28 juni 2024

Supporting reasoning skills with heuristic trees in pre-university mathematics

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Abstract

This research investigates the design of heuristic trees in supporting pre-university students in mathematical reasoning, as well as finding and formulating proofs within Dutch secondary education. The study addresses the challenge of improving students' reasoning skills, particularly focusing on the clarity and structure of their reasoning processes. The research employed an iterative approach where three iterations have been carried out with a total of 62 students, refining heuristic tree design based on student feedback and observed outcomes. Each iteration involved designing, refining, and implementing heuristic trees with distinct phases – orientation, elaboration, formulation, and completion – to guide students through proof-based tasks. Findings across iterations demonstrated improvements in students' reasoning and proof-writing skills. Enhanced reasoning skills were visible in identifying assumptions, structuring proofs, and justifying reasoning steps. Students showed increased engagement with heuristic tree components, with varying levels of interaction indicating adaptability to individual learning needs. It can be concluded that this four-phases design of heuristic trees supports pre-university students in mathematical reasoning and proof formulation. Future research could explore adaptations for different educational levels and further refine the digital environment to optimize heuristic tree usability for mathematical reasoning.

Key words: heuristic trees, mathematical reasoning, proofs, secondary education

Introduction

Reasoning skills hold significant importance in the secondary education mathematics curriculum (Arshad et al., 2017). Gunhan (2014) highlights the significance of secondary school students being proficient in mathematical reasoning, engaging in deductive and inductive reasoning through the formulation of mathematical assertions, and actively developing and sustaining their reasoning skills. This is particularly of importance in proofs, as Gunhan found that good mathematical reasoning skills are imperative to proof-writing performance. Furthermore, educational research has shed light on the intricate relationship between reasoning and proof. When students grapple with open problems, they often employ reasoning activities to construct a conjecture, establishing a cognitive bridge between the reasoning and proof (Pedemonte, 2018). Therefore, a connection between reasoning and proof becomes evident.

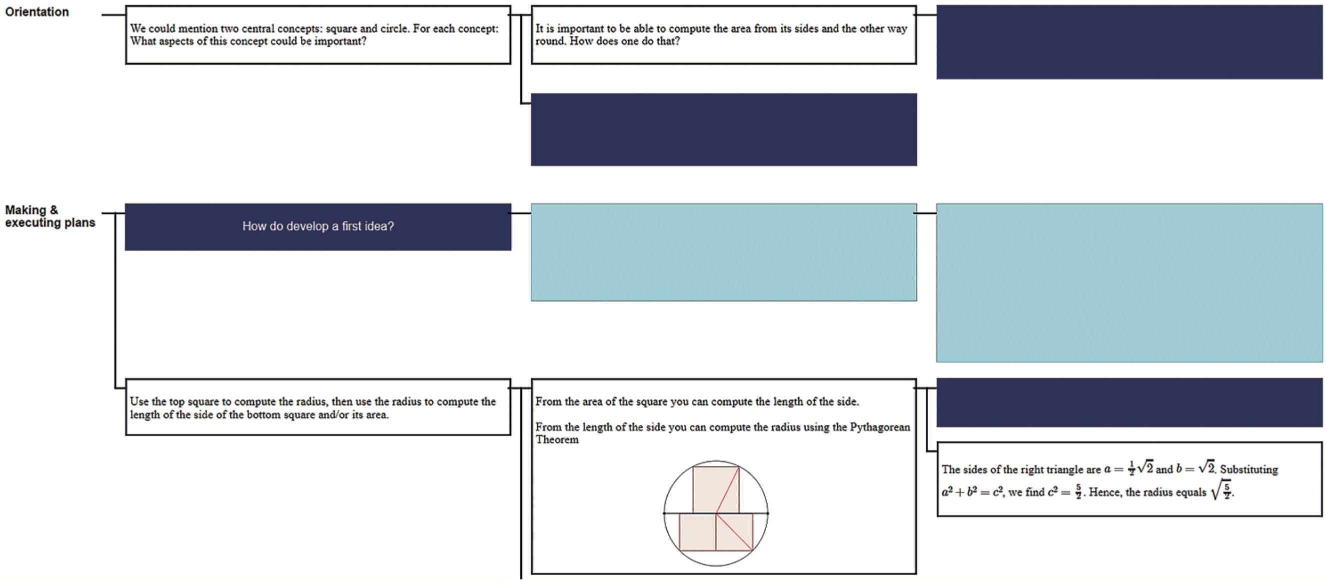
Research literature shows that proving and reasoning comes with many challenges (Stavrou, 2014). Firstly, high school students' competencies of doing proofs are found below the desired level, attributed to the difficulties encountered in teaching and learning mathematics (Köğce et al., 2010; Ubi et al., 2018). Secondly, many students cannot find a starting point in a proof and cannot identify correct arguments with respect to the specific context of a proof (Reiss et al., 2001). The lack of understanding among students in handling mathematical proofs underscores the crucial need for them to comprehend the processes involved. However, a problem is that proving and reasoning seems to require intense teacher supervision, which is not always available (Lester, 2013). Therefore, there is a need for strategies specifically directed at addressing challenges in proving in the classroom, as the importance of providing a proof lies in promoting understanding of abstract notions (Hanna, 1995; Stylianides, 2018).

We would like to explore heuristic trees as a means to support students' reasoning in the absence of a teacher. Heuristic trees were successfully implemented to support problem-solving, which of course involves reasoning (Bos & Van den Bogaart, 2022a). However, heuristic trees with the specific aim of supporting reasoning have not been explored. Learning from heuristic trees for problem-solving areas within mathematics is one of the directions for proving questions like those above in mathematics. Heuristic trees are interactive pages designed to offer a structured collection of hints presented in a tree format, aiding users in addressing a particular problem (Bos & Van den Bogaart, 2022b). Such a tree consists of three interconnected sub-trees, aligned with Pólya's (1945) stages: orientation, making and executing plans, and completion (figure 1). This enables students to work freely and actively engage with mathematical issues. Additionally, students learn to concentrate on broad heuristic strategies during mathematical problem-solving tasks, as intended by the tree structure of a heuristic tree.

However, heuristic trees have only been applied to problem-solving mathematical areas so far. Bearing that in mind, it is not known whether heuristic trees are suitable for supporting, finding and formulating (techniques for) mathematical reasoning. Furthermore, no design guidelines exist yet for heuristic trees in the reasoning field of pre-university mathematics education. With that in mind, the aim of this research is to find out whether heuristic trees can be a suitable support for reasoning in the process of finding and formulating proofs for students in secondary education. As design guidelines for heuristic trees will play a crucial role in the research, a design study will be conducted. To pursue this aim, we will design heuristic trees for mathematical reasoning based on existing literature and iteratively refine them based on results from three iterations. In each iteration, a class of students will carry out proof-based tasks.

Figure 1

An Example of the Upper Section of a Heuristic Tree



Note. See https://edspace.nl/htree/heuristiekboom.php?boom_id=146# for the complete heuristic tree. Please note that closed cards are represented in blue, with dark blue cards indicating the cards to be opened next.

Theoretical Background

As the research aims to find out whether heuristic trees can be a suitable support in finding and formulating mathematical proofs for students in secondary education, the challenges are outlined through introducing encountered difficulties by students in reasoning and proving. Secondly, the potential solutions are outlined through the exploration of heuristic trees and their possible applicability in the proof field of mathematics. The concept of heuristic trees and their design criteria will be discussed. Specifically, the requirements for heuristic trees needed for assisting with mathematical problem-solving tasks will be explained, based on the already available literature.

Defining mathematical proof and reasoning

A mathematical proof is a deductive argument that demonstrates a mathematical statement, ensuring that the given assumptions logically lead to the conclusion. In a formal proof, the argument relies on other previously established statements, such as theorems or lemmas or axiomatic systems (Clapham et al., 2014; Cupillari, 2005; Gossett, 2009). On the other hand, the term of a mathematical reasoning is not always clearly defined; it is generally assumed that everyone has an intuitive understanding of it and the way in which a mathematical reasoning is described in various documents tends to be vague, unsystematic, and sometimes even contradictory from one document to another (Jeannotte & Kieran, 2017). Therefore, the term mathematical reasoning is not rigid. In this study we will define mathematical reasoning as a form of proving as the focus will be on proof-based tasks. We will follow the previous description of an “argument that demonstrates a mathematical statement, ensuring that the given assumptions logically lead to the conclusion”. However, it is not on the level of a formal mathematical proof, so argumentation based on theorems, lemmas and axiomatic systems are not required. From now on, the terms reasoning and proof are used interchangeably, where proof does not refer to a formal proof.

Difficulties in reasoning and proving

As mentioned before, understanding the concept of proofs and their construction is fundamental for reasoning skills in mathematics education (Çetin & Dikici, 2021). A proof, which is more than mere examples, entails a structured logical explanation of why a mathematical statement is true. However, students often struggle with constructing proofs due to inadequate understanding and application of mathematical language, notation, and proof processes. These challenges manifest in various forms, such as difficulty in expressing definitions, conceptual understanding, and initiating the proof process. Instead of engaging in proof construction and writing, students may resort to informal approaches which hinders their ability to grasp the overarching logical structure of the proof process. For example, if students do not know how to construct a proof, they try using examples to prove something (Raman, 2002). In geometry, this may imply that students often believe a pictorial representation suffices as proof, or that empirical examples provided by students constitute a valid proof for the arbitrary case (Schoenfeld, 2013; Fischbein & Kedem, 1982).

Additional challenges are also found by Stavrou (2014) as students frequently make one of these four mistakes: they assume the conclusion in order to prove it; they prove general assertions with specific instances; they fail to prove both conditions in a biconditional statement; and they misuse definitions. Moreover, many students use superfluous examples to support valid proofs, leaving many assignments blank with comments like "I'm not sure how to start the proof". Two prevalent misconceptions add complexity: the belief that a single counterexample is not sufficient to refute false statements, and the misconception that a few confirming cases are adequate to establish the truth of a mathematical generalization (Stylianides, 2018).

In light of these challenges, consider the following question in geometry, often tackled by students in secondary education:

"Prove that the sum of the angles of a triangle is 180 degrees"

Although this classic problem does not require a lot of prerequisite knowledge or a robust understanding of formal proving, students often struggle here. For example, one may use a square that can be divided into two right triangles as example to prove it. A pictorial representation of such a divided square may likely suffice as argumentation by the student. However, this is considered a confirming case but not a generalized proof.

Heuristic trees in problem-solving

The concept of heuristic trees and their potential in aiding proving skills might help to address the challenges mentioned in the previous section. Heuristic trees strongly engage students in problem-solving, allowing them to maintain ownership of the solution methods. Hence, it is hypothesized that a heuristic tree may serve as potential support in addressing and enhancing proving skills in secondary education.

The fundamental theoretical concepts regarding the problem-solving phase and the compression of mathematical knowledge are reflected in the design principles of heuristic trees (Bos & van den Bogaart, 2022b). Compression is a cognitive process of reorganizing mathematical information because it is typified by a change in focus from phenomena to common aspects of those phenomena (Thurston, 1990; Sfard, 2008). Compression applies to objects, procedures, and statements (Bos & van den Bogaart, 2022b). Two types of compression can be distinguished in mathematics: compression on cases and compression on steps. When many items are considered as examples of a single overarching category, this is known as compression on cases. When knowledge is compressed on steps, distinct steps in a line of reasoning are viewed as a single, cohesive process.

According to Bos (2017), heuristic trees in problem-solving adhere to six design principles. Compression-Decompression Ordering (P1) emphasizes progressing from general to concrete concepts. This can be noticed in figure 1 where opened cards from left to right become increasingly more concrete each step. Logical Ordering (P2) ensures a coherent structure, separating main and side issues. Problem Solving Phases Ordering (P3) organizes branches in alignment with orientation, plans, and completion. Independence (P4) guarantees self-contained information in each branch. Rationing (P5) provides just enough help per click, and Revelation (P6) hints at content without explicit disclosure. Overall, these principles are very valuable to the study since they can serve as a basis for proof-based areas of mathematics.

Existing models of the proving process

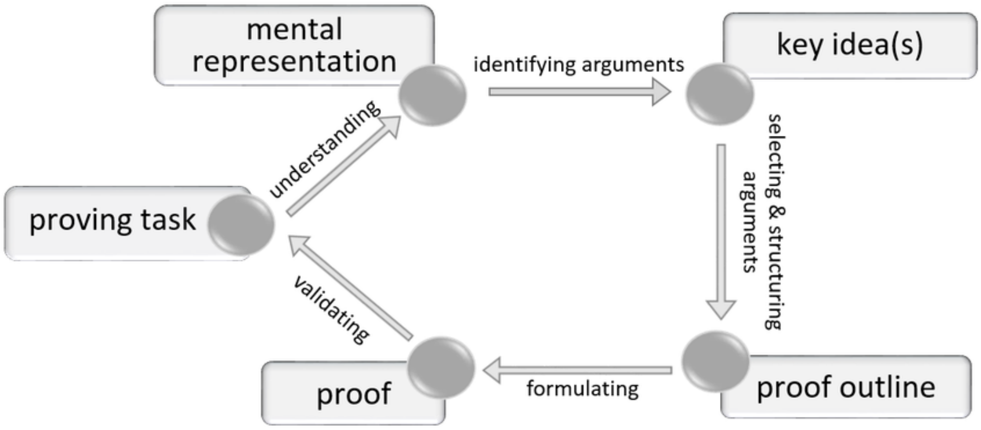
After having outlined heuristic trees in general and having connected the principles of designing heuristic trees to problem solving, the following will show how it differs with proving. For this reason, existing proof models are explained first, providing grounds for a different approach in designing heuristic trees for proving.

In the landscape of existing models of the proving process, prominently characterized by Stein (1984) and Boero (1999), the emphasis has traditionally been on problem exploration, particularly within open-ended problem areas. Meanwhile, the proving cycle can also frequently start with a proving task centered on a statement estimated to be true, aligning with the challenges encountered by secondary students in proving (Kirsten, 2018). The proving cycle (figure 2), attuned to the challenges of pre-university students, unfolds through a series of phases guiding learners from initial exploration to final proof validation:

- Exploration: explore the statement area, discover key ideas, and identify reasons for the validity of the statement.
- Selection and Structuring: select promising ideas, work out details, and structure single arguments in a deductive order, resulting in a proof outline.
- Revision and Finalization: fill gaps, revise linguistic and formal arrangement of the proof to meet community standards.
- Validation: validate the final proof by reviewing content, structure, and linguistics.

Figure 2

A Visual Representation of the Modified Proving Cycle (Kirsten, 2018)



The purpose of incorporating this proof cycle in the research is to provide a structured framework that secondary students can use to systematically approach and construct proofs. By guiding students through these phases, the proof cycle helps them navigate the complexities of proving tasks and enhances their ability to develop mathematical reasoning.

Drawing on this model for proofs from Kirsten (2018) and the structure of heuristic trees for problem-solving, a four-phased proving cycle (figure 3) may serve as a promising starting point to design heuristic trees to enhance the understanding and application of mathematical proofs. The first two stages are one of the most crucial stages to include and design here as it lets students grasp ideas to prove the task. During an initial exploratory phase students need to experience freedom and flexibility (Durand-Guerrier et al., 2012). As some students tend to make justifications based on specific examples or figures resulting from their own actions, a carefully developed exploration can help making them aware such a specific case encourages the generation of conjectures, but it does not itself constitute a justification; rather, it merely serves to support one (Hsieh et al., 2012). Besides a divergent exploratory phase, students need to experience a convergent validating one too. Through this process, students acquaint themselves with the inherent openness of exploration, characterized by its flexibility in generating and executing ideas. Simultaneously, they grasp the necessity of a more stringent approach required for formulating comprehensive proofs, emphasizing the precise utilization of language, structure, and content (Durand-Guerrier et al., 2012).

Figure 3

A Four-Phased Proving Cycle that may serve as a Starting Point for designing Heuristic Trees

1. **Understanding & Identification**
   ⇨ Focus on comprehending the statement to be proven and identifying key arguments.

2. **Proof structuring**
   ⇨ Involves organizing key ideas and structuring arguments in a deductive order, resulting in a proof outline.

3. **Elaboration & Formulation**
   ⇨ Includes working out the details and formulating the complete proof.

4. **Finalization & Validation**
   ⇨ Encompasses filling in any gaps, revising the linguistic and formal arrangement of the proof, and validating the final proof by reviewing content, structure, and language.

The last phase in the model includes validating activities, which can be compared to a certain extent to Pólya's (1945) stage of looking back. Although this four-phase proof model shares some resemblance with the structure observed in heuristic trees for problem-solving (orientation/making and executing plans/completion), substantial deviations are noted in most aspects. Therefore, the construction of heuristic trees for problem-solving partially aligns with the theoretical underpinnings of the four-phased proving cycle (figure 3), both providing a starting point for designing heuristic trees for proving.

Research question

Three main concepts have been made clear: the challenges students face in proving in secondary mathematics education, the potential of heuristic trees as a solution and an existing model of the process of proving. Reasoning skills are crucial for effective mathematical understanding, particularly in proofs. Common challenges and misconceptions in pupils' reasoning abilities in proofs underscore the multifaceted nature of difficulties. Heuristics can be implemented to enhance proving skills and conceptual understanding in mathematics. This involves utilizing heuristic trees, providing an interactive framework to support mathematical problem-solving through structured hints. With a four-phased proving cycle drawing on existing models and aligning with the outline of heuristic trees, it offers a structured and systematic approach to guide the construction and understanding of proofs in the upper secondary education.

As can be noticed in the description of a heuristic tree, heuristic trees have only been designed and applied for problem-solving areas of mathematics education. No research has been done on the exploration of heuristic trees in reasoning in mathematics education. Thus, it is unclear whether heuristic trees are useful for students to assist in addressing, finding, and formulating their proofs, let alone if there are specific design standards for heuristic trees in the secondary mathematics education proof field. Hence, the main question of the research is:

How can heuristic trees support pre-university students in mathematical reasoning, as well as finding and formulating proofs in the Dutch secondary mathematics education?

Methods

A qualitative design study was conducted where heuristic trees were designed for tasks in mathematical reasoning and proofs. These heuristic trees were iteratively tested and updated with students.

Design: heuristic trees for proofs

Taking the four-phased proving cycle (figure 3) as a foundation, while (partially) adhering to the six design principles, provided a starting point for developing heuristic trees in the field of proofs. Unlike the heuristic trees developed for problem-solving tasks, the heuristic trees for mathematical reasoning were developed with a different structure. Instead of three phases for problem-solving tasks, these heuristic trees explicitly have four phases, based on the four-phased proving cycle (figure 3). The following four phases were developed with accompanying initial questions:

- Orientation Phase: How do I get an idea for my proof?
- Elaboration Phase: How do I make my proof?
- Formulation & Finalization Phase: How do I write my proof down?
- Completion Phase: What did I learn and how can I use this knowledge in other settings?

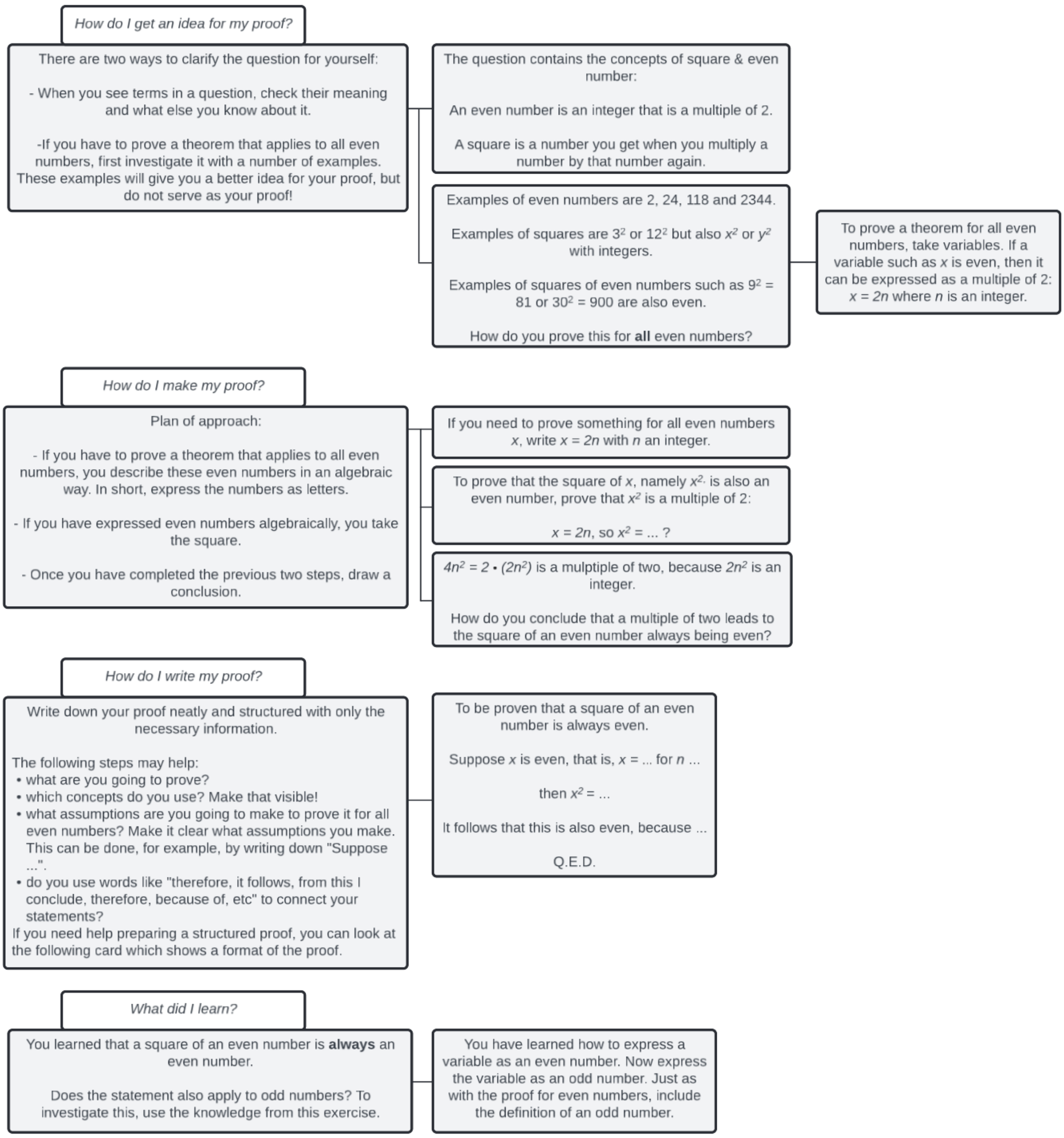
The orientation phase encompasses the first phase from figure 3, while the elaboration phase corresponds to the second and third one and the formulation phase corresponds to the latter two phases. The completion phase is not based on the four-phased proving cycle but is instead based on the previously developed heuristic trees for problem-solving tasks (Bos & van den Bogaart, 2022a).

Three heuristic trees have been designed. As an example, figure 4 shows a final version of a fully opened designed heuristic tree with its questions that appear in an unopened branch. This tree is designed for the algebraic task of proving that the square of an even number is always even. Some design principles established for heuristic trees in problem-solving tasks are present (Bos & van den Bogaart, 2022a):

- P1: From general to concrete steps. For example, in the elaboration phase, the general plan of approach is made concrete with a suitable expression.
- P4: Self-contained information in each branch. Each phase is independently accessible.
- P5 & P6: Just enough help per click & content without explicit disclosure.

Figure 4

A Fully Opened Designed Heuristic Tree

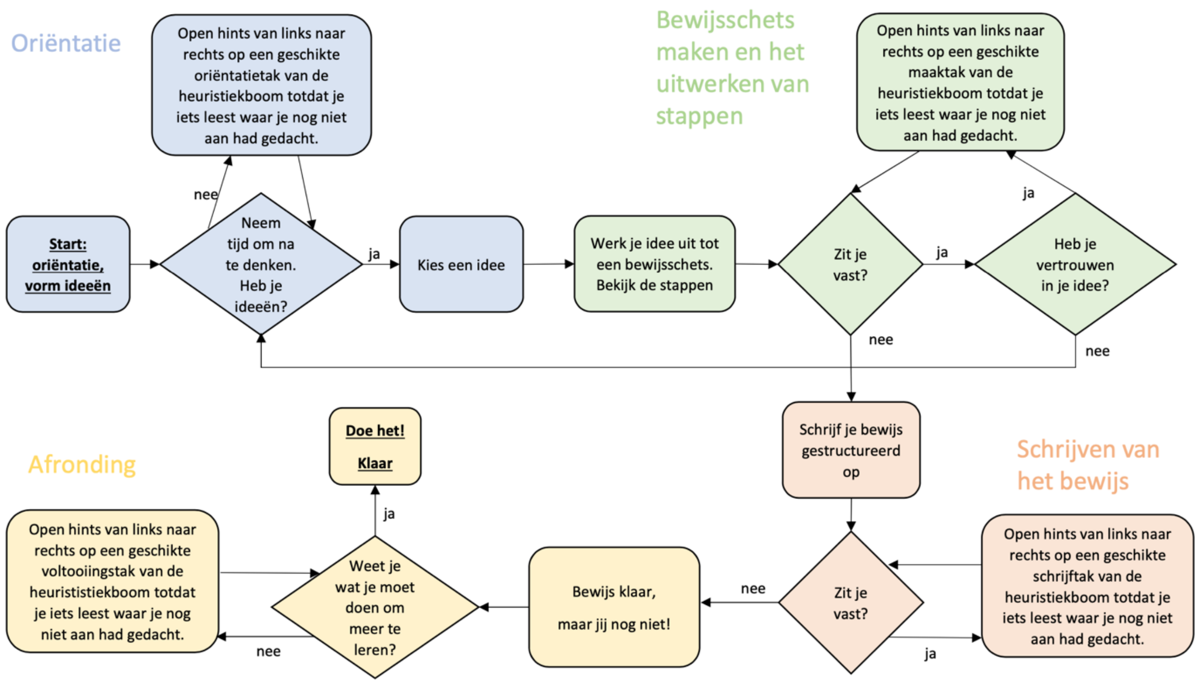


Note. The questions that are visible in the unopened tree are in white boxes. The problem to which the heuristic tree applies is: prove that a square of an even number is always even.

Design: flowchart

The students used a help-seeking flowchart (figure 5) to support the help-seeking process (Lemmink, 2019). The chart has been modified for mathematical reasoning problems in proving. Therefore, it has been extended with a third section about how to write your proof. The chart provides guidance on when and how to transition between the orientation, elaboration, formulation, and completion phases. It also covers the transition from heuristic support to more specific hints.

Figure 5



The Help-Seeking Flowchart

Note. The blue boxes represent the orientation phase, the green boxes the elaboration phase, the red boxes the formulation phase, and the yellow boxes the completion phase.

Design of the study

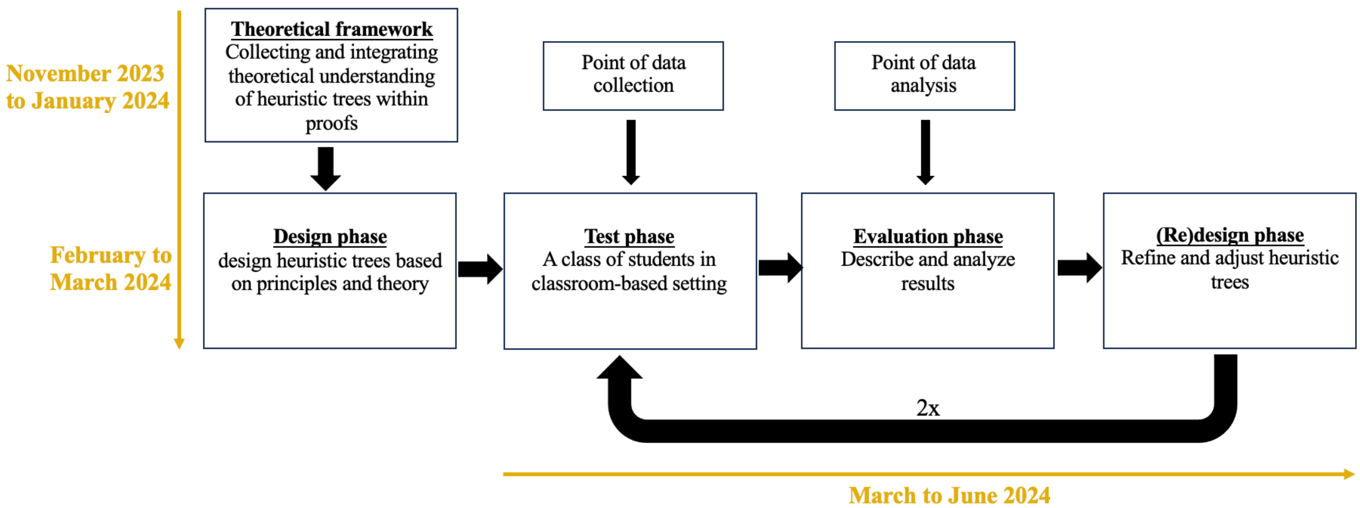
The study had the following design research cycle (Bakker, 2018; for a timeline, see figure 6):

- Theoretical framework on reasoning, difficulties in proofs, heuristic trees, and existing proof models
- Designing heuristic trees for three different proof tasks
- Testing the tasks with a class of students
- Analysis of the iteration
- Making theoretical conclusions and steps towards further design iteration

The study consisted of three iterations. Each iteration took place in a classroom-based setting at two high schools in Utrecht. All three iterations were conducted between March and May of 2024. All cards from the heuristic trees were labelled with a letter and a number. The letter corresponds to the phase (A for orientation, B for elaboration or formulation, C for completion), while the number indicates the ordering of the cards within each phase. The heuristic trees were adjusted after the first and second iterations based on the results and optimized for the next iteration. The heuristic tree from figure 4 was used for the final iteration. The details of the adjustments per iteration are described in the results section.

Figure 6

An Illustrative Summary of the Research Cycle of the Study



Each iteration took 45 minutes. During each iteration, 25-35 minutes were available to work on the tasks individually. For this reason, the desks were separated, and each row of students received a different problem. Printed flowcharts and pawns were distributed to students for this part of the iteration. During the proving process, students were asked to move the pawn along the flowchart in accordance with their current state. On their laptops, they had access to the heuristic trees and the tasks. The students received a worksheet with a wide column for their work and a narrow column for noting the opened cards. Every 3 minutes, students were asked to draw a line under their work and the noted opened cards. In the second and third iterations, the students received a short 10-minute instruction beforehand in which an explanation was provided on how to use heuristic trees and the flowchart and what valid mathematical reasoning/proof consists of. Finally, during each iteration, 5-10 minutes were given to answer two questions about the experience of using the heuristic tree for the process of proving.

Instruments

As previously mentioned, three heuristic trees were designed, with each heuristic tree pertaining to a different branch of mathematics:

1. Logical reasoning: who ate the cookie?
2. Geometry: the sum of the angles in a triangle is 180 degrees.
3. Algebra: the square of an even number is even.

The third one is shown fully opened with all cards in figure 4. All cards start the elaboration phase with card B1, but for the formulation phase the logical and geometric problem start with card B7 whereas for the algebraic problem this is card B5.

All final versions of the designed heuristic trees can be found via the following links:

https://edspace.nl/hboom/heuristiekboom.php?boom_id=361

https://edspace.nl/hboom/heuristiekboom.php?boom_id=360

https://edspace.nl/hboom/heuristiekboom.php?boom_id=358

The heuristic trees are only available in Dutch, as the target group was Dutch students. In addition to the developed heuristic trees and flowchart, students used a worksheet, supplemented with two questions to be answered at the end of the iteration. The following two questions were provided to the students:

1. *Based on your experience with the heuristic tree:*

- *Give two tops if you thought the heuristic tree was a useful addition.*
- *Give two tips if you did noy find the heuristic tree a useful addition.*
- *Give one tip and one top if you found the help both useful and not useful.*

For example, discuss finding an idea for your proof, creating your proof or writing down your proof.

2. *Write down one (mathematical) proof or reasoning technique that you learned today and will use again. Explain why.*

Participants

The participants were 62 students in total from third class at the gymnasium level, with three teachers from each class collaborating. In the first iteration, 22 students from Stedelijk Gymnasium Utrecht (SGU) participated. This school uses the mathematics book series "Getal & Ruimte." In the second and third iterations, 21 and 19 students from Christelijk Gymnasium Utrecht (CGU) participated, respectively. A different class participated in the third iteration than in the second iteration. CGU uses the book series "De Wageningse Methode".

Data analysis

A multiple case study was carried out in combination with a bottom-up approach in document analysis. Denscombe (2017) emphasizes that case studies focus on a limited number of instances of a particular phenomenon and aligns with qualitative research methodologies, especially fitting for small-scale projects. As for our case, the implementation and impact of heuristic trees on the process of finding and formulating mathematical proofs was a previously unexplored case. Therefore, a case study was a valuable way of retrieving in-depth information about the impact of heuristic trees in proof-based mathematics. The adoption of a multiple case study design was particularly advantageous as it allowed for a nuanced examination of how pre-university students engage with and benefit from heuristic trees in the process of finding and formulating mathematical proofs, providing valuable insights into the varied dynamics and potential impact of heuristic trees (Gustafsson, 2017).

In addition, since document analysis is very useful for qualitative case studies, it was employed (Bowen, 2009). All obtained data from the students' work underwent the steps of skimming, reading, and interpretation. Specifically, we applied thematic analysis, where the students' work was analyzed and categorized (Fereday & Muir-Cochrane, 2006). Since the idea was to let patterns emerge from the students' work, a bottom-up approach was appropriate, making document analysis suitable.

Results

1st iteration

Three different heuristic trees were distributed among the students. Looking at the results for each heuristic tree (see figure 7 for a legenda), there were differences. Almost everyone correctly provided a proof to the logical reasoning problem. Most students who received this problem noted that the task was too easy. Since the problem was perceived as too easy, it was not used for further analysis.

From a bottom-up approach in analysis of the work of students for the other two tasks, five categories became apparent to assess the students on. Indicators how to assess these, are shown below:

- *Proof direction*
  - ⇨ Has the student provided a reasoning that points toward a valid proof? A student can demonstrate the wrong thing or demonstrate something with an (informal) example. Has the student provided any proof direction at all?
- *Assumptions*
  - ⇨ Are assumptions made explicitly visible? For example, this could include assumptions being stated as "suppose that …". Are they visible, but implicitly?
- *Use of conjunctions in connecting steps*
  - ⇨ Is each step logically connected with conjunctions to give meaning to the progression of the reasoning steps? For example, this can include a "suppose that …, then it follows …" or a "This means that …".
- *Structure of the proof*
  - ⇨ Is there a transition from a messy or chaotic elaboration to a concrete (compact) step-by-step elaboration? Are neat compact versions of the elaboration visible? Some kind of revised version? A version without notes and random pictures/examples? Is the proof structured in general?
- *Justification*
  - ⇨ Are all steps clearly explained in terms of why something is done, why it is the way it is, how it follows from previous steps, or why it can be assumed?

Figure 7

The Legend the Tables of the Results are based on

Legend:

Green = correct proof direction // assumptions explicitly visible // conjunctions, structure and/or justification visible
➔ Made visible with a "yes"

Orange = wrong proof direction // assumptions implicitly visible // conjunctions, structure and/or justification partly visible
➔ Made visible with a "wrong", "implicitly", or "partly"

Red = no proof direction visible or present // assumptions not present // conjunctions, structure and/or justification not present
➔ Made visible with a "no"

It should be noted that a green space in table 1 and figure 7 does not necessarily indicate a correct assumption, justification, et cetera. It merely means the student has provided a visible argument explaining why they did something within their mathematical reasoning. Therefore, the attempt to correctly develop and formulate the proof is important, rather than the correctness of the solution itself.

From table 1, it becomes evident that students struggled with constructing a mathematical argument. While almost everyone employed a proof strategy, they often used an incorrect one, typically relying on informal examples. Most students did not use assumptions, structure, or connectives in their proofs. Despite these shortcomings, about half of the students were able to explain their reasoning with an argument, regardless of its correctness.

Table 1

Results of the Students from the First Iteration

Task	Student	Proof direction	Assumptions	Conjunctions	Structure	Justification
algebraic	1	wrong	implicitly	partly	yes	no
	2	wrong	no	no	no	no
	3	wrong	no	yes	partly	yes
	4	wrong	yes	yes	partly	yes
	5	no	no	no	no	no
	6	wrong	no	no	no	no
	7	no	no	no	no	no
	8	no	no	no	no	no
	9	wrong	no	no	no	yes
geometric	10	wrong	no	no	no	no
	11	wrong	no	no	no	no
	12	wrong	no	no	no	yes
	13	wrong	no	no	no	yes
	14	no	no	no	no	no
	15	wrong	no	no	no	yes
	16	wrong	no	no	no	yes
	17	wrong	no	no	no	yes
	18	no	no	no	no	no
	19	wrong	no	no	no	yes
	20	wrong	no	no	yes	yes
	21	wrong	no	no	no	no
	22	yes	implicitly	no	no	partly

To assess the extent to which the heuristic tree has supported the student in mathematical reasoning, a closer examination can be made for each problem.

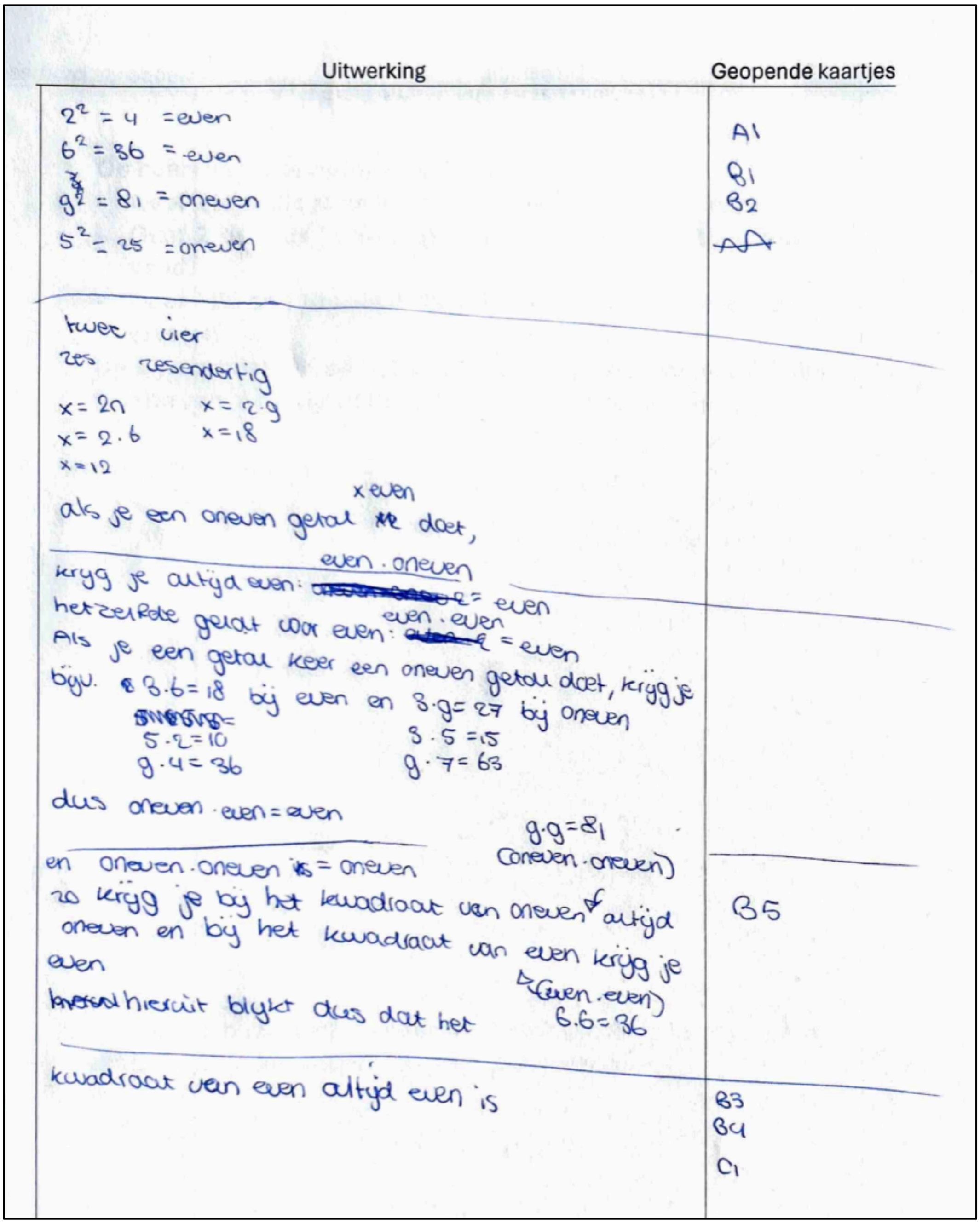
Algebraic problem

Many used numerical examples instead of transitioning to algebraic expressions. Almost everyone who wrote $x = 2n$, failed to connect it to $x^2 = 2 \cdot (2n^2)$.

In figure 8, it is visible how a student has used at least one card from each phase (namely A1, B1, B5, C1). Although this student was not able to generalize to $x = 2n$, it is noticeable that their method of writing changed due to card B5. However, the use of the heuristic tree did not lead this student to form an idea for a correct proof approach.

Figure 8

The Work of a Student from the First Iteration on the Algebraic Task



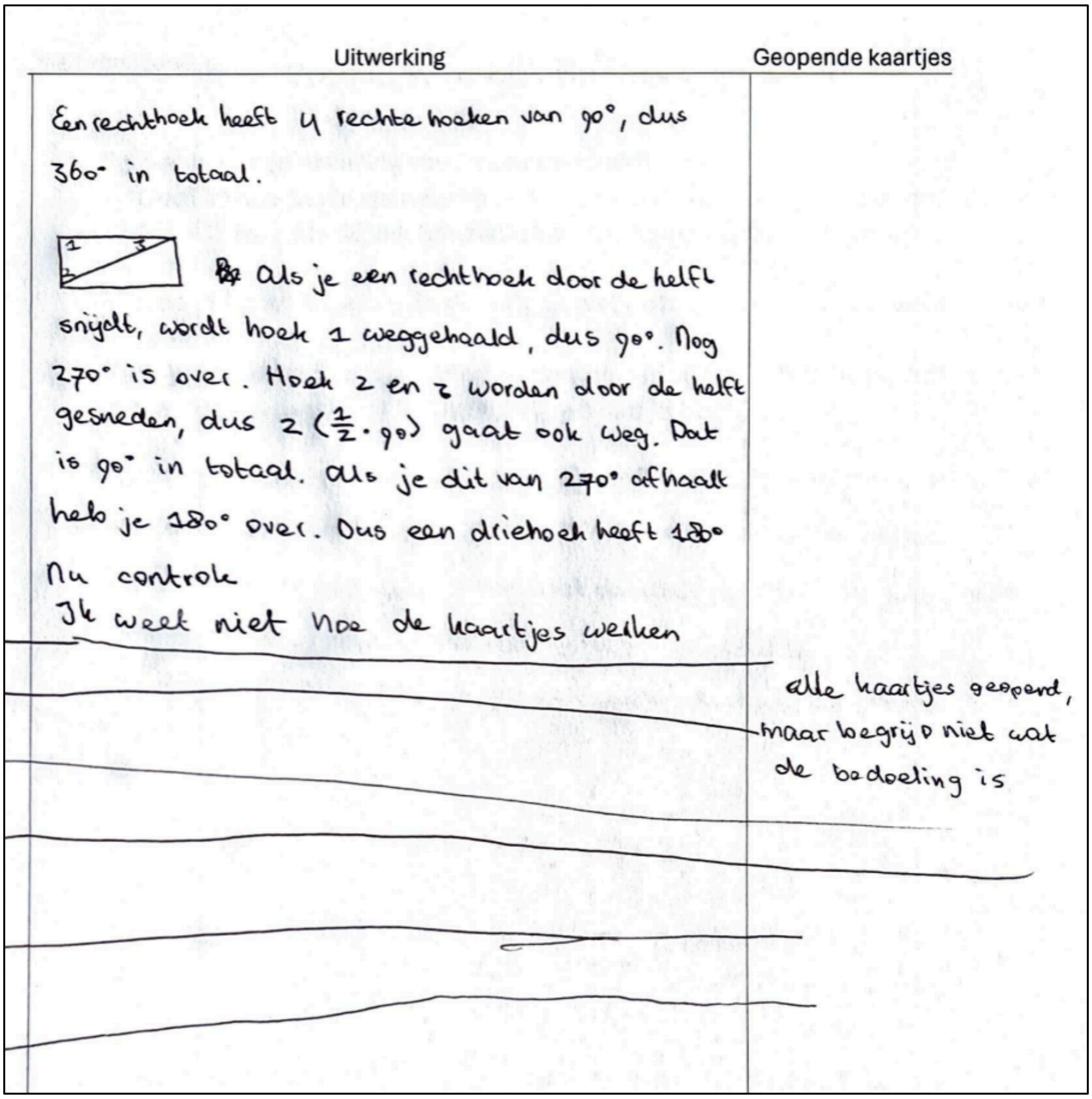
Uitwerking	Geopende kaartjes
$2^2 = 4$ = even $6^2 = 36$ = even $9^2 = 81$ = oneven $5^2 = 25$ = oneven	A1 B1 B2 ~~A1~~
twee vier zes zesendertig x = 2n x = 2·9 x = 2·6 x = 18 x = 12 als je een oneven getal x even doet,	
krijg je altijd even: even · oneven = even hetzelfde geldt voor even: even · even = even Als je een getal keer een oneven getal doet, krijg je bijv. 3·6 = 18 bij even en 3·9 = 27 bij oneven 5·2 = 10 3·5 = 15 9·4 = 36 9·7 = 63 dus oneven · even = even	
en oneven · oneven = oneven 9·9 = 81 (oneven · oneven) zo krijg je bij het kwadraat van oneven altijd oneven en bij het kwadraat van even krijg je even (even · even) 6·6 = 36 hieruit blijkt dus dat het	B5
kwadraat van even altijd even is	B3 B4 C1

Geometric problem

Almost everyone used an informal example of taking a rectangle or square and constructing two triangles from it. One student did apply a correct method, but the caveat is that there are deficiencies in writing this proof. For instance, there's a lack of mention of using Z-angles. Although many consulted the orientation cards and applied the straight angle, they often missed the connection to using auxiliary lines and Z-angles, primarily due to opening cards from the first column, notably B1 and B7 (both by opening and not opening the “elaboration”-cards). Figure 9 illustrates how a student is convinced that their proof is complete and correct. However, they proved it just for right triangles.

Figure 9

The Work of a Student from the First Iteration on the Geometric Task



Alterations made for 2nd iteration

As the logical reasoning problem was below the appropriate level for seeking help through a heuristic tree, it was decided to reuse only the remaining two problems for the second iteration.

The first iteration showed that students did not understand how to construct a mathematical argument. This was evident in both the given tips and their solutions. Nearly all students incorrectly provided proofs. Based on numerous tips highlighting vagueness and confusion about using the heuristic tree and the flowchart, it was decided to provide a brief 10 min instruction for the second iteration covering:

- The lesson's goal (practicing mathematical reasoning)
- What a heuristic tree is and how to use it
- When something counts as a proof (the difference between an example/conjecture and a proof)

- The phases of mathematical reasoning
- An explanation of what to expect (details about the provided papers)

Based on the results from the first iteration, the following adjustments were made in the heuristic trees:

- The language was made clearer. For instance, "algebraically" was explained as "use letters".
- In the phase "How do I write down my proof/reasoning?" it was made more concrete that the second card contains a format for a structured elaboration.
- For the algebraic problem, greater emphasis was placed on using letters in the orientation phase for forming ideas.
- For the geometric problem, a more concrete hint was provided on the "how do I make my proof?" card. The first card now mentions both backward and forward reasoning.

2nd iteration

Table 2 shows that instruction and more concrete hints contributed to better support for proof construction. Although only a small portion of the entire group applied their proofs in the correct direction, this marked progress. Notably, there is visible improvement in the use of assumptions, conjunctions, and structure. In the first iteration, these elements were scarcely present in the students' work, whereas now, the majority shows at least some (albeit limited) use of one or more of these elements. For assumptions, this was mostly still implicit. Regarding justification, there weren't a lot more students providing sufficient argumentation, but the number of students without any justification decreased. More students now provided (limited) argumentation.

Table 2

Results of the Students from the Second Iteration

Task	Student	Proof direction	Assumptions	Conjunctions	Structure	Justification
algebraic	1	yes	yes	yes	yes	yes
	2	wrong	no	partly	yes	partly
	3	yes	implicitly	yes	yes	yes
	4	yes	implicitly	no	no	partly
	5	wrong	yes	yes	yes	partly
	6	wrong	implicitly	partly	no	no
	7	wrong	implicitly	partly	no	partly
	8	wrong	implicitly	yes	yes	yes
	9	wrong	no	no	no	yes
	10	wrong	no	yes	no	yes
	11	no	no	no	no	no
geometric	12	yes	implicitly	no	partly	yes
	13	no	implicitly	no	no	no
	14	wrong	no	no	no	yes
	15	wrong	no	no	no	no
	16	wrong	no	no	no	no
	17	wrong	no	partly	partly	yes
	18	yes	implicitly	no	no	partly
	19	yes	implicitly	yes	yes	yes
	20	yes	yes	yes	no	yes
	21	yes	implicitly	no	partly	no

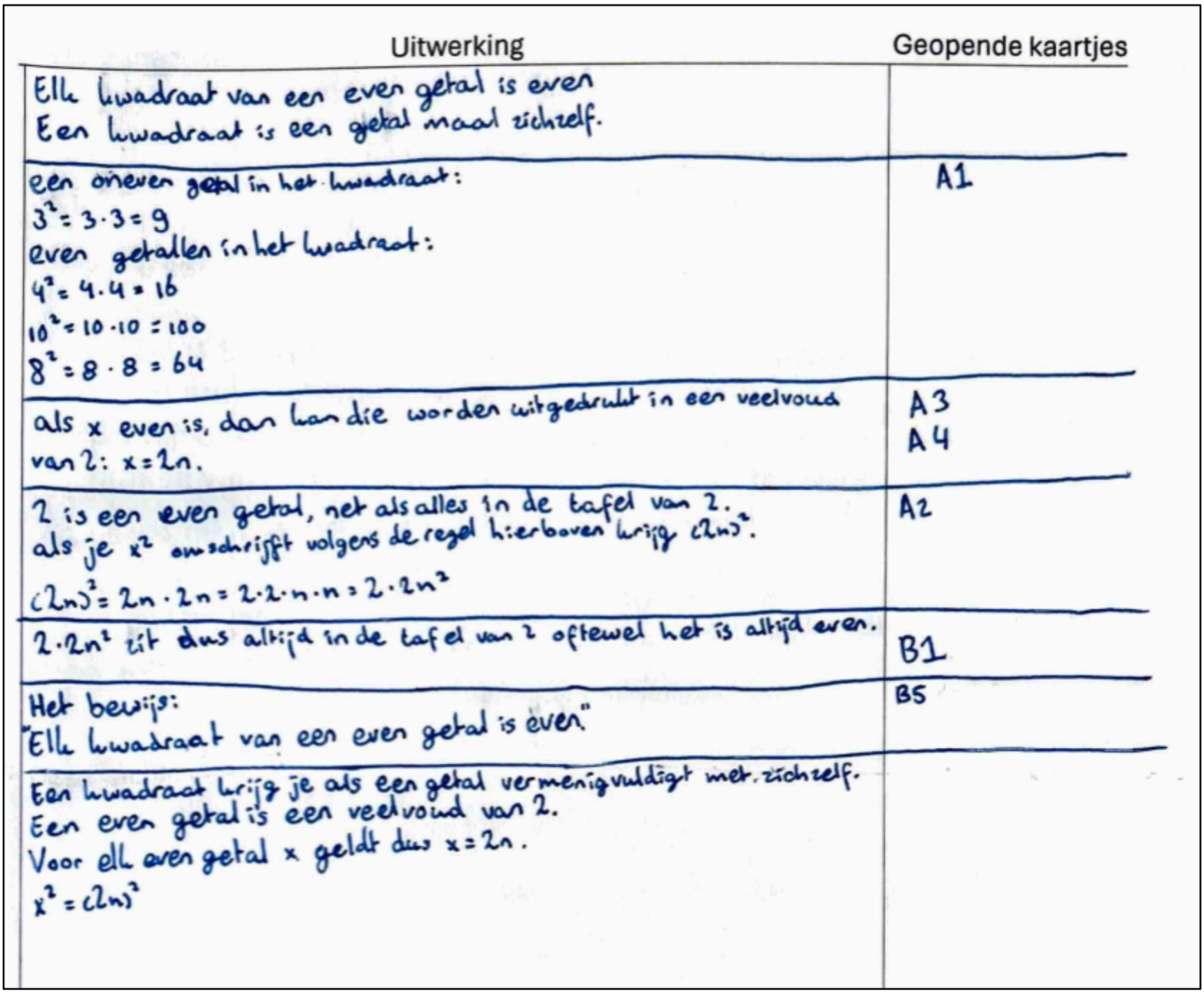
Algebraic task

Around a quarter of the students have correctly interpreted the question. Among the students who have made mistakes, almost everyone has used an algebraic expression but was unable to correctly simplify or argue it. For instance, solutions contain expressions like $4n^2 = 2 \cdot (2n^2)$ (or something similar), but there was a lack of adequate explanation. Almost every student began with an explanation of the concepts and provided examples with numbers. Almost all of them transitioned to an algebraic expression, especially after opening cards A4 and/or B2 (which provided the hint $x = 2n$).

Looking at figure 10, it is clear how the orientation cards (A-cards) guide a student step by step to the generalizing step $x = 2n$ to prove the problem. Starting with developing concepts and providing examples, the student realizes the direction in which the proof can be given after opening card A4. Additionally, it is visible that after opening a formulation card (B5-card), the student begins again writing down their proof, but now including both the statement to be proven and the assumption with its justification. In short, the heuristic tree has supported the student in both forming a correct proof strategy and writing it down in a structured manner.

Figure 10

The Work of a Student from the Second Iteration on the Algebraic Task



Uitwerking	Geopende kaartjes
Elk kwadraat van een even getal is even Een kwadraat is een getal maal zichzelf.	
een oneven getal in het kwadraat: $3^2 = 3 \cdot 3 = 9$ even getallen in het kwadraat: $4^2 = 4 \cdot 4 = 16$ $10^2 = 10 \cdot 10 = 100$ $8^2 = 8 \cdot 8 = 64$	A1
als x even is, dan kan die worden uitgedrukt in een veelvoud van 2: $x = 2n$.	A3 A4
2 is een even getal, net als alles in de tafel van 2. als je x^2 omschrijft volgens de regel hierboven krijg $(2n)^2$. $(2n)^2 = 2n \cdot 2n = 2 \cdot 2 \cdot n \cdot n = 2 \cdot 2n^2$	A2
$2 \cdot 2n^2$ zit dus altijd in de tafel van 2 oftewel het is altijd even.	B1
Het bewijs: "Elk kwadraat van een even getal is even"	B5
Een kwadraat krijg je als een getal vermenigvuldigt met zichzelf. Een even getal is een veelvoud van 2. Voor elk even getal x geldt dus $x = 2n$. $x^2 = (2n)^2$	

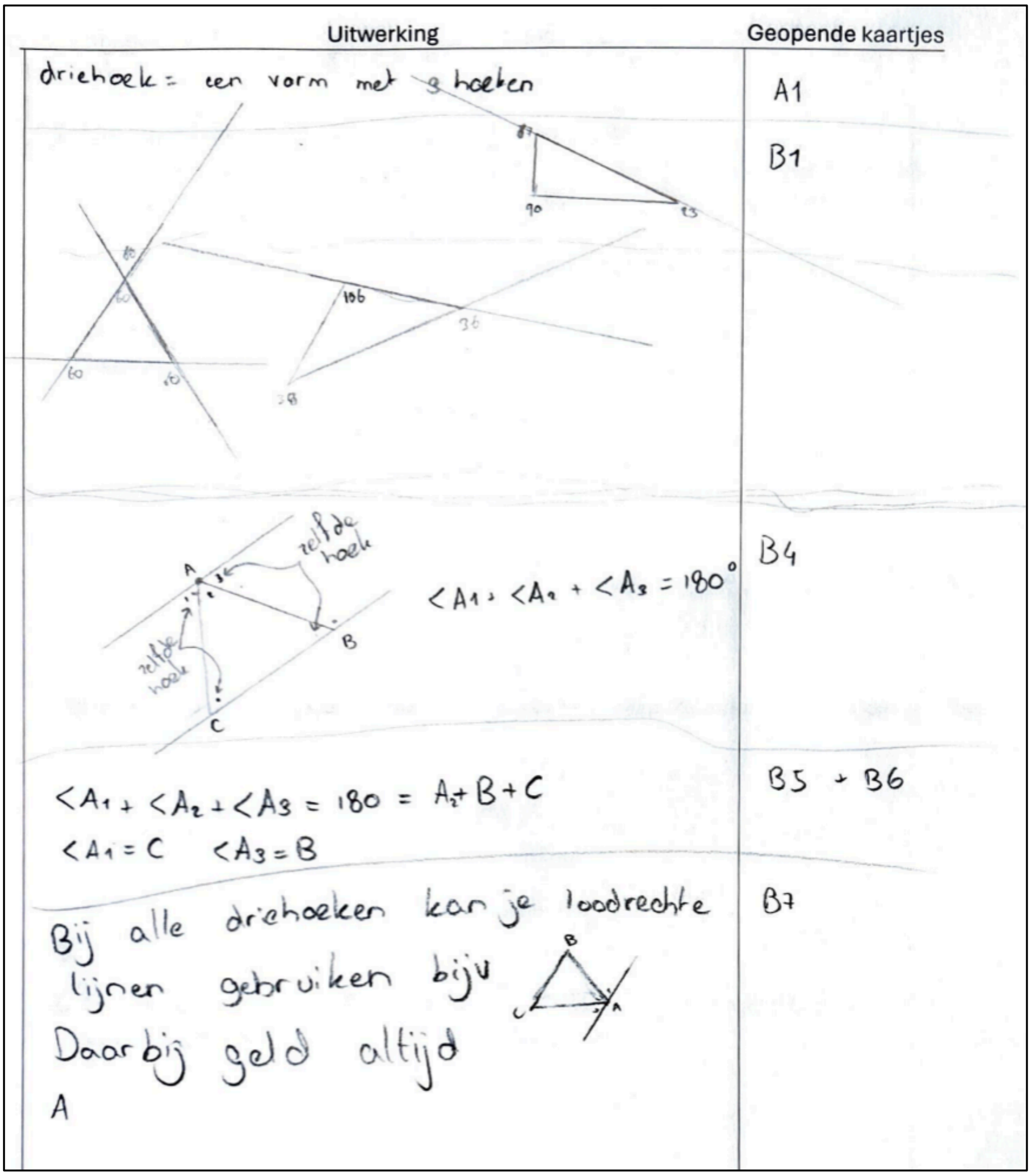
Geometric task

Approximately half of the students have carried out the geometric task using a correct proof technique. However, it is noticeable that there is still a lack of formulation and structure in their solutions. For example, the use of Z-angles is not explicitly stated. There has been little use of informal examples, such as a right triangle, as evidence. A fifth of the students have stuck to the example from the orientation phase for their own proof.

In figure 11, it can be observed that a student changes their proof approach after opening a B-card. Auxiliary lines and Z-angles become implicitly visible, and the argument that the straight angle is equal to the sum of the angles of the triangle via Z-angles is partially visible. However, even after opening a writing card (B7-card), the student is still unable to deliver a structured elaboration. In short, the heuristic tree has supported the student in guiding them towards a correct proof strategy but has insufficiently supported them in justification and structure.

Figure 11

The Work of a Student from the Second Iteration on the Geometric Task



Alterations made for 3rd iteration

Although there was improvement compared to the first iteration, issues persisted. Structuring the proof and making use of conjunctions was still not applied that much. Additionally, there was still a notable lack of explicit notation regarding the use of concepts and/or assumptions. Therefore, the intention for the third iteration was to focus more on the formulation phase.

The following adjustments were made:

- Language formulations were made more concrete, shortened where necessary, and simplified. Text portions were reduced to better highlight the core of the hint. Terms like "class of objects" were removed to make the hint more concrete and easier to understand.
- In the A1 and B1 cards, an explicit mentioning of an example not being a valid proof for all cases is noted.
- In the phase "How do I write my proof/reasoning?", adjustments were made to the first card. It now includes the essence of writing your proof neatly and structured. Additionally, it specifies that made assumptions and used concepts should be explicitly stated. It also asks whether conjunctions are used to logically connect statements.
- There were not any significant changes in instruction. However, emphasis was placed on the difference between providing a proof and an informal example or conjecture. It was emphasized that using a right triangle in a proof cannot be generalized to any triangle.
- In the questions at the end, the second question now explicitly asks for a "learned *mathematical* reasoning technique" instead of a "learned reasoning technique".

3rd iteration

Table 3 shows that a subtle progression is noticeable compared to the 2nd iteration. The most striking improvements are found in the components of proof direction, assumptions, and structure. The improvements may be attributed to a clearer hint on a card in the writing phase of the heuristic tree, which highlights assumptions, conjunctions, and structure more clearly.

As for learned mathematical reasoning techniques, about 30% of students did not write anything down or were unable to indicate what they had learned. The most common written down learned techniques for the geometric problem were:

- Examples do not count as evidence
- Look for special angles
- Draw a picture

For the algebraic problem, these were:

- Express numbers as letters for generalization
- An even number can be expressed as “2n”
- Write down all your thoughts and then turn them into a clear story

Algebraic problem

Six out of nine students correctly interpreted and applied the algebraic problem, while two out of nine applied it incorrectly, and one student didn't provide any reasoning. Almost every student began with an explanation of the concepts and gave examples with numbers. Almost all transitioned to an algebraic expression, especially after opening cards A4 and/or B2 (which provide the hint $x = 2n$). Regarding the correctly carried out proofs, it's notable that only two out of six students made their assumptions explicit, while the rest were implicit.

Table 3

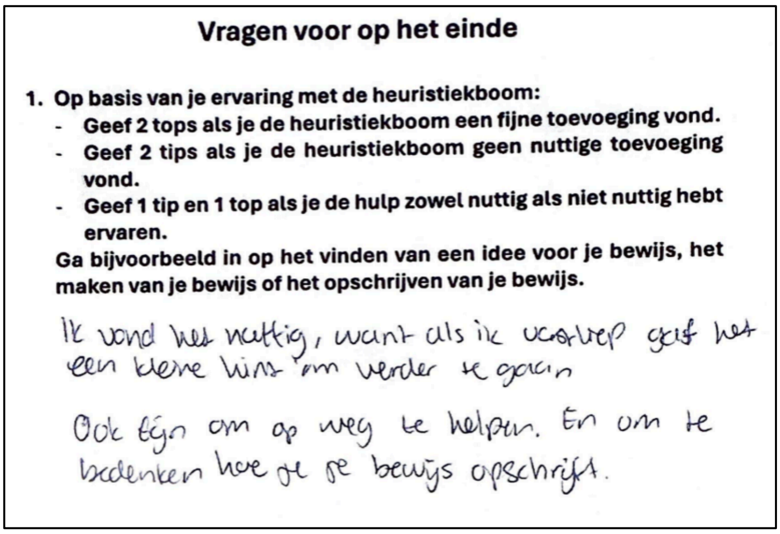
Results of the Students from the Third Iteration

Task	Student	Proof direction	Assumptions	Conjunctions	Structure	Justification
algebraic	1	yes	implicitly	yes	yes	partly
	2	yes	implicitly	yes	yes	yes
	3	yes	implicitly	yes	yes	yes
	4	yes	implicitly	yes	yes	partly
	5	yes	yes	yes	yes	yes
	6	yes	yes	yes	yes	yes
	7	no	no	yes	no	no
	8	wrong	no	no	no	yes
	9	wrong	no	no	no	yes
geometric	10	no	no	no	no	no
	11	wrong	no	no	no	no
	12	yes	yes	yes	yes	yes
	13	yes	implicitly	no	yes	partly
	14	yes	yes	yes	yes	partly
	15	yes	yes	no	yes	no
	16	yes	implicitly	partly	yes	partly
	17	wrong	no	no	yes	partly
	18	yes	yes	no	no	partly
	19	wrong	no	no	no	no

Looking at figure 13, it can be noticed how a student, with the help of the A- and B-cards, gradually moves closer to the generalizing step $x = 2n$ to prove the problem. Starting with providing examples, the student understands the direction of the proof after opening card B2. Additionally, after opening a formulation card (B5 card), the student starts over with writing down their proof. In this revision, the student explicitly mentions assumptions, conjunctions, and justifications for the steps taken. In short, the heuristic tree supported the student in both forming a correct proof strategy and writing in a structured manner. The effectiveness of the formulation card is also apparent from the given top in figure 12, where the student indicates that the heuristic tree was helpful for formulating their proof.

Figure 12

The Provided Feedback of a Student from the Third Iteration on the Algebraic Task



Vragen voor op het einde

1. Op basis van je ervaring met de heuristiekboom:
   - Geef 2 tops als je de heuristiekboom een fijne toevoeging vond.
   - Geef 2 tips als je de heuristiekboom geen nuttige toevoeging vond.
   - Geef 1 tip en 1 top als je de hulp zowel nuttig als niet nuttig hebt ervaren.

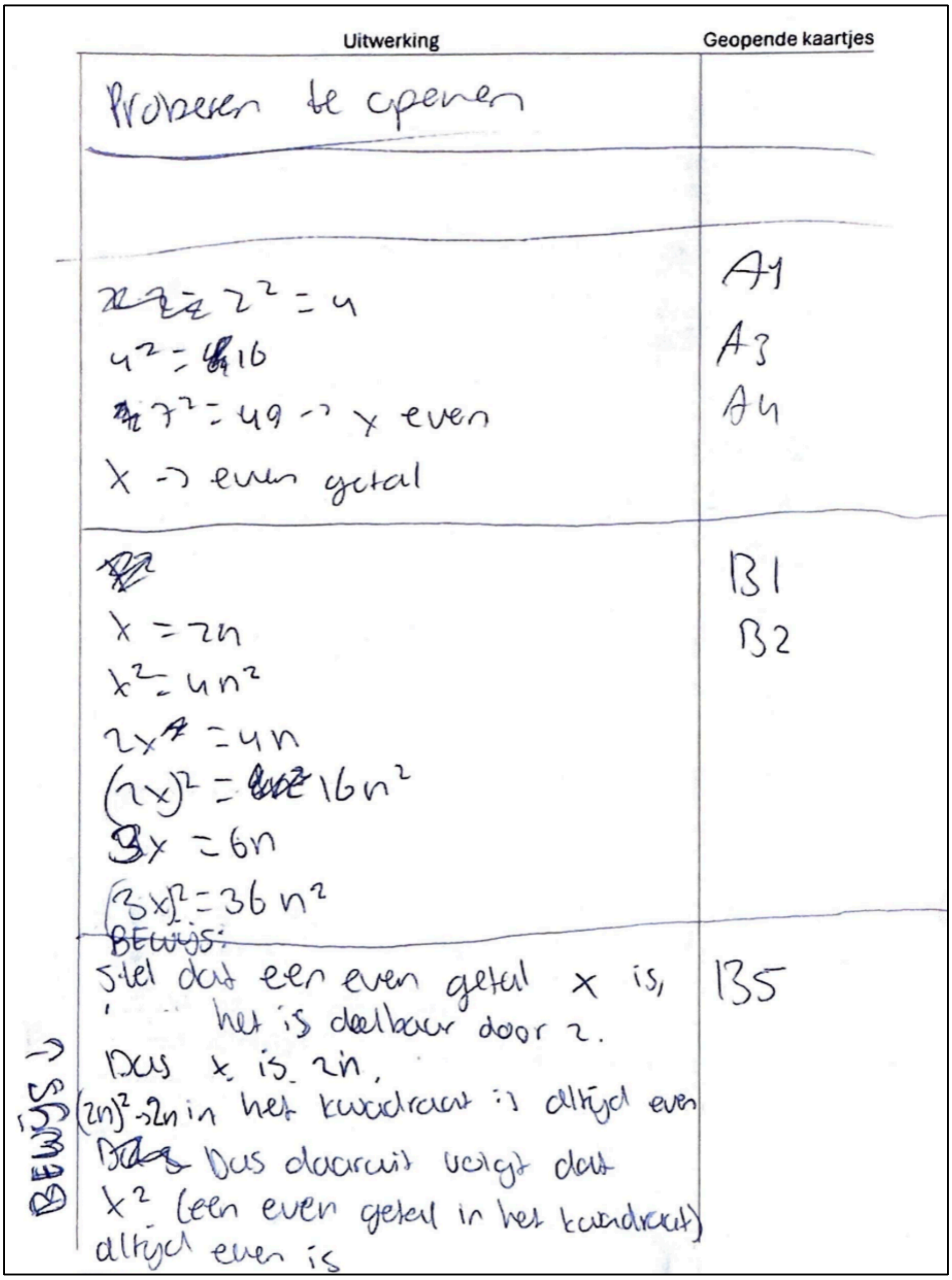
   Ga bijvoorbeeld in op het vinden van een idee voor je bewijs, het maken van je bewijs of het opschrijven van je bewijs.

Ik vond het nuttig, want als ik vastliep gaf het een kleine hint om verder te gaan

Ook fijn om op weg te helpen. En om te bedenken hoe je je bewijs opschrijft.

Figure 13

The Work of a Student from the Third Iteration on the Algebraic Task



Geometric problem

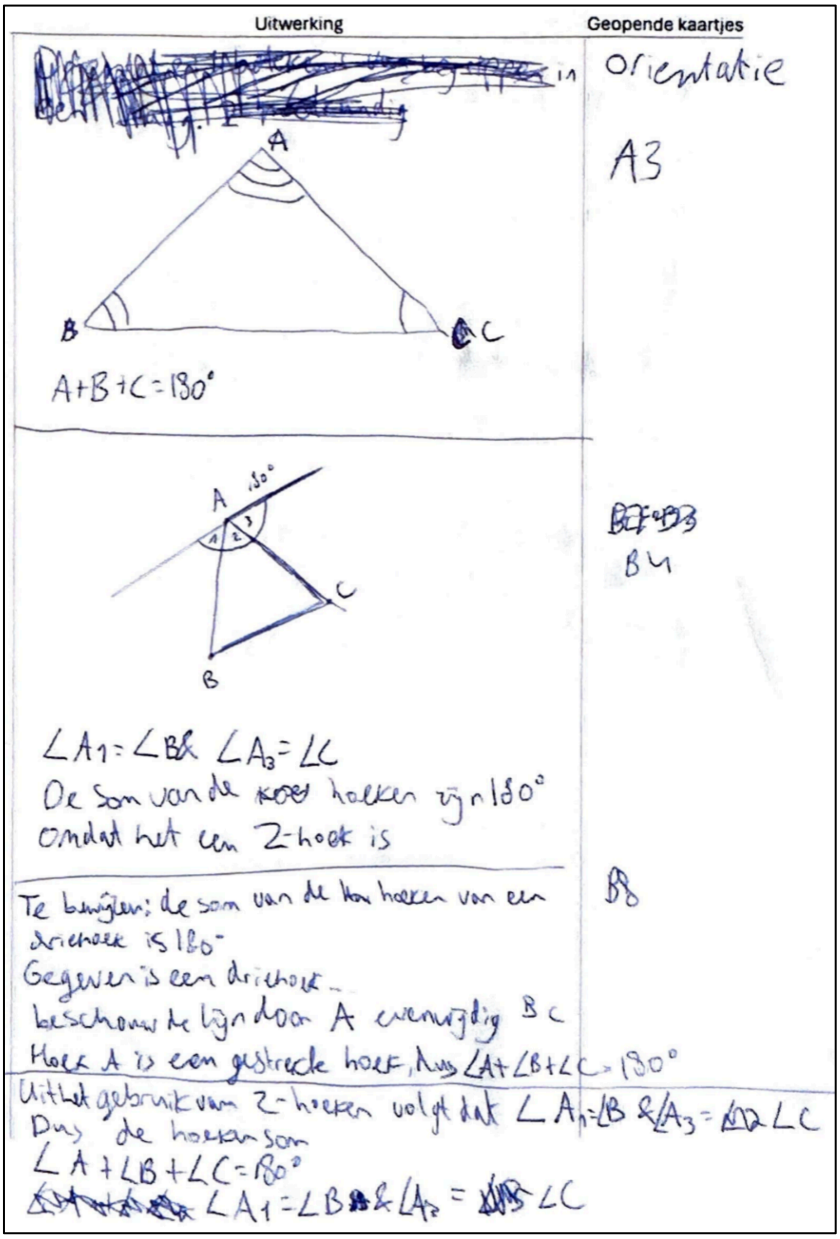
Out of the ten students in table 3, six carried out their problem using a correct proof technique, while three used a wrong one, and one didn't provide a proof direction. In six out of ten cases, there's a clear transition visible in the formulation of the proof, regardless of whether the question is answered correctly or not. Regarding this transition, students progress from solutions without a clear starting and ending point, with numerous drawings and examples, to more concise solutions with justification.

However, there's still a noticeable lack of justification and explicit mention of assumptions and concepts. Only four out of ten students explicitly mentioned their assumptions, with some students being implicit. Additionally, six out of ten students justified their solutions to some extent, but the majority are incomplete and/or informal. One student flawlessly completed the problem without any errors, even explicitly mentioning assumptions and Z-angles.

In figure 14, it can be observed that after opening a B-card, a student changes their proof approach (noted as B4, but this is via B1). Auxiliary lines and Z-angles become implicitly visible but are explicitly mentioned after opening a writing card (B8-card, but via B7). In a revision of their work, the student explicitly mentions assumptions, conjunctions, and justifications for the steps taken. In short, the heuristic tree supported the student in developing a correct proof strategy, explicitly mentioning crucial concepts, and structuring their work.

Figure 14

The Work of a Student from the Third Iteration on the Geometric Task



Tips and tops

In table 4 the total number of all given tips and tops are found. Firstly, the number of positive remarks regarding the clarity of card explanations increased over the iterations, from three in the first iteration to eight in the third. This suggests that improvements made to the clarity of the cards were well-received and effective. Secondly, the usefulness of having their task divided into small steps/phases remained relatively stable each iteration. This indicates consistent value in the phased approach. Thirdly, the positive feedback for helping to write down their proof emerged only in the third iteration. This suggests a late but successful improvement of features that support students in the formulation of their proofs. Lastly, it is visible that many more tops are given than tips. This indicates that students generally experienced the heuristic tree as a positive contribution. Overall, the feedback indicates notable improvements as there is increased positive feedback on card clarity and specific helpful features, such as breaking tasks into smaller steps and support in proof writing, in addition to a decrease of negative feedback on vagueness or unclarity.

Table 4

Received Feedback of Students from all Iterations on the Use of the Heuristic Trees

Tips or tops	Given feedback	1st iteration	2nd iteration	3rd iteration
tops	Clear explanation of cards	3	6	8
	Nice to be helped without getting an immediate answer	8	-	2
	Helpful if you get stuck or have no idea	7	9	4
	Useful with different (small) steps and/or phases	3	4	4
	Helpful for writing down the proof	-	-	3
	Language is comprehensible	-	2	-
tips	Unclear or vague card	8	5	5
	Heuristic tree is confusing to work with	3	-	1
	Language is difficult	1	-	-

Opened cards

Percentages from table 5 show how many cards per student were opened on average per phase. For example, in the elaboration phase of the third iteration, a student has opened an average of 53% of the cards. This suggests that not everyone clicks through and that some more are satisfied with just a general heuristic.

Table 5

Average Percentage of Cards Opened per Phase of the Heuristic Tree

Iteration	No. students	Orientation phase	Elaboration phase	Formulation phase	Completion phase
First	22	62,5%	33,3%	43,2%	22,5%
Second	21	70,2%	49,0%	52,4%	21,9%
Third	19	60,5%	53,1%	60,5%	21,4%

The results highlight that the orientation and completion phases are opened roughly the same number of times in each iteration. In contrast, the elaboration and formulation phases have an increase in the number of opened cards. The elaboration phase has stabilized at around 50%, while the formulation phase has surged with each iteration. Notably, in the last iteration, there is a consistent need for each phase, except for the completion phase, which is seldom used in any iteration. It is noticeable that the formulation phase is opened more often in each iteration than the elaboration phase. Therefore, there is a need for support with the formulation of their proof.

Examining this in more detail, we can observe the number of cards opened during the third iteration. For the algebraic task, this was:

A1: 6, A2: 2, A3: 6, A4: 6
B1: 7, B2: 7, B3: 6, B4: 6, B5: 6, B6: 4
C1: 1, C2: 1

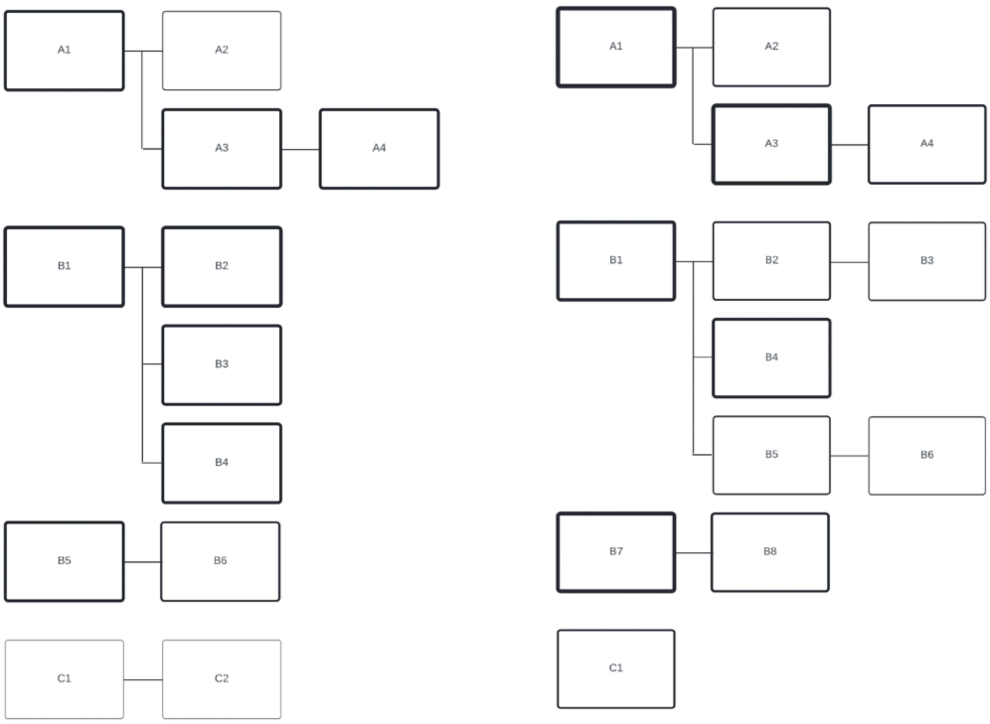
The number of opened cards for the geometric task was:

A1: 9, A2: 4, A3: 8, A4: 5
B1: 7, B2: 4, B3: 3, B4: 6, B5: 3, B6: 2, B7: 8, B8: 5
C1: 4

From figure 15 it is visible that there was no decrease in the opening of cards further to the right in the branch, and all the phases were opened almost proportionally (except for the completion phase). This is different for the geometric problem as cards A1, A3, B1, and B7 were opened the most. The need for students to be supported both in finding and guiding their proof and in structuring their proof is thus evident. Furthermore, this again suggests that not everyone clicks through and that some are satisfied with just a general heuristic instead of more concrete hints further in the branch.

Figure 15

An Illustration of the Number of Opened Cards per Heuristic Tree for the Third Iteration



Note. The left tree is the algebraic task and the right tree the geometric task. The illustration is designed by adjusting the thickness of the card to the number of times the card has been opened. For example, a card that has been opened 4 times will have a thickness of 4 px.

Conclusions

This research explored how heuristic trees can support pre-university students in mathematical reasoning, as well as finding and formulating proofs in the Dutch secondary mathematics education. The study involved designing heuristic trees for proofs with three iterations, each refining the approach based on feedback and observed outcomes.

The findings from the first iteration showed that students struggled with finding and formulating mathematical proofs. Most relied on informal examples and common issues included the lack of explicit assumptions, structured steps, and logical connections in their proofs. Feedback revealed that students found the heuristic tree instructions unclear and confusing, leading to underutilization of the provided support.

For this reason, the second iteration included a short instructional session to get students acquainted with heuristic trees and the phases of mathematical reasoning. Heuristic tree cards were made clearer and more concrete, resulting in better guidance for students. Big improvements were observed in students' proof direction, the use of assumptions, and overall structure. However, justification, explicit use of assumptions and formulation still posed challenges. Feedback indicated fewer issues with card vagueness, suggesting increased clarity and effectiveness.

Further refinements in the third iteration emphasized the importance of assumptions, conjunctions, and structured proof writing. Continued improvement was observed, with more students demonstrating well-structured proofs and explicitly stating their assumptions and reasoning. For the first time, some students successfully completed all aspects of the tasks correctly. Students more frequently documented learned mathematical reasoning techniques, particularly for algebraic problems.

When it comes to the design of the heuristic trees, progress is visible in the different elements of the proof with each iteration. In the second iteration, it became evident how the heuristic tree contributed to an improved proof direction and use of assumptions and conjunctions. However, by placing even more emphasis on the formulation phase in the third iteration, the results showed progress in proof direction, assumptions, and structure. Therefore, by extending the heuristic trees into four phases for reasoning, with a clear distinction between carrying out and formulating a proof, students were able to develop a deeper understanding of mathematical reasoning and find and formulate structured proofs.

When it specifically comes to formulating a proof, the results indicate that the cards from the formulation phase were beneficial for the students. Firstly, the cards were accessed proportionately to the orientation and elaboration phases, suggesting that students found them necessary. Secondly, several students' elaborations demonstrate how their proofs were rewritten into more compact versions with a clear beginning and end, incorporating assumptions, conjunctions, and justifications. Thirdly, in the third iteration, students frequently noted that the cards for formulating their proofs were useful. Therefore, the addition of the formulation phase can be considered an effective component of heuristic trees aimed at mathematical reasoning.

It is important to note that not every student clicked through all the heuristic tree cards. This is a positive sign, indicating that some students were able to find sufficient guidance with the initial, more generic hints provided by the cards. This suggests that the heuristic trees were flexible enough to cater to varying levels of student need, allowing those who required less support to proceed efficiently while still offering detailed assistance for those who needed it.

The study demonstrated that the iterative process of refining the heuristic trees in four distinct phases and the instructional approach led to progressive enhancements in students' ability to find and formulate mathematical proofs. The improvements in students' proof-writing skills across the iterations indicate that the heuristic trees developed in this study can effectively support students in their skills in proving. By progressively clarifying and refining the guidance provided by the heuristic trees, students were better equipped to construct well-reasoned and coherent proofs. Thus, this indicates that we have developed heuristic trees that can support pre-university students in their mathematical reasoning, as well as finding and formulating proofs. To support students in this, the following four phases are necessary:

- *Orientation Phase*
  ⇨ Let students get a grip on the problem and get an idea how to provide a proof.
- *Elaboration Phase*
  ⇨ Let students make a plan for their proof and guide them with carrying it out.
- *Formulation & Finalization Phase*
  ⇨ Guide students how to formulate their proof in a structured manner and remind them to check missing details.
- *Completion Phase*
  ⇨ Let students reflect on what they have learnt and how to apply it in other settings.

Discussion

Interpretations and implications

The results of the first iteration show that students had much more difficulty with mathematical proofs than anticipated. Problems were intentionally chosen that required little prior mathematical knowledge, and the required knowledge had already been covered in the first year of their high school career. Therefore, if students already struggle with problems that require little prerequisites, it raises the question of how well heuristic trees can support students when much more mathematical knowledge is required.

Additionally, the first iteration shows that the results align with the literature: students who did not know how to construct a proof, tried using (informal) examples to prove something (Raman, 2002). Many students provided a pictorial representation as proof (a square with two triangles in it) or gave empirical examples (even squared numbers) that would constitute as a valid proof for the arbitrary case (Schoenfeld, 2013; Fischbein & Kedem, 1982). Furthermore, some students assumed the conclusion in order to prove it (Stavrou, 2014).

Where Bos & van den Bogaart (2022a; 2022b) have already delved into the design and use of heuristic trees in mathematics education, this research has contributed to their study. Before the start of this research, no research had been done on the exploration of heuristic trees in proofs in mathematics education, just problem-solving. Thus, it was unclear whether heuristic trees were useful for students to assist in finding and formulating proofs. This research has shown that heuristic trees can indeed support students in their mathematical reasoning, albeit in a modified form compared to the known problem-solving heuristic trees. Therefore, this research has extended and enriched the previous study

Limitations

Although it is observed that the developed heuristic trees have contributed to students' reasoning skills in mathematics, there are also some limitations:

Firstly, it should be noted that the first iteration was conducted at a different school (SGU) compared to the second and third iterations (CGU). The SGU uses the mathematics book series "Getal & Ruimte" which is known for its problem-solving approach. The CGU uses the

"Wageningse Methode", an innovative book series that focuses more on understanding and self-discovery (van Smaalen, 2011). This means that the Wageningse Methode places more emphasis on mathematical reasoning than Getal & Ruimte. Although the addition of a short instruction and redesign of the heuristic trees contributed to improved results in the second and third iterations, it is also worth considering the influence of the mathematics book series on the results. If the Wageningse Methode indeed focuses more on mathematical reasoning, it would not be surprising that this also has a positive effect on the results. Therefore, it must be taken into account that two different mathematics book series are involved in the research.

Secondly, three gymnasium classes participated in the research. Although it is good to maintain the same educational level in all iterations, it raises the question of whether heuristic trees for mathematical reasoning work equally effective for other levels of education such as atheneum or HAVO, designed in the format of this study. It may well be that at other levels of education the design of the heuristic tree for mathematical reasoning is different. For example, some phases may need to be more prominent than others.

A final limitation lies in the layout of the digital environment in which the heuristic trees were developed. The site was initially developed for creating heuristic trees for problem-solving tasks in mathematics. Therefore, the heuristic trees could only be made with the three phases: orientation, making and executing plans, and completion. The layout of the website could not clearly separate the phases of elaboration and formulation in this research. This may have influenced their use by the students.

Future work

Future research, focusing on developing suitable tasks and instructions for learning mathematical reasoning and proofs, can take this study as a starting point. Although the focus of this study has been on developing heuristic trees for mathematical reasoning, designing suitable tasks with customized instruction is just as important. This can be covered to improve the coherence of learning to find and formulate proofs.

Another great challenge that can be investigated is the difference in the influence of the two mathematical book series. It is advisable for future research to delve into this to see how much of a difference it makes.

Some small future developments could focus on redesigning the digital environment so that heuristic trees can be developed with customizable phases. Additionally, it would be beneficial to create an extra intermediate step in the formulation phase. Currently, two cards are visible, but the transition between the two is still quite abrupt. An intermediate step could streamline the transition. Furthermore, it would be interesting to see if the developed heuristic tree for mathematical reasoning can be as effective in the upper levels of secondary education or if some of the phases need more or less attention.

Data availability

The data supporting the findings of this study are available by contacting the author and supervisor.

Code availability

The web-app for heuristic trees is freely available at https://edspace.nl/hboom/index.php and published under a creative commons license. The source code can be obtained from the author.

Ethics statement

The data collection process adhered to the ethical guidelines set forth by the Freudenthal Institute at Utrecht University. All procedures involving participants were conducted with their informed consent, ensuring confidentiality, anonymity, and respect for their rights throughout the study.

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