Temperature Dependence of Holographic Shear Viscosity

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Abstract

Recent analysis of heavy ion collisions has shown clear signs of a temperature dependence of the shear viscosity to entropy ratio. However, it is very difficult to compute the shear viscosity theoretically in strongly coupled plasmas using traditional methods. One alternative method is to use holographic correspondence. However, the standard holographic computation yields a universal value for the shear viscosity to entropy ratio that is independent of temperature. In this thesis, we modify the bottom-up QCD-like holographic theories with the goal of matching the T-dependence of the shear viscosity to entropy ratio to the one observed for quark-gluon plasma. For this, we use the Bayesian analysis of heavy ion collisions.

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Contents

1 Introduction

The strong nuclear force is the fundamental interaction that binds protons and neutrons in the atom's nucleus. The strength of this interaction is what causes the nucleus to store a great amount of energy, around a million times more than the energy stored in electrons bound in the typical atom. The prevailing theory describing the strong force is Quantum Chromodynamics (QCD)[1]. In this theory, the strong force is a result of fundamental particles called quarks and gluons interacting together. The strength of the interaction is encapsulated in the coupling constant, a fundamental parameter of the theory. One of the most striking facts of Quantum Field Theory (QFT) is that the value of this constant depends on the energy scale -or equivalently length scale- of the experiment that probes the theory.

For QCD the coupling constant turns out to be small at high energies and becomes large at low energies [2]. This presents a big challenge when doing theoretical calculations as the traditional physics approach of calculating small perturbations around an exact solution, no longer works. However, there are a number of other approaches for doing calculations in QCD which can be summarized into two camps: Effective Field Theory and Lattice QCD. The former is based on an effective description of the degrees of freedom in hadrons interacting weakly and the latter is based on doing computer simulations of the theory on a lattice. Even though both approaches are very successful in their own right, they both have limitations. The effective field theory approach breaks down close to the QCD energy scale and the lattice field theory is only effective at very small baryon chemical potential. This leaves many unanswered questions regarding the phases of nuclear matter.

One such extreme phase of nuclear matter is Quark-Gluon Plasma (QGP). This is a phase produced at the Relativistic Heavy Ion Collider (RHIC) and Large Hadron Collider (LHC) in heavy ion collision experiments where heavy nuclei usually Au or Pb collide at very high energies. This creates a hot, dense, strongly interacting plasma of quarks and gluons that is not confined inside hadrons as we experience in everyday life. Studies have shown that this plasma can be modeled very well as a relativistic fluid using hydrodynamics [3, 4]. The important property of a fluid that distinguishes it from an ideal gas is dissipation. That is the return of the fluid to equilibrium after a perturbation. For example, throwing a pebble at a pond creates waves that propagate in the fluid which then quickly decay, and the fluid returns to equilibrium. This effect is encapsulated in a parameter called *viscosity*. However, the shear viscosity cannot be computed from hydrodynamics. Instead, it depends on the microscopic details of the fluid and must be given as input to the theory. But as mentioned before, this state of matter is strongly interacting, making it very hard to perform this calculation from first principles. However, there is another technique called holographic correspondence, otherwise known as AdS/CFT or gauge-gravity duality.

The holographic correspondence [5, 6, 7] is the idea that two seemingly quite different theories are equivalent descriptions of an underlying physical system. On the one side, we have a semi-classical gravitational theory in five dimensions, and on the other side a four dimensional quantum field theory in four dimensions. The upside of this duality is that when the the QFT is strongly coupled, the dual gravitational theory is weakly curved allowing us to make calculations for QFT observables at strong coupling using the dual theory. One of the most elegant results of this method to this day remains the calculation of the *shear viscosity to entropy ratio* for strongly coupled plasmas[8, 9]:

$$\frac{\eta}{s} = \frac{\hbar}{4\pi k_B}. \tag{1.1}$$

This simple result was very successful when compared with initial data from accelerators and lattice QCD. What is remarkable, is that it is a generic prediction of a wide class of gravitational theories. However, we don't expect this result to be accurate at all temperatures. In particular, the viscosity is roughly proportional to the mean free path which becomes longer as the coupling becomes weaker. Thus since the QCD coupling becomes smaller as energy increases, one would expect the shear viscosity to increase. This intuitive argument is supported by recent results coming from Bayesian analysis of heavy ion collisions that have shown a temperature dependence for the shear viscosity to entropy ratio [15, 17].

Motivated by this recent analysis, in this thesis, we explore the temperature dependence of the shear and bulk viscosities by including higher derivative corrections to existing holographic models. We constrain the nature of these higher derivative corrections by fitting to the Bayesian results for the shear viscosity and we show that the only two derivative term contributing to the shear viscosity is the Riemann squared term, even when the higher derivative terms are coupled to the dilaton. We calculate the bulk viscosity to entropy ratio for a class of theories with non-minimal dilaton coupling and compare it to data from heavy ion collisions. Finally, we explore the UV and IR asymptotics of the modified theory and show that under certain assumptions the theory still exhibits confinement.

This thesis is organized as follows. The second chapter is a brief review of the holographic viscosity. The third and fourth chapters discuss the calculation of the shear viscosity to entropy ratio for higher derivative theories and the fitting to the data from Bayesian analysis. The fifth chapter discusses the general curvature squared action. The sixth chapter is devoted to the calculation of the bulk viscosity and its comparison to Bayesian data. Finally, the seventh chapter explores the equations of motion with a focus on the asymptotic behavior of the theory and confinement.

2 Holographic viscosity

2.1 The Classic Viscosity Picture

An excellent introduction into AdS/CFT and applications is the book by Natsuume "AdS/CFT Duality User Guide" [32]. Much of what is covered in this subsection follows this book.

Let's start by recalling the classical physics picture for the definition of shear viscosity in Figure 1. We have two plates separated by a distance L, with a viscous fluid between them. If we give

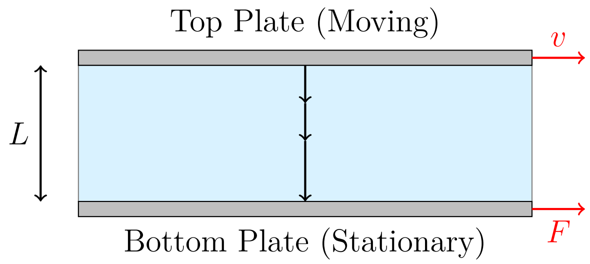

Figure 1: Classical picture for the definition of viscosity. Two plates are separated by a fluid, the top plate is moving and the bottom plates feels a force due to friction in the fluid.

some velocity to the top plate v, then due to momentum transferring between the molecules this will manifest as a force acting on the bottom plate. This force per unit area will be proportional to the velocity v and inversely proportional to the distance L with a proportionality constant η:

$$\frac{F}{A} = \eta \frac{v}{L}. \tag{2.2}$$

This proportionality constant η is the shear viscosity. So the higher η is the more force the bottom plate experiences for the same top plate velocity. Now let's do some dimensional analysis to the formula (2.2). η has units of $[\eta] = kg \times m^{-1} \times s^{-1}$. So supposing that it depends on the mass density ρ, mean particle velocity $\bar{v}$ and mean free path l then by dimensional analysis:

$$\eta \sim \rho \, \bar{v} \, l. \tag{2.3}$$

So the shear viscosity is proportional to the mean free path, this means the more strongly interacting a fluid is the lower it's mean free path and thus the smaller it's viscosity. Later when we come to talk about quark-gluon-plasma we will discuss the famous result of $\eta/s = 1/4\pi$. This is the lowest such ratio compared to other fluids in nature due to the strongly coupled nature of QGP. In addition, when considering the fact that QCD becomes weakly coupled at high energies then intuitively we expect the shear viscosity to increase as temperature increases for QGP. This will be discussed in depth in the following sections of this thesis.

The next logical step is considering the viscosity for a relativistic fluid. But first let's recall the definition of the energy momentum tensor for the perfect fluid:

$$T^{\mu v} = (\epsilon + P) u^{\mu} u^{\nu} + P \eta^{\mu\nu}, \tag{2.4}$$

where ϵ is the energy density, P is the pressure density and $\eta_{\mu\nu}$ is the Minkowski metric*. If we go to the rest frame of the fluid then the four velocity is simply $u^{\mu} = (1, 0, 0, 0)$. This means that the enrgy momentum tensor take the simple form:

$$T^{\mu\nu} = \begin{pmatrix} \epsilon & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{pmatrix} \tag{2.5}$$

*We use the (-+++) convention.

Notice that here $T^{00} = \epsilon$ and due to the rest frame $P^i = T^{0i} = 0$. Also all the off-diagonal elements are zero which, roughly speaking, means that we don't have any friction between fluid layers. Instead $T^{ij} = P\delta^{ij}$ i.e. we have an isotropic fluid, meaning that the pressure is the same in all directions.

But we can go a step further instead of just including functions of the four velocity we can include also first derivatives of the four velocity. This will account for dissipation within the fluid since now we will also include the variation of the four-velocity. The energy-momentum tensor now takes the form:

$$T^{\mu v} = (\epsilon + P)u^\mu u^\nu + P\eta^{\mu\nu} + \tau^{\mu\nu}, \tag{2.6}$$

where $\tau^{\mu\nu}$ is a new symmetric tensor we introduce to account for dissipation. To simplify things we go to the rest frame, defining $T^{00} = \epsilon$ as before. Now the extra term we introduced can depend only on first derivatives of u^μ and it has to be symmetric, therefore there are only two possible terms we can include:

$$\tau_{ij} = -\underbrace{\eta\left(\partial_i u_j + \partial_j u_i - \frac{2}{3}\delta_{ij}\partial_k u^k\right)}_{traceless} - \underbrace{\zeta\delta_{ij}\partial_k u^k}_{trace}. \tag{2.7}$$

The coefficients multiplying these terms can be generic but we write them in such a way to split the traceless with the trace part of τ^{ij}. The constants η,ζ that appear are called transport coefficients. What's more, this formula gives a natural interpretation of the physics each coefficient is related to. The shear viscosity η is associated with off-diagonal components accounting for loss of energy to adjacent fluid layers. One the other hand, the bulk viscosity ζ is only associated with diagonal components and accounts for loss of energy due to volume deformations such as expansion or compression of the fluid. Finally, this derivative expansion can continue to include second derivatives of u^μ which will give rise to second order transport coefficients, however, this analysis will not be needed for the purposes of this thesis.

Looking at the expression for τ^{ij} we see that η,ζ are input parameters to the theory. From the perspective of QFT, hydrodynamics is an effective theory whose parameters must be determined from the microscopic details of the fluid. Of course in practice, this is done by using experiments to measure these parameters. Still, one would like to be able to calculate these coefficients from first principles. In the case of quark-gluon plasma, the fluid is so strongly interacting that this calculation is very hard to do using QCD. There is however a different method one can use called holographic correspondence.

2.2 Basics of Holography

The holographic correspondence [5],[6],[7] conjectures a duality between two seemingly different theories. On the one side, a gravitational theory in d+1 dimensions admitting semi-classical corrections and on the other side a QFT in d dimensions in the large N limit. This limit refers to taking the rank of the gauge group (in the case of QCD the number of colors) to infinity. The upside of this duality is that when the the QFT is strongly coupled, the dual gravitational theory is weakly curved and vice-versa. This allows us to make calculations for QFT observables at strong coupling from classical gravity computations. The precise formulation is given by the GKPW formula:

$$\left\langle \exp\left(i\int\phi_0\mathcal{O}\right)\right\rangle = e^{i\underline{S}\left[\phi|_{u=0}=\phi_0\right]}, \tag{2.8}$$

where the left side of the formula refers to the generating functional of the gauge theory and the right side refers to the generating functional of the gravitational theory. $\underline{S}$ represents the on-shell action and $u = 0$ is the boundary of space. This boundary is where the QFT "lives".

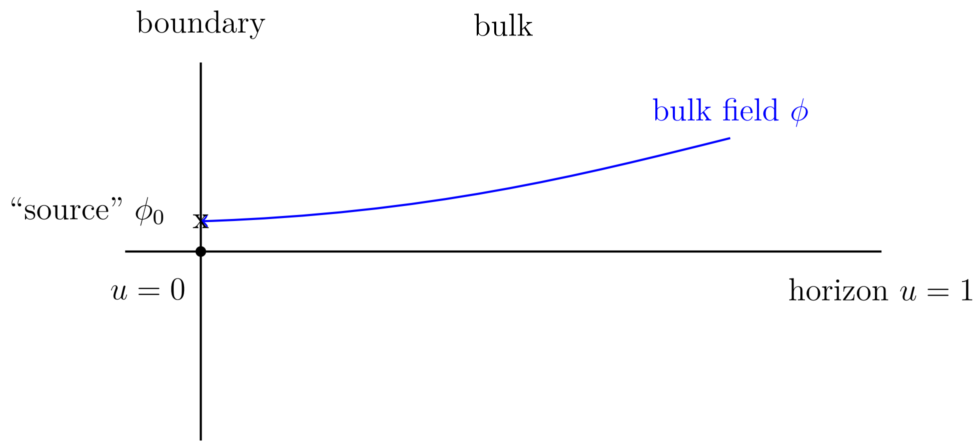

Figure 2: Illustration of the duality between a bulk scalar field and and a boundary scalar operator. The bulk field ϕ evaluated at the boundary $u = 0$ becomes a source for a scalar operator $\mathcal{O}$ in the dual quantum theory.

The bulk field ϕ evaluated at the boundary $u = 0$ becomes a source for a scalar operator $\mathcal{O}$ in the dual quantum theory, as illustrated in Figure 2. This example is for a scalar field but this relation also holds for other fields. A bulk theory gauge field is dual to a boundary theory conserved current and the bulk theory metric is dual to the boundary theory energy-momentum tensor as shown in the following table:

$$\begin{array}{ccc} \text{Boundary operators} & & \text{External sources} \\ \mathcal{O} & \leftrightarrow & \phi \\ J^{\mu} & \leftrightarrow & A_{\mu} \\ T^{\mu\nu} & \leftrightarrow & h_{\mu\nu} \end{array} \tag{2.9}$$

Now what happens at finite temperature? Well in that case the temperature in the QFT is identified with the Hawking temperature of a black hole in the dual gravity description. For example, a CFT at finite temperature is dual to an AdS black hole solution. This is because black holes are actually thermodynamic systems admitting both a temperature and an entropy. Consider a generic black hole solution given by the following metric:

$$ds^2 = -f(r)\, dt^2 + f^{-1}(r)\, dr^2 + d\mathbf{x}^2. \tag{2.10}$$

Then the temperature and entropy of the black hole are given by the following simple formulas

$$T = \frac{f'(r_h)}{4\pi}, \quad S = \frac{A}{4G_5} = \frac{1}{4G}, \tag{2.11}$$

where A is the area of the black hole horizon and r_h is the value of the r coordinate at the horizon. The derivation of the temperature formula is performed in the following chapter. The fact that the entropy is proportional to the area of the horizon gives us a hint that the physics can be described by a dual statistical system in one lower dimension.

Now for the purposes of this thesis, we are interested in calculating the shear viscosity at finite temperature. But how does the notion of viscosity make sense for a black hole? Well, Figure 3 illustrates an intuitive picture for this. When an object falls into a black hole, the horizon becomes irregular but the effect quickly dissipates and the black hole returns to its original state. This is in analogy with hydrodynamics, throwing a pebble into a pond creates waves that propagate in the fluid but which quickly decay and the fluid returns to its original state. These are both relaxation phenomena associated with viscosity and the holographic correspondence helps us make this intuitive picture precise.

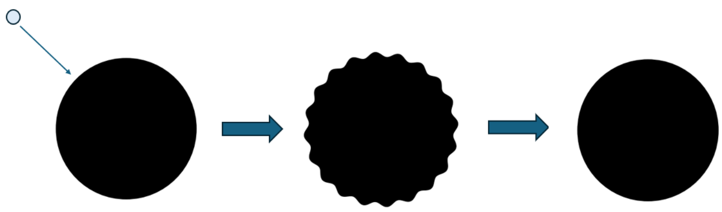

Figure 3: A black hole horizon fluctuates when an object falls into it. The decay of this fluctuation is analogous to a fluid relaxing after a perturbation.

2.3 Holographic QCD

The topic of holographic QCD theories includes a vast literature with many interesting and successful approaches. We refer the reader to [20] and the references therein for a review on the subject. We will not delve into this vast topic in this section, instead we introduce the specific holographic models used in this thesis.

The first model we introduce is called Improved Holographic QCD (ihQCD)[25],[26]. This is a holographic model for Yang-Mills theory and will also serve as the backbone for the next theory we introduce, V-QCD. The first thing to consider as motivation, is that QCD is not a CFT. Instead, there is a scale Λ_{QCD} associated with the theory which can be thought of as the cut-off scale introduced in the re-normalization procedure. This mean that, its dual gravitational description should not have pure AdS geometry as in AdS/CFT. Instead the geometry should asymptote to AdS in the UV where QCD becomes asymptotically free. In this model, this is achieved by the introduction of a scalar field called the dilaton. The inspiration for this is non-critical string theory. In this formulation, the introduction of the dilation allows for the theory to exist in a non-critical dimension. The effective gravitational action is then obtained by the condition that the beta function is set to zero. The action defining the model is five dimensional Einstein-dilaton gravity:

$$S_{ihQCD} = M_p^3 N_c^2 \int d^5x \sqrt{-g} \left[R - \frac{4}{3} \frac{(\partial \lambda)^2}{\lambda^2} + V(\lambda) \right]. \tag{2.12}$$

Where the following dualities are established between fields and operators.

$$g_{\mu\nu} \leftrightarrow T_{\mu\nu}, \quad e^{\Phi} = \lambda \leftrightarrow \mathrm{Tr}[G_{\mu\nu} G^{\mu\nu}]. \tag{2.13}$$

$G_{\mu\nu}$ is the YM field strength tensor and the trace is over group indices. The zero temperature solution of the action in domain wall coordinates is given by the following metric:

$$ds^2 = du^2 + e^{2A(u)} \left(-dt^2 + d\mathbf{x}^2 \right), \quad \Phi = \Phi(u). \tag{2.14}$$

So the energy scale of the four dimensional spacetime slice is given by $e^{2A(u)}$. Thus in ihQCD there is a natural identification between the logarithm of the QFT energy scale and the scale factor A encoding the bulk geometry.

$$\log E \longleftrightarrow A(u). \tag{2.15}$$

An important property of the theory which we briefly mention is that it exhibits confinement. The test for confinement is an area law behaviour for the Wilson loop. The typical way this is achieved is that the geometry ends at a finite value r_0 of the holographic coordinate. This is called a "hard-wall" model, but ihQCD is a "soft-wall" model in the sense that there is no hard end of space, but rather the dilaton becomes very large for $r > r_0$ thus the world-sheet can only extend infinitesimally past this point.

V-QCD

Now we move on to an improved version of ihQCD called V-QCD [21], see [20] for a review. This theory has many improved properties including fermions and a mechanism for chiral symmetry breaking. It is divided into two sectors, a gluon sector and a flavor sector:

$$S_{VQCD} = S_{ihQCD} + S_f. \tag{2.16}$$

The gluon sector takes the form of ihQCD which we have already introduced:

$$S_{ihQCD} = M_p^3 N_c^2 \int d^5x\sqrt{-g}\left[R - \frac{4}{3}\frac{(\partial\lambda)^2}{\lambda^2} + V_g(\lambda)\right]. \tag{2.17}$$

And the flavor action takes the following form:

$$S_f = -x_f M_p^3 N c^2 \int d^5x V_{f0}(\lambda)\, e^{-\tau^2}\sqrt{-\det\left(g_{\mu\nu} + \kappa(\lambda)\,\partial_\mu\tau\partial_\nu\tau + w(\lambda)\,\hat{F}_{\mu\nu}\right)}. \tag{2.18}$$

This sector is based on a setup of two space filling $D4 - \bar{D4}$ branes. The brane action includes a tachyon DBI action along with a Chern-Simons action. This last term however will not be important for this work. The tachyon field τ is dual to the quark mass operator ($\tau \leftrightarrow \bar{q}q$) and accounts for the breaking of chiral symmetry in QCD. The tachyon approaches zero in the boundary limit of the theory which corresponds to the UV limit of QCD, thus restoring chiral symmetry. Here we have assumed that the tachyon is proportional to the identity matrix and so the quarks have equal masses. In fact, for the purposes of the thesis we will be setting $\tau = 0$ since we are interested in the chiraly symmetric phase. $\hat{F}_{\mu\nu}$ is the Abelian component of the field strength tensor. This also approaches zero in the UV and will not be important for calculating the viscosity. The potentials $\kappa(\lambda), w(\lambda), V_{f0}(\lambda)$ appearing in S_f are assumed to only depend on λ and are not derived from a specific brane setup but instead are determined by fitting various properties of QCD.

An important aspect of the theory, which is where this theory gets it's name from is the Veneziano limit. In this limit, in addition to the large-N limit which is standard in Holography, we also take the number of flavors to infinity, while keeping their ratio fixed. In summary, the Veneziano limit is the following:

$$N_c \to \infty\,, \quad N_f \to \infty\,, \quad x_f \equiv \frac{N_f}{N_c} = \text{fixed}\,, \quad \lambda \equiv g^2 N_c = \text{fixed}. \tag{2.19}$$

The reason this limit is important is that if we were not taking $N_f \to \infty$ then the ratio $x_f \to 0$. However, we know that in QCD there are 3 colors and 2-3 light quarks and so this number is actually close to 1. Therefore, it is expected that important aspects of flavor physics are not captured without this limit. Notice also that the flavor action is of the same order as the glue action because of this limit. The potentials appearing in the action V_g, V_f take the following form

$$\begin{aligned} V_g(\lambda) &= 12\left[1 + V_1\lambda + \frac{V_2\lambda^2}{1+\lambda/\lambda_0} + V_{IR}\, e^{-\lambda_0/\lambda}(\lambda/\lambda_0)^{4/3}\sqrt{\log(1+\lambda/\lambda_0)}\right], \\ V_{f0}(\lambda) &= W_0 + W_1\lambda + \frac{W_2\lambda^2}{1+\lambda/\lambda_0} + 12 W_{IR} e^{-\lambda_0/\lambda}(\lambda/\lambda_0)^2. \end{aligned} \tag{2.20}$$

In is important to note that all the parameters that appear in the potentials are fixed. On the UV side, they are matched to the RG flow of QCD perturbation theory and on the IR side they are determined by comparing to lattice data. We discuss the exact values that we use in Section 4.3.

3 Temperature Dependence of the Shear Viscosity

In this section we calculate the shear viscosity to entropy ratio for Einstein-dilaton gravity, including higher derivative corrections. The shear viscosity is extracted from the Kubo formula:

$$\eta = - \lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} G^R_{xy,xy} \left(\omega, k = 0\right). \tag{3.21}$$

where the retarded Green's function is given by

$$G^R_{xy,xy}(\omega, \mathbf{k} = 0) = -i \int dt d\mathbf{x} e^{i\omega t} \theta(t) \left\langle [T_{xy}(x), T_{xy}(0)] \right\rangle. \tag{3.22}$$

The action we consider is of the form [14]:

$$S = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[R - 2(\nabla\Phi)^2 + V(\Phi) + \ell^2 \beta G(\Phi) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right]. \tag{3.23}$$

This is the same form as the ihQCD action introduced earlier, with the addition of the Riemann squared term coupled to the dilaton. As we discuss below, this term is crucial for obtaining a non-trivial temperature flow. Here, β is a parameter that we take to be predicatively small. We are interested in shear viscosity of the strongly coupled plasma dual to this theory. Using holography, this is calculated by introducing a shear metric fluctuation h_{xy} to our black hole background solution. This will produce a response in the dissipative part of the energy momentum tensor thus allowing us to find its Green's function. We introduce the following metric perturbation:

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}, \tag{3.24}$$

where $g^{(0)}_{\mu\nu}$ is a general background metric. This action will not give rise to $SAdS_5$ solution since the scalar field acts as a source term and thus the Einstein field equations do not admit vacuum solutions. We expand the metric fluctuation in Fourier modes:

$$h_x^{\ y} = \int d^4k \phi_k \left(u\right) e^{-i\omega t + ikz}. \tag{3.25}$$

Now we expand the action perturbatively up to second order in ϕ_k to obtain an effective action from which we can read-off the shear stress tensor fluctuation T_{xy}. We start with the determinant term: $\sqrt{-g}$. We make use of the well-known formula:

$$ln\left(detM\right) = Tr\left(lnM\right), \tag{3.26}$$

to write (in matrix notation):

$$\sqrt{-g} = e^{log(\sqrt{-g})} = e^{\frac{1}{2}log\left(-g^{(0)} - h\right)} = \sqrt{-g^{(0)}} \left(e^{\frac{1}{2}log\left(1 + \left(g^{(0)-1}h\right)\right)} \right) = \sqrt{-g^{(0)}} \left(e^{\frac{1}{2}Trlog\left(1 + \left(g^{(0)-1}h\right)\right)} \right), \tag{3.27}$$

where in the last step we take the trace of the log of the matrices. Expanding the logarithm up to second order in h and subsequently the exponential up to second order in h gives the following result:

$$\sqrt{-g} = \sqrt{-g^{(0)}} \left(1 + \frac{1}{2} Tr\left(g^{(0)-1}h\right) - \frac{1}{4} Tr\left(\left(g^{(0)-1}h\right)^2 \right) + \frac{1}{8} \left(Tr\left(g^{(0)-1}h\right)\right)^2 \right), \tag{3.28}$$

which in index notation reads:

$$\sqrt{-g} = \sqrt{-g^{(0)}}\left(1 + \frac{1}{2}h^\mu_\mu - \frac{1}{4}h^\mu_\nu h^\nu_\mu + \frac{1}{8}h^\mu_\mu h^\nu_\nu\right). \tag{3.29}$$

In our case the trace of the perturbation is zero thus only the second term contributes:

$$\sqrt{-g} = -\frac{1}{2}\sqrt{-g^{(0)}}\left(h^x_y\right)^2. \tag{3.30}$$

So we see that the kinetic and potential terms of the dilaton will have a contribution to the effective action proportional to $\phi(u)^2$. As we will see later terms of this type will not contribute directly to the viscosity. Thus adding a free field to the action will not affect the shear viscosity. In fact, to get a temperature flow one needs to add a non-trivial dilaton potential along with non-minimaly coupled terms. The next step in the calculation is inserting the perturbation into the action and keeping terms up to second order in h.

Effective Action
After expanding all the terms we come to an effective action for $\phi(u)$

$$\begin{aligned} S_{eff} = \int \frac{d^4k}{(2\pi)^4}du\Big[&A(u)\phi_k''\phi_{-k} + B(u)\phi_k'\phi_{-k}' + C(u)\phi_k'\phi_{-k} \\ &+D(u)\phi_k\phi_{-k} + E(u)\phi_k''\phi_{-k}'' + F(u)\phi_k''\phi_{-k}'\Big]. \end{aligned} \tag{3.31}$$

This is the general form of the effective action for any two derivative theory. The coefficients $A, B, .., F$ encode the information of our background solution. They are given in Appendix A. The background solution used for our purposes is a black-brane solution parameterized in the following way:

$$ds^2 = -a^2(u)\,dt^2 + c^2(u)\,du^2 + b^2(u)\,d\mathbf{x}^2, \quad \Phi = \varphi(u). \tag{3.32}$$

We can expand the metric functions around the horizon by assuming a first order zero in g_{tt} and a corresponding first order pole in g_{uu},

$$\begin{aligned} a(u)^2 &= a_0(1-u) + a_1(1-u)^2 + a_2(1-u)^3 + \ldots \\ b(u)^2 &= b_0(1 + (1-u) + \ldots) \\ c(u)^2 &= c_0(1-u)^{-1} + c_1 + c_2(1-u) + \ldots \\ \varphi(u) &= \varphi_h + \varphi_1(1-u) + \varphi_2(1-u)^2 + \ldots \end{aligned} \tag{3.33}$$

This will be of use as we intend to formulate $\frac{\eta}{s}$ completely in terms of the horizon information. Our goal now is to derive a formula that gives us the shear viscosity in terms of the $A, B, .., E$ coefficients [11]:

$$\eta = \frac{1}{8\pi G_5}\left[\sqrt{-\frac{g_{uu}}{g_{tt}}}\left(A - B + \frac{F'}{2}\right) + \left(E\left(\sqrt{-\frac{g_{uu}}{g_{tt}}}\right)'\right)'\right]\Bigg|_{u=u_h}. \tag{3.34}$$

where primes denote u derivatives and the whole expression is evaluated at the horizon. This is is quite important as it allows us to expand our metric functions close to the horizon and express the final answer in terms a_0, a_1, c_0, c_1 and so on. The following subsection is devoted to proving this relation starting from (3.31).

3.1 From the Kubo relation to corrections to $1/4\pi$

This subsection follows [10, 11]. To calculate the viscosity we shall make use of the Kubo relation. This gives us the viscosity in terms of the retarded Green's function of the Energy momentum tensor.

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{xy,xy}\left(\omega, k=0\right). \tag{3.35}$$

To find the Green's function we need to evaluate the effective action (3.31) on shell and according to the holographic correspondence our field ϕ will be a source for the dual theory energy momentum tensor component T_{xy}. Thus the on-shell action evaluated at the boundary will allow us to compute the right hand side of (3.35).

With this goal in mind we wish to obtain the equation of motion for ϕ, so we begin by varying the action (3.31):

$$\delta S = \int d\mathbf{k}du\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi'}\delta\phi' + \frac{\partial\mathcal{L}}{\partial\phi''}\delta\phi''\right), \tag{3.36}$$

where $\mathbf{dk} = \frac{d^4k}{(2\pi)^4}$. In our case:

$$\begin{aligned}\frac{\partial\mathcal{L}}{\partial\phi_{-k}} &= A\left(u\right)\phi_k'' + C\left(u\right)\phi_k' + 2D\left(u\right)\phi_k,\\ \frac{\partial\mathcal{L}}{\partial\phi_{-k}'} &= 2B\left(u\right)\phi_k' + C\left(u\right)\phi_k + F\left(u\right)\phi_k'',\\ \frac{\partial\mathcal{L}}{\partial\phi_{-k}''} &= A\left(u\right)\phi_k + 2E\left(u\right)\phi_k'' + F\left(u\right)\phi_k'.\end{aligned} \tag{3.37}$$

Notice that we can rescale $\vec{k}\to-\vec{k}$ in the integral to obtain our result only in terms of ϕ_k. We can integrate (3.36) by parts to obtain

$$\begin{aligned}\delta S = &\int d\mathbf{k}du\left(\frac{\partial\mathcal{L}}{\partial\phi}\delta\phi - \left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)'\delta\phi + \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)''\delta\phi\right)\\ &+\int d\mathbf{k}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\delta\phi + \frac{\partial\mathcal{L}}{\partial\phi''}\delta\phi' - \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)'\delta\phi\right)\Bigg|_{boundary}.\end{aligned} \tag{3.38}$$

The first term we recognise as the equation of motion $\frac{\partial S}{\partial\phi} = 0$. In our case we have :

$$\begin{aligned}&\frac{d^2}{du^2}\left(A\left(u\right)\phi_k'' + C\left(u\right)\phi_k' + 2D\left(u\right)\phi_k\right) - \frac{d}{du}\left(2B\left(u\right)\phi_k' + C\left(u\right)\phi_k + F\left(u\right)\phi_k''\right)\\ &+ A\left(u\right)\phi_k + 2E\left(u\right)\phi_k'' + F\left(u\right)\phi_k' = 0.\end{aligned} \tag{3.39}$$

However, an issue arises with the second row of (3.38). Even though the variation of the field can be taken to vanish in the boundary as per usual $\delta\phi|_{bdry} = 0$, this not necessarily true for the derivative of the variation $\delta\phi'$. Thus as it stands the variation problem is not well defined. To remedy this we introduce a generalized Gibbons-Hawking term: a boundary term that exactly cancels our problematic $\delta\phi'$ term. This of course will modify our original action. The generalized Gibbons-Hawking term should satisfy:

$$\delta S_{GH} \sim -\int d\mathbf{k}\frac{\partial\mathcal{L}}{\partial\phi''}\delta\phi'\Bigg|_{boundary}, \tag{3.40}$$

plus terms proportional to $\delta\phi$. In our case the generalized Gibbons-Hawking term takes the following form:

$$S_{GH} = \int d\mathbf{k} \left(-A\phi_k\phi'_{-k} - \frac{F}{2}\phi'_k\phi'_{-k} + E\left(p_1\phi'_k + 2p_0\phi_k\right)\phi'_{-k} \right)\Bigg|_{boundary}, \tag{3.41}$$

where p_0 , p_1 are defined as follows. Notice that in (3.39) we can always move terms of order β to the right and divide by the coefficient of ϕ'' to write.

$$\phi''_k + p_1\phi'_k + p_0\phi_k = \mathcal{O}\left(\beta\right). \tag{3.42}$$

Keep in mind that the F and E coefficients will always be of order β since they come form our $R^2_{\mu\nu\rho\sigma}$ term, thus third and fourth order derivatives of ϕ will be moved to the right. The exact form of p_0 , p_1 does not matter for our purposes but formulating the EoM in this way allows us to define the variational problem perturbatively and cancel the boundary term proportional to E. Indeed, upon varying the GH term we find:

$$\delta S_{GH} = \int d\mathbf{k}\left(-A\phi'_k + 2p_0E\phi_k\right)\delta\phi + \left(-A\phi_k - F\phi'_k + 2E\left(p_1\phi'_k + p_0\phi_k\right)\right)\delta\phi'\Bigg|_{boundary}. \tag{3.43}$$

Thus we see that all terms cancel with the original action except for the E term which becomes:

$$\int d\mathbf{k}\; 2E\left(\phi''_k + p_1\phi'_k + p_0\phi_k\right)\delta\phi'\Bigg|_{boundary}. \tag{3.44}$$

But according to (3.42) the term in parenthesis is of order β and so is E. Thus up to linear order in β this term vanishes and we are saved.

Now that the action is properly defined we move to evaluate the on-shell action. We can write the action as:

$$2S_{eff} = \int d\mathbf{k}du\left(\frac{\partial\mathcal{L}}{\partial\phi}\phi + \frac{\partial\mathcal{L}}{\partial\phi'}\phi' + \frac{\partial\mathcal{L}}{\partial\phi''}\phi''\right) + 2S_{GH}. \tag{3.45}$$

In classic fashion, we integrate by parts bringing the action to the following form:

$$\begin{aligned} 2S_{eff} &= \int d\mathbf{k}du\left(\frac{\partial\mathcal{L}}{\partial\phi} - \left(\frac{\partial\mathcal{L}}{\partial\phi'}\right)' + \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)''\right)\phi \\ &+ \int d\mathbf{k}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\phi + \frac{\partial\mathcal{L}}{\partial\phi''}\phi' - \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)'\phi\right)\Bigg|_{boundary} + 2S_{GH}. \end{aligned} \tag{3.46}$$

The first line is the EoM and vanishes on shell. The second line gives the on-shell action. This is usually written as $S = \int d\mathbf{k}\,\mathcal{F}_k|_{boundary}$ where $\mathcal{F}_k$ is called a flux factor. In our case:

$$\mathcal{F}_k = \frac{1}{2}\int d\mathbf{k}\left(\frac{\partial\mathcal{L}}{\partial\phi'}\phi + \frac{\partial\mathcal{L}}{\partial\phi''}\phi' - \left(\frac{\partial\mathcal{L}}{\partial\phi''}\right)'\phi\right) + S_{GH}. \tag{3.47}$$

Furthermore, evaluating this expression gives :

$$2\mathcal{F}_k = \left(B - A\right)\phi'_k\phi_{-k} + \frac{1}{2}\left(C - A'\right)\phi_k\phi_{-k} - E'\phi''_k\phi_{-k} - E\phi'''_k\phi_{-k} - \frac{F'}{2}\phi'_k\phi_k + Ep_0\phi_k\phi'_{-k} \tag{3.48}$$

We can discard the second term in this expansion as it is real and will not contribute to the viscosity. In addition, the final term is of order $\mathcal{O}\left(\omega^2\right)$ and can be discarded in our limit.

The remaining expression is in-fact independent of u in the low frequency approximation and so we can evaluate it at any point we wish. To generate the nice analytic formula (3.34) we evaluate it at the black-brane horizon. The next important step in the calculation is considering in-falling boundary conditions. An in-falling observer at the horizon perceives it as a regular point, the seeming divergence is only a matter of our coordinate system. To get a clear picture of what happens at the horizon we use Eddington-Finklestein coordinates. In particular the v coordinate is defined as:

$$dv = dt + \sqrt{\frac{g_{uu}}{-g_{tt}}}du, \tag{3.49}$$

which is non-singular at $u = u_h$. So an in-falling observer should observe that a scalar field at the horizon only depends on this specific non-singular combination of u, t. In other words, $\phi(r, t, \mathbf{x}) \to \phi(v, \mathbf{x})$. This implies that [12]

$$\partial_u \phi(u_h, t) = \sqrt{\frac{g_{uu}}{-g_{tt}}}\partial t \phi(u_h, t), \tag{3.50}$$

which upon Fourier transform gives:

$$\begin{aligned} \partial_u \phi(u_h, \omega) &= -i\omega\sqrt{\frac{g_{uu}}{-g_{tt}}}\phi(u_h), \\ \partial_u^2 \phi(u_h, \omega) &= -i\omega\partial_u\left(\sqrt{\frac{g_{uu}}{-g_{tt}}}\right)\phi(u_h), \\ \partial_u^3 \phi(u_h, \omega) &= -i\omega\partial_u^2\left(\sqrt{\frac{g_{uu}}{-g_{tt}}}\right)\phi(u_h). \end{aligned} \tag{3.51}$$

All the pieces that we need are now in place, all that's left is to substitute the above expression into our formula for the flux factor $\mathcal{F}_k$ (3.47) to write it in terms of ϕ and not derivatives of ϕ. We find the following:

$$2\mathcal{F}_k = i\omega\sqrt{\frac{g_{uu}}{-g_{tt}}}\left(A - B + \frac{F}{2}\right)\phi_k\phi_{-k}\Big|_{u=u_h} + i\omega\left(E\left(\sqrt{\frac{g_{uu}}{-g_{tt}}}\right)'\right)'\phi_k\phi_{-k}\Big|_{u=u_h}. \tag{3.52}$$

According to the holographic correspondence the retarded Green's function is now:

$$G^R_{xy,xy}(\omega, k) = -\frac{2\mathcal{F}_k}{\phi_k\phi_{-k}}\Big|_{u=u_h}, \tag{3.53}$$

and the Kubo relation reads:

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}G^R_{xy,xy}(\omega, k=0). \tag{3.54}$$

Combining these three equations we obtain the desired formula (3.34)

$$\eta = \frac{1}{8\pi G_5}\left[\sqrt{-\frac{g_{uu}}{g_{tt}}}\left(A - B + \frac{F'}{2}\right) + \left(E\left(\sqrt{-\frac{g_{uu}}{g_{tt}}}\right)'\right)'\right]\Big|_{u=u_h}. \tag{3.55}$$

Shear viscosity for our theory

Evaluating the formula (3.34) using (3.32) along with the coefficients in the Appendix and taking the horizon limit by expanding with (3.33) yeilds the following result [13]:

$$\eta = \frac{b_0^{3/2}}{16\pi G_5}\left[1 - \frac{3a_1c_0 - a_0c_1 + a_0c_0}{a_0c_0^2}\ell^2\beta G(\phi_h) - \frac{2a_0c_0}{a_0c_0^2}\ell^2\beta\phi_1 G'(\phi_h)\right]. \tag{3.56}$$

3.2 Entropy of a black-brane from Wald's formula

One of the major discoveries of black hole physics has to do with their entropy. For a regular black-hole the entropy is simply given by the area of the black-hole horizon, thus one can take the horizon limit $u \to 1$ and calculate the induced metric $\gamma_{\alpha\beta}$ and the entropy will be proportional to $\sqrt{\gamma}$. However, in our case this no longer holds as there are corrections to this term stemming from our higher curvature term. In order to compute these corrections, we calculate the entropy by using Wald's formula:

$$S = -2\pi \oint_\Sigma d^3x \sqrt{-h} \frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} \epsilon_{\mu\nu} \epsilon_{\rho\sigma}, \tag{3.57}$$

where h is the induced metric after applying Stokes's theorem in the t direction. The tensor $\epsilon_{\mu\nu}$ is binormal to Σ and is normalized as $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$ and $\epsilon_{\mu\nu} = -\epsilon_{\nu\mu}$. We have

$$\mathcal{L} = \frac{1}{16\pi G_5} \left[R - 2(\nabla\Phi)^2 + V(\Phi) + \ell^2 \beta G(\Phi) R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \right]. \tag{3.58}$$

We can immediately see that only the first and last term will contribute as only they depend on the Riemann tensor. We can write the Ricci scalar as

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\nu} g^{\alpha\rho} R_{\rho\mu\alpha\nu}. \tag{3.59}$$

So we find:

$$\frac{\delta \mathcal{L}}{\delta R_{\mu\nu\rho\sigma}} = \frac{1}{16\pi G_5} \left[g^{\nu\sigma} g^{\rho\mu} + 2\ell^2 \beta G(\Phi) R^{\mu\nu\rho\sigma} \right]. \tag{3.60}$$

and using (3.32) we calculate $h = \left(b^2(u)\right)^3$, so $\sqrt{h} = b^3(u) \to \sqrt{h} = b_0^{3/2}$ where higher order terms in (3.33) vanish at the horizon $u = 1$. Putting this all together we find:

$$S = \frac{b_0^{3/2}}{4G_5} V_3 \left[1 - \ell^2 \beta G(\phi_h) R^{\mu\nu\rho\sigma} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} \right], \tag{3.61}$$

where we use $\epsilon_{\mu\nu}\epsilon^{\mu\nu} = -2$ and $\int d^3x = V_3$ is the 3-dimensional volume that diverges. We are however interested in the entropy density $s = \frac{S}{V_3}$ so we don't have to worry about this factor. Since the only non-zero components of $\epsilon_{\mu\nu}$ are $\epsilon_{ut} = -\epsilon_{tu}$ we can simplify the above expression to express the entropy density as:

$$s = \frac{b_0^{3/2}}{4G_5} \left[1 - 4\ell^2 \beta G(\phi_h) R^{utut} \epsilon_{ut}^2 \right]. \tag{3.62}$$

Since the correction term is already of order β we use only the background metric to find $R^{utut}\epsilon_{ut}^2$.Using the defining property of $g^{\mu\rho} g^{\nu\sigma} \epsilon_{\mu\nu} \epsilon_{\rho\sigma} = -2$ along with the fact that $\epsilon_{\mu\nu}$ is binormal to Σ we have that:

$$g^{uu} g^{tt} \epsilon_{ut}^2 = -1 \Rightarrow \epsilon_{ut}^2 = -g_{uu} g_{tt} \Rightarrow \epsilon_{ut} = a(u)\, c(u) \tag{3.63}$$

Thus after calculating the desired Riemann component and evaluating at the horizon using (3.33) we find the final form of the entropy density[13]:

$$s = \frac{b_0^{3/2}}{4G_5} \left[1 - \frac{3a_1 c_0 - a_0 c_1}{a_0 c_0^2} \ell^2 \beta G(\phi_h) \right]. \tag{3.64}$$

Previously we found (3.56)

$$\eta = \frac{b_0^{3/2}}{16\pi G_5} \left[1 - \frac{3a_1 c_0 - a_0 c_1 + a_0 c_0}{a_0 c_0^2} \ell^2 \beta G(\phi_h) - \frac{2a_0 c_0}{a_0 c_0^2} \ell^2 \beta \phi_1 G'(\phi_h) \right]. \tag{3.65}$$

Thus dividing the two expressions and keeping terms up to first order yields[13, 14]:

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \frac{\beta\ell^2}{c_0}\left(G\left(\phi_h\right) + 2\phi_1 G'\left(\phi_h\right)\right)\right]. \tag{3.66}$$

An aside: calculating the temperature of the black-brane
The fastest and most elegant way to compute the Hawking temperature of our solution is by requiring that the Euclidean version of the metric be smooth. By performing a Wick rotation $\tau = it$ to the metric (3.32) we obtain:

$$ds^2 = a^2\left(u\right)d\tau^2 + c^2\left(u\right)du^2 + b^2\left(u\right)d\mathbf{x}^2. \tag{3.67}$$

Moreover we can expand this close to the horizon using (3.33):

$$ds^2 = a_0\left(1-u\right)d\tau^2 + \frac{c_0}{1-u}du^2 + b_0 d\mathbf{x}^2. \tag{3.68}$$

Now we introduce a new coordinate $\rho = 2\sqrt{(1-u)c_0}$. This transforms the metric as follows:

$$ds^2 = \rho^2\frac{a_0}{4c_0}d\tau^2 + d\rho^2 + b_0 d\mathbf{x}^2. \tag{3.69}$$

Notice that for $\mathbf{x} = const$ this metric is simply the plane in polar coordinates. However our "angular" coordinate needs to have a periodicity of 2π otherwise the metric has a conical singularity at $\rho = 0$. Thus $\frac{a_0}{4c_0}d\tau^2 = d\left(\frac{1}{2}\sqrt{\frac{a_0}{c_0}}\tau\right)^2$ must have a periodicity of 2π. So the periodicity of τ is $\beta = 4\pi\sqrt{\frac{c_0}{a_0}}$ which the inverse of the temperature so we obtain:

$$T = \frac{1}{4\pi}\sqrt{\frac{a_0}{c_0}}. \tag{3.70}$$

As a final note, using the Hawking temperature one can calculate the free energy of the system as $F = T\bar{S}$ where the bar denotes the action is on-shell. In addition, this is an alternative way of calculating the entropy by making use of $S = -\partial_T F$.

3.3 Connecting to the diatonic brane solutions

The corrections to the shear viscosity to entropy ratio in (3.66) are already of order β so we don't have to solve for the full action to find the behaviour of $c\left(u\right)$ and $\phi\left(u\right)$ at the horizon. Instead we may ignore the higher order corrections and relate the near horizon expansions to the known black brane ansatz for the minimally coupled dilaton.

We introduce a new radial coordinate to write the metric as follows:

$$ds^2 = f^{-1}\left(r\right)dr^2 + e^{2A(r)}\left(d\mathbf{x}^2 - f\left(r\right)dt^2\right), \quad \Phi = \Phi\left(r\right). \tag{3.71}$$

where we can expand the metric functions close to the horizon :

$$\begin{aligned} A(r) &= A_h + A_1\left(r - r_h\right) + \cdots \\ f(r) &= f_1\left(r - r_h\right) + \cdots \\ \Phi(r) &= \Phi_h + \Phi_1\left(r - r_h\right) + \cdots \end{aligned} \tag{3.72}$$

Now to connect this near horizon expansion to the previous one (3.32) we match the two metric function expansions at the horizon. Looking at the $d\mathbf{x}^2$ coefficient we find:

$$b_0\left(1 + \left(1-u\right)\right) = e^{2A_h}\left(1 + 2A_1\left(r - r_h\right)\right), \tag{3.73}$$

which yields $b_0 = e^{2A_h}$ and $r - r_h = \frac{1}{2A_1}(1-u)$. Thus the transformation $du = -2A_1 dr$ connects the two expansions at the horizon and we find:

$$\varphi_h = \Phi_h, \quad \varphi_1 = \frac{1}{2A_1}\Phi_1, \quad c_0 = \frac{1}{2f_1 A_1}. \tag{3.74}$$

Substituting back to (3.66) we can relate η/s to f_1, A_1, Φ_h

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - 2f_1 A_1 \beta \ell^2 \left(G(\phi_h) + \frac{\Phi_1}{A_1} G'(\phi_h)\right)\right]. \tag{3.75}$$

3.4 Phase Variables

Having connected the two solutions at the horizon, our goal now is to express A_1 and f_1 in terms of thermodynamic properties of the black brane, in particular we will derive a formula that writes (3.56) in terms of the potential $V(\Phi)$ and $V'(\Phi)$. Of course in the end, we want to express the result in terms of temperature to connect it to the data. This can be achieved by *relating the horizon value of the dilaton to the temperature.* For simple exponential potentials this can be done analytically.

The Einstein and dilaton equations of motion can be reformulated into five differential equations with the help of the phase variables method developed in [24]. We define the functions X, Y:

$$X(\Phi) \equiv \frac{\zeta}{4}\frac{\Phi'}{A'}, \quad Y(\Phi) \equiv \frac{1}{4}\frac{f'}{fA'}. \tag{3.76}$$

To solve for X, Y we only need to solve two coupled first order differential equations:

$$\begin{aligned} \frac{dX}{d\Phi} &= -\zeta\left(1 - X^2 + Y\right)\left(1 + \frac{1}{2\zeta}\frac{1}{X}\frac{d\log V}{d\Phi}\right), \\ \frac{dY}{d\Phi} &= -\zeta\left(1 - X^2 + Y\right)\frac{Y}{X}, \end{aligned} \tag{3.77}$$

and the rest of the functions are simply determined in terms of X,Y as follows:

$$\begin{aligned} \frac{dA}{dr} &= -\frac{1}{\ell}e^{-\zeta\int_0^\Phi X(t)dt}, \\ \frac{d\Phi}{dr} &= -\frac{4}{\ell\zeta}X(\Phi)e^{-\zeta\int_0^\Phi X(t)dt}, \\ \frac{1}{f}\frac{df}{dr} &= -\frac{4}{\ell}Y(\Phi)e^{-\zeta\int_0^\Phi X(t)dt}, \end{aligned} \tag{3.78}$$

where ζ is a factor that depends on the normalization of the kinetic term, in our case $\zeta = \sqrt{\frac{8}{3}}$. The temperature and entropy are given completely in terms of the dilaton horizon value.

$$\begin{aligned} T(\Phi_h) &= \frac{\ell}{12\pi}e^{A(\Phi_h)}V(\Phi_h)\, e^{\zeta\int_0^{\Phi_h} X(\Phi)d\Phi}, \\ S &= \frac{1}{4G_5}e^{3A(\Phi_h)}. \end{aligned} \tag{3.79}$$

Combining these two expressions we can solve for the variable X and relate it to the scalar potential evaluated at the horizon:

$$e^{-\zeta\int_0^{\Phi_h} X(\Phi)d\Phi} = C\frac{S^{\frac{1}{3}}}{T}V(\Phi_h). \tag{3.80}$$

where $C = \frac{\ell(4\pi)^{-\frac{4}{3}}}{3M_p}$ and the Planck mass is $M_p = (16\pi G_5)^{-\frac{1}{3}}$.

Combining (3.78) and (3.80) we find that close to the horizon:

$$\begin{aligned} \frac{dA}{dr} &= A_1 = -\frac{C}{\ell}\frac{S^{\frac{1}{3}}}{T} V(\Phi_h), \\ \frac{d\Phi}{dr} &= \Phi_1 = \frac{3C}{4\ell}\frac{S^{\frac{1}{3}}}{T} V'(\Phi_h), \end{aligned} \tag{3.81}$$

where to obtain the second equation we use the boundary conditions for X which follow from demanding regularity at the horizon. Namely at $\Phi \sim \Phi_h$ we have $X(\Phi) = -\frac{1}{2\zeta}\frac{V(\Phi_h)}{V'(\Phi_h)}$. In addition, we have that

$$4\pi T = -f'(r_h)\, e^{A(r_h)}, \tag{3.82}$$

which after substituting for the entropy (3.79) gives us

$$f_1 = -M_p(4\pi)^{\frac{4}{3}} \frac{T}{S^{\frac{1}{3}}}. \tag{3.83}$$

Substituting A_1, Φ_1 and f_1 back into our expression for η/s (3.75) we find

$$\boxed{\frac{\eta}{s} = \frac{1}{4\pi}\left[1 + \frac{2}{3}\beta\ell^2\left(-G(\Phi_h)V(\Phi_h) + \frac{3}{4}G'(\Phi_h)V'(\Phi_h)\right)\right]} \tag{3.84}$$

This is the main result of our analysis and the formula that we will use for matching to the Bayesian analysis of heavy ion collisions. Notice that the final result only depends on the behaviour of the potentials G, V evaluated at the horizon.

4 Fitting the data from Bayesian Analysis of Heavy Ion collisions

Recent analysis of heavy ion collisions in [15, 16, 17] has revealed interesting results for the behaviour of transport coefficients in quark gluon plasma. The authors of [15] determined that there is a clear tendency for η/s to increase as temperature increases away from the QCD critical temperature. For low enough temperatures, it appears that the value of η/s is consistent with $1/4\pi$. However, it seems that higher order corrections need to be included capture the temperature flow away from the constant result of the simple Einstein-Hilbert action. In this model, the temperature flow will be determined by the choice of dilaton potential along with the dilaton coupling (3.84). Below, we analyze the thermodynamics of different potentials and fit their prediction for η/s to the data from heavy ion collisions. We start from the simplest potential and progress to more realistic ones that capture various properties of QCD.

Bayesian analysis of Heavy Ion Collisions
The authors of [15] parameterize η/s as

$$\frac{\eta}{s} = a + b(T - T_c)\left(\frac{T}{T_c}\right)^c. \tag{4.1}$$

where $Tc = 154 \times 10^{-3}$GeV and they determine

$$a = 0.065^{+0.038}_{-0.040}, \quad b = 0.9^{+0.81}_{-0.90}\,\text{GeV}^{-1}, \quad c = -0.04^{+0.75}_{-0.85}. \tag{4.2}$$

They plot this along with a 90% confidence band in Figure 4.

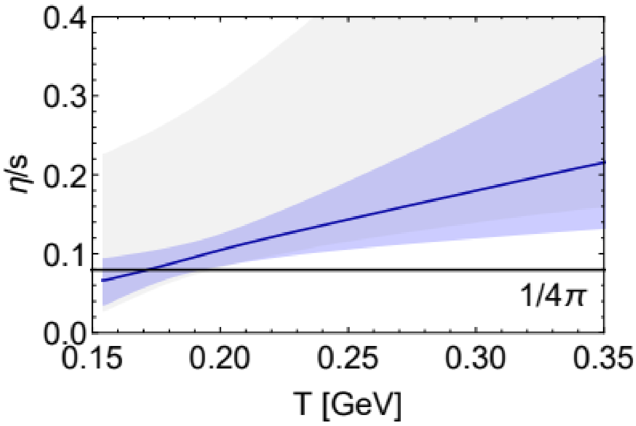

Figure 4: Posterior distribution for the shear viscosity to entropy ratio versus temperature with a 90% confidence band in blue.

4.1 Thermodynamics of the Chamblin-Reall black brane

The first potential we look at has the advantage of being analytically solvable and serves as a simplistic model for obtaining a temperature flow. The Chamblin-Reall black brane has an exponential potential of the form:

$$V(\Phi) = \frac{V_0}{\ell^2} e^{\alpha\Phi}, \tag{4.3}$$

where V_0 is positive and dimensionless and ℓ sets the spacetime length scale. We choose $\alpha > 0$. We have $\frac{d\log V}{d\Phi} = \alpha$ so using the phase variables formalism one can see that the equation for X in (3.77):

$$\frac{dX}{d\Phi} = -\zeta\left(1 - X^2 + Y\right)\left(1 + \frac{1}{2\zeta}\frac{1}{X}\frac{d\log V}{d\Phi}\right), \tag{4.4}$$

is solved by

$$X(\Phi) = x_0 = -\frac{\alpha}{2\zeta}. \tag{4.5}$$

From this we can calculate the background functions, in particular we are interested in A:

$$A(\phi) = A(\Phi_c) + \frac{\zeta}{4}\int_{\Phi_c}^{\Phi}\frac{d\tilde{\Phi}}{X} = A(\Phi_c) + \frac{\zeta}{4x_0}(\Phi - \Phi_c). \tag{4.6}$$

Using A we can now calculate the temperature and entropy as a function of the horizon value of the dilaton field Φ_h using the formula (3.79). We find:

$$T(\Phi_h) = T_0 e^{\frac{\zeta}{4x_0}\left(1-4x_0^2\right)\Phi_h}, \quad S(\Phi_h) = S_0 e^{\frac{3\zeta}{4x_0}\Phi_h}, \tag{4.7}$$

where we defined

$$T_0 = \frac{V_0}{12\pi\ell} e^{A_c - \frac{\zeta\Phi_c}{4x_0}}, \quad S_0 = \frac{1}{4G_N} e^{3A_c - 3\frac{\zeta}{4x_0}\Phi_c}. \tag{4.8}$$

Inverting the temperature formula and solving for Φ_h yields

$$\Phi_h = \frac{4x_0}{\zeta\left(1 - 4x_0^2\right)}\log\left(\frac{T}{T_0}\right) = -\frac{6\alpha}{3 - 3\alpha^2}\log\left(\frac{T}{T_0}\right). \tag{4.9}$$

This allows us to write the entropy as a function of temperature namely:

$$S = S_0\left(\frac{T}{T_0}\right)^{\frac{3}{1-4x_0^2}}. \tag{4.10}$$

For the system to be thermally stable the specific heat $C = T\frac{dT}{dS}$ must be positive. To check this we note that both S_0 and T_0 are positive so we have

$$C \propto \left(\frac{3}{1-4x_0^2}\right) T^{\frac{3}{1-4x_0^2}} > 0, \tag{4.11}$$

which translates to

$$x_0^2 < \frac{1}{4}, \quad \Rightarrow \quad 0 < \alpha < \sqrt{\frac{8}{3}}. \tag{4.12}$$

For this potential to exhibit confinement we need $\alpha \geq \sqrt{\frac{8}{3}}$. So for this type of potential confinement and thermal stability are mutually exclusive.

Fitting the data for a simple exponential coupling

For our first attempt at making contact with reality we pick the simple coupling function $G(\Phi) = e^{\gamma\Phi}$. Substituting into our general expression for η/s (3.84) we find

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \frac{2}{3}V_0\left(1 - \frac{3}{4}\gamma\alpha\right)e^{(\alpha+\gamma)\Phi_h}\right]. \tag{4.13}$$

Plugging in (4.9) we can replace Φ_h for T to obtain:

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \frac{2}{3}\beta V_0\left(1 - \frac{3}{4}\gamma\alpha\right)\left(\frac{T}{T_0}\right)^{-\frac{6\alpha(\alpha+\gamma)}{8-3\alpha^2}}\right]. \tag{4.14}$$

Without loss of generality we take $T_0 = T_c$. This prediction has three free parameters $\{\beta V_0, \alpha, \gamma\}$ which we can fit to our experimental curve:

$$\frac{\eta}{s} = a + b\,(T - T_c)\left(\frac{T}{T_c}\right)^c, \tag{4.15}$$

with a, b, c and their uncertainties given in (4.2).

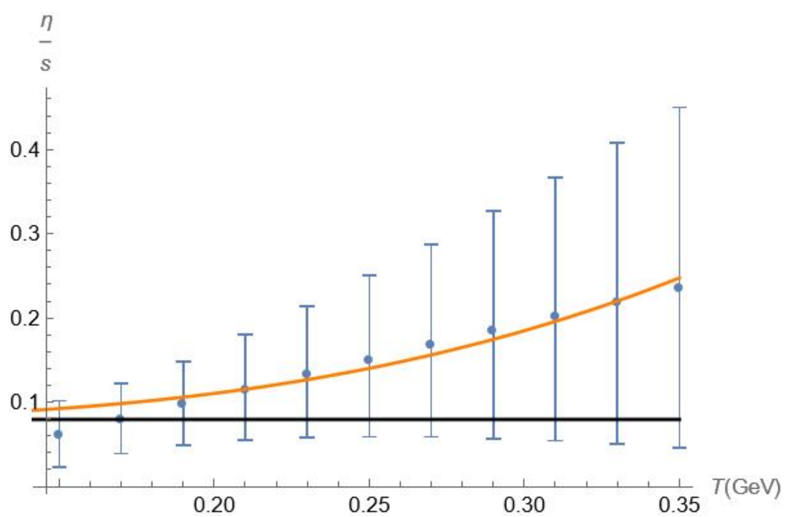

Figure 5: Best fit for the shear viscosity to entropy ratio for $G(\Phi) = e^{\gamma\Phi}$.The yellow line shows the theoretical curve while the black line is $\eta/s = 1/4\pi$ for reference. Optimal values for the parameters where found to be: $\{\beta V_0 = -0.239, \alpha = 2.755, \gamma = -0.052\}$

.

Figure 5 shows the best fit for this setup. There are a few things to note about this fit:

- After fixing the free parameters our function looks like: $\eta/s = 0.080 + 4.02\,T^{3.02}$.
- We have that $\alpha > \sqrt{\frac{8}{3}}$ which means that we are in the confining but thermally unstable region of the potential.
- With this formula the theoretical curve will asymptote to $1/4\pi$ for $T \to 0$ when $\alpha > \sqrt{\frac{8}{3}}$ and $-\gamma > \alpha$.
- We were not able to determine a good fit in the thermally stable region since there, the exponent of T goes negative and $1/4\pi$ is approached from below. For low temperatures the curve falls off to minus infinity.

Furthermore, we investigated the case of $\gamma = 0$ which amounts to a constant correction from the $R^2_{\mu\nu\rho\sigma}$ term and minimal coupling of the dilaton. The be best fit in this case is similar and can be seen in figure 6. The same general comments hold in this case as well.

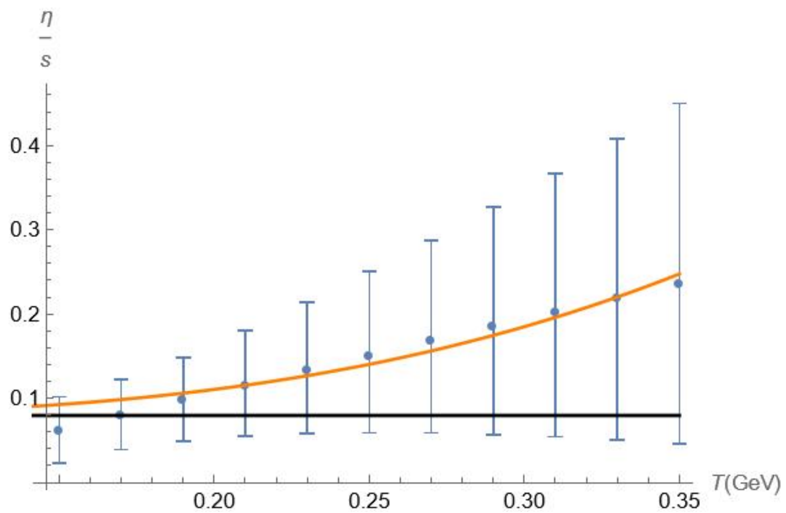

Figure 6: Best fit for the shear viscosity to entropy ratio for $G(\Phi) = 1$. The yellow line shows the theoretical curve while the black line is $\eta/s = 1/4\pi$ for reference. Optimal values for the parameters where found to be: $\{\beta V_0 = -0.264, \alpha = 2.805\}$

4.2 Determining the optimal $G(\Phi)$ function

Instead of arbitrarily picking functions for G, we would like to determine what the the optimal function would be so as to get a better fit with the data. To achieve this, we solve (3.84) as a differential equation in G. If we re-arrange some terms we can write it as:

$$G'(\Phi_h) + P(\Phi_h)\,G(\Phi_h) = R(\Phi_h)\,, \tag{4.16}$$

where

$$P(\Phi_h) = -\frac{4}{3}\frac{V(\Phi_h)}{V'(\Phi_h)}\,, \quad R(\Phi_h) = \left[4\pi\left(\frac{\eta}{s}\right) - 1\right]\frac{2}{\ell^2 \beta V'(\Phi_h)} \tag{4.17}$$

This is a first order differential equation and the general solution is given by:

$$G(\Phi_h) = F^{-1}(\Phi_h)\int d\Phi_h F(\Phi_h)\,R(\Phi_h) + c_1 F^{-1}(\Phi_h)\,, \tag{4.18}$$

where $F(\Phi_h) = e^{\int P(\Phi_h)d\Phi_h}$. In place of η/s we plug in the experimental curve (4.1) where we substitute T for Φ_h according to the relation determined by the thermodynamics of the chosen

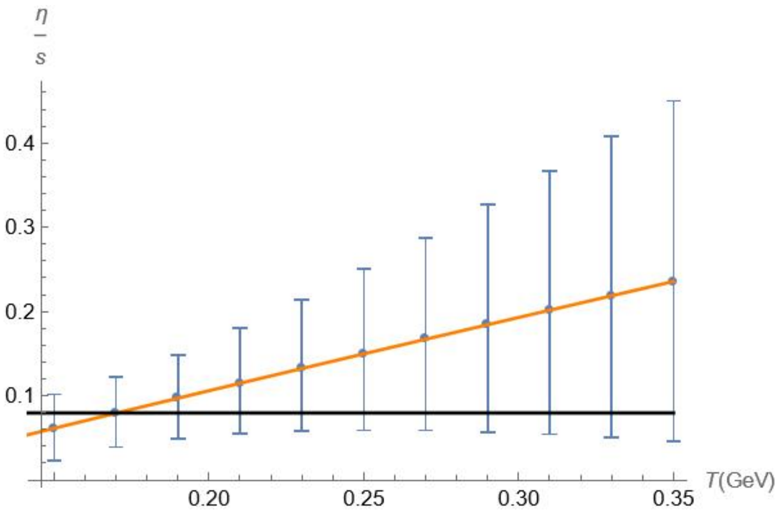

Figure 7: Best fit for the shear viscosity to entropy ratio for $G(\Phi)$ given in (4.20). The yellow line shows the theoretical curve while the black line is $\eta/s = 1/4\pi$ for reference.

potential. The integration constant c_1 which multiplies the homogeneous solution and will not play a role in the viscosity since the homogeneous solution drops out in (3.84).

In the case of the CR brane $V(\Phi) = \frac{V_0}{\ell^2}e^{\alpha\Phi}$ this solution is not too hard to find, we have:

$$F = e^{\frac{-4\Phi_h}{3\alpha}}, \quad R = \frac{2e^{-\alpha\Phi_h}}{\beta V_0 \alpha}\left[4\pi\left(a + b\left(e^{c\Phi_h\left(\frac{\alpha}{2}-\frac{4}{3\alpha}\right)}\right)\left(-1 + e^{\Phi_h\left(\frac{\alpha}{2}-\frac{4}{3\alpha}\right)}\right)\right) - 1\right], \tag{4.19}$$

where a, b, c are given in (4.2). Using this we can obtain the general solution for G:

$$G(\Phi_h) = c_1 e^{\frac{4\Phi_h}{3\alpha}} + \frac{48b\pi T_c e^{\Phi_h(\alpha' c - \alpha)}}{V_0\beta}\left[\frac{3e^{-\Phi_h\alpha' c}(1-4a\pi)}{96b\pi T_c} + \frac{e^{\Phi_h \alpha'}}{(3\alpha^2-8)(c+1)-8} - \frac{e^{\Phi_h\alpha' c}}{(3\alpha^2-8)c-8}\right], \tag{4.20}$$

with $\alpha' = \frac{3\alpha^2-8}{6\alpha}$. Note that for our correction to remain perturbative we must have $V_0 >> 1$. The advantage of this method is that once we have handpicked our coupling, the curve we obtain for η/s is identical to the one we are attempting to fit. In Figure 7 we plot the shear viscosity for this coupling G.

4.3 Shear viscosity of improved holographic QCD

In this subsection we consider a more realistic model, where the dilaton potential is designed to capture certain properties of QCD. In particular, we will look at a potential of the form:

$$V(\Phi) = V_0 e^{Q\Phi}\Phi^P. \tag{4.21}$$

Using the phase variables formalism we wish to solve for $X(\Phi)$ and using that to find the metric functions and the temperature and entropy as a function of Φ_h. From (3.77) we have

$$\frac{dX}{d\Phi} = -\zeta\left(1 - X^2 + Y\right)\left(1 + \frac{1}{2\zeta}\frac{1}{X}\frac{d\log V}{d\Phi}\right). \tag{4.22}$$

We have $\frac{d\log V}{d\Phi} = Q + \frac{P}{\Phi}$, so in the region where $\Phi >> 1$ we have an approximate solution of [19] :

$$X(\Phi) = -\frac{1}{2\zeta}\left(Q + \frac{P}{\Phi}\right) + \mathcal{O}\left(\frac{1}{\Phi^2}\right). \tag{4.23}$$

Using this we can calculate the metric function A:

$$A(\Phi) = A(\Phi_c) - \frac{\zeta^2}{2}\int_{\Phi_c}^{\Phi} \frac{\Phi}{Q\Phi + P} d\Phi, \tag{4.24}$$

which gives

$$A(\Phi) = A_0 - \frac{\zeta^2}{2}\frac{Q\Phi - P\log(P + Q\Phi)}{Q^2}, \quad A_0 = A(\Phi_c) + \frac{\zeta^2}{2}\frac{Q\Phi_c - P\log(P + Q\Phi_c)}{Q^2}. \tag{4.25}$$

The temperature and entropy are now given by (3.79) and read:

$$\begin{aligned} T(\Phi_h) &= T_0 e^{\Phi_h\left(\frac{Q}{2} - \frac{\zeta^2}{2Q}\right)} \Phi_h^{\frac{\zeta^2 P}{2Q^2} + \frac{P}{2}}, \\ S(\Phi_h) &= S_0 e^{\frac{-3\zeta^2\Phi_h}{2Q}} \Phi_h^{\frac{3\zeta^2 P}{2Q^2}}. \end{aligned} \tag{4.26}$$

where for the integral of $X(\Phi)$ to not diverge we require $P > 0$. Notice also we had to perform the integral $e^{\zeta\int_0^{\Phi_h} X(\Phi)d\Phi}$ where we know our approximation won't hold for $\Phi \in [0, 1]$. We proceed assuming that this will be a small correction to the result. T_0 and S_0 are positive constants:

$$S_0 = e^{3A_0} Q^{\frac{3\zeta^2 P}{2Q^2}}, \quad T_0 = \frac{\ell}{12\pi} e^{A_0} Q^{\frac{\zeta^2 P}{2Q^2}} V_0. \tag{4.27}$$

To determine the thermal stability of this potential we look at the specific heat:

$$C = T\frac{dS}{dT} = T\frac{dS}{d\Phi_h}\frac{d\Phi_h}{dT} > 0. \tag{4.28}$$

Since the temperature is positive we must have both $\frac{dS}{d\Phi_h}$ and $\frac{d\Phi_h}{dT}$ either positive or negative. From the previous equation for the temperature if we suppose $\frac{dT}{d\Phi_h} < 0$, we find

$$\frac{dT}{d\Phi_h} = T_0 e^{\Phi_h\left(\frac{Q}{2} - \frac{\zeta^2}{2Q}\right)} \Phi_h^{\frac{\zeta^2 P}{2Q^2} + \frac{P}{2}} \left(\frac{Q}{2} - \frac{\zeta^2}{2Q} + \left(\frac{\zeta^2 P}{2Q^2} + \frac{P}{2}\right)\frac{1}{\Phi_h}\right) < 0. \tag{4.29}$$

This is equivalent to

$$\frac{Q}{2} - \frac{\zeta^2}{2Q} + \left(\frac{\zeta^2 P}{2Q^2} + \frac{P}{2}\right)\frac{1}{\Phi_h} < 0 \quad \Rightarrow \quad 0 < \left(\frac{\zeta^2}{Q} + 1\right)\frac{P}{\Phi_h} < -Q + \frac{\zeta^2}{Q}. \tag{4.30}$$

So we find

$$Q < \sqrt{\frac{8}{3}}. \tag{4.31}$$

So if we have $\frac{dS}{d\Phi_h} < 0$ this would mean that $Q < \sqrt{\frac{8}{3}}$ and the region of thermal stability is outside the confining region. However if we have $Q \geq \sqrt{\frac{8}{3}}$ and $\frac{dS}{d\Phi_h} > 0$ then we have stability and confinement. If we check this condition for the entropy we find:

$$\frac{dS}{d\Phi_h} = S_0 e^{\frac{-3\zeta^2\Phi_h}{2Q}} \Phi_h^{\frac{3\zeta^2 P}{2Q^2}} \frac{3\zeta^2}{2Q}\left(-1 + \frac{P}{Q\Phi_h}\right) > 0. \tag{4.32}$$

This sets an upper bound for the dilaton horizon value:

$$\Phi_h < \frac{P}{Q}, \quad \Rightarrow \quad \Phi_h < \frac{3P}{8}, \tag{4.33}$$

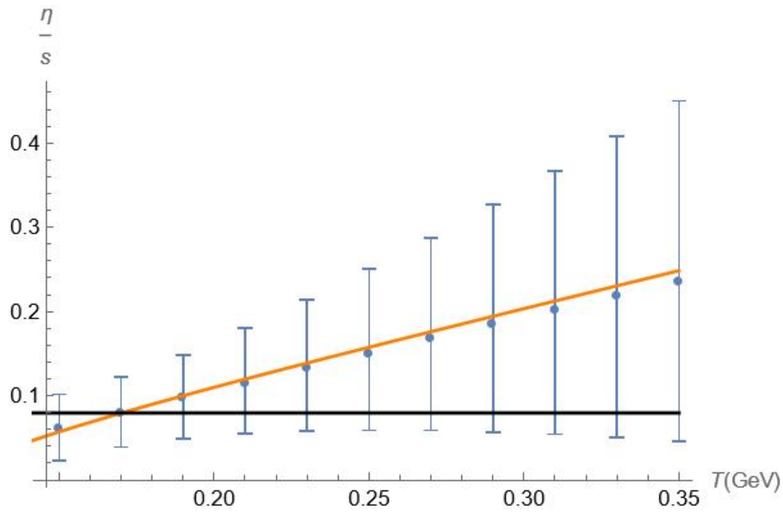

Figure 8: Best fit for the shear viscosity to entropy ratio for ihQCD with $Q = \sqrt{\frac{8}{3}}$, $P = 1/2$ and $G(\Phi)$ given in (4.36). The yellow line shows the theoretical curve while the black line is $\eta/s = 1/4\pi$ for reference.

suggesting that our original approximation of $\Phi >> 1$ breaks down and determining the thermal stability would require a numerical treatment of the potential. Nevertheless, we proceed with finding a fit for η/s in this setup. The most interesting case is that of $Q = \sqrt{\frac{8}{3}}$ and $P = \frac{1}{2}$. This is because these parameter choices exhibit the best fit best with the QCD data. In this case Φ_h is simply expressed as:

$$\Phi_h = \left(\frac{T}{T_0}\right)^2. \tag{4.34}$$

Using (4.18) we obtain the optimal coupling for this potential:

$$\begin{aligned} G(\Phi_h) = e^{\sqrt{\frac{2}{3}}\Phi_h}(\frac{3}{2} + 2\sqrt{6}\Phi_h)^{-1/4}\Big(c_1 + \\ \int_1^{\Phi_h} d\Phi \, \frac{e^{-\sqrt{6}\Phi}\Phi^{1/2}(\frac{3}{2} + 2\sqrt{6}\Phi)^{-3/4}\left(-125 + 500\mathrm{a}\pi + 77b\pi\Phi^{c/2}\left(-1 + \Phi^{1/2}\right)\right)}{125\ell^2\beta \mathrm{V}_0}\Big). \end{aligned} \tag{4.35}$$

If we approximate $(\frac{3}{2} + 2\sqrt{6}\Phi)^{-1/4} \simeq (2\sqrt{6}\Phi)^{-1/4}$ and $\Phi^{c/2} \simeq 1$ since $c \sim -0.04$, we can solve the integral exactly and get a closed form for G:

$$G(\Phi_h) = \frac{e^{\sqrt{\frac{2}{3}}\Phi_h}}{(\frac{3}{2} + 2\sqrt{6}\Phi_h)^{1/4}}\left[c_1 + \frac{12}{125\ell^2\beta V_0}\left(0.079 + 43.45\Phi_h^{3/4}E_{\frac{1}{4}}\left(\sqrt{6}\Phi_h\right) - 12.83\Gamma\left(5/4, \sqrt{6}\Phi_h\right)\right)\right]. \tag{4.36}$$

where $\Gamma(s,x) = \int_x^\infty t^{s-1}e^t \, dt$ is the incomplete gamma function and $E_n(x) = \int_1^\infty \frac{e^{-xt}}{t^n}dt$ is the exponential integral function. In Figure 8 we plot the viscosity for this coupling. As can be seen our approximations have only slightly shifted the theoretical curve.

4.4 Shear viscosity of V-QCD

In principle, we could perform our analysis for ihQCD again numerically and try to find a thermally stable fit. However, since this is the point where analytic calculations stop, we will move to an improved version of this theory called V-QCD[21], see [20] for a review. This theory has many improved properties including fermions which we have so far neglected. It is divided into two sectors, a gluon sector and a flavor sector:

$$S_{VQCD} = S_{ihQCD} + S_f. \tag{4.37}$$

The gluon sector takes the form of ihQCD which we have already studied:

$$S_{ihQCD} = M_p^3 N_c^2 \int d^5x\sqrt{-g}\left[R - \frac{4}{3}\frac{(\partial\lambda)^2}{\lambda^2} + V_g(\lambda) + \ell^2\beta G(\lambda)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]. \tag{4.38}$$

where to keep up with convention we have changed the dilaton kinetic term normalization to 4/3 and written the action in terms of $\lambda = e^{\Phi}$. The curvature squared term is not part of "standard" vQCD but is there to serve our temperature dependence needs as discussed previously. In addition, this action will also include a generalized Gibbons-Hawking term as in (3.41). The flavor action takes the following form:

$$S_f = -x_f M_p^3 N c^2 \int d^5x V_{f0}(\lambda)\, e^{-\tau^2}\sqrt{-\det\left(g_{\mu\nu} + \kappa(\lambda)\,\partial_\mu\tau\partial_\nu\tau + w(\lambda)\,\hat{F}_{\mu\nu}\right)}. \tag{4.39}$$

This sector is based on a setup of two space filling $D4 - \bar{D}4$ branes. The brane action includes a tachyon DBI action along with a Chern-Simons action. The tachyon field is dual to the quark mass operator ($\tau \leftrightarrow \bar{q}q$) and roughly speaking accounts for the breaking of chiral symmetry in QCD. The potentials $\kappa(\lambda), w(\lambda), V_{f0}(\lambda)$ appearing in S_f are assumed to only depend on λ and are not derived from a specific brane setup but instead are determined by fitting various properties of QCD.

This theory is in Veneziano limit where in addition to the large-N limit which is standard in Holography we also take the number of flavors to infinity, while keeping their ratio fixed. In summary, the Veneziano limit is the following:

$$N_c \to \infty\,, \quad N_f \to \infty\,, \quad x_f \equiv \frac{N_f}{N_c} = \text{fixed}\,, \quad \lambda \equiv g^2 N_c = \text{fixed}. \tag{4.40}$$

For the physics of QGP that we are interested in, we have $N_c = 3$ and $N_f = 3$ accounting for 3 light quarks {u,d,s}. Recall that the next heaviest quark is charm with a mass of order 1 GeV thus we don't expect it to contribute in the energies that we are interested in namely $T \sim (0.15 - 0.35)$ GeV. So for the rest of the analysis we keep $x_f = 1$.

The non-confining, chirally symmetric part of QCD corresponds to the near boundary behaviour of the gravitational theory. It turns out [21] that the tachyon vanishes near the boundary while the behaviour of $\hat{F}_{\mu\nu}$ is not relevant for the viscosity. Thus, we may set them both to zero $\tau = \hat{F}_{\mu\nu} = 0$, in which case the flavor part of the action simply reduces to a "correction" for the dilaton potential of ihQCD. In other words, our action reduces back to (3.23) with an effective potential:

$$V_{eff}(\lambda) = V_g(\lambda) - x_f V_{f0}(\lambda)\,. \tag{4.41}$$

These two potentials take the following form

$$\begin{aligned} V_g(\lambda) &= 12\left[1 + V_1\lambda + \frac{V_2\lambda^2}{1+\lambda/\lambda_0} + V_{IR}\, e^{-\lambda_0/\lambda}\,(\lambda/\lambda_0)^{4/3}\sqrt{\log(1+\lambda/\lambda_0)}\right], \\ V_{f0}(\lambda) &= W_0 + W_1\lambda + \frac{W_2\lambda^2}{1+\lambda/\lambda_0} + 12 W_{IR} e^{-\lambda_0/\lambda}\,(\lambda/\lambda_0)^2\,. \end{aligned} \tag{4.42}$$

It is important to stress that all the parameters that appear above are fixed. On the UV side, they are matched to the RG flow of QCD perturbation theory and on the IR side they are determined by comparing to lattice data. Below we present the parameter values that we use, they are potential set 7a in [20] which is an intermediate variant of the V-QCD equation of state:

$$V_1 = \frac{11}{27\pi^2}\,, \quad V_2 = \frac{4619}{46656\pi^4}\,, \quad W_1 = \frac{8+3W_0}{9\pi^2}\,, \quad W_2 = \frac{6488+999W_0}{15552\pi^4}$$
$$\lambda_0 = 8\pi^2/3\,, \quad V_{IR} = 2.05\,, \quad W_0 = 2.5\,, \quad W_{IR} = 0.9\,, \quad \Lambda_{UV} = 211\text{MeV}. \tag{4.43}$$

Fitting to the data from Bayesian analysis
Now that we have explained our setup we wish to fit the prediction of this theory to the Bayesian data for η/s. The procedure we follow is the same: We relate the horizon value of the dilaton (or equivalently λ_h) to the temperature using the thermodynamics of our potential. In Figure 9 we plot this function which was determined by numerically solving for the background functions. Note that the critical temperature denoted in the plot is $T_c = 120$MeV which is different form the QCD crossover temperature of 154 MeV used in (4.2).

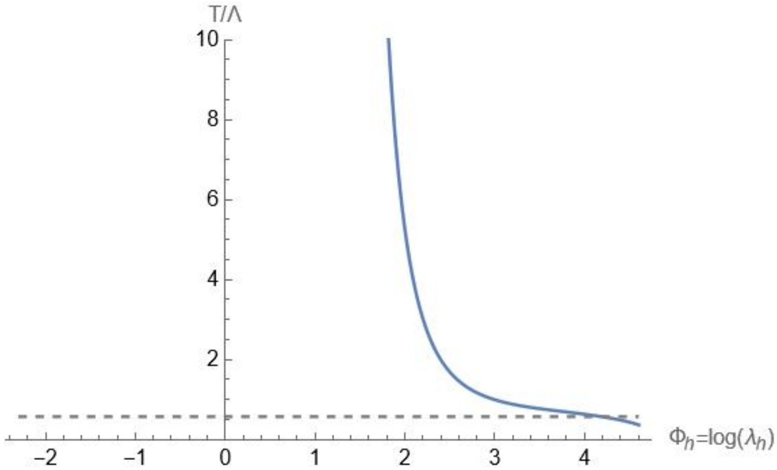

Figure 9: Temperature as a function of the dilaton horizon value in VQCD. The dotted line is T_c/Λ.

Our next order of business is determining the optimal function for the coupling $G(\Phi_h)$. We start with the simplest possible case of $G = 1$. In that case the theoretical prediction for the shear viscosity is:

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \frac{2}{3}\beta V(\Phi_h)\right]. \tag{4.44}$$

were we set $\ell = 1$. Since $V(\Phi_h)$ is completely fixed we only have β as a free parameter. In Figure 10 we plot the shear viscosity for this setup. As can be seen, $G(\Phi) = 1$ does not provide a good fit. Since the potential is fixed our parameter β simply sets the scale of the correction and does not change the shape of the potential. This indicates that a simple curvature correction term with constant coefficients is not adequate to explain the data and we require a non-trivial dilaton coupling to the higher order term.
Since the fit for the simplest case was not very successful we move on to the general case by solving the differential equation for $G(\Phi_h)$ (4.16) numerically to determine the optimal coupling function. We take $\beta = 0.1$ without loss of generality. In Figure 11 we present the best fit determined for $G(\Phi_h)$ and in in Figure 12 we plot the shear viscosity for this setup. As a reminder, the theoretical curve for η/s is given by

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 + \frac{2}{3}\beta\left(-G(\Phi_h)V(\Phi_h) + \frac{3}{4}G'(\Phi_h)V'(\Phi_h)\right)\right]. \tag{4.45}$$

A few remarks about this fit:

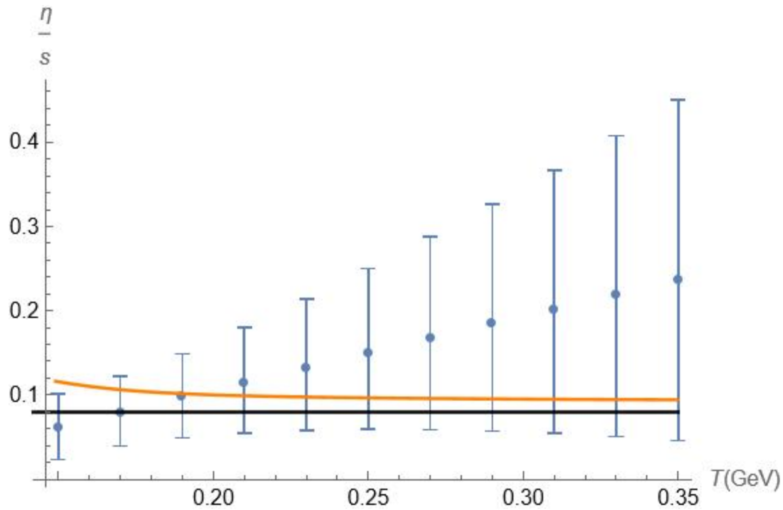

Figure 10: Best fit for the shear viscosity to entropy ratio for VQCD with $G(\Phi) = 1$. The yellow line shows the theoretical curve while the black line is $\eta/s = 1/4\pi$ for reference. The optimal value for the parameter was determined to be $\{\beta = -0.264\}$.

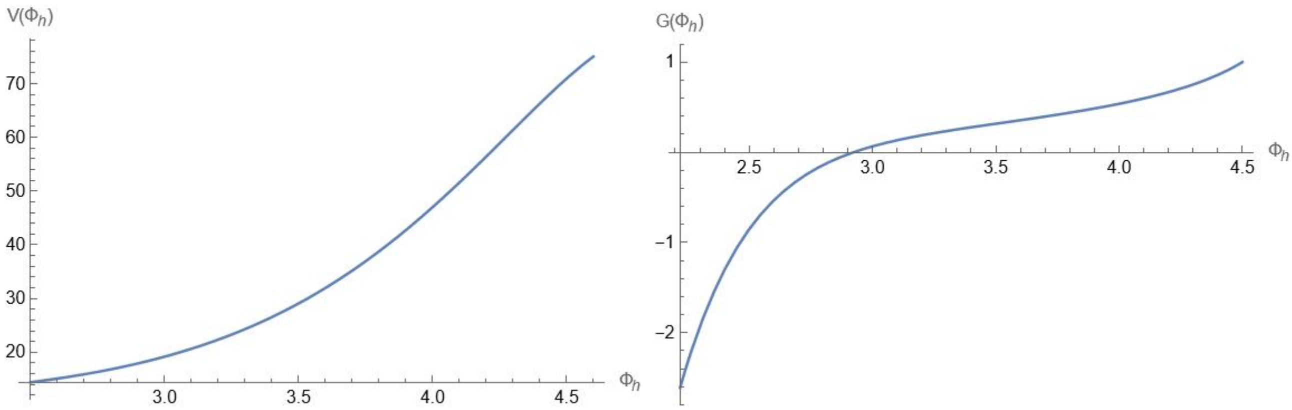

Figure 11: Dilaton potential and dilaton coupling as functions of the dilaton horizon value.

- The coupling function is in the range: $-0.26 < \beta G(\Phi_h) < 0.1$ meaning that this remains a perturbative correction.
- As mentioned before, our choice of initial conditions for $G(\Phi_h)$ does not matter for η/s since the homogeneous solution drops out. We choose $G(\Phi_h = 4.5) = 1$, this may matter for other fits.
- We see that G crosses zero at around $T \sim 0.17$ GeV where $\eta/s \sim 1/4\pi$ so the dilaton decouples at around this universal value.

5 Curvature squared corrections: The full action

One might wonder why we have not included all possible couplings to R^2 corrections in (3.23). The most general action we could consider at this order would be:

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[R - 2(\nabla\Phi)^2 + V(\Phi) + \ell^2\beta\left(G_1(\Phi)\,R^2 + G_2(\Phi)\,R_{\mu\nu}R^{\mu\nu} + G_3(\Phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]. \tag{5.1}$$

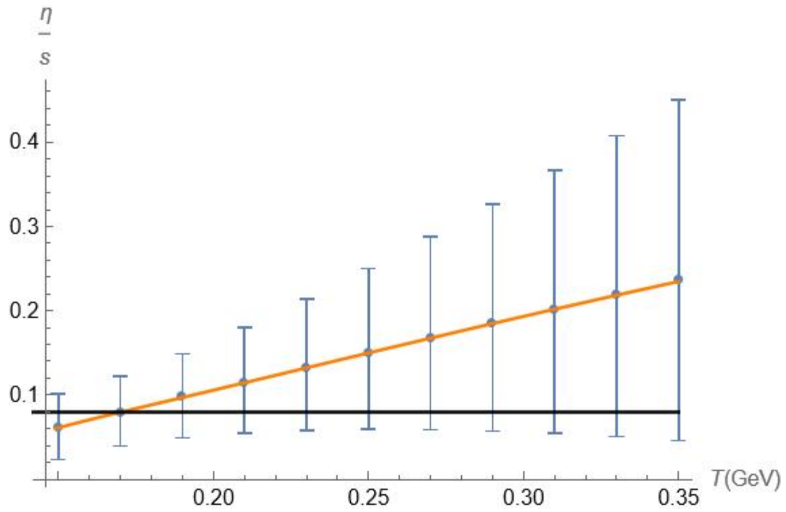

Figure 12: Best fit for the shear viscosity to entropy ratio for VQCD with $G(\Phi)$ shown in Figure (11). The yellow line shows the theoretical curve while the black line is $\eta/s = 1/4\pi$ for reference. We set $\beta = 0.1$ without loss of generality.

It turns out that the G_1,G_2 terms will not contribute to η/s. Even though they will effect the shear viscosity and entropy individually, taking the fraction precisely cancels the extra contributions and we get the same result as before. For constant coefficients $G_1 = \lambda_1$, $G_2 = \lambda_2$ these two terms can be absorbed into a field redefinition for the metric. This explains the irrelevance of the terms as η/s is a universal result independent of coordinates. The question is why does this also happen in the coupled case?

This is because what we are actually interested in is the horizon behaviour of the couplings since our entire calculation for both η and s is evaluated at the horizon. As we will argue later the extra terms at the horizon effectively act as constants thus not changing the result. We will start first with the constant coupling case, and build up our argument afterwards.

5.1 Warm up: Shear viscosity for constant coefficients

We examine the following action [18]

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[R - 2(\nabla\Phi)^2 + V(\Phi) + \ell^2\beta\left(\lambda_1 R^2 + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]. \tag{5.2}$$

Following the same process as Section 1, using the formula:

$$\eta = \frac{1}{8\pi G_5}\left[\sqrt{-\frac{g_{uu}}{g_{tt}}}\left(A - B + \frac{F'}{2}\right) + \left(E\left(\sqrt{-\frac{g_{uu}}{g_{tt}}}\right)'\right)'\right]\Bigg|_{u=u_h}, \tag{5.3}$$

we find:

$$\eta = \frac{b_0^{3/2}}{16\pi G}\left[1 - \ell^2\beta\left(\frac{6a_0c_0 + 3a_1c_0 - a_0c_1}{a_0c_0^2}\lambda_1 + \frac{3\left(a_0 + a_1\right)c_0 - a_0c_1}{2a_0c_0^2}\lambda_2 + \frac{a_0c_0 + 3a_1c_0 - a_0c_1}{a_0c_0^2}\lambda_3\right)\right]. \tag{5.4}$$

The Lagrangian in this case is

$$\mathcal{L} = \frac{1}{16\pi G_5}\left[R - 2(\nabla\Phi)^2 + V(\Phi) + \ell^2\beta\left(\lambda_1 R^2 + \lambda_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right)\right]. \tag{5.5}$$

so the entropy density using Wald's formula (3.57) is:

$$s = 2\pi\sqrt{h}\left[1 + 4\lambda_1 R - 2\lambda_2 R^{bd}\epsilon^c_{\ b}\epsilon_{cd} - 2\lambda_3 R^{abcd}\epsilon_{ab}\epsilon_{cd}\right]\Big|_{u=u_h}. \tag{5.6}$$

Evaluating this as before and expanding around the horizon yeilds:

$$s = \frac{b_0^{3/2}}{4G}\left[1 - \ell^2\beta\left(\frac{6a_0c_0 + 3a_1c_0 - a_0c_1}{a_0c_0^2}\lambda_1 + \frac{3\left(a_0 + a_1\right)c_0 - a_0c_1}{2a_0c_0^2}\lambda_2 + \frac{3a_1c_0 - a_0c_1}{a_0c_0^2}\lambda_3\right)\right]. \tag{5.7}$$

Dividing the two expressions and keeping terms up to order β we find:

$$\frac{\eta}{s} = \frac{1}{4\pi}\left[1 - \frac{\beta\ell^2}{c_0}\lambda_3\right] = \frac{1}{4\pi}\left[1 - 2f_1A_1\beta\ell^2\lambda_3\right] = \frac{1}{4\pi}\left[1 - \frac{2}{3}\beta\ell^2\lambda_3 V\left(\phi_h\right)\right]. \tag{5.8}$$

So it turns out that the final result only depends on λ_3. This can be explained as in our original Lagrangian we can absorb the λ_2 and λ_1 terms into a field redefinition for the metric.

$$g_{\mu\nu} \to g_{\mu\nu} + ag_{\mu\nu}R + bR_{\mu\nu}. \tag{5.9}$$

where a and b are $\mathcal{O}\left(\beta\right)$. To find how the inverse of the metric transforms we take the ansatz:

$$g^{\mu\nu} \to g^{\mu\nu} + a'g^{\mu\nu}R + b'R^{\mu\nu}. \tag{5.10}$$

Then we use the fact that $g_{\mu\nu}g^{\nu\rho} = \delta^\rho_\mu$. Expanding this we find:

$$\left(g_{\mu\nu} + ag_{\mu\nu}R + bR_{\mu\nu}\right)\left(g^{\nu\rho} + a'g^{\nu\rho}R + b'R^{\nu\rho}\right) = \delta^\rho_\mu. \tag{5.11}$$

Performing the multiplication and discarding higher order terms yields:

$$\delta^\rho_\mu = \delta^\rho_\mu + \left(a + a'\right)R\delta^\rho_\mu + \left(b + b'\right)R^\rho_\mu. \tag{5.12}$$

Thus since the Kronecker delta maps to itself in the field redefinition we have $a = -a'$ and $b = -b'$ and we find that the inverse of the metric transforms as follows:

$$g^{\mu\nu} \to g^{\mu\nu} - ag^{\mu\nu}R - bR^{\mu\nu}. \tag{5.13}$$

The next order of business is to to calculate how the determinant of the metric transforms under this field redefinition. We make use of the formula:

$$ln\left(detM\right) = Tr\left(lnM\right). \tag{5.14}$$

to write (in matrix notation):

$$\sqrt{-g} = e^{log\left(\sqrt{-g}\right)} = e^{\frac{1}{2}log\left(-g-agR-b\mathcal{R}\right)} = \sqrt{-g}\left(e^{\frac{1}{2}log\left(1+aR+bg^{-1}\mathcal{R}\right)}\right) = \sqrt{-g}\left(e^{\frac{1}{2}Trlog\left(1+aR+bg^{-1}\mathcal{R}\right)}\right). \tag{5.15}$$

Expanding the logarithm and exponential up to first order in a, b gives:

$$\sqrt{-g} \to \sqrt{-g}\left(1 + \frac{1}{2}Tr\left(aR + bg^{-1}\mathcal{R}\right)\right). \tag{5.16}$$

or in index notation:

$$\sqrt{-g} \to \sqrt{-g}\left(1 + \frac{5a+b}{2}R\right), \tag{5.17}$$

where the 5 comes from tracing over an implied Kronecker delta in the first term. The higher derivative terms in the action are already of order β therefore we don't need to calculate how they transform and we keep only the zeroth order term. Finally, the only term that's left to calculate is the Ricci scalar. We find:

$$R \to R - aR^2 - bR_{\mu\nu}R^{\mu\nu} \tag{5.18}$$

This brings the action to the following form [18] (we absorb $\ell^2\beta$ into λ for simplicity) :

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[\frac{R}{2\kappa} - 2(\nabla\Phi)^2 + V(\Phi) + \tilde{\lambda}_1 R^2 + \tilde{\lambda}_2 R_{\mu\nu}R^{\mu\nu} + \lambda_3 R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right] \tag{5.19}$$

where

$$\frac{1}{\kappa} = 2 + (5a+b)\left(V(\Phi) - 2(\nabla\Phi)^2\right), \quad \tilde{\lambda}_1 = \lambda_1 + \frac{3a+b}{2}, \quad \tilde{\lambda}_2 = \lambda_2 - b. \tag{5.20}$$

We can pick $b = \lambda_2$ and $a = \frac{-2\lambda_1-\lambda_2}{3}$ such that we set $\tilde{\lambda}_1 = \tilde{\lambda}_2 = 0$ and therefore eliminating these terms from the action. The re-scaling of the Ricci scalar will not produce a deviation from the universal result of $1/4\pi$ since the shear viscosity does not depend on κ. This is because in formula (3.34) the field redefinition simply amounts to a re-scaling of the A,B, F and E coefficients. If we define

$$A = A_0 + \beta A_1, \quad B = B_0 + \beta B_1, \tag{5.21}$$

then we have the following rescaling:

$$\begin{aligned} A_0 &\to A_0\left[1 + \frac{5a+b}{2}\left(V(\Phi) - 2(\nabla\Phi)^2\right)\right], \\ B_0 &\to B_0\left[1 + \frac{5a+b}{2}\left(V(\Phi) - 2(\nabla\Phi)^2\right)\right]. \\ A_1 &\to \tilde{A}_1(\lambda_3), \quad B_1 \to \tilde{B}_1(\lambda_3) \end{aligned} \tag{5.22}$$

The other two relevant coefficients, F, E also only depend on λ_3 as they come from the higher derivative corrections. If we write $\eta = \eta_0 + \beta\eta_1$, then shear viscosity will change to:

$$\begin{aligned} \eta_0 &\to \eta_0 + \frac{b_0^{3/2}}{16\pi G}\frac{5a+b}{2}\left(V(\Phi) - 2(\nabla\Phi)^2\right)\Big|_{u=u_h}, \\ \eta_1 &\to \tilde{\eta}_1(\lambda_3). \end{aligned} \tag{5.23}$$

If we write $s = s_0 + \beta s_1$, it's easy to see using Wald's formula that the entropy changes to:

$$s_0 \to s_0 + \frac{b_0^{3/2}}{4G}\frac{5a+b}{2}\left(V(\Phi) - 2(\nabla\Phi)^2\right)\Big|_{u=u_h}, \tag{5.24}$$

where again $\tilde{s}_1$ only depends on λ_3. Dividing the two expressions we have that:

$$\frac{\eta}{s} = \frac{\eta_0 + \beta\eta_1}{s_0 + \beta s_1} = \frac{\eta_0 + \beta\eta_1}{s_0}\left(1 - \beta\frac{s_1}{s_0}\right). \tag{5.25}$$

But importantly if we define $x = \frac{b_0^{3/2}}{4G}\frac{5a+b}{2}\left(V(\Phi) - 2(\nabla\Phi)^2\right)\Big|_{u=u_h}$ we have that:

$$\frac{\eta_0}{s_0} = \frac{\eta_0 + x/4\pi}{s_0 + x} = \frac{\eta_0 + x/4\pi}{s_0}\left(1 - \frac{x}{s_0}\right). \tag{5.26}$$

Now making use of the fact that $\eta_0 = s_0/4\pi$ and that x is order β we finally find:

$$\frac{\eta}{s}\Big|_0 \to \frac{\eta}{s}\Big|_0, \quad \frac{\eta}{s}\Big|_1 = \frac{\tilde{\eta}}{s}\Big|_1(\lambda_3). \tag{5.27}$$

So the entire dependence on λ_1 and λ_2 is shifted to κ multiplying R. However, η/s does not depend on κ since taking the ratio of the two cancels the factor in front. Therefore η/s will only depend on the Riemann squared term.

5.2 Shear viscosity with all terms coupled: why only $R^2_{\mu\nu\rho\sigma}$contributes

Now we turn to the question of the dilaton couplings. As was mentioned before, what we are interested in is the horizon behaviour of $G_1(\Phi)$ and $G_2(\Phi)$ since this is where η/s is evaluated. If we assume the couplings are regular at the horizon, we can expand them in the following way:

$$\begin{aligned} G_1(\Phi) &= G_1(\phi_h) + \phi_1 G_1'(\phi_h)(1-u) + \ldots \\ G_2(\Phi) &= G_2(\phi_h) + \phi_1 G_2'(\phi_h)(1-u) + \ldots \\ G_3(\Phi) &= G_3(\phi_h) + \phi_1 G_3'(\phi_h)(1-u) + \ldots \end{aligned} \tag{5.28}$$

So we see that the second term in the expansion will vanish in the exact $u \to 1$ limit. In that case $G = G(\Phi_h)$ and the coupling is effectively constant at the horizon. We can see immediately see from this that the entropy density will not be effected by $\lambda \to G(\Phi)$. In fact the only chance this has of effecting the result is if a derivative of G appears somewhere. In that case at the horizon $G_i'(\Phi) = -\phi_i G_i'(\phi_h)$ deviating from the case of λ.

Recall the formula for the shear viscosity (3.34)

$$\eta = \frac{1}{8\pi G_5}\left[\sqrt{-\frac{g_{uu}}{g_{tt}}}\left(A - B + \frac{F'}{2}\right) + \left(E\left(\sqrt{-\frac{g_{uu}}{g_{tt}}}\right)'\right)'\right]\Bigg|_{u=u_h}. \tag{5.29}$$

derivatives appear of the F, E functions so this is where there is a chance of contributing for G_1 and G_2 (and also where the G_3' contribution comes from). Thus the only chance to deviate from the constant result will come from these terms. The calculation of the effective action shows that G_1 does not contribute to F, E. G_2 contributes to both F, E but the total contribution after some calculations turns out to be

$$\eta \sim \frac{b_0^{3/2}\ell^2\beta\phi_1 G_2'(\phi_h)(u-1)\left(3-3u+u^2\right)^{3/2}}{32\pi G\left(c_0 + (u-1)\left(c_1 + c_2(u-1)\right)\right)}. \tag{5.30}$$

which at $u = 1$ is vanishing. This indicates that the metric redefinition is still valid even if the coefficients are not constant.

6 Temperature Dependence of the Bulk Viscosity

In this section we calculate the bulk viscosity for the higher derivative theory we are considering. In the first two subsections we work out the bulk viscosity to entropy ratio for some toy models and motivate the fit to the Bayesian data. After deriving the formula for ζ/s we present the zeroth order prediction of V-QCD for the bulk viscosity.

6.1 Bulk viscosity of a CR brane

The authors of [15] parameterized ζ/s as an unnormalized Cauchy distribution

$$\frac{\zeta}{s} = d\left[\frac{e^2}{(T-f)^2 + e^2}\right]. \tag{6.31}$$

where they determined:

$$d \equiv (\zeta/s)_{max} = 0.0060^{+0.0058}_{-0.0060}, \quad e \equiv (\zeta/s)_{width} = 0.10^{+0.15}_{-0.08}\,\text{GeV}, \quad f \equiv (\zeta/s)_{T_0} = 0.202^{+0.047}_{-0.039}\,\text{GeV}. \tag{6.32}$$

This is plotted along with a 90% confidence band in Figure 13. As can be seen, the value of ζ/s is very small and even consistent with zero hinting at a nearly conformal plasma. We note also that the results of Jetscape [17] show a viscosity around an order of magnitude larger, however both results have big uncertainties and are only in statistical disagreement for low temperatures.

The most convenient way to calculate the bulk viscosity to entropy ratio is by use of the

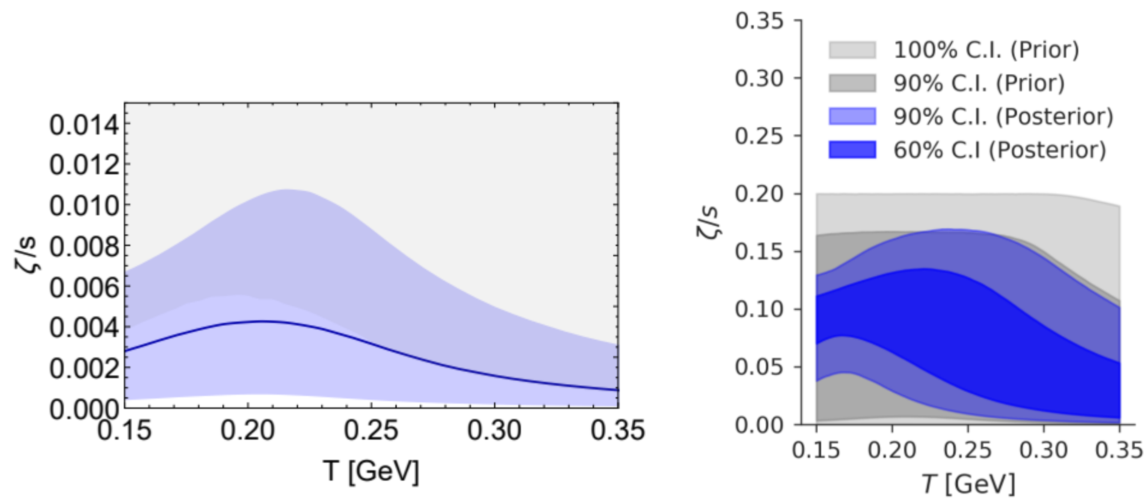

Figure 13: Posterior distribution for the bulk viscosity to entropy ratio versus temperature with a 90% confidence band in blue. The left result is that of Trajectum [15] and the right is that of Jetscape[17].

Eling-Oz formula [22, 23]:

$$\frac{\zeta}{s} = \frac{1}{4\pi}\left(s\frac{\partial\Phi_h}{\partial s}\right)^2 . \tag{6.33}$$

which relates ζ/s to derivatives of the dilaton horizon value with respect to entropy. This formula was derived with a different normalization for the dilaton kinetic term: $-\frac{1}{2}(\nabla\Phi)^2$. To normalize it correctly to our case we can re-scale the dilaton as:

$$\Phi \to 2\Phi\,, \quad \Longrightarrow -\frac{1}{2}(\nabla\Phi)^2 \to -2(\nabla\Phi)^2. \tag{6.34}$$

So this will change the formula in our case to:

$$\frac{\zeta}{s} = \frac{1}{\pi}\left(s\frac{\partial\Phi_h}{\partial s}\right)^2 . \tag{6.35}$$

For the moment we have set $\beta = 0$ and we will just look at the simple case of the CR brane analyzed in Section 2.1. We determined

$$S = S_0 e^{\frac{3\zeta}{4X_0}\Phi_h}, \tag{6.36}$$

where the constant $\zeta = \sqrt{8/3}$ in the exponent is for correct normalization, not to be confused with bulk viscosity. Thus we find:

$$\frac{\partial\Phi_h}{\partial s} = \frac{2X_0}{\sqrt{6}}\frac{1}{s}. \tag{6.37}$$

So in this toy example we find a constant ζ/s:

$$\frac{\zeta}{s} = \frac{2X_0^2}{3\pi} = \frac{\alpha^2}{16\pi}. \tag{6.38}$$

The thermal stability for the CR brane requires $X_0^2 < 1/4$ so this sets an upper bound on the bulk viscosity:

$$\frac{\zeta}{s} < \frac{1}{6\pi} \simeq 0.053. \tag{6.39}$$

Comparing this to Figure 13 we see that the mean value peaks at just below that value. It seems like in this simple case with $\alpha = \sqrt{8/3}$ which is just the threshold for confinement we have tension with the data as temperature increases. This is similar to the shear viscosity where as temperature increases the tendency for η/s is to increase away from $1/4\pi$. This also agrees with the result of [28].

6.2 Bulk viscosity in the adiabatic approximation

Using the formula for the bulk viscosity, we would like in the case of V-QCD to relate it to the dilaton potential, we have:

$$\frac{\zeta}{s} = \frac{1}{4\pi}\left(s\frac{\partial \Phi_h}{\partial s}\right)^2. \tag{6.40}$$

This formula was derived with a the following normalization for the dilaton kinetic term: $-\frac{1}{2}(\nabla\Phi)^2$ to normalize it correctly to the case of V-QCD we can rescale the dilaton as:

$$\Phi \to \sqrt{\frac{8}{3}}\Phi\,, \quad \Longrightarrow -\frac{1}{2}(\nabla\Phi)^2 \to -\frac{4}{3}(\nabla\Phi)^2. \tag{6.41}$$

So this will change the formula in our case to:

$$\frac{\zeta}{s} = \frac{2}{3\pi}\left(s\frac{\partial \Phi_h}{\partial s}\right)^2. \tag{6.42}$$

Now making use of the Bekenstein-Hawking entropy formula $S = \frac{1}{4G_5}e^{3A(\Phi_h)}$ we can write:*

$$\frac{\partial \Phi_h}{\partial s} = \frac{\partial \Phi_h}{\partial A}\frac{\partial A}{\partial s} = \frac{\partial \Phi_h}{\partial A}\frac{1}{3s}. \tag{6.43}$$

Thus we find

$$\frac{\zeta}{s} = \frac{2}{27\pi}\left(\frac{\partial \Phi_h}{\partial A}\right)^2 = \frac{2}{27\pi}\left(\frac{\partial \Phi(r)}{\partial A}\Big|_{r=r_h}\right)^2. \tag{6.44}$$

Now we will make use of the phase variables formalism to write this in terms of the potential, recall that from equation (3.78) we have:

$$\begin{aligned} \frac{dA}{dr} &= A_1 = -\frac{C}{\ell}\frac{S^{\frac{1}{3}}}{T}V(\Phi_h)\,, \\ \frac{d\Phi}{dr} &= \Phi_1 = \frac{2C}{\zeta^2\ell}\frac{S^{\frac{1}{3}}}{T}V'(\Phi_h)\,, \end{aligned} \tag{6.45}$$

with ζ depending on the kinetic term normalization, for the standard ($-4/3$ we have $\zeta = 4/3$). Thus we can write

$$\frac{\partial \Phi(r)}{\partial A}\Big|_{r=r_h} = \frac{\partial \Phi(r)}{\partial r}\frac{\partial r}{\partial A}\Big|_{r=r_h} = \frac{\Phi_1}{A_1} = -\frac{9}{8}\frac{V'(\Phi_h)}{V(\Phi_h)}. \tag{6.46}$$

*This chain rule is not strictly correct, this is an approxiamtion.

Finally we find that the bulk viscosity to entropy ratio is given by [28]:

$$\frac{\zeta}{s} = \frac{3}{32\pi} \left(\frac{V'(\Phi_h)}{V(\Phi_h)} \right)^2 \tag{6.47}$$

Using this formula we plot ζ/s for the V-QCD potential, in comparison with the Trajectum results in Figure 14. As can be seen the agreement is not very good especially in the low temperature range.

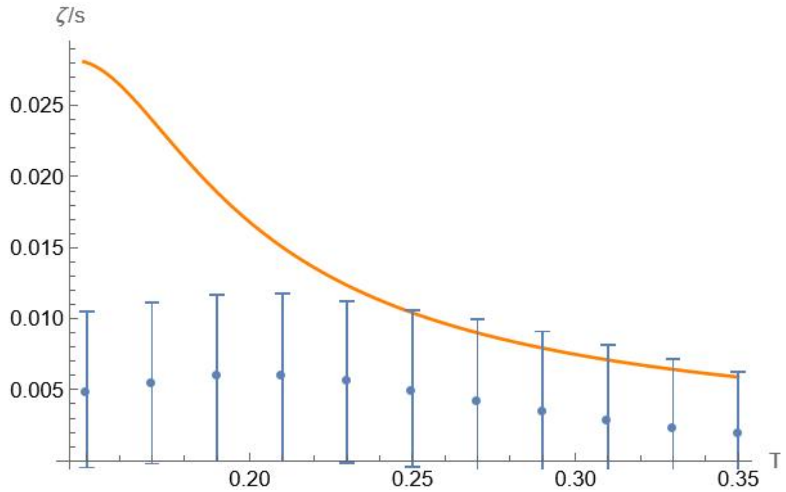

Figure 14: Bulk viscosity of VQCD without higher derivative corrections in the adiabatic approximation.

Adiabatic approximation

The adiabatic approximation assumes that the potential is slowly varying or in other words $\frac{V'}{V} \sim$ constant. We checked this explicitly for V, G in the region we are interested in for the Baysian data. This is plotted in Figure 15.

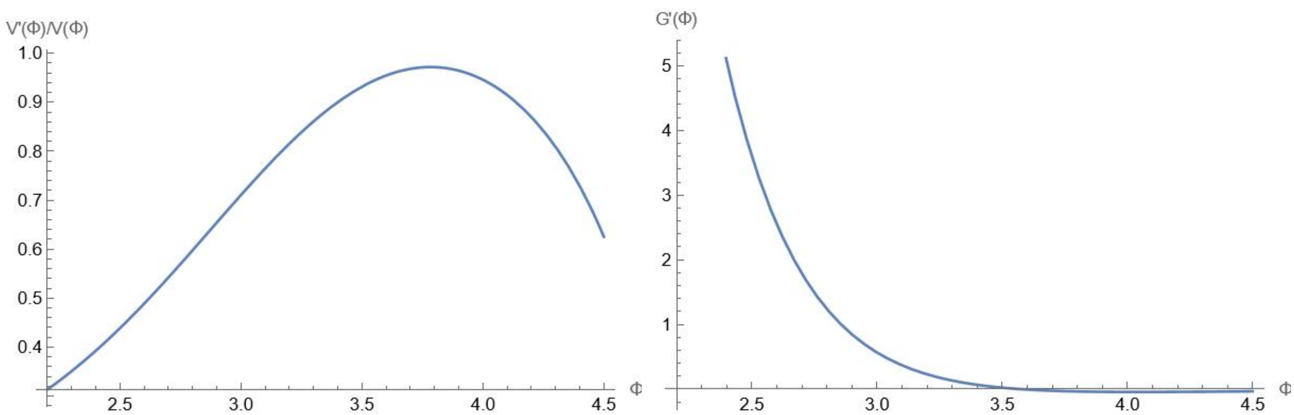

Figure 15: An explicit check of the adiabatic approximation for the dilaton potentials. For V we used the fixed V-QCD potential while for G we used an analytic function given in Section 4.4.

While this is not a terrible approximation if one seeks an order of magnitude calculation, it will certainly have a noticeable effect on the result. So to make contact with the data we would need a more careful treatment of the fluctuations.

6.3 Bulk viscosity with higher derivative corrections

In this subsection we employ a more careful treatment of the problem and derive an analytic formula for the bulk viscosity to entropy ratio for the theory we are considering:

$$S = \frac{1}{16\pi G_5}\int d^5x\sqrt{-g}\left[R - \frac{4}{3}(\nabla\Phi)^2 + V(\Phi) + \ell^2\beta G(\Phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}\right]. \tag{6.48}$$

We follow the analysis of [29],[30], extracting the bulk viscosity from a radially conserved current. We extend the results of [30] to a class of theories with non-minimal dilaton coupling. As we have argued in previous sections, this setup allows us to capture the temperature flow of the QGP transport coefficients as we move away from the critical point. We start first by presenting the formula and comparing it to the one derived in [30] and subsequently move on to present the main steps of the calculation. Starting with (6.48) we find the following:

$$\frac{\zeta}{s} = \frac{8z_0^2}{27\pi}\left[1 + \frac{1}{2}\ell^2\beta\left(-\frac{4}{3}GV + \frac{G(V')^2}{V} + G'V'\right)\right]. \tag{6.49}$$

Where the entire expression is evaluated at the horizon. z_0 is the gauge invariant fluctuation of the scalar field Φ at lowest order in the hydrodynamic expansion [30]. Shifting $V \to -V + 12$ and $\Phi \to \sqrt{\frac{3}{8}}\Phi$ to match the conventions of [30], we find:

$$\frac{\zeta}{s} = \frac{z_0^2}{9\pi}\left[1 + \frac{2}{3}\ell^2\beta\left(G\left(V-12\right) - \frac{2G(V')^2}{V-12} - 2G'V'\right)\right]. \tag{6.50}$$

This formula reduces to the result of [30] for $a_1 = 0,\ a_2 = 0,\ G = a_3$.

Deriving the bulk viscosity to entropy ratio

We start from a generic metric ansatz:

$$ds^2 = -c_1\left(r\right)^2 dt^2 + c_2^2\left(r\right)d\mathbf{x}^2 + c_3^2\left(r\right)dr^2, \quad \Phi = \Phi(r). \tag{6.51}$$

Varying the action with respect to the metric, one obtains the background equations of motion, see the next chapter for more details. Substituting this ansatz in the equations of motion we obtain the following set of background equations:

$$\begin{aligned}
&\Phi'' + \frac{c_1'\Phi'}{c_1} + \frac{3c_2'\Phi'}{c_2} - \frac{c_3'\Phi'}{c_3} + \frac{3}{8}c_3^2V' = 0\,,\\
&\left(\Phi'\right)^2 + \frac{3}{4}c_3^2V - \frac{9c_1'c_2'}{2c_1c_2} - \frac{9(c_2')^2}{2c_2^2} = 0\,,\\
&c_2'' + \frac{\left(c_2'\right)^2}{c_2} - \frac{c_2'c_3'}{c_3} + \frac{2}{9}c_2\left(\Phi'\right)^2 - \frac{1}{6}c_2c_3^2V = 0\,,\\
&c_1'' + \frac{2c_1'c_2'}{c_2} - \frac{c_1'c_3'}{c_3} + \frac{c_1(c_2')^2}{c_2^2} - \frac{1}{6}c_1c_3^2V + \frac{2}{9}c_1\left(\Phi'\right)^2 = 0.
\end{aligned} \tag{6.52}$$

We present the result for the zeroth order equation as the β components are too cumbersome to present here. To compute the relevant retarded Green's function we expand the metric and dilaton, considering SO(3) invariant perturbations:*

$$\begin{aligned}
&c_1\left(r,t\right) \to c_1\left(r\right)\left(1 + \lambda\frac{h_{tt}\left(r,t\right)}{c_1^2\left(r\right)}\right)^{1/2},\\
&c_2\left(r,t\right) \to c_2\left(r\right)\left(1 + \lambda\frac{h_{11}\left(r,t\right)}{c_2^2\left(r\right)}\right)^{1/2},\\
&c_3\left(r,t\right) \to c_3\left(r\right),\\
&\Phi\left(r,t\right) \to \Phi\left(r\right) + \lambda\Psi\left(r,t\right).
\end{aligned} \tag{6.53}$$

*Our definition for h_{tt} differs from [30] by a minus sign

With this choice we fix the axial gauge by taking $h_{rr} = h_{rt} = 0$. We introduce new variables H_{11}, H_{tt}, assuming harmonic time dependence:

$$h_{tt}(r,t) = e^{-i\omega t} c_1^2 H_{00}(r) \,, \quad h_{11}(r,t) = e^{-i\omega t} c_2^2 H_{11}(r) \,. \tag{6.54}$$

and the gauge invariant scalar fluctuations Z are defined as:

$$\Psi(r,t) = e^{-i\omega t}\left(Z(r) + \frac{\Phi'(r)\, c_2(r)}{2c_2'(r)} H_{11}(r)\right) \tag{6.55}$$

Applying these expansions to the equations of motion and keeping terms up to $\mathcal{O}(\lambda)$, we derive the equations of motion for the fluctuations:

$$H' + \frac{8c_2c_3\Phi'}{9c_2'} Z + \beta\{...\} = 0\,, \quad H_{11} = \frac{c_2'}{c_2c_3} H, \tag{6.56}$$

$$\begin{aligned} &H_{00}' - \frac{8c_2\Phi'}{9c_2'} Z' + \frac{c_2}{27c_1(c_2')^2}\left(24c_1'c_2'\Phi' - c_1c_3^2\left(9c_2'V' + 8c_2V\Phi'\right)\right) Z \\ &+ \left(\frac{\omega^2 c_3}{c_1^2} - \frac{1}{3}c_3 V + \frac{(c_1')^2}{c_3c_1^2} + \frac{3c_1'c_2'}{c_1c_2c_3}\right) H + \beta\{...\} = 0, \end{aligned} \tag{6.57}$$

coming from the 12 and 11 components of Einsteins equations respectively, where we made use of the background equations of motion to simplify the above expressions. As before, we only provide the zeroth order results as the first order expressions are too cumbersome to present here. Remarkably, the equation for Z decouples from the other fluctuations once the constraints above are used in tandem with the background equations. This yields the following equation:

$$Z'' + \left(\frac{c_1'}{c_1} + \frac{3c_2'}{c_2} - \frac{c_3'}{c_3}\right) Z' + c_3^2\left(\frac{\omega^2}{c_1^2} + \frac{4}{3}V - \frac{2c_2^2c_3^2V^2}{9(c_2')^2} + \frac{4c_2Vc_1'}{3c_1c_2'} + \frac{2c_2V'\Phi'}{3c_2'} + \frac{3}{8}V''\right) Z + \beta\{...\} = 0. \tag{6.58}$$

Where also the β terms are only functions of Z. Following the same expansion we derive the effective action, following the prescription of [30]:

$$S_c = \frac{1}{16\pi G_5}\int dr \mathcal{L}_c\{h_{11,\omega}, h_{00,\omega}, p_\omega, h_{11,\omega}^*, h_{00,\omega}^*, p_\omega^*\}, \tag{6.59}$$

where we define

$$h_{tt}(t,r) = e^{-i\omega t} h_{00,\omega}(r)\,, \quad h_{11}(t,r) = e^{-i\omega t} h_{11,\omega}(r)\,, \quad \psi(t,r) = e^{-i\omega t} p_\omega(r)\,. \tag{6.60}$$

Since the expression is very lengthy and does not add to the understanding of the derivation, we do not present it here, we refer the reader to the citation for more details on the derivation. In the following we suppress the ω indices and instead use "$*$" to refer to conjugate fields. We calculate the conserved current by integrating the effective action of the fluctuation by parts as in (3.38), thus the conserved current is given by:

$$\begin{aligned} J_\omega = &\frac{\partial\mathcal{L}}{\partial h_{11}^{*\prime}}\delta h_{11}^* + \frac{\partial\mathcal{L}}{\partial h_{11}^{*\prime\prime}}\delta h_{11}^{*\prime} - \left(\frac{\partial\mathcal{L}}{\partial h_{11}^{*\prime\prime}}\right)'\delta h_{11}^* \\ &+ \frac{\partial\mathcal{L}}{\partial h_{00}^{*\prime}}\delta h_{00}^* + \frac{\partial\mathcal{L}}{\partial h_{00}^{*\prime\prime}}\delta h_{00}^{*\prime} - \left(\frac{\partial\mathcal{L}}{\partial h_{00}^{*\prime\prime}}\right)'\delta h_{00}^* \\ &+ \frac{\partial\mathcal{L}}{\partial p^{*\prime}}\delta p^* + \frac{\partial\mathcal{L}}{\partial p^{*\prime\prime}}\delta p^{*\prime} - \left(\frac{\partial\mathcal{L}}{\partial p^{*\prime\prime}}\right)'\delta p^*. \end{aligned} \tag{6.61}$$

After some lengthy but straightforward calculations, we find:

$$
\begin{aligned}
J_\omega =& \frac{1}{12c_1^4 c_2^2 c_3}\Bigg[9c_2^5 h_{00}^* h_{00} c_1' - 9c_1^2 c_2^3 h_{00}^* h_{11} c_1' - 3c_1 c_2^5 (h_{00} h_{00}^*{}' + h_{00}^* h_{00}') + \\
& c_1^5 \left(9c_2 h_{11} h_{11}^*{}' + 9h_{11}^* \left(h_{11} c_2' - c_2 h_{11}'\right) + 16 c_2^5 p^* p' + 24 c_2^3 p^* h_{11} \Phi'\right) + \\
& c_1^3 c_2^2 \left(-9h_{11}^* h_{00} c_2' + c_2 \left(9h_{00} h_{11}^*{}' + 9h_{11} h_{00}^*{}' + 8c_2^2 p^* h_{00} \Phi'\right)\right)\Bigg] + \beta\{...\}
\end{aligned}
\tag{6.62}
$$

We specifically need the current in the hydrodynamic approximation thus we expand the fluctuations as follows:

$$
Z = \left(\frac{c_1}{c_2}\right)^{-i\mathfrak{w}} (z_0 + i\mathfrak{w} z_1), \quad H = H_0 + i\mathfrak{w} H_1, \quad H_{00} = H_{00,0} + i\mathfrak{w} H_{00,1}, \tag{6.63}
$$

where $\mathfrak{w} = \frac{\omega}{2\pi T}$. We discuss boundary conditions later when we come to the solution of the z equation. We note that the Noether current associated with the $U(1)$ symmetry of $\mathcal{L}_c$ is actually given by:

$$
J_N = -i\left(J_\omega - J_{-\omega}\right) = 2\mathrm{Im} J_\omega. \tag{6.64}
$$

where by $J_{-\omega}$ we refer to the conserved current coming from the integration by parts of the $\{h_{11}, h_{00}, p\}$ action. The last equality holds because $J_{-\omega} = (J_\omega)^*$. To extract the bulk viscosity using the Kubo relation, we will need only the order ω, imaginary part of the current J_1:

$$
J_\omega = J_0 - i\mathfrak{w} J_1. \tag{6.65}
$$

After making use of the equations of motion for H, H_{00} we find that the expression simplifies significantly and the conserved current only depends on z_0, z_1:

$$
\begin{aligned}
J_1 =& \frac{4c_2^2 \left(c_1 z_0^2 c_2' - c_2 \left(z_0^2 c_1' + c_1 z_1 z_0' - c_1 z_0 z_1'\right)\right)}{3c_3} \\
&+ \frac{16\ell^2\beta}{27 c_1^2 c_3^3 c_2'^2}\left(c_1 z_0^2 c_2' - c_2\left(z_0^2 c_1' + c_1 z_1 z_0' - c_1 z_0 z_1'\right)\right)\Big[108 c_2^2 G(\Phi) c_1'^2 c_2'^2 - \\
& 6c_1 c_2 c_1' c_2' \left(4c_2^2 c_3^2 G(\Phi) V(\Phi) - 21 G(\Phi) c_2'^2 + 6c_2 c_2' G'(\Phi)\Phi'\right) + \\
& c_1^2 \left(c_2^4 c_3^4 G(\Phi) V(\Phi)^2 - 9c_2^2 c_3^2 G(\Phi) V(\Phi) c_2'^2 + 18 G(\Phi) c_2'^4 - \right. \\
& \left. 18 c_2 c_2'^3 G'(\Phi)\Phi' + c_2^3 c_3^2 c_2' \left(5V(\Phi) G'(\Phi) + 2G(\Phi) V'(\Phi)\right)\Phi'\right)\Big].
\end{aligned}
\tag{6.66}
$$

A key observation in this calculation is that since the current is radially conserved one may evaluate it at any point in the holographic direction and in particular evaluate it at the black hole horizon. This simplifies the expression and gives ζ only in terms of horizon data. We expand around the horizon as in (3.33):

$$
\begin{aligned}
c_1^2(u) &= a_0(1-u) + a_1(1-u)^2 + a_2(1-u)^3 + \ldots \\
c_2^2(u) &= b_0(1 + (1-u) + \ldots) \\
c_3^2(u) &= c_0(1-u)^{-1} + c_1 + c_2(1-u) + \ldots \\
\Phi(u) &= \Phi_h + \Phi_1(1-u) + \Phi_2(1-u)^2 + \ldots
\end{aligned}
\tag{6.67}
$$

Regularity at the horizon gives the following constraints on the horizon functions:

$$
c_0 = \frac{3}{2V_h}, \quad \Phi_1 = -\frac{9V_h'}{16V_h}, \quad c_1 = \frac{15}{4V_h} + \frac{81\,(V_h')^2}{128 V_h^3}, \quad a_1 = a_0\left(\frac{1}{2} + \frac{9(V_h')^2}{64 V_h^2}\right), \tag{6.68}
$$

Expanding the current at the horizon and making use of the constraints, we find the following:

$$J_{1h} = \frac{2}{3}\sqrt{\frac{a_0}{c_0}} b_0^{3/2} z_0^2 \left(1 + \frac{\ell^2 \beta}{2} V' \left(G' + \frac{GV'}{V}\right)\right), \tag{6.69}$$

where the entire expression is evaluated at the horizon. The bulk viscosity is given by the Kubo relation:

$$\zeta = -\frac{4}{9} \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} G_R = -\frac{1}{18\pi G_5} \lim_{\omega \to 0} \frac{1}{\omega} \text{Im} J_\omega \tag{6.70}$$

where the factor in front comes from the normalization of the Green's function:

$$G_R(\omega) = -i \int dt d\mathbf{x} e^{i\omega t} \theta(t) \left\langle \left[\frac{1}{2} T_i^i(t, \mathbf{x}), \frac{1}{2} T_j^j(0, \mathbf{0})\right] \right\rangle, \tag{6.71}$$

since the fluctuations of the metric have a 1/2 factor in front. Thus we find for the bulk viscosity:

$$\zeta = \frac{\mathfrak{w}}{27\pi G_5} \sqrt{\frac{a_0}{c_0}} b_0^{3/2} z_0^2 \left(1 + \frac{\ell^2 \beta}{2} V' \left(G' + \frac{GV'}{V}\right)\right). \tag{6.72}$$

The temperature for this metric is given by (3.70): $T = \frac{1}{4\pi}\sqrt{\frac{a_0}{c_0}}$, which was obtained from the conical singularity. The entropy is given by Wald's formula (3.64):

$$s = \frac{b_0^{3/2}}{4G_5} \left[1 - \frac{3a_1 c_0 - a_0 c_1}{a_0 c_0^2} \ell^2 \beta G\right] = \frac{b_0^{3/2}}{4G_5} \left[1 + \frac{2\ell^2 \beta}{3} GV\right]. \tag{6.73}$$

Taking the ratio between the two expressions and keeping terms up to order β we arrive at eq. (6.49):

$$\frac{\zeta}{s} = \frac{8 z_0^2}{27\pi} \left[1 + \frac{1}{2}\ell^2 \beta \left(-\frac{4}{3} GV + \frac{G(V')^2}{V} + G'V'\right)\right]. \tag{6.74}$$

6.4 Numerical solution of the fluctuation and fitting the Bayesian data

Having presented the main steps in the derivation of (6.49) we now wish to make use of the formula to fit the data from Bayesian analysis in Figure 13. As a starting point, we will set β to zero and calculate the 0^{th} order result for the V-QCD potential. There are a few steps we take to simplify the numerical calculation starting from changing coordinates as follows:

$$c_1 \to \sqrt{f(r)} e^{A(r)}, \quad c_2 \to e^{A(r)}, \quad c_3 \to \frac{e^{A(r)}}{\sqrt{f(r)}}. \tag{6.75}$$

In addition, we introduce a new variable $q(A)$ to replace the scale factor $A(r)$ and change variables to the scale factor instead of radial coordinate:

$$q(A) = \frac{e^A(r)}{A'(r)} \tag{6.76}$$

The first step in computing the fluctuation is solving the background equations at zeroth order in β. We expand the functions in β using:

$$\begin{aligned} f &= f_0 + \beta f_1, \quad q = q_0 + \beta q_1, \\ \lambda &= \lambda_0 + \beta \lambda_1, \quad z_0 = z_{00} + \beta z_{01}. \end{aligned} \tag{6.77}$$

The three equations we solve are the following:

$$\begin{aligned}
&\frac{f_0''(A)}{\sqrt{f_0(A)}} + \frac{q_0(A)^2 V(\lambda_0) f_0'(A)}{3 f_0(A)^{3/2}} - \frac{f_0'(A)^2}{f_0(A)^{3/2}} = 0,\\
&q_0(A)\left(-\frac{f_0'(A)}{f_0(A)} - 4\right) + \frac{q_0(A)^3 V(\lambda_0)}{3 f_0(A)} + q_0' = 0,\\
&\frac{1}{2}\sqrt{\frac{9 f_0'(A) + 36 f_0(A) - 3 q_0(A)^2 V(\lambda_0)}{f_0(A)}} + \frac{\lambda_0'(A)}{\lambda_0(A)} = 0.
\end{aligned} \tag{6.78}$$

They are solved subject to the following horizon boundary conditions:

$$\begin{aligned}
f_0 &= f_{00}(A - A_h) + f_{01}(A - A_h)^2 + ...\\
\lambda_0 &= \lambda_{00} + \lambda_{01}(A - A_h) + ..\\
q_0 &= q_{00} + q_{01}(A - A_h) + ...
\end{aligned} \tag{6.79}$$

Regularity at the horizon in these coordinates implies:

$$q_{00}^2 = \frac{3 f_{00}}{V_h}. \tag{6.80}$$

Notice that in this form equations (6.78) are invariant under a resealing of the type:

$$f_0 \to \alpha f_0\,, \quad q_0 \to \sqrt{\alpha} q_0. \tag{6.81}$$

We can use this fact to our advantage in the numerical solution as we want $f_0 \to 1$ at the UV boundary such that the Hawking temperature is correctly defined. Thus the procedure we follow is:

- Solve the system of equations numerically assuming the horizon behaviour (6.79) where we take $f_{00} = 1$ for simplicity.
- Rescale $f \to f/f_{UV}\,, q \to q/\sqrt{f_{UV}}$ using (6.81) so that $f_{0,UV} \to 1$.

In this way the solution is correctly normalized. The next step is solving the z_0 equation numerically, in these coordinates this equation reads:

$$\begin{aligned}
&z_{00}'' + z_{00}'\left(\frac{f_0'}{f_0} - \frac{q_0'}{q_0} + 4\right) +\\
&z_{00}\frac{q_0^2\left(3 f_0\left(64 V(\lambda_0) + (16\lambda_0' + 9\lambda_0) V'(\lambda_0) + 9\lambda_0^2 V''(\lambda_0) - 16 V(\lambda_0)\left(q_0^2 V(\lambda_0) - 3 f_0'\right)\right)\right.}{72 f_0^2} = 0.
\end{aligned} \tag{6.82}$$

We assume that z_{00} is regular at the horizon admitting an expansion of the form:

$$z_{00} = c_0 + c_1(A - A_h)\,. \tag{6.83}$$

Plugging this into the z_{00} equation along with the horizon expansion (6.79), we relate c_0 to c_1 by demanding that the pole at the horizon vanishes such that the z_{00} equation is well defined. We find:

$$c_1 = c_0\frac{9}{8}\left(\frac{V'(\Phi_h)^2}{V(\Phi_h)^2} - \frac{V''(\Phi_h)}{8V(\Phi_h)}\right) \tag{6.84}$$

In addition we must require that the dilaton perturbation Ψ is not sourced, since only the metric should be sourced to compute the bulk viscosity. This sets the leading behavior of z_{00} towards the boundary to be:

$$z_{00}\Big|_{UV} \sim 1/2A \tag{6.85}$$

Using these two conditions we solve the fluctuation equation using the following procedure:

- We solve the equation making use of the background solution along with (6.84), where we take $c_0 = 1$.
- We rescale the solution appropriately such that z_{00} satisfies (6.85). Notice that (6.82) is invariant under the rescaling of z_{00}.

In this way we correctly normalize our solution for z_{00} to match the boundary conditions. Following this process we solve the background and fluctuation equations at zeroth order for different values of the dilaton horizon value. Using the temperature relation depicted in figure 9, we can translate this into solving the equations for different temperatures. The bulk viscosity is then given by the simple formula:

$$\frac{\zeta}{s} = \frac{8 z_{00} \left(A_h\right)^2}{27\pi} \tag{6.86}$$

We plot this in figure 16 for the the values of Φ_h relevant for comparison to data. In addition,

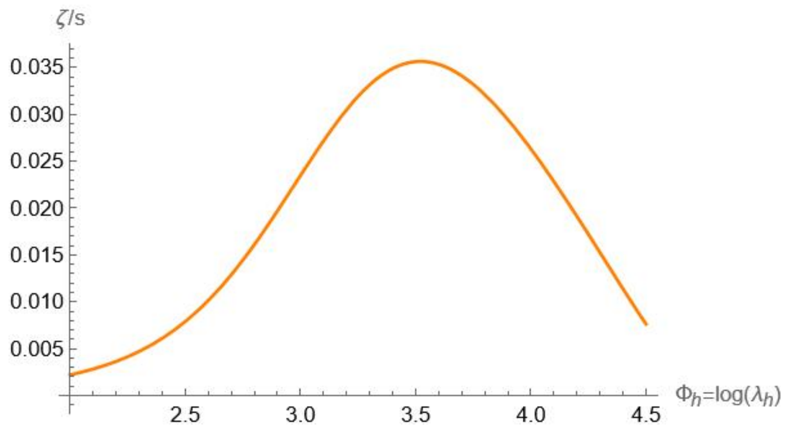

Figure 16: Bulk viscosity as a function of the dilaton horizon value for the V-QCD potential.

as a check for our numerical calculation we compare this result with the result of the Elling-Oz formula [22, 23] which reads:

$$\frac{\zeta}{s} = \frac{2}{3\pi} \left(s \frac{\partial \Phi_h}{\partial s} \right)^2 \tag{6.87}$$

We plot this comparison in figure 17. As can be seen, the two formulas are in agreement with a $\sim 1\%$ discrepancy. The two methods were originally shown to agree in [31] for ihQCD. However the agreement is not expected to hold as higher derivative corrections are included [30].
Finally, we come to the main motivation of this calculation which is to compare this result with the Bayesian analysis of heavy ion collisions. In figure 18 we plot the bulk viscosity as a function of temperature along with the 90% confidence bands of the two data sets. Some comments about this fit:

- The fit has zero additional free parameters as the V-QCD potential used is completely fixed by other demands.
- We observe a better agreement with the data from Jetscape for the zeroth order result. Thus the zeroth order prediction coming from V-QCD is that the bulk viscosity is not heavily suppressed compared to the shear viscosity hinting at a soft but not negligible break of conformality.
- We expect the first order correction to be important as it was for the shear viscosity and thus change the agreement with the data.

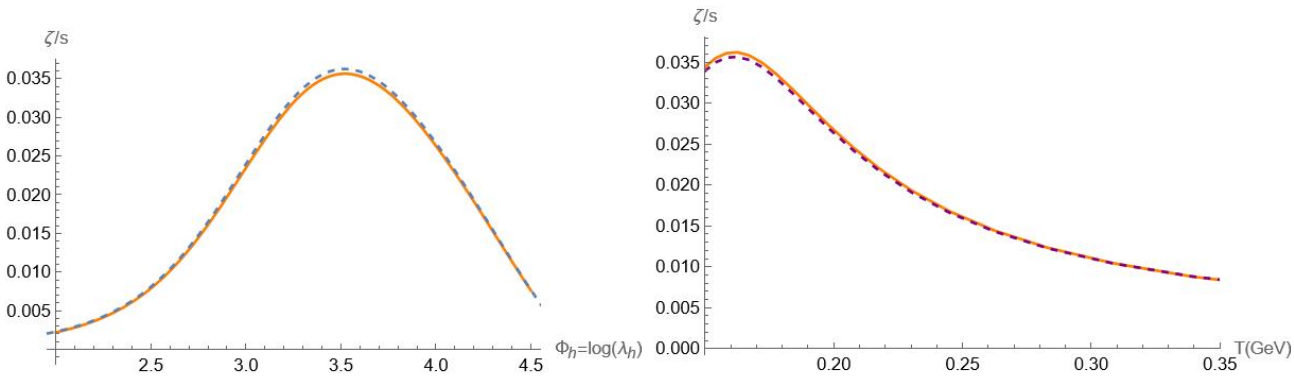

Figure 17: Comparison between the conserved current and EO methods of deriving ζ/s for the V-QCD potential. The conserved current method is shown in orange and the Elling-Oz method is shown as a dotted line.

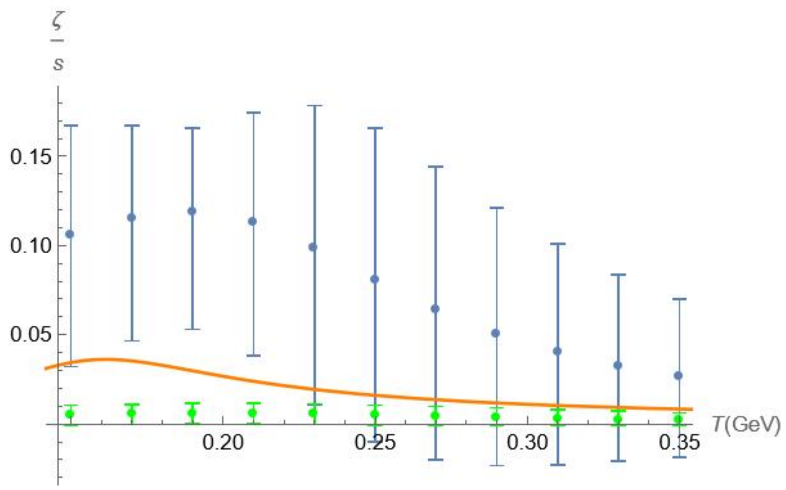

Figure 18: Bulk viscosity as a function of temperature for V-QCD. The orange line is the prediction coming from V-QCD. The blue error-bars are data from the Jetscape collaboration and the green errobars are data from the Trajectum collaboration.

7 Equations of Motion

So far in our analysis we have assumed that the higher derivative corrections are small compared to the background geometry and thus we would not expect it to change the properties of the holographic-QCD theories dramatically. In our analysis of the shear viscosity the correction to the background geometry is a second order effect and thus was unimportant. However, corrections to the asymptotics of background functions will be linear in β and so it is natural to ask how confinement will be affected. In addition, corrections to the thermodynamics will also be a first order effect and so it is natural to ask how the thermal stability will be affected. Answering these questions will hopefully constrain the function $G(\Phi)$ with the ultimate goal in mind of giving the theory predictive power.

Beyond this, it is also important to note that we don't expect these corrections to be small in the entire spectrum of energies. In particular, in the UV limit we know that the supergravity approximation breaks down and we have to think of the potentials as effectively resuming all possible corrections by matching to the QCD behaviour. For this reason, it is important to have a full analysis of the action (3.23), starting of course with the equations of motion.

We start with the EoM for Φ. The least action principle $\frac{\delta S_\phi}{\delta \phi}=0$ gives:

$$\frac{8}{3}\nabla_\mu\nabla^\mu\Phi+\ell^2\beta R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}G'(\Phi)+V'(\Phi)=0. \tag{7.1}$$

And by varying the action with respect to the metric we obtain Einstein's equations:

$$\begin{aligned}&R_{\mu\nu}-\frac{1}{2}g_{\mu\nu}R-\frac{1}{2}g_{\mu\nu}V(\Phi)-\frac{4}{3}(\nabla_\mu\Phi)(\nabla_\nu\Phi)+\frac{2}{3}g_{\mu\nu}(\nabla_\sigma\Phi)(\nabla^\sigma\Phi)+\\ &\ell^2\beta\Big[2G(\Phi)R_\mu{}^{\rho\sigma\alpha}R_{\nu\rho\sigma\alpha}-\frac{1}{2}g_{\mu\nu}G(\Phi)R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}+2G(\Phi)\nabla^\rho\nabla^\sigma R_{\mu\rho\nu\sigma}+2G(\Phi)\nabla^\sigma\nabla^\rho R_{\mu\rho\nu\sigma}+\\ &4(\nabla^\rho\Phi)\nabla^\sigma R_{\mu\rho\nu\sigma}G'(\Phi)+4(\nabla^\rho\Phi)(\nabla^\sigma R_{\mu\sigma\nu\rho})G'(\Phi)+2R_{\mu\rho\nu\sigma}(\nabla^\sigma\nabla^\rho\Phi)G'(\Phi)+\\ &+2R_{\mu\rho\nu\sigma}(\nabla^\rho\nabla^\sigma\Phi)G'(\Phi)+4R_{\mu\rho\nu\sigma}(\nabla^\rho\Phi)(\nabla^\sigma\Phi)G''(\Phi)]=0.\end{aligned} \tag{7.2}$$

It's easy to see that setting $\beta=0$ to the equations above reduces to the Einstein-Dilaton equation of motion. Now we look for background solutions with $T=0$ using the following ansatz:

$$ds^2=e^{2A(r)}\left(dr^2-dt^2+d\mathbf{x}^2\right), \quad \Phi=\Phi(r). \tag{7.3}$$

Adding the rr and tt components of Einstein's equations and substituting this ansatz yields:

$$\begin{aligned}&e^{-2A(r)}\Big(108\ell^2\beta G(\Phi)\dot{A}(r)^4-36\ell^2\beta\dot{A}(r)^3G'(\Phi)\dot{\Phi}(r)-9e^{2A(r)}\ddot{A}(r)+48\ell^2\beta G(\Phi)\ddot{A}(r)^2+\\ &3\dot{A}(r)^2\left(3e^{2A(r)}-52\ell^2\beta G(\Phi)\ddot{A}(r)\right)-4\dot{\Phi}(r)^2\left(e^{2A(r)}-3\ell^2\beta\ddot{A}(r)G''(\Phi)\right)\\ &+12\ell^2\beta G'(\Phi)\ddot{A}(r)\ddot{\Phi}(r)+24\ell^2\beta G'(\Phi)\dot{\Phi}(r)A^{(3)}(r)-36\ell^2\beta\dot{A}(r)\Big(G'(\Phi)\dot{\Phi}(r)\ddot{A}(r)\\ &+G(\Phi)A^{(3)}(r)\Big)+12\ell^2\beta G(\Phi)A^{(4)}(r)\Big)=0,\end{aligned} \tag{7.4}$$

where primes denote Φ derivatives and dots denote radial derivatives. In this combination of equations the potential drops out and we can analyze the asymptotics of the rest of the functions. Subtracting the rr and tt components of Einstein's equations gives:

$$\begin{aligned}&e^{-2A(r)}\Big(e^{4A(r)}V(\Phi)-3e^{2A(r)}\ddot{A}(r)-\dot{A}(r)^2\left(9e^{2A(r)}+20\ell^2\beta G(\Phi)\ddot{A}(r)\right)-36\ell^2\beta G(\Phi)\dot{A}(r)^4-\\ &12\ell^2\beta\dot{A}(r)^3G'(\Phi)\dot{\Phi}(r)+4\ell^2\beta\dot{\Phi}(r)^2\dot{A}(r)G''(\Phi)+4\ell^2\beta G'(\Phi)\ddot{A}(r)\ddot{\Phi}(r)+8\ell^2\beta G'(\Phi)\dot{\Phi}(r)A^{(3)}(r)+\\ &20\ell^2\beta\dot{A}(r)\left(G'(\Phi)\dot{\Phi}(r)\ddot{A}(r)+G(\Phi)A^{(3)}(r)\right)+4\ell^2\beta G(\Phi)A^{(4)}(r)\Big)=0.\end{aligned} \tag{7.5}$$

In the $\beta=0$ limit equations (7.4) and (7.5) reduce to equation (2.11) of [26] as expected.

7.1 UV Asymptotics

In the UV QCD becomes asymptotically free and conformal. It is natural then to assume that on the dual side, the geometry asymptotes to AdS_5. So in the limit of $r\to 0$ we have:

$$A(r)\to-\log(r/\ell). \tag{7.6}$$

Substituting this into (7.3) we see that the metric indeed goes to AdS_5:

$$ds^2=\frac{\ell^2}{r^2}\left(dr^2-dt^2+d\mathbf{x}^2\right). \tag{7.7}$$

Plugging these asymptotics for $A(r)$ into equation (7.4) we find that it reduces to the following:

$$-\dot{\Phi}(r)^2+3\beta\left[\dot{\Phi}(r)^2G''(\Phi)+G'(\Phi)\left(\frac{2\dot{\Phi}(r)}{r}+\ddot{\Phi}(r)\right)\right]=0. \tag{7.8}$$

As can be seen, only derivatives of $G(\Phi)$ appear in the equation and thus a simple solution is $G(\Phi) \to \textit{constant}$ at the UV. This then reduces the EoM to simply: $\dot{\Phi}(r)^2 = 0$ in which case we can take $\Phi(r) \to \textit{constant}$ which solves the EoM and we asymptote to pure AdS. However, if we want to preserve the ihQCD asymptotics in the UV namely: $\Phi(r) \to -\log(-\log(r\Lambda))$ we need to include higher order terms in the expansion of the scale factor, we look at this case below.

Now looking at the potential equation (7.5) and plugging in $A(r) \to -\log(r/\ell)$ we find the following:

$$\frac{\ell^2 V(\Phi) - 12}{r^2} + \frac{8\beta G(\Phi)}{r^2} + \frac{4\beta G'(\Phi)\left(-6\dot{\Phi}(r) + r\ddot{\Phi}(r)\right)}{r} + 4\beta\dot{\Phi}^2 G''(\Phi) = 0. \tag{7.9}$$

Solving this equation at zeroth order, assuming $G \to G_0$ gives a β correction to the AdS cosmological constant:

$$V(\Phi) \to V_0 = \frac{12}{\ell^2} - \frac{8\beta G_0}{\ell^2}. \tag{7.10}$$

The magnitude of this correction is determined by the asymptotic value of $G(\Phi)$.

Preserving ihQCD in the UV

We now take the following expansion for the scale factor [26]:

$$A(r) = -\log(r/\ell) + \frac{4}{9\log(r\Lambda)} + \mathcal{O}\left(\frac{-\log(\log(r\Lambda))}{\log(r\Lambda)^2}\right). \tag{7.11}$$

Plugging this into the equations of motion (7.4), we find the following up to leading order in the $r \to 0$ limit:

$$\begin{aligned} &\frac{4}{9(\log(\Lambda r))^8 r^2}\Big[3(\log(\Lambda r))^2\left((\log(\Lambda r))^4 - (\log(\Lambda r))^6 r^2\dot{\Phi}^2(r)\right) - 4\beta G(\Phi)(\log(\Lambda r))^6 \\ &3\beta G'(\Phi)\left(6(\log(\Lambda r))^6\dot{\Phi}(r) + 9\log(\Lambda r)^6 r\ddot{\Phi}(r)\right) + 9\beta G''(\Phi)(\log(\Lambda r))^6\Big] = 0. \end{aligned} \tag{7.12}$$

A simple solution to this equation is:

$$\begin{aligned} &G(\Phi) = G_0, \\ &\Phi(r) = -\log(-\log(r\Lambda))\left(1 - \frac{4}{3}\beta G_0\right). \end{aligned} \tag{7.13}$$

Giving us a β correction to the asymptotics of the dilaton. It's important to stress that this is not a perturbative result in β.

Moving on to the potential equation, to first order we find the same result. This is expected since in this combination of the EoM the $\dot{\Phi}(r)$ term drops out. For completeness, we find the following for the potential equation to first order in r:

$$\frac{\ell^2 V(\Phi) - 12}{r^2} + \frac{8\beta G(\Phi)}{r^2} \sim G'(\Phi) + G''(\Phi). \tag{7.14}$$

So as before (7.10) we find a correction to the cosmological constant or AdS length-scale:

$$V(\Phi) \to V_0 = \frac{12}{\ell^2} - \frac{8\beta G_0}{\ell^2}. \tag{7.15}$$

7.2 IR Asymptotics and Confinement

In this subsection we discuss the IR Asymptotics of the theory. We want to preserve desirable properties of ihQCD in the IR such as confinement. So our starting approach is to assume that ihQCD asymptotics in [25, 26] hold and admit small β corrections. Ideally, we would like to leave $V(\Phi)$ unchanged, therefore we will start by looking at equation (7.4). We have the following asymptotics:

$$\begin{aligned} &A(r) \to -Cr^{\alpha} + \beta\delta A(r)\,, \quad \alpha > 0\,, C > 0 \\ &\Phi(r) \to -\frac{3}{2}A(r) + \frac{3}{4}\log\left|\dot{A}(r)\right| + \beta\delta\Phi(r)\,. \end{aligned} \tag{7.16}$$

Now we ask the following question: How should $G(\Phi)$ behave in the $\Phi \to \infty$ limit such that the equations of motion are still satisfied? And how should $\delta A(r)$ and $\delta\Phi(r)$ behave in the $r \to \infty$ limit so as to not cause any issues? Plugging (7.16) into (7.4) and keeping only terms linear in β, we find the following for the leading order terms:

$$\begin{aligned} &\frac{e^{2Cr^{\alpha}}}{r^4}\Big(144C^5\ell^2\alpha^5\beta G(\Phi)\,r^{4\alpha} - 72C^5\ell^2\alpha^5\beta G'(\Phi)\,r^{4\alpha} + 36C^4\ell^2\alpha^4\beta G''(\Phi)\,r^{3\alpha}(\alpha-1)\Big) = \\ &\frac{1}{r^2}\Big(3C\alpha - 6C\alpha^2 + 3C\alpha^3 + 6\beta r^{1-\alpha}\delta\dot{A}(r)\left(1-2\alpha+\alpha^2\right) + 16C^2\alpha^2\beta r^{1+\alpha}\delta\dot{\Phi}(r) \\ &+ 6\beta r^{2-\alpha}\delta\ddot{A}(r)(1-\alpha)\Big). \end{aligned} \tag{7.17}$$

We observe that δA and $\delta\Phi$ are exponentially suppressed in the EoM. Now we proceed by solving the equation order by order in β. At zeroth order we recover the ihQCD result and the equation is solved up to an $1/r^2$ term.

$$\frac{1}{r^2}\left(3C\alpha - 6C\alpha^2 + 3C\alpha^3\right) \to 0. \tag{7.18}$$

Loking at the first order terms we observe that the left side of the equation is multiplied by an exponential that diverges in the $r \to \infty$ limit. In addition we expect δA and $\delta\Phi$ to be polynomial or logarithmic functions such that they are actually sub-leading in this regime. Therefore we take the following ansatz for G:

$$G(\Phi) = G_{IR}e^{\gamma\Phi}\Phi^{\delta}. \tag{7.19}$$

Using the Φ asymptotics (7.16), to first order in β and r this gives:

$$\beta G(r) = \beta G_{IR}(C\alpha)^{\frac{3}{4}\gamma}(-C)^{\delta}\,e^{\frac{3}{2}\gamma Cr^{\alpha}}r^{(\alpha-1)3\gamma/4+\alpha\delta}. \tag{7.20}$$

Plugging this back into the equation and keeping the leading terms in r and β we find the following:

$$\begin{aligned} &\beta\tilde{G}_{IR}e^{Cr^{\alpha}\left(2+\frac{3}{2}\gamma\right)}r^{(\alpha-1)(3\gamma/4+4)+\alpha\delta} = 6\beta r^{-\alpha-1}\delta\dot{A}(r)\left(1-2\alpha+\alpha^2\right) \\ &+ 16C^2\alpha^2\beta r^{\alpha-1}\delta\dot{\Phi}(r) + 6\beta r^{-\alpha}\delta\ddot{A}(r)(1-\alpha)\,. \end{aligned} \tag{7.21}$$

where $\tilde{G}_{IR} = 72C^5\ell^2\alpha^2 G_{IR}(-C)^{\delta}(C\alpha)^{3\gamma/4}$. If we want the left side to behave well in the IR we need the exponential to converge thus have to impose

$$\gamma \leq -\frac{4}{3}. \tag{7.22}$$

For the strictly smaller case we don't have a restriction on δ since the left side decays exponentially in the $r \to \infty$ limit. We analyze the special case of $\gamma = -\frac{4}{3}$ below. As for the right side we have a few possible solutions. We start be assuming a polynomial behaviour for $\delta\Phi$ and δA:

$$\begin{aligned} \delta A(r) &= C_A r^{\zeta}, \\ \delta\Phi(r) &= C_{\Phi}r^{\eta}. \end{aligned} \tag{7.23}$$

Plugging in these assumptions we find the following for the $\mathcal{O}(\beta)$ equation:

$$6\beta C_A r^{\zeta-\alpha-2}\left(\alpha^2\zeta-\alpha\zeta+\zeta^2(1-\alpha)\right)+16C_\Phi\eta C^2\alpha^2\beta r^{\alpha-2+\eta}+\text{vanishing}=0. \tag{7.24}$$

Here there's two perspectives we can take, one is that again we can have the equation solved up to a term of $1/r^2$ in which case we impose $\zeta\le\alpha$ and $\eta\le-\alpha<0$. Notice that this requires that $\delta\Phi$ vanishes in the IR. The other option, is to solve the equation exactly. In that case we equate the exponents of r which gives

$$\zeta-\eta=2\alpha. \tag{7.25}$$

Special case of $\gamma=-\frac{4}{3}$

Now, we look at the case of $\gamma=-\frac{4}{3}$ which exactly cancels the exponential and gives, to leading order:

$$\beta G(r)=\beta\tilde{G}_{IR}\frac{e^{-2Cr^\alpha}}{C\alpha}(-C)^\delta r^{1+\alpha(\delta-1)}. \tag{7.26}$$

Now asyming the same polynomial behaviour for δA, $\delta\Phi$ (7.23), to leading order, we find the following for the $\mathcal{O}(\beta)$ equation

$$6\beta C_A r^{\zeta-\alpha-2}\left(\alpha^2\zeta-\alpha\zeta+\zeta^2(1-\alpha)\right)+16C_\Phi\eta C^2\alpha^2\beta r^{\alpha-2+\eta}=\beta\tilde{G}_{IR}r^{3(\alpha-1)+\alpha\delta}. \tag{7.27}$$

Now solving this equation exactly requires matching the coefficients to leading order, the expression above simplifies to:

$$6\beta C_A r^{\zeta-\alpha}\left(\alpha^2\zeta-\alpha\zeta+\zeta^2(1-\alpha)\right)+16C_\Phi\eta C^2\alpha^2\beta r^{\alpha+\eta}=\beta\tilde{G}_{IR}r^{3\alpha-1+\alpha\delta}. \tag{7.28}$$

To proceed, we take two cases, if $\zeta-\eta<2\alpha$ then the η term is dominant and we find:

$$\delta=\frac{\eta+1-2\alpha}{\alpha}. \tag{7.29}$$

On the other hand, if $\zeta-\eta\ge2\alpha$ than the ζ term dominates and we find

$$\delta=\frac{\zeta+1-4\alpha}{\alpha}. \tag{7.30}$$

Asymptotics of the Dilaton Potential

We investigate now equation (7.5) which includes $V(\Phi)$. We want to find out if the same asymptotics of ihQCD hold and if there are any more contraints for the δA, $\delta\Phi$ functions. We start by substituting the same ansatz (7.16). For $V(\Phi)$ we assume the following asymptotic behaviour:

$$V(\Phi)\to V_{IR}e^{Q\Phi}\Phi^P. \tag{7.31}$$

Keeping only the higher order terms in the $r\to\infty$ limit we find the following:

$$\begin{aligned}&\frac{e^{2Cr^\alpha}}{r^4}\Big(-36C^4\ell^2\alpha^4\beta G(\Phi)\,r^{3\alpha}+18C^4\ell^2\alpha^4\beta G'(\Phi)\,r^{3\alpha}\Big)=\\&\frac{1}{r^4}\Big[-3C\alpha r^2+3C\alpha^2r^2-9C^2\alpha^2r^{2+\alpha}+18C\alpha\beta r^3\delta A'(r)-3\beta r^{4-\alpha}\delta A''(r)+\\&\tilde{V}_{IR}e^{\frac{1}{2}C(-4+3Q)r^\alpha}r^{4-\alpha+\alpha P+\frac{3Q}{4}(\alpha-1)}\left(1+\frac{4-3Q}{2}\beta\delta A(r)+Q\beta\delta\Phi(r)-\frac{3Q}{4C\alpha}r^{1-\alpha}\beta\delta A'(r)\right)\Big].\end{aligned} \tag{7.32}$$

where $\tilde{V}_{IR}=V_{IR}\,4^{-P}(6C)^P(-C\alpha)^{3Q/4}$. We proceed to solve the equation order by order in β. To zeroth order we recover the ihQCD result:

$$-3C\alpha r^{-2}+3C\alpha^2r^{-2}-9C^2\alpha^2r^{\alpha-2}+\tilde{V}_{IR}e^{\frac{1}{2}C(-4+3Q)r^\alpha}r^{4-\alpha+\alpha P+\frac{3Q}{4}(\alpha-1)}=0. \tag{7.33}$$

The third term in the equation is an exponential coming from the potential. In order to cancel this we have to impose $Q = \frac{4}{3}$. After imposing this, we are left with two terms of $\mathcal{O}(1)$ which are potentially divergent, the first one is proportional to $r^{\alpha-2}$ and the other coming from the potential is proportional to $r^{\alpha P-1}$. If we want these two terms to cancel we require $P = \frac{\alpha-1}{\alpha}$. After this we see that the equations of motion are solved up to a term of $1/r^2$. Thus we find that the same result of [26] holds:

$$Q = \frac{4}{3}, \quad P = \frac{\alpha - 1}{\alpha}. \tag{7.34}$$

There are still some subtleties regarding the behaviour of the β corrections, after fixing Q, P we look at the $\mathcal{O}(\beta)$ equation:

$$\begin{aligned} &\beta \tilde{G}_{IR} e^{Cr^\alpha\left(2+\frac{3}{2}\gamma\right)} r^{3\alpha-4+(\alpha-1)3\gamma/4+\alpha\delta} = \\ &18C\alpha\beta r^{-1}\delta A'(r) - 3\beta r^{-\alpha}\delta A''(r) + \frac{4}{3}\tilde{V}_{IR} r^{\alpha-2}\beta\delta\Phi(r) - \frac{\tilde{V}_{IR}}{C\alpha} r^{-1}\beta\delta A'(r). \end{aligned} \tag{7.35}$$

We found before that $\gamma \leq -4/3$. For the strictly smaller case the first term is subleading. Substituting the asymptotic behaviour for $\delta A, \delta\Phi$ (7.23) we find the following

$$\left(18C\alpha C_A - \frac{\tilde{V}_{IR}}{C\alpha}\right)\beta\zeta r^{\zeta-2} - 3\beta C_A\zeta(\zeta-1) r^{\zeta-\alpha-2} + \frac{4}{3}\tilde{V}_{IR}\beta C_\Phi r^{\alpha-2+\eta} = 0. \tag{7.36}$$

Since $\alpha > 0$ then the second term is sub-leading, thus equating the first and third term gives:

$$\zeta - \eta = \alpha. \tag{7.37}$$

However, combining this with out previous constraint from the other equation $\zeta - \eta = 2\alpha$ we find $\alpha = 0$ which is a contradiction. Therefore for this solution to hold the first term in the equation above must vanish. This fixes the coeficient C_A such that:

$$C_A = \frac{\tilde{V}_{IR}}{18(C\alpha)^2}. \tag{7.38}$$

Equating the two terms that are left, we find the same condition for ζ, η

$$\zeta - \eta = 2\alpha. \tag{7.39}$$

Finally, the special case of $\gamma = -4/3$ gives the same result as the previous equation, after fixing C_A.

So it seems that if G falls off fast enough in the IR the EoM are still satisfied with ihQCD asymptotics. In Figure 19 we present two fits for $G(\Phi_h)$ in VQCD where G drops off when approaching the IR.

Confinement

Technically speaking, confinement is defined as an area law for the Wilson loop. As explained in [26], for a generic 5D metric, the large L behavior of the quark-antiquark potential is:

$$E(L) \sim T_f e^{2A_S(r_*)} L. \tag{7.40}$$

which exhibits an area law if $A_S(r_*)$ is finite. A_S is the string-frame scale factor given by:

$$A_S(r) = A(r) + \frac{2}{3}\Phi(r). \tag{7.41}$$

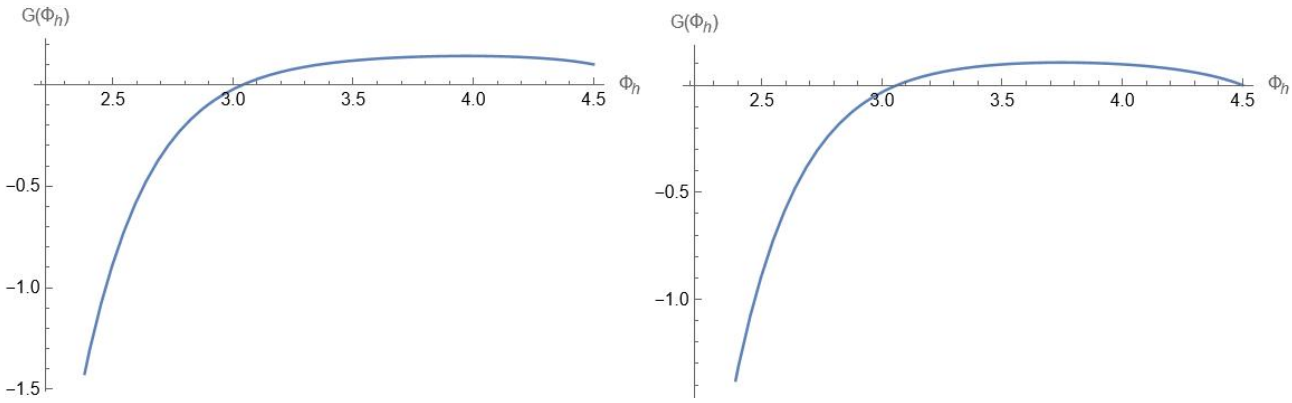

Figure 19: Dilaton coupling as a function of the dilaton horizon value. The plots have $G(4.5) = 0.1$ and $G(4.5) = 0$ as initial conditions, giving a fit where G drops to zero in the IR

In our case assuming the IR asymptotics (7.16) it is given by

$$A_S(r) = \frac{1}{2}\log\left|\dot{A}(r)\right| + \frac{2}{3}\beta\delta\Phi(r). \tag{7.42}$$

In the limit of $r \to 0$ we have: $A(r) \sim -\log(r/\ell) \to +\infty$ as discussed in the UV asymptotics. Therefore we want A_S to *not* asymptote to $-\infty$ at $r \to \infty$ such that e^{2A_S} does not go to zero in the IR limit and we have confinement. If we expand (7.42) according to (7.16), we find:

$$A_S(r) = \frac{1}{2}(\alpha - 1)\log\left(\frac{r}{R}\right) - \frac{1}{2}\frac{\beta\delta\dot{A}(r)}{C\alpha r^{\alpha-1}} + \frac{2}{3}\beta\delta\Phi(r). \tag{7.43}$$

where $C = R^{-\alpha}$. We see that the logarithm is still the leading term in the β expansion and we find that $\alpha > 1$ still leads to confinement. And for $\alpha < 1$ there is no confinement. However, there is a difference with ihQCD for the $\alpha = 1$ case where now the sub-leading terms are no longer constants and we don't necessarily reduce to 5D flat space.

Instead, assuming the asymptotics (7.23) we find:

$$A_S(r) = \frac{1}{2}(\alpha - 1)\log\left(\frac{r}{R}\right) - \beta\frac{\zeta\tilde{V}_{IR}}{36(C\alpha)^3}r^{\zeta-\alpha} + \frac{2}{3}\beta C_\Phi r^\eta. \tag{7.44}$$

Substituting $\alpha = 1$ and $\zeta = 2\alpha + \eta$ we find that the leading behaviour in this case is given by:

$$A_S(r) = -\beta\frac{\zeta\tilde{V}_{IR}}{36(C\alpha)^3}r^{\eta+1} = \beta\frac{\zeta V_{IR}}{36(C\alpha)^2}\left(\frac{3C}{2}\right)^P r^{\eta+1}. \tag{7.45}$$

So for confinement we need $\beta\zeta V_{IR} > 0$, and assuming $\beta V_{IR} > 0$ this gives the requirement of $\zeta > 0$ or equivalently $\eta > -2$. The case of $\zeta = \alpha = 0$ is also confining as A_S goes to a constant. This special case however will not matter for our fits since the physical case is $\alpha = 2$ which corresponds to $P = \frac{1}{2}$ in the V-QCD potential.

7.3 Alternative fits for $G(\Phi)$

In the previous fits for $G(\Phi_h)$ the function crosses zero and changes sign. If for some string theoretic reason or for some other reason such as causality or unitarity, we want G to be positive or negative definite then these fits won't suffice. In Figure 20 we provide some alternative fits for G where it is either positive or negative definite. The right graph clearly looks like an exponential, however recall that we set $\beta = 0.1$ so G is now of the same order as V meaning that perturbation theory breaks down. In that case, it's not clear even if the formula used for the fitting would still hold since in deriving it we set β^2 terms to zero. Even the variation problem for the shear fluctuations is defined up to linear order in β in (3.41).

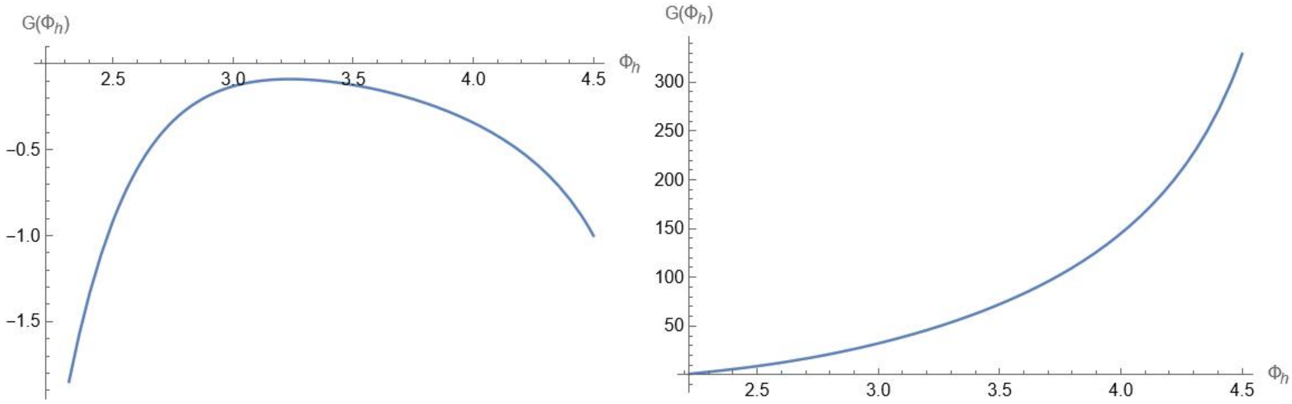

Figure 20: Dilaton coupling as a function of the dilaton horizon value. The plots have $G(4.5) = -1$ and $G(2.22) = 1$ as initial conditions, giving a fit where G is negative definite and positive definite respectively.

What violation of $1/4\pi$ implies for $\mathbf{G}\left(\boldsymbol{\Phi}_{\mathbf{h}}\right)$

The Bayesian analysis data of Trajectum have the mean value of η/s drop below the universal $1/4\pi$. However, $1/4\pi$ is well within the 90% confidence so it is still not certain if the viscosity bound is violated. The implication that this has on our fits as we will show is that either G must be zero at this point or grow exponentially.

To see this, we start by re-writing formula (3.84) as:

$$\frac{\eta}{s} - \frac{1}{4\pi} = \frac{\beta \ell^2}{6\pi}\left(-G\left(\Phi_h\right)V\left(\Phi_h\right) + \frac{3}{4}G'\left(\Phi_h\right)V'\left(\Phi_h\right)\right) \tag{7.46}$$

Now if we assume that η/s breaks the bound then this would mean that there would be a point where this function is zero. If $\Phi_h = \Phi_v$ at that point then this would imply:

$$G\left(\Phi_v\right)V\left(\Phi_v\right) = \frac{3}{4}G'\left(\Phi_v\right)V'\left(\Phi_v\right) \tag{7.47}$$

This has two possible solutions:

$$G\left(\Phi_v\right) = G'\left(\Phi_v\right) = 0 \quad \text{or} \quad \frac{G'\left(\Phi_v\right)}{G\left(\Phi_v\right)} = \frac{4}{3}\frac{V\left(\Phi_v\right)}{V'\left(\Phi_v\right)} \tag{7.48}$$

This explains the two types of fits we were producing since we are fitting to the mean value which indeed goes below $1/4\pi$.

7.4 An analytic example for $G\left(\Phi\right)$

Given the previous discussion in this chapter, one would like to find an analytic function $G\left(\Phi\right)$ that satisfies the constraints imposed by the asymptotics in the IR and UV and also fits to the shear viscosity data in the intermediate region. Here we focus on a simple example of one such function. Consider :

$$G\left(\Phi\right) = \frac{e^{\alpha\Phi} + \gamma}{1 + e^{\delta\Phi - \zeta}}, \quad \alpha, \delta > 0. \tag{7.49}$$

with $\alpha, \gamma, \delta, \zeta$ being constants. This function in the UV for $\Phi \to -\infty$ goes to

$$G_{UV} \sim \gamma. \tag{7.50}$$

In the IR for $\Phi \to +\infty$ this function behaves like

$$G_{IR}(\Phi) \sim e^{(\alpha-\delta)\Phi}. \tag{7.51}$$

demanding that

$$\alpha - \delta \leq -4/3. \tag{7.52}$$

Such that the equations of motion are satisfied in the IR. By fitting to the Bayesian data in the intermediate region we estimate the optimal values for these parameters to be:

$$\{\alpha = 1.8145, \quad \gamma = -275.28, \quad \delta = 3.1485, \quad \zeta = 2.5514\}. \tag{7.53}$$

We find that the optimal fit gives an $(\alpha - \delta) \sim -1.3340$ close but slightly smaller than $-4/3 \sim -1.3333$. In Figure 21 we plot this function with the optimal parameter values. We see that indeed it falls off exponentially in the IR and goes to a constant in the UV.

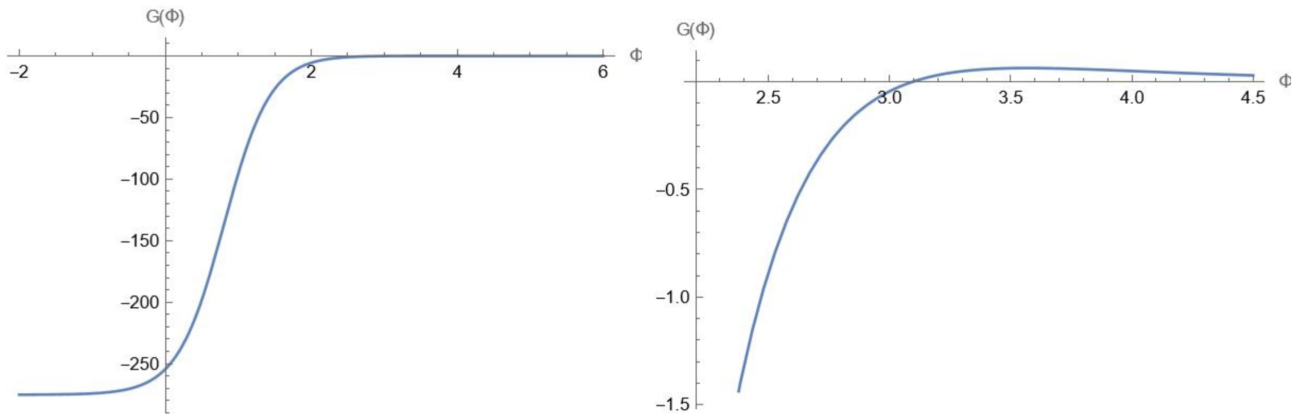

Figure 21: Example of an analytic function $G(\Phi) = \frac{e^{\alpha\Phi}+\gamma}{1+e^{\delta\Phi-\zeta}}$ fitting all the constraints. The right plot is the region of Φ relevant for the shear viscosity fit.

In Figure 22 we present the shear viscosity to entropy ratio as a function of temperature for this function. As in previous fits we transition from functions of Φ_h to functions of temperature by numerically relating the two through the VQCD thermodynamics.
As a final note, looking at the equation for the λ correction (7.10), using this fit we find that the correction is positive and in fact large with $\beta G_0 = \beta\gamma \sim -27.5$. This gives $V_0 = \frac{232}{\ell^2}$.

7.5 Domain wall coordinates and holographic c-theorem

An important idea in holography is that the radial (or holographic) direction in the $5D$ theory serves as a measure of energy in the $4D$ theory. In the case of AdS/CFT it is conjectured that $E \leftrightarrow 1/r$. This gives a natural correspondence between UV divergences and geometric singularities. In Einstein-dilaton gravity it is argued in [25] that the natural identification is

$$\log E \longleftrightarrow A(u). \tag{7.54}$$

where $A(u)$ is the scale factor in domain wall coordinates defined by the following metric:

$$ds^2 = du^2 + e^{2A(u)}\left(-dt^2 + d\mathbf{x}^2\right), \quad \Phi = \Phi(u). \tag{7.55}$$

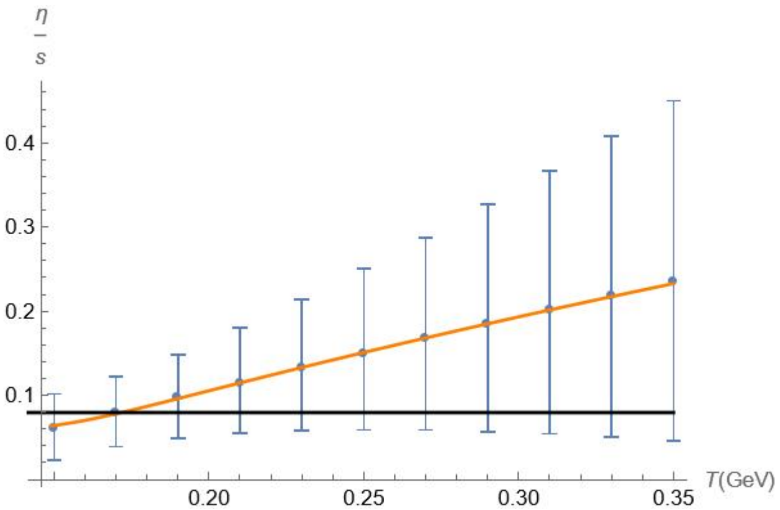

Figure 22: Best fit for the shear viscosity to entropy ratio for VQCD with $G(\Phi) = \frac{e^{\alpha\Phi}+\gamma}{1+e^{\delta\Phi-\zeta}}$. The optimal values for the parameters are given in (7.53). The yellow line shows the theoretical curve while the black line is $\eta/s = 1/4\pi$ for reference. We set $\beta = 0.1$ without loss of generality.

This is related to the conformal coordinates (7.3) by the transformation: $dr = e^{-2A(u)}du$. Thus the argument follows from the fact that in (7.55), the energy scale of the four dimensional slice is given by $e^{2A(u)}$.

It is a fact of nature that many different systems are described by the same low energy physics. A striking example is hydrodynamics. It describes both water and honey equally well, even though they are made of different molecules. In terms of the renormalization group this is explained by the fact that many different theories "flow" to the same IR conformal point. The reason for this is that the microscopic degrees of freedom are irrelevant at low energies. These universal points are characterised by zeros of the beta function.

Now connecting this to holography, if we propose a geometric theory as being dual to a QFT, this would would imply that, given (7.54), we should have some function of the radial coordinate u that is monotonic. This function would then serve as the renormalization group flow between a UV fixed point and an IR fixed point of the dual QFT. To construct such a function we can look at the equations of motion in domain wall coordinates. Combining the rr and tt components gives the following:

$$-\frac{4}{3}\dot{\Phi}(u)^2 = 3\ddot{A}(u) - \ell^2\beta G''(\Phi)\left(4\dot{A}(u)^2\dot{\Phi}(u)^2 + 4\dot{\Phi}(u)^2\ddot{A}(u)\right) + \ell^2\beta G'(\Phi)\Big[-4\dot{A}(u)^2\ddot{\Phi}(u) + 4e^{-2A(r)}\left(6+e^{2A(u)}\right)\dot{A}(u)^3\dot{\Phi}(u) + 8e^{-2A(u)}\left(-3+4e^{2A(u)}\right)\dot{A}(u)\dot{\Phi}(u)\ddot{A}(u) - 4\ddot{A}(u)\ddot{\Phi}(u) - 8\dot{\Phi}(u)A^{(3)}(u)\Big] - 4e^{-4A(u)}\ell^2\beta G(\Phi)\Big[6\left(-3+5^{2A(u)}\right)\dot{A}(u)^4 + \left(18-37e^{2A(u)}+12e^{4A(u)}\right)\dot{A}(u)^2\ddot{A}(u) - e^{2A(u)}\left(-2+5e^{2A(u)}\right)\dot{A}(u)A^{(3)}(u) + e^{4A(u)}A^{(4)}(u)\Big]. \tag{7.56}$$

We can write this as:

$$-\frac{4}{3}\dot{\Phi}(u)^2 = 3\ddot{A}(u) - \ell^2\beta\tilde{G}(u). \tag{7.57}$$

And we see that the equations of motion give:

$$3\ddot{A}(u) - \ell^2\beta\tilde{G}(u) \leq 0. \tag{7.58}$$

Thus the following function is monotonically decreasing.

$$f(u) = 3\dot{A}(u) - \ell^2\beta \int du\, \tilde{G}(u) \,. \tag{7.59}$$

What we have found are β corrections to the familiar result of $\ddot{A}(u) \leq 0$ which is shown to be equivalent to the Null Energy Condition [27]. In the IR where G decays exponentially we see that this function tends to the ihQCD one. However in the UV where corrections become large we expect deviations from this result. This is also what is expected from an RG flow perspective. In our case it's hard to construct an explicit function since one would have to know A, G as functions of u to perform the integral. In addition, because of the higher derivative term it's not clear what the proper definition of the energy momentum tensor is in this case. If one wanted to retain the condition $\ddot{A}(u) \leq 0$ then one would have to prove that $\ell^2\beta\tilde{G}(u) \leq 0$ which appears a daunting task especially because many of the fits we are producing for G are not positive or negative definite.

8 Conclusions and Outlook

In this thesis, we explored the temperature dependence of the shear and bulk viscosities for holographic QCD theories. We argued that higher derivative corrections are needed to obtain a temperature dependence consistent with results from Bayesian analysis of heavy ion collisions. We constrained the form of the higher derivative corrections by fitting to these results. We derived the full equations of motion and analyzed the UV and IR asymptotics of the modified theory showing that under certain assumptions the theory still exhibits confinement. Finally, we derived an analytic formula for the bulk viscosity to entropy ratio that includes higher derivative corrections coupled to the dilaton and analyzed its zeroth order solution for the V-QCD potential.

There are numerous directions for the improvement of the work presented here. Obvious examples include the effect of the higher curvature terms on the thermodynamics of the theory including the phase structure and phase transitions. Since in our analysis, the higher-order corrections were quite large, we would expect non-trivial changes to the thermodynamics. In addition, we would also expect non-trivial effects on particle spectra.

Our analysis of the equations of motion and asymptotic behavior was by no means extensive. An interesting direction for future work would be constructing a general solution for the equations with numerical checks for the fitting functions used.

For the same reasons, the β correction to the bulk viscosity is expected to be important and change our fit significantly. Furthermore, the other two curvature squared correction terms in this order will contribute to ζ/s, unlike the case of η/s. These calculations are the subject of future work.

9 Appendix

9.1 Effective Action coefficients

The coefficients of the effective action (3.31) starting from the original action (3.23) are the following:

$$A(u) = \frac{2a(u)\,b^2(u)\left(b(u)\,c^3(u) + 4\ell^2\beta G(\Phi)\left(b'(u)\,c'(u) - c(u)\,b''(u)\right)\right)}{c(u)^4},$$

$$B(u) = \frac{b(u)}{2a(u)\,c^5(u)}\Big[4\ell^2\beta b^2(u)\,c^2(u)\,G(\Phi)\,(a'(u))^2 + a^2(u)\Big(16\ell^2\beta c^2(u)\,G(\Phi)\,(b'(u))^2$$
$$+ b^2(u)\left(3c^4(u) + 4\ell^2\beta G(\Phi)\,(c'(u))^2\right) - 8\ell^2\beta b(u)\,c(u)\,G(\Phi)\left(b'(u)\,c'(u) + c(u)\,b''(u)\right)\Big)\Big],$$

$$C(u) = -\frac{1}{c^5(u)\,a(u)}\Big[-a(u)\,b^3(u)\,c^4(u)\,a'(u) + 4\ell\beta b^2(u)\,c^2(u)\,G(\Phi)\,(a'(u))^2\,b'(u) +$$
$$+ a^2(u)\Big(8\ell^2\beta c^2(u)\,G(\phi)\,(b')^3 + b^3(u)\,c^3(u)\,c'(u) - 8\ell^2\beta b(u)\,c(u)\,G(\Phi)\,b'(u)\left(b'(u)\,c'(u) - c(u)\,b''(u)\right)$$
$$- 4b^2(u)\left(c^4(u)\,b'(u) - \ell^2\beta G(\Phi)\,b'(u)\,(c'(u))^2 + \ell^2\beta c(u)\,G(\Phi)\,c'(u)\,b''(u)\right)\Big)\Big],$$

$$D(u) = \frac{1}{2c^5(u)\,a(u)\,b(u)}\Big[2a(u)\,b^3(u)\,c^3(u)\left(-b(u)\,a'(u)\,c'(u) + c(u)\left(3a'(u)\,b'(u) + b(u)\,a''(u)\right)\right) -$$
$$4\ell^2\beta b^2(u)\,G(\Phi)\Big(b^2(u)\,(a'(u))^2\,(c'(u))^2 - 2b^2(u)\,c(u)\,a'(u)\,c'(u)\,a''(u) +$$
$$+ c^2(u)\left(3\,(a'(u))^2\,(b'(u))^2 + b^2(u)\,(a''(u))^2\right) + a^2(u)\Big(b^4(u)\,c^6(u)\left(2\partial_u\Phi\partial^u\Phi - V(\Phi)\right) -$$
$$- 12\ell^2\beta c^2(u)\,G(\Phi)\,(b')^4 + 6b^3(u)\,c^3(u)\left(-b'(u)\,c'(u) + c(u)\,b''(u)\right) + 6b^2(u)\Big(c^4(u)\,(b'(u))^2 -$$
$$- 2\ell^2\beta G(\Phi)\,(b'(u))^2\,(c'(u))^2 + 4\ell^2\beta c(u)\,G(\Phi)\,b'(u)\,c'(u)\,b''(u) - 2\ell^2\beta c^2(u)\,G(\Phi)\,(b''(u))^2\Big)\Big)\Big)\Big],$$

$$F(u) = \frac{4\ell^2\beta a(u)\,b^2(u)\,G(\Phi)\left(2c(u)\,b'(u) - b(u)\,c'(u)\right)}{c^4(u)},$$

$$E(u) = \frac{2\ell^2\beta a(u)\,b^3(u)\,G(\Phi)}{c^3(u)}. \tag{9.60}$$

where we have already applied the relevant limit of $\omega \to 0$ and $k = 0$.

9.2 Summary of the asymptotic behaviour

	UV	IR
r	$r \to 0$	$r \to +\infty$
$A(r)$	$-\log(r/\ell) + \frac{4}{9\log(r\Lambda)}$	$-Cr^\alpha + \beta\delta A(r)\,, \quad \alpha, C > 0$
$\Phi(r)$	$-\log(-\log(r\Lambda))\left(1 - \frac{4}{3}\beta G_0\right)$	$-\frac{3}{2}A(r) + \frac{3}{4}\log\left\|\dot{A}(r)\right\| + \beta\delta\Phi(r)$
$\delta\Phi(r)$	-	$C_\Phi r^\eta\,, \quad \eta = \zeta - 2\alpha$
$\delta A(r)$	-	$C_A r^\zeta\,, \quad C_A = \frac{\tilde{V}_{IR}}{18(C\alpha)^2}$
$G(\Phi)$	G_0	$G_{IR} e^{\gamma\Phi}\Phi^\delta\,, \quad \gamma \leq -\frac{4}{3}$
$V(\Phi)$	$V_0 = \frac{12}{\ell^2} - \frac{8\beta G_0}{\ell^2}$	$V_{IR} e^{Q\Phi}\Phi^P\,,\ Q = \frac{4}{3}\,,\ P = \frac{\alpha-1}{\alpha}$

Table 1: Summary of the asymptotic behaviour, assuming polynomial ansatz for $\delta A, \delta\Phi$.

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