Type-free Predicational Theories of Ground

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Contents

Abstract

The aim of this thesis is to extend predicational theories of metaphysical grounding. In the existing predicational theories of ground (Korbmacher [1], [2]), the coding function is restricted such that the terms corresponding to sentences that contain the ground or the truth predicate cannot be arguments of the truth or ground predicates. In this thesis, I relax these restrictions.

This makes the theories more expressive, but it also make possible to derive paradoxes of self-referentiality for the ground predicate (Korbmacher [3], [2]), analogous to the *Liar Paradox* for the truth predicate, and predicational versions of *Puzzles of Ground* (Fine [4], Krämer [5]). In this thesis, I develop type-free theories of ground that avoid these paradoxes and puzzles. In section 1, I introduce the topic of metaphysical grounding, the operational and predicational approaches, the existing predicational theories, iterated ground and the paradoxes and puzzle previously mentioned. In section 2, I define the technical framework and notation I will use. In section 3, first, I develop a non-classical (K_3) model for the ground and truth predicate in the style of Kripke fixed point semantics (Kripke, [6]) that avoids the paradoxes of self-referentiality. Second, I develop an axiomatic theory for this model inspired by the *Kripke-Feferman* theory of truth. In section 4, first, I highlight the fact that, if some plausible principles about the interaction between the ground and truth predicate are added to the theories previously developed, they become inconsistent due to a version of *Fine's Puzzles of Ground* (Fine, [4]). Second, I develop a semantic model that avoids this inconsistency. In section 5, I draw some philosophical conclusions about the formal results of the previous two sections. I explain why a non-classical approach to the paradox of self-referentiality for the ground predicate is philosophically justified and interesting. I compare my solution to Fine's puzzle with the solutions proposed by Fine ([4, pp. 103-115]).

1 Metaphysical Grounding

In this section, I introduce *metaphysical grounding* and the most relevant philosophical and conceptual issues related to it which will be relevant for this paper. First, in section 1.1, I explain what *metaphysical grounding* is, I make some conceptual distinctions between different notions of grounding and I explain which ones I will consider in this paper and why. Then, in section 1.2, I will introduce two different kinds of formal theories of grounding, operational and predicational ones. I will list the *pros* and *cons* of adopting one approach with respect to the other. In section 1.3, I will delve into predicational theories of grounding, which is the topic of this paper. I will introduce the existing theories in the literature and their most relevant results. I will state in full detail the base predicational theory of partial ground PG, which will be relevant for the next sections. In section 1.4, I will introduce the notion and main theories of iterated ground. One of the objectives of this paper is to develop a theory of iterated ground in a predicational setting. In section 1.5, I will introduce two inconsistency results that threaten theories of grounding: the *paradox of self-referentiality for ground* and *Fine's puzzle*. I will show that, under certain assumptions, the predicational setting is expressive enough to derive both these puzzles. In the rest of the paper, I will formulate theories which aim to avoid them.

1.1 The Notion of Grounding

Metaphysical grounding is a kind of metaphysical explanation, in which the *explanans* and the *explanandum* are connected through some constitutive form of determination (Fine [7, p. 37]). Intuitively, a metaphysically grounds b when b holds *in virtue of* a, b is the case *because of* a, or b is *made true by* a. Typical examples of such phenomenon are:

(1) The fact that the ball is red and round obtains in virtue of the fact that it is red and the fact that it is round;

(2) Universals exist in virtue of their having exemplifiers;

(3) The fact that the particle is accelerating obtains in virtue of the fact that it is being acted upon by some net positive force;

(4) Mental facts obtain in virtue of neurophysiological facts;

(5) Normative facts are grounded in natural facts;

(6) Semantic properties are exemplified in virtue of certain non-semantic properties being exemplified;

(7) The existence of a non-empty set is grounded in the existence of its members;

(8) The existence of a whole is grounded in the existence of its parts. (Correia [8, pp. 251-252], Fine [7, p. 38])

As the examples above suggest, metaphysical grounding is a philosophically very relevant and interesting notion because it is widespread across many topics in philosophy and science. Many philosophical and scientific claims and theories are spelled out in terms of grounding. In fact, examples (1) and (2) above suggest that logically complex formulas are grounded in simpler ones, example (3) suggests that scientific claims can be expressed using the notion of grounding, examples (4)-(8) suggest that claims in various subfields of philosophy (respectively, philosophy of mind, ethics, semantics, philosophy of mathematics and ontology) are spelled out in terms of grounding relations.

There are different ways of conceiving the notion of metaphysical grounding. The following five distinctions are important for the rest of the paper[1]. The first distinction is between *factive* and *non-factive* conceptions of ground. On the factive conception, both what explains (the *ground*) and what is explained (the *grounded*) in a grounding explanation must be factive, i.e. a fact or a true term, proposition or sentence. Instead, on the non-factive conception, the grounds and the grounded can also be something non-factive, such as hypothetical facts or false sentences. In this paper, I will only focus on factive ground. The reasons why I restrict my analysis to factive ground are more practical than theoretical. The question about which between factive and non-factive ground is the more primitive notion is open[2]. However, the predicational theories of ground that this paper aims to further develop are all about factive ground. While developing a theory to non-factive ground is for sure a interesting open topic, in this paper I aim stick more closely to the existing predicational theories of ground and extend them in a few directions. Moreover, predicational theories that formalise factive ground create a natural environment to study the relationship between metaphysical grounding and truth. Studying their relation is an interesting philosophical question which will be one of the main focuses of this paper.

The second distinction is between *full* and *partial* conceptions of ground. a fully grounds b iff a, on its own, is the ground of the grounded b. a partially grounds b iff a, on its own or with some other c, is the ground of the grounded b. There is general consensus in the literature that it is possible to define partial ground in terms of full ground, but not the other way around (e.g. Fine [7, p. 50])[3]. Thus, in this sense, full ground is a more fundamental notion than partial ground. Despite this, in this paper, I will focus on partial ground. The reason for this choice is that, as I explain in section 1.3, developing a theory of full ground in a predicational setting raises some technical issues that make its formalisation more complex. In fact, the only fully developed predicational theories of grounding are theories of partial ground [1, 2]. However, for the objectives of this paper, it is sufficient to work with theories of partial ground. Therefore, I will stick to the simpler framework of theories of partial ground.

The third distinction is between *mediate* and *immediate* conceptions of ground. a immediately grounds b iff a grounds b and the explanatory relation between them is direct, in the sense that

[1] See Fine [7, pp. 48-54] for a complete overview.

[2] For example, Fine [7, pp. 49-50] claims that factive ground is more fundamental than non-factive ground, while Litland [9] argues for the opposite position.

[3] An exception to this consensus is Trogdon and Witmer [10], who propose a definition of full ground in terms of partial ground.

it does not depend on some c such that a grounds c and c grounds b. a mediately grounds b iff a immediately grounds b or the fact that a grounds b can be obtained by chaining relationships of immediate ground. Thus, mediate ground, but not immediate ground, is transitive. In this paper, I will only focus on mediate ground. This seems a natural choice because the intuitive notion of grounding that the examples above exemplify is one of mediate ground. In fact, most of the literature has focused on mediate ground. Moreover, mediate ground is the more primitive notion because immediate ground can be defined in terms on mediate ground, but not the other way around (Fine, [7, pp. 50-51]).

The fourth distinction is between *weak* and *strict* conceptions of ground. On the weak notion, things should ground themselves. Thus, weak ground is reflexive, a grounds a. Instead, we can think of the strict notion as relating the grounded only with grounds which are at a lower level in the explanatory hierarchy. Thus, strict ground is irreflexive, a does not ground a. In this paper, I work with strict ground. It is possible to both define weak ground in terms of strict ground and the latter in terms of the former one (Fine, [7, p. 52]). However, strict ground is a more intuitive notion than weak ground, it is the one which has been more extensively analysed in the literature and it is generally considered more fundamental than weak ground[4]. Fine [7, pp. 53-54] shows that full and partial and weak and strict ground can be combined in various ways to form three different notions of strict/partial ground (*partial strict* ground, *strict partial* ground and *part strict* ground). These distinctions do not matter for the theories I am going to develop in this paper.

The fifth distinction is between *conceptualist* and *worldly* conceptions of ground. Given a statement, we can distinguish between its *worldly* and its *conceptual content*. The worldly content is just a matter of the way it represents the world, while the conceptual content is also a matter of how it represents that content (Fine [11, pp. 685-686]). The worldly conception of ground does not distinguish between facts that represent the world in the same way, or, in other words, it is only sensitive to differences in *worldly* or *factual content*. Instead, the conceptualist conception of ground also distinguishes facts based on the concepts they use to describe the world. In other words, it is sensitive to the *conceptual content* of facts, which we may also call *propositions* (Correia, [8, pp. 256-259]). There is room for disagreement about whether two facts or propositions are the same. However, it is widely held that propositions are more fine-grained than worldly facts, in the sense that there can be different propositions describing the same fact, but not the other way around.

[4] Fine [7, pp. 52-53] defends the opposite view that weak ground is the more fundamental than strict ground.

Typical examples of different propositions expressing the same worldly fact are:

1) 'a is a water molecule' and 'a is an H_2O molecule' ([8, p.256]),

2) 'France is east of Argentina' and 'Argentina is west of France ([8, p.256]),

3) 'The ball is red' and 'it is not the case that it is not the case that the ball is red',

4) 'The ball is red' and 'The ball is red and it is red',

5) 'The ball is red' and 'The ball is red or it is red'.

In examples (1) and (2), it is commonly held that the two propositions describe or are about the same worldly fact (the fact that the molecule a has certain characteristics and the fact that two places are in a certain spatial relation with respect to each other), even if they employ different concepts to do so (*being a water molecule* and *being an H_2O molecule*, *being east of* and *being west of*).

Examples (3)-(5) are about logically equivalent sentences which present the same *factual content* in two different ways. In this sense, we can say that each of the pairs of sentences in examples (3)-(5) describes the same fact but expresses a different *conceptual content.* Recently, there has been a growing interest in *hyperintensionality*[5]. A context is hyperintensional iff it does not respect (classical) logical equivalence (Cresswell [14, p. 25]). Thus, according to an hyperintensional account of propositions, equivalence between propositions is more fine-grained than classical logical equivalence, in the sense that there are classically equivalent sentences expressing different propositions. The literature on grounding naturally fits within the one on hyperintensionality because theories of grounding claim the existence of grounding relations between classically equivalent sentences. Thus, theories of grounding argue for a distinction between some classically equivalent sentences on the base of the grounding relations they satisfy.

Thus, given that propositions are more fine-grained than worldly facts, conceptualist ground is also more fine-grained than worldly ground. The latter, but not the former, can distinguish between two grounding statements that differ only because the *grounds* and/or the *grounded* that constitute them describe the same worldly facts with different propositions. Thus, for example, in a worldly conception of ground, one cannot distinguish between:

'The ball is red grounds it is not the case that it is not the case that the ball is red' and

[5] See Nolan [12] and Berto and Nolan [13] for an introduction.

'It is not the case that it is not the case that the ball is red grounds the ball is red'.

Instead, in conceptualist conception, the former but not the latter is an intuitively true ground statement. Analogous arguments hold for the sentences in examples (4) and (5).

In this paper, I will work conceptualist theories of ground. The reasons for this choice are both practical and theoretical. Practically, the existing predicational theories of ground are theories of conceptualist ground (Korbmacher [1], [2]), and the purposes of this paper is to extend the existing predication theories. Theoretically, it is easier to develop a theory of worldly ground starting from a theory of conceptualist ground than the other way around. Given that conceptualist theories are more fine-grained, the results of a worldly theory should naturally follow from the principles conceptualist ground after we added to them the preferred view about which propositions are about the same facts.

1.2 Formal Theories of Ground

This paper is about *formal theories of ground.* It aims to build satisfactory logical theories and semantics for the notion exemplified in the previous sub-section. An important sub-topic of formal theories of ground is *the logic of ground.* The logic of ground is concerned with the logical consequence relation between ground statements, or, in other words, which ground statements follows *by logic* from other (possibly none) statements. The logic of ground can be divided into two parts: the *pure* or *structural* logic of ground and the *impure* or *applied* one (Fine [7, pp. 54-71]). The pure logic of ground is concerned with what follows from statements of ground without regard to the internal structure of the *ground* or the *grounded.* A typical example of such logical relations is that grounding is transitive, i.e. if a grounds b and b grounds c, then a grounds c. Instead, the impure logic of ground is concerned with the logical relations which holds in virtue of the internal structure of the *ground* and the *grounded.* For example, impure logic of ground typically state that a grounds $a \vee b$. The theories I develop in this paper contains both a pure and an impure logic of ground.

Ground can be formalised in two different ways on depending on the grammatical form used to express statements of ground. Two different kinds of theories follow from these two approaches: the operational and the predicational ones. Operational theories formalise grounding as an operator between sentences. Thus, in this case, ground is an operator from sentences to a sentence. The operational approach is common for other formal theories in logic and philosophy, such as logical theories of modality, where the operator $\Box$ takes a sentence (e.g. p) to from a new one (e.g. $\Box p$), or theories of the conditional, where the binary connective $\to$ takes two sentences (e.g. p and q) to from a new one (e.g. $p \to q$). Predicational theories formalise grounding as a predicate between some entities, usually understood as facts, sentences, propositions or truths. Thus, the ground predicate takes terms denoting some entities to form a sentence. This is the usual approach used in theories of truth, where truth is formalised as a unary predicate that is applied to terms that denote sentences. For example, given a sentence p, $Tr(\ulcorner p \urcorner)$ (where $\ulcorner p \urcorner$ is the name of the sentence p) is a formalisation of the sentences that says that the sentence p is true.

This difference in grammatical form has relevant theoretical and philosophical implications. At first hand, it is not clear which approach is the superior one. There are various *pros* and *cons* in adopting an approach with respect to adopting the other. On the one hand, the predicational approach is more ontologically demanding. The predicate approach needs terms denoting the relata of ground and so, by Quine's criterion of ontological commitment, it commits us to the existence of the relata of ground. Thus, since we are committed to the existence of the relata of ground, we also need a background theory for them. The predicational approach presupposes the existence of an ontology of entities, such as facts, propositions or truths (Fine [7, p. 47], Correia and Schneider [15, p. 11], Correia [8, p. 254]). This is particularly worrying if the entities under consideration are facts, because it should in principle be possible to make claims of grounding without being committed to an ontology of facts (Correia [8, p. 254], Korbmacher [1, p. 166]). Instead, on the operator approach, all this is not necessary.

On the other hand, predicational theories are more expressive than operational ones. This is because, on the predicational approach, it is natural to quantify over the entities under consideration and express quantified claims about the ground and the grounded of a ground statement. For example, on the predicational approach, the principle that a sentence is true iff its truth is either fundamental or grounded in some other truth can be straightforwardly formalized as:

$$\forall x(Tr(x) \leftrightarrow (Fund(x) \vee \exists y(y \lhd x))),$$

where Tr and $Fund$ are, respectively, unary predicates for true and fundamental sentences.

Formally, it is possible to achieve similar results in terms of expressivity on the operator approach by using quantification into sentence position or *propositional quantification*. However, quantification over sentences is not usually considered as legitimate (Correia and Schneider [15, pp. 11-12]). Moreover, even if one is willing to accept it, propositional quantification implies a significant deviation from classical logic. Instead, in the predicational approach, the same results can be achieved without deviating from classical logic. Thus, the predicational approach is advantageous for expressing this kind of principles. This also allows to define ground-theoretic concepts directly in the object language (Korbmacher [1, p. 164]). Moreover, as I will extensively discuss in this paper, adopting a predicational approach creates a natural setting to study the connections between ground and truth. In fact, theories of truth are usually formalised in a predicational setting. Thus, it is possible to use the same formal framework to develop theories of both ground and truth. Moreover, the expressivity of predicational theories allows to naturally express principles about the relation of ground and truth. Analogously to Korbmacher [1, 2], a substantial part of this paper will be focused on developing a theory of truth and ground in a predicational setting.

Another interesting result about predicational theories of grounding is that they are expressive enough to formulate paradoxes of self-referentiality for ground (Korbmacher [3]), analogous to the *Liar Paradox* for the truth predicate, which cannot be expressed in an operational framework. Moreover, they can formulate a predicational version of the puzzles of ground presented by Fine [4] and Krämer [5], which I will name *Fine's puzzle*[6]. Thus, the predicational setting implies more stringent criteria to establish the consistency of principles of grounding.

In conclusion, the question about which is the correct or most fruitful grammatical form to express ground statements remains open. Most of the recent literature focused on operational theories[7], while predicational theories have not been developed in the same detail yet. In this paper, I aim to expand the existing theories of predicational ground to replicate some results of the operational theories and derive results which cannot be achieved in an operational setting. In this paper, I do not aim to provide a conclusive answer to the question about which of the two approaches is superior. However, this paper is clearly sympathetic to the predicational approach. It provides evidence for the thesis that some interesting results about grounding can only be derived by predicational theories.

1.3 Predicational Theories of Ground

The only papers where a predicational theory of ground is fully developed are Korbmacher [1, 2]. In these two papers, Korbmacher develops axiomatic theories of partial, factive, mediate, strict conceptual ground. These theories are formulated using the same formal framework as so-called *axiomatic theories of truth* (Halbach [19]). The idea is, roughly, that we can use the language of

[6] See section 1.5 for a detailed presentation of the paradoxes of self-referentiality and *Fine's puzzle*.
[7] For example, see Fine [16], Correia [17] and deRosset and Fine [18].

arithmetic, i.e. the language of natural numbers, to talk about syntax via the method of Gödel numbering, which allows us to use numerals as names for individual sentences. This means that we can add a relational predicate for grounding to the language of arithmetic, analogously to the addition of a truth predicate in axiomatic theories of truth[8]. Thus, we obtain a unique framework for the investigation of theories of grounding, truth, and their relation.

Predicational theories of truth show that there are strong links between metaphysical grounding and truth. In particular, factive theories of grounding can be thought of as theories of how some truths ground others. Grounding relations between truths can be understood as principles that, given some truths, allow us to derive further ones. Thus, given that ground and truth are formalised in the same formal setting, it is natural to expect that results about the relation about theories can be proven. In fact, some of the main results of Korbmacher [1] and [2] are about the newly developed theories of ground and existing theories of truth.

More precisely, in [1], Korbmacher develops the base predicational theory of partial ground PG, which I now state in full detail.

Definition 1 (PG)**.** *The axioms of PG are the axioms of $PATG$[9] plus:*

Basic Ground Axioms:

G_1 $\forall x \neg(x \lhd x)$

G_2 $\forall x \forall y \forall z((x \lhd y \land y \lhd z) \rightarrow x \lhd z)$

G_3 $\forall x \forall y(x \lhd y \rightarrow Tr(x) \land Tr(y))$

Basic Truth Axioms:

T_1 $\forall s \forall t(Tr(s \underset{\cdot}{=} t) \leftrightarrow s^\circ = t^\circ)$

T_2 $\forall s \forall t(Tr(s \underset{\cdot}{\neq} t) \leftrightarrow s^\circ \neq t^\circ)$

T_3 $\forall x(Tr(x) \rightarrow Sent(x))$

Upward Directed Axioms:

U_1 $\forall x(Tr(x) \rightarrow x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x)$

U_2 $\forall x \forall y((Tr(x) \rightarrow x \lhd x \underset{\cdot}{\lor} y) \land (Tr(y) \rightarrow y \lhd x \underset{\cdot}{\lor} y))$

U_3 $\forall x \forall y(Tr(x) \land Tr(y) \rightarrow (x \lhd x \underset{\cdot}{\land} y) \land (y \lhd x \underset{\cdot}{\land} y))$

U_4 $\forall x \forall y((Tr(\underset{\cdot}{\neg} x) \rightarrow \underset{\cdot}{\neg} x \lhd \underset{\cdot}{\neg}(x \underset{\cdot}{\land} y)) \land (Tr(\underset{\cdot}{\neg} y) \rightarrow \underset{\cdot}{\neg} y \lhd \underset{\cdot}{\neg}(x \underset{\cdot}{\land} y)))$

[8] See section 2 for a detailed introduction of the technical framework.
[9] See section 2 for an introduction $PATG$ and for clarifications about the notation.

U_5 $\forall x\forall y(Tr(\underset{\cdot}{\neg}x)\wedge Tr(\underset{\cdot}{\neg}y)\to(\underset{\cdot}{\neg}x\lhd\underset{\cdot}{\neg}(x\underset{\cdot}{\vee}y))\wedge(\underset{\cdot}{\neg}y\lhd\underset{\cdot}{\neg}(x\underset{\cdot}{\vee}y)))$

U_6 $\forall x\forall t\forall v(Tr(x(t/v))\to x(t/v)\lhd\underset{\cdot}{\exists}vx)$

U_7 $\forall x\forall v(\forall t Tr(\underset{\cdot}{\neg}x(t/v))\to\forall t(\underset{\cdot}{\neg}x(t/v)\lhd\underset{\cdot}{\neg}\underset{\cdot}{\exists}vx))$

U_8 $\forall x\forall v(\forall t(Tr(x(t/v))\to\forall t(x(t/v)\lhd\underset{\cdot}{\forall}vx)))$

U_9 $\forall x\forall t\forall v(Tr(\underset{\cdot}{\neg}x(t/v))\to\underset{\cdot}{\neg}x(t/v)\lhd\underset{\cdot}{\neg}\underset{\cdot}{\forall}vx)$

Downward Directed Axioms:

D_1 $\forall x(Tr(\underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\to x\lhd\underset{\cdot}{\neg}\underset{\cdot}{\neg}x)$

D_2 $\forall x\forall y(Tr(x\underset{\cdot}{\vee}y)\to(Tr(x)\to x\lhd x\underset{\cdot}{\vee}y)\wedge(Tr(y)\to y\lhd x\underset{\cdot}{\vee}y))$

D_3 $\forall x\forall y(Tr(x\underset{\cdot}{\wedge}y)\to(x\lhd x\underset{\cdot}{\wedge}y)\wedge(y\lhd x\underset{\cdot}{\wedge}y))$

D_4 $\forall x\forall y(Tr(\underset{\cdot}{\neg}(x\underset{\cdot}{\wedge}y))\to(Tr(\underset{\cdot}{\neg}x)\to\underset{\cdot}{\neg}x\lhd\underset{\cdot}{\neg}(x\underset{\cdot}{\wedge}y))\wedge(Tr(\underset{\cdot}{\neg}y)\to\underset{\cdot}{\neg}y\lhd\underset{\cdot}{\neg}(x\underset{\cdot}{\wedge}y)))$

D_5 $\forall x\forall y(Tr(\underset{\cdot}{\neg}(x\underset{\cdot}{\vee}y))\to(\underset{\cdot}{\neg}x\lhd\underset{\cdot}{\neg}(x\underset{\cdot}{\vee}y))\wedge(\underset{\cdot}{\neg}y\lhd\underset{\cdot}{\neg}(x\underset{\cdot}{\vee}y)))$

D_6 $\forall x(Tr(\underset{\cdot}{\exists}vx(v))\to\exists t(x(t/v)\lhd\underset{\cdot}{\exists}vx))$

D_7 $\forall x\forall v(Tr(\underset{\cdot}{\neg}\underset{\cdot}{\exists}vx)\to\forall t(\underset{\cdot}{\neg}x(t/v)\lhd\underset{\cdot}{\neg}\underset{\cdot}{\exists}vx))$

D_8 $\forall x\forall v(Tr(\underset{\cdot}{\forall}vx)\to\forall t(x(t/v)\lhd\underset{\cdot}{\forall}vx))$

D_9 $\forall x\forall v(Tr(\underset{\cdot}{\neg}\underset{\cdot}{\forall}vx)\to\exists t(\underset{\cdot}{\neg}x(t/v)\lhd\underset{\cdot}{\neg}\underset{\cdot}{\forall}vx))$

Korbmacher [1] proves some important results about PG. First, PG is adequate with respect the operational theory developed by Schneider in [20], in the sense that all the sentences proved by the latter theory are also proved by the former one. Second, PG is a proof-theoretically conservative extension of the theory of *positive truth* PT and, so, it is proven to be a theory of ground and truth. Third, PG is consistent and so, by the completeness theorem for first-order logic, there is a first-order model of PG. Then, Korbmacher constructs the model of PG using *grounding-trees* and proves its correctness.

In [2], Korbmacher extends the base theory PG to PGA_α, the typed theory of partial ground with Aristotelian principles. The Aristotelian principles state that:

1) If ϕ is a true sentence, then $Tr(\ulcorner\phi\urcorner)$[10] holds either wholly or partially in virtue of ϕ.

2) If $\neg\phi$ is a true sentence, then $\neg Tr(\ulcorner\phi\urcorner)$ holds either wholly or partially in virtue of $\neg\phi$.

If the Aristotelian principles are formalised and added to PG, the resulting theory gives rise to a version of the puzzles presented by Fine [4] and Krämer [5] (*Fine's puzzle*[11]). To overcome the inconsistency, Korbmacher develops a theory of ground and typed truth. In a nutshell, he substitutes

[10] See section 2 for clarifications about the notation.
[11] See section 1.5 for a detailed presentation of the puzzle.

the truth predicate in PG with a Tarskian hierarchy of truth predicates. He develops a hierarchy of theories PGA_α for any ordinal $0 \leq \alpha < \epsilon_0$ in which, at every level α, PGA_α contains a new truth predicate in the Tarskian hierarchy. First, he proves that the theory PGA_α is a proof-theoretically conservative extension of the theory of *positive ramified truth*, PRT_α. Second, he proves that PGA_α is consistent. Third, he extends the model for the base theory PG to the new typed theory and proves its correctness. Finally, he relaxes the restriction that the ground predicate cannot be iterated (in the sense that terms denoting sentences containing the ground predicate cannot be arguments for the ground or truth predicate) and adds plausible principles about the relation between the ground and truth predicate. He shows that, if we are not careful when relaxing this restriction, we run into another paradox, the *paradox of self-referentiality for ground*[12]. This paradox can be understood as the analogous for ground of other paradoxes of self-reference, such as Tarski's paradox for predicational theories of truth (*Liar Paradox*) and Montague's paradox for predicational theories of modality.

These theories of predicational ground have some limitations. First, they are only theories of *partial* ground and not of *full* ground. As, I mentioned in section 2.1, this is a relevant limitation because there full ground is more primitive than partial ground. The reason for this is that, in a predicational setting, partial ground is a *one-to-one* relation, while full ground is a *many-to-one* relation. Thus, partial ground can be naturally expressed as a binary relation. Instead, expressing full ground would require either grouping the grounds together into a single entity and, so, applying ground predicate to sets of sentences, or introducing *multigrade predicates*, i.e. predicates that can take an arbitrary number of terms (even infinite) as arguments. However, both strategies bear some complications. The former one leads to cardinality issues for the coding function. The latter one implies a departure from standard first-order logic and the use of *plural logic.* Thus, extending the existing predicational theories of partial ground to full ground is a non-trivial matter (Korbmacher [1, pp. 189-190]). However, I think the results of this paper could be generalised without particular issues to predicational theories of full ground. Thus, I stick to easier and already well-developed framework of theories of partial ground in this paper. I leave the formulation of a predicational theory of full ground for future research.

Another limitation is that the existing predicational theories of ground is that they are expressed in a restricted language. The language in which the base theory of partial ground PG is formulated

[12] See section 1.5 for a detailed presentation of the paradox.

in Korbmacher [1] does not allow terms that denote sentences that contain either the truth of the ground predicate to be arguments of the truth of ground predicate. The theory in Korbmacher [2] relaxes this constraint for the truth predicate by typing the language and hints how to extend this solution to the ground predicate. The motivation for these restrictions is that, as I will show in section 1.5, relaxing them makes the theory inconsistent with very plausible principles about ground and truth due to paradoxes of self-refentiality and *Fine's puzzle*. The main objective of this paper is to overcome these issues and to formulate theories of ground in an unrestricted untyped language.

1.4 Theories of Iterated Ground

One of the objectives of this paper is to formulate a *theory of iterated ground* in a predicational setting. By *theory of iterated ground*, I mean a theory that derives results about which ground relations ground statements are part of. This is a natural question to ask because, given that a theory of ground aims to establish ground relations between facts or truths, 'a grounds b' can itself be a fact or truth and, so, be part of further ground relations as the *ground* or *grounded*. The formal framework in which the existing predicational theories of grounding are formulated excludes the possibility of deriving statements of iterated ground because the domain coding function is L (Korbmacher [1]) or L_{Tr} (Korbmacher [2]). Thus, there are not terms denoting statements of ground that can be arguments of the ground predicate.

There are a couple conceptual worries linked to the idea of iterated ground. First, suppose that statements of ground are not grounded by anything. A widespread view is that, if a fact is not grounded by anything, then it is fundamental. Also, another common view is that, if a fact is fundamental, so are its components. Thus, under these assumptions, all statements of ground are fundamental, and so are the *grounds* and *grounded* that compose them (Trogdon [21, pp. 115-116]). Thus, if the previous assumptions hold, it follows that all the facts that are part of relations of grounding are fundamental. This contrasts with the idea that grounding relations describe a hierarchical structure of reality where some fundamental facts ground derivative ones, which is popular among philosophers interested in metaphysical grounding. Thus, this argument suggests that advocates of metaphysical grounding should look for a theory of iterated ground. Second, Rabin and Rabern [22] investigate whether the idea that statements of ground are grounded is compatible with the thesis that the structure of ground is 'well-founded', in the sense that there exists a fundamental level of basic facts on which all derivative facts are grounded. The *wellfoundedness* of ground is common to many accounts of grounding. Thus, if it was incompatible with iterated ground, then a theory of iterated ground would need to reject one of the most popular principles of grounding. However, Rabin and Rabern [22] develop three distinct notions of *wellfoundedness* of a ground structure and show that neither of them is inconsistent with the fact that statements of ground are grounded.

From a formal point of view, the idea that statements of ground are part of ground relations naturally fits with both the operational and the predicational approach. On the operational perspective, statements of ground are expressed by sentences which are the results of applying the ground operator to other sentences. Formally, there is no reason why the ground operator should not apply to all the sentences of the language, including the ones that contain the ground operator. On the predicational perspective, we could more easily avoid iterated ground by restricting the domain of application of the ground predicate to a set of entities that do not contain facts, truths, propositions or sentences that correspond to statements of ground. However, in lack of an argument to exclude them, the natural choice is to take the most liberal position and allow iterated ground among the arguments of the ground predicate.

Litland [9, 23] develops an operational pure logic of iterated full ground that contains plausible principles of iterated ground. In summary, Litland's results are the following. Non-factive ground statements are zero-grounded in the sense of Fine [7, p. 48], while factive ground statements are grounded in the corresponding non-factive statements and on their *grounds* being the case. Thus, the basic notion of ground is non-factive and factive ground statements are derivative from non-factive ones. Also, non-factive ground statements hold because they are linked with a certain kind of explanatory arguments from their *grounds* to their *grounded.* In this paper, I am not interested in the non-factive ground. Thus, I aim to incorporate in a predicational framework only the factive part of Litland's logic of iterated ground, i.e. the fact that a ground statement is (partially) grounded in its *grounds* being the case. I call this principle *GG principle*[13]:

If ϕ holds in virtue of ψ, then $\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$[14] *holds in virtue of ϕ.*

Intuitively, it means that the *ground* grounds *that the ground grounds the grounded.* Formally, given two sentences ϕ and ψ, this principle claims that, if $\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$, then $\ulcorner\phi\urcorner \lhd \ulcorner\phi \underset{\cdot}{\lhd} \psi\urcorner$. This principle is also defended by other accounts of iterated ground, such as deRosset [24] and Bennett [25], and it is also named the *superinternality account.*

[13] The name *GG* reminds the fact that the *Ground Grounds* the whole ground statement it is part of.
[14] See section 2 for clarifications about the notation.

1.5 Paradoxes of Ground

Three potential sources of inconsistency will be relevant for the theories I am going to develop in sections 3 and 4. First, there is the the well-known Tarski theorem about self-referentiality of the truth predicate, or the *liar paradox*. Second, the *paradox of self-referentiality for the ground predicate*, which, roughly, is the analogous of the *liar paradox* for ground predicate. Third, a predicational version of the so called *Fine's puzzle*, which is an inconsistency that derives from some intuitively plausible principles of ground. In this section, I formally state and prove three theorems that show under which assumptions these paradoxes and puzzles are derived.

I state the three syntax conditions needed to derive the two paradoxes of self-reference (for the *liar paradox* the first two are sufficient). First, we need a theory T that is a sufficiently strong to talk about its sentences. More precisely, T proves that we have a unique name $\ulcorner\phi\urcorner$ for every sentence ϕ, in the sense that, for all sentences ϕ and ψ, $T \vdash \ulcorner\phi\urcorner = \ulcorner\psi\urcorner$ only if $\phi = \psi$[15]. Second, we need that T proves the *diagonal lemma*.

Lemma 1 (Diagonal Lemma). *A theory T proves the Diagonal Lemma iff, for all formulas $\phi(x)$ with exactly one free variable, there exists a sentence δ such that $T \vdash \delta \leftrightarrow \phi(\ulcorner\delta\urcorner)$.*

Third, we need a theory T that proves that we have a function symbol $\underset{\cdot}{\neg}$ that represents the syntactic operation $\neg$ of negation, in the sense that, for all sentences ϕ, $T \vdash \underset{\cdot}{\neg}\ulcorner\phi\urcorner = \ulcorner\neg\phi\urcorner$.

The reason why the paradoxes of self-referentiality are theoretically interesting is that these syntax conditions are rather weak. In fact, any standard background theory of syntax, such as Robinson arithmetic Q, satisfies all three of them. As we will see in section 2, The standard theory of arithmetic PA in which theories of truth and theories of grounding are developed satisfies these three syntax conditions because it is stronger that Q. Moreover, these conditions are not only results of some formal theories, but they reflect characteristics of natural languages. In fact, they are also able to talk about their own sentences and paradoxical sentences in formal languages have intuitive equivalents in natural ones.

We can now state the *liar paradox*:

Theorem 1 (Liar Paradox). *Take any theory T in the language L that satisfies the first two syntax conditions above and a unary predicate $Tr \in L$ such that, for every sentence ϕ of the language L, the T-scheme $T \vdash \phi \leftrightarrow Tr(\ulcorner\phi\urcorner)$ holds, then T is inconsistent.*

[15] See section 2 for clarifications about the notation.

Proof. Take the formula $\phi(x) = \neg Tr(x)$. By the *diagonal lemma*, there is λ such that $T \vdash \lambda \leftrightarrow \neg Tr(\ulcorner \lambda \urcorner)$.

1) $T \vdash \lambda \rightarrow \neg Tr(\ulcorner \lambda \urcorner)$ by *diagonal lemma*

2) $T \vdash \lambda \rightarrow Tr(\ulcorner \lambda \urcorner)$ by assumption

3) $T \vdash \lambda \rightarrow \bot$ by 1) and 2)

4) $T \vdash \neg \lambda$ by 3)

5) $T \vdash \neg \lambda \rightarrow Tr(\ulcorner \lambda \urcorner)$ by *diagonal lemma*

6) $T \vdash \neg \lambda \rightarrow \neg Tr(\ulcorner \lambda \urcorner)$ by assumption

7) $T \vdash \neg \lambda \rightarrow \bot$ by 5) and 6)

8) $T \vdash \bot$ by 4) and 7) $\square$

The sentence λ is not only a formal construction, but a natural language sentence with an analogous meaning can be easily stated. For example, consider:

1) 'The sentence (1) at p. 16 of this thesis is false'.

Assuming that the truth predicate in natural language is interpreted according to the T-scheme, it can be easily checked that it is impossible to consistently assign a truth-value to (1). If (1) is true, then what (1) asserts is the case, so (1) is false. If (1) is false, then what (1) asserts is not the case, so (1) is true.

To derive the analogous paradox of self-referentiality for the ground predicate, we need to assume the following two principles of ground, which are also axioms of PG (see section 1.3).

$$U_1 : \forall x(Tr(x) \rightarrow x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x)$$

$$G_3 : \forall x \forall y(x \lhd y \rightarrow Tr(x) \wedge Tr(y))$$

Both U_1 and G_3 are very plausible principles of a theory of factive ground. U_1 states that, if a sentence is true, it grounds its double negation, G_3 states that $\lhd$ is a predicate of *factive* ground, i.e. both grounds and the grounded of a grounding relation are true.

Then, we also assume two principles about the interaction between the ground and truth predicates:

$$G^+ : \forall s \forall t(Tr(s \underset{\cdot}{\lhd} t) \leftrightarrow s^\circ \lhd t^\circ),$$

$$G^- : \forall s \forall t(Tr(\underset{\cdot}{\neg}(s \underset{\cdot}{\lhd} t)) \leftrightarrow \neg(s^\circ \lhd t^\circ)).$$

These axioms states that the ground predicate behaves as one expects with respect to the truth predicate. In fact, they are just the instances for the ground predicate of the general scheme of axioms that one normally adds to a theory containing the truth predicate when a new relation symbol R is added:

$$R^+ : \forall t_1, ..., t_n(Tr(\underset{\cdot}{R}(t_1, ..., t_n)) \leftrightarrow R(t_1^\circ, ..., t_n^\circ)),$$
$$R^- : \forall t_1, ..., t_n(Tr(\underset{\cdot}{\neg}\underset{\cdot}{R}(t_1, ..., t_n)) \leftrightarrow \neg R(t_1^\circ, ..., t_n^\circ)).$$

We can now state the paradox of self-referentiality for the ground predicate[16]:

Theorem 2 (**Paradox of self-referentiality for ground**). *Any theory T that satisfies the three syntax conditions above, U_1, G_3, G^+ and G^- is inconsistent.*

Proof. Take the formula $\phi(x) = \neg(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)$. By the *diagonal lemma*, there is σ such that $T \vdash \sigma \leftrightarrow \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$.

1) $T \vdash \neg\sigma \rightarrow \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner$ by *diagonal lemma*

2) $T \vdash \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner \rightarrow Tr(\ulcorner\sigma\urcorner)$ by G_3

3) $T \vdash Tr(\ulcorner\sigma\urcorner) \rightarrow \sigma$ by G^-[17]

4) $T \vdash \neg\sigma \rightarrow \sigma$ by (1), (2) and (3)

5) $T \vdash \sigma$ by (4)

6) $T \vdash \sigma \rightarrow Tr(\ulcorner\sigma\urcorner)$ by G^-

7) $T \vdash Tr(\ulcorner\sigma\urcorner) \rightarrow \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner$ by U_1

8) $T \vdash \sigma \rightarrow \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner$ by (7) and (8)

9) $T \vdash \sigma \rightarrow \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$ by *diagonal lemma*

10) $T \vdash \sigma \rightarrow \bot$ by (8) and (9)

11) $T \vdash \bot$ by (4) and (10) □

It is important to note that using U_1 is not necessary to derive the contradiction. It is possible to prove the result in the exact same way using grounding principles for other connectives, such as $\ulcorner\phi\urcorner \lhd \ulcorner\phi \vee \phi\urcorner$ or $\ulcorner\phi\urcorner \lhd \ulcorner\phi \wedge \phi\urcorner$.

As for the *liar sentence*, we can find a natural language sentence that corresponds to the paradoxical sentence σ. For example, consider:

2) 'The sentence (2) at p. 17 of this thesis does not ground its own double negation'.

[16] See Korbmacher [3, pp. 4-5], [2, pp. 219-221] for analogous results.
[17] Note that I apply G^- and not G^+ because $\sigma = \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$.

If we try to determine whether what (2) asserts is the case and we reason assuming that the assumptions of Theorem 2 hold, we run into a paradox. In fact, if (2) grounds its own double negation, then what (2) asserts is not the case. But if (2) grounds its own double negation, then (2) is true because only true claims can ground something else, then what (2) asserts is the case and (2) does not ground its own double negation. If (2) does not ground its own double negation, then what (2) asserts is the case, then (2) is true, then (2) grounds its own double negation because all true claims ground their double negation. Then, what (2) asserts is the case and (2) does not ground its double negation.

The third inconsistency I will deal with is *Fine's puzzle.* Differently from the previous two, this inconsistency does not depend on self-referentiality, but on the fact that some intuitively valid principles of ground are inconsistent between each other. Fine [4] and Krämer [5] present various versions of the inconsistency. Here, I present a version of the puzzle in a predicational setting analogous to the one in Korbmacher [2, pp. 198-199]. The principles of ground needed to derive the contradiction are the following. First, one of the two so-called *Aristotelian principles* AP_T and AP_F:

$$AP_T : \forall x(Tr(x) \to x \lhd \underset{\cdot}{Tr}(x))$$
$$AP_F : \forall x(Tr(\underset{\cdot}{\neg} x) \to \underset{\cdot}{\neg} x \lhd \underset{\cdot}{\neg}\underset{\cdot}{Tr}(x)).$$

Intuitively, AP_T states that, if a sentence ϕ is true, then ϕ is true in virtue of what it says being the case, and AP_F states that, if ϕ is false, then ϕ is not true in virtue of what it says not being the case. The motivation for these principles traces back to Aristotle's *Metaphysics*, where he claims that:

> *It is not because we think truly that you are pale, that you are pale; but because you are pale we who say this have the truth.* (Metaphysics 1051b6–9)

Then, we need to assume G_3 and other three plausible principles of grounding, which are also axioms of PG (see section 1.3). First, that an existential is grounded in one of its true instances, given that the instance under consideration is true. Let x be a formula, v a variable and t a term, then:

$$U_6 : \forall x \forall t \forall v(Tr(x(t/v)) \to x(t/v) \lhd \underset{\cdot}{\exists} vx).$$

Second, that grounding is transitive:

$$G_2 : \forall x \forall y \forall z((x \lhd y \land y \lhd z) \to x \lhd z).$$

Third, that grounding is irreflexive:

$$G_1 : \forall x \neg (x \lhd x).$$

Theorem 3 (Fine's puzzle). *A theory T that contains G_3, AP_T, U_6, G_2 and G_1 and proves the truth of an arbitrary sentence ϕ is inconsistent.*

Proof. [18]

ϕ is an arbitrary sentence.

1) $T \vdash Tr(\ulcorner \phi \urcorner)$ by assumption

2) $T \vdash \ulcorner \phi \urcorner \lhd \ulcorner Tr(\ulcorner \phi \urcorner) \urcorner$ by AP_T

3) $T \vdash Tr(\ulcorner Tr(\ulcorner \phi \urcorner) \urcorner)$ by G_3

4) $T \vdash \ulcorner Tr(\ulcorner \phi \urcorner) \urcorner \lhd \ulcorner \exists x Tr(x) \urcorner$ by U_6

5) $T \vdash Tr(\ulcorner \exists x Tr(x) \urcorner)$ by G_3

6) $T \vdash \ulcorner \exists x Tr(x) \urcorner \lhd \ulcorner Tr(\ulcorner \exists x Tr(x) \urcorner) \urcorner$ by AP_T

7) $T \vdash Tr(\ulcorner Tr(\ulcorner \exists x Tr(x) \urcorner) \urcorner)$ by G_3

8) $T \vdash \ulcorner Tr(\ulcorner \exists x Tr(x) \urcorner) \urcorner \lhd \ulcorner \exists x Tr(x) \urcorner$ by U_6

9) $T \vdash \ulcorner \exists x Tr(x) \urcorner \lhd \ulcorner \exists x Tr(x) \urcorner$ by (6), (8) and G_2

10) $T \vdash \neg(\ulcorner \exists x Tr(x) \urcorner \lhd \ulcorner \exists x Tr(x) \urcorner)$ by G_1

11) $T \vdash \bot$ by (9) and (10) $\square$

It is important to note that is it possible to derive the contradiction in the exact same way by using the other Aristotelian principle AP_F instead of AP_T.

Intuitively, the contradiction derives because the theory T proves two plausible claims in about ground, which, intuitively, mean:

1) 'The fact there exists something true grounds that it is true that there exists something true',
2) 'It is true that there exists something true grounds the fact that there exists something true'.

(1) and (2) are clearly contradictory if grounding is transitive and irreflexive.

2 Technical Background and Notation

In this section, I will describe the technical tools and introduce the notation needed to develop the theories in the following sections[19]. First, in section 2.1, I introduce the technical framework

[18] This proof is analogous to Korbmacher [2, pp. 198-199].
[19] A similar introduction to this technical background can be found in Halbach [19, pp. 29-38] and Korbmacher [1, pp. 167-169]

in which the base theory of partial ground PG is formulated in [1] (which is the same as the one used by Halbach [19] to develop axiomatic theories of truth). PG will be the base to develop the semantics in sections 3.2 and 4.3 and I will develop the axiomatic theory of section 3.3 within the same framework of PG. More precisely, in this section, I define the languages I will work with, I briefly describe the technique of Gödel-numbering, I define the base theory PA of arithmetic and its useful extensions, I define useful recursive functions and sets, I briefly mentions a few useful remarks about the valuation function and the models I will work with and I will define PG^*, which is a slightly modified version of PG.

Second, in section 2.2, I will introduce the Strong Kleene (K_3) logic and its evaluation schemata that I will use to develop the semantics in sections 3.2 and 4.3 and I will provide an alternative definition of the formulas of a language based on an induction over literals instead of atomic sentences, which will be useful to prove some results in the rest of the paper.

2.1 Technical Framework of Axiomatic Theories of Ground

PG and the axiomatic theory of section 3.3 are developed within the background theory of arithmetic PA. The language of PA, which I name L, is composed of the standard arithmetic vocabulary: the individual constant 0, the unary function symbol s, the binary function symbols $+$ and $\times$, the binary relation $=$ and the logical symbols of first-order logic. For every natural number n, the numeral $\overline{n}$ is the n-fold application of s to the constant 0. Thus, the numeral $\overline{n}$ is a term of L which denotes the number n. The language of truth is $L_{Tr} = L \cup \{Tr\}$, where Tr denotes the unary truth predicate. The language of predicational ground is $L_{\lhd} = L \cup \{\lhd\}$, where $\lhd$ denotes the binary grounding relation. The language of ground and truth $L^{\lhd}_{Tr}$ is $L \cup \{Tr\} \cup \{\lhd\}$. In section 3.3, I will also the unary falsity predicate F and the not grounding predicate $\not\lhd$ and I will extend the language accordingly when needed.

It is well known that, through the technique of Gödel-numbering, the theory of arithmetic is also a theory of syntax, in the sense that PA can talk about the sentences of its own languages. In fact, the technique of Gödel-numbering is used to obtain names for every expression of the language. In particular, # is an injective function from a string of symbols σ of the language to a natural number $\#\sigma$. $\ulcorner\sigma\urcorner$ is the numeral that denote the Gödel-numbering $\#\sigma$ of a string of symbols σ. Thus, $\ulcorner\sigma\urcorner$ is a term that correspond to the name of a string of symbols σ. In Korbmacher [1, 2], axiomatic theories of ground are formulated in $L^{\lhd}_{Tr}$, but with restrictions on the coding function to subsets of $L^{\lhd}_{Tr}$ (either L, $L_{\lhd}$, or L_{Tr}). In the following sections, I will relax this constraint and allow the coding function over $L^{\lhd}_{Tr}$.

The theory PA of arithmetic consists of the standard axioms for zero, the successor function, addition, and multiplication, plus all the instances of the induction scheme:

$$\phi(0) \wedge \forall x(\phi(x) \rightarrow \phi(s(x))) \rightarrow \forall x \phi(x)$$

over formulas $\phi(x)$ in the language L. The theory PAT extends PA with the missing instances of the induction scheme over L_{Tr}. The theory PAG extends PA with all the missing instances of the induction scheme over $L_{\lhd}$. The theory $PATG$ extends PA with all the missing instances of the induction scheme over $L^{\lhd}_{Tr}$.

Derivability in a theory T is expressed by $T \vdash$ or $\vdash_T$. It is well-known that PA can represent any recursive function, in the sense that, if f is a recursive function, then there is a formula $\phi(x, y)$ such that, for all natural numbers n, m:

$$f(n) = m \text{ iff } \vdash_{PA} \forall x(\phi(\overline{n}, x) \leftrightarrow x = \overline{m}).$$

Many syntactic functions on the codes of expressions are recursive and thus representable. In particular, the functions that correspond to the logical operations and relations between Gödel numbers are recursive. For example, the function that maps the code $\#\phi$ of a formula ϕ to the code $\#\neg\phi$ of its negation is recursive, as it is the function that maps the codes $\#\phi$ and $\#\psi$ of the formulas ϕ and ψ to the code $\#(\phi \wedge \psi)$ of their conjunction, and so on for the other logical operators. Moreover, the function that maps the code $\#t$ of a term t to the code $\#Tr(t)$ of the atomic formula $Tr(t)$ is recursive, as it is the function that maps the codes $\#s$ and $\#t$ of two terms to the code $\#(s \lhd t)$ of the atomic formula $s \lhd t$, and analogously for $=$. For convenience, I define function symbols for these functions. If f is a recursive functions, I use $\underset{\cdot}{f}$ as a symbol for it. Thus:

$\vdash_{PA} \underset{\cdot}{\neg}\ulcorner\phi\urcorner = \ulcorner\neg\phi\urcorner$ $\quad$ $\vdash_{PA} \ulcorner\phi\urcorner \underset{\cdot}{\wedge} \ulcorner\psi\urcorner = \ulcorner\phi \wedge \psi\urcorner$ $\quad$ $\vdash_{PA} \ulcorner\phi\urcorner \underset{\cdot}{\vee} \ulcorner\psi\urcorner = \ulcorner\phi \vee \psi\urcorner$

$\vdash_{PA} \underset{\cdot}{\exists}(\ulcorner v\urcorner, \ulcorner\phi\urcorner) = \ulcorner\exists v\phi\urcorner$ $\quad$ $\vdash_{PA} \underset{\cdot}{\forall}(\ulcorner v\urcorner, \ulcorner\phi\urcorner) = \ulcorner\forall v\phi\urcorner$ $\quad$ $\vdash_{PA} \ulcorner s\urcorner \underset{\cdot}{=} \ulcorner s\urcorner = \ulcorner s = t\urcorner$

$\vdash_{PA} \underset{\cdot}{Tr}(\ulcorner t\urcorner) = \ulcorner Tr(t)\urcorner$ $\quad$ $\vdash_{PA} \ulcorner s\urcorner \underset{\cdot}{\lhd}(\ulcorner t\urcorner) = \ulcorner s \lhd t\urcorner$.

PA can also (strongly) represent every recursive set, in the sense that, if S is a recursive set, then there is a formula $\phi(x)$ such that for all natural numbers n:

$n \in S$ iff $\vdash_{PA} \phi(\overline{n})$ $\quad$ $n \notin S$ iff $\vdash_{PA} \neg\phi(\overline{n})$.

$Sent$ abbreviates the formula that represents the recursive set of codes sentences in L, $Sent^{\lhd}_{Tr}$ abbreviates the formula that represents the codes of sentences in $L^{\lhd}_{Tr}$ and analogously for all the other languages defined above. Similarly, Var and $ClTerm$ are abbreviations for the formulas that represent the sets of (codes of) variables and closed terms. As an abbreviation for $\forall x(Var(x) \rightarrow \phi(x))$, we write $\forall v\phi(v)$ and, as an abbreviation for $\forall x(ClTerm(x) \rightarrow \phi(x))$, we write $\forall t\phi(t)$.

I will use the symbol $^\circ$ for the evaluation function, i.e. the function from closed terms of the language to their value. Note that, if there are function symbols for certain primitive recursive functions in the language, there cannot be a symbol representing this function (Halbach [19, p. 16]). Nevertheless, I will use $x^\circ = y$ to express that the value of x is y and $s^\circ = t^\circ$ to express that s and t coincide in their values.

In the following sections, I will work with a modified version of PG, under two respects. First, in its original formulation in Korbmacher [1], PG is expressed in the language $L^{\lhd}_{Tr}$ with the coding function restricted to L. Instead, I extend the coding function to $L^{\lhd}_{Tr}$. This modification allows us to generate more terms that can be arguments of the truth or ground predicate. In particular, we add terms that denote sentences that contain the ground and truth predicates. Second, I need to weaken axiom T_3. The reason for this is that the main aim of the theories that I am going to develop is to extend PG to derive sentences with iterations of the truth and/or ground predicates. In PG, by T_3, it follows that:

$$PG \vdash \forall x(\neg Sent(x) \rightarrow \neg Tr(x))$$

and by G_3 and T_3:

$$PG \vdash \forall x(\neg Sent(x) \vee \neg Sent(x) \rightarrow \neg(x \lhd y)).$$

Thus, PG proves that, if a sentence is not in L, then it cannot be the argument of the truth or ground predicates. This means that all iterations of the truth and/or ground predicates are false. Thus, I substitute the axiom T_3 with:

$$T_3^* : \forall x(Tr(x) \rightarrow Sent^{\lhd}_{Tr}(x))$$

and I call the resulting theory PG^*. In section 3.2, I will prove that stronger than PG^* theories are consistent. Therefore, it follows that PG^* is also consistent.

2.2 Strong Kleene Logic

In sections 3.2 and 4.3, I develop semantic models for truth and ground based on the *Strong Kleene* logic K_3. K_3 is a three-valued logic with the extra truth-value i other than the classical ones 0 and 1. i is interpreted as an intermediate truth-value between truth and falsity. Intuitively, i means *undetermined* or *neither true nor false*. The three truth-value give rise to an ordering, where $0 \leq i \leq 1$. The consequence relation $\vDash_{K_3}$ is defined as the preservation of the designated value 1:

$$\Gamma \vDash_{K_3} \phi \text{ iff, for any valuation } v \text{ such that } v(\psi) = 1 \text{ for all } \psi \in \Gamma,\ v(\phi) = 1.$$

The most relevant difference between K_3 and classical logic is that the *law of excluded middle* is not valid in K_3, i.e. $\nvDash_{K_3} \phi \vee \neg\phi$. Thus, K_3 admits truth-value gaps, sentences which are neither true nor false. Instead, K_3 is not a paraconsistent logic, the principle of *explosion* is valid in K_3 i.e. for all ϕ, ψ, $\phi \wedge \neg\phi \vDash_{K_3} \psi$ because the premise $\phi \wedge \neg\phi$ is always $\neq 1$. Thus, K_3 does not admit truth-values gluts, sentences that that are both true and false.

A first order K_3-model M over a language L is a structure (D, I, v), where the domain D is a non-empty set, the universe of discourse over which the variables of the language range, I is an interpretation function that assigns meanings to the symbols of the language and v is K_3-valuation function that assign truth-values to the formulas of the language according to the K_3-truth tables for the connectives and quantifiers. More precisely:

- I assigns the element $I(c)$ of the domain D to each constant symbol c in L,
- I assign the n-ary function $f^M : D^n \to D$ to each n-ary function symbol f in L,
- I assign the extension $P^+ \subseteq D^n$ and the anti-extension $P^- \subseteq D^n$, with $P^+ \cap P^- = \emptyset$, to each n-ary predicate symbol P in L.

The valuation function v assign truth-values to the atomic sentences of L as:

$$v(P(c_1, ..., c_n)) = \begin{cases} 1 & \text{if } \langle v(c_1), ..., v(c_n)\rangle \in P^+ \\ 0 & \text{if } \langle v(c_1), ..., v(c_n)\rangle \in P^- \\ i & \text{otherwise} \end{cases}$$

and to complex formulas as:

$$v(\neg\phi) = \begin{cases} 1 & \text{if } v(\phi) = 0 \\ i & \text{if } v(\phi) = i \\ 0 & \text{if } v(\phi) = 1 \end{cases}$$

- $v(\phi \wedge \psi) = min\{v(\phi); v(\psi)\}$
- $v(\phi \vee \psi) = max\{v(\phi); v(\psi)\}$
- $v(\exists x \phi(x)) = max\{v(\phi(\overline{n})) : n \in D\}$
- $v(\forall x \phi(x)) = min\{v(\phi(\overline{n})) : n \in D\}$

The connective $\to$ can be defined in terms of $\neg$ and $\vee$ as:

$$\phi \to \psi \text{ iff } \neg\phi \vee \psi.$$

Differently with respect to classical logic, the interpretation of a predicate symbol P is not its extension only, because its anti-extension cannot be implicitly defined as the complement of the extension. Instead, the anti-extension is explicitly defined as a subset of the domain P^- disjoint from the extension P^+. In fact, there can be objects of the domain that are neither in the extension nor in the anti-extension of a predicate P. This is the semantic counterpart of the fact the *law of excluded middle* is not valid in K_3 logic. In the limit case in which $P^+ \cup P^- = D$ for all predicate symbols, then the K_3 model is a classical one. In fact, in this case, all atomic formulas have truth-value 0 or 1 and it is easy to check by induction that, if this is the case, then all the formulas have truth-value 0 or 1.

In the next sections, I will work with a K_3-model M in the language $L_{Tr}^{\triangleleft}$ with the natural numbers $\mathbb{N}$ as domain and the interpretation function I which assigns:

- the natural number 0^M to the only constant symbol 0,
- the function $s^M : \mathbb{N} \rightarrow \mathbb{N}$, where $s^M(n) = n + 1$, to the unary function s,
- the functions $+^M : \mathbb{N}^2 \rightarrow \mathbb{N}$ and $\times^M : \mathbb{N}^2 \rightarrow \mathbb{N}$, interpreted as the usual addition and multiplication, to the binary functions $+$ and $\times$,
- the relation $=^M \subseteq \mathbb{N}^2$, interpreted as the usual identity relation, to the predicate symbol $=$,
- the extension $R \subseteq \mathbb{N}^2$ and anti-extension $\overline{R} \subseteq \mathbb{N}^2$, with $R \cap \overline{R} = \emptyset$, to the ground predicate $\triangleleft$,
- the extension $S \subseteq \mathbb{N}$ and anti-extension $\overline{S} \subseteq \mathbb{N}$, with $S \cap \overline{S} = \emptyset$, to the truth predicate Tr.

It is important to highlight two remarks. First, the interpretation of the arithmetical symbols of L is classical. In particular, the anti-extension of the identity $=$ is the complement of its extension of its extension $=^M$. Therefore, there no formulas in L with truth-value i and the logical truths of classical logic holds for the sub-language L. Thus, in the K_3 models I will develop, the properties of K_3 models only holds for the portion of the language that extends the language of arithmetic. Instead, the structure that interprets the language of arithmetic maintains its classical properties. For this reason, the syntactic properties of the classical theory of arithmetic PA still holds in my K_3 model. Second, I will exclusively work in the context of the standard model of PA. This model of L has the set $\mathbb{N}$ of the natural numbers as its domain and I defined above as interpretation function on L. I do not allow for nonstandard interpretations of the arithmetic vocabulary.

In the next sections, I will use the following *Strong Kleene* evaluation schema. The semantic clauses for literals are:

$M \vDash_{K_3} t = s$ iff $\mathbb{N} \vDash t = s$,

$M \vDash_{K_3} \neg(t = s)$ iff $\mathbb{N} \vDash \neg(t = s)$,

$M \vDash_{K_3} Tr(\ulcorner\phi\urcorner)$ iff $\#\phi \in S$,

$M \vDash_{K_3} \neg Tr(\ulcorner\phi\urcorner)$ iff $\#\phi \in \overline{S}$,

$M \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ iff $\langle\#\phi; \#\psi\rangle \in R$,

$M \vDash_{K_3} \neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$ iff $\langle\#\phi; \#\psi\rangle \in \overline{R}$.

The semantic clauses for complex sentences are:

$M \vDash_{K_3} \neg\neg\phi$ iff $M \vDash_{K_3} \phi$,

$M \vDash_{K_3} \phi \wedge \psi$ iff $M \vDash_{K_3} \phi$ and $M \vDash_{K_3} \psi$,

$M \vDash_{K_3} \neg(\phi \wedge \psi)$ iff $M \vDash_{K_3} \neg\phi$ or $M \vDash_{K_3} \neg\psi$,

$M \vDash_{K_3} \phi \vee \psi$ iff $M \vDash_{K_3} \phi$ or $M \vDash_{K_3} \psi$,

$M \vDash_{K_3} \neg(\phi \vee \psi)$ iff $M \vDash_{K_3} \neg\phi$ and $M \vDash_{K_3} \neg\psi$,

$M \vDash_{K_3} \forall x\phi(x)$ iff for all $n \in \mathbb{N}$, $M \vDash_{K_3} \phi(\ulcorner n\urcorner)$,

$M \vDash_{K_3} \neg\forall x\phi(x)$ iff there is $n \in \mathbb{N}$, $M \vDash_{K_3} \neg\phi(\ulcorner n\urcorner)$,

$M \vDash_{K_3} \exists x\phi(x)$ iff there is $n \in \mathbb{N}$, $M \vDash_{K_3} \phi(\ulcorner n\urcorner)$,

$M \vDash_{K_3} \neg\exists x\phi(x)$ iff for all $n \in \mathbb{N}$, $M \vDash_{K_3} \neg\phi(\ulcorner n\urcorner)$.

The formulas of a language are usually defined by the induction over atomic formulas. In this paper, I use a slightly different definition based on induction over *literals* instead of atomic formulas, where a literal is either an atomic formulas or its negation.

Definition 2 (**Formulas of language** L)**.** *Given a language L, the set of formulas $Form_L$ of L is defined by induction:*

- *if ϕ is a literal then $\phi \in Form_L$,*
- *if $\phi = \neg\neg\psi$ and $\psi \in Form_L$, then $\phi \in Form_L$,*
- *if $\phi = \psi \wedge \delta$, $\psi \in Form_L$ and $\delta \in Form_L$, then $\phi \in Form_L$,*
- *if $\phi = \psi \vee \delta$, $\psi \in Form_L$ and $\delta \in Form_L$, then $\phi \in Form_L$,*
- *if $\phi = \neg(\psi \wedge \delta)$, $\psi \in Form_L$ and $\delta \in Form_L$, then $\phi \in Form_L$,*
- *if $\phi = \neg(\psi \vee \delta)$, $\psi \in Form_L$ and $\delta \in Form_L$, then $\phi \in Form_L$,*
- *if $\phi = \exists v\psi$ and $\psi \in Form_L$, then $\phi \in Form_L$,*
- *if $\phi = \forall v\psi$ and $\psi \in Form_L$, then $\phi \in Form_L$,*
- *if $\phi = \neg\exists v\psi$ and $\psi \in Form_L$, then $\phi \in Form_L$,*
- *if $\phi = \neg\forall v\psi$ and $\psi \in Form_L$, then $\phi \in Form_L$,*

- *nothing else is a formula of L.*

In the next sections, when I show that a result holds for all sentences of a language, I will first prove that it holds for all literals of the language and then that it is preserved trough the inductive steps of Definition 2.

3 Untyped Theory of Partial Ground and Truth

In this section, starting from the base theory PG, I develop theories that derive plausible instances of iteration of the ground and the truth predicates in a type-free language. First, in section 3.1, I motivate the theoretical and philosophical relevance of the theories I am going to develop and explain the technical approach I will use. In particular, I explain in which sense these theories extend PG by deriving claims that contain iteration of the ground and the truth predicates, I explain the advantages of adopting of an type-free setting and I provide more details about the technical framework I will apply in sections 3.2 and 3.3.

Second, in section 3.2, I develop a K_3-semantic model that satisfies plausible instances of iteration of the ground and the truth while avoiding the paradoxes of self-referentiality of Theorems 1 and 2. More precisely, I define a sequence of models in the style of Kripke's fixed point semantics (Kripke [6]) and show that it has fixed points. I take the extensions and anti-extensions at the fixed points as the extensions and anti-extensions of the truth and ground predicates. Then, I prove that the fixed points are not inconsistent because of the paradoxes of self-referentiality.

Third, in section 3.3, I develop the axiomatic theory of *untyped partial ground and truth* $PUGT$. To do so, I first construct a classical model based on one of the fixed points of the construction of section 3.2. To replicate the results of the K_3-model in a classical setting, I add to the language a falsity predicate and a predicate of *non-grounding.* Then, I prove that $PUGT$ is sound with respect this model, that it proves the *Kripke-Feferman* theory of truth KF and that $PUGT$ proves the same theorems of KF in the language L_{Tr} with the coding function over L_{Tr}.

It is important to underline that the theories developed in this section do not incorporate the Aristotelian principles:

If ϕ *is a true sentence, then* $Tr(\ulcorner\phi\urcorner)$ *holds either wholly or partially in virtue of* ϕ*,*

If $\neg\phi$ *is a true sentence, then* $\neg Tr(\ulcorner\phi\urcorner)$ *holds either wholly or partially in virtue of* $\neg\phi$*.*

nor the GG principle:

If ϕ holds in virtue of ψ, then $\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ holds in virtue of ϕ.

The reason for this is that, as I will show in more detail in section 4.1, naively adding the Aristotelian principles to the theories developed in sections 3.2 and 3.3 gives rise an inconsistency due to Fine's puzzle (Th. 3). Also, in section 4.2, I will show that adding the GG principle results in a similar inconsistency involving the ground predicate instead of the truth predicate. In section 4.3, I will provide a solution to add the Aristotelian and GG principles to a theory of ground inspired by the one in section 3.2.

3.1 Approach and Motivation

In sections 3.2 and 3.3, I develop a semantic model and an axiomatic theory that extend the base predicational theory of partial ground PG. More precisely, I develop a theory that formalises iterations of the ground predicate, iterations of the truth predicate, and combinations of both. By iteration of the ground predicate, I mean sentences in which a ground statement is among the arguments of another ground relation, as in:

$$(\ulcorner\ulcorner 0 = 0\urcorner \lhd \ulcorner\neg\neg 0 = 0\urcorner\urcorner) \lhd (\ulcorner\neg\neg(\ulcorner 0 = 0\urcorner \lhd \ulcorner\neg\neg 0 = 0\urcorner)\urcorner) \text{ or}$$
$$(\ulcorner 0 = 0\urcorner \lhd \ulcorner\neg\neg 0 = 0\urcorner) \lhd (\ulcorner\ulcorner 0 = 0\urcorner \lhd \ulcorner\neg\neg 0 = 0\urcorner\urcorner \wedge 1 = 1).$$

By iteration of the truth predicate, I mean sentences in which a statement containing the truth predicate is the argument of the truth predicate, as in:

$$Tr(\ulcorner Tr(\ulcorner 0 = 0\urcorner)\urcorner) \text{ or}$$
$$Tr(\ulcorner\neg Tr(\ulcorner 0 = 1\urcorner) \wedge 0 = 0\urcorner).$$

By combinations of the of both, I mean sentences in which both the previous two cases realise, as in:

$$\ulcorner Tr(\ulcorner 0 = 0\urcorner)\urcorner \lhd \ulcorner\neg\neg Tr(\ulcorner 0 = 0\urcorner)\urcorner \text{ or}$$
$$Tr(\ulcorner\ulcorner 0 = 0\urcorner \lhd \ulcorner\neg\neg 0 = 0\urcorner\urcorner).$$

The main difference between my approach and the existing literature on this topic (in particular, Korbmacher [2]) is that I develop my theory in a type-free setting, i.e. one in which the language contains only one truth and one ground predicate. The alternative approach would be to use a typed setting, i.e. one in which there is a Tarskian hierarchy of languages L_0, L_1, L_2, ... , L_α, ... for every ordinal $0 \leq \alpha < \epsilon_0$, each of whom contains new truth and ground predicates of 'higher level'. Thus, such hierarchy is:

$$L_0 = L,$$
$$L_1 = L \cup \{Tr_1\} \cup \{\lhd_1\},$$
$$L_2 = L_1 \cup \{Tr_2\} \cup \{\lhd_2\},$$
$$...$$
$$L_\alpha = L_{\alpha-1} \cup \{Tr_\alpha\} \cup \{\lhd_\alpha\},$$
$$...$$

The idea is that, at each level α, L_α takes the previous language $L_{\alpha-1}$ in the hierarchy as its object-language. Given a level α, only terms of the language $L_{\alpha-1}$ can be arguments of Tr_α and $\lhd_\alpha$. Thus, Tr_α and $\lhd_\alpha$ can only be applied to terms that contain truth and ground predicates up to $Tr_{\alpha-1}$ and $\lhd_{\alpha-1}$ in the hierarchy. On the one hand, the main advantage of the typed approach is that it immediately provides a solution to the paradoxes and puzzles of section 1.5. In particular, the typed solution to the *Liar paradox* (Th. 1) is the well-known Tarski's solution to the paradox. A typed solution to the *paradox of self-referentiality for ground* (Th. 2) can be developed analogously. Korbmacher [2] provides a typed solution to *Fine's puzzle* (Th. 3).

On the other hand, the type-free approach has several advantages over the typed one. First, in the type-free approach, it is possible to express intuitively legitimate claims that cannot be expressed in a typed language. In fact, the typed approach takes the drastic solution to exclude the truth and ground predicates from the object-language. In other worlds, these predicates can never apply to themselves, but only to predicates of 'lower level'. However, there are relevant portions of both natural and formal languages where the application of these predicates to themselves does not seem problematic. Examples in formal languages are the ones at the beginning of this section. Examples in natural languages are:

1) 'It is true that it is true that snow is white',

2) 'If God grounds everything, then God grounds the fact that God grounds everything',

Intuitively, examples (1) and (2) and the examples above about the iteration of truth and/or ground are unproblematic because their truth-value ultimately depends on facts independent of the truth and ground predicate (respectively, the colour of snow, the properties of God, arithmetical facts)[20]. Unlike the typed approach, the type-free approach does not exclude from the outset the expressibility of these intuitively legitimate claims.

Second, as claimed by Kripke [6, pp. 694-699], the typed approach is suspicious as an analysis of our intuitions, especially about natural languages. It does not seem that natural languages contain

[20] See Kripke [6] and Leitgeb [26] for accounts of why sentences like these ones are not paradoxical.

any hierarchy of predicates, neither of truth, nor of ground. Advocates of the typed approach claim that such predicates are systematically ambiguous, in the sense that there is an implicit reference to a certain level of the hierarchy every time we use one of them. However, this claim is not convincing. Kripke shows that there are sentences containing the truth predicate to which, intuitively, we can unambiguously assign truth-values, but whose truth predicates cannot be assigned any level. For example, consider two sentences: the first, uttered by Dean, asserts:

(1) 'All of Nixon's utterances about Watergate are false'.

The second, uttered by Nixon, asserts:

(2) 'Everything Dean says about Watergate is false'[6, pp. 695-696].

Clearly, if we try to assign level to the truth predicates of these sentences, the level of the truth predicate in (1) would depend on the levels of truth predicates the Nixon's utterances. In particular, it would need to be higher than the level of truth predicates in any the Nixon's utterances. However, among Nixon's utterances, there is also (1), which makes choice of a level for the truth predicate impossible. An analogous reasoning follows it we start by trying to assign a level to the truth predicate in (2).

In section 3.2, I construct a K_3-model (see section 2 for an introduction) for ground and truth that avoids the *Liar paradox* (Th. 1) and the *paradox of self-referentiality for ground* (Th. 2). These paradoxes are avoided because using K_3 logic admits truth-value gaps when the extension and anti-extension of the predicates Tr and $\lhd$ are defined. Thus, the paradoxical sentences will lack truth-value and, so, the paradoxes will be blocked.

K_3 is not the only logic that can block the paradoxes. For example, I could have use a logic that admits truth-value gluts, such as LP[21] or FDE[22], and avoid the paradoxes by assigning both truth values to paradoxical sentences. The main reason why I choose the logic K_3 will become clearer in sections 4.3 and 5.1. In fact, in section 4, I will show that, when the Aristotelian or GG principles are added to the model of section 3.2, the latter becomes inconsistent because of a version of *Fine's puzzle*. One of the solution Fine proposes to his puzzle, the *impredicativist compromise* solution (Fine, [4, pp. 109-115]), includes the use a K_3-model. In section 4.3, I will build K_3 model than is consistent with the Aristotelian or GG principles and compare it with Fine's compromise solution. In section 5, I will argue that K_3 provides an intuitively very plausible logical framework

[21] Logic of Paradoxes
[22] First-Degree Entailment

to formalise theories of ground. Another convenient feature of K_3 is that is closer to classical logic than other logics that can be used to block the paradoxes. In fact, it only admits truth-value gaps and not truth-value gluts. Moreover, as I will show in section 3.2, K_3 gives rise to intuitively persuasive recursive clauses in the construction of the extension and anti-extension of truth and ground predicates. In any case, a semantics that meets the objectives of the K_3-semantics of section 3.2 can also be developed based on different logics by using a similar strategy as the one developed for K_3.

The extension and anti-extension of Tr and $\lhd$ are defined through the recursive construction of a sequence of models for any ordinal α, in the style of Kripke's fixed point semantics for the truth predicate (Kripke [6]). At level 0, the extensions and anti-extensions are empty. Then, at any level α, they are defined based on the model at the previous level $\alpha - 1$. I will show that the extensions and anti-extensions are increasing in α and that, at some level, they reach a fixed point[23]. At the fixed point, they are the same as the ones at the next level. Thus, we can take the extensions and anti-extensions at the fixed point as the interpretation of the predicates $\lhd$ and Tr. Note that, at all the stages of the construction, even at the fixed points, the extensions and anti-extension are not exhaustive of the domain of the model. There are truth-value gaps. In particular, the recursive clause of the extensions and anti-extension are such that they never include the numbers of paradoxical sentences into them.

Intuitively, the recursive clauses for the extensions and anti-extensions are defined as follows. The ones for the truth predicate are defined analogously to Kripke [6]. Thus, at each level α, the extension of the truth predicate S_α contains the numbers of the sentences which were true at the previous level, while the anti-extension $\overline{S_\alpha}$ contains the numbers of the sentences which were false at the previous level. Intuitively, the extension of the ground predicate is defined by applying, at every level α, the axioms of PG^* over the truths of the model at the previous level $\alpha - 1$. The pairs of numbers that correspond to the *ground* and the *grounded* of the ground statements that are proven by such theory form the extension of the ground predicate at level α. Thus, for example, if $M_{\alpha-1} \vDash_{K_3} \ulcorner \overline{0} = \overline{0} \urcorner \lhd \ulcorner \neg\neg\overline{0} = \overline{0} \urcorner$, then $\langle \#(\ulcorner \overline{0} = \overline{0} \urcorner \lhd \ulcorner \neg\neg\overline{0} = \overline{0} \urcorner); \#(\neg\neg(\ulcorner \overline{0} = \overline{0} \urcorner \lhd \ulcorner \neg\neg\overline{0} = \overline{0} \urcorner)) \rangle \in R_\alpha$ by U_1. The anti-extension of the ground predicate is defined analogously by applying, at every level α, the axioms of PG^* over the falsities of the model at the previous level $\alpha - 1$. Then, the pairs

[23] Note that there exist multiple fixed points, but, for simplicity, here I implicitly only refer to the least one. Its properties generalise also to all other fixed points.

of numbers that correspond to the negative ground statements that are proven by such theory form the anti-extension of the ground predicate. For example, if $M_{\alpha-1} \vDash_{K_3} \neg\phi$ for some $\phi \in L^{\lhd}_{Tr}$, then, for any $n \in \mathbb{N}$, $\langle \#\phi; n \rangle \in \overline{R_\alpha}$ and $\langle n; \#\phi \rangle \in \overline{R_\alpha}$. Towards the end of section 3.2, I will add to the inductive construction of the extension and anti-extension of the ground predicate some *base ground* statements. However, this does not substantially change the inductive construction described above.

In section 3.3, I develop the axiomatic theory of *untyped partial ground and truth PUGT*. To do so, I first construct a classical model based on one of the fixed points of the semantics developed in section 3.2. To replicate the results of the K_3-model in a classical setting, I add two predicates to the language: a falsity predicate F and a predicate of *not grounding* $\not\lhd$. More precisely, the extension of the classical ground predicate $\lhd$ will correspond to the extension R of ground predicate at the fixed point K_3-model, the extension of the classical *not ground* predicate $\not\lhd$ will correspond to the anti-extension $\overline{R}$ of ground predicate at the fixed point K_3-model, the extension of the classical truth predicate Tr will correspond to the extension S of truth predicate at the fixed point K_3-model and the extension of the classical falsity predicate F will correspond to the anti-extension $\overline{S}$ of truth predicate at the fixed point K_3-model. Then, I will simplify the language by defining the falsity predicate in terms of the truth predicate and negation as:

$$\forall x(F(x) \leftrightarrow Tr(\dot{\neg}x)).$$

Then, I develop axioms for this model in the style of the the *Kripke-Feferman* theory of truth KF (Halbach [19, pp. 181-188]), which is an axiomatic theory for Kripke fixed-point semantics. The basic ideas are: 1) to add axioms that correspond to the properties of the extension and anti-extension of the truth and ground predicates at the fixed points of the construction of section 3.2, 2) to add consistency axioms that correspond to the fact that the extension and anti-extension of the truth and ground predicates are disjoint and 3) to add the *not ground* predicate to the axioms of PG in order to distinguish between the statements in which the *not ground* predicate holds and the ones in which the ground predicate does not hold. I show that $PUGT$ does not derive the inconsistencies resulting from the paradoxes of self-referentiality in Theorems 1 and 2. Then, I prove that $PUGT$ is sound with respect to the model previously described.

Given that the semantics developed in section 3.2 contains Kripke fixed-point semantics, it is natural to expect that $PUGT$ also contains the *Kripke-Feferman* theory of truth KF. I will prove two results about the relation between $PUGT$ and KF. First, that $PUGT$ proves KF. This means that all the theorems of KF are also theorems of $PUGT$. From this, it follows that $PUGT$ proves the full T-scheme for *T-positive formulas*, which are formulas in which the truth predicate Tr does not occur in the scope of an odd number of negation symbols. Then, I prove that $PUGT$ proves the same theorems of KF in the language L_{Tr} with coding function over L_{Tr}. This means that the further theorems that $PUGT$ proves with respect to KF are ground statements or sentences that express the truth of a ground statement.

3.2 Semantics

A model for untyped partial ground in the language $L^{\lhd}_{Tr}$ is a tuple $M = \langle \mathbb{N}; R; \overline{R}; S; \overline{S} \rangle$, where $R, \overline{R} \subseteq \mathbb{N}^2$ and $S, \overline{S} \subseteq \mathbb{N}$. R is the set of ordered pairs such that, according to the model, the first number is the Gödel code of the *ground* of ground relation and the second number is its *grounded*. $\overline{R}$ is the set of of ordered pairs such that, according to the model, the first number is the Gödel code of a sentence that does not ground the sentence whose Gödel code is the second number. S is the set of codes of true sentences according to the model. $\overline{S}$ is the set of codes of false sentences according to the model.

I construct the extensions and anti-extensions $(R; \overline{R}; S; \overline{S})$ of the model M by applying the Knaster-Tarski theorem.

Theorem 4 (**Knaster–Tarski Theorem**). *Let $(L, \leq)$ be a complete lattice and let $f : L \to L$ be an order-preserving (monotonic) function w.r.t. $\leq$. Then, the set of fixed points of f in L forms a complete lattice under $\leq$.*

Since the set of fixed points is a complete lattice, it follows that there must exist at least a fixed point and, in particular, a least fixed point and greatest fixed point.

I now define some concepts and notation that will be useful for the development of the model. The *theory of a model M_α* is the set of sentences satisfied by M_α, i.e. $Th(M_\alpha) = \{\phi \in L^{\lhd}_{Tr} : M_\alpha \vDash_{K_3} \phi\}$. To apply the *Knaster–Tarski theorem*, I define a operator $\Phi : Th(M) \to Th(M)$, which I will later prove to be monotone. Note that $\Phi(Th(M_\alpha)) = Th(M_{\alpha+1})$. I will use $\Phi(M_\alpha)$ to refer to the model whose theory is $\Phi(Th(M_\alpha))$. Note that $\Phi(M_\alpha) = M_{\alpha+1}$. In the formulation of the model, I will use the following notation: the *positive truth set of M_α* is the set of sentences that express that all the sentences in $Th(M_\alpha)$ are true, i.e. $Tr^+(M_\alpha) = \{Tr(\ulcorner\phi\urcorner) : M_\alpha \vDash_{K_3} \phi\}$, the *negative truth set of M_α* is the set of sentences that express that all the sentences whose negation is in $Th(M_\alpha)$ are not true, i.e. $Tr^-(M_\alpha) = \{\neg Tr(\ulcorner\phi\urcorner) : M_\alpha \vDash_{K_3} \neg\phi\}$. For simplicity, for every level α, I will name the

$Tr^+(M_\alpha) \cup Tr^-(M_\alpha) \cup PG^*$ as PG^*_α.

Finally, M_{PG} is a model of the theory PG (Korbmacher [1, pp. 180-187]). I will define as $BG(M_{PG})$ the set of *base ground truths* of a model M_{PG} of PG. By base ground truths, I mean the ground predicate true literals of a model M_{PG} such that both the *ground* and the *grounded* are sentences in the language of arithmetic L. Thus, they are *basic truths* in the sense that they do not contain any iteration of the truth and/or ground predicate. I name the set of positive base ground truths $BG^+(M_{PG})$ and the set of negative ones $BG^-(M_{PG})$. Thus, formally:

$$BG(M_{PG}) = BG^+(M_{PG}) \cup BG^-(M_{PG}) = \{\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner : M_{PG} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\} \cup \{\neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) : \phi, \psi \in L, M_{PG} \vDash \neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)\}^{24}.$$

The next paragraphs are divided in two stages. I formulate two different versions of the operator Φ, where the second one is stronger because it also proves claims about the base ground truths of a model M_{PG}. For each stage, I prove two key results. First, I show that the operators satisfy the assumptions of the *Knaster–Tarski theorem* (Th. 4), so there exist fixed points for them. Second, I show that the fixed points are consistent, so that they are K_3 models and they are not trivialized by explosion in K_3, i.e. for all ϕ and ψ, $\phi \wedge \neg\phi \vDash_{K_3} \psi$. The first stage one is a bit simpler than second one, so presenting the former first is instrumental also for presenting the latter. Moreover, both models are of theoretical interest, so it is worth to develop them both.

Definition 3 (Construction 1st stage). *The construction at the first stage is:*

$$M_0 = (\mathbb{N}; R_0; \overline{R_0}; S_0; \overline{S_0}) = (\mathbb{N}; \emptyset; \emptyset; \emptyset; \{n : PA \vdash \neg Sent^{\lhd}_{Tr}(\overline{n})\})$$

$$M_{\alpha+1} = (\mathbb{N}; R_{\alpha+1}; \overline{R_{\alpha+1}}; S_{\alpha+1}; \overline{S_{\alpha+1}})$$
$$R_{\alpha+1} = \{\langle \#\phi; \#\psi \rangle : PG^*_\alpha \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\}$$
$$\overline{R_{\alpha+1}} = \{\langle \#\phi; \#\psi \rangle : PG^*_\alpha \vdash \neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)\}$$
$$S_{\alpha+1} = \{\#\phi : M_\alpha \vDash_{K_3} \phi\}$$
$$\overline{S_{\alpha+1}} = \{\#\phi : M_\alpha \vDash_{K_3} \neg\phi\}$$

$$M_\alpha = (\mathbb{N}; R_\alpha; \overline{R_\alpha}; S_\alpha; \overline{S_\alpha}),\ \alpha \text{ limit ordinal}$$
$$R_\alpha = \bigcup_{\beta<\alpha} R_\beta$$
$$\overline{R_\alpha} = \bigcup_{\beta<\alpha} \overline{R_\beta}$$
$$S_\alpha = \bigcup_{\beta<\alpha} S_\beta$$
$$\overline{S_\alpha} = \bigcup_{\beta<\alpha} \overline{S_\beta}$$

[24] The condition that $\phi, \psi \in L$ is superfluous for $BG^+(M_{PG})$ because it follows from PG.

At the level 0, the model satisfies the truths the standard model of arithmetic $\mathbb{N}$. The only addition to it is that the anti-extension of the truth predicate contains all the numbers that are codes of strings of symbols that are not sentences of $L^{\lhd}_{Tr}$. Thus, if a string of symbols is not sentences of $L^{\lhd}_{Tr}$, it is not true at level 0. At level α, $S_{\alpha+1}$ contains the Gödel-numbers of the sentences ϕ such that $M_\alpha \vDash_{K_3} \phi$, $\overline{S_{\alpha+1}}$ contains the Gödel-numbers of the sentences ϕ such that $M_\alpha \vDash_{K_3} \neg\phi$. These are the same definitions as in Kripke's semantics [6]. The extension and anti-extension of the ground predicate are determined by the ground literals that are proven by the theory $Tr^+(M_\alpha) \cup Tr^-(M_\alpha) \cup PG^*$. Intuitively, we apply the axioms of PG^* over the set of truths and falsities of the model at the previous level α. The truths of the model at level 1 are determined by applying the axioms of PG^* over the truths of arithmetic. Thus, it aims to derive the ground relations between truths of arithmetic, analogously to a model M_{PG} of PG, except for two differences. First, M_1 is a K_3 model, while models of PG are classical models. Second, in the models of PG, if the *grounds* or the *grounded* are not sentences of L, then the ground statement is false by axiom T_3. This restriction is relaxed for all the models M_α in the construction.

I now prove a useful result about the sequence of models M_α, i.e. that the $Th(M_\alpha)$ is increasing in α. In the other words, at every level, the model satisfies at least as many sentences as the previous levels.

Lemma 2. *$Th(M_\alpha)$ is increasing in α. For α and for all $\phi \in L^{\lhd}_{Tr}$, if $\phi \in Th(M_\alpha)$, then $\phi \in Th(M_{\alpha+1})$.*

Proof. By induction on α. Given α, I show that, if $\phi \in Th(M_\alpha)$, then $\phi \in Th(M_{\alpha+1})$.

I consider literals first.

- Suppose $\phi = Tr(\ulcorner\psi\urcorner)$. If $M_\alpha \vDash_{K_3} Tr(\ulcorner\psi\urcorner)$, then $\#\psi \in S_\alpha$, so $M_{\alpha-1} \vDash_{K_3} \psi$. By induction hypothesis, $M_\alpha \vDash_{K_3} \psi$, which implies $\psi \in S_{\alpha+1}$, which implies $M_{\alpha+1} \vDash_{K_3} Tr(\ulcorner\psi\urcorner)$.
- Suppose $\phi = \neg Tr(\ulcorner\psi\urcorner)$. If $M_\alpha \vDash_{K_3} \neg Tr(\ulcorner\psi\urcorner)$, then $\#\psi \in \overline{S_\alpha}$, so $M_{\alpha-1} \vDash_{K_3} \neg\psi$. By induction hypothesis, $M_\alpha \vDash_{K_3} \neg\psi$, which implies $\psi \in \overline{S_{\alpha+1}}$, which implies $M_{\alpha+1} \vDash_{K_3} \neg Tr(\ulcorner\psi\urcorner)$.
- Suppose $\phi = \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$. If $M_\alpha \vDash_{K_3} \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$, then $Tr^+(M_{\alpha-1}) \cup Tr^-(M_{\alpha-1}) \cup PG^* \vdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$. If $Tr(\ulcorner\gamma\urcorner) \in Tr^+(M_{\alpha-1})$, then $M_{\alpha-1} \vDash \gamma$, then $M_\alpha \vDash \gamma$ by induction hypothesis, then $Tr(\ulcorner\gamma\urcorner) \in Tr^+(M_\alpha)$. Thus, $Tr^+(M_{\alpha-1}) \subseteq Tr^+(M_\alpha)$. If $\neg Tr(\ulcorner\gamma\urcorner) \in Tr^-(M_{\alpha-1})$, then $M_{\alpha-1} \vDash \neg\gamma$, then $M_\alpha \vDash \neg\gamma$ by induction hypothesis and $\neg Tr(\ulcorner\gamma\urcorner) \in Tr^-(M_\alpha)$. Thus, $Tr^-(M_{\alpha-1}) \subseteq Tr^-(M_\alpha)$. Thus, if $Tr^+(M_{\alpha-1}) \cup Tr^-(M_{\alpha-1}) \cup PG^* \vdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$, then $Tr^+(M_\alpha) \cup Tr^-(M_\alpha) \cup PG^* \vdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$

and $M_{\alpha+1} \models_{K_3} \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$.

- Suppose $\phi = \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$. If $M_\alpha \models_{K_3} \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$, then $Tr^+(M_{\alpha-1}) \cup Tr^-(M_{\alpha-1}) \cup PG^* \vdash \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$. By an analogous argument as before, $Tr^+(M_\alpha) \cup Tr^-(M_\alpha) \cup PG^* \vdash \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$ and $M_{\alpha+1} \models_{K_3} \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$.
- Suppose ϕ is atomic and $\phi \in L$. Then, there are four cases.
  - M_α and $M_{\alpha+1}$ are both consistent. If M_α is consistent, then $M_\alpha \models_{K_3} \phi$ iff $\mathbb{N} \models \phi$. ($\Leftarrow$) If $\mathbb{N} \models \phi$, then $M_0 \models_{K_3} \phi$, then $M_\alpha \models_{K_3} \phi$ by induction hypothesis. ($\Rightarrow$) Suppose $M_\alpha \models_{K_3} \phi$ and $\mathbb{N} \not\models \phi$. Then, $\mathbb{N} \models \neg\phi$ and, by induction hypothesis, $M_\alpha \models_{K_3} \neg\phi$, which contradicts the assumption that M_α is consistent. Then, if $M_{\alpha+1}$ is also consistent then $M_{\alpha+1} \models_{K_3} \phi$ iff $\mathbb{N} \models \phi$ iff $M_\alpha \models_{K_3} \phi$. If $M_{\alpha+1}$ is also consistent, then $M_{\alpha+1} \models_{K_3} \psi$ for all $\psi \in L^{\lhd}_{Tr}$ and the claim is trivially proven.
  - M_α is consistent and $M_{\alpha+1}$ is inconsistent, then the claim trivially follows because $M_{\alpha+1}$ proves all sentences.
  - If M_α is inconsistent then also $M_{\alpha+1}$ is inconsistent because, for example, if $M_\alpha \models_{K_3} \phi$ and $M_\alpha \models_{K_3} \neg\phi$, then $M_{\alpha+1} \models_{K_3} Tr(\ulcorner\phi\urcorner)$ and $M_{\alpha+1} \models_{K_3} \neg Tr(\ulcorner\phi\urcorner)$. Thus, $Th(M_\alpha) = Th(M_{\alpha+1})$.

For complex formulas, the claim follows from the fact that the true literals of M_α are a subset of the true literals of $M_{\alpha+1}$ and that the same semantic clauses are applied to both models. $\square$

I now show that every model M_α of the construction is consistent. This fact is necessary for M_α to be a K_3-model because K_3 is not a paraconsistent logic and it does not admit truth values gluts. The fact that M_α is consistent plus the use of the Strong Kleene evaluation scheme defined in section 2 imply that M_α is a K_3-model. From the fact the every M_α is consistent, it also follows that, if there exist fixed points of Φ, then they are consistent.

Theorem 5. *For all α, M_α is consistent. In other words, for all $\phi \in L^{\lhd}_{Tr}$, it is not the case that $M_\alpha \models_{K_3} \phi$ and $M_\alpha \models_{K_3} \neg\phi$.*

Proof. By induction on α. For $\alpha = 0$, M_0 is clearly consistent because, for all $\phi \in L^{\lhd}_{Tr}$, $M_0 \models_{K_3} \phi$ iff $\mathbb{N} \models \phi$. Then, we need to show that, if M_α is consistent, then $M_{\alpha+1}$ is consistent. First, we consider atomic sentences.

- $\phi = Tr(\ulcorner\psi\urcorner)$ and M_α is consistent. Suppose $M_{\alpha+1} \models_{K_3} Tr(\ulcorner\psi\urcorner)$ and $M_{\alpha+1} \models_{K_3} \neg Tr(\ulcorner\psi\urcorner)$, then $\#\phi \in S_{\alpha+1}$ and $\#\phi \in \overline{S_{\alpha+1}}$, then $M_\alpha \models_{K_3} \psi$ and $M_\alpha \models_{K_3} \neg\psi$, which contradicts the hypothesis that M_α is consistent.

- $\phi = \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$. By assumption, $\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner \notin Th(M_\alpha)$ or $\neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner) \notin Th(M_\alpha)$, or, equivalently, $PG^*_{\alpha-1} \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$ or $PG^*_{\alpha-1} \nvdash \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$. We want to show that the same holds for $M_{\alpha+1}$, i.e. $PG^*_\alpha \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$ or $PG^*_\alpha \nvdash \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$. This step of the proof is not trivial because, from the fact that $PG^*_{\alpha-1} \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$, it does not follow that $PG^*_\alpha \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$ because PG^*_α is a stronger theory than $PG^*_{\alpha-1}$ because $Tr^+(M_{\alpha-1}) \subseteq Tr^+(M_\alpha)$ and $Tr^-(M_{\alpha-1}) \subseteq Tr^-(M_\alpha)$. Thus, from the fact that $PG^*_{\alpha-1} \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$ or $PG^*_{\alpha-1} \nvdash \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$, it does not follow that $PG^*_\alpha \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$ or $PG^*_\alpha \nvdash \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$. We prove this by constructing a classical model M^{CL}_α for PG^*_α and, so, show that it is consistent. I prove this with following lemmas. Before doing this, we briefly check the remaining cases of the proof.
- If ϕ is atomic and $\phi \in L$, then, if M_α is consistent, $M_\alpha \vDash_{K_3} \phi$ iff $\mathbb{N} \vDash \phi$. $M_{\alpha+1}$ does not prove any new literal in the language of arithmetic unless it is inconsistent and it proves any sentence. Thus, if $M_{\alpha+1}$ is inconsistent, the inconsistency must be due to some $\psi \notin L$.

If ϕ is a complex formula and the set of literals of $M_{\alpha+1}$ is consistent, then applying the K_3 semantic clauses does not generate any contradiction. Thus, assuming that the step involving involving atomic ground sentences is valid, the theorem is proved. □

I now develop in detail the proof of the claim above for atomic ground statements by proving that PG^*_α is consistent. First, I define a complexity function for all $\phi \in L^{\lhd}_{Tr}$.

Definition 4 (**Complexity**). *For all $\phi \in L^{\lhd}_{Tr}$, I define the complexity function $c(\phi)$ as:*

if ϕ is a literal, then $c(\phi) = 0$,

if $\phi = \neg\neg\psi$, then $c(\phi) = c(\psi) + 1$,

if $\phi = \psi \circ \delta$, with $\circ = \wedge, \vee$, then $c(\phi) = max\{c(\psi); c(\delta)\} + 1$,

if $\phi = \neg(\psi \circ \delta)$, with $\circ = \wedge, \vee$, then $c(\phi) = max\{c(\neg\psi); c(\neg\delta)\} + 1$,

if $\phi = Qv\psi$ with $Q = \forall, \exists$, then $c(\phi) = max\{c(\psi(d))\} + 1$,

if $\phi = \neg Qv\psi$ with $Q = \forall, \exists$, then $c(\phi) = max\{c(\neg\psi(d))\} + 1$.

I now construct the classical model $M^{CL}_\alpha = (\mathbb{N}; S^{CL}_\alpha; R^{CL}_\alpha)$, where $S^{CL}_\alpha = \{\#\phi : M^{CL}_\alpha \vDash Tr(\ulcorner\phi\urcorner)\}$ is the (classical) extension of the truth predicate and $R^{CL}_\alpha = \{\langle\#\phi; \#\psi\rangle : M^{CL}_\alpha \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\}$ is the (classical) extension of the ground predicate. Then, I prove M^{CL}_α is a model for PG^*_α.

Definition 5 (M^{CL}_α). *$M^{CL}_\alpha = (\mathbb{N}; S^{CL}_\alpha; R^{CL}_\alpha)$ is defined as:*

$S^{CL} = \{\#\phi : \phi \in Th(M_\alpha)\} = S_{\alpha+1}$;

$R^{CL} = \{\langle \#\phi; \#\psi \rangle : \phi \in Th(M_\alpha), \psi \in Th(M_\alpha), c(\phi) < c(\psi)\}$.

Lemma 3. *M_α^{CL} is a model for PG_α^*.*

Proof. We need to check that, if $PG_\alpha^* \vdash \phi$, then $M_\alpha^{CL} \vDash \phi$. Thus, we need to check that M_α^{CL} satisfies $Tr^+(M_\alpha)$, $Tr^-(M_\alpha)$ and all the axioms of PG^*.

- M_α^{CL} satisfies $Tr^+(M_\alpha)$. If $Tr(\ulcorner\phi\urcorner) \in Tr^+(M_\alpha)$, then $\phi \in Th(M_\alpha)$, $\#\phi \in S_\alpha^{CL}$ and $M_\alpha^{CL} \vDash Tr(\ulcorner\phi\urcorner)$.
- M_α^{CL} satisfies $Tr^-(M_\alpha)$. If $\neg Tr(\ulcorner\phi\urcorner) \in Tr^-(M_\alpha)$, then $\neg\phi \in Th(M_\alpha)$. Given that M_α is consistent, $\phi \notin Th(M_\alpha)$, $\#\phi \notin S_\alpha^{CL}$ and $M_\alpha^{CL} \nvDash Tr(\ulcorner\phi\urcorner)$, or $M_\alpha^{CL} \vDash \neg Tr(\ulcorner\phi\urcorner)$.
- $G_1 : \forall x \neg(x \lhd x)$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M_\alpha^{CL} \vDash \neg(\overline{n} \lhd \overline{n})$. Clearly, $c(\phi) \nless c(\phi)$. Thus, $\langle n; n\rangle \notin R_\alpha^{CL}$ and $M_\alpha^{CL} \vDash \neg(\overline{n} \lhd \overline{n})$.
- $G_2 : \forall x \forall y \forall z(x \lhd z \wedge z \lhd y \rightarrow x \lhd y)$. This means that, for all $n, m, k \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$ and $k = \#\delta$, $M_\alpha^{CL} \vDash \overline{n} \lhd \overline{k} \wedge \overline{k} \lhd \overline{m} \rightarrow \overline{n} \lhd \overline{m}$. If $M_\alpha^{CL} \vDash \overline{n} \lhd \overline{k}$, then $\phi \in Th(M_\alpha)$, $\delta \in Th(M_\alpha)$ and $c(\phi) < c(\delta)$. If $M_\alpha^{CL} \vDash \overline{k} \lhd \overline{m}$, then $\delta \in Th(M_\alpha)$, $\psi \in Th(M_\alpha)$ and $c(\delta) < c(\psi)$. Thus, $\phi \in Th(M_\alpha)$, $\psi \in Th(M_\alpha)$ and $c(\phi) < c(\psi)$, so $\langle n; m\rangle \in R_\alpha^{CL}$ and $M_\alpha^{CL} \vDash \overline{n} \lhd \overline{m}$.
- G_3, T_1, T_2 and T_3^* are trivial.

I now prove the claim for some exemplifying cases of the downward and upward axioms. The remaining ones can be proved with analogous arguments.

- $U_1 : \forall x(Tr(x) \rightarrow x \lhd \underset{.}{\neg}\underset{.}{\neg} x)$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M_\alpha^{CL} \vDash Tr(\overline{n}) \rightarrow \overline{n} \lhd \underset{.}{\neg}\underset{.}{\neg}\overline{n}$. If $M_\alpha^{CL} \vDash Tr(\overline{n})$, then $\phi \in Th(M_\alpha)$ and $\neg\neg\phi \in Th(M_\alpha)$ by K_3 logic. Also, $c(\phi) < c(\phi) + 1 = c(\neg\neg\phi)$. Thus, $M_\alpha^{CL} \vDash \overline{n} \lhd \underset{.}{\neg}\underset{.}{\neg}\overline{n}$.
- $D_3 : \forall x \forall y(Tr(x \underset{.}{\wedge} y) \rightarrow (x \lhd x \underset{.}{\wedge} y) \wedge (y \lhd x \underset{.}{\wedge} y))$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $M_\alpha^{CL} \vDash Tr(\overline{n} \underset{.}{\wedge} \overline{m}) \rightarrow (\overline{n} \lhd \overline{n} \underset{.}{\wedge} \overline{m}) \wedge (\overline{m} \lhd \overline{n} \underset{.}{\wedge} \overline{m})$. If $M_\alpha^{CL} \vDash Tr(\overline{n} \underset{.}{\wedge} \overline{m})$, then $\phi \wedge \psi \in Th(M_\alpha)$, then $\phi \in Th(M_\alpha)$ and $\psi \in Th(M_\alpha)$ by K_3 logic. $c(\phi) < max\{c(\phi); c(\psi)\} + 1 = c(\phi \wedge \psi)$ and $c(\psi) < max\{c(\phi); c(\psi)\} + 1 = c(\phi \wedge \psi)$. Thus, $M_\alpha^{CL} \vDash \overline{n} \lhd \overline{n} \underset{.}{\wedge} \overline{m}$, $M_\alpha^{CL} \vDash \overline{m} \lhd \overline{n} \underset{.}{\wedge} \overline{m}$ and $M_\alpha^{CL} \vDash (\overline{n} \lhd \overline{n} \underset{.}{\wedge} \overline{m}) \wedge (\overline{m} \lhd \overline{n} \underset{.}{\wedge} \overline{m})$.
- $U_6 : \forall x \forall t \forall v(Tr(x(t/v)) \rightarrow x(t/v) \lhd \underset{.}{\exists} v x)$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $M_\alpha^{CL} \vDash Tr(\overline{n}(\overline{m}/v)) \rightarrow \overline{n}(\overline{m}/v) \lhd \underset{.}{\exists} v \overline{n}$. If $M_\alpha^{CL} \vDash Tr(\overline{n}(\overline{m}/v))$, then $\phi(\overline{m}/v) \in Th(M_\alpha)$ and $\exists v \phi \in Th(M_\alpha)$ by K_3 logic. $c(\phi(\overline{m}/v)) < max\{c(\phi(d))\} + 1 = c(\exists v \phi)$. Thus, $M_\alpha^{CL} \vDash \overline{n}(\overline{m}/v) \lhd \underset{.}{\exists} v \overline{n}$.

- $D_8 : \forall x \forall v(Tr(\dot{\forall} vx) \rightarrow \forall t(x(t/v) \lhd \dot{\forall} vx))$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M_\alpha^{CL} \vDash Tr(\dot{\forall} v\overline{n}) \rightarrow \forall t(\overline{n}(t/v) \lhd \dot{\forall} v\overline{n})$. If $M_\alpha^{CL} \vDash Tr(\dot{\forall} v\overline{n})$, then $\forall v\phi \in Th(M_\alpha)$ and, for all t, $\phi(t/v) \in Th(M_\alpha)$ by K_3 logic. For all t, $c(\phi(\overline{n}(t/v)) < max\{c(\phi(d))\} + 1 = c(\forall v\phi)$. Thus, $M_\alpha^{CL} \vDash \forall t(\overline{n}(t/v) \lhd \dot{\forall} v\overline{n})$.

□

I now show that there exist fixed points for Φ by applying the *Knaster–Tarski theorem* (Th. 4). Before proving this, I need to show the operator Φ is monotone with respect to inclusion, i.e. if $A \subseteq B$, then $\Phi(A) \subseteq \Phi(B)$.

Lemma 4. *If, for some α and β, $Th(M_\alpha) \subseteq Th(M_\beta)$, then $\Phi(Th(M_\alpha)) \subseteq \Phi(Th(M_\beta))$.*

Proof. Suppose $Th(M_\alpha) \subseteq Th(M_\beta)$ and $\Phi(Th(M_\alpha)) \nsubseteq \Phi(Th(M_\beta))$. Thus, there is $\phi \in L_{Tr}^{\lhd}$ such that $\phi \in \Phi(Th(M_\alpha))$ and $\phi \notin \Phi(Th(M_\beta))$. First, suppose ϕ is a literal.

- Suppose $\phi = Tr(\ulcorner\psi\urcorner)$. If $Tr(\ulcorner\psi\urcorner) \in \Phi(Th(M_\alpha))$, then $\psi \in Th(M_\alpha)$, then $\psi \in Th(M_\beta)$ by assumption. Thus, $Tr(\ulcorner\psi\urcorner) \in \Phi(Th(M_\beta))$. An analogous argument holds for $\phi = \neg Tr(\ulcorner\psi\urcorner)$.
- Suppose $\phi = \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$. If $\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner \notin \Phi(Th(M_\beta))$, then $PG_\beta^* \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$. If $Th(M_\alpha) \subseteq Th(M_\beta)$, then $Tr^+(M_\alpha) \subseteq Tr^+(M_\beta)$ and $Tr^-(M_\alpha) \subseteq Tr^-(M_\beta)$. Thus, if $PG_\beta^* \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$, then $PG_\alpha^* \nvdash \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$. Thus, $\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner \notin \Phi(Th(M_\alpha))$. An analogous argument holds for $\phi = \neg(\ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner)$.
- Suppose ϕ is atomic and $\phi \in L$. If M_α is inconsistent, then $\Phi(M_\alpha)$ is also inconsistent because, if $M_\alpha \vDash_{K_3} \phi$ and $M_\alpha \vDash_{K_3} \neg\phi$, then $\Phi(M_\alpha) \vDash_{K_3} \phi$ and $\Phi(M_\alpha) \vDash_{K_3} \neg\phi$ by Lemma 1. If M_α is inconsistent, then M_β and $\Phi(M_\beta)$ are also inconsistent. Thus, $\Phi(Th(M_\alpha)) = \Phi(Th(M_\beta))$. Suppose M_α is consistent and M_β is inconsistent. Then, by Theorem 5, $\Phi(M_\alpha)$ is consistent and the claim trivially follows because $\Phi(M_\beta)$ is inconsistent. Suppose both M_α and M_β are consistent. Then, $\Phi(Th(M_\alpha))$ and $\Phi(Th(M_\beta))$ are consistent by Theorem 5. Then, $\phi \in \Phi(Th(M_\alpha))$ iff $\mathbb{N} \vDash \phi$ iff $\phi \in \Phi(Th(M_\beta))$, so $\Phi(Th(M_\alpha)) = \Phi(Th(M_\beta))$.

Suppose ϕ is a complex formula. I previously showed that, if $Th(M_\alpha) \subseteq Th(M_\beta)$, then the literals of $\Phi(Th(M_\alpha))$ are a subset of the literals of $\Phi(Th(M_\beta))$. Given that the semantic clauses of $\Phi(M_\alpha)$ and $\Phi(M_\beta)$ are the same, it follows that $\Phi(Th(M_\alpha)) \subseteq \Phi(Th(M_\beta))$. □

I can now prove the main claim that there exist fixed points for Φ.

Theorem 6. *The Knaster–Tarski theorem (Th. 4) applies to the construction above. From this, it follows that there exists a least fixed point, i.e. a model M_I such that $Th(M_I) = \Phi(Th(M_I))$.*

Proof. I check that the assumption of the *Knaster–Tarski theorem* are satisfied for the lattice $(\mathcal{P}(A); \subseteq)$, where $A = \{\phi : PA \vdash Sent^{\lhd}_{Tr}(\ulcorner\phi\urcorner)\}$ is the set of all sentences of $L^{\lhd}_{Tr}$ and the operator Φ.

First, I show $(\mathcal{P}(A); \subseteq)$ is a complete lattice. $(\mathcal{P}(A); \subseteq)$ is a partially ordered set because $\subseteq$ trivially satisfies reflexivity, transitivity and antisymmetry. Moreover, for all $a, b \in (\mathcal{P}(A))$, $sup\{a, b\} = a \cup b \in (\mathcal{P}(A))$ and $inf\{a, b\} = a \cap b \in (\mathcal{P}(A))$. Thus, $(\mathcal{P}(A); \subseteq)$ is a lattice. For all $S \subseteq (\mathcal{P}(A))$, $sup(S) = \bigcup\{x : x \in S\} \in (\mathcal{P}(A))$ and $inf(S) = \bigcap\{x : x \in S\} \in (\mathcal{P}(A))$. Thus, $(\mathcal{P}(A); \subseteq)$ is a complete lattice.

Note that $Th(M_\alpha) \subseteq \mathcal{P}(A)$. Thus, $\Phi : \mathcal{P}(A) \to \mathcal{P}(A)$. The last condition to show before applying the theorem is that Φ order-preserving with respect to $\subseteq$. This follows from Lemma 2.

Thus, there exists a complete lattice of fixed points of Φ in $\mathcal{P}(A)$ and, in particular, a least fixed point, which I name M_I. □

I now proceed to the second stage.

Definition 6 (Construction 2nd stage). *The new construction at the second stage is:*

$$M_0 = (\mathbb{N}; R_0; \overline{R_0}; S_0; \overline{S_0}) = (\mathbb{N}; \emptyset; \emptyset; \emptyset; \{n : PA \vdash \neg Sent^{\lhd}_{Tr}(\overline{n})\})$$

$$M_{\alpha+1} = (\mathbb{N}; R_{\alpha+1}; \overline{R_{\alpha+1}}; S_{\alpha+1}; \overline{S_{\alpha+1}})$$
$$R_{\alpha+1} = \{\langle \#\phi; \#\psi \rangle : BG(M_{PG}) \cup PG^*_\alpha \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\}$$
$$\overline{R_{\alpha+1}} = \{\langle \#\phi; \#\psi \rangle : BG(M_{PG}) \cup PG^*_\alpha \vdash \neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)\}$$
$$S_{\alpha+1} = \{\#\phi : M_\alpha \vDash_{K_3} \phi\}$$
$$\overline{S_{\alpha+1}} = \{\#\phi : M_\alpha \vDash_{K_3} \neg\phi\}$$

$M_\alpha = (\mathbb{N}; R_\alpha; \overline{R_\alpha}; S_\alpha; \overline{S_\alpha})$, *$\alpha$ limit ordinal*
$$R_\alpha = \textstyle\bigcup_{\beta<\alpha} R_\beta$$
$$\overline{R_\alpha} = \textstyle\bigcup_{\beta<\alpha} \overline{R_\beta}$$
$$S_\alpha = \textstyle\bigcup_{\beta<\alpha} S_\beta$$
$$\overline{S_\alpha} = \textstyle\bigcup_{\beta<\alpha} \overline{S_\beta}$$

The only difference from the previous stage is that I added $BG(M_{PG})$ to the clauses for R_α and $\overline{R_\alpha}$. $BG(M_{PG})$ contains the *base ground truths* of a model M_{PG} or PG. The reason why I add them is that I want to remain agnostic about which base ground sentences (among the ones consistent with PG) are true. There are many ground atomic sentences, with the *ground* and *grounded* in L, such that PG does not prove them, nor their negation, e.g. $PG \nvdash \ulcorner \overline{0} = \overline{0} \urcorner \lhd \ulcorner \overline{1} = \overline{1} \urcorner$ and $PG \nvdash \neg(\ulcorner \overline{0} = \overline{0} \urcorner \lhd \ulcorner \overline{1} = \overline{1} \urcorner)$. I want the model to be compatible with different choices about which base ground sentences are true, as long as this choice is consistent with the axioms of PG. Thus, I formalise a family of constructions, such that each of them satisfies the *base ground truths* of a different model M_{PG} at the first level.

I now adapt Theorem 5 and Theorem 6 to the new construction. Note that the same exact proofs of Lemma 2 and Lemma 4 of the first stage also holds for the second one.

Theorem 7 (2^{nd} stage analogous of Th. 5)**.** *For all α, M_α is consistent. In other words, for all $\phi \in L^{\lhd}_{Tr}$, it is not the case that $M_\alpha \vDash_{K_3} \phi$ and $M_\alpha \vDash_{K_3} \neg\phi$.*

The only step that needs to be changed involves atomic grounds sentences. We need to change the definition of the model M^{CL}_α because it can happen that the the *base ground truth* do not respect the complexity condition. It can happen that the *grounded* in a *base ground truth* is not more complex than its *ground*. Thus, we define M^{CL}_α as $M^{CL}_\alpha = (\mathbb{N}; S^{CL}_\alpha; R^{CL}_\alpha)$, where:

$S^{CL}_\alpha = \{\#\phi : \phi \in Th(M_\alpha)\} = S_{\alpha+1}$

$R^{CL}_\alpha = \{\langle \#\phi; \#\psi \rangle : \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner \in BG^+(M_{PG})\} \cup \{\langle \#\phi; \#\psi \rangle : \phi \in Th(M_\alpha) \setminus Th(\mathbb{N}), \psi \in Th(M_\alpha) \setminus Th(\mathbb{N}), c(\phi) < c(\psi)\} \cup \{\langle \#\phi; \#\psi \rangle : \phi \in Th(\mathbb{N}), \psi \in Th(M_\alpha) \setminus Th(\mathbb{N})\}$

Lemma 5 (2^{nd} stage analogous of Lemma 3)**.** *M^{CL}_α is a model of $BG(M_{PG}) \cup PG^*_\alpha$.*

Proof. We need to check that, if $BG(M_{PG}) \cup PG^*_\alpha \vdash \phi$, then $M^{CL}_\alpha \vDash \phi$. Thus, we check that M^{CL}_α satisfies $BG(M_{PG})$, $Tr^+(M_\alpha)$, $Tr^-(M_\alpha)$ and all the axioms of PG^*.

- M^{CL}_α satisfies $BG(M_{PG})$.
  - If $\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner \in BG(M_{PG})$, then $\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner \in BG^+(M_{PG})$ and $\langle \#\phi; \#\psi \rangle \in R^{CL}_\alpha$.
  - If $\neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) \in BG(M_{PG})$, we need to check that $\langle \#\phi; \#\psi \rangle \notin R^{CL}_\alpha$ and, so, that $\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ does not follow from any of the three conditions in the definition of R^{CL}. From $\neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) \in BG(M_{PG})$, it follows that $\neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) \in BG^-(M_{PG})$ and, so, $\phi \in L, \psi \in L$ and $M_{PG} \vDash \neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$. Thus, 1) $M_{PG} \nvDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ because M_{PG} is a classical model. Suppose $\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $\psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $c(\phi) < c(\psi)$. $\phi \in L$, so $\phi \in Th(M_\alpha)$ iff $\phi \in Th(\mathbb{N})$. Thus, 2) $\phi \notin Th(M_\alpha) \setminus Th(\mathbb{N})$. Analogously for ψ. Suppose $\phi \in Th(\mathbb{N})$, $\psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$.

For the same reasoning, 3) $\psi \notin Th(M_\alpha) \setminus Th(\mathbb{N})$. Thus, none of the three conditions in the definition of R^{CL}_α can be satisfied and $\langle \#\phi; \#\psi \rangle \notin R^{CL}_\alpha$.

- For $Tr^+(M_\alpha)$ and $Tr^-(M_\alpha)$ the proof is the same as in Lemma 3.
- $G_1 : \forall x \neg (x \lhd x)$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M^{CL}_\alpha \vDash \neg(\overline{n} \lhd \overline{n})$. $M_{PG} \nvDash (\overline{n} \lhd \overline{n})$ because M_{PG} satisfies G_1, so $\overline{n} \lhd \overline{n} \notin BG^+(M_{PG})$. If $\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$, then $c(\phi) \nless c(\phi)$. Thus, $\langle n; n \rangle \notin R^{CL}_\alpha$ and $M^{CL}_\alpha \vDash \neg(\overline{n} \lhd \overline{n})$.
- $G_2 : \forall x \forall y \forall z (x \lhd z \wedge z \lhd y \rightarrow x \lhd y)$. This means that, for all $n, m, k \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, and $k = \#\delta$, $M^{CL}_\alpha \vDash \overline{n} \lhd \overline{k} \wedge \overline{k} \lhd \overline{m} \rightarrow \overline{n} \lhd \overline{m}$.
  - If $\overline{n} \lhd \overline{k} \in BG^+(M_{PG})$, $\overline{k} \lhd \overline{m} \in BG^+(M_{PG})$, then $M_{PG} \vDash \overline{n} \lhd \overline{k}$ and $M_{PG} \vDash \overline{k} \lhd \overline{m}$, then $M_{PG} \vDash \overline{n} \lhd \overline{m}$ because M_{PG} satisfies G_2, $\overline{n} \lhd \overline{m} \in BG^+(M_{PG})$ and $\langle n; m \rangle \in R^{CL}_\alpha$, $M^{CL}_\alpha \vDash \overline{n} \lhd \overline{m}$.
  - If $\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $\delta \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $\psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $c(\phi) < c(\delta)$, $c(\delta) < c(\psi)$, then $c(\phi) < c(\psi)$ and $\langle n; m \rangle \in R^{CL}_\alpha$, $M^{CL}_\alpha \vDash \overline{n} \lhd \overline{m}$.
  - Suppose $M_{PG} \vDash \overline{n} \lhd \overline{k}$ and $\delta \in Th(\mathbb{N}), \psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$. Then, $\phi \in Th(\mathbb{N})$, so $M^{CL}_\alpha \vDash \overline{n} \lhd \overline{m}$.
  - If $\phi \in Th(\mathbb{N})$, $\delta \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $\psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $c(\delta) < c(\psi)$, then $\langle n; m \rangle \in R^{CL}_\alpha$, $M^{CL}_\alpha \vDash \overline{n} \lhd \overline{m}$.
  - If $M_{PG} \vDash \overline{n} \lhd \overline{k}$, then $\phi \in Th(\mathbb{N})$, $\delta \in Th(\mathbb{N})$. Thus, it cannot be that $\delta \in Th(M_\alpha) \setminus Th(\mathbb{N})$
  - It cannot be that $\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $\delta \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $\delta \in Th(\mathbb{N})$, $\psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$.
  - It cannot be that $\phi \in Th(\mathbb{N})$, $\delta \in Th(M_\alpha) \setminus Th(\mathbb{N})$, $\delta \in Th(\mathbb{N})$, $\psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$.
- G_3, T_1, T_2 and T^*_3 are trivial.

I now prove the claim for some exemplifying cases of the downward and upward axioms. The remaining ones can be proved with analogous arguments.

- $U_1 : \forall x (Tr(x) \rightarrow x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x)$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M^{CL}_\alpha \vDash Tr(\overline{n}) \rightarrow \overline{n} \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}\overline{n}$. If $M^{CL}_\alpha \vDash Tr(\overline{n})$, then $\phi \in Th(M_\alpha)$ and $\neg\neg\phi \in Th(M_\alpha)$. If $\phi \in Th(\mathbb{N})$, then $\neg\neg\phi \in Th(\mathbb{N})$ and $M_{PG} \vDash \phi \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}\phi$. If $\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$, then $\neg\neg\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$ and $c(\phi) < c(\neg\neg\phi)$. Thus, $M^{CL}_\alpha \vDash \overline{n} \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}\overline{n}$.
- $D_3 : \forall x \forall y (Tr(x \underset{\cdot}{\wedge} y) \rightarrow (x \lhd x \underset{\cdot}{\wedge} y) \wedge (y \lhd x \underset{\cdot}{\wedge} y))$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $M^{CL}_\alpha \vDash Tr(\overline{n} \underset{\cdot}{\wedge} \overline{m}) \rightarrow (\overline{n} \lhd \overline{n} \underset{\cdot}{\wedge} \overline{m}) \wedge (\overline{m} \lhd \overline{n} \underset{\cdot}{\wedge} \overline{m})$. If $M^{CL}_\alpha \vDash Tr(\overline{n} \underset{\cdot}{\wedge} \overline{m})$, then $\phi \wedge \psi \in Th(M_\alpha)$, then $\phi \in Th(M_\alpha)$ and $\psi \in Th(M_\alpha)$ by K_3 logic. If $\phi \wedge \psi \in Th(\mathbb{N})$, then

$\phi \in Th(\mathbb{N})$, then $\overline{n} \lhd \overline{n} \dot{\wedge} \overline{m} \in BG(M_{PG})$ because PG satisfies D_3. Suppose $\phi \wedge \psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$ and $\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$. $c(\phi) < max\{c(\phi); c(\psi)\} + 1 = c(\phi \wedge \psi)$. Thus, $M_\alpha^{CL} \vDash \overline{n} \lhd \overline{n} \dot{\wedge} \overline{m}$. Suppose $\phi \wedge \psi \in Th(M_\alpha) \setminus Th(\mathbb{N})$ and $\phi \in Th(\mathbb{N})$, then $M_\alpha^{CL} \vDash \overline{n} \lhd \overline{n} \dot{\wedge} \overline{m}$. Analogously for $\overline{m} \lhd \overline{n} \dot{\wedge} \overline{m}$. Thus, $M_\alpha^{CL} \vDash (\overline{n} \lhd \overline{n} \dot{\wedge} \overline{m}) \wedge (\overline{m} \lhd \overline{n} \dot{\wedge} \overline{m})$.

- $U_6 : \forall x \forall t \forall v(Tr(x(t/v)) \to x(t/v) \lhd \dot{\exists} vx)$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $M_\alpha^{CL} \vDash Tr(\overline{n}(\overline{m}/v)) \to \overline{n}(\overline{m}/v) \lhd \dot{\exists} v\overline{n}$. If $M_\alpha^{CL} \vDash Tr(\overline{n}(\overline{m}/v))$, then $\phi(\overline{m}/v) \in Th(M_\alpha)$ and $\exists v\phi \in Th(M_\alpha)$ by K_3 logic. If $\phi(\overline{m}/v) \in Th(\mathbb{N})$ and $\exists v\phi \in Th(\mathbb{N})$, then $\overline{n}(\overline{m}/v) \lhd \dot{\exists} v\overline{n} \in BG(M_{PG})$ because PG satisfies U_6. $\phi(\overline{m}/v) \in Th(M_\alpha) \setminus Th(\mathbb{N})$ iff $\exists v\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$. Given that $M_\alpha \vDash_{K_3} \phi(\overline{m}/v)$, $c(\phi(\overline{m}/v)) < max\{c(\phi(d)) : M_\alpha \vDash_{K_3} \phi(d)\} + 1 = c(\exists v\phi)$. Thus, $M_\alpha^{CL} \vDash \overline{n}(\overline{m}/v) \lhd \dot{\exists} v\overline{n}$.
- $D_8 : \forall x \forall v(Tr(\dot{\forall} vx) \to \forall t(x(t/v) \lhd \dot{\forall} vx))$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M_\alpha^{CL} \vDash Tr(\dot{\forall} v\overline{n}) \to \forall t(\overline{n}(t/v) \lhd \dot{\forall} v\overline{n})$. If $M_\alpha^{CL} \vDash Tr(\dot{\forall} v\overline{n})$, then $\forall v\phi \in Th(M_\alpha)$ and, for all t, $\phi(t/v) \in Th(M_\alpha)$ by K_3 logic. If $\phi(\overline{m}/v) \in Th(\mathbb{N})$ and $\forall v\phi \in Th(\mathbb{N})$, then $\overline{n}(\overline{m}/v) \lhd \dot{\forall} v\overline{n} \in BG(M_{PG})$ because PG satisfies U_6. $\phi(\overline{m}/v) \in Th(M_\alpha) \setminus Th(\mathbb{N})$ iff $\forall v\phi \in Th(M_\alpha) \setminus Th(\mathbb{N})$. Given that $M_\alpha \vDash_{K_3} \phi(t/v)$ for all t, $c(\phi(\overline{n}(t/v)) < max\{c(\phi(d)) : M_\alpha \vDash_{K_3} \phi(d)\} + 1 = c(\forall v\phi)$. Thus, $M_\alpha^{CL} \vDash \forall t(\overline{n}(t/v) \lhd \dot{\forall} v\overline{n})$.

□

Given that $BG(M_{PG}) \cup PG_\alpha^*$ is consistent, then the rest of proof of Theorem 7 is analogous to the one of Theorem 5 at the first stage.

Theorem 8 (2^{nd} stage analogous of Th. 6)**.** *The Knaster–Tarski theorem (Th. 4) applies to the construction above. From this, it follows that there exists a least fixed point, i.e. a model M_I such that $Th(M_I) = \Phi(Th(M_I))$.*

Proof. Given that Lemma 4 and Theorem 7 holds, the proof follows exactly as in the first stage. □

Observation 1. *Given any fixed point M_{FP}, it holds that $\Phi(M_{FP}) = M_{FP}$ and $\Phi(R_{FP}; \overline{R_{FP}}; S_{FP}; \overline{S_{FP}}) = (R_{FP}; \overline{R_{FP}}; S_{FP}; \overline{S_{FP}})$. Thus, it follows that:*

- $\Phi(R_{FP}) = \{\langle \#\phi; \#\psi \rangle : BG(M_{PG}) \cup PG_{FP}^* \vdash \ulcorner \phi \urcorner \lhd \ulcorner \psi \urcorner\} = R_{FP}$
- $\Phi(\overline{R_{FP}}) = \{\langle \#\phi; \#\psi \rangle : BG(M_{PG}) \cup PG_{FP}^* \vdash \neg(\ulcorner \phi \urcorner \lhd \ulcorner \psi \urcorner)\} = \overline{R_{FP}}$
- $\Phi(S_{FP}) = \{\#\phi : M_{FP} \vDash_{K_3} \phi\} = S_{FP}$
- $\Phi(\overline{S_{FP}}) = \{\#\phi : M_{FP} \vDash_{K_3} \neg\phi\} = \overline{S_{FP}}$.

I now show how a fixed point of the previous construction deals with the two paradoxes of self-reference: the *liar paradox* (Theorem 1) and the *paradox of self-referentiality for ground* (Theorem 2). I prove these results for the least fixed point M_I, but they can be generalized with analogous arguments to any fixed point M_{FP}. Before this, I prove the useful lemma that M_I satisfies factivity, i.e. that, if M_I satisfies a ground statement, then M_I also satisfies that its *ground* and its *grounded* are true.

Lemma 6. *For all* α *and for all* $\phi, \psi \in L^{\lhd}_{Tr}$, *if* $M_\alpha \models_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$, *then* $M_\alpha \models_{K_3} Tr(\ulcorner\phi\urcorner)$ *and* $M_\alpha \models_{K_3} Tr(\ulcorner\psi\urcorner)$.

Proof. This follows from Lemma 5. Given that $M^{CL}_{\alpha-1}$ is a model of $BG(M_{PG}) \cup PG^*_{\alpha-1}$, it follows that, if $BG(M_{PG}) \cup PG^*_{\alpha-1} \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$, then $\phi \in Th(M_{\alpha-1})$ and $\psi \in Th(M_{\alpha-1})$. $BG(M_{PG}) \cup PG^*_{\alpha-1} \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ iff $M_\alpha \models_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$. Thus, if $M_\alpha \models_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$, then $\phi \in Th(M_{\alpha-1})$ and $\psi \in Th(M_{\alpha-1})$. If $\phi \in Th(M_{\alpha-1})$ and $\psi \in Th(M_{\alpha-1})$, then $M_\alpha \models_{K_3} Tr(\ulcorner\phi\urcorner)$ and $M_\alpha \models_{K_3} Tr(\ulcorner\psi\urcorner)$. Thus, the claim is proved. □

Proposition 1. λ *is the liar sentence of Theorem 1. Then,* $(i) M_I \not\models_{K_3} \lambda$ *and* $(ii) M_I \not\models_{K_3} \neg\lambda$.

Proof. From Observation 1, it follows that $S_I = \{\#\phi : M_I \models_{K_3} \phi\}$. Thus, for all $\phi \in L^{\lhd}_{Tr}$, $M_I \models_{K_3} Tr(\ulcorner\phi\urcorner)$ iff $M_I \models_{K_3} \phi$. From the *Diagonal Lemma* (Lemma 1), $PA \vdash \lambda \leftrightarrow \neg Tr(\ulcorner\lambda\urcorner)$. Thus, $\mathbb{N} \models \lambda \leftrightarrow \neg Tr(\ulcorner\lambda\urcorner)$ and $M_I \models_{K_3} \lambda \leftrightarrow \neg Tr(\ulcorner\lambda\urcorner)$. From Theorem 7, we know that M_I is a K_3-model and, so, it cannot be that $\#\lambda \in S_I$ and $\#\lambda \in \overline{S_I}$. Thus, three cases are possible:

- $\#\lambda \in S_I$ and $\#\lambda \notin \overline{S_I}$. Then, $M_I \models_{K_3} Tr(\ulcorner\lambda\urcorner)$, $M_I \models_{K_3} \lambda$ because M_I is a fixed point and $M_I \models_{K_3} \neg\lambda$ by *Diagonal Lemma*. This contradicts the fact M_I is a K_3-model (Th. 7). Thus, (i) follows.
- $\#\lambda \notin S_I$ and $\#\lambda \in \overline{S_I}$. Then, $M_I \models_{K_3} \neg Tr(\ulcorner\lambda\urcorner)$, $M_I \models_{K_3} \neg\lambda$ because M_I is a fixed point and $M_I \models_{K_3} \lambda$ by *Diagonal Lemma*. This contradicts the fact M_I is a K_3-model (Th. 7). Thus, (ii) follows.
- $\#\lambda \notin S_I$ and $\#\lambda \notin \overline{S_I}$. From this, it follows that $M_I \not\models_{K_3} Tr(\ulcorner\lambda\urcorner)$ and $M_I \not\models_{K_3} \neg Tr(\ulcorner\lambda\urcorner)$. Then, $M_I \not\models_{K_3} \lambda$ and $M_I \not\models_{K_3} \neg\lambda$ because M_I is a fixed point. Thus, $M_I \models_{K_3} \lambda \leftrightarrow \neg Tr(\ulcorner\lambda\urcorner)$ holds.

□

Proposition 2. σ *is the paradoxical sentence of Theorem 2. Then,* $(i) M_I \not\models_{K_3} \sigma$ *and* $(ii) M_I \not\models_{K_3} \neg\sigma$.

Proof. From Observation 1, it follows that $R_I = \{\langle \#\phi; \#\psi \rangle : BG(M_{PG}) \cup PG^*_I \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\}$. Thus, for all $\phi \in L^{\lhd}_{Tr}$, $M_I \vDash_{K_3} \phi$ iff $Tr(\ulcorner\phi\urcorner) \in Tr^+(M_I)$ iff $PG^*_I \vdash \ulcorner\phi\urcorner \lhd \ulcorner\neg\neg\phi\urcorner$ iff $M_I \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\neg\neg\phi\urcorner$. From the *Diagonal Lemma* (Lemma 1), $PA \vdash \sigma \leftrightarrow \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$. Thus, $\mathbb{N} \vDash \sigma \leftrightarrow \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$ and $M_I \vDash_{K_3} \sigma \leftrightarrow \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$. From Theorem 7, we know that M_I is a K_3-model and, so, it cannot be that $\#\sigma \in R_I$ and $\#\sigma \in \overline{R_I}$. Thus, three cases are possible:

- $\#\sigma \in R_I$ and $\#\sigma \notin \overline{R_I}$. Then, $M_I \vDash_{K_3} \ulcorner\sigma\urcorner \lhd \ulcorner\sigma\urcorner$, $M_I \vDash_{K_3} \neg\sigma$ by *Diagonal Lemma*, $M_I \vDash_{K_3} Tr(\ulcorner\sigma\urcorner)$ by Lemma 6 and $M_I \vDash_{K_3} \sigma$ because M_I is a fixed point. This contradicts the fact M_I is a K_3-model (Th. 7). Thus, (i) follows.
- $\#\sigma \notin R_I$ and $\#\sigma \in \overline{R_I}$. Then, $M_I \vDash_{K_3} \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\sigma\urcorner)$, $M_I \vDash_{K_3} \sigma$ by *Diagonal Lemma*, then $M_I \vDash_{K_3} \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner$ and $M_I \vDash_{K_3} \neg\sigma$ because M_I is a fixed point. This contradicts the fact M_I is a K_3-model (Th. 7). Thus, (ii) follows.
- $\#\sigma \notin R_I$ and $\#\sigma \notin \overline{R_I}$. From this, it follows that $M_I \nvDash_{K_3} \ulcorner\sigma\urcorner \lhd \ulcorner\sigma\urcorner$ and $M_I \nvDash_{K_3} \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\sigma\urcorner)$. Then, $M_I \nvDash_{K_3} \sigma$ and $M_I \nvDash_{K_3} \neg\sigma$ because M_I is a fixed point. Thus, $M_I \vDash_{K_3} \sigma \leftrightarrow \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\sigma\urcorner)$ holds.

□

It is important to highlight a few remarks. First, the results for the *liar* sentence are the same as in Kripke's fixed-point semantics (Kripke [6]). Second, there is a strong symmetry between the results for the paradoxical sentence σ and the ones for the *liar* sentence. Third, the model avoids the paradoxes of self-referentiality of both Theorems 1 and 2 by weakening classical logic. Again, the same strategy is used in Kripke's fixed-point semantics for the *liar* only.

3.3 Axioms

In this section, I aim to develop a classical axiomatic theory for the semantics of section 3.2. More precisely, I will focus the semantics defined in first stage of section 3.2. This is because the base ground statements added at the second stage are derived from a model. Instead, in this section, I am interested in the theorems that can be proven by strengthening PG^* with axioms that derive instances of iteration of ground and truth. The basic idea is, first, to reproduce the results of a fixed point K_3-models $M_{FP} = (\mathbb{N}; R_{FP}; \overline{R_{FP}}; S_{FP}; \overline{S_{FP}})$ in a classical setting. In order to do so, I define four predicates.

- The binary predicate $\lhd$, such that, given two terms s and t, $s \lhd t$ means that s *grounds* t. Intuitively, we want to extension of this predicate to be equivalent to the extension of R_{FP}.
- The binary predicate $\not\lhd$, such that, given two terms s and t, $s \lhd t$ means that s *does not ground* t. Intuitively, we want to extension of this predicate to be equivalent to the extension of $\overline{R_{FP}}$.
- The truth predicate Tr, such that, given a term s, $Tr(s)$ means s *is true*. Intuitively, we want to extension of this predicate to be equivalent to the extension of S_{FP}.
- The falsity predicate F, such that, given a term s, $F(s)$ means s *is false*. Intuitively, we want to extension of this predicate to be equivalent to the extension of $\overline{S_{FP}}$.

Note that, if we want to reproduce in a classical setting the distinction between $M_{FP} \vDash_{K_3} \neg Tr(s)$ and $M_{FP} \nvDash_{K_3} Tr(s)$ and between $M_{FP} \vDash_{K_3} \neg(s \lhd t)$ and $M_{FP} \nvDash_{K_3} s \lhd t$, we need to add two new predicates. More precisely, we add the predicate $\not\lhd$ such that, given an axiomatic theory T, $T \vdash s \not\lhd t$ is the proof-theoretic counterpart of $M_{FP} \vDash_{K_3} \neg(s \lhd t)$. Instead, the proof-theoretic counterpart of $M_{FP} \nvDash_{K_3} s \lhd t$ will be $T \vdash \neg(s \lhd t)$. Analogously, the proof-theoretic counterpart of $M_{FP} \nvDash_{K_3} \neg(s \lhd t)$ will be $T \vdash \neg(s \not\lhd t)$. Intuitively, we can think of the difference between the previous statement as: $T \vdash s \not\lhd t$ can be interpreted as *s does not ground t*, $T \vdash \neg(s \lhd t)$ as *it is not the case that s grounds t*[25] and $T \vdash \neg(s \not\lhd t)$ as *it is not the case that s does not ground t*.

A similar reasoning holds for the truth and falsity predicates. We add the falsity predicate F such that $T \vdash F(s)$ is the proof-theoretic counterpart of $M_{FP} \vDash_{K_3} \neg Tr(s)$, $T \vdash \neg Tr(s)$ is the proof-theoretic counterpart of $M_{FP} \nvDash_{K_3} Tr(s)$ and $T \vdash \neg F(s)$ is the proof-theoretic counterpart of $M_{FP} \nvDash_{K_3} \neg Tr(s)$. Intuitively, we can think of the difference between the previous statement as: $T \vdash F(s)$ can be interpreted as *s is false*, $T \vdash \neg Tr(s)$ as *s is not true* and $T \vdash \neg F(s)$ as *s is not false*.

Thus, formally, I define a classical model $M = (\mathbb{N}; R; \overline{R}; S; \overline{S})$ over the language $L^{\lhd,\not\lhd}_{Tr,F} = L^{\lhd}_{Tr} \cup \{\not\lhd\} \cup \{F\}$. The semantic clauses for the predicates $\lhd$, $\not\lhd$, Tr and F are:

- $M \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ iff $\langle \#\phi; \#\psi \rangle \in R$
- $M \vDash \neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$ iff $\langle \#\phi; \#\psi \rangle \in \overline{R}$
- $M \vDash Tr(\ulcorner\phi\urcorner)$ iff $\#\phi \in S$
- $M \vDash F(\ulcorner\phi\urcorner)$ iff $\#\phi \in \overline{S}$

The extension of the predicates is defined on the K_3-model M_{FP} as:

- $\langle \#\phi; \#\psi \rangle \in R$ iff $\langle \#\phi; \#\psi \rangle \in R_{FP}$

[25] I justify the legitimacy of this distinction in section 5.1.

- $\langle \#\phi; \#\psi \rangle \in \overline{R}$ iff $\langle \#\phi; \#\psi \rangle \in \overline{R_{FP}}$
- $\#\phi \in S$ iff $\#\phi \in S_{FP}$
- $\#\phi \in \overline{S}$ iff $\#\phi \in \overline{S_{FP}}$.

Note that, for all $\phi \in L^{\lhd}_{Tr}$ and all M_α of the semantics in section 3.2, $\#\phi \in \overline{S_\alpha}$ iff $\#\neg\phi \in S_\alpha$. Thus, $\#\phi \in \overline{S_{FP}}$ iff $\#\neg\phi \in S_{FP}$ and $M_{FP} \vDash_{K_3} \neg Tr(\ulcorner \phi \urcorner)$ iff $M_{FP} \vDash_{K_3} Tr(\ulcorner \neg\phi \urcorner)$. Thus, in the classical model, the extension of the falsity predicate $\overline{S}$ is implicitly defined by the extension of the truth predicate S. I particular, the number of a sentence is in $\overline{S}$ iff the number its negation is in S. Therefore, we can work with a simplified version of the model $M = (\mathbb{N}; R; \overline{R}; S)$, over the language $L^{\lhd, \not\lhd}_{Tr} = L^{\lhd}_{Tr} \cup \{\not\lhd\}$. The language can be simplified by defining the falsity predicate in terms of the truth predicate as:

$$\forall x(F(x) \leftrightarrow Tr(\underset{\cdot}{\neg} x)).$$

Now that I have translated the results of section 3.2 in a classical framework, I can state in full detail the theory of *untyped ground and truth PUGT*.

Definition 7 (**PUGT**). *The axioms of the predicational theory untyped partial ground and truth (PUGT) are: PATG plus all the missing instances of the induction scheme over $L^{\lhd, \not\lhd}_{Tr}$ plus the axioms of PG^*, with the only differences that G_1 becomes:*

G^*_1 $\forall x(x \not\lhd x)$

and that T^*_3 becomes:

T^{**}_3 $\forall x(Tr(x) \rightarrow Sent^{\lhd; \not\lhd}_{Tr}(x))$

plus the following extra Ground Axiom:

G^-_3 $\forall x \forall y(Tr(\underset{\cdot}{\neg} x) \vee Tr(\underset{\cdot}{\neg} y) \rightarrow x \not\lhd y)$

plus the following extra Truth Axioms:

T_4 $\forall s(Tr(\underset{\cdot}{Tr}(s)) \leftrightarrow Tr(s^\circ))$

T_5 $\forall s(Tr(\underset{\cdot}{\neg}\underset{\cdot}{Tr}(s)) \leftrightarrow Tr(\underset{\cdot}{\neg} s^\circ))$

T_6 $\forall s \forall t(Tr(s \underset{\cdot}{\lhd} t) \leftrightarrow s^\circ \lhd t^\circ)$

T_7 $\forall s \forall t(Tr(\underset{\cdot}{\neg}(s \underset{\cdot}{\lhd} t)) \leftrightarrow s^\circ \not\lhd t^\circ)$

and Consistency Axioms:

$Cons_{Tr}$ $\forall x(\neg(Tr(x) \wedge Tr(\underset{\cdot}{\neg} x)))$

$Cons_{\lhd}$ $\forall x \forall y(\neg(x \lhd y \wedge x \not\lhd y))$.

The intuitive motivation why I modified the axiom G_1 to G_1^* is that I want to distinguish between the ground statements which the principles of ground prove they do not hold and the ones that the principles of ground do not prove to hold. On the semantic side, the former ones correspond to the ones in the extension of $\overline{R_{FP}}$, while the latter ones correspond to the ones which are not in the extension of R_{FP}. Note that $Cons_{\lhd}$ implies that $\forall x \forall y(x \not\lhd y \rightarrow \neg(x \lhd y))$. Thus, G_1^* and $Cons_{\lhd}$ imply G_1.

The intuitive motivation why I added the axiom T_3^{**} instead of T_3^* is analogous to why I added T_3^* instead of T_3 in PG^* in section 2.1. $PUGT$ will prove theorems with the *not ground* predicate such as, for example, $PUGT \vdash \ulcorner 0=0 \urcorner \not\lhd \ulcorner 0=0 \urcorner$. We also want that $PUGT$ to prove that these theorems are true, so, for example, $PUGT \vdash Tr(\ulcorner \ulcorner 0=0 \urcorner \not\lhd \ulcorner 0=0 \urcorner \urcorner)$. Thus, in order to do so, we need to relax the condition that only terms that denote sentences in $L_{Tr}^{\lhd}$ can be arguments of the truth predicate and allow also terms that denote sentences with the *not ground* predicate to do so.

Axioms $Cons_{Tr}$ and $Cons_{\lhd}$ guarantee that the theory is consistent, i.e. no term is both true and false and there are not two terms such that one grounds and does not ground the other. These axioms are the proof-theoretic equivalent of the semantic fact that M_{FP} is a K_3-model which does not admit overlap between S_{FP} and $\overline{S_{FP}}$ and between R_{FP} and $\overline{R_{FP}}$.

Axioms T_4 and T_5 are the axioms which need to be added to the theory of *positive truth PT* in order to get the *Kripke-Fefereman KF* theory of truth (together with relaxing the restriction on the truth predicate analogously to what I did from T_3 to T_3^*[26]). These axioms are the proof-theoretic counterpart of the results in Observation 1 for S_{FP} and $\overline{S_{FP}}$. As mentioned in section 1.5, when a new relation symbol is added to a theory, we expect it to relate to the truth predicate according to the general schemes:

$$R^+ : \forall t_1, ..., t_n(Tr(\underset{\cdot}{R}(t_1, ..., t_n)) \leftrightarrow R(t_1^\circ, ..., t_n^\circ)),$$
$$R^- : \forall t_1, ..., t_n(Tr(\underset{\cdot}{\neg}\underset{\cdot}{R}(t_1, ..., t_n)) \leftrightarrow \neg R(t_1^\circ, ..., t_n^\circ)),$$

which, if instantiated with the truth predicate itself, would result in:

$$T^+ : \forall s(Tr(\underset{\cdot}{T}r(s)) \leftrightarrow Tr(s^\circ)),$$
$$T^- : \forall s(Tr(\underset{\cdot}{\neg}\underset{\cdot}{T}r(s)) \leftrightarrow \neg Tr(s^\circ)).$$

Note that T^+ is axiom T_4, while, T^- is different from axiom T_5. Intuitively, adding T_5 instead of T^- makes a distinction between *being false* and *being not true* and, by $Cons_{Tr}$, *being false*

[26] See Halbach [19, pp. 181-188] for a detailed presentation of KF.

implies *being not true*, but not the converse. Adding T^- instead of axiom T_5 would make the theory inconsistent because it would prove the *Liar Paradox* (Th. 1)[27].

I now prove how $PUGT$ deals with the *Liar Paradox*. In particular, I prove that $PUGT$ proves the *liar sentence* λ[28].

Proposition 3. *λ is the liar sentence, i.e. $PA \vdash \ulcorner \lambda \urcorner = \ulcorner \neg Tr(\ulcorner \lambda \urcorner) \urcorner$. Then: $(i) PUGT \cup \{Tr(\ulcorner \lambda \urcorner)\} \vdash \bot$, $(ii) PUGT \cup \{Tr(\ulcorner \neg \lambda \urcorner)\} \vdash \bot$ and $(iii) PUGT \vdash \lambda$.*

Proof. Note that, from axiom $Cons_{Tr}$, it follows by logic that $\forall x (Tr(\underset{\cdot}{\neg} x) \rightarrow \neg Tr(x))$.

- $PUGT \vdash Tr(\ulcorner \lambda \urcorner) \leftrightarrow Tr(\ulcorner \neg Tr(\ulcorner \lambda \urcorner) \urcorner) \leftrightarrow Tr(\ulcorner \neg \lambda \urcorner) \rightarrow \neg Tr(\ulcorner \lambda \urcorner)$. Thus, $PUGT \cup \{Tr(\ulcorner \lambda \urcorner)\} \vdash \neg Tr(\ulcorner \lambda \urcorner)$, so (i) follows.
- $PUGT \vdash Tr(\ulcorner \neg \lambda \urcorner) \leftrightarrow Tr(\ulcorner \neg\neg Tr(\ulcorner \lambda \urcorner) \urcorner) \leftrightarrow Tr(\ulcorner Tr(\ulcorner \lambda \urcorner) \urcorner) \leftrightarrow Tr(\ulcorner \lambda \urcorner)$. $PUGT \vdash Tr(\ulcorner \neg \lambda \urcorner) \rightarrow \neg Tr(\ulcorner \lambda \urcorner)$. Thus, $PUGT \cup \{Tr(\ulcorner \neg \lambda \urcorner)\} \vdash Tr(\ulcorner \lambda \urcorner) \wedge \neg Tr(\ulcorner \lambda \urcorner)$, so (ii) follows.
- From (i), it follows that $PUGT \vdash \lambda$. This does not lead to a contradiction because $PUGT \vdash \neg Tr(\ulcorner \lambda \urcorner) \leftrightarrow \neg Tr(\ulcorner \neg Tr(\ulcorner \lambda \urcorner) \urcorner) \leftrightarrow \neg Tr(\ulcorner \neg \lambda \urcorner)$, which means that the *liar* is not true iff it is not false. Since we distinguished $\neg Tr(s)$ and $Tr(\neg s)$, this is not contradictory.

Note that a theory T with the same axioms of $PUGT$ except for T^- instead of T_5 would be inconsistent. $PUGT \vdash \lambda \leftrightarrow \neg Tr(\ulcorner \lambda \urcorner) \leftrightarrow \neg Tr(\ulcorner \neg Tr(\ulcorner \lambda \urcorner) \urcorner) \leftrightarrow \neg\neg Tr(\ulcorner \lambda \urcorner) \leftrightarrow Tr(\ulcorner \lambda \urcorner) \leftrightarrow \neg \lambda$. □

Axioms T_6 and T_7 are the analogous for the ground predicate of axioms T_4 and T_5 for truth. Instantiating the general schemes R^+ and R^- with the ground predicate would result in:

$$G^+ : \forall s \forall t (Tr(s \underset{\cdot}{\lhd} t) \leftrightarrow s^\circ \lhd t^\circ),$$
$$G^- : \forall s \forall t (Tr(\underset{\cdot}{\neg}(s \underset{\cdot}{\lhd} t)) \leftrightarrow \neg(s^\circ \lhd t^\circ)).$$

Analogously to above for the truth predicate, G^+ is the same as axiom T_6, while G^- and axiom T_7 are different. T_6 and T_7 distinguish between *it is not the case that x grounds y*, which is formalised as $\neg(x \lhd y)$, from *x does not ground y*, which is formalised as $x \not\lhd y$. $Cons_{\lhd}$, $\forall x \forall y (x \not\lhd y \rightarrow \neg(x \lhd y))$, but not the converse. Note that, by axiom T_6, $\forall x \forall y (\neg Tr(x \underset{\cdot}{\lhd} y) \leftrightarrow \neg(x \lhd y))$ and, by axiom T_7, $\forall x \forall y (Tr(\underset{\cdot}{\neg}(x \underset{\cdot}{\lhd} y)) \leftrightarrow x \not\lhd y)$. Thus, it follows that $\forall x \forall y (\neg Tr(x \underset{\cdot}{\lhd} y) \rightarrow Tr(\underset{\cdot}{\neg}(x \underset{\cdot}{\lhd} y)))$, which is consistent with the previous definitions of the truth and falsity predicates. Adding G^- instead of axiom G_5 would make the theory inconsistent because of the *paradox of self-referentiality for ground* (Th. 2).

[27] See also Halbach [19, p. 183].
[28] The same results also holds for KF plus the consistency axiom $Cons_{Tr}$. See Halbach [19, p. 201].

The last axiom I need to motivate is G_3^-. The reason I add it is that, under the new interpretation of the predicates Tr and $\lhd$, the contrapositive of G_3 state that, for some terms s and t, if the theory proves that either s or t is not true, then it does not prove that s grounds t or it does not prove that t grounds s. However, G_3^- states the more precise claim that, if either s or t is false, then s does not ground t and t does not ground s.

I now prove how $PUGT$ deals with the *paradox of self-referentiality for ground*. Similarly to the result of Proposition 3, I prove that $PUGT$ proves the paradoxical sentence σ.

Proposition 4. *σ is the paradoxical sentence of Theorem 2. By Diagonal Lemma (Lemma 1), $PA \vdash \ulcorner\sigma\urcorner = \ulcorner\neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)\urcorner$. Then: $(i) PUGT \cup \{\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner\} \vdash \bot$, $(ii) PUGT \cup \{\ulcorner\sigma\urcorner \not\lhd \ulcorner\neg\neg\sigma\urcorner\} \vdash \bot$ and $(iii) PUGT \vdash \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$.*

Proof. Note that, from axiom $Cons_{\lhd}$, it follows by logic that $\forall x \forall y(x \not\lhd y \to \neg(x \lhd y))$.

- $PUGT \vdash \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner \leftrightarrow Tr(\ulcorner\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner\urcorner) \leftrightarrow Tr(\ulcorner\neg\sigma\urcorner) \to \neg Tr(\ulcorner\sigma\urcorner))$. $PUGT \vdash \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner \to Tr(\ulcorner\sigma\urcorner)$. Thus, $PUGT \cup \{\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner\} \vdash Tr(\ulcorner\sigma\urcorner) \wedge \neg Tr(\ulcorner\sigma\urcorner)$, so (i) follows.
- $PUGT \vdash \ulcorner\sigma\urcorner \not\lhd \ulcorner\neg\neg\sigma\urcorner \leftrightarrow Tr(\ulcorner\neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)\urcorner) \leftrightarrow Tr(\ulcorner\sigma\urcorner) \leftrightarrow \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner$. $PUGT \vdash \ulcorner\sigma\urcorner \not\lhd \ulcorner\neg\neg\sigma\urcorner \to \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$. Thus, $PUGT \cup \{\ulcorner\sigma\urcorner \not\lhd \ulcorner\neg\neg\sigma\urcorner\} \vdash \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner \wedge \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$, so (ii) follows.
- From (i), it follows that $PUGT \vdash \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)$, which is equivalent to $PUGT \vdash \sigma$. From (ii), it follows that $PUGT \vdash \neg(\ulcorner\sigma\urcorner \not\lhd \ulcorner\neg\neg\sigma\urcorner)$. Thus, it is not the case that σ grounds $\neg\neg\sigma$, nor it is the case that σ does not ground $\neg\neg\sigma$.

Note that a theory T with the same axioms of $PUGT$ except for G^- instead of T_6 would be inconsistent. $PUGT \vdash \sigma \leftrightarrow \neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner) \leftrightarrow Tr(\ulcorner\neg(\ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner)\urcorner) \leftrightarrow Tr(\ulcorner\sigma\urcorner) \leftrightarrow \ulcorner\sigma\urcorner \lhd \ulcorner\neg\neg\sigma\urcorner$. □

I now prove that the axiomatic theory $PUTG$ is sound with respect to the model $M = (\mathbb{N}; R; \overline{R}; S)$ described above. In other words, I show that all the theorems of $PUGT$ are true in the model M constructed on an arbitrary fixed point M_{FP} of the semantics of section 3.2.

Theorem 9. *Given a fixed point $M_{FP} = (\mathbb{N}; R_{FP}; \overline{R_{FP}}; S_{FP}; \overline{S_{FP}})$ of section 3.2 and a classical model $M = (\mathbb{N}; R; \overline{R}; S)$ such that $R_{FP} = R$, $\overline{S_{FP}} = \overline{R}$ and $S_{FP} = S$, then $M \vDash PUGT$.*

Proof. We need to check that M satisfies all the axioms of $PUGT$.

- $G_1^*: \forall x(x \not\lhd x)$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M \vDash \overline{n} \not\lhd \overline{n}$. $M \vDash \overline{n} \not\lhd \overline{n}$ iff $\langle n; n\rangle \in \overline{R}$ iff $\langle n; n\rangle \in \overline{R_{FP}}$ iff $PG^*_{FP-1} \vdash \neg(\overline{n} \lhd \overline{n})$, which holds because G_1 is an axiom of PG^*_{FP-1}.
- $G_2: \forall x \forall y \forall z(x \lhd z \wedge z \lhd y \rightarrow x \lhd y)$. This means that, for all $n, m, k \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$ and $k = \#\delta$, $M \vDash \overline{n} \lhd \overline{k} \wedge \overline{k} \lhd \overline{m} \rightarrow \overline{n} \lhd \overline{m}$. $M \vDash \overline{n} \lhd \overline{k}$ iff $\langle n; k\rangle \in R$ iff $\langle n; k\rangle \in R_{FP}$ iff $PG^*_{FP-1} \vdash \overline{n} \lhd \overline{k}$. An analogous derivation holds for $PG^*_{FP-1} \vdash \overline{k} \lhd \overline{m}$. Thus, $PG^*_{FP-1} \vdash \overline{n} \lhd \overline{m}$ because G_2 is an axiom of PG^*_{FP-1}, $\langle n; m\rangle \in R_{FP}$, $\langle n; m\rangle \in R$ and $M \vDash \overline{n} \lhd \overline{m}$.
- $G_3: \forall x \forall y(x \lhd y \rightarrow Tr(x) \wedge Tr(y))$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$ and $m = \#\psi$, $M \vDash (\overline{n} \lhd \overline{m} \rightarrow Tr(\overline{n}) \wedge Tr(\overline{m}))$. $M \vDash \overline{n} \lhd \overline{m}$ iff $\langle n; m\rangle \in R$ iff $\langle n; m\rangle \in R_{FP}$ iff $M_{FP} \vDash_{K_3} \overline{n} \lhd \overline{m}$. Then, by Lemma 6, $M_{FP} \vDash_{K_3} Tr(\overline{n})$ and $M_{FP} \vDash_{K_3} Tr(\overline{m})$. $M_{FP} \vDash_{K_3} Tr(\overline{n})$ iff $\#n \in S_{FP}$ iff $\#n \in S$ iff $M \vDash Tr(\overline{n})$. An analogous derivation holds for $M \vDash Tr(\overline{m})$.
- $G_3^-: \forall x \forall y(Tr(\underset{\cdot}{\neg} x) \vee Tr(\underset{\cdot}{\neg} y) \rightarrow x \not\lhd y)$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$ and $m = \#\psi$, $M \vDash Tr(\underset{\cdot}{\neg}\overline{n}) \vee Tr(\underset{\cdot}{\neg}\overline{m}) \rightarrow \overline{n} \not\lhd \overline{m}$. Suppose $M \vDash Tr(\underset{\cdot}{\neg}\overline{n})$, then $n \in \overline{S}$, $n \in \overline{S_{FP}}$, $M_{FP} \vDash_{K_3} \neg Tr(\overline{n})$. Note that, for all α, if $M_\alpha \vDash_{K_3} \neg Tr(\overline{n})$, then $\neg Tr(\overline{n}) \in Tr^-(M_{\alpha-1})$. Thus, by axiom G_3, $PG^*_{\alpha-1} \vdash \neg(\overline{n} \lhd \overline{m})$ and $M_\alpha \vDash_{K_3} \neg(\overline{n} \lhd \overline{m})$. Thus, $M_{FP} \vDash_{K_3} \neg(\overline{n} \lhd \overline{m})$, $\langle n; m\rangle \in R_{FP}$, $\langle n; m\rangle \in R$ and $M^{CL}_\alpha \vDash \overline{n} \not\lhd \overline{m}$. An analogous argument holds if $M \vDash Tr(\underset{\cdot}{\neg}\overline{m})$.
- T_1, T_2 and T_3^{**} are trivial.
- $T_4: \forall s(Tr(\underset{\cdot}{T}r(s)) \leftrightarrow Tr(s^\circ))$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M \vDash Tr(\underset{\cdot}{T}r(\overline{n})) \leftrightarrow Tr(\overline{n}))$. $M \vDash Tr(\overline{n})$ iff $\#\overline{n} \in S$ iff $\#\overline{n} \in S_{FP}$ iff $M_{FP} \vDash_{K_3} Tr(\overline{n})$ iff $\#Tr(\overline{n}) \in \Phi(S_{FP})$ iff $\#Tr(\overline{n}) \in S_{FP}$ by Observation 1 iff $\#Tr(\overline{n}) \in S$ iff $M \vDash (Tr(\underset{\cdot}{T}r(\overline{n}))$.
- $T_5: \forall s(Tr(\underset{\cdot}{\neg}\underset{\cdot}{T}r(s)) \leftrightarrow Tr(\underset{\cdot}{\neg}s^\circ))$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M \vDash Tr(\underset{\cdot}{\neg}\underset{\cdot}{T}r(\overline{n})) \leftrightarrow Tr(\underset{\cdot}{\neg}\overline{n}))$. $M \vDash Tr(\underset{\cdot}{\neg}\overline{n})$ iff $\#\neg\phi \in S$ iff $\#\neg\phi \in S_{FP}$ iff $n \in \overline{S_{FP}}$ iff $M_{FP} \vDash_{K_3} \neg Tr(\overline{n})$ iff $\#\neg Tr(\overline{n}) \in \Phi(S_{FP})$ iff $\#\neg Tr(\overline{n}) \in S_{FP}$ by Observation 1 iff $\#\neg Tr(\overline{n}) \in S$ iff $M \vDash (Tr(\underset{\cdot}{\neg}\underset{\cdot}{T}r(\overline{n}))$.
- $T_6: \forall s \forall t(Tr(s \underset{\cdot}{\lhd} t) \leftrightarrow s^\circ \lhd t^\circ)$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $M \vDash Tr(\overline{n} \underset{\cdot}{\lhd} \overline{m}) \leftrightarrow \overline{n} \lhd \overline{m}$. $M \vDash \overline{n} \lhd \overline{m}$ iff $\langle n; m\rangle \in R$ iff $\langle n; m\rangle \in R_{FP}$ iff $M_{FP} \vDash_{K_3} \overline{n} \lhd \overline{m}$ iff $\#\overline{n} \lhd \overline{m} \in \Phi(S_{FP})$ iff $\#\overline{n} \lhd \overline{m} \in S_{FP}$ by Observation 1 iff $\#\overline{n} \lhd \overline{m} \in S$ iff $M \vDash Tr(\overline{n} \underset{\cdot}{\lhd} \overline{m})$.
- $T_7: \forall s \forall t(Tr(\underset{\cdot}{\neg}(s \underset{\cdot}{\lhd} t)) \leftrightarrow s^\circ \not\lhd t^\circ)$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $M \vDash Tr(\underset{\cdot}{\neg}(\overline{n} \underset{\cdot}{\lhd} \overline{m})) \leftrightarrow \overline{n} \not\lhd \overline{m}$. $M \vDash \overline{n} \not\lhd \overline{m}$ iff $\langle n; m\rangle \in \overline{R}$ iff $\langle n; m\rangle \in \overline{R_{FP}}$ iff $M_{FP} \vDash_{K_3} \neg(\overline{n} \lhd \overline{m})$ iff $\#(\neg(\overline{n} \lhd \overline{m})) \in \Phi(S_{FP})$ iff $\#(\neg(\overline{n} \lhd \overline{m})) \in S_{FP}$ by Observation 1 iff $\#(\neg(\overline{n} \lhd \overline{m})) \in S$ iff $M \vDash Tr(\underset{\cdot}{\neg}(\overline{n} \underset{\cdot}{\lhd} \overline{m}))$.

- $Cons_{Tr}: \forall x(\neg(Tr(x) \wedge Tr(\underset{\cdot}{\neg} x)))$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M \vDash Tr(\overline{n}) \rightarrow \neg Tr(\underset{\cdot}{\neg}\overline{n})$. $M \vDash Tr(\overline{n})$ iff $n \in S$ iff $n \in S_{FP}$. If $n \in S_{FP}$, then $n \notin \overline{S_{FP}}$ by Theorem 7. $n \notin \overline{S_{FP}}$ iff $\#\neg\phi \notin S_{FP}$ iff $\#\neg\phi \notin S$ iff $M \nvDash Tr(\underset{\cdot}{\neg}\overline{n})$ iff $M \vDash \neg Tr(\underset{\cdot}{\neg}\overline{n})$.
- $Cons_{\lhd}: \forall x \forall y(\neg(x \lhd y \wedge x \not\lhd y))$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $M \vDash \overline{n} \lhd \overline{m} \rightarrow \neg(\overline{n} \not\lhd \overline{m})$. $M \vDash \overline{n} \lhd \overline{m}$ iff $\langle n; m\rangle \in R$ iff $\langle n; m\rangle \in R_{FP}$. If $\langle n; m\rangle \in R_{FP}$, then $\langle n; m\rangle \notin \overline{R_{FP}}$ by Theorem 7. $\langle n; m\rangle \notin \overline{R_{FP}}$ iff $\langle n; m\rangle \notin \overline{R}$ iff $M \nvDash \overline{n} \not\lhd \overline{m}$ iff $M \vDash \neg(\overline{n} \not\lhd \overline{m})$.
- $U_1: \forall x(Tr(x) \rightarrow x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x)$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M \vDash Tr(\overline{n}) \rightarrow \overline{n} \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}\overline{n}$. $M \vDash Tr(\overline{n})$ iff $n \in S$ iff $n \in S_{FP}$ iff $M_{FP} \vDash_{K_3} Tr(\overline{n})$ iff $M_{FP-1} \vDash_{K_3} \phi$. Thus, $PG^*_{FP-1} \vdash \overline{n} \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}\overline{n}$ by axiom U_1, $\langle n; \#\neg\neg\phi\rangle \in R_{FP}$, $\langle n; \#\neg\neg\phi\rangle \in R$ and $M \vDash \overline{n} \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}\overline{n}$.
- $D_2: \forall x \forall y(Tr(x \underset{\cdot}{\vee} y) \rightarrow (Tr(x) \rightarrow x \lhd x \underset{\cdot}{\vee} y) \wedge (Tr(y) \rightarrow y \lhd x \underset{\cdot}{\vee} y))$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $M \vDash (Tr(\overline{n} \underset{\cdot}{\vee} \overline{m}) \rightarrow (Tr(\overline{n}) \rightarrow \overline{n} \lhd \overline{n} \underset{\cdot}{\vee} \overline{m}) \wedge (Tr(\overline{m}) \rightarrow \overline{m} \lhd \overline{n} \underset{\cdot}{\vee} \overline{m}))$. $M \vDash Tr(\overline{n} \underset{\cdot}{\vee} \overline{m})$ iff $\#(\phi \vee \psi) \in S$ iff $\#(\phi \vee \psi) \in S_{FP}$ iff $M_{FP} \vDash_{K_3} Tr(\overline{n} \underset{\cdot}{\vee} \overline{m})$ iff $M_{FP-1} \vDash_{K_3} \phi \vee \psi$. An analogous derivation holds for $M_{FP-1} \vDash_{K_3} \phi$. Thus, $PG^*_{FP-1} \vdash \overline{n} \lhd \overline{n} \underset{\cdot}{\vee} \overline{m}$, $\langle n; \#(\phi \vee \psi)\rangle \in R_{FP}$, $\langle n; \#(\phi \vee \psi)\rangle \in R$ and $M \vDash \overline{n} \lhd \overline{n} \underset{\cdot}{\vee} \overline{m}$. An analogous derivation holds for $M \vDash \overline{m} \lhd \overline{n} \underset{\cdot}{\vee} \overline{m}$.
- The proof for the other upward and downward axioms is analogous.

□

I now prove some results about the truth-theoretic commitments and the proof-theoretic strength of the theory $PUGT$. First, I prove that $PUGT$ proves the Kripke-Fefereman theory of truth KF. Second, from the previous theorem, it follows that $PUGT$ proves the full T-scheme for T-positive formulas, which are formulas in which the truth predicate Tr does not occur in the scope of an odd number of negation symbols. Third, I prove that $PUGT$ proves the same theorems as KF in the language L_{Tr} with coding function over L_{Tr}. This means that the further theorems that $PUGT$ proves with respect to KF are ground statements or sentences that express the truth of a ground statement.

Definition 8 (**Kripke-Fefereman** KF)**.** *The consistent theory Kripke-Fefereman KF on the language L_{Tr} consists of the axioms of PAT and the three base truth axioms T_1, T_2 plus the following axioms [19, pp. 181-203]:*

T_1 $\forall s \forall t(Tr(s \underset{\cdot}{=} t) \leftrightarrow s^\circ = t^\circ)$

T_2 $\forall s \forall t(Tr(s \underset{\cdot}{\neq} t) \leftrightarrow s^\circ \neq t^\circ)$

T_3^{Tr} $\forall x(Tr(x) \rightarrow Sent_{Tr}(x))$[29]

KF_1 $\forall x(Tr(x) \leftrightarrow Tr(\underset{\cdot}{\neg}\underset{\cdot}{\neg}x))$

KF_2 $\forall x\forall y(Tr(x\underset{\cdot}{\vee}y) \leftrightarrow Tr(x) \vee Tr(y))$

KF_3 $\forall x\forall y(Tr(x\underset{\cdot}{\wedge}y) \leftrightarrow Tr(x) \wedge Tr(y))$

KF_4 $\forall x\forall y(Tr(\underset{\cdot}{\neg}(x\underset{\cdot}{\wedge}y)) \leftrightarrow Tr(\underset{\cdot}{\neg}x) \vee Tr(\underset{\cdot}{\neg}y))$

KF_5 $\forall x\forall y(Tr(\underset{\cdot}{\neg}(x\underset{\cdot}{\vee}y)) \leftrightarrow Tr(\underset{\cdot}{\neg}x) \wedge Tr(\underset{\cdot}{\neg}y))$

KF_6 $\forall x\forall v(Tr(\underset{\cdot}{\exists}vx) \leftrightarrow \exists t Tr(x(t/v)))$

KF_7 $\forall x\forall v(Tr(\underset{\cdot}{\neg}\underset{\cdot}{\exists}vx) \leftrightarrow \forall t Tr(\underset{\cdot}{\neg}x(t/v)))$

KF_8 $\forall x\forall v(Tr(\underset{\cdot}{\forall}vx) \leftrightarrow \forall t Tr(x(t/v)))$

KF_9 $\forall x\forall v(Tr(\underset{\cdot}{\neg}\underset{\cdot}{\forall}vx) \leftrightarrow \exists t Tr(\underset{\cdot}{\neg}x(t/v)))$

T_4 $\forall s(Tr(\underset{\cdot}{T}\underset{\cdot}{r}(s)) \leftrightarrow Tr(s^\circ))$

T_5 $\forall s(Tr(\underset{\cdot}{\neg}\underset{\cdot}{T}\underset{\cdot}{r}(s)) \leftrightarrow Tr(\underset{\cdot}{\neg}s^\circ))$

$Cons_{Tr}$ $\forall x(\neg(Tr(x) \wedge Tr(\underset{\cdot}{\neg}x)))$

Theorem 10. *$PUGT \vdash KF$.*

Proof. We derive all the axioms of KF from the axioms of $PUGT$.

- T_1, T_2, T_4, T_5 and $Cons_{Tr}$ are also axioms of $PUGT$.
- KF_1-KF_9 follow from the G_3 and the upward axioms U_1-U_9 and downward axioms D_1-D_9 of $PUGT$ as in Proposition 2 in Korbmacher [1, p. 176] that states that $PG \vdash PT$, where PT is the theory of *positive truth*. In fact, KF_1-KF_9 as also axioms of PT. I use the derivation of KF_3 as an illustrative example:

  1) $PUGT \vdash \forall x\forall y(Tr(x) \wedge Tr(y) \rightarrow (x \lhd x\underset{\cdot}{\wedge}y)$ by U_3

  2) $PUGT \vdash \forall x\forall y(x \lhd x\underset{\cdot}{\wedge}y \rightarrow Tr(x\underset{\cdot}{\wedge}y))$ by $G3$

  3) $PUGT \vdash \forall x\forall y(Tr(x) \wedge Tr(y) \rightarrow Tr(x\underset{\cdot}{\wedge}y))$ by 1) and 2)

  4) $PUGT \vdash \forall x\forall y(Tr(x\underset{\cdot}{\wedge}y) \rightarrow (x \lhd x\underset{\cdot}{\wedge}y) \wedge (y \lhd x\underset{\cdot}{\wedge}y))$ by D_3

  5) $PUGT \vdash \forall x\forall y(x \lhd x \wedge y \rightarrow Tr(x))$ by G_3

  6) $PUGT \vdash \forall x\forall y(y \lhd x \wedge y \rightarrow Tr(y))$ by G_3

  7) $PUGT \vdash \forall x\forall y(Tr(x\underset{\cdot}{\wedge}y) \rightarrow Tr(x) \wedge Tr(y))$ by 4), 5) and 6)

  8) $PUGT \vdash \forall x\forall y(Tr(x\underset{\cdot}{\wedge}y) \leftrightarrow Tr(x) \wedge Tr(y))$ by 7)

[29] T_3^{Tr} is the equivalent of axiom T_3 of PG extended to allow the names of sentences of L_{Tr} to be the argument of the truth predicate, analogously to what we did before with T_3^* and $L_{Tr}^{\lhd}$.

- If the language is L_{Tr}, then $PA \vdash \forall x(Sent^{\lhd,\ntriangleleft}_{Tr}(x) \leftrightarrow Sent_{Tr})$ since there no sentences containing $\lhd$ or $\ntriangleleft$. Thus, T_3^{**} implies T_3^{Tr}.

□

Proposition 5. *For all formulas $\phi(v_1, ..., v_n) \in L^{\lhd,\ntriangleleft}_{Tr}$ in which the truth predicate Tr does not occur in the scope of an odd number of negation symbols, also called T-positive formulas:*

$$PUGT \vdash \forall t_1, ..., \forall t_n(Tr(\ulcorner\phi(t_1, ..., t_n)\urcorner \leftrightarrow \phi(t_1, ..., t_n))$$

Proof. By induction on the length of ϕ. First, I consider atomic sentences:

- If $\phi(v_1, ..., v_n)$ is of the form $v_1 = v_2$, then the claim follows by T_1.
- If $\phi(v_1, ..., v_n)$ is of the form $Tr(v)$, then the claim follows by T_4.
- If $\phi(v_1, ..., v_n)$ is of the form $v_1 \lhd v_2$, then the claim follows by T_6.

I now consider complex statements.

- If $\phi(v_1, ..., v_n)$ is of the form $\neg\neg\psi(v_1, ..., v_n)$, then, by induction hypothesis, $PUGT \vdash \forall t_1, ..., \forall t_n(Tr(\ulcorner\psi(t_1, ..., t_n)\urcorner) \leftrightarrow \psi(t_1, ..., t_n))$. Then, by Theorem 10 and KF_1, $PUGT \vdash \forall t_1, ..., \forall t_n (Tr(\ulcorner\psi(t_1, ..., t_n)\urcorner) \leftrightarrow Tr(\ulcorner\neg\neg\psi(t_1, ..., t_n)\urcorner))$. By logic, $PUGT \vdash \forall t_1, ..., \forall t_n(\psi(t_1, ..., t_n) \leftrightarrow \neg\neg\psi(t_1, ..., t_n))$. Thus, $PUGT \vdash \forall t_1, ..., \forall t_n(Tr(\ulcorner\neg\neg\psi(t_1, ..., t_n)\urcorner) \leftrightarrow \neg\neg\psi(t_1, ..., t_n))$.
- If $\phi(v_1, ..., v_n)$ is of the form $(\psi\vee\delta)(v_1, ..., v_n)$, then, by induction hypothesis, $PUGT \vdash \forall t_1, ..., \forall t_n (Tr(\ulcorner\psi(t_1, ..., t_n)\urcorner) \leftrightarrow \psi(t_1, ..., t_n))$ and $PUGT \vdash \forall t_1, ..., \forall t_n(Tr(\ulcorner\delta(t_1, ..., t_n)\urcorner) \leftrightarrow \delta(t_1, ..., t_n))$. Thus, it follows that $PUGT \vdash \forall t_1, ..., \forall t_n(Tr(\ulcorner\psi(t_1, ..., t_n)\urcorner)\vee Tr(\ulcorner\delta(t_1, ..., t_n)\urcorner) \leftrightarrow (\psi\vee\delta)(t_1, ..., t_n))$. By Theorem 10 and KF_2, $PUGT \vdash \forall t_1, ..., \forall t_n(Tr(\ulcorner\psi(t_1, ..., t_n)\urcorner)\vee Tr(\ulcorner\delta(t_1, ..., t_n)\urcorner) \leftrightarrow Tr(\ulcorner(\psi\vee\delta)(t_1, ..., t_n)\urcorner))$. Thus, $PUGT \vdash \forall t_1, ..., \forall t_n((\psi \vee \delta)(t_1, ..., t_n) \leftrightarrow Tr(\ulcorner(\psi \vee \delta)(t_1, ..., t_n)\urcorner))$.
- The proof for the other connectives and quantifiers is analogous.

Note that, if $\phi(v_1, ..., v_n)$ is not a *T-positive formula*, then the claim does not hold. For example, if $\phi(v_1, ..., v_n)$ is of the form $\neg Tr(v)$, then, by T_5, it follows that $PUGT \vdash \forall t(Tr(\ulcorner\neg Tr(t)\urcorner) \leftrightarrow Tr(\neg t))$. By $Cons_{Tr}$, $PUGT \vdash \forall t(Tr(\neg t) \rightarrow \neg Tr(t))$. Thus, $PUGT \vdash \forall t(Tr(\ulcorner\neg Tr(t)\urcorner) \rightarrow \neg Tr(t))$, but not the converse. An analogous reasoning holds if $\phi(v_1, ..., v_n)$ is of the form $v_1 \lhd v_2$.

□

Theorem 11. *PUGT proves the same theorems as KF in the language L_{Tr} with coding function over L_{Tr}.*

Proof. We define the translation function $\pi : L_{Tr}^{\lhd,\not\lhd} \to L_{Tr}$ recursively by saying that:

$\pi(\phi) = \phi$ if $\phi \in L$ is atomic

$\pi(Tr(\ulcorner\phi\urcorner)) = Tr(\ulcorner\pi(\phi)\urcorner)$

$$\pi(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) = \begin{cases} Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner) & \text{if } c(\phi) < c(\psi) \\ \bot & \text{otherwise} \end{cases}$$

$$\pi(\ulcorner\phi\urcorner \not\lhd \ulcorner\psi\urcorner) = \begin{cases} Tr(\ulcorner\neg\pi(\phi)\urcorner) \vee Tr(\ulcorner\neg\pi(\psi)\urcorner) & \text{if } c(\phi) < c(\psi) \\ \top & \text{otherwise} \end{cases}$$

$\pi(\neg\phi) = \neg\pi(\phi)$

$\pi(\phi \circ \psi) = \pi(\phi) \circ \pi(\psi), \circ = \wedge, \vee$

$\pi(Qx\phi) = Qx(\pi(\phi)), Q = \forall, \exists$

We want to show that, if $PUGT \vdash \phi$, then $KF \vdash \pi(\phi)$. If $\phi \in L_{Tr}$ with the coding function restricted to L_{Tr}, then $\pi(\phi) = \phi$. We need to show that the translation of the ground-theoretic axioms of $PUGT$ can be derived from KF.

- G_1^* : We want to show that $KF \vdash \pi(\forall x(x \not\lhd x))$, or $KF \vdash \forall x \pi(x \not\lhd x)$. For all $n \in \mathbb{N}$ such that $n = \#\phi$, $KF \vdash \pi(\ulcorner\phi\urcorner \not\lhd \ulcorner\phi\urcorner)$. $c(\phi) = c(\phi)$, so the latter is equivalent to $KF \vdash \top$, which holds by logic.
- G_2 : We want to show that $KF \vdash \pi(\forall x \forall y \forall z(x \lhd z \wedge z \lhd y \to x \lhd y))$, or $KF \vdash \forall x \forall y \forall z(\pi(x \lhd z) \wedge \pi(z \lhd y) \to \pi(x \lhd y))$. For all $n, m, k \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$ and $k = \#\delta$, $KF \vdash \pi(\ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner) \wedge \pi(\ulcorner\delta\urcorner \lhd \ulcorner\psi\urcorner) \to \pi(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$. If $c(\phi) < c(\delta)$ and $c(\delta) < c(\psi)$, then $c(\phi) < c(\psi)$ and $KF \vdash Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\delta)\urcorner) \wedge Tr(\ulcorner\pi(\delta)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner) \to Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner)$, which holds by logic. If $c(\phi) \not< c(\delta)$ or $c(\delta) \not< c(\psi)$, then $KF \vdash \bot \to \pi(\ulcorner\pi(\phi)\urcorner \lhd \ulcorner\pi(\psi)\urcorner)$, which holds by logic.
- G_3 : We want to show that $KF \vdash \pi(\forall x \forall y(x \lhd y \to Tr(x) \wedge Tr(y)))$, or $KF \vdash \forall x \forall y(\pi(x \lhd y) \to \pi(Tr(x)) \wedge \pi(Tr(y)))$. For all $n, m \in \mathbb{N}$ such that $n = \#\phi$ and $m = \#\psi$, $KF \vdash (\pi(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) \to Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner))$. If $c(\phi) < c(\psi)$, then $KF \vdash Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner)) \to Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner))$, which holds by logic. If $c(\phi) \not< c(\psi)$, then $KF \vdash \bot \to Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner))$, which also holds by logic.
- G_3^- : We want to show that $KF \vdash \pi(\forall x \forall y(Tr(\dot\neg x) \vee Tr(\dot\neg y) \to x \not\lhd y))$, or $KF \vdash \forall x \forall y(\pi(Tr(\dot\neg x)) \vee \pi(Tr(\dot\neg y)) \to \pi(x \not\lhd y))$. For all $n, m \in \mathbb{N}$ such that $n = \#\phi$ and $m = \#\psi$, $KF \vdash Tr(\ulcorner\neg\pi(\phi)\urcorner) \vee Tr(\ulcorner\neg\pi(\psi)\urcorner) \to \pi(\ulcorner\phi\urcorner \not\lhd \ulcorner\psi\urcorner)$. If $c(\phi) < c(\psi)$, then $KF \vdash Tr(\ulcorner\neg\pi(\phi)\urcorner) \vee Tr(\ulcorner\neg\pi(\psi)\urcorner) \to$

$Tr(\ulcorner\neg\pi(\phi)\urcorner) \vee Tr(\ulcorner\neg\pi(\psi)\urcorner)$, which holds by logic. If $c(\phi) \not< c(\psi)$, then $KF \vdash Tr(\ulcorner\neg\pi(\phi)\urcorner) \vee Tr(\ulcorner\neg\pi(\psi)\urcorner) \to \top$, which also holds by logic.

- T_1, T_2, T_3^{**}, T_4, T_5 and $Cons_{Tr}$ are trivial.
- T_6 : We want to show that $KF \vdash \pi(\forall s\forall t(Tr(s\dot{\lhd}t) \leftrightarrow s^\circ \lhd t^\circ))$, or $KF \vdash \forall s\forall t(\pi(Tr(s\dot{\lhd}t)) \leftrightarrow \pi(s^\circ \lhd t^\circ))$. For all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $KF \vdash \pi(Tr(\ulcorner\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\urcorner)) \leftrightarrow \pi(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$, $KF \vdash Tr(\ulcorner\pi(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)\urcorner) \leftrightarrow \pi(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$. If $c(\phi) < c(\psi)$, the latter becomes $KF \vdash Tr(\ulcorner Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner)\urcorner) \leftrightarrow Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner)$, which is equivalent to $KF \vdash Tr(\ulcorner Tr(\ulcorner\pi(\phi) \wedge \pi(\psi)\urcorner)\urcorner) \leftrightarrow Tr(\ulcorner\pi(\phi) \wedge \pi(\psi)\urcorner)$ by KF_3 and holds by T_4. If $c(\phi) \not< c(\psi)$, then $KF \vdash Tr(\ulcorner\bot\urcorner) \leftrightarrow \bot$. The latter holds because the scheme:

$$KF \vdash \forall t_1, ..., \forall t_n(Tr(\ulcorner\phi(t_1, ..., t_n)\urcorner \leftrightarrow \phi(t_1, ..., t_n))$$

  for all formulas $\phi(x_1, ..., x_n)$ in which the truth predicate Tr does not occur in the scope of an odd number of negation symbols, also called *T-positive formulas.* (see Lemma 15.4 in Halbach [19, p. 187] or Cantini [27]).
- T_7 : We want to show that $KF \vdash \pi(\forall s\forall t(Tr(\dot\neg(s\dot{\lhd}t)) \leftrightarrow s^\circ \not\lhd t^\circ))$, or $KF \vdash \forall s\forall t(\pi(Tr(\dot\neg(s\dot{\lhd}t))) \leftrightarrow \pi(s^\circ \not\lhd t^\circ))$. For all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $KF \vdash \pi(Tr(\ulcorner\neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)\urcorner) \leftrightarrow \ulcorner\phi\urcorner \not\lhd \ulcorner\psi\urcorner)$, $KF \vdash Tr(\ulcorner\neg(\pi(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner))) \leftrightarrow \pi(\ulcorner\phi\urcorner \not\lhd \ulcorner\psi\urcorner)$. If $c(\phi) < c(\psi)$, the latter becomes $KF \vdash Tr(\ulcorner\neg(Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner))\urcorner) \leftrightarrow Tr(\ulcorner\neg\pi(\phi)\urcorner \vee Tr(\ulcorner\neg\pi(\psi)\urcorner)$, which is equivalent to $KF \vdash Tr(\ulcorner\neg Tr(\ulcorner\pi(\phi) \wedge \pi(\psi)\urcorner)\urcorner) \leftrightarrow Tr(\ulcorner\neg(\pi(\phi) \wedge \pi(\psi))\urcorner)$ by KF_3 and KF_4 holds by T_5. If $c(\phi) \not< c(\psi)$, then $KF \vdash Tr(\ulcorner\neg\bot\urcorner) \leftrightarrow \top$, which is equivalent to $KF \vdash Tr(\ulcorner\top\urcorner) \leftrightarrow \top$. The latter holds because of the scheme introduced above in the proof of T_6 for *T-positive formulas.*
- $Cons_\lhd$: We want to show that $KF \vdash \pi(\forall x\forall y(\neg(x \lhd y \wedge x \not\lhd y)))$, or $KF \vdash \forall x\forall y(\neg(\pi(x \lhd y) \wedge \pi(x \not\lhd y)))$. For all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $KF \vdash \neg(\pi(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) \wedge \pi(\ulcorner\phi\urcorner \not\lhd \ulcorner\psi\urcorner))$. If $c(\phi) < c(\psi)$, the latter becomes $KF \vdash \neg(Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\psi)\urcorner) \wedge (Tr(\ulcorner\neg\pi(\phi)\urcorner) \vee Tr(\ulcorner\neg\pi(\psi)\urcorner)))$, which is equivalent to $KF \vdash \neg(Tr(\ulcorner\pi(\phi) \wedge \pi(\psi)\urcorner) \wedge Tr(\ulcorner\neg(\pi(\phi) \wedge \pi(\psi))\urcorner))$ by KF_3 and KF_4 holds by $Cons_{Tr}$. If $c(\phi) \not< c(\psi)$, then $KF \vdash \neg(\bot \wedge \top)$, which holds by logic.
- U_1 : We want to show that $KF \vdash \pi(\forall x(Tr(x) \to x \lhd \dot\neg\dot\neg x))$, or $KF \vdash \forall x(\pi(Tr(x)) \to \pi(x \lhd \dot\neg\dot\neg x))$. For all $n \in \mathbb{N}$ such that $n = \#\phi$, $KF \vdash \pi(Tr(\ulcorner\phi\urcorner)) \to \pi(\ulcorner\phi\urcorner \lhd \ulcorner\neg\neg\phi\urcorner)$, $KF \vdash Tr(\ulcorner\pi(\phi)\urcorner) \to \pi(\ulcorner\phi\urcorner \lhd \ulcorner\neg\neg\phi\urcorner))$. $c(\phi) < c(\neg\neg\phi)$, so $KF \vdash Tr(\ulcorner\pi(\phi)\urcorner) \to Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\neg\neg\phi)\urcorner)$. $KF \vdash Tr(\ulcorner\pi(\phi)\urcorner) \to Tr(\ulcorner\pi(\phi)\urcorner)$ by logic and $KF \vdash Tr(\ulcorner\pi(\phi)\urcorner) \to Tr(\ulcorner\pi(\neg\neg\phi)\urcorner)$ by KF_1.
- D_2 : We want to show that $KF \vdash \pi(\forall x\forall y(Tr(x\dot\vee y) \to (Tr(x) \to x \lhd x\dot\vee y) \wedge (Tr(y) \to y \lhd x\dot\vee y)))$, or $KF \vdash \forall x\forall y(\pi(Tr(x\dot\vee y)) \to (\pi(Tr(x)) \to \pi(x \lhd x\dot\vee y) \wedge (\pi(Tr(y)) \to \pi(y \lhd x\dot\vee y))))$. For all

$n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$, $KF \vdash \pi(Tr(\ulcorner\phi \vee \psi\urcorner)) \rightarrow (\pi(Tr(\ulcorner\phi\urcorner)) \rightarrow \pi(\ulcorner\phi\urcorner \lhd \ulcorner\phi \vee \psi\urcorner) \wedge (\pi(Tr(\ulcorner\psi\urcorner)) \rightarrow \pi(\ulcorner\psi\urcorner \lhd \ulcorner\phi \vee \psi\urcorner)))$, or $KF \vdash Tr(\ulcorner\pi(\phi \vee \psi)\urcorner) \rightarrow (Tr(\ulcorner\pi(\phi)\urcorner) \rightarrow \pi(\ulcorner\phi\urcorner \lhd \ulcorner\phi \vee \psi\urcorner)) \wedge (Tr(\ulcorner\pi(\psi)\urcorner) \rightarrow (\pi(\ulcorner\psi\urcorner \lhd \ulcorner\phi \vee \psi\urcorner)))$. $c(\phi) < c(\phi \vee \psi)$, so $KF \vdash Tr(\ulcorner\pi(\phi \vee \psi)\urcorner) \rightarrow (Tr(\ulcorner\pi(\phi)\urcorner) \rightarrow Tr(\ulcorner\pi(\phi)\urcorner) \wedge Tr(\ulcorner\pi(\phi \vee \psi)\urcorner)) \wedge (Tr(\ulcorner\pi(\psi)\urcorner) \rightarrow Tr(\ulcorner\pi(\psi)\urcorner) \wedge Tr(\ulcorner\pi(\phi \vee \psi)\urcorner))$, which holds by logic.

- The proof for the other upward and downward axioms is analogous.

□

4 Aristotelian and GG Principles

As mentioned in section 3, it not possible to naively add the Aristotelian principles to the semantics of section 3.2 because of the inconsistency of Fine's puzzle (Th. 3). Also, adding the *GG* principle results in an analogous inconsistency involving the ground predicate instead of the truth predicate.

In section 4.1, I explain with more detail why adding the Aristotelian principles to the semantics of section 3.2 leads to the Fine's puzzle. I try various strategies to incorporate the Aristotelian principles into the semantics of section 3.2 without restricting its truths. I formally show that this cannot be done because it leads to contradictory results. Therefore, I conclude some restrictions to the principles of the previous theories are needed to consistently add the Aristotelian principles to them.

In section 4.2, I show that an analogous reasoning holds if we try to add the *GG* principle to the semantics of section 3.2. I state and prove an analogous theorem to Fine's puzzle (Th. 3) using the ground predicate instead of the truth predicate. Then, I show analogous issues as before arise if we try to add the *GG* principle to the semantics of section 3.2. Thus, I conclude that some restrictions are needed also to add the *GG* principle to the theories of the previous sections. Moreover, I argue that the inconsistency that follows from *GG* originates because of the same theoretical reasons that give rise to Fine's puzzle. Thus, it is possible and desirable to develop a common solution to consistently incorporate both of them into a theory of ground.

In section 4.3, I develop a semantic model in the style of the ones of section 3.2 that incorporates the Aristotelian and *GG* principles. I develop the model in two stages. First, I only consider the propositional part of PG^*, which, as I hint at the end of section 4.2, is consistent with the Aristotelian and *GG* principles. Then, I close it with the axioms of PG^* about the quantifiers and derive ground relations whose *ground* or *grounded* is a term denoting a quantified claim. I show that the model so constructed is consistent and, so, it avoids all the versions of Fine's puzzle. In section 5.2, I will compare this two-stage model with the solutions proposed by Fine to his puzzle and add philosophical motivation in support of this approach.

4.1 Aristotelian Principles

The aim of this section is to try to add the Aristotelian principles to the semantic model of section 3.2[30] The Aristotelian principles state that:

If ϕ is a true sentence, then $Tr(\ulcorner\phi\urcorner)$ holds either wholly or partially in virtue of ϕ,

If $\neg\phi$ is a true sentence, then $\neg Tr(\ulcorner\phi\urcorner)$ holds either wholly or partially in virtue of $\neg\phi$.

Thus, I aim to change the recursive clause for the extension of ground predicate:

$$R_{\alpha+1} = \{\langle \#\phi; \#\psi\rangle : PG^*_\alpha \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\},$$

to add all the ordered pairs of the kind $\langle \#\phi; \#Tr(\ulcorner\phi\urcorner)\rangle$ for all ϕ such that $M_{\alpha+1} \vDash_{K_3} Tr(\ulcorner\phi\urcorner)$ and of the kind $\langle \#\neg\phi; \#\neg Tr(\ulcorner\phi\urcorner)\rangle$ for all ϕ such that $M_{\alpha+1} \vDash_{K_3} Tr(\ulcorner\neg\phi\urcorner)$.

In general, two strategies are possible. First, we can add two axioms to PG^* that derive sentences of the kind $\ulcorner\phi\urcorner \lhd \ulcorner Tr(\ulcorner\phi\urcorner)\urcorner$ and $\ulcorner\neg\phi\urcorner \lhd \ulcorner\neg Tr(\ulcorner\phi\urcorner)\urcorner$. Second, we can add some semantic clauses to the definition of $R_{\alpha+1}$ that derive these additional pairs.

I examine the former strategy first. The natural candidates to be added to PG^* as formalisation of the Aristotelian principles are:

$$AP_T : \forall x(Tr(x) \to x \lhd \underset{\cdot}{T}r(x)),$$
$$AP_F : \forall x(Tr(\underset{\cdot}{\neg}x) \to \underset{\cdot}{\neg}x \lhd \underset{\cdot}{\neg}\underset{\cdot}{T}r(x)).$$

However, it is easy to check that this strategy fails because of Fine's puzzle (Th. 3). In fact, Theorem 3 derives an inconsistency from axioms G_1, G_2, G_3, U_6 (which are all axioms of PG^*) and either AP_T or AP_F. Thus, also the model constructed the following recursive clause on the extension of the ground predicate:

$$R_{\alpha+1} = \{\langle \#\phi; \#\psi\rangle : PG^*_\alpha + AP_T/AP_F \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\}$$

is inconsistent.

Instead, the second strategy seems more promising. Natural candidates for the semantic clauses that need to be added to the recursive definition of $R_{\alpha+1}$ are:

[30] I take the semantics of the first stage of section 3.2 for simplicity. The exact same reasoning applies to the one at the second stage.

$$\{\langle \#\phi; \#Tr(\ulcorner\phi\urcorner)\rangle : M_\alpha \vDash_{K_3} \phi\},$$
$$\{\langle \#\neg\phi; \#\neg Tr(\ulcorner\phi\urcorner)\rangle : M_\alpha \vDash_{K_3} \neg\phi\}.$$

Also, we want the recursive clause on the extension of the ground predicate to satisfy transitivity in the sense that, if $\langle n; k\rangle \in R_{\alpha+1}$ and $\langle k; m\rangle \in R_{\alpha+1}$, then $\langle n; m\rangle \in R_{\alpha+1}$. Thus, we need to add the Transitive Closure TC of the clauses in the recursive definition of $R_{\alpha+1}$.

Thus, the resulting recursive definition of $R_{\alpha+1}$ is:

$$R^A_{\alpha+1} = \{\langle \#\phi; \#\psi\rangle : PG^*_\alpha \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\} \cup \{\langle \#\phi; \#Tr(\ulcorner\phi\urcorner)\rangle : M_\alpha \vDash_{K_3} \phi\} \cup \{\langle \#\neg\phi; \#\neg Tr(\ulcorner\phi\urcorner)\rangle : M_\alpha \vDash_{K_3} \neg\phi\}^{TC}.$$

However, the model constructed in this way is inconsistent.

Lemma 7. *There is some α such that the model M^A_α that results from the construction of the first stage in section 3.2 with $R^A_{\alpha+1}$ instead of $R_{\alpha+1}$ is inconsistent. This means that there is some ϕ such that $M^A_\alpha \vDash_{K_3} \phi \wedge \neg\phi$.*

Proof. For example, consider the following derivation[31]:

1) $M^A_0 \vDash_{K_3} \overline{0} = \overline{0}$

2) $\#(\overline{0} = \overline{0}) \in S_1$

3) $M^A_1 \vDash_{K_3} Tr(\ulcorner\overline{0} = \overline{0}\urcorner)$

4) $M^A_1 \vDash_{K_3} \exists x Tr(x)$ by K_3 logic

5) $\#(\exists x Tr(x)) \in S_2$

6) $M^A_2 \vDash_{K_3} Tr(\ulcorner\exists x Tr(x)\urcorner)$

7) $Tr(\ulcorner Tr(\ulcorner\exists x Tr(x)\urcorner)\urcorner) \in Tr^+(M_2)$

8) $M^A_3 \vDash_{K_3} \ulcorner Tr(\ulcorner\exists x Tr(x)\urcorner)\urcorner \lhd \ulcorner\exists x Tr(x)\urcorner$ by axiom U_6 and (7)

9) $M^A_2 \vDash_{K_3} \exists x Tr(x)$ by Lemma 2[32]

10) $M^A_3 \vDash_{K_3} \ulcorner\exists x Tr(x)\urcorner \lhd \ulcorner Tr(\ulcorner\exists x Tr(x)\urcorner)\urcorner$ by the new semantic causes in $R^A_{\alpha+1}$ and (9)

11) $M^A_3 \vDash_{K_3} \ulcorner\exists x Tr(x)\urcorner \lhd \ulcorner\exists x Tr(x)\urcorner$ by Transitivity Closure

12) $M^A_3 \vDash_{K_3} \neg(\ulcorner\exists x Tr(x)\urcorner \lhd \ulcorner\exists x Tr(x)\urcorner)$ by G_1

13) $M^A_3 \vDash_{K_3} \ulcorner\exists x Tr(x)\urcorner \lhd \ulcorner\exists x Tr(x)\urcorner \wedge \neg(\ulcorner\exists x Tr(x)\urcorner \lhd \ulcorner\exists x Tr(x)\urcorner)$ from (11) and (12) □

In conclusion, there is not a trivial way to add the Aristotelian principles to the semantics of section 3.2. In section 4.3, I will restrict some of its principles and construct a new K_3 semantics that incorporates the Aristotelian principles.

[31] Where not specified, the steps follow from the previous ones how the construction in Definition 3 is defined.

[32] We know that $M^A_2 \vDash_{K_3} \exists x Tr(x)$ because we know by Lemma 2 that $M_2 \vDash_{K_3} \exists x Tr(x)$ and M^A_2 extends M_2. Alternatively, we can also prove the analogous of Lemma 2 for the new sequence of models M^A by induction on α.

4.2 GG Principle

Another principle that I aim to incorporate into the semantics of section 3.2 is the principle that the *ground* grounds *that the ground grounds the grounded*, or *GG* principle, which was introduced in section 1.4. As mentioned above, this is a common principle of theories of iterated ground, such as Bennett [25], deRosset [24] and Litland [9], [23], which has not yet been formalised in a predicational setting. In particular, this is a special case of iterated ground where a ground statements is the *grounded* of its *ground*. An exemplifying instance of this principle is:

$$\ulcorner 0 = 0 \urcorner \lhd (\ulcorner\ulcorner 0 = 0 \urcorner \lhd \ulcorner \neg\neg 0 = 0 \urcorner\urcorner).$$

In this section, I show that adding the *GG* principle to the theory of sections 3.2 and 3.3 give rise to similar inconsistencies to the ones derived above when the Aristotelian principles are added (Th. 3 and Lemma 7). First, I show that any axiomatic theory of ground that satisfy some rather weak principles of ground and truth is inconsistent with the *GG* principle. The principle *GG* can be formalised axiomatically as:

$$GG : \forall x \forall y (x \lhd y \to x \lhd (x \underset{\cdot}{\lhd} y)).$$

To derive the inconsistency, we also need the left to right direction of the T-scheme for T-positive formulas (see Lemma 5). Thus, we assume the axiom T^{++} as:

$$T^{++} : \forall t_1, ..., \forall t_n (Tr(\ulcorner \phi(t_1, ..., t_n) \urcorner \leftarrow \phi(t_1, ..., t_n)),$$

for all $\phi(x_1, ..., x_n)$ T-positive formulas.

Theorem 12 (**Fine's Puzzle for Ground Predicate**). *A theory T that contains U_1, U_6, T^{++} and GG and proves the truth of an arbitrary sentence ϕ is inconsistent.*

Proof. ϕ is an arbitrary true sentence.

1) $T \vdash Tr(\ulcorner \phi \urcorner)$ by assumption

2) $T \vdash \ulcorner \phi \urcorner \lhd \ulcorner \neg\neg\phi \urcorner$ from (1) by U_1

3) $T \vdash \exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x)$ from (2) by logic

4) $T \vdash Tr(\ulcorner \exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner)$ from (3) by T^{++}

5) $T \vdash \ulcorner \exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner \lhd \ulcorner \neg\neg\exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner$ from (4) by U_1

6) $T \vdash Tr(\ulcorner\ulcorner \exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner \lhd \ulcorner \neg\neg\exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner\urcorner)$ from (5) by T^{++}

7) $T \vdash \ulcorner (\ulcorner \exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner \lhd \ulcorner \neg\neg\exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner) \urcorner \lhd \ulcorner \exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner$ from (6) by U_6

8) $T \vdash \ulcorner \exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner \lhd \ulcorner (\ulcorner \exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner \lhd \ulcorner \neg\neg\exists x (x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x) \urcorner) \urcorner$ from (5) by GG

9) $T \vdash \ulcorner \exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner \lhd \ulcorner \exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner$ by (7), (8) and G_2

10) $T \vdash \neg(\ulcorner \exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner \lhd \ulcorner \exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner)$ by G_1

11) $T \vdash \bot$ by (9) and (10) □

Intuitively, as for the Fine's puzzle in Theorem 3, the contradiction derives from the fact that the theory T proves two plausible claims about ground. Translated in natural language, they state that:

1) 'The fact that something grounds its own double negation grounds that the fact that something grounds its own double negation grounds its own double negation.'

by the GG principle, and:

2) 'The fact that something grounds its own double negation grounds it own double negation grounds the fact that something grounds its own double negation.'

by axiom U_6. By combining these (1) and (2) with the transitivity and irreflexivity of ground relations, a contradiction clearly follows.

Note that we could have derived the inconsistency also with other axioms of ground instead of U_1. For example, consider the sentence:

$$\exists x(x \lhd x \underset{.}{\vee} \ulcorner \overline{0} = \overline{0} \urcorner)$$

instead of:

$$\exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)$$

and axiom U_2 instead of U_1. Analogously as in the proof above, we can derive:

$$\ulcorner \exists x(x \lhd x \underset{.}{\vee} \ulcorner \overline{0} = \overline{0} \urcorner)\urcorner \lhd \ulcorner \exists x(x \lhd x \underset{.}{\vee} \ulcorner \overline{0} = \overline{0} \urcorner) \underset{.}{\vee} \ulcorner \overline{0} = \overline{0} \urcorner \urcorner$$

by GG and:

$$\ulcorner \exists x(x \lhd x \underset{.}{\vee} \ulcorner \overline{0} = \overline{0} \urcorner) \underset{.}{\vee} \ulcorner \overline{0} = \overline{0} \urcorner \urcorner \lhd \ulcorner \exists x(x \lhd x \underset{.}{\vee} \ulcorner \overline{0} = \overline{0} \urcorner)\urcorner$$

by U_2. From, these two, a contradiction originates as in Theorem 12.

I now prove an analogous result for the semantics in section 3.2. More precisely, I show that, if GG is added to the recursive clause of $R_{\alpha+1}$, an inconsistency is derived. As for the Aristotelian principles in section 4.1, we can either add GG as an axiom of PG^* or as an added semantic clause on $R_{\alpha+1}$. In the former case, $R_{\alpha+1}$ becomes:

$$R^G_{\alpha+1} = \{\langle \#\phi; \#\psi \rangle : PG^*_\alpha + GG \vdash \ulcorner \phi \urcorner \lhd \ulcorner \psi \urcorner\}.$$

In the latter one, $R_{\alpha+1}$ becomes:

$$R^{GG}_{\alpha+1} = \{\{\langle \#\phi; \#\psi\rangle : PG^*_\alpha \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\} \cup \{\#\phi; \#(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) : M_{\alpha+1} \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\}\}^{TC}.$$

Lemma 8. *There is some α such that the models M^G_α and M^{GG}_α that results from the construction of the first stage in section 3.2 with $R^G_{\alpha+1}$ and $R^{GG}_{\alpha+1}$ instead of $R_{\alpha+1}$ are inconsistent.*

Proof. I start by proving the inconsistency of M^G_α:

1) $M^G_0 \vDash_{K_3} \overline{0} = \overline{0}$

2) $Tr(\ulcorner\overline{0} = \overline{0}\urcorner) \in Tr^+(M^G_0)$ from (1)

3) $M^G_1 \vDash_{K_3} \ulcorner\overline{0} = \overline{0}\urcorner \lhd \ulcorner\neg\neg\overline{0} = \overline{0}\urcorner$ from (2) by U_1

4) $M^G_1 \vDash_{K_3} \exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)$ from (3) by K_3 logic

5) $Tr(\ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner) \in Tr^+(M^G_1)$ from (4)

6) $M^G_2 \vDash_{K_3} \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\neg\neg\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner$ from (5) by U_1

7) $M^G_3 \vDash_{K_3} \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\neg\neg\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner$ from (6) by Lemma 2[33]

8) $M^G_3 \vDash_{K_3} \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\neg\neg\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner\urcorner$ from (7) by GG

9) $Tr(\ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\neg\neg\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner) \in Tr^+(M^G_2)$ from (6)

10) $M^G_3 \vDash_{K_3} \ulcorner\ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\neg\neg\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner\urcorner \lhd \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner$ from (9) by U_6

11) $M^G_3 \vDash_{K_3} \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner$ from (8) and (10) by G_2

12) $M^G_3 \vDash_{K_3} \neg(\ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner)$ by G_1

13) $M^G_3 \vDash_{K_3} \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \wedge \neg(\ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner \lhd \ulcorner\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg}x)\urcorner)$ from (11) and (12)

Instead, if we consider M^{GG}_α, then the derivation is the exactly the same except for the fact that we use the new semantic clause instead of GG and the transitive closure TC instead of G_2. $\square$

Thus, as for the case of the Aristotelian principles, it is not possible to trivially add the GG principle to the semantics of section 3.2. Intuitively, the reason why, in both cases, adding these principles leads to a contradiction is the following. The axioms of PG state that every quantified claim Qvx, with $Q = \exists, \forall$, is grounded in every of its true instances $x(t/v)$. Given that we are working in a type-free language without any restrictions on the coding function, t can be Qvx itself. However, when $x(t/v)$ has a certain form, the general scheme of the Aristotelian and GG principles state that, for every term t (including Qvx), t grounds $x(t/v)$. Therefore, allowing quantified claims

[33] It is easy to check that Lemma 2 holds for M^G and M^{GG} with an analogous proof by induction on α.

to be grounded in every of their true instances is a strong principle. The bottom line is that we need to be careful when we add principles that derive that ground relations in which quantified claims are the *grounds*, because it is likely we will derive that a quantified claim ground one of its true instances, as I showed in Theorems 3 and 12 for the Aristotelian and GG principles. The semantics I develop in section 4.3 restricts some of the principles of the one in section 3.2 in order to incorporate both the Aristotelian and GG principles.

Note that the, in all the Theorems and Lemmas above, I derived the inconsistency using the existential quantifier. However, analogous results can be proved for the universal quantifier. For example, we assume the same assumptions of Theorem 3 except for U_8 and U_2 instead of U_6, plus the classical logical truth:

$$\forall x(Tr(x) \vee \neg Tr(x)).$$

Then, applying an analogous reasoning as Theorem 3, we can derive:

$$1)\ T \vdash \ulcorner Tr(\ulcorner \phi \urcorner) \vee \neg Tr(\ulcorner \phi \urcorner)\urcorner \lhd \ulcorner \phi \urcorner, \text{ where } \phi = \forall x(Tr(x) \vee \neg Tr(x))$$

by applying U_8 at the final step of the derivation of (1) and:

$$2)\ T \vdash \ulcorner \phi \urcorner \lhd \ulcorner Tr(\ulcorner \phi \urcorner)\urcorner,$$
$$3)\ T \vdash \ulcorner Tr(\ulcorner \phi \urcorner)\urcorner \lhd \ulcorner Tr(\ulcorner \phi \urcorner) \vee \neg Tr(\ulcorner \phi \urcorner)\urcorner, \text{ where } \phi = \forall x(Tr(x) \vee \neg Tr(x))$$

by applying, respectively, the Aristotelian principle AP_T and U_2 at the final step of the derivation of (2) and (3). Then, it is easy to show that 1), 2) and 3) are inconsistent with transitivity and irriflexivity of ground as in Theorem 3.

4.3 Semantics

I will develop the model in two stages. First, I develop a model using only the propositional portion of PG^*. I construct a K_3 model in the style of the ones developed in section 3.2 which is based on the propositional axioms of PG^* and the Aristotelian and GG principles. By proving that this model is consistent, I provide evidence for the hint suggested at the and of the previous section that Fine's puzzle originates because the grounding principles on the quantifiers in PG are not compatible with the Aristotelian or GG principles. Instead, the propositional part of PG does not give rise to the same problems when the Aristotelian or GG principles are added to it.

To develop the first stage, first, I define the axiomatic theory that I will use in the models in the next paragraphs. I define the propositional base theory of partial ground PG^*_P. Then, I define the theory PG^{APG} by adding to PG^*_P the axioms that formalise the Aristotelian and GG principles.

Definition 9 (PG^*_P). *The propositional base theory of partial ground PG^*_P is formed with all the axioms of PG^* except for the ones about the quantifiers: $U_6 - U_9$ and $D_6 - D_9$.*

Definition 10 (PG^{APG}_α). *The theory PG^{APG} is formed with the axioms of PG^*_P plus two axioms that formalise that Aristotelian principles:*

$$AP_T : \forall x(Tr(x) \land Tr(\underset{\cdot}{Tr}(x)) \to x \lhd \underset{\cdot}{Tr}(x)),$$
$$AP_F : \forall x(Tr(\underset{\cdot}{\neg} x) \land Tr(\underset{\cdot}{\neg}\underset{\cdot}{Tr}(x)) \to \underset{\cdot}{\neg} x \lhd \underset{\cdot}{\neg}\underset{\cdot}{Tr}(x)).$$

and one that formalises the GG principle:

$$GG : \forall x \forall y(Tr(x) \land Tr(x \underset{\cdot}{\lhd} y) \to x \lhd (x \underset{\cdot}{\lhd} y)).$$

PG^{APG}_α is defined as $Tr^+_l(M_\alpha) \cup Tr^-(M_\alpha) \cup PG^{APG}$, where $Tr^+_l(M_\alpha)$ is defined as $\{Tr(\ulcorner\phi\urcorner) : M_\alpha \vDash_{K_3} \phi$ and ϕ is a literal$\}$[34].

I defined the Aristotelian and GG principles as both upward and downward axioms, in the sense that the antecedent states that both the *ground* and the *grounded* must be true. The reason for this is that I want to stick to the formal framework of an increasing sequences of models of section 3.2. At every stage, I define the ground relations based on a larger set of truths and the *ground* and the *grounded* are in this set of truths. Instead, if I take the upward version of the axioms, such as:

$$AP^U_T : \forall x(Tr(x) \to x \lhd \underset{\cdot}{Tr}(x)),$$

and the set of truths $Tr(M_\alpha)$ at level α, the fact that, for some term t, the *ground* $Tr(t) \in Tr(M_\alpha)$ does not imply by K_3 logic that the *grounded* $Tr(\underset{\cdot}{Tr}(t)) \in Tr(M_\alpha)$, as happens with the other upward and downward axioms of PG^*. Moreover, from $Tr(t)$, we can derive $Tr(\underset{\cdot}{Tr}(t))$ by AP^U_T and G_3 and this derivation can be iterated to $Tr(\underset{\cdot}{Tr}(\underset{\cdot}{Tr}(t)))$, and so on. An analogous reasoning holds for the upward versions AP^U_F and GG^U of AP_F and GG. Thus, the idea of generating the ground relations based on a given set of truths at every level is incompatible with use the upward version of the axioms because they derive truths outside the initial set.

I now define the inductive definition of a sequence of models in the style of the ones developed in Definitions 3 and 6 in section 3.2. Then, I prove that every model in the sequence is consistent.

Definition 11 (**Propositional Construction Aristotelian and** GG **Principles**). *I define a monotone operator $\Lambda : M \to M$:*

$$M_0 = (\mathbb{N}; R_0; \overline{R_0}; S_0; \overline{S_0}) = (\mathbb{N}; \emptyset; \emptyset; \emptyset; \{n : PA \vdash \neg Sent^{\lhd}_{Tr}(\overline{n})\})$$

[34] $Tr^-(M_\alpha) = \{\neg Tr(\ulcorner\phi\urcorner) : M_\alpha \vDash_{K_3} \neg\phi\}$ as in section 3.2.

$M_{\alpha+1} = (\mathbb{N}; R_{\alpha+1}; \overline{R_{\alpha+1}}; S_{\alpha+1}; \overline{S_{\alpha+1}})$

$R_{\alpha+1} = \{\langle \#\phi; \#\psi \rangle : PG^{APG}_{\alpha} \vdash \ulcorner \phi \urcorner \lhd \ulcorner \psi \urcorner\}$

$\overline{R_{\alpha+1}} = \{\langle \#\phi; \#\psi \rangle : PG^{APG}_{\alpha} \vdash \neg(\ulcorner \phi \urcorner \lhd \ulcorner \psi \urcorner)\}$

$S_{\alpha+1} = \{\#\phi : PG^{APG}_{\alpha} \vdash Tr(\ulcorner \phi \urcorner)\}$

$\overline{S_{\alpha+1}} = \{\#\phi : PG^{APG}_{\alpha} \vdash Tr(\ulcorner \neg\phi \urcorner)\}$

$M_{\alpha} = (\mathbb{N}; R_{\alpha}; \overline{R_{\alpha}}; S_{\alpha}; \overline{S_{\alpha}})$, *$\alpha$ limit ordinal*

$R_{\alpha} = \bigcup_{\beta<\alpha} R_{\beta}$

$\overline{R_{\alpha}} = \bigcup_{\beta<\alpha} \overline{R_{\beta}}$

$S_{\alpha} = \bigcup_{\beta<\alpha} S_{\beta}$

$\overline{S_{\alpha}} = \bigcup_{\beta<\alpha} \overline{S_{\beta}}$

Intuitively, at every level α, we take the literals proven by PG^{APG}_{α} as the true literals of the model M_{α}, except for the condition on $\overline{S_{\alpha}}$, in which, for every $\phi \in L^{\lhd}_{Tr}$, if $PG^{APG}_{\alpha-1} \vdash Tr(\ulcorner \neg\phi \urcorner)$, then $M_{\alpha} \vDash_{K_3} \neg Tr(\ulcorner \phi \urcorner)$. The reason for this is that, for example, $PG^{APG} \vdash Tr(\ulcorner \neg(\overline{0} = \overline{1}) \urcorner)$, but $PG^{APG} \nvdash \neg Tr(\ulcorner \overline{0} = \overline{1} \urcorner)$. For example, we want to derive $M_2 \vDash_{K_3} \ulcorner \neg(\overline{0} = \overline{1}) \urcorner \lhd \ulcorner \neg Tr(\ulcorner \overline{0} = \overline{1} \urcorner) \urcorner$, and this can be done only with $Tr(\ulcorner \neg Tr(\ulcorner \overline{0} = \overline{1} \urcorner) \urcorner)$ as assumption. It is easy to see that the latter can be derived by PG^{APG}_{1} with the modified clause on $\overline{S_{\alpha+1}}$.

Lemma 9. *$Th(M_{\alpha})$ is increasing in α. For α and for all $\phi \in L^{\lhd}_{Tr}$, if $\phi \in Th(M_{\alpha})$, then $\phi \in Th(M_{\alpha+1})$.*

Proof. By induction on α. Given some α, I show that, if $\phi \in Th(M_{\alpha})$, then $\phi \in Th(M_{\alpha+1})$.

If $PG^{APG}_{\alpha-1} \vdash \phi$, then $Tr^{+}_{l}(M_{\alpha-1}) \cup Tr^{-}(M_{\alpha-1}) \cup PG^{APG} \vdash \phi$. By induction hypothesis, $Th(M_{\alpha-1}) \subseteq Th(M_{\alpha})$, so $Tr^{+}_{l}(M_{\alpha-1}) \subseteq Tr^{+}_{l}(M_{\alpha})$ and $Tr^{-}(M_{\alpha-1}) \subseteq Tr^{-}(M_{\alpha})$. Thus, $Tr^{+}_{l}(M_{\alpha}) \cup Tr^{-}(M_{\alpha}) \cup PG^{APG}\phi$ and $PG^{APG}_{\alpha} \vdash \phi$.

Given the previous fact, it is easy to check that, if ϕ is a literal, then, if $\phi \in Th(M_{\alpha})$, then $\phi \in Th(M_{\alpha+1})$. Then, the claim generalizes to all sentence as in Lemma 2. □

Definition 12 (**ω-complexity**)**.** *For all α, I define an ω-complexity function $c_{M_{\alpha}}$ relative to a model M_{α} as a partial function from formulas to ordinals such that, for all $\phi \in Th(M_{\alpha})$:*

if ϕ is a literal and $\phi \in L$, then $c_{M_{\alpha}}(\phi) = 0$,

if $\phi = \neg(\ulcorner \psi \urcorner \lhd \ulcorner \delta \urcorner)$, then $c_{M_{\alpha}}(\phi) = 0$,

if $\phi = Tr(\ulcorner \psi \urcorner)$, then $c_{M_{\alpha}}(\phi) = \omega + c_{M_{\alpha}}(\psi)$, *if* $\phi = \neg Tr(\ulcorner\psi\urcorner)$*, then* $c_{M_\alpha}(\phi) = \omega + c_{M_\alpha}(\neg\psi)$*,*

if $\phi = \ulcorner\psi\urcorner \lhd \ulcorner\delta\urcorner$*, then* $c_{M_\alpha}(\phi) = \omega + max\{c_{M_\alpha}(\psi) : M_\alpha \vDash_{K_3} \psi; c_{M_\alpha}(\delta) : M_\alpha \vDash_{K_3} \delta\}$*,*

if $\phi = \neg\neg\psi$*, then* $c_{M_\alpha}(\phi) = c_{M_\alpha}(\psi) + 1$*,*

if $\phi = \psi \vee \delta$*, then* $c_{M_\alpha}(\phi) = max\{c_{M_\alpha}(\psi) : M_\alpha \vDash_{K_3} \psi; c_{M_\alpha}(\delta) : M_\alpha \vDash_{K_3} \delta\} + 1$*,*

if $\phi = \psi \wedge \delta$*, then* $c_{M_\alpha}(\phi) = max\{c_{M_\alpha}(\psi); c_{M_\alpha}(\delta)\} + 1$*,*

if $\phi = \neg(\psi \wedge \delta)$*, then* $c_{M_\alpha}(\phi) = max\{c_{M_\alpha}(\neg\psi) : M_\alpha \vDash_{K_3} \neg\psi; c_{M_\alpha}(\neg\delta) : M_\alpha \vDash_{K_3} \neg\delta\} + 1$*,*

if $\phi = \neg(\psi \vee \delta)$*, then* $c_{M_\alpha}(\phi) = max\{c_{M_\alpha}(\neg\psi); c_{M_\alpha}(\neg\delta)\} + 1$*.*

Theorem 13. *For all* α*,* M_α *is consistent. In other words, for all* $\phi \in L^{\lhd}_{Tr}$*, it is not the case that* $M_\alpha \vDash_{K_3} \phi$ *and* $M_\alpha \vDash_{K_3} \neg\phi$*.*

Proof. The structure of the proof is similar to the one of Theorem 5. By induction on α, M_0 is clearly consistent, then, assuming that M_α is consistent, we need to show that $M_{\alpha+1}$ is also consistent. We prove that the claim holds for all atomic sentences with the truth and ground predicate with the next Lemmas, in the sense that, for all terms s and t, if $M_\alpha \nvDash_{K_3} Tr(s) \wedge \neg Tr(s)$, then $M_{\alpha+1} \nvDash_{K_3} Tr(s) \wedge \neg Tr(s)$ and, if $M_\alpha \nvDash_{K_3} s \lhd t \wedge \neg(s \lhd t)$, then $M_{\alpha+1} \nvDash_{K_3} s \lhd t \wedge \neg(s \lhd t)$. Then, the claim generalises to all sentences as in Theorem 5. □

To check that the claim holds for all atomic sentences, I first show that the classical theory PG^{APG}_α is consistent. To show this, I construct a model for it. I construct a classical model M^{CL}_α as $M^{CL}_\alpha = (\mathbb{N}; R^{CL}_\alpha; S^{CL}_\alpha)$ in the style of Lemma 3. I define the extension of the ground predicate R^{CL}_α and the extension of the truth predicate S^{CL}_α as:

$R^{CL}_\alpha = \{\langle\#\phi; \#\psi\rangle : \#\phi \in S^{CL}_\alpha, \#\psi \in S^{CL}_\alpha, c_{M_\alpha}(\phi) < c_{M_\alpha}(\psi), c_{M_\alpha}(\phi) \text{ and } c_{M_\alpha}(\psi) \text{ are defined}\}$

$S^{CL}_\alpha = \{\#\phi : Tr(\ulcorner\phi\urcorner) \in Tr^+(M_\alpha)\}$[35]

From the consistency of PG^{APG}_α, it follows that the claim holds for atomic sentences with the ground predicate. Then, to prove it also holds for the atomic sentences with the truth predicate, I prove that, for all $\phi \in L^{\lhd}_{Tr}$, it is not that case that $PG^{APG}_\alpha \vdash Tr(\ulcorner\phi\urcorner)$ and $PG^{APG}_\alpha \vdash Tr(\ulcorner\neg\phi\urcorner)$. Then, the claim follows by the clause on $\overline{S_{\alpha+1}}$.

Lemma 10. M^{CL}_α *is a model for* PG^{APG}_α*.*

Proof. We need to check that M^{CL}_α is a model for PG^{APG}_α, i.e., for all $\phi \in L^{\lhd}_{Tr}$, if $PG^{APG}_\alpha \vdash \phi$, then $M^{CL}_\alpha \vDash \phi$. Thus, we need to check that M^{CL}_α satisfies $Tr^+_l(M_\alpha)$, $Tr^-(M_\alpha)$ and all the axioms of PG^{APG}.

[35] $Tr^+(M_\alpha) = \{Tr(\ulcorner\phi\urcorner) : M_\alpha \vDash_{K_3} \phi\}$ as in section 3.2.

- First, we check that, if $PG_\alpha^{APG} \vdash Tr(\ulcorner\phi\urcorner)$ because $Tr(\ulcorner\phi\urcorner) \in Tr_l^+(M_\alpha)$, then $M_\alpha^{CL} \models Tr(\ulcorner\phi\urcorner)$. $Tr_l^+(M_\alpha) \subseteq Tr^+(M_\alpha)$ and, by definition of S_α^{CL}, if $Tr(\ulcorner\phi\urcorner) \in Tr^+(M_\alpha)$, then $\#\phi \in S_\alpha^{CL}$ and $M_\alpha^{CL} \models Tr(\ulcorner\phi\urcorner)$.
- Second, we check that, if $PG_\alpha^{APG} \vdash \neg Tr(\ulcorner\phi\urcorner)$ because $\neg Tr(\ulcorner\phi\urcorner) \in Tr^-(M_\alpha)$, then $M_\alpha^{CL} \models \neg Tr(\ulcorner\phi\urcorner)$. If $\neg Tr(\ulcorner\phi\urcorner) \in Tr^-(M_\alpha)$, then $M_\alpha \models_{K_3} \neg\phi$, then $M_\alpha \not\models_{K_3} \phi$ because M_α is consistent, $Tr(\ulcorner\phi\urcorner) \notin Tr^+(M_\alpha)$ and $\#\phi \notin S_\alpha$, $M_\alpha^{CL} \not\models Tr(\ulcorner\phi\urcorner)$ and $M_\alpha^{CL} \models \neg Tr(\ulcorner\phi\urcorner)$.
- $G_1 : \forall x \neg(x \lhd x)$, easy, by the complexity condition in R_α^{CL}.
- $G_2 : \forall x \forall y \forall z(x \lhd z \wedge z \lhd y \rightarrow x \lhd y)$, easy, by the complexity condition in R_α^{CL} and the condition that the numbers of the *ground* and the *grounded* in a true ground relation are in S_α^{CL}.
- $G_3 : \forall x \forall y(x \lhd y \rightarrow Tr(x) \wedge Tr(y))$, easy, by the condition in R_α^{CL} that the numbers of the *ground* and the *grounded* in a true ground relation are in S_α^{CL}.
- T_1, T_2 and T_3^* are trivial.
- $AP_T : \forall x(Tr(x) \wedge Tr(\dot{T}r(x)) \rightarrow x \lhd \dot{T}r(x))$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M_\alpha^{CL} \models Tr(\ulcorner\phi\urcorner) \wedge Tr(\ulcorner Tr(\ulcorner\phi\urcorner)\urcorner) \rightarrow \ulcorner\phi\urcorner \lhd \ulcorner Tr(\ulcorner\phi\urcorner)\urcorner$. If $M_\alpha^{CL} \models Tr(\ulcorner Tr(\ulcorner\phi\urcorner)\urcorner)$ and $M_\alpha^{CL} \models Tr(\ulcorner\phi\urcorner)$, then $\#Tr(\ulcorner\phi\urcorner) \in S_\alpha^{CL}$ and $\#\phi \in S_\alpha^{CL}$. $c_{M_{\alpha+1}}(\phi) < \omega + c_{M_\alpha}(\phi) = c_{M_\alpha}(Tr(\ulcorner\phi\urcorner))$. Note that $c_{M_\alpha}(\phi)$ and $c_{M_\alpha}(Tr(\ulcorner\phi\urcorner))$ are defined because $Tr(\ulcorner\phi\urcorner) \in Tr^+(M_\alpha)$ and $Tr(\ulcorner Tr(\ulcorner\phi\urcorner)\urcorner) \in Tr^+(M_\alpha)$, so $M_\alpha \models_{K_3} \phi$ and $M_\alpha \models_{K_3} Tr(\ulcorner\phi\urcorner)$. Thus, $M_\alpha^{CL} \models \ulcorner\phi\urcorner \lhd \ulcorner Tr(\ulcorner\phi\urcorner)\urcorner$. Analogously for AP_F.
- $GG : \forall x \forall y(Tr(x) \wedge Tr(x \dot{\lhd} y) \rightarrow x \lhd (x \dot{\lhd} y))$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$ $M_\alpha^{CL} \models Tr(\ulcorner\phi\urcorner) \wedge Tr(\ulcorner\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\urcorner) \rightarrow \ulcorner\phi\urcorner \lhd (\ulcorner\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\urcorner)$. If $M_\alpha^{CL} \models Tr(\ulcorner\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\urcorner)$ and $M_\alpha^{CL} \models Tr(\ulcorner\phi\urcorner)$, then $\#(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) \in S_\alpha^{CL}$ and $\#\phi \in S_\alpha^{CL}$. $c_{M_\alpha}(\phi) < \omega + max\{c_{M_\alpha}(\phi); c_{M_\alpha}(\psi)\} = c_{M_\alpha}(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$. Note that $c_{M_\alpha}(\phi)$ and $c_{M_\alpha}(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$ are defined because $Tr(\ulcorner\phi\urcorner) \in Tr^+(M_\alpha)$ and $Tr(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner) \in Tr^+(M_\alpha)$, so $M_\alpha \models_{K_3} \phi$ and $M_\alpha \models_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$. Thus, $M_\alpha^{CL} \models \ulcorner\phi\urcorner \lhd (\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)$.

I now prove the claim for some exemplifying cases of the downward and upward axioms. The remaining ones can be proved with analogous arguments.

- $U_1 : \forall x(Tr(x) \rightarrow x \lhd \dot{\neg}\dot{\neg}x)$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M_\alpha^{CL} \models Tr(\ulcorner\phi\urcorner) \rightarrow \ulcorner\phi\urcorner \lhd \ulcorner\neg\neg\phi\urcorner$. If $M_\alpha^{CL} \models Tr(\ulcorner\phi\urcorner)$, then $\#\phi \in S_\alpha^{CL}$, $Tr(\ulcorner\phi\urcorner) \in Tr^+(M_\alpha)$, $M_\alpha \models_{K_3} \phi$ and, by K_3 logic, $M_\alpha \models_{K_3} \neg\neg\phi$, then $Tr(\neg\neg\phi) \in Tr(M_\alpha)$ and $\#\neg\neg\phi \in S_\alpha^{CL}$. Also, $c_{M_\alpha}(\phi) < c_{M_\alpha}(\phi) + 1 = c_{M_\alpha}(\neg\neg\phi)$. Thus, $M_\alpha^{CL} \models \ulcorner\phi\urcorner \lhd \ulcorner\neg\neg\phi\urcorner$.
- $D_3 : \forall x \forall y(Tr(x \dot{\wedge} y) \rightarrow (x \lhd x \dot{\wedge} y) \wedge (y \lhd x \dot{\wedge} y))$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$,

$m = \#\psi$, $M^{CL}_\alpha \vDash Tr(\ulcorner\phi\wedge\psi\urcorner) \to (\ulcorner\phi\urcorner \lhd \ulcorner\phi\wedge\psi\urcorner) \wedge (\ulcorner\psi\urcorner \lhd \ulcorner\phi\wedge\psi\urcorner)$. If $M^{CL}_\alpha \vDash Tr(\ulcorner\phi\wedge\psi\urcorner)$, then $\#(\phi\wedge\psi) \in S^{CL}_\alpha$, $Tr(\ulcorner\phi\wedge\psi\urcorner) \in Tr(M_\alpha)$, $M_\alpha \vDash_{K_3} \phi\wedge\psi$ and, by K_3 logic, $M_\alpha \vDash_{K_3} \phi$ and $M_\alpha \vDash_{K_3} \psi$, $Tr(\ulcorner\phi\urcorner) \in Tr(M_\alpha)$ and $Tr(\ulcorner\psi\urcorner) \in Tr(M_\alpha)$ and $\#\phi \in S^{CL}_\alpha$ and $\#\psi \in S^{CL}_\alpha$. $c_{M_\alpha}(\phi) < max\{c_{M_\alpha}(\phi); c_{M_\alpha}(\psi)\} + 1 = c_{M_\alpha}(\phi\wedge\psi)$ and $c_{M_\alpha}(\psi) < max\{c_{M_\alpha}(\phi); c_{M_\alpha}(\psi)\} + 1 = c_{M_\alpha}(\phi\wedge\psi)$. Thus, $M^{CL}_\alpha \vDash \ulcorner\phi\urcorner \lhd \ulcorner\phi\wedge\psi\urcorner$ and $M^{CL}_\alpha \vDash \ulcorner\psi\urcorner \lhd \ulcorner\phi\wedge\psi\urcorner$ and $M^{CL}_\alpha \vDash (\ulcorner\phi\urcorner \lhd \ulcorner\phi\wedge\psi\urcorner) \wedge (\ulcorner\psi\urcorner \lhd \ulcorner\phi\wedge\psi\urcorner)$.

□

Lemma 11. *For all $\phi \in L^{\lhd}_{Tr}$, it is not that case that $PG^{APG}_\alpha \vdash Tr(\ulcorner\phi\urcorner)$ and $PG^{APG}_\alpha \vdash Tr(\ulcorner\neg\phi\urcorner)$.*

Proof. Suppose $PG^{APG}_\alpha \vdash Tr(\ulcorner\phi\urcorner)$ and $PG^{APG}_\alpha \vdash Tr(\ulcorner\neg\phi\urcorner)$. By the previous Lemma 10, we know that, if $PG^{APG}_\alpha \vdash Tr(\ulcorner\phi\urcorner)$, then $Tr(\ulcorner\phi\urcorner) \in Tr^+(M_\alpha)$ and $M_\alpha \vDash_{K_3} \phi$. Analogously, if $PG^{APG}_\alpha \vdash Tr(\ulcorner\neg\phi\urcorner)$, then $Tr(\ulcorner\neg\phi\urcorner) \in Tr^+(M_\alpha)$ and $M_\alpha \vDash_{K_3} \neg\phi$. By inductive hypothesis, we know that M_α is consistent. Thus, it cannot be that $PG^{APG}_\alpha \vdash Tr(\ulcorner\phi\urcorner)$ and $PG^{APG}_\alpha \vdash Tr(\ulcorner\neg\phi\urcorner)$. □

Note that, at every level α, there are true quantified sentences. For example, at level 0, all the quantified truths of the standard model of arithmetic are true and, for every $\alpha \geq 1$, $M_\alpha \vDash_{K_3} \exists x Tr(x)$ follows from the fact that there is at least one truth is satisfied, e.g. $Tr(\ulcorner\overline{0} = \overline{0}\urcorner)$. However, it is important to highlight that, in all models of the construction, there are not true ground statements with the numeral of a quantified claim as an arguments. Intuitively, this follows from the fact that PG^{APG}_α contains the truth set of the true literals at the previous level, not of all sentences as in section 3.2. Thus, it does not prove the truth of quantified sentences and it does not prove ground relations based on them.

Observation 2. For all α and $\phi \in L^{\lhd}_{Tr}$, $c_{M_\alpha}(Qv\phi)$ and $c_{M_\alpha}(\neg Qv\phi)$, with $Q = \forall, \exists$, are not defined. Thus, for all $\psi \in L^{\lhd}_{Tr}$, $M^{CL}_\alpha \nvDash \ulcorner\psi\urcorner \lhd \ulcorner Qv\phi\urcorner$, $M^{CL}_\alpha \nvDash \ulcorner Qv\phi\urcorner \lhd \ulcorner\psi\urcorner$, $M^{CL}_\alpha \nvDash \ulcorner\psi\urcorner \lhd \ulcorner\neg Qv\phi\urcorner$, $M^{CL}_\alpha \nvDash \ulcorner\neg Qv\phi\urcorner \lhd \ulcorner\psi\urcorner$. By Lemma 10, for all α, if $M_{\alpha+1} \vDash_{K_3} \phi$, then $M^{CL}_\alpha \vDash \phi$. Thus, for all α and $\phi, \psi \in L^{\lhd}_{Tr}$, $M_{\alpha+1} \nvDash_{K_3} \ulcorner\psi\urcorner \lhd \ulcorner Qv\phi\urcorner$, $M_{\alpha+1} \nvDash_{K_3} \ulcorner Qv\phi\urcorner \lhd \ulcorner\psi\urcorner$, $M_{\alpha+1} \nvDash_{K_3} \ulcorner\psi\urcorner \lhd \ulcorner\neg Qv\phi\urcorner$, $M_{\alpha+1} \nvDash_{K_3} \ulcorner\neg Qv\phi\urcorner \lhd \ulcorner\psi\urcorner$.

The existence of fixed points for this construction can be proved analogously to the constructions in section 3.2. I now take a fixed point of the construction[36] and close its theory with the axioms of PG^* about the quantifiers in order to derive ground relations between quantified sentences. Formally, the idea to define the quantified base theory of partial ground PG^*_Q by taking only the upward and

[36] I will take the least fixed point M_I for simplicity, but the reasoning can be generalised analogously to all fixed points.

downward axioms that are about the quantifiers. Then, I build a new sequence of models by the inductively applying the axioms of PG^*_Q starting from the theory of the fixed point M_I.

Definition 13 (PG^*_Q). *The quantified base theory of partial ground PG^*_Q is formed with all the axioms of PG^* except for the ones about the propositional connectives: $U_1 - U_5$ and $D_1 - D_5$.*

Definition 14 (**Closure under the quantifiers**). *Given the fixed point M_I of Definition 11, its closure under the quantifiers is defined with the following operator $\Lambda^Q : M \to M$:*

$M^Q_0 = (\mathbb{N}; R^Q_0; \overline{R^Q_0}; S^Q_0; \overline{S^Q_0}) = (\mathbb{N}; \{\langle \#\phi; \#\psi\rangle : M_I \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\}; \{\langle \#\phi; \#\psi\rangle : M_I \vDash_{K_3} \neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)\}; \{\#\phi : M_I \vDash_{K_3} Tr(\ulcorner\phi\urcorner)\}; \{\#\phi : M_I \vDash_{K_3} \neg Tr(\ulcorner\phi\urcorner)\})$

$M^Q_{\alpha+1} = (\mathbb{N}; R^Q_{\alpha+1}; \overline{R^Q_{\alpha+1}}; S^Q_{\alpha+1}; \overline{S^Q_{\alpha+1}})$
$R^Q_{\alpha+1} = \{\langle \#\phi; \#\psi\rangle : PG^Q_\alpha$[37] $\vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner\}$
$\overline{R^Q_{\alpha+1}} = \{\langle \#\phi; \#\psi\rangle : PG^Q_\alpha \vdash \neg(\ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner)\}$
$S^Q_{\alpha+1} = \{\#\phi : PG^Q_\alpha \vdash Tr(\ulcorner\phi\urcorner)\}$
$\overline{S^Q_{\alpha+1}} = \{\#\phi : PG^Q_\alpha \vdash Tr(\ulcorner\neg\phi\urcorner)\}$

$M^Q_\alpha = (\mathbb{N}; R^Q_\alpha; \overline{R^Q_\alpha}; S^Q_\alpha; \overline{S^Q_\alpha})$, *$\alpha$ limit ordinal*
$R^Q_\alpha = \bigcup_{\beta<\alpha} R^Q_\beta$
$\overline{R^Q_\alpha} = \bigcup_{\beta<\alpha} \overline{R^Q_\beta}$
$S^Q_\alpha = \bigcup_{\beta<\alpha} S^Q_\beta$
$\overline{S^Q_\alpha} = \bigcup_{\beta<\alpha} \overline{S^Q_\beta}$

Theorem 14. *For allα, M^Q_α is consistent.*

Proof. The proof is similar to the one of Theorem 13. By induction on α. M^Q_0 is consistent because M_I is consistent. For an arbitrary α, if M^Q_α is consistent, we want to show that $M^Q_{\alpha+1}$ is consistent. The interesting step is to check that the claim holds for atomic sentences. To prove this for atomic ground sentences, we prove that PG^Q_α is consistent. From this fact, we use an analogous proof of Lemma 11 to prove the claim also for atomic truths. To prove the consistency of the theory PG^Q_α, we we build a classical model for it. $\phi \in L^P$ means that ϕ is in the propositional part of the language $L^{\lhd}_{Tr}$, in the sense that it does not contain symbols of quantifiers $\exists$ or $\forall$. $\phi \notin L^P$ means that ϕ contains either $\exists$ or $\forall$. I define the model $M^{CL}_\alpha = (\mathbb{N}; R^{CL}_\alpha; S^{CL}_\alpha)$ for PG^Q_α as:

[37] PG^Q_α is defined as in section 3.2 as $Tr^+(M^Q_\alpha) \cup Tr^-(M^Q_\alpha) \cup PG^*_Q = \{Tr(\ulcorner\phi\urcorner) : M^Q_\alpha \vDash_{K_3} \phi\} \cup \{\neg Tr(\ulcorner\phi\urcorner) : M^Q_\alpha \vDash_{K_3} \neg\phi\} \cup PG^*_Q$.

$R^{CL}_{\alpha} = \{\langle \#\phi ; \#\psi \rangle : M_I \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner, \phi \in L^P, \psi \in L^{P}$[38]$\} \cup \{\langle \#\phi ; \#\psi \rangle : \phi \in Th(M^Q_{\alpha}), \psi \in Th(M^Q_{\alpha}), \phi \in L^P, \psi \notin L^P\} \{\langle \#\phi ; \#\psi \rangle : \phi \in Th(M^Q_{\alpha}), \psi \in Th(M^Q_{\alpha}), c(\phi) < c(\psi), \phi \notin L^P, \psi \notin L^P\}$,

$S^{CL}_{\alpha} = \{\#\phi : \phi \in Th(M^Q_{\alpha})\}$.

I now check that M^{CL}_Q is a model for $Tr^+(M^Q_{\alpha}) \cup Tr^-(M^Q_{\alpha}) \cup PG^*_Q$.

- $Tr^+(M^Q_{\alpha})$ and $Tr^-(M^Q_{\alpha})$ can be check analogously to Lemma 10.
- $G_1 : \forall x \neg (x \lhd x)$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M^{CL}_{\alpha} \vDash \neg(\ulcorner\phi\urcorner \lhd \ulcorner\phi\urcorner)$. Then, we need to check the three clauses of R^{CL}_{α}. Respectively, $M_I \nvDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\phi\urcorner$ because G_1 is an axiom of PG^{APG}; it cannot be that $\phi \in L^P$ and $\phi \notin L^P$; $c(\phi) \nless c(\phi)$. Thus, $\langle \#\phi ; \#\phi \rangle \notin R^{CL}_{\alpha}$, $M^{CL}_{\alpha} \nvDash \ulcorner\phi\urcorner \lhd \ulcorner\phi\urcorner$ and $M^{CL}_{\alpha} \vDash \neg(\ulcorner\phi\urcorner \lhd \ulcorner\phi\urcorner)$.
- $G_2 : \forall x \forall y \forall z (x \lhd z \wedge z \lhd y \rightarrow x \lhd y)$. This means that, for all $n, m, k \in \mathbb{N}$ such that $n = \#\phi$, $m = \#\psi$ and $k = \#\delta$, $M^{CL}_{\alpha} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner \wedge \ulcorner\delta\urcorner \lhd \ulcorner\psi\urcorner \rightarrow \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$. Then, there are several possibilities depending on which of the three clauses of R^{CL}_{α} make true each of the two ground relations:
  - Both $M^{CL}_{\alpha} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner$ and $M^{CL}_{\alpha} \vDash \ulcorner\delta\urcorner \lhd \ulcorner\psi\urcorner$ because of the first clause of R^{CL}_{α}. Then, $PG^{APG}_{I-1} \vdash \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner$ and $PG^{APG}_{I-1} \vdash \ulcorner\delta\urcorner \lhd \ulcorner\psi\urcorner$ and $PG^{APG}_{I-1} \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ because G_2 is an axiom of PG^{APG}, $M_I \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ and $M^{CL}_{\alpha} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$.
  - Both $M^{CL}_{\alpha} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner$ and $M^{CL}_{\alpha} \vDash \ulcorner\delta\urcorner \lhd \ulcorner\psi\urcorner$ because of the third clause of R^{CL}_{α}. Then, $\phi \in Th(M^Q_{\alpha})$, $\psi \in Th(M^Q_{\alpha})$, $c(\phi) < c(\delta) < c(\psi)$, $\phi \notin L^P$ and $\psi \notin L^P$, which imply $M^{CL}_{\alpha} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$.
  - $M^{CL}_Q \vDash \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner$ because of the first clause of R^{CL}_Q and $M^{CL}_Q \vDash \ulcorner\delta\urcorner \lhd \ulcorner\psi\urcorner$ because of the second one. Then, $\phi \in L^P$, $\psi \in Th(M^Q_{\alpha})$, $\psi \notin L^P$. Also, if $M_I \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner$, then $PG^{APG}_{I-1} \vdash \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner$, $Tr(\phi) \in Tr^+(M_{I-1})$ by Lemma 10, $M_{I-1} \vDash_{K_3} \phi$, $M_{I-1} \vDash_{K_3} \phi$ by Lemma 9 and $\phi \in Th(M^Q_{\alpha})$ because $Th(M_I) \subseteq Th(M^Q_{\alpha})$[39]. Thus, $M^{CL}_Q \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ because of the second clause of R^{CL}_Q.
  - $M^{CL}_Q \vDash \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner$ because of the second clause of R^{CL}_Q and $M^{CL}_Q \vDash \ulcorner\delta\urcorner \lhd \ulcorner\psi\urcorner$ because of the third one. Then, $\phi \in Th(M^Q_{\alpha})$, $\phi \in L^P$, $\psi \in Th(M^Q_{\alpha})$, $\psi \notin L^P$. Thus, $M^{CL}_Q \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ because of the second clause of R^{CL}_Q.
  - For all the remaining five possible combinations of the three clauses of R^{CL}_Q, the antecedent of $M^{CL}_Q \vDash \ulcorner\phi\urcorner \lhd \ulcorner\delta\urcorner \wedge \ulcorner\delta\urcorner \lhd \ulcorner\psi\urcorner \rightarrow \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ is false because it implies that $\delta \in L^P$ and

[38] Note that, if $M_I \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$, then $\phi \in L^P$ and $\psi \in L^P$ by Observation 2.

[39] Note that the analogous of Lemma 9 that $Th(M^Q_{\alpha})$ is increasing in α can be proven using an analogous argument.

$\delta \notin L^P$.

- $G_3 : \forall x \forall y(x \lhd y \to Tr(x) \land Tr(y))$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, and $m = \#\psi$, $M_Q^{CL} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner \to Tr(\ulcorner\phi\urcorner) \land Tr(\ulcorner\psi\urcorner)$. If $M_Q^{CL} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ because $M_I \vDash_{K_3} \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ and $\phi \in L^P$ and $\psi \in L^P$, then $PG_{I-1}^{APG} \vdash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$, $PG_{I-1}^{APG} \vdash Tr(\ulcorner\phi\urcorner)$ by axiom G_3 of PG^{APG} and $M_I \vDash_{K_3} \phi$. Thus, $\phi \in Th(M_\alpha^Q)$ (see footnote 39), $\#\phi \in S_Q^{CL}$ and $M_Q^{CL} \vDash Tr(\ulcorner\phi\urcorner)$. An analogous derivation holds for $M_Q^{CL} \vDash Tr(\ulcorner\psi\urcorner)$. If $M_Q^{CL} \vDash \ulcorner\phi\urcorner \lhd \ulcorner\psi\urcorner$ because of the second or the third clause of R_Q^{CL}, then $\phi \in Th(M_\alpha^Q)$, $\psi \in Th(M_\alpha^Q)$ and $M_Q^{CL} \vDash Tr(\ulcorner\phi\urcorner)$, $M_Q^{CL} \vDash Tr(\ulcorner\psi\urcorner)$.
- T_1, T_2 and T_3^* are trivial.

I now prove the claim for some exemplifying cases of the downward and upward axioms. The remaining ones can be proved with analogous arguments.

- $U_6 : \forall x \forall t \forall v(Tr(x(t/v)) \to x(t/v) \lhd \dot{\exists} vx)$. This means that, for all $n, m \in \mathbb{N}$ such that $n = \#\phi$, $M_Q^{CL} \vDash Tr(\overline{n}(\overline{m}/v)) \to \overline{n}(\overline{m}/v) \lhd \dot{\exists} v\overline{n}$. If $M_Q^{CL} \vDash Tr(\overline{n}(\overline{m}/v))$, then $\phi(\overline{m}/v) \in Th(M_\alpha^Q)$ and $\exists v\phi \in Th(M_\alpha^Q)$ by K_3-logic. $\exists v\phi \notin L^P$. If $\phi(\overline{m}/v) \in L^P$, then $M_Q^{CL} \vDash \overline{n}(\overline{m}/v) \lhd \dot{\exists} v\overline{n}$. If $\phi(\overline{m}/v) \notin L^P$, then $M_Q^{CL} \vDash \overline{n}(\overline{m}/v) \lhd \dot{\exists} v\overline{n}$ because $c(\phi(\overline{m}/v)) < max\{c(\phi(d))\} + 1 = c(\exists v\phi)$.
- $D_8 : \forall x \forall v(Tr(\dot{\forall} vx) \to \forall t(x(t/v) \lhd \dot{\forall} vx))$. This means that, for all $n \in \mathbb{N}$ such that $n = \#\phi$, $M_Q^{CL} \vDash Tr(\dot{\forall} v\overline{n}) \to \forall t(\overline{n}(t/v) \lhd \dot{\forall} v\overline{n})$. If $M_Q^{CL} \vDash Tr(\dot{\forall} v\overline{n})$, then $\forall v\phi \in Th(M_\alpha^Q)$ and, for all t, $\phi(t/v) \in Th(M_\alpha^Q)$ by K_3 logic. $\forall v\phi \notin L^P$. Since t is a numeral, it does not matter for $\phi(t/v)$ being or not in L^P. Thus, either $\phi(t/v) \in L^P$ for all t and $M_Q^{CL} \vDash \forall t(\overline{n}(t/v) \lhd \dot{\forall} v\overline{n})$. Or $\phi(t/v) \in L^P$ for all t, $c(\phi(\overline{n}(t/v)) < max\{c(\phi(d))\} + 1 = c(\forall v\phi)$ and $M_Q^{CL} \vDash \forall t(\overline{n}(t/v) \lhd \dot{\forall} v\overline{n})$.

□

The existence of fixed points for this construction can be proved analogously to the constructions in section 3.2. Note that, as mentioned above, for example, $M_I \vDash_{K_3} \exists x Tr(x)$. The novelty of the closure under the quantifiers is that, for example, $M_1^Q \vDash_{K_3} \ulcorner Tr(\ulcorner\overline{0} = \overline{0}\urcorner)\urcorner \lhd \ulcorner\exists x Tr(x)\urcorner$. Thus, also $M_1^Q \vDash_{K_3} Tr(\ulcorner\exists x Tr(x)\urcorner)$ by axiom G_3 and, so, it follows that $M_2^Q \vDash_{K_3} \ulcorner Tr(\ulcorner\exists x Tr(x)\urcorner)\urcorner \lhd \ulcorner\exists x Tr(x)\urcorner$. Therefore, one of the two 'problematic' sentences that give rise to Fine's puzzle (Th. 3) is true in the model. More precisely, it is true the one that follows from the grounding axioms about the quantifiers in PG^*, while the one that follows from the Aristotelian principles is not true because I restricted their application to the propositional part of PG^*. Note that the same does not hold for the sentences that give rise to Fine's puzzle with the GG principle (Th. 12). This is because, when I closed the propositional theory with the quantifiers, I closed it with the theory

PG^*_Q, which does not contain the axioms of PG^* about the propositional connectives. For example, it cannot derive $\ulcorner \exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner \lhd \ulcorner \neg\neg\exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner$. However, this sentence is true in the classical model for PG^Q_α of Theorem 14. Thus, even if PG^Q_α does not prove these kind of sentences, they are not inconsistent with PG^Q_α. Therefore, it is possible build a closure based on a stronger theory that prove them and, so, also proves one of the two 'problematic' sentences that give rise to the puzzle of Theorem 12 (e.g. that proves $\ulcorner\ulcorner \exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner \lhd \ulcorner \neg\neg\exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner\urcorner \lhd \ulcorner \exists x(x \lhd \underset{.}{\neg}\underset{.}{\neg} x)\urcorner$).

5 Philosophical Discussion

5.1 Strong Kleene Logic and Theories of Ground

One of the main results of the theories developed in section 3 is to show that there are strong symmetries between the type-free solutions to the *Liar paradox* (Th. 1) and the type-free solutions to the *paradox of self-referentiality for the ground predicate* (Th. 2). In this section, I focus on whether there are specific advantages in adopting a type-free non-classical approach for theories of ground, other than the generic *pros* and *cons* of type-free theories I described in section 3.1. In the context of theories of truth, the main disadvantage of adopting a type-free solution to the *Liar paradox* is that it requires the use of a non-classical logic and the rejection of the *Principle of Bivalence* of classical logic that asserts that there are only two exhaustive and mutually exclusive truth values (*true* and *false*) for all propositions. In a formal setting based on K_3 logic, this implies that there exists a third truth value intermediate between truth and falsity that can be interpreted as *undetermined* or *neither true nor false*. From this, the distinction between *being not true* and *being false* introduced in section 3.3 follows.

The application of the same formal setting to theories of ground and to the ground predicate results in the fact that, give two terms s and t, there exists a third intermediate possibility between 's ground t' and 's does not ground t' and that the negation of the former one (i.e. 'it is not the case that s grounds t') is not equivalent to the latter one. At first glance, this distinction seems unjustified because it is natural to assume that either something grounds something else, or it does not ground it, and there is not an intermediate state between these two possibilities. However, the principles of grounding formalised in theories of ground, both predicational and operational ones, do not aim to derive all the true ground relations and their true negations. They derive the ground relations that hold because of the general principles of ground (axioms $G_1 - G_3$ of PG) and because of the ground principles between logically complex truths and their simpler parts (upward and downward axioms of PG). Also, they derive the negations of ground relations that hold because of these principles. For simplicity, we can call these principles the *logical principles of ground* because they are either principles of the *pure logic* of ground ($G_1 - G_3$) or of the *impure* one (upward and downward axioms)[40].

It is natural to assume that not all true ground relations holds 'by logic'. For example, ground relations can follow from metaphysical principles. The Aristotelian and GG principles analysed in the previous sections are examples of this fact. They derive ground relations which do not hold 'by logic', but because we adhere to the truth of certain metaphysical principles. In general, many philosophical and scientific theories implicitly or explicitly establish ground relations between facts which are not supposed to hold because of the *logical principles* mentioned above[41]. Thus, given two terms s and t, we can interpret the ground predicate in the K_3 theories of section 3 as 's grounds t "by logic"' and its negation as 's does not ground t "by logic"'. Therefore, it is natural to assume that there is third domain of pairs of sentences for which the *logical principles* do not prove a ground relation between them exists, nor they prove it does not exist, but such that other theories or principle can prove that a ground relation or its negation holds between them.

Therefore, the use of K_3 models as formalisation of theories of ground has intuitive plausibility because there are pairs of sentences for which the principles of the theory do not determine whether one grounds the other or not. This case is different from the one in which the principle of the theory determine that a ground relation does not hold. Thus, the distinction between 'it is not the case that s grounds t' and 's does not ground t' has intuitive plausibility. Also, the other non-classical logics mentioned in section 3.1 that can solve the paradoxes of self-referentiality, such as the ones with truth-value gluts (LP or FDE), do not have the intuitive plausibility of K_3 because they would imply that there are cases for which a ground relation holds and does not hold. Instead, the truth value gaps of K_3 fits well with the idea that there are ground relations not determined by the theory.

Two possible replies to this interpretation of theories of grounding are the following ones. First, one might argue that, when further principles than the *logical* ones are added to the theory (e.g the Aristotelian and GG in section 4), then the source of indeterminacy is removed and using classical logic becomes the most natural choice. However, given that ground relations can be established by

[40] See section 2.1 for an introduction to the pure and impure logic of ground.
[41] See the examples at the beginning of in section 1.1 for some illustrative cases.

many theories and principles of different disciplines[42], it seems implausible to be able to find the principles that derive all ground relations and express all of them in a single theory. Moreover, this would be substantial philosophical claim and it is better if formal theories of ground do not depend on it. Second, one might claim that the *logical principles* are the only true principles of ground. Again, this would be a substantial philosophical claim and it is better to formulate theories that do not take its truth as an assumption and are also compatible with other views. Note that a K_3 theory is compatible with the possibility that its principles derive all true ground relations. This is the case in which all the *undetermined* ground relations turn out not to hold.

In conclusion, I highlight two further facts in favour of this interpretation of theories of ground which are specific to predicational ones. First, the base theory of ground PG can also be interpreted in this way. In fact, since PG extends PA, it is incomplete because of Gödel incompleteness theorem. In particular, PG does not prove many ground statement, nor their negation, even basic ones such as $PG \nvdash \ulcorner \overline{0} = \overline{0} \urcorner \lhd \ulcorner \overline{1} = \overline{1} \urcorner$ and $PG \nvdash \neg(\ulcorner \overline{0} = \overline{0} \urcorner \lhd \ulcorner \overline{1} = \overline{1} \urcorner)$[43]. In fact, there is not only one classical model valid for PG, but a family of models that assign different truth values to the grounding claims not proven by PG. Thus, they are models of all the different ways ground relations would be determined if all principles of ground were known. Instead, the idea behind using a K_3 model is to explicitly leave as *undetermined* the truth value of the ground relations which are not decided by the theory and, eventually, decide them when further principles are added to it. In this sense, K_3 theories more explicitly formalise the the idea that theories of ground formalise some, but not all, principles of ground, and they do not aim to determine all ground relations.

Second, the use of the arithmetical framework in which all existing predicational theories of ground are developed is not motivated by the aim of deriving all the true ground relations in this specific domain, but to formulate some basic principles of ground that are valid in general. In fact, the use of arithmetic is not motivated by a special interest in the specific ground relations that hold or do not hold between arithmetic facts, but to use a well-developed and relatively simple framework to formalise general principles which aim to be valid for all the domains in which the notion of grounding matters. Even if there were ground relations that only hold in the domain of natural numbers, they would not be very interesting for the purposes of the theory and they should

[42] The fact that the domain of the theories of this paper is the natural numbers limits the kinds of principles of ground that can be added to them. Nevertheless, even in this specific domain, different principles can be added, such as the Aristotelian and GG principles.

[43] See also section 3.2.

not be formalised in it. Therefore, the possibility of undetermined ground relations is implicit in the motivation for developing these theories.

5.2 Solutions to Fine's Puzzle and the Hierarchy of Truths

Fine's ([4, pp. 103-107]) diagnosis of why the puzzles described in sections 4.1 and 4.2 originate is the following. Once we assume some plausible assumptions (see Fine [4, pp. 99-101]), which, in the formal setting of this paper, correspond to the assumptions of Theorems 3 and 12, there is a conflict between two aspects which are constitutive of the classical approach to logic. On the one hand, there is a plausible metaphysical view that classical truth-conditions are to be interpreted as conditions of ground. This view is composed by two claims. First, it states that every logically complex truth (with the exception of negations of atomic truths) is grounded in simpler ones (this principles is called *Complex Ground*). Second, classical truth-conditions provide a guide to ground, in the sense that one direction of the truth-conditions (the one from logically simpler to more complex truths) expresses which are the grounds of a logical complex truth (this principles is called *Classicality*). In the formal setting of this paper, this reading of the classical truth-conditions is formalised with the upward and downward axioms of PG. For example, axiom U_3 states that, if both conjuncts are true, a conjunction is grounded by each of its conjuncts.

On the other hand, under the previously mentioned assumptions, this reading of the classical truth-conditions as conditions of ground is inconsistent with some classical logical truths such as, for example:

$$1)\ \exists x(Tr(x) \vee \neg Tr(x)),$$

$$2)\ \forall x(Tr(x) \vee \neg Tr(x)).$$

I have already sketched at the end of section 4.2 how we can derive a contradiction from 2), the assumptions of Theorem 3 and the axioms of PG. A contradiction can be derived with an analogous argument assuming 1) instead of 2). Thus, the previously introduced view about truth-conditions is in conflict with the adherence to classical logic and the acceptance of its logical truths.

Fine [4, pp. 107-115] proposes four options to overcome the conflict between the view of classical truth-conditions as conditions of ground and classical logical truths. The first option is to adopt a form of predicativism. In the context of this thesis, this would be equivalent to adopt a typed-setting that limit the expressivity of the language in a way that avoids the contradiction without restricting neither of the two principles in conflict. A solution to Fine's puzzle of this kind is developed by Korbmacher [2]. In this paper, I focus one the impredicativist options and stick within an type-free setting. The three impredicavist options: (*i*) rejecting classical logic and adopting a logic in which all the classical logical truths that give rise to the puzzle do not hold. For example, Fine suggests to use a *weak Kleene* three-valued logic; (*ii*) rejecting the view of classical truth-conditions as conditions of ground; (*iii*) adopting a compromise position in which we adopt a logic (K_3) in which only the logical truth 1) holds and the view of truth-conditions as grounding principles is restricted, but not fully rejected. Solution (*ii*) is not very interesting in the context of this paper because rejecting the view of truth-condition as conditions of ground implies rejecting most of the axioms of PG on which the theories developed in the previous sections are based. Also, I do not delve into solutions in the style of (*i*) because I want to adopt a formal framework as close as possible to classical logic and not to reject very plausible truths such as 1). Instead, it is interesting to compare the compromise solutions (*iii*) with the theories developed in this paper especially because they are both developed in a K_3 logical framework.

More precisely, on the one hand, Fine's compromise solution consists in using K_3 logic in order to reject logical truth 2) and all the versions of the puzzle that involve the universal quantifier. On the other hand, it avoids the versions of the puzzle that involve the existential quantifier by adopting a weaker reading of the truth-conditions as principles of ground with respect to the usual one (which is the one formalised by the upward and downward axioms of PG). The ground principles that he proposes to weaken are the ones for disjunction and the existential quantifier. Intuitively, the new ground principles state that:

3) 'Any disjunctive truth is grounded by a disjunct.'
4) 'Any existential truth is grounded by an instance.' (Fine [4, p. 108])

Thus, these new version of the ground principle state that complex disjunctive and existential truths are grounded, respectively, by at least one of their disjuncts and at least one of their true instances, and not by all of them as it was for the original principles formalised in PG.

Fine ([4, pp. 110-115]) also argues that there is a relation between the impredicativist solutions to his puzzle and Kripke's fixed point semantics. In particular, for the compromise solution (*iii*), the idea is to restrict ground relations only to couples of sentences such that the second one (the *grounded*) is not at a lower level compared to the first one (the *ground*) in hierarchy of truths that follows from Kripke's construction. For example, in the semantics of section 3.2, $\overline{0} = \overline{0}$ and

$\overline{0} = \overline{0} \vee Tr(\ulcorner \overline{0} = \overline{0} \urcorner)$ are true at level 0, while $Tr(\ulcorner \overline{0} = \overline{0} \urcorner)$ is true at level 1. Thus, according to this principle, $\ulcorner \overline{0} = \overline{0} \urcorner \lhd \ulcorner \overline{0} = \overline{0} \vee Tr(\ulcorner \overline{0} = \overline{0} \urcorner) \urcorner$ is an admissible ground relation, while $Tr(\ulcorner \overline{0} = \overline{0} \urcorner) \urcorner \lhd \ulcorner \overline{0} = \overline{0} \vee Tr(\ulcorner \overline{0} = \overline{0} \urcorner) \urcorner$ is not. Analogously to the latter one, $\ulcorner Tr(\ulcorner \exists x Tr(x) \urcorner) \urcorner \lhd \ulcorner \exists x Tr(x) \urcorner$ is not admissible because $Tr(\ulcorner \exists x Tr(x) \urcorner)$ is true at level 2, while $\exists x Tr(x)$ is true at level 1. It is easy to check that the original and stronger version of ground principles for disjunction and the existential quantifier do not hold in this new approach. This is because there are ground relations between true disjucts and true instances of a formula and, respectively, their disjunction and existential claim that are not admissible according to the new approach. However, the weaker versions of the principles holds because there is always at least one true disjuct or true instances of a formula that is true at the same level of the disjunction or existential claim or lower and, so, that can ground them according to this new approach.

Fine's compromise approach can be easily developed starting from the semantics of section 3.2 by imposing the required restrictions of the disjunctive and existential ground relations. The main advantage of this approach would be that it takes the dependence relation between the levels of Kripke's construction seriously, in the sense that it does not allow sentences true at a new level of the hierarchy to ground sentences which were already true at the previous one. Intuitively, this approach means that, once a truth is grounded or made true by others at the same or lower levels, it cannot be made true again by sentences which will become true at later stages. All the true ground relations must respect this dependence relation from lower to higher levels in Kripke's construction. Nevertheless, adopting this strategy implies strong restrictions on the admissible ground relations. For example, as mentioned before, it implies that sentences like $Tr(\ulcorner \overline{0} = \overline{0} \urcorner) \urcorner \lhd \ulcorner \overline{0} = \overline{0} \vee Tr(\ulcorner \overline{0} = \overline{0} \urcorner) \urcorner$ are not true. As I showed in section 4.3, it not necessary to add restrictions on the disjunctive ground relations to block Fine's puzzle. The ground relations on the existential quantifier are also severely restricted. For example, given that $\exists x Tr(x)$ is true at level 1, it cannot be grounded by any sentence which become true at a level higher than 1, e.g. $Tr(Tr(\ulcorner \overline{0} = \overline{0} \urcorner))$, even if the resulting ground relation does not give rise to Fine's puzzle.

In the semantics I develop in section 4.3 as a solution to Fine's puzzle, I take a different approach. First, I take a different and less strict interpretation of the dependence relation between the levels in Kripke's construction. In all true ground relations of the semantics of section 4.3 (and also of section 3.2), each *grounded* has a *ground* which is true at its same level or at a lower one. However, I do not exclude the possibility that a sentence that becomes true at a certain level grounds another one which has already been made true at a previous one. Second, I aim not to restrict ground relations unless needed to avoid Fine's puzzle. Thus, for example, I combine the unrestricted propositional axioms of PG and Aristotelian and GG principles and derive all ground relations that follow from them. Third, instead of weakening the truth-conditional principles of ground as Fine suggests in his solution (*iii*), I restrict the application of the Aristotelian and GG principles to the propositional part of the theory. The philosophical motivation for this choice is the principle that quantified claims always supervene all their instances, in the sense that they are determined only after and on the base of all their instances being determined. The application of the Aristotelian and GG principles to quantifies claims is in conflict with this principle because it implies that there are quantified claims that ground some of their instances (e.g. $\ulcorner \exists x Tr(x)$ and $Tr(\ulcorner \exists x Tr(x) \urcorner)$ and $\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x)$ and $\ulcorner \exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x)\urcorner \lhd \ulcorner \neg\neg\exists x(x \lhd \underset{\cdot}{\neg}\underset{\cdot}{\neg} x)\urcorner$).

In this paper, I do not delve into the philosophical motivation in favor of the thesis quantified claims supervene their instances. However, I showed in section 4.3 that adhering to this principles allows us to develop an alternative solution to Fine's puzzle. This solution has some intuitive plausibility and the theoretical advantage of minimizing the needed restrictions on the grounding principles and on the ground relations that can be derived within this approach.

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