Structural Similarity in Inverse Problems

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Abstract

This thesis provides a theoretical basis for applying structural similarity to joint inverse problems with gradient-based variational regularization. This study develops an overarching formulation for these types of problems which have been successfully applied in geophysics, image enhancement, and medical imaging in prior research. Via the Direct Method from the calculus of variations, the study identifies lower semi-continuity and coerciveness as essential properties for the well-posedness of the variational problem with regularizers that are integrals over an integrand specifying structural similarity. Informed by practice, well-posedness of the coupled inverse problem is proven for previously used specific integrands with solutions in $W^{m,p}, BV, SBV$, and the space of finite Radon measures $\mathcal{M}$. Specifically, the use of gradient-difference, cross-gradient or Schatten norms as structural similarity quantifiers is theoretically justified. A generalized form of the cross-gradient that inherently works on N coupled problems is introduced and is proven to lead to a well-posed problem. Additionally, quasiconvex relaxation and compensated compactness are explored as alternative methods that provide insight when the Direct Method fails, in particular for the case of using a dot-product regularizer. This thesis also shows that both new and existing structural regularizers outperform traditional TV regularization in RGB image deblurring problems.

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Contents

1 Introduction

In mathematics, we can often phrase a problem in such a way that we apply an operator K to some object b resulting in some other object d. Without paying attention to the precise meaning of these terms we have the operator equation

$$K(b) = d.$$

This is called a forward problem, since in applications given an observed quantity b we apply the forward operator K to get data d. The inverse problem is then given by the opposite, where we have some data d and aim to find a corresponding b. Inverse problems are common in all disciplines where data is gathered: including all physical sciences, finance and engineering. Take for example b and d as vectors and K a matrix, then the inverse problem can be solved by calculating the inverse matrix K^{-1}. Even in this simple case already difficulties arise, as a matrix is not always invertible. For non-linear operators K, which are used for example in scattering problems or fluid mechanics, an inverse operator K^{-1} can be ill-defined.

This is why solving an inverse problem (see [23] for general theory) is often difficult as the problems can be ill-posed, ill-conditioned and there is often noise present. To combat these problems the notion of a well-posed problem is introduced. We call an inverse problem well-posed if there exists a solution and there is some continuous dependence of this solution based on the initial data. Sometimes, also uniqueness of the solution is required to call a problem well-posed but we disregard this here as proving this is only possible in specific cases. Where possible, the conditions for uniqueness of a solution are mentioned throughout.

Often in application, multiple measurements via similar or different data-gathering techniques are performed of the same physical system. A relatively recent idea [35] is joint inversion, where we can introduce a coupling between inverse problems with different data $d_1, \ldots, d_N$ that describe the same system b. The inverse problems can be the same except for noise when the system is measured via the same method, or of different form when multiple methods are used. There have already been practical applications in geophysical reconnaissance [60, 1, 30, 31, 32, 33, 35, 34, 38, 47, 48, 56, 57], medical imaging [7, 8, 10, 24, 25, 41, 43, 64] and image enhancement [37, 21, 45], but also in mathematical problems in spectral theory like determining the singular values of compact operators [29]. There are also theoretical indications for why joint inversion improves on separate inversion, but a complete theory does not exist. In [29] the authors prove that for self-adjoint compact operators, the joint problem is at most as ill-posed as the worst separate problem, and often less ill-posed.

In broad terms there are two approaches to joint problems; Model Fusion (MF) and Structural Similarity (SS). The former is where we use one or more of the gathered data-sets $d_i, i = 1, \ldots, N$ corresponding to ill-posed problems to introduce additional constraints to enrich the other better posed problems [53]. Instead of looking for the best solution in some solution space we can also model a joint inverse problem statistically. We regard the data and solution as random variables and search for the solution with the highest likelihood given the data. Here, instead of using constraints, the information gained from knowing a (partial) solution of one of the problems can be used in the other problems. More concretely, this statistical influence is used to adjust the Bayesian priors of the other problems [8, 10, 34, 64]. Model fusion is then also called Mutual Information (MI). Mathematically, MF leads to a direct encapsulation of given data in the inverse problem to be solved and must be taken on a case to case basis.

The latter approach (SS) assumes a priori structural similarities between the problems and is the focus of this thesis. For example, inverting MRI anatomical information together with PET-scan functional information to create a 2/3D representation of a brain [25, 22, 24], where there is some complicated relation between the two imaging methods. Structural similarity is also used in multi-channel imaging, where we can regard RGB-valued images as three quantities b_1, b_2, b_3 corresponding to the same image [37, 21], or jointly invert a low resolution gray-valued image with a high resolution pan-chromatic image [46, 6]. This resemblance in structure can then be used to define a coupled inverse problem based on all data-sets. It can be expected that the added information of similar structure, when true, leads to a better solution to the coupled inverse problem compared to the solutions when solving separately. This has been established quantifiably for many specific problems (for examples, see [57, 34, 32]). The practical details can be completely different within disciplines and across applications. In this thesis, we aim for a general view of inverse problems with structural similarity and see that they can be modelled via a similar approach.

Using MF, only one separate inverse problem is solved in the end and no coupled solutions are computed. It cannot be easily fit in the same framework as structural similarity and we only discuss it briefly in relation to SS in this thesis.

In geophysics and medical imaging the measurements are often performed via complicated machinery where a large amount of physical parts and sensors are used. Since inverse problems like this often come from real-world data, the ability to deal with noisy measurements is essential. An ill-posed problem can only be solved unsatisfactorily via numerical methods, as the solutions might differ enormously when using noisy data.

Also, there might be no solution or there could be no numerically robust way of solving it. Before being able to proof convergence of a solving algorithm, existence of a solution is necessary. This is where well-posedness is important. Also, well-posed problems can properly deal with noisy measurements as estimations of the continuous dependence on the initial data of the solution can be used to give upper bounds on the possible error. To change an ill-posed problem into an well-posed one, one often adds regularization terms [23, 15]. Explicit regularization is then used to promote solutions with specific properties such as sparseness or limited growth [44]. We can formalize a joint inverse problem in a variational problem where we want to solve

$$\operatorname{argmin}_{b\in\mathcal{B}} \|Kb - d\|_{\mathcal{H}}^2 + \alpha J(b). \tag{1}$$

Where $K : \mathcal{B} \to \mathcal{H}$ is an operator, d the corresponding data, $\alpha > 0$ and $J : \mathcal{B} \to \mathbb{R}$ the regularization term. The classical case is Tikhonov-type regularization which is closely linked to the spectral values of K where we take

$$J(b) := \frac{1}{2}\|b\|_{\mathcal{B}}^2. \tag{2}$$

Throughout this report, to get some grasp on the existence of solution to the problem, we will assume $\mathcal{B}$ a Banach space and $\mathcal{H}$ a Hilbert space with some corresponding topologies $\tau_{\mathcal{B}}, \tau_{\mathcal{H}}$. We will define these aggregate $\mathcal{B}, \mathcal{H}$ using direct products as $\mathcal{B} := \bigoplus \mathcal{B}_i, \mathcal{H} := \bigoplus \mathcal{H}_i, i = 1, \ldots, N$ such that we can look at the components $\mathcal{B}_i, \mathcal{H}_i$ separately for each of the $N \in \mathbb{N} = \{1, 2, \ldots, \}$ data-gathering techniques.

Furthermore, we will be assuming that all components of K given by $K_i : \mathcal{B}_i \to \mathbb{R}$ are linear operators with two different types of particular choices for the Banach spaces $\mathcal{B}_i$. Namely, $\mathcal{B}_i$ being a function space of vector-valued functions $b : \Omega \to \mathbb{R}$ such as

$$L^p(\Omega), W^{m,p}(\Omega), BV(\Omega), SBV(\Omega),\ p \in [1, \infty], m \in \mathbb{N},$$

or the space of vector-valued finite Radon measures $b : \Omega \to \mathbb{R}^N$ given by

$$[\mathcal{M}(\Omega)]^N, N \in \mathbb{N},$$

with Ω a nice subset of $\mathbb{R}^n$. In these settings, our regularization term $J : b := (u_1, \ldots, u_N) \mapsto \bar{\mathbb{R}}$ is some integral that quantifies the structural similarity between the problems like

$$J(u_1, \ldots, u_N) := \int_\Omega f(x, u_1(x), \ldots, u_N(x), \nabla u_1(x), \ldots, \nabla u_N(x))dx.$$

With $\mathcal{M}(\Omega)$ treated slightly differently as Radon-Nikodym derivatives need to be used instead of gradients. Here we use $J : \mu \mapsto \bar{\mathbb{R}}$ as

$$J(\mu) := \int_\Omega f\left(\frac{d\mu^a}{d\mathcal{L}^1}\right) dx.$$

For certain integrands $f(\cdot)$, the Direct method of the Calculus of Variations will provide necessary and sufficient conditions for well-posedness of the variational problem defined in Equation (4). In the case as outlined above, the relevant notions for using the Direct method for J with vector-valued f are (mean) coercivity, lower semi-continuity, quasiconvexity, and the growth at small and large values of $f(\nabla b(x)), x \in \Omega$. The theory of compensated compactness and (quasi)convex relaxation come into play in settings where the Direct method has lacking answers. Throughout this thesis, we will prove results in the general $N \in \mathbb{N}$ case when possible. Currently, the applied mathematical literature almost exclusively works with $N = 2$ and hence the specific $f(\cdot)$ that are used are only defined in this case. One of the aims of this report is developing a more thorough mathematical understanding of the general case, which could lead experts in more applied fields such as geoscience and computational imaging to develop new regularizers that lead to well-posed inverse problems.

Coupled inverse problems have up till now been mainly approached from a problem-based perspective. This can be seen in the current literature on joint inversion, where only stratified methodological sources exist [33, 7, 64] and these are mainly focused on practical performance. Theoretical underpinning of joint inversion in variational problems is, where present, exclusively heuristic and informal. To the best knowledge of the author, this thesis rectifies these gaps in knowledge in two ways.

Firstly, applying the theory of Calculus of Variations to provide an explicit characterisation of the necessary properties for well-posedness of coupled inverse problems for $N, n \in \mathbb{N}$. This leads to a more delicate understanding of the variational problem. The broader framework is then immediately useful, as gathering joint integrands $f(\cdot)$ from all application domains that use joint inversion in one place and investigating well-posedness for specific $f(\cdot)$ is now possible. Before only subject-specific gathering of joint inversion techniques were written [33, 37, 64].

Secondly, in problems that are not well-posed, like when measuring structural similarity of two vectors by the magnitude of the dot-product (where we take the integrand f in J to be dependent on the dot-product), the application of (quasi)convex relaxation and compensated compactness can still give reasons why structural similarity is fruitful in some settings. Calculating the quasiconvex envelopes of integrand functions can lead to new quasiconvex regularizers in a similar way as convex relaxation is used to develop lower semi-continuous integral representations. Additionally, understanding the topological spaces where non-linear expressions are compact with respect to the weak convergence leads to verifiable conditions for regimes where the variational problem is well-posed. This can provide insight into the type of applications where dot-product regularization should work well and where it will not. Both these methods, although thoroughly understood, have not been used for structural similarity integrals before.

Additionally, when considering coupled inverse problems with more than two components $N > 2$, up till now structural regularizers have been considered as

$$J(u_1, \dots, u_N) := \sum_{i=1}^{N} \sum_{j=1}^{N} J(u_i, u_j), \tag{3}$$

where $J(u_i, u_j)$ measures the similarity between a pair of problems. A direct definition of $J(u_1, \dots, u_N)$ not based on pairs u_i, u_j is newly introduced here, where we have used the quasiconvexity of certain specific integrands f based on the cross-product for $N = 2$ to inform us about the correct form for $N > 2$. The numerical experiments conducted in this thesis are the first using a structural regularizer defined in this way.

The remainder of this thesis is organised as follows. In Section 2 the general theory for existence and uniqueness of solutions for a coupled inverse problem is discussed. After a brief discussion of the possible choices for the pair of the Hilbert space with a topology $(\mathcal{H}, \tau_{\mathcal{H}})$, a thorough exposition of the possible pairs $(\mathcal{B}, \tau_{\mathcal{B}})$ of a Banach space $\mathcal{B}$ with a given topology $\tau_{\mathcal{B}}$ where we are searching for a solution is included. In Section 3 we determine the general necessary and sufficient conditions for the integral functionals J that lead to a well-posed Equation (4). In Section 4 we prove or disprove these properties for many specific integrands $f(\cdot)$ set forth in the literature over the last few decades. For the quasiconvex integrands f_{GD}, f_{CG}, we extend the definition from $N = 2$ to $N \in \mathbb{N}$. Section 5 exists of the statements and proofs of well-posedness when having $\mathcal{B}$ as $[W^{1,p}(\Omega)]^N, [BV(\Omega)]^N, [SBV(\Omega)]^N$, or $\mathcal{M}$. In Section 6, we take a closer look at the integrands that are separately convex but not quasiconvex. A version of the Div-Curl lemma is used to find a subspace of $W^{1,2}$ where J is l.s.c. when using integrands that are convex transformations of the dot-product. Also, we investigate if (quasi)convex relaxation can be used to define better regularizers $\bar{J}$ for structurally similar problems. Finally to put the theory to the test, in Section 7, an algorithmic approach to solving inverse problems that have both TV regularization and a joint structural component is developed. Using this method, we numerically asses the integrands that lead to well-posed problems in the case $N = 3, n = 2$ for vector-valued image enhancements.

2 Problem setting

The variational joint inverse problem that we study is given by

$$\operatorname{argmin}_{b\in\mathcal{B}} \|Kb-d\|_{\mathcal{H}}^2 + \alpha J(b). \tag{4}$$

With $\mathcal{B} = \mathcal{B}_1 \bigoplus \cdots \bigoplus \mathcal{B}_N$ where $\mathcal{B}_i \in \{L^p(\Omega), W^{m,p}(\Omega), \mathcal{M}(\Omega), BV(\Omega), SBV(\Omega)\}, i = 1,\dots,N$ are Banach, $\mathcal{H}$ Hilbert, $K : \mathcal{B} \to \mathcal{H}$ is an operator, d the corresponding data, $\alpha > 0$ the regularization parameter, and $J : \mathcal{B} \to \mathbb{R}$ the regularization term in our case given by

$$J(u_1,\dots,u_N) := \int_\Omega f(x, u_1(x),\dots,u_N(x), \nabla u_1(x),\dots,\nabla u_N(x))dx. \tag{5}$$

Although most of the theory is applicable to any finite number $N \in \mathbb{N}$ of data-gathering techniques, it is sensible to first look at how to couple two inverse problems for $N = 2$. To explicitly include the two different problems we take $\mathcal{H}_1, \mathcal{H}_2$ Hilbert spaces, $\mathcal{B}_1, \mathcal{B}_2$ Banach spaces over $\mathbb{R}$. Then $\mathcal{H} = \mathcal{H}_1 \bigoplus \mathcal{H}_2$ is a Hilbert space with the inner product given by

$$\langle h_1, h_2\rangle_{\mathcal{H}} := \langle h_1^1, h_2^1\rangle_{\mathcal{H}_1} + \langle h_1^2, h_2^2\rangle_{\mathcal{H}_2}, \tag{6}$$

for $h_1 = h_1^1 \bigoplus h_1^2, h_2 = h_2^1 \bigoplus h_2^2 \in \mathcal{H}$. Similarly for $1 \le p < \infty$, $\mathcal{B} = \mathcal{B}_1 \bigoplus_p \mathcal{B}_2$ is a Banach space with the norm given by

$$\|b\|_{\mathcal{B}} := \left(\|b_1\|_{\mathcal{B}_1}^p + \|b_2\|_{\mathcal{B}_2}^p\right)^{\frac{1}{p}}. \tag{7}$$

For $b = b_1 \bigoplus b_2 := (b_1, b_2) \in \mathcal{B}$. We then have

$$\|Kb-d\|_{\mathcal{H}}^2 = \|K_1b_1 - d_1\|_{\mathcal{H}_1}^2 + \|K_2b_2 - d_2\|_{\mathcal{H}_2}^2.$$

In most cases this construction is sufficient, as in applications often only two inverse problems are coupled. However, we can generalise the construction of $\mathcal{B}, \mathcal{H}$ also for the more general case over $N \in \mathbb{N}$. In this report we almost exclusively use the direct sum $\bigoplus_1$ with $p = 1$ as the norm belonging to $\mathcal{B}$ is the linear sum of the component norms, more explicitly

$$\|b\|_{\mathcal{B}} := \sum_{i=1}^N \|b_i\|_{\mathcal{B}_i}, \tag{8}$$

$$\langle h_1, h_2\rangle_{\mathcal{H}} := \sum_{i=1}^N \langle h_1^i, h_2^i\rangle_{\mathcal{H}_i}. \tag{9}$$

In this section, we state the general theorem of existence of the variational optimalisation problem formulated in Equation (4) and in particular for a coupled system of size $N \in \mathbb{N}$ as described in the introduction. The existence is dependent on functional analytical properties; the choice of specific spaces $\mathcal{B}, \mathcal{H}$, the topologies $\tau_{\mathcal{B}}, \tau_{\mathcal{H}}$, and convex-analytic properties of J. We postpone the discussion concerning J to Section 3. Here we discuss the possible choices of $(\mathcal{B}, \tau_{\mathcal{B}})$ and comment on $(\mathcal{H}, \tau_{\mathcal{H}})$. Specifically, which topologies $\tau_{\mathcal{B}}$, domains Ω, and other properties are necessary for solutions in $\mathcal{B}$. Only some basic prerequisite knowledge of $L^p(\Omega)$ spaces is necessary as the relevant choices for the Banach space $\mathcal{B}$ (Sobolev, BV, SBV, $\mathcal{M}$) are introduced from definition.

2.1 Existence of a minimizer

First we define some properties of functions.

Definition 1 (Lower semi-continuity)**.** *Let $f : \mathbb{R}^n \to \bar{\mathbb{R}} = \mathbb{R} \cup \{\infty, -\infty\}$ a function. It is lower semi-continuous at $x_0 \in \mathbb{R}^n$ if for all $x \to x_0 \in \mathbb{R}^n$ converging in $\mathbb{R}^n$ we have*

$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$

A function is called l.s.c. if it is lower semi-continuous at all $x_0 \in \mathbb{R}^n$.

Definition 2 (Coercivity)**.** *Let $f : \mathcal{X} \to \bar{\mathbb{R}}$ a function from a normed vector space $\mathcal{X}$. It is called coercive if for every sequence $(x_n)_n \subset \mathcal{X}$ that has $\|x_n\|_{\mathcal{X}} \to \infty$ we have*

$$\lim_{n \to \infty} f(x_n) = \infty.$$

For variational regularization we have the following result.

Theorem 1 (Variational minimiser)**.** *[23] Let $\mathcal{B}$ a Banach space and $\mathcal{H}$ a Hilbert space associated with the topologies $\tau_{\mathcal{B}}, \tau_{\mathcal{H}}$. Assume the pair $(\mathcal{B}, \tau_{\mathcal{B}})$ has the property that bounded sequences have $\tau_{\mathcal{B}}$-convergent subsequences. Moreover, assume the norm on $\mathcal{H}$ is $\tau_{\mathcal{H}}$-l.s.c. and that the operator $K : \mathcal{B} \to \mathcal{H}$ is linear and sequentially continuous with respect to the topologies $\tau_{\mathcal{B}}$ and $\tau_{\mathcal{H}}$. The functional $J : \mathcal{B} \to \mathbb{R}^+ \cup \{\infty\}$ is proper and $\tau_{\mathcal{B}}$-l.s.c. Additionally, let either J be coercive or the pair (K, J) are (mean) coercive in the sense of Lemma 1. Let $d \in \mathcal{H}, \alpha > 0$.*
If all these conditions are met, then the minimisation problem defined by Equation (4) given by

$$\text{argmin}_{b \in \mathcal{B}} \|Kb - d\|^2_{\mathcal{H}} + \alpha J(b),$$

has a minimiser.

Remark 1. *Here l.s.c. stands for lower semi-continuity, see Definition 1. Note that the inclusion of ∞ in the range of J implies that J is still well-defined if we have elements $b \in \mathcal{B}$ such that $J(b) = \infty$. In the remainder we write $\mathbb{R}^+ \cup \{\infty\}$ as $[0, \infty]$. This is similar to the extended real line $\bar{\mathbb{R}} := \{\mathbb{R}, \infty, -\infty\}$ but now we have only added one element ∞ instead of two.*

Lemma 1 (Mean coercivity)**.** *[23] Let* $K : \mathcal{B} \to \mathcal{H}$ *linear,* $J : \mathcal{B} \to [0, \infty]$ *and* $d \in \mathcal{H}$*. Let* $p_0 \in \mathcal{B}^*$ *the topological dual space of continuous linear functionals on* $\mathcal{B}$ *and* $b_0 \in \mathcal{B}$*, such that* $\langle p_0, b_0 \rangle = 1$*, and* $b_0 \notin \mathcal{N}(K)$ *(the null space of* K*), chosen such that* J *is coercive on*

$$\mathcal{B}_0 := \{b \in \mathcal{B} : b \in \mathcal{N}(p_0)\},$$

in the sense that for $b \in \mathcal{B}$

$$\|b - \langle p_0, b \rangle b_0\|_{\mathcal{B}} \to \infty \implies J(b) \to \infty.$$

If these choices can be made we say that the pair (K, J) *is mean coercive.*
If this is the case, the requirement of coercivity of J *in Theorem 1 can be substituted by the requirement that the pair* (K, J) *is mean coercive.*

Remark 2. *This lemma seems quite technical but is necessary already in basic situations. For example, when standard TV regularisation is used on* $BV([0, 1])$ *with* $J(b) = TV[b]$*, already* J *is not coercive on the entire space. Namely, a sequence of constant functions running off to infinity has zero total variation and hence* J *is not coercive. However, taking* $p_0(b) := \int_0^1 b(x)dx$ *as the continuous linear functional computing the mean, then the corresponding subspace* $\mathcal{B}_0 \subset BV([0, 1])$ *are the functions that have zero mean. Now* J *is (mean) coercive on* $\mathcal{B}_0$ *and satisfies the assumptions as required in Theorem 1 with* $\mathcal{B} := \mathcal{B}_0$*. In addition to TV regularisation, the matrix-norm regularisers that are considered in Section 4 will also turn out to be mean coercive.*

Now to more explicitly regard a coupled problem, the general form in Theorem 1 is equivalent to the following for joint inversion.

Corollary 1 (Joint construction, $N = 2$)**.** *Follow the construction in Equations* (6) *and* (7)*. Let* K_1, K_2 *be linear operators that are sequentially continuous with respect to topologies on* $\mathcal{B}_1, \mathcal{H}_1$ *and* $\mathcal{B}_2, \mathcal{H}_2$ *respectively. Let the norms on* $\mathcal{H}_1, \mathcal{H}_2$ *be* $\tau_{\mathcal{H}_1}, \tau_{\mathcal{H}_2}$ *lower semi-continuous. Let* $(\mathcal{B}_1, \tau_{\mathcal{B}_1}), (\mathcal{B}_2, \tau_{\mathcal{B}_2})$ *have the property that bounded sequences have convergent sub-sequences. Let* $K : \mathcal{B} \mapsto \mathcal{H}, (u_1, u_2) \mapsto (K_1u_1, K_2u_2)$ *be an operator that is linear and sequentially continuous with respect to* $\tau_{\mathcal{B}}, \tau_{\mathcal{H}}$*. Then the existence of a minimizer of the Tikhonov type regularization is only dependent on the properties of* J *and on possibly whether* (K, J) *is (mean) coercive.*

Proof. Let $\tau_{\mathcal{B}}$ be the product topology of $\tau_{\mathcal{B}_1}, \tau_{\mathcal{B}_2}$. We have assumed that $\mathcal{B}_1, \mathcal{B}_2$ are compact spaces (wrt the respective topologies), since the finite direct sum of compact spaces is compact we have that $\mathcal{B}$ is compact wrt $\tau_{\mathcal{B}}$. Since l.s.c. works piece-wise also the norm on $\mathcal{H}$ is l.s.c. wrt to the product topology $\tau_{\mathcal{H}}$. By construction, K is linear and sequentially continuous wrt the topologies. All other necessary conditions in Theorem 1 are related to J and the result follows. $\square$

The relevant properties of J are properness, non-negativity, $\tau_{\mathcal{B}}$-l.s.c. and (mean) coercivity. We will deal with these in Section 3. We state Lemma 1 more explicitly in the case where $\mathcal{B} := \mathcal{B}_1 \bigoplus_p \mathcal{B}_2$ is given below. There is some nuance with taking duals of a space built from a direct sum.

Lemma 2 (Dual of direct sum)**.** *[28] Take $1 \leq p < \infty$. Let $\mathcal{B} = \mathcal{B}_1 \bigoplus_p \mathcal{B}_2$ be a Banach space. Then*

$$\mathcal{B}^* \text{ isometrically isomorphic to } \mathcal{B}_1^* \bigoplus_q \mathcal{B}_2^*,$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

The following technical result can be informally understood as the functional $J(b)$ being mean coercive on a function space $\mathcal{B}$ if $\|\nabla b\| \to \infty \implies J(b) \to \infty$.

Lemma 3 (Mean Coercivity of Direct Sum)**.** *Let $K : \mathcal{B} \to \mathcal{H}$ linear, $J : \mathcal{B} \to [0, \infty]$ and $d \in \mathcal{H}$. Let $p_0 = (p_u, p_v) \in \mathcal{B}^*$ and $b_0 = (u_0, v_0) \in \mathcal{B}$ such that for all $b = (u, v)$, $\langle p_u, u_0 \rangle = \langle p_v, v_0 \rangle = 1$ and $(u_0, v_0) \notin \mathcal{N}(K)$. Chosen such that J is coercive on*

$$\mathcal{B}_0 := \left\{ (u, v) \in B_1 \bigoplus_p B_2 : u \in \mathcal{N}(p_u), v \in \mathcal{N}(p_v) \right\}.$$

in the sense that for all $b \in \mathcal{B}$,

$$\|u - \langle p_u, u \rangle u_0\|_{\mathcal{B}_1} \text{ or } \|v - \langle p_v, v \rangle v_0\|_{\mathcal{B}_2} \implies J(b) \to \infty.$$

Then the variational problem given in Equation (4) *can also be called (mean) coercive.*

Proof. The only actual change from Lemma 1 is the condition that for all (u, v) we have $\langle p_u, u_0 \rangle = \langle p_v, v_0 \rangle = 1$ instead of $\langle p_0, b_0 \rangle = 1$. This can be changed since in the proof of Lemma 1 we actually only need that any b can be split into $b = b_1 + b_2$ where $b_1 \in \mathcal{B}_0$ and $b_2 \in \operatorname{span} b_0$. Heuristically, think of these two parts as the result of splitting the function b into a multiple of the 1 function (times the mean), which is in span b_0, and the remaining oscillations around the mean $\in \mathcal{B}_0$. The condition $\langle p_0, b_0 \rangle = 1$ makes sure that this also holds for $b := b_0$ as then $\langle p_0, b_0 - \langle p_0, b_0 \rangle b_0 \rangle \rangle = 0$. This can also be done element-wise for u, v. By the definition of p_0 we have $b - \langle p_0, b \rangle b_0 \in \mathcal{B}_0$ for all b iff $u - \langle p_u, u \rangle u_0 \in \mathcal{N}(p_u)$ and $v - \langle p_v, v \rangle v_0 \in \mathcal{N}(p_v)$ for all (u, v). This follows from the assumption $\langle p_u, u_0 \rangle = \langle p_v, v_0 \rangle = 1$. Hence if we pick (p_u, p_v) in such a way that the conditions are true, we have the same result as in Lemma 1 via

$$\| \cdot \|_{\mathcal{B}} = \| \cdot \|_{\mathcal{B}_1} + \| \cdot \|_{\mathcal{B}_2}.$$

So the coercivity condition can also be split over the two sub-spaces. □

Note that in Corollary 1 we have skipped over the difficulty of determining if K is sequentially continuous with respect to to the topologies by assuming this is the case. Also the dimensionality and structure of $\mathcal{N}(K)$ plays a role via Lemma 3 but we skip over this for now. Even in practical examples these things are not always straightforward, especially when considering problems over general Banach spaces with non-trivial topologies. If the topologies are the weak topologies over function spaces $\mathcal{B}$, then the situation is much simpler as continuity in the strong topologies implies continuity in the weak topologies. It is well-known that continuity in the norm topology is equivalent to boundedness of a linear operator K and this is frequently verifiable in practice. The structure of $\mathcal{N}(K)$ does not need to be complicated, as in some applications we have injective operators such that $\mathcal{N}(K) = \{0\}$ and mean coercivity is straightforward. Also skipped over but equally important is the fact that Theorem 1 only tells us something over the existence of a minimizer but nothing about the uniqueness or dimensionality of the set of minimizers. In the case of integral operators J as described in the introduction, uniqueness comes either from injectivity of K or strict convexity of J [23] as can be seen from Lemma 5 below.

One can easily extend Corollary 1 to any finite but arbitrary number $N \in \mathbb{N}$.

Corollary 2 (Joint construction, $N \in \mathbb{N}$)**.** *Take $\mathcal{B} = \mathcal{B}_1 \bigoplus_1 \cdots \bigoplus_1 \mathcal{B}_N$, $\mathcal{H}$ constructed via Equation 9. Let $K_1, \ldots, K_N$ be linear operators that are sequentially continuous with respect to topologies on $(\mathcal{B}_1, \mathcal{H}_1), \cdots, (\mathcal{B}_N, \mathcal{H}_N)$ respectively. Let the norms on $\mathcal{H}_1, \ldots, \mathcal{H}_N$ be $\tau_{\mathcal{H}_1}, \ldots, \tau_{\mathcal{H}_N}$ lower semi-continuous. Let $(\mathcal{B}_1, \tau_{\mathcal{B}_1}), \ldots, (\mathcal{B}_N, \tau_{\mathcal{B}_N})$ have the property that bounded sequences have convergent sub-sequences. Let $K : \mathcal{B} \mapsto \mathcal{H}, (u_1, \ldots, u_N) \mapsto (K_1 u_1, \ldots, K_N u_N)$ be an operator that is linear and sequentially continuous wrt $\tau_{\mathcal{B}}, \tau_{\mathcal{H}}$. Then the existence of a minimizer of the Tikhonov-type regularization is only dependent on the properties of J and on whether (K, J) is mean coercive.*

Proof. The proof is along similar lines as the $N = 2$ case and is omitted. □

For uniqueness of the minimizer for a minimization problem with regularisation we have the following. Define the functional $E : \mathcal{B} \to \bar{\mathbb{R}}$ with $\alpha > 0$ as

$$E(b) := \|Kb - d\|_{\mathcal{H}}^2 + \alpha J(b). \tag{10}$$

Lemma 4 (Injectivity implies strict convexity)**.** *[23] Let $\mathcal{B}$ be a Banach space and $\mathcal{H}$ a Hilbert space. Furthermore, let $K : \mathcal{B} \to \mathcal{H}$ be linear, $d \in \mathcal{H}$. Then*

$$\|Kb - d\|_{\mathcal{H}}^2$$

is convex in $d \in \mathcal{B}$. Furthermore, it is strictly convex if and only if K is injective.

Lemma 5 (Uniqueness of minimizer)**.** *[23] Assume that the functional E has at least one minimiser and either K is injective or J is strictly convex. Then the minimiser is unique.*

2.2 Choice of Hilbert space

In light of Corollary 2 there are multiple considerations from the functional analytical perspective. Firstly, there is a choice to be made considering the $\mathcal{H}$-norm and topology as they appear in the data fidelity term given by

$$\|Kb - d\|_{\mathcal{H}}^2.$$

In the continuous setting where the physical system $b := (u_1, \ldots, u_N) : \mathcal{B} \to \mathbb{R}^N$ and the observed parameter field $d := (d_1, \ldots, d_N) \in \mathcal{H}$ are continuous function, we consider the L^2 norm on the components as

$$\|Kb - d\|_{\mathcal{H}}^2 := \sum_{i=1}^{N} \|K_i u_i - d_i\|_{L^2(\Omega)}^2,$$

where $\Omega \subset \mathbb{R}^n$. In regards to function spaces, the $L^p(\Omega)$ spaces are the easiest to define and best understood, they form the natural choice for $\mathcal{H}$. Note that we have to take $p = 2$ since this the only L^p space that is Hilbert. With this choice Corollary 2 is applicable and we can consider $b \in \mathcal{B}$ in any Banach space. This requirement of $\|\cdot\|_{\mathcal{H}}$ corresponding to a Hilbert norm is non-trivial as many choices for normed spaces $\mathcal{H}$ are not complete or the norm is not induced by an inner product. In addition to $L^2(\Omega)$, the Sobolev space $W^{1,2}(\Omega)$ as defined in Definition 5 is also known to be Hilbert and hence an appropriate choice.

If we have non-continuous parameter fields $d_1, \ldots, d_N$ the choice is easier since the consideration of d existing in a Hilbert space is only relevant when $d(x), x \in \Omega$ is non-discrete. This is because taking a particular $\mathcal{H}$ is mainly a modelling choice dictated by the application setting of the coupled inverse problem. In image enhancement, it is natural to choose b and d both as matrices in $\mathbb{R}^{l_1 \times l_2}$ as an image of length l_1 by $l_2 \in \mathbb{N}$ is already discretized(with constant value on each pixel of the image). Via this model, we are no longer interested in deciding which space of functions $\mathcal{H}$ is as we can take the norm from the inner product on $\mathbb{R}^{l_1 \times l_2}$. Even for geophysical and medical applications, data d is often only gathered at a finite amount of points such that we can regard d as a multi-dimensional array (a matrix in 2-d or tensor for n-d, $n > 2$) in a general Euclidean space.

We are interested in solving the inverse problem numerically and we have discretized our region Ω, we can only look at a finite number of values $x_j, j = 1, \dots, k \in \Omega$. Then while carefully matching the ordering of multi-dimensional $b(x_j), d(x_j)$ and only looking at differences in the same location, we can take the norm as

$$\| \cdot \|_{\mathcal{H}}^2 := \sum_{i=1}^{N} \sum_{j=1}^{k} \|K_i(u_i(x_j)) - d_i(x_j)\|_{\mathcal{H}_i}^2,$$

where we have taken $\| \cdot \|_{\mathcal{H}_i}$ a norm on the space of multi-dimensional arrays. The difference being that now any norm $\| \cdot \|_{\mathcal{H}_i}$ that arises from an inner product on $\mathcal{H}_i$ can be chosen instead of only the L^2 norm as any finite-dimensional inner product space is automatically Hilbert.

With these choices, in both the continuous ($\mathcal{H}_i = L^p(\Omega), W^{1,2}(\Omega)$) and the discrete ($\mathcal{H}_i \sim \mathbb{R}^{Nk}$) case the norms on $\mathcal{H}_i, i = 1, \dots, N$ are by definition $\tau_{\mathcal{H}_i}$ lower semi-continuous. Hence we know how we can deal with the norm over $\mathcal{H}$ in the variational problem.

2.3 Choice of Banach space

Throughout this thesis, we assume that our Banach spaces $\mathcal{B}_i, i = 1, \dots, N$ are in particular function spaces over some subset $\Omega \subset \mathbb{R}^n$ such that $u_i : \Omega \to \mathbb{R}, i = 1, \dots, N$. Here instead of the general case, we assume one-dimensional range of u_i, as these function are commonly parameter fields such as temperature or conductivity this is a valid choice. Even then, note that this is not restrictive, as for vector-valued co-domains of u_i we can take a bigger N via the component-restrictions. It is now a natural question to ask which of those Banach spaces have the property that bounded sequences have convergent sub-sequences with respect to the considered topology (compactness). Disregarding the analytical properties of J for now we can say the following; having dealt with the Hilbert space, in order to have existence of a minimizer of joint regularization via the Direct method (Corollary 1), we have to verify two things; does the pair $(\mathcal{B}, \tau_{\mathcal{B}})$ have the property that bounded sequences have convergent sub-sequences and is the functional J $\tau_{\mathcal{B}}$-l.s.c? We deal with the former below, and deal with J entirely in Section 3.

Definition 3 (Compactness)**.** *We call a Banach space $\mathcal{B}$ compact with respect to the topology $\tau_{\mathcal{B}}$ if any bounded (wrt $\| \cdot \|_{\mathcal{B}}$) sequence in $\mathcal{B}$ has a $\tau_{\mathcal{B}}$- convergent sub-sequence.*

Remark 3. *Note that this is the usual definition of a compact space when we take $\tau_{\mathcal{B}}$ to be the norm topology. There is nuance however, as different topologies can be considered on the normed vector space $(\mathcal{B}, \| \cdot \|_{\mathcal{B}})$ unrelated to the norm topology. Determining compactness wrt these different (often weaker) topologies is not always easy.*

The celebrated Banach-Alaoglu theorem provides an answer towards determining if a pair is compact.

Theorem 2 (Sequential Banach - Alaoglu)**.** *[14] Let $\mathcal{B}$ be a separable normed vector space. Then every bounded sequence $(u_k)_k \subset \mathcal{B}^*$ has a weak$-*$ convergent subsequence. If $\mathcal{B}$ is also reflexive, then every bounded sequences in $\mathcal{B}$ has a weakly convergent subsequence and on $\mathcal{B}^*$ the weak and the weak$-*$ topologies coincide.*

The statement above gives a sufficient condition for compactness with respect to the weak$-*$ topology, namely $\mathcal{B}$ being a separable reflexive normed space. This is not a necessary condition as the Banach spaces $\mathcal{M}(\Omega), BV(\Omega)$, and $SBV(\Omega)$ which are defined later in this section do not satisfy these conditions. However, we can still get compactness of the topology $\tau_{\mathcal{B}}$ of a subspace $\mathcal{B}' \subset \mathcal{B}$ by adding additional constraints on the considered elements in $\mathcal{B}'$. For example, in $BV(\Omega)$ we can consider any $\mathcal{B}' \subset BV(\Omega)$ where we have a uniform bound $C \geq 0$ such that $u \in \mathcal{B}' \implies \|u\|_{BV(\Omega)} \leq C$.

Note that via our 1-direct sum construction, if we assume $\mathcal{B}_1 = \cdots = \mathcal{B}_N$, then separability and reflexivity of the components provides it for $\mathcal{B}$ as well. Proving separability and reflexivity of normed spaces is often not straightforward. Since the details are rather technical and not that enlightening, I have opted to put them in Appendix A. In short, the admittable $(\mathcal{B}, \tau_{\mathcal{B}})$ are given in the following paragraph. The formal definitions and minor results are the topic of the rest of this section.

Let $\Omega \subset \mathbb{R}^n$. Assume $\mathcal{B}_1 = \cdots = \mathcal{B}_N, \tau_{\mathcal{B}_1} = \cdots = \tau_{\mathcal{B}_N}$, then the pair $(\mathcal{B}, \tau_{\mathcal{B}})$ has the property that bounded sequences have convergent sub-sequences if we have $(\mathcal{B}_i, \tau_{\mathcal{B}_i})$ as in one of the following cases;

1. $L^p(\Omega),\ p \in (1, \infty)$ with the weak topology.
2. A bounded and equi-integrable set $F \subset L^1(\Omega)$ with the weak topology.
3. For Ω Lipschitz, $W^{m,p}(\Omega)$ for $p \in (1, \infty), m \in \mathbb{N}$ with the weak topology.
4. $\mathcal{M}(\Omega)$ with its weak$-*$ topology and considering uniformly bounded sequences $(\mu_k)_k$.
5. For Ω bounded and Lipschitz, $BV(\Omega)$ with its weak$-*$ topology and considering uniformly bounded sequences $(u_k)_k$.
6. For Ω bounded and Lipschitz, $SBV(\Omega)$ with its weak$-*$ topology and the conditions in Theorem 37.

Remark 4. *By taking the* 1*-direct sum of these $\mathcal{B}_i, i = 1, \ldots, N$ implies that we have $\mathcal{B} = [L^p(\Omega)]^N, \ldots, [SBV(\Omega)]^N$ via the usual notation.*

Remark 5. *The assumption $\mathcal{B}_1 = \cdots = \mathcal{B}_N$ is not necessary and it is possible to have $\mathcal{B}_i \neq \mathcal{B}_j$ since the coupling in the $\mathcal{H}$ norm is indirect and every proof works component-wise for $\mathcal{B}$. It is possible to mix and match choices in the list above to construct $\mathcal{B}$.*

A quick introduction of these spaces follows.

Remark 6. *Throughout this report, we denote by $\mathcal{L}^N$ (or $\mathcal{L}$) the Lebesgue measure on $\mathbb{R}^N$. With abuse of notation, we denote by the same notation $\mathcal{L}^N$ the restricted Lebesgue measure on any $\Omega \subset \mathbb{R}^N$. $\mathbb{B}(\Omega)$ is the corresponding Borel σ-algebra over $\Omega \subseteq \mathbb{R}^N$.*

Definition 4 (L^p spaces)**.** *Let $(\Omega, \mathbb{B}(\Omega), \mathcal{L})$ be a measure space and $p \in [1, \infty)$. The space $L^p(\Omega)$ consists of equivalence classes of measurable functions $f : \Omega \to \bar{\mathbb{R}}$ such that*

$$\int_\Omega |f|^p d\mathcal{L} < \infty,$$

where two measurable functions are equivalent if they are equal $\mathcal{L}$-a.e. The $L^p(\Omega)$-norm is defined as

$$\|f\|_{L^p(\Omega)} := \left(\int_\Omega |f|^p d\mathcal{L} \right)^{\frac{1}{p}}. \tag{11}$$

We are not only interested in the L^p spaces but also in the more general Sobolev spaces, as often our regularizers depend on the size of the gradients and not only on the function values themselves.

Definition 5 (Sobolev spaces)**.** *[2] Let $m \in \mathbb{N}, p \in [1, \infty)$, then*

$$W^{m,p}(\Omega) := \{u \in L^p(\Omega) : D^\alpha u \in L^p(\Omega) \text{ for all } 1 \leq |\alpha| \leq m\}, \tag{12}$$

with D^α the derivative with respect to multi-index α, together with

$$\| \cdot \|_{m,p,\Omega} := \left\{ \sum_{0 \leq |\alpha| \leq m} \|D^\alpha u\|^p_{L^p(\Omega)} \right\}^{\frac{1}{p}} \tag{13}$$

is called the Sobolev space corresponding to L^p with order m. For $p = \infty$, we take the essential supremum instead of $(f^p)^{\frac{1}{p}}$. The closure of $C_0^\infty(\Omega)$ in $W^{m,p}(\Omega)$ is denoted by $W_0^{m,p}(\Omega)$.

Definition 6 (Weak convergence in $W^{m,p}$)**.** *Let $\Omega \subset \mathbb{R}^n, p \in [1, \infty)$ then $(u_h)_h$ converges to u weakly in $W^{m,p}(\Omega)$ (denoted $u_h \rightharpoonup u$) if for all multi-indices $1 \leq |\alpha| \leq m$, $(D^\alpha u_h)_h$ weakly converges to $D^\alpha u$ in $L^p(\Omega)$ and $(u_h)_h$ strongly converges to u in $L^p(\Omega)$.*

Note that in particular for $m = 1$ we need $\nabla u_h \rightharpoonup \nabla u$ weakly in $L^p(\Omega)$ and $u_h \to u$ strongly in $L^p(\Omega)$ for weak convergence of $u_h \to u$ in $W^{1,2}(\Omega)$.

In variational problems even the Sobolev functions are often too restrictive. The extension to functions of bounded variation (BV) is often possible and necessary. Although historically the theory of Sobolev functions and BV functions are not directly related, nowadays good links between the two are established. From Definition 5, a function f belongs to $W^{m,p}$ if its corresponding weak derivatives $f^\alpha, |\alpha| \leq m$ are in L^p. There are functions for which the weak derivatives are not functions themselves, but only defined measure-theoretically as distributions. The classical example is the Heaviside function

$$H(x) := \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

with weak derivative δ_0 as Dirac measure/distribution. One can extend $W^{1,1}(\Omega)$ (also called the space of absolutely continuous functions) $\subset L^1(\Omega)$ to also consider these objects. First we introduce the space of finite Radon measures $\mathcal{M}(\Omega)$, on which we can also perform minimization problems. Our BV functions then are not elements in this space, but have weak derivatives in $\mathcal{M}(\Omega)$.

Definition 7 (Radon measure)**.** *[4] Let $(\Omega, \mathbb{B}(\Omega))$ be a measure space with $\Omega \subset \mathbb{R}^n$ and $\mathbb{B}(\Omega)$ its corresponding Borel σ-algebra. Then a measure $\mu : \mathbb{B}(\Omega) \to \mathbb{R}^N$ is called a finite Radon measure if it is finite on all compact Borel subsets of Ω. The space of all $\mathbb{R}^N$-valued finite Radon measures on Ω is denoted as $[\mathcal{M}(\Omega)]^N$.*

Remark 7. *There is a more general definition of Radon measure over metric spaces, but since Ω is always an open subset of some $\mathbb{R}^n$ and hence is always locally compact and Hausdorff we take this easier definition throughout.*

Theorem 3 (Riesz)**.** *[4] The dual space of the Banach space of all continuous linear functionals on Ω denoted by $[C_0(\Omega)]^N$ with the supremum norm is the space $[\mathcal{M}(\Omega)]^N$ with $|\mu|(\Omega)$ the variation of a measure on some set E defined as*

$$|\mu|(E) := \sup \left\{ \sum_{h=0}^{\infty} |\mu(E_h)| : E_h \in E \textit{ pairwise disjoint, } E = \cup_{h=0}^{\infty} E_h \right\}, \tag{14}$$

as norm.

An application of Theorems 2 and 3 results in the following.

Theorem 4 (De La Vallée - Poussin)**.** *[4] Firstly, $([\mathcal{M}(\Omega)]^N, |\mu|)$ is a Banach space. Secondly, when only taking into account uniformly bounded sequences $(\mu_h)_h$ with $\sup_h |\mu_h|(\Omega) < \infty$, this space is weakly$-*$ compact and the map $\mu \mapsto |\mu|(\Omega)$ is l.s.c. with respect to the weak$-*$ convergence.*

Hence we can take $\mathcal{B}_1 = [\mathcal{M}(\Omega)]^N$ with $\tau_{\mathcal{B}_1}$ as the weak$-*$ topology. Instead of considering the finite Radon measures themselves, we can also look at functions in $L^1(\Omega)$ that have weak derivatives that are finite Radon measures.

Definition 8 (BV functions)**.** *[4] Let $\Omega \subset \mathbb{R}^n$ open. Let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if the distributional derivative Du is representable by a finite Radon measure in Ω, i.e. if*

$$\int_\Omega u \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi dD_i u,\ \forall \varphi \in C_c^\infty(\Omega),\ i = 1, \dots, n. \tag{15}$$

For vector-valued functions $u \in [BV(\Omega)]^N$ we have

$$\int_\Omega u^\alpha \frac{\partial \varphi}{\partial x_i} dx = -\int_\Omega \varphi dD_i^\alpha u, \forall \varphi \in C_c^\infty(\Omega), i = 1, \dots, n, \alpha = 1, \dots, N. \tag{16}$$

The measure being Radon and finite are the required properties for the distributional derivative being compatible with the topology on Ω. Another equivalent characterisation is the following.

Theorem 5 (Characterization of BV)**.** *[4] $u \in [L^1(\Omega)]^N$ belongs also to $[BV(\Omega)]^N$ iff $V(u, \Omega) < \infty$. Where the variation V is defined to be*

$$V(u, \Omega) := \sup \left\{ \sum_{\alpha=1}^N \int_\Omega u^\alpha \ div\ \varphi^\alpha dx : \varphi \in [C_c^1(\Omega)]^{Nn}, \|\varphi\|_\infty \leq 1 \right\}. \tag{17}$$

Note that BV is a strict subspace of L^1 and a Banach space if we take the norm

$$\|u\|_{BV} := \int_\Omega |u| dx + V(u, \Omega). \tag{18}$$

We can see that $u \in L^1(\Omega) \setminus BV(\Omega)$ only if $V(u, \Omega) = \infty$.

With the aim of extending our integral operators J to functions on BV it is prudent to understand the distributional derivatives Du for functions in $u \in BV$.

Definition 9 (Hausdorff measure)**.** *Let $k \in [0, \infty)$, $\Omega \subset \mathbb{R}^n$. Then the k-dimensional Hausdorff measure of Ω is given by*

$$\mathcal{H}^k(\Omega) := \lim_{\delta \downarrow 0} \mathcal{H}_\delta^k(\Omega),$$

where for $0 < \delta \leq \infty$, $\mathcal{H}_\delta^k(\Omega)$ is defined by

$$\mathcal{H}_\delta^k(\Omega) := \pi^{\frac{k}{2}} \frac{\Gamma\left(1 + \frac{k}{2}\right)}{2^k} \inf \left\{ \sum_{i \in I} [diam(E_i)]^k : diam(E_i) < \delta, E \subset \cup_{i \in I} E_i \right\}, \tag{19}$$

for finite or countable covers $(E_i)_I$ of Ω.

Remark 8. *The definition above is technical, this is necessary as the Hausdorff measure can be seen as quantifying the k-dimensional area of a set with k not necessarily integer. The scaling in front of the infimum is to have the property that* $\mathcal{H}^n(\Omega) = \mathcal{L}^n(\Omega)$ *for all* $\Omega \subset \mathbb{R}^m$.

Any measure μ can be split into three parts, the absolutely continuous part μ^a, the purely atomic part μ^j and a diffuse singular part μ^c. Note that $W^{1,1}$ is the space of functions where the weak derivative only consists of μ^a. We say that $u \in BV(\Omega)$ is a jump function if $Du = D^j u$ and denote by J_u the set of jump discontinuities (or atoms of Du).

Theorem 6 (Du in BV)**.** *[4] Let* $u \in [BV(\Omega)]^N$. *Then* $Du \in [\mathcal{M}(\Omega)]^N$ *can be composed into its absolutely continuous part* $D^a u$, *its jump part* $D^j u$, *and its Cantor part* $D^c u$ *as*

$$Du = D^a u + D^j u + D^c u.$$

Additionally, these parts are given by

$$D^a u := \nabla u(x) \mathcal{L}^n \tag{20}$$

the approximate differential,

$$D^j u := Du|_{J_u}, \tag{21}$$

with J_u *the jump set of* u, *and*

$$D^c u := Du|_{\Omega \setminus S_u}, \tag{22}$$

with S_u *the discontinuity set.*

There are additional formulas for computation of $D^c u$ and $D^j u$ but as we are mainly focused on SBV (where $D^c u = 0$) we do not state them. The intuitive understanding of the different parts is as follows: $D^a u = \nabla u$ and non-zero for $x \in \Omega$ where we can take a proper limit of linear approximations. It is n-dimensional. $D^j u(x) = (u^+(x) - u^-(x))\mathcal{H}^{n-1}$ as value between the left and right limits towards x in the $n-1$ dimensional Hausdorff measure. The Cantor part is harder to understand intuitively and it leads to analytical difficulties since its dimensionality is somewhere in $(n-1, n)$. In variational optimalisation there is always a trade-off between the functional setting, where we decide which objects to consider and what convergence looks like, and the analytical side, where we want to do computations of limits and operators. The main difficulty for minimization in BV is not the functional side, but the fact that too few regularizers J depending on Du are l.s.c. due to the analytical difficulties with computing $D^c u$. This is why often a less general function space is chosen, where we have easier to check conditions for l.s.c. In the case of BV we use SBV.

Definition 10 (SBV)**.** *A function $u \in BV(\Omega)$ is a special function of bounded variation, denoted by $u \in SBV(\Omega)$ if $D^c u = 0$.*

Since SBV is a subset of L^1, the weak topology is defined in the same way. The following chain of inclusions is true.

Lemma 6 (Space inclusions)**.** *Let Ω be open. Then*

$$W^{1,1}(\Omega) \subset SBV(\Omega) \subset BV(\Omega) \subset L^1(\Omega).$$

Hence SBV is a useful extension of the Sobolev functions, where we still have a grasp on the distributional derivatives.

3 Properties of the Regularizer

In light of Theorem 1, we are interested in the properties of a regularizer $J(u_1, \dots, u_N)$ defined over functions $(u_1, \dots, u_N) \in \mathcal{B}$, with respect to some topology $\tau_{\mathcal{B}}$. We will assume that we have picked an admissible pair $(\mathcal{B}, \tau_{\mathcal{B}})$ as discussed in Section 2.3. We discuss the major case distinctions for the dimension of the domain/co-domain and introduce the framework for l.s.c. of integral operators. Next, we state the theorems that provide l.s.c. in $W^{1,p}, BV, SBV$ and their application domains. Then, we clearly define the possible types of convexity and explain pathways to prove quasiconvexity for specific integrands f. In Section 5, we will use these results together with Section 2 to prove or disprove well-posedness of the variational regularization of many integrands used in the literature.

3.1 Method

We wish to frame the regularisation in the form of Calculus of Variations and answer multiple questions about integral regularizers of the form

$$J(u_1, \dots, u_N) := \int_\Omega f(x, u_1, \dots, u_N, \nabla u_1, \dots, \nabla u_N) dx.$$

We have with $m_1, \dots, m_N, k_1, \dots, k_N \in \mathbb{N}$ as the dimensionality of respectively

$$u_1, \dots, u_N, \nabla u_1, \dots, \nabla u_N,$$

the general integrand function

$$f(x, u_1, \dots, u_N, s_1, \dots, s_N) : \Omega \times \mathbb{R}^{m_1 + \dots + m_N + k_1 + \dots + k_N}.$$

In practice we collect equal m-dimensional observed quantities $u_1(x), \dots, u_N(x)$ with x in some n-dimensional space $\Omega \subset \mathbb{R}^n$, so we can assume $m_1 = \dots = m_N = m$ such that $m_1 + \dots + m_N = Nm$. Note also that as $(s_1, \dots, s_N) := (\nabla u_1, \dots, \nabla u_N)$ this now implies that $k_1 = \dots = k_N = \mathrm{Dim}(\Omega) m := nm$. So $f : \Omega \times \mathbb{R}^{Nm} \times \mathbb{R}^{Nnm}$. Of course, this can also be written as $f : \Omega \times \mathbb{R}^{Nm(n+1)}$ with an appropriate mapping. For ease of notation, we assume $d = 1$ throughout. Note that for $m > 1 \in \mathbb{N}$, we can project down to $m = 1$ by considering each component in $u_i(x) \in \mathbb{R}^m, i = 1, \dots, N$ as a different quantity giving $u_i^1(x), \dots, u_i^m(x)$. From Corollary 2 we need to answer the following questions for existence of a minimizer in regards to J.

1. Is J non-negative and proper?

2. Is J coercive or is (K, J) mean coercive?

3. Is J $\tau_{\mathcal{B}}$ weakly lower semi-continuous?

When we require uniqueness, it comes either from injectivity of K or strict convexity of J via Lemma 5. Non-negativity, properness and (mean) coercivity can be checked relatively easily when a specific $f(\cdot)$ is given. One of the main problems is that for integrands f_s specifying structural similarity, (K, J) is not mean coercive. This is because weak convergence of $u_h \rightharpoonup u, v_h \rightharpoonup v$ does not imply $\langle u_h, v_h \rangle \rightharpoonup \langle u, v \rangle$. Most of the structural similarity measures f_s only consist of these types of cross-terms. The div-curl lemma (Theorem 22) and the theory of compensated compactness deals with integrands of this type, and we will discuss this in Section 6.1. For practical purposes, if we have a non-coercive structural similarity regularizer J_S, a mean coercivity term of the form

$$J_C(u_1 \ldots, u_N) := \sum_{i=1}^{N} \alpha_i \mathrm{TV}(u_i), \alpha_i > 0, \tag{23}$$

is added such that $J := J_S + J_C$ is mean coercive by construction. Using this, we do not need to worry about the coercivity of the sole structural similarity term. The third question however, has been an active object of study for a long time with still an incomplete theory. The lower semi-continuity of integrals in general Banach spaces is still an area of study, but there are partial answers in our case of vectorial integrands from $\mathbb{R}^{N \times n} \to \bar{\mathbb{R}}$ in function spaces $\mathcal{B}$. Based on the theory there is a case distinction.

(i) $N = 1$ or $n = 1$.

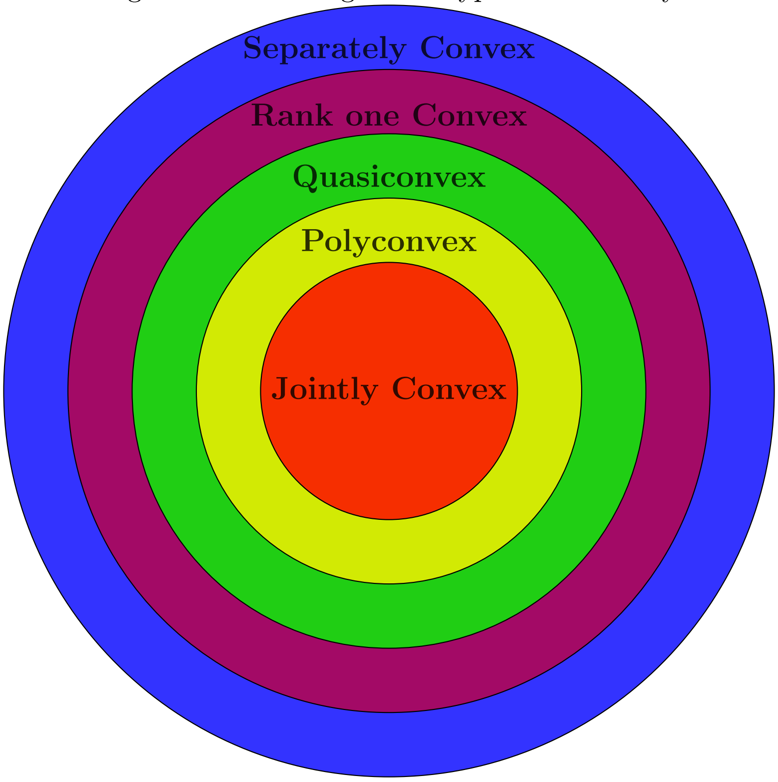
(ii) $N = n = 2$.

(iii) $N = 2, n > 2$.

(iv) $N \geq 3, n \geq 2$.

Taking $N = 1$ is just separate inversion. We can assume $N \geq 2$ since we are doing joint inversion. However most structural similarity metrics are defined only for $N = 2$ such that cases $(ii), (iii)$ are most relevant. For one-dimensional domain or co-domain (case (i)), it is a core fact of variational calculus that convexity of the integrand provides l.s.c. of the integral. For functions with vectorial input and vector-valued output, different notions of convexity exists (joint, separate, quasi, poly, rank one) and there are inherent links between them and l.s.c. In essence, for this more general class of functions, convexity is no longer necessary and the broader class of quasiconvex functions $f(\cdot)$ result in J that are l.s.c. The case distinction above over N, n is between the equivalence of the different types of convexity (defined in Section 3.2) which in general has the following implications shown in Figure 1.

Figure 1: Venn diagram of types of convexity



Whereas separate convexity plays no role in the l.s.c. of J, it is a necessary requirement for convergence of numerical solutions in most algorithms (See Section 7.3). Different types of convexity are necessary for l.s.c. in the different cases. The larger the phase space $\mathbb{R}^{N\times n}$, the easier it is for an integral to be l.s.c. and consequently a less strict notion of convexity is necessary. In case (i), joint convexity of f is necessary and sufficient, for (ii) it is still unknown if rank one convexity is necessary and sufficient. This is because there are strong indications that quasiconvexity and rank one convexity are equivalent in this case [18, 50, 63]. In cases (iii) and (iv) we have strict separation between all the different convexities and quasiconvexity is equivalent to l.s.c. The definitions and exact results are the topic of the next section.

3.2 Types of convexity

Let us define the other forms of convexity that play a role in the vectorial Calculus of Variations. We take our definitions from Dacorogna [18]

Definition 11 (Joint convexity)**.** *A function* $f : \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{+\infty\}$ *is said to be jointly convex if*

$$f(\lambda\xi + (1-\lambda)\eta) \leq \lambda f(\xi) + (1-\lambda)f(\eta),$$

for every $\lambda \in [0,1], \xi, \eta \in \mathbb{R}^{N\times n}$.

Definition 12 (Polyconvexity). *A function* $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ *is said to be polyconvex if there exists* $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R} \cup \{+\infty\}$ *jointly convex, such that*

$$f(\xi) = F(T(\xi)),$$

where $T : \mathbb{R}^{N\times n} \to \mathbb{R}^{\tau(n,N)}$ *is such that*

$$T(\xi) := (\xi, \operatorname{adj}_2 \xi, \cdots, \operatorname{adj}_{n\wedge N} \xi). \tag{24}$$

In the preceding definition, $\operatorname{adj}_s \xi$ *stands for the matrix of all* $s \times s$ *minors of the matrix* $\xi \in \mathbb{R}^{N\times n}$ *(See Definition 20),* $2 \le s \le n \wedge N := \min\{n, N\}$ *and*

$$\tau(n,N) := \sum_{s=1}^{n\wedge N} \sigma(s), \text{ where } \sigma(s) := \begin{pmatrix} N \\ s \end{pmatrix} \begin{pmatrix} n \\ s \end{pmatrix} = \frac{N!n!}{(s!)^2(N-s)!(n-s)!}. \tag{25}$$

Definition 13 (Quasiconvexity). *A Borel measurable and locally bounded function* $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ *is said to be quasiconvex if*

$$f(\xi) \le \frac{1}{\mathcal{L}^n(D)} \int_D f(\xi + \nabla\varphi(x))dx, \tag{26}$$

for every bounded open set $D \subset \mathbb{R}^n$, *for every* $\xi \in \mathbb{R}^{N\times n}$ *and for every* $\varphi \in W_0^{1,\infty}(D)$.

Definition 14 (Rank one convexity). *A function* $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{+\infty\}$ *is said to be rank one convex if*

$$f(\lambda\xi + (1-\lambda)\eta) \le \lambda f(\xi) + (1-\lambda) f(\eta)$$

for every $\lambda \in [0,1], \xi, \eta \in \mathbb{R}^{N\times n}$ *with* $\operatorname{rank}\{\xi - \eta\} \le 1$.

Definition 15 (Separate convexity). *A function* $f : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ *is said to be separately convex, or convex in each variable, if the function* $x_i \to f(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_m)$ *is jointly convex for every* $i = 1, \cdots, m$, *for every fixed* $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_m) \in \mathbb{R}^{m-1}$.

Remark 9. *Quasiconvexity is not defined for functions taking values at* ∞. *This is not a problem since in Theorem 12 we have assumed* f *to be Carathéodory.*

Joint convexity is the most basic and states that the value of the function f between two points is everywhere equal to or less than the linear interpolation of the endpoints values. Polyconvex functions are functions that are jointly convex from a certain point of view. Namely, the function is jointly convex when considering the new variable set $T(\xi)$ instead of ξ based on the minors of ξ. Quasiconvexity is not intuitive as Equation (26) is not straightforward. One can see this non-local condition as a generalisation of the idea that there is no way to add a function φ locally around a point ξ to decrease the average value around $f(\xi)$. However, rank one convexity can easily be characterized by saying that these are the functions that are jointly convex along all rank one directions in ξ.

The conditions for quasiconvexity are in practice very difficult to prove since we need to look at all the Lipschitz perturbations $\nabla\varphi$ over all possible domains D at all possible points ξ. There are results that make it a bit easier, as it is proven that we can fix $D = B_1$ the open ball of radius 1 [18] like in Theorem 17. Even so, proving quasiconvexity via the definition can only be done in specific easy cases. This is why rank one convexity and polyconvexity are often used to establish if a function is quasiconvex. In addition to Figure 1 above, we have the following relations.

Theorem 7 (Relations between convexities)**.** *[18] Let* $f : \mathbb{R}^{N\times n} \to \mathbb{R}$*, then*

$$\textit{jointly convex} \implies \textit{polyconvex} \implies \textit{quasiconvex} \implies \textit{rank one convex}\ .$$

With all counter-implications false for general $N, n \in \mathbb{N}$*. For* $\min\{N, n\} = 1$*,*

$$\textit{jointly convex} \iff \textit{polyconvex} \iff \textit{quasiconvex} \iff \textit{rank one convex}\ .$$

In cases (ii), (iii), (iv) [18, 3]

$$\textit{polyconvex} \not\Rightarrow \textit{jointly convex},$$

and

$$\textit{quasiconvex} \not\Rightarrow \textit{polyconvex}.$$

In cases (iii), (iv), [59]

$$\textit{rank one convex} \not\Rightarrow \textit{quasiconvex}.$$

It is a well-known open conjecture by Morrey [40] stating that in case (ii), the so-called planar case where $N = n = 2$, we have that rank one convexity and quasiconvexity are equivalent.

For functions in $C^2(\Omega)$, rank one convexity is equivalent to ellipticity of the Euler-Lagrange (E-L) equations. The E-L equations for a C^2 function f are elliptic if the Legendre-Hadamard (L-H) condition given by

$$\sum_{i,j=1}^{N} \sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f(\xi)}{\partial \xi^i_\alpha \partial \xi^i_\beta} \lambda^i \lambda^j \mu_\alpha \mu_\beta \geq 0, \tag{27}$$

is satisfied for all $\lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^n, \xi \in \mathbb{R}^{N\times n}$.

Lemma 7 (Legendre-Hadamard conditions)**.** *[18] Let $f : \mathbb{R}^{N\times n} \to \bar{\mathbb{R}} \in C^2(\Omega)$. Then f is rank one convex iff Equation* (27) *holds for all $\lambda \in \mathbb{R}^N, \mu \in \mathbb{R}^n, \xi \in \mathbb{R}^{N\times n}$.*

A general way to disprove quasiconvexity is then as follows. We find parameter values λ, μ, ξ for a counterexample of the Legendre-Hadamard conditions, which implies that the function is not rank one convex. Then as all quasiconvex functions are rank one convex, we immediately disprove it. For the cases (ii), (iii) with $N = 2$ and assuming that partial derivatives commute, the L-H conditions are given as

$$\sum_{\alpha,\beta=1}^{n} \frac{\partial^2 f(\xi)}{\partial\xi_\alpha^1 \partial\xi_\beta^1} \left(\lambda^1\right)^2 \mu_\alpha\mu_\beta + 2\frac{\partial^2 f(\xi)}{\partial\xi_\alpha^1 \partial\xi_\beta^2}\lambda^1\lambda^2\mu_\alpha\mu_\beta + \frac{\partial^2 f(\xi)}{\partial\xi_\alpha^2 \partial\xi_\beta^2} \left(\lambda^2\right)^2 \mu_\alpha\mu_\beta \geq 0. \tag{28}$$

For all $\lambda \in \mathbb{R}^2, \mu \in \mathbb{R}^n, \xi \in \mathbb{R}^{2\times n}$.

An additional result that will be used characterizes equivalence between rank one convexity, quasiconvexity and polyconvexity for quadratic forms.

Theorem 8 (Convexity for quadratic forms)**.** *[18] Let M be a symmetric matrix in $\mathbb{R}^{(N\times n)\times(N\times n)}$. Let*

$$f(\xi) := \langle M\xi; \xi\rangle,$$

where $\xi \in \mathbb{R}^{N\times n}$ and $\langle\cdot;\cdot\rangle$ denotes the scalar product in $\mathbb{R}^{N\times n}$. The following statements then hold.

1. *f is rank one convex if and only if f is quasiconvex.*

2. *If $N = 2$ or $n = 2$, then f polyconvex $\Leftrightarrow$ f quasiconvex $\Leftrightarrow$ f rank one convex.*

3. *If $N, n \geq 3$, then in general f rank one convex $\not\Rightarrow$ f polyconvex.*

So in all cases, a quadratic integrand is equivalently rank one convex and quasiconvex. Additionally, in our most prominent case (ii) also polyconvexity is equivalent.

3.3 Lower semi-continuity

We wish to find sufficient properties of integrands f for the functional J to be $\tau_{\mathcal{B}}$ weakly l.s.c. depending on the dimension n of Ω and the amount of observed quantities N. We first need some minimal measure-theoretical requirements on f such that we can integrate it against the Lebesgue measure.

Definition 16 (Normal and Carathéodory functions)**.** *[4] Let $f : \Omega \times \mathbb{R}^d \to \bar{\mathbb{R}}$ be a function. We say that f is normal if f is $\mathcal{L}^N \times \mathbb{B}(\mathbb{R}^d)$ measurable and $s \to f(x, s)$ is lower semi-continuous in $\mathbb{R}^d$ for $\mathcal{L}^N$ almost every $x \in \Omega$. We say that f is Carathéodory if f is real-valued, $\mathcal{L}^N \times \mathbb{B}(\mathbb{R}^d)$ measurable and $s \to f(x, s)$ is continuous in $\mathbb{R}^d$ for $\mathcal{L}^N$ almost every $x \in \Omega$.*

As structural quantifiers depend on the gradients of the functions $u_1, \ldots, u_N$ we do not state results for l.s.c. in $L^p(\Omega)$ spaces but immediately concern us with Sobolev spaces.

3.3.1 $W^{1,p}(\Omega)$

Sufficient but not necessary conditions for general N, n are given in the following classical result.

Theorem 9 (Ioffe)**.** *[4] Let $\Omega \subset \mathbb{R}^n$ an open and bounded set, $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^k \to [0, \infty]$ be a normal function. Assume that $z \to f(x, y, z)$ is convex in $\mathbb{R}^k$ for all $x \in \Omega$ and all $y \in \mathbb{R}^N$. Then*

$$\int_\Omega f(x, y, z)\ dx$$

is lower semi-continuous for $(y_h)_h \subset [L^1(\Omega)]^N$ strongly converging to y and $(z_h)_h \subset [L^1(\Omega)]^k$ weakly converging to z.

If we want to use this result with $y := (u_1, \ldots, u_N)$ and $z := (\nabla u_1, \ldots, \nabla u_N)$ we need to take $k = nN$. Note that the strong/weak convergence here is defined component-wise. We immediately state the most general result for the two parts $N = 1, n = 1$ of case (i) in a Sobolev space.

Theorem 10 (l.s.c. for case $n = 1$)**.** *[4, 12] Let $\Omega \subset \mathbb{R}$ an open interval, $N \in \mathbb{N}^+$. Let $p \in (1, \infty]$. Define*

$$J(u_1, \ldots, u_N) := \int_\Omega f(x, \vec{u}, \vec{\nabla} u).$$

If

- *f is normal and non-negative,*
- *f is jointly convex in the variables $\nabla u_1, \ldots, \nabla u_N$,*

then J is weakly l.s.c. on $W^{1,p}(\Omega)$.

Theorem 11 (l.s.c. for case $N = 1$)**.** *[16, 18] Let $\Omega \subset \mathbb{R}^n$ be open and bounded with boundary $\partial\Omega$ Lipschitz. Let $p \in (1, \infty]$. Let $f : \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ a function. Define*

$$J(u) := \int_\Omega f(x, u, \nabla u)$$

If

- *f is Carathéodory and bounded from below,*
- *The mapping $\xi \mapsto f(\cdot, \cdot, \xi)$ is jointly convex,*

then J is weakly l.s.c. on $W^{1,p}(\Omega)$.

Remark 10. *There is some nuance with the two different cases, where the topological requirements of the possible f are slightly different. Normal vs. Carathéodory and non-negative vs. bounded from below. This difference is only relevant when considering if and only if statements of the same results. In this thesis we are (almost) always in the less general case of non-negative Carathéodory functions and only care about sufficient properties, so we can forget about this nuance.*

These results are based on the structure of $W^{1,p}(\Omega)$ as weak convergence here implies strong $L^p(\Omega)$ convergence of the functions themselves with weak $L^p(\Omega)$ convergence of the gradients. For Sobolev spaces $W^{m,p}(\Omega)$ with higher order $m > 1 \in \mathbb{N}$ but still $n = 1$ or $N = 1$ we can use the Rellich-Kandrochov Sobolev embeddings given in Theorem 35 . From this theorem, regardless of dimension of a Lipschitz Ω, for $p \in [1, \infty), m > 0 \in \mathbb{N}$ we have a compact embedding of

$$W^{1+m,p}(\Omega) \hookrightarrow W^{1,p}(\Omega).$$

So for J to be weakly l.s.c. on $W^{1+m,p}(\Omega)$ we have exactly the same conditions as for $W^{1,p}(\Omega)$ as the weak convergence in $W^{1+m,p}$ comes naturally from the weak topology of $W^{1,p}$.

In a multi-dimensional domain Ω such that $n > 1$, joint convexity is no longer a necessary condition and there is a larger choice of regularization functionals that are l.s.c. We have informally the following (formally Theorem 12).

$$J \text{ weakly l.s.c.} \iff f \text{ Carathéodory and quasiconvex.}$$

Note that in addition to this weaker type of convexity, there is also a switch from more general normal functions to Carathéodry ones as from Theorem 10 to Theorem 11. Ideally, we would like to consider the normal functions which can also have values equal to infinity and only l.s.c. instead of full continuity of f is needed. However, if we consider normal functions instead the statement is no longer if and only if, quasiconvexity is necessary but no longer sufficient (Coming from the difficulty of quasiconvexity with values at ∞) [18].

Quasiconvexity is a fundamental notion because of Theorem 12 below, which provides l.s.c. for functions only dependent on the derivative. Note that throughout the report we often take $\xi := \nabla b = (\nabla u_1, \ldots, \nabla u_N) \in \mathbb{R}^{N\times n}$ as a (Jacobian) matrix. Then to cut down on excessive notation we write

$$|\xi| := \|\xi\| = \|\xi\|_2 = \|\xi\|_{\mathbb{R}^{Nn}} = \sqrt{\sum_{i,j}\left(\xi_i^j\right)^2}. \tag{29}$$

Remark 11. *Even further in the report we will also use the equivalent Schatten 2-norm notation $\|\xi\|_{l^p}$ defined in Definition 21.*

Definition 17 (Growth conditions)**.** *[18] Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ and $1 \le p \le \infty$. Then f is said to satisfy growth condition (C_p) if*

1. *(C_∞) When $p = \infty$*

$$|f(\xi)| \le \eta(|\xi|) \text{ for every } \xi \in \mathbb{R}^{N\times n},$$

   where η is a continuous and increasing function;

2. *(C_p) When $1 < p < \infty$*

$$-\alpha\left(1+|\xi|^q\right) \le f(\xi) \le \alpha\left(1+|\xi|^p\right) \text{ for every } \xi \in \mathbb{R}^{N\times n},$$

   for some $\alpha \ge 0, 1 \le q < p$;

3. *(C_1) When $p = 1$*

$$|f(\xi)| \le \alpha(1+|\xi|)$$

   for every $\xi \in \mathbb{R}^{N\times n}$, where $\alpha \ge 0$.

Theorem 12 (l.s.c. in $W^{1,p}$)**.** *[18] Let $p \in [1, \infty]$. Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be Carathéodory, quasiconvex and satisfying growth condition (C_p). Let $\Omega \subset \mathbb{R}^n$ be a bounded open set and*

$$J(b) := \int_\Omega f(\nabla u_1, \ldots, \nabla u_N)dx.$$

*Then J is weakly lower semi-continuous in $W^{1,p}(\Omega)$ (weak * lower semi-continuous if $p = \infty$), i.e.*

$$\liminf_{b_\nu \to b} J(b_\nu) \ge J(b).$$

Hence we have sufficient conditions for all cases when our Banach space $\mathcal{B}$ is assumed to be a Sobolev space. Note that there are also some minor conditions on Ω, namely it being bounded and open. There are results that also hold for unbounded Ω but the growth restrictions become more cumbersome in this case [18]. For Sobolev spaces with higher order, we can apply the same result.

3.3.2 $\mathcal{M}(\Omega)$

Unfortunately, the theory of l.s.c. of regularizers as integrals over some integrand function f where we take $\mathcal{B} = [\mathcal{M}(\Omega)]^N$ is rather limited. This has to do with the complicated ways that measures can be approximated by other measures. Namely, for a general measure μ we can weakly$-*$ approximate any part of it (be it absolutely continuous, atomic, or Cantor) by a sequence of any other part ($\mu_h^{ac}, \mu_h^{\text{atomic}}$ or μ_h^c) or any combination of them. The only concrete results are concerned with the special case $n = 1$ where Ω is one-dimensional, as here convexity and weak$-*$ convergence of these approximations are manageable. These are explored below.

From [4] we have the following result on the space of $\mathbb{R}^N$-valued finite Radon measures on Ω.

Theorem 13. *(Radon-Nikodym)[4] Let μ be a finite measure on Ω. Then there is a unique pair of $\mathbb{R}^N$-valued measures μ^a, μ^s such that $\mu^a << \mathcal{L}^1, \mu^s \perp \mathcal{L}^1$ and $\mu = \mu^a + \mu^s$. There is a unique function $f \in [L^1(\Omega)]^N$ such that $\mu^a = f\mathcal{L}^1$. This function f is called the density of μ wrt $\mathcal{L}^1$ (or Radon-Nikodym derivative) and is denoted by $\frac{d\mu^a}{d\mathcal{L}^1}$.*

The Radon-Nikodym derivative acts on measures as the usual derivative acts on functions and is the natural definition.

Theorem 14. *[4] Let $\Omega \subset \mathbb{R}$ be one-dimensional and open, $N \in \mathbb{N}$. Define $J : [\mathcal{M}(\Omega)]^N \to \mathbb{R}$ as*

$$J(\mu) := \int_\Omega f(\varphi_\mu)\, dx$$

where φ_μ is $\frac{d\mu^a}{d\mathcal{L}^1}$. Where $f : \mathbb{R}^N \to [0, \infty]$ is Borel. Then J is strongly l.s.c. iff f is l.s.c. Also J is weakly l.s.c. iff $f \neq \infty$, l.s.c., jointly convex and it has non-negative recession function. i.e.

$$f_\infty(x) := \lim_{t\to\infty} \frac{f(tx) - f(0)}{t} \geq 0, \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Remark 12. *The condition on the recession function f_∞ is quite weak as it says that eventually the function f is non-decreasing in every direction.*

3.3.3 $BV(\Omega)$

As discussed above, the complicated ways we can approximate measures dictate that weak l.s.c. of our regularization terms J can only be proved in easy settings. In the space $[\mathcal{M}(\Omega)]^N$ we only have Theorem 14 where we restrict to the special case $n = 1$. This result can be used to prove a similar result for functions in $[BV(\Omega)]^N$ as the weak derivatives of such functions are by definition in $[\mathcal{M}(\Omega)]^N$. As this is how l.s.c. is proved via the Direct method, there are no general existence results in the full multi-variate case for BV that are derived from this approach. It turns out that the only integrands $f(\cdot)$ for which we can get weak l.s.c. of J are the ones where we can regard our function as defined on a one-dimensional domain. Or equivalently, where our values of $f(\cdot)$ can be parameterised with only one variable.

This can be understood intuitively as follows. For a function $b \in [BV(\Omega)]^N, \Omega \subset \mathbb{R}^n$ we have $\nabla b \in [\mathcal{M}(\Omega)]^N$. As we only have a result in $[\mathcal{M}(D)]^N$ if $\text{Dim}(D) = 1$ we can understand l.s.c. in $[BV(\Omega)]^N$ if the corresponding $[\mathcal{M}(\Omega)]^N$ can be equivalently understood as $[\mathcal{M}(D)]^N$ with one-dimensional D. Hence there is some mapping ψ that reduces the information on n-dimensional Ω to a 1-dimensional D, say

$$\psi(x) : \Omega \to D.$$

Then if this ψ is well-behaved and $f(\psi(x))$, $\psi^{-1}(f(x))$ satisfy certain properties we can use our result on $[\mathcal{M}(D)]^N$ for $[BV(\Omega)]^N$. It turns out that if f is isotropic, so $\psi(x) = |x|$ this preserves l.s.c. of the integral functional J. The isotropy transforms our vectorial integrand into a function with essentially one-dimensional input domain.

Theorem 15 (l.s.c. in BV)**.** *[4] Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $f : [0,\infty) \to [0,\infty]$. Let $J : BV(\Omega) \to \mathbb{R}$ be given by*

$$J(b) := \int_\Omega f(|\nabla b|)dx.$$

Then J is sequentially weakly$-*$ l.s.c. in $BV(\Omega)$ iff f is increasing, l.s.c., and convex in the one variable.

Remark 13. *Formally in [4] the authors have an additional limit condition on the recession function f_∞, but since in their case $\beta \in [0,\infty]$ and the limit exists, this always holds.*

Note the similarity to Theorem 10.

3.3.4 $SBV(\Omega)$

The conditions necessary for compactness of a subspace $SBV' \subset SBV$ stated in Theorem 37 also provide enough regularity on the behaviour of D^a, D^j for an application of the Direct method and Theorem 10.

Definition 18 (SBV minorant)**.** *Let $K \subset \mathbb{R}^n$ be a compact set. A function $\varphi : K \times K \times \mathbb{R}^{N\times n} \to [0,\infty]$ is called a SBV minorant for some function f if it is jointly convex in the third variable, l.s.c., and increasing with super-linear growth for some $c > 0$. Also*

$$\varphi(i,j,p) \geq c|p|, \forall i,j \in K, i \neq j, p \in \mathbb{R}^{N\times n},$$

and

$$f(x, s, z) \geq \varphi(x, s, z),$$

for all (x, s, z).

Theorem 16 (Existence minimizer in SBV). *[4] Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded,* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0, \infty]$ *be normal and convex in the third variable. Additionally, there exists a SBV minorant* φ *for* f. *Then*

$$\min\{J(b) : b \in [SBV(\Omega)]^N, b(x) \in K \textit{ for } \mathcal{L}^n \textit{ a.e } x \in \Omega\},$$

has at least one solution.

Note the convexity requirement, but the weaker assumption that $b(x) \in K$ is not necessary for all $x \in \Omega$ such that we do not need uniformly bounded b, but only boundedness a.e. Theorem 16 is quite restrictive as φ has strong requirements and f must be jointly convex in the gradients. There is a result relaxing this convexity to quasiconvexity, but due to the more complicated gradients for SBV functions we need to introduce additional measure-theoretic conditions. Due to technical reasons [4, 18, 54] (L^q bounds on L^p oscillations of ∇u near jump sets J_u of functions $u \in SBV$) only results for functions f with super-linear growth (C_p with $p > 1$) are known for SBV. With the exception of the matrix-based generalisations of total variation (VTV, TNV, TSV), the regularizations that are used in joint inversion have super-linear growth. Note that this is a problem, as we can prove that there exists such a φ as necessary in Theorem 16 in the $f_{VTV}, f_{TNV}, f_{TSV}$ cases. For the case where there is super-linear growth we have the following.

Theorem 17 (l.s.c. in SBV). *[4] Let* $f : \Omega \times \mathbb{R}^N \times \mathbb{R}^{N\times n} \to [0, \infty)$ *be Carathéodory with*

$$c|\xi|^p \leq f(x, b, \xi) \leq a(x) + \psi(|b|)(1 + |\xi|^p), \tag{30}$$

for all $(x, u, \xi) \in B_1 \times \mathbb{R}^N \times \mathbb{R}^{N\times n}$ *for some* $p > 0, c > 0, a \in L^1(\Omega)$ *and an increasing function* $\psi : [0, \infty) \to [0, \infty)$. *Here* B_1 *is the Euclidean unit ball in* $\mathbb{R}^n$. *If* f *is quasiconvex with respect to* ξ *for all* $x \in \Omega, b \in \mathbb{R}^N$ *then*

$$J(b) := \int_\Omega f(x, b, \nabla b)$$

is weakly lower semi-continuous for sequences $b_h \in [SBV(\Omega)]^N$ *converging to* $b \in [SBV(\Omega)]^N$ *with Hausdorff measure* $\sup_h \mathcal{H}^{N-1}(J_{b_h}) < \infty$.

Remark 14. *For functions* f *only depending on the gradient* ξ *the upper bound in Equation* (30) *is equivalent to the upper bound of the* C_p *growth condition. Whereas* *when working in* $W^{m,p}$, *for* C_p *we can have negative* $q-$*growth with* $q < p$, *for functions in* SBV *the lower bound in the equation above is much more strict. However, applying Theorem 17 to*

$$f_\varepsilon(x, b, \xi) := \varepsilon|\xi|^r + f(x, b, \xi) \tag{31}$$

and then taking the limit $\varepsilon \downarrow 0$ *we get the equivalent lower bound*

$$0 \leq f(x, b, \xi).$$

So for non-negative functions f *only depending on the gradient, we have* $C_p \implies$ *Equation* (30) *with the given* p.

4 Specific Integrands

Now that the general framework for regularization with J as an integral with respect to some integrand function f has been established, we will take a look at the explicit forms of integrands that have been used in applications. One of the core aims of this thesis is to look at how joint inversion is used in different application domains and which specific form this takes. In light of this, an exhaustive literature review was performed, and consequently many different integrands have been found that are used in multiple disciplines that are adjacent to mathematics. To already have something in mind when we think about how to measure similarity, we first define all specific integrands under consideration. To motivate the introduction of these numerous integrands, we discuss in Section 4.2 the past results on how model fusion and structural similarity have been used before in medical and geoscientific imaging, and image enhancement. This discussion will underline why there are many different integrands as the desired (analytical, numerical, heuristic) properties of these functions f differ wildly in the applications. Afterwards, in Section 4.3, we prove or disprove the desired properties as discussed in Theorem 1 using the types of convexity discussed Section 3.2. Finally in Section 4.4 the integrands f that are quasiconvex for $n \in \mathbb{N}, N = 2$ are given generalised definitions for $N \in \mathbb{N}$. An overview and tabulation of the found properties is given in Table 1 at the end of the section.

4.1 Definition of integrands

The explicit integrands are only defined for $N = 2$. For ease of notation we take $\xi = (\xi^1, \dots, \xi^N) := (\nabla u_1, \dots, \nabla u_N)$. We omit J as it is assumed to be of the form

$$J(\nabla b) = J(\xi) = \int_\Omega f(\nabla u_1, \dots, \nabla u_N), \tag{32}$$

with integrand $f : \mathbb{R}^{N \times n} \to [0, \infty)$ and $\nabla b \in \mathcal{B}$. For the case where we have finite Radon measures $\mathcal{M}(\Omega)$, we will define our functions only when necessary in Section 5.2. This is because care must be taken with defining functions and derivatives of functions with measure-valued input. See Section 4.2 for the contexts in which the f below are used and some cursory notes on performance. As the specific forms for the integrands have been taken from the existing literature, the definitions below are all for $N = 2$. With exception of the matrix-norm based $f_{VTV}, f_{TNV}, f_{TSV}$ (defined below) and the new $f_{gCG}, f_{g\text{Nambu}}$.

We stress that although the forms of the functions can be quite different, the reasons behind the definition are largely the same. Namely, we aim to quantify the similarity of two n-dimensional vectors ξ^1, ξ^2. As these describe the change in value of our parameter fields u_1, u_2, similarity in $\mathbb{R}^n$ of the gradients $\nabla u_1, \nabla u_2$ is equivalent to structural similarity in the actual values. As there is no standard way to compare two vectors, the importance of different properties of these vectors lead us to define different quantifiers. For example, should we only look at the angle between the vectors or is the difference in magnitude also importance? Also, should we only do an element-wise comparison or is there some relation between different directions? Another possibility is regarding the whole ξ as a matrix and looking at the eigenvalues/vectors.

We can look at the (element-wise) difference between the gradients

$$f_{\mathrm{GD}}(\xi) := \sum_{i=1}^{n} |\xi_i^1 - \xi_i^2|^2, \tag{33}$$

or a variant of this called the matched difference

$$f_{\mathrm{mGD}}(\xi) := \min_{w \in \mathbb{R}^2} \|w_1 \xi^1 - \xi^2\|_2^2 + \|\xi^1 - w_2 \xi^2\|_2^2, \tag{34}$$

where an optimal ω-weighting between directions is included. Some integrands are based on the dot-product,

$$f_{\mathrm{DOT}}(\xi) := \langle \xi^1, \xi^2 \rangle^2, \tag{35}$$

such as the adapted dot product

$$f_{\mathrm{aDOT}}(\xi) := |\langle \xi^1, \xi^2 \rangle|, \tag{36}$$

or the normalised dot product

$$f_{\mathrm{nDOT}}(\xi) := \left| \left\langle \frac{\xi^1}{|\xi^1|}, \frac{\xi^2}{|\xi^2|} \right\rangle \right|^2. \tag{37}$$

We can also use the angle between vectors to measure similarity via the cosine similarity

$$f_{\cos}(\xi) := 1 - \cos(\theta) = 1 - \frac{\langle \xi^1, \xi^2 \rangle^2}{|\xi^1|^2 |\xi^2|^2}, \tag{38}$$

where θ is the angle between ξ^1 and ξ^2. Several can be fit in the general framework by Arridge et al. [25] of integrating over

$$f_{\varphi,\psi}(\xi) = \varphi(\psi(|\xi^1||\xi^2|)) - \psi(|\langle \xi^1, \xi^2 \rangle|), \tag{39}$$

with arbitrary strictly increasing functions $\varphi, \psi : [0, \infty) \to [0, \infty)$. In particular, linear parallel level sets:

$$f_{\mathrm{LP}}(\xi) := |\xi^1||\xi^2| - \langle \xi^1, \xi^2 \rangle, \tag{40}$$

quadratic parallel level sets:

$$f_{\mathrm{QP}}(\xi) := \sqrt{1 + |\xi^1|^2 |\xi^2|^2} - \langle \xi^1, \xi^2 \rangle^2, \tag{41}$$

and the cross-gradient:

$$f_{CG}(\xi) := |\xi^1 \times \xi^2|^2 = |\xi^1|^2|\xi^2|^2 - \langle \xi^1, \xi^2 \rangle^2. \tag{42}$$

A normalized cross-gradient is also sometimes used as

$$f_{nCG}(\xi) := \left| \frac{\xi^1}{|\xi^1|} \times \frac{\xi^2}{|\xi^2|} \right|^2. \tag{43}$$

However, if we work out the brackets, we will find exactly the same form as f_{cos}. So throughout the report we take $f_{cos} = f_{nCG}$ and only worry about the first form. An adapted version of f_{CG} is called the Nambu functional.

$$f_{\text{Nambu}}(\xi) := |\xi^1 \times \xi^2| = \sqrt{\|\xi^1\|^2\|\xi^2\|^2 - \langle \xi^1, \xi^2 \rangle^2}. \tag{44}$$

As we will prove later in this section, the only functions that result in a well-posed problem via the Direct method (as they are quasiconvex) are f_{CG}, f_{Nambu}, and f_{GD}. We will introduce generalised versions of f_{CG} and f_{Nambu} for general N that remain quasiconvex. For $N > 2 \in \mathbb{N}$, joint regularisers are mostly used in image processing and are generalised vectorial total variation norms (see [37] for an excellent overview of the properties). We take a look at vectorial/joint total variation (VTV/JTV), which is sometimes also called the total Frobenius variation (TFV)

$$f_{\text{VTV}}(\xi) = f_{\text{TFV}}(\xi) := \sqrt{\sum_{i=1}^{N} |\xi^i|^2}, \tag{45}$$

total nuclear variation (TNV),

$$f_{\text{TNV}}(\xi) := \|\xi\|_*, \tag{46}$$

where with singular values of ξ given by $\sigma_1(\xi), \dots,$

$$\|\xi\|_* := \sum_i |\sigma_i(\xi)|, \tag{47}$$

the nuclear norm of matrix ξ. Finally, we also regard total spectral variation (TSV)

$$f_{\text{TSV}}(\xi) := \|\xi\|_\infty, \tag{48}$$

where

$$\|\xi\|_\infty := \max_i \{\sigma_i(\xi)\}, \tag{49}$$

the spectral norm of matrix ξ.

4.2 Joint inversion in applications

Incorporating structural similarity into regularizers is commonplace in many fields. We can roughly put the applications in three groups; medical imaging, geoscientific reconnaissance, and image enhancement. The method of incorporating structure can also be done in three different ways; first solving one inverse problem and then using the solution to directly influence the other inverse problem(s) (a priori information / Model Fusion (MF)), using information gained from solving one inverse problem to adjust Bayesian priors for the other inverse problem(s) (Mutual Information (MI)), and the variational coupling of inverse problems via a joint regularization. In practice, this categorization is too rigid as results and ideas in different methodologies influence each-other. Although we are expressively only interested in the variational method, this is why we discuss results of all methods.

Although most of the time the medical and geoscientific problems are solving for 2- or 3-dimensional images there is a distinct difference between them and image enhancement. As the image enhancement inverse problem takes as input an image d that is corrupted by an operator K^{-1} and solves for the uncorrupted image b. In medical and geoscientific applications, the inputs are physical parameter fields such as electrical conductivity or DC resistivity and the inverse problem is for solving an image of the physical system b. We discuss below which integrands f have been used in the variational approach by researchers and how they compared based on their practicality and usefulness.

4.2.1 Medical Imaging

Structural information is mostly assimilated via an anatomical prior based on MRI scans, as the image quality of MRI scans are good but the relevant functional properties can best be captured via different image methods.

When using MF, MRI priors in the form of undersampled Fourier data have been used in 2-dimensional electrical impedance tomography (EIT) [41], positron emission tomography (PET) [24, 43], and magnetic particle imaging (MPI) [7] reconstruction. Different methods using the MRI information are used based on the form of the imaging data to be inverted. Defining the finite element mesh [41], introducing a conditional regularization based on dot-product and cross-gradient functions [24] or weighted TV [41] based on the MRI signal strength have all been shown to improve the image reconstruction.

From the statistical perspective (MI), Vunckx et al. [64] compares different statistical methods for combined MRI-PET scans with Bowsher's prior [10] performing the best. Ehrhardt et al [24] also compares conditional regularization to Bowsher's prior, with the structural regularization outperforming Bowsher. Mutual information is also used for multi-spectral MRI [8]. Since the variational joint inversion that our results are about can be regarded as an indirect form of structural priors, this performance indicates that further gains can be made.

In this sense, different integrands f_{LP}, f_{QP}, f_{VTV} have been compared when doing joint inversion of PET functional information with MRI anatomical information in [25]. In this paper, joint inversion improves on separate inversion with f_{LP} performing best, followed by f_{QP}, f_{VTV}, and lastly separate TV.

4.2.2 Geosciences

Gallardo and Meju [33] is an excellent source of using different methods based on joint structure in geoscience. Instead of MF, in the geoscientific literature the term structure-coupled joint inversion of multi-physics data is used, the variational approach is called structural cooperative inversion [33]. Model Fusion and Mutual Information approaches are difficult for geoscientific applications [34] because of the different scales involved ($10^{-3} - 10^{6}$ meters) and the lack of understanding of the exact petrophysics. There is also a lack of good computational techniques for solving the inverse problems when framed in a MF or MI language due to the highly non-linear and non-convex functions involved.

For MF, constraints based on the cross-gradient [30, 32, 1], or similarity of the Hessians [56] are introduced. Noteworthy is [31] for pioneering MF with cross-gradient (CG) constraints with $N = 4$. Where-after further gains where made using real world data and quantitative comparison to the ground truth [60, 47, 48]. Incorporating structure using CG constraints leads to better imaging.

In the variational framework, $f_{DOT}, f_{nDOT}, f_{CG}, f_{nCG}, f_{GD}$ and $\frac{1}{f_{cos}}$ are all used in geoscience [33]. Originally, [35] used a smooth threshold operator where we use a step function to either include a cost when the structure is different or no cost when the structure is similar. Concrete results come from [34], where the authors compare f_{CG} and f_{VTV} for a DC resistivity and borehole tomography problem. The variational approach performs better than separate inversion with f_{CG} and f_{VTV} behaving similarly. Also, [38] shows that using f_{CG} improves joint seismic and EM imaging.

4.2.3 Image Enhancement

In image enhancement, total variation regularization is of standard use as it has been shown to significantly improve image quality. This is because TV regularization leads to sharper edges and more pronounced structures in the image, and omitting it leads to smoothed edges which visually look fuzzy. For multi-channel ($N > 1$) data such as RGB or spectral imagery multiple generalizations for TV regularization can be defined. Holt [37] compares the different generalizations $f_{VTV}/f_{TFV}, f_{TNV}, f_{TSV}$ that fit within our variational framework to each other and to so-called l^1 and global l^2 (Color TV in [21]) pooling.

$$J_{l^1}(\xi) = \sum_{i=1}^{N} TV[\xi^i], \tag{50}$$

$$J_{l^2}(\xi) = \sqrt{\sum_{i=1}^{N} TV[\xi^i]^2}. \tag{51}$$

Only f_{VTV} has all desired theoretical properties for a joint structure regularizer in multi-channel images and performs best quantitatively for denoising as well. Followed by f_{VTV} and f_{TSV}, with J_{l^1}, J_{l^2} lacking the most theoretical properties and having the worst quantitative results. Ehrhardt et al [21] compares f_{Nambu}, f_{LP}, and J_{l^2} to a non-structure based state-of-the-art approach called non-local means. For denoising, the methods perform similarly for low noise with the structure-based approaches performing better as noise increases. For demoisaicking, J_{l^2} is worst overall with f_{LP} also outperforming f_{Nambu} at high noise level.

Remark 15. *In a synthetic setting, Scherders [57] compares different methods for structural regularization with computing inverse acoustic and EM wave fields. With applications in non-destructive material testing, bio-medics, and geophysical exploration. f_{GD} and f_{cos} are compared with separate inversion, where improvement in EM but not in acoustic data was found.*

4.3 Properties of integrands

We consider different structural similarity integrands put forth in the literature and consider if we can apply Corollary 2. For a given integrand f, we need non-negativity, properness, a C_p growth condition and quasiconvexity. Coercivity or Lemma 3 is also necessary, and we will prove it when possible. Else we will assume our inclusion of the coercivity regularizer J_c as defined in Equation (23). The specific p in the growth condition then tells us which Sobolev space we can consider in the variational problem. The focus is here on the Sobolev spaces as the well-posedness in other spaces only has stricter requirements. For the other considered function spaces $\mathcal{B}$, we refer to Section 2 giving us the topology for which the problem has a minimizer after establishing well-posedness in $W^{1,p}$. Note that properness and non-negativity both are automatically satisfied by our assumption on the form of J in Equation (32). However, the non-negativity of the integrand $f(\cdot)$ such that it is admissible in this assumption is sometimes not directly evident, a proof is then given. In Section 5, all assumptions and the precise settings where the inverse problem is well-posed is described for each quasiconvex integrand f. An overview can be found in Table 1. Note that we abuse notation and use $|\cdot|$ as Euclidean norm on $\mathbb{R}^n, \mathbb{R}^{Nn}$ and $\mathbb{R}^N$ interchangeably based on the length of the argument inside.

4.3.1 Gradient Difference

Lemma 8. *The function defined as*

$$f_{GD}(\xi) := \sum_{i=1}^{n} \left|\xi_i^1 - \xi_i^2\right|^2 = \sum_{i=1}^{n} (\xi_i^1)^2 + (\xi_i^2)^2 - 2\xi_i^1\xi_i^2,$$

satisfies C_2, is polyconvex and separately convex for $n \in \mathbb{N}$.

Proof. Since f_{GD} is a sum of squares, it is non-negative. The lower bound C_p is then trivially true for any $\alpha \geq 0, q \geq 1$. For the upper bound, the highest order terms in f_{GD} are of order 2, hence

$$f_{GD}(\xi) \leq \sum_{j=1,2} \sum_{i=1}^{n} \left(\xi_i^j\right)^2 = |\xi|^2.$$

Take $\alpha = 1$ and we have the upper bound such that f_{GD} satisfies C_2.
f_{GD} is a quadratic form with a corresponding symmetric matrix. We can apply Theorem 8. Take the specific ordering $\xi = (\xi_1^1, \xi_2^1, \dots, \xi_n^2, \xi_n^2) \in \mathbb{R}^{2\times 2n}$. Then the symmetric matrix M is given by

$$M := \begin{pmatrix} 1 & -1 & 0 & 0 & \dots & 0 \\ -1 & 1 & -1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 0 & -1 & 1 \end{pmatrix}$$

with ones on the main diagonal and -1's on each lower and upper diagonal. Then $f_{GD}(\xi) = \langle M\xi; \xi\rangle$. From Theorem 8, rank one convexity is equivalent to quasiconvexity for all (N, n) and poly-convexity for $N = 2$ or $n = 2$. We have assumed $N = 2$ implicitly. We proof rank one convexity via the Legendre-Hadamard conditions, which is allowed since $f_{GD} \in C^2$. We compute for $\alpha, \beta = 1, \dots, n$

$$\frac{\partial^2 f}{\partial \xi_\beta^2 \partial \xi_\alpha^1} = \begin{cases} -2 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

$$\frac{\partial^2 f}{\partial \xi^1_\alpha \partial \xi^1_\beta} = \begin{cases} 2 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

$$\frac{\partial^2 f}{\partial \xi^2_\alpha \partial \xi^2_\beta} = \begin{cases} 2 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}$$

Summing the terms we can simplify our sum over $\alpha, \beta = 1, \ldots, n$ to a sum over $\alpha = 1, \ldots, n$ since the cross-contributions (where $\alpha \neq \beta$) are 0. Then the L-H condition is

$$\sum_{\alpha=1}^{n} \left(2(\lambda^1)^2 - 4\lambda^1\lambda^2 + 2(\lambda^2)^2\right) \mu_\alpha^2 = \sum_{i=1}^{n} 2(\lambda^1 - \lambda^2)^2 \mu_\alpha^2 \geq 0$$

for all $\lambda \in \mathbb{R}^2, \mu \in \mathbb{R}^n$. This is true since every term is a product of squares, so our function is rank one convex via Lemma 7. From Theorem 8 polyconvexity follows since we have $N = 2$.
Polyconvexity implies separate convexity so the lemma follows.

□

4.3.2 mGD

Lemma 9. *The function defined as*

$$f_{mGD}(\xi) := \min_{w \in \mathbb{R}^2} \|w_1\xi^1 - \xi^2\|^2 + \|\xi^1 - w_2\xi^2\|^2,$$

does not satisfy any C_p, is not rank one convex and not separately convex for $n \in \mathbb{N}$.

We first rewrite f_{mGD} in a more explicit form.

$$f_{mGD}(\xi) = \min_{\omega \in \mathbb{R}^2} \sum_{i=1}^{n} (\omega_1\xi^1_i - \xi^2_i)^2 + \sum_{i=1}^{n} (\xi^1_i - \omega_2\xi^2_i)^2$$

$$= \min_{\omega \in \mathbb{R}^2} \sum_{i=1}^{n} (\omega_1\xi^1_i)^2 + (\xi^2_i)^2 - 2\omega_1\xi^1_i\xi^2_i + (\xi^1_i)^2 + (\omega_2\xi^2_i)^2 - 2\omega_2\xi^1_i\xi^2_i$$

$$= \min_{\omega \in \mathbb{R}^2} \sum_{i=1}^{n} (\omega_1^2 + 1)(\xi^1_i)^2 + (\omega_2^2 + 1)(\xi^2_i)^2 - 2\xi^1_i\xi^2_i(\omega_1 + \omega_2)$$

$$= \min_{\omega \in \mathbb{R}^2} (\omega_1^2 + 1)|\xi^1|^2 + (\omega_2^2 + 1)|\xi^2|^2 - 2(\omega_1 + \omega_2) \sum_{i=1}^{n} \xi^1_i\xi^2_i.$$

Proof. For $n \in \mathbb{N}$, we differentiate the argument with respect to $\omega_j, j = 1, 2$.

$$\frac{\partial}{\partial \omega_j} f_{mGD}(\xi) = 2\omega_j \sum_{i=1}^{n} (\xi^j_i)^2 - 2 \sum_{i=1}^{n} \xi^1_i\xi^2_i.$$

Equating $\frac{\partial}{\partial \omega_j} f_{mGD}(\xi)$ to zero for non-zero ξ^j gives

$$\omega_j = \frac{\sum_{i=1}^n \xi_i^1 \xi_i^2}{|\xi^j|^2}.$$

With $\omega_j \in \mathbb{R}$ arbitrary if $\xi^j = \vec{0}$. The critical points are at $(\omega_1, 0), \omega_1 \in \mathbb{R}$ if $\xi^1 = \vec{0}$, at $(0, \omega_2), \omega_2 \in \mathbb{R}$ if $\xi^2 = \vec{0}$ and at $\left(\frac{\sum_{i=1}^n \xi_i^1 \xi_i^2}{|\xi^1|^2}, \frac{\sum_{i=1}^n \xi_i^1 \xi_i^2}{|\xi^2|^2}\right)$ else. Then the value of the function at the special cases $\xi^j = \vec{0}$ is given by

$$f_{mGD}(\xi) = \begin{cases} |\xi^2|^2 \text{ if } |\xi^1| = 0, |\xi^2| \neq 0 \\ |\xi^1|^2 \text{ if } |\xi^1| \neq 0, |\xi^2| = 0 \end{cases}$$

and for the non-special cases equal to

$$f_{mGD}(\xi) = \sum_{k=1}^n (\xi_k^1)^2 + (\xi_k^2)^2 - \frac{\left(\sum_{i=1}^n \xi_i^1 \xi_i^2\right)^2}{|\xi^1|^2} - \frac{\left(\sum_{i=1}^n \xi_i^1 \xi_i^2\right)^2}{|\xi^2|^2}.$$

Equivalently for the non-special cases,

$$f_{mGD}(\xi) = |\xi|^2 - \langle \xi^1, \xi^2 \rangle^2 \cdot \left(\frac{1}{|\xi^1|^2} + \frac{1}{|\xi^2|^2} \right).$$

We have a counter-example for rank one convexity using an interpolation point in the n-dimensional subspace where $|\xi^j|^2 = 0$. Let

$$\lambda = \frac{1}{2}, \xi = \begin{pmatrix} 1 & 0 & \dots \\ 1 & 0 & \dots \end{pmatrix}, \eta = \begin{pmatrix} 1 & 0 & \dots \\ -1 & 0 & \dots \end{pmatrix}.$$

Then $\xi - \eta$ is a rank one matrix. Now since $\xi^1, \xi^2, \eta^1, \eta^2 \neq 0$,

$$\frac{1}{2}\left(2 - 1^2 \left(\frac{1}{1} + \frac{1}{1}\right)\right) + \frac{1}{2}\left(2 - (-1)^2 \left(\frac{1}{1} + \frac{1}{1}\right)\right) = 0.$$

However, the component $(\lambda\xi + (1-\lambda)\eta)^2 = \vec{0}$ such that we are in the special case and

$$f_{mGD}(\lambda\xi + (1-\lambda)\eta) = 1^2 = 1.$$

Which is the reverse inequality necessary for rank one convexity.

This function is not separately convex, there is complete symmetry in ξ^1 and ξ^2, without loss of generality we fix all $\xi_i^2, i = 1, \dots, n$, $\xi_i^1, i \neq j$ for some $j \in \{1, \dots, n\}$. We then have a one argument function $\bar{f}_{mGD}$ as

$$\bar{f}_{mGD}(\xi_j^1) = |\xi^2|^2 + \sum_{i \neq j} (\xi_i^1)^2 + (\xi_j^1)^2 - \left(\sum_{i \neq j} \xi_i^1 \xi_i^2 + \xi_j^2 \xi_j^1 \right)^2 \left(\frac{1}{\sum_{i \neq j} (\xi_i^1)^2 + (\xi_j^1)^2} + \frac{1}{|\xi^2|^2} \right).$$

As counterexample we fix $\xi_j^2 = 1, \xi_i^2 = 0, i \neq j$, then for some $k \neq j \in \{1, \dots, n\}$ fix $\xi_k^1 = 1, \xi_i^1 = 0, i \neq j, k$. Then

$$\bar{f}_{mGD}(x) = 1^2 + 1^2 + x^2 - (0 + x)^2 \left(\frac{1}{1 + x^2} + \frac{1}{1} \right) = 2 - \frac{x^2}{1 + x^2}.$$

This is not a convex function, from $\lambda = \frac{1}{2}, x_1 = -1, x_2 = 1$

$$\frac{1}{2}\bar{f}_{mGD}(-1) + \frac{1}{2}\bar{f}_{mGD}(1) = (2 - \frac{1}{2}) < 2 = \bar{f}_{mGD}(0).$$

Due to asymptotic behaviour at small $|\xi^j|$ values, this function does not satisfy any growth condition $C_p, p \in [1, \infty]$ as our function is unbounded in the directions where $\lim_{|\xi^j| \to 0} f_{mGD}(\xi)$. □

4.3.3 Dot, aDOT, nDOT

Heuristically, using the dot-product in regularization makes sense as it quantifies both the size of the angle and the magnitudes of two n-dimensional vectors. This is why it is disappointing that, as Lemma 10 shows, solving the problem in Corollary 2 via the Direct method in Sobolev spaces is not guaranteed to be well-posed for regularizations based on the dot-product. However, in particular for f_{DOT} and f_{aDOT}, there is still some hope as described in Section 6.1. There we find that under some additional (minor) functional analytical assumptions on the solution space, variational minimization problems having a dot-product regularization term are well-posed.

For now, we provide proofs for the properties in Table 1. We first state a more general result under which both f_{DOT} and f_{aDOT} fall. It tells us that any sensible transformation h of the dot-product can never be a rank one convex function.

Lemma 10. *Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{\infty\}$ be a function given by*

$$f(\xi) = h(\langle \xi^1, \xi^2 \rangle) = h\left(\sum_{i=1}^{n} \xi_i^1 \xi_i^2\right),$$

for some $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$. Additionally, assume that there exists some $z \neq 0 \in \mathbb{R}$ such that

$$h(z) > h(0).$$

Then f is not rank one convex.

This is the simplest version of the statement, a more general version is given by the following corollary.

Corollary 3. *Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{\infty\}$ be a function given by*

$$f(\xi) = h\left(a \sum_{i=1}^{n} g\left(\xi_i^1 \xi_i^2\right)\right) + c,$$

with $h : \mathbb{R} \to \mathbb{R}, a \neq 0 \in \mathbb{R}, c \in \mathbb{R}$*. Assume* $g : \mathbb{R} \to \mathbb{R}$ *with for some* $z \neq 0$ $g(z) > g(0)$ *and*

$$h\left(a\sum_{i=1}^{n} g(0)\right) < h\left(a\sum_{i=1}^{n} g(0) + ag(z)\right).$$

Then f *is not rank one convex.*

The proof of this version follows straightforwardly from the proof of Lemma 10 with the additional assumptions.

Proof. We can prove that f is not rank one convex by providing a counterexample. Let

$$\lambda = \frac{1}{2}, \xi = \begin{pmatrix} x & 0 & \dots \\ 0 & 0 & \dots \end{pmatrix}, \eta = \begin{pmatrix} 0 & 0 & \dots \\ y & 0 & \dots \end{pmatrix}, x, y \neq 0 \in \mathbb{R}.$$

Then $\operatorname{rank}(\xi - \eta) = 1$. Then

$$f(\lambda\xi + (1-\lambda)\eta) = h\left(\sum_{i=2}^{n} 0 + \frac{xy}{4}\right),$$

and

$$\lambda f(\xi) + (1-\lambda) f(\eta) = h\left(\sum_{i=1}^{n} 0\right).$$

Hence for this to be a counterexample the inequality below must be true for some $x, y \neq 0 \in \mathbb{R}$.

$$h\left(\frac{xy}{4}\right) \geq h(0).$$

We can now retroactively choose x, y such that $z = \frac{xy}{4}$. Then our assumption on h provides this directly. □

Lemma 11. *The function defined as*

$$f_{DOT}(\xi) := \langle \xi^1, \xi^2 \rangle^2,$$

satisfies C_4*, is not rank one convex and is separately convex for* $n \in \mathbb{N}$*.*

Proof. f_{DOT} satisfies C_4 as it is non-negative (such that any lower bound is satisfied) and using Jensen's inequality

$$f_{DOT}(\xi) := \left(\sum_{i=1}^{n} \xi_i^1 \xi_i^2\right)^2 \leq \sum_{i=1}^{n} \left(\xi_i^1\right)^2 \left(\xi_i^2\right)^2 \leq \sum_{i=1}^{n} \max(\left(\xi_i^1\right)^4, \left(\xi_i^2\right)^4) \leq \sum_{i=1}^{n} \left(\xi_i^1\right)^4 + \sum_{i=1}^{n} \left(\xi_i^2\right)^4$$

$$\leq \left(\sum_{i=1}^{n} (\xi_i^1)^2\right)^2 + \left(\sum_{i=1}^{n} (\xi_i^2)^2\right)^2 \leq \left(\sum_{i=1}^{n} (\xi_i^1)^2 + \sum_{i=1}^{n} (\xi_i^2)^2\right)^2 = |\xi|^4,$$

such that C_4 holds.

Although a proof via the Legendre-Hadamard conditions is possible, we can also simply choose $h(x) = x^2$ in Lemma 10 from which non rank one convexity follows immediately. This function is separately convex, there is complete symmetry in ξ^1 and ξ^2, without loss of generality we fix all $\xi_i^2, i = 1, \dots, n$, $\xi_i^1, i \neq j$ for some $j \in \{1, \dots, n\}$. We then have a one argument function $\bar{f}_{DOT}$ as

$$\bar{f}_{DOT}(\xi_j^1) = \left(\sum_{i \neq j} \xi_i^1 \xi_i^2 + \xi_j^2 \xi_j^1 \right)^2 .$$

This is a second degree polynomial in ξ_j^1 with a positive second order coefficient $(\xi_j^2)^2$. This is a convex function. □

Lemma 12. *The function defined as*

$$f_{aDOT}(\xi) := \left| \langle \xi^1, \xi^2 \rangle \right| ,$$

satisfies C_2, is not rank one convex and is separately convex for $n \in \mathbb{N}$.

Proof. From the absolute value, we have non-negativity of f_{aDOT} such that any lower bound for C_p is satisfied. As established before in the proof of Lemma 11, we have $\langle \xi^1, \xi^2 \rangle = \mathcal{O}\left(|\xi|^2\right)$. Such that C_2 is satisfied.

We apply Lemma 10 with $h(x) = |x|$ to disprove rank one convexity of f_{aDOT}.

This function is separately convex, there is complete symmetry in ξ^1 and ξ^2, without loss of generality we fix all $\xi_i^2, i = 1, \dots, n$, $\xi_i^1, i \neq j$ for some $j \in \{1, \dots, n\}$. We then have a one argument function $\bar{f}_{aDOT}$ as

$$\bar{f}_{aDOT}(\xi_j^1) = \left| \sum_{i \neq j} \xi_i^1 \xi_i^2 + \xi_j^2 \xi_j^1 \right| .$$

This is a linear function and hence is convex. □

We generalize Lemma 10 to include normalization.

Corollary 4. *Let $f : \mathbb{R}^{N \times n} \to \mathbb{R} \cup \{\infty\}$ be a function given by*

$$f(\xi) = h\left(\frac{\langle \xi^1, \xi^2 \rangle}{|\xi^1||\xi^2|} \right) = h\left(\frac{1}{|\xi^1||\xi^2|} \sum_{i=1}^{n} \xi_i^1 \xi_i^2 \right),$$

for some $h : \mathbb{R} \to \mathbb{R} \cup \{\infty\}$ with $h(z) > h(0)$ for some $z \in [-1, 1]$. Then f is not rank one convex.

Proof. We can prove that f is not rank one convex by providing a counterexample. Let

$$\lambda = \frac{1}{2}, \xi = \begin{pmatrix} x & 0 & 0 & \dots \\ 0 & y & 0 & \dots \end{pmatrix}, \eta = \begin{pmatrix} 0 & x & 0 & \dots \\ y & 0 & 0 & \dots \end{pmatrix}, x, y \neq 0 \in \mathbb{R}.$$

Then $\operatorname{rank}(\xi - \eta) = 1$. Then

$$f\left(\frac{1}{2}\xi + \frac{1}{2}\eta\right) = h\left(\frac{1}{|(\frac{\xi}{2} + \frac{\eta}{2})^1||(\frac{\xi}{2} + \frac{\eta}{2})^2|} \sum_{i=1}^{n} \left(\frac{\xi}{2} + \frac{\eta}{2}\right)^1_i \left(\frac{\xi}{2} + \frac{\eta}{2}\right)^2_i\right)$$

$$= h\left(\frac{1}{\sqrt{\frac{x^2}{4} + \frac{y^2}{4}}\sqrt{\frac{x^2}{4} + \frac{y^2}{4}}}\left(\frac{xy}{4} + \frac{xy}{4}\right)\right) = h\left(\frac{\frac{xy}{2}}{\frac{x^2+y^2}{4}}\right) = h\left(\frac{2xy}{x^2 + y^2}\right)$$

and

$$\frac{1}{2}f(\xi) + \frac{1}{2}f(\eta) = \frac{1}{2}h\left(\frac{1}{2 \cdot 2} \cdot (0)\right) + \frac{1}{2}h\left(\frac{1}{2 \cdot 2} \cdot (0)\right) = h(0).$$

Then our assumption on h provides this directly as the range of $\frac{2xy}{x^2+y^2}$ is $[-1, 1]$. As we can pick x, y such that this fraction is equal to z. □

Lemma 13. *The function defined as*

$$f_{nDOT}(\xi) := \left\langle \frac{\xi^1}{|\xi^1|}, \frac{\xi^2}{|\xi^2|} \right\rangle^2 = \left(\frac{1}{|\xi^1||\xi^2|} \sum_{i=1}^{n} \xi_i^1 \xi_i^2\right)^2$$

satisfies C_1, is not rank one convex and not separately convex for $n \in \mathbb{N}$.

Proof. The function is non-negative since our last operation on the argument is a square. Hence any lower bound in C_p is satisfied. For the upper bound, we look at the orders (in $|\xi|$) of the terms in f_{nDOT}. We have $\langle \xi^1, \xi^2 \rangle = \mathcal{O}(|\xi|^2)$ and $|\xi^1| \cdot |\xi^2| = \mathcal{O}(|\xi^1|) \cdot \mathcal{O}(|\xi^2|) = \mathcal{O}(|\xi|) \cdot \mathcal{O}(|\xi|) = \mathcal{O}(|\xi|^2)$. Hence dividing and then squaring them gives

$$f_{nDOT}(\xi) = \left(\frac{\mathcal{O}(|\xi|^2)}{\mathcal{O}(|\xi|^2)}\right)^2 = \mathcal{O}(1).$$

We can bound f_{nDOT} from above by some number $\alpha > 0$ and C_1 is satisfied.

The function f_{nDOT} can be written as in Lemma 4 with the function $h(x) = x^2$. Then $h(1) > h(0)$ and the lemma disproves rank one convexity. This function is not separately convex, there is complete symmetry in ξ^1 and ξ^2, without loss of generality we fix all $\xi_i^2, i = 1, \dots, n$, $\xi_i^1, i \neq j$ for some $j \in \{1, \dots, n\}$. We then have a one argument function $\bar{f}_{\text{nDOT}}$ as

$$\bar{f}_{\text{nDOT}}(\xi_j^1) = \left(\frac{\sum_{i \neq j} \xi_i^1 \xi_i^2 + \xi_j^1 \xi_j^2}{\sqrt{\sum_{i \neq j} (\xi_i^1)^2 + (\xi_j^1)^2} \|\xi^2\|}\right)^2.$$

As counterexample we fix $\xi^2_j = 1, \xi^2_i = 0, i \neq j$ and for some $k \neq j \in \{1, \dots, n\}, \xi^1_k = 1, \xi^1_i = 0, i \neq j, k$. Then

$$\bar{f}_{\text{nDOT}}(\xi^1_j) = \left(\frac{0 + 1 \cdot \xi^1_j}{\sqrt{1^2 + \left(\xi^1_j\right)^2 \cdot 1}} \right)^2 = \frac{\left(\xi^1_j\right)^2}{\left(\xi^1_j\right)^2 + 1}.$$

This is not a convex function as the second derivative is not everywhere positive.

$$\bar{f}''_{\text{nDOT}}(\xi^1_j) = \frac{2 - 6\left(\xi^1_j\right)^2}{\left(\left(\xi^1_j\right)^2 + 1\right)^3}, \ \bar{f}''_{\text{nDOT}}(1) = \frac{-1}{2} < 0$$

□

4.3.4 LP

Lemma 14. *The function defined as*

$$f_{LP}(\xi) := |\xi^1||\xi^2| - \langle \xi^1, \xi^2 \rangle,$$

satisfies C_2, is not rank one convex and is separately convex for $n \in \mathbb{N}$.

Proof. This is as in Equation (39) with $\varphi(s) = \psi(s) = s$. Since $|\xi^1||\xi^2| \geq \langle \xi^1, \xi^2 \rangle$, we have non-negativity of f_{LP}. Hence we satisfy any lower bound for the C_p growth conditions. Our function satisfies C_2, because we have general inequalities

$$|\xi^1||\xi^2| < |\xi^1|^2 + |\xi^2|^2 = |\xi|^2,$$

and the dot product component is also of $\mathcal{O}\left(|\xi|^2\right)$.
We disprove rank one convexity via the Legendre-Hadamard conditions. The terms for α, β are

$$\frac{\partial^2 f_{\text{LP}}}{\partial \xi^2_\beta \partial \xi^1_\alpha} = \begin{cases} \frac{\xi^1_\alpha \xi^2_\beta}{|\xi^1||\xi^2|} & \alpha \neq \beta \\ \frac{\xi^1_\alpha \xi^2_\alpha}{|\xi^1||\xi^2|} - 1 & \alpha = \beta \end{cases}$$

$$\frac{\partial^2 f_{\text{LP}}}{\partial \xi^1_\alpha \partial \xi^1_\beta} = \begin{cases} -\frac{\xi^1_\alpha \xi^1_\beta |\xi^2|}{|\xi^1|^3} & \alpha \neq \beta \\ \frac{\xi^1_\alpha \xi^1_\alpha |\xi^2|}{|\xi^1|^3} & \alpha = \beta \end{cases}$$

$$\frac{\partial^2 f_{\text{LP}}}{\partial \xi^2_\alpha \partial \xi^2_\beta} = \begin{cases} -\frac{\xi^2_\alpha \xi^2_\beta |\xi^1|}{|\xi^2|^3} & \alpha \neq \beta \\ \frac{\xi^2_\alpha \xi^2_\alpha |\xi^1|}{|\xi^2|^3} & \alpha = \beta \end{cases}$$

For all $\lambda \in \mathbb{R}^2, \mu \in \mathbb{R}^n, \xi \in \mathbb{R}^{2\times n}$. We can find a counterexample by only considering one non-cross term in μ_α. Take $\mu_i = 0, i \neq 1,2$ and $\mu_1 = \mu_2 = 1$, the LHS of the L-H conditions is then given by

$$\sum_{\alpha=1}^{2} \frac{\xi^1_\alpha \xi^1_\alpha |\xi^2|}{|\xi^1|^3}(\lambda^1)^2 + 2\lambda^1\lambda^2 \left(\frac{\xi^1_\alpha \xi^2_\alpha}{|\xi^1||\xi^2|} - 1\right) + (\lambda^2)^2 \frac{\xi^2_\alpha \xi^2_\alpha |\xi^1|}{|\xi^2|^3}$$

$$+ \sum_{(\alpha,\beta)\in\{(1,2),(2,1)\}} -\frac{\xi^1_\alpha \xi^1_\beta |\xi^2|}{|\xi^1|^3}(\lambda^1)^2 + 2\lambda^1\lambda^2 \frac{\xi^1_\alpha \xi^2_\beta}{|\xi^1||\xi^2|} - \frac{\xi^2_\alpha \xi^2_\beta |\xi^1|}{|\xi^2|^3}(\lambda^2)^2.$$

Now taking $\lambda^1 = \lambda^2 = 1$ and keeping $\xi^1_\beta, \xi^2_\beta, \beta \neq 1,2$ arbitrary, we get

$$= \sum_{\alpha=1,2} \frac{(\xi^1_\alpha)^2 |\xi^2|}{|\xi^1|^3} + 2\left(\frac{\xi^1_\alpha \xi^2_\alpha}{|\xi^1||\xi^2|} - 1\right) + \frac{(\xi^2_\alpha)^2 |\xi^1|}{|\xi^2|^3}$$

$$+ \sum_{(\alpha,\beta)\in\{(1,2),(2,1)\}} -\frac{\xi^1_\alpha \xi^1_\beta |\xi^2|}{|\xi^1|^3} + 2\frac{\xi^1_\alpha \xi^2_\beta}{|\xi^1||\xi^2|} - \frac{\xi^2_\alpha \xi^2_\beta |\xi^1|}{|\xi^2|^3}.$$

We can rewrite this by combining terms as

$$\frac{|\xi^2|}{|\xi^1|}\left(1 - \frac{2\xi^1_1\xi^1_2}{|\xi^1|^2}\right) + \frac{|\xi^1|}{|\xi^2|}\left(1 - \frac{2\xi^2_1\xi^2_2}{|\xi^2|^2}\right) + \frac{(\xi^1_1 + \xi^1_2)(\xi^2_1 + \xi^2_2)}{|\xi^1||\xi^2|} - 4.$$

Picking $\xi^1_2 = \xi^1_2 = \xi^2_1 = 1, \xi^1_1 = 10$ and substitution gives

$$= \frac{\sqrt{2}}{\sqrt{101}}\left(1 - \frac{20}{101}\right) + \frac{\sqrt{101}}{\sqrt{2}}\left(1 - \frac{2}{2}\right) + \frac{11\cdot 2}{\sqrt{2}\sqrt{101}} - 4 \approx -2.34 < 0.$$

So there is no rank one convexity and consequently neither all other forms of convexity. This function is separately convex, there is complete symmetry in ξ^1 and ξ^2, without loss of generality we fix all $\xi^2_i, i = 1,\ldots,n$, $\xi^1_i, i \neq j$ for some $j \in \{1,\ldots,n\}$. We then have a one argument function as

$$\bar{f}_{\mathrm{LP}}(\xi^1_j) = \sqrt{\sum_{i\neq j}(\xi^1_i)^2 + (\xi^1_j)^2}|\xi^2| - \left(\sum_{i\neq j}\xi^1_i\xi^2_i + \xi^1_j\xi^2_j\right).$$

We can write this with variables $a, b \geq 0, c, d \in \mathbb{R}$ depending on the fixed variables as

$$\bar{f}_{\mathrm{LP}}(x) = \sqrt{x^2 + a}\cdot b - (c + dx).$$

Convexity can relatively easily be proven via the second derivative.

$$\bar{f'}_{\mathrm{LP}}(x) = \frac{bx}{\sqrt{x^2 + a}} - d,$$

$$\bar{f''}_{\mathrm{LP}}(x) = \frac{ab}{(x^2 + a)^{\frac{3}{2}}}.$$

Now since $a, b \geq 0$ we have $\bar{f''}_{\mathrm{LP}} \geq 0$ and f_{LP} is separately convex. □

4.3.5 QP

Lemma 15. *The function defined as*

$$f_{QP}(\xi) := \sqrt{1 + |\xi^1|^2|\xi^2|^2} - \langle \xi^1, \xi^2 \rangle^2,$$

satisfies C_p *for* $p > 4$*, is not rank one convex and not separately convex for* $n \in \mathbb{N}$.

Proof. This is as in Equation (39) with $\varphi(s) = \sqrt{1+s}, \psi(s) = s^2$. Since $\langle a, b\rangle = |a||b|\cos(\theta)$ and $\cos(\theta) \in [0,1]$ we have $|a||b| \geq \langle a, b\rangle$. Our function satisfies $C_p, p \geq 4$, because we have at most $\mathcal{O}\left(|\xi|^4\right)$ terms from

$$|\xi^1|^2|\xi^2|^2 < |\xi^1|^4 + |\xi^2|^4 = \mathcal{O}\left(|\xi|^4\right),$$

this implies

$$\sqrt{1 + |\xi^1|^2|\xi^2|^2} = \mathcal{O}(|\xi|^2)\,.$$

We also have

$$\langle \xi^1, \xi^2 \rangle = \mathcal{O}\left(|\xi|^2\right),$$

squaring it is then of order $\mathcal{O}\left(|\xi|^4\right)$. By positivity of the first term and non-positivity of the second term in f_{QP} we can bound by

$$f_{\mathrm{QP}}(\xi) \geq -\langle \xi^1, \xi^2 \rangle^2 = -\mathcal{O}(|\xi|^4)$$

and

$$f_{\mathrm{QP}}(\xi) \leq \sqrt{1 + |\xi^1||\xi^2|} = \mathcal{O}(|\xi|^2).$$

The strict inequality of $q < p$ necessary between the lower bound and the upper bounds in C_p makes it such that in this case the lower bound has the tightest constraint. Resulting in $C_p, p > 4$ being satisfied.

We disprove rank one convexity via the Legendre-Hadamard conditions. The terms for α, β are

$$\frac{\partial^2 f}{\partial \xi^2_\beta \partial \xi^1_\alpha} = \begin{cases} \frac{\xi^1_\alpha \xi^2_\beta}{\sqrt{1+|\xi^1|^2|\xi^2|^2}}\left[2 - \frac{|\xi^1|^2|\xi^2|^2}{1+|\xi^1|^2|\xi^2|^2}\right] - 2\xi^1_\beta \xi^2_\alpha & \alpha \neq \beta \\ \frac{\xi^1_\alpha \xi^2_\alpha}{\sqrt{1+|\xi^1|^2|\xi^2|^2}}\left[2 - \frac{|\xi^1|^2|\xi^2|^2}{1+|\xi^1|^2|\xi^2|^2}\right] - 2\left(\xi^1_\alpha \xi^2_\alpha - \langle \xi^1, \xi^2\rangle\right) & \alpha = \beta \end{cases}$$

$$\frac{\partial^2 f}{\partial \xi^1_\alpha \partial \xi^1_\beta} = \begin{cases} -\frac{\xi^1_\alpha \xi^1_\beta |\xi^2|^2}{(1+|\xi^1||\xi^2|)^{\frac{3}{2}}} - 2\xi^2_\alpha \xi^2_\beta & \alpha \neq \beta \\ \frac{|\xi^2|^2}{\sqrt{1+|\xi^1||\xi^2|}}\left[1 - \frac{(\xi^1_\alpha)^2|\xi^2|^2}{1+|\xi^1|^2|\xi^2|^2}\right] - 2(\xi^2_\alpha)^2 & \alpha = \beta \end{cases}$$

$$\frac{\partial^2 f}{\partial \xi^2_\alpha \partial \xi^2_\beta} = \begin{cases} -\frac{\xi^2_\alpha \xi^2_\beta |\xi^1|^2}{(1+|\xi^1||\xi^2|)^{\frac{3}{2}}} - 2\xi^1_\alpha \xi^1_\beta & \alpha \neq \beta \\ \frac{|\xi^1|^2}{\sqrt{1+|\xi^1||\xi^2|}}\left[1 - \frac{(\xi^2_\alpha)^2|\xi^1|^2}{1+|\xi^1|^2|\xi^2|^2}\right] - 2(\xi^1_\alpha)^2 & \alpha = \beta \end{cases}$$

For all $\lambda \in \mathbb{R}^2, \mu \in \mathbb{R}^n, \xi \in \mathbb{R}^{2\times n}$. We can find a counterexample by only considering the non-cross-terms in μ_α. Take all $\mu_i = 0, i \neq 1$ and $\mu_1 = 1, \lambda^1 = \lambda^2 = 1$, the LHS of the L-H conditions is then given by

$$\sum_{\alpha=1}^{n} \frac{|\xi^2|^2}{\sqrt{1+|\xi^1||\xi^2|}}\left[1-\frac{(\xi^1_\alpha)^2|\xi^2|^2}{1+|\xi^1|^2|\xi^2|^2}\right] - 2(\xi^2_\alpha)^2 + \frac{2\xi^1_\alpha\xi^2_\alpha}{\sqrt{1+|\xi^1|^2|\xi^2|^2}}\left[2-\frac{|\xi^1|^2|\xi^2|^2}{1+|\xi^1|^2|\xi^2|^2}\right]$$
$$-4\left(\xi^1_\alpha\xi^2_\alpha - \langle\xi^1,\xi^2\rangle\right) + \frac{|\xi^1|^2}{\sqrt{1+|\xi^1||\xi^2|}}\left[1-\frac{(\xi^2_\alpha)^2|\xi^1|^2}{1+|\xi^1|^2|\xi^2|^2}\right] - 2(\xi^1_\alpha)^2.$$

Then taking $\xi^1 = \vec{0}$

$$= \sum_{\alpha=1}^{n} |\xi^2|^2\left[1-\frac{0}{1}\right] - 2(\xi^2_\alpha)^2 + \frac{0}{1}\left[2-\frac{0}{1}\right] - 4\,(0-0) + \frac{0}{\sqrt{1}}\left[1-\frac{0}{1}\right] - 2\cdot 0^2$$
$$= \sum_{\alpha=1}^{2} |\xi^2|^2 - 2(\xi^2_\alpha)^2.$$

Picking $\xi^2_\beta = 0, \beta \neq 1, , \xi^2_1 = \frac{1}{2}$ we get

$$\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right)^2 = -\frac{15}{16} < 0.$$

So there is no rank one convexity and consequently neither all other forms of convexity.

This function is not separately convex, there is complete symmetry in ξ^1 and ξ^2, without loss of generality we fix all $\xi^2_i, i = 1, \ldots, n$, $\xi^1_i, i \neq j$ for some $j \in \{1, \ldots, n\}$. We then have a one argument function $\bar{f}_{\mathrm{QP}}$ as

$$\bar{f}_{\mathrm{QP}}(\xi^1_j) = \sqrt{1 + \left(\sum_{i\neq j}(\xi^1_i)^2 + (\xi^1_j)^2\right)|\xi^2|^2 - \left(\sum_{i\neq j}\xi^1_i\xi^2_i + \xi^1_j\xi^2_j\right)^2}.$$

Setting $\xi^1_i = \xi^2_i = 0, i \neq j$ and $\xi^2_j = 1$ we have

$$\bar{f}_{\mathrm{QP}}(x) = \sqrt{1+x^2} - x^2.$$

This is not a convex function as can be seen from the second derivative

$$\bar{f}''_{\mathrm{QP}}(x) = \frac{1}{(1+x^2)^{\frac{3}{2}}} - 2, \bar{f}''_{\mathrm{QP}}(0) = -1 < 0.$$

□

4.3.6 CG

Lemma 16. *The function defined as*

$$f_{CG}(\xi) := (\xi^1 \times \xi^2)^2 := \|\xi^1\|^2\|\xi^2\|^2 - \langle\xi^1,\xi^2\rangle^2,$$

satisfies C_4, is polyconvex and separately convex for $n \in \mathbb{N}$.

The proof is given after Lemma 17 below. Note that we take the second expression to be the definition of $(\xi^1 \times \xi^2)^2$ in general n-dimensional vector spaces. This is because the vectorial cross-product as mostly used in the physical sciences is exclusively defined in $\mathbb{R}^3$. The generalized definition above comes usually from the fact that the cross-product in $\mathbb{R}^3$ is equivalent to the exterior product $u \wedge v$ of two vectors $u, v \in \mathbb{R}^3$, which can be written as Gramian

$$u \wedge v := G(u, v) := \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix}.$$

Now using the same formula and generalizing to $u, v \in \mathbb{R}^n$ we get

$$(u \times v)^2 := u \wedge v = \begin{vmatrix} \langle u, u\rangle & \langle u, v\rangle \\ \langle u, v\rangle & \langle v, v\rangle \end{vmatrix} = \|u\|^2\|v\|^2 - \langle u, v\rangle^2,$$

by the basic properties of the inner product.

We first state some results that are used in the proof of Lemma 16. From Definition 12 we know that for f_{CG} to be polyconvex, we need to have a convex function $F : \mathbb{R}^{\tau(n,2)} \to \mathbb{R} \cup \{\infty\}$ such that

$$f_{CG}(\xi) = F(T(\xi)),$$

with $T : \mathbb{R}^{2\times n} \to \mathbb{R}^{\tau(n,2)}$ and

$$T(\xi) := (\xi, \mathrm{adj}_2 \xi).$$

Here our assumption that $N = 2$, makes the computations more tractable since T only depends on the first two minors of ξ. Whereas in general it would depend on the first $\min\{N, n\}$ minors. We quickly unpack the construction of these adjugate matrices of order s as denoted $\mathrm{adj}_s \xi$ for general $N \times n$ matrices ξ.

Definition 19 (Increasing Tuples)**.** *Let $n \in \mathbb{N}, 1 \le s \le n$. Then define*

$$I_s^n := \{(\alpha_1, \dots, \alpha_s) \in \mathbb{N}^s : 1 \le \alpha_1 < \alpha_2 < \cdots < \alpha_s \le n\},$$

to be the set of all increasing s-tuples up to n.

We introduce a backwards inverse lexicographical ordering on I_s^n as follows.

$$\alpha \succ \beta$$

if and only if for the largest integer $k \le s$ such that $\alpha_k \ne \beta_k$ and $\alpha_l = \beta_l$ for all $l > k$ we have

$$\alpha_k < \beta_k.$$

Then there is a unique bijection $\varphi_s^n : \{1, \dots, \binom{n}{2}\} \to I_s^n$ that respects this ordering [18].

Definition 20 (Adjugate matrices). *The adjugate matrix of order s, $adj_s\xi \in \mathbb{R}^{\binom{N}{s}\times\binom{n}{s}}$ is defined to be*

$$adj_s\xi = \begin{pmatrix} (adj_s\xi)_1^1 & \dots & (adj_s\xi)_{\binom{n}{s}}^1 \\ \vdots & \ddots & \vdots \\ (adj_s\xi)_1^{\binom{N}{s}} & \dots & (adj_s\xi)_{\binom{n}{s}}^{\binom{N}{s}} \end{pmatrix}.$$

Where

$$(adj_s\xi)_\alpha^i = (-1)^{i+\alpha} \det \begin{pmatrix} \xi_{\alpha_1}^{i_1} & \dots & \xi_{\alpha_s}^{i_1} \\ \vdots & \ddots & \vdots \\ \xi_{\alpha_1}^{i_s} & \dots & \xi_{\alpha_s}^{i_s} \end{pmatrix},$$

and $(i_1, \dots, i_s), (\alpha_1, \dots, \alpha_s)$ are such that $\varphi_s^N(i) = (i_1, \dots, i_s), \varphi_s^n(\alpha) = (\alpha_1, \dots, \alpha_s)$.

Lemma 17 (Lagrange's identity). *[9] For any two sets $(a_1, \dots, a_n), (b_1, \dots, b_n), n \in \mathbb{N}$ real (or complex) numbers we have*

$$\sum_{i=1}^n a_i^2 \sum_{i=1}^n b_i^2 - \left(\sum_{i=1}^n a_i b_i\right)^2 = \sum_{1\le i<j\le n} (a_i b_j - b_i a_j)^2. \tag{52}$$

Proof. (Lemma 16) From Lemma 17 we see that f_{CG} can be written as a sum of squares, and thus is always non-negative. Any lower bound for C_p is satisfied. We have

$$|\xi|^4 = \left(\sum_{i=1}^n |\xi_i^1|^2 + |\xi_i^2|^2\right)^2 = \sum_{i=1}^n \left(|\xi_i^1|^2 + |\xi_i^2|^2\right)^2 + \sum_{i,j=1, i\neq j}^n \left(|\xi_i^1|^2 + |\xi_i^2|^2\right)\left(|\xi_j^1|^2 + |\xi_j^2|^2\right)$$

$$\geq \sum_{i,j=1}^n \|\xi^1\|^2 \|\xi^2\|^2 \geq f_{CG}(\xi).$$

Hence taking into account the upper bound, we have $C_p, p \geq 4$.

Since implicitly $N = 2, n > 1$, we get $s = \min\{N, n\} = 2$ and $\tau(n, 2) = \binom{2}{1}\binom{n}{1} + \binom{2}{2}\binom{n}{2} = 2n + \frac{n(n-1)}{2}$. In particular we only need to worry about the minors $\text{adj}_2\xi$ of order 2. We have variables

$$T(\xi) = \left(\xi, (\text{adj}_2\xi)_1^1, \dots, (\text{adj}_2\xi)_{\binom{n}{2}}^1\right) \in \mathbb{R}^{2n+\frac{n(n-1)}{2}}.$$

From Definition 20 we have in particular $I_2^2 = \{(1, 2)\}$ with $|I_2^2| = 1$ and I_2^n the set of all $\binom{n}{2}$ increasing 2-tuples up to n. Hence the upper indices of our 2×2 minors are fixed and equal to $(1, 2)$. Let $\alpha \in \{1, \dots, \binom{n}{2}\}$ such that $\varphi_2^n(\alpha) = (\alpha_1, \alpha_2)$ with $(\alpha_1, \alpha_2) \in I_2^n$. Then

$$(\text{adj}_2\xi)_\alpha^1 = (-1)^{1+\alpha} \det \begin{pmatrix} \xi_{\alpha_1}^1 & \xi_{\alpha_2}^1 \\ \xi_{\alpha_1}^2 & \xi_{\alpha_2}^2 \end{pmatrix},$$

and

$$\left((\text{adj}_2\xi)_\alpha^1\right)^2 = \left(\xi_{\alpha_1}^1 \xi_{\alpha_2}^2\right)^2 + \left(\xi_{\alpha_2}^1 \xi_{\alpha_1}^2\right)^2 - 2\xi_{\alpha_1}^1 \xi_{\alpha_2}^1 \xi_{\alpha_1}^2 \xi_{\alpha_2}^2$$

$$= \left(\xi^1_{\alpha_1}\xi^2_{\alpha_2} - \xi^1_{\alpha_2}\xi^2_{\alpha_1}\right)^2.$$

Note that this is the same form as in Lemma 17. It can be seen that the terms under the sum in the RHS of Equation (52) are the adjugate minors of the matrix

$$\begin{pmatrix} a_1 & \dots & a_n \\ b_1 & \dots & b_n \end{pmatrix}.$$

Our claim is that we can take $F : \mathbb{R}^{\tau(n,2)} \to \mathbb{R} \cup \{\infty\}$ as $F(T(\xi)) = \sum_{\alpha \in I_2^n} \left((\text{adj}_2\xi)^1_\alpha\right)^2$. It is a sum of squares of some of our variables $(\xi, \text{adj}_2\xi)$. Since $x \mapsto x^2$ is convex and a sum of convex function is convex we have that F is a convex function in $T(\xi)$. The only difference between Equation (52) and $F(T(\xi))$ is the summation over $\alpha \in I_s^n$, from the definition of I_2^n we have an equivalent index set as

$$I_2^n = \{(i,j) \in \mathbb{N}^2, 1 \leq i < j \leq n\}.$$

From Definition 12 we have that f_{CG} is polyconvex. Polyconvexity implies separate convexity so the lemma follows.

□

4.3.7 Nambu

Lemma 18. *The function defined as*

$$f_{Nambu}(\xi) = |\xi^1 \times \xi^2|,$$

satisfies C_2, is polyconvex and separately convex for $n \in \mathbb{N}$.

Notably, using the same algebraic definition as discussed in Lemma 16, namely

$$|\xi^1 \times \xi^2| := \sqrt{\|\xi^1\|^2\|\xi^2\|^2 - \langle \xi^1, \xi^2 \rangle^2},$$

f_{Nambu} is the square root of f_{CG}.

Proof. We can write

$$f_{Nambu} = \sqrt{f_{CG}},$$

and f_{CG} satisfies C_4, so we can take roots on the lower and upper bound of f_{CG} and immediately get that f_{Nambu} satisfies $C_{\sqrt{4}} = C_2$. Plugging in,

$$f_{\text{Nambu}}(\xi) = \sqrt{\sum_{\alpha \in I_2^n} \left((\text{adj}_2)^1_\alpha\right)^2},$$

We can rephrase this in terms of a norm. If we define $\|\cdot\|_M$ to be the Euclidean norm on $(\text{adj}_2\xi) \in \mathbb{R}^{\binom{n}{2}}$. Now extend this function via the standard embedding to $\mathbb{R}^{2n} \times \mathbb{R}^{\binom{n}{2}}$. Then

$$f_{\text{Nambu}}(\xi) = \|T(\xi)\|_M, \text{ where } T(\xi) \in \mathbb{R}^{2n} \times \mathbb{R}^{\binom{n}{2}}.$$

We can take $F_{\text{Nambu}} = \|\cdot\|_M$, which is a convex function since it is a (projected) norm. So directly from the definition we have f_{Nambu} polyconvex. Polyconvexity implies separate convexity so the result follows. $\square$

Remark 16. *From Lemma 16, we have polyconvexity of the cross-gradient. By definition this gives a convex function F_{CG} in terms of $(\xi, adj_2\xi)$. So*

$$f_{Nambu}(\xi) = \sqrt{F_{CG}(T(\xi))}.$$

Unfortunately the composition is the wrong way, as we would have $F_{CG}(T(\sqrt{\xi}))$ also polyconvex. However,the specific form of F_{CG} is also conducive for the inverse composition.

4.3.8 VTV, TNV, TSV

Before proving the properties of these regularizers, we give some additional motivation about their construction. Compared to the other regularizers that are defined only for $N = 2$, these are defined for general $N, n \in \mathbb{N}$. Also they are mainly used in linear programming instead of variational optimization and are developed from a different mathematical view. The three structural regularizers $f_{VTV}, f_{TNV}, f_{TSV}$ can be regarded as different versions of a more general form. Where we regard $\xi \in \mathbb{R}^{N \times n}$ as a matrix and take

$$f_{\text{Norm}}(\xi) := \|\xi\|_{\cdot}, \tag{53}$$

where $\|\xi\|_{\cdot}$ is some matrix norm defined on $\mathbb{R}^{N \times n}$. In particular, f_{VTV} comes from the Frobenius norm, f_{TNV} from the nuclear norm, and f_{TSV} from the spectral norm. Although we are not restricted in choosing these particular norm, there are good reasons why these would work best. In applications where TV regularization is used, the most natural extension of single TV regularization defined as

$$TV[u] = \int_\Omega |\nabla u|,$$

is instead of taking the absolute value over a vector $\nabla u \in \mathbb{R}^n$, we take the Nn-dimensional norm over N vectors $u_1, \ldots, u_N$ as

$$TV[u_1, \ldots, u_N] := \int_\Omega \sqrt{\sum_{i=1}^N |\nabla u_i|^2}.$$

This is exactly the same formula as f_{VTV}, or taking the Frobenius norm of the matrix $\xi := [\nabla u_1, \ldots, \nabla u_N]^T$. Note the difference between the definition above and $J_{l^2}(\xi)$ as defined in Equation (51) where the integrals are taking inside the square root instead of the other way around. This makes f_{VTV} a natural choice in vectorial settings for application domains that use TV regularization [25, 34, 37].

From the perspective of looking at ξ as a matrix with the gradients of parameter fields as rows, structural similarity can be quantified via looking at $\text{rank}(\xi)$. As having small rank is equivalent to having the gradients be linear combinations of each-other, and hence they act the same structurally. In fact, if we have $\text{rank}(\xi) = 1$ the behaviour of all gradients can be predicted from only one parameter field. In contrast, if $\text{rank}(\xi) = \min\{N, n\}$ then in general the gradients act independently from each other and there is low structural similarity. However, rank minimization problems have been proven to be NP-hard [19, 61] and hence intractable for usage in realistic applications where we have to minimize $\text{rank}(\xi(x))$ at many points x for each iteration. However, approximating the solution of minimizing the rank function could still lead to satisfactory regularization [27]. We have the following result.

Theorem 18 (Anderson)**.** *[5] Let $G \in \mathbb{R}^{N \times n}$. Let $\sigma_1, \ldots$ be the singular values of G. Then the nuclear norm*

$$\|G\|_* := \sum_i |\sigma_i|,$$

is the convex envelope of the function rank(G) over the unit ball B_1 in $\mathbb{R}^{N \times n}$.

Taking the convex envelope of the rank function makes sense, as taking an envelope is a natural approximation to a function. Additionally, as convex envelopes are convex we get automatically l.s.c. of the integral when taking $\| \cdot \|_*$ as integrand. We can extend this envelope affinely to any other bounded set in the space of matrices by the following procedure. Let B_1 be the unit ball in 2-norm over the space of matrices. Let B_r be the ball of radius $r \geq 1$ in the 2-norm. The convex envelope on B_r is then given by $\frac{1}{r}\| \cdot \|_*$. As $f_{TNV}(\xi) = \|\xi\|_*$, this is a natural choice for minimization problems with matrices.

Finally, note that f_{TNV} and f_{VTV} are not only matrix norms, but in particular Schatten p-norms with $p = 1, 2$ respectively.

Definition 21 (Schatten Norm)**.** *Let $p \in [1, \infty)$. Let G a matrix with $\sigma_1, \ldots$ its singular values. Then the Schatten p-norm is given by*

$$\|G\|_{l^p} := \sum_i (\sigma_i)^p.$$

For $p = \infty$ we have

$$\|G\|_{l^\infty} := \max\{\sigma_1, \ldots, \}.$$

Regarding the Schatten ∞-norm results in f_{TFV} [37].

Lemma 19. *The functions defined as*

$$f_{TNV}(\xi) = \|\xi\|_{l^1},$$

$$f_{VTV}(\xi) = \|\xi\|_{l^2},$$

$$f_{TSV}(\xi) = \|\xi\|_{l^\infty},$$

satisfy C_1, are jointly convex and separately convex for $N, n \in \mathbb{N}$.

Proof. By definition $|\xi| := \sqrt{\sum_{i,j} |\xi_j^i|^2} = \|\xi\|_{l^2}$. The C_1 condition in this case asks for existence of an $\alpha \geq 0$ such that for all $G \in \mathbb{R}^{N\times n}$ the following holds.

$$|\|G\|_{l^p}| \leq \alpha(1 + |G|).$$

Note that both $|G|$ and $\|G\|_{l^p}$ are matrix norms on a finite dimensional vector space, hence there is equivalence between them. By non-negativity of the norm, we automatically satisfy all lower bounds for C_p. The equivalence of matrix norms gives a constant $c_p \geq 0$ such that

$$\|G\|_{l^p} \leq c_p|G|, \forall G \in \mathbb{R}^{N\times n}.$$

Picking $\alpha = c_p$ implies that condition C_1 holds. There are exact values known of these constants c_p, but we omit them here since only existence is necessary. $\|\xi\|_{l^p}$ is trivially jointly convex, since any norm on a vector space is convex. Joint convexity implies separate convexity. □

4.4 Extended definitions

In the previous subsections we have found that the integrand functions that are quasiconvex are $f_{GD}, f_{CG}, f_{Nambu}, f_{VTV}, f_{TNV}$, and f_{TSV}. There are some differences between them regarding where they are defined for our parameters N, n. Namely, where $f_{VTV}, f_{TNV}, f_{TSV}$ are defined on the entire space $\mathbb{N}^2$ we have that f_{CG}, f_{Nambu} and f_{GD} are only defined for $n \in \mathbb{N}, N = 2$. We remark that f_{CG} and f_{Nambu} both evaluate to the zero function for $n = 1$.

There is a standard way of considering joint structural similarity regularizers in cases where there are more than two channels ($N > 2$) [37, 21]. This is the naive way of comparing each pair over all channels and adding the contributions. Namely, if we have an $N \times n$ dimensional $b = (u_1, \ldots, u_N) \in \mathcal{B}$ and the regularizer comparing $u_i, u_j, i, j = 1, \ldots, N, i \neq j$ is given by

$$\mathcal{R}_{i,j}(u_i, u_j),$$

then the final regularizer is given by

$$\mathcal{R}(b) := \sum_{i \neq j = 1, \ldots, N} R_{i,j}(u_i, u_j) = \frac{1}{2} \sum_{i=1,\ldots,N} \sum_{j=1,\ldots,i} R_{i,j}(u_i, u_j).$$

Here we only have to compute certain terms by the symmetry of $\mathcal{R}$ in (i, j). We choose to define our generalized f_{GD} in this way as

$$f_{gGD} := \frac{1}{2} \sum_{i=1,\ldots,N} \sum_{j=1,\ldots,i} f_{GD}(\xi^i, \xi^j). \tag{54}$$

This is well-defined and equivalent to the construction above because of the linearity of our integral J with respect to f. We do the same with f_{CG} and f_{Nambu} for completeness sake. However we denote them by f_{jCG}, f_{jNambu} with "j" for joint as we define different generalisations f_{gCG}, f_{gNambu} via a novel non-naive approach.

$$f_{jCG}(\xi) := \frac{1}{2} \sum_{i=1,\ldots,N} \sum_{j=1,\ldots,i} f_{CG}(\xi^i, \xi^j), \tag{55}$$

$$f_{jNambu}(\xi) := \frac{1}{2} \sum_{i=1,\ldots,N} \sum_{j=1,\ldots,i} f_{Nambu}(\xi^i, \xi^j). \tag{56}$$

As seen from the proof of polyconvexity of f_{CG}, f_{Nambu} in Lemma 16 and Lemma 18 we can write a cross-product of two vectors $\xi^1 \times \xi^2$ as convex function of the adjugate matrices of $\xi = (\xi^1, \xi^2)^T$. In particular

$$f_{CG} := \sum_{\alpha \in I_2^n} \left((\mathrm{adj}_2 \xi)^1_\alpha \right)^2.$$

This is a sum over all adjugate matrices of order 2 of ξ. This begs the following question, what is the natural way of generalizing this sum for $N \neq 2$? The parameter n is already neatly incorporated in the expression above in the I_2^n term. However, we have a choice about how to incorporate N, do we only look at the the adjugate matrices of order N, at all adjugate matrices up to order N or up to $\min\{n, N\}$? Also, do we incorporate another sum over I_2^N as this is implicitly already in included in the expression as $|I_2^2| = 1$?

The author would argue that simplicity and symmetry are two key qualities that we would like this integrand to have. To have a function that is symmetric in (N, n) is natural because this implies a duality between domain $\mathbb{R}^n$ and co-domain $\mathbb{R}^N$. Fundamentally, we can view the parameter fields $u_1, \ldots, u_N$ and the locations $x \in \Omega \subset \mathbb{R}^n$ also in this dual way. We can take each of the notion as foundational, where every location has N different corresponding quantities or dually where every parameter field has a value at each location. For symmetry, both sums over I_s^N and I_s^n are necessary and we use as upper bound for the order s the term $\min\{n, N\}$. To have the simplest function, we argue that looking only at one order s instead of multiple $s = 1, \dots, \min\{n, N\}$ is key. This does not hamstring our function f_{gCG} in an informational perspective as all adjugate matrices of a fixed order s incorporate all coefficients of ξ. It is arguable that additionally using the adjugate matrices of order $< s$ only muddles the impact of a given ξ_j^i on the final result. As then every coefficient is included in more than one of the adjugate contributions and the explicit formula, where we sum over all order up to s, only increases in complexity. Taking the $\min\{n, N\}$-th order results in the largest products of coefficients (since it includes the most cross-terms) where our generalized definition is still equal to f_{CG} for $N = 2$. Hence we have chosen to define the generalised version f_{gCG} for $\xi \in \mathbb{R}^{N \times n}$ as

$$f_{gCG}(\xi) := \sum_{\beta \in I^N_{\min\{N,n\}}} \sum_{\alpha \in I^n_{\min\{N,n\}}} \left((\mathrm{adj}_{\min\{N,n\}} \xi)^\beta_\alpha \right)^2 . \tag{57}$$

Taking different notions than simplicity and symmetry as core will naturally lead to a different definition of f_{gCG}, here investigation on the best form is still possible.

Lemma 20. *The function f_{gCG} satisfies $C_{\min\{N,n\}^2}$, is polyconvex and separately convex for $N, n \in \mathbb{N}$.*

Proof. The definition of adjugate matrices $(\mathrm{adj}_s \xi)^\beta_\alpha$ (Definition 20) gives that it is $(-1)^{\beta+\alpha}$ times the determinant of a $s \times s$ matrix. From a function-based perspective we can see $(\mathrm{adj}_s \xi)^\beta_\alpha$ as a polynomial of variables $\xi_j^i, i = 1, \dots, N, j = 1, \dots n$ of order s as we know how to compute determinants. Now in f_{gCG} we have that each term is a square of a multi-variable polynomials of order $\min\{N, n\}$. Hence we have f_{gCG} a multi-variable polynomial of order $\min\{N, n\}^2$. From each variable ξ_j^i having

$$\xi_j^i \leq |\xi|,$$

we get for any arbitrary product of length $\min\{N, n\}^2$ of coefficients of ξ denoted by $x_k, k = 1, \dots, \min\{N, n\}$ where every x_k is equal to some coefficient ξ_j^i the following bound.

$$\prod_{k=1,\dots,\min\{N,n\}^2} x_k \leq |\xi| \cdot \dots \cdot |\xi| = |\xi|^{\min\{N,n\}^2}.$$

So each polynomial is bounded from above by $|\xi|^{\min\{N,n\}^2}$ and adding up all contributions we get that f_{gCG} satisfies $C_{\min\{N,n\}^2}$ with $\alpha = |I^N_{\min N,n}||I^n_{\min N,n}|$.

By construction this is a sum of squares over $(\mathrm{adj}_{\min N,n}\xi)^{\beta}_{\alpha}$. By the definition of adjugate matrices these variables above are the coefficients of $\mathrm{adj}_{\min N,n}\xi$ with $\beta = 1,\ldots,\binom{N}{\min\{N,n\}}, \alpha = 1,\ldots,\binom{n}{\min\{N,n\}}$. By the definition of polyconvexity we have a convex function $F : \mathbb{R}^{\tau(n,N)} \to \mathbb{R}$ in $T(\xi) = (\xi,\ldots,\mathrm{adj}_{\min\{N,n\}}\xi)$. Hence f_{gCG} is polyconvex. Since polyconvexity implies separate convexity, we get this for free. □

We define f_{gNambu} in exactly the same correspondence to f_{gCG} as f_{Nambu} is to f_{CG}.

$$f_{gNambu}(\xi) := \sqrt{\sum_{\beta\in I^N_{\min\{N,n\}}} \sum_{\alpha\in I^n_{\min\{N,n\}}} \left(\left(\mathrm{adj}_{\min\{N,n\}}(\xi)\right)^{\beta}_{\alpha}\right)^2} \tag{58}$$

Doing it this way, we get the same properties for f_{gNambu} as for f_{Nambu}.

Lemma 21. *The function f_{gNambu} satisfies $C_{\min\{N,n\}}$, is polyconvex and separately convex for $N, n \in \mathbb{N}$.*

Proof. The proof is the same as for Lemma 18. Namely, we can regard the square root of the double sum over I^N_s, I^n_s with $s = \min\{N,n\}$ as a norm over $(\mathrm{adj}_s\xi) \in \mathbb{R}^{\binom{N}{s}\binom{n}{s}}$. Then extend it to the entire space $T(\xi) \in \mathbb{R}^{\tau\{N,n\}}$ as projected norm. Then f_{gNambu} is a norm over this Euclidean space and hence convex. Since the variables are given in terms of the coefficients of the adjugate minors this gives polyconvexity of f_{gNambu}. Polyconvexity implies separate convexity. In addition, the order of f_{gNambu} is given by a square root of the order of f_{gCG}. So via a similar reasoning as in Lemma 20 of each term under the square root we have f_{gNambu} satisfying $C_{\min\{N,n\}}$. □

Table 1: Properties of specific integrands f

	Coercive	C_p	Convex	Poly	Quasi	Rank One	Separate
GD	-	2	-	X	X	X	X
mGD	-	-	-	-	-	-	-
Dot	-	4	-	-	-	-	X
aDot	-	2	-	-	-	-	X
nDot	-	1	-	-	-	-	-
LP	-	2	-	-	-	-	X
QP	-	> 4	-	-	-	-	-
CG	-	4	-	X	X	X	X
Nambu	-	2	-	X	X	X	X
VTV	X	1	X	X	X	X	X
TNV	X	1	X	X	X	X	X
TSV	X	1	X	X	X	X	X
gGD	-	2	-	X	X	X	X
jCG	-	4	-	X	X	X	X
jNambu	-	2	-	X	X	X	X
gCG	-	$\min\{N, n\}^2$	-	X	X	X	X
gNambu	-	$\min\{N, n\}$	-	X	X	X	X

5 Well-posedness

Here we combine all the properties found in Section 4.3 with the different cases in Sections 2.3 and 2.3 for $(\mathcal{B}, \tau_{\mathcal{B}})$ to determine for which specific integrands f our minimization problem is well-posed.

5.1 $W^{1,p}(\Omega)$

As before, we only look at $W^{1,p}(\Omega)$ since $W^{m,p}(\Omega)$ is compactly embedded in it for $m > 1$.

Theorem 19 (Well-posedness in $W^{1,p}(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}^n$ be open, bounded and Lipschitz, $p \in (1, \infty)$. Take as Banach space $\mathcal{B} = \bigoplus_{i=1}^N W^{m,p}(\Omega)$ with the corresponding weak topology on each component. Let $\mathcal{H} = \bigoplus_{i=1}^N \mathcal{H}_i$ where $\mathcal{H}_i$ are Hilbert spaces with $\tau_{\mathcal{H}_i}$ l.s.c. norms. Let $K : \mathcal{B} \to \mathcal{H}$ be a linear operator that is sequentially continuous wrt $\tau_{\mathcal{B}}, \tau_{\mathcal{H}}$. $d \in \mathcal{H}, b = (u_1, \dots, u_N) \in \mathcal{B}, \alpha > 0$. Let $J : \mathcal{B} \to \mathbb{R} \cup \{\infty\}$ be given by*

$$J(u_1, \dots, u_N) := \int_\Omega f(\nabla u_1, \dots, \nabla u_N),$$

for $f(\cdot) \in \{f_{VTV}, f_{TNV}, f_{TSV}\}$ or

$$J(u_1, \dots, u_N) := \int_\Omega f_s(\nabla u_1, \dots, \nabla u_N)dx + \sum_{i=1}^N \alpha_i TV[u_i], \alpha_i > 0,$$

for $f_s(\cdot) \in \{f_{gGD}, f_{jCG}, f_{jNambu}, f_{gCG}, f_{gNambu}\}$. Additionally, let K be such that (K, J) is mean coercive. Then

$$argmin_{b \in \mathcal{B}} \|Kb - d\|_{\mathcal{H}}^2 + \alpha J(b)$$

has a minimizer. Furthermore, this minimizer is unique if K is injective.

Proof. The core argument rests on Theorem 1. It is necessary to check all conditions in this statement. Via Corollary 2 we can transport our properties on the components of $\bigoplus_{i=1}^N \mathcal{B}_i$ and $\bigoplus_{i=1}^N \mathcal{H}_i$ to properties of $\mathcal{B}, \mathcal{H}$. From the discussion in Section 2.3, we know that our combination of choosing as Banach space $W^{m,p}(\Omega), p > 1, m \in \mathbb{N}$ with the weak topology and Lipschitz Ω satisfies the required properties of Corollary 2 (and consequently Theorem 1). Our assumptions on $\mathcal{H}_i, \tau_{\mathcal{H}_i}$ also give the required properties of Theorem 1 for the Hilbert space.

Note that by construction all f, f_s are real-valued and continuous on $\bar{\Omega}$ and since Ω is bounded, we have that J is proper (and non-negative by non-negativity of f or f_s).

For $f \in \{f_{VTV}, f_{TNV}, f_{TSV}\}$ we have that they satisfy the necessary properties for weak l.s.c. in $W^{1,p}(\Omega)$ in Theorem 12 from Lemma 19. Here we can directly decide mean coercivity of (K, J) depending on $\mathcal{N}(K)$ since $J(b)$ is coercive in ∇b.

For $f_s \in \{f_{gGD}, f_{gCG}, f_{gNambu}\}$ we get weak l.s.c. in $W^{1,p}(\Omega)$ from Lemma 22 below. Since there is no way to get mean coercivity from the structural part f_s we add a mean coercive TV part. We can write the total variation inside the integral as

$$J(b) = \int_\Omega f_s(\nabla b) + \sum_{i=1}^N |\nabla b_i| dx.$$

Now we can decide mean coercivity depending on $\mathcal{N}(K)$ since this $J(b)$ is coercive in ∇b.

Our assumptions on K are the same as in Theorem 1. Hence with the discussion above, the theorem can be applied and results in existence of a minimizer for the variational problem.

For uniqueness of the minimizer, we need strict convexity of either $\|Kb - d\|_{\mathcal{H}}^2$ or $J(b)$. Since in all choices for f, f_s, there is no strict convexity of $J(b)$ in either b or ∇b, we only have to look at the first term. Applying Lemma 4 gives uniqueness if K is injective. $\square$

Noteworthy is the fact that if K is injective, we can easily prove mean coercivity of (K, J) in the cases above as then $\mathcal{N}(K) = \{0\}$ and we choose $b_0 = b_\Omega$ as the mean over our domain in Lemma 3. We have chosen here for the more general case, as K is not always injective and it is sometimes worth the effort to prove mean coercivity of (K, J) explicitly.

Remark 17. *It is also possible to get uniqueness of the minimizer from strict convexity of $J(b)$. It is not clear how to define strict quasiconvexity in a similar way as strict joint convexity. The core argument of uniqueness is that J strictly convex implies $J\left(\frac{u+v}{2}\right) < \frac{J(u)+J(v)}{2}$. If we naively define strict quasiconvexity to be*

$$f(\xi) < \frac{1}{|D|} \int_D f(\xi + \nabla\varphi) dx,$$

we do not have the same property, so this does not translate.

Lemma 22 (TV contribution preserves l.s.c.)**.** *Let $p \in [1, \infty], \Omega \subset \mathbb{R}^n$ be open and bounded. Let $f_s : \mathbb{R}^{N\times n} \to \mathbb{R}$ be as in Theorem 12 such that*

$$\int_\Omega f_s(\xi) dx,$$

is l.s.c. in $W^{1,p}(\Omega; \mathbb{R}^N)$. Then the function given by

$$J(\xi) := \int_\Omega f_s(\xi) dx + \sum_{i=1}^N \alpha_i TV[\xi_i]$$

is also l.s.c in $W^{1,p}(\Omega)$. In addition, the same is true when we replace $W^{1,p}(\Omega; \mathbb{R}^N)$ by $SBV(\Omega)$.

Proof. We can write the total variation also as an integral over Ω via

$$TV[u_i] = \int_\Omega |\nabla u_i(x)| dx,$$

or

$$TV[u_i] = \int_\Omega |\xi_i|,$$

So our result follows if we can apply Theorem 12 to the function $f = f_s + f_c := f_s + \sum_{i=1}^N |\xi_i|$. We need f to be Carathéodory, quasiconvex and satisfy C_p.

From Lemma 6 we have $W^{1,p}(\Omega) \subset BV(\Omega)$ such that $TV[u_i] < \infty$ for all $i = 1, \ldots, N$ and the sum $0 \leq \sum_{i=1}^N \alpha_i TV[u_i] < \infty$. So $f_s(\xi) \leq f(\xi) < \infty$ for all $\xi \in \mathbb{R}^{N \times n}$, in particular it is real-valued. From the explicit form of the continuous contribution $\sum_{i=1}^N |\xi_i|$ we can easily see f to be $\mathcal{L}^N \times \mathbb{B}(\mathbb{R}^{N\times n})$ measurable and continuous. Together this implies f to be Carathéodory.

The new contribution f_c is affine in ξ, this implies that it is convex and in particular quasiconvex. Now, the sum of quasiconvex functions is quasiconvex via

$$f(\xi) = f_s(\xi)+f_c(\xi) \leq \frac{1}{|D|}\int_D f_s(\xi+\nabla\varphi)dx+\frac{1}{|D|}\int_D f_c(\xi+\nabla\varphi)dx = \frac{1}{|D|}\int_D f(\xi+\nabla\varphi)dx$$

for all $D \subset \mathbb{R}^n$ bounded and open, $\xi \in \mathbb{R}^{N\times n}$ and $\varphi \in W_0^{1,\infty}(D)$. So f is quasiconvex. We prove that f_c satisfies C_1.

$$0 \leq f_c(\xi) = \sum_{i=1}^N \alpha_i |\xi_i| \leq \max_i \alpha_i \cdot \sum_{i=1}^N \sqrt{\sum_{j=1}^n (\xi_i^j)^2}.$$

Note that $\sqrt{\cdot}$ is a concave function, then applying Jensen's inequality for concave functions with $\phi(x) = \sqrt{x}$ with $x_i = \sum_{j=1}^n (\xi_i^j)^2$ we have

$$\phi\left(\frac{\sum_{i=1}^N x_i}{N}\right) \geq \frac{\sum_{i=1}^N \phi(x_i)}{N}.$$

Substitution gives

$$\sqrt{\frac{\sum_{i=1}^N \sum_{j=1}^n (\xi_i^j)^2}{N}} \geq \frac{\sum_{i=1}^N |\xi_i|}{N}.$$

Writing this as a norm we get for all $\xi \in \mathbb{R}^{N\times n}$

$$\sqrt{N}|\xi| \geq f_c(\xi).$$

Such that with $\alpha = \sqrt{N}\max_i \alpha_i \geq 0$ we have C_1 for f_c.

Let f_s satisfy $C_p, p \in (1, \infty)$. Then

$$f_s(\xi) + f_c(\xi) \leq \alpha(1 + |\xi|^p) + \bar{\alpha}|\xi| \leq \max\{\alpha, \bar{\alpha}, 1\}(1 + |\xi|^p + |\xi|),$$

for some $\alpha \geq 0, \bar{\alpha} = \sqrt{N} \max_i \alpha_i$. For all ξ with $|\xi| > 1$ we have $|\xi|^p > |\xi|$ so

$$f(\xi) \leq \max\{\alpha, \bar{\alpha}, 1\}(1 + |\xi|^p).$$

The set of all ξ with $|\xi| \leq 1$ is compact in $\mathbb{R}^{N \times n}$. Since f is continuous it is bounded on this set, denote the upper bound by $C > 0 \in \mathbb{R}^+$. Then for all ξ we have

$$f(\xi) \leq \max\{C, \max\{\alpha, \bar{\alpha}, 1\}(1 + |\xi|^p)\} \leq a(1 + |\xi|^p)$$

with $a = \max\{C, \alpha, \bar{\alpha}, 1\}$ and f satisfies the upper bound of C_p. For the lower bound we notice that $f_c \geq 0$ such that the same q lower bound holds for f as for f_s. Together we have that f satisfies C_p.

If f_s satisfies C_1 we have easily

$$|f(\xi)| \leq |f_s(\xi) + f_c(\xi)| = (\alpha + \bar{\alpha})(1 + |\xi|),$$

such that f satisfies C_1. If f_s satisfies C_∞, then the new $\bar{\eta}(|\xi|) := \eta(|\xi|) + \bar{\alpha}(1 + |\xi|)$ implies C_∞ for f. We have all necessary properties of Theorem 12. So we can apply it to $f := f_s + f_c$ and the lemma follows in $W^{1,p}(\Omega)$.

The conditions for l.s.c. of $SBV(\Omega)$ are also Carathéodory, quasiconvexity and a growth condition as described in Theorem 17. We can use the same arguments to conclude the result for $SBV(\Omega)$ instead of $W^{1,p}(\Omega)$ since the growth condition necessary is a special case of C_p as discussed in the remark after Theorem 17. □

5.2 $\mathcal{M}(\Omega)$ with $n = 1$

The notation in this case is not straightforward as our Banach space $\mathcal{B}$ is now given by $[\mathcal{M}(\Omega)]^N$ with the variation norm as seen in Section 2.3. The integrand f as in Theorem 14 is still a vector-based function but now with one-dimensional domain. Care must be taken as our integrands f as in Section 4.1 are only defined on function spaces. We define new corresponding functions $f_.$ such that we can take $f(\mu) = f_{gGD}(\mu), f_{VTV}(\mu), f_{TNV}(\mu), f_{TSV}(\mu)$. We want to define integrands $f(\cdot) : [\mathcal{M}(\Omega)]^N \to [0, \infty]$ that measure structural similarity. For a measure $\mu \in [\mathcal{M}(\Omega)]^N$ we can assume that the structure is encoded in $\frac{d\mu^a}{d\mathcal{L}^1} \in [L^1(\Omega)]^N$. We can define our $f(\cdot)$ in the natural way where we consider the Radon-Nikodym derivatives similarly as the weak derivatives ∇u for $u \in W^{1,p}(\Omega)$. We use similar notation as from the context we can see which definition is used.

Definition 22 (Matrix norms on $\mathcal{M}(\Omega)$)**.** *Define for $\Omega \subset \mathbb{R}, \mu \in [\mathcal{M}(\Omega)]^N$ the following $f_. : [\mathcal{M}(\Omega)]^N \to [0, \infty)$;*

$$f_{gGD}(\mu) := \frac{1}{2} \sum_{i=1,\dots,N} \sum_{j=1,\dots,i} \left(\left(\frac{d\mu^a}{d\mathcal{L}^1} \right)^i - \left(\frac{d\mu^a}{d\mathcal{L}^1} \right)^j \right)^2,$$

$$f_{TNV}(\mu) := \left\| \frac{d\mu^a}{d\mathcal{L}^1} \right\|_{l^1},$$

$$f_{VTV}(\mu) := \left\| \frac{d\mu^a}{d\mathcal{L}^1} \right\|_{l^2},$$

$$f_{TSV}(\mu) := \left\| \frac{d\mu^a}{d\mathcal{L}^1} \right\|_{l^\infty}.$$

Lemma 23. *The functions $f_{gGD}, f_{TNV}, f_{VTV}, f_{TSV}$ as in the definition above are jointly convex for $n = 1, N \in \mathbb{N}$.*

Proof. Note that taking a linear combination is well-defined for Radon-Nikodym derivatives. So that convexity is defined by the usual condition as in Definition 11. For $n = 1$ we have equivalence between rank one convexity and convexity. Taking into account the proofs of Section 4.3 the only integrands that possibly are convex are

$$f_{gGD}, f_{(j)(g)CG}, f_{(j)(g)Nambu}, f_{VTV}, f_{TNV}, f_{TSV}.$$

The cross-product in one dimension is equal to 0, so both $f_{(j)(g)CG}$ and $f_{(j)(g)Nambu}$ are equal to the zero function. Although this is a convex function, it is the trivial case and does not act as a regularizer anymore, we do not consider this trivial case. By the equivalence of convexity we have $f_{gGD}, f_{TNV}, f_{VTV}, f_{TSV}$ convex for $n = 1$. □

Lemma 23 states that these newly defined functions are l.s.c. and convex. The Schatten norms can be simplified as $\frac{d\mu^a}{d\mathcal{L}^1}$ is a function in $[L^1(\Omega)]^N$ and can be considered a vector in $\mathbb{R}^N$ for a fixed $x \in \Omega$. The singular values of a vector are given by its 2-norm followed by all 0's. Plugging this into the Schatten norms, we get

$$f_{VTV}(\mu) = \sum_{i=1}^{N} \left\| \left(\frac{d\mu^a}{d\mathcal{L}^1} \right)^i \right\|_{L^1(\Omega)}^2,$$

and

$$f_{TNV}(\mu) = f_{TSV}(\mu) = \sqrt{\sum_{i=1}^{N} \left\| \left(\frac{d\mu^a}{d\mathcal{L}^1} \right)^i \right\|_{L^1(\Omega)}^2}.$$

Theorem 20. *[Well-posedness in $\mathcal{M}(\Omega)$] Let $\Omega \subset \mathbb{R}$ be open, $N \in \mathbb{N}$. Take as Banach space $\mathcal{B} = [\mathcal{M}(\Omega)]^N$ with the corresponding weak$-*$ topology. Let $\mathcal{H} = \bigoplus_{i=1}^N \mathcal{H}_i$ where $\mathcal{H}_i$ are Hilbert spaces with $\tau_{\mathcal{H}_i}$ l.s.c norms. Let $K : \mathcal{B} \to \mathcal{H}$ be a linear operator that is sequentially continuous wrt $\tau_{\mathcal{B}}, \tau_{\mathcal{H}}$. $d \in \mathcal{H}, \mu \in \mathcal{B}, \alpha > 0$. Let $J : \mathcal{B} \to \mathbb{R} \cup \{\infty\}$ be given by*

$$J(\mu) := \int_\Omega f(\mu)\, d\mathcal{L}^1,$$

for $f(\cdot) \in \{f_{gGD}, f_{VTV}, f_{TNV}, f_{TSV}\}$*. Additionally, let* K *be such that* (K, J) *is mean coercive. Then*

$$argmin_{\mu\in\mathcal{B}}\|K\mu - d\|_{\mathcal{H}}^2 + \alpha J(\mu)$$

has a minimizer. Furthermore, this minimizer is unique if $f = f_{gGD}$ *or* K *is injective.*

Proof. As with all these proofs for well-posedness, we want to apply Theorem 1. It is necessary to check all conditions in this statement. Via Corollary 2 we can transport our properties on the components of $\bigoplus_{i=1}^N \mathcal{B}_i$ and $\bigoplus_{i=1}^N \mathcal{H}_i$ to properties of $\mathcal{B}, \mathcal{H}$. From the discussion in Section 2.3, we know that our combination of choosing as Banach space $\mathcal{B} = [\mathcal{M}(\Omega)]^N$ with the variation topology gives the required properties for Theorem 1. Our assumptions on $\mathcal{H}_i, \tau_{\mathcal{H}_i}$ also give the required properties of Theorem 1 for the Hilbert space.

Note that by the definitions above all f are well-defined and real-valued. Additionally, by non-negativity of f and boundedness of Ω we have that J is non-negative and proper. The (mean) coercivity of J in $\frac{d\mu^a}{d\mathcal{L}^1}$ comes from the proofs in Section 4.3.

Our assumptions on K are the same as in Theorem 1 and we have assumed mean coercivity of (K, J).

Only left is the weak l.s.c of J. If we take $f(\mu) = \varphi\left(\frac{d\mu^a}{d\mathcal{L}}\right)$, we need to prove l.s.c., joint convexity and the recession condition to apply Theorem 14. Lemma 23 gives joint convexity of our functions f. Since norms are continuous, f_{TNV}, f_{VTV} are l.s.c. in particular. As f_{gGD} is a polynomial in $[L^1(\Omega)]^N$ it is also continuous.

We compute the recession functions for functions of type $f := \|\cdot\|_{l^p}$. By homogeneity of a norm , for non-trivial $\frac{d\mu^a}{d\mathcal{L}^1}$ we have

$$f_\infty\left(\frac{d\mu^a}{d\mathcal{L}^1}\right) = \lim_{t\to\infty} \frac{\|t\frac{d\mu^a}{d\mathcal{L}^1}\|_{l^p} - \|0\|_{l^p}}{t} = \lim_{t\to\infty} \frac{|t|\|\frac{d\mu^a}{d\mathcal{L}^1}\|_{l^p}}{t} = \left\|\frac{d\mu^a}{d\mathcal{L}^1}\right\|_{l^p} > 0.$$

For $f\left(\frac{d\mu^a}{d\mathcal{L}^1}\right) = f_{gGD}(\mu)$ in the non-trivial case we have

$$f_\infty\left(\frac{d\mu^a}{d\mathcal{L}^1}\right) = \lim_{t\to\infty} \sum_{i=1,\dots,N} \sum_{j=1,\dots,i} \frac{\left(\left(\frac{d\mu^a}{d\mathcal{L}^1}\right)^i - \left(\frac{d\mu^a}{d\mathcal{L}^1}\right)^j\right)^2 - f(0)}{t}$$

$$= \lim_{t\to\infty} \sum_{i=1,\dots,N} \sum_{j=1,\dots,i} t\left(\left(\left(\frac{d\mu^a}{d\mathcal{L}^1}\right)^i\right)^2 + \left(\left(\frac{d\mu^a}{d\mathcal{L}^1}\right)^j\right)^2\right) + 2\left(\frac{d\mu^a}{d\mathcal{L}^1}\right)^i\left(\frac{d\mu^a}{d\mathcal{L}^1}\right)^j = \infty.$$

So the conditions of Theorem 14 are satisfied and we consequently get weakly$-*$ l.s.c. of J.

Finally, we can apply Theorem 1 and get existence of a minimizer for the variational problem. By the strict convexity of f_{gGD} we immediately get a unique minimizer via Lemma 5. Note that a norm is never strictly convex by homogeneity, so in the other cases only the injectivity of K can imply uniqueness of the minimizer. $\square$

5.3 $BV(\Omega)$ with isotropic integrand

Based on Theorem 15, our challenge of determining the well-posedness of the Tikhonov-type regularization problem is given by determining which functions $f_{\cdot}$ defined in Section 4.1 are isotropic (this means that we can write it is a function of $|\nabla b|$) and test if they satisfy the properties necessary in the theorem.

We can quickly forget about most of the integrands considered in this thesis, as we can rule out isotropy immediately in most cases. If we take $\xi \in \mathbb{R}^{2\times n}$ (or $\mathbb{R}^{N\times n}$) and consider

$$\xi = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \eta = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix},$$

with additional zeros in the case $N \neq 2$, then the element-wise norms are equal as

$$|\xi| = |\eta| = 2,$$

however

$$|\xi^1||\xi^2| = 2 \neq 0 = |\eta^1||\eta^2|,$$

and

$$\langle \xi^1, \xi^2 \rangle = 2 \neq 0 = |\eta^1||\eta^2|.$$

So any function $f(\cdot)$ including any of these terms cannot be isotropic as we can find two matrices $\xi \neq \eta$ with equal norm that have different $f(\cdot)$ value. Every function except the Schatten norms f_{TSV}, f_{VTV}, and f_{TNV} belong to this class and hence cannot be isotropic. In regards to these matrix norms we have that $f_{VTV}(\xi) = |\xi|$ and hence is the trivial case, this is still interesting as using f_{VTV} as integrand results in a well-posed problem over $[BV(\Omega)]^N$. Although f_{TNV} and f_{TSV} are equivalent to f_{VTV} when regarded as norms a proof of an increasing mapping between them as necessary for Theorem 15 could not be found or constructed. We conjecture that such a function does not exist based on the complex behaviour of the Schatten norms with small perturbations in the singular values. As the only structural integrand giving a well-posed problem in this setting is the trivial one given by f_{VTV}, we have opted to not include a lemma or proof of this fact.

5.4 $SBV(\Omega)$

The complexity of the structure of all compact subspaces of the space $SBV(\Omega)$ as discussed in Appendix A increases the explicit conditions for well-posedness of the variational problem. However,

Theorem 21 (Well-posedness in $SBV(\Omega)$)**.** *Let $\Omega \subset \mathbb{R}$ be open, bounded and Lipschitz, $N \in \mathbb{N}$. Take as Banach space $\mathcal{B}$ a subspace of the Banach space $[SBV(\Omega)]^N$ with the corresponding weak$-*$ topology Additionally, assume there is some l.s.c. increasing $\theta : (0, \infty) \to (0, \infty]$ with*

$$\lim_{t \to 0} \frac{\theta(t)}{t} = \infty,$$

where

$$\sup_{b \in \mathcal{B}} \left\{ \int_{J_u} \theta \left(|u^+ - u^-| \right) d\mathcal{H}^{n-1} \right\} < \infty. \tag{59}$$

Let $\mathcal{H} = \bigoplus_{i=1}^N \mathcal{H}_i$ where $\mathcal{H}_i$ are Hilbert spaces with $\tau_{\mathcal{H}_i}$ l.s.c norms. Let $K : \mathcal{B} \to \mathcal{H}$ be a linear operator that is sequentially continuous wrt $\tau_{\mathcal{B}}, \tau_{\mathcal{H}}$. $d \in \mathcal{H}, b = (u_1, \ldots, u_N) \in \mathcal{B}, \alpha > 0$. Let $J : \mathcal{B} \to \mathbb{R} \cup \{\infty\}$ be given by

$$J(b) := \int_{\Omega} f_s(\nabla u_1, \ldots, \nabla u_N) dx + \sum_{i=1}^{N} \alpha_i TV[u_i], \alpha_i > 0$$

for $f_s(\cdot) \in \{f_{gGD}, f_{jCG}, f_{jNambu}, f_{gCG}, f_{gNambu}\}$. Additionally, let K be such that (K, J) is mean coercive. Then

$$argmin_{b \in \mathcal{B}} \|Kb - d\|^2_{\mathcal{H}} + \alpha J(b)$$

has a minimizer. Furthermore, this minimizer is unique if K is injective.

For compactness of our Banach space $\mathcal{B}' \subset [SBV(\Omega)]^N$ we see in Theorem 37 that for each uniformly bounded sequence $(u_h)_h \subset [SBV(\Omega)]^N$ we can find corresponding φ, θ such that we have

$$\sup_h \left\{ \int_{\Omega} \varphi(|\nabla u_h|) dx + \int_{J_{u_h}} \theta(|u_h^+ - u_h^0|) d\mathcal{H}^{n-1} \right\} < \infty.$$

As we will see in the proof below, we can forget about the φ since C_p with $p > 1$ implies its existence. Now instead of a θ defined on the whole of $\mathcal{B}$, we can also take a collection of $(\theta_k)_{k \in K}$ for some index set K with for each k satisfying the properties for some $\mathcal{B}^k \subset SBV(\Omega)$. Then taking $\mathcal{B} = \bigcup_{k \in K} \mathcal{B}^k$ by considering these subspaces simultaneously is permitted.

Proof. The proof is almost equivalent to the proof of the second J in Theorem 19 with $f_s + TV$. The only difference being the additional functional conditions of $\mathcal{B}$ and that we need l.s.c. in $SBV(\Omega)$ instead of $W^{1,p}(\Omega)$. Here taking as integrands $f_{VTV}, f_{TNV}, f_{TSV}$ do not lead to well-posed problems as these have linear growth and subsequently cannot satisfy the lower bound $c|\xi|^p \leq f(\xi)$ with $p > 1$ for some $c > 0$.

Instead of Lemmata 22 and 11 for l.s.c in $W^{1,p}(\Omega)$ of the two cases for f we now use Lemmata 22 and 17 to get weak l.s.c. in $[SBV(\Omega)]^N$.

We need compactness of $\mathcal{B}$ with respect to the weak$-*$ topology. This is the case when we satisfy the conditions of Theorem 37. In the assumptions above, a satisfactory θ provides two things; a bound on both the size of the jump set $J_b, b \in \mathcal{B}$ and magnitude of the jumps $|b^+ - b^-|$ in J_b. We still need to prove existence of a suitable φ. The claim is that the growth condition necessary for the l.s.c. in $SBV(\Omega)$ provides us with a φ. By the l.s.c. in $SBV(\Omega)$ we have for some $p > 1$

$$0 \leq f(\xi) \leq \alpha(1 + |\xi|^p).$$

Take $\varphi : [0, \infty) \to [0, \infty]$ as $\varphi(x) = \alpha(1 + |x|^p)$. Then this is easily seen to be lower semi-continuous and increasing in x. Also

$$\lim_{t \to \infty} \frac{\alpha(1 + |t|^p)}{t} = \infty,$$

since $\alpha > 0, p > 1$. Finally, let $u_h \in SBV(\Omega)$ be a uniformly bounded sequences with $\|u_h\|_\infty < C$ for all h. Then since Ω is bounded,

$$\sup_h \left\{ \int_\Omega \varphi(|\nabla b_h|) dx \right\} = \sup_h \left\{ \int_\Omega \alpha(1 + |\nabla b_h|^p) dx \right\} \leq \int_\Omega \alpha(1+C^p) dx \leq \alpha(1+C^p)|\Omega| < \infty.$$

Hence we can dispense of the need to provide adequate φ for compactness of $\mathcal{B}$. For the rest of the argument we lead to reader towards the proof of Theorem 19. $\square$

6 Relaxation

We are primarily interested in functions f_s that are separately convex but are not jointly convex as by Theorem 28 it is necessary for numerical convergence. Functions that are quasiconvex give l.s.c. of the integral regularizer J via the Direct method. From Table 1 we see that these are f_{DOT}, f_{aDOT}, and f_{LP}. There are largely two main avenues of approach in adapting our joint inverse problem such that it becomes well-posed for non quasiconvex integrands.

The first being compensated compactness, explored in Section 6.1, where we can provide weak continuity of non-linear expressions $f_s(\nabla b)$ using additional functional analytic assumptions.

The second utilizing (quasi)-convex relaxation, explored in Section 6.2. This is a method where we define a new integral regularizer $\bar{J}$ called the relaxed problem as

$$\bar{J}(b) := \int_\Omega Qf(\nabla b(x))dx, \tag{60}$$

where Qf is the quasiconvex envelope of f, similarly defined as in convex analysis or used in Γ-convergence.

Definition 23 (Envelopes)**.** *For L = C(convex), P(polyconvex), Q(quasiconvex), R(rank-one-convex) we have the L-convex envelope of a function f given by*

$$Lf(\xi) := \sup\{g(\xi) \leq f(\xi) : g \text{ is } L - convex\}.$$

With these definitions, we can apply the well-posedness results from Section 5 with joint structural regularizer $\bar{J}$ as the Direct method is applicable to the quasiconvex integrand Qf.

6.1 Compensated compactness

In general, nonlinear expressions do not commute with weak$-*$ limits. However, there are specific cases where weak$-*$ convergence of the variables does guarantee weak continuity of the given expression. A prominent example are the minors of a gradient matrix ξ with in particular the determinant [54]. Providing the compatibility of nonlinear functions with weak($-*$) limits is the core of the theory of compensated compactness, with its most important results being concerned with products. We have the following div-curl lemma.

Definition 24 (Precompactness in a metric space)**.** *Let X be a metric space. A subset $Y \subset X$ is called precompact (or relatively compact) if any sequence of elements $(y_n)_n \in Y$ has a subsequence $(y_{n_j})_j \in Y$ that metrically converges to some element $y \in X$.*

Theorem 22 (Murat-Tartar)**.** *[54] Assume that the sequences* $(u_j), (v_j) \subset L^2(\Omega)$ *are such that* $u_j \rightharpoonup u, v_j \rightharpoonup v$ *in* L^2 *and that* $(div\ u_j)_j, (curl\ v_j)_j$ *are precompact in* $W^{-1,2}_{loc}(\Omega)$*. Then*

$$u_j \cdot v_j \rightharpoonup u \cdot v \ in \ L^1_{loc}.$$

Remark 18. *There is a generalisation for* $1 < p < \infty$ *and* q *its conjugate on* $(u_j) \subset L^p(\Omega), (v_j) \subset L^q(\Omega)$ *[51] along similar lines.*

Let $(u_j)_j \in W^{1,p}(\Omega)$ with Δu_j relatively compact. Since we are working in a metric space we have $\{\Delta u_j\}_j$ precompact iff any sequence in the set has a norm-convergent subsequence.

If we pick for the precompact sets in particular the null space of the differential operators, we have the conditions: curl $v_j = 0$ and div $u_j = 0$. Now in particular we are interested in sequences of gradients that weakly convergence such that we apply Theorem 22 to sequences $\bar{u}_j := \nabla u_j$ and $\bar{v}_j := \nabla v_j$ for some $(u_j), (v_j) \in W^{1,p}_{\mathrm{loc}(\Omega)}$. From standard calculus we have that the curl of a gradient field is always zero. Hence the condition on v_j is non-restrictive and the condition on u_j (div $\nabla u_j = \Delta u_j = 0$) characterizes the set of harmonic functions. Taking the Laplacian of discontinuous functions is not well-defined in the classical calculus sense, we remedy this by using the usual weak forms defined on Sobolev spaces.

Definition 25 (Weak Laplacian)**.** *Let* Ω *an open bounded domain with* C^1 *boundary. The strong solution of the Dirichlet problem is given by solving for* u *in some function space*

$$\Delta u = f, \ \ in \ \Omega,$$

and

$$u = 0, \ \ in \ \partial\Omega,$$

for given $f \in L^2(\Omega)$*. We call* $u \in W^{1,2}(\Omega)$ *a weak solution if*

$$\int_\Omega \nabla u \cdot \nabla v = \langle f, v \rangle; \ \forall v \in W^{1,2}_0(\Omega).$$

Remark 19. *We have only defined the Dirichlet problem with zero boundary condition for ease of understanding, we can extend the definition and all results to non-zero boundary conditions.*

Using the fact that $W^{1,2}(\Omega)$ is Hilbert, via the Lax-Milgram Lemma [49] (or Riesz representation Theorem 3) we have a function $-\Delta u \in W^{-1,2}(\Omega)$ such that

$$\langle -\Delta u, v \rangle = \langle f, v \rangle; \forall v \in W^{1,2}_0(\Omega).$$

This is meant when stating that $-\Delta u = f$ in the distributional sense. This distributional solution is now taken to be the definition of a Laplacian on $W^{1,2}(\Omega)$.

Lemma 24 (Div-Curl for gradient fields)**.** *Assume we have sequences $u_j, v_j \in W^{1,2}(\Omega)$ such that $\nabla u_j \rightharpoonup \nabla u, \nabla v_j \rightharpoonup \nabla v$ in $L^2(\Omega)$. Additionally assume that for all j, $\Delta u_j \in \mathcal{K}$ for some precompact set $\mathcal{K} \subset W^{-1,2}(\Omega)$. Then*

$$\nabla u_j \cdot \nabla v_j \rightharpoonup \nabla u \cdot \nabla v \text{ in } L^1_{loc}(\Omega).$$

Proof. By definition, $u_j, v_j \in W^{1,2}(\Omega)$ implies $\nabla u_j, \nabla v_j \in L^2(\Omega)$. For weak differential operators we still have the usual identities $\operatorname{curl} \nabla u = 0$ and $\operatorname{div} \nabla u = \Delta u$ in the distributional sense. Hence, following the discussion below Theorem 22 leads to a direct application and we immediately get the result. □

This also explains the nomenclature as we regard only a subspace of the possible gradients, in particular those that have $\Delta u_j \in \mathcal{K}$. This additional topological condition then "compensates" for the lack of compactness through the dot product in general. We apply the lemma above to some specific structural regularizers $J(u, v)$.

Lemma 25 (Weak l.s.c of convex functions)**.** *[42] Let $F : \mathbb{R} \to \mathbb{R}$ convex. Then if $u_n \rightharpoonup u$ in $L^1(\Omega)$ we have*

$$\int F(u(x))dx \leq \liminf_n \int F(u_n(x))dx.$$

Theorem 23 (L.s.c using div-curl lemma)**.** *Let $\Omega \subset \mathbb{R}^n$ bounded. Let $(u_j, v_j)_j \subset [W^{1,2}(\Omega)]^2$ with $u_j \rightharpoonup u, v_j \rightharpoonup v \in W^{1,2}(\Omega)$. Assume $(\Delta u_j)_j$ a precompact set in $W^{-1,2}(\Omega)$. Let $J : [W^{1,2}(\Omega)]^2 \to \mathbb{R}$ given by*

$$J(u, v) := \int_\Omega F\left(\nabla u(x) \cdot \nabla v(x)\right) dx, \tag{61}$$

with $F : \mathbb{R} \to \mathbb{R}$ a convex function. Then, with possibly taking a subsequence

$$\liminf_{j \to \infty} J((u_j, v_j)_j) \geq J(u, v).$$

Proof. Let $(u_j, v_j)_j \in [W^{1,2}(\Omega)]^2$, then $\nabla u_j, \nabla v_j \in L^2(\Omega)$. By the definition of the $W^{1,2}(\Omega)$ norm, we have strong convergence of $\nabla u_j \to \nabla u, \nabla v_j \to \nabla v$ in $L^2(\Omega)$. Note that strong convergence implies weak convergence such that $\nabla u_j \rightharpoonup \nabla u, \nabla v_j \rightharpoonup \nabla v$ in $L^2(\Omega)$ up to some sub-sequence. Applying Lemma 24 we have that weakly $\nabla u_j \cdot \nabla v_j \to \nabla u \cdot \nabla v$ in $L^1_{\text{loc}}(\Omega)$ up to sub-sequence. Denote this possible sub-sequence by the same notation $(u_j, v_j)_j$. For bounded Ω we have $b_j \rightharpoonup b$ in $L^1(\Omega)$ iff $b_j \rightharpoonup b$ in $L^1_{\text{loc}}(\Omega)$. Then we can apply Lemma 25 to $(b_j)_j := (u \cdot v)_j \in L^1(\Omega)$ with F and get

$$\liminf_j J(u_j, v_j) = \liminf_j \int_\Omega F\left(\nabla u_j(x) \cdot \nabla v_j(x)\right) \geq \int_\Omega F\left(\nabla u \cdot \nabla v\right) = J(u, v).$$

□

Note that in particular we can apply Theorem 23 to using integrands f_s that are convex transformations of the dot-product. Specifically f_{DOT} and f_{aDOT}.

6.2 Quasiconvex relaxation

Computing the (quasi)convex envelopes of functions is of interest to minimization problems because of the reasons. Firstly, as explained earlier, computing the quasiconvex envelope Qf of a function f enables the usage of the Direct method to an inverse problem with the relaxed integral regularizer $\bar{J}$ as in Equation (60). Secondly, when not dealing with a data fidelity term and interested solely in the minimization of an energy functional $J(\nabla b)$, the infimum cannot by attained using minimizing sequences of J in the set of admissible functions. However, with some coercivity conditions, we can find a minimizing sequence attaining the infimum the relaxed problem $\bar{J}$.

In this thesis, we are mainly interested in the first application but our results on the envelopes of specific integrands f can also be applied in the direct minimization problems in the second application. Where we can also deduce properties of the minimizers of system reliant on energy functionals using f_{DOT} and f_{aDOT}.

Definition 26 (Relaxed problem). *Let $p \in [0, \infty]$. Let a variational problem called (P) be given by*

$$\inf \left\{ \int_\Omega f(\nabla b) dx, u \in u_0 + W_0^{1,p}(\Omega) \right\}.$$

Then the relaxed problem (QP) is given by

$$\inf \left\{ \int_\Omega Qf(\nabla b) dx, u \in u_0 + W_0^{1,p}(\Omega) \right\}.$$

Theorem 24 (Relaxation). *[18] Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be Borel measurable function satisfying, for $1 \leq p < \infty$ and all $\xi \in \mathbb{R}^{N\times n}$*

$$g(\xi) \leq f(\xi), |g(\xi)|, |f(\xi)| \leq \alpha(1 + |\xi|^p),$$

where $g : \mathbb{R}^{N\times n} \to \mathbb{R}$ is quasiconvex and $\alpha > 0$ is a constant, while for $p = \infty$ it is assumed that f is locally bounded and bounded below by g. Then

$$\inf(P) = \inf(QP).$$

More precisely, for every $p \leq q \leq \infty$ and $u \in W^{1,q}(\Omega)$, there exists a sequence $\{u_\nu\}_\nu \subset u + W_0^{1,q}(\Omega)$ such that $u_\nu \to u$ in $L^q(\Omega)$ and

$$\int_\Omega f(\nabla u_\nu(x)) dx \to \int_\Omega Qf(\nabla u(x)) dx.$$

Remark 20. *Note that this convergence is in general not in any Sobolev space. However, if we also have p-undergrowth of f with $p > 1$, we can get convergence of the minimizing sequence u_ν to u in $W^{1,p}(\Omega)$ [18].*

From the nested structure (as seen in Figure 1) of the different types of convexity, we have

$$Cf \leq Pf \leq Qf \leq Rf \leq f.$$

In the light of the reasons outlined above, we are particularly interested in computing the convex and quasiconvex envelope for the given functions. We have the following representation for Qf.

Theorem 25 (Representation Qf)**.** *[18] Let $f : \mathbb{R}^{N\times n} \to \mathbb{R}$ be locally bounded and Borel measurable. Let $g : \mathbb{R}^{N\times n} \to \mathbb{R}$ be quasiconvex and such that $f(\xi) \geq g(\xi)$ for every $\xi \in \mathbb{R}^{N\times n}$. Then, for every ξ,*

$$Qf(\xi) = \inf\left\{\frac{1}{\mathcal{L}^n(D)}\int_D f(\xi + \nabla\phi(x))dx : \phi \in W_0^{1,p}(D)\right\}, \tag{62}$$

where $D \subset \mathbb{R}^n$ is a bounded open set. In particular it is independent of the choice of D.

From the definition, we see that the global relaxation can be regarded as point-wise over ξ and can be looked at locally. However, similarly as in the definition of quasiconvexity (Definition 13), the right-hand-side is difficult to compute explicitly due to the large search space. This is why the rank one convex envelope Rf is used as a first approximation of Qf.

6.2.1 Rank one envelope

If we can prove that $Rf(\xi) = 0$ for a non-negative f such that $f(\xi) \geq 0$, then $Qf \leq Rf$ implies that also $Qf = 0$. There are multiple representation formulas for Rf, we will be working with the one below, based on repeated laminations.

Theorem 26 (Representation Rf)**.** *[18] Let $f : \mathbb{R}^{N\times n} \to \mathbb{R} \cup \{\infty\}$. Let $g : \mathbb{R}^{N\times n} \to \mathbb{R}\cup\{\infty\}$ be rank one convex and such that $f(\xi) \geq g(\xi)$ for every $\xi \in \mathbb{R}^{N\times n}$. Let $R_0 f := f$ and for $k \in \mathbb{N}$ define inductively*

$$R_{k+1}f(\xi)$$

$$:= \inf_{\lambda\in[0,1], A,B\in\mathbb{R}^{N\times n}} \{\lambda R_k f(A) + (1-\lambda)R_k f(B) : \lambda A + (1-\lambda)B = \xi \text{ with } rank\{A-B\} \leq 1\}.$$

Then point-wise for ξ, $Rf(\xi) = \lim_{k\to\infty} R_k f(\xi) = \inf_{k\in\mathbb{N}} R_k f(\xi)$.

We will use Theorem 26 to prove that the rank one convex envelopes of f_{DOT} and f_{aDOT} are both equal to the zero function. In fact, while a priori there is no exact k such that $R_k f = Rf$, we will see in the proof that already for $k = 2$ we have $R_2 f = 0$, this together with the non-negativity of f_{DOT}/f_{aDOT} implies that $Rf = R_2 f = 0$. We first need a technical lemma about a particular type of approximation of $R_k f$. It tells us that we can "rank one convexify" via repeated lamination of a majorant function in each step. We can see the $\bar{R}_k$ as a kind of forcing function that "drives" R_k to be below it in each step.

Lemma 26 (Forcing $R_k f$)**.** *Let $\xi \in \mathbb{R}^{N\times n}$. Let $f : \mathbb{R}^{N\times n} \to [0, \infty]$. Define inductively $\bar{R}_0 f(\xi) := f(\xi)$,*

$$\bar{R}_{k+1} f(\xi) := \lambda_k \bar{R}_k f(A_k) + (1 - \lambda_k)\bar{R}_k f(B_k)$$

with some $\lambda_k \in [0, 1], A_k, B_k \in \mathbb{R}^{N\times n}$ with

$$\xi = \lambda_k A_k + (1 - \lambda_k)B_k,$$

and

$$\text{rank}\{A_k - B_k\} \leq 1.$$

(Note this is not a uniquely determined value and depends on the choice of $\lambda_l, (A_l, B_l)$ in each step $l \leq k$) Then for all $k \in \mathbb{N}$ and any choice of $\lambda_1, \ldots, \lambda_k, A_1, B_1, \ldots, A_k, B_k$ we have point-wise for ξ

$$R_k f(\xi) \leq \bar{R}_k f(\xi).$$

Proof. Let $\xi \in \mathbb{R}^{N\times n}$. For $k = 1$ we have

$$\bar{R}_1 f(\xi) = \lambda_0 f(A_0) + (1 - \lambda_0)f(B_0),$$

for some A_0, B_0 with $\text{rank}\{A_0 - B_0\} \leq 1$ and $\xi = \lambda_0 A_0 + (1 - \lambda_0)B_0$. By definition of $R_1 f(\xi)$, these λ_0, A_0, B_0 are in the admissible set that the infimum is taken over. By the definition of an infimum over a set, we have

$$R_1 f(\xi) \leq \bar{R}_1 f(\xi).$$

Assume the statement holds for all values up to and including $k \in \mathbb{N}$. Then for $k+1$ we have

$$R_{k+1} f(\xi)$$

$$:= \inf_{\lambda\in[0,1],A,B\in\mathbb{R}^{N\times n}} \{\lambda R_k f(A)+(1-\lambda)R_k f(B) : \lambda A+(1-\lambda)B = \xi \text{ with } \text{rank}\{A-B\} \leq 1\}.$$

For a point-wise majorant $g(x) \geq l(x) \geq 0$ for all $x \in C$ with C a set, we have in general

$$\inf_x l(x) \leq \inf_x g(x).$$

By positivity of f and λ and since $R_k f \leq \bar{R}_k f$ by assumption we can take $g = \bar{R}_k f$ and $l = R_k f$, then take $C := \{\lambda \in [0,1], A, B \in \mathbb{R}^{N\times n} : \lambda A + (1-\lambda)B = \xi \text{ with } \operatorname{rank}\{A - B\} \leq 1\}$. We get

$$\begin{aligned} &\inf_C \{\lambda R_k f(A) + (1-\lambda) R_k f(B)\} \\ &\leq \inf_C \{\lambda \bar{R}_k f(A) + (1-\lambda) \bar{R}_k f(B)\}. \end{aligned}$$

Now picking a particular choice of admissible A_k, B_k, λ_k can only result in a bigger value than the infimum over C and hence

$$\leq \lambda_k \bar{R}_k(A_k) + (1-\lambda_k)\bar{R}_k(B_k) =: \bar{R}_{k+1} f(\xi).$$

Which proves the claim. □

Lemma 27 ($Rf = 0$)**.** *For $n = 2$, we have $Rf = 0$ for $f = f_{DOT}, f = f_{aDOT}$.*

Proof. Let $\xi \in \mathbb{R}^{2\times 2}$. The proof consists of two parts, in the first part we define $A_\lambda, B_\lambda \in \mathbb{R}^{2\times 2}$ such that $\operatorname{rank}\{A_\lambda - B_\lambda\} \leq 1$ and for any $\lambda \in (0,1)$ we have $\lambda A_\lambda + (1-\lambda)B_\lambda = \xi$. Now defining $C := \{\lambda \in [0,1], A, B \in \mathbb{R}^{N\times n} : \lambda A + (1-\lambda)B = \xi \text{ with } \operatorname{rank}\{A - B\} \leq 1\}$ in the definition of $\bar{R}_k f$, we can take the limit $\lambda \to 0$ with A_λ, B_λ such that we stay inside C. This limit will then be proven to be

$$\lim_{\lambda \downarrow 0} \lambda f(A_\lambda) + (1-\lambda) f(B_\lambda) = \min\{|\xi_1^1 \xi_1^2|, |\xi_2^1 \xi_2^2|\}.$$

Since we have done one lamination over A_λ, B_λ, we get by definition of $\bar{R}_k$,

$$\bar{R}_1 f(\xi) \leq \min\{|\xi_1^1 \xi_1^2|, |\xi_2^1 \xi_2^2|\}.$$

This holds for any ξ, and since the zero function is a rank one convex minorant for f, we have that $Rf \geq 0$. In the second step we will prove that based on this choice of $\bar{R}_1$ we have $\bar{R}_2 f \leq 0$. Via Lemma 26 we get $0 \leq R_2 f \leq \bar{R}_2 f \leq 0$, which implies $R_2 f = 0$ and finally $Rf = \inf_{k\in\mathbb{N}} R_k f = R_2 f = 0$.

Let $\mu \in \mathbb{R} \setminus \{0\}$. Define

$$A_\lambda^\mu := \begin{pmatrix} 0 & \xi_2^1 - \frac{\xi_1^1}{\mu} \\ \frac{\xi_1^2}{\lambda} & \xi_2^2 + \frac{\xi_1^2}{\mu}\left(\frac{1}{\lambda} - 1\right) \end{pmatrix} \tag{63}$$

$$B_\lambda^\mu := \begin{pmatrix} \frac{\xi_1^1}{1-\lambda} & \xi_2^1 + \frac{\lambda \xi_1^1}{(1-\lambda)\mu} \\ 0 & \xi_2^2 - \frac{\xi_1^2}{\mu} \end{pmatrix} \tag{64}$$

Note that A_λ^μ and B_λ^μ both have a zero entry, this tells us that they are on axes. We now have two free parameters, $\lambda \in (0,1)$ tells us how far along ξ is along the line of between $A^\mu_\lambda, B^\mu_\lambda$ and μ adjusts the position on the ξ^1_1, ξ^2_1 axes where the points $A^\mu_\lambda, B^\mu_\lambda$ lie in such a way that ξ is still on the line between them. We compute

$$\lambda A^\mu_\lambda + (1-\lambda)B^\mu_\lambda = \begin{pmatrix} 0 & \lambda\xi^1_2 - \frac{\lambda\xi^1_1}{\mu} \\ \xi^2_1 & \lambda\xi^2_2 + \frac{\xi^2_1}{\mu}(1-\lambda) \end{pmatrix} + \begin{pmatrix} \xi^1_1 & (1-\lambda)\xi^1_2 + \frac{\lambda\xi^1_1}{\mu} \\ 0 & (1-\lambda)\xi^2_2 - \frac{(1-\lambda)\xi^2_1}{\mu} \end{pmatrix}$$

$$= \begin{pmatrix} \xi^1_1 & \xi^1_2 \\ \xi^2_1 & \xi^2_2 \end{pmatrix} = \xi.$$

Also

$$\operatorname{rang}\{A^\mu_\lambda - B^\mu_\lambda\} = \operatorname{rang}\begin{pmatrix} -\frac{\xi^1_1}{1-\lambda} & \xi^1_2 - \frac{\xi^1_1}{\mu} - \xi^1_2 - \frac{\lambda\xi^1_1}{(1-\lambda)\mu} \\ \frac{\xi^2_1}{\lambda} & \xi^2_2 + \frac{\xi^2_1}{\mu}\left(\frac{1}{\lambda}-1\right) - \xi^2_2 + \frac{\xi^2_1}{\mu} \end{pmatrix}$$

$$= \operatorname{rang}\begin{pmatrix} -\frac{\xi^1_1}{1-\lambda} & -\frac{\xi^1_1}{1-\lambda}\frac{\lambda(1-\lambda)}{\mu} \\ \frac{\xi^2_1}{\lambda} & \frac{\xi^2_1}{\lambda}\frac{\lambda(1-\lambda)}{\mu} \end{pmatrix} = 1.$$

Where the coefficient between the columns is given by $\frac{\lambda(1-\lambda)}{\mu}$. So $\lambda, A^\mu_\lambda, B^\mu_\lambda$ are valid choices for computing $\bar{R}_1 f(\xi)$. Now for $f := f_{aDOT}$;

$$\lambda f(A^\mu_\lambda) + (1-\lambda) f(B^\mu_\lambda)$$

$$= \lambda\left|0 + \left(\xi^1_2 - \frac{\xi^1_1}{\mu}\right)\left(\xi^2_2 + \frac{\xi^2_1}{\mu}\left(\frac{1}{\lambda}-1\right)\right)\right| + (1-\lambda)\left|0 + \left(\xi^1_2 + \frac{\lambda\xi^1_1}{(1-\lambda)\mu}\right)\left(\xi^2_2 - \frac{\xi^2_1}{\mu}\right)\right|$$

$$= \lambda\left|\xi^1_2\xi^2_2 - \frac{1}{\mu}\left(\xi^1_1\xi^2_2 + \left(\xi^1_2\xi^2_1 - \frac{\xi^1_1\xi^2_1}{\mu}\right)\left(\frac{1}{\lambda}-1\right)\right)\right| + (1-\lambda)\left|\xi^1_2\xi^2_2 + \frac{\lambda}{1-\lambda}\left(\frac{\xi^1_1\xi^2_2}{\mu} - \xi^1_1\xi^2_1\right) - \frac{\xi^1_2\xi^2_1}{\mu}\right|.$$

Note that we can pick $\mu \in \mathbb{R}\setminus\{0\}$ freely, as we let $\mu \to \infty$, we note that all the terms with $\frac{1}{\mu} \to 0$ as our values for ξ^i_j are fixed. Looking at the order of the terms in μ and λ we have

$$\lambda\left|\xi^1_2\xi^2_2 - \frac{1}{\mu}\left(\xi^1_1\xi^2_2 + \left(\xi^1_2\xi^2_1 - \frac{\xi^1_1\xi^2_1}{\mu}\right)\left(\frac{1}{\lambda}-1\right)\right)\right| + (1-\lambda)\left|\xi^1_2\xi^2_2 + \frac{\lambda}{1-\lambda}\left(\frac{\xi^1_1\xi^2_2}{\mu} - \xi^1_1\xi^2_1\right) - \frac{\xi^1_2\xi^2_1}{\mu}\right|$$

$$= \lambda\left|\xi^1_2\xi^2_2 - \mathcal{O}\left(\frac{1}{\mu}\right) - \mathcal{O}\left(\frac{1}{\lambda\mu}\right) - \mathcal{O}\left(\frac{1}{\mu^2}\right)\right| + (1-\lambda)\left|\xi^1_2\xi^2_2 + \frac{\lambda}{1-\lambda}\left(\mathcal{O}\left(\frac{1}{\mu}\right) + \mathcal{O}(1)\right) - \mathcal{O}\left(\frac{1}{\mu}\right)\right|.$$

We end up with arbitrarily small contribution of the terms with an order involving $\frac{1}{\mu}$. When taking μ to ∞ we have some ε depending on ξ, μ such that

$$\lambda f(A^\mu_\lambda) + (1-\lambda) f(B^\mu_\lambda) = \lambda\left|\xi^1_2\xi^2_2\right| + (1-\lambda)\left|\xi^1_2\xi^2_2 - \frac{\lambda}{1-\lambda}\mathcal{O}(1)\right| + \varepsilon.$$

Now since $\frac{\lambda}{1-\lambda} \to 0$ as $\lambda \to 0$, we can make the rhs equal to

$$= |\xi^1_2\xi^2_2| + \varepsilon,$$

with arbitrarily small $\varepsilon > 0$. Now by the definition of $\bar{R}_k$ we have choices of λ_0, A_0, B_0 such that $\bar{R}_1 f(\xi) = |\xi^1_2\xi^2_2| + \varepsilon$.

Via the same procedure but taking $A^\mu_\lambda, B^\mu_\lambda$ to have zero components in the second column instead of the first, and adjusting the other entries such that $A^\mu_\lambda, B^\mu_\lambda$ stay admissible to compute $\bar{R}_1 f$ we get choices for λ_0, A_0, B_0 such that

$$\bar{R}_1 f(\xi) = |\xi_1^1 \xi_1^2| + \varepsilon.$$

From the definition, we can choose our value for $\bar{R}_1 f$ based on which ξ we have got. Choosing the smallest value between the two constructions above, we can take $\bar{R}_1 f(\xi) := \min\{|\xi_1^1\xi_1^2|, |\xi_2^1\xi_2^2|\} + \varepsilon$ to get our forcing function as small as possible.

For the second lamination we take $A^\mu_\lambda, B^\mu_\lambda$ as defined above again. Then by construction we have admissible choices for $\lambda_1, A_1 := \lim_{\mu\to\infty} A^\mu_{\lambda_1}, B_1 := \lim_{\mu\to\infty} B^\mu_{\lambda_1}$ for $\bar{R}_2 f(\xi)$ and we have

$$\bar{R}_2 f(\xi) = \lambda \bar{R}_1 f(\xi) + (1-\lambda)\bar{R}_1 f(\xi) = \lambda(\min\{|A_1^1 A_1^2|, |A_2^1 A_2^2|\} + \varepsilon) + (1-\lambda)(\min\{|B_1^1 B_1^2|, |B_2^1 B_2^2|\} + \varepsilon)$$

$$= \lambda(0+\varepsilon) + (1-\lambda)(0+\varepsilon) = \varepsilon.$$

We can take $\varepsilon > 0$ arbitrarily small. Now via Lemma 26 we have

$$R_2 f \leq \lim_{\varepsilon\to 0} \bar{R}_2 f = 0,$$

and we have our claim as outlined in the beginning of the proof.

For $f := f_{DOT}$ a similar reasoning holds. Via the same line of arguing we can take $\bar{R}_1 f(\xi) := \min\{(\xi_1^1\xi_1^2)^2, (\xi_2^1\xi_2^2)^2\} + \varepsilon$. Afterwards then $\bar{R}_2 f \leq \varepsilon$ for any $\varepsilon > 0$. So the same claim follows. □

6.3 Further application

Both techniques that have been explored in this section, compensated compactness and quasiconvex relaxation, have only been put to minor use in this thesis. It is very likely that further results are possible through these methods as only surface-level techniques are explored, what follows is a brief discussion on possible avenues that can lead to stronger results.

In regards to compensated compactness, there are much more general results of the Murat-Tartar theorem (Theorem 22) [54]. The setup is as follows, let $\mathcal{A}$ be a homogeneous first order linear PDE operator with constant coefficients. Define the wave cone $\Lambda_{\mathcal{A}} := \cup_{\xi\in\mathbb{S}^{d-1}} \ker A(\xi)$ where $A(\xi)$ is the symbol of $\mathcal{A}$. Which are all non-elliptic directions. Then weak convergence can now be taken through any quadratic form $q : \mathbb{R}^N \to \mathbb{R}$ if we can find a corresponding $\mathcal{A}$ such that

$$q(A) \geq 0 \text{ for all } A \in \Lambda_{\mathcal{A}}.$$

Taking $\mathcal{A} :=$ curl and $q : \mathbb{R}^{2n} \to \mathbb{R}$ as $q(A_1, \ldots, A_{2n}) = \sum_{i=1}^{n} A_i A_{n+i}$ results in Theorem 22. Finding new versions of these correspondences that satisfy the condition above will give for $b \in W^{1,2}(\Omega)$

$$\int_\Omega \phi q(\nabla b) dx \leq \liminf_j \int_\Omega \phi q(\nabla b_j) dx, \forall \phi \in C_0(\Omega). \tag{65}$$

This can be used to get similar statements as in Theorem 23 but for structural integrands based on different non-linear expressions than the dot-product.

In regards, to quasiconvex relaxation we state the following conjecture;

Conjecture 1. *For $n \in \mathbb{N}$, we have $Rf = 0$ for $f = f_{DOT}$ or $f = f_{aDOT}$.*

Using clever direct computations of more repeated laminations $R_k f$ this seems doable, but the author has not found the correct avenue of approach yet per date of publication. Furthermore, computing Rf with $f = f_{LP}$ might lead to new insights as it is not evident that Rf_{LP} is necessarily zero. Finally, new methods of numerical computations and approximations of rank one or quasiconvex envelopes are still being developed [63, 20] and could be applied to structural integrands to find potential quasiconvex candidates.

7 Numerical Experiments

In addition to the theoretical part where we look at well-posedness of the joint inverse problem with structural similarity regularization we also perform some practical numerical experiments.

We will use the newly proposed f_{gCG}, f_{gNambu} in Section 4.4 and compare them with existing methods $f_{gGD}, f_{jCG}, f_{jNambu}$ in the case $(N = 3, n = 2)$. This is inspired by the work in geophysics, where good methods of 3-dimensional imaging can have a large impact and the work by Arridge et al. [21] on RGB-images using parallel level sets (f_{LP}, f_{QP}). The application is for image enhancement with RGB-channels. Our optimization problem is to minimize over $u = (u_1, u_2, u_3)$ with $u_i : \Omega \subset \mathbb{R}^2 \to \mathbb{R}, i = 1, 2, 3$. With the cost functional being given by

$$\|K_1u_1 - d_1\|^2 + \|K_2u_2 - d_2\|^2 + \|K_3u_3 - d_3\|^2+$$
$$+\alpha_1 TV[u_1] + \alpha_2 TV[u_2] + \alpha_3 TV[u_3] + \beta \int_\Omega f_s(\xi), \tag{66}$$

with $\alpha_1, \alpha_2, \alpha_3, \beta > 0$, K_i linear operators and $f_s \in \{f_{gGD}, f_{jCG}, f_{jNambu}, f_{gCG}, f_{gNambu}\}$. We can roughly write this as a functional over $[0, \infty)$ where for simplicity we assume that the TV regularization parameters are equal $(\alpha_1 = \alpha_2 = \alpha_3 = \alpha)$ as

$$\Phi(u) := \|Ku - d\|^2 + \alpha TV[u] + \beta J(\xi). \tag{67}$$

The problem is well-posed via Corollary 2 for $\mathcal{B} = [W^{1,p}(\Omega)]^3$ or an appropriate $\mathcal{B}' \subset [SBV(\Omega)]^3$ since all conditions are satisfied for Theorems 19 and 21. While this is true, in this section we lay aside the exact theoretical setting and focus on the practical concerns with solving such a problem numerically. We have a discretized bounded domain $\Omega \subset \mathbb{R}^2$ with a bounded co-domain $[0, 255]$ as in standard image practices. The exact function space where we consider u is not important as we have a uniform bound on solutions and only look at a finite number of values $x_1, \ldots, x_K \in \Omega, K \in \mathbb{N}$. When sampled in this way, where we have a discrete grid, as discussed in Section 2, any norm will is admissible. Throughout the implementation, the Euclidean (L^2) norm is used. As we have p-growth with $p = 2$ or 4 (see Table 1) for our integrands we can take $\mathcal{B}_i = W^{1,2}(\Omega), W^{1,4}(\Omega)$ or $SBV(\Omega), i = 1, 2, 3$ if pressed for an answer.

We can see this minimization problem over Φ from two perspectives. Define the following sub-problems,

$$A_i(u_i) := \|K_iu_i - d_i\|^2 + \alpha_i TV[u_i],\ i = 1, 2, 3, \tag{68}$$

$$A(u) := \sum_{i=1}^{3} A_i(u_i) = \|Ku - d\|^2 + \alpha TV[u], \tag{69}$$

$$B(u) := \|Ku - d\|^2 + \beta J(u). \tag{70}$$

Either we solve for three vector variables $u_1, u_2, u_3 \in \mathbb{R}^K$ a problem

$$\Phi(u_1, u_2, u_3) = \sum_{i=1}^{3} \|K_i u_i - d_i\|^2 + \alpha_i TV[u_i] + \beta J(u_1, u_2, u_3),$$

or we solve for one vector variable $u \in \mathbb{R}^{3K}$

$$\Phi(u) = A(u) + \beta J(u).$$

It is a valid approach [15] to solve Φ directly as a non-convex optimization problem. However, the splitting of the problem in this way via sub-problems may prove fruitful as A_i, A are now linear inverse problems with TV regularization. These problems are well-researched and good algorithmic approaches have been developed. In this thesis, the Bregman method is used in particular. Note that B is in general a non-convex problem (since it is only quasiconvex) such that another algorithm is needed to solve it. Note that $J(\cdot)$ is separably convex so a Block Coordinate Descent algorithm is a valid approach [13]. Specifically, for non-convex optimization problems with a large number of variables, one of the main algorithms used is L-BFGS. From the second perspective and solving for A, B, there is an iteration component in both problems. A priori it is unclear which order of solving the sub-problems (and which view to take) gives the best result. We assume that each channel has an equal amount of information about the structure and take $\alpha := \alpha_1 = \alpha_2 = \alpha_3$. We include in each sub-problem the data fidelity term since this is the main inverse problem and the other terms are just additive regularization terms.

After defining the algorithms below, the results of 6 experiments of increasing complexity are included in Subsection 7.1. Afterwards, a discussion on the numerical convergence and implementation details is given in the final Subsections 7.3 and 7.4. Most figures can be found in Appendix C as an increased size possible and side-by-side comparison is most convenient. All code can be found in the authors GitHub: `https://github.com/schilperoortteun/Structural-similarity-in-inverse-problems`.

7.1 Experimental setups

Before taking a look at a more realistic application, a comparison between algorithmic approaches is useful as per the above discussion. When solving Φ or B a L-BFGS algorithm is used due to the non-convexity of the problem and the large amount of variables. When solving a particular sub-problem A_i or A we use existing numerical methods for solving TV regularization. For the exact implementation details and an overview of the programming packages used we refer to reader to Section 7.4. Since both solving methods work using iterations that compare a new cost with an old cost we can use this to our advantage by additionally iterating over different methods to get the best of both worlds. In a small example we will compare the following algorithmic approaches based on speed and performance.

(A) Solve entire problem $\Phi(u)$.

(B) Solve $A_1(u_1)$, then iterate $\Phi(u)$, then $A_2(u_2)$, then $\Phi(u)$, then $A_3(u_3)$ then $\Phi(u)$, $\ldots$.

(C) Solve $A_1(u_1)$, then iterate $B(u)$, then $A_2(u_2)$, then iterate $B(u)$, $\ldots$.

(D) Solve $A(u)$, then iterate $\Phi(u)$.

(E) Solve $A(u)$, then iterate $B(u)$.

(F) Solve $\Phi(u)$, then $A(u)$.

(G) Solve $B(u)$, then $A(u)$.

We can look at algorithms F and G as solving a TV regularized problem after a pre-regularization has been done via the continuous problem using structural similarity. The algorithms D and E pre-regularize via the structural similarity and afterwards solve a TV problem. As exit condition deciding when to switch solving methods, it makes sense to use the same condition that is used when solving the respective algorithms itself. Specifically when only minor improvements are made between the new and the old cost of a solution. Heuristically, ending with an inverse problem that involves the TV term makes sense as one hopes to get "flatter" and more defined objects in the picture. However, there is also an argument to be made that one can best end with a continuous optimization solver, as strict bounds on the variables can be present and these can be better implemented when using continuous methods. We will compare these approaches visually and using PSNR and SSIM measures where we take the mean SSIM over the channels. Note that this test is not for finding the optimal image reconstruction in a quantitative sense and only uses the measures as an indicator of performance together with computing time, RAM usage and other implementational challenges.

7.2 Results

Six experiments have been performed:

1. A square with equal changes in magnitude for $N = 2$ channels.
2. A square with equal changes in magnitude for $N = 3$ channels.
3. A square with unequal changes in magnitude for $N = 3$ channels.
4. A more complex image with multiple structures of unequal overlap and unequal changes in magnitude for $N = 3$ channels.
5. Our method applied to a realistic large-scale image.
6. Comparison between different integrands f_s.

The quantitative results are included in this section in tables, but for ease of the reading the image results are included in Appendix C.

7.2.1 Two channels

As the reader will see in Experiments 2 and 3, pre-regularization followed by continuous optimization is the best algorithmic approach. However, an even more trivial example is possible and investigated here. We only test algorithms D and E to each-other and to solely using TV regularization. The most base example we can have is similar structure in $N = 2$ channels, we have used similar code as in the upcoming RGB examples, but now only looking at the R and G channels. Structurally, a square is interesting since it has sharp edges and corners and a region inside without any changes in value. These are features that TV regularization has problems with. The two channels have exactly the same structure in this example.

With $f = f_{gCG}$ we get the following results for different values of β. The benchmark is given for $\beta = 0$ with the TV regularization with PSNR = 25.73 and SSIM = 0.7691. Higher and lower values for β than included in the table give worse performance.

Table 2: 2 Channels, β

Algorithmn	β	PSNR (CG)	SSIM (CG)
D	1e-3	26.82	0.8873
E	1e-3	26.82	0.8873
D	1e-2	27.10	0.8779
E	1e-2	27.10	0.8779
D	1e-1	26.56	0.7435
E	1e-1	26.56	0.7435

As we can see there is substantial improvement of including a structural regularizer.

Note that there is no difference between D and E up to the fourth significant figure. This is because the inclusion of the small parameter α for the problem to remain coercive in algorithm D has little to no impact because it is overshadowed by the cost of J. A visual comparison of the deblurred image at its cross-sections is included in Figure 2. Visually there does not seem to be much improvement from TV, this makes sense as the relative change in PSNR and SSIM values is relatively small. A close look at the horizontal and vertical section tells another story. For TV regularization we get negative pixel values (included via the implementation but possible to cutoff manually) and at the regions outside the square where there is a constant background, TV introduces a small increase and decreases at some points resulting in distinctive bands. The structural regularization afterwards smoothes out these bands.

7.2.2 Equal channels

We deblur a picture of a square where at the same points in all RGB channels there is an equal change in magnitude. Immediately clear from the visuals in Figure 3 are the deformities in algorithms B, C, F, G, and TV. For the TV regularization, these white regions are slightly negative grey-scale values, and consequently not that different from the black background. This is why the PSNR and SSIM values are not largely impacted by this minor difference in value. For the others, it appears that there are numerical instabilities introduced somewhere along the solving method. This is likely due to solving a TV regularization problem after the L-BFGS-B algorithm as this is the common denominator across the approaches. As there are approaches with adequate results, the mechanics of these instabilities are not explored further.
Algorithmn A has only minor changes to the blurred image, this is why one of the reasons why decoupling both components of the inverse problem, the TV and the structural term, is necessary. Algorithms D and E lead to marked improvements in PSNR and SSIM values in Table 3 and are visually comparable to TV regularization when we have deleted negative values. .

7.2.3 Unequal channels

Now we change the image from the previous channel such that between background and foreground, the blue channel value increases from null to maximum, while in the other channels a smaller jump from null to half is made. (see Figure 4)

The results in Tabel 4 are similar as in the case of equal channels with algorithms D and E leading to improvements. Noteworthy is that the relative and absolute improvement

Table 3: Equal channels

Algorithmn	PSNR (CG)	SSIM (CG)
A	23.0	0.45
B	20.5	0.61
C	-6.7	0.00
D	29.4	0.84
E	29.3	0.83
F	8.7	0.03
G	8.7	0.03
TV	28.5	0.76

Table 4: Unequal channels

	PSNR (CG)	SSIM (CG)
A	17.9	0.39
B	17.3	0.55
C	-8.0	0.00
D	24.1	0.72
E	24.0	0.70
F	1.4	0.02
G	1.4	0.02
TV	23.4	0.67

in the quantitative measures is decreasing as the picture becomes more complicated.

7.2.4 Qualitative comparison

Based on the previous experiments, the best algorithms D and E first solve a TV regularization problem, where-after taking the structural regularization into account via Φ or B to further augment our variables. To quantitatively test whether using the structure leads to a better result we define the following Algorithm S:

1. Solve $A(u)$
2. Solve $S(u)$

where we have defined a new problem that is essentially sub-problem $B(u)$ given by

$$S(u) := \|Ku - d\|^2 + \beta J(u) + \gamma TV_\varepsilon[u],$$

with $1 >> \gamma > 0$ included to grant coercivity of the problem. Note that without the $\gamma TV[u]$ and if we have fixed all $u_j, j \neq i$ but one of the channels $u_i, i = 1, \ldots, 3$, we have a convex function $S(u_i)$. The question on whether the incorporation of J helps has now been reduced to comparing the solution after the first step of the algorithm to the final solution after solving $S(u)$. For the bigger data set we tested on one picture and determined $\alpha \in [0.0001, 0.001]$ performs best for problem A. We fix this across the data set. We tested different $\gamma < \alpha$ with the best performance with $\gamma = 10^{-6}$. From Table 5 we see that this new algorithmic approach S has better PSNR and similar SSIM as the others. In Figure 5 we see that Algorithmn S works better in the parts where there is overlap and from the sections we see that there are also sharper edges. There is some larger intermingling between the channels solely using TV regularization resulting in distinctive patches of mixed colour.

Table 5: Multi-structure

	PSNR (CG)	SSIM (CG)
D	21.6	0.82
E	21.6	0.82
S	22.0	0.82
TV	20.9	0.71

7.2.5 Large-scale image

Instead of constructed small-scale examples as in the experiments above, here we have sampled three different realistic images of significant size taken from the Skimage data set. Since the sizes of the images change and we do not have normalized the cost function J, a new optimal value for β needs to be chosen for each image. The blurring kernels are given by Gaussian kernels with parameter $\frac{1}{\sigma^2} = 0.005$. Of note, is the increased computation time necessary for these larges images compared to the earlier experiments. The growth in computation time is definitely non-linear in both memory usage and number of computations. Below a table denoting the sizes and parameter chosen.

Table 6: Large-Scale images

Image	Size	β	PSNR(TV)	SSIM(TV)	PSNR(CG)	SSIM(CG)
Cat	300 x 415	10^{-8}	26.8	0.67	29.7	0.80
Coffee	400 x 600	10^{-9}	20.3	0.58	25.1	0.68
Rocket	427 x 640	10^{-9}	25.8	0.77	28.7	0.83

7.2.6 Comparison structural integrands

As found in Section 5, there are several different structural integrand functions that lead to well-posed problems and consequently should lead to improved numerical performance. All previous comparisons have been between TV and cross-gradient regularization since during the implementation stage it was discovered that the particular form of f_s did not have a large impact on the final results. The functions $f_s := f_{CG}, f_{GD}, f_N, f_{jCG}, f_{jN}$ have been implemented as can be found in Section 7.4. In Figures 9 and 10 a quantitative comparison across these different choices can be found. The data used is the Cat image but with a smaller Gaussian blur than the large-scale image experiment is used to speed up computation time. Note that only a small part of the y-axis is included to highlight the minor differences between methods, but performance is similar and better compared to TV regularization with in this case PSNR equal to 33.2 and SSIM equal to 0.90.

7.3 Convergence

We are interested in the convergence of numerical methods applied to an inverse problem of the form in Equation (66). This can be regarded more generally as an inverse problem with both separate (TV) and joint regularization (structural similarity). Formally, this is a functional minimization problem over $b = (u_1, \ldots, u_N) \in \mathcal{B}$ with cost $\Phi : \mathcal{B} \to [0, \infty]$ given by:

$$\Phi(b) := \|Kb - d\|_{\mathcal{H}}^2 + \sum_{i=1}^{N} R_i(u_i) + R(u_1, \ldots, u_N), \; b \in B. \tag{71}$$

The right hand side is given by a data fidelity term $\|Kb - d\|^2$, single regularizers $R_i : \mathcal{B}_i \to [0, \infty]$ and a joint regularizer $R : \mathcal{B} \to [0, \infty]$. In our RGB deblurring problem we have taken $R_i = \alpha_i TV, R = \beta J$.

Setting aside the specific method used to minimize a sub-problem of $\Phi(b)$ and only assuming that it is numerically sound we take a look at the class of methods that take advantage of the interplay between the components of $b = (u_1, \ldots, u_N)$. The most common approach for solving these types of problems is called alternating minimization [17, 13] for $N = 2$ and (block) coordinate descent for $N > 2$. Where the overarching property of such algorithms are that they are iterative and only a subset of the total amount of equations is used in each respective iteration. In our case this is done by considering only one $u_i, i = 1, \ldots, N$ at the time and fixing all others. Define the Block Coordinate Descent (BCD) algoritmn as cyclically solving for z the minimization problem given by $\Phi(z, u_2, \ldots, u_N), \Phi(u_1, z, \ldots, u_N), \ldots, \Phi(u_1, \ldots, u_{N-1}, u_N)$. In the case where $\Phi(b)$ is jointly convex, we have Theorem 27 for convergence. For non-convex $\Phi(u)$ we get Theorem 28. The need to introduce the sets D_i are to limit our search space for values of $u_i(x) \in \mathbb{R}$. As we can see, the convergence of solutions depend on two factors; analytical properties of the function Φ and topological properties of the ranges of the components.

Definition 27 (Range)**.** *For $i = 1, \dots, N$, let $D_i \subset \mathbb{R}$ be the set of possible values for $u_i(x), x \in \Omega, i = 1, \dots, N$.*

Theorem 27 (Convergence convex case)**.** *[58] If $\Phi(\cdot)$ is jointly convex and differentiable, D_i closed and convex, and when looking at the one-vector projection*

$$\Phi(u_1, \dots, z, \dots, u_N)$$

this attains a unique minimum $\bar{z}$ for all $z = u_1, \dots, u_N$. Then every limit point of a sequence of solutions when using BCD is a minimizer.

Theorem 28 (Convergence non-convex case)**.** *[58] If $\Phi(\cdot)$ is continuously differentiable, D_i closed and convex, and when looking at the one-vector projection $\Phi(u_1, \dots, z, \dots, u_N)$ this attains a unique minimum $\bar{z}$ for all $z = u_1, \dots, u_N$ and is monotonically non-increasing in the interval (u_i, z) or $(z, u_i) \subset \bar{D}_i$. Then every limit point of a sequence of solutions when using BCD is a minimizer.*

In our case when using the structural similarity regularization $R := \beta J$, we have generally R a non-convex functional such that Theorem 28 is the most relevant one. Theorem 27 can be used for TV regularization or when having a convex R. We can find conditions on R_i, R such that Theorem 28 can be applied.

Lemma 28. *Assume $R(u_1, \dots, u_N)$ continuously differentiable and separately convex, and that $R_i(u_i)$ is convex and continuously differentiable for all $i = 1, \dots, N$. Assume there exist $a, b \in \mathbb{R}^N$ with for all $x \in \Omega, u_i(x) \in [a_i, b_i] \in \mathbb{R}$ for all $i = 1, \dots, N$. Then the minimisation problem given by Equation* (71) *converges (up to subsequence) under BCD.*

Proof. All three terms in the inverse problem given in Equation (71) are continuously differentiable, so Φ is continuously differentiable. By Theorem 4 we have convexity of the data fidelity term so we only have to look at the other contributions. The $D_i = [a_i, b_i]$'s being intervals are closed and convex. Since we have assumed $R_i(u_i)$ to be a convex function, the second term will also not lead to any problems. We apply Theorem 28, note that separate convexity of R in each component u_i gives the unique maximum and monotone non-increasing behaviour on each interval D_i. □

Remark 21. *Note that the condition that R is separately convex is stronger than the necessary condition in Theorem 28 that it is separately monotonically non-increasing having a unique minimum. Since separate convexity comes automatically for quasiconvex functions and we only use regularizers of this type, this more complicated condition is superfluous for this report.*

We can almost take $R_i(u_i) := \alpha_i TV[u_i], R(u) := \beta J(u)$ with J chosen quasiconvex and directly apply the lemma above. The only hiccup is that TV is not continuously differentiable at zero. As described in the implementation (Section 7.4), and through a commonly used method in numerical applications, we have slightly changed TV at small values such that it becomes smooth. With this in mind we have the following result for convergence of Equation (67).

Lemma 29. *Let TV_ε be a continuously differentiable variant of TV. Let J be continuously differentiable and separately convex. Let $b \in \mathcal{B}$ with compact image in $\mathbb{R}^N$. Then every limit point of a sequence of solutions using BCD is a minimizer of*

$$\|Kb - d\|^2 + \alpha TV_\varepsilon[b] + \beta J(\nabla b).$$

Proof. This follows almost immediately from Lemma 28. Note that the compact image $K(b)$ of b can be separated into closed and bounded intervals $D_i \in \mathbb{R}, i = 1, \ldots, N$. □

7.4 Implementation

We want to solve Equation (67) with a numerical algorithm. Our problems A_i and A defined in Equation (69) only have the first two terms and constitute a linear inverse problem with TV regularization in $\mathbb{R}^K$ or $\mathbb{R}^{3K}, K \in \mathbb{N}$ respectively. This can be solved via standard techniques [26]. PyLops [52] has a Split-Bregman implementation that can handle L^1 and L^2-regularizers with a linear operator. We use this when solving A or A_i in algorithms B - G. Within the PyLops implementation, parameters $n_{\text{outer}}, n_{\text{inner}}$ for the amount of inner and outer loops of the Bregman optimization need to be given. From the documentation "A small number of inner iterations is generally sufficient and for many applications optimal efficiency is obtained when only one iteration is performed." [52]. Small values of n_{inner} were tested in the multi-structure case (experiment 4) with $n_{\text{inner}} = 1$ performing best, increasing n_{inner} leads to increasingly worse results. We have tested multiple values for n_{outer}, which should not be too big to keep the computation tractable, and chose $n_{\text{outer}} = 150$ to strike the balance between performance and computation time. Noteworthy is that PSNR / SSIM are not monotonically increasing functions in

n_{outer}, there is a sharp increase in performance for small values but choosing a large $n_{\text{outer}} >\approx 200$ eventually leads to a decrease.

Now that we have a method for the sub-problems A_i, A, we can take a look at the joint problem Φ. It is not jointly convex so the Bregman method cannot be used and non-convex optimisation is necessary. In our case, we have chosen the structural regularizer J in such a way that it is an integral with a quasiconvex integrand f. By linearity of integrals this implies that J is separately convex in $u_i, i = 1, \ldots, N$ when all other $u_j, j \neq i$ are fixed. As our benchmark paper is [21] we have chosen for a similar L-BFGS implementation using quasi-Newton line search. In particular, the SciPy [62] implementation for L-BFGS-B where we have supplied an approximated gradient $D\Phi(u)$ as described below. No explicit form of the gradient is required an sich in the L-BFGS-B implementation as linear (or quadratic) approximations of the gradient can be computed from the objective function $\Phi(u)$. However, the computation time required to approximate $D\Phi(u)$ blows up due to the large amount of variables we require (one for each pixel and channel in the image) and the fact that computing $J(u)$ is time-consuming as it is an integral. We use a NumPy implementation to calculate J, together with the increased accuracy this makes it worthwhile to calculate give an explicit version of $D\Phi(u)$.

Equation (67) can be written as a PDE (in time) of

$$\partial_t \Phi(u) = -D\Phi(u) = -A^*(Au - d) + \alpha D[TV[u]] + \beta D[J(\nabla u)], \tag{72}$$

where A^* is the adjoint of A and D is the Gateaux differential. Using Taylor approximation and Green's identity, we can compute $D[TV[u]]$ and $D[J(\nabla u_1, \nabla u_2, \nabla u_3)]$ with $\nabla u_2, \nabla u_3$ fixed.

Theorem 29 (Gateaux formula)**.** *[21] Let $f : \mathbb{R}^N \to \mathbb{R}$ be a twice continuously differentiable function. Then the Gateaux derivative of $J : C^1(\Omega) \to \mathbb{R}$ at $u_1 \in C^1(\Omega)$ with direction $h : C^1(\Omega)$ is given by $DJ_{u_1} : C^1(\Omega) \to \mathbb{R}$ with*

$$DJ_{u_1}(h) = -\int_\Omega h \ div \ [\nabla f(\nabla u_1)] + \int_{\partial\Omega} h \langle \nabla f(\nabla u_1), n \rangle,$$

where n is the outer normal vector of Ω.

Using image extension (or setting $\nabla u = 0$ at $\partial\Omega$) it is practically easy to disregard the surface integral over $\partial\Omega$. Note the straightforward extension of $f(\nabla u_1)$ to $f(\nabla u)$ via channel-wise differential operations. Then

$$DJ_u(h) = [DJ_{u_1}, DJ_{u_2}, DJ_{u_3}](h) = -\int_\Omega h \ \text{div} \ [\nabla f(\nabla u)].$$

Our aim is the compute the right-hand side for a chosen

$$f_s \in \{f_{gGD}, f_{jCG}, f_{jNambu}, f_{gCG}, f_{gNambu}\}$$

such that we can plug it in Equation (72) and apply L-BFGS. To obtain the Gateaux derivatives we need to approximate the norms $\|\xi\|, \|\xi^{\cdot}\|, \|\xi_{\cdot}\|$ smoothly. We take for $\varepsilon > 0$

$$\|\xi\|_\varepsilon := \sqrt{\|\xi\| + \varepsilon^2}.$$

Other than the numerical instabilities present when dividing by zero when using the exact norm $\|\xi\|$, there is an additional gain in TV regularization when using this method. Where we define the differentiable approximation of TV used in Lemma 29 by

$$TV_\varepsilon[u] := \sqrt{|\nabla u|^2 + \varepsilon},$$

it is well-known that using this formula gives an absence of "staircasing" [39]. This is a phenomena that can be found when using TV regularization on images that smoothes out the oscillations into blocks with zero gradients. Additionally, for robustness at small gradients, we have a trade-off to make because of the two following desired behaviours; ε is such that if there are only small changes in the parameter field, so $\|\xi\| \sim 0$ then $\|\xi\|_\varepsilon$ is close to zero. This would imply a small ε. As we will see in the computation below, we also need to be able to have a nice expression for $\lim_{\|\xi\|\to 0} \frac{1}{\|\xi\|_\varepsilon}$. Where the limit is finite and large but in such a way having a few of these values in a solution u does not overshadow the other contributions to $D\Phi(u)$. This implies a not too small ε, we have used $\varepsilon = 0.1$ for the small-scale and $\varepsilon = 1$ for the large-scale experiments.

For the actual calculations of $D[TV[u]], D[J[u]]$ we refer to Appendix B. Note that we have an explicit form for the gradient $D\Phi(u)$ we take a look at the discretization of the domain Ω. We have three functions $u_1, u_2, u_3 : \mathbb{R}^2 \to [0, 255]$, since we already have discretized images we want to know the values of these functions at $M \in \mathbb{N}$ different pixels. We label $x_{m,l}, l, m = 1, \ldots, \frac{M}{2}$ as a 2-dimensional grid. For implementation of BFGS, we want to know the gradient in direction $DJ_{u_k(x_{m,l})}$ for $k = 1, 2, 3, m, l = 1, \ldots, \frac{M}{2}$. Note that although J is defined as an energy functional on Ω, the derivative DJ is a local function depending on $x_{m,l}$. We use a central difference formula for the components of ξ. As a concrete example to show the computation we take $f_s = f_{gCG}$. Using the results from Appendix B, $DJ_{u_k(x_{m,l})}$ is given by

$$DJ_{u_k(x_{m,l})} = -\frac{1}{2}(|\nabla u_i(x_{m,l})|^2 + |\nabla u_j(x_{m,l})|^2), i \neq j \neq k.$$

We now estimate $\nabla u_i(x_{m,l}) \in \mathbb{R}^2$ in each direction m, l via a central difference approximation, i.e.

$$\nabla u_i(x_{m,l}) = \frac{1}{2}\left(u_i(x_{m+1,l}) - u_i(x_{m-1,l}), u_i(x_{m,l+1}) - u_i(x_{m,l-1})\right).$$

Substitution of this approximation above in $DJ_{u_k}(x_{(m,l)})$ gives the gradient for J used in the code and likewise for the other integrands f_s.

The PSNR of a given image $\bar{u}$ compared to the original image u is given by

$$\mathrm{PSNR}(\bar{u}) = 20 \log \left(\frac{\mathrm{MAX}(u)}{\sqrt{\mathrm{MSE}(u, \bar{u}}} \right). \tag{73}$$

Where $\mathrm{MAX}(u)$ is the maximum value. For SSIM the "structural similarity" method of scikit-image [55] has been used.

8 Summary and Outlook

The core aim of this thesis was to develop a fundamental understanding of the necessary conditions for well-posedness of coupled inverse problem. More specifically, to investigate in regards to joint inverse problems, where the coupling of the different components is via a variational regularization that quantifies structural similarity. Using existing theory of optimization problems over Banach spaces and taking into account the application domains of structural similarity we have found settings such that Theorem 1 can be applied. These settings lead to a well-posed minimization problem with an additional condition for uniqueness of solution.

Roughly, two types of conditions where investigated, the functional analytic, where it was determined that the following settings lead to Banach spaces where bounded sequences have convergent sub-sequences and hence can be used to optimize over:

1. For Ω Lipschitz, $W^{m,p}(\Omega)$ for $p \in (1, \infty), m \in \mathbb{N}$ with the weak topology.
2. $\mathcal{M}(\Omega)$ with its weak$-*$ topology and considering uniformly bounded sequences $(\mu_k)_k$.
3. For Ω bounded and Lipschitz, $BV(\Omega)$ with its weak$-*$ topology and considering uniformly bounded sequences $(u_k)_k$.
4. For Ω bounded and Lipschitz, $SBV(\Omega)$ with its weak$-*$ topology and the conditions in Theorem 37.

For the other type, the necessary convex analytic properties belonging to the structural regularization functional J are given by non-negativity, (mean) coercivity, C_p growth, and quasiconvexity. After a thorough review of the three different application where structural similarity is used (medical imaging, geophysical reconnaissance, and image enhancement) a catalog of different structural integrands f_s with their properties was constructed. Especially of note are $f_{GD}, f_{CG}, f_{Nambu}$ which are quasiconvex when restricted to two coupled inverse problems. In addition to the usual way of generalizing to N inverse problems via summing all contributions over pairs ($f_{gGD}, f_{jCG}, f_{jNambu}$), explicit generalized definitions f_{gCG}, f_{gNambu} that preserve quasiconvexity have been newly defined in this thesis. On top of this, the Schatten p-norms $f_{VTV}, f_{TNV}, f_{TSV}$ can also be used in the most general setting as these are (quasi)convex.

The two types of conditions are combined to prove well-posedness of the variational problem in specific cases. In particular, Theorems 19, 20 and 21 outlining existence of solution in respectively $W^{1,p}, \mathcal{M}$, and $SBV(\Omega)$ are new.

For structural integrands depending on the dot product that are separately convex but not quasiconvex there are proof of concept results (Theorem 23 and 27) based on the theory of compensated compactness and relaxation. Specifically, when working on the Hilbert space $[W^{1,2}(\Omega)]^N$ and restricting to sequences that have weak Laplacians Δ in a precompact set, there is l.s.c of J with these types of structural regularizers. Clever computation of repeated laminations and forcing functions are used to prove that the rank one envelopes of f_{DOT}, f_{aDOT} are the zero functions for $n = 2$.

Finally, structural similarity was also investigated numerically. Using RGB-valued images, qualitative and quantitative improvement is discovered when using structural similarity after total variation regularization. The best algorithmic practice was discovered to be Algorithm S described by first minimizing over $u \in \mathcal{B}$

$$\|Ku - d\|^2 + \alpha TV[u],$$

via conventional TV optimization, then afterwards using this solution u as the initial condition to minimize (with appropriate parameters)

$$\|Ku - d\|^2 + \beta J(u) + \gamma TV_\varepsilon[u].$$

This algorithm convergences to a local minimum and leads to substantial advancement on large-scale realistic images.

Concerning further work, as discussed in Section 6, more technical theory can likely be applied to get additional results about l.s.c structural regularizers not predicated on the Direct method. Additionally, a brief exploration in results from Mumford-Shah functionals over $BV(\Omega)$ that is not included seems to indicate that they can be used to get well-posedness or the existence of minimizing sequences for non-isotropic integrands. Furthermore, there is the hope that the explicit conditions on J can be used by researchers to define better structural similarity quantifiers and that applied scientists can sleep better at night knowing their variational problems are well-posed when using the cross-gradient.

A Fine Properties of Function Spaces

In this appendix we state the results that allows us to take a particular Banach space $\mathcal{B}$ with a topology $\tau_{\mathcal{B}}$ as an admittable choice in the main variational problem. This is by no means a comprehensive account of the statement and proofs and only aims to shed light on the necessary theory to confirm admissibility. Compactness either comes via Banach - Alaoglu (Theorem 2) for $L^p, W^{m,p}, p > 1, m \in \mathbb{N}$ or is proved directly in the case of BV, SBV. The definition of the spaces and their topologies can be found in Section 2.

It is a well-known result that $L^p, 1 < p < \infty$ is reflexive and separable. This follows from the following line of statements.

Definition 28 (Uniform convexity). *A normed space X is called uniformly convex if for all $\varepsilon > 0$ there exists some $\delta > 0$ so that for x, y in the unit ball B_1, $\|x + y\| \geq 2 - \delta$ implies $\|x - y\| < \varepsilon$.*

Theorem 30 (Uniform convexity of L^p). *[36] Let $1 < p < \infty$. Then L^p is uniformly convex.*

Theorem 31 (Milman-Pettis). *[11] Every uniformly convex Banach space is reflexive.*

Theorem 32 (Separablility of L^p). *The space L^p is separable for $1 \leq p < \infty$.*

Hence the $L^p(\Omega)$-space with its weak topology is compact via Theorem 2. Note that this holds for any measure space on arbitrary Ω with any positive measure μ, there are no restrictions on the shape of Ω and μ does not necessarily be equal to $\mathcal{L}$. For the edge cases $p = 1$ and $p = \infty$, we do not have reflexivity of L^p. Since we do not even have separability of L^∞, variational problems are difficult to solve and we omit this case. However, another characterization for separability exists for $p = 1$.

Definition 29 (Equi-integrability). *Let (Ω, R, μ) be a positive measure space. A set $F \subset L^1(\Omega)$ is called equi-integrable if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that*

$$\int_E |f| d\mu < \varepsilon,$$

for $f \in F$ and $\mu(E) < \delta$. In the case where μ is finite and F bounded, equi-integrability is equivalent to

$$F \subset \left\{ f \in L^1(\Omega, \mu) : \int_\Omega \varphi(|f|) d\mu \leq 1 \right\},$$

for some increasing continuous function $\varphi : [0, \infty) \to [0, \infty]$ with super-linear growth.

The first formulation is the classical one, but the second is more useful when we consider in particular $BV, SBV \subset L^1$ as we will do later.

Theorem 33 (Dunford-Pettis)**.** *[4] Let $F \subset L^1(\Omega)$ be a bounded subset. Then it is relatively weakly sequentially compact if and only if F is equi-integrable.*

This result gives us the possibility of choosing $(\mathcal{B}, \tau_{\mathcal{B}}) := (F, \tau_{L^1(\Omega)})$ with F bounded and equi-integrable. A relatively straight-forward corollary of Theorem 30.

Theorem 34. *[2] $W^{m,p}(\Omega)$ is separable for $1 \leq p < \infty$ and reflexive and uniformly convex for $1 < p < \infty$.*

Another important statement are the Rellich-Kondrachov Sobolev embedding theorems. There is some nuace with the topological properties of Ω and which embeddings are valid. We will assume Ω bounded and being a Lipschitz domain but we note that most embeddings work for unbounded domains and a significantly weaker topological notion than Lipschitz called the "weak cone property" [2].

Definition 30 (Lipschitz domain)**.** *Let $\Omega \subset \mathbb{R}^n$ be bounded. We say that Ω is locally Lipschitz, or a Lipschitz domain if for all $x \in \partial\Omega$ we have a neighbourhood U_x where $U_x \cap \partial\Omega$ is the graph of a Lipschitz continuous functions.*

In the theorem below we have $n = \text{Dim}\Omega$.

Theorem 35 (Rellich-Kondrachov, $m \geq 1$)**.** *[2] Let $\Omega \in \mathbb{R}^n$ be Lipschitz. Let $j \geq 0, m \in \mathbb{N}, 1 \leq p < \infty$.*

- *If $mp > n$ or $m = n, p = 1$ then*

$$W^{j+m,p}(\Omega) \hookrightarrow C^j(\bar{\Omega}), \tag{74}$$

in particular for $j = 0$,

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), p \leq q \leq \infty.$$

- *If $mp = n$, then*

$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), p \leq q \leq \infty, \tag{75}$$

in particular for $j = 0$,

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), p \leq q \leq \infty.$$

- *If $mp < n$ then*

$$W^{j+m,p}(\Omega) \hookrightarrow W^{j,q}(\Omega), p \leq q \leq \frac{np}{n - mp}, \tag{76}$$

in particular for $j = 0$,

$$W^{m,p}(\Omega) \hookrightarrow L^q(\Omega), p \leq q \leq \frac{np}{n - mp}.$$

Remark 22. *The same embeddings are true with arbitrary domains* Ω *if we take* $W_0^{m,p}(\Omega)$. *If we consider bounded* Ω, *then we also have the embeddings for* $1 \leq q < p$. *Rellich-Kondrachov implies that for Lipschitz* Ω, *bounded sequences in* $W^{m,p}(\Omega)$ *have convergent subsequences in* $L^p(\Omega)$. *From Definition 6, weakly convergent sequences in* $W^{m,p}(\Omega)$ *converge strongly in* $L^p(\Omega)$, *so the function space* $W^{m,p}(\Omega)$ *is compact with respect to its weak topology.*

For weak$-*$ compactness in $\mathcal{M}(\Omega)$ we have the previously stated Theorem 4 (De La Vallée - Poussin).

To think about compactness in BV, we notice that a Sobolev space $W^{m,p}(\Omega), p > 1$ is compact with respect to its weak topology. As $W^{1,1} \subset BV$, we want a similar statement for $p = 1$. However the space $W^{1,1}(\Omega)$ is not compact with respect to its weak topology due to the difficulties with compactness in L^1, it is necessary to extend $W^{1,1}$ to BV for a similar compactness result. This provides an additional reason why BV is an interesting choice in variational problems.

Theorem 36 (Compactness in BV)**.** *[4] Let* $\Omega \subset \mathbb{R}^n$ *be bounded and Lipschitz. Let* $(u_k)_k \subset [BV(\Omega)]^N$ *be a bounded sequence in* $[BV(\Omega)]^N$, *then there exists a sub-sequence* $(u_h(k))_k$ *weakly*$-*$ *converging in* $[L^1(\Omega)]^N$ *towards some* $u \in [BV(\Omega)]^N$.

Note that this is a not too restrictive setting and would invite us to work on the space BV for our minimization problem. See the problems discussed after Theorem 6 for reasons why we restrict our setting to SBV. This restriction leads us into having some kind of grasp on the weak derivatives of our functions. This grasp is not trivial however, since all L^1 vector fields are the gradient of a SBV function and L^1 functions are badly behaved in general. As we have seen before in Theorem 33, our desired property; compactness and closure of $L^1(\Omega)$ wrt some topology is not straightforward. This carries over to $SBV(\Omega)$, which is in general an unbounded subset of $L^1(\Omega)$, and in the equi-integrability condition we integrate. Here non-zero Cantor parts D^c can appear. As in BV, we consider the weak$-*$ topology on $SBV(\Omega)$. The necessary conditions come from a natural application of Dunford-Pettis (Theorem 33) when considering $F = SBV$. They are essentially boundedness and conditions on $D^a u$ and $D^j u$ similar to the second formulation of equi-integrability in Definition 29. In most cases we take $\theta = 1$, and subsequently only measure the size of J_{u_h} in Hausdorff measure.

Theorem 37 (Compactness of $SBV(\Omega)$)**.** *[4] Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded. Let* $\varphi : [0, \infty) \to [0, \infty], \theta : (0, \infty) \to (0, \infty]$ *be lower semi-continuous increasing functions with*

$$\lim_{t \to \infty} \frac{\varphi(t)}{t} = \infty, \lim_{t \to 0} \frac{\theta(t)}{t} = \infty.$$

Let $(u_h)_h \subset SBV(\Omega)$, with $\|u_h\|_\infty$ uniformly bounded in h. Additionally,

$$\sup_h \left\{ \int_\Omega \varphi(|\nabla u_h|)dx + \int_{J_{u_h}} \theta(|u_h^+ - u_h^-|)d\mathcal{H}^{n-1} \right\} < \infty,$$

where the jump set J_{u_h} is the set where $D^j u_h \neq 0$. Then there exists a subsequence $(u_{h(k)})_k$ weakly$-*$ converging in $BV(\Omega)$ to $u \in SBV(\Omega)$. Additionally, the approximate gradients $\nabla(u_h)_h$ weakly converge to ∇u in $[L^1(\Omega)]^N$ and $(D^j u_h)_h$ weakly$-*$ converge to $D^j u$ in Ω.

So any subset $(u_h)_h \subset SBV(\Omega)$ that is uniformly bounded in ∞-norm with corresponding functions φ, θ that satisfy the conditions above is an admissible $\mathcal{B}_1$ with the weak-$*$ topology. The conditions in Theorem 37 on φ, θ can be used to determine if the two separate parts of the weak gradient D^a, D^j converge to their respective parts or if there is cross-over in the limit.

B Computation Gateaux Derivatives

Necessary for the implementation of the numerical experiment is and explicit expressions for $D[TV[u]]$. This is equivalent to computing the Gateaux derivative as given in Theorem 29 with $f_{TV}(u) := \sum_{i=1}^{3} |\nabla u_i|$. Let $k \in \{1,2,3\}$, pick $i \neq j \neq k \in \{1,2,3\}$ the other two values. Then we have for some constant $C(i,j) \in \mathbb{R}$

$$\nabla f_{TV}(\xi^k) = \nabla \left[\sqrt{\left(\xi_1^k\right)^2 + \left(\xi_2^k\right)^2} + C(i,j) \right] = \left(\frac{\xi_1^k}{\|\xi^k\|}, \frac{\xi_2^k}{\|\xi^k\|} \right).$$

Remark 23. *Again, we abuse notation and use* $\|\xi\|, \|\xi^{\cdot}\|, \|\xi_{\cdot}\|$ *all as the Euclidean norm that makes sense given the vector/matrix inside.*

Note that via our notation, the gradient operator ∇ acts over the lower indices $\mathbb{R}^n = \mathbb{R}^2$ in this case with the upper indices of ξ denoting the channel. Computing the divergence over ξ_1^k, ξ_2^k gives

$$\text{div } [\nabla f(\xi^k)] = \frac{(\xi_2^k)^2}{\|\xi^k\|^{\frac{3}{2}}} + \frac{(\xi_1^k)^2}{\|\xi^k\|^{\frac{3}{2}}} = \frac{1}{\|\xi^k\|}.$$

Because of the outlined difficulties with the norms $\|\xi\|$ as described in the implementation details in Section 7.4, we use

$$D(TV[u]) = -\sum_{k=1}^{3} \frac{1}{\|\nabla u_k\|_\varepsilon}.$$

For the computations below, we assume that we have a well-defined $\frac{1}{\|\xi\|}$ as in the end we will plug in the smooth approximation $\|\xi\|_\varepsilon$ of the norm. For the k-th channel we have

$$\nabla f_{gGD}(\xi^k) = \nabla \left[\frac{1}{2} \left((\xi^i - \xi^k)^2 + (\xi^j - \xi^k)^2 + (\xi^j - \xi^i)^2 \right) \right] = \frac{1}{2}[4\xi_1^k - 2(\xi_1^i + \xi_1^j), 4\xi_2^k - 2(\xi_2^i + \xi_2^j)].$$

$$\text{div } \nabla f_{gGD}(\xi^k) = \frac{1}{2}(4+4) = 4.$$

For each k this is the same, so the function is constant for all values $\xi \in \mathbb{R}^{3\times 2}$ and given by

$$\text{div } \nabla f_{gGD}(\xi) = 4.$$

A more involved calculation gives

$$\nabla f_{jCG}(\xi^k) = \nabla \left[\frac{1}{2} \left(|\xi^k|^2|\xi^i|^2 + |\xi^k|^2|\xi^j|^2 + |\xi^i|^2|\xi^j|^2 - \langle \xi^k, \xi^i \rangle^2 - \langle \xi^k, \xi^j \rangle^2 - \langle \xi^i, \xi^j \rangle^2 \right) \right]$$

$$= \frac{1}{2} \left[\xi_1^k \left(|\xi^i|^2 + |\xi^j|^2 - \xi_1^i - \xi_1^j \right), \xi_2^k \left(|\xi^i|^2 + |\xi^j|^2 - \xi_2^i - \xi_2^j \right) \right].$$

With for a particular channel

$$\text{div } \nabla f_{jCG}(\xi^k) = 2(|\xi^i|^2 + |\xi^j|^2) - \xi_1^i - \xi_2^i - \xi_1^j - \xi_2^j.$$

In total

$$\text{div } \nabla f_{jCG}(\xi) = 4(\|\xi^1\|^2 + \|\xi^2\|^2 + \|\xi^3\|^2) - 2\sum_{l,m} \xi_m^l.$$

Using the chain rule we have

$$\nabla f_{jNambu}(\xi^k) = \nabla \left(\sqrt{f_{jCG}(\xi^k)} \right)$$

$$= \frac{1}{4f_{jNambu}(\xi^k)} \left[\xi_1^k \left(|\xi^i|^2 + |\xi^j|^2 - \xi_1^i - \xi_1^j \right), \xi_2^k \left(|\xi^i|^2 + |\xi^j|^2 - \xi_2^i - \xi_2^j \right) \right],$$

then using the quotient rule for

$$\frac{\left[\xi_1^k \left(|\xi^i|^2 + |\xi^j|^2 - \xi_1^i - \xi_1^j \right), \xi_2^k \left(|\xi^i|^2 + |\xi^j|^2 - \xi_2^i - \xi_2^j \right) \right]}{4f_{jNambu}} =: \frac{[g_1(\xi^k), g_2(\xi^k)]}{4f_{jNambu}},$$

we get

$$\text{div } \nabla f_{jNambu}(\xi^k) = \sum_{l=1,2} \frac{\frac{\partial g_l}{\partial \xi_l^k} \left(f_{jNambu} - \frac{g_l}{2f_{jNambu}} \right)}{f_{jNambu}^2},$$

with

$$\frac{\partial g_l}{\partial \xi_l^k} = (|\xi^i|^2 + |\xi^j|^2) - \xi_l^i - \xi_l^j = \frac{\partial}{\partial \xi_l^k} \nabla f_{jCG}(\xi^k), l = 1, 2.$$

This can be computed for a given ξ as we know $\nabla f_{jCG}(\xi^k)$.

The explicit form of the generalised cross-gradient defined in Definition 57 for $N = 3, n = 2$ is given by

$$f_{gCG}(\xi)$$

$$= \left(\xi_1^1\xi_2^2\right)^2 + \left(\xi_1^1\xi_2^3\right)^2 + \left(\xi_1^2\xi_2^1\right)^2 + \left(\xi_1^2\xi_2^3\right)^2 + \left(\xi_1^3\xi_2^1\right)^2 + \left(\xi_1^3\xi_2^2\right)^2$$

$$-2(\xi_1^1\xi_2^1\xi_1^2\xi_2^2 + \xi_1^1\xi_2^1\xi_1^3\xi_2^3 + \xi_1^2\xi_2^2\xi_1^3\xi_2^3), \ \xi \in \mathbb{R}^{3\times 2}.$$

For the k-th channel with $C(i,j) \in \mathbb{R}$ a constant, we have the gradient

$$\nabla f_{gCG}(\xi^k)$$

$$= \nabla \left[\left(\xi_1^k\right)^2 \left(\left(\xi_2^i\right)^2 + \left(\xi_2^j\right)^2 \right) + \left(\xi_2^k\right)^2 \left(\left(\xi_1^i\right)^2 + \left(\xi_1^j\right)^2 \right) - 2\left(\xi_1^k\xi_2^k \left(\xi_1^i\xi_2^i + \xi_1^j\xi_2^j \right) \right) + C(i,j) \right]$$

$$= 2\left(\xi_1^k\|\xi_2\|^2 - \xi_2^k \left(\xi_1^i\xi_2^i + \xi_1^j\xi_2^j \right), \xi_2^k\|\xi_1\|^2 - \xi_1^k \left(\xi_1^i\xi_2^i + \xi_1^j\xi_2^j \right) \right).$$

Such that taking the divergence results in

$$\text{div}[\nabla f_{gCG}(\xi^k)] = 2\left[\|\xi^i\|^2 + \|\xi^j\|^2 \right].$$

The chain rule and quotient rule also holds channel-wise for f_{gCG} and f_{gNambu} as for f_{jCG}, f_{jNambu} before. So

$$\operatorname{div}[\nabla f_{gNambu}(\xi^k)] = \sum_{l=1,2} \frac{\frac{\partial g_l(\xi^k)}{\partial \xi_l^k} \cdot \left(f_{gNambu}(\xi^k) - \frac{g_l(\xi^k)}{2 f_{gNambu}(\xi^k)} \right)}{f_{gNambu}(\xi^k)^2},$$

with

$$(g_1(\xi^k), g_2(\xi^k)) := \nabla f_{gCG}(\xi^k) = \left(2\|\xi_2\|^2, 2\|\xi_1\|^2 \right).$$

C Figures

Included here the image results for the numerical experiments performed throughout Section 7.

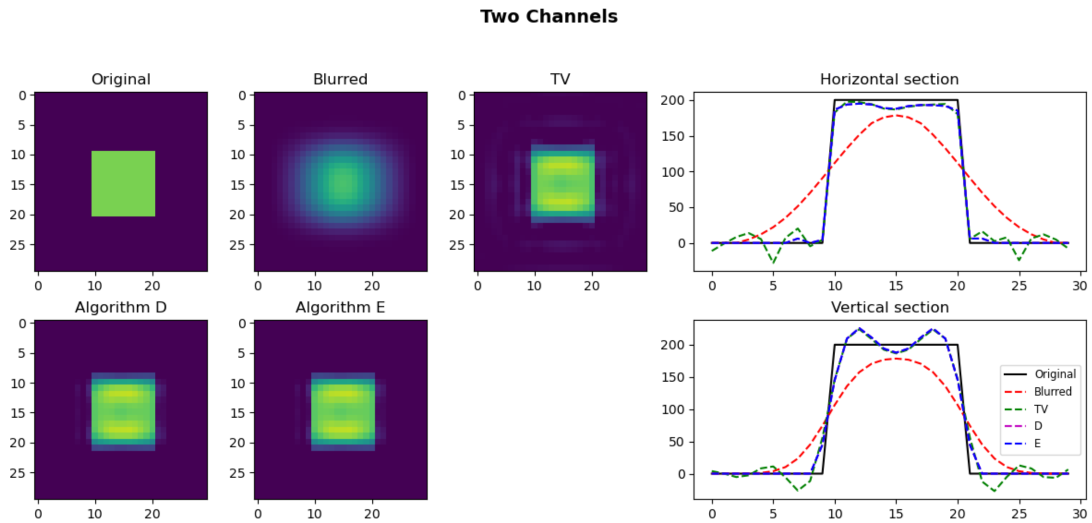


Figure 2: Two Channel Deblurring

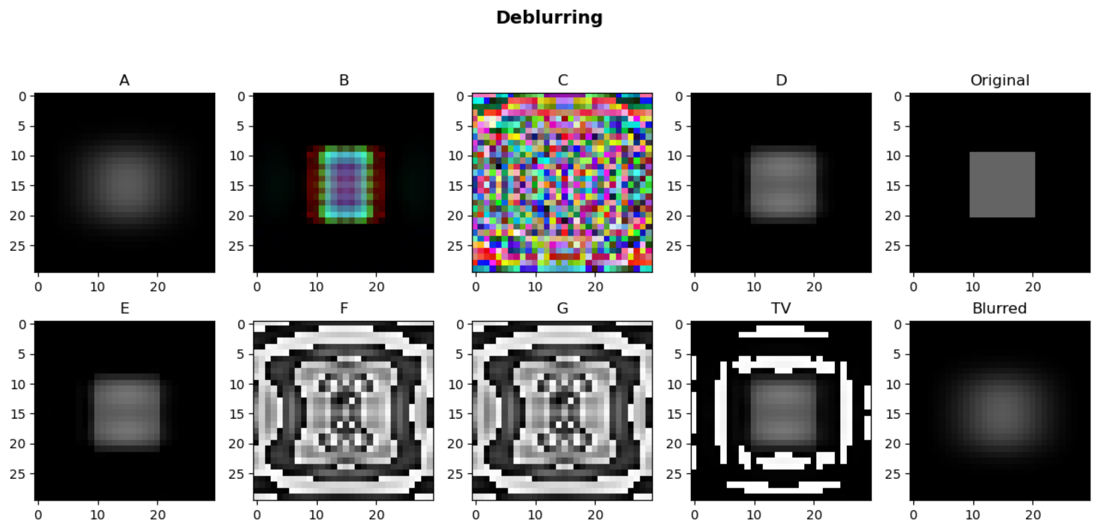


Figure 3: Equal Channels

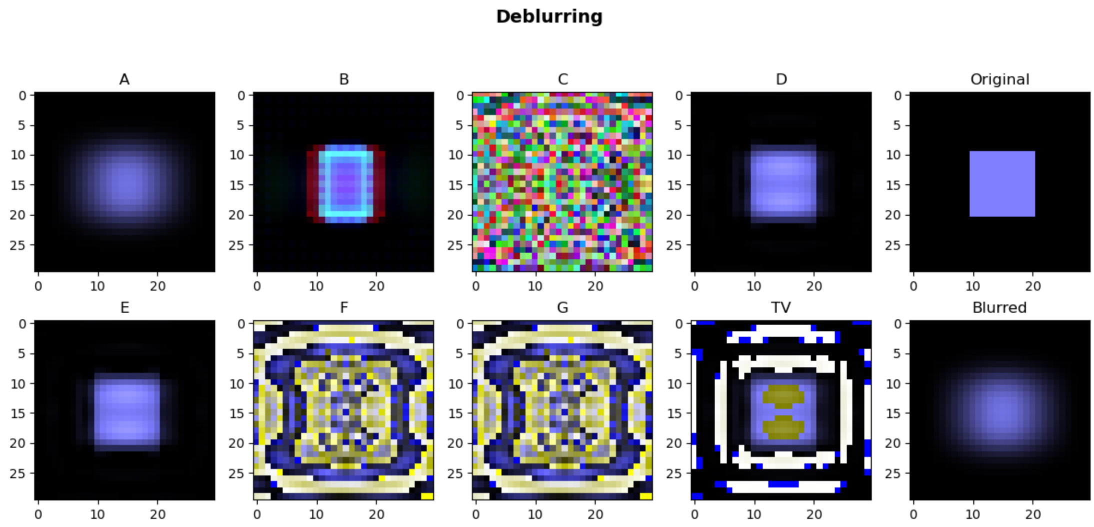


Figure 4: Unequal Channels

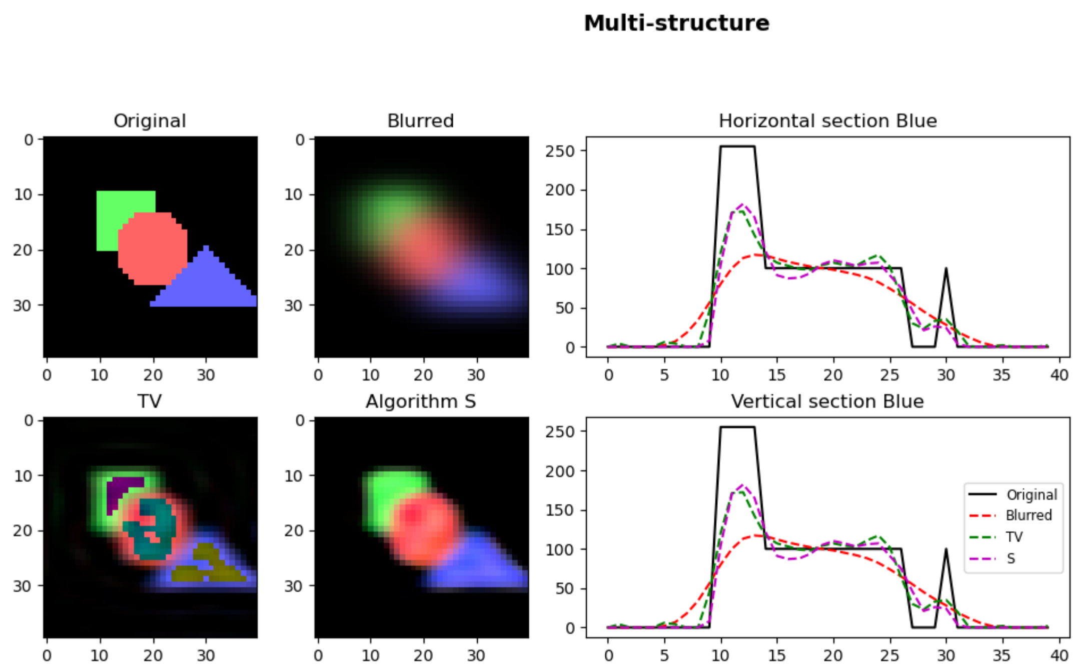


Figure 5: Multi-structure

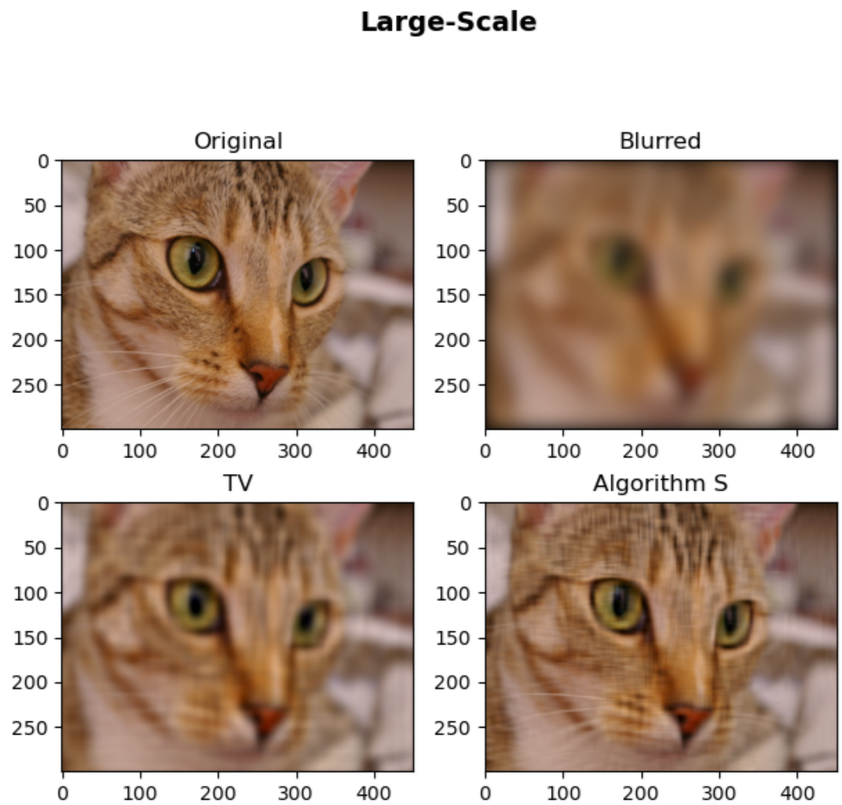

Large-Scale
Original
Blurred
TV
Algorithm S
0
50
100
150
200
250
0
100
200
300
400

Figure 6: Cat

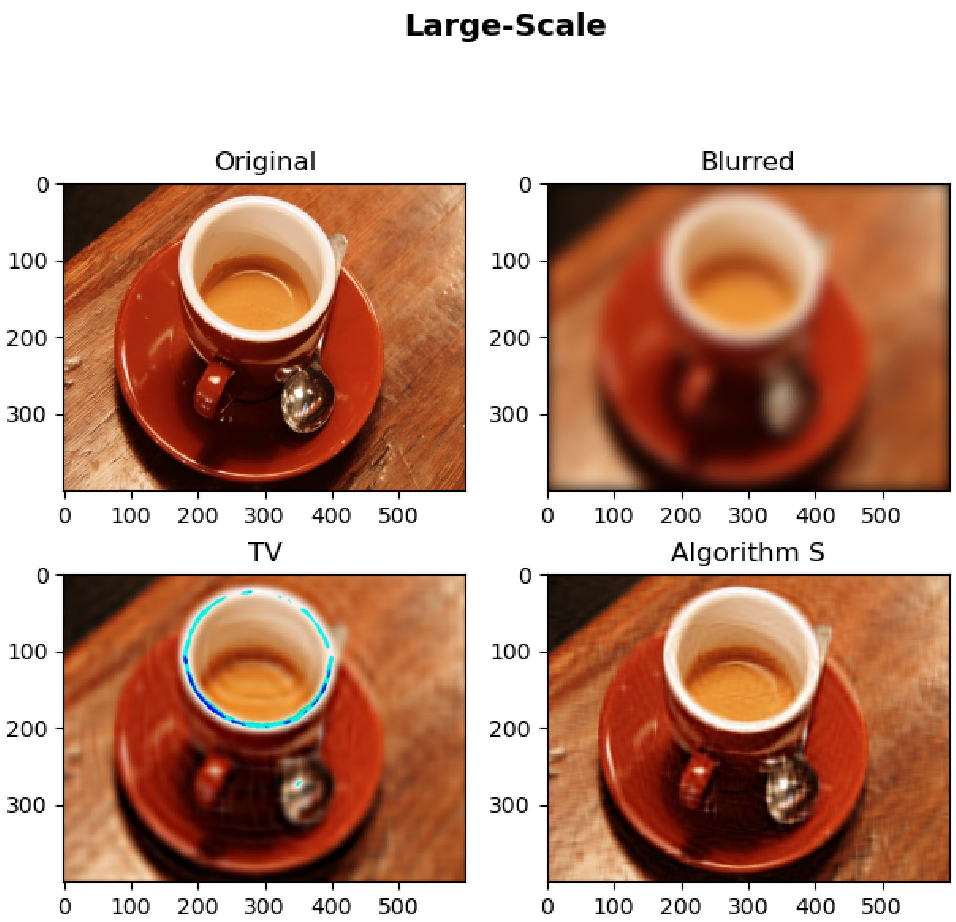


Figure 7: Coffee

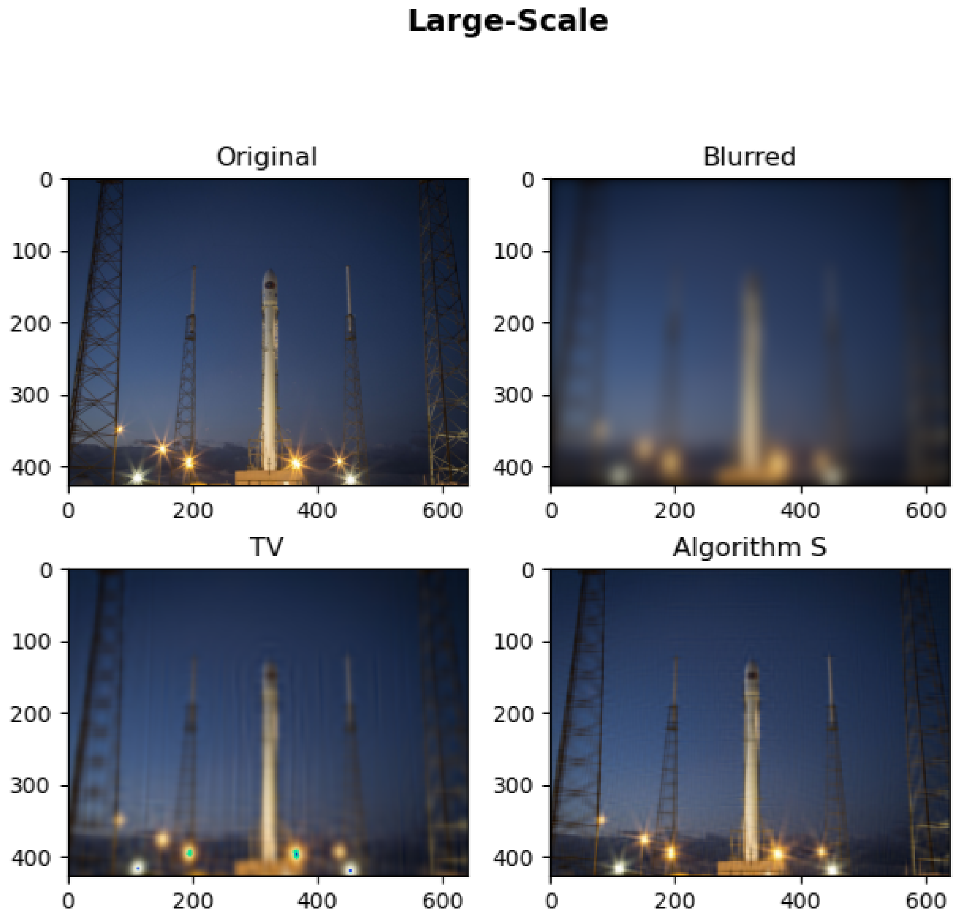


Figure 8: Rocket

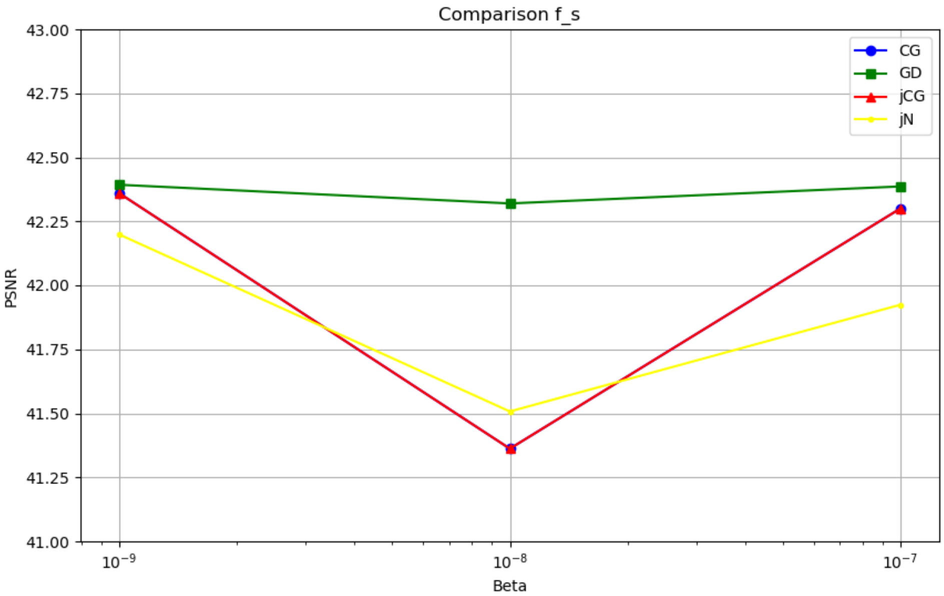


Figure 9: Comparison PSNR

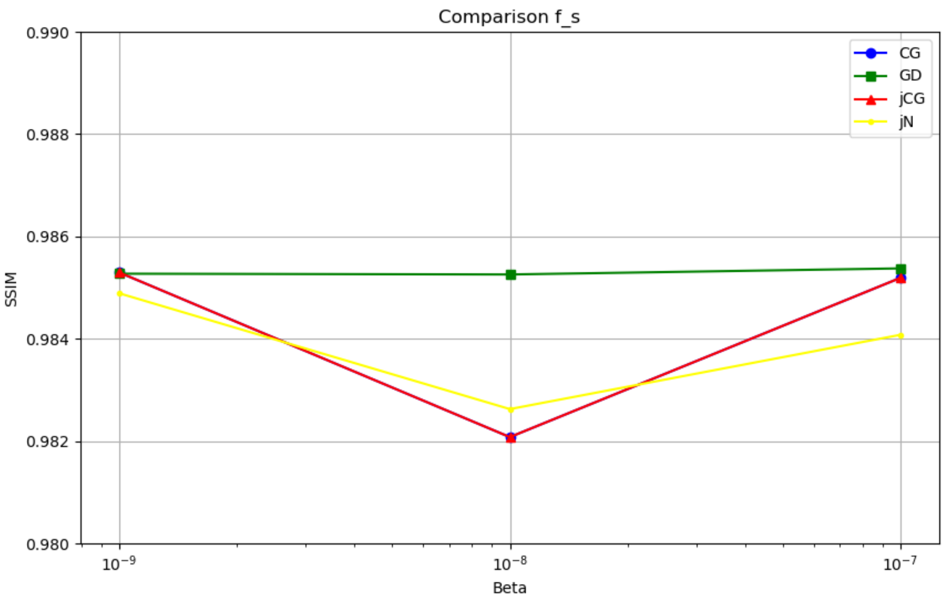


Figure 10: Comparison SSIM

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