Homotopical models of UF and their relationship to ∞-topoi

MSc thesis

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Abstract

Homotopy Type Theory (HoTT) is a formal system for constructive mathematical reasoning. The Univalent Foundations (UF) program enhances HoTT with the Univalence Axiom and Higher Inductive Types (HITs), proposing it as an alternative to ZFC for the foundations of mathematics. My thesis focuses on the categorical semantics of UF.

The semantics of the elimination principle of the identity type correspond to lifting properties, a fundamental component of model categories, which serve as an abstract framework for homotopy theory. In 2013, it was demonstrated that Kan Complexes, or equivalently ∞-groupoids, can support a model of UF. These findings suggest a profound connection between logic and homotopy theory / higher category theory. Over the past decade, numerous attempts have been made to expand on these results.

In my thesis, I investigate two ways in which models of HoTT relate to higher categories. The primary link is $\mathrm{Ho}_\infty(-)$, which produces a homotopy ∞-category of a category with weak equivalences (and perhaps additional structure). Using an incarnation of $\mathrm{Ho}_\infty(-)$, one can simultaneously introduce and characterize $(\infty, 1)$-topoi in two models for higher categories.

We showcase three main results. The first starts with a type theory obeying certain rules and produces a locally cartesian closed ∞-category. The second and third state, that every ∞-topos, (respectively presentable and locally cartesian closed ∞-category) can be presented by a model category that models univalent HoTT (respectively HoTT+FunExt). We also prove that two of these constructions are compatible in a certain sense.

Contents

1 Introduction

The practice of forming collections of given mathematical objects of interest dates back to antiquity and has permeated mathematics ever since. Understanding collections as sets and their systematic study as such dates back to the 1870s in the work of Cantor.[1] Even in its infantile state, set theory was already capable of demonstrating the presence of more than one infinity and posing deep questions such as the *generalized continuum hypothesis*. Especially in combination with the work of Bourbaki, set theory quickly presented itself as a strong candidate for robust and rigorous foundations of mathematics. Thus, Frege, viewing sets and their "calculus" as belonging to the discipline of logic, was led to assert that all of mathematics was reducible to logical principles. This position became known as logicism. A primary ingredient of his philosophy was the *(full) Comprehension Principle*, which asserted that *any* property P defines a *set* of objects that satisfy said property

$$\mathcal{P} := \{ \ x \mid P(x) \ \}.$$

The full comprehension principle turned out to be *too expressive*, expressive enough to create paradoxes. Famously, if we let $P(x) := x \notin x$ and obtain $\Omega := \{ \ x \mid x \ \notin \ x \ \}$ then,

$$\Omega \in \Omega \iff \Omega \notin \Omega$$

Paradoxes such as this created a deep crisis in the foundations of mathematics. There were three philosophical doctrines that emerged as an answer. Logicism (renewed), Formalism, and Intuitionism, see [Fer04].

Formalism proposed a recasting of set theory in a 1^{st} order formal language. The precise axiomatization of set theory is due to Zermelo and Fraenkel and notably includes the axiom of choice and a restriction of the full comprehension principle. The main counter-proposal was Intuitionism. Its main tenet is that we re-imagine mathematical entities and the very practice of doing mathematics as *mental constructions*. Another notable proposal was Russel's theory of ramified types whose conceptual grandchild with Intuitionism will be a protagonist of this thesis.

For many years formalism was the clear victor of the debate and much, arguably all, of mathematics was based on ZFC for many decades. When an everyday working mathematician wants to define his or her favorite object they simply posit a set and declare their desired list of conditions.

One of the most popular fields of mathematics that emerged was Algebraic Topology which proposed the study of space via algebraic means, notably the fundamental groups, originating

[1]Some authors, like the greatly respected Fereiros, also credit Dedekind, see [Fer08].

with [Poi95]. In the service of algebraic topology, Saunders and MacLane introduce *Category Theory*, see [ME45][2]

In modern geometry, differential or algebraic, it has consistently proven to be of benefit to define/study things locally. Indeed the very definition of a manifold boils down to a smooth[3] passage to a local setting. This was one of the motivating ideas that led to the definition of a Grothendieck Topos in [GV72]. Presheaves and sheaves were studied in depth by Grothendieck and his collaborators. It did not take too long to notice that categories of sheaves behaved much like the category of sets. For instance, one could form versions of disjoint unions and cartesian products and, even more crucially, characteristic functions.

Isolating the precise properties that established this similarity and axiomatizing them led to the definition of elementary topoi. Their inherent similarity with the category of sets allowed them to replicate many constructions that are central to mathematical logic. The study of such phenomena was termed categorical logic. Two of its main protagonists were Lawvere [Law00], [Law05] and Tierney [Tie73]. It turns out that one can import enough logical structure into an elementary topos, to allow it to serve as a setting where a substantial part of set theory, and thus most of mathematics, can be internalized.

Back in the '30s, the λ-calculus was introduced as an attempt to make the notion of an "effective procedure" or more simply, an "algorithm", mathematically precise. It was quickly realized that it could be productively combined with type-theoretic ideas. This provided a link with computational/ constructive mathematics. Building on these ideas Per Martin Löf introduced an Intuitionistic Type Theory, called *Martin Löf Dependent Type Theory*, or MLDTT for short, [Mar75a] thus effectively combining two of the answers to the foundational crisis.

The inherent structure present in Sets, elementary topoi, and intuitionistic (potentially higher order) type theories made all three reasonable proposals as *foundations of Mathematics*. But, this common denominator also made it possible to construct passages from one framework of foundations to another. For example, there is a bidirectional correspondence between higher-order type theories and elementary topoi, outlined in [LS86]. Furthermore, one can obtain (constructive) set theory out of an elementary topos [MM94] and out of a type theory [Acz78].

[2]Commonly considered the foundational document of category theory. On the first page it notes: *Presented to the Society, September 8, 1942; received by the editors May 15, 1945.* The second date is exactly a week after the end of WW2 in Europe.

[3]pun intended

$$\begin{tikzcd}
 & \text{Constructive Set Theory} & \\
\text{Intuitionistic Type Theory} \arrow[ur] \arrow[rr, shift left] & & \text{(Elementary) Topos Theory} \arrow[ul] \arrow[ll, shift left]
\end{tikzcd}$$

Another crucial element to the discussion is *homotopical mathematics*. The philosophical motivation is that quite often in mathematical practice *equality* proves too restrictive and thus it is productive to adopt different, less rigid, notions of sameness.

An important example, and coincidentally the birthplace of homotopical mathematics, is the work of Poincaré on the fundamental group of a topological space, see [Poi95]. This led to further algebraic invariants of topological spaces, like the higher homotopy groups and (co)homology theories. Quite interestingly their algebraic laws hold only *up to coherent homotopy*. In a precise sense, sets are too discrete and thus do not offer a natural framework to capture and work with these higher data.

Instead of working in set-theoretic foundations and having to resort to non-trivial constructions to define "up to homotopy" algebraic structures, one would rather have a natively homotopical environment where *everything* is defined up to coherent homotopy and then simply consider their favourite algebraic object. Categories whose structure is natively given "up to coherent homotopy" are called **weak higher categories**. In this thesis, we will exclusively focus on a special case where almost all of the "higher data" is better behaved, namely, invertible. The so called $(\infty, 1)$-categories are treated in section 2. Notably in subsubsection 2.3.3 we showcase the elementary definitions of $(\infty, 1)$-categories in the setting of ∞-categories. Even in that special case, as we'll discuss later there are more than one ways to "model" what an $(\infty, 1)$-category should be. We note that we'll use the term "$(\infty, 1)$-categories" as the *model independent* one, to refer to properties or features of "$(\infty, 1)$-categories" no matter how they are incarnated. The first model we'll be interested in takes place in *simplicial categories*, so we will call it the simplicial model. The second model, and our preferred one, consists of simplicial sets satisfying the *weak Kan condition*. Simplicial sets satisfying this condition will be called "∞-categories". Here we must remark that some authors use the term "∞-categories" as a synonym for "$(\infty, 1)$-categories" and others to refer exclusively to these simplicial sets like Groth [Gro15].

However, a bare ∞-category, although inherently interesting, can't do much by itself. So one wonders what kind of additional structure could one impose. Jacob Lurie, much like Grothendieck, was motivated by the development of *derived* algebraic geometry. Namely, a version where one uses these homotopy coherent algebraic structures, like simplicial rings or E_∞ rings, to provide local charts for schemes. In [Lur09] Lurie generalised the notion of Grothendieck topoi to that of an ∞-topos, the topic of subsection 2.4.

We will build things up mostly in a self-contained manner for the longest duration possible. Unfortunately, as Lurie's two [Lur09], [Lur18] (out of three [Lur17]) $\approx$ 1000+ page documents on the topic showcase, one cannot hope to provide a self-contained treatment of higher category theory and higher topos theory in the confines of an MSc thesis. We should also note that our motivations differ drastically from Lurie's and therefore homotopy coherent algebra serves only to provide historical context and motivations but will not be treated mathematically at all.

Much like (constructive) sets are central to the non-homotopical setting, here we need a framework that is inherently homotopical, like spaces. Spaces and their homotopy theory cannot be understood in isolation. They must be studied in comparison to other homotopy theories. This leads us to consider **abstract homotopy theories**. We do that by introducing and analyzing various ways to do that. We survey each in order of increasing structure, starting from homotopical categories, fibration categories and model categories. In the last case, we also look into various extra structures one might place on a model category or some useful properties they might have. This is the topic of section 3

The last vertex is Intuitionistic Type Theory. In the beginning, people were mostly focused on *extensional* MLDTT. The extensional version essentially satisfies a truncation principle that trivialises all higher homotopical data. Slowly but surely the community discovered non-truncated models in the context of model categories and higher categories. The archetypal example is Voevodsky's simplicial model [KLV12]. It has since become clear that HoTT/UF exhibits many of the behaviours one would expect from a "homotopy coherent" intuitionistic type theory. We treat these ideas along with a quick rundown of the historical development of HoTT in section 5. The situation thus far can be captured in a diagram:

$$\begin{tikzcd}
& \text{Abstract Homotopy Theory} \arrow[dl, dash] \arrow[dr, dash] & \\
\text{Homotopy Type Theory} \arrow[rr, dash] & {} & \infty - \text{(Elementary) Topos Theory} \\
& \text{Constructive Set Theory} \arrow[u, Rrightarrow, "\text{Homotopy Coherence}"'] & \\
\text{Intuitionistic Type Theory} \arrow[ur] \arrow[rr, shift left] & & \text{(Elementary) Topos Theory} \arrow[ul] \arrow[ll, shift left]
\end{tikzcd}$$

We must remark that the theory of *elementary ∞-topoi* is still very much under development. There is a proposal for a definition by Rasekh [Ras22] and some work around it. At the moment, the definition is not yet universally accepted. Lurie's ∞-topoi are much better understood and are the topic of one of the main theorems in this thesis, see subsection 5.9. Now, we focus on the upper part of the diagram and explain what the arrows are. We must note that arrows here don't (always) stand for actual functors between categories, merely conceptual passages from one setting to another. This diagram captures the main topic of this thesis. To study the relationship of "models of HoTT", in the guise of appropriately structured model categories, and ∞-categories.

$$
\begin{tikzcd}
\textit{strict}\text{ models} & & \\
& \text{Structured Model Categories} \arrow[ul, "\text{local universes method}"'] \arrow[dr, "\mathrm{Ho}_\infty(-)"] & \\
\text{Homotopy Type Theory} \arrow[ur, "\text{models}"] \arrow[rr, dashed] & & \text{Structured } \infty - \text{Categories} \arrow[ul, bend right, "\text{presentations}"']
\end{tikzcd}
$$

The first construction we showcase, originating in [Kap14], simply asks us to start with a type theory with Σ, id and Π -types that satisfy FunExt and consider its "classical" category of contexts $\mathrm{Cl}(\mathbb{T})$. By studying $\mathbb{T}$ internally we get natural candidates for notions of weak equivalences and fibrations making $\mathrm{Cl}(\mathbb{T})$ into a *fibration* category. Then one can compute its ∞-category of frames, see [Szu14], and find that for such a $\mathbb{T}$, $N_f(\mathrm{Cl}(\mathbb{T}))$ is a locally cartesian closed ∞-category.

In parallel, we explain the ideas that led to *homotopical semantics* of HoTT thus motivating an arrow HoTT/UF $\to$ mCat, model categories. We note that here, we don't view "HoTT/UF" as a category of some sort only as a collection and hence the arrow is not a functor. In due time mCat will become a category. In due time, mCat will be a category

We note that models like these suffer from a *coherence problem* and thus require a *strictification* theorem to obtain an *actual* model of HoTT. For the model categorical models of interest, this is dealt with in appropriate generality by the *local universes method* of [LW15]. Then, we characterise the model categories that can support a model of HoTT+FunExt as the right proper Cisinski model categories and those that admit a model of full HoTT/UF as Type Theoretic Model Topoi. Lastly, through results established in section 4 we know that every presentable locally cartesian closed ∞-category is *presented by* a right proper Cisinski model category and that every ∞-topos is presented by a *model topos*, which, up to Quillen equivalence, is indistinguishable from a TTMT. Thus:

(1) Every ∞-topos can be presented by a model category that can in turn be strictified to model HoTT/ UF

(2) Every presentable and locally cartesian closed ∞-category can be presented by a model category that can, in turn, be strictified to model HoTT+FunExt.

We close off with subsection 5.10 by establishing that a pair of these constructions are "compatible with one another", in the sense that if one starts with an ∞-topos $\mathfrak{X}$, takes a TTMT that presents it $\mathscr{E}$, canonically extracts a (non enriched) fibration category out of it, E_0^f, then $N_f(E_0^f) \simeq \mathfrak{X}$. This showcases an important theme of homotopy theory, that the (homotopy type of the) *underlying ∞-category* of a homotopical category depends only on the weak equivalences. These are left untouched by the passage from $\mathscr{E}$ to E_0^f, and we can thus retrieve $\mathrm{Ho}_\infty(E_0^f) \simeq N_f(E_0^f) \simeq \mathfrak{X}$ up to equivalence of ∞-categories.

The nature of this thesis is mostly expository. Through it, we hoped to create an, as self-contained as possible, introduction and exposition, to what we consider to be one of the most intriguing and fascinating phenomena being actively researched at the moment, the relation between homotopical models of HoTT/UF and $(\infty, 1)$-categories. The last section contains original work in verifying that the methods Kapulkin uses in his PhD thesis [Kap14] can be applied to the objects Shulman uses in [Shu19] and produce essentially the same results.

2 Higher categories

2.1 Introduction

Ever since its conception category theory has proven to be exceptionally effective as an organizational framework for much of modern mathematics. It is truly hard to overestimate the degree to which category-theoretic methods have infiltrated some of the most important fields of mathematics. Indeed one could even assert that category theory is the language in which modern algebraic geometry and topology are written.

Yet, perhaps surprisingly, category-theoretic methods have limitations, particularly in studying *homotopical* phenomena. Both in algebraic topology but also in logic, mere categories prove insufficient, in that they offer no immediate way to record homotopies between morphisms, thus pushing us towards non-trivial constructions. In a very fundamental way, this is the appeal of higher category theory. It provides a natural framework and machinery where one can *natively* do homotopy coherent mathematics.

To illustrate this we recall the construction of the fundamental groupoid of a topological space, $\Pi_1(X)$, whose objects are points of X and the arrows stand for paths between points. We should picture $\pi_1(X, x)$ sitting over any point $x \in X$, and an additional rich world of (non-loop) paths between them. The natural candidate for a "group" operation is concatenation. Unfortunately, the corresponding group laws hold only up to homotopy. In the past, for instance in 1895 when Henri Poincaré introduced the fundamental group in [Poi95], this was dealt with by quotienting out by the equivalence relation induced by "homotopy", so that homotopic paths were literally identified. But any quotient, by its very nature, leads to loss of information which may be undesirable. We wanted to study the homotopy type of a space and in order to do that we've thrown out most of the homotopical information!

The very idea behind the fundamental group(oid) of a space is to record more information than mere existence of a path between two points. In classical mathematics existence of an object is recorded by a binary choice, yes or no. Instead, in *proof relevant* constructive mathematics we form the collection of all proofs and reason about it mathematically. Proof relevance originates with Kleene's realizers. To treat sentences of a logic proof relevantly amounts precisely to considering the collection of their constructive proofs and considering functions, and other operators, like quantifiers, between such collections. Now the constructive proofs of the statement "there exists a path $x \rightsquigarrow y$" are precisely the paths γ with endpoints x, y. In other words, at the very core of algebraic topology, the theory of homotopic paths and of fundamental groups lies a principle originating in the constructive philosophy of mathematics. When these ideas are applied to the statement above, after quite a bit of work, one has produced a powerful algebraic invariant, a fundamental tool in the study of topological spaces.

In the fundamental groupoid, we only retain the information of whether two paths are homotopic or not. That means we were proof-relevant, but just for a single step. What if we could be more proof-relevant? What if a pair of paths is homotopic in two, three or uncountably many ways? What if two homotopies between a fixed pair of paths are themselves homotopic in some appropriate sense? The loss of information resulting from the quotient we took leaves us unable to tackle, or even pose such questions.

The proposed solution of higher category theory is to enrich the fundamental objects of a category as follows: instead of just objects and morphisms, we add a notion of 2-morphism between regular (1-)morphisms, to play the role of recording homotopies between them. This idea is the first stepping stone toward higher category theory. Of course, there is no reason to stop at level 2. Indeed in the topological example we started with one could well have a homotopy between homotopies to be recorded by a 3-arrow etc. As far as our motivating example was concerned we can use these higher arrows to record higher homotopies, namely homotopies between homotopies and so on. In doing so we produce a category with higher arrows, where all higher arrows are homotopies and therefore invertible. Thus we call the resulting category fundamental ∞-groupoid of a space, $\Pi_\infty(X)$. In some of its incarnations $\Pi_\infty(X)$ captures the entire homotopy type of X, see [Cis06]. In fact, among homotopy theorists, it is generally accepted that the converse is also true, namely that every $(\infty, 1)$-groupoid arises in this way.

As in the case of the fundamental group(oid), the algebraic laws of concatenation of paths, like the existence of identities, their unitality with respect to concatenation, associativity et cetera, hold only up to homotopy. Concretely, instead of an equality $\gamma \star id_x = \gamma$, in a great illustration of the proof relevant spirit of doing mathematics, we get a 2-arrow *witnessing* the identity $\gamma \star \mathrm{const}_x \Rightarrow \gamma$. This illustrates a fundamental theme of higher category theory, whose nature is inherently proof relevant: *equalities of ordinary categories hold only up to "higher morphism"*. In other words higher category theory, especially in the guise of ∞-categories, boils down to doing category theory treating composition *proof relevantly*, in that higher arrows serve as witnesses of equalities between lower arrows.

Riehl, in her article, [Rie23], considers the problem of transferring the structure of a G-space across a homotopy equivalence $X \simeq Y$, and faces a similar situation. The point here is that as soon as you specify many pieces of low-dimensional data, say two composable 1-arrows, you immediately have to specify another 1-arrow, their weak composition, and a 2-arrow witnessing that composition. And as soon as you have more composable 1-arrows and 2-arrow-witnesses, you have to specify a 3-arrow witnessing associativity of composition, and then a 4-arrow relating multiple pieces of associativity data and so on. Furthermore, none of these choices are a priori unique. Thankfully they are unique in a weaker sense, namely unique up to homotopy in that the *space* of all choices is *contractible*. All that information can be arranged in what is called a *homotopy coherent diagram*.

This need for homotopy coherent algebra and functors motivates our wish for *homotopy coherent category theory*. Indeed, that's exactly how we think of the theory of higher categories. Furthermore, one of the main appeals of the theory of ∞-categories, indeed their very definition, axiomatically ensures the existence and coherence of all these data. Now doing category theory up to coherent homotopy may sound great in theory, but unfortunately, the practice is a bit more convoluted. Just look at all the pieces of data we have to specify and coherently relate to one another by specifying even more data and so on.

These new and complex algebraic structures we call higher categories. As explained above, some of their fundamental features are the existence of higher morphisms and a weakening of identities. One consequence is that in this new setting to define even simple objects, such a morphism composition, often requires an infinite amount of data. A definition that used to be about objects and arrows now also has to ensure that all the potentially infinite higher data are coherently interrelated and compatible. A way to battle the complexity produced by the requirement for an infinite amount of data is to postulate that any morphisms of dimension at least 2 that exist, will be invertible. These are called $(\infty, 1)$-categories and will be the focus of this thesis.

At the moment we imagine an $(\infty, 1)$-category $\mathscr{C}$, as a collection of objects, and for each pair of objects a collection of morphisms $\mathrm{Hom}_{\mathscr{C}}(X, Y)$. The latter collection should come equipped with a notion of a k-arrow playing the role of "homotopy between $(k-1)$-arrows". This in particular implies that if $k > 1$ all k-arrows are "homotopies" and therefore invertible. Hence $\mathrm{Hom}_{\mathscr{C}}(X, Y) \in \infty\text{-Gpd}$ and thus can be equivalently thought of as (the homotopy type of) a topological space. This leads to an interesting blend between a category-theoretic and topological object.

There are many ways to productively achieve this blend and thus multiple models of $(\infty, 1)$-categories. One might propose to work directly with topological categories, whose mapping spaces are "concrete" topological spaces and we can therefore work with $\Pi_\infty(\mathrm{Map}_{\mathscr{C}}(X, Y))$. Unfortunately, this approach has various technical disadvantages. Thankfully, simplicial homotopy theory ensures us that instead of topological spaces we can alternatively, and equivalently, work with some special simplicial sets, Kan complexes. They are much more tractable, essentially algebraic objects that reside in a *very* well-behaved ambient setting. There are two main ways to achieve that. The ambiguity rests solely on whether we'll take our collection of objects and add simplicial sets as mapping spaces between them or if we'll include the collection of objects in the simplicial set.

The first approach incarnates in *simplicial categories*, the first model of $(\infty, 1)$-categories that we'll look into. Instead of Hom-*sets* between objects, we'll ask for a Hom-*simplicial set*. Now out of all simplicial sets we choose the more "topological looking" or ∞-groupoidal, the Kan complexes. Thus our first model of $(\infty, 1)$-categories are categories enriched in $\mathscr{K}an$.

Our other model of interest is ∞-categories.[4]There, we take a single simplicial set and impose axioms that precisely axiomatise the existence of coherent higher homotopical data. The axioms also allow us to import elementary constructions of category theory in an "obvious" way. Only after some elementary work will we be able to show that $\mathrm{Map}_C(x, y)$ is an ∞-groupoid.

Between the two models, ∞-categories derive their greater popularity both from inherent mathematical advantages of the theory and the great development it met in the hands of Joyal, for example in [Joy02], [Joy08a], and even more so by Lurie in the monumental [Lur09] and [Lur18]. In this thesis, we will look into both models mainly with the object of comparing them. In fact this comparative study of the two models is one of the central ingredients in Lurie's approach in [Lur09][5].

One of the main points we wish to make with this thesis stems from this interaction and we believe is at the heart of the connection between homotopical models of HoTT and ∞-topoi. There is a comparison functor $\mathfrak{N} : \mathrm{sCat} \to \mathrm{sSet}$, which turns out to be a Quillen equivalence, an equivalence of *homotopy theories*. This functor restrict to one between models of $(\infty, 1)$-categories namely if $\mathscr{C}$ is $\mathscr{K}an$-enriched then $\mathfrak{N}(\mathscr{C})$ is an ∞-category, see proposition 43. In the same way, for many ∞-categorical objects or behaviours, one can detect "$\mathfrak{N}$-inverse images", namely sCat-properties that "map on" the ∞-categorical ones via $\mathfrak{N}$. In particular, this can be done for all the notions that go into the definition and characterisation of ∞-topoi. Thus Rezk obtains the notion of a *model topos* [Rez10]. The analogy is precise enough to allow for a simultaneous development of higher topos theory in the two settings, with each notion in the ∞-setting having a precise analogue in the simplicial one. This analogy is expanded upon in section 4.

The first subsection of this chapter deals with some intuition about how simplicial categories can be thought of as a model of $(\infty, 1)$-categories. In the second, we turn to the other model of interest, ∞-categories. Using this model we develop elementary ∞-category theory. We begin by introducing the theory of simplicial sets. Having defined ∞-categories, we look into the *proof relevant composition* discussed in the introduction. Then, we turn to the theory of limits and

[4]Also known as quasicategories or quategories.

[5]For more information we refer to Tim Campion's answer in this math overflow question

colimits in ∞-categories and adjoint functors. We proceed with one of the key pieces of the puzzle, the theory of presentable ∞-categories.

A presentable ∞-category, the object of study in subsubsection 2.3.7, is one where we have a set of generators for objects via the formation of colimits. Thus, every object is a colimit of these generators which moreover, are all taken to be κ-compact. One can exploit this to obtain a variety of useful theorems, for example, the ∞-version of Freyd's adjoint functor theorem resulting in characterisations of representable functors. As a result, presentable ∞-categories enjoy particularly nice formal properties. In addition to that, presentable ∞-categories provide a robust link with the simplicial model since an ∞ category is presentable exactly when it is presented by a simplicial combinatorial model category [Proposition A.3.7.6 [Lur09]].

Armed with presentability, in subsection 2.4, we turn to study Lurie's ∞-topoi. They are defined as reflective subcategories of presheaf categories. We survey their characterisation as presentable ∞-categories with universal colimits and descent and lastly look into one of the most basic constructions of ∞-topoi, universes, the ∞-generalisation of subobject classifiers.

2.2 Simplicial categories as a model of $(\infty, 1)$-categories

In ordinary category theory, one mainly deals with locally small categories, those with $\mathrm{Hom}(X, Y) \in \mathrm{Set}$. However, many categories found in nature have extra structure on their Hom-sets. For example, if we consider modules over some ring, the Hom sets naturally inherit the structure of an abelian group. **Enrichment** is a way to account for and systematically study, this phenomenon.

Definition 1. Let $\mathcal{D}$ be a symmetric monoidal category with tensor product $\otimes$. For a textbook treatment we refer to section 11 of [Lan78]. A $\mathcal{D}$-enriched-category $\mathcal{C}$ is given by the following data:

(1) A class of objects $\mathrm{ob}\mathcal{C}$

(2) For each pair of objects $X, Y \in \mathrm{ob}\mathcal{C}$, a $\mathrm{Hom}_{\mathcal{C}}(X, Y) \in \mathrm{ob}\mathcal{D}$, a $\mathcal{D}$-object of morphisms from X to Y.

(3) A composition map $\mathrm{Hom}_{\mathcal{C}}(X, Y) \otimes \mathrm{Hom}_{\mathcal{C}}(Y, Z) \to \mathrm{Hom}_{\mathcal{C}}(X, Z)$.

(4) For each object $X \in \mathrm{ob}\mathcal{C}$ an element $* \xrightarrow{\mathrm{id}_X} \mathrm{Hom}_{\mathcal{C}}(X, X)$

(5) The data above are required to satisfy an associativity and a unitality diagram.

There are complementary definitions of functors and natural transformations of $\mathcal{D}$-enriched categories. Below we give the definition for our special case of interest.

Frequently enrichment amounts to endowing the Hom-sets of a category $\mathcal{C}$ with algebraic features such as additive (and/or) multiplicative structure. For example, the category of vector spaces is enriched over itself. The same is true for Ab and R-Mod. Functions inherit the operations/actions pointwise. Among potentially large categories, small categories can be seen as those enriched in sets.

As discussed in the introduction, our aim in developing this theory is to model $(\infty, 1)$-categories. Thus we'd like our mapping spaces to behave like topological spaces. Relying on the ideas of simplicial homotopy theory we might equivalently ask that the mapping spaces behave like a $\Pi_\infty(X)$. Thus we are particularly interested in enriching categories with the algebraic structure of a simplicial set.

Definition 2. A **simplicially enriched category** $\mathscr{E}$, or a **simplicial category**[6] is a category enriched in the cartesian monoidal structure of simplicial sets presented in the example below.

(1) a collection ob$\mathscr{E}$.

(2) Elements of the set $\mathrm{Hom}_{\mathscr{E}}(X, Y)_0$ are called morphisms and elements of the set $\mathrm{Hom}_{\mathscr{E}}(X, Y)_1$ are called homotopies.

(3) For every triple of objects $X, Y, Z \in \mathscr{E}$ we have a map of simplicial sets playing the role of the composition law which is additionally required to be unital and associative.

$$C_{X,Y,Z} : \mathrm{Hom}_{\mathscr{E}}(X, Y) \times \mathrm{Hom}_{\mathscr{E}}(Y, Z) \to \mathrm{Hom}_{\mathscr{E}}(X, Z)$$

(4) For each $X \in \mathrm{Ob}\mathscr{E}$ a morphism $\mathrm{id}_X \in \mathrm{Map}(X, X)_0$.

A **functor of simplicial categories** $F : \mathscr{D} \to \mathscr{E}$ consists of the following data:

(1) A function $\mathrm{Ob}\mathscr{D} \to \mathrm{Ob}\mathscr{E}$

(2) For any two objects $X, Y \in \mathrm{Ob}\mathscr{D}$ a map of simplicial sets $\mathrm{Map}_{\mathscr{D}}(X, Y) \to \mathrm{Map}_{\mathscr{E}}(FX, FY)$, compatible with identities and composition in the obvious way.

The category of small simplicial categories and simplicial functors between them will be denoted by sCat. Then, to specify a simplicial functor one has to supply ob$\mathscr{D}^2$-many maps of simplicial sets. Needless to say that for non-boring $\mathscr{D}$ that's a lot of data that must be specified.

[6]In this thesis by "simplicial category" we will exclusively refer to a category enriched in simplicial sets, and *not* a "simplicial object in Cat".

Example 3. The category sSet is enriched over itself. sSet being a presheaf topos, carries lots of extra gadgets, notably the ability to form well-behaved products and a product– exponent adjunction. These are all valuable features that have the following consequences:

Firstly, the structure of the cartesian product, which is given canonically for all presheaf topoi, makes $\mathbf{sSet} = \hat{\Delta}$ into a monoidal category. We also have a canonical exponential presheaf given by

$$\mathrm{Fun}(X, Y) := Y^X := \mathrm{Hom}_{\hat{\Delta}}(\Delta^n \times X, Y)$$

and an adjunction

$$- \times X \dashv (-)^X$$

The assignment $(X, Y) \mapsto Y^X$ is functorial in both variables. Moreover, we observe that $Y_0^X = \mathrm{Hom}_{\hat{\Delta}}(\Delta^0 \times X, Y)$ but $\Delta^0 \times X \cong X$ and so $Y_0^X = \mathrm{Hom}_{\hat{\Delta}}(X, Y)$. The composition law is taken to be composition of functors. Taking these internal homs to be our mapping spaces we see that the axioms listed above are satisfied. In [Rie14] section 3.3 one sees that any monoidal $\mathcal{V}$ with right adjoints against all product functors has a canonical $\mathcal{V}$ enrichment. One exploits the adjunction to obtain a unique bifunctor $\underline{\mathrm{Hom}}_{\mathcal{V}}(-,-) : \mathcal{V}^{\mathrm{op}} \times \mathcal{V} \to \mathcal{V}$ of **internal homs**.

In the introduction above we have explained that higher categories are a generalisation of categories where the fundamental notions are expanded to include 2-arrows between morphisms, 3-arrows between 2-arrows etc. A simplicial category naturally fits into this framework, since it comes equipped with a simplicial set of morphisms between any two objects denoted $\mathrm{Hom}_{\mathscr{E}}(X, Y)$. Vertices of this simplicial set play the role of morphisms of $\mathscr{C}$. Take $f, g \in \mathrm{Map}_{\mathscr{C}}(X, Y)_0$. Take a $\sigma \in \mathrm{Map}_{\mathscr{C}}(X, Y)_1$ such that its face operators satisfy $d_1(\sigma) = f, d_0(\sigma) = g$. This is a 2-arrow between f and g. Similarly, we can define explicitly the 3-arrows etc. As discussed above, an (∞, n)-category is one where all the n-morphisms, for $n \geqslant 1$, are invertible. We are exclusively interested in the case $n = 1$, captured by the next definition.

Definition 4. We say $\mathscr{E} \in \mathrm{sCat}$ is **locally Kan** if all of its mapping spaces are Kan complexes (See definition 17). Equivalently, one could say that $\mathscr{E}$ is enriched in Kan complexes.[7]

A central result in Joyal's development of the theory of ∞-categories is that Kan complexes are the same as ∞-groupoids, namely higher categories with exclusively invertible (higher) arrows. Indeed this already appears in his first published work on the topic as Corollary 1.4 in [Joy02]

[7] The full subcategory of Kan complexes is closed under the formation of products and exponentials so the internal hom restricts to $\mathscr{K}an$. In particular, $\mathscr{K}an$ is monoidal.

Then, a locally Kan simplicial category $\mathscr{C}$ can be used to model $(\infty, 1)$-categories and it does so in a way that is very close to the intuition of what $(\infty, 1)$-categories should be. The main advantage of developing the theory in this way is that one immediately obtains some very interesting examples as sSet above.

The disadvantage of working in this model is twofold. First, to define a simplicial functor requires a great amount of data and lots of coherencies between them. Secondly, all these identities hold up to equality which is against the philosophy of homotopical mathematics we are trying to develop. This is not just a philosophical objection. Simplicial functors are too rigid and don't behave as we would like $(\infty, 1)$-functors to behave. In addition to that, in the simplicial model, many constructions don't immediately return the "homotopically correct" object forcing one to keep taking (black-box) cofibrant replacements. For more information, see here or here.

The higher arrows present in a simplicial category can be used to record homotopies between lower arrows. For instance, we can think of a 1-simplex $\sigma \in \mathrm{Map}_{\mathscr{C}}(X, Y)_1$ as a witness that $\mathscr{E}$ arrows $d_1(\sigma) = f, d_0(\sigma) = g$ are homotopic.

With that intuition in mind, we can define:

Definition 5. The homotopy category of a simplicial category $\mathscr{E}$ has:

(1) $\mathrm{ObHo}\mathscr{E} = \mathrm{Ob}\mathscr{E}$

(2) $\mathrm{Hom}_{\mathrm{Ho}\mathscr{E}}(X, Y) = \pi_0(\mathrm{Map}_{\mathscr{E}}(X, Y))$

Recall that the action of π_0 on simplicial sets is to identify $x, y \in X_0$ that are connected by a $f : x \to y$. In this case, vertices of the simplicial set stand for morphisms between objects and paths between them stand for homotopies. Therefore we are identifying homotopic arrows.

In this way, we can see simplicial categories from a homotopic point of view. Having adopted such a perspective we obtain a notion of sameness of simplicial categories, by comparing their respective homotopy theories. It is important to note that the first part of the definition would only compare a "quotiented" out part of the information, and would thus be insufficient. The second part ensures a comparison of higher data too.

Definition 6. A simplicial functor F is a **sDK-equivalence** when:

(1) it induces an equivalence of homotopy categories $\mathrm{Ho}(F) : \mathrm{Ho}\mathscr{C} \to \mathrm{Ho}\mathscr{D}$ is an equivalence of ordinary categories

(2) for any pair of objects $X, Y \in \mathrm{Ob}\mathscr{C}$ the induced map of simplicial sets is a weak equivalence in $\mathrm{sSet}_{\mathrm{KQ}}$,

$$\mathrm{Hom}_{\mathscr{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}}(FX, FY)$$

2.3 ∞-categories as a model of $(\infty, 1)$ categories

∞-categories were first discovered by Broadman and Vogt during their study of homotopy invariant algebraic structures in [BV73] in 1973. It was Joyal who recognised that these objects can be used to develop category theory. He developed this theory first in [Joy02], and later in unpublished work that is still canonical reference [Joy08c] and [Joy08b].

Once again, we recall that an $(\infty, 1)$-category is a structurally richer category whose goal is to help us to category theory up to coherent homotopy. In particular, this implies that the mapping spaces should have the appropriate machinery to obtain a notion of homotopy between morphisms. Instead of working with topologically enriched categories, we rely on simplicial homotopy theory that ensures that Kan complexes are essentially the same as topological spaces. In the previous section, we adopted an "enriched approach" and asked for a collection of objects, and for each pair thereof, a Kan complex's worth of morphisms between them.

The theory of ∞-categories brings the objects inside the simplicial set. By imposing the *weak Kan condition* on simplicial sets, we obtain ∞-categories. This axiom makes the vertices C_0 of the simplicial set behave like objects, and its edges C_1 like arrows of a category, thus internalising the two notions.

The "imagery" associated to the very definition of ∞-categories provides an "obvious" way to mimic the fundamental definitions of a "category", such as composition, associativity and more. Indeed, as Joyal himself said in an online talk: "I must say this was not really difficult. Because...it just worked! I mean, look, it's kind of amazing, that it works. [...] you sit down and you work out the proof and it just works. It's miraculous! Category Theory can be extended to higher category theory and it's easy." Notably, the weak Kan condition is *tautologous* with the existence of coherent higher homotopical data discussed in the introduction.

This is, of course, a proper extension and in this new setting one really develops homotopy coherent category theory. The approach taken by Joyal is mostly to rewrite category theory in this new setting and obtain meaningful results. And indeed, he did so with great success in a series of notes and papers mentioned above. One of the most notable results of this development is the Joyal lifting theorem whose primary consequence ∞-groupoids are the same as Kan complexes. Another was the establishment of the "Joyal model structure"[8] on simplicial sets whose bifibrant objects are exactly the ∞-categories and thus captures the homotopy theory of higher categories.

Around the time when the second result was made public, Lurie became interested in ∞-categories. His contribution to the development of the theory of ∞-categories with [Lur09] and [Lur18] is monumental. His approach differs from Joyal's in an important aspect. Instead

[8] for more information see Section 137

of developing the theory exclusively internally, Lurie also focuses and exploits the comparison with a different model, that of simplicial categories, which we have just presented.

One of the main advantages of choosing the model of ∞-categories is the very well-behaved ambient setting. Indeed, sSet is a presheaf topos and so in particular it is complete and cocomplete. Furthermore, it is symmetric monoidal and enriched over itself. One of the disadvantages is that it becomes hard to obtain concrete examples of ∞-categories. A notable exception is the nerve functor that fully faithfully embeds Cat $\hookrightarrow$ sSet. Unfortunately, such examples are too well behaved and hence their behaviour is not representative of the general case. Lurie's philosophy and its incarnation in the homotopy coherent nerve Quillen equivalence provides an abundance of examples that are more representative of the general case.

We proceed by developing from scratch the elementary and more advanced aspects of the theory required for this thesis. First, we offer a brief introduction to simplicial sets. Then we turn to the task of justifying how these "horn-filling conditions" can be used to obtain a homotopy coherent version of the axioms of category theory. Then we turn to the theory of (co)limits and adjoint functors, presentable ∞-categories and finally ∞-topoi.

2.3.1 Simplicial sets

This exposition[9] is largely based on Jacob Lurie's online resource Kerodon on higher category theory and homotopy theory [Lur18].

Definition 7. We define the **simplex category** Δ by setting

- $\mathrm{ob}(\Delta) := \{[n] \mid n \in \mathbb{N}\}$.
- $\mathrm{Hom}_\Delta([m],[n]) := \{f \in \mathrm{Hom}_{\mathrm{Set}}([m],[n]) \mid f \text{ is order-preserving}\}$,

where $[n] := \{0, 1, \ldots, n\}$ denotes the unique totally ordered set of n elements. Identities and composition of morphisms are given by the set-theoretic functions in the obvious way.

Recall that any preorder $(X, \leqslant)$ can be interpreted as a category with X as the collection of objects in which no parallel arrows exist, for every pair $x, y \in X$, there is a unique arrow $x \to y$ if and only if $x \leqslant y$. Here, reflexivity implies the existence of identities and transitivity encodes the existence of composites. Hence, for $n \in \mathbb{N}$, we can view $[n]$ as a category. From

[9]Much of the code for this section was written in collaboration with Ioli Tente, Melle van Merle, Nathan van den Berg and Sara Rousta for the project *An introduction to simplicial sets and Higher Categories* under the supervision of Gijs Heuts in the UU course *Orientation to Mathematical Research*.

this point of view, we can, for example, represent

$$[2]: \quad \begin{tikzcd} & 2 & \\ 0 \arrow[ur] \arrow[rr] & & 1 \arrow[ul] \end{tikzcd} \qquad [3]: \quad \begin{tikzcd} & 2 \arrow[d] & \\ & 3 & \\ 0 \arrow[uur] \arrow[ur] \arrow[rr] & & 1 \arrow[uul] \arrow[ul] \end{tikzcd}$$

as a triangle and tetrahedron respectively. Note that the uniqueness of arrows ensures that the arrow $0 \to 2$ coincides with the composition $0 \to 1 \to 2$. It is not difficult to see that for any $n, m \in \mathbb{N}$, an order-preserving map

$$f : \{0 \to 1 \to \cdots \to m\} \longrightarrow \{0 \to 1 \to \cdots \to n\}$$

induces a functor from the category $[m]$ to the category $[n]$ and conversely that every functor $f : [m] \to [n]$ must be an order-preserving function on the underlying sets. We can thus also view Δ as a category whose objects are the *categories* of the form $[n]$ for $n \in \mathbb{N}$, and whose morphisms are functors between said categories.

An important feature of Δ is that it has a smaller set of special morphisms that can be used to factor an arbitrary order-preserving map. This is a version of the epi-mono factorisation present in all elementary topoi, or even any pretopos. For any $n \in \mathbb{N}$ and $0 \leqslant i \leqslant n$ we introduce the notation δ^i for the **i-th coface map**, which is the unique arrow from $[n-1]$ to $[n]$ in Δ that 'skips over' only the value i in its image,

$$\delta^i : [n-1] \to [n], \;\; j \mapsto \begin{cases} j, & j < i; \\ j+1, & j \geqslant i. \end{cases} \tag{2.1}$$

Likewise, for any $n \in \mathbb{N}$, and $0 \leqslant i \leqslant n$, let σ^i denote the **i-th codegeneracy map**, which is the unique arrow $[n+1]$ to $[n]$ in Δ that repeats only the value i,

$$\sigma^i : [n+1] \to [n], \;\; j \mapsto \begin{cases} j, & j \leqslant i; \\ j-1, & j > i. \end{cases} \tag{2.2}$$

Unfolding the definitions we get the **cosimplicial identities**,

$$\begin{cases} \delta^j\delta^i = \delta^i\delta^{j-1}, & i < j; \\ \sigma^j\delta^i = \delta^i\sigma^{j-1}, & i < j; \\ \sigma^j\delta^j = \mathrm{id} = \sigma^j\delta^{j+1}; & \\ \sigma^j\delta^i = \delta^{i-1}\sigma^j, & i > j+1; \\ \sigma^j\sigma^i = \sigma^i\sigma^{j+1}, & i \leqslant j, \end{cases} \tag{2.3}$$

which together with the cofaces and codegeneracies generate Δ. Now our first important definition is that of a simplicial set.

Definition 8. A **simplicial set** is a presheaf on Δ, i.e. a functor $X : \Delta^{\mathrm{op}} \to \mathrm{Set}$. We denote the image of an object $[n]$ under X by X_n.

To give the data of a simplicial set X it is enough to evaluate the functor on the generating maps $\delta^i : [n-1] \to [n]$ and $\sigma^i : [n+1] \to [n]$ to obtain morphisms $d_i : X_n \to X_{n-1}$ and $s_i : X_n \to X_{n+1}$ called the i**-th face map** and the i**-th degeneracy map** respectively. We have by the functoriality the **simplicial identities**,

$$\begin{cases} d_jd_i = d_id_{j-1}, & i < j; \\ s_jd_i = d_is_{j-1}, & i < j; \\ s_jd_j = 1 = s_jd_{j+1}; & \\ s_jd_i = d_{i-1}s_j, & i > j+1; \\ s_js_i = s_is_{j+1}, & i \leqslant j. \end{cases} \tag{2.4}$$

The category of simplicial sets is denoted by $\mathrm{sSet} = \mathrm{Set}_\Delta := \hat{\Delta} = \mathrm{Hom}_{\mathrm{Cat}}(\Delta^{\mathrm{op}}, \mathrm{Set})$ and a morphism of simplicial sets $f : X \to Y$ is thus a natural transformation of such functors. Explicitly, it is a collection of functions $\{f_n : X_n \to Y_n\}_{n\geqslant 0}$ which commute with the arrows of Δ. This suffices to be checked on the faces and degeneracies. The examination of this category gives rise to some important examples of simplicial sets.

Proposition 9. Every order-preserving map $f : [n] \to [m]$ can be factored, in an essentially unique way, as a composition of s_i followed by a composition of d_j for some choice of indices.

Corollary 10. To determine that an n-indexed collection of functions $f_n : X_n \to Y_n$ is a natural transformation between S and T it suffices to check the naturality squares for the face and degeneracy maps.

Example 11. Consider the representable functor

$$\Delta^n := \mathrm{Hom}_\Delta(-, [n]) : \Delta^{\mathrm{op}} \to \mathrm{Set},$$

operating on morphisms by precomposition. This simplicial set is called the **standard** n-**simplex**. Note that by the Yoneda lemma, for any simplicial set X, we have a natural isomorphism

$$\mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \cong X_n$$

between the n-**simplices** or n-**cells** X_n of X and the set of simplicial morphisms of the form $\Delta^n \to X$.

2.3.2 Important sub-complexes and the Kan condition

Definition 12. Given $S \subseteq [n]$ we define $\Delta^S \subseteq \Delta^n$ to be $\Delta^{n,S}_m = \{f : [m] \to [n] : \mathrm{im} f \subseteq S\}$

For instance, $\Delta^{\{k\}} \simeq *$ and $\Delta^{\{0,2\}} =$

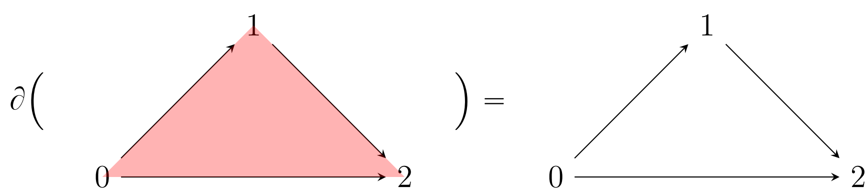
$$\begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

Definition 13. Let $n \in \mathbb{N}$. We define the simplicial set $\partial\Delta^n : \Delta^{\mathrm{op}} \to \mathrm{Set}$, called the **boundary** of Δ^n, by setting

$$\partial\Delta^n(-) = \bigcup_{0 \leqslant i \leqslant n} \delta^i(-)$$

Morally, we just have to rely on our topological intuition on boundaries. We take the boundary to be the union of all the faces, meaning we just remove the interior of the simplex.

Pictorially:



An alternative, and of course equivalent, definition is

$$\partial\Delta^n_m = \bigcup_{i=0}^{n} \Delta^{[n]\setminus i} = \{f \in \mathrm{Hom}_\Delta([m], [n]) \mid [n] \nsubseteq f([m])\}$$

so that $\partial\Delta^n([m])$ is the set of all non-surjective morphisms from $[m]$ to $[n]$.

Definition 14. Let $n \in \mathbb{N}$ and $0 \leqslant k \leqslant n$. We define the simplicial set $\Lambda^n_i : \Delta^{\mathrm{op}} \to \mathrm{Set}$, called the i-**th horn in** Δ^n, by setting

$$\Lambda^n_k := \bigcup_{i \neq k} \delta^i(-)$$

We call Λ^n_k an **inner horn** if $0 < k < n$, and an **outer horn** if $k = 0$ or $k = n$.

The k^{th} horn is the full simplex minus the interior and the face opposite the k^{th} vertex. We immediately see that $\Lambda^n_k \subseteq \partial\Delta^n$. For example,

$$\Lambda^2_1 : \quad \begin{tikzcd} & 1 \arrow[dr] & \\ 0 \arrow[ur] & & 2 \end{tikzcd} \qquad \Lambda^2_0 : \quad \begin{tikzcd} & 1 & \\ 0 \arrow[ur] \arrow[rr] & & 2 \end{tikzcd}$$

Now that we've defined these subcomplexes it is important to also explain the data required to define a map $\partial\Delta^n \to S$ or $\Lambda^n_k \to S$. A boundary is an n-simplex minus the interior. Intuitively, we may understand this as a collection of $n-1$-faces all glued together appropriately along their intersections, a *collatable* collection of $n-1$ faces. For example, $\partial\Delta^2$ above, an empty triangle, can be seen as three intervals glued along their endpoints (boundaries) in a specific way. This intuition can be generalised. Following this line of reasoning to determine a map $\partial\Delta^2 \to X_2$ it suffices to determine the images of the three aforementioned intervals and to make sure that their images will too be *collatable*. Λ^n_k and $\partial\Delta^n$ differ only by a face, so to determine a map $\Lambda^n_k \to S$ on has to determine an *incomplete*, but still a collatable sequence of $n-1$ faces. This intuition is formalised in the following propositions.

Proposition 15. The map given by pre-composition by δ_l is injective

$$\mathrm{Hom}_{\mathrm{sSet}}(\partial\Delta^n, S) \simeq (S_{n-1})^n$$

and its image consists of those sequences $(\sigma_0, \ldots, \sigma_n)$ where $\sigma_i \in (S_{n-1})^n$ satisfy the *collatability condition*: $d_i(\sigma_j) = d_j(\sigma_i)$

Proposition 16. The map given by pre-composition by δ_l, restriction to the $l_t h$ face, is injective

$$\mathrm{Hom}_{\mathrm{sSet}}(\Lambda^n_k, S) \simeq (S_{n-1})^{n-1}$$

and its image consists of those sequences $(\sigma_0, \ldots, \sigma_{k-1}, \bullet, \sigma_{k+1}, \sigma_n)$ where $\sigma_i \in S_{n-1}$ satisfy a *collatability condition*: $d_i(\sigma_j) = d_j(\sigma_i)$ for $i, j \in [n] \backslash \{k\}$ and $i < j$.

Proof. [Lur18, Tag 050F] $\square$

Definition 17. Let X be a simplicial set. We call X a **Kan complex** if for any horn Λ^n_k, each map $\sigma : \Lambda^n_k \to X$, can be extended to $\tau : \Delta^n \to X$.

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\tau"] \arrow[d, hook] & X \\ \Delta^n \arrow[ur, dashed, "\exists t"'] & \end{tikzcd}$$

Equivalently, we demand that pre-composition with the horn inclusion is surjective on Hom sets:

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\tau"] \arrow[d, hook, "i"'] & X \\ \Delta^n \arrow[ur, dashed, "\exists t"'] & \end{tikzcd} \qquad\qquad \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n, X) \xrightarrow{-\circ i} \mathrm{Hom}_{\mathrm{sSet}}(\Lambda^n_k, X)$$

Definition 18. Let X be a simplicial set. We call X an ∞-**category** if for any **inner horn**, Λ^n_k, with $0 < k < n$, any map $\sigma : \Lambda^n_i \to X$, can be extended $\tau : \Delta^n \to X$.

These two definitions can be generalised to maps of simplicial sets and indeed they constitute some of the central objects of study of simplicial homotopy theory.

Definition 19. A map of simplicial sets $f : X \to Y$ is a **Kan fibration** if it **admits lifts** against all horn inclusions if all commutative squares as below admit a diagonal lift s producing two commutative triangles. Kan fibrations will be denoted with a double-headed arrow, $\twoheadrightarrow$.

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d, two heads, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dashed, "s"] & Y \end{tikzcd}$$

The intuition for Kan fibrations is the following. Say we have an extension problem as given by the top left triangle of the square. We have a $\sigma : \Lambda^n_k \to X$ and wish to extend it to a $s : \Delta^n \to X$. If we have *any map* $f : X \to Y$ we can compose and obtain $f \circ \sigma : \Lambda^n_k \to Y$. Commutativity of the outer squares ensures that the bottom map $\Delta^n \to Y$ is a solution of the extension problem for $f \circ \sigma$. f is a Kan fibration exactly when that solution **can be lifted through p** and solve the original extension problem. Thus the very definition of Kan fibrations captures the intuitive idea that to solve an extension, or horn-filling problem in X it suffices to solve it inside the image of a Kan fibration $p(X) \subseteq Y$. Stated differently, Kan fibrations are precisely the maps that allow solutions to extension problems to be transferred backwards.

Here are some important variants of this definition.

Definition 20. An **inner fibration** of simplicial sets $f : X \to Y$ is a map that admits lifts against all inner horn inclusions. This is the same definition as above with $0 < k < n$. We sometimes denote an inner fibration as $X \twoheadrightarrow_i Y$.

Corollary 21. A simplicial set K is a Kan complex $\iff$ the unique map $K \twoheadrightarrow *$ is a Kan fibration. Similarly, a simplicial set C is an ∞-category $\iff$ the unique map $C \twoheadrightarrow_i *$ is an inner fibration.

Remark 22. With elementary means one can show that inner fibrations are stable under pullback. Indeed, it is a special case of proposition 102. In particular, this means that fibers of an inner fibration, namely base changes over a point, are ∞-categories themselves. Therefore one can reasonably think of an inner fibration $X \twoheadrightarrow_i Y$ as a Y_0-parametrised family of ∞-categories.

Using this *horn-filling condition* one can internalise most of category theory inside ∞-categories. This is what the next section is devoted to.

2.3.3 Basic category theory ∞-categories

In this first section, we look at why and how imposing weak Kan condition on a simplicial set makes it behave like a category. Namely, we look at how one can internalise proof-relevant version of the axioms of category theory in this new setting. We proceed by surveying the internalisation of more elaborate classic category theoretic constructions such as (co)limits, and adjoint functors. We use these to introduce locally cartesian closed ∞-categories and eventually discuss Grothendieck ∞-topoi.

∞-categories are simplicial sets satisfying the weak Kan condition. We interpret C_n as the collection of n-morphisms. In particular, we call a 0-simplex a **vertex** or an **object** of C and a 1-simplex and **edge** or a **morphism**. The simplicial operators d_0, d_1 pick out the two endpoints of an arrow $f \in C_1$ and therefore serve as the **target** and **source** maps. All this can be done in a general simplicial set. What distinguishes ∞-categories is the ability to define well-behaved **homotopy coherent composition, associativity, and so on.** This is the raison d'être of the definition of ∞-categories. Recall,

Definition 23. A simplicial set C is an ∞-category any map $\sigma : \Lambda^n_k \to C$, with $0 < k < n$, admits an extension to Δ^n

$$\begin{tikzcd} \Lambda^n_k \arrow[r, "\sigma"] \arrow[d, hook] & C \\ \Delta^n \arrow[ur, dashed] & \end{tikzcd}$$

For $n = 2$ we obtain the following picture in C:

$$\Lambda^2_1 \to C : \qquad \begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed] & & z \end{tikzcd}$$

The map $\Lambda^2_1 \to C$ corresponds to the data given by the solid arrows. An extension of it to Δ^2 corresponds to providing an additional 1-simplex with endpoints x, z and a 2-simplex filling the triangle. With this idea in mind, we define composition accordingly. As we explained in the Introduction, in the higher setting, what used to be equalities in 1-categories now hold only up to higher morphism. We proceed by making this mathematically precise.

Definition 24. Let C be an ∞-category and take $f, g \in C_1$. We call f, g **right** (respectively left homotopic) $\iff$ $\exists$ a 2-simplex σ as below: Left homotopy on the left and right homotopy on the right. For $c \in C_0$ we suggestively write $\mathrm{id}_c := s_0(c)$.

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & & x \end{tikzcd} \qquad \begin{tikzcd} & x \arrow[dr, "f"] & \\ y \arrow[ur, "\mathrm{id}_y"] & & y \arrow[ll, "g"'] \end{tikzcd}$$

Proposition 25. (Proposition 11.6 in [Rez22]) In an ∞-category the notions of left and right homotopy coincide and induce an equivalence relation on C_1. Write $f \sim g$ for homotopic maps and $[f]$ to denote the equivalence class of $f \in C_1$.

The next two propositions are an excellent example of how the good old axioms of category theory hold only up to homotopy in the higher setting.

Definition 26. Let C be an ∞-category. We say that $h \in C_1$ is a **composition** of $f, g \in C_1$ exactly when there exists a $\tau : \Delta^2 \to C$ with $d_2(\tau) = f, d_0(\tau) = g$ and $d_1(\tau) = h$. In that case we say that τ *witnesses* the equality $[g] \circ [f] = [h]$

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

Of course, it can be shown (Lemma 11.8 in [Rez22]) that the homotopy class of $[f] \circ [g]$ is independent of the choice of representatives.

Proposition 27. (Lemma 13.9 (exercise) in [Rez22]) For arbitrary $c \in C_0$ we write $\mathrm{id}_c := s_0(c) \in C_1$ for the unique degenerate arrow induced by the vertex. One can think of this as the "canonical constant loop" over a point. Then for arbitrary $f \in C_1$ we have

$$[f] \circ [\mathrm{id}_c] = [f] = [\mathrm{id}_c] \circ [f]$$

Equipped with all these propositions we see that the following construction is well defined.

Definition 28. Let C be an ∞-category. Then we can define an ordinary category, its **homotopy category** hC by setting:

$$\mathrm{ob}hC := C_0 \quad \& \quad \mathrm{Hom}_{hC}(x,y) = {}^{\mathrm{Hom}_C(x,y)}\!\big/\!_{\sim}$$

Definition 29. A morphism $f \in C_1$ will be called an isomorphism $\iff_{\mathrm{df}} [f]$ is an isomorphism in hC.

In short, we used inner 2-horn fillers to record the data that some morphism h is "equal" to the composite of some f, g. So the axiom for ∞-categories for $n = 2$ asserts that for any pair of composable morphisms can be composed and their composition is unique up to homotopy.

What about outer fillers? We could for example consider,

$$\Lambda^2_0 \to C : \qquad \begin{tikzcd} & y \arrow[dr, dashed] & \\ x \arrow[ur, "f"] \arrow[rr, "\mathrm{id}_x"'] & & x \end{tikzcd}$$

Filling this horn would correspond to obtaining a morphism $g : y \to x$ such that $[g] \circ [f] = [\mathrm{id}_x]$, i.e. a left inverse for f. Similarly, if we can fill $\Lambda^2_2 \to C$ we can obtain right inverses for arrows f. Then, again via elementary means, and potentially by filling a 3-horn or two, these left and right inverses can be shown to be equal. Then, the ability to fill outer horns allows us to obtain inverses to morphisms. For example, assume that $\mathcal{G}$ is a Kan complex. Take an arbitrary morphism f. Using outer horn fillers as indicated above we can show that f is an isomorphism. Therefore $\mathcal{G}$ is an ∞-groupoid. Therefore Kan complexes are ∞-groupoids. In fact, with a bit more work, one can also show the converse of this statement and obtain:

Theorem 30. Let C be an ∞-category. The following are equivalent:

(1) C is an ∞-groupoid

(2) hC is a groupoid

(3) C is a Kan complex

The only non-trivial implication is $\big((1) \iff_{\mathrm{df}} (2)\big) \implies (3)$. It is a corollary of the *Joyal lifting theorem*

Theorem 31. Let $f : C \to D$ be an inner fibration between ∞-categories. Then,

(1) $\sigma(\Delta^{\{0,1\}})$ is an iso in C then $(\Lambda^n_0 \hookrightarrow \Delta^n)$ lifts against f.

(2) $\tau(\Delta^{\{n-1,n\}})$ is an iso in C then $(\Lambda^n_n \hookrightarrow \Delta^n)$ lifts against f.

The proof rests on, among other things, the notion of an isofibration, which we record because of its independent interest.

Definition 32. A map of simplicial sets $p : C \to D$ is an **isofibration** $\iff_{\text{df}}$ it is an inner fibration and if we have an iso in D_1 and we can lift its starting point through p then we can lift the whole isomorphism, meaning, for any $x \in C_0$ and any isomorphism $f : p(x) \to d$ in D_1 there exists $y \in C_0$ and an isomorphism $f' : x \to y \in C_1$ such that $p(f') = f$.

A crucial ingredient for this thesis is functor ∞-categories. Recall that sSet being a presheaf topos has exponential objects. In general they are given by

$$\mathrm{Fun}(X, Y) := Y^X := \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \times X, Y)$$

This formula is derivable from the presheaf structure. These exponential objects play the role of internal homs. The collection of ∞-categories is closed under the formation of exponential objects [Joy08a], as the next theorem records.

Theorem 33. If K is an arbitrary simplicial set and C is an ∞-category, then $\mathrm{Fun}(K, C)$ is an ∞-category

Note that vertices of $\mathrm{Fun}(X, Y)$ are natural transformations between the underlying simplicial sets X, Y. They are the ∞-functors of ∞-categories.

Using the internal homs we can finally show in what sense do ∞-categories satisfy the intuition required from all $(\infty, 1)$-categories regarding having a *space*, or ∞-groupoid, of morphisms between any two objects. First, we define the **arrow ∞-category** of an ∞-category to be $\mathrm{Fun}(\Delta^1, C)$. Then, we obtain the source and target maps by restricting along $i : \partial\Delta^1 = \Delta^0 \sqcup \Delta^0 \hookrightarrow \Delta^1$. This induces $\mathrm{Fun}(\Delta^1, C) \to C \times C$. Then, if we take the fiber of this map over a pair of vertices (x, y), we obtain the mapping space $\mathrm{Map}_C(x, y)$. Now the theory of anodyne morphisms and of enriched lifting problems ensures that the map i above is anodyne and that restriction along an anodyne morphism is a trivial Kan fibration. Those being stable under pullback, we obtain that $\mathrm{Map}_C(x, y)$ is a contractible ∞-groupoid.

Thinking along the same lines we can provide an elegant characterisation of ∞-categories. The discussion after the definition of ∞-categories motivates how we can think of maps $\tau : L^2_1 \to C$ as picking out a composable pair of morphisms in C. Then, maps $\sigma : \Delta^2 \to C$ with $\sigma|_{\Lambda^2_1} = \tau$ correspond to a composition thereof. This leads us to consider the map induced by restriction along $L^2_1 \subset \Delta^2$ and obtain $\mathrm{Fun}(\Delta^2, C) \to \mathrm{Fun}(\Lambda^2_1, C)$. A fiber of this map over a pair of composable maps (f, g) denotes the space of compositions of f, g. Reasoning similarly as above we obtain that this fiber is a contractible ∞-groupoid/ space.

Proposition 34. A simplicial set C is an ∞-category if and only if the morphism as above is a trivial fibration.

We can interpret this to mean that C is an ∞-category exactly when it has compositions that are unique *up to homotopy.* Observe that the appearance of internal homs implicates all the coherent higher data.

We proceed with a discussion of equivalences of ∞-categories. We *define* natural transformations between such functors to be the elements of the set $\mathrm{Fun}(X, Y)_1$.

Definition 35. Let C, D be ∞-categories. Let f_0, f_1 be a pair of functors between them. Then a **natural transformation** between f_i is defined to be a morphism $\alpha : f_0 \to f_1$ in $\mathrm{Fun}(C, D)$

Definition 36. A natural transformation $\alpha : f_0 \to f_1$ is a **natural isomorphism** exactly when α is an isomorphism in the ∞-category $\mathrm{Fun}(C, D)$.

Proposition 37. (Theorem 5.14 in [Joy08b]) a natural transformation $a : u \to v \in \mathrm{Fun}(A, X)$ is a natural isomorphism if and only if for all vertices x of A, the induced $a_x : u(x) \to v(x)$ is an isomorphism in X.

Definition 38. A functor $f : C \to D$ between ∞-categories is said to be a **categorical equivalence** when there exists a functor $g : D \to C$ such that gf is naturally isomorphic to id_C in $\mathrm{Fun}(C, C)$ and fg is naturally isomorphic to id_D in $\mathrm{Fun}(D, D)$.

By generalising appropriately the notion of fully faithful and essentially surjective we rediscover the well-known characterisation of equivalences of ordinary categories.

Definition 39. A map of simplicial sets is called a **weak homotopy equivalence** exactly when its geometric realization $|f|$ is a weak homotopy equivalence of topological spaces.

Remark 40. We have just seen two ways in which a pair of ∞-categories can be said to be "almost the same". The first is when there is an invertible-up-to-homotopy functor between them. This is a direct generalisation of the notion of an equivalence of categories. In the second we care about the homotopy type of the topological space captured by the simplicial sets. Later, we shall see that these two different ways of comparing simplicial sets give rise to two classes of *weak equivalences* which in turn participate in two *model structures* on simplicial sets. These two model structures capture the homotopy theory of higher categories and spaces respectively.

2.3.4 Examples of ∞-categories

We've developed part of the theory of ∞-categories but we haven't yet touched on examples. This is because getting internal examples of ∞-categories is one of the main disadvantages of the theory. The examples we can readily obtain concern small 1-categories via the nerve construction.

Let $\mathcal{C}$ be a small category. We define a simplicial set $N\mathcal{C}$ whose n-simplices are functors $[n] \to \mathcal{C}$. Since $[n] = \{0 < 1 < \cdots < n\}$ such a functor corresponds to an n-tuple of composable morphisms in $\mathcal{C}$. For example $N\mathcal{C}_0 = \mathrm{ob}\mathcal{C}$ and $N\mathcal{C}_1 = \mathrm{Arr}\mathcal{C}$. We can additionally take the face operators to compose two consecutive maps and thus produce an $n-1$-simplex. Similarly, degeneracy operators add an identity on some object of the string thus producing an $n+1$ simplex. With these definitions $N\mathcal{C} \in \mathrm{sSet}$. Moreover it is not hard to prove that for any $\mathcal{C}$, $N\mathcal{C}$ is an ∞-category [Lur18, Tag 0031]. In fact, the nerve construction induces a fully faithful embedding $\mathrm{Cat} \hookrightarrow \mathrm{sSet}$, taking values in ∞-categories.

Because of this embedding, we think of nerves of small categories as 1-categories seen as "trivial" ∞-categories. Thus, the benefit of the nerve construction is that it serves as a reality check for all the higher categorical constructions we are doing in the new setting. For example, one can prove that $N\mathcal{C}$ is a Kan complex, i.e. an ∞-groupoid $\iff$ $\mathcal{C}$ is a groupoid. Results of a similar philosophy abound.

The problem is that nerves of small categories are too well behaved as ∞-categories and therefore are not indicative of the general case. For example, $S \cong N\mathcal{C} \iff S$ has *unique* horn filler for all inner horns.

This brings us back to our need for examples of ∞-categories. The solution proposed and exploited by Lurie amounts to functorially importing examples from the other model of $(\infty, 1)$-categories we've discussed, locally Kan simplicial categories. This functorial construction is the homotopy coherent nerve.

Intuitively we understand this construction as follows. For an insightful and concise exposition of this phenomenon, we refer to [Rie23]. As has been previously emphasised, the theory of ∞-categories essentially amounts to doing *category theory up to coherent homotopy* where all the composition/associativity data of the category are given by making *coherent* choices of higher and higher homotopies. That means we require homotopies recording compositions $h_{i,j,k} : f_{i,k} \simeq f_{j,k} \circ f_{i,j}$, a 2-homotopy specifying associativity data between three composable morphisms et cetera.

Historically, the development of such ideas was in the context of the category of topological spaces which is enriched in itself. Later topological spaces were replaced with the more tractable and algebraic in nature of Kan complexes, transferring these considerations in the realm of simplicial categories. By carefully examining the data that go into such a structure we realise they can be organised in a functorial construction. A **homotopy coherent diagram of shape A in $S \in$ sCat** is given by a diagram $\mathfrak{C}A \to S$.

For the purposes of developing homotopy coherent category theory the shapes we are interested in are essentially captured by the Δ^n, Δ^2 capturing homotopy-commutative triangles, Δ^3 capturing up-to-homotopy associativity data and so on. It turns out that the functor $\mathfrak{C}$: sSet $\to$ sCat admits a right adjoint, the **homotopy coherent nerve**$\mathfrak{N}$sCat $\to$ sSet. We survey parts of this construction below. The natural bijection of the adjunction asserts that

$$\{\mathfrak{C}\Delta^n \to S\} \cong \{\Delta^n \to \mathfrak{N}S\}$$

That means that the *homotopy coherent diagrams of category-theoretic interest in S can be arranged as simplices of the simplicial set $\mathfrak{N}S$*

To turn these ideas in formal mathematics, one begins by defining a functor $\Delta \to$ sCat. By the free cocompletion property of sSet this extends to a functor $\mathfrak{C}$: sSet $\to$ sCat that admits an adjoint. It is that adjoint we are interested in. In subsection 3.5 we will see this adjunction from a different light, as asserting that two ways of presenting the homotopy theory of $(\infty, 1)$-categories are equivalent. In fact, this right adjoint, the *homotopy coherent nerve* lies at the heart of a deep analogy. One of the main theorems we survey asserts that every ∞-topos arises as the simplicial set of homotopy coherent diagrams of a model category that models HoTT/UF.

Definition 41. Following Cisinski in [Cis16], for each $n \geqslant 0$ we define a simplicial category $\mathfrak{C}[\Delta^n]$ as follows: The objects are the natural numbers $0 \leqslant k \leqslant n$. For a pair of integers let $\mathrm{Hom}_n(k,l)$ denote the $S \subseteq [n]$ with $\min S = k$ and $\max S = l$. Then, $\mathrm{Hom}_n(k,l)$ is included in the poset $(\mathcal{P}([n]), \subseteq)$ and thus it is a poset itself. Thus we can canonically view it as a category. Then, we define the hom-simplicial-sets of our simplicial category to be

$$\mathrm{Map}_{\mathfrak{C}[\Delta^n]}(k,l) := N\Big(\mathrm{Hom}_n(k,l)\Big)$$

Example 42. We look into the simplicial category $\mathfrak{C}[\Delta^2]$. It has the same objects as Δ^2, namely $0, 1, 2$. Now we compute the mapping spaces. Spaces of the form $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(i, i{+}1) \cong \Delta^0$ because there is a unique chain between consecutive edges. On the contrary if $Q = [2]$, then to compute $\mathrm{Map}_{\mathfrak{C}[\Delta^2]}(0,2)$ we notice that there are exactly two chains in $[2]$ with the desired endpoints, $0 < 1 < 2$ and $0 < 2$. Of course $0 < 1 < 2 \supseteq 0 < 2$ and therefore the mapping space is $N([1]) \cong \Delta^1$. So, $\mathfrak{C}(\Delta^2)$ is depicted below:

$$\begin{tikzcd}
& 1 \arrow[dr, "1<2"] & \\
0 \arrow[ur, "0<1"] \arrow[rr, bend left, "0<1<2"] \arrow[rr, bend right, "0<2"'] \arrow[rr, phantom, "\Downarrow"] & & 2
\end{tikzcd}$$

With $(0 < 1) \circ (1 < 2) = 0 < 1 < 2 \neq 0 < 2$. So the triangle doesn't commute strictly, instead, it commutes only up to homotopy.

It can be shown that sCat is cocomplete. So, by the free cocompletion property of $\mathrm{sSet} = \hat{\Delta}$ the functor $\mathfrak{C}$ that was defined on the representable Δ^n can be extended to all of sSet in a colimit preserving way. Moreover, it induces an adjunction:

$$\mathfrak{C} : \mathrm{sSet} \rightleftarrows \mathrm{sCat} : \mathfrak{N}$$

We also obtain a formula for the adjoint $\mathfrak{N}(\mathscr{C})_n := \mathrm{Hom}_{\mathrm{sCat}}(\mathfrak{C}\Delta^n, \mathscr{C})$.

For example, since one can easily compute that for $i = 0, 1$, $P[i] = [i]$ and therefore $\mathfrak{N}(C)_i = NC_i = C_i$. More importantly, through the computation in example **??** above, we get that $\sigma \in \mathfrak{N}(\mathscr{C})_2$ determines a diagram in $\mathscr{C}$:

$$\begin{tikzcd}
& C_1 \arrow[dr, "f_{1,2}"] & \\
C_0 \arrow[ur, "f_{0,1}"] \arrow[rr, bend left, "f_{1,2} \circ f_{0,1}"] \arrow[rr, bend right, "f_{0,2}"'] \arrow[rr, phantom, "\Downarrow\alpha"] & & C_2
\end{tikzcd}$$

Conversely, any such diagram determines a unique 2 simplex in the coherent nerve. If we substitue $f_{i,i+1} = \mathrm{id}_{C_i}$, then we see that two arrows f, g are connected by a 2-arrow in $C \iff$ they are homotopic in the ∞-categorical sense of Definition 25.

The intuition here is the following: A homotopy-commutative triangle in the simplicial category $\mathscr{C}$ is given by a pair of maps, here $f_{0,1}$ and $f_{1,2}$, another map $f_{0,2}$ and a homotopy $h : f_{0,2} \simeq f_{1,2} \circ f_{0,1}$. In the simplicial category $\mathscr{C}$ we have a notion of strict composition and a notion of a homotopy. So we obtain a homotopy commutative triangle for each map homotopic to the strict composition $f_{1,2} \circ f_{0,1}$. This can be seen as a sort of "fattening-up"[10] of the strict composite $f_{1,2} \circ f_{0,1}$ in an entire homotopy class of composites matching the up-to-homotopy composition we have in the context of ∞-categories.

[10] This is the actual scholarly term.

As discussed above, an important use of the homotopy coherent nerve is to produce examples of ∞-categories out of locally Kan simplicial categories:

Theorem 43. If $\mathscr{C} \in \mathrm{sCat}$ is locally Kan, then $\mathfrak{N}(\mathscr{C})$ is an ∞-category.

Proof. For a proof we refer the reader to Prop 1.1.5.10 in [Lur09]. $\square$

For example, nerves of ordinary categories arise as $\mathfrak{N}(\underline{C})$ homotopy coherent nerve of the *constant simplicial category on* C, see [Lur18, Tag 00KZ], whose hom-simplicial-sets is the constant simplicial set on $\mathrm{Hom}_C(X, Y)$.

We obtain some of the most important ∞-categories in this way. For example, let $\mathscr{K}\mathrm{an}$ denote the full subcategory of sSet spanned by the Kan complexes. This is a *simplicial* subcategory of sSet. In fact, using the theory of internal lifting problems, it can be shown that it is closed under the formation of internal homs, namely if K is a Kan complex, then so is $\mathrm{Fun}(S, K)$ for any S. This means that the subcategory $\mathscr{K}\mathrm{an}$ is locally Kan. For another proof of this fact using the theory of simplicial model categories, one could observe that $\mathscr{K}\mathrm{an} = \mathrm{sSet}^{\circ}_{KQ}$, see 178.

Definition 44. We define the ∞-category of spaces $\mathcal{S} := \mathfrak{N}(\mathscr{K}\mathrm{an})$

Let qCat denote the full subcategory of sSet spanned by ∞-categories. The internal homs make this into a simplicial category. In general, this simplicial category is not locally Kan. To remedy that, instead of taking internal homs we take their maximal sub-∞-groupoid, $\mathrm{Fun}(X, Y)^{\cong}$. This is now a locally Kan simplicial category. We denote it by Qcat

Definition 45. Define $\mathrm{Cat}_\infty = \mathfrak{N}(\mathrm{Qcat})$. We think of Cat_∞ as the ∞-category of small ∞-categories

2.3.5 Limits and colimits

To obtain a notion of limits and colimits we generalise *cones* on a functor, we consider the ∞-category of such cones, and ask that it has a terminal/initial object respectively. The way to generalise cones in the setting of ∞-categories comes from the work of Joyal and amounts to obtaining a *join* $\dashv$ *slice* adjunction. It turns out that it is easier to describe the join functor and explain the slice via the adjunction, so that is how we proceed.

Definition 46. Let $\mathcal{C}, \mathcal{D}$ be categories. We define their **join** to be the category resulting from "juxtaposing" $\mathcal{C}, \mathcal{D}$ and adding exactly one arrow from every object of $\mathcal{C}$ to every object in $\mathcal{D}$.

This immediately generalises to simplicial sets.

Definition 47. Let $X, Y \in \mathrm{sSet}$. Define their **join** to be the simplicial set whose n- simplices are:

$$X \star Y = \coprod_{i+j+1=n} K_i \times L_j$$

Examples 48. *(1)* It is a classical result that $\Delta^n \star \Delta^m = \Delta^{m+n+1}$.

(2) Take an ∞-category C and form the ∞-category $C^{\rhd} := C \star \Delta^0$. Then $C_1^{\rhd}$ consists of the arrows of C and $\forall x \in C_0$ a unique arrow $x \to * \in \Delta_0^0$. Thus $C^{\rhd}$ formally adjoins a terminal object to an ∞-category C. Dually, we think of $C^{\lhd} := \Delta^0 \star C$ as formally adjoining an initial object to C.

In the context of the previous example it is also important to consider what do maps $p : C^{\rhd} \to D$ look like. If $C^{\rhd}$ is $C+$ terminal object we might suspect, and correctly so, that to specify such a map it suffices to provide the following data:

(1) A map $f : C \to D$.

(2) A vertex $v \in D_0$.

(3) $\forall$ vertex $x \in C_0$, an arrow $d_x : f(x) \to v \in D_1$

(4) The maps d_x above should be compatible with one another in the sense that for any map $a : x \to y$ in C_1 we should get a 2 simplex in D witnessing the commutative triangle $d_x = d_y \circ f(a)$.

(5) Higher coherencies should be treated similarly.

We have a canonical inclusion $i : C \hookrightarrow C^{\rhd}$ and we observe that $p|_C = p \circ i = f$. Thus we say that such a map p extends f. Using this imagery we are indeed very close to obtaining a notion of cocone on f. Let $f : I \to C$ be a diagram whose colimit we'd like to compute. The point is that the data of a map $f : I^{\rhd} \to C$ that extends f captures exactly what our intuition of a cocone on f is. Indeed, by the discussion above, such a map is given by f, a choice of a vertex d in C, the *nadir* of the cocone, and compatible maps into the nadir d.

As explained in the introduction of this subsection, the slice construction is functorial and admits a left adjoint. Via this adjunction, it turns out that maps of the form $I \star \Delta^0 \to C$ transpose to maps $\Delta^0 \to C_{/f}$ and thus we think of the latter as the ∞-category of cocones on $f : I \to C$. One can then generalise the notion of terminal and initial object in an ∞-category. Lastly, we define a colimit for $f : I \to C$ to be a terminal object in the ∞-category of cocones $C_{/f}$.

Definition 49. The join construction can be arranged in a functor $S \star - : \mathrm{sSet} \to {}^{S}\!\diagup_{\mathrm{sSet}}$

In the presheaf topos, $\hat{\Delta}$ all (co)limits are computed componentwise. Moreover, the product functor admits a canonically given right adjoint, the internal hom, and therefore commutes with colimits. Using these two facts and the degree-wise definition of join on can show that $S \star -$ preserves colimits. That makes us suspect it might admit a right adjoint. It turns out that is indeed the case and we can construct it explicitly. We'll see why we can interpret this construction as "the slice ∞-categories". We will use them to capture the "∞-categories of (co)cones on some diagram".

Denote it by: $(p : S \to C) \mapsto C_{p/} : {}^{S}\!\diagup_{\mathrm{sSet}} \to \mathrm{sSet}$. The adjunction means we have bijective correspondences between hom-sets as seen below:

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[d] & C \\ S \star K \arrow[ur, dashed] & \end{tikzcd} \qquad \cong \qquad \begin{tikzcd} K \arrow[r, dashed] & C_{/p} \end{tikzcd}$$

To determine the simplices of $C_{p/}$ we can equivalently look at maps $\Delta^n \to C_{p/}$. Using the adjunction above we get

$$(C_{p/})_n \simeq \mathrm{Hom}_{\mathrm{sSet}}(S \star \Delta^n, C) \quad (C_{/p})_n \simeq \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \star T, C)$$

It turns out that if C is an ∞-category then so are $C_{p/}, C_{/q}$ for any $p, q : S \to C$, for a proof we refer to Proposition 1.2.9.3 in [Lur09] or Proposition 29.3 [Rez22].

Now that we have at least a basic understanding of these simplicial sets we'll look into how to use them. We will use them in exactly two ways. Firstly, by taking $p = \hat{x} : \Delta^0 \to C$ we'll obtain a characterisation of the terminal (or dually initial) object on C. Moreover, when $K = \Delta^0$ and $p : I \to C$ we can interpret $C_{/p}$ as the ∞-category of cones on $p : I \to C$. Combined with the previous we'll obtain the notion of (co)limits for ∞-categories.

<u>1st use:</u> We want to obtain a notion of terminal and initial objects. For ordinary categories, we have that an object $T \in ob\mathcal{C}$ is terminal exact when $\forall Y \in ob\mathcal{C} : \exists! f : Y \to T$. Then, one can deduce that the ordinary slice ${}^{\mathcal{C}}\!\diagup_{T} \simeq \mathcal{C}$. The intuition behind this fact is that since all objects come with a unique map into T to look at maps into T is equivalent to looking at objects of $\mathcal{C}$. This is the intuition behind the ∞-categorical generalisation of the notion. In an attempt to mimic the phenomenon above we want to look into the slice ∞-category over a vertex $x \in C_0$, thus we let $p = \hat{x} : \Delta^0 \to X$. Ultimately we'd like to compute this ∞-categories n-simplices so we also let $K = \Delta^n$. Let the last isomorphism come from 48. Then, the naturally bijective homs above specialise to:

$$(C_{/\hat{x}})_n \simeq \Delta^n \to C_{/\hat{x}} \simeq \mathrm{Hom}_{\mathrm{sSet}}(\Delta^n \star \Delta^0, C) \simeq \mathrm{Hom}^{\hat{x}}_{\mathrm{sSet}}(\Delta^{n+1}, C)$$

By the $\hat{x}$ "exponent" on the right we mean to indicate that we take those maps of simplicial sets that extend $\hat{x} : \Delta^0 \to C$. Another way to put this is to say that n-simplices of this ∞-category are given by maps $\sigma : \Delta^{n+1} \to C$ with $\sigma|_{\Delta^{\{n+1\}}} = x$.

Now as a "reality-check" we can verify that n-simplices of $C_{/\hat{x}}$ agree with our intuition of slices in ordinary categories. Vertices are intervals that extend $\hat{x}$, namely arrows $f : y \to x$. Similarly arrows in the ∞-slice correspond to 2-simplices of C "over" the vertex x, namely diagrams of the form:

$$\begin{tikzcd} \bullet_1 \arrow[rr] \arrow[dr] & & \bullet_2 \arrow[dl] \\ & x & \end{tikzcd}$$

An important consequence of this construction is that we obtain a canonical map $\pi : C_{/x} \to C$ that takes these diagrams and "forgets" the vertex x along with the maps going into it. For example $\pi_0(y \to x) = y$ and π_1 applied to the diagram above would output the map $\bullet_1 \to \bullet_2$. Then, one shows that this map is always a right fibration of simplicial sets.

A small digression is in order here. The study of presheaves is a central aspect of category theory and the homotopy coherent generalisation is no different. ∞-presheaves are defined as $\mathrm{Psh}_\infty(\mathscr{C}) := \mathrm{Fun}(\mathscr{C}, \mathcal{S})$ where $\mathcal{S} = N^{hc}(\mathscr{K}an) = N^{hc}(\mathrm{sSet}^\circ_{\mathrm{KQ}})$. A functor of ∞-categories $\mathscr{C} \to N^{hc}(\mathscr{K}an)$ transposes to one of simplicial categories: $\mathfrak{C}\mathscr{C} \to \mathscr{K}an$ which corresponds to a *homotopy coherent diagram*. Now, diagrams such as these involve writing down an infinite amount of coherence data. This makes the study of ∞-presheaves rather unmanageable. The method for overcoming this, Lurie's ∞-Grothendieck construction, or straightening-unstraightening is what some call "technical heart" of Lurie's Higher Topos Theory, see here and here. One can then study "coCartesian" fibrations over $\mathscr{C}$ instead of presheaves on $\mathscr{C}$. The former environment is substantially better behaved. As much I would have enjoyed delving deeper into these ideas they were too much of an undertaking on top of my other goals with this thesis. A non-published account I've found enjoyable is Jaco Ruit's Msc Thesis on the topic available here. We should also note that Lurie's correspondence relies on the model structure for *marked simplicial sets* whose treatment lies outside the scope of this thesis.

Definition 50. A vertex $x \in C_0$ will be called **terminal** in C when the forgetful map $\pi : C_{/x} \to C$ is a trivial fibration. Dually, a vertex is initial exactly when the forgetful map $\pi : C_{x/} \to C$ is a trivial fibration.

Since these maps are already right or left fibrations, additionally being a trivial fibration is equivalent to being an equivalence of ∞-categories and we therefore find our desired caracterisation. Essential surjectivity captures that for any vertex y there exists an n-simplex going into t. Fully faithfulness captures the uniqueness of such a map. Indeed one can show that the full subcategory spanned by terminal (respectively) initial objects is either empty or equivalent to the terminal ∞-category $\simeq \Delta^0$, generalising the fact of ordinary category theory that if there exists a terminal object then it is unique up to unique isomorphism, see Proposition 29.7 [Rez22].

<u>2nd use:</u> As previously discussed, we can think of maps $p : I^{\rhd} \to C$ as cocones on $p : I \to C$ whose colimit we'd like to compute.

Definition 51. In the setting above, a colimit for a diagram $p : I \to C$ is an initial object in $C_{p/}$, the ∞ category of cocones on p.

This is the topic of section 31 in [Rez22]. Using the adjunction *join* $\dashv$ *slice* we can get equivalent formulations for lifting problems. In fact, we get operations on arrows called "pushout-join" and "pullback-slice". Moreover, many properties of maps of simplicial sets have various stability properties with respect to these operations. Section 30 of [Rez22] is devoted to the study of how these notions interact. Consequences of this interaction include that if C is an ∞-category then so are the $C_{p/}, C_{/p}$. When it comes to our example of interest above we get an explicit formulation of the requirement for a cocone to be colimiting. Explicitly, a colimit of $p : I \to C$ is given by a map $\tilde{p} : I^{\rhd} \to C$ which extends p, such that any diagram as below admits a lift. The map $\tilde{p}$ is the transpose of $\Delta^0 \to C_{/p}$ that picks out the colimiting cocone. Asking for a lift amounts to the forgetful $\pi : C_{/p} \to C$ being a trivial fibration.

$$
\begin{tikzcd}
I^{\rhd} \arrow[d, tail] \arrow[dr] & \\
I \star \partial\Delta^n \arrow[r] \arrow[d, tail] & C \\
I \star \Delta^n \arrow[ur, dashed] &
\end{tikzcd}
$$

Definition 52. Let C be an ∞-category. Consider the span $\bullet \leftarrow \bullet \rightarrow \bullet \cong \Lambda^2_1$. A **pullback** in C is a limit for a functor $\Lambda^2_1 \to C$.

Developing the theory in this way one retrieves many, if not all, usual category theoretic facts about (co)limits.

2.3.6 Adjoints

In general, when we want to generalise a notion to the ∞-categorical setting it is just a matter of choosing the correct 1-categorical definition to generalise. For adjoints, there are many available candidates. A prime candidate would be to say $f : C \leftrightarrows D : g$ is an adjunction when we have isomorphic or better, homotopy equivalent, objects $\mathrm{Map}(f(-), -) \cong \mathrm{Map}(-, g(-)) : C \times C^{\mathrm{op}} \to \mathcal{S}$. As discussed in the digression of the previous section, space-valued presheaves are hard to understand directly. We understand them indirectly via their "classifying (left or right) fibrations". A treatment of these ideas lies outside the scope of this thesis. It is the topic of section 2.2.1 of [Lur09]. Instead, we'll use a definition that doesn't require additional technical machinery. So we'll use the *Kerodon* definition generalising the triangle equalities.

Definition 53. Let $F : C \leftrightarrows D : G$ be functors between ∞-categories. We say $F \dashv G$ when there exist $\begin{cases} \eta : \mathrm{id}_C \to GF \in \mathrm{Fun}(C, C)_1 \\ \epsilon : FG \to \mathrm{id}_D \in \mathrm{Fun}(D, D)_1 \end{cases}$ and two 2-simplices as below:

$$\begin{tikzcd} & FGF \arrow[dr, "\epsilon \circ \mathrm{id}_F"] & \\ F \arrow[ur, "\mathrm{id}_F \circ \eta"] \arrow[rr, "\mathrm{id}_F"'] & & F \end{tikzcd} \qquad \begin{tikzcd} & GFG \arrow[dr, "\mathrm{id}_G \circ \epsilon"] & \\ G \arrow[ur, "\eta \circ \mathrm{id}_G"] & & G \arrow[ll, "\mathrm{id}_G"] \end{tikzcd}$$

In any case, we are just providing a definition for completion. Effectively we will be black-boxing this notion and any technical proofs involving it.

We can immediately give the definition of what it means to be a locally cartesian closed ∞-category.

Definition 54. An ∞-category is **cartesian closed** when it has finite products and for any object $x \in C$, the product functor $x \times - : C \to C$ has a right adjoint.

Definition 55. An ∞-category is **locally cartesian closed** when for any object $x \in C$ the slice category $C_{/x}$ is cartesian closed.

In the presence of *presentability*, the topic of the next section, locally cartesian closed ∞-categories admit a tractable characterisation.

2.3.7 Presentable ∞-categories

The notion of presentability in ∞-category theory is essentially a straightforward generalisation of the corresponding notion for 1-categories[11]. Structurally, a presentable ∞-category is one that has small colimits and where one has a small set S of "well behaved" objects, and that any object in the ∞-category can be obtained as a colimit formed from objects of S. This statement

[11] see section 3.8.1

correctly echoes the well-known slogan of ordinary category theory, "every presheaf is a colimit of representables". Then the objects of S can be thought of as "generators" for $\mathscr{C}$.

Now whenever one has generators with respect to some operation, one gets an induced "induction principle". This induction principle says that to define a "operation" preserving function out of a set with generators for that same operation, it suffices to provide the values of the function on the generators. Making this precise would require a treatment of accessible ∞-categories which lies outside the scope of this thesis. But, we do record the universal properties of the *idempotent completion* $\mathrm{Ind}_\kappa(\mathscr{C})$, a sort of formal κ-filtered cocompletion of $\mathscr{C}$. Then we get the universal property of Proposition 5.3.5.10 in [Lur09] that asserts that if $\mathscr{D}$ admits κ-filtered colimits, then, one has the following equivalence of ∞-categories for the functor ∞-category of κ-filtered-colimit-commuting functors

$$\mathrm{Fun}^\kappa(\mathrm{Ind}_\kappa(\mathscr{C}), \mathscr{D}) \simeq \mathrm{Fun}(\mathscr{C}, \mathscr{D})$$

Thus armed, one can prove various good properties for presentable ∞-categories. Perhaps the most important one is a characterisation proved by Simpson that characterises presentable ∞-categories as exactly those that arise as reflective subcategories of ∞-presheaf categories. Another is the *representability criterion* that asserts that a presheaf out of a presentable ∞-category preserves colimits $\iff$ it is representable! This in particular implies that presentable categories are complete and enjoy an adjoint functor theorem. For a comprehensive account consult Section 5, in particular 5.5 of [Lur09]. A concis exposition is given in [Lur03].

We begin the section by briefly looking into κ-accessible ∞-categories, those that have a set of well-behaved generators with respect to colimits. Then we define presentable ∞-categories and sketch a proof of their main characterisation. We conclude by studying some of the very strong formal properties enjoyed by presentable ∞-categories.

But in what sense are the generators well behaved?

In ordinary 1-category theory, a category is filtered when any pair of objects and any pair of parallel arrows have "an upper bound". This "upper boundedness" makes colimits indexed over filtered categories enjoy nice properties. It is exactly this idea echoed in the following definition.

Definition 56. (Definition 5.3.1.7 [Lur09]) Let κ be a regular cardinal and $\mathscr{C}$ an ∞-category. We call $\mathscr{C}$ κ-filtered exactly when any diagram $f : K \to \mathscr{C}$ extends to $\tilde{f} : K^{\triangleright} \to \mathscr{C}$.

Definition 57. Let $\mathscr{C}$ be an ∞-category which admits small κ-filtered colimits.

(1) A functor $f : \mathscr{C} \to \mathscr{D}$ will be called κ**-continuous** when it preserves κ-filtered colimits.

(2) An object will be called κ-compact if the functor corepresented by it is κ continuous namely if $\mathrm{Hom}_{\mathscr{C}}(C, -)$ preserves κ-filtered colimits.

(3) We denote by $\mathscr{C}_\kappa$ the full subcategory spanned by the κ-compact objects.

Being κ-compact can be understood in the following way. If we let $C = *$ we find $\mathrm{Map}_{\mathscr{C}}(*, \mathrm{colim}_i \mathscr{C}_i) \simeq \mathrm{colim}_i \mathrm{Map}_{\mathscr{C}}(*, \mathscr{C}_i)$, namely that to "pick a point" in a "disjoint union" is equivalent to "picking a point" in "one of the constituents". Moreover, it is a classic slogan of topology that compact sets behave much like points which motivates the name.

Definition 58. Let κ be an uncountable cardinal. An ∞-category $\mathscr{C}$ is **essentially-κ-small** if it is κ-compact as an object of Cat_∞. It is equivalent to asking that it is categorically equivalent to a κ-small simplicial set.

Definition 59. An ∞-category $\mathscr{C}$ is **essentially small** when it is essentially-κ-small for some small κ.

Definition 60. An ∞-category $\mathscr{C}$ is **locally small** when $\forall$ objects X, Y of $\mathscr{C}$ the mapping space $\mathrm{Map}_{\mathscr{C}}(X, Y)$ is essentially small.

Definition 61. Let $\mathscr{C}$ be an ∞-category. $\mathscr{C}$ will be called **κ-accessible** when it admits small κ-filtered colimits and contains an essentially small full subcategory $\mathscr{C}' \subseteq \mathscr{C}$ which consists of κ-compact objects and generates $\mathscr{C}$ under small κ-filtered colimits.

This is in some sense analogous to requiring a basis $\mathscr{B}$ for a topological space $\mathscr{X}$.

Proposition 62. (Proposition 5.4.2.2 & remark 5.4.2.13 [Lur09]) If $\mathscr{C}$ is κ-accessible then $\mathscr{C}_\kappa$ is essentially small.

Definition 63. An ∞-category $\mathscr{C}$ is **presentable** if it is accessible and admits small colimits.

In summary, an ∞-category is **presentable** if it is locally small, admits small colimits and there exists a regular cardinal κ and a "small" set S of κ-compact objects of $\mathscr{C}$ such that every object of $\mathscr{C}$ can be built as a κ-filtered colimit of objects from S.

Example 64. For an locally small ∞-category $\mathscr{C}$, $\mathrm{Psh}_\infty(\mathscr{C})$ is a presentable category.

Indeed, by example 5.4.2.7 [Lur09] it is κ-accessible for some κ. Since spaces $\mathcal{S}$ are cocomplete and colimits in $\mathrm{Fun}(\mathscr{C}, \mathcal{S})$ are computed componentwise ∞-presheaves are also cocomplete.

Moreover, we have the next important characterization.

Theorem 65. (Theorem 5.5.1.1. in [Lur09] and Theorem 1 in [BÄR]) The following conditions are equivalent for an ∞-category $\mathscr{C}$.

(1) $\mathscr{C}$ is presentable.

(2) $\mathscr{C}$ is cocomplete and there exists a small set $S \subset \mathscr{C}_\kappa$ that generates $\mathscr{C}$ under κ-filtered colimits.

(3) There is a small ∞-category $\mathscr{E}$ and an adjunction

$$\mathscr{C} \underset{i}{\overset{L}{\leftrightarrows}} \mathrm{Fun}(\mathscr{E}, \mathcal{S})$$

(4) There exists a regular cardinal κ and $S \subset \mathscr{C}_\kappa$ that *detects equivalences*, namely a morphism $u : A \to B$ in $\mathscr{C}$ is an equivalence exactly when $\mathrm{Map}(\Sigma, u)$ is an equivalence for all $\Sigma \in S$.

Proof. (1) $\iff$ (2) is just the definition.

(2) $\iff$ (3) Assume $\mathscr{C}$ is presentable. Take κ such that $\mathscr{C}$ is κ-accessible. Let $\mathscr{C}_\kappa$ denote the full subcategory of κ-comapct objects. As discussed above there exists a small set $S \subseteq \mathscr{C}_\kappa$ that generates $\mathscr{C}$ via κ-filtered colimits. If we think of $\mathrm{Psh}(\mathscr{C}_\kappa)$ as the free cocompletion of $\mathscr{C}_\kappa$ it is reasonable to expect $\mathscr{C}$ to arise as a subcategory. For formal reasons it is reflective.

Part of Proposition 1.3.11 of [Lur03] or Proposition 5.4.2.2 [Lur09] is that if $\mathscr{C}$ is κ-accesssible then $\mathscr{C}_\kappa$ is essentially small. Consider $\mathrm{Psh}(\mathscr{C}_\kappa)$. There is an obvious inclusion $\mathscr{C}_\kappa \hookrightarrow \mathscr{C}$ which extends uniquely to a continuous functor $\mathrm{Psh}(\mathscr{C}_\kappa) \to \mathscr{C}$. In the spirit of Kan extensions, the latter is left adjoint to the Yoneda embedding, which is of course fully faithful.

Now we assume $\mathscr{C}$ is of this form and wish to show it is presentable. By Prop 5.3.5.12 in [Lur09], $\mathrm{Psh}(\mathscr{E})$ is accessbile. This is a refinement of the statement that every presheaf is a colimit of representables. By Proposition 13.21 [Lur03] we obtain that so is the essential image of L, namely $\mathscr{C}$. To see that $\mathscr{C}$ admits small colimits, just form the colimit in $\mathrm{Psh}(\mathscr{E})$ and apply the right-adjoint L, which preserves them.

(2) $\implies$ (4) Take the $S \subset \mathscr{C}_\kappa$ of (2). Take a $u : A \to B$. Assume $\mathrm{Map}(\Sigma, u)$ is an equivalence for all $\Sigma \in S$. Consider the class of objects $C \in \mathscr{C}$ such that $\mathrm{Map}(C, u)$ is an equivalence. Using the universal property of colimits we see that this class is stable under colimits. Because S was taken to generate $\mathscr{C}$, the class above contains all objects of $\mathscr{C}$. Therefore $\mathrm{Map}(C, u)$ equivalence for any $C \in \mathscr{C}$ and therefore u equivalence.

For the proof that (4) $\implies$ (2) we refer to Theorem 1 ((7) $\implies$ (6)) in [BÄR]. $\square$

Remark 66. Condition (4) above is particularly illustrative of the close relationship of presentable ∞-categories admitting presentations by *S-Bousfield localisations* of model categories. Indeed, such a localisation enlarges the class of weak equivanelnces $\mathcal{W}$ of a model category $\mathcal{M}$ so as to ensure S-local objects become weakly equivalent. These localisations are the topic of Section 3.6.3

Presentable ∞-categories enjoy very strong formal properties that our next theorems record. The first are internal or structural. The next are relational.

Theorem 67. (Corollary 5.5.2.4 in [Lur09] but the proof is taken from Corollary 5 in [BÄR]) A presentable ∞-category $\mathscr{C}$ admits all small limits.

Proof. Consider $\mathscr{C}$ as the full subcategory of local objects in $\mathrm{Psh}_\infty(\mathscr{D})$ from some localisation $i : \mathscr{C} \leftrightarrows \mathrm{Psh}_\infty(\mathscr{D}) : L$ and let $p : I \to \mathscr{C}$ be a diagram. The crucial observation is that local objects are closed under limits in $\mathrm{Psh}_\infty(\mathscr{D})$. This implies that the limit of $i \circ p$ is a local object. Full faithfulness of i allows us to deduce that the limit cone of $i \circ p$ factors through i and gives a limit cone in $\mathscr{C}$. □

Theorem 68. (Proposition 5.5.2.2 [Lur09]) Let $\mathscr{C}$ be a presentable ∞-category and $F \in \mathrm{Psh}(\mathscr{C})$. Then F is representable if and only if F preserves small limits.

Theorem 69. Let $F : \mathscr{C} \to \mathscr{D}$ be a functor between presentable ∞-categories. Then,

(*1*) F has a right adjoint $\iff$ it preserves small colimits.

(*2*) F has a left adjoint $\iff$ it is accessible and preserves small limits.

Corollary 70. Let $\mathscr{C}$ be a presentable ∞-category. Then $\mathscr{C}$ is locally cartesian closed exactly when *colimits are universal*, namely pullback functors preserve colimits, see Proposition 75

Proof. $\mathscr{C}$ is locally cartesian closed exactly when pullback functors have right adjoint. $\mathscr{C}$ has universal colimits when pullback functors preserve colimits. The previous theorem asserts the two conditions are equivalent. □

2.4 Grothendieck ∞-topoi

In modern times the study of geometry has found it advantageous to study surfaces or manifolds *locally*. Indeed, the inquiry into manifolds or Riemann surfaces is one where things are at least (very) well-behaved locally. Apart from the surfaces themselves we also want to study functions out of them. This naturally leads us to want to study functions defined locally. This is the motivating idea behind sheaves and how they were employed by Grothendieck, who has shaped modern Algebraic Geometry.

Take $X \in \mathrm{Top}$ and let $\mathcal{O}(X)^{\mathrm{op}}$ be its poset of open sets. Over each open set U we want to consider the collection of functions with domain U. Of course, when the codomain of the functions has some algebraic structure as a group or ring, then the collection of functions naturally inherits that structure pointwise. So, if we are thinking of real-valued functions over each open U, we should provide a ring R_U to be thought of as the ring of functions defined on U. Given an inclusion of open sets $V \subseteq U$ one may also require that there is a restriction function $R_U \to R_V$ defined by $f \mapsto f|_V$. This data can be concisely packaged by requiring a functorial assignment $R : \mathcal{O}(X)^{\mathrm{op}} \to \mathrm{Ring}$.

Restriction of functions can be seen as a global to local operation. The outcome is certainly at least more local. A general distinctive feature of sheaves is a sort of inverse to this operation. In short, given a *compatible* family of functions we want to be able to *glue* them together to a global one, which in turn can be restricted to retrieve the pieces. Of course, if there is any hope to do this, the functions should agree wherever they overlap, otherwise, how could we canonically decide the value of the global function on the domain of disagreement? Take an open cover $\{U_i\}_i$ of X, and functions $f_i \in R_{U_i}$. If these functions agree on overlaps of the various U_j, that is if $f_i|_{U_i \cap U_j} = f_j|_{U_i \cap U_j}$ then we postulate the existence of an $f \in R_X$ such that $f|_{U_i} = f_i$. The preshaves satisfying such a condition are called *sheaves* and are denoted $\mathrm{Sh}(X)$. This is a *local to global* condition. In fact, it says that if you take all these pieces you can glue them together (local $\to$ global) in such a way that if then you restrict again (global $\to$ local) you retrieve the original pieces.

It wasn't long before it was understood that (pre)sheaves of sets instead of rings are much more general and enjoy *fantastic* formal properties. Indeed $\mathrm{Psh}(\mathcal{C})$ is always a topos. In addition to that Grothendieck realised that the topological spaces he wanted to study, schemes with the Zariski topology, didn't have enough open sets. So he generalised the setting he was working in. In the study of sheaves, we require the notion of an open cover, i.e. special collections $\{U_i \to X\}_i$ that are declared to be *covering*. Imposing some reasonable axioms on when such collections should be thought of as covering we obtained the notion of a *Grothendieck topology* J. A category $\mathcal{C}$ equipped with such a topology is called a *site*.

Through it, once again, we define what it means to be a sheaf and obtain a collection $\mathrm{Sh}(\mathcal{C})$ on a site $(\mathcal{E}, J)$. Morphisms of sheaves are nothing but natural transformations between them, this makes the embedding $\mathrm{Sh}(\mathcal{C}) \overset{i}{\hookrightarrow} Psh(\mathcal{C})$ full. In addition to that, there is a functorial process, *sheafification* or the $a{+}{+}$ construction, turning a presheaf into a sheaf. For a comprehensive account see [MM94]. One can prove that $a \dashv i$, making $\mathrm{Sh}(\mathcal{C})$ a *reflective* subcategory of $Psh(\mathcal{C})$, and moreover that a preserves finite limits or is *exact*. A Grothendieck 1-topos is one that is equivalent to a category of sheaves on a site. This is the definition we generalise to the ∞-categorical setting.

Definition 71. An ∞-category $\mathfrak{X}$ is an ∞-topos exactly when there exists another ∞-category $\mathscr{C}$ so that

$$\mathfrak{X} \underset{\hookrightarrow}{\overset{a}{\longleftarrow}} \mathrm{Psh}_\infty(\mathscr{C})$$

Where $a \dashv i$ and a preserves finite limits.

2.4.1 Descent

While this definition appeals to our intuition of what a Grothendieck ∞-topos should be, it is very hard to use. Indeed, if we had an ∞-category $\mathfrak{X}$ and wanted to verify it is an ∞-topos we'd have to construct an entire ∞-category $\mathscr{C}$ and a special pair of adjoints. This definition is extrinsic and difficult to manage. Instead, we'd like a definition *internal* to $\mathfrak{X}$.

Such a characterisation is achieved through the notion of *descent* and universal colimits. We have:

Theorem 72. An ∞-category $\mathfrak{X}$ is an ∞-topos exactly when it is presentable as an ∞-category, its colimits are universal and it satisfies descent.

As already discussed, an ∞-category being presentable is equivalent to it being a reflective subcategory of some presheaf category. The two extra properties will help us prove that the left adjoint reserves finite limits. Confusingly, since descent is rarely present without universal colimits, many authors use the term both for the property and the pair of properties. We will not adopt this terminology.

Descent is a generalisation of the "sheaf condition" to the ∞-setting. It is a pair of requirements that together ensure a good interaction between pullbacks and the formation of colimits.

Descent has many equivalent formulations. We will use different ones according to our purposes in each section. Presently, we want to emphasize descent as a condition mediating a good interaction between *local* and *global* treatment of our objects. To that end, we introduce strong descent. Next, we wish to discuss object classifiers in ∞-topoi for which a seemingly weaker version of descent is better suited. in the presence of universal colimits, as in an ∞-topos, the two are equivalent, see proposition [**2descents**].

Consider a diagram $X_\bullet : I \to \mathfrak{X}$ and $X \cong \mathrm{colim}_i X_i$. We think of X_i as the local pieces being glued together to form X. We are also interested in maps into X captured by the slice category ${}^{\mathfrak{X}}\diagup_{\mathrm{colim}_i X_i}$.

Definition 73. We say that an ∞-category $\mathscr{C}$ has **strong descent** when the self indexing functor $s : \mathscr{C}^{\mathrm{op}} \to \mathrm{Cat}_\infty$ takes colimits to limits,

$$ {}^{\mathscr{C}}\diagup_{\mathrm{colim}_i X_i} \cong \lim_i {}^{\mathscr{C}}\diagup_{X_i} $$

The right-hand side above is a limit of a functor taking values in Cat_∞. In HTT section 3.3 Lurie gives some computational tools for such limits. In the result below we find ${}^{\mathscr{C}^I_{\mathrm{cart}}}\diagup_{X_\bullet}$. The denotation "cart" signifies that we only consider the natural transformations whose naturality squares are all pullbacks.

Proposition 74. We have an equivalence $\lim_i {\mathscr{C}}\big/ {X_i} \cong {\mathscr{C}^I_{\mathrm{cart}}}\big/ {X_\bullet}$.

We want to compute the limit of a functor taking values in Cat_∞ According to Corollary 3.3.3.2 of [Lur09] the limit is computed by $\lim_i F \circ X \cong \Gamma_I(E_{F\circ X})$, the ∞-category of *cartesian sections* of $E_{F\circ X} \to I$, where $E_{F\circ X}$ is the pullback of $F \circ X : I \to \mathrm{Cat}_\infty$ against the *universal fibration.* We can compute $E_{F\circ X} \cong \mathscr{C}$ so that the space of cartesian sections is exactly ${\mathscr{C}^I_{\mathrm{cart}}}\big/ {X_\bullet}$.

So, restricting to cartesian or equifibered natural transformation ${\mathscr{C}^I_c}\big/ {X_\bullet}$ was imposed by the computation of limits in Cat_∞. It has the following pleasant consequences.

In an *equi-fibered* natural transformation all components of the natural transformation have equivalent fixed fibers over a given point: Explicitly, $\mathcal{X}_{c_i,y} \cong \mathcal{X}_{c_j,\varphi(y)}$, reminiscent of the "compatible intersections" condition of 1-Grothendieck topoi.

$$\begin{tikzcd}
\mathcal{X}_y \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d, "c_i"] & X_j \arrow[d, "c_j"] \\
\Delta^0 = \{y\} \arrow[r] & Y_i \arrow[r, "\varphi"] & Y_j
\end{tikzcd}$$

We will describe the equivalence of Theorem 73 via an adjoint equivalence. We define the pair of functors going in opposite directions. First,

$$\mathrm{colim} : {\mathscr{C}^I_{\mathrm{cart}}}\big/ {X_\bullet} \to {\mathscr{C}}\big/ {\mathrm{colim}_i X_i}$$

where given an equifirebed natural transformation with components $a_i : Y_i \to X_i$, we can take colimits horizontally and obtain a map $\mathrm{colim} a_i : \mathrm{colim}_i Y_i \to \mathrm{colim}_i X_i$. In the other direction, we define,

$$\mathrm{cst} : {\mathscr{C}}\big/ {\mathrm{colim}_i X_i} \to {\mathscr{C}^I_{\mathrm{cart}}}\big/ {X_\bullet}$$

To be given by $\mathrm{cst}(Y \xrightarrow{g} \mathrm{colim}_i X_i) = X_i \times_{\mathrm{colim} X_i} Y \to X_i$. In Set, a pullback against an inclusion is an inverse image. Here we have $X \cong \mathrm{colim}_i X_i$ so we think of X as the product of "gluing" together the pieces X_i, and then take the span $X_i \hookrightarrow X \leftarrow B$. So, the pullback is computed by $g^{-1}(X_i)$.

Using the universal property of pullback and other elementary means one establishes that these functors are adjoints.

$${\mathscr{C}}\big/ {\mathrm{colim}_i X_i} \leftrightarrows {\mathscr{C}^I_{\mathrm{cart}}}\big/ {X_\bullet}$$

On the left hand side we have ${}^{\mathscr{C}}\!\diagup_{\mathrm{colim}_i X_i}$. Since we have a formed colimit, we have glued the pieces, it makes sense to think of this as the *global* object. In forming ${}^{\mathscr{C}}\!\diagup_{\mathrm{colim}_i X_i}$ we look at arrows with a "global" codomain. On the right-hand side, we have ${}^{\mathscr{C}^I_{\mathrm{cart}}}\!\diagup_{X_\bullet} \cong \lim_i {}^{\mathscr{C}}\!\diagup_{X_i}$, so a family of arrows $Y_i \to X_i$ into the *local* pieces X_i, with the addition of the compatibilities enforced by the definition of Y_i and the formation of the limit.

Naturally, the colim functor is (local $\to$ global) and the cst is (global $\to$ local). An ∞-category has a strong descent exactly when this adjunction is an equivalence. That is equivalent to both adjoints being fully faithful and both the unit and counit to comprise of natural isomorphisms.

We then get two statements. Each corresponds exactly to one of the two properties we are interested in. The first is *universal colimits*.

$$Y \to \mathrm{colim} X_i \cong \mathrm{colim}\big(\mathrm{cst}(Y \to \mathrm{colim} X_i)\big) = \mathrm{colim}(X_i \times_{\mathrm{colim} X_i} Y \to X_i) \implies$$

$$Y \cong \mathrm{colim}(X_i \times_{\mathrm{colim} X_i} Y)$$
$$g : Y \to \mathrm{colim} X_i \cong \mathrm{colim}\big(g_i : X_i \times_{\mathrm{colim} X_i} Y \to X_i\big)$$

So if we start with a map into a colimit the cst functor lets us break both the domain and the map itself into pieces coherently with the pieces of $X = \mathrm{colim} X_i$. Then strong descent can be

$$Y_\bullet \to X_\bullet \cong \mathrm{cst}\big(\mathrm{colim}(Y_\bullet \to X_\bullet)\big) = \mathrm{cst}\big(\mathrm{colim} Y_i \to \mathrm{colim} X_i\big) = \{X_i \times_{\mathrm{colim} X_i} \mathrm{colim} Y_i \to X_i\}_i$$

And therefore, in particular, $Y_i \cong X_i \times_{\mathrm{colim} X_i} \mathrm{colim} Y_i$. So given a collection of compatible pieces, namely an equifibered natural transformation $Y_\bullet \to X_\bullet$, we can glue them together and then take them apart again and retrieve the original pieces, up to equivalence of fibrations, a categorical equivalence compatible with a base.

In addition to that, in doing so we keep the fibers intact. Indeed,

$$\begin{tikzcd}
\mathcal{X}_y \arrow[r] \arrow[d] & Y_i \arrow[r, "\simeq"] \arrow[d, "c_i"'] & X_i \times_{\mathrm{colim} X_i} \mathrm{colim} Y_i \arrow[r] \arrow[d, "\pi_i"] & \mathrm{colim} Y_i \arrow[d] \\
\Delta^0 = \{y\} \arrow[r] & X_i \arrow[r, "\simeq"] & X_i \arrow[r] & \mathrm{colim} X_i
\end{tikzcd}$$

so that any fiber of any c_j arises as a fiber of $\mathrm{colim} Y_i \to \mathrm{colim} X_i$.

Now that the (local to [global to local) to global] character of descent has been emphasized we turn to the version of descent and universal colimits that is best suited for applications, like our next section on object classifiers.

Proposition 75. The functor *cst* is fully faithful $\iff$ for any $f : T \to S$ the pullback functor $f^* : {}^{\mathscr{C}}\!/_S \to {}^{\mathscr{C}}\!/_T$ preserves colimits.

Proof. The proof below is adapted from *Lecture 3* of [Rez]. We begin by reformulating some things in a more convenient way.

(UC1): The functor *cst* is fully faithful when $\mathrm{colim} Y_i \simeq Y$ over X.

$$\begin{tikzcd} X_i \times_X Y =: Y_i \arrow[r] \arrow[d] & Y \arrow[d] \\ X_i \arrow[r, hook] & \mathrm{colim} X_i \end{tikzcd}$$

(UC2): The pullback functor induced by $f : T \to S$ preserves colimits when: $f^*(\mathrm{colim} X_i \to S) := (\mathrm{colim} X_i) \times_S T \to T$ is isomorphic to $\mathrm{colim}_i\big(X_i \times_S T\big)$. So in the situation of the diagram below:

$$\begin{tikzcd} X_i \times_X Y =: Y_i \arrow[r] \arrow[d] & Y \arrow[r] \arrow[d] & T \arrow[d, "f"] \\ X_i \arrow[r, hook] & \mathrm{colim} X_i = X \arrow[r] & S \end{tikzcd}$$

we can deduce $\mathrm{colim}_i Y_i = \mathrm{colim}_i\big(X_i \times_S T\big)$ is iso to $Y = (\mathrm{colim} X_i) \times_S T \to T$. With these reformulations we can immediately see that $UC2 \implies UC1$ for $X = S$ and $T = Y$

The implication $UC1 \implies UC2$ is essentially by the pullback pasting lemma. Let $Y \to X := f^*\big(g : \mathrm{colim} X_i \to S\big)$, the middle vertical map in the diagram above. This map further pulls back to $Y_i \ = to X_i$. By our assumption that UC1 holds, we get $\mathrm{colim} Y_i$ is isomorphic to Y over X. But, by the pasting lemma, $Y_i \to X_i$ are isomorphic over X_i to $f^*(X_i \hookrightarrow X \to S)$. Combining these two facts:

$$f^*(\mathrm{colim} X_i \to S) := Y \to T \underset{UC1}{\simeq} \mathrm{colim} Y_i \to T := \mathrm{colim}(X_i \times_X Y) \to T \underset{PBL}{\simeq} f^*(X_i \hookrightarrow X \to S)$$

□

Definition 76. $\mathscr{C}$ has descent when the core of selfindexing functor, $cs : \mathscr{C}^{\mathrm{op}} \to \mathcal{S}$ with $X \mapsto ({}^{\mathscr{C}}\!/_X)^{\simeq}$, takes colimits to limits.

Proposition 77. In the presence of universal colimits, strong descent is equivalent to descent.

Proof. Strong descent specialises in descent for $\mathcal{S} \subset \mathrm{Cat}_\infty$. For the converse observe that strong descent in particular means that the functor "colim" has a fully faithful right adjoint, cst, making colim a localisation functor in the sense of Lurie. But a localisation functor is an equivalence exactly when it is conservative, which is exactly what is asserted by universal colimits. □

2.4.2 Caracterising ∞-topoi

This subsection is devoted to sketching the proof of

Theorem 78. An ∞-category $\mathscr{C}$ is an ∞-topos exactly when it is presentable, and has universal colimits and descent.

Proof. The proof presented here is a slightly adapted version of the one given by Rezk here. To show that all ∞-topoi satisfy these properties one argues as follows:
The first step is to show $\mathcal{S} = \mathfrak{N}(\mathrm{sSet}^{\circ}_{KQ})$ satisfies these properties. This appears as Proposition 6.1.3.14 in [Lur09], with a rather different, but of course equivalent, formulation.

Firstly, we've seen how all ∞-presheaf categories are presentable. We will later see in greater detail how various properties, or structures, of simplicial model categories lift through $\mathfrak{N}$. The point is that it suffices to show corresponding model-categorical properties in sSet_{KQ}. One then directly proves that sSet_{KQ} has universal *homotopy colimits* and descent for *homotopy colimits*. A key ingredient for achieving that amounts to compatibly replacing morphisms of a homotopy-(co)limit with fibrations or cofibrations in which case the diagram is an actual (co)limit in sSet. This uses arguments that appeared in [KL18].

Having established that spaces $\mathcal{S}$ have descent, one can then prove that since limits and colimits in $\mathrm{Psh}_\infty(\mathscr{C})$ are computed componentwise, these properties lift.

Lastly, as in Proposition 6.1.3.15 in [Lur09], we find that having universal colimits is closed under forming left exact reflective subcategories. Rezk notes that obtaining the same for descent is substantially more demanding but doable. Now for the opposite direction, suppose you have a presentable ∞-category $\mathscr{C}$ with universal colimits and descent. We've already seen how being presentable already places $\mathscr{C}$ as a reflective subcategory of $\mathrm{Psh}_\infty(\mathscr{C}^\kappa)$ for some κ. The other two properties go into showing that the left adjoint, call it L, is left exact.

The key case is showing it preserves pullbacks. Take a pullback in $\mathrm{Psh}(\mathscr{C}^\kappa)$. Let $P := X \times_B Y$. We must show we get an induced iso $LP \simeq LX \times_{LB} LY$. As discussed in the previous section, descent and universal colimits ensure a smooth passage between treating objects locally and globally, in the guise of a good interaction between colimits and pulling back against inclusions. How could we apply these ideas here? Since we are working with presheaves, there is a "canonical" step. Write the middle vertex of the span as a colimit of representables.

$$\begin{tikzcd} P \arrow[r] \arrow[d] & Y \arrow[d] \\ X \arrow[r] & B = \mathrm{colim}_i y_{B_i} \end{tikzcd}$$

Pullback against the inclusions $y_{B_i} \hookrightarrow \mathrm{colim}_I B_i$ and thus obtain "local pullbacks" for each i.

They are also compatible with one another in the sense that for each morphism $i \to j$ in I we get, by the universal property of pullback for P_j, a map $P_i \to P_j$, same with X, Y. By the pullback pasting lemma we get pullbacks

$$\begin{tikzcd}
P_i \arrow[r] \arrow[d] & X_i \arrow[r] \arrow[d] & y_{B_i} \arrow[d] \\
P_j \arrow[r] & X_j \arrow[r] & y_{B_j}
\end{tikzcd}$$

The same happens for Y.

In these local pullbacks the middle vertex of the span is representable. $\mathscr{E}$ having universal colimits ensures that forming the colimit over i of each vertex in these local pullbacks will produce the original object, for instance, $P \simeq \mathrm{colim} P_i$. One then shows separately that L preserves pullbacks whose middle vertex is representable. Thus L commutes with the pullbacks above. Moreover, L being a left adjoint means it preserves colimits so $LP \simeq L\mathrm{colim} P_i \simeq \mathrm{colim} LP_i$, similarly for X, Y. By descent in $\mathscr{E}$, we get pullbacks:

$$\begin{tikzcd}
LP_i \arrow[r] \arrow[d] & LP \arrow[d] \\
LX_i \arrow[r] \arrow[d] & LX \arrow[d] \\
Ly_{B_i} \arrow[r] & LB
\end{tikzcd}$$

Combining with those for LY we get:

$$\begin{tikzcd}
LP_i \arrow[rr] \arrow[dr] \arrow[dd] & & LY_i \arrow[dr] \arrow[dd] & \\
& LX_i \arrow[rr] \arrow[dd] & & Ly_{B_i} \arrow[dd] \\
LP \arrow[rr] \arrow[dr] & & LY \arrow[dr] & \\
& LX \arrow[rr] & & LB
\end{tikzcd}$$

where every face but the bottom one is a pullback.

By universal colimits in $\mathscr{E}$, the colimit of pullbacks is a pullback of colimits, implying that LP is a pullback of $LX \to LB \leftarrow LY$, as desired.

$\square$

2.5 Object classifiers and universes

In good old regular ZFC set theory, it is well known that the information that specifies a subset $A \subseteq B$, or equivalently an inclusion, $A \hookrightarrow B$ can be encoded in an arrow $\chi_A : A \to \{0,1\}$. In the category Set this can be rephrased by saying that any inclusion arises as the pullback of a unique map against $\{1\} \hookrightarrow \{0,1\}$.

This definition can be very naturally generalised for arbitrary categories. First, we generalise the notion of a subset. We want a definition of what a subobject is. We consider the collection of monos with a fixed codomain X. Of course, if two such monos are connected by an iso in the slice category they should represent the same subobject. So, we define, $\mathrm{Sub}(X) = \{A \rightarrowtail X\}_{\sim}$. We call each representative of an equivalence class a subobject of X. If the ambient category $\mathcal{C}$ is locally small, this is always a set. If, in addition, $\mathcal{C}$ admits finite limits, then $\mathrm{Sub}(-)$ acts on morphisms by pullback and is indeed a presheaf. A category is said to have a subobject classifier exactly when this functor is representable, that is if there exists a special object $* \rightarrowtail \Omega$ such that $\mathrm{Sub}(-) \cong \mathrm{Hom}(-, \Omega)$. The existence of a subobject classifier is a very powerful property. The structure it induces lies at the heart of the study of logical aspects of elementary topoi. There are many classical references. The canonical one is [MM94]. Another is [Gol14].

Indeed, we can use some archetypal objects that can be formed in virtually any category of interest such as the inclusion $0 \hookrightarrow 1$ whose character we define to be $\neg : 1 \to \Omega$. In a topos, we can also mimic intersections and unions. For a comprehensive account see Goldblat. All these operations impose algebraic structure on $\mathrm{Sub}(X)$ making it a lattice with a meet and join operation. It turns out that the complement operation given by the arrow $\neg$ is a *pseudocomplement* making $\mathrm{Sub}(X)$ a Heyting Algebra, instead of a Boolean one. Heyting Algebras classically serve as the algebraic framework in which we interpret intuitionistic logic in contrast to classical one. There are also internal properties of a topos that ensure that $\mathrm{Sub}(X)$ is a Boolean Algebra. Such topoi are called classical or Boolean topoi. For example, if $\mathfrak{f}$ or $\mathfrak{t} : 1 \to \Omega$ have a complement in $\mathrm{Sub}(\Omega)$ or if $i_1 : 1 \to 1+1$ is a subobject classifier, then the topos is Boolean. The last statement is particularly illuminating in that it can be interpreted to mean that the archetypal subobject classifier we have from Set really characterises the class of Boolean topoi.

What form does the subobject classifier take in an ∞-topos? The first step is to reformulate the above in a way that matches the machinery of ∞-categories we have developed thus far, namely in terms of slice categories and terminal objects. For any mono $A \hookrightarrow B$ there exists a unique pullback square whose other vertical map is $\mathfrak{t} \hookrightarrow \Omega$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, hook] & \mathfrak{t} \arrow[d, hook] \\ B \arrow[r] & \Omega \end{tikzcd}$$

Recall that the arrow category $\mathrm{Arr}\mathscr{C} = \mathrm{Fun}(\Delta^1, \mathscr{C})$, or to follow Lurie's notation for the ∞-setting, $\mathcal{O}_{\mathscr{C}}$ or just $\mathcal{O}$ when the ambient category is clear, has objects the morphisms of $\mathscr{C}$ and commutative squares for arrows. We want to describe $\mathrm{Sub}(X)$ so we fix the codomain and restrict to monomorphisms. Let $\mathcal{O}^M$ be the full subcategory spanned by monomorphisms. Let $\mathcal{O}^{(M)}$ have the same objects but where arrows are only pullbacks.

There is an obvious projection $\mathcal{O} \to \mathcal{C}$ taking an arrow to its domain. Fibers of this map are precisely slice categories. Similarly, fibers of $\mathcal{O}^M \to \mathcal{C}$ pick out the full subcategory of the slice spanned by the monos with a fixed codomain. Then, fibers of $\mathcal{O}^{(M)} \to \mathcal{C}$ only retain the isomorphisms between monos, namely $\left(\mathcal{C}/X\right)^{M,\cong}$. Lastly, to obtain $\mathrm{Sub}(X)$ we have to identify isomorphic objects by taking a skeleton of the category of every X. This relies on the axiom of choice.

Let M denote the collection of monos in $\mathscr{C}$. This defines a full subcategory of $\mathcal{O}$. A subobject classifier is exactly a terminal object of $\mathcal{O}^{(M)}$, it is an object, $T \to \Omega$ such that for any mono there exists a unique morphism, a pullback square, with codomain $T \to \Omega$.

The above formulation suggests that we could try to find a classifying object for other classes of morphisms $\mathcal{J}$, by asking that $\mathcal{O}^{(\mathcal{J})}$ has a terminal object. In 1-category theory we immediately run into an issue. Let $f : Y \to X$ be a non-mono arrow. Then the corresponding object of $\mathcal{C}/X$ may have non-trivial automorphisms, namely arrows $\alpha : Y \to Y$ with $f \circ \alpha = f$.

The condition that f is mono is precisely the statement that objects of $\mathcal{C}/X$ have exclusively identity automorphisms, making $\mathcal{O}^M$ a discrete category. Now if f is not a mono and such a non-trivial automorphism exists in $\mathcal{C}/X$ then, because of the supposed bijection with $\mathrm{Hom}(X, \Omega)$ we should get a non-trivial automorphism of $\chi_f \in \mathrm{Hom}(X, \Omega)$. But that is impossible since the latter is a set and therefore a discrete category. The only way to record such automorphisms is for $\mathrm{Hom}(-,-)$ to have extra structure, like that of a groupoid or, even better, an ∞-groupoid, precisely as is the case for ∞-categories.

So in 1 categories the limited structure of $\mathrm{Hom}(-,-)$ only allowed us to classify monomorphisms. Using this classifying object one is able to develop and study in depth the logical aspects of elementary 1-topoi. In the ∞-setting this obstruction is removed. So, we can reasonably hope for an even stronger condition, an object classifier Ω_∞, classifying any arrow $Y \to X$. Take an ∞-category $\mathscr{C}$. If $\mathscr{C}$ has finite limits we get the self indexing functor $\mathscr{C} \to \mathrm{Cat}_\infty$ that associates $X \mapsto \mathscr{C}/X$. We can also consider the core of self indexing $\mathscr{C} \to \mathcal{S}$ given by

$X \mapsto {}^{\mathscr{C}}\!\diagup_{X}^{\cong}$. The hope is that the functor will be representable. The situation is elucidated by the following proposition which appears as Proposition 6.1.6.3 [Lur09].

Proposition 79. Let $\mathscr{C}$ be a presentable ∞-category in which colimits are universal. Let S be a class of morphisms that is stable under pullbacks. Then, there exists a classifying object for S exactly when

(1) The class S is local, meaning the right fibration $\mathcal{O}^{(S)} \to \mathscr{C}$ is classified by a colimit preserving functor.

(2) For every object X, the full subcategory of ${}^{\mathscr{C}}\!\diagup_{X}$ spanned by S is essentially small.

As discussed in subsubsection 2.3.7 an ∞-category $\mathscr{C}$ being presentable has very strong formal consequences. One of the most important ones is that it allows for a potent characterisation of representable functors. Indeed, according to Corollary 5.5.2.2, a small-valued presheaf out of a presentable category is representable exactly when it preserves small limits. These two conditions are equivalent to those above.

It is also important to note that local classes are connected with descent. Indeed Theorem 6.1.3.9 asserts that for a presentable ∞-category with universal colimits, the condition of descent is equivalent to the class of all morphisms being local. Since we've seen that ∞-topoi have descent, all that remains is to partition $\mathrm{mor}\mathfrak{X}$ in *bounded* local pieces, namely pieces that are all local and such that full subcategories of slices spanned by said pieces are always small.

It becomes clear that our hope for a unique object that classifies all morphisms simultaneously was too optimistic. According to the previous proposition, the existence of such an object would force all slices ${}^{\mathscr{C}}\!\diagup_{X}$ to be small. There is no reason why this should be true in general. This essentially technical difficulty can be circumvented by a cardinality-based stratification. Instead of considering all maps into X, we just take those that are *relatively-κ*-compact. Recall that a simplicial set K is κ-compact when its corepresentable functor commutes with κ-filtered colimits.

Definition 80. Let $\mathscr{C}$ be a presentable ∞-category. We call a morphism **relatively-κ-compact** if its base changes have the property that if the codomain is κ-compact then so is the domain. So in a pullback square as below, Y' is κ-compact $\implies$ X' is κ-compact.

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r] & Y \end{tikzcd}$$

Theorem 81. Let $\mathscr{C}$ be a presentable ∞-category that has universal colimits and let S be a local class of morphisms in $\mathscr{C}$. For each regular cardinal κ let $S_\kappa := S \cap \{\text{relatively-}\kappa\text{ -compact morphisms}\}$. If κ is sufficiently large then S_κ has a classifying object.

Proof. One has to choose a sufficiently large cardinal κ to ensure that the class of κ-compact objects is stable under pullback. To show that S_κ has a classifying object we use the previous proposition. To show that S_κ is local requires an equivalent characterisation not given here. To see that the full subcategory of ${}^{\mathscr{C}}\!\diagup_{X}$ spanned by the relatively κ compact morphisms is essentially small it suffices to recall that $\mathfrak{X}^\kappa$ is and therefore so are its slices. $\square$

3 Abstract homotopy theory

3.1 Introduction

When should we consider two topological spaces the *same?*

The classical answer was "when there is a bicontinuous bijective correspondence between them", called a homeomorphism. It turned out that this answer was too strict and unintuitive in the sense that it doesn't always align with our intuitive understanding of when two spaces "have the same shape." Instead, we might think of the specific "shape" of a topological space as merely a "coincidental" incarnation of its intrinsic properties. The internal topology of the space should be independent of its specific form. Thus, if two topological spaces "have the same shape," they are essentially different manifestations of the same internal structure.

With this idea in mind, we'd much prefer to classify spaces according to their internal topology than just compare the arbitrary incarnations they've assumed. Then we need a criterion for determining when two spaces have the same intrinsic topology. The answer offered by algebraic topology is: We consider two spaces the same when one can be *continuously deformed on the other*. This happens exactly when there are continuous maps between the two spaces that are inverses to each other up to homotopy. Such maps are called **homotopy equivalences**.

Through the notion of homotopy, we've also obtained some powerful algebraic invariants: the fundamental group and the higher homotopy groups, which in turn paved the way for homology. This led to another notion of "sameness" even weaker than the previous. Two topological spaces are to be considered the same when their fundamental group is the same in every degree. A continuous map that induces isomorphisms between fundamental groups for all n is called a **weak homotopy equivalence**. Two spaces that are weak homotopy equivalent are indistinguishable as far as the fundamental groups can see.

If two spaces are homotopy equivalent then they're also weakly equivalent. But the converse fails. The situation can fixed for sufficiently nice spaces. The celebrated *Whitehead Theorem*, that first appeared in [Whi49], asserts that if we restrict our attention to $\mathcal{CW}$ complexes these two classes of maps coincide. Moreover, it can also be shown that any space is weakly equivalent to a $\mathcal{CW}$-complex, thus from the point of view of the homotopy theory of topological spaces, $\mathcal{CW}$-complexes are sufficient to capture all homotopy types.

(Weak) Homotopy equivalences are not isomorphisms in Top but behave much as if they were. For example, they satisfy 2-out-of-3. One might observe that they are the preimage of isomorphisms via some functor(s). In the first case, the functor is the projection on homotopy classes, and in the second the various π_n. Last, but not least, these maps *capture a notion of sameness of topological spaces*.

Instead of just considering homotopy equivalent spaces as being "almost the same", the homotopy theorist would like them to be literally the same so that he or she can work with spaces *up to homotopy*. That amounts to obtaining a new category, canonically induced from the first, where the (weak/homotopy) equivalences have been turned into literal isomorphisms. We have successfully turned a "philosophical" problem into a mathematical one.

Given a category and a set of morphisms, one can always forcefully invert them. This is analogous to ring-theoretic localization, where one formally inverts a multiplicative set of elements S. Unfortunately, the result of this rather violent process, the category of fractions, is usually badly behaved, since one loses control of the collection of maps between two objects which may no longer even be a set. The details and origins of this construction can be found in [GZ12].

A reasonable question then is: *what structure can I impose on my starting category so that the result of this process is better behaved?* This is the question answered by Quillen in [Qui67]. This extra structure is precisely that of a **model category**. Indeed, to a big extent, the *raison d'être* of the axioms of a model category (that are not about $\mathcal{W}$) is to control the construction of $\mathcal{M}[\mathcal{W}^{-1}]$ and make it better behaved.

In doing so, model categories provide an axiomatic framework for abstract homotopy theory, partly by abstracting out from the classical homotopy theory of topological spaces. Model categories provide a rich toolkit to the homotopy theorist. For example, we can explicitly construct the homotopy category $\mathrm{Ho}(\mathcal{M})$ which can be shown to satisfy the universal property required of the localisation, $\mathcal{M}[\mathcal{W}^{-1}]$.

In the general spirit of category theory, to fully understand a kind of object one must also study the "structure-preserving maps" between said objects. Another related and central question of the theory is the study of how to lift functors $F : \mathscr{M} \to \mathscr{N}$ between model categories to functors between their respective homotopy categories. We especially care about lifting adjoint pairs and equivalences. This leads us to the study of **Quillen Adjoints** and in turn **Quillen Equivalences**.

In the first subsection, we will take an abstract look in weak factorisation systems and model categories. Then, we will lay out some basic constructions for model categories and explore various interesting properties or additional structure on model categories. Much of these extra "gadgets" will play an important role when the time comes to interpret HoTT inside model categories. For a survey of the above ideas and more see [Rie20]. Standard references for model categories used throughout this chapter include [DHK97], [May99], [Hov99], [DS95].

Throughout the chapter, we will look at a variety of possible answers to the question "What is a homotopy theory?". We will survey some models and pay special attention to model categories. Some models have some deficiencies which others remedy. For example:

$$\text{hCat} \longrightarrow \text{mCat} \longrightarrow \text{smCat}$$

$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X,Y)$may be a class higher data is only implicit

We begin, in subsection 3.2, by introducing the notion of a homotopical category.[12] Seeing that the construction of the localization at weak equivalences is ill-behaved it is clear we need to import more structure from the archetypal example of topological spaces. The canonical answer is the structure of a model category. In the literature, there exists an intermediate notion, of a fibration category which we present first, in subsection 3.3. Then, in subsubsection 3.4.2, we indeed turn our attention to model categories. We survey the basics of how to do homotopy theory with model categories, the construction of the homotopy category, a model of the localisation at the class of weak equivalences, and, in subsubsection 3.4.4 the theory of under what circumstances we can lift functors, adjunctions and adjoint equivalences between model categories to their respective homotopy categories. Then in subsection 3.6 we look at three important ways of constructing new model categories from old. Lastly, we look at some more internal properties of model categories and a simplicially enriched variation of the initial definition, which will play an important role in this thesis.

3.2 Homotopical categories

The discussion above emphasized the central role of weak homotopy equivalences in the study of the homotopy theory of topological spaces. In a sense, a homotopy theory amounts to answering *in what sense should two things be considered the same?* Since when it comes to topological spaces, two objects are connected by a (zig-zag) of (weak) homotopy equivalences exactly when they have the same (weak) homotopy type, mathematically, to determine a homotopy theory, it suffices to specify the class of weak equivalences.

This is what the first model of a homotopy theory is centred around. The definition requires hardly anything, which is an advantage since examples abound. On the other hand, there's not much to work with and, as we'll see, the resulting constructions one would like to perform are, in general, ill-behaved.

Definition 82. A category $\mathcal{C}$ is called a **homotopical category** when it comes equipped with $\mathcal{W} \subset \mathrm{mor}(\mathcal{C})$ such that

[12]The term in common use in the literature is *relative* category. We follow Kapulkin [Kap14] in adopting a more illuminating term.

(1) We always have $\mathrm{id}_X \in \mathcal{W}$

(2) $\mathcal{W}$ is closed under composition.

A natural step for the homotopy theory of topological spaces is to be able to consider spaces *up to homotopy*, i.e. identify spaces that have the same homotopy type. In our more abstract context, this amounts to identifying weakly equivalent objects. A naive way to do that is to try to construct a new category where the weak equivalences are made into isomorphisms.

One makes this precise by constructing the **localization** of $\mathcal{C}$ at $\mathcal{W}$, denoted by $\mathcal{C}[\mathcal{W}^{-1}]$. This procedure is well understood in many cases of interest. The similarity to ring theoretic localization, where one forcefully inverts a multiplicative set of elements, rightfully comes to mind. These ideas first appeared in [GZ12].

Definition 83. Let $\mathcal{C}$ be a category and $\mathcal{W} \subset \mathrm{mor}(\mathcal{C})$. Then a **localization of $\mathcal{C}$ at $\mathcal{W}$** is a category and a functor into it

$$L : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$$

such that L maps the elements of $\mathcal{W}$ to isomorphisms and it is initial among functors that do that.

(1) $L(w)$ iso for $w \in \mathcal{W}$

(2) If $F : \mathcal{C} \to \mathcal{D}$ also maps $w \in \mathcal{W}$ to isos then it uniquely factors through $\mathcal{C}[\mathcal{W}^{-1}]$

Proposition 84 (Proposition 2.1.2 [Rie20])**.** The restriction along $L : \mathcal{C} \to \mathcal{C}[\mathcal{W}^{-1}]$ induces a fully faithful embedding, which in turn gives an isomorphism of functor categories $\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{M}) \cong \mathrm{Fun}_{\mathcal{W}\mapsto\cong}(\mathcal{C}, \mathcal{M})$. Namely, functors out of the localization correspond bijectively to functors that map $\mathcal{W}$ to isomorphisms.

$$\begin{tikzcd}
\mathrm{Fun}(\mathcal{C}[\mathcal{W}^{-1}], \mathcal{M}) \arrow[rr, hook] \arrow[dr] & & \mathrm{Fun}(\mathcal{C}, \mathcal{W}) \\
& \mathrm{Fun}_{\mathcal{W}\mapsto\cong}(\mathcal{C}, \mathcal{M}) \arrow[ur] &
\end{tikzcd}$$

Remark 85. It can be shown that for any pair $(\mathcal{C}, \mathcal{W})$ there is always a unique localization $\mathcal{C}[\mathcal{W}^{-1}]$. $\mathcal{C}[\mathcal{W}^{-1}]$ is obtained by momentarily treating $\mathcal{C}$ as a directed graph, adjoining inverses to the arrows $w \in \mathcal{W}$ and then quotienting out appropriately to enforce the desired identities, and then "freely completing" the resulting directed graph into a category again.

The universal property satisfied by these two constructions determines $\mathcal{C}[\mathcal{W}^{-1}]$ up to unique isomorphism. whilst introducing a very important theme. It allows us to lift functors between relative categories to functors between the respective homotopy categories.

Definition 86. Let $\mathcal{C}, \mathcal{D}$ be homotopical categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is called **homotopical** if it preserves weak equivalences.

Definition 87. Let hCat denote the category of homotopical categories and homotopical functors between them.

Remark 88. Homotopical functors induce functors between the respective homotopy categories.

Unfortunately, and perhaps unsurprisingly, the construction of $\mathcal{C}[\mathcal{W}^{-1}]$ is not well-behaved in general. First, we may run into size issues. Indeed, the localization of a locally small category may be a large category. Moreover, between adding inverses, quotienting out by relations and then freely completing into a category, we lose any understanding we could have of $\mathrm{Hom}_{\mathcal{C}[\mathcal{W}^{-1}]}(X, Y)$. Thankfully, one can introduce appropriate axioms so that this construction becomes better behaved. These are the axioms of model categories.

3.3 Fibration categories

A fibration category is an intermediate notion between homotopical categories and model categories. Since, as we will later see in greater detail, types are interpreted as fibrations, it is an environment nicely suited for interpreting type theory without redundant axioms. In addition to that, and for the same reasons, it makes sense to ask for various pieces of structure to exist only with respect to fibrations, as we'll do for local cartesian closure below.

Definition 89. A **fibration category** is a category $\mathcal{M}$ with two distinguished classes of wide subcategories the fibrations, $\mathcal{F}$ and the weak equivalences $\mathcal{W}$, satisfying the following axioms.

(1) $\mathcal{W}$ satisfies the 2-of-6 property.

(2) All isomorphisms are both fibrations and weak equivalences, or acyclic fibrations.

(3) Pullbacks along fibrations exist. Fibrations are stable under pullback.

(4) $\mathcal{M}$ has a terminal object and $\forall X \in \mathcal{M}$, $X \to 1 \in \mathcal{F}$

The obvious example, and the most important for our purposes, is that given a model category $\mathcal{M}$, the full subcategory on fibrant objects $\mathcal{M}_f$ is a fibration category. In particular qCat and $\mathscr{K}$an are fibration categories. Perhaps surprisingly, not all fibration categories arise this way, see example 5.1.4 [Kap14], where it is remarked that a category of C^* algebras arising in operator theory is a fibration category that does not fit in a model structure.

Using these axioms we can construct products where the projections are fibrations, show right properness, obtain path objects, define a notion of homotopy between maps and more. Another axiomatisation with essentially the same class of examples and consequences is being pursued by van den Berg in articles such as [Ber18].

Definition 90. Let $\mathcal{M}, \mathcal{N}$ be fibration categories. A functor $F : \mathcal{M} \to \mathcal{N}$ will be called exact when it preserves the structure present in a fibration category, explicitly, if it preserves (acyclic) fibrations, pullbacks along fibrations and the terminal object.

We saw how slice categories inherit a model structure from the base. We would like to do the same for fibration categories. Let $\mathcal{M}$ be a fibration category and $A \in \mathcal{M}$. Then ${}^{\mathcal{M}}\!\diagup_{A}$ does not have to be a fibration category since not all objects are fibrant (there existnon-fibrationn morphisms $B \to A$). To correct for this we define:

Definition 91. By $\mathcal{M}(A)$ we will denote the full subcategory of ${}^{\mathcal{M}}\!\diagup_{A}$ spanned by the fibrations. This is a fibration category.

Proposition 92. Let $f : X \to Y$ be a morphism in a fibration category $\mathcal{M}$. Then the pullback functor $f^* : \mathcal{M}(Y) \to \mathcal{M}(X)$ is exact.

Proof. We've asked that fibrations are pullback-stable so this definition makes sense. The terminal in the slice is the identity over the base. Pullback of an identity is an identity so $f^*(-)$ preserves the terminal object. The functionality of pullbacks implies that pullbacks are preserved. Stability of fibrations under pullback and right properness give preservation of (acyclic) fibrations. □

Definition 93. Let fCat denote the category of fibration categories and exact functors between them

Proposition 94. Every exact functor is homotopical. Hence there is an obvious forgetfull functor $U_{\text{fh}} : \text{fCat} \to \text{hCat}$.

Proof. The proof amounts to showing that if a functor between fibration categories sends acyclic fibrations to weak equivalences then it preserves all weak equivalences. This is exactly Lemma 4.1 in [Bro73]. □

Using the notion of *right homotopic* and *path objects* that we'll define for model categories in subsection 3.4 one gets a notion of a *homotopy category* of a fibration category. An exact map between fibration categories will be called an **equivalence** if it induces an equivalence between induced homotopy categories.

Definition 95. We call $\mathcal{M}$ a **locally cartesian closed fibration category** when

(1) All objects are cofibrant.

(2) For any fibration $p : B \to A$, the pullback functor $f^* : \mathcal{M}(A) \to \mathcal{M}(B)$ in addition to being exact, also has a homotopical right adjoint.

Remark 96. We've asked for right adjoints to pullback functors *only against fibrations*, therefore a locally cartesian closed fibration category need not be locally cartesian closed.

Remark 97. All fibration slices $\mathcal{M}(A)$ are cartesian closed. Moreover, for a fibration $p : B \to A$, the product functor $p \times - : \mathcal{M} \to \mathcal{M}$ is exact and has a homotopical right adjoint.

3.4 Model categories

3.4.1 Weak factorisation systems

A **weak factorisation system** in a category $\mathcal{C}$ consists of an interrelated pair of classes of morphisms, $\mathcal{L}, \mathcal{R}$. Using these two classes one can factor any arrow of the category. They are important because if we restrict our attention to them we can always solve certain **lifting problems**. Throughout the following let $\mathcal{C}$ denote an ambient category.

Definition 98. A **lifting** problem in $\mathcal{M}$ is a commutative square as seen below. A **solution** to the lifting problem is an arrow $s : C \to D$ splitting the square into two commutative triangles.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "e"'] & B \arrow[d, "m"] \\ C \arrow[r, "v"'] \arrow[ur, dashed, "s"] & D \end{tikzcd}$$

The solution s may also be denoted $e \boxslash m$. If such a solution exists, we may say that e has the **left lifting property** with respect to m or, equivalently, that m has the **right lifting property** with respect to e.

Definition 99. Let $\mathcal{K} \subset \mathrm{mor}\mathcal{M}$ be a class of morphisms in $\mathcal{M}$. Then we define:

i $\mathcal{K}^{\boxslash} = \mathrm{LLP}(\mathcal{K}) := \left\{ f \in \mathrm{mor}\mathcal{M} \middle| \quad (\forall k \in \mathcal{K}) \quad \exists s = k \boxslash f \right\}$ $\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "k"'] & B \arrow[d, "f"] \\ C \arrow[r, "v"'] \arrow[ur, dashed, "s"] & D \end{tikzcd}$

ii ${}^{\boxslash}\mathcal{K} = \mathrm{RLP}(\mathcal{K}) := \left\{ g \in \mathrm{mor}\mathcal{M} \middle| \quad (\forall k \in \mathcal{K}) \quad \exists g \boxslash k \right\}$

Anticipating the role maps from $\mathcal{L}$ and $\mathcal{R}$ will play in later Sections we will start indicating that an arrow belongs to one of the classes by decorating it $\rightarrowtail \in \mathcal{L}$ and $\twoheadrightarrow \in \mathcal{R}$.

Definition 100. Let $\mathcal{L}, \mathcal{R} \subset \mathrm{mor}\mathcal{M}$ be two distinguished classes of morphisms of $\mathcal{M}$. We say that $\mathcal{L}$ has the left lifting property against $\mathcal{R}$ when

$$(\forall l \in \mathcal{L})(\forall r \in \mathcal{R}) \quad \exists s := l \boxslash r, \qquad \begin{tikzcd} A \arrow[r, "u"] \arrow[d, tail, "l"] & B \arrow[d, two heads, "r"] \\ C \arrow[r, "v"'] \arrow[ur, dashed, "s"] & D \end{tikzcd}$$

If that is the case, we may write $\mathcal{L} \boxslash \mathcal{R}$ We note that $\mathcal{L} \boxslash \mathcal{R} \iff \mathcal{L} \subseteq {}^{\boxslash}\mathcal{R} \quad \& \quad \mathcal{R} \subseteq \mathcal{L}^{\boxslash}$

Definition 101. Two classes of maps form a **lifting pair** when $\mathcal{L} = {}^{\boxslash}\mathcal{R} \quad \& \quad \mathcal{R} = \mathcal{L}^{\boxslash}$

So, to show that some $f \in \mathcal{L}$ we can instead show it lifts against an arbitrary map $r \in \mathcal{R}$. Primarily using this argument and other elementary means one can prove the following closure properties for lifting pairs.

Proposition 102. Let $(\mathcal{L}, \mathcal{R})$ be a lifting pair in a category $\mathcal{E}$. Then:

(*1*) Both $\mathcal{L}$ and $\mathcal{R}$ contain all isomorphisms.

(*2*) $\mathcal{R}$ is closed under composition: for composable $f, g \in \mathcal{R} \implies g \circ f \in \mathcal{R}$

(*3*) $\mathcal{R}$ is closed under pullbacks. For any $a \in \mathrm{mor}\mathcal{E}$ with the same codomain as f, $f \in \mathcal{R} \implies a^*(f) \in \mathcal{R}$, where $a^*(-)$ denotes pullback against a.

(*4*) $\mathcal{R}$ is closed under forming products of maps: $f, g \in \mathcal{R} \implies f \times g \in \mathcal{R}$.

(*5*) $\mathcal{R}$ is closed under forming retracts. $f \in \mathcal{R}$ and g is a retract of $f \implies g \in \mathcal{R}$. g is said to be a retract of f when there is a diagram:

$$\begin{tikzcd} A \arrow[r] \arrow[d, "g"] \arrow[rr, bend left, "\mathrm{id}"] & C \arrow[r] \arrow[d, "f"] & A \arrow[d, "g"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & D \arrow[r] & B \end{tikzcd}$$

Proof. Straightforward proof. For the details see [Nor17] $\square$

Remark 103. The dual properties hold for $\mathcal{L}$ namely, $\mathcal{L}$ is closed under pushouts, pushouts of maps and retracts. A class of maps enjoying this set of properties is sometimes called **saturated**. These, play an important role in producing examples of model categories. More information in subsubsection 3.8.2

Examples 104. (*1*) In **Set**, $(\mathcal{I}, \mathcal{S}) = (\{\text{injections}\}, \{\text{surjections}\})$ form a lifting pair. Point of interest: this relies on the axiom of choice.

(2) In **Top** we can take $\mathcal{L}$ to be the quotient maps and $\mathcal{R}$ to be continuous inclusions.

Definition 105. A pair of distinguished classes of morphisms, $\mathcal{L}, \mathcal{R}$ **factors** a category $\mathcal{E}$, when $\forall f \in \text{mor}\mathcal{E} : (\exists l \in \mathcal{L})(\exists r \in \mathcal{R})$ such that $f = r \circ i$

Definition 106. This factorisation is functorial when its data are given by a functor which, put concisely, is a section to the composition functor:

$$\mathfrak{c} : \mathcal{C}^{\Delta[2]} \to \mathcal{C}^{\Delta[1]}$$

Unpacking this definition we ask for a functor $\mathfrak{f} : \mathcal{C}^{\Delta[1]} \to \mathcal{C}^{\Delta[2]}$ such that $\mathfrak{c} \circ \mathfrak{f} = \text{id}_{\mathcal{C}^{\Delta[1]}}$. Unraveling even further, the functor fact comprises of two functors,

$$r : \text{mor}\mathcal{E} \to \text{mor}\mathcal{E} : f \mapsto r_f \in \mathcal{R}$$

$$l : \text{mor}\mathcal{E} \to \text{mor}\mathcal{E} : f \mapsto l_f \in \mathcal{L}$$

satisfying identities that ensure that the domains and codomains of the arrows above are as desired: $\text{dom} \circ l = \text{dom}$, i.e. the domain of l_f is the same as that of f, $\text{cod} \circ l = \text{dom} \circ p$ and $\text{cod} \circ p = \text{cod}$. We indeed have $\mathfrak{c} \circ \mathfrak{f} = \text{id}_{\mathcal{C}^{\Delta[1]}}$ since $\mathfrak{c} \circ \mathfrak{f}(f) = \mathfrak{c}(l_f, r_f) = f$.

Definition 107. A **Weak Factorisation System**, abbreviated WFS, in a category $\mathcal{C}$ distinguishes two classes of morphisms of $\mathcal{L}, \mathcal{R} \subset \text{mor}\mathcal{E}$ such that

(1) $\mathcal{L}, \mathcal{R}$ factor $\mathcal{C}$ meaning that any map $f : X \to Y$ can be factored as a left map followed by a right map.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, tail, "l_f"'] & & Y \\ & M \arrow[ur, two heads, "r_f"'] & \end{tikzcd}$$

(2) $\mathcal{L} = {}^{\boxslash}\mathcal{R}$ & $\mathcal{R} = \mathcal{L}^{\boxslash}$. In particular, any lifting problem as below has a solution, namely a map $s : C \to B$ producing two commutative triangles.

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, tail, "l"'] & B \arrow[d, two heads, "r"] \\ C \arrow[r, "v"'] \arrow[ur, dashed, "s"] & D \end{tikzcd}$$

A WFS if functorial when its factorisation is.

3.4.2 Definition and first properties

As already discussed, the naive approach in abstract homotopy theory doesn't get us too far. If we wish to go further, we must import more structure from Top in our abstract setting. The only question is what structure exactly. In 1967, Quillen made a particularly successful proposal in [Qui67].

He chose to import the structure of two intertwined weak factorisation systems and required the presence of limits and colimits. The two factorisation systems come with a choice of 3 classes of morphisms. Firstly, the *weak equivalences* $\mathcal{W}$ play the role of weak homotopy equivalences. We decorate these arrows as $\xrightarrow{\sim} \ \in \mathcal{W}$. The crucial feature is that two spaces connected by a (zig-zag of) weak equivalence are thought of as having the same homotopy type. This intuition is formalised by the fact that it is precisely then when two spaces become identified in the homotopy category. Secondly and thirdly, we ask for classes of *fibrations*, and *cofibrations*, which must be thought of as classes of *nice* surjections and injections respectively. These classes determine collections of objects where our constructions become particularly tractable and indeed make the outcome much better behaved.

Model categories come with a significant amount of extra structure, which facilitates many constructions. First and foremost, one can closely mimic much of the classical homotopy theory of spaces in **Top** in an abstract setting. Indeed, we can talk about homotopies of maps, and see and reinvent many known results including HEP and the (dual) Whitehead theorem. The basics are covered in subsubsection 3.4.3. The most important application is the construction of the *homotopy Category*, a model for the localisation of $\mathcal{M}$ at the class of weak equivalences, $\mathcal{W}$. This is also covered in subsubsection 3.4.3. We then turn, in subsubsection 3.4.4, to the theory of lifting functors and adjunctions from the level of model categories to their respective homotopy categories. This will lead to the notion of Quillen Equivalences, the correct notion of equivalence of homotopy theories. The rest of the chapter is devoted to studying constructions that produce new model categories from old, in subsection 3.6, studying additional properties a model category may have and variants of the definition that give model categories even more structure.

Definition 108. A category $\mathcal{E}$ has a model structure when:

(1) It has all (small) limits and colimits

(2) It comes equipped with three distinguished classes of morphisms $(\mathcal{C}, \mathcal{W}, \mathcal{F})$, called cofibrations, weak equivalences and fibrations respectively, such that:

(a) $\mathcal{W}$ satisfies the **two out of three** property, given a composable pair of morphisms,

f, g, whenever two out of $f, g, f \circ g$ are weak equivalences, so is the third.

(b) $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ and $(\mathcal{C}, \mathcal{W} \cap \mathcal{F})$ are both WFS on $\mathcal{E}$.

Maps in $\mathcal{C} \cap \mathcal{W}$ are called trivial or acyclic cofibrations and maps in $\mathcal{W} \cap \mathcal{F}$ are called trivial or acyclic fibrations.

To facilitate reasoning with diagrams it is customary in the literature to decorate arrows when they belong to one of these classes. In what follows we will denote

$$\rightarrowtail \in \mathcal{C} \qquad \xrightarrow{\sim} \in \mathcal{W} \qquad \twoheadrightarrow \mathcal{F}$$

It is instructive to illustrate explicitly what the condition of the two WFS entails. For any $f \in \mathcal{E}$ we have two factorisations:

$$\begin{tikzcd} X \arrow[r, tail] \arrow[d, tail, "\sim"'] \arrow[dr, "f"] & F(f) \arrow[d, two heads, "\sim"] \\ C(f) \arrow[r, two heads] & Y \end{tikzcd}$$

Observe that both $F(f), C(f)$ are connected to X by weak equivalences and therefore have the same homotopy type.

Moreover, all lifting problems as below admit solutions,

$$\begin{tikzcd} A \arrow[r] \arrow[d, tail, "\sim"'] & C \arrow[d, two heads] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r] \arrow[d, tail] & C \arrow[d, two heads, "\sim"'] \\ B \arrow[r] \arrow[ur, dashed] & D \end{tikzcd}$$

Definition 109. Recall, that a model category is bicomplete (for small (co)limits) thus in particular it has both a terminal and an initial object. An object $X \in \mathcal{E}$ is called **fibrant** exactly when the unique map $X \overset{!}{\twoheadrightarrow} 1$ is a fibration. Dually, it is called cofibrant when the unique arrow $\varnothing \overset{!}{\rightarrowtail} X$ is a cofibration. When an object is both fibrant and cofibrant we sometimes call it bifibrant.

Example 110. One of the archetypal examples for model structures comes from the *classical model structure on Topological spaces.* We distinguish the following classes of maps:

(1) $\mathcal{C}$ = "generated" [13] by $I_{\text{Top}} = \{S^{n-1} \hookrightarrow D^n\}_{n \in \mathbb{N}}$.

(2) $\mathcal{F}$ = the Serre fibrations, that are defined as the right complement to the acyclic cofibrations.

[13] The meaning of these statements will be made precise in the section for cofibrantly generated model categories.

(3) $\mathcal{W} = \{$weak homotopy equivalences$\}$.

Acyclic cofibrations are those maps that are simultaneously a weak equivalence and a cofibration. In this case they are also generated by $J_{\text{Top}} = \{D^n \hookrightarrow D^n \times I\}_{n\in\mathbb{N}}$. With these definitions all objects are fibrant. The cofibrant (and therefore bifibrant) objects are precisely $\mathcal{CW}$-complexes. A classical result asserts that any space is weakly equivalent to a $\mathcal{CW}$-complex so that they comprise a class of representatives of the homotopy types with particularly nice formal properties.

Example 111. The category **sSet** admits the **Kan-Quillen** model structure by distinguishing the following classes of morphisms. If we want to emphasize that we consider the category of simplicial sets equipped with this model structure we write $\mathbf{sSet}_{\text{KQ}}$.

(1) $\mathcal{C} = \{$ monomorphisms $\}$.

(2) $\mathcal{F} = \{$ Kan Fibrations $\}$.

(3) $\mathcal{W} = \{$ weak homotopy equivalences $\}$.

Furthermore, we note that the acyclic/trivial cofibrations are $\mathcal{C} \cap \mathcal{W} = \{$ Anodyne Maps $\}$ and the trivial fibrations $\mathcal{F} \cap \mathcal{W} = \{$ Trivial Kan Fibrations $\}$. As above, $\mathcal{C}$ is generated by $\{\partial\Delta^n \hookrightarrow \Delta^n\}$ and $\mathcal{C} \cap \mathcal{W}$ by $\{\Lambda^n_k \hookrightarrow \Delta^n\}$. By definition, the fibrant objects of $\mathbf{sSet}_{\text{KQ}}$ are the Kan Complexes. Since maps of the form $\varnothing \to X$ are always monic = cofibrations, we get that all simplicial sets are cofibrant.

A classical result of Joyal, corollary 1.4 in [Joy02], is that a simplicial set is a Kan complex $\iff$ it is an ∞-groupoid. Thus, this model category captures the homotopy theory of ∞-groupoids. In item 3, we will see that there is an *equivalence of homotopy theories*, in some appropriate sense, to the homotopy theory of topological spaces. This is but a restatement of the idea that an ∞-groupoid is the correct algebraic gadget to capture the homotopy type of a space. This idea originates with Quillen [Qui67] and Grothendieck [Gro21].

Example 112. The category **sSet** can be equipped with another model structure, the Joyal model structure, denoted $\mathbf{sSet}_{\text{Joy}}$, obtained by putting

(1) $\mathcal{C} = \{$ monomorphisms $\}$.

(2) $\mathcal{F} = \{$ categorical fibrations $\}$.

(3) $\mathcal{W} = \{$ categorical equivalences $\}$.

Furthermore, we note that by Proposition 39.2 in [Rez22] $\mathcal{F} \cap \mathcal{W} = \{$ Trivial Kan Fibrations $\}$. To determine the fibrant objects first we note that $*$ is an ∞-groupoid and therefore a Kan Complex and therefore an ∞-category. Moreover, we note corollary 2.4.6.5 in [Lur09] given a map whose target is an ∞-category, then f is categorical fibration $\iff$ it is an isofibration. Moreover, by definition, an isofibration is, in particular, an inner fibration so, we immediately get that fibrant objects are *precisely* the ∞-categories. As above, since cofibrations are the monos, we get that all objects are cofibrant. The (bi)fibrant objects being precisely ∞-categories we can say that this model category captures the homotopy theory of ∞-categories.

3.4.3 Homotopy theory in model categories

When doing homotopy theory in Top, we make very frequent use of cylinder objects $\mathrm{Cyl}(X) := X \times I$ and path spaces X^I, where $I = [0,1]$. The axioms for model categories chosen by Quillen in [Qui67] naturally allow us to mimic these constructions. One can then use them to define a notion of homotopical maps f, g in a model category.

Definition 113. A **cylinder object for** X is given by

$$X \underset{i_1}{\overset{i_0}{\rightrightarrows}} \mathrm{Cyl}X \xrightarrow[p]{\sim} X$$

With $p \circ i_k = \mathrm{id}_X$. Observe that by 2-of-3 we get that $i_k \in \mathcal{W}$. Intuitively we think of i_k as $X \hookrightarrow X \times \{k\}$ so it should not be surprising they are weak equivalences.

$\mathrm{Cyl}(X)$ is **good** if $i_1 + i_2 : X \sqcup X \rightarrowtail X$. It is **very good** if $p \in \mathcal{F} \cap \mathcal{W}$. By factoring the folding map $X \sqcup X \hookrightarrow X$ we get a very good cylinder object for arbitrary X.

Definition 114. A **left homotopy** between $f, g : X \to Y$ is a map $h : \mathrm{Cyl}(X) \to X$ with $h \circ i_0 = f$ and $h \circ i_1 = g$.

Definition 115. Dually, a **path object** for X is given by

$$X \xrightarrow[i]{\sim} P(X) \underset{p_1}{\overset{p_0}{\rightrightarrows}} X$$

All the definitions above dualise. We factor the diagonal map $\Delta : X \to X \times X$ to obtain a very good path space for any X.

Remark 116. We should note, that the left/right homotopy is not an equivalence relation in general. However, if we place assumptions on X and Y the situation improves. X cofibrant implies that left homotopy gives a right homotopy. The dual holds for a fibrant Y. When both these requirements are satisfied, and we get an equivalence relation $\simeq$ on $\mathrm{Ho}_{\mathcal{M}}(X, Y)$. This is a characteristic instance of the benefits of restricting to a subcategory of bifibrant objects. Another such instance is the following Proposition

Proposition 117. (Proposition 3.3.10 in [Rie20]) A map $f : A \to B$ between bifibrant A, B is a weak equivalence if and only if it admits a homotopy inverse.

An important ingredient is a procedure to **replace** objects with bifibrant ones. This can be done functorially in two steps. We successively apply a fibrant replacement and then a cofibrant one. The fibrant replacement of X produces a fibrant object KX *weakly equivalent* to X. In fact, it is a natural transformation of endofunctors $\eta : \mathrm{id}_{\mathcal{M}} \Rightarrow K$ which is pointwise a weak equivalence. The dual/same is true for the cofibrant replacement. Therefore any object is two weak equivalences away from a bifibrant one. Thus, for any X there exists a bifibrant object BX with the same homotopy type. Moreover, these constructions interact nicely with homotopies. By an elementary 2-of-3 argument, one can prove that fibrant and cofibrant replacements are homotopical functors. Therefore so is their composition. For a comprehensive account see section 14.3 of [May99]. For an object X, we write BX for a bifibrant replacement.

Definition 118. Given a model category $\mathcal{M}$ we define its **homotopy category** $\mathrm{Ho}\mathcal{M}$ as having the same objects and let the hom-sets be the homotopy equivalence classes between their bifibrant replacements.

$$\mathrm{Hom}_{\mathrm{Ho}\mathcal{M}}(X, Y) := {\mathrm{Hom}_{\mathcal{M}}(BX, BY)}\big/_{\simeq}$$

We get a canonical functor $\gamma : \mathcal{M} \to \mathrm{Ho}\mathcal{M}$

Remarks 119. *(1)* Since BX, BY are bifibrant, $\simeq$ is an equivalence relation and therefore the definition above makes sense.

(2) The Hom-sets of $\mathrm{Ho}(\mathcal{M})$ depend solely on functions between bifibrant objects and homotopies thereof.

The following theorem formalises the intuition that the homotopy category construction does actually compute a model for the localisation of the model category at the weak equivalences. For a full presentation of the proofs of these results, quite a bit of additional machinery is required. The reader is therefore referred to section 14 of [May99] for the proofs and the prerequired theory. Here we will only give sketches of the proofs.

Theorem 120. The category $\mathrm{Ho}\mathcal{M}$ is a model for $\mathcal{M}\mathcal{W}^{-1}$.

Proof. We prove that $\mathrm{Ho}\mathcal{M}$ has the universal property of localisation. First, we must show it sends weak equivalences to isomorphisms. Take $w : A \to B \in \mathcal{W}$. Since the bifibrant replacement functor is homotopical we get $Bw : BA \to BB$. Now we have a weak equivalence between bifibrant objects. By Proposition 117, Bw admits a homotopy inverse, making it an isomorphism in Ho$\mathcal{M}$.

Now we must verify the universal property of Ho$\mathcal{M}$. Take arbitrary $F : \mathcal{M} \to \mathcal{E}$ that inverts weak equivalences. We want to define $\tilde{F} : \text{Ho}\mathcal{M} \to \mathcal{E}$ giving a commutative triangle and showing it is unique. Since γ is bijective on objects, $F, \tilde{F}$ agree on objects. By naturality of (co)(bi) fibrant replacement, we obtain a natural transformation $\alpha : F \Rightarrow FB$. These natural transformations where point-wise weak equivalences and F inverts them, making α a natural isomorphism. Now, take a representative of a homotopy class of some $h : BX \to BY$. Define,

$$\tilde{F}h := \qquad FX \xrightarrow{\alpha_X} FBX \xrightarrow{FBh} FBY \xrightarrow{\alpha_Y^{-1}} FY$$

Again elementarily we show that $\tilde{F}$ is functorial and that $\tilde{F}\gamma = F$. For uniqueness, observe that any map $h : BX \to BY$ in Ho$\mathcal{M}$ is isomorphic to one of the form $\gamma(h) = B(h)$. But the value of $\tilde{F}$ on the latter is uniquely determined by $F(h)$, because it is equal to it. Thus the value $\tilde{F}(h)$ is uniquely determined by $F(h)$. □

Proposition 121. $f \in \mathcal{W} \iff \gamma(f)$ iso in Ho$\mathcal{M}$. Use B and proposition 117.

3.4.4 Quillen adjunctions and equivalences

We have determined, how to pass from model categories to 1-categories, their homotopy categories. A natural next question is to ask under what conditions can we lift functors in homotopy categories.

Let F be a homotopical functor between model categories. Then the composition $\gamma_{\mathcal{N}} \circ F$ sends weak equivalences of $\mathcal{M}$ to isomorphisms. By universal property of localisation we get a unique functor $\tilde{F} : \text{Ho}\mathcal{M} \to \text{Ho}\mathcal{N}$.

The functors found in nature are rarely homotopical, so this definition although natural is not very useful. Instead, one can determine a weaker set of conditions that when satisfied, allow a functor to be lifted to homotopy categories on the left or on the right. These are called **derived** functors. Take $F : \mathcal{M} \to \mathcal{N}$. A left derived functor for F is $\mathbb{L}F : \text{Ho}\mathcal{M} \to \mathcal{N}$ and a natural transformation $\mu : LF \circ \gamma \Rightarrow F$ that is terminal among such pairs. Namely for any other $K : \text{Ho}(\mathcal{M}) \to \mathcal{N}$ that comes equipped a $\xi : K \circ \gamma \to F$ $\exists! \sigma K \to \mathbb{L}F$ such that $\mu \circ (\sigma * \gamma)$ coincides with ξ.[14]

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"'] & \mathcal{N} \\
\text{Ho}(\mathcal{M}) \arrow[ur, dashed, "\mathbb{L}F"] \arrow[ur, bend right, "K"'] \arrow[ur, phantom, "\sigma\Uparrow"]
\end{tikzcd}$$

[14] $\sigma * \gamma$ denotes *whiskering of natural transformations.*

We often drop μ from the notation. There is a dual notion of **right derived functors**.

Proposition 122. (Proposition 16.1.3-4 in [May99]) If $F : \mathcal{M} \to \mathcal{N}$ takes acyclic cofibrations between cofibrant objects to isomorphisms, then there exists a left derived functor $\mathbb{L}F$. Furthermore, if X is cofibrant, $\mu\mathbb{L}FX \to FX$ is an isomorphism. Dually, if F takes acyclic fibrations between fibrant objects to isos, then there exists a right derived $\mathbb{R}F$. If X is fibrant, we get an iso $FX \to \mathbb{R}FX$.

This abstract formulation is unnecessarily obtuse. The proof of the statement above makes everything practically much more tractable and direct. We can construct *a model for* the (left) right derived functor by precomposing with a (co)fibrant replacement. This is yet another instance where the better-behaved (co)fibrant objects largely simplify a situation. We follow Rezk's section 4.2 in [Rez10]. Recall that by factoring an arbitrary map $\varnothing \to X$ and $Y \to *$ and obtaining the factorisation we get from the axioms of a model category we get (co)fibrant replacement functors.

$$\begin{tikzcd}
\varnothing \arrow[rr] \arrow[dr, tail] & & X \\
& X^c = QX \arrow[ur, two heads, "\sim"] &
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[rr, two heads] \arrow[dr, tail, "\sim"] & & * \\
& Y^f = RY \arrow[ur, two heads] &
\end{tikzcd}$$

Proof. (of proposition 122. Adapted from [May99])

Let $F : \mathcal{M} \to \mathcal{N}$ be a functor that takes weak equivalences between cofibrant objects to isomorphisms. Take $w : X \xrightarrow{\sim} Y$. It is easy to see that a functorial cofibrant replacement Q is homotopical, explicitly $Q(w) : Q(X) = X^c \to Y^c = Q(Y)$ is again a weak equivalence. But then the assumption on F applies. Then, $F \circ Q$ sends weak equivalences of $\mathcal{M}$ to isomorphisms in $\mathcal{N}$. Let $\mathbb{L}F$ be the functor obtained by the universal property of γ applied on $F \circ Q$. Then, $\mathbb{L}F \circ \gamma = F \circ Q$. Define $\mu_X := Fq : FQX \to FX$. If X was already cofibrant, then q is a weak equivalence between cofibrant objects and thus Fq is an isomorphism.

□

Moreover, the second conclusion of the proposition simply becomes $\mathbb{L}F \approx F(QX) = F(X^c)$. Dually, if F takes weak equivalences between fibrant replacements to isomorphisms, it admits a right derived functor $\mathbb{R}F : \mathrm{Ho}(\mathcal{M}) \to \mathcal{N}$ for which $\mathbb{R}F(Y) \approx F(Y^f) = FRY$.

We return to our original question of lifting functors between model categories to the homotopy categories. The situation looks like this:

$$\begin{tikzcd}
\mathcal{M} \arrow[r, "F"] \arrow[d, "\gamma"] & \mathcal{N} \arrow[d, "\delta"] \\
\mathrm{Ho}\mathcal{M} \arrow[r, dashed] & \mathrm{Ho}\mathcal{N}
\end{tikzcd}$$

We seek $\mathrm{Ho}F : \mathrm{Ho}\mathcal{M} \to \mathrm{Ho}\mathcal{N}$. It suffices to obtain a left-derived functor for $\delta \circ F$. Since δ takes weak equivalences to isomorphisms, we see that F taking cofibrations between cofibrant objects to *weak equivalences* suffices to guarantee a left derived functor for $\delta \circ F$. This is precisely the motivation for the axioms of *Quillen adjunctions.* The desired fact is an immediate consequence of these axioms.

Definition 123. Let $\mathcal{M}, \mathcal{N}$ be model categories and $F : \mathcal{M} \rightleftarrows \mathcal{N} : G$ be a pair of adjoints. We call them **Quillen adjoints** or a **Quillen pair**, if one of the following equivalent conditions is satisfied:

(*1*) F preserves cofibrations and acyclic cofibrations

(*2*) G preserves fibrations and trivial fibrations.

(*3*) F preserves cofibrations and G preserves fibrations

(*4*) F preserves acyclic cofibrations and G preserves acyclic fibrations.

To see that the conditions are indeed equivalent is a straightforward model-theoretic diagram chase. Condition 4 above immediately implies that F, G have total derived functors. But something stronger holds:

Proposition 124. (Proposition 16.2.2 in [May99]) A Quillen adjunction $F \dashv G$ between $\mathcal{M}$ and $\mathcal{N}$ lifts to an adjunction $\mathbb{L}F \dashv \mathbb{R}G$ between $\mathrm{Ho}(\mathcal{M})$ and $\mathrm{Ho}(\mathcal{N})$.

Now that we have an adjunction $\mathbb{L}F : \mathrm{Ho}(\mathcal{M}) \leftrightarrows \mathrm{Ho}\mathcal{N} : \mathbb{R}G$ we can also wonder when is the (co)unit an iso. If both the unit and the counit were isos, we'd have an adjoint equivalence. This leads us to define: The pair $L \dashv R$ is said to be a **Quillen equivalence** exactly when either $\mathbb{L}, \mathbb{R}$ is an equivalence. A Quillen equivalence is the correct notion of sameness of Homotopy Theories.

Definition 125. Let $F : \mathcal{M} \leftrightarrows \mathcal{N} : G$ be a Quillen Adjunction. The following are equivalent and when satisfied we call (F, G) a **Quillen Equivalence**

(*1*) $\mathbb{L}F$ is an equivalence between homotopy categories.

(*2*) $\mathbb{R}G$ is an equivalence between homotopy categories

(*3*) For any cofibrant object C in $\mathcal{M}$ and fibrant object F of $\mathcal{N}$ we have that

$$f : C \xrightarrow{\sim} R(F) \iff \bar{f} : L(C) \xrightarrow{\sim} F$$

Remark 126. It is important to note that Quillen Equivalence is not a "symmetric relation". That's because the pair of functors in a Quillen pair are dual to one another, and therefore different in general. For example, one of the adjoints preserves fibrations and the other cofibrations. Hence, if someone asserts that $F : \mathcal{M} \to \mathcal{N}$ is part of a Quillen Equivalence we must also ask if F is the left or right adjoint.

This is precisely the condition required to make the lifted adjunction an adjoint equivalence. Indeed,

Proposition 127. (Proposition 16.2.2 in [May99]) $L \dashv R$ is a Quillen Equivalence exactly when $\mathbb{L} \dashv \mathbb{R}$ is an adjoint equivalence between homotopy categories.

Proof. (following the reference). Recall that $\mathbb{L}F = FQ$ and $\mathbb{R}G = GR$.

To see that the lifted adjunction is an equivalence we show that both the **derived unit** and **derived counit** comprise of isomorphisms. Unraveling the definitions we find that the derived unit is given by

$$QD \xrightarrow{\eta_{QD}} GF(QD) \xrightarrow{R_{FQ(D)}} G(R(FQ(D)))$$

Where the first map is the unit of the non-derived Quillen pair on component QD, and the second is $G(-)$ of the arrow $R_{FQ(D)} : FQ(D) \overset{\sim}{\rightarrowtail} RFQ(D)$, the trivial cofibration of a fibrant replacement of $FQ(D)$. Since everything is precomposed with $Q(-)$, we might as well suppose we have a cofibrant X. Then the derived unit is $G(R_F) \circ \eta_X$. By rules of the adjunctions, this map transposes to $r : FX \to RFX$, which is a weak equivalence by the construction of fibrant replacement. Thus the derived unit is made up of weak equivalences which become isos in the homotopy category. A dual argument applies to the derived counit. The two combined prove a Quillen equivalence induces an equivalence between homotopy categories. □

Examples 128. *(1)* Let $\mathcal{C} \in \mathrm{mCat}$. We have a trivial Quillen Equivalence $\mathrm{id} : \mathcal{C} \leftrightarrows \mathcal{C} : \mathrm{id}$. Immediate by the definition.

(2) For any model category we have $\mathrm{incl} : \mathcal{M}^f \leftrightarrows \mathcal{M} : E$ is a Quillen Equivalence for any fibrant replacement functor such that for all X, $\epsilon : X \xrightarrow{\sim} EX$. The bijection of hom-sets is given by composition with ϵ. In that triangle, the 2-of-3 property of weak equivalences corresponds exactly to the definition of QE 125.

(3) Write $\mathrm{Top}_{\mathrm{Quillen}}$ for the classical model structure on topological spaces. Then we have a QE:

$$\mathrm{Sing} : \mathrm{Top}_{\mathrm{Quillen}} \leftrightarrows \mathrm{sSet}_{\mathrm{KQ}} : |-|$$

The latter is the historic statement known as the *Homotopy Hypothesis*. The equivalence mentioned above is due to Daniel Quillen and appeared in [TV05]. For more information see [Def19]

We now record some properties of Quillen adjunctions that will prove useful later on in the thesis. The following are taken from nLab.

Lemma 129. Let $f : X \to Y$ be an arrow in $\mathcal{C}$. Then there is a commutative diagram

$$\begin{tikzcd} & X \arrow[d, two heads, "\sim r"] \\ X \arrow[ur, "\mathrm{id}_X"] \arrow[r, "j"] \arrow[dr, "f"'] & Z \arrow[d, two heads, "g"] \\ & Y \end{tikzcd}$$

We will not prove this lemma. The proof can be found in *factorisation lemma* nLab entry.

Lemma 130. Given a Quillen adjunction $L \dashv R$ we get that

(*1*) L preserves weak equivalences between cofibrant objects

(*2*) R preserves weak equivalences between fibrant objects

Proof. We do the proof only for R, since this is the one we'll need. The other proof is dual. Take a pair of fibrant objects and a weak equivalence between them. Since R preserves fibrations and $R(*) = *$ because it is a right adjoint and therefore preserves limits. Therefore $R(F), R(G)$ are also fibrant objects. By the factorisation lemma, we get a diagram:

$$\begin{tikzcd} & F \arrow[d, two heads, "\sim r"] \\ F \arrow[ur, "\mathrm{id}_F"] \arrow[r, "j"] \arrow[dr, "w"'] & Z \arrow[d, two heads, "g"] \\ & G \end{tikzcd} \qquad \begin{tikzcd} RF \arrow[r, "Rj"] \arrow[dr, "Rw"'] & RZ \arrow[d, two heads, "\sim Rg"] \\ & RG \end{tikzcd}$$

We want to show that $Rw \in \mathcal{W}$. We will use 2-of-3. In the top triangle below we know that identities are in particular weak equivalences. Moreover, we've assumed that r is a trivial fibration and in particular a weak equivalence. By 2-of-3 we obtain that $j : F \to Z \in \mathcal{W}$. So if we now focus on the bottom triangle, we've assumed that $w \in \mathcal{W}$. Again by 2-of-3, we get that $g \in \mathcal{W}$ making g a trivial fibration. By the axioms for Right Quillen functors, we get that so is Rg. Now, if we apply R to the top triangle below, we get id_{RF} and Rr. Again, since R is right Quillen, Rr is again a trivial fibration, and in particular a weak equivalence. The same is true for id_{RF}. By 2-of-3 we obtain that $Rj \in \mathcal{W}$. We can finally deduce that $Rw \in \mathcal{W}$ as desired. Note that the assumption that the objects are fibrant is essential for the construction of the factorisation. □

Corollary 131. It is immediate, that if (F, G) is a Quillen pair, the left Quillen functor admits a left derived functor and the right Quillen functor a right derived one.

Another important point is that we would like to have a category of model categories and forgetful functors towards the category of fibration categories and homotopical categories. Moreover, we'd like them to be compatible with the forgetful functor $U_{\mathrm{fh}} : \mathrm{fCat} \to \mathrm{hCat}$. Recall that morphisms in fCat are the *exact* functors between fibration categories. One immediately sees that the properties of exact functors correspond exactly to those of right Quillen functors. Moreover, exact functors between fibration categories are homotopical. In general, a right Quillen functor preserves weak equivalences only between fibrant objects. Let mCat_f be the subcategory whose objects are "full subcategories of fibrant objects in a model category", denoted by $\mathcal{M}^f$, and whose morphisms are restricted right Quillen functors. Then $\mathrm{mCat}_f \hookrightarrow \mathrm{fCat}$. This provides a canonical way to obtain a fibration category out of a model category. We could also compose with the forgetful functor $U_{\mathrm{fh}} : \mathrm{fCat} \to \mathrm{hCat}$.

We constructed two ways to turn a model category to a homotopical one. The two do not commute up to equality but they do commute up to hDK equivalences of homotopical categories. Given a model category $\mathcal{M}$ with weak equivalences $\mathcal{W}$, we can form the homotopical category $(\mathcal{M}, \mathcal{W})$ or obtain a canonical fibration category $(\mathcal{M}_f, \mathcal{W}_f, \mathcal{F}_f)$ and then the homotopical $(\mathcal{M}_f, \mathcal{W}_f)$. By Proposition 191 the two outcomes are "hDK-equivalent", weakly equivalent as homotopical categories.

3.4.5 Homotopy (co)limits

Now that we've established a more robust notion of an abstract homotopy theory and of a notion of morphisms between them, we can start asking ourselves how the various standard constructions of ordinary categories interact with these definitions. We quickly run into a problem. (Co)limits are not compatible with the homotopy theory. For example, one can have two spans where the "vertices" are pointwise homotopy equivalent but their pullback is not.

The solution to this problem is one of the main applications of the theory of derived functors previously defined. As discussed, derived functors are universal homotopical approximations to given functors. Thus we reasonably hope that by taking a derived version of the (co)limit functor we'll solve the problem outlined above.

A derived functor does not always exist. For instance, a condition is found in Proposition

122. Furthermore, the implicit assumption in that proposition is that the domain of the functor has a model structure. In our case that proves problematic because colim : $\mathcal{M}^I \to \mathcal{M}$ and there is no obvious way to make $\mathcal{M}^I$ into a model category. Thankfully, under proper assumptions on $\mathcal{M}$, there are non-obvious ways. This problem will be treated in more detail in subsubsection 3.6.2. Here we just record:

Theorem 132. (Theorem 11.6.1. in [Hir03a]) If $\mathcal{M}$ is a *cofibrantly generated* model category, see subsubsection 3.8.2, then one can determine that same structure for $\mathcal{M}^I$, for any small I, the *projective* model structure.

Having given $\mathcal{M}^I$ a model structure we immediately find,

Theorem 133. (Theorem 11.6.8 in [Hir03a]) With the hypotheses above, we obtain a Quillen pair colim : $\mathcal{M}^I \leftrightarrows \mathcal{M}$: cs, the constant diagram functor. Moreover, the colim functor preserves weak equivalences between cofibrant objects.

Proof. Whenever $\mathcal{M}$ is cocomplete we always have the adjunction. In 3.6.2 we'll see that weak equivalences and fibrations of $\mathcal{M}^I$ are defined objectwise, namely $X \to Y \in \mathcal{M}^I$ is a weak equivalence exactly when for all $m \in \mathcal{M}$: $X_m \xrightarrow{\sim} Y_m$. The same is true for fibrations. Then, we immediately see that the constant diagram functor preserves both fibrations and trivial fibrations making the adjunction in a Quillen pair. The second conclusion appears misleadingly simple. One must be wary of the non-trivial definition of cofibrations in the projective model structure. □

Since the two functors form a Quillen pair we obtain a left derived functor $\mathbb{L}\mathrm{colim} =:$ hocolim. By the discussion of derived functors, we also get a canonical way to compute it, $\mathrm{hocolim}X \approx \mathrm{colim}X'$ where X' is a cofibrant replacement of X in the projective model structure. Unfortunately, that cofibrant replacement is obtained via a small object argument and therefore, for practical purposes, is like a black box.

One can obtain another model structure on $\mathcal{M}^I$, which rests on a stronger hypothesis. If $\mathcal{M}$ is *sheafifiable*, see [BEK00], then $\mathcal{M}^I$ admits the *injective* model structure that has pointwise weak equivalences and cofibrations. Whenever this model structure exists, a dual argument shows that lim : $\mathcal{M}^I \to \mathcal{M}$ is right Quillen and thus admits a right-derived functor.

3.5 Two models for the homotopy theory of higher categories

In this section, we look into an important application of the theory of model categories and Quillen Equivalences. In Definition 6 we saw a reasonable definition for a notion of weak equivalence between simplicial categories. This definition requires the homotopy categories to be equivalent as ordinary categories and the mapping spaces to be equivalent as simplicial sets. Thus we turned sCat into a homotopical category. It turns out we can do better. Here we present a model structure for sCat. This was introduced by Julia Bergner in [Ber05]. A survey-style treatment is in [Rie20]. Before we proceed with the definition of the model structure we require the following definition.

Definition 134. In $\mathscr{C} \in \mathrm{sCat}$ a morphism $f : X \to Y$ in $\mathscr{C}$ will be called an isomorphism if and only if it is an iso in $\mathrm{Ho}\mathscr{C}$.

Theorem 135. *The category sCat of simplicial categories and simplicial functors between admits a model structure defined as follows:*

(1) The weak equivalences are taken to be the sDK equivalences discussed above.

(2) The fibrations are taken to be simplicial fuctors $F : \mathscr{C} \to \mathscr{D}$ such that:

   (a) The induced map of simplicial sets: $\mathrm{Hom}_{\mathscr{C}}(X, Y) \twoheadrightarrow \mathrm{Hom}_{\mathscr{D}}(FX, FY)$ is a Kan fibration.

   (b) Given any $X \in \mathscr{C}$ and an isomorphism $f : FX \to D$ in $\mathscr{D}$ there exists an isomorphism $\tilde{f}$ and an object Y with $\tilde{f} : X \to Y$ and $F(\tilde{f}) = f$.

(3) Cofibrations are just taken to be the set of maps with RLP against trivial fibrations as defined above.

It is important to record a characterization for fibrant objects in $\mathrm{sCat}_{\mathrm{Berg}}$.

Proposition 136. *The fibrant objects are precisely $\mathscr{K}an$-enriched categories.*

Proof. First we note that the terminal simplicial category $\mathscr{T}$ is whose set of objects is a singleton and has the terminal simplicial set as its mapping space. So it has a point in each dimension. We ask ourselves when is a simplicial functor $F : K \to *$ a fibration as above. We immediately notice that the second condition becomes trivial. There is a unique map of the form $f : FX \to D$ and that is the identity $\mathrm{id}_* : * \to *$. Therefore, there always exists a lifting for this isomorphism, namely the identity of X.

The second condition is not trivial. As we noted above, $\mathrm{Hom}_{\mathscr{T}}(*, *) =$ the terminal presheaf. Hence, the second condition is that for any objects of $\mathscr{C} : X, Y$ we have $\mathrm{Hom}_{\mathscr{C}}(X, Y) \twoheadrightarrow *$ which is exactly the definition of $\mathscr{C}$ being "locally Kan". □

But we know how to think of a locally Kan simplicial category as a $(\infty, 1)$-category. Thus the model category $\mathrm{sCat}_{\mathrm{Berg}}$ captures the homotopy theory of $(\infty, 1)$ categories.

The Joyal model structure is another model for the homotopy theory of $(\infty, 1)$-categories in the model of ∞-categories. It was introduced in [Joy08b]. It is also treated in section 2.2.5 of [Lur09]. For a lecture notes style treatment one can consult section 47 of [Rez22].

Theorem 137. sSet admits a model structure where

(1) $\mathcal{W} = \{$ categorical equivalences $\}$

(2) $\mathcal{C} = \{$ monomorphisms $\}$

(3) $\mathcal{F} = \{$ categorical fibrations $\}$

Remark 138. We denote this model structure by $\mathrm{sSet}_{\mathrm{Joy}}$. Since cofibrations are monomorphisms, all objects are cofibrant. That means bifibrant objects are the fibrant ones. In $\mathrm{sSet}_{\mathrm{Joy}}$ fibrations into an ∞-category are precisely the isofibrations and therefore (bi)fibrant objects are precisely the ∞-categories. Hence, $\mathrm{sSet}_{\mathrm{Joy}}$ captures the homotopy theory of $(\infty, 1)$-categories. The weak equivalences capture the appropriate notion of sameness of ∞-categories, equivalences of ∞-categories.

We have just presented two distinct candidates for the homotopy theory of $(\infty, 1)$-categories. It turns out the two homotopy theories are equivalent, in the appropriate sense.

Theorem 139. The adjoint pair $(\mathfrak{C}, \mathfrak{N})$ is a Quillen equivalence between the Bergner model structure on simplicial categories and the Joyal model structure on simplicial sets

$$\mathfrak{C} : \mathrm{sSet}_{\mathrm{Joy}} \leftrightarrows_{\mathrm{QE}} \mathrm{sCat}_{\mathrm{Berg}} : \mathfrak{N}$$

For a proof of this fact we refer to Theorem 2.2.5.1 in [Lur09].

So, for example, Theorem 43 can be reformulated as stating that $\mathfrak{N}$ takes fibrant objects to fibrant objects, namely locally Kan simplicial categories to ∞-categories, or one model of $(\infty, 1)$-categories to another. In the literature, one can find two more models of ∞-categories, Segal categories and Complete Segal Spaces. Each of these can be exhibited as bifibrant objects of some model structure. Thus we have four models for the homotopy theory of $(\infty, 1)$-categories. Nevertheless, there is no ambiguity. They are pairwise connected by Quillen equivalences. For more information see [Ber06]. The homotopy theory of (∞, n)-categories displays a bit of ambiguity, see [BS20].

3.6 New model categories from old

3.6.1 Model structure in the slice category

Let $\mathcal{M}$ be a model category. For the rest of this section, we fix an object $X \in \mathrm{ob}\mathcal{M}$.

Proposition 140. The slice category inherits a model structure from $\mathcal{M}$ in the obvious way. A map $\tilde{f} : (A \to X) \to (B \to X)$ is in one of the classes $\mathcal{C}, \mathcal{F}, \mathcal{W}$ if and only if its image under the canonical forgetful functor is, namely, $\tilde{f} \in - \iff (f : A \to B) \in -$.

Due to this definition lots of properties from $\mathcal{M}$ lift straightforwardly to ${}^{\mathcal{M}}\!\diagup_{X}$. For example,

Proposition 141. If $\mathcal{M}$ is cofibrantly generated then so is ${}^{\mathcal{M}}\!\diagup_{X}$. The new generating acyclic cofibrations are the image of $(\mathrm{ac})\mathrm{cof}_{\mathcal{M}}$ under $X \sqcup -$

Dual results hold for the coslice category.

3.6.2 Model structure in diagram categories

Consider the functor category $[\mathcal{M}, \mathcal{N}]$ for a pair of model categories. The question we address in this section is how does $[\mathcal{M}, \mathcal{N}]$ inherit a model structure from that of $\mathcal{M}, \mathcal{N}$. This is also treated in section A.2.8 of [Lur09]. In fact, for our purposes it suffices to consider the case where $\mathcal{N} = \mathrm{sSet}$ so that we restrict our attention to **Simplicial Presheaves**, namely functor categories of the form $\mathrm{sPsh}(\mathcal{M}) := [\mathcal{M}^{\mathrm{op}}, \mathrm{sSet}]$.

We want to obtain a model structure on $\mathrm{sPsh}(\mathcal{M})$. The naive approach would be to define a natural transformation to be a weak equivalence (respectively a (co) fibration) $\iff$ all of its components are. Unfortunately, this is problematic.

Take a commutative square of natural transformations whose left vertical n.t. is componentwise an acyclic cofibration and the right vertical one is a componentwise fibration. Take components and obtain a family of squares in sSet. Then for each square, there is a lift. But there is no a priori reason why this collection of lifts is arranged into a natural transformation.

So we have to construct the desired model structure in a more roundabout way. We will exploit the fact that one can lift a model structure through a functor or an adjoint pair.

First, we present the theorems in full generality and then specialize to our case of interest.

Consider the following more general setting. Let M be a small category and $\mathcal{N}$ be a model category. Take $U : M \to \mathcal{N}$. We want to lift the model structure of $\mathcal{N}$ through U so that M will become a model category. As above, the naive approach would be to declare $f \in \mathrm{mor}M$ to be in some class whenever $U(f)$ is. If we take a square in M with vertical maps as desired, apply U to take into $\mathcal{N}$ consider the lift obtained in $\mathcal{N}$, then there is no a priori reason why the lift is in the U image. This is essentially the same reasoning as above. The solution to this problem is to restrict one of the classes we want to define on M to where the problem is no more.

Definition 142. For any functor $U : \mathcal{M} \to \mathcal{N}$ where $\mathcal{N}$ is a model category we define a morphism in $\mathcal{M}$ to be:

(*1*) A U- weak equivalence (respectively U-(co)fibration) if its image under U is.

(*2*) A **projective cofibration** if it has the left lifting property against all U-acyclic fibrations.

(*3*) A **projective acyclic cofibration** if it has the left lifting property against all U-fibrations.

(*4*) A **injective fibration** if it has the left lifting property against all U-acyclic cofibrations.

(*5*) A **injective acyclic fibration** if it has the left lifting property against all U-cofibrations.

In general it is *not* the case that the triples $(\mathcal{W}_U, \mathcal{F}_U, \mathcal{C}_{\text{proj}})$ and $(\mathcal{W}_U, \mathcal{F}_{\text{inj}}, \mathcal{C}_U)$ are model categories. If they exist, the first is called the **projective** model structure and the second is called **injective**.

The following proposition records some conditions under which the model structures exist and inherit properties from that of $\mathcal{N}$.

Proposition 143. (Proposition 8.2 in [Shu19]) Let $\mathcal{N}$ be a model category and $U : \mathcal{M} \to \mathcal{N}$ be a functor that has both adjoints $F \dashv U \dashv G$ such that the adjunction $UF \dashv UG$ is Quillen. Then:

(*1*) If $\mathcal{N}$ is cofibrantly generated then the projective model structure exists and is cofibrantly generated itself.

(*2*) If $\mathcal{N}$ is combinatorial and $\mathcal{M}$ is locally presentable, then both the projective and injective model structures exist and are combinatorial.

(*3*) Every projective cofibration is a U-cofibration and every injective fibration is a U-fibration.

(*4*) The projective/injective model structures are right or left proper if $\mathcal{N}$ is.

(*5*) If $\mathcal{N}, \mathcal{M}$ and the adjunctions are simplicially enriched then so is the projective/injective model structure on the category of diagrams.

Now we seek a functor U to apply the theorem to. To that end, recall that the product of model categories canonically becomes a model category by defining everything componentwise. In our case we can consider $\prod_{\mathrm{ob}\mathcal{A}} \mathrm{sSet}$. Moreover, there exists a forgetful functor

$$\mathrm{sPsh}(\mathcal{A}) := \mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}} \xrightarrow{U} \mathrm{sSet}^{\mathrm{ob}\mathcal{A}} \cong \prod_{\mathrm{ob}\mathcal{A}} \mathrm{sSet}$$

Then, since sSet is both complete and cocomplete, U admits both left and right Kan extensions. Thus we obtain $L \dashv U \dashv R$

$$\begin{tikzcd} \mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}} \arrow[r, "U" description] & \mathrm{sSet}^{\mathrm{ob}\mathcal{A}^{\mathrm{op}}} \arrow[l, bend right, "L" description] \arrow[l, bend left, "R" description] \end{tikzcd}$$

One can directly construct formulas for L, R. A $X \in \prod_{\mathrm{ob}\mathcal{A}} \mathrm{sSet}$ is an ob$\mathcal{A}$-parametrised family of simplicial sets. Then $L(X)(A) = \coprod_{a:A \to B} X_B$ and dually, $R(X)(A) = \prod_{a:C \to A} X_C$.

Proposition 144. The adjunction $UL \dashv UR$ is Quillen.

Proof. (Adapted from Example 7.46 in [HM22]) The fact that $L \dashv U \dashv R$ gives $UL \dashv UR$ is a classic exercise on adjoint functors. To show the latter forms a Quillen pair it suffices to show UL preserves both cofibrations and trivial cofibrations. Since sSet is combinatorial cofibrations are monomorphisms. UL is pointwise L and the latter's action on morphisms is by composition. Both the class of monos and weak equivalences (hence also their intersection) are closed by composition, which completes the proof. □

Theorem 145. Let $\mathcal{A}$ be a small, simplicially enriched category. Then $\mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}}$ admits both the projective and injective model structure. Moreover, that model structure is left-right proper, cofibrantly generated, combinatorial and simplicially enriched.

Proof. $\mathrm{sSet}_{\mathrm{KQ}}$ satisfied the requirements of Proposition 143. □

Since both model structures have the same weak equivalences they capture the same homotopy theory. So we shouldn't be surprised that we can obtain a Quillen equivalence

$$\mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{proj}} \leftrightarrows_{\mathrm{QE}} \mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{inj}}$$

Remark 146. The fibrations in $\mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{proj}}$ are pointwise fibrations in sSet. Therefore the fibrant objects of $\mathrm{sPsh}(\mathcal{A})$ are those diagrams taking values in Kan complexes. The dual is true for $\mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{inj}}$. Since all objects of sSet are cofibrant, the same is true for $\mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{proj}}$.

3.6.3 Bousfield localisation

Thus far we've inquired into homotopical categories and model categories as ways of capturing a "homotopy theory". An all-important construction is that of the homotopy category, the universal category where objects with the same homotopy type, objects connected by a zig-zag of weak equivalences, have become identified, isomorphic. Thus the homotopy category $\mathrm{Ho}(\mathcal{M})$ is one where we are treating our objects "up to homotopy". In forming the hom-sets of $\mathrm{Ho}(\mathcal{M})$ restrict to bifibrant objects so that "homotopy" becomes an equivalence relation and then we quotient out by it. This is akin to the construction of the fundamental groupoid of a space. Instead, we'd like to arrange that information in a richer algebraic object like an ∞-groupoid. We will explore these ideas further in section 4. Here, we will just borrow the intuitions we require.

Inside a model category $\mathcal{M}$, there is a natural construction of a simplicial set, whose role is to capture the homotopy type of the Hom-∞-groupoid $\mathrm{Hom}_{\mathcal{M}}(X, Y)$. This *homotopy coherent function space* is called the homotopy function complex or derived mapping space. It is a construction functorial up to homotopy, that depends on two objects and encodes important homotopical information. Notably, its path components encode the Hom-set between the two objects in the homotopy category.

Having established that construction one realises that certain maps, the so-called *local* maps, behave much like weak equivalences without literally being so. It becomes natural then to ask if one can enlarge the class of weak equivalences of $\mathcal{M}$ to include them. Of course, we'd like to do so without changing the model category too much. This is doable under some conditions on $\mathcal{M}$. The process is called (Left) Bousfield Localisation. The outcome is a model structure on the same underlying category, with a larger set of weak equivalences but the same cofibrations. For a comprehensive account, one can consult the canonical reference [Hir03b]. Another textbook reference is [HM22]. The following exposition is also based on [Rap] and [Def19].

For $X, Y \in \mathrm{ob}\mathcal{M}$ we obtain a simplicial set $\mathrm{map}_{\mathcal{M}}(X, Y)$. The precise definition of this mapping space requires many preliminaries which do not contribute conceptually to this thesis and are therefore omitted. For details, we refer the interested reader to [Def19]. As discussed, the *homotopy functions complexes* capture important information about the model category $\mathcal{M}$.

Theorem 147. For $X, Y \in \mathrm{ob}\mathcal{M}$ there is an isomoprhism

$$\pi_0 \mathrm{map}_{\mathcal{M}}(X, Y) \cong \mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X, Y)$$

Example 148. In $\mathrm{sSet}_{\mathrm{KQ}}$ the internal homs $\mathrm{Fun}(X, Y)$ play the role of homotopy function complexes. Then, we obtain an isomorphism $\pi_0 \mathrm{Fun}(X, Y) \cong \mathrm{Hom}_{hC}(X, Y)$.

Another important property is that they interact nicely with weak equivalences so that g is a weak equivalence in $\mathcal{M} \iff g^*, g_*$, post- and precomposition with g are weak equivalences of simplicial sets between homotopy function complexes. This fact can be interpreted as saying that weakly equivalent spaces should also have weakly equivalent hom-∞-groupoid mapping space. It is also a homotopical version of the fact that the Yoneda embedding reflects isomorphisms.

This is an important property and generalises to the next definition. An object is S-local if it makes the maps in S behave as if they were weak equivalences, in the sense above. Or as Dugger puts it in [Dug01b] "In other words, the S-local objects are the ones which see every map in S as if it were a weak equivalence. The S-local equivalences are those maps which are seen as weak equivalences by every S-local object."

Definition 149. Let $\mathcal{C}$ be a model category and $S \subset \mathrm{mor}\mathcal{C}$

(*1*) An object W is **S-local** when it is fibrant and for all $f : A \to B \in S$ we get an induced weak equivalence $f^* : \mathrm{map}(B, W) \to \mathrm{map}(A, W)$.

(*2*) A map $g : X \to Y$ in $\mathcal{C}$ is an **S -local equivalence** if for all S-local objects we get a weak equivalence $g^* : \mathrm{map}(Y, W) \to \mathrm{map}(X, W)$.

Notice that by thinking of homotopy function complexes like homotopy coherent function spaces one would expect that a weak equivalence in $\mathcal{C}$ is also an S-local equivalence for any S. Moreover, S-local equivalences between S-local objects coincide with weak equivalences.

These maps behave like weak equivalences, but they are not. The Left-Bousfiled localization is a construction that enlarges the class of weak equivalences to include the S-local equivalences for some S.

Definition 150. Let $\mathcal{C}$ be a model category and S be a class of maps. The **Left Bousfield Localisation** of $\mathcal{C}$ with respect to S, if it exists, is a model structure on $\mathcal{C}$, denoted $\mathcal{L}_{\mathrm{S}}(\mathcal{C})$ with the following properties:

(*1*) It has the same underlying category, $\mathcal{C}$.

(*2*) The weak equivalences $\mathcal{W}_{\mathrm{S}}$ are the S-local equivalences.

(*3*) The two model structures have the same cofibrations.

(*4*) The fibrations of $\mathcal{L}_{\mathrm{S}}(\mathcal{C})$ are precisely the morphisms that have the right lifting property against the acyclic S-cofibrations.

Example 151 (Proposition 3.3.3 in [Hir03a])**.** Here, nLab authors offer a computation of the various classes of fibrant/cofibrant objects of $\mathcal{L}_S(\mathcal{M})$ comparatively to those of $\mathcal{M}$. The two model categories have the same cofibrations and therefore the same trivial fibrations. $\mathcal{L}_S(\mathcal{M})$ has more weak equivalences and therefore fewer fibrations than $\mathcal{M}$.

Example 152. (Example d) page 324 in [HM22]) $\mathrm{sSet}_{\mathrm{KQ}} = \mathcal{L}_S(\mathrm{sSet}_{\mathrm{Joy}})$ for $S = \{\Lambda^n_k \hookrightarrow \Delta^n\}$. Therefore $\mathrm{Ho}(\mathrm{sSet}_{\mathrm{KQ}})$ is full subcategory of $\mathrm{Ho}(\mathrm{sSet}_{\mathrm{Joy}})$ for which the inclusion has a left adjoint.

Proof. First, we determine the S-local objects. $K \in \mathscr{K}an \iff \forall i : \Lambda^n_k \hookrightarrow \Delta^n : \mathrm{Fun}(\Delta^n, K) \xrightarrow{\sim} \mathrm{Fun}(\Lambda^n_k, K)$. ($\implies$) is done using the theory of enriched lifting problems, specifically one can apply (3) of Proposition 22.2 in [Rez22]. The inverse is implied by categorical equivalences being essentially surjective. Hence the S-local objects are exactly the Kan complexes. Then, $\mathcal{W}_{KQ}$ can be equivalently defined as $\{f : \forall K \in \mathscr{K}an : f^* \in \mathcal{W}_{Joy}\}$, see section 52 in [Rez22]. □

Proposition 153. We have a Quillen Adjunction $\mathrm{id}_! : \mathcal{C} \leftrightarrows \mathcal{L}_{\mathrm{S}}(\mathcal{C}) : \mathrm{id}^*_{\mathcal{C}}$. It induces a reflection on the homotopy categories, namely, the inclusion $\mathrm{Ho}(\mathcal{L}_{\mathrm{S}}(\mathcal{M})) \hookrightarrow \mathrm{Ho}(\mathcal{M})$ has a left adjoint.

Proof. Suffices to show $\mathrm{Ho}(id_!)$ is fully faithful which is equivalent to the derived counit being comprised of isos. That essentially amounts to showing that the counit comprises of weak equivalences. The derived counit can be computed in terms of the pair of adjoints and (co)fibrant replacements in the model category. Both adjoints are identities, so we are left with the replacements. Without loss of generality, we can assume we start with a fibrant object. Then, the only non-trivial functor is a cofibrant replacement, which by definition comprises of trivial fibrations, in particular weak equivalences. The proof in more detail can also be found here.

□

As discussed above, given a model category and a class of maps, the left Bousfield localization doesn't always exist. The situation is rectified for a class of model categories that are very relevant to the material we are presenting. The following is a theorem recording some of the properties that are stable under the formation of Left Bousfield Localisations.

Theorem 154 (Raptis 4.6 refers to Hirchorn)**.** Let $\mathscr{C}$ be a simplicially enriched, left proper, and combinatorial[15] model category and S a set of morphisms in $\mathscr{C}$. Then the left Bousfield Localization exists and is itself simplicial, combinatorial and left proper.

[15] see section 3.7

3.7 Properness

Proposition 155. (Proposition 15.4.2 in [May99] or for the second part, Proposition A.2.4.2-3 in [Lur09]) Let $\mathcal{M}$ be a model category. Then, weak equivalences between fibrant objects are stable under pullback against a fibration. Dually, weak equivalences between cofibrant objects are stable under pushout against a cofibration.

It is common for model categories to exhibit similar behaviour without all objects being (co)fibrant. That's what the next definition records. When its condition is satisfied we can think that the objects of the model category are "kind of (co)fibrant".

Definition 156. A model category is called **right proper** if its weak equivalences are stable under pullback against fibrations. Dually, a model category is left proper when its weak equivalences are stable under pushout against cofibrations.

An immediate corollary of proposition is that:

Corollary 157. If $\mathcal{M}$ is a model category with $\mathcal{C} = \mathrm{Mon}$, then it is left proper.

Proof. The map $\varnothing \hookrightarrow X$ is always a mono. Therefore every object is cofibrant and thus $\mathcal{M}$ is left proper. $\square$

Corollary 158. Both $\mathrm{sSet}_{\mathrm{KQ}}$ and $\mathrm{sSet}_{\mathrm{Joy}}$ are left proper.

Remark 159. $\mathrm{sSet}_{\mathrm{Joy}}$ is *not* right proper. The inclusion $\Lambda^2_1 \hookrightarrow \Delta^2$ is a categorical equivalence but it doesn't remain one after pulling back against the fibration $\Delta^{\{0,2\}} \twoheadrightarrow \Delta^2$.

We conclude this section with an important characterization due to Charles Rezk. To motivate it we record that from every morphism $f : X \to Y$ we can obtain an associated Quillen pair: $f_! = \Sigma_f \dashv f^*$

$$\Sigma_f : \mathcal{M}\big/_X \leftrightarrows \mathcal{M}\big/_Y : f^*$$

The left adjoint is postcomposition. Postcomposition just changes the base of the slice leaving the horizontal map untouched. Therefore it preserves cofibrations and trivial cofibrations, making it left Quillen.

Theorem 160. (Proposition 2.5 in [Rez00]) Let $f : X \to Y$ be a weak equivalence of $\mathcal{M}$. Then, $\mathcal{M}$ is right proper $\iff_{\mathrm{df}}$ the induced adjoint pair $f_! : \mathcal{M}\big/_X \rightleftarrows \mathcal{M}\big/_Y : f^*$ is a Quillen equivalence.

Proof. The proof can be found at Proposition 2.7 in this nLab article

$\square$

3.8 Combinatorial model categories

While model categories may be large and complicated, in some nice cases we can pinpoint a small set of well-behaved generators for the objects or arrows, in particular the (acyclic) cofibrations of the category. In each case we may distinguish a small set of objects or arrows and any other object or (acyclic) cofibration respectively can be obtained by "glueing" these generators. These properties make the model category particularly tractable.

The next two sections analyse these two cases. In the last subsection, we see an elegant characterisation of combinatorial model categories.

3.8.1 Locally presentable

Morally, a category is called **presentable** there exists a small set of (special) objects such that any object of the category can be given as a colimit over that set of objects. Essentially, there is a set of generators for the objects of the category. Objects in that set are like building blocks for all objects of our category. Indeed, "glueing" these generators can give us any other object of the category. This is the ordinary version of the presentability for ∞-categories we saw at subsubsection 2.3.7. For a comprehensive account see [AR94].

Definition 161. Let λ be a regular cardinal.

(1) A poset is λ-directed if every set of cardinality $< \lambda$ has an upper bound.

(2) An object is λ-compact if and only if the hom functor $\mathrm{Hom}(K, -)$ preserves λ-directed colimits.

(3) A category $\mathcal{C}$ is called λ-accessible if it is closed under λ-directed colimits and it has a set $\mathcal{G}$ of λ-compact objects such that any object in the category can be obtained as a λ-directed limit of them.

(4) A category $\mathcal{C}$ will be called λ-**presentable** if it is cocomplete and λ-accessible.

Remark 162. (2) above has an illuminating interpretation. It is asserting that

$$\mathrm{Hom}(K, \mathrm{colim}_i X_i) \simeq \mathrm{colim}_i \mathrm{Hom}(K, X_i)$$

The data of a map $f : K \to \mathrm{colim}_i X_i$ can be given by a map $f_j : K \to X_j$ for some j. Or, a map into the colimit $\mathrm{colim}_i X_i$ factors through one of the constituents X_i.

Example 163. In Set, the point is λ-compact for any λ. Indeed, we can even think of the point as the archetypal compact object, which follows the slogan of topology that compact sets behave like points.

3.8.2 Cofibrantly generated

Under the homotopy hypothesis we have $\mathrm{Ho}(\mathrm{sSet}_{\mathrm{KQ}})$ is Quillen Equivalent to $\mathrm{Ho}(\mathrm{Top}_{\mathrm{KQ}})$. It is a classical fact that the $\mathcal{CW}$ complexes suffice to capture all weak homotopy types of spaces. The class of $\mathcal{CW}$ complexes is defined as the minimal class closed under some operations. We start with a collection of points. The operation under which we "close" our set is "gluings of n-disks along their boundary". To show a topological space is a $\mathcal{CW}$ complex amounts to showing how it can be obtained via such "gluings". It is well known that in category theory we model "gluings" using pushout squares or, more generally, coproducts and colimits.

In summary, the class of $\mathcal{CW}$ complexes can be described by a simpler collection of morphisms, the inclusions of boundaries of disks in disks, and the formation of "glueings" which is made precise with closure under some colimits.

Keeping the Quillen equivalence of topological spaces and Kan complexes in mind, it is unsurprising that we recover a similar situation in sSet. This is exactly the content of the *skeletal filtration*. It provides a concrete recipe for writing a simplicial set as a series of skeleta $\mathrm{sk}_k(X_\bullet)$ where the next skeleton is obtained by its successor by "glueing" (i.e. forming a pushout square) standard simplices. This is but a restatement of the construction of $\mathcal{CW}$ complexes in the context of sSet. It is with that intuition that we should interpret the next definition. We demand that **saturated** classes are "stable under gluings". Perhaps more importantly, that means that the codomains of said maps can be formed by successive gluings of a collection of objects. In the case of $\mathcal{CW}$ complexes, we glue spheres and disks along their boundaries. In that of sSets, we'll glue horns or standard simplices.

This is the feature that we also want to obtain for model categories. We wish to consider relatively simple classes of maps and "close" them under gluings. As explained above, this also means that we get a particularly tractable description of the collection of their codomains, as certain colimits. We will ask that that the closure of a class of maps, its **saturation**, described the class of (trivial) cofibrations. Then by looking at their codomains we obtain a description of cofibrant objects in $\mathcal{M}$.

For the theory to work, we also need to add closure under retracts which unfortunately eludes the above explanation but plays an important formal role.

In the other direction, we might wonder if one of the classes of our model category admits a description as some smaller class of generators closed under gluings. It is then reasonable to note that taking saturations leaves the right complement of a class of maps unaffected.

Definition 164. A class of monomorphisms $\mathcal{M}$ is called **saturated** when

(*1*) it contains all isomorphisms

(2) it is closed under forming pushouts i.e. for a pushout square with $i \in \mathcal{M} \implies \tilde{i} \in \mathcal{M}$

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & C \arrow[d, "\tilde{i}"] \\ B \arrow[r] & D \end{tikzcd}$$

(3) closed under retracts

(4) closed under coproducts: if $\forall l : i_l : A_l \to B_l \in \mathcal{M} \implies \sum_l i_l : \sum_l A_l \to \sum_l B_l \in \mathcal{M}$

(5) closed under $\omega-$ composites. Given a diagram $I : \omega \to \mathcal{E}$ with $I(j \to j+1) = i_j : X^j \to X^{j+1} \in \mathcal{M}$ then $\tilde{i} =$ the $\omega-$ composite of $\{i_j\} \in \mathcal{M}$.

$$\begin{tikzcd} X^0 \arrow[r, "i_0"] \arrow[drr, "\tilde{i} \in \mathcal{M}"'] & X^1 \arrow[r, "i_1"] \arrow[dr] & \cdots \arrow[r] & X^\omega \arrow[dl] \\ & & \operatorname{colim}_{j \in \omega} X^j & \end{tikzcd}$$

Proposition 165. Let $\mathcal{K}$ be arbitrary. Then LLP($\mathcal{K}$) is saturated. This is essentially the dual statement to Proposition 102.

Definition 166. Denote by M_S the intersection of all saturated classes containing $M \subset$ mon. We say that M_S is generated by M.

Then an important feature is that taking the saturation of a class doesn't change RLP(-).

More importantly, we have the following quite general result that is central to producing model categories.

Proposition 167 (section 16 in [Rez22])**.** (Small Object Argument) Fix a set of morphisms in **sSet** $M = \{s_i : A_i \to B_i\}$. Then any map $f : X \to Y$ can be factored as $f = pj$ with $j \in M_S$ and $p \in \mathrm{RLP}(M)$.

This allows us to take a pair of classes of morphisms $I, J \subset \mathrm{mor}\mathcal{C}$ form their saturations $\bar{I}, \bar{J}$ and hopefully to be able to consider them as the classes of cofibrations and trivial cofibrations of a model category. When that happens, the resulting model category is called cofibrantly generated.

Definition 168. A model category $\mathcal{M}$ is called **cofibrantly generated** when there exists a pair of classes $I, J \subset \mathrm{mor}\mathcal{M}$ such that $\bar{I} = \mathcal{C}$ and $\bar{J} = \mathcal{C} \cap \mathcal{W}$.

3.8.3 Dugger's theorem

Definition 169. A **combinatorial** model category is one that is locally presentable as a category and the model structure is cofibrantly generated.

Example 170. For example, the projective model structure on simplicial presheaves is combinatorial.

Definition 171. A combinatorial model category whose underlying category is a Gorthendieck topos and the cofibrations are precisely the monomorphisms is called a **Cisinksi model category.**

In Theorem 65 we saw that presentable ∞-categories correspond to reflective subcategories of presheaf categories. Here we find an analogous structural characterisation for combinatorial model categories due to Dugger. These ideas were developed in 3 articles [Dug01a] [Dug01b] [Dug01c].

Theorem 172. Let $\mathcal{M}$ be a combinatorial model category. Then there is a small category $\mathcal{A}$ so that, a set of maps S, in $\mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}}$ and a Quillen Equivalence

$$\mathcal{L}_S\big(\mathrm{sSet}^{\mathcal{A}^{\mathrm{op}}}_{\mathrm{proj}}\big) \leftrightarrows \mathcal{M}$$

Proof. Following Raptis [Rap] we give an outline of the proof and refer to [Dug01a] for more details.

Much like in the case of presentations of Abelian groups, one finds a surjection $\mathbb{Z}^n \twoheadrightarrow A$ and determines the kernel R. Then we obtain $\mathbb{Z}^n \big/ R \cong A$. Given a small category $\mathcal{C}$ one can see $\mathrm{sSet}^{\mathcal{C}^{\mathrm{op}}}$ as the free model category on $\mathcal{C}$.

Say we want to obtain a presentation for $\mathcal{M}$. Then we seek a *homotopically surjective map* $F : \mathrm{sSet}^{\mathcal{C}^{\mathrm{op}}} \leftrightarrows \mathcal{M} : G$. So Dugger's proof comes to defining what a *homotopically surjective map* is. Given such a map one immediately obtains the desired presentation. The tricky part of the proof is in obtaining the homotopical surjection. This is doable by *carefully choosing the correct* $\mathcal{C}$. Following Dugger's notation, let $c\mathcal{M}$ be the category of *cosimplicial resolutions* of $\mathcal{M}$. Let CR be its full subcategory consisting of those resolutions A_* such that $A^n \in \mathcal{M}_\lambda$[16] for all n. The inclusion of CR in $c\mathcal{M}$ induces a $\mathrm{sSet}^{CR^{\mathrm{op}}}$ which Dugger shows is homotopically surjective. □

Remark 173. Dugger's theorem gives us what Rezk, in [Rez10], calls a small presentation. For technical reasons, it is important to replace this with a small *simplicial* presentation. Namely to replace the small $\mathcal{A}$ above with a $\mathscr{A}$ that is enriched in Kan complexes.

[16] this denotes the λ-compact objects of $\mathcal{M}$.

Proposition 174. A small presentation can be promoted to a small simplicial one.

Proof. First, we can canonically turn any small category into a simplicial one by taking constant simplicial sets for all of $\mathrm{Hom}_{\mathcal{C}}(X, Y)$. Then, all face and degeneracy maps are identities. Using the *collatability conditions* one can easily fill all horns making $\underline{\mathcal{C}}$ locally Kan. Because of the naturality squares for these faces/degeneracies a map $C \to \mathrm{sSet}$ fully determines one $\underline{\mathcal{C}} \to \mathrm{sSet}$ and vice versa. $\square$

3.9 Simplicial model category

Lastly, we will see that all this extra structure on the hom sets interacts nicely with the Kan-Quillen model structure on **sSet**. When the two structures coexist on a category and are compatible then we call that category a **simplicial model category**.

Definition 175. A **simplicial model category** is one where the simplicial enrichment interacts nicely with the model structure. That is, for a cofibration $i : A \rightarrowtail B$ and a fibration $p : X \twoheadrightarrow Y$ we require that we obtain a fibration:

$$\mathrm{Fun}(B, X) \xrightarrow{i^* \times p_*} \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B, Y)$$

Additionally, this fibration is a weak equivalence if either i or p is.

First, we interpret what this property means.

Remark 176. Fix a pair of maps $i : A \to B$ and $p : X \to Y$. Consider the pullback from above:

$$\begin{tikzcd}
\mathrm{Fun}(B, X) \arrow[dr, "i^* \times p_*"] & & \\
& \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B, Y) \arrow[r] \arrow[d] & \mathrm{Fun}(B, Y) \arrow[d, "i^*"] \\
& \mathrm{Fun}(A, X) \arrow[r, "p_*"] & \mathrm{Fun}(A, Y)
\end{tikzcd}$$

By the interpretation of pullback in Set we see that a vertex in the pullback consists of a pair of maps $(u, v) \in \mathrm{Fun}(A, X) \times \mathrm{Fun}(B, Y)$ such that $p_*(u) = i^*(v) \iff pu = vi$ i.e. a commutative square as below.

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "i"'] & X \arrow[d, "p"] \\
B \arrow[r, "v"'] \arrow[ur, dashed] & Y
\end{tikzcd}$$

Moreover, an $s \in \mathrm{Fun}(B, X)$ with $i^* \times p_*(s) = (u, v)$ amounts to a map $S : B \to X$ such that $si = u \& ps = v$ i.e. a filler for the diagram!

Furthermore, if i or p are as in Proposition 175 then we observe the following. First and foremost a trivial Kan fibration is in particular surjective on vertices. With the interpretation offered above that translates to there always being lifts in squares with vertical maps i, p. Moreover, observe that a fiber of this map corresponds to a choice of a specific square. Recall that fibers of trivial Kan fibrations are contractible. We interpret this to mean that, while solutions to lifting problems are not unique per se, they are unique up to *contractible ambiguity*, or unique up to homotopy.

A most important consequence of this property is:

Proposition 177. Let $\mathscr{E}$ be a smCat. Let X, Y be a pair of objects. If X is cofibrant and Y is fibrant, then $\mathrm{Hom}_{\mathscr{E}}(X, Y)$ is a Kan Complex.

Proof. By definition we have a cofibration $i : \varnothing \hookrightarrow X$ and a fibration $p : Y \twoheadrightarrow *$. Applying said property we get a fibration

$$\mathrm{Hom}_{\mathscr{E}}(X, Y) \twoheadrightarrow \mathrm{Hom}_{\mathscr{E}}(\varnothing, Y) \times_{\mathrm{Hom}_{\mathscr{E}}(\varnothing, *)} \mathrm{Hom}_{\mathscr{E}}(X, *) \simeq *$$

□

Corollary 178. Let $\mathscr{E}$ be a smCat. Then $\mathscr{E}^\circ$ is locally Kan.

Proof. Take $X, Y \in \mathscr{E}^\circ$. Recall that $\mathscr{E}^\circ$ denotes restriction to bifibrant objects so that both X, Y are bifibrant. Then we may apply the previous proposition.

□

Proposition 179. The statement above holds for **sSet**. The map we claim is a fibration is sometimes called the **Leibniz Exponential** and is denoted by $i \pitchfork p$.

Proof. Suffices to show the map lifts against an arbitrary horn inclusion. Form the diagram:

$$\begin{tikzcd}
\Lambda^n_k \arrow[rr] \arrow[d, hook, "r"] & & \mathrm{Fun}(B, X) \arrow[d, "(i^*, p_*)"] \\
\Delta^n \arrow[r, "q"] & \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A, Y)} \mathrm{Fun}(B, Y) &
\end{tikzcd}$$

The map $q : \Delta^n \to \mathrm{Fun}(A, X) \times_{\mathrm{Fun}(A,Y)} \mathrm{Fun}(B, Y)$ corresponds to a pair of maps q_i into the two function complexes. Transposing them and r we obtain a triple of maps satisfying various conditions. Simply by unpacking the definitions, we see that this data is precisely what is needed to determine a diagram like below:

$$\begin{tikzcd}
(\Lambda^n_k \times A) \cup_{\Lambda^n_k \times B} (\Delta^n \times B) \arrow[r] \arrow[d, "j"] & X \arrow[d, two heads, "p"] \\
\Delta^n \times A \arrow[r] \arrow[ur, dashed, "s"] & Y
\end{tikzcd}$$

The key to the proof lies in the fact that j is anodyne, we get a lift s that, in turn, induces the desired lift in the original diagram. To argue that j is anodyne see Jardine corollary 4.6 page 20. $\square$

Corollary 180. *(1)* If $p : X \twoheadrightarrow Y$ is a fibration then so is $p_* : \mathrm{Fun}(K, X) \twoheadrightarrow \mathrm{Fun}(K, Y)$.

(2) If X is fibrant then the induced map $i^* : \mathrm{Fun}(L, X) \twoheadrightarrow \mathrm{Fun}(K, X)$ is a fibration.

In subsection 5.9 we will require a simplicial model category to simultaneously be *simplicially locally cartesian closed*. This just amounts to all pullback $f^* : {}^{\mathscr{E}}\!/_{Y} \to {}^{\mathscr{E}}\!/_{X}$ to admit a *simplicially enriched* right adjoint.

4 Higher homotopy categories

As discussed, homotopical, or relative categories, fibration categories, model categories and the latter's simplicially enriched counterpart, provide abstract frameworks to capture a homotopy theory. The key ingredient, and indeed the only ingredient present in all, is the class of weak equivalences. Intuitively we think of a pair of objects connected by a zig-zag of weak equivalences as having the same homotopy type. In short, any category with weak equivalences captures a homotopy theory. The additional structure is only auxiliary so as, for instance, to make potentially important constructions better behaved.

One of the most important constructions in homotopy theory is that of the homotopy category, which must be thought of as the category where we deal with objects *up to homotopy*. In constructing it one of the key steps it that we restrict attention to *bifibrant* objects. Then, the notion of *homotopy* is definable from the axioms of a model[17] category, becomes an equivalence relation on Hom-sets. Then,

$$\mathrm{Hom}_{\mathrm{Ho}(\mathcal{M})}(X,Y) := {}^{\mathrm{Hom}_{\mathcal{M}}(BX,BY)}\diagup_{\simeq}$$

This makes the homotopy category akin to the fundamental groupoid of a topological space. In the Introduction, we emphasised how the loss of homotopical information that accompanies taking quotients may be undesirable. We thus motivated the passage to $\Pi_\infty(X) \in \infty\text{-}\mathcal{GPD}$.

We'd like to do the same in the abstract setting above. In this section, we'll show multiple, but homotopically equivalent, ways to *explicitly* capture the homotopical data present in $\mathcal{M}$. Namely, we'll look into constructing *homotopy* $(\infty,1)$*-categories* from a category with weak equivalences and potentially additional structure.

We begin with the most general construction, the *hammock localisation*. Starting just with a category with weak equivalences one constructs a simplicial category, $L^H(\mathcal{M},\mathcal{W})$ properly contains $\mathcal{M}[\mathcal{W}^{-1}]$. We think of it as the simplicial version of the localisation. We remark that every model category or fibration category has an obvious underlying homotopical category. We repeat that the passage to the underlying homotopical category is not functorial. The cause is that the morphisms in hCat are the homotopical functors. Nevertheless if, for example, $\mathcal{M}$ is a model category we can, and should, think of $L^H(\mathcal{M},\mathcal{W})$ as the simplicial homotopy category of $\mathcal{M}$.

[17]One can replicate this construction in a fibration category using path objects. A homotopical category doesn't offer an analogous construction.

4.1 Homotopy simplicial category

The first step in that direction was made in a series of articles by Dwyer & Kan [DK80a], [DK80b], [DK80c], in the guise of the *hammock localisation*. Other standard references include [BK12], [TV05] and the more modern [Hin15]. Given a homotopical category $(\mathcal{M}, \mathcal{W})$, they produce a simplicial one, its hammock localisation $L^H(\mathcal{M}, \mathcal{W})$, which generalises the ordinary homotopy category. The construction depends only on the weak equivalences and homotopically equivalent inputs produce equivalent outputs. Furthermore, the mapping spaces of $L^H(\mathcal{M}, \mathcal{W})$ have the correct *homotopy type*, in that they are weakly equivalent to the *homotopy function complexes* we saw in subsubsection 3.6.3. That also justifies the intuition we presented there of the homotopy function complexes capturing the homotopy type of the "Hom-∞-groupoid" between a pair of objects. The following ideas and results are recorded in the context of homotopical categories, where they originate, but it should not be forgotten that they are just as applicable to model categories.

Definition 181. Let $(\mathcal{C}, \mathcal{W})$ be a homotopical category. Using it one can construct a simplicial category, which we denote $L^H(\mathcal{C}, \mathcal{W})$, as follows. The two categories have the same set of objects, those of $\mathcal{C}$. Given $X, Y \in \mathrm{Ob}\mathcal{C}$ the n-simplices of $\mathrm{Map}_{L^H\mathcal{C}}(X, Y)$ are given by diagrams:

$$\begin{tikzcd}
 & C_{1,1} \arrow[r, dash] \arrow[d, "\sim"] & C_{1,2} \arrow[r, dash] \arrow[d, "\sim"] & \cdots \arrow[r, dash] & C_{1,k} \arrow[d, "\sim"] \arrow[ddr] & \\
 & C_{1,2} \arrow[r, dash] \arrow[d, "\sim"] & C_{2,2} \arrow[r, dash] \arrow[d, "\sim"] & \cdots \arrow[r, dash] & C_{2,k} \arrow[d, "\sim"] \arrow[dr] & \\
X \arrow[uur, dash] \arrow[ur, dash] \arrow[dr, dash] & \cdots \arrow[d, "\sim"] & \cdots \arrow[d, "\sim"] & & \cdots \arrow[d, "\sim"] & Y \\
 & C_{n,1} \arrow[r, dash] & C_{n,2} \arrow[r, dash] & \cdots \arrow[r, dash] & C_{n,k} \arrow[ur] &
\end{tikzcd}$$

subject to the following conditions:

(1) $n \geqslant 0$.

(2) All vertical maps are in $\mathcal{W}$.

(3) In each column, all horizontal maps have the same direction. If they point to the left, they are in $\mathcal{W}$.

(4) Maps in adjacent columns point in opposite directions.

(5) no column contains exclusively identities.

Face operators are given omitting a row and composing the arrows the vertical arrows. Degeneracy operators are given by duplicating a row and connecting it with identities.

Historically, in [DK80a], the hammock localization was introduced as an improvement to the "standard simplicial localisation L". The main disadvantage of this construction was that it was difficult to determine the homotopy type of the simplicial sets $L\mathrm{Map}_C(X,Y)$. The hammock localization of a homotopical category is weakly equivalent to its standard localisation and rectifies this problem. The next proposition records some important properties of the hammock localization.

Proposition 182. (Proposition 2.2 in [DK80b]) Let $(\mathcal{C},\mathcal{W})$ be a homotopical category. The two constructions above produce weakly equivalent outcomes, $L^H\mathcal{C} \xleftarrow{\sim} \cdots \xrightarrow{\sim} L\mathcal{C}$.

As discussed in the Introduction the hammock localisation is a simplicial category that generalises, or "properly contains" the ordinary homotopy category.

Proposition 183. (Proposition 3.1 in [DK80a]) Taking path components extracts the localization of $\mathcal{C}$ at $\mathcal{W}$, $\pi_0 L^H\mathcal{C} = \mathcal{C}[\mathcal{W}^{-1}]$.

Furthermore, the hammock localisation equips the homotopical category with homotopy function complexes, in the following sense.

Proposition 184. (Proposition 4.4 in [DK80b]) If $\mathcal{M}$ is a model category we can construct the homotopy function complexes $\mathrm{Map}(X,Y)$. Then, $\mathrm{Hom}_{L^H\mathcal{M}}(X,Y) \sim \mathrm{Map}(X,Y)$ as simplicial sets. Thus the Hom-sSets of the hammock localisation have the same homotopy type as its derived mapping spaces.

In subsection 3.4 we defined Quillen pairs as those adjunctions between model categories that lift to adjunctions between homotopy categories. It turns out that we can replicate this construction modulo a small modification. A restriction is placed because we consider only homotopical functors between homotopical categories. A right Quillen functor is homotopical only between fibrant objects. Only then does it induce a functor between homotopical categories. The situation is explained in full in [Maz15]. Here we record two relevant results.

Proposition 185. A Quillen pair $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ induces a weak equivalence $\mathrm{Hom}_{L^H\mathcal{C}}(X, G(Y)) \approx \mathrm{Hom}_{L^H\mathcal{D}}(F(X), Y)$.

Proposition 186. A Quillen equivalence $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ induces two weak equivalences:

(1) $L^H(F^c) : L^H(\mathcal{C}^c) \to L^H(\mathcal{D}^c)$

(2) $L^H(G^f) : L^H(\mathcal{D}^f) \to L^H(\mathcal{C}^f)$

Moreover, restricting to (co)fibrant objects produces hDK-equivalent homotopical categories, see Proposition 191 and its dual. But hDK-equivalent homotopical categories are exactly those that produce sDK-equivalent hammock localisations. Thus we can deduce that from a Quillen equivalence as above we can deduce $L^H(\mathcal{C})$ is sDK equivalent to $L^H(\mathcal{D})$. This is a very natural requirement from a homotopy category. Equivalent homotopy theories should produce equivalent homotopy categories. The same is true for these simplicial homotopy categories.

Proposition 187. (Proposition 4.5 in [DK80b]) A Quillen equivalence between model categories induces a weak equivalence of simplicial homotopy theories. In particular, we obtain weak equivalences of simplicial sets $\mathrm{Hom}_{\mathcal{M}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{N}}(LX, LY)$

Hopefully, all these results are convincing enough towards thinking of $L^H(\mathcal{C}, \mathcal{W})$ as a simplicial homotopy category of a homotopical category $(\mathcal{C}, \mathcal{W})$. Now something rather interesting happens if one applies this construction to a model category that is already simplicially enriched.

Proposition 188. (Proposition 4.8 in [DK80b]) If $\mathscr{M}$ is a simplicial model category then, forgetting the enrichment and applying L^H, produces $L^H(\mathscr{M}^0, \mathcal{W}) \sim \mathscr{M}^\circ$, the full simplicial subcategory of bifibrant objects.

This means that we can think of $\mathscr{E}^\circ \subset \mathscr{E}$ as (a model for) the simplicial homotopy category of a simplicially enriched model category.

4.1.1 Homotopical categories revisited

Many of the new *notions of sameness* we've introduced and so productively used come not from comparing two objects directly but instead from comparing their images under a functor of interest. In the same spirit, using the construction of L^H, we can produce a new *notion of sameness* between homotopical categories. In the end, this relaxed notion of sameness will produce yet another homotopy theory on the category of homotopical categories and homotopical functors between them. The hammock localization helps us give some important definitions of homotopical categories. Recall we call a functor $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{U})$ between homotopical categories, *homotopical functor* when it preserves weak equivalences, i.e. $F(\mathcal{W}) \subseteq \mathcal{U}$.

Definition 189. A homotopical functor $F : (\mathcal{C}, \mathcal{W}) \to (\mathcal{D}, \mathcal{U})$ will be called an **hDK equivalence** exactly when the induced functor $L^H(F)$ is an sDK equivalence. Explicitly, we require two things,

(1) We get an equivalence of categories $\mathrm{Ho}F : \mathrm{Ho}\mathcal{C} \to \mathrm{Ho}\mathcal{D}$.

(2) The induced map of simplicial sets: $L^H(F)_{X,Y} : \mathrm{Hom}_{L^H(\mathcal{C})}(X,Y) \xrightarrow{\sim} \mathrm{Hom}_{L^H(\mathcal{D})}(FX,FY)$ is a weak equivalence in $\mathrm{sSet}_{\mathrm{KQ}}$.

Proposition 190. (Example 3.1.23 [Kap14]) Let $F : \mathcal{M} \to \mathcal{N}$ be a right Quillen functor which is part of a Quillen Equivalence. Then, in general, F is not a hDK equivalence of underlying homotopical categories but $F^f : \mathcal{M}^f \to \mathcal{N}^f$ is.

Proof. By definition of QE we know that $\mathbb{F}$, the right derived functor of F, is an equivalence of homotopy categories. But by definition, the right derived functor is the functor precomposed with a fibrant replacement, i.e. the same as F^f. Therefore $\mathrm{Ho}F^f$ is an equivalence of homotopy categories. The remaining weak equivalence of simplicial sets is immediately obtained by applying Proposition 185 to F. □

Proposition 191. For any model category $\mathcal{M}$ the full inclusion $\mathcal{M}^f \hookrightarrow \mathcal{M}$ is an hDK equivalence.

Proof. Apply the previous proposition to the trivial Quillen Equivalence 1. Observe that id^f is the same thing as $\mathcal{M}^f \hookrightarrow \mathcal{M}$. □

Thus far we've seen how a homotopical category $(\mathcal{C}, \mathcal{W})$ or a simplicial category $\mathscr{E}$ are possible answers to the question "What is a homotopy theory?". In this section, we introduce a natural notion of equivalence between simplicial categories that makes sCat into a homotopical category itself! Then, we'll see how the hammock localization can be used to "lift" that notion of "equivalence" to the collection of homotopical categories, hCat, making it a homotopical category as well.

Definition 192. Let **hCat** denote the category of homotopical categories and homotopical functors between them. Then, hCat is itself a homotopical category where $\mathcal{W}_{\mathrm{hCat}} = \{ \text{ hDK equivalences } \}$.

Definition 193. Let **sCat** denote the category of simplicial categories and simplicial functors between them. Then, sCat is a homotopical category with $\mathcal{W}_{\mathrm{sCat}} = \{ \text{ sDK equivalences } \}$. We will later see that this is the underlying homotopical category of a model category that will play an important role in what is to come.

With those definitions, we can finally make precise the intuition that the hammock localisation gives a simplicial category $L^H(\mathcal{C}, \mathcal{W})$ with the *same homotopy theory as* $(\mathcal{C}, \mathcal{W})$.

Proposition 194. $L^H : hCat \to sCat$ is an hDK equivalence between homotopical categories.

4.2 Homotopy $(\infty, 1)$-categories

In the previous section, we saw how to produce a simplicial homotopy category $L^H(\mathcal{M}, \mathcal{W})$ for a homotopical category $L^H(\mathcal{M}, \mathcal{W})$. But in subsubsection 2.3.4 we saw a robust connection between simplicial categories and ∞-categories. Thus, in this section, we expand on the previous one by exploiting that connection to obtain a *homotopy ∞-category* of (at least) a homotopical category.

From the previous section, we define the simplicial homotopy category of a homotopical category $\mathrm{Ho}_S(\mathcal{M}, \mathcal{W}) = L^H(\mathcal{M}, \mathcal{W})$. We'd like to apply the homotopy coherent nerve functor to canonically obtain an ∞-category. Unfortunately, $L^H(\mathcal{M}, \mathcal{W})$ is not always the homotopically correct object, it need not be enriched in Kan complexes. Thus we are forced to define:

Definition 195. Let $(\mathcal{M}, \mathcal{W}) \in \mathrm{hCat}$. It's ∞-homotopy category is $\mathrm{Ho}_\infty(\mathcal{M}, \mathcal{W}) = \mathfrak{N}(L^H(\mathcal{M}, \mathcal{W})^*)$ where $*$ denotes a Kan replacement in $\mathrm{sCat}_{\mathrm{Berg}}$.

One can provide a more direct construction of the homotopy type of $\mathrm{Ho}_\infty(\mathcal{M}, \mathcal{W})$ via the theory of localisations of ∞-categories. We present it briefly following Chapter 9 in Cisinski's [Cis16]. The essential point is that we take the universal property of the standard localisation and "stick ∞ in front of everything" to obtain the corresponding statement for ∞-categories.

Definition 196. Let $S \subset C$ denote an ∞-subcategory of C. Let $\mathrm{Fun}_S(C, X)$ denote the full subcategory of $\mathrm{Fun}(C, X)$ that send all arrows of S to isomorphisms in X.

Definition 197. A localisation of C at S is a functor $\gamma C \to S^{-1}C$ that:

(1) for any $f \in S \subseteq C$, $\gamma(f)$ is an iso in $S^{-1}C$.

(2) For all ∞-categories X precomposition by γ induces an equivalence of ∞-categories: $\mathrm{Fun}(S^{-1}C, X) \to \mathrm{Fun}_S(C, X)$.

Remark 198. If $C = NC'$ for an ordinary category C', the homotopy category, in the sense of definition – of $h(S^{-1}C)$ is canonically equivalent to $C[S^{-1}]$, the category of fractions of [GZ12]. Thus we can think of $S^{-1}C$ as a $\mathscr{K}an$-enriched version of the latter.

This intuition can be expanded on by the following description of the same construction offered by Kapulkin, in paragraph 3.1.45 of [Kap14]. Consider the various $f \in S \subset C$. By Yoneda lemma, we know that $f \in N(C)_1 \leftrightsquigarrow \Delta^1 \to N(C)$. Thus we obtain a map $\bigsqcup_{f \in S} \Delta^1 \to N(C)$. Similarly we obtain $\bigsqcup_{f \in S} \Delta^1 \to \mathscr{J}$, the *walking isomorphism*. Morally, to form $S^{-1}C$ amounts to adding an inverse to each arrow $f \in S$ or to "glue" a $\mathscr{J}$ over each $\hat{f} : \Delta^1 \to N(C)$ Using our intuition for the pushout of ordinary categories, that amounts to forming the pushout:

$$\begin{tikzcd} \coprod_{f\in S} \Delta^1 \arrow[r] \arrow[d] & N(C) \arrow[d] \\ \coprod_{f\in S} \mathscr{J} \arrow[r] & S^{-1}\bar{N}(C) \end{tikzcd}$$

Unfortunately, as above, this construction too fails to produce the homotopically correct object, only its homotopy type. Thus we define,

Definition 199. Given a homotopical category $(\mathcal{C}, \mathcal{W})$ we define its **standard localisation**, $L(\mathcal{C}, \mathcal{W})$ as a $\mathrm{sSet}_{\mathrm{Joy}}$-fibrant replacement of $S^{-1}\bar{N}(C)$ above.

Proposition 200. (Proposition 8.7 in [HS01]) The functor $L : \mathrm{hCat} \to \mathrm{qCat}$ is an hDK equivalence. Moreover, for any $\mathcal{M} \in \mathrm{hCat}$ there exists an equivalence, $L(\mathcal{M}) \simeq \mathfrak{N}(L^H(\mathcal{M})^*)$.

It is also important to note what happens when we apply all these ideas to a simplicial model category $\mathscr{E} \in \mathrm{mCat}$. As recorded in Proposition 182, we find $L^H(\mathscr{E}_0, \mathcal{W}) \simeq \mathscr{E}^\circ$, the full simplicial subcategory on bifibrant objects. It is a particularly convenient property of simplicial model categories that we can find the simplicial homotopy category, or rather its homotopy type, in such a direct and clear way. Then, the definition above becomes:

Definition 201. Let $\mathscr{E} \in \mathrm{smCat}$. Then we can think of $\mathscr{E}^\circ$ as a homotopy simplicial category for $\mathscr{E}$. We define the homotopy ∞-category of $\mathscr{E}$ to be $\mathrm{Ho}_\infty(\mathscr{E}) := \mathfrak{N}(\mathscr{E}^\circ)$.

Other phases commonly used is that the model category $\mathscr{E}$ *presents* $\mathscr{C} = \mathrm{Ho}_\infty(\mathscr{E})$ or that $\mathscr{C}$ is the *underlying* ∞*-category of* $\mathscr{E}$. In the next two sections, we will apply all these constructions to two specific cases that become very relevant when it comes to interpreting HoTT.

4.3 Quasicategory of frames of a fibration category

In this subsection, we look at another direct and functorial construction for a model of $\mathrm{Ho}_\infty(-)$ of a fibration category, the ∞-category of frames.

We start by building up some machinery. The construction originates in [Szu14]. It is the main tool used by Kapulkin in his PhD thesis, [Kap14] and [Kap17], to settle Joyal's conjecture. We start with some preliminary definitions required to introduce the main object of Kapulkin's work, the ∞-category of frames of a fibration category, $N_f(-) : \mathrm{fCat} \to \mathrm{qCat}$. We proceed by surveying the most crucial properties of this construction.

Definition 202. *(1)* A category J will be called **inverse** if it comes equipped with a degree function $\deg : \mathrm{ob}J \to \mathbb{N}$ such that for any non identity arrow $j \to j'$ we have $\deg(j') < \deg(j)$. Let J be an inverse category:

(2) Take $j \in J$. The **matching category** $\partial(j/J)$ is the full subcategory of the slice category ${}^{j}\!/_{J}$ consisting of all objects except the identity. There is a canonical functor cod : $\partial(j/J) \to J$.

(3) Let $X : J \to \mathcal{C}$ and $j \in J$. The **mathcing object** of X at j is the following limit: $M_j(X) := \lim(\partial(j/J) \to J \to X)$.

(4) Let $\mathcal{M}$ be a fibration category. A diagram is called **Reedy fibrant** if $\forall j \in J$ there exists a matching object $M_j(X)$ and the canonical map $X(j) \twoheadrightarrow M_j(X)$ is a fibration.

Proposition 203. By induction on the minimal degree of J one can show that one can always take the limit of a Reedy fibrant diagram.

Definition 204. We denote by $[n]$ the linearly ordered set considered as a homotopical category with only trivial weak equivalences.

We can then construct a new homotopical category $D[n]$ and a homotopical functor $p : D[n] \to [n]$. It has objects pairs $([k], \phi : [k] \to [n])$. A map $f : ([k], \phi : [k] \to [n]) \to ([l], \psi : [l] \to [n])$ is given by an injective, order preserving $\tilde{f} : [k] \hookrightarrow [l]$ so that ψ factors through ϕ. We put $p : D[n] \to [n]$ be given by $\big(([k], \phi) \mapsto \phi(k)\big)$ and put $\mathcal{W}_{D[n]} := p^{-1}(\mathcal{W}_{[n]})$ so that p becomes homotopical. This construction readily generalises to arbitrary homotopical categories K.

Definition 205. Let $\mathcal{M}$ be a fibration category. We define the ∞**-category of frames of a fibration category** $\mathcal{C}$ to be the following simplicial set:

$$N_f(\mathcal{M})_n := \mathrm{Fun}^{\mathrm{hr}}(D[n]^{\mathrm{op}}, \mathcal{M})$$

So that n-simplices are homotopical and Reedy fibrant diagrams $D[n]^{\mathrm{op}} \to \mathcal{M}$.

We conclude this section by recording some of the basic properties of this construction. Unfortunately, the proof of most of the following results makes use of complete Segal spaces. Therefore, their treatment lies outside the scope of this thesis.

First and foremost, the ∞-category of frames is *another model for the underlying ∞- category of the relative/homotopical category of the fibration category $\mathcal{M}$.*

Theorem 206. (Theorem 9.1.2 in [Kap14]) For any fibration category $\mathcal{M}$ the ∞-categories $L(\mathcal{M})$ and $N_f(\mathcal{M})$ are equivalent.

Theorem 207. (Theorem 3.2 in [Szu14] or Theorem 5.3.12 in [Kap14]) The functor $N_f : \mathcal{F}\text{ib} \to$ sSet takes values in finitely complete ∞-categories and is an exact functor between fibration categories[18] $\mathcal{F}\text{ib} \to \text{qCat}$.

Proof. The first part of the proof is done by showing that the functor $N_f(-)$ preserves pullbacks along fibrations and the terminal object. Preservation of (trivial) fibrations is significantly more technical. $\square$

Proposition 208. (Proposition 3.5 in [Szu14]) Let $\mathcal{M}$ be a fibration category and K a simplicial set. Then, there is a natural bijection:

$$K \to N_f(\mathcal{M}) \quad \leftrightarrow \quad DK^{\mathrm{op}} \to \mathcal{M}$$

In some appropriate sense, the ∞-category of frames construction preserves slices and adjoints as the next two theorems record. One could say these are the main two tools one has for working with the ∞-category of frames. Kapulkin introduces and makes use of the ∞-category of frames with the aim of establishing some conditions on a fibration category $\mathcal{C}$ that, when satisfied will make $N_f(\mathcal{C})$ a locally cartesian closed ∞-category. Recall that the latter involves the pullback functor between slice ∞-categories always having a right adjoint. The next two properties relate 1-categorical slices and adjoints between fibrations categories to ∞-categorical slices and adjoints between their respective ∞-categories of frames. This is, of course, analogous to lifting Quillen pairs between model categories to homotopy categories. This provides the intuition that homotopical and exact functors play the same role as Quillen pairs between model categories.

Theorem 209. (Theorem 7.2.2 [Kap14]) Let $\mathcal{M}$ be a fibration category and $A : D[0]^{\mathrm{op}} \to \mathcal{M}$ be a 0-simplex in $N_f(\mathcal{M})$. Then there is an equivalence of ∞-categories:

$$N_f(\mathcal{M})_{/A} \simeq N_f(\mathcal{M}(A_0))$$

Recall that $\mathcal{M}(A_0)$ is the notation we have used for the full subcategory of $\mathcal{M}_{/A_0}$ spanned by the fibrations of $\mathcal{M}$.

Theorem 210. (Theorem 7.3.9 in [Kap14]) Let $F : \mathcal{C} \leftrightarrows \mathcal{D} : G$ be an adjunction between fibration categories. If all objects are cofibrant, F is homotopical and G is exact, then we obtain an adjunction of ∞-categories, $N_f(F) : N_f(\mathcal{C}) \leftrightarrows N_f(\mathcal{D}) : N_f(G)$.

[18] There is a category of Fibration Categories and exact functors between them. In fact, this is a fibration category where the weak equivalences are maps that lift to equivalences between homotopy categories and fibrations are those functors that lift against (pseudo) factorisations and isofibrations.

Proposition 211. (Proposition 9.2.3 in [Kap14]) If $\mathcal{C}$ is a locally cartesian closed fibration category, then for each $A \in \mathcal{C}$ the category $\mathcal{C}(A)$ is cartesian closed. Additionally, for any fibration $p : B \twoheadrightarrow A$ the product functor in the slice, $p \times - : \mathcal{C}(A) \to \mathcal{C}(A)$ is exact and its right adjoint is homotopical.

The two theorems above play a crucial role in showing one of Kapulkin's stepping stones towards the main theorem, Theorem 9.3.17, of his thesis [Kap14].

Theorem 212. (Theorem 9.2.8 in [Kap14]) If $\mathcal{M}$ is a locally cartesian closed fibration category then $N_f(\mathcal{M})$ is a locally cartesian closed ∞-category.

Proof. Much like in 1-category theory, if an ∞-category $\mathscr{C}$ has a terminal object and pullbacks then it is locally cartesian closed exactly when the pullback functor against a morphism f admits a right adjoint classically denoted Π_f. By Proposition 207 $N_f(\mathcal{M})$ is finitely complete and therefore has a terminal object and admits pullbacks. So it remains to show that all slices are cartesian closed. Take a vertex A and form ${}^{N_f(\mathcal{M})}\!\diagup_{A}$. By Proposition 209 that's equivalent to $N_f(\mathcal{M}(A_0))$. In Proposition 9.2.3 of [Kap14] Kapulkin shows that the product/exponential adjunction in $\mathcal{M}(A_0)$ satisfies the requirements of 210. $\square$

4.4 Model categorical presentations of ∞-categories

In this section, we recall one of the main players in this thesis, the homotopy coherent nerve and the adjunction it participates in:

$$\mathfrak{C} : \mathrm{sSet} \leftrightarrows \mathrm{sCat} : \mathfrak{N}$$

Thus far we've provided two distinct interpretations for this functor. Firstly, in the previous sections, we constructed a functor $L^H : \mathrm{hCat} \to \mathrm{sCat}$ that we interpreted as producing a simplicial homotopy category for a category with weak equivalences $(\mathcal{C}, \mathcal{W})$. Then, as $\mathfrak{N}$ mediates the passage from simplicial to ∞-categories, we used it to define the homotopy ∞-category of a homotopical category. As hCat admits forgetful functors from smCat, mCat, fCat this construction extends together one of these settings. This makes precise the slogan that the homotopy theory is fully determined by the class of weak equivalences.

Secondly, at the beginning of the thesis, we looked into categories enriched in Kan Complexes and ∞-categories as two models for the theory of $(\infty, 1)$-categories. Moreover, in subsection 3.5 we endowed the ambient categories of each with a model structure and exhibited each as the full subcategory of bifibrant objects. In fact, in both model structures, all objects are cofibrant, so the previous statement reduces to subcategories of *fibrant* objects.

For instance, we saw how $\mathscr{K}an$-enriched categories arise as precisely the fibrant objects of $\mathrm{sCat}_{\mathrm{Berg}}$ and ∞-categories as the fibrant objects of $\mathrm{sSet}_{\mathrm{Joy}}$. Thus both model categories capture the homotopy theory of $(\infty, 1)$-categories. Through the discussion of the previous section and the paragraph above it becomes clear that we can view $\mathrm{qCat} = \mathrm{sSet}^{\circ}_{\mathrm{Joy}} = \mathrm{Ho}_{\infty}(\mathrm{sSet}_{\mathrm{Joy}})$ as the homotopy ∞-category of $\mathrm{sSet}_{\mathrm{Joy}}$. Then, the Quillen equivalence $\mathfrak{C} \dashv \mathfrak{N}$ establishes the intuition that the development of $(\infty, 1)$-categories in the two models yields homotopically equivalent results.

In the following section, we look deeper and find that using $\mathfrak{N}$ we can construct a very clear dictionary between for notions of $(\infty, 1)$-categories in the two models. Specifically, we find that one can simultaneously develop $(\infty, 1)$-topos theory in the two settings, with each ∞-categorical piece of the definition having a precise smCat-egorical counterpart. These ideas have their root in the work of Simpson, for instance, [Sim99], and Toën and Vezossi, in [TV05], but took their present form through the work of Rezk, notably in [Rez10].

The upshot is that we can characterise the exact form of simplicial model categories $\mathscr{M}$, such that $\mathfrak{N}(\mathscr{M}^{\circ})$ is an ∞-topos. Moreover, any ∞-topos can be described in this way.

4.5 Model categorical presentations of ∞-topoi

Thus far, we have defined and studied two models of $(\infty, 1)$ categories, locally Kan simplicial categories and ∞-categories. The comparison between the two is mediated by the homotopy coherent nerve functor $\mathfrak{N} : \mathrm{sCat} \to \mathrm{sSet}$. We've offered two distinct but interrelated conceptual interpretations for this functor. The first is, $\mathfrak{N}$ as a Quillen equivalence between two model structures capturing the homotopy theory of $(\infty, 1)$-categories developed in the two different models. This formalises the intuition that these two ways of developing the theory of $(\infty, 1)$-categories are homotopically equivalent. The second, view is $\mathfrak{N}$ as computing a homotopy ∞-category of a simplicial model category. In this section, we will see how this equivalence goes significantly deeper.

We will see how, through $\mathfrak{N}$, each ∞-categorical notion that goes into the definition of ∞-topoi has a clear analogue in the enriched model categorical setting, thus allowing us to simultaneously define and characterise $(\infty, 1)$-topoi in the two settings. All these relationships are recorded in Table 1 at the end of the section.

Grothendieck (∞)-topoi were defined as left exact reflective subcategories of presheaf categories. We will use the theory we've developed so far to determine the model categorical versions of the ∞-categorical notions that go in defining and characterising ∞-topoi. This will be done in 3 steps. First, we'll see that simplicial presheaves present ∞-presheaves. Then, Bousfield localisations, that, under Proposition 153 induce a reflection between homotopy categories, also induce a reflection between homotopy ∞-categories. This is made precise by the fact that Quillen pairs lift through $\mathfrak{N}$. Lastly, since homotopy (co)limits also present their ∞-categorical counterparts, we get that the reflector between homotopy categories is left exact with respect to homotopy (co)limits exactly when the reflector between homotopy ∞-categories is left exact.

Our first task is to identify how to present the ∞-category $\mathrm{Psh}_\infty(\mathscr{A}) = \mathrm{Fun}(\mathscr{A}, \mathcal{S})$. The slogan here is that *simplicial presheaves present ∞-presheaves*. This means that in the same way that *spaces* $\mathcal{S}$ play the role of sets in the setting of ∞-categories, the same is true for $\mathrm{sSet}_{\mathrm{proj}}$ in the simplicial setting. Presheaves valued in sSet in the simplicial model of $(\infty, 1)$-categories correspond to presheaves valued in spaces for the ∞-categorical model.

Theorem 213. (Proposition 4.2.4.4. in [Lur09][19])

Let $\mathscr{C}$ be a $\mathscr{K}an$ enriched category. Recall that we write $\mathrm{sPsh}(\mathscr{C})$ for $\mathrm{sSet}_{\mathrm{proj}}^{\mathscr{C}^{\mathrm{op}}}$

Then we have an equivalence of ∞-categories:

$$\mathfrak{N}(\mathrm{sPsh}(\mathscr{C})^\circ) \simeq \mathrm{Fun}(\mathfrak{N}(\mathscr{C}), \mathfrak{N}(\mathrm{sSet}^\circ_{\mathrm{KQ}})) = \mathrm{Fun}(\mathfrak{N}(\mathscr{C}), \mathcal{S}) = \mathrm{Psh}_\infty(\mathfrak{N}(\mathscr{C}))$$

Thus, simplicial presheaves present ∞-presheaves. The next step towards the definition of ∞-topoi is to consider reflective subcategories thereof. We have already supplied an important characterisation of such categories. Recall that a presentable ∞-category is one when one has a small set of generators and all objects can be built by glueing these generators. Along the same line, a combinatorial model category is one where the same is true for objects and cofibrations of the model structure. It shouldn't be surprising that they model the same $(\infty, 1)$-categories.

Theorem 214. Let $\mathscr{C}$ be a presentable ∞-category. Then there exists a small ∞-category $\mathscr{E}$, a fully faithful inclusion i that admits a left adjoint L.

$$\mathscr{C} \underset{i}{\overset{L}{\leftrightarrows}} \mathrm{Fun}(\mathscr{E}, \mathscr{S})$$

Proof. We already presented and discussed this in Theorem 65 $\square$

Theorem 215. (Proposition A.3.7.6 [Lur09]) An ∞-category $\mathscr{C}$ is presentable exactly when there exists a simplicial combinatorial model category $\mathscr{A}$ and an equivalence of ∞-categories $\mathscr{C} \simeq \mathfrak{N}(\mathscr{A}^\circ)$

Proof. For the proof we refer to the source. Instead, we just note how $\mathscr{A}$ is obtained. We know that $\mathscr{C}$ is equivalent to a reflective subcategory of a ∞-presheaf category on the small and full ∞-subcategory $\mathscr{C}^\kappa$ on κ-comapct objects, see Theorem 65. Recall that by $\mathfrak{C}$ we denote the left Quillen adjoint of $\mathfrak{N}$. Let $\mathcal{A} := \mathfrak{C}((\mathscr{C}^\kappa)^{\mathrm{op}}) \in \mathrm{sCat}$. Then, $\mathscr{A} := \mathrm{sPsh}(\mathcal{A})_{\mathrm{inj}}$. $\square$

[19]For a clearer exposition see also the nLab page

In addition to that it is important to recall Dugger's Theorem:

Theorem 216. Let $\mathscr{E}$ be a simplicial combinatorial model category. A **small presentation** for $\mathscr{E}$ is a small category D, a set of maps S in $\mathrm{sSet}^{D^{\mathrm{op}}}_{\mathrm{proj}}$ and a Quillen Equivalence

$$\mathscr{E} \leftrightarrows \mathcal{L}_S(\mathrm{sSet}^{D^{\mathrm{op}}}_{\mathrm{proj}})$$

As Rezk notes in paragraph 5.4 of [Rez10], a model category $\mathcal{M}$ admits a small presentation if and only if it admits a small *simplicial* presentation. See Proposition 174. Combining all these facts we see that the notion of a combinatorial model category mediates the following analogy.

A presentable ∞-category $\mathscr{C}$ arises as a reflective subcategory of $\mathrm{Psh}_\infty(\mathscr{C}^\kappa)$. Presentable ∞-categories are presented by combinatorial simplicial model categories. Up to Quillen equivalence, the latter arise as $\mathcal{L}_S\left(\mathrm{sSet}^{\mathscr{E}^{\mathrm{op}}}_{\mathrm{proj}}\right)$.

The last step towards the definition of ∞-topoi is to require the left adjoint of the fully faithful inclusion to be left exact, namely to preserve finite limits. The notion of "left exact reflector" has a straightforward simplicial model-categorical analogue.

We say the left Bousfield localisation with respect to S is **left exact** when the left derived functor of $a : \mathcal{M} \to \mathcal{L}_S\mathcal{M}$ preserves finite homotopy limits. Since it always preserves the terminal object it suffices for it to preserve homotopy pullbacks. It is important to note that after Corollary 1.5.2 of [Hin15] we know that homotopy (co)limits present the ∞-categorical ones[20]. Thus, there is a natural candidate for a model categorical presentation of an ∞-topos. We define:

Definition 217. Let $\mathscr{C}$ be a small simplicial category and S a set of maps in $\mathrm{sPsh}(\mathscr{C})$. We say that a model category is a **model topos** when it is equivalent to a left exact left Bousfield localisation of simplicial presheaves.

Theorem 218. Any ∞-topos is presented by a model topos $\mathscr{E} =_{QE} \mathcal{L}_S\left(\mathrm{sSet}^{\mathscr{M}^{\mathrm{op}}}_{\mathrm{proj}}\right)$ for a simplicial $\mathscr{M}$ and left exact S-localisation. Namely, there exists an equivalence of ∞-categories $\mathfrak{X} \simeq \mathfrak{N}(\mathscr{E}^\circ)$.

Then model topoi are the $(\infty, 1)$-topoi of the simplicial model for $(\infty, 1)$-categories. We further this intuition by noting that model topoi also admit a Giraud-type characterisation in perfect analogy with Theorem 72.

[20]To prove that one shows that one can lift Quillen pairs between model categories to their underlying ∞-categories. And applies that to the two Quillen adjunctions we defined in subsubsection 3.4.5

Theorem 219. (Theorem 6.9 in [Rez10]) A model category $\mathcal{E}$ is a model topos if and only if

(*1*) $\mathcal{E}$ admits a small presentation

(*2*) $\mathcal{E}$ has universal homotopy colimits and descent for homotopy colimits.

Proof. For the proof of the theorem, we refer to [Rez10]. Some remarks are in order. For ($\Longrightarrow$) we remark that one shows directly that $\mathrm{sSet}_{\mathrm{KQ}}$ has descent. Moreover, since homotopy (co)limits are computed componentwise in simplicial presheaves, it is also the case that (2) above lifts. Lastly, one shows that (2) is also stable under the formation of left Bousfield localisation.

For the converse, we are satisfied with noting that the small presentation exhibits $\mathcal{E}$ as a localisation of simplicial presheaves. Note that simplicial presheaves form a combinatorial and simplicial model category. Then, by Theorem 215, it presents a presentable ∞-category, which in turn is a reflective subcategory of an ∞-presheaf category. As in the case of ∞-topoi, Property (2) goes into showing that the resulting reflector is left exact.

Hence we see that not only the statements of the two theorems are in perfect analogy but so are their proofs. This speaks to the depth of the analogy mediated by $\mathfrak{N}$. □

In a very similar spirit, one can also characterise the model categories that present (presentable) locally cartesian closed ∞-categories in the following way. We note that the following theorem appears as (Theorem 4.1.25 [Kap14]).

Theorem 220. Let $\mathscr{C}$ be an ∞-category. The following are equivalent.

(*1*) $\mathscr{C}$ is presentable and locally cartesian closed.

(*2*) $\mathscr{C}$ admits a presentation as a *right proper* left Bousfield localisation of the injective model structure on simplicial presheaves.

(*3*) $\mathscr{C}$ admits a presentation by a right proper Cisinksi model category

Encyclopedically, we note that the latter condition is also equivalent to admitting a presentation by a combinatorial locally cartesian closed model category.

Proof. The equivalence of (1) $\iff$ (3) is given in the post and comments of Shulman's n-Category Cafe blog post. Shulman's proof that (3) $\implies$ (1) is instructive so we include it here. We want to show the ∞-pullback functor admits a right adjoint. We will lift the corresponding adjunction through $\mathfrak{N}$. It suffices to show $g^* \dashv \Pi_g$ form a Quillen pair. Since to obtain the right derived $\mathbb{R}g^*$ we'll "precompose" with a fibrant replacement, it suffices to consider the case where g is a fibration, namely a fibrant object in the slice model structure. To show the adjunction above is a Quillen pair we show it preserves cofibrations and trivial fibrations. In a Cisinksi model category, the cofibrations are monomorphisms which are, of course, pullback stable. Since we assumed g is a fibration and the model category is right proper, pullback against g also preserves weak equivalences, completing the proof. $\square$

The following table records notions that correspond to one another for the two models of $(\infty, 1)$-categories of interest to us. They are the notions that go into the definition of an $(\infty, 1)$-topos, known as an ∞-topos in the model of ∞-categories and as a model topos in the context of simplicial model categories.

Model A: ∞-categories	Model B: smCat	Result
$\mathrm{Psh}_\infty(\mathscr{A}) = \mathrm{Fun}(\mathscr{A}, \mathcal{S})$	$\mathrm{sPsh}(\mathscr{B})_{\mathrm{proj}}$	Prop. 4.2.4.4. [Lur09]
$\mathscr{A}$ is a presentable ∞-category i.e. a reflective subcategory of an ∞-presheaf category $\mathscr{A} \simeq \mathscr{P} \leftrightarrows \mathrm{Psh}_\infty(\mathscr{E})$	$\mathscr{E} \in$ Comb-mCat so $\mathscr{E} \simeq \mathcal{L}_S\big(\mathrm{sPsh}(\mathcal{E})_{\mathrm{proj}}\big)$	153
The left adjoint above is left exact	The left adjoint $a : \mathrm{sPsh}(\mathscr{C}) \to \mathcal{L}_S\big(\mathrm{sPsh}(\mathscr{C})\big)$ is left exact	[Rez10], [TV05]
$\mathfrak{X}$ is an ∞-topos	Model Topos	[Rez10], [TV05]
$\mathfrak{X}$ is presentable, has descent and universal ∞-colimits	small presentation, descent and universal homotopy colimits	[Rez10], [TV05]
$\mathscr{C}$ is presentable and locally cartesian closed ∞-category	right proper $C \in$ Comb-mCat and $C = \mathcal{L}_S\big(\mathrm{sPsh}(S)_{\mathrm{inj}}\ \big)$	220

Table 1: The analogy

5 Models of type theories and higher categories

5.1 Martin Löf Dependent Type Theory

Around the 1930s there where various proposals for a mathematically precise notion of computation. These proposals included a version of Church's λ-Calculus [Chu32], which was shown to be equivalent to the other two main proposals, Gödel's general recursive functions and Turing Machines. After Gödel successfully translated the meta-mathematics of Peano Arithmetic inside his theory of general recursive functions it was clear that these models of computation had a robust link with logical systems.

Following this line of inquiry, it was observed that the rules of λ-calculus were in perfect analogy with the rule of Gentzen's Natural Deduction. This was made mathematically precise as the *Curry-Howard correspondence* in [How80], or the *propositions as types* paradigm. Therein, it was proposed that terms of types of λ-calculus behave exactly the same way as proofs of propositions in natural deduction. Moreover, the correspondence was established in such a way that each step of computation in a λ-term corresponds exactly to an application of a logical rule.

Martin Löfs' Dependent Type Theory (MLDTT) was intended as a formal system of for Intuitionistic/Constructive mathematical reasoning, see [Mar75b], [Mar84], [Mar98]. It turned out to be so much more. From a logical point of view, the system deals with terms of types, which can be interpreted as proofs of propositions. MLDDT makes the addition of *dependent types*, $x : A \vdash B(x) : \text{Type}$, to be thought of as a family of types indexed by $x : A$. If types are propositions, a type that depends on an argument is a "dependant" proposition, namely a predicate.

The next step is to add *quantifiers* to our language. We want to maintain the constructive character of our system and thus we pay special attention to adding constructive quantifiers. For instance, we introduce a *sum-type* that plays the role of the existential quantifier. In constructive mathematics the proof of an existential statement $\exists x : P(x)$ consists of a construction of a specific object and a and a proof that "a indeed has property $P(-)$". Thus to construct a term of $\Sigma x : AB(x)$ one must supply a pair $(a : A, p : B(a))$. Similarly, to prove a $\forall x : P(x)$-statement in a constructive setting we must provide an "algorithm" or an "effective procedure" that on input a produces $p_a : P(a)$. In terms of our set-theoretic intuition, these can be thought of as functions whose codomain depends on the input. The standard example is $n : \text{nat} \mapsto \vec{0} \in \mathbb{R}^n$.

The last addition is the *identity type*, which really turns out to be the culprit for much of the phenomena that interest us in this thesis. MLDTT comes naturally equipped with "definitional equality", a syntactic equivalence relation "generated by abbreviatory definitions, changes of bound variables and the principle of substituting equals for equals" [21]. Now this equality proves too strict for practical use. For instance, one can't prove $\mathrm{add}(0,n) \doteq n$. Thus, we further extend the type theory with an *internal notion of equality*, a type $\mathrm{Id}_A(x,y)$ or $x =_A y$ standing for the proposition "x, y are equal terms of type A". A term $p : x =_A y$ stands for a proof that x, y are indeed equal. We call that *propositional equality*.

But how should we construct or define this type? Well, in full generality the only thing that should always hold is that x can always be expected to be equal to itself. Thus, we postulate terms

$$\frac{a : A}{r_a : a =_A a}$$

Not having much else to work with, we postulate this bi-parametrised family of types to be *freely generated* by the r_x's[22]. Whenever something is freely generated via an operation applied to a collection of generators, to define an operation-preserving function out of it, it suffices to supply the data for the values of the generators. The same is true in this context. This is the *induction principle for identity types* or *path induction*.

$$\frac{p : x =_A y, \qquad x : A \vdash d(x) : B(x,x,r_x)}{J(d,x,y,p) : B(x,y,p)}$$

It must be read as: "Given arbitrary $x, y : A$ and $p : x =_A y$, if you supply a family $d(x) : B(x,x,r_x)$ depending on $x : A$ you may obtain $J(d,x,y,p) : B(x,y,p)$." This phenomenon, namely that to define a potentially complicated function it suffices to specify its values in the most trivial of cases, makes path induction very valuable from a practical standpoint.

5.2 The groupoid interpretation

Having introduced the identity type one wonders about its internal structure. In the beginning, it was thought that the identity type was a simple as it could be, namely that for $p, q : x =_A y$ then it should be derivable that $p =_{x=_A y} q$, the so-called *Uniqueness of Identity Proofs*. Indeed it was shown to be derivable in some special case type theories. Unfortunately, UIP destroys some good properties of the system such as *decidability of type checking*. This motivated the pursuit of models that invalidate it. The first such instance was provided by Hofmann and Streicher, [HS94] and [HS02], who produced a categorical model of MLDTT that didn't satisfy UIP, thus finally settling the question.

[21] p.16 in [Mar75a]

[22] This is very common practice in MLDTT. Such inductive definitions allow us to express potentially infinite structures with finite means. This is a common "trick" among constructivists who value their methods but are not willing to give up on, for example, the natural numbers.

The truly crucial part is *how* the question was settled. Before Hofmann and Streicher the community thought of type theory in a rather "set-theoretic" manner. And indeed in a "discrete" setting such as Set the reflection principle is validated. Instead, by interpreting type theory in a setting where there is *higher data* present, where entities are more structured than sets, UIP fails. In retrospect, UIP is a *truncation* principle, and indeed the strictest truncation. To invalidate it, it suffices to produce a non-completely truncated example. Hofmann and Streicher proposed a notion of an abstract model of MLDTT, categories with families, and produced this structure in the category of groupoids.

Recall a groupoid is a category with exclusively invertible arrows. Every set can be seen as a *discrete* groupoid with only identity arrows. Types, and contexts[23] will be interpreted as groupoids and terms as objects of the groupoid. Morphisms of the groupoid model propositional equality, the identity type. Namely, let $\mathcal{G}$ stand for the type A. Then, $p : x =_A y \leftrightsquigarrow p \in \mathrm{Hom}_{\mathcal{G}}(X, Y)$. Composition of arrows plays the role of transitivity of equality or concatenation. Identity morphisms stand for the reflexivity proofs.

The collection of (small) groupoids and functors between them can be arranged in a category $\mathcal{GPD}$. We think of this as the universe of types. Internally in type theory a dependant family on $\Gamma, x : A \vdash B(x)$ can equivalently be thought of as a function $f : A \to \mathcal{U}$. Thus, we interpret type families as $B : A \to \mathcal{GPD}$.

Then, the dependent family of types $x =_A y$ is interpreted as a "presheaf" $I_A : A \times A \to \mathcal{GPD}$. It's values are the sets $\mathrm{Hom}_A(X, Y)$ seen as discrete groupoids and the action is $I_A(q_1, q_2)(s) = q_2 \circ s \circ q_1^{-1}$. Observe that to provide the data of an identification, $p : x =_A y$ one must provide a triple $x, y : A$ and $p : I_A(x, y)$, i.e. a morphism between x, y in the groupoid A. The collection of such triples is thus isomorphic to the arrow groupoid $A^{\to}$. This foreshadows the general phenomenon of interpreting identity types as *path objects*.

Next, we interpret the J rule, path induction. Let C be a family over the groupoid $A^{\to}$. The goal is: only given the data of $d_a : C(a, a, r_a)$ to define the entire *dependant object* $J(A, C, d)(x, y, s) : C(x, y, s)$. The trick is to use a Yoneda style argument to express the value of an arbitrary s in terms of id_{a_1}. Observe that we always have an $A^{\to}$-morphism $(\mathrm{id}_a, s) : (a, a, \mathrm{id}_a) \to (a, b, s)$ given by the commutative square

$$\begin{tikzcd} a \arrow[r, "\mathrm{id}_a"] \arrow[d, "\mathrm{id}_a"'] & a \arrow[d, "s"] \\ a \arrow[r, "s"'] & b \end{tikzcd}$$

Then, we let $J(A, C, d)(x, y, s) := (\mathrm{id}_a, s) \cdot d(a)$. Hence, the value on an arbitrary s is

[23] dependent lists of types.

expressed in terms of id_a, the interpretation of r_a, and $d(a)$ as required.

Fix a, b and consider the fiber of $A^{\rightarrow} \to A \times A$. In the case of simplicial sets, this is how we defined $\mathrm{Map}(a, b)$. That fiber is precisely $I_A(a, b)$. To invalidate UIP we seek $p \neq q : x =_A y$. So it suffices to observe that there exist groupoids with *non-trivial* $A^{\rightarrow}$*- automorphisms of an arrow* $x \to y$. In a set-theoretic interpretation these objects are collected in a set which, being discrete, doesn't have non-identity automorphisms.[24] However, following Theorem 5.1 in [HS02], if we take the groupoid $\mathbb{Z}_2$ with a single object $\star$ and two *distinct morphisms* $\mathrm{id}_\star, p$, then UIP would produce a term of $I(I(A, \star, \star), p, \mathrm{id}_\star)$, which is impossible since $\mathrm{id}_\star \neq p$.

One must moreover note that this groupoid model satisfies a "higher" version of UIP. Indeed, even in a groupoid $\mathcal{G}$ we have a set of morphisms between two $f, g \in \mathcal{G}^{\rightarrow}$. Ultimately this brings us back to the same situation just one level higher. Thus UIP is a 1-*truncation principle* and groupoids are 1-truncated objects while sets are 0-truncated.

Of course, groupoids are tightly linked with the homotopy theory of topological spaces, via the fundamental groupoid of a topological space $\Pi_1(X)$. This was the first clear indication of a link between MLDTT and models of a homotopical higher flavour.

5.3 Homotopical models

The next substantial advance was made by Awodey and Warren in [AW09]. The key insight here is that the diagram corresponding to the semantics of *path induction* can be recast as a lifting property like one of the weak factorisation systems and model categories. More specifically, if we interpret MLDTT in a *contextual category* path induction takes the form seen below:

$$\begin{tikzcd}
{[a : A]} \arrow[r, "{[x,y,p,d(a)]}"] \arrow[d, tail, "{[a,a,r_a]}"] & {[x, y : A, p : x =_A y, z : P(x,y,p)]} \arrow[d, two heads] \\
{[x, y : A, p : x =_A y]} \arrow[r, no head, "\mathrm{id}"'] \arrow[ur, dashed, "{[x,y,p,J_d(x,y,p)]}"] & {[x, y : A, p : x =_A y]}
\end{tikzcd}$$

The contextual category has contexts for objects and *context morphisms* between them. Context morphisms consist of lists of terms definable in the domain context and belonging in the types of the codomain context. Objects are contexts but we'd like to work with types. Maps as the one on the right in the diagram above serve to pick out the types again. In this instance the map $\Delta, P \twoheadrightarrow \Delta = [x, y : A, p : x =_A y]$ picks out that type P out of the context Δ. In general, such a context morphism or *display map* $\pi_A : \Gamma.A \to \Gamma$ simply projects out the

[24]The presence of non-trivial automorphisms is precisely the same obstruction that limited what arrows we could classify in an elementary 1-topos. Its removal is the common cause for the invalidation of UIP and the object classifiers we get in an ∞-topos.

variables of Γ. So we think of the display map on the right as defining a dependent family $z : P(x, y, p)$ with respect to the data of the context $[x, y : A, p : x =_A y]$.

The other vertical map picks out the reflexivity term $[a, a, r_a]$. By commutativity of the square, the non-trivial horizontal map specifies an element $d(a) : P(a, a, r_a)$. According to path induction that is sufficient data to obtain a judgment of the form $[x, y : A, p : x =_A y] \vdash J_d(x, y, p) : P(x, y, p)$, equivalently specifying a section of the display map on the right. Commutativity of the top-left triangle, whose last component is $J_d(x, y, p) \circ r_a = d(a)$, is exactly the computation rule, namely that if we evaluate J on r_a we'll retrieve d_a.

So display maps are meant to distinguish types out of contexts. Since they appear on the right of these diagrams we expect types to be related to fibrant objects/fibrations. Indeed, the authors postulate that $\vdash A : \mathrm{Type}$ corresponds to a fibrant object and $x : A \vdash B(x)$ to a fibration $f : B \twoheadrightarrow A$. We'd like to interpret terms of a type as its *elements*. But if the type is captured by an entire map $A \twoheadrightarrow *$ or $f : B \twoheadrightarrow A$, we should take elements in a slice category. The terminal object of the slice is the identity and slice morphisms out of the identity correspond to *sections*. Thus, terms of a type correspond to sections of the corresponding fibration.

Lastly, the identity type has a canonical interpretation as a path object for A. Recall, that this is a canonical construction from the axioms of a model category, or even a WFS, with at least binary products. Take $A \in ob\mathcal{M}$. Form the diagonal map $\Delta : A \to A \times A$. According to the axioms of the WFS, this admits a factorisation:

$$\begin{tikzcd} & PA \arrow[dr, two heads] & \\ A \arrow[ur, tail, "\sim"] \arrow[rr, "\delta_A"'] & & A \times A \end{tikzcd}$$

We interpret the map $A \rightarrowtail A \times A$ as $a \mapsto r_a$ and the fibration $PA \twoheadrightarrow A \times A$, the free path space fibration, as $\mathrm{Id}_A(a, b)$ over (a, b). Postcomposing with the projections we can even obtain source and target maps $(s, t) : PA \rightrightarrows A$. This ensures us that we can indeed think of identifications $p : x =_A y$ as paths between x, y. More on that in the next section.

These ideas were pushed further in a similar vein. A notable such instance is [BG11], where the authors, Benno van den Berg and Richard Garner, axiomatise the existence of path objects in what they call a *path object category* and observe that such a structure exists in topological spaces, groupoids, chain complexes and simplicial sets. This is the first instance of truly "exotic" models of identity types. Another important construction was given by Gambino and Garner in [GG08] who produced a weak factorisation system on the *classifying category*, or the *category of contexts* of a type theory.

5.4 Higher structures in type theory

Through Hofman and Streicher's groupoid interpretation, we saw that UIP, the strictest truncation principle, is invalidated and thus there should be *some higher data present* over identity types. This suspicion was strengthened via Awodey and Warren's homotopical models. Now that we've established the presence of higher data the next natural question is their quantity and internal structure. The answer to the second question is more immediate.

For a moment we return to working with identity types internally in the type theory. Recall that given two terms $x, y : A$ in arbitrary A we can form a new type $\mathrm{Id}_A(x, y)$ whose terms serve as proofs of " $x =_A y$. We think of this bi-parametrised family of types as being freely generated by the reflexivity proofs $r_x : x =_A x$. This induces a rule governing how we form functions *out of* an identity type, path induction, which states that to define a function out of the identity type it suffices to provide the values of the function on the generators, the r_x's.

The remark regarding the identity type's internal structure is that when defining the identity type of an arbitrary A there were no assumptions placed on A. This in particular implies that we are free to iterate this construction and obtain $\mathrm{Id}_{\mathrm{Id}_A(x,y)}(p, q)$, the type of identifications $\alpha : p \Rightarrow q$, $\mathrm{Id}_{\mathrm{Id}_{\mathrm{Id}_A(x,y)}(p,q)}(\alpha, \beta)$ of identifications between $\chi : \alpha \Rrightarrow \beta$ and so on. Thus in principle, if we wanted to capture this data in a category we'd require n-arrows for all n.

The second observation is that one can prove the properties we list below. We note that in the following properties, the type A is arbitrary and we could therefore substitute $\mathrm{Id}_A(x, y)$ or $\mathrm{Id}_{\mathrm{Id}_A(x,y)}(p, q)$ if we'd like. This means that the *groupoidal laws* hold for the higher data. For a comprehensive account, we refer to [Rij22].

$$\mathrm{concat} : \prod_{x,y,z:A} (x = y) \to \Big((y = z \to (x = z)\Big)$$
$$\mathrm{inv} : \prod_{x,y:A} (x = y) \to (y = x)$$

It can be similarly shown that inv behaves as an inversion operation with respect to concatenation. Interestingly enough, concatenation is not associative on the nose. Instead one can only construct an identification, the *associator* $\alpha : (p \cdot q) \cdot r \Rightarrow p \cdot (q \cdot r)$.

Thus we obtain groupoidal laws for identity types. Moreover, the presence of the associator strongly brings to mind the $\Pi_\infty(X)$ of a topological space.

It becomes clearer and clearer that (higher) identifications over a type behave much like (higher) homotopies over a topological space.

It is exactly this line of reasoning followed by Benno van den Berg and Richard Garner in

[BG10]. Notably similar ideas were investigated by Warren [War08] and Lumsdaine [Lum10] in their respective PhD theses.

The main result of [BG10] can be informally stated as: ...if $\mathbb{T}$ is a dependently typed calculus admitting [identity types], then each type A therein gives rise to a weak ω-groupoid whose objects are elements of A, and whose higher cells are elements of the iterated identity types on A".

Unfortunately, the authors make use of a different notion of a higher category than we've been using and thus a more in-depth look at their work would extend outside the scope of this thesis. Van den Berg and Garner argue that Batanin's [Bat98] notion of a higher category matches the type theory most closely because of its essentially algebraic nature. It is important to note that as the main theorem of Cisinski's [Cis06] asserts, one can retrieve the full homotopy type of $X \in \mathrm{Top}$ from its associated Batanin ∞-groupoid.

5.5 The simplicial model

A truly landmark moment in the development of the semantics of HoTT was Voevodsky's model in the model category $\mathrm{sSet}_{\mathrm{KQ}}$.

His contribution was great mainly, but not exclusively, for the following reasons. His method of universes provided semantics for the hitherto uninterpreted internal universes of type theory. In doing so he also provided a general solution to the *coherence problem* mentioned above. Lastly, but perhaps *most* importantly, in constructing this model he observed an equivalence between the identity type of the universe and "an object of internal equalities". Type theoretically this equivalence takes the form $(A =_{\mathcal{U}} B) \simeq (A \simeq B)$. Voevodsky proposed to add this as an axiom to MLDTT. Philosophically this produces a system that captures the mathematical practice of identifying isomorphic objects. Moreover, it has very pleasant type-theoretic consequences. MLDTT with the addition of such an axiom is termed *Univalent Foundations*.

Now as has been greatly elaborated on in this thesis, $\mathrm{sSet}_{\mathrm{KQ}}$ plays an integral role in the theory of ∞-categories and it presents the archetypal ∞ topos of spaces $\mathcal{S} = \mathfrak{N}(\mathrm{sSet}_{\mathrm{KQ}}^{\circ})$. In this section, we look a bit deeper into the constructions that go into establishing this model. Thus we not only provide an example for the considerations of the previous subsections but also get the chance to look into some of the constructions that Shulman generalises to obtain "a model of HoTT in any ∞-topos". This exposition is largely based on [KL18] and [Rie].

First and foremost we record that types are interpreted as Kan complexes, the fibrant objects of $\mathrm{sSet}_{\mathrm{KQ}}$. Then, we interpret identity types. As discussed, the identity type is interpreted as a path object. In $\mathrm{sSet}_{\mathrm{KQ}}$ this takes a particularly clear form,

$$\begin{tikzcd}
 & A^{\Delta^1} = \mathrm{Fun}(\Delta^1, A) \arrow[dr, two heads, "{(s,t)}"] & \\
A \arrow[ur, tail, "\mathrm{refl}\sim"] \arrow[rr, "\delta_A"'] & & A \times A
\end{tikzcd}$$

Where s, t stand for evaluation at $0, 1$ and are interpreted as source and target maps. The path object captures the entire type family so if we want to produce the identity type for two specific terms, $x =_A y$ we take the fiber of $A^{\Delta^1} \twoheadrightarrow A \times A$ against $\Delta^0 \overset{(x,y)}{\to} A \times A$. By pullback stability of Kan fibrations that fiber is too a Kan complex and is thus "allowed" to capture a type.

To interpret path induction we exploit this correspondence of lifting problems.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail, "r\sim"] & E \arrow[d, two heads, "p"] \\
A^{\Delta^1} \arrow[r, "f"'] & B
\end{tikzcd}
\quad \rightsquigarrow \quad
\begin{tikzcd}
A \arrow[r, "d"] \arrow[d] \arrow[rr, bend left] & P \arrow[r] \arrow[d, two heads] & E \arrow[d, "p"'] \\
A^{\Delta^1} \arrow[r, no head, "id"'] \arrow[ur, dashed, "J"] & A^{\Delta^1} \arrow[r, "f"'] & B
\end{tikzcd}$$

Recall that a type family depending on $p : x =_A y$ is given by a fibration $P \twoheadrightarrow A^{\Delta^1}$ and a term thereof by a section of this fibration.

The heart of the matter is that (Proposition 1.1.2 [Rie]) to define such a section, it suffices to define a partial section over the subspace $A \rightarrowtail A^{\Delta^1}$ of loops.

$$\begin{tikzcd}
A \arrow[r, "d"] \arrow[d, tail, "\sim r"'] & P \arrow[d, two heads] \\
A^{\Delta^1} \arrow[r, no head, "id"'] \arrow[ur, dashed, "J"] & A^{\Delta^1}
\end{tikzcd}$$

sSet being a presheaf topos is in particular locally cartesian closed and thus comes equipped with for any $f : \Delta \to \Gamma$, a triple adjunction $\Sigma_f \dashv f^* \dashv \Pi_f$.

$$\begin{tikzcd}
\mathrm{sSet}_{/\Delta} \arrow[r, bend right, "\Pi_f"'] & \mathrm{sSet}_{/\Gamma} \arrow[l, bend right, "\Sigma_f"'] \arrow[l, "f^*"]
\end{tikzcd}$$

We note that all the following constructions can be adapted to work over arbitrary contexts. Σ and Π types are interpreted via this triple adjunction. Let A be a type, so $a : A \twoheadrightarrow *$. A dependent family over A, $x : A \vdash B(x)$, is given by $p : B \twoheadrightarrow A$. Then, the interpretation of $\Sigma_{a:A} B(a)$ is given by $\Sigma_a(p : B \twoheadrightarrow A) = B \twoheadrightarrow A \twoheadrightarrow *$. This is a fibration over a point and thus can indeed be thought of as the interpretation of a type.

So, we'd like to think of Σ Π as *operators* that take fibrations to Kan complexes. Thus it becomes crucial that they *preserve fibrations*. That's exactly the content of Lemma 1.3.2. in [Rie]. The statement for Σ is immediate since fibrations are closed under composition. The fact that Π_f preserves fibrations depend on the ambient model category being *right proper*.

The proof is by a general fact concerning adjoints in model categories, see proposition 2.7 of this nLab article. A left adjoint preserves trivial cofibrations exactly when the right adjoint preserves fibrations. Here, in $\mathrm{sSet}_{\mathrm{KQ}}$ cofibrations are monomorphisms and thus are always pullback stable, and, by right properness, weak equivalences are stable under pullback against a fibration. Thus, when f is a fibration, f^* preserves trivial cofibrations.

Under these interpretations, one obtains a lovely pictorial interpretation of the constructive quantifiers. For a type in empty context, $A \twoheadrightarrow *$, Σ_A produces the total space of the fibration and Π_A the space of sections.

5.5.1 Univalence in simplicial sets

As discussed in the introduction univalence weakly identifies two different notions of sameness present in HoTT. The first is about identifications in the universe, a type of all (small) types. The second is a notion of equivalence of types one can develop internally. To explain the latter we begin with the notion of contractible types which are defined as $\mathrm{isContr}(A) := \sum_{c:A} \prod_{x:A} c =_A x$. The interpretation of the internal type-theoretic contractibility matches well the one present in $\mathrm{sSet}_{\mathrm{KQ}}$. A type in context is given by a fibration $p : A \twoheadrightarrow \Gamma$. This is a trivial fibration exactly when $\mathrm{isContr}(p)$ has a section (namely the corresponding type is inhabited)(Lemma 1.3.7 in [Rie])

Exploiting this we can produce a notion of equivalence of types. Again internally to type theory, we call a function an equivalence when we can inhabit $\mathrm{isEquiv}(f) := \prod_{y:B} \mathrm{isContrfib}_f(y)$, where $\mathrm{fib}_f(y) := \sum_{x:A} fx = y$. Again, this notion matches well that of $\mathrm{sSet}_{\mathrm{KQ}}$, (Lemma 1.3.11) A map between fibrations over Γ is a weak equivalence if and only if $\mathrm{isEquiv}(f) \in {}^{\mathrm{sSet}}\!/_{\Gamma}$ has a section.

Now we may turn our attention to universes, the heart of Voevodsky's construction and univalence. The overarching outline for Voevodsky's work we review here is as follows: Declare a model of type theory to be a "contextual category" $\mathcal{C}$, an essentially algebraic structure. We thus sidestep much of the cumbersome syntactical bookkeeping. Observe it has a "coherence problem", namely substitution in $\mathbb{T}$ is stricter than its interpretation in $\mathcal{C}$. Show that if all morphisms of interest are taken as "chosen pullbacks" of a specific morphism you can both solve coherence and obtain internal universes for the modelled type theory. Construct such a morphism in $\mathrm{sSet}_{\mathrm{KQ}}$.

For reasons similar to why the object classifier of ∞-topoi could not classify all morphisms at once, it is unreasonable to hope to simultaneously classify all Kan fibrations. Instead, we introduce a cardinality-based stratification. Let α be a regular cardinal. Voevodsky constructs a *weakly* universal Kan fibration, $p_a : \tilde{\mathcal{U}}_\alpha \to \mathcal{U}_\alpha$. This universe will classify α-small fibrations, fibrations whose fibers always have cardinality $< \alpha$.

Again the presence of non-trivial automorphisms poses an issue. Here they are removed. To that effect, one uses AC to well-order all fibers. Then given two maps in the slice over X we only consider morphisms between them that are compatible with the base, *and the well ordering*. Since there exists at most one isomorphism between well-ordered sets, there exists at most one isomorphism between two objects in the slice.

Given $X \in$ sSet they define $\mathbb{U}_\alpha(X)$ to be the isomorphism classes of α-small-well-ordered fibrations $Y \to X$. To show this functor is representable Voevodsy exhibits it as a subfunctor, defined by a pullback, of $\mathbb{W}_\alpha(X)$ of *all* α-small-well-ordered morphisms $Y \to X$, which is independently shown to be representable in Lemma 2.1.5. With not much additional work, Lemma 2.1.11, they show that $\mathbb{U}_\alpha$ is representable too. Let $\mathscr{U}$ be the representing object of $\mathbb{U}_\alpha(-)$. Then, every α-small-well-ordered fibration $Y \to X$ corresponds to a $\ulcorner f \urcorner : X \to \mathscr{U}_\alpha$.

Recall that type theoretically, the univalence axiom asserts that "equality is equivalent to equivalence" or

$$(A =_{\mathcal{U}} B) \simeq (A \simeq B)$$

To be more precise, we don't merely assert that there exists an equivalence. Instead, we ask that canonical map id-to-equiv, which canonically produces an equivalence out of an identification and is in an equivalence, admits a homotopy inverse. For the contents of this section, we suppress the regular or inaccessible cardinal α from the notation and implicitly assume, as is common practice, that we fix a specific universe $\mathcal{U}_\alpha$ and work internally therein.

The goal is direct. Write the statement above in the *internal language* of $\mathrm{sSet}_{\mathrm{KQ}}$ and show id-to-equiv is a weak equivalence. The left-hand side is well understood. We know that path objects are of the form $\mathcal{U}^{\Delta^1} = \mathrm{Fun}(\Delta^1, \mathcal{U})$ and we take a fiber over $\Delta^0 \overset{(A,B)}{\to} \mathcal{U} \times \mathcal{U}$ to obtain $\mathrm{Map}(A, B)$. If A, B were definable in context Γ we'd form the pullback against $\Gamma \overset{(A,B)}{\to} \mathcal{U} \times \mathcal{U}$.

An internalisation of $(A \simeq B)$ is trickier. As before we have a pair of types in context Γ captured by a generalised element $\Gamma \overset{(A,B)}{\to} \mathcal{U} \times \mathcal{U}$.

$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d, two heads] & \tilde{\mathcal{U}} \times \mathcal{U} \arrow[r] \arrow[d, "\pi_i^*(u)"'] & \tilde{\mathcal{U}} \arrow[d, two heads, "u"] \\
\Gamma \arrow[r, "{(A,B)}"] \arrow[rr, bend right, "\bullet"'] & \mathcal{U} \times \mathcal{U} \arrow[r, "\pi_i"'] & \mathcal{U}
\end{tikzcd}$$

We can thus see $\pi_i(u)$ as capturing the data of the i-th Kan complex/type out of a pair of types (A, B). We denote this pullback by $\mathscr{U}_i$ to simplify the notation.

Observe that sSet being locally cartesian closed, means all slice categories admit a right adjoint to their product functor, and thus have an exponential object. Thus we can for the internal hom in the slice over $\mathcal{U} \times \mathcal{U}$, $\mathrm{Map}_{\mathcal{U}\times\mathcal{U}}(\mathscr{U}_1, \mathscr{U}_2)$. Using the adjunction we can characterise its simplices. Take a map $\Delta^n \to \mathrm{Map}_B(E_1, E_2)$ transposing and making use of the universal property of the pullback we find that it corresponds to a choice of a simplex $b : \Delta^n \to B$ and a map $u : b^*E_1 \to b^*E_2$. A crucial ingredient is *the local character of Kan complexes.* Many properties of Kan complexes-Kan fibrations can be detected *locally*, in fibers over simplices. For instance,

Lemma 221. (Lemma 3.2.4 in [KLV12]) Let $f : E_1 \to E_2$ be a morphism over a base B. If for all n-simplices b the induced map $f_b : b^*E_1 \to b^*E_2$ is a weak equivalence then so is f.

Exploiting this local character thus we can define a simplicial subset $\mathrm{Eq}_B(E_1, E_2)$ to consist of those simplices where in the explicit description above the map $u : b^*E_1 \xrightarrow{\sim} b^*E_2$ is a weak equivalence. In our case of interest we can define $\mathrm{Eq}(\mathcal{U}) := \mathrm{Eq}_{\mathcal{U}\times\mathcal{U}}(\mathscr{U}_1, \mathscr{U}_2)$, and the explicit description becomes,

$$\mathrm{Eq}(\mathcal{U})_n \cong (b_1, b_2 \in \mathcal{U}_n, w : b_1^*\tilde{\mathcal{U}} \xrightarrow{\sim} b_2^*\tilde{\mathcal{U}})$$

Then, we can immediately define id-to-equiv $:= \delta_{\tilde{\mathcal{U}}} : \tilde{\mathcal{U}} \to \mathrm{Eq}(\tilde{\mathcal{U}})$ by $u \in \tilde{\mathcal{U}}_n \mapsto (u, u, \mathrm{id}_{u^*\tilde{\mathcal{U}}})$

Definition 222. A fibration is univalent exactly when $\delta_{\tilde{\mathcal{U}}}$ is a weak equivalence.

Whenever that is the case, since δ is additionally a monomorphism, we obtain a factorisation of the diagonal in a trivial cofibration followed by fibration, thus exhibiting $\mathrm{Eq}(\tilde{\mathcal{U}})$ as a path object for $\mathcal{U}$.

$$\begin{tikzcd}
\mathcal{U} \arrow[rr, "\delta"] \arrow[dr, tail, "\Delta\sim"'] & & \mathcal{U} \times \mathcal{U} \\
& \mathrm{Eq}(\tilde{\mathcal{U}}) \arrow[ur, two heads] &
\end{tikzcd}$$

We close this section off, by adding a particularly concise explanation by Shulman, which appears in p.84 of [Shu15]:

> The univalence axiom, when interpreted in a model category, is a statement about a "universe object" $\mathcal{U}$, which is fibrant and comes equipped with a fibration $\pi : \tilde{\mathcal{U}} \twoheadrightarrow \mathcal{U}$, that is generic, in the sense that any fibration with "small fibers" is a pullback of π. In homotopy theory, it would be natural to ask for the stronger property that $\mathcal{U}$ is a classifying space for small

fibrations, i.e. that homotopy classes of maps $\Gamma \to \mathcal{U}$ are in bijection with (rather than merely surjecting onto) equivalence classes of small fibrations over A. The univalence axiom is a further strengthening of this: it says that the path space of $\mathcal{U}$ is equivalent to the "universal space of equivalences" between fibers of π... In particular, therefore, if two pullbacks of π are equivalent, then their classifying maps are homotopic.

Diagrammatically,

$$\begin{tikzcd}
A \arrow[r, squiggly, "\simeq"] \arrow[rr, bend right=10] \arrow[dr] & B \arrow[r] \arrow[d] & \tilde{\mathcal{U}} \arrow[d, "p"] \\
& \Gamma \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[r, phantom, "\wr"] & \mathcal{U}
\end{tikzcd}$$

A generalised element as on the left below captures an equivalence of types $A \simeq B$, which under univalence corresponds to a morphism in $\mathcal{U}^{\Delta^1}$ between their classifying maps a, b. Furthermore, an equivalence in $\mathrm{sSet}_{\mathrm{KQ}}$ also associates the "homotopy coherent higher data" of each Kan complex.

$$\begin{tikzcd}
& \mathrm{Eq}(\tilde{\mathcal{U}}) \arrow[d, two heads] \arrow[r, leftrightsquigarrow] & a \arrow[r, "\varepsilon"] & b \in \mathcal{U}^{\Delta^1} \\
\Gamma \arrow[r, "{(A,B)}"] \arrow[ur, dashed, "e"] & \mathcal{U} \times \mathcal{U}
\end{tikzcd}$$

In [KLV12] a substantial amount of work goes towards showing that the fibration $u : \tilde{\mathcal{U}} \twoheadrightarrow \mathcal{U}$ is indeed univalent.

5.6 Local universes method

One of Voevodsky's motivations in developing his universes was to obtain models for the internal universes of the type theory, but also to solve the *coherence problem*, an asymmetry in the *strictness* of certain type-theoretic operations in comparison to their interpretation.

Voevodsky made use of *contextual categories* as his notion of an abstract model of type theory. They are defined with an axiomatisation that abstracts out from the structure present in the *category of contexts* of a type theory. Substitution is interpreted by pullback.

Now, all induction principles, like path induction mentioned above, naturally come with *computation rules*. These computation rules assert something well-known to anyone having done elementary undergraduate mathematics. To define a function $f : \mathbb{N} \to \mathbb{N}$ by induction it suffices to supply the data $f(0) = a$ and $f(S(n)) = g(n)$ for some g. Those pieces of data produce a unique function $\tilde{f}$. It should definitely be the case that $\tilde{f}(0) = a$. Similarly, if we use the datum $d_a : D(a, a, r_a)$ to produce a $J(x, y, p) : D(x, y, p)$ and then evaluate it at a we should find d_a. In type theory, we assert that this is the case *up to definitional equality*. Since most interesting functions are defined via these methods it is quite common to have substitutions into a certain function satisfying an "equation" up to definitional equality. Similarly, many type-theoretic operations declare *strict stability* under substitution.

The problem stems from the fact that in contextual categories one interprets substitution via an appropriate pullback operation, which is pseudofunctorial, meaning its values are defined only up to isomorphism. Yet if we want to model the type theory faithfully, we would like pullback to be strictly functorial and to strictly preserve all the logical structure.

Voevodsky solves the coherence problem using his universes by *choosing* a distinguished pullback out of each isomorphism class. So, to *categorical* models of a formal system for *constructive* reasoning, we had to *choose* pullbacks, in order to make certain constructions *evil*[25].

As Lusmdaine and Warren mention in the introduction to their [LW15],

> The present work arose from a careful reading of Voevodsky's model in simplicial sets [KLV12]. Universes are used there for two distinct purposes: Firstly to obtain coherence of the model, and secondly to become type-theoretic universes within the model. It turned out that not only may the two aspects be entirely disentangled, but moreover, the coherence construction may be modified to work without a universe.

The following are based on the introduction to [LW15]. In their paper, the authors use *yet another* notion of an abstract model of type theory, a *comprehension category*. The key ingredient of the definition is a *cloven Grothendieck fibration* of categories $P : \mathcal{T} \to \mathcal{C}$ meant to capture the idea of types being fibered over contexts. The fiber over a given context $\Gamma \in \mathcal{C}$ is all the types $A \in \mathcal{T}$ definable in context Γ. A split comprehension category is one where this fibration is split. A split fibration is, unsurprisingly, one where one has a strictly functorial interaction of the fibration with pullbacks, see [Str23]. Split comprehension categories are models of an essentially algebraic theory. The crux of [LW15] is to provide a canonical way to turn a comprehension category with weak stability into a split one, capturing the strict honest and faithful model of the type theory.

Now given a comprehension category $\mathcal{C}$ they manage to replace it with an equivalent but split $\mathcal{C}_!$. Certain additional structure is required to lift all the logical constructors present in $\mathcal{C}$, if any, to $\mathcal{C}_!$. The point is that this structure is implied by local cartesian closure, which in particular is present in all homotopical models of interest.

[25] *Evil* refers to categorical constructions not invariant under equivalence of categories. They're called so because stability under equivalence of categories is a central point in the philosophy of category theory.

The central definition is then,

Definition 223. (Definition 4.2.1 in [LW15]) A **logical weak factorisation system** on $\mathcal{E}$ is a WFS such that:

(*1*) fibrations are exponentiable, namely for $p : X \twoheadrightarrow Y$ the pullback functor p^* has a right adjoint.

(*2*) left maps are preserved by pullback along fibrations.

(*3*) any left map i between fibrations over a common base Γ remains a left map after having pulled back against any $f : \Gamma' \to \Gamma$. Such a WFS is called **semi-logical** if the bases of all fibrations are additionally assumed to be fibrant objects.

Their main theorem is:

Theorem 224. Let $\mathcal{E}$ be finitely complete, with stable finite coproducts, equipped with a WFS. If the WFS is semi-logical, then $(\mathcal{E}_f)_!$ models type theory with Π, Σ, unit, Id- and finite sum-types. If the WFS is logical so does $\mathcal{E}_!$.

And the central examples we are interested in are: (Examples 4.2.5 in [LW15])

(*1*) Any right proper Cisinski model category

(*2*) $\mathcal{M}$ is cofibrantly generated and $\mathcal{J}$ is arbitrary then both the projective and injective model structures on $\mathcal{M}^{\mathcal{J}}$ are logical.

5.7 Joyal's conjecture

The past few sections have achieved two important goals. First, we've established a robust connection between models of HoTT and higher categories/∞-groupoids. In addition to that we've seen how these structures don't directly model HoTT but must be further strictified. The local universes method provides a robust method of doing so that covers all the examples of interest of this thesis.

Thus we can finally begin to look deeper into exactly *which* ∞-categories can support a model of HoTT or, equivalently, what kind of structure must be present in an ∞-category to model some type of theoretic phenomena. The first result of this kind we present, and indeed one of the main results that contribute to "the relation between models of HoTT and ∞-categories", the topic of this thesis, is Joyal's conjecture.

Joyal's Conjecture asserts, in a mathematically precise way, that Martin–Löf dependent type theory, with Σ, Π and identity types, gives rise to locally cartesian closed ∞-category. This is the main result established in Kapulkin's PhD thesis. The central theorem that establishes this result is:

Theorem 225. (Theorem 9.2.8 in [Kap14]) If $\mathcal{M}$ is a locally cartesian closed fibration category then $N_f(\mathcal{M})$ is a locally cartesian closed ∞-category.

Thus to settle Joyal's Conjecture it suffices to show that if $\mathbb{T}$ has Σ, Π-types that satisfy FunExt, and identity types, then $\mathrm{Cl}(\mathbb{T})$ is a locally cartesian closed fibration category. First, we identify the fibrations and weak equivalences that provide the "fibration category" structure. The homotopical structure of $\mathrm{Cl}(\mathbb{T})$ is clearer at the level of types instead of that of general contexts. It turns out that if we assume the η-rule[26] for Σ-types we can show any context to be equivalent to a context of length 1. This can be done by iterating the following equivalence:

$$[x : A, b : B(x)] \simeq [p : \sum_{x:A} B(x)]$$

The η rule implies that the maps $<,>: [x : A, y : B(x)] \to [p : \sum_{x:A} B(x)]$ and that $\pi_1, \pi_2 : [p : \sum_{x:A} B(x)] \to [x : A, b : B(x)]$ are inverses of each other. So Σ types can be thought of as internalising contexts, and any context can be equivalently given as an iterated Σ type. If we restrict our attention to contexts of length 1, types, then a context morphism is just a function type. We declare such a morphism to be a weak equivalence exactly when it is a weak equivalence in the type-theoretic sense when all of its homotopy fibers are contractible types if the type $\mathrm{isEquiv}(f)$ is inhabited. One can easily show that the resulting $\mathcal{W}$ defines a wide subcategory that contains all identities.

In addition to that, we define $\mathcal{F}$ to be the canonical projections p_Γ or maps isomorphic to them in the arrow category. We note that maps of the form π_Γ were defined as $\Gamma.A \to \Gamma$ or in the case of a dependant family $\Gamma.A.B \to \Gamma$. If we internalise as above they become projections maps out of the Σ type, $\pi : \sum_{x:A} B(x) \to A$. Then, internally to HoTT one can show that $\mathrm{fib}_\pi(a) \simeq B(a)$.

With these definitions the verification of all the axioms that make $(\mathrm{Cl}(\mathbb{T}), \mathcal{W}, \mathcal{F})$ in a fibration category is essentially an elementary verification. It is given both in section 9 of [Kap14] and section 3 of [AKL15]. One point worth noting is that the fibration structure on $\mathrm{Cl}(\mathbb{T})$ is not part of a model structure, see 9.3.12 in [Kap14]. It is important to note that all objects are cofibrant (Lemma 3.2.14 [AKL15])

Theorem 226. If $\mathbb{T}$ is a type theory with Σ, Π, identity types that satisfies function extensionality, then $\mathrm{Cl}(\mathbb{T})$ is a locally cartesian closed fibration category.

[26]If c is a term of the sum-type then the η-rule asserts that if we pair the two projections applied to c we retrieve c up to definitional equality.

Hence if one takes such a type theory $\mathbb{T}$ one can draw the diagram below which also appears in the introduction to Stenzel's PhD thesis [Ste19].

$$\begin{tikzcd} & \mathrm{Cl}(\mathbb{T}) \arrow[dr] & \\ \mathbb{T} \arrow[ur] \arrow[rr, dashed] & & N_f(\mathrm{Cl}(\mathbb{T})) \approx Ho_\infty(\mathrm{Cl}(\mathbb{T})) \end{tikzcd}$$

It is important to remark on the direction of the arrows. In this construction, we start from a type theory obeying certain rules. We obtain its category of contexts, an essentially algebraic model for the type theory. Then, internal considerations of the type theory induce the structure of a fibration category on $\mathrm{Cl}(\mathbb{T})$. Lastly, we consider its homotopy ∞-category, or more accurately we obtain a model for its homotopy type, via Kapulkin's ∞-category of frames. Kapulkin then shows that the latter is a locally cartesian closed ∞-category.

In the next two sections we record two results "in the opposite direction".

5.8 Locally cartesian closed ∞-categories and HoTT+FunExt

In this section, we record a partial inverse result to Joyal's conjecture. This comes as the combination of two important results.

Theorem 227. Any presentable and locally cartesian closed ∞-category admits a model categorical presentation by a right proper Cisinski model category.

Theorem 228. (Lemma 5.9 in [SHU14]) Function extensionality holds in the internal type theory of a type-theoretic fibration category if and only if dependent products along fibrations preserve trivial fibrations.

Intuitively, this property is important because it makes $g^* \dashv \Pi_g$ into a Quillen pair and thus the adjunction lifts at the level of homotopy ∞-categories. But g being a fibration is the same as having precomposed with a fibrant replacement thus, in reality, this is the homotopy limit. Therefore it presents the ∞-pullback. Hence, it is the ∞-pullback that has a right adjoint making $\mathscr{C}$ a locally cartesian closed ∞-category.

We saw in subsection 5.5 how that property relates to right proneness. Thus the requirement above is satisfied in all right proper Cisinski model categories. Thus, combining the two we reach another of the central results we aimed to present with this thesis.

Theorem 229. Every presentable and locally cartesian closed ∞-category interprets HoTT+FunExt.

Again we emphasise the "direction" of the construction. Here we start with a presentable locally cartesian closed ∞-category, obtain a model categorical presentation and then assert that model category models HoTT+ Funext.

5.9 ∞-topoi, type theoretic model topoi and univalent HoTT

HoTT/UF refers to type theories with the ordinary Σ, Π and id-types, and the addition of the intriguing higher inductive types that allow one to do synthetic homotopy theory. Lastly, we add the Univalence axiom to obtain UF. It didn't take too long to establish interpretations for all these types in any presentable locally cartesian closed ∞-category.

What remained elusive was the univalence axiom. The analogy between the ∞-topos object classifiers and universes in a model category is clear. Rezk (Theorem 6.1.6.8 [Lur09]) caracterised ∞-topoi as precisely the presentable ∞-categories $\mathfrak{X}$ with universal colimits[27] and such that for all sufficiently large cardinals κ, the class of relatively κ compact morphisms in $\mathfrak{X}$ has a classifying object. Given this characterisation, it was expected that model categories with univalent universes for κ-small fibrations should present ∞-topoi.

Slowly but surely, mainly Shulman with notable contributions from Cisinksi and Lumsdaine, achieved univalence in some special cases of $(\infty, 1)$-topoi. The general result was established by Shulman in [Shu19], where he showed the second "main theorem" of this thesis

Theorem 230. (Theorem 11.1 in [Shu19]) Every Grothendieck ∞-topos can be presented by a Type Theoretic Model Topos.

This section is devoted to looking into what *Type Theoretic Model Topoi* are and how one achieves to present any ∞-topos with them.

Definition 231. A **Type Theoretic Model Topos** is a model category $\mathscr{E}$ satisfying the following:

(*1*) The underlying category is a Grothendieck 1-Topos.

(*2*) The model category is right proper and its cofibrations are precisely the monomorphisms, making it left proper as well.

(*3*) It is combinatorial, meaning it is locally presentable as a category and cofibrantly generated as a model category. This means that both the objects of $\mathscr{E}$ and its model structure can be obtained from a small set of generators under "gluings".

[27] Recall that in the presence of presentability universal colimits are equivalent to local cartesian closure.

(4) $\mathscr{E}$ is a simplicial model category. This ensures compatibility of the simplicial enrichment and the model structure.

(5) It is also simplicially locally cartesian closed, meaning the simplicial enrichment interacts nicely with the cartesian closure of the Grothendieck 1-topos, specifically the right adjoint of the pullback functor is simplicially enriched.

(6) There is a locally representable and relatively acyclic notion of fibred structure $\mathbb{F}$ on $\mathscr{E}$ such that $|\mathbb{F}|$ is the class of all fibrations.

We place 3 structures on the $\mathscr{E}$. First, a topos structure, thus ensuring local cartesian closure, which is used to model Σ and Π types. Moreover, we ask that cofibrations are precisely the monomorphisms which are known to enjoy particularly nice properties inside a topos. Thirdly, we ask for a model structure. That is necessary for interpreting identity types. Fourthly, we ask for a simplicial enrichment. This axiom is designed to get us closer to "higher structures". In a way, having asked that the model structure is combinatorial, by Dugger's theorem[28], we have already implicitly asked for a simplicial enrichment.

In addition to these, there are axioms that ensure that all three structures are compatible with one another, in the following way.

$$\begin{tikzcd}
 & \text{simplicial enrichment} \arrow[dl, dash, "\text{simpl. lcc}"'] \arrow[dr, dash, "\text{simpl. model category}"] & \\
\text{topos structure} \arrow[rr, dash, "\text{right proper}"] & & \text{model structure}
\end{tikzcd}$$

We saw that right properness came into showing that Π_f preserved fibrations/fibrant objects. This has the type-theoretic interpretation of Π-types *outputing* types.

What remains mysterious is the 6$^{\text{th}}$ axiom above, which will guarantee univalent universes. Shulman asks for a "locally representable and relatively acyclic notion of fibred structure $\mathbb{F}$ on $\mathscr{E}$ such that $|\mathbb{F}|$ is the class of all fibrations". Many of these notions reside in a bicategorical setting. For the appropriate background, we refer for example to [Bén67].

Definition 232. A notion of **fibred structure** on $\mathscr{E}$ is a discrete fibration $\phi : \mathbb{F} \to \mathbb{E}$ in $\mathcal{PSH}(\mathscr{E})$ with small fibers.

Above $\mathbb{E}$ stands for the **core of self-indexing** of $\mathscr{E}$, with a pseudofunctorial pullback action. Thus the fiber over any $f : A \to \Gamma$ is a small set. Any pullback square with vertical maps f, f' induces a map between these fibers.

[28] Recall Dugger's theorem asserts that any combinatorial model category is Quillen equivalent to a left Bousfield localization of simplicial presheaves. The latter are simplicially enriched.

$$\mathbb{E} : \mathscr{E}^{\mathrm{op}} \to \mathcal{GPD} : \Gamma \mapsto ({}^{\mathscr{E}} \diagup_{\Gamma})^{\cong}$$

The archetypal notion of fibred structure that we abstract out from in the axiomatisation is the restriction on fibrations over a given object Γ.

$$\mathbb{F} : \mathscr{F}^{\mathrm{op}} \to \mathcal{GPD} : \Gamma \mapsto ({}^{\mathscr{F}} \diagup_{\Gamma})^{\cong}$$

Of course, the type-theoretic interpretation here is clear. Fibers over a given Γ capture the collection of types over a context. For any regular cardinal κ we can consider the full subgroupoids consisting of relatively κ presentable morphisms. Recall they are stable under pullback and thus define a subfunctor $\mathbb{E}^{\kappa}$. Then we can also define $\mathbb{F}^{\kappa} := \mathbb{F} \times_{\mathbb{E}^{\kappa}} \mathbb{E}$.

Definition 233. A notion of fibred structure is **locally representable** if the discrete fibration is representable, namely for any $Z \in \mathscr{E}$ and a weak bicategorical pullback

$$\begin{tikzcd}
\mathbb{P} \arrow[r] \arrow[d] & \mathbb{F} \arrow[d] \\
\mathrm{Hom}_{\mathscr{E}}(-, Z) \arrow[r] & \mathbb{E}
\end{tikzcd}$$

We obtain that $\mathbb{P}$ is also represented by some $X \in \mathscr{E}$.

Crucially this generalises the *local character of Kan complexes*, see example 3.16 [Shu19]. For the notion of fibred structure of fibrations in a presheaf category, local presentability is equivalent to the statement that a map is a fibration exactly when its pullbacks over representables, namely its fibers, are fibrations.

We proceed with a discussion of Shulman's notion of universes and how it is obtained. Ideally, as in simplicial sets, a universe U in $\mathscr{E}$ for κ-small fibrations would be a representing object for $\mathbb{F}ib^{\kappa}$. Shulman replaces it with a "weakly equivalent" one, a sort of "cofibrant replacement", that enjoys better properties. Since $\mathcal{PSH}(\mathscr{E})$ does not have a model structure, one lifts some definitions via the Yoneda embedding $\mathscr{E} \to \mathcal{PSH}(\mathscr{E})$. So, for example we call $\mathbb{X} \to \mathbb{Y}$ an **acyclic fibration** if it has the right lifting property against all $\mathrm{Hom}_{\mathscr{E}}(-, j) : \mathrm{Hom}_{\mathscr{E}}(-, A) \to \mathrm{Hom}_{\mathscr{E}}(-, B)$ for all $\mathscr{E}$-cofibrations $j : A \to B$.

In the special case that the morphism is additionally representable, we get a version of pullback stability for these acyclic cofibrations. Then, $\mathbb{X} \to \mathbb{Y}$ is an acyclic fibration in the sense above exactly when the induced $W \to Z$ is an acyclic fibration in $\mathscr{E}$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{E}}(-, W) \arrow[r] \arrow[d] & \mathbb{X} \arrow[d] \\
\mathrm{Hom}_{\mathscr{E}}(-, Z) \arrow[r] & \mathbb{Y}
\end{tikzcd}$$

Recall that with this last axiom for Type Theoretic Model Topoi, we are trying to obtain universes in appropriately structured model categories $\mathscr{E}$. Naively, we'd like a representing object of $\mathbb{F}$. Instead, we ask for a coifbrant object $\mathcal{U}$ and a trivial fibration $\mathrm{Hom}_{\mathscr{E}}(-, U) \overset{\sim}{\twoheadrightarrow} \mathbb{F}$ which we think of as a cofibrant replacement.

By the bicategorical Yoneda lemma, κ-small fibrations $p : E \twoheadrightarrow B$ are in bijective correspondence with pseudonatural transformations $\mathrm{Hom}_{\mathscr{E}}(-, B) \to \mathbb{F}^\kappa$. Similarly, we get such a p.n.t. corresponding to $\pi_{\mathcal{U}}$. We call it π.

We'd like to think of pullbacks against π_U as fibrations with κ-small fibers and hope that pi_U classifies all such fibrations. This amounts to π being surjective. Now if q is classified by π_U we think of q as having κ-small fibers. That means that whenever q arises as the pullback of some other fibration r then the fibers of r, or at least the fibers in the image of q inside r, are also κ-small since, by the pullback lemma, they are fibers of q as well.

This would make the fibers of r as small as those of q and thus we expect that r is classified by π_U as well. This is captured by the *realignement property*. It admits a concise statements in term of $p : \mathrm{Hom}(-, U) \to \mathbb{F}^\kappa$.

Definition 234. In a model category $\mathscr{E}$ with all objects cofibrant, a small fibration $\pi_U : \tilde{U} \to U$ is a **universal small fibration** if the pseudo-natural transformation $p : \mathrm{Hom}(-, U) \to \mathbb{F}^\kappa$ is an acyclic fibration, meaning that for any cofibration (i.e. mono) $i : A \rightarrowtail B$ the diagram below admits a lift.

$$\begin{tikzcd}
\mathrm{Hom}(-, A) \arrow[r, "h"] \arrow[d, "i"'] & \mathrm{Hom}(-, U) \arrow[d, "\pi"] \\
\mathrm{Hom}(-, B) \arrow[r, "p"'] \arrow[ur, dashed] & \mathbb{F}^\kappa
\end{tikzcd}$$

By the correspondence of small fibrations and pseudo-natural transformations into $\mathbb{F}^\kappa$, and the fact that these functors act by pullback, we can see that this amounts to:

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr] \arrow[d, two heads, "q"'] & & \tilde{U} \arrow[d] \\
A \arrow[rr] \arrow[dr, tail, "i"'] & Y \arrow[ur, dashed] \arrow[d, two heads, "p"] & U \\
 & B \arrow[ur, dashed] &
\end{tikzcd}$$

So that whenever q is simultaneously a pullback of the universe and some other fibration p then p itself must be a pullback of the universe.

Lastly, we connect the abstract notion of fibred structure with the actual class of fibrations via the following axiom.

Definition 235. Consider the image of $\phi : \mathbb{F} \to \mathbb{E}$. This gives a full[29] notion of fibered structure $|\mathbb{F}| \hookrightarrow \mathbb{E}$. We call $\mathbb{F}$ **relatively acyclic** when it induces an acyclic fibration $\mathbb{F} \overset{\sim}{\twoheadrightarrow} |\mathbb{F}|$, meaning that for any cofibration $i : A \rightarrowtail B$ we get a lift

$$\begin{tikzcd} \mathrm{Hom}_{\mathscr{E}}(-,A) \arrow[r] \arrow[d, "i_*"] & \mathbb{F} \arrow[d, two heads, "\sim"] \\ \mathrm{Hom}_{\mathscr{E}}(-,B) \arrow[r] \arrow[ur, dashed] & |\mathbb{F}| \end{tikzcd}$$

Then, Shulman proceeds to show that if the model category $\mathscr{E}$ is structured enough, as it certainly is for TTMT, one can produce a univalent universe.

He does that in multiple steps. In each step, he requires a piece of additional structure from the model category to achieve the next one. Firstly, he introduces the notion of a **stack of cell complexes**, groupoid valued pseudofunctors that interact well with certain colimits. In Lemma 5.7 he shows that all locally representable notions of fibred structure are also a stack for cell complexes. Then, in Theorem 5.9 with the extra assumption that $\mathscr{E}$ is combinatorial, he shows that one can adapt the small object argument in this setting and obtain, for arbitrary $f : \mathrm{Hom}_{\mathscr{E}}(-,A) \to \mathbb{X}$ a factorisation as a cofibration followed by a trivial fibration in $\mathcal{PSH}(\mathscr{E})$ as defined above.

Then, as Corollary 5.10, if $\mathscr{E}$ is a Grothendieck 1-topos, with a combinatorial model structure, with monomorphisms as cofibrations, then all locally representable notions of fibred structure $\mathbb{F}$, have a universe. He obtains that by using the previous result to factor the trivial map $\mathrm{Hom}_{\mathscr{E}}(-,\varnothing) \to \mathbb{F}$. Lastly, to obtain a universe for $|\mathbb{F}|$ we use the corollary to obtain one for $\mathbb{F}$ and then simply compose the two acyclic fibrations

$$\mathrm{Hom}_{\mathscr{E}}(-,U) \overset{\sim}{\twoheadrightarrow} \mathbb{F} \overset{\sim}{\twoheadrightarrow} |\mathbb{F}|$$

After some additional work, which we omit, Shulman obtains,

Theorem 236. (Theroem 5.22 in [Shu19] as presented in [Rie24].) Let $\mathscr{E}$ be a right proper simplicial Cisinski category, and $\mathbb{F}$ a locally presentable and relatively acyclic notion of fibred

[29] a subfunctor.

structure with $|\mathbb{F}| = \mathscr{F}_{\mathscr{E}}$. Then, there exists a regular cardinal λ such that for all regular cardinals κ[30] there exists a relatively κ-presentable fibration $\pi : \tilde{\mathcal{U}} \to \mathcal{U}$ such that

(1) Every κ-presentabe fibration arises as a pullback of π.

(2) $\mathcal{U}$ is a fibrant object.

(3) π satisfies the univalence axiom.

Of course, the archetypal example, $\mathrm{sSet}_{\mathrm{KQ}}$, is a TTMT. If all these constructions are performed there, we retrieve Voevodsky's simplicial model. All the axioms for a TTMT, except the last one, are such that include all the properties of a model category that were previously observed to contribute in the achieving semantics for increasing fragments of HoTT. In that sense, TTMT enjoy a "maximal" set of good properties. For example, Shulman collects all the properties from [AW09], [AK11], [SHU14] and [LS19]. As the name suggests type-theoretic model topoi are linked to Rezk's model topoi. For instance,

Theorem 237. (Theorem 6.4 in [Shu19])
A TTMT $\mathscr{E}$ has descent and universal homotopy colimits in the sense of [Rez10]. It also admits a small simplicial presentation. Therefore every type-theoretic model topos is a model topos.

Shulman also shows that the collection of TTMT is closed under various constructions like taking slices over objects and products but most crucially:

(1) Let $\mathscr{E}$ be a TTMT and $\mathscr{D}$ small and simplicially enriched. Then, $\mathscr{E}_{\mathrm{inj}}^{\mathscr{D}^{\mathrm{op}}}$ is a TTMT.

(2) Let $\mathscr{E}$ be a TTMT and S a set of morphisms such that the left Bousfield localisation is left exact. Then, $\mathcal{L}_S\mathscr{E}$ is again a TTMT.

As a corollary, we get that all model categories of the form $\mathcal{L}_S\Big(\mathrm{sPsh}(\mathscr{D})_{\mathrm{inj}}\Big)$ are TTMT. Recall that a model topos is a model category *Quillen equivalent* to one as above, with the projective model structure. We note that the corollary does *not* mean we obtain that all model topoi are TTMT. The reason is that Quillen equivalence does *not* preserve the property that "cofibrations are exactly the monomorphisms". However, it does demonstrate that any model topos is Quillen Equivalent to a type-theoretic one. Therefore,

Theorem 238. (Theorem 11.1 in [Shu19]) Every ∞-topos $\mathfrak{X}$ can be presented by a type theoretic model topos $\mathscr{E}$.

[30]With a certain relationship to λ

Proof. Every ∞-topos can be presented by a model topos. Every model topos is, by definition, Quillen equivalent to a model category of the form $\mathcal{L}_S\Big(\mathrm{sPsh}(\mathscr{D})_{\mathrm{inj}}\Big)$. We saw that the latter are all type-theoretic model topoi. $\square$

5.10 $N_f(E^0_f)$ is an Elementary Higher Topos

The three theorems we've presented that outline the relationship of models of HoTT and ∞-categories came in two flavours. First, Kapulkin's result started with a $\mathbb{T}$, recasted it in an essentially algebraic model $\mathrm{Cl}(\mathbb{T})$ and produced an ∞-category $N_f(\mathrm{Cl}(\mathbb{T}))$. On the other hand, the other two constructions were in a different direction. They started with a structured ∞-category, an ∞-topos or a presentable locally cartesian closed ∞-category respectively. Recall presentability is equivalent to admitting a presentation by a simplicial model category. Then one shows that the additional structure of the ∞-category makes the model category presenting it structured in an appropriate way to host a model of HoTT.

In this section, we compose the two constructions in the case of a type-theoretic model topos $\mathscr{E}$. This section also serves as an application of many of the results and ideas in the thesis.

Before we proceed we must briefly introduce the notion of an *Elementary Higher Topos* due to Rasekh [Ras22]. As discussed in the introduction, topoi originate and were popularised through the work of Grothendieck as categories of sheaves on a site. It was quickly realised that these categories behave much like the category Set. The properties that make it so were isolated and used to define the more general *Elementary topoi*. After some work, it was realised that those same properties were in very close analogy with structures present in intuitionistic logical systems. Indeed such logical systems are in a robust correspondence with elementary topoi, see [LS86]. Having seen how HoTT is in such a robust link with structured ∞-categories one can reasonably hope for a notion of an elementary ∞-topos whose *internal language* is HoTT/ UF.

A reasonable place to start is to take Lurie's (Grothendieck) ∞-topoi and try to isolate the more "logical" axioms. Given the constructive nature of the internal logic of ordinary elementary topoi one must insist on the finite nature of their defining axioms. As the nLab authors note "In general, the problem with "elementary-izing" the notion of $(\infty, 1)$-topos is that Grothendieck $(\infty, 1)$-toposes have many properties that are not reflected in type theory due to the finitary nature of type theory; the question is to find appropriate "finitary shadows" of them."

An elementary topos is a locally cartesian closed category with finite limits and a subobject classifier. Rasekh proposes the natural generalisation, that an *elementary ∞-topos* should be a finitely complete and locally cartesian closed ∞-category with an additional property. As was discussed in length in subsection 2.5 the notion of a subobject/ object classifier in ∞-topoi a more complicated than in the case of ordinary categories. What we gain is that in the ∞-setting the richer structure of $\mathrm{Hom}(-,-)$ objects that are ∞-groupoids instead of sets, allows for *every* morphism to be classified, not only the monomorphisms. What we lose is that we no longer have a single classifying morphism but an entire collection of them. Each morphism is classified by a universe depending on the cardinality of the morphism's fibers. So we obtain a collection of universes $\mathscr{U}_\kappa$.

Whatever the size of the corresponding fibers is one can consider the -1-truncated objects. In the context of HoTT they are called *Propositions*. This can be done for each universe separately and thus we obtain Prop_κ. Thus, a natural question arises: "What is the exact relationship between all the $\mathscr{U}_\kappa$". In HoTT one is thus motivated to postulate Axiom 3.5.5 [Uni13]: The natural map $\mathrm{Prop}_i \to \mathrm{Prop}_{i+1}$ is an equivalence. This principle is called *propositional resizing* and it motivates Rasekh to ask for a unique subobject classifier alongside sufficient universes.

Definition 239. (2.5 in [Ras22]) Let $\mathscr{C}$ be finitely complete ∞-category. Let $\mathrm{Sub}_{\mathscr{C}} : \mathscr{C}^{op} \to Set$ be the composition:

$$\mathscr{C}^{op} \to \mathrm{Cat}_\infty \to Set$$
$$C \mapsto {}^{\mathscr{C}}\!\big/_{C} \to \tau_{-1}\left({}^{\mathscr{C}}\!\big/_{C}\right) \in Set$$

A subobject classifier is a representing object for $\mathrm{Sub}_{\mathscr{C}}$.

Having sufficient universes refers to each map being classified by *some* universe. Then Rasekh defines,

Definition 240. An ∞-category $\mathscr{E}$ is an elementary ∞-topos if and only if it:

(*1*) is finitely complete

(*2*) is locally Cartesian closed

(*3*) has a subobject classifier

(*4*) has sufficient universes

As discussed in the introduction of this section, the goal here is to compose the two constructions we presented in this thesis.

$$\begin{tikzcd}
& \text{TTMT} \quad \mathscr{E} \arrow[dl, bend right] & \\
\text{fibration category} \quad E_f^0 \arrow[rr, bend right, "N_f"'] & & \infty - \text{topoi} \arrow[ul, bend right, "\text{presented by}"']
\end{tikzcd}$$

Theorem 241. Let $\mathfrak{X}$ be an ∞-topos and $\mathscr{E}$ a TTMT that presents it. Let E^0_f be the fibration category obtained by $\mathscr{E}$ by forgetting the simplicial enrichment and restricting to fibrant objects. Then, $N_f(E^0_f)$ is an elementary higher topos. In fact, it is a Grothendieck one, equivalent to $\mathfrak{X}$.

The phrasing of the theorem may seem a bit weird. Shortly, we will show that $N_f(E^0_f) \simeq \mathfrak{N}(\mathscr{E}^\circ) \simeq \mathfrak{X}$. So why claim it is an *elementary* higher topos? The intention is to showcase how the machinery developed for $N_f(-)$ takes care of two out of four requirements. We defined:

We'll obtain properties 1 & 2 through the theory around $N_f(-)$. The first comes for free from Szumiłos construction. We already recorded that as Theorem 207. So we move on to the second. By Theorem 212 it suffices to show that E^0_f is a locally cartesian closed fibration category. Indeed,

Proposition 242. Let $\mathscr{E}$ be a TTMT. Then E^0_f is a locally cartesian closed fibration category.

Proof. We verify the definition 95 of a locally cartesian closed fibration category. Firstly, we know that in a TTMT cofibrations are monos and therefore all objects are cofibrant.

Now, we proceed by showing that for any fibration $p : B \twoheadrightarrow A$ pullback functor $p^* : E^0_f(A) \to E^0_f(B)$ is an exact functor that has a homotopical right adjoint. The pullback functor is exact by Lemma 6.1.5 in [Kap14].

Now $\mathscr{E}$ being a TTMT means that it is in particular simplically locally cartesian closed. By definition, this means the adjunction $g^* \dashv \Pi_g$ is Quillen. We must show that Π_g is homotopical. This is not the case in general but in E^f_0 we've conveniently restricted our attention to fibrant objects. By Ken Brown's Lemma 130, Π_g being right Quillen means it preserves weak equivalences between fibrant objects, thus completing the proof. □

It remains to show that $N_f(E^0_f)$ has a subobject classifier and sufficient universes. This is certainly true of all Grothendieck ∞-topoi. We know that $N_f(-)$ is another model for the homotopy simplicial category of a fibration category. In the rest of the section, we verify that even if we start from $\mathfrak{N}(\mathscr{E}^\circ) \simeq \mathfrak{X}$ the non-enriched and restricted to fibrant objects E^0_f maintains sufficient information to retrieve $\mathfrak{X}$ up to equivalence of ∞-categories. This is yet another instance of a central slogan of homotopy theory, that the homotopy (∞-) category depends exclusively on the weak equivalences.

$$\begin{tikzcd}
\mathrm{smCat} \arrow[rr, "-^{\circ}"] \arrow[d] \arrow[ddddr, dotted] & & \mathrm{sCat}_{\mathrm{Berg}} \arrow[ddr, "\mathfrak{N}"] & \\
hCat \arrow[rr, "L^H(-)'"] \arrow[d, "sDK{-}^f"'] & & \mathrm{sCat}_{\mathrm{Berg}} \arrow[d, "sDK"'] \arrow[dr, "\mathfrak{N}"] & \\
hCat \arrow[rr, "L^H(-^f)'"] \arrow[d] & & \mathrm{sCat}_{\mathrm{Berg}} \arrow[r, "\mathfrak{N}"] & \mathrm{sSet}_{\mathrm{Joy}} \\
hCat \arrow[urrr, "L(-^f)"'] & & & \\
 & fCat \arrow[ul] \arrow[uurr, "N_f(-^f)"'] & &
\end{tikzcd}$$

We note that to view smCat as a category with forgetful functors towards fCat, hCat that is also compatible with the forgetful functor $U_{\mathrm{fh}} : \mathrm{fCat} \to \mathrm{hCat}$ we must restrict to (simplicial) right Quillen functors between simplicial model categories. We only care about what happens on objects so this is not much of an obstruction.

(1) $\mathfrak{N}(\mathscr{M}^{\circ}) = \mathrm{Ho}_{\infty}(\mathscr{M}) : smCat \to qCat$ is what it means for a simplicial model category to *present* an ∞-category.

(2) The second row is the two-step process that produces the homotopy ∞-category of a homotopical category. We start with a homotopical category (M, W). Using the *hammock localization* we promote it to a simplicial category. Then we get an ∞-category using the right derived coherent nerve, i.e. we apply the functor to a fibrant replacement of $L^H(M, W)$, as above.

(3) We restrict to fibrant objects and weak equivalences among them.

(4) We apply the *standard localization*

(5) We start with the fibration category $M^f(W^f, F^f)$ and apply the ∞-*category of frames* construction.

We claim the diagram commutes up to appropriate weak equivalence.

Proof. First, we show that the three first rows produce equivalent ∞-categories or, equivalently, that they produce weakly equivalent ∞-categories in $\mathrm{sSet}_{\mathrm{Joy}}$. As shown in proposition 139 $\mathfrak{N}$ is part of a Quillen Equivalence $\mathfrak{N} : \mathrm{sCat}_{\mathrm{Berg}} \leftrightarrows \mathrm{sSet}_{\mathrm{Joy}} : \mathfrak{C}$ so in particular it is right Quillen. Therefore, by proposition 130, it preserves weak equivalences among fibrant objects. Thus, to obtain equivalent ∞-categories it suffices to produce a zig-zag of weak equivalences among fibrant objects in $\mathrm{sCat}_{\mathrm{Berg}}$. We claim we have

$$\begin{tikzcd} L^H(E^f) \arrow[r] \arrow[d] & L^H(E) \arrow[r] \arrow[d] & \bullet & \mathscr{E}^\circ \arrow[l] \\ L^H(E^f)' \arrow[r] & L^H(E)' & & \end{tikzcd}$$

where all arrows are weak equivalences in $\mathrm{sCat}_{\mathrm{Berg}}$.

First, we focus on the square:
In proposition 191 we've seen that for any model category M the full inclusion $M^f \hookrightarrow M$ is an hDK equivalence. By 194, $L^H : \mathrm{hCat} \to \mathrm{sCat}_{\mathrm{Berg}}$ is itself a DK equiv of homotopical categories, by definition, it will produce sDK equivalent simplicial Categories. Therefore $L^H(E^f) \xrightarrow{\sim} L^H(E)$. Moreover, by a simple 2-of-3 argument, we can see that weakly equivalent objects have weakly equivalent fibrant replacements. Therefore the vertical maps are weak equivalences. The fact that $L^H(E)$ is weakly equivalent to $\mathscr{E}^\circ$ is given in proposition 1.3.5 in [Hin15].

Moreover, recall that the fibrant objects of $\mathrm{sCat}_{\mathrm{Berg}}$ are the locally Kan simplicial categories. In proposition 177 we've shown that for all simplicial model categories, we have $\mathscr{E}^\circ$ is locally Kan. In Proposition 3.1.46 in [Kap17] we see that for any homotopical category $\mathcal{C}$ we have an equivalence $L(\mathcal{C}) \simeq N^{\mathrm{hc}}(L^H(\mathcal{C})')$. Lastly, from one of Kapulkins main theorems, Theorem 9.1.2 [Kap17] we get that for any fibration category, $L(\mathcal{C}) \simeq N_f(\mathcal{C})$.

□

Theorem 243. Let $\mathscr{E}$ be a TTMT that presents an ∞-topos $\mathfrak{X}$. Then $N_f(E_0^f) \simeq \mathfrak{X}$.

Proof. By definition, $\mathscr{E} \in \mathrm{smCat}$ and $\mathfrak{N}(\mathscr{E}) \simeq \mathfrak{X}$. Let E_0 be the model category obtained by $\mathscr{E}$ if we forget the simplicial enrichment. Let E_0^f be the fibration category we canonically obtain by E_0 by restricting to fibrant objects. Then, $N_f(E_0^f) \simeq \mathfrak{X}$ as ∞-categories. The result is essentially the commutativity of the diagram above. We saw how:

$$N_f(E_0^f) \simeq L(E_0^f) \simeq \mathfrak{N}(L^H(E_0^f)') \simeq \mathfrak{N}(\mathscr{E}^\circ) \simeq \mathfrak{X}$$

□

References

[Acz78] Peter Aczel. “The Type Theoretic Interpretation of Constructive Set Theory**The ideas for this paper were germinating during a visit to Caltech, supported by NSF grant MPS 75-07562, and came to fruition while visiting Utrecht University. The results were first announced in an abstract for the 1976 Oxford Logic Colloquium.” In: *Logic Colloquium ’77*. Ed. by Angus Macintyre, Leszek Pacholski, and Jeff Paris. Vol. 96. Studies in Logic and the Foundations of Mathematics. Elsevier, 1978, pp. 55–66. DOI: `https://doi.org/10.1016/S0049-237X(08)71989-X`. URL: `https://www.sciencedirect.com/science/article/pii/S0049237X0871989X`.

[AK11] Peter Arndt and Krzysztof Kapulkin. “Homotopy-Theoretic Models of Type Theory”. In: (2011), pp. 45–60. ISSN: 1611-3349. DOI: `10.1007/978-3-642-21691-6_7`. URL: `http://dx.doi.org/10.1007/978-3-642-21691-6_7`.

[AKL15] JEREMY AVIGAD, KRZYSZTOF KAPULKIN, and PETER LEFANU LUMSDAINE. “Homotopy limits in type theory”. In: *Mathematical Structures in Computer Science* 25.5 (Jan. 2015), pp. 1040–1070. ISSN: 1469-8072. DOI: `10.1017/s0960129514000498`. URL: `http://dx.doi.org/10.1017/S0960129514000498`.

[AR94] J. Adamek and J. Rosicky. *Locally Presentable and Accessible Categories*. London Mathematical Society Lecture Note Series. Cambridge University Press, 1994.

[AW09] Steve Awodey and Michael A. Warren. “Homotopy theoretic models of identity types”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 146.1 (Jan. 2009), pp. 45–55. ISSN: 1469-8064. DOI: `10.1017/s0305004108001783`. URL: `http://dx.doi.org/10.1017/S0305004108001783`.

[BÄR] P. BÄRNREUTHER. *Presentalbe ∞ categories-I*. URL: `https://graptismath.net/higher_category_theory/presentable_infinity-cats1-notes.pdf`.

[Bat98] M.A. Batanin. “Monoidal Globular Categories As a Natural Environment for the Theory of Weakn-Categories”. In: *Advances in Mathematics* 136.1 (1998), pp. 39–103. ISSN: 0001-8708. DOI: `https://doi.org/10.1006/aima.1998.1724`. URL: `https://www.sciencedirect.com/science/article/pii/S0001870898917248`.

[BEK00] TIBOR BEKE. “Sheafifiable homotopy model categories”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 129.3 (Nov. 2000), pp. 447–475. ISSN: 0305-0041. DOI: `10.1017/s0305004100004722`. URL: `http://dx.doi.org/10.1017/S0305004100004722`.

[Bén67] Jean Bénabou. "Introduction to bicategories". In: *Reports of the Midwest Category Seminar*. Berlin, Heidelberg: Springer Berlin Heidelberg, 1967, pp. 1–77. ISBN: 978-3-540-35545-8.

[Ber05] Julia E. Bergner. *A model category structure on the category of simplicial categories*. 2005. arXiv: `math/0406507 [math.AT]`.

[Ber06] Julia E. Bergner. *A survey of* $(\infty, 1)$*-categories*. 2006. arXiv: `math/0610239 [math.AT]`. URL: `https://arxiv.org/abs/math/0610239`.

[Ber18] Benno van den Berg. *Univalent polymorphism*. 2018. URL: `https://arxiv.org/abs/1803.10113`.

[BG10] Benno van den Berg and Richard Garner. "Types are weak ω -groupoids". In: *Proceedings of the London Mathematical Society* 102.2 (Oct. 2010), pp. 370–394. ISSN: 0024-6115. DOI: `10.1112/plms/pdq026`. URL: `http://dx.doi.org/10.1112/plms/pdq026`.

[BG11] Benno van den Berg and Richard Garner. *Topological and simplicial models of identity types*. 2011. arXiv: `1007.4638 [math.LO]`. URL: `https://arxiv.org/abs/1007.4638`.

[BK12] C. Barwick and D. M. Kan. "A characterization of simplicial localization functors and a discussion of DK equivalences". In: *Indag. Math. (N.S.)* 23.1-2 (2012), pp. 69–79. ISSN: 0019-3577,1872-6100. DOI: `10.1016/j.indag.2011.10.001`. URL: `https://doi.org/10.1016/j.indag.2011.10.001`.

[Bro73] Kenneth S. Brown. "Abstract Homotopy Theory and Generalized Sheaf Cohomology". In: *Transactions of the American Mathematical Society* 186 (1973), pp. 419–458. ISSN: 00029947. URL: `http://www.jstor.org/stable/1996573` (visited on 07/11/2024).

[BS20] Clark Barwick and Christopher Schommer-Pries. *On the Unicity of the Homotopy Theory of Higher Categories*. 2020. arXiv: `1112.0040 [math.AT]`. URL: `https://arxiv.org/abs/1112.0040`.

[BV73] J. M. Boardman and R. M. Vogt. *Homotopy invariant algebraic structures on topological spaces*. Vol. Vol. 347. Lecture Notes in Mathematics. Springer-Verlag, Berlin-New York, 1973, pp. x+257.

[Chu32] Alonzo Church. "A Set of Postulates for the Foundation of Logic". In: *Annals of Mathematics* 33.2 (1932), pp. 346–366. ISSN: 0003486X, 19398980. URL: `http://www.jstor.org/stable/1968337` (visited on 07/06/2024).

[Cis06] Denis-Charles Cisinski. *Batanin higher groupoids and homotopy types*. 2006. arXiv: `math/0604442 [math.AT]`. URL: `https://arxiv.org/abs/math/0604442`.

[Cis16] Denis-Charles Cisinski. "Catégories supérieures et théorie des topos". In: *Astérisque* 380 (2016), Exp. No. 1097, 263–324. ISSN: 0303-1179,2492-5926.

[Def19] Gian Deflorin. "The Homotopy Hypothesis". In: (2019).

[DHK97] W. Dwyer, Philip Hirschhorn, and Daniel Kan. "Model categories and more general abstract homotopy theory: a work in what we like to think of as progress". In: (Apr. 1997), p. 55.

[DK80a] W. G. Dwyer and D. M. Kan. "Calculating simplicial localizations". In: *J. Pure Appl. Algebra* 18.1 (1980), pp. 17–35. ISSN: 0022-4049,1873-1376. DOI: `10.1016/0022-4049(80)90113-9`. URL: `https://doi.org/10.1016/0022-4049(80)90113-9`.

[DK80b] W. G. Dwyer and D. M. Kan. "Function complexes in homotopical algebra". In: *Topology* 19.4 (1980), pp. 427–440. ISSN: 0040-9383. DOI: `10.1016/0040-9383(80)90025-7`. URL: `https://doi.org/10.1016/0040-9383(80)90025-7`.

[DK80c] W. G. Dwyer and D. M. Kan. "Simplicial localizations of categories". In: *J. Pure Appl. Algebra* 17.3 (1980), pp. 267–284. ISSN: 0022-4049,1873-1376. DOI: `10.1016/0022-4049(80)90049-3`. URL: `https://doi.org/10.1016/0022-4049(80)90049-3`.

[DS95] W.G. Dwyer and J. Spalinski. "CHAPTER 2 - Homotopy Theories and Model Categories". In: *Handbook of Algebraic Topology*. Ed. by I.M. JAMES. Amsterdam: North-Holland, 1995, pp. 73–126. ISBN: 978-0-444-81779-2. DOI: `https://doi.org/10.1016/B978-044481779-2/50003-1`. URL: `https://www.sciencedirect.com/science/article/pii/B9780444817792500031`.

[Dug01a] Daniel Dugger. "Combinatorial model categories have presentations". In: *Adv. Math.* 164.1 (2001), pp. 177–201. ISSN: 0001-8708,1090-2082. DOI: `10.1006/aima.2001.2015`. URL: `https://doi.org/10.1006/aima.2001.2015`.

[Dug01b] Daniel Dugger. "Replacing model categories with simplicial ones". In: *Trans. Amer. Math. Soc.* 353.12 (2001), pp. 5003–5027. ISSN: 0002-9947,1088-6850. DOI: `10.1090/S0002-9947-01-02661-7`. URL: `https://doi.org/10.1090/S0002-9947-01-02661-7`.

[Dug01c] Daniel Dugger. "Universal homotopy theories". In: *Adv. Math.* 164.1 (2001), pp. 144–176. ISSN: 0001-8708,1090-2082. DOI: `10.1006/aima.2001.2014`. URL: `https://doi.org/10.1006/aima.2001.2014`.

[Fer04] José Ferreirós. “An episode of the crisis of foundations, 1904”. In: *Gac. R. Soc. Mat. Esp.* 7.2 (2004), pp. 449–467. ISSN: 1138-8927.

[Fer08] J. Ferreirós. *Labyrinth of Thought: A History of Set Theory and Its Role in Modern Mathematics*. Birkhäuser Basel, 2008. ISBN: 9783764383503. URL: `https://books.google.gr/books?id=vrQLbbxGNMsC`.

[GG08] Nicola Gambino and Richard Garner. “The identity type weak factorisation system”. In: *Theoretical Computer Science* 409.1 (Dec. 2008), pp. 94–109. ISSN: 0304-3975. DOI: `10.1016/j.tcs.2008.08.030`. URL: `http://dx.doi.org/10.1016/j.tcs.2008.08.030`.

[Gol14] R. Goldblatt. *Topoi: The Categorial Analysis of Logic*. ISSN. Elsevier Science, 2014. ISBN: 9781483299211. URL: `https://books.google.nl/books?id=AprOBQAAQBAJ`.

[Gro15] Moritz Groth. *A short course on ∞-categories*. 2015. arXiv: `1007.2925 [math.AT]`. URL: `https://arxiv.org/abs/1007.2925`.

[Gro21] Alexander Grothendieck. *Pursuing Stacks*. 2021. arXiv: `2111.01000 [math.CT]`.

[GV72] A. Grothendieck and J. L. Verdier. *Théorie des Topos et Cohomologie Étale des Schémas. Séminaire de Géométrie Algébrique du Bois-Marie 1963-1964 (SGA 4). Tome 1*. 1st ed. Vol. 269. Lecture Notes in Mathematics. Springer Berlin, Heidelberg, 1972. DOI: `10.1007/BFb0081551`. URL: `https://doi.org/10.1007/BFb0081551`.

[GZ12] P. Gabriel and M. Zisman. *Calculus of Fractions and Homotopy Theory*. Ergebnisse der Mathematik und ihrer Grenzgebiete. 2. Folge. Springer Berlin Heidelberg, 2012. ISBN: 9783642858444. URL: `https://books.google.gr/books?id=ySvqCAAAQBAJ`.

[Hin15] V. Hinich. *Dwyer-Kan localization revisited*. 2015. arXiv: `1311.4128 [math.QA]`.

[Hir03a] Philip S. Hirschhorn. “Model categories and their localizations”. In: (2003). URL: `https://api.semanticscholar.org/CorpusID:118799523`.

[Hir03b] Philip S. Hirschhorn. *Model categories and their localizations*. Vol. 99. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2003, pp. xvi+457. ISBN: 0-8218-3279-4. DOI: `10.1090/surv/099`. URL: `https://doi.org/10.1090/surv/099`.

[HM22] Gijs Heuts and Ieke Moerdijk. *Simplicial and dendroidal homotopy theory*. Vol. 75. Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics]. Springer, Cham, [2022] ©2022, pp. xx+612. ISBN: 978-3-031-10446-6; 978-3-031-10447-3. DOI: 10.1007/978-3-031-10447-3. URL: https://doi.org/10.1007/978-3-031-10447-3.

[Hov99] Mark Hovey. *Model categories*. Vol. 63. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1999, pp. xii+209. ISBN: 0-8218-1359-5.

[How80] William Alvin Howard. "The Formulae-as-Types Notion of Construction". In: *To H. B. Curry: Essays on Combinatory Logic, Lambda Calculus, and Formalism*. Ed. by Haskell Curry et al. Academic Press, 1980.

[HS01] André Hirschowitz and Carlos Simpson. *Descente pour les n-champs (Descent for n-stacks)*. 2001. arXiv: math/9807049 [math.AG]. URL: https://arxiv.org/abs/math/9807049.

[HS02] Martin Hofmann and Thomas Streicher. "The Groupoid Interpretation of Type Theory". In: (Apr. 2002).

[HS94] M. Hofmann and T. Streicher. "The groupoid model refutes uniqueness of identity proofs". In: *Proceedings Ninth Annual IEEE Symposium on Logic in Computer Science*. 1994, pp. 208–212. DOI: 10.1109/LICS.1994.316071.

[Joy02] A. Joyal. "Quasi-categories and Kan complexes". In: *Journal of Pure and Applied Algebra* 175.1 (2002). Special Volume celebrating the 70th birthday of Professor Max Kelly, pp. 207–222. ISSN: 0022-4049. DOI: https://doi.org/10.1016/S0022-4049(02)00135-4. URL: https://www.sciencedirect.com/science/article/pii/S0022404902001354.

[Joy08a] A. Joyal. "The theory of quasicategories and applications". In: (2008). URL: http://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf.

[Joy08b] André Joyal. *Notes on quasicategories*. 2008. URL: http://www.math.uchicago.edu/~may/IMA/Joyal.pdf.

[Joy08c] André Joyal. *The Theory of Quasi-Categories and its Applications*. 2008. URL: https://mat.uab.cat/~kock/crm/hocat/advanced-course/Quadern45-2.pdf.

[Kap14] Krzysztof Kapulkin. *Joyal's Conjecture in Homotopy Type Theory*. Sept. 2014. URL: http://d-scholarship.pitt.edu/21718/.

[Kap17] Krzysztof Kapulkin. "Locally cartesian closed quasi-categories from type theory". In: *Journal of Topology* 10.4 (Nov. 2017), pp. 1029–1049. ISSN: 1753-8424. DOI: 10.1112/topo.12031. URL: http://dx.doi.org/10.1112/topo.12031.

[KL18] Chris Kapulkin and Peter LeFanu Lumsdaine. “The Simplicial Model of Univalent Foundations (after Voevodsky)”. In: (2018). arXiv: `1211.2851 [math.LO]`.

[KLV12] Chris Kapulkin, Peter Lumsdaine, and Vladimir Voevodsky. *The Simplicial Model of Univalent Foundations*. Nov. 2012.

[Lan78] Saunders Mac Lane. *Categories for the Working Mathematician*. Vol. 5. Graduate Texts in Mathematics. Springer New York, NY, 1978. DOI: `10.1007/978-1-4757-4721-8`. URL: `https://doi.org/10.1007/978-1-4757-4721-8`.

[Law00] F. William Lawvere. “Comments on the development of topos theory”. In: *Development of mathematics 1950–2000*. Birkhäuser, Basel, 2000, pp. 715–734. ISBN: 3-7643-6280-4.

[Law05] F. William Lawvere. “An elementary theory of the category of sets (long version) with commentary”. In: *Repr. Theory Appl. Categ.* 11 (2005). Reprinted and expanded from Proc. Nat. Acad. Sci. U.S.A. **52** (1964) [MR0172807], With comments by the author and Colin McLarty, pp. 1–35.

[LS19] PETER LEFANU LUMSDAINE and MICHAEL SHULMAN. “Semantics of higher inductive types”. In: *Mathematical Proceedings of the Cambridge Philosophical Society* 169.1 (June 2019), pp. 159–208. ISSN: 1469-8064. DOI: `10.1017/s030500411900015x`. URL: `http://dx.doi.org/10.1017/S030500411900015X`.

[LS86] J. Lambek and P. J. Scott. *Introduction to Higher Order Categorical Logic*. USA: Cambridge University Press, 1986. ISBN: 0521246652.

[Lum10] Peter LeFanu Lumsdaine. “Weak omega-categories from intensional type theory”. In: *Logical Methods in Computer Science* Volume 6, Issue 3 (Sept. 2010). ISSN: 1860-5974. DOI: `10.2168/lmcs-6(3:24)2010`. URL: `http://dx.doi.org/10.2168/LMCS-6(3:24)2010`.

[Lur03] Jacob Lurie. *On Infinity Topoi*. 2003. arXiv: `math/0306109 [math.CT]`.

[Lur09] Jacob Lurie. *Higher Topos Theory (AM-170)*. Princeton University Press, 2009. ISBN: 9780691140490. URL: `http://www.jstor.org/stable/j.ctt7s47v` (visited on 12/07/2023).

[Lur17] Jacob Lurie. *Higher Algebra*. eBook PDF (1553 pages). Harvard University, 2017. URL: `https://www.math.harvard.edu/~lurie/papers/HA.pdf`.

[Lur18] Jacob Lurie. *Kerodon*. `https://kerodon.net`. 2018.

[LW15] Peter Lefanu Lumsdaine and Michael A. Warren. "The Local Universes Model: An Overlooked Coherence Construction for Dependent Type Theories". In: *ACM Transactions on Computational Logic* 16.3 (July 2015), pp. 1–31. ISSN: 1557-945X. DOI: 10.1145/2754931. URL: http://dx.doi.org/10.1145/2754931.

[Mar75a] Per Martin-LÃ¶f. "An Intuitionistic Theory of Types: Predicative Part". In: *Logic Colloquium '73*. Ed. by H.E. Rose and J.C. Shepherdson. Vol. 80. Studies in Logic and the Foundations of Mathematics. Elsevier, 1975, pp. 73–118. DOI: https://doi.org/10.1016/S0049-237X(08)71945-1. URL: https://www.sciencedirect.com/science/article/pii/S0049237X08719451.

[Mar75b] Per Martin-Löf. "An intuitionistic theory of types: predicative part". In: *Logic Colloquium '73 (Bristol, 1973)*. Vol. Vol. 80. Stud. Logic Found. Math. North-Holland, Amsterdam-Oxford, 1975, pp. 73–118.

[Mar84] Per Martin-Löf. *Intuitionistic type theory*. Vol. 1. Studies in Proof Theory. Lecture Notes. Notes by Giovanni Sambin. Bibliopolis, Naples, 1984, pp. iv+91. ISBN: 88-7088-105-9.

[Mar98] Per Martin-Löf. "An intuitionistic theory of types". In: *Twenty-five years of constructive type theory (Venice, 1995)*. Vol. 36. Oxford Logic Guides. Oxford Univ. Press, New York, 1998, pp. 127–172. ISBN: 0-19-850127-7.

[May99] J.P. May. "A Concise Course in Algebraic Topology". In: Chicago Lectures in Mathematics (1999). URL: https://books.google.gr/books?id=g8SG03R1bpgC.

[Maz15] Aaron Mazel-Gee. *Quillen adjunctions induce adjunctions of quasicategories*. 2015. arXiv: 1501.03146 [math.CT]. URL: https://arxiv.org/abs/1501.03146.

[ME45] Saunders MacLane and Samuel Eilenberg. "General Theory of Natural Equivalences". In: *Transactions of the American Mathematical Society* (1945), pp. 231–294.

[MM94] S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Introduction to Topos Theory*. Universitext. Springer New York, 1994. ISBN: 9780387977102. URL: https://books.google.nl/books?id=SGwwDerbEowC.

[Nor17] Paige Randall North. "Type theoretic weak factorization systems". In: (2017). DOI: 10.17863/CAM.11207. URL: https://www.repository.cam.ac.uk/handle/1810/265152.

[Poi95] Henri Poincare. “Analysis situs”. In: *Journal de l’École Polytechnique* 174.1 (1895), pp. 1–123. URL: `https://gallica.bnf.fr/ark:/12148/bpt6k4337198/f1.vertical`.

[Qui67] Daniel G. Quillen. “Homotopical Algebra”. In: (1967).

[Rap] George Raptis. *Notes on combinatorial model categories*. URL: `https://graptismath.net/files/notes-comb-model-cat.pdf`.

[Ras22] Nima Rasekh. *A Theory of Elementary Higher Toposes*. 2022. arXiv: `1805.03805 [math.CT]`.

[Rez] Charles Rezk. *Lectures on Higher Topos Theory*. URL: `https://rezk.web.illinois.edu/leeds-lectures-2019.pdf`.

[Rez00] Charles Rezk. *Every homotopy theory of simplicial algebras admits a proper model*. 2000. arXiv: `math/0003065 [math.AT]`.

[Rez10] Charles Rezk. “Toposes and homotopy toposes (version 0.15)”. In: (Jan. 2010).

[Rez22] Charles Rezk. “Introduction to Quasicategories”. In: (2022). URL: `https://rezk.web.illinois.edu/595-sp22/math-595-sp22.html`.

[Rie] Emily Riehl. *Homotopy Types as Homotopy Types*. URL: `https://emilyriehl.github.io/files/hopkins65.pdf`.

[Rie14] Emily Riehl. *Categorical homotopy theory*. Vol. 24. New Mathematical Monographs. Cambridge University Press, Cambridge, 2014, pp. xviii+352. ISBN: 978-1-107-04845-4. DOI: `10.1017/CBO9781107261457`. URL: `https://doi.org/10.1017/CBO9781107261457`.

[Rie20] Emily Riehl. *Homotopical categories: From model categories to $(\infty, 1)$-categories*. 2020. arXiv: `1904.00886 [math.AT]`.

[Rie23] Emily Riehl. *Homotopy coherent structures*. 2023. arXiv: `1801.07404`.

[Rie24] Emily Riehl. “On the ∞-topos semantics of homotopy type theory”. In: *Bulletin of the London Mathematical Society* 56.2 (Feb. 2024), pp. 461–517. ISSN: 1469-2120. DOI: `10.1112/blms.12997`. URL: `http://dx.doi.org/10.1112/blms.12997`.

[Rij22] Egbert Rijke. *Introduction to Homotopy Type Theory*. 2022. arXiv: `2212.11082 [math.LO]`.

[SHU14] MICHAEL SHULMAN. “Univalence for inverse diagrams and homotopy canonicity”. In: *Mathematical Structures in Computer Science* 25.5 (Nov. 2014), pp. 1203–1277. ISSN: 1469-8072. DOI: `10.1017/s0960129514000565`. URL: `http://dx.doi.org/10.1017/S0960129514000565`.

[Shu15] Michael Shulman. *The univalence axiom for elegant Reedy presheaves*. 2015. arXiv: `1307.6248 [math.AT]`. URL: `https://arxiv.org/abs/1307.6248`.

[Shu19] Michael Shulman. "All $(\infty, 1)$-toposes have strict univalent universes". In: (2019). arXiv: `1904.07004 [math.AT]`.

[Sim99] Carlos Simpson. *A Giraud-type characterization of the simplicial categories associated to closed model categories as ∞-pretopoi*. 1999. arXiv: `math/9903167 [math.AT]`. URL: `https://arxiv.org/abs/math/9903167`.

[Ste19] Raffael Stenzel. "On univalence, Rezk Completeness and presentable quasi-categories". Mar. 2019. URL: `https://etheses.whiterose.ac.uk/24342/`.

[Str23] Thomas Streicher. *Fibered Categories a la Jean Benabou*. 2023. arXiv: `1801.02927 [math.CT]`. URL: `https://arxiv.org/abs/1801.02927`.

[Szu14] Karol Szumiło. *Two Models for the Homotopy Theory of Cocomplete Homotopy Theories*. 2014. arXiv: `1411.0303 [math.AT]`.

[Tie73] M. Tierney. "Axiomatic sheaf theory: some constructions and applications". In: *Categories and commutative algebra (Centro Internaz. Mat. Estivo (C.I.M.E.), III Ciclo, Varenna, 1971)*. Centro Internazionale Matematico Estivo (C.I.M.E.) Ed. Cremonese, Rome, 1973, pp. 249–326.

[TV05] Bertrand Toën and Gabriele Vezzosi. "Homotopical algebraic geometry. I. Topos theory". In: *Adv. Math.* 193.2 (2005), pp. 257–372. ISSN: 0001-8708,1090-2082. DOI: `10.1016/j.aim.2004.05.004`. URL: `https://doi.org/10.1016/j.aim.2004.05.004`.

[Uni13] The Univalent Foundations Program. *Homotopy Type Theory: Univalent Foundations of Mathematics*. Institute for Advanced Study: `https://homotopytypetheory.org/book`, 2013.

[War08] Michael A. Warren. "Homotopy theoretic aspects of constructive type theory". In: (2008). URL: `http://mawarren.net/papers/phd.pdf`.

[Whi49] J. H. C. Whitehead. "Combinatorial homotopy. I". In: *Bulletin of the American Mathematical Society* 55.3.P1 (1949), pp. 213–245.