epsilon_{k-k'} - \mu)} - 1} - \frac{1}{e^{\beta\hbar v_F|k'|} - 1} \right] \frac{1}{i\omega_n - \frac{\epsilon_{k-k'} - \mu}{\hbar} - v_F|k'|} \\
&\quad + \int \frac{\mathrm{d}k'}{2\pi} \left[\frac{1}{e^{i\beta\hbar\omega_n} e^{-\beta(\epsilon_{k-k'} - \mu)} - 1} - \frac{1}{e^{-\beta\hbar v_F|k'|} - 1} \right] \frac{1}{i\omega_n - \frac{\epsilon_{k-k'} - \mu}{\hbar} + v_F|k'|} \\
&= \int \frac{\mathrm{d}k'}{2\pi} \left[\frac{1}{e^{-\beta(\epsilon_{k-k'} - \mu)} + 1} + \frac{1}{e^{\beta\hbar v_F|k'|} - 1} \right] \frac{1}{i\omega_n - \frac{\epsilon_{k-k'} - \mu}{\hbar} - v_F|k'|} \\
&\quad - \int \frac{\mathrm{d}k'}{2\pi} \left[\frac{1}{e^{-\beta(\epsilon_{k-k'} - \mu)} + 1} + \frac{1}{e^{-\beta\hbar v_F|k'|} - 1} \right] \frac{1}{i\omega_n - \frac{\epsilon_{k-k'} - \mu}{\hbar} + v_F|k'|} \\
&= \int \frac{\mathrm{d}k'}{2\pi} \left[1 - \frac{1}{e^{\beta(\epsilon_{k-k'} - \mu)} + 1} + \frac{1}{e^{\beta\hbar v_F|k'|} - 1} \right] \frac{1}{i\omega_n - \frac{\epsilon_{k-k'} - \mu}{\hbar} - v_F|k'|} \\
&\quad - \int \frac{\mathrm{d}k'}{2\pi} \left[1 - \frac{1}{e^{\beta(\epsilon_{k-k'} - \mu)} + 1} - 1 - \frac{1}{e^{\beta\hbar v_F|k'|} - 1} \right] \frac{1}{i\omega_n - \frac{\epsilon_{k-k'} - \mu}{\hbar} + v_F|k'|} \\
&= \int \frac{\mathrm{d}k'}{2\pi} \left[1 - N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|) \right] \frac{1}{i\omega_n - \frac{\epsilon_{k-k'} - \mu}{\hbar} - v_F|k'|} \\
&\quad + \int \frac{\mathrm{d}k'}{2\pi} \left[N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|) \right] \frac{1}{i\omega_n - \frac{\epsilon_{k-k'} - \mu}{\hbar} + v_F|k'|},
\end{aligned}
\tag{A.1}
$$

where we used $e^{i\beta\hbar\omega_n} = -1$.

Following (3.3), we perform a Wick rotation $i\omega_n \to \omega$, enabling the use of the $i\epsilon$ trick.

$$\begin{aligned}
&\int \frac{\mathrm{d}k'}{2\pi} \left[1 - N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|)\right] \frac{1}{i\omega_n - (\epsilon_{k-k'} - \mu)/\hbar - v_F|k'|} \\
&\quad + \int \frac{\mathrm{d}k'}{2\pi} \left[N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|)\right] \frac{1}{i\omega_n - (\epsilon_{k-k'} - \mu)/\hbar + v_F|k'|} \\
&= \int \frac{\mathrm{d}k'}{2\pi} \left[1 - N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|)\right] \frac{1}{\omega - (\epsilon_{k-k'} - \mu)/\hbar - v_F|k'| + i\epsilon} \\
&\quad + \int \frac{\mathrm{d}k'}{2\pi} \left[N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|)\right] \frac{1}{\omega - (\epsilon_{k-k'} - \mu)/\hbar + v_F|k'| + i\epsilon}.
\end{aligned} \tag{A.2}$$

Since we are only concerned with the imaginary part of (3.1), we can compute it term by term using the following identities:

$$\mathrm{Im}\left\{\frac{1}{x + i\epsilon}\right\} = -\pi\delta(x), \qquad \text{and} \qquad \delta(g(x)) = \sum_i \frac{\delta(x - x_i)}{|g'(x_i)|}, \tag{A.3}$$

where x_i are the roots of $g(x) = 0$. Using these, we find

$$\begin{aligned}
&\mathrm{Im} \int \frac{\mathrm{d}k'}{2\pi} \left[1 - N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar|k'|)\right] \frac{1}{\omega - (\epsilon_{k-k'} - \mu)/\hbar - v_F|k'| + i\epsilon} \\
&= -\pi \int_0^\infty \frac{\mathrm{d}k'}{2\pi} \left[1 - N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar k')\right] \delta\left(\omega - \frac{\epsilon_{k-k'} - \mu}{\hbar} - v_F k'\right) \\
&\quad - \pi \int_0^\infty \frac{\mathrm{d}k'}{2\pi} \left[1 - N_{FD}(\epsilon_{k+k'}) + N_{BE}(v_F\hbar k')\right] \delta\left(\omega - \frac{\epsilon_{k+k'} - \mu}{\hbar} - v_F k'\right) \\
&= -\pi \int \frac{\mathrm{d}k'}{2\pi} \left[1 - N_{FD}(\epsilon_{k-k'}) + N_{BE}(v_F\hbar k')\right] \Theta(k') \\
&\quad \times \left[\frac{\delta\left(k' - \sqrt{\frac{2m\omega}{\hbar} + 2k_F^2 - 2kk_F} + k_F - k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar} + 2k_F^2 - 2kk_F}} + \frac{\delta\left(k' + \sqrt{\frac{2m\omega}{\hbar} + 2k_F^2 - 2kk_F} + k_F - k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar} + 2k_F^2 - 2kk_F}}\right] \\
&\quad - \pi \int \frac{\mathrm{d}k'}{2\pi} \left[1 - N_{FD}(\epsilon_{k+k'}) + N_{BE}(v_F\hbar k')\right] \Theta(k') \\
&\quad \times \left[\frac{\delta\left(k' - \sqrt{\frac{2m\omega}{\hbar} + 2k_F^2 + 2kk_F} + k_F + k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar} + 2k_F^2 + 2kk_F}} + \frac{\delta\left(k' + \sqrt{\frac{2m\omega}{\hbar} + 2k_F^2 + 2kk_F} + k_F + k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar} + 2k_F^2 + 2kk_F}}\right].
\end{aligned} \tag{A.4}$$

Now, we take the $T = 0$ approximation, which corresponds to $\beta \to \infty$. At zero temperature, $N_{BE}(v_F\hbar|k'|) = 0$ and $1 - N_{FD}(x) = \Theta(x)$. Additionally, the step function imposes the conditions $|k-k'|>k_F$ for the first term and $|k+k'|>k_F$ for the second term on the range of k' under integration. Thus, we have

$$
\begin{aligned}
&(\text{A.4})\\
&= -\pi \int \frac{\mathrm{d}k'}{2\pi}\Theta(\epsilon_{k-k'}-\mu)\Theta(k')\left[\Theta(k-k_F-k')+\Theta(-k-k_F+k')\right]\\
&\quad\times\left[\frac{\delta\left(k'-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}+k_F-k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}}+\frac{\delta\left(k'+\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}+k_F-k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}}\right]\\
&\quad-\pi \int \frac{\mathrm{d}k'}{2\pi}\Theta(\epsilon_{k+k'}-\mu)\Theta(k')\left[\Theta(-k-k_F-k')+\Theta(k-k_F+k')\right]\\
&\quad\times\left[\frac{\delta\left(k'-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}+k_F+k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}}+\frac{\delta\left(k'+\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}+k_F+k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}}\right]\\
&= -\frac{1}{2}\frac{m/\hbar}{\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}}\left[\Theta\left(\omega-v_F\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}+k-k_F\right)\right)\right.\\
&\quad\times\Theta\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}+k-k_F\right)\Theta\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}-2k_F\right)\\
&\quad+\Theta\left(\omega-v_F\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}+k-k_F\right)\right)\\
&\quad\left.\times\Theta\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}+k-k_F\right)\Theta\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}\right)\right]\\
&\quad-\frac{1}{2}\frac{m/\hbar}{\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}}\left[\Theta\left(\omega-v_F\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-k-k_F\right)\right)\right.\\
&\quad\times\Theta\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-k-k_F\right)\Theta\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-2k_F\right)\\
&\quad+\Theta\left(\omega-v_F\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-k-k_F\right)\right)\\
&\quad\left.\times\Theta\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-k-k_F\right)\Theta\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}\right)\right].
\end{aligned}
\tag{A.5}
$$

Similarly, the second term is given by

$$
\begin{aligned}
&\mathrm{Im}\int_{-\infty}^{\infty}\frac{\mathrm{d}k'}{2\pi}\left[N_{FD}(\epsilon_{k-k'})+N_{BE}(v_F\hbar|k'|)\right]\frac{1}{\omega-(\epsilon_{k-k'}-\mu)/\hbar+v_F|k'|+i\epsilon}\\
&=-\pi\int_{-\infty}^{\infty}\frac{\mathrm{d}k'}{2\pi}\left[N_{FD}(\epsilon_{k-k'})+N_{BE}(v_F\hbar|k'|)\right]\delta\left(\omega-\frac{\epsilon_{k-k'}-\mu}{\hbar}+v_F|k'|\right)
\end{aligned}
\tag{A.6}
$$

$$
\begin{aligned}
=&-\pi\int_0^\infty \frac{\mathrm{d}k'}{2\pi}\left[N_{FD}(\epsilon_{k-k'})+N_{BE}(v_F\hbar k')\right]\delta\left(\omega-\frac{\epsilon_{k-k'}-\mu}{\hbar}+v_Fk'\right)\\
&-\pi\int_0^\infty \frac{\mathrm{d}k'}{2\pi}\left[N_{FD}(\epsilon_{k+k'})+N_{BE}(v_F\hbar k')\right]\delta\left(\omega-\frac{\epsilon_{k+k'}-\mu}{\hbar}+v_Fk'\right)\\
=&-\pi\int \frac{\mathrm{d}k'}{2\pi}\left[N_{FD}(\epsilon_{k-k'})+N_{BE}(v_F\hbar k')\right]\Theta(k')\\
&\times\left[\frac{\delta\left(k'-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-k_F-k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}}+\frac{\delta\left(k'+\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-k_F-k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}}\right]\\
&-\pi\int \frac{\mathrm{d}k'}{2\pi}\left[N_{FD}(\epsilon_{k+k'})+N_{BE}(v_F\hbar k')\right]\Theta(k')\\
&\times\left[\frac{\delta\left(k'-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}-k_F+k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}}+\frac{\delta\left(k'+\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}-k_F+k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}}\right].
\end{aligned}
\tag{A.7}
$$

Under the $T=0$ approximation, $N_{BE}(v_F\hbar|k'|)=0$ and $N_{FD}(x)=\Theta(-x)$. This imposes $|k-k'|<k_F$ for the first part and $|k+k'|<k_F$ for the second part, leading to

$$
\begin{aligned}
&\text{(A.7)}\\
=&-\pi\int \frac{\mathrm{d}k'}{2\pi}N_{FD}(\epsilon_{k-k'})\Theta(k')\Theta(k+k_F-k')\Theta(k_F-k+k')\\
&\times\left[\frac{\delta\left(k'-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-k_F-k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}}+\frac{\delta\left(k'+\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}-k_F-k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}}\right]\\
&-\pi\int \frac{\mathrm{d}k'}{2\pi}N_{FD}(\epsilon_{k+k'})\Theta(k')\Theta(-k+k_F-k')\Theta(k+k_F+k')\\
&\times\left[\frac{\delta\left(k'-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}-k_F+k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}}+\frac{\delta\left(k'+\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}-k_F+k\right)}{\frac{\hbar}{m}\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}}\right]\\
=&-\frac{1}{2}\frac{m/\hbar}{\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}}\Theta\left(-\omega-v_F\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}+k+k_F\right)\right)\\
&\times\Theta\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}+k+k_F\right)\Theta\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}\right)\\
&\times\Theta\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2+2kk_F}+2k_F\right)+\text{continue to next page ...}
\end{aligned}
$$

...

$$
\begin{aligned}
&-\frac{1}{2}\frac{m/\hbar}{\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}}\Theta\left(\omega-v_F\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}-k+k_F\right)\right)\\
&\quad\times\Theta\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}-k+k_F\right)\Theta\left(\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}\right)\\
&\quad\times\Theta\left(-\sqrt{\frac{2m\omega}{\hbar}+2k_F^2-2kk_F}+2k_F\right). \qquad \text{(A.8)}
\end{aligned}
$$

This concludes the calculation.

Chapter B

Detailed calculations in Chapter 4

B.1 Wick-rotation and real-time Green's function

For convenience in calculations, we transition to using the real-time parameter t rather than τ. This involves performing a Wick rotation, where $\tau \to it$ and simultaneously $i\omega_n \to \omega$. This approach is particularly useful when working with the zero-temperature approximation. Since $\omega_n = \frac{2n\pi}{\hbar\beta}$, in the limit $T \to 0^+$, $\beta \to \infty$, and ω becomes a real continuous variable. Consequently, the summation over ω_n transforms into an integral over ω, facilitating calculations that would otherwise be challenging, as we will demonstrate later.

One immediate consequence of the Wick rotation is that the Green's function satisfies the relation $G_0(x,t;x',t') = iG_0(x,\tau;x',\tau')$. This can be derived directly from Eq. (4.13), which states

$$\begin{aligned}
G_0(x,\tau;x',\tau') &= \lim_{\beta\to\infty}\int\frac{\mathrm{d}k}{2\pi}\sum_n\frac{-i}{2\pi}\frac{i2\pi}{\hbar\beta}\frac{-\hbar}{-i\hbar\omega_n+\epsilon_k-\mu}e^{ik(x-x')}e^{-i\omega_n(\tau-\tau')} \\
&= -i\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}\omega}{2\pi}\frac{-\hbar}{-\hbar\omega+\epsilon_k-\mu}e^{ik(x-x')}e^{-i\omega(t-t')} \\
&= -iG_0(x,t;x',t'). \qquad \text{(B.1)}
\end{aligned}$$

Another important implication is that the integral over ω requires careful handling. Specifically, the integral must follow the contour depicted in Fig. B.1. An example calculation illustrating this conclusion is provided in Appendix B.2. Despite the additional complexity introduced by the contour, Wick rotation enables us to evaluate many integrals that are otherwise intractable using Matsubara summation.

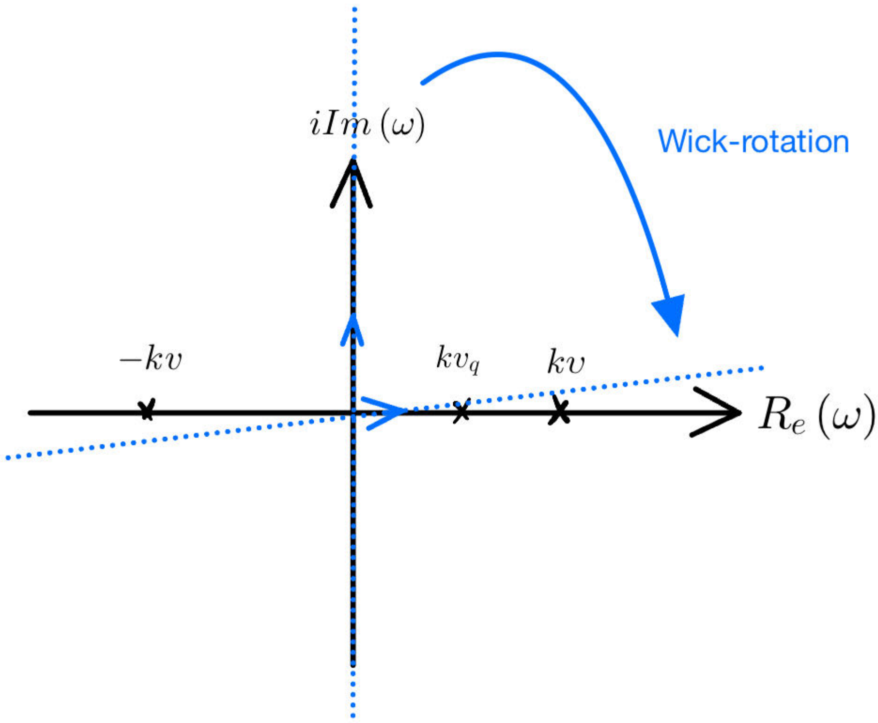


Figure B.1: Integration over ω

B.2 Integration under Wick-rotation

In this section, we demonstrate an example calculation that utilizes both the Matsubara sum and the integral over the contour depicted in Fig. B.1. This serves to validate the chosen contour for integration.

We aim to evaluate the expression

$$\frac{i}{\hbar\beta}\sum_{n=-\infty}^{\infty}\frac{-\omega_n^2-k^2v_F^2}{(i\omega_n+kv)(i\omega_n-kv)(i\omega_n-v_qk)^2}, \tag{B.2}$$

which appears as part of the calculations in Chapter 4. The Wick-rotated counterpart is given by

$$\lim_{\epsilon\to 0}\int\frac{d\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega+kv)(\omega-kv)(\omega-v_qk)^2}. \tag{B.3}$$

To compute Eq. (B.2), we note that the summand converges as $|\omega_n|\to\infty$. This allows us to employ contour integration and the commonly used Matsubara summation technique. The sum can be expressed as a contour integral with the contour chosen as shown in Fig. B.2 (two solid lines). The integral is then written as

$$\frac{i}{\hbar\beta}\sum_{n=-\infty}^{\infty}\frac{-\omega_n^2-k^2v_F^2}{(i\omega_n+kv)(i\omega_n-kv)(i\omega_n-v_qk)^2}=\frac{1}{2\pi i}\oint_C \mathrm{d}z\,\frac{z^2-k^2v_F^2}{(z+kv)(z-kv)(z-v_qk)^2}\frac{-1}{e^{\hbar\beta z}+1}. \tag{B.4}$$

The contour C encircles the poles of the Fermi-Dirac distribution, $z = i\omega_n$, and excludes the poles of the denominator function. This ensures the correctness of the calculation under Wick-rotation.

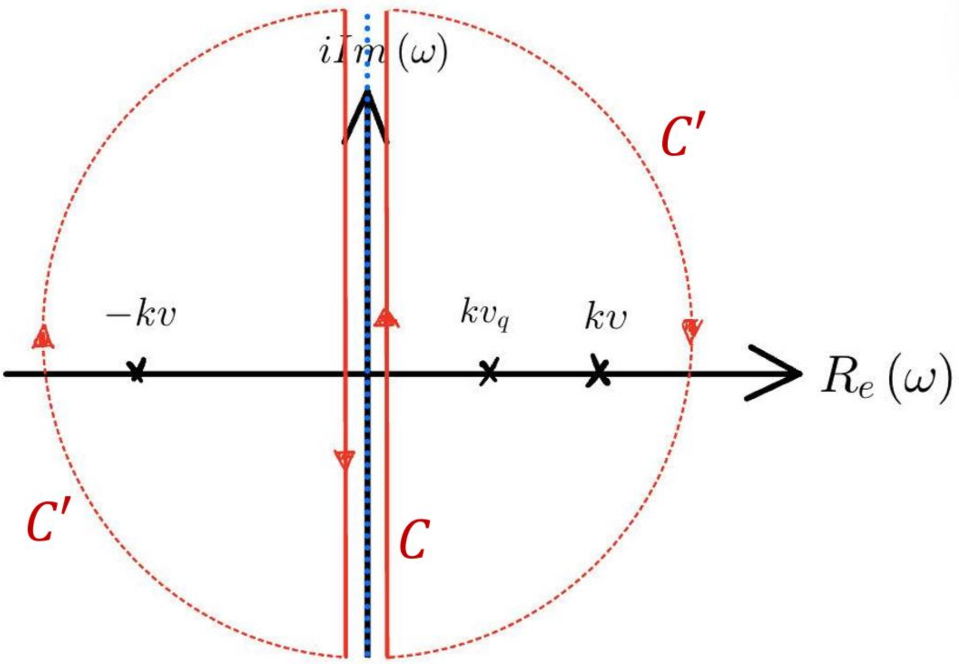


Figure B.2: Diagrammatic illustration of Matsubara sum

Now we can include a vanishing contour C', which does not alter the result. This transforms the integral into

$$\begin{aligned}
&\frac{i}{2\pi i}\int_{C+C'}\mathrm{d}z\frac{z^2-k^2v_F^2}{(z+kv)(z-kv)(z-v_qk)^2}\frac{-1}{e^{\hbar\beta z}+1}\\
&=-i\left(\mathrm{Res}(kv)+\mathrm{Res}(-kv)+\mathrm{Res}(v_qk)\right)\\
&=\frac{v^2-v_F^2}{2kv(v-v_q)^2}\frac{i}{e^{\hbar\beta kv}+1}-\frac{v^2-v_F^2}{2kv(v+v_q)^2}\frac{i}{e^{-\hbar\beta kv}+1}+\lim_{z\to v_qk}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2-k^2v_F^2}{z^2-k^2v^2}\frac{i}{e^{\hbar\beta z}+1}\\
&=\frac{v^2-v_F^2}{2kv(v-v_q)^2}\frac{i}{e^{\hbar\beta kv}+1}-\frac{v^2-v_F^2}{2kv(v+v_q)^2}\frac{i}{e^{-\hbar\beta kv}+1}\\
&\quad+\left(\frac{2v_q(v_F^2-v^2)}{k(v_q^2-v^2)^2}\right)\frac{i}{e^{\hbar\beta v_qk}+1}-\frac{v_q^2-v_F^2}{v_q^2-v^2}\frac{i\hbar\beta e^{\hbar\beta v_qk}}{(e^{\hbar\beta v_qk}+1)^2}.
\end{aligned} \tag{B.5}$$

Finally, we take the limit $\beta\to\infty$ to compare the result with the integral approach.

Under this approximation,

$$\lim_{\beta\to\infty}\frac{i}{e^{\hbar\beta kv}+1}=\lim_{\beta\to\infty}\frac{i}{e^{\hbar\beta v_q k}+1}=\begin{cases}0, & k>0,\\ i, & k<0,\end{cases}$$

$$\lim_{\beta\to\infty}\frac{i}{e^{-\hbar\beta kv}+1}=\begin{cases}i, & k>0,\\ 0, & k<0,\end{cases}$$

$$\lim_{\beta\to\infty}\frac{i\hbar\beta e^{\hbar\beta v_q k}}{(e^{\hbar\beta v_q k}+1)^2}=0.$$

Thus, the final result becomes

$$\begin{aligned}&\lim_{\beta\to\infty}\frac{i}{\hbar\beta}\sum_{n=-\infty}^{\infty}\frac{-\omega_n^2-k^2v_F^2}{(i\omega_n+kv)(i\omega_n-kv)(i\omega_n-v_qk)^2}\\ &=\Theta(k)\left[-i\frac{v^2-v_F^2}{2kv(v+v_q)^2}\right]+\Theta(-k)\left[i\frac{v^2-v_F^2}{2kv(v-v_q)^2}+i\frac{2v_q(v_F^2-v^2)}{k(v_q^2-v^2)^2}\right]\\ &=\Theta(k)\left[-i\frac{v^2-v_F^2}{2kv(v+v_q)^2}\right]+\Theta(-k)\left[i\frac{v^2-v_F^2}{2kv(v+v_q)^2}\right].\end{aligned}\tag{B.6}$$

Now we calculate equation (B.3) using a contour as shown in Fig.B.1. Utilizing this contour is equivalent to shifting the poles on the negative real axis infinitesimally above and the poles on the positive real axis infinitesimally below. This is achieved by rewriting equation (B.3) as

$$\begin{aligned}&\lim_{\epsilon\to0}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega+kv-i\mathrm{sgn}(k)\epsilon)(\omega-kv+i\mathrm{sgn}(k)\epsilon)(\omega-v_qk+i\mathrm{sgn}(k)\epsilon)^2}\\ =&\Theta(k)\lim_{\epsilon\to0}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega+kv-i\epsilon)(\omega-kv+i\epsilon)(\omega-v_qk+i\epsilon)^2}\\ &+\Theta(-k)\lim_{\epsilon\to0}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega-|kv|+i\epsilon)(\omega+|kv|-i\epsilon)(\omega+|v_qk|-i\epsilon)^2}.\end{aligned}\tag{B.7}$$

Now, for the first term, we have

$$\begin{aligned}&\lim_{\epsilon\to0}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega+kv-i\epsilon)(\omega-kv+i\epsilon)(\omega-v_qk+i\epsilon)^2}\\ &=\lim_{\epsilon\to0}2\pi i\,\mathrm{Res}(-kv+i\epsilon)\\ &=-i\frac{v^2-v_F^2}{2kv(v+v_q)^2}.\end{aligned}\tag{B.8}$$

For the second term, we have

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega-|kv|+i\epsilon)(\omega+|kv|-i\epsilon)(\omega+|v_qk|-i\epsilon)^2}\\
&=\lim_{\epsilon\to 0}2\pi i\left\{\mathrm{Res}(-|kv|+i\epsilon)+\mathrm{Res}(-|kv_q|+i\epsilon)\right\}\\
&=\lim_{\epsilon\to 0} i\left\{\frac{(kv+i\epsilon)^2-k^2v_F^2}{(2kv+2i\epsilon)(kv-kv_q)^2}+\lim_{z\to kv_q+i\epsilon}\frac{\mathrm{d}}{\mathrm{d}z}\frac{z^2-k^2v_F^2}{(z+kv+i\epsilon)(z-kv-i\epsilon)}\right\}\\
&=i\left\{\frac{v^2-v_F^2}{2kv(v-v_q)^2}+\lim_{z\to kv_q+i\epsilon}\frac{2zk^2(v_F^2-v^2)}{(z^2-k^2v^2)^2}\right\}\\
&=i\left\{\frac{v^2-v_F^2}{2kv(v-v_q)^2}+\frac{2v_q(v_F^2-v^2)}{k(v_q^2-v^2)^2}\right\}\\
&=i\frac{(v^2-v_F^2)(v_q+v)^2-4vv_q(v^2-v_F^2)}{2kv(v_q^2-v^2)^2}\\
&=i\frac{(v^2-v_F^2)(v_q-v)^2}{2kv(v_q^2-v^2)^2}\\
&=i\frac{v^2-v_F^2}{2kv(v+v_q)^2}.
\end{aligned}
\tag{B.9}
$$

Hence, equation (B.3) becomes

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega+kv-i\mathrm{sgn}(k)\epsilon)(\omega-kv+i\mathrm{sgn}(k)\epsilon)(\omega-v_qk+i\mathrm{sgn}(k)\epsilon)^2}\\
&=i\left\{\Theta(-k)\frac{v^2-v_F^2}{2kv(v+v_q)^2}-\Theta(k)\frac{v^2-v_F^2}{2kv(v+v_q)^2}\right\}.
\end{aligned}
\tag{B.10}
$$

Comparing equation (B.6) with equation (B.10), we observe that they are exactly the same. This confirms that the integral over ω on a contour in Fig.B.1 yields the same result as the Matsubara sum in the zero-temperature limit.

B.3 Solution for the phase factor

The equation to be solved is

$$
\hbar\frac{\partial}{\partial t}\theta(x,t)+\frac{\hbar^2 q}{m}\frac{\partial}{\partial x}\theta(x,t)+\kappa(x,t)=0. \tag{B.11}
$$

The solution to this equation can be found using the following method. First, we apply the Fourier transform to both $\theta(x,t)$ and $\kappa(x,t)$,

$$\begin{aligned}\theta(x,t) &= \int \frac{\mathrm{d}k}{2\pi}\int \frac{\mathrm{d}\omega}{2\pi}\theta(k,\omega)e^{ikx}e^{-i\omega t},\\ \kappa(x,t) &= \int \frac{\mathrm{d}k}{2\pi}\int \frac{\mathrm{d}\omega}{2\pi}\kappa(k,\omega)e^{ikx}e^{-i\omega t}.\end{aligned} \tag{B.12}$$

Substituting the Fourier transforms of $\theta(x,t)$ and $\kappa(x,t)$ into the original equation, we get

$$\int \frac{\mathrm{d}k}{2\pi}\int \frac{\mathrm{d}\omega}{2\pi}\left[\left(-i\hbar\omega + i\frac{\hbar^2 qk}{m}\right)\theta(k,\omega) + \kappa(k,\omega)\right]e^{ikx}e^{-i\omega t} = 0. \tag{B.13}$$

This implies that

$$\left(-i\hbar\omega + i\frac{\hbar^2 qk}{m}\right)\theta(k,\omega) + \kappa(k,\omega) = 0, \tag{B.14}$$

or equivalently,

$$\theta(k,\omega) = \frac{-i\kappa(k,\omega)}{\hbar(\omega - v_q k)}, \tag{B.15}$$

where $v_q = \frac{\hbar q}{m}$. Substituting this expression back into the Fourier transform (B.12), we obtain

$$\theta(x,t) = \int \frac{\mathrm{d}k}{2\pi}\int \frac{\mathrm{d}\omega}{2\pi}\frac{-i\kappa(k,\omega)}{\hbar(\omega - v_q k)}e^{ikx}e^{-i\omega t}. \tag{B.16}$$

B.4 Two-Point correlation function of the auxiliary field supplement

Firstly,

$$\begin{aligned}&\left\langle [\theta(x,t) - \theta(x',t')]^2 \right\rangle_\kappa \\ &= \langle \theta(x,t)\theta(x,t)\rangle_\kappa + \langle \theta(x',t')\theta(x',t')\rangle_\kappa - \langle \theta(x,t)\theta(x',t')\rangle_\kappa - \langle \theta(x',t')\theta(x,t)\rangle_\kappa .\end{aligned} \tag{B.17}$$

Next,

$$\begin{aligned}&\langle \theta(x,t)\theta(x,t)\rangle_\kappa \\ &= \int \frac{\mathrm{d}k}{2\pi}\int \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}k'}{2\pi}\int \frac{\mathrm{d}\omega'}{2\pi}\frac{-\langle \kappa(k,\omega)\kappa(k',\omega')\rangle_\kappa}{\hbar^2(\omega - v_q k)(\omega' - v_q k')}e^{i[(k+k')x-(\omega+\omega')t]}.\end{aligned} \tag{B.18}$$

In momentum space, the most general action for κ can be expressed as (by Fourier transform the general action to momentum space)

$$S[\kappa] = \int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \kappa(k,\omega) G_\kappa^{-1}(k,\omega) \kappa(-k,-\omega). \tag{B.19}$$

Clearly, this approach also applies to the effective action in our case. This implies momentum conservation, and

$$\left\langle \kappa(k,\omega)\kappa(k',\omega') \right\rangle_\kappa = \delta(k+k')\delta(\omega+\omega') \left\langle \kappa(k,\omega)\kappa(-k,-\omega) \right\rangle_\kappa . \tag{B.20}$$

Substituting this back into (B.18), we get

$$\begin{aligned} &\langle \theta(x,t)\theta(x,t) \rangle_\kappa \\ &= -\int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\langle \kappa(k,\omega)\kappa(-k,\omega) \rangle_\kappa}{\hbar^2(\omega - v_q k)^2}. \end{aligned} \tag{B.21}$$

The same result holds for $\langle \theta(x',t')\theta(x',t') \rangle_\kappa$ because of translational symmetry.

Secondly,

$$\begin{aligned} &\left\langle \theta(x,t)\theta(x',t') \right\rangle_\kappa \\ &= \int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \int \frac{\mathrm{d}k'}{2\pi} \int \frac{\mathrm{d}\omega'}{2\pi} \frac{- \left\langle \kappa(k,\omega)\kappa(k',\omega') \right\rangle_\kappa}{\hbar^2(\omega - v_q k)(\omega' - v_q k')} e^{i(kx + k'x' - \omega t + \omega' t')} \\ &= -\int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\langle \kappa(k,\omega)\kappa(-k,-\omega) \rangle_\kappa}{\hbar^2(\omega - v_q k)^2} e^{i[k(x-x') - \omega(t-t')]}. \end{aligned} \tag{B.22}$$

Similarly,

$$\begin{aligned} &\left\langle \theta(x',t')\theta(x,t) \right\rangle_\kappa \\ &= -\int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\langle \kappa(k,\omega)\kappa(-k,-\omega) \rangle_\kappa}{\hbar^2(\omega - v_q k)^2} e^{-i[k(x-x') - \omega(t-t')]}. \end{aligned} \tag{B.23}$$

Adding these two expressions together, we get

$$\begin{aligned} &\left\langle [\theta(x,t) - \theta(x',t')]^2 \right\rangle_\kappa \\ &= -2 \int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\langle \kappa(k,\omega)\kappa(-k,\omega) \rangle_\kappa}{\hbar^2(\omega - v_q k)^2} \left[1 - \cos(k(x-x') - \omega(t-t')) \right]. \end{aligned} \tag{B.24}$$

B.5 Details in section 4.3

For the first part, we consider

$$\int \frac{\mathrm{d}k}{2\pi} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - v_F^2 k^2}{\omega^2 - v^2 k^2} \frac{1}{(\omega - v_q k)^2}, \tag{B.25}$$

which can be computed using contour integration (assuming $q > 0$), with the contour described in appendix B.1. We obtain

$$\begin{aligned} &\lim_{\epsilon\to 0} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - k^2 v_F^2}{(\omega + kv - \mathrm{sgn}(k)i\epsilon)(\omega - kv + \mathrm{sgn}(k)i\epsilon)(\omega - v_q k + \mathrm{sgn}(k)i\epsilon)^2} \\ =&\Theta(k) \lim_{\epsilon\to 0} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - k^2 v_F^2}{(\omega + kv - i\epsilon)(\omega - kv + i\epsilon)(\omega - v_q k + i\epsilon)^2} \\ +&\Theta(-k) \lim_{\epsilon\to 0} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - k^2 v_F^2}{(\omega - |kv| + i\epsilon)(\omega + |kv| - i\epsilon)(\omega + |v_q k| - i\epsilon)^2}. \end{aligned} \tag{B.26}$$

We close the contour in the upper half-plane and obtain

$$\begin{aligned} &\lim_{\epsilon\to 0} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\omega^2 - k^2 v_F^2}{(\omega + kv - i\epsilon)(\omega - kv + i\epsilon)(\omega - v_q k + i\epsilon)^2} \\ =&\lim_{\epsilon\to 0} 2\pi i \mathrm{Res}(-kv + i\epsilon) \\ =&\lim_{\epsilon\to 0} i \left\{ \lim_{\omega\to -kv+i\epsilon} \frac{\omega^2 - k^2 v_F^2}{(\omega - kv + i\epsilon)(\omega - v_q k + i\epsilon)^2} \right\} \\ =&\lim_{\epsilon\to 0} i \left\{ \frac{(-kv + i\epsilon)^2 - k^2 v_F^2}{(-2kv + 2i\epsilon)(-kv - kv_q + 2i\epsilon)^2} \right\} \\ =&-i \frac{v^2 - v_F^2}{2kv(v + v_q)^2} \equiv \frac{i}{k} c_1(q), \end{aligned} \tag{B.27}$$

and

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega-|kv|+i\epsilon)(\omega+|kv|-i\epsilon)(\omega+|v_qk|-i\epsilon)^2}\\
=&\lim_{\epsilon\to 0}2\pi i\left\{\mathrm{Res}(-|kv|+i\epsilon)+\mathrm{Res}(-|kv_q|+i\epsilon)\right\}\\
=&\lim_{\epsilon\to 0} i\left\{\frac{(kv+i\epsilon)^2-k^2v_F^2}{(2kv+2i\epsilon)(kv-kv_q)^2}+\lim_{z\to kv_q+i\epsilon}\frac{\mathrm{d}}{dz}\frac{z^2-k^2v_F^2}{(z+kv+i\epsilon)(z-kv-i\epsilon)}\right\}\\
=&i\left\{\frac{v^2-v_F^2}{2kv(v-v_q)^2}+\lim_{z\to kv_q+i\epsilon}\frac{2zk^2(v_F^2-v^2)}{(z^2-k^2v^2)^2}\right\}\\
=&i\left\{\frac{v^2-v_F^2}{2kv(v-v_q)^2}+\frac{2v_q(v_F^2-v^2)}{k(v_q^2-v^2)^2}\right\}\\
=&i\frac{(v^2-v_F^2)(v_q+v)^2-4vv_q(v^2-v_F^2)}{2kv(v_q^2-v^2)^2}\\
=&i\frac{(v^2-v_F^2)(v_q-v)^2}{2kv(v_q^2-v^2)^2}\\
=&i\frac{v^2-v_F^2}{2kv(v+v_q)^2}=-\frac{i}{k}c_1(q).
\end{aligned}
\tag{B.28}
$$

Hence, (B.25) is equal to

$$
\begin{aligned}
&\lim_{\epsilon\to 0}\int \frac{\mathrm{d}\omega}{2\pi}\frac{\omega^2-k^2v_F^2}{(\omega+kv-i\mathrm{sgn}(k)\epsilon)(\omega-kv+i\mathrm{sgn}(k)\epsilon)(\omega-v_qk+i\mathrm{sgn}(k)\epsilon)^2}\\
=&\frac{i}{k}c_1(q)\left[\Theta(-k)-\Theta(k)\right].
\end{aligned}
\tag{B.29}
$$

For the second part, we calculate

$$
\begin{aligned}
\int \frac{\mathrm{d}\omega}{2\pi}\Bigg[&\frac{1}{k}\left(\frac{c_1(q)}{\omega+vk-\mathrm{sgn}(k)i\epsilon}+\frac{c_2(q)}{\omega-vk+\mathrm{sgn}(k)i\epsilon}+\frac{c_3(q)}{\omega-v_qk+\mathrm{sgn}(k)i\epsilon}\right)\\
&+\frac{c_4(q)}{(\omega-v_qk+\mathrm{sgn}(k)i\epsilon)^2}\Bigg]\times\cos(k\bar{x}-\omega\bar{t}).
\end{aligned}
\tag{B.30}
$$

The first term is computed as (here we assume $\epsilon\to 0$ at the end, so we omit the limit symbol)

$$
\int \frac{\mathrm{d}\omega}{2\pi}\frac{\cos(k\bar{x}-\omega\bar{t})}{\omega-vk+\mathrm{sgn}(k)i\epsilon}=\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{2}\frac{e^{i(k\bar{x}-\omega\bar{t})}+e^{-i(k\bar{x}-\omega\bar{t})}}{\omega-vk+\mathrm{sgn}(k)i\epsilon}.
\tag{B.31}
$$

Assuming $\bar{t}>0$, $e^{i(k\bar{x}-\omega\bar{t})}$ converges in the lower half-plane, and $e^{-i(k\bar{x}-\omega\bar{t})}$ converges in the upper half-plane.

Thus, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{1}{2}\frac{e^{i(k\bar{x}-\omega\bar{t})}}{\omega-vk+\mathrm{sgn}(k)i\epsilon}\\
=&\Theta(k)\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{1}{2}\frac{e^{i(k\bar{x}-\omega\bar{t})}}{\omega-vk+i\epsilon}+\Theta(-k)\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{1}{2}\frac{e^{i(k\bar{x}-\omega\bar{t})}}{\omega+|vk|-i\epsilon}\\
=&\Theta(k)(-\frac{i}{2}\mathrm{Res}(kv-i\epsilon))+\Theta(-k)*0\\
=&\Theta(k)(-\frac{i}{2}e^{ik(\bar{x}-v\bar{t})}),
\end{aligned}\tag{B.32}$$

and

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{\mathrm{d}\omega}{2\pi}\frac{1}{2}\frac{e^{-i(k\bar{x}-\omega\bar{t})}}{\omega-vk+\mathrm{sgn}(k)i\epsilon}\\
=&\Theta(-k)(\frac{i}{2}\mathrm{Res}(kv+i\epsilon))\\
=&\Theta(-k)(\frac{i}{2}e^{-ik(\bar{x}-v\bar{t})}).
\end{aligned}\tag{B.33}$$

Adding these two parts together, we obtain

$$\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{k}\frac{\cos(k\bar{x}-\omega\bar{t})}{\omega-vk+\mathrm{sgn}(k)i\epsilon}=\frac{i}{2}\left(\frac{\Theta(-k)e^{-ik(\bar{x}-v\bar{t})}-\Theta(k)e^{ik(\bar{x}-v\bar{t})}}{k}\right).\tag{B.34}$$

Similarly, we have

$$\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{k}\frac{\cos(k\bar{x}-\omega\bar{t})}{\omega+vk-\mathrm{sgn}(k)i\epsilon}=\frac{i}{2}\left(\frac{\Theta(k)e^{-ik(\bar{x}+v\bar{t})}-\Theta(-k)e^{ik(\bar{x}+v\bar{t})}}{k}\right),\tag{B.35}$$

and

$$\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{k}\frac{\cos(k\bar{x}-\omega\bar{t})}{\omega-v_qk+\mathrm{sgn}(k)i\epsilon}=\frac{i}{2}\left(\frac{\Theta(-k)e^{-ik(\bar{x}-v_q\bar{t})}-\Theta(k)e^{ik(\bar{x}-v_q\bar{t})}}{k}\right).\tag{B.36}$$

Lastly, we calculate the term with a second-order pole

$$\begin{aligned}
&\lim_{\epsilon\to0}\int\frac{\mathrm{d}\omega}{2\pi}\frac{\cos(k\bar{x}-\omega\bar{t})}{(\omega-v_qk+i\mathrm{sgn}(k)\epsilon)^2}\\
=&\lim_{\epsilon\to0}\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{2}\frac{e^{i(k\bar{x}-\omega\bar{t})}+e^{-i(k\bar{x}-\omega\bar{t})}}{(\omega-v_qk+i\mathrm{sgn}(k)\epsilon)^2}\\
=&\Theta(k)\left(-\frac{i}{2}\lim_{\omega\to v_qk-i\epsilon}\frac{\mathrm{d}}{\mathrm{d}z}e^{i(k\bar{x}-z\bar{t})}\right)+\Theta(-k)\left(\frac{i}{2}\lim_{\omega\to v_qk+i\epsilon}\frac{\mathrm{d}}{\mathrm{d}z}e^{-i(k\bar{x}-z\bar{t})}\right)\\
=&-\frac{\bar{t}}{2}\left(\Theta(k)e^{ik(\bar{x}-v_q\bar{t})}+\Theta(-k)e^{-ik(\bar{x}-v_q\bar{t})}\right).
\end{aligned}\tag{B.37}$$

For $\bar{t} < 0$,

$$\int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{k} \frac{\cos(k\bar{x} - \omega\bar{t})}{\omega - vk + \mathrm{sgn}(k)i\epsilon} = \frac{i}{2} \left(\frac{\Theta(-k)e^{ik(\bar{x}-v\bar{t})} - \Theta(k)e^{-ik(\bar{x}-v\bar{t})}}{k} \right),$$

$$\int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{k} \frac{\cos(k\bar{x} - \omega\bar{t})}{\omega + vk - \mathrm{sgn}(k)i\epsilon} = \frac{i}{2} \left(\frac{\Theta(k)e^{ik(\bar{x}+v\bar{t})} - \Theta(-k)e^{-ik(\bar{x}+v\bar{t})}}{k} \right),$$

$$\int \frac{\mathrm{d}\omega}{2\pi} \frac{1}{k} \frac{\cos(k\bar{x} - \omega\bar{t})}{\omega - v_q k + \mathrm{sgn}(k)i\epsilon} = \frac{i}{2} \left(\frac{\Theta(-k)e^{ik(\bar{x}-v_q\bar{t})} - \Theta(k)e^{-ik(\bar{x}-v_q\bar{t})}}{k} \right),$$

$$\lim_{\epsilon\to 0} \int \frac{\mathrm{d}\omega}{2\pi} \frac{\cos(k\bar{x} - \omega\bar{t})}{(\omega - v_q k + i\mathrm{sgn}(k)\epsilon)^2} = \frac{\bar{t}}{2} \left(\Theta(k)e^{-ik(\bar{x}-v_q\bar{t})} + \Theta(-k)e^{ik(\bar{x}-v_q\bar{t})} \right). \quad \text{(B.38)}$$

B.6 Coefficients of partial decomposition

The goal is to decompose the following function

$$\frac{\omega^2 - v_F^2 k^2}{(\omega^2 - v^2k^2)(\omega - v_q k)^2} = \frac{A}{\omega + vk} + \frac{B}{\omega - vk} + \frac{C}{\omega - v_q k} + \frac{D}{(\omega - v_q k)^2}. \quad \text{(B.39)}$$

We start by combining the four terms on the right-hand side of equation (B.39) and then compare it to the left-hand side.

Thus,

$$\begin{aligned}
&\frac{A}{\omega + vk} + \frac{B}{\omega - vk} + \frac{C}{\omega - v_q k} + \frac{D}{(\omega - v_q k)^2} \\
=&\frac{1}{(\omega^2 - v^2k^2)(\omega - v_q k)^2} \{A(\omega - vk)(\omega - v_q k)^2 + B(\omega + vk)(\omega - v_q k)^2 \\
&+ C(\omega^2 - v^2k^2)(\omega - v_q k) + D(\omega^2 - v^2k^2)\} \\
=&\frac{1}{(\omega^2 - v^2k^2)(\omega - v_q k)^2} \{A(\omega^3 - 2\omega^2 k v_q + \omega k^2 v_q^2 - \omega^2 kv + 2\omega k^2 v v_q - k^3 v v_q^2) \\
&+ B(\omega^3 - 2\omega^2 k v_q + \omega k^2 v_q^2 + \omega^2 kv - 2\omega k^2 v v_q + k^3 v v_q^2) \\
&+ C(\omega^3 - \omega^2 k v_q - \omega k^2 v^2 + k^3 v^2 v_q) + D(\omega^2 - k^2 v^2)\} \\
=&\frac{1}{(\omega^2 - v^2k^2)(\omega - v_q k)^2} \{\omega^3(A + B + C) + \omega^2(-(kv + 2kv_q)A \\
&+ (kv - 2kv_q)B - kv_q C + D) + \omega((k^2 v_q^2 + 2k^2 v v_q)A \\
&+ (k^2 v_q^2 - 2k^2 v v_q)B - k^2 v^2 C) + (k^3 v v_q^2(-A + B) + k^3 v^2 v_q C - k^2 v^2 D)\}. \quad \text{(B.40)}
\end{aligned}$$

From this, we obtain four equations:

$$\begin{aligned}
&1.\ A+B+C=0 \Rightarrow C=-(A+B),\\
&2.\ -2kv_q(A+B)-kv(A-B)-kv_qC+D=1,\\
&3.\ k^2v_q^2(A+B)+2k^2vv_q(A-B)=k^2v^2C,\\
&4.\ kvv_q^2(A-B)-kv^2v_qC)+v^2D=v_F^2.
\end{aligned} \tag{B.41}$$

From the third equation, we obtain

$$A-B=\frac{v^2+v_q^2}{2vv_q}C. \tag{B.42}$$

From the fourth equation, we derive

$$D=\frac{kv_q(v^2-v_q^2)}{2v^2}C+\frac{v_F^2}{v^2}. \tag{B.43}$$

Using the second equation along with the previous results, we find C to be

$$C=-\frac{2v_q(v^2-v_F^2)}{k(v_q^2-v^2)^2}. \tag{B.44}$$

Thus, the coefficients are

$$A=-\frac{v^2-v_F^2}{2kv(v_q+v)^2}, \qquad B=\frac{v^2-v_F^2}{2kv(v_q-v)^2}, \qquad D=\frac{v_q^2(v^2-v_F^2)}{v^2(v_q^2-v^2)}. \tag{B.45}$$

B.7 Non-interacting Green's function

The non-interacting Green's function defined in equation (4.13) can be computed as follows:

$$\begin{aligned}
&G_0(x,t;x',t')\\
&=\int\frac{\mathrm{d}k}{2\pi}\int\frac{\mathrm{d}\omega}{2\pi}\frac{1}{\omega-\frac{\hbar q^2}{2m}-v_qk'+\frac{\mu}{\hbar}}e^{i(q+k)(x-x')}e^{-i\omega(t-t')}\\
&=-ie^{iq\bar{x}}\int\frac{\mathrm{d}k}{2\pi}e^{ik\bar{x}}\Bigg\{\Theta(\bar{t})\Theta\left(\frac{\hbar q^2}{2m}+v_qk-\frac{\mu}{\hbar}\right)e^{-i\left\{\frac{\hbar q^2}{2m}+v_qk-\frac{\mu}{\hbar}\right\}\bar{t}}\\
&\quad-\Theta(-\bar{t})\Theta\left(-\frac{\hbar q^2}{2m}-v_qk+\frac{\mu}{\hbar}\right)e^{-i\left\{\frac{\hbar q^2}{2m}+v_qk-\frac{\mu}{\hbar}\right\}\bar{t}}\Bigg\}.
\end{aligned} \tag{B.46}$$

Using the formula

$$\int \mathrm{d}k\ \Theta(k)e^{ikx} = \pi\delta(x) + \frac{1}{ix}, \tag{B.47}$$

we can show

$$\int \mathrm{d}k\ \Theta(ak+b)e^{ikx} = \pi e^{-i\frac{b}{a}x}\delta(x) = \pi\delta(x), \tag{B.48}$$

where we omit the finite principal value, as it will only contribute to the real part of the momentum-space Green's function.

Thus, the Green's function becomes

$$G_0(x,t;x',t') = ie^{iq\bar{x}}e^{-i\left\{\frac{\hbar q^2}{2m}-\frac{\mu}{\hbar}\right\}\bar{t}}\delta(\bar{x}-v_q\bar{t})\left\{\Theta(-\bar{t})-\Theta(\bar{t})\right\}. \tag{B.49}$$

B.8 Proof of identity (4.57)

Using the following known results from Fourier transforms (see Wikipedia's table of important Fourier transforms)

$$\int \mathrm{d}x\ e^{i\omega x}f(x+a) = e^{-ia\omega}\int \mathrm{d}x\ e^{i\omega x}f(x) \tag{B.50}$$

and

$$\int \mathrm{d}x\ e^{i\omega x}(\pm ix)^{-\alpha} = \frac{2\pi}{\Gamma(\alpha)}\Theta(\pm\omega)(\pm\omega)^{\alpha-1} \qquad \text{for} \qquad 0<\alpha<1, \tag{B.51}$$

we can now prove the following identity

$$\begin{aligned}\mathcal{F}_{\bar{t}}\left[(1+iW\bar{t})^{-\alpha}\right] =& \int \mathrm{d}\bar{t}\ e^{i\omega\bar{t}}(1+iW\bar{t})^{-\alpha} \\ =&\frac{1}{|W|}\int \mathrm{d}u\ e^{i\frac{\omega}{|W|}u}(1+\mathrm{sgn}(W)iu)^{-\alpha} \\ =&\frac{1}{|W|}\int \mathrm{d}u\ e^{i\frac{\omega}{|W|}u}\left[\mathrm{sgn}(W)i(-\mathrm{sgn}(W)i+u)\right]^{-\alpha} \\ =&e^{-\frac{\omega}{W}}\frac{1}{|W|}\int \mathrm{d}u\ e^{i\frac{\omega}{|W|}u}\left[\mathrm{sgn}(W)iu\right]^{-\alpha} \\ =&\frac{1}{|W|}\frac{2\pi}{\Gamma(\alpha)}\Theta(\frac{\omega}{W})\left(\frac{\omega}{W}\right)^{\alpha-1}e^{-\frac{\omega}{W}}.\end{aligned} \tag{B.52}$$

Acknowledgment

As I write this page, I find myself at the end of a transformative 2.5-year journey in the Netherlands. Achievements like this are never accomplished alone, and I would like to take this opportunity to express my sincere gratitude to everyone who has supported me along the way and to reflect on my time at Utrecht University.

First and foremost, I want to thank my supervisor, Professor Stoof, for providing such an intriguing and challenging research topic. I've often told my friends that if someone had shown me my thesis a year ago, I wouldn't have believed I was capable of completing these calculations. Over the course of this one-year project, I've learned far more than I could have ever imagined—not just about physics, but also about perseverance, patience, and embracing the fear of the unknown while exploring it. These experiences have given me the confidence to believe I can tackle questions that truly interest me in the future.

This journey would not have been possible without the unwavering support of my parents, who have borne incredible sacrifices to pave the way for my education. Words cannot express how grateful I am for their boundless support throughout my life. They've always ensured I had everything I needed, even at great personal cost. Their generosity allowed me to pursue my interests, such as studying physics at university, without the pressure of financial concerns—something I didn't fully appreciate until I came to the Netherlands, where the cost of living is so much higher than in China.

During these 2.5 years, I've faced both bright moments and dark times, often questioning whether I made the right decision to study abroad, given the financial burden it placed on my family. Last week, I calculated that my studies here have cost €85,000, with a remaining €38,000 loan from the bank. Considering my parents earn less than €800 per month, their dedication to supporting my education is extraordinary. I am profoundly lucky to have such selfless parents. I know I've been selfish in following my dreams, often prioritizing my interests above practicality. But it is thanks to their sacrifices that I've had the freedom to chase those dreams, and for that, I will be forever grateful.

I've also been fortunate to meet so many enthusiastic and inspiring friends during this journey. For the first time, I experienced the pure joy of tackling theoretical physics challenges together, even amid immense stress. I want to thank Hugo (who shares my love of Chinese food and with whom I've cooked many meals), Rimo, Taylor (my first-year roommate), Marc (whose sense of humor and shared struggles in quantum theory made the journey lighter), and Nujia (whose research interests align closely with mine).

In my second year, when life became more challenging, I was lucky to have the support of Sander (who took all the same courses with me), Yoram (who listened to my problems and offered constant encouragement), Giorgos (my steadfast friend who even invited me to Greece), Giovanni, Thomas and his girlfriend Sylia, Uma, and Duojie (with whom I had many discussions about Chinese education and shared mutual support).

Throughout my years of studying physics, I faced more voices urging me to abandon my dream of becoming a theoretical physicist than those encouraging me to persevere. Deep down, I've always doubted whether I could truly succeed. Yet, I could never accept the idea of settling for a less fulfilling, mundane life doing something else. Unlike those driven purely by curiosity about nature, my motivation has often stemmed from my admiration for the unique personalities of certain theoretical physicists and the hope of living a life as extraordinary and adventurous as theirs. I'm not sure if this is the best reason to pursue physics, but it's the reason that keeps me moving forward.

I owe profound gratitude to this part of myself—the part that refuses to give up. It has carried me through countless moments where giving up would have seemed easier or more rational. Without this persistence, I wouldn't be standing here today. While I can't say it's an unquestionably good trait, it feels like the most natural one for me. Thanks to this resilience, I've made it all the way to this moment.

Many people may think that science belongs to all humankind, but in reality, it is a luxury that only a small fraction of the population can truly access. The opportunities to pursue it are shaped by factors such as the country of one's birth, nationality, gender, and even the region where one is raised. For someone like me, coming from a small town with limited resources, it feels almost like a miracle to have overcome these obstacles and arrived at this point. I am not certain how long I will be able to continue on this path, but the fact that I have completed this thesis and this program is something I deeply cherish. I feel incredibly fortunate to have had this experience, and I am confident that the knowledge and lessons I've gained will have lasting value in my future. My heartfelt gratitude goes to all the people and coincidences that made this journey possible.

Finally, I want to thank Oscar for thoroughly reviewing my thesis draft and providing invaluable suggestions for improvement.

To everyone who has played a part in this journey, thank you from the bottom of my heart.

Boyang,

January 2025, Utrecht

Bibliography

[1] M. Khodas, M. Pustilnik, A. Kamenev, and L. I. Glazman, "Fermi-luttinger liquid: Sprectral function of interacting one-dimensional fermions," 2007. DOI: 10.1103/PhysRevB.76.155402. [Online]. Available: https://link.aps.org/doi/10.1103/PhysRevB.76.%20155402.

[2] S. Smit, E. Mauri, L. Bawden, *et al.*, "Momentum-dependent scaling exponents of nodal self-energies measured in strange metal cuprates and modelled using semi-holography," 2021. DOI: 10.48550/ARXIV.2112.06576. [Online]. Available: https://arxiv.org/abs/2112.06576.

[3] H. T. C. Stoof, K. B. Gubbels, and D. B. M. Dickerscheild, *Ultracold Quantum Fields*, 1st. Springer, 2009, ISBN: 1402087624. [Online]. Available: http://www.worldcat.org/%20isbn/1402087624.

[4] G. D. Mahan, *Many-Particle Physics*, 3rd. Boston, MA: Springer, 2000, p. 82, ISBN: 978-0306463389.

[5] S. Laboratory. "Cuprate superconductors." (2025), [Online]. Available: https://arpes.stanford.edu/research/quantum-materials/cuprate-superconductors.