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Understanding holonomy and the Ambrose-Singer theorem using Lie algebroids.

Master Thesis

Mathematical Sciences

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Abstract

Holonomy groups are Lie groups associated to Ehresmann connections on surjective submersions. Elements of the holonomy groups are generated via parallel transport along loops in the base manifold. The Ambrose-Singer theorem relates the Lie algebras of these holonomy groups to the curvature of the connection.

The theory of holonomy fits naturally inside the framework of Lie groupoids and Lie algebroids. A Lie groupoid is a generalisation of a Lie group, and its infinitesimal counterpart is a Lie algebroid. In contrast with the Lie group-Lie algebra correspondence, not every Lie algebroid is integrable to a Lie groupoid. Besides an extensive discussion of basic groupoid and algebroid theory, we discuss this integrability problem. We focus on the case of transitive algebroids acting freely on surjective submersions.

We also discuss in detail the theory of transitive algebroids and the associated Atiyah sequences. We assign notions of connections and curvature to transitive algebroids, and show that for integrable transitive algebroids, these notions correspond to principal bundle connections and curvature.

Mackenzie ([Mac05]) first described holonomy as a branch of Lie groupoid theory. In this thesis, we follow a slightly different approach, initialized by Crainic. We show that the bundle of Lie algebras of the holonomy groups together with the tangent bundle of the base manifold, form a Lie algebroid. This algebroid acts on the surjective submersion and is integrable. Using its groupoid integration, we attempt to prove the Ambrose-Singer theorem again. Although our proof is still missing an important step, the approach described in this thesis gives more insight into the theory of holonomy and connections. We show that under certain conditions on the connection, the original surjective submersion is isomorphic to a fibered product over a principal bundle. This is a strong result which is not evident from the classical approach.

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Contents

1 Introduction

This thesis combines two subjects from differential geometry, namely holonomy and Lie algebroids, in an attempt to give more insight into the Ambrose-Singer theorem on holonomy groups.

In 1952, Ambrose and Singer stated and proved a theorem on holonomy for principal bundles in [AS53]. Holonomy groups are Lie groups arising from parallel transport along connections, for example on principal bundles. In their paper, Ambrose and Singer proved that these groups are indeed Lie groups, computed the identity component of these Lie groups, and most importantly gave a description of their Lie algebras, all in the context of principal bundles. The core idea of the theorem is that holonomy arising from a connection is generated by the curvature associated to that connection.

Holonomy groups encode information about the connection that they come from; for example, the holonomy groups are trivial if and only if the connection is flat. But holonomy groups can also encode information about the manifolds involved; for certain Riemannian manifolds with a Levi-Civita connection, Berger's classification of holonomy classifies all possible holonomy groups, simultaneously classifying possible structures on the base manifold. This is one example where holonomy is an interesting object of study in differential geometry. The study of holonomy groups, including the Ambrose-Singer theorem, is discussed extensively in the important book 'Foundations of differential geometry' by Kobayashi and Nomizu [KN96].

The other subject that this thesis is concerned with is the theory of Lie groupoids and Lie algebroids. Groupoids were first introduced by Brandt, and their smooth version (which we now call Lie groupoids) by Ehresmann in the 1950's. Groupoids can be interpreted as a generalisation of groups: while a group is endowed with a binary operation defined on all pairs of group elements, for a groupoid this operation is only partially defined.

A generalisation of Lie groups of course asks for a generalisation of Lie algebras. The corresponding notion of Lie algebroids, together with the Lie theory for groupoids and algebroids, was first developed by Pradines in the 1960's. At first, Pradines believed that every Lie algebroid was integrable to a Lie groupoid, similar to the Lie algebra-Lie group correspondence. However, this was proved to be false when Almeida and Molino found the first example of a non-integrable Lie algebroid. After that, more and more work was done in this field, resulting for example in the extensive description of groupoids by Mackenzie in [Mac87]. This book focused mainly on locally trivial groupoids and algebroids.

New interest in the subject arose when Weinstein, Karasëv and Zakrzewski (independently) saw the need for a general integrability theory of Lie algebroids arising in Poisson geometry. It had already been observed that a Poisson structure induces an algebroid, the cotangent algebroid. The new observation was that symplectic realisations of Poisson structures correspond to symplectic integrations of these cotangent algebroids. Since symplectic groupoids are not necessarily locally trivial, the focus shifted to finding integrability conditions for general Lie algebroids. This shift is also reflected by the theory discussed in Mackenzie's second book on groupoids and algebroids ([Mac05]), which has more focus on the general case than his first book.

The integrability problem of Lie algebroids was clearly not a simple generalisation of that of Lie algebras, as examples of non-integrable Lie algebroids had been found. For certain classes of Lie algebroids (for example, algebroids arising from foliations), the integrability problem was positively solved. Finally, the most general answer to this problem was introduced in [CF11], where Crainic and Fernandes give conditions for any Lie algebroid to be integrable.

Moving back to the subject of this thesis, we look for a relationship between holonomy and algebroids. A relationship between these subjects is also discussed by Mackenzie in his second book. His observation is that holonomy groups arising from a connection should be viewed inside groupoid theory. He constructs a groupoid that naturally contains the holonomy groups, and by computing the corresponding algebroid he proves the original Ambrose-Singer theorem.

This 'groupoid point of view' also formed the basis for a talk by Marius Crainic, in memory of Mackenzie. The notes based on this talk look for a relationship between holonomy and algebroids as well, but from a slightly different perspective than Mackenzie. However, the notes were never finished, and the goal of this thesis is to give a correct, detailed account of the ideas presented there.

Part I of this thesis will be an overview of Lie groupoid and Lie algebroid theory. We will focus on the integrability of transitive Lie algebroids by giving an overview of the conditions introduced in [CF11] and proving an integrability theorem for transitive algebroids that admit a free algebroid action.

In Part II, we will discuss the general theory of connections, holonomy, and the Ambrose-Singer theorem. The Ambrose-Singer theorem was originally stated for principal bundles, and many texts on the subject of holonomy also focus on this case. We will generalise to proper fibrations in this thesis.

Finally in Part III, the theory developed on groupoids and algebroids will be applied to the discussion on holonomy. Associated to a connection, we will find a transitive algebroid, and show it is integrable. Then using this algebroid and its integration, we will attempt to prove the Ambrose-Singer theorem again. At the very least, we will gain additional insight into the structure of the original fibration and the connection. We will prove that the fibration is isomorphic to a fibered product over a principal bundle, and that the original connection is equivalent to a connection on this principal bundle.

Part I
Groupoids and algebroids

We begin by discussing some generalities on (Lie) groupoids and Lie algebroids. We will start in Section 2 with groupoids and Lie algebroids, and discuss morphisms, homotopies, integrability and actions.

In Section 3, we discuss one specific example of a groupoid, namely the general linear groupoid. We compute its algebroid, and see that this example is closely related to groupoid and algebroid representations, just as in the classical case of Lie group and Lie algebra representations.

In Section 4, we discuss connections. Here we focus on a specific kind of algebroid, namely transitive algebroids, and define connections and curvature. We will see that this is closely related to connections on princpal bundles.

We will return to these concepts in Part III, where we use the theory of transitive algebroids and algebroid actions to discuss holonomy. The general linear groupoid/algebroid discussed in Section 3 can be considered as a detailed example and this section is less important for the rest of the thesis.

2 (Lie) groupoids and Lie algebroids

As the name groupoid suggests, a groupoid can be interpreted as a generalization of a group. A group is a set of objects with a binary operation that is defined for any pair of group elements. In a groupoid, this binary operation is only defined for certain pairs of elements. The definition of a groupoid that we use in this thesis will be stated in terms of category theory terminology, but it is equivalent to this intuitive way of generalising groups.

Groups also have a smooth version, namely Lie groups, and Lie groups come with a Lie algebra. Similarly, we can define Lie groupoids, and define the corresponding Lie algebroid. The Lie algebroid will be a natural extension of Lie algebras.

This section is based mostly on [CF11], [CFM21] and [Mei17]. More or less detailed descriptions and examples can be found there. Basically all of the theory discussed in Part I is also discussed in [Mac05], which includes many detailed examples. Mackenzie's work is very important for this thesis, but the notation is very different. Our notation is much more alike the first three sources, so we will mainly refer to those.

2.1 Groupoids

The shortest definition of a groupoid is:

Definition 2.1. A **groupoid** is a small category in which all arrows are invertible.

We expand this definition, and fix notation for the structure maps involved.

Definition 2.2. A **groupoid** $\mathcal{G} \rightrightarrows M$ consists of a set of objects M, a set of arrows $\mathcal{G}$, and the following five structure maps:

1. $s : \mathcal{G} \to M$ and $t : \mathcal{G} \to M$, the source and target map (giving the source and target of an arrow).
2. $m : \mathcal{G} \times_M \mathcal{G} \to \mathcal{G}$, denoted $m(g, h) = gh$, the multiplication map. Here $\mathcal{G} \times_M \mathcal{G}$ denotes the set of composable arrows:
$$\mathcal{G} \times_M \mathcal{G} := \{ (g, h) \mid s(g) = t(h) \} \subset \mathcal{G} \times \mathcal{G}.$$
3. $u : M \to \mathcal{G}$, the unit map sending $x \in M$ to the unit arrow 1_x.
4. $i : \mathcal{G} \to \mathcal{G}$, the inversion map, sending an arrow g to its inverse denoted $i(g) = g^{-1}$.

These maps satisfy the following axioms:

1. For g and h composable arrows, $t(gh) = t(g)$ and $s(gh) = s(h)$.
2. For g, h and k pairwise composable arrows, $(gh)k = g(hk)$.
3. For an arrow g with source x and target y, and 1_x, 1_y the corresponding identity arrows, $1_y g = g = g 1_x$.
4. For an arrow g from x to y, $gg^{-1} = 1_y$ and $g^{-1}g = 1_x$.

An arrow $g \in \mathcal{G}$ is interpreted as an arrow between two points in M, starting at its source $s(g)$ and pointing to its target $t(g)$. We will use the notation $g : s(g) \to t(g)$, or simply $g : x \to y$, indicating that x is the source of g and y the target.

We can now list the first examples of groupoids. The first two can be considered 'extreme cases'; the first groupoid is determined solely by a group, and the second is determined by a set.

Example 2.3. A group G is the same thing as a groupoid over a single point $G \rightrightarrows \{\cdot\}$. The space of arrows of this groupoid is the set of group elements. The source and target maps are trivial, and the other structure maps encode the group structure.

Example 2.4. Given a set M, the **pair groupoid** is denoted $M \times M \rightrightarrows M$, and it is the groupoid containing a unique arrow between any pair of points in M. The set of arrows is thus $M \times M$. The source map is projection onto the second coordinate and the target map is projection on the first. The reason for this seemingly backwards notation, is that the multiplication map is now very straightforward. Consider the composable arrows (z, y) and (y, x) drawn below:

$$z \xleftarrow{(z,y)} y \xleftarrow{(y,x)} x$$

The multiplication map simply 'contracts' the point y:

$$m((z,y),(y,x)) = (z,x),$$

resulting in an arrow from x to z as expected. Finally, the unit map sends x to (x,x), and the inversion map sends (y,x) to (x,y).

Example 2.5. Given a group G which is acting on a set M, we define the **action groupoid** $G \times M \rightrightarrows M$. The space of arrows is $G \times M$, with source map $s(g,x) = x$ and target map $t(g,x) = gx$. The multiplication map is $m((h,gx),(g,x)) = (hg,x)$, the unit sends x to (e,x) (e being the identity in G) and the inversion sends (g,x) to (g^{-1},gx).

To investigate other properties of these groupoids, we will need the following notions. The **source fiber** of $\mathcal{G}$ at a point $x \in M$ is $s^{-1}(x)$, its preimage under the source map. Similarly, the **target fiber** of $\mathcal{G}$ at $x \in M$ is $t^{-1}(x)$.

A group G comes with a binary operation $G \times G \to G$, which can be seen as an action of G on itself. In the case of a groupoid, such a multiplication map $\mathcal{G} \times \mathcal{G} \to \mathcal{G}$ is not well-defined, as not all arrows are composable. We have to define the left and right action of a groupoid element on the groupoid using the source and target fibers, in order to make it well-defined. To that end, let $g \in \mathcal{G}$ be an arrow from x to y. Then for any arrow $h \in t^{-1}(x)$, the pair (g,h) is composable and we define left multiplication by g as

$$L_g : t^{-1}(x) \to t^{-1}(y),\ \ h \mapsto gh.$$

For an arrow $k \in s^{-1}(x)$, the pair (k,g) is composable and we define right multiplication by g as

$$R_g : s^{-1}(y) \to s^{-1}(x),\ \ k \mapsto kg.$$

Using the source and target fibers, we also define orbits and isotropy groups.

Definition 2.6. Given a groupoid $\mathcal{G} \rightrightarrows M$ and a point $x \in M$, the **isotropy group** at x is

$$\mathcal{G}_x := s^{-1}(x) \cap t^{-1}(x) = \{\, g \in G \mid s(g) = t(g) = x \,\}.$$

The **orbit** through x is

$$\mathcal{O}_x := t(s^{-1}(x)) = s(t^{-1}(x)) = \{\, y \in M \mid \exists g \in \mathcal{G} \text{ s.t. } s(g) = x,\, t(g) = y \,\}.$$

The **orbit set** $M/\mathcal{G}$ of the groupoid is the collection of orbits. Equivalently, it is a quotient of M, where two points are identified if they lie in the same orbit:

$$M/\mathcal{G} := \{\, \mathcal{O}_x \mid x \in M \,\} = M/\sim_{\mathcal{G}}\ .$$

The isotropy group at x has a clear group structure, where the unit arrow 1_x is the identity, and composition of arrows gives the group operation. Note the groupoid multiplication is well-defined on $\mathcal{G}_x \times \mathcal{G}_x$.

Example 2.7. For the group over a point $G \rightrightarrows \{\cdot\}$, the isotropy group and orbit are defined at only one point, the basepoint. The isotropy group is G, as all arrows have the basepoint as source and target. The orbit is the basepoint itself, and the orbit space contains one element.

Example 2.8. Consider the pair groupoid $M \times M \rightrightarrows M$ over a set M, and a point $x \in M$. By definition, there is a unique arrow between any pair of points; this implies there can only be one arrow in $\mathcal{G}_x$, which must be the unit 1_x. The orbit at x is M, as there is an arrow between any two points in M, and again the orbit space contains one element.

We see again that these examples are the two extreme cases of groupoids; the first has the largest possible isotropy groups, and an orbit with only one element, while the second has the smallest possible isotropy groups and the largest possible orbits.

In the next example, we see that the isotropy groups and orbits of groupoids are indeed a natural extension of the isotropy groups and orbits from group theory.

Example 2.9. For the action groupoid $G \times M \rightrightarrows M$, the isotropy group at a point $x \in M$ is

$$\mathcal{G}_x = \{ (g, x) \in G \times M \mid x = gx \}$$

and this agrees with the isotropy group arising from the group action of G on M. Similarly, the orbit is

$$\mathcal{O}_x = \{ gx \mid g \in G \}$$

which agrees with the group action orbit.

We now introduce morphisms of groupoids. Since we defined groupoids as certain categories, a morphism is simply a functor of categories. In more detail:

Definition 2.10. A **morphism of groupoids** from $\mathcal{G} \rightrightarrows M$ to $\mathcal{H} \rightrightarrows N$ is a pair of maps (Φ, ϕ), where $\Phi : \mathcal{G} \to \mathcal{H}$ is a map of arrows, $\phi : M \to N$ is a map of objects, such that they are compatible with all the structure maps. Explicitly, they satisfy:

- An arrow $g : x \to y$ in $\mathcal{G}$ is mapped by Φ to an arrow $\Phi(g) : \phi(x) \to \phi(y)$.
- For $g, h \in \mathcal{G}$ composable arrows, $\Phi(gh) = \Phi(g)\Phi(h)$.
- For $x \in M$, $\Phi(1_x) = 1_{\phi(x)}$.
- For $g \in \mathcal{G}$, $\Phi(g^{-1}) = \Phi(g)^{-1}$.

Definition 2.11. A **subgroupoid** of a groupoid $\mathcal{G} \rightrightarrows M$ is a groupoid $\mathcal{H} \rightrightarrows N$ together with an injective groupoid morphism $i : \mathcal{H} \to \mathcal{G}$.

We now introduce two more examples of groupoids. The first one is a subgroupoid of the pair groupoid. The second example, the gauge groupoid, will play a central role in the theory of transitive groupoids.

Example 2.12. Consider a submersion $\mu : M \to N$. This gives rise to the **submersion groupoid**, denoted $M \times_\mu M \rightrightarrows M$. The set of objects is M, and the set of arrows is

$$M \times_\mu M := \{ (x, y) \mid \mu(x) = \mu(y) \}.$$

This is a subgroupoid of the pair groupoid, with $i : M \times_\mu M \to M \times M$ the obvious inclusion. The structure maps of the submersion groupoid are simply those of the pair groupoid, restricted to $M \times_\mu M$. The isotropy groups of the submersion groupoid are still trivial. The orbit through a point $x \in M$ is precisely the fiber $M_{\mu(x)} = \mu^{-1}(\mu(x))$.

Example 2.13. Let G be a Lie group, and $\mathrm{pr} : P \to M$ a principal G-bundle. Consider the pair groupoid $P \times P \rightrightarrows P$. By quotienting the pair groupoid by the G-action, we find the **gauge groupoid**:

$$\mathcal{G} := (P \times P)/G \rightrightarrows P/G = M.$$

The equivalence relation on $P \times P$ is given by $(p, q) \sim (pg, qg)$ for $g \in G$. The source map of this groupoid is $s([p, q]) = \mathrm{pr}(q)$ and the target map is $t([p, q]) = \mathrm{pr}(p)$. Since the fibers of a principal bundle are G-torsors, these maps are well-defined. The unit above x is $[(p, p)]$ where p is any element in P_x (again this is well-defined as all such (p, p) lie in the same equivalence class). The inversion map is $[(p, q)] \mapsto [(q, p)]$.

To compute the isotropy groups of the gauge groupoid, let $x \in M$. Then $\mathcal{G}_x$ consists of all arrows with source and target x:

$$\mathcal{G}_x = (P_x \times P_x)/G = \{ [(p, q)] \mid \mathrm{pr}(p) = \mathrm{pr}(q) = x \}.$$

Fix a point p_0 in P_x. Then any class in $\mathcal{G}_x$ can be written as $[(p_0, p_0 g)]$ for some $g \in G$, since P_x is a G-torsor. We see

$$\mathcal{G}_x = \{ [(p_0, p_0 g)] \mid g \in G \} \cong G,$$

so for gauge groupoids, all isotropy groups are isomorphic to G.

From the fact that the pair groupoid $P \times P$ has one orbit (namely P), we see that the gauge groupoid also has one orbit, which is M.

Definition 2.14. A groupoid is called **transitive** if it has one orbit.

So far, M and $\mathcal{G}$ have been sets. In this thesis, we will consider mostly smooth groupoids.

Definition 2.15. A **Lie groupoid** is a groupoid $\mathcal{G} \rightrightarrows M$ with a smooth structure on M and $\mathcal{G}$, and where s and t are submersions and all structure maps are smooth.

Remark 2.16.

1. All smooth manifolds in this thesis are assumed to be second countable and Hausdorff. The only exception is the space of arrows of a groupoid $\mathcal{G}$, which is not required to be Hausdorff. For example, a regular foliation of a manifold induces a groupoid that is already non-Hausdorff, and this is not a very extreme case at all (this example will be discussed in Section 2.8). The source and target fibers in $\mathcal{G}$ will always be Hausdorff due to the source and target map being submersions, so we will not encouter many issues with this. For more detail on this and the precise way to handle this we refer to Chapter 13 in [CFM21].

2. The condition that s and t are submersions implies that the set of composable arrows $\mathcal{G} \times_M \mathcal{G}$ is a submanifold of $\mathcal{G} \times \mathcal{G}$ by the submersion theorem. This shows that it makes sense to impose that m is a smooth map.

3. The condition that both s and t are submersions is actually redundant. If either s or t is a submersion, it already follows that the other one is as well, since they are related via the inversion map.

Finally, for Lie groupoids we will use the following facts. For the proof, we refer to Theorem 5.4 of [MM03].

Proposition 2.17. *For a Lie groupoid $\mathcal{G}$, the following hold.*

- *The unit map $u : M \to \mathcal{G}$ is an embedding.*
- *The orbits $\mathcal{O}_x \subset \mathcal{G}$ for $x \in M$ are immersed submanifolds.*
- *The isotropy groups $\mathcal{G}_x \subset \mathcal{G}$ are Lie groups.*

Recall that a transitive groupoid is a groupoid with a single orbit. For transitive Lie groupoids, we have the following useful characterisation.

Proposition 2.18. *Any transitive Lie groupoid is a gauge groupoid.*

Proof. Let $\mathcal{G} \to M$ be a transitive groupoid. Fix some point $x \in M$, consider $s^{-1}(x) \subset \mathcal{G}$. By transitivity, the restriction of the target map $t : s^{-1}(x) \to M$ is surjective, and since $\mathcal{G}$ is a Lie groupoid, t is a submersion. Furthermore, $\mathcal{G}_x$ acts on $s^{-1}(x)$ by composition of arrows. We find that $s^{-1}(x) \to M$ is a principal $\mathcal{G}_x$-bundle.

From the principal bundle $s^{-1}(x)$, we can form a gauge groupoid. We claim that this gauge groupoid is isomorphic to the original groupoid $\mathcal{G}$:

$$\mathcal{G} \cong (s^{-1}(x) \times s^{-1}(x))/\mathcal{G}_x =: \mathcal{G}_{gauge}.$$

First define the mapping $\mathcal{G} \to \mathcal{G}_{gauge}$. Let $g \in \mathcal{G}$ be an arrow $g : y \to z$. Since $\mathcal{G}$ is transitive, there exists some arrow $h : x \to z$. Now define

$$\mathcal{G} \to \mathcal{G}_{gauge},\ g \mapsto [(h, hg^{-1})].$$

Note both h and hg^{-1} lie in $s^{-1}(x)$. It is easily checked that this map commutes with the groupoid structures.

Next define the inverse mapping $\mathcal{G}_{gauge} \to \mathcal{G}$ by $[(h, g)] \mapsto hg^{-1}$. Both h and g have source x, thus h and g^{-1} are composable. Again, it is easily checked that this map commutes with the group structure, and is an inverse to the map defined above. □

2.2 The Lie algebroid of a Lie groupoid

A natural question to ask after defining Lie groupoids is whether there is a corresponding infinitesimal version, similar to the Lie algebra of a Lie group. This is the notion of Lie algebroids. We will introduce Lie algebroids from this perspective by 'differentiating' Lie groupoids, in a similar manner as one does for Lie groups. However, there is one big difference: not all Lie algebroids are integrable to (or come from) a Lie groupoid. Still, to define them, it is natural to start from this perspective of differentiating Lie groupoids.

We first recall the procedure that describes the Lie algebra of a Lie group, and then imitate this for the case of Lie groupoids. Let G be a Lie group. Then the underlying vector space of its Lie algebra is $\mathfrak{g} = T_eG$, the tangent space at the identity. The Lie bracket on $\mathfrak{g}$ is determined by identifying T_eG with $\mathfrak{X}_{inv}(G)$, the space of right-invariant vector fields on G. This identification works as follows: a vector field $X \in \mathfrak{X}_{inv}(G)$ is mapped to $X(e) \in T_eG$, and a vector $v \in T_eG$ determines a vector field X which is defined by $X_g = d_eR_g(v)$. This mapping is an isomorphism. The vector fields on G are naturally equipped with a Lie bracket, and the right-invariant vector fields are closed under the bracket. So by this identification we find a Lie bracket on T_eG, making it into a Lie algebra, specifically the Lie algebra $\mathfrak{g}$ of G.

We now start with a Lie groupoid $\mathcal{G} \rightrightarrows M$, and we want to define its Lie algebroid in a similar way. The first step in defining $\mathfrak{g}$ was taking the tangent space at the identity element. However, in a groupoid, there is not a single identity element; for every $x \in M$, there is a unit 1_x. This suggests that we should consider not a vector space, but a vector bundle over M. We expect that the fiber above x of this vector bundle will be $T_{1_x}\mathcal{G}$.

We then look for an identification with right-invariant vector fields, i.e., vector fields invariant under right translation. Here the second issue comes up; right translation for groupoids is only defined on the source fibers. This suggests that instead of fibers $T_{1_x}\mathcal{G}$, we should only consider vectors that are tangent to the source fibers, so $T_{1_x}s^{-1}(x)$. With this motivation, we define the underlying vector bundle of the Lie algebroid of $\mathcal{G}$. We will use the notation $T^s\mathcal{G} := \mathrm{Ker}(ds) \subset T\mathcal{G}$.

Definition 2.19. For a groupoid $\mathcal{G} \rightrightarrows M$, the vector bundle associated to its Lie algebroid is $A \to M$, defined by

$$A := T^s\mathcal{G}|_M = u^*(T^s\mathcal{G}).$$

Here, when we write $|_M$, we identify M with its embedding $u(M) \subset \mathcal{G}$.

The Lie bracket of an algebroid will be defined on its space of sections $\Gamma(A)$. We will define this by identifying $\Gamma(A)$ with certain right-invariant vector fields on the groupoid, a notion that we have to introduce first. To this end, consider first any arrow $h : y \to z$ in $\mathcal{G}$. Then $T^s_h\mathcal{G}$ is just the tangent space to $s^{-1}(y)$ at h. Recall that for an arrow $g : x \to y$, we have defined $R_g : s^{-1}(y) \to s^{-1}(x)$, so by taking the differential at $h \in s^{-1}(y)$ we find a map

$$d_hR_g : T_hs^{-1}(y) \to T_{hg}s^{-1}(x).$$

This is equivalently written as

$$d_hR_g : T^s_h\mathcal{G} \to T^s_{hg}\mathcal{G}.$$

Then a vector field $X \in \mathfrak{X}(\mathcal{G})$ is right-invariant tangent to the source fibers, if $X_{hg} = d_hR_g(X_h)$ for composable arrows h and g. The space of such vector fields is denoted $\mathfrak{X}^s_{inv}(\mathcal{G})$. Note that this space is closed under the Lie bracket, following from the same argument that shows the right-invariant vector fields on Lie groups are closed under the Lie bracket.

Lemma 2.20. *There is a 1-1 correspondence between* $\Gamma(A)$*, the sections of the associated vector bundle* A *of a groupoid* $\mathcal{G}$*, and the right-invariant vector fields on* $\mathcal{G}$ *tangent to the source fibers* $\mathfrak{X}^s_{inv}(\mathcal{G})$*.*

Proof. Let $\alpha \in \Gamma(A)$, and define pointwise the vector field $\alpha^R_g = d_{1_{t(g)}}R_g(\alpha_{t(g)})$. This is clearly smooth. It is right-invariant by the following computation:

$$\begin{aligned}\alpha^R_h g &= d_{1_{t(hg)}}R_hg(\alpha_{t(hg)})\\ &= d_hR_g(d_{1_{t(h)}}R_h(\alpha_{t(h)}))\\ &= d_hR_g(\alpha^R_h).\end{aligned}$$

It is also tangent to the source fibers from the following computation, where we set $k := R_g\left(1_{t(g)}\right)$:

$$d_ks\left(d_{1_{t(g)}}R_g(\alpha_{t(g)})\right) = d_k\left(s \circ R_g\right)\left(\alpha_{t(g)}\right) = 0,$$

since $s \circ R_g$ is the constant map $s^{-1}(t(g)) \to M$, $h \mapsto s(g)$.

In the other direction, let $X \in \mathfrak{X}^s_{inv}(\mathcal{G})$, and let g be an arrow from x to y. Note that $R_g(1_y)$ is well-defined. We have that

$$X_g = d_{1_y}R_g(X_{1_y}),$$

so the vector field X is fully determined by its values at the unit arrows. Now defining a section $\alpha \in \Gamma(A)$ by $\alpha_x = X_{1_x}$, we clearly have a 1-1 correspondence $\Gamma(A) \simeq \mathfrak{X}^s_{inv}(\mathcal{G})$. □

Definition 2.21. The Lie bracket on $A \to M$ is induced by the correspondence $\Gamma(A) \simeq \mathfrak{X}^s_{inv}(\mathcal{G})$:

$$[\alpha, \beta]^R = [\alpha^R, \beta^R].$$

With this, all steps in the procedure of finding the Lie algebra of a Lie group are imitated for Lie groupoids. However, an additional element is needed to define Lie algebroids. This final element is the anchor map, which is a vector bundle map $\rho : A \to TM$. The anchor map should always be related to the Lie bracket via a Leibniz rule, described in Proposition 2.23. We first give some intuition on how to interpret the anchor. Consider that we now have two vector bundles over M: the tangent bundle TM, and the algebroid A. The algebroid can be interpreted as carrying information of a certain structure over M, for example the infinitesimal data encoding a groupoid. In other words, A can be interpreted as 'alternative tangent directions' on M, that are relevant to the structure of interest. The anchor map translates these to actual tangent directions. For example, the anchor map corresponding to an action groupoid will be precisely the infinitesimal Lie algebra action (see Example 2.28). With this interpretation in mind, we now define the anchor of a Lie algebroid of a Lie groupoid, which will be induced by the target map.

Definition 2.22. The anchor map of $A \to M$ is the vector bundle map $\rho_A : A \to TM$ given by the restriction of $dt : T\mathcal{G} \to TM$ to $A = T^s\mathcal{G}|_M \subset T\mathcal{G}$.

Proposition 2.23. *The following Leibniz rule holds for the anchor map: for all $\alpha, \beta \in \Gamma(A)$ and $f \in C^\infty(M)$,*

$$[\alpha, f\beta] = f[\alpha, \beta] + \mathcal{L}_{\rho(\alpha)}(f)\beta.$$

Proof. In the computation, we will use the following property for $f \in C^\infty(M)$ and $\beta \in \Gamma(A)$, with β^R the vector field as in Lemma 2.20:

$$(f\beta)^R_g = d_{1_{t(g)}}R_g((f\beta)_{t(g)}) = f(t(g))d_{1_{t(g)}}R_g(\beta_{t(g)}), \text{ so we have } (f\beta)^R = (f \circ t)\beta^R.$$

We now use the identification $\Gamma(A) \cong \mathfrak{X}^s_{inv}(\mathcal{G})$, and then apply the usual Leibniz rule for the Lie bracket of vector fields to find:

$$\begin{aligned}
[\alpha, f\beta]^R &= [\alpha^R, (f\beta)^R] \\
&= [\alpha^R, (f \circ t)\beta^R] \\
&= (f \circ t)[\alpha^R, \beta^R] + \mathcal{L}_{\alpha^R}(f \circ t)\beta^R \\
&= (f[\alpha, \beta])^R + \mathcal{L}_{\alpha^R}(f \circ t)\beta^R.
\end{aligned}$$

We now compute $\mathcal{L}_{\alpha^R}(f \circ t)$ seperately as a function of $\mathcal{G}$. Let $g \in \mathcal{G}$, then

$$\begin{aligned}
\mathcal{L}_{\alpha^R}(f \circ t)(g) &= d_g(f \circ t)\alpha^R(g) \\
&= d_{t(g)}f(d_g t(\alpha^R(g))) \\
&= d_{t(g)}f(\rho(\alpha))_{t(g)} \\
&= \mathcal{L}_{\rho(\alpha)}(f)(t(g)).
\end{aligned}$$

All together, we have

$$[\alpha, f\beta]^R = (f[\alpha, \beta])^R + \big(\big(\mathcal{L}_{\rho(\alpha)}(f)\big) \circ t\big)\, \beta^R$$

which indeed implies

$$[\alpha, f\beta] = f[\alpha, \beta] + \big(\mathcal{L}_{\rho(\alpha)}(f)\big)\, \beta.$$

□

Proposition 2.24. *The induced map on the space of sections $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra morphism.*

These two properties will hold for any Lie algebroid. For the algebroid associated to a groupoid, the Leibniz rule follows from the identification $\Gamma(A) \cong \mathfrak{X}^s_{inv}(\mathcal{G})$, but for general Lie algebroids this condition has to be imposed in the definition. The fact that the anchor map is a Lie algebra morphism follows from the Jacobi identity for the Lie bracket, combined with the Leibniz rule. Since this property will hold in general, we postpone the proof to the general case in Proposition 2.30.

Gathering all the intermediate steps, we can finally define fully the Lie algebroid of a Lie groupoid.

Definition 2.25. Given a Lie groupoid $\mathcal{G} \rightrightarrows M$, its **Lie algebroid** is the vector bundle $A = T^s\mathcal{G}|_M \to M$, together with the Lie bracket on its space of sections induced by $\Gamma(A) \cong \mathfrak{X}^s_{inv}(\mathcal{G})$, and with the anchor map $\rho = dt|_A : A \to TM$.

The algebroid of a Lie groupoid $\mathcal{G}$ will also be denoted by $\mathcal{L}ie(\mathcal{G})$.

Example 2.26. Consider the Lie group over a point $G \rightrightarrows \{\mathrm{pt}\}$. There is one point in the base, so the vector bundle A becomes simply a vector space. Since $s^{-1}(\mathrm{pt}) = G$, this vector space is T_eG. The Lie bracket is simply the usual bracket of the Lie algebra of G, and the anchor map is 0. This example shows that indeed Lie algebroids are a generalisation or extension of the usual Lie algebras corresponding to Lie groups.

Example 2.27. Consider the pair groupoid $M \times M \rightrightarrows M$. In this case, A will be a vector bundle over M. To compute the fiber A_x for $x \in M$, we consider the source fiber $s^{-1}(x)$. This is precisely $M \times \{x\}$, as all possible pairs in $M \times M$ are connected by a unique arrow in the pair groupoid. Taking the tangent space to $s^{-1}(x)$ at the unit above x, we find

$$A_x = T_{(x,x)}(M \times \{x\}) \cong T_xM,$$

and we see that the associated vector bundle of the pair groupoid is the tangent bundle. To find the Lie bracket on its space of sections $\Gamma(A)$, we have to compute the right-invariant vector fields on $M \times M$ that are tangent to the source fibers. We claim that $\mathfrak{X}^s(M \times M)^{inv} \cong \mathfrak{X}(M)$.

Consider a vector field $X \in \mathfrak{X}^s(M \times M)^{inv}$. Let $h : y \to z$ be an arrow, also denoted $h = (z, y)$. Denote $X_h = X_{(z,y)} = (X^1_{(z,y)}, X^2_{(z,y)})$. Since X is tangent to the source fibers, we must have $X^2_{(z,y)} \in T_y\{y\}$. We see that the second component of source-tangent vector fields is always trivial, and we can write $X_{(z,y)} = (X^1_{(z,y)}, 0_{(z,y)})$. Next, to describe right-invariance, consider an arrow $g : x \to y$. This induces right translation

$$R_g : M \times \{y\} \to M \times \{x\}$$

and, since hg is composable,

$$d_hR_g : T_zM \times T_y\{y\} \to T_zM \times T_x\{x\}.$$

Applying this to X_h yields

$$d_h R_g(X_h) = d_h R_g(X^1_{(z,y)}, 0_{(z,y)}) = (X^1_{(z,y)}, 0_{(z,x)}).$$

Being right-invariant then translates to the following condition:

$$(X^1_{(z,y)}, 0_{(z,x)}) = (X^1_{(z,x)}, 0_{(z,x)}), \text{ i.e., } X^1_{(z,y)} = X^1_{(z,x)} \, \forall x, y, z \in M.$$

In other words, the first component of $X_{(z,y)}$ only depends on z, and the second component is always zero. We find an isomorphism $\mathfrak{X}^s(M \times M)^{inv} \cong \mathfrak{X}(M)$, which also endows $\Gamma(A) \cong TM$ with the usual Lie bracket of vector fields on M.

Finally, the anchor map is given by the differential of the target map. The target map is projection onto the first coordinate: $t : M \times M \to M$, $t(y, x) = y$. Its differential restricted to A is

$$d_{(x,x)}t : T_x M \times T_x\{x\} \to T_x M, \; (v, 0) \mapsto v$$

and we see that the anchor $\rho : A \cong TM \to TM$ is the identity map. So the algebroid of the pair groupoid is precisely the tangent bundle.

Example 2.28. Consider now the action groupoid $G \times M \rightrightarrows M$. To compute the fibers of the associated vector bundle, we compute the source fiber above $x \in M$.

$$s^{-1}(x) = \{ (g, y) \mid s(g, y) = x \} = \{ (g, x) \mid g \in G \} \cong G.$$

We see that each fiber of A is isomorphic to the Lie algebra $\mathfrak{g}$ of G.

To compute the bracket, we consider the space of sections $\Gamma(A) \cong \Gamma(\mathfrak{g} \times M)$. We see $f \in \Gamma(A)$ is equivalent to a smooth map $f : M \to \mathfrak{g} \times M$ such that $f(x) = (v, x)$ for some $v \in \mathfrak{g}$: we see $\Gamma(A) \cong C^\infty(M; \mathfrak{g})$. Each $f \in C^\infty(M; \mathfrak{g})$ is identified with a right-invariant vector field f^R on $G \times M$ by

$$f^R(g, x) = d_{1_{gx}} R_g(f(gx)),$$

and at $g = 1_x$ we see

$$f^R(1_x, x) = d_{(e,x)} R_e(f(x)).$$

Now the bracket on $\Gamma(A)$ is defined by

$$[f, g]_A(x) = [f^R, g^R]_{\mathfrak{g}}(1_x).$$

For constant functions (denoted c_1 and c_2), we can take 1_x into the bracket to find

$$[c_1, c_2]_A(x) = [d_{(e,x)} R_e(c_1(x)), d_{(e,x)} R_e(c_2(x))]_{\mathfrak{g}} = [c_1(x), c_2(x)]_{\mathfrak{g}}.$$

In general, by writing any section as the sum of constant sections and applying the Leibniz rule, we find the following Lie bracket on $\Gamma(A)$:

$$[f, g]_A(x) = [f(x), g(x)]_{\mathfrak{g}} + (\mathcal{L}_{\rho(f(x))} g)(x) - (\mathcal{L}_{\rho(g(x))} f)(x). \tag{1}$$

Finally the anchor map simply encodes the infinitesimal action, as the target map $t : G \times M \to M$ encodes the group action and $\rho = dt|_A$.

In this example, we could have also started with a Lie algebra action, which is a Lie algebra morphism $a : \mathfrak{g} \to \mathfrak{X}(M)$. This gives rise to an algebroid by defining the vector bundle $\mathfrak{g} \times M \to M$, with anchor $\rho = a$ and with the bracket on its space of sections $\Gamma(\mathfrak{g} \times M) \cong C^\infty(M; \mathfrak{g})$ defined by Equation (1). This algebroid is called the **action algebroid** associated to the Lie algebra action a. It follows that the action algebroid associated to a Lie algebra action is precisely the algebroid associated to the action groupoid of the integrating Lie group action.

In Section 3, we will discuss another example of a groupoid and a large part of the chaper is devoted to computing its algebroid. Then in Section 4 we will compute the algebroid of the gauge groupoid.

2.3 Lie algebroids

In the previous section we have introduced the Lie algebroid of a Lie groupoid, but Lie algebroids can (and do) also exist without a groupoid. From the discussion in the previous section, we already have a natural notion of Lie algebroids.

Definition 2.29. A **Lie algebroid** over a manifold M is a vector bundle $A \to M$, together with a Lie bracket $[\cdot, \cdot]_A$ on the space of sections $\Gamma(A)$ and a vector bundle map $\rho : A \to TM$ called the anchor map, such that the following Leibniz rule holds:

$$[\alpha, f\beta] = f[\alpha, \beta] + \mathcal{L}_{\rho(\alpha)} f\beta \; \forall \alpha, \beta \in \Gamma(A), \; f \in C^\infty(M).$$

From this definition, it follows that the anchor map is a Lie algebra morphism.

Proposition 2.30. *For any Lie algebroid $A \to M$, the anchor map on the space of sections $\rho : \Gamma(A) \to \mathfrak{X}(M)$ is a Lie algebra homomorphism:*

$$\rho([\alpha, \beta]_A) = [\rho(\alpha), \rho(\beta)].$$

Proof of Proposition 2.24 and Proposition 2.30. To prove this identity, we apply the Leibniz rule and the Jacobi identity to the term $[[\alpha, \beta]_A, f\gamma]_A$ (for α, β, $\gamma \in \Gamma(A)$):

$$\begin{aligned}
[[\alpha, \beta]_A, f\gamma]_A =& f[[\alpha, \beta]_A, \gamma]_A + \mathcal{L}_{\rho([\alpha,\beta]_A)} f\gamma \text{ (by Leibniz)} \\
[[\alpha, \beta]_A, f\gamma]_A =& [[\alpha, f\gamma]_A, \beta]_A + [[f\gamma, \beta]_A, \alpha]_A \text{ (by Jacobi)} \\
=& [f[\alpha, \gamma]_A, \beta]_A + [\mathcal{L}_{\rho(\alpha)} f\gamma, \beta]_A - [f[\beta, \gamma], \alpha]_A - [\mathcal{L}_{\rho(\beta)} f\gamma, \alpha]_A \text{ (by Leibniz)} \\
=& - f[\beta, [\alpha, \gamma]_A]_A - \mathcal{L}_{\rho(\beta)} f[\alpha, \gamma]_A - \mathcal{L}_{\rho(\alpha)} f[\beta, \gamma]_A - \mathcal{L}_{\rho(\beta)} \left(\mathcal{L}_{\rho(\alpha)} f\right) \gamma + \\
& f[\alpha, [\beta, \gamma]]_A + \mathcal{L}_{\rho(\alpha)} f[\beta, \gamma] + \mathcal{L}_{\rho(\beta)} f[\alpha, \gamma]_A + \mathcal{L}_{\rho(\alpha)} \left(\mathcal{L}_{\rho(\beta)} f\right) \gamma \\
=& f([\alpha, [\beta, \gamma]_A]_A - [\beta, [\alpha, \gamma]_A]_A) + \mathcal{L}_{[\rho(\alpha), \rho(\beta)]} f\gamma \\
=& f([[\alpha, \beta]_A, \gamma]_A) + \mathcal{L}_{[\rho(\alpha), \rho(\beta)]} f\gamma
\end{aligned}$$

Comparing the first and the last lines of this computation, we see

$$\mathcal{L}_{\rho([\alpha,\beta]_A)} f\gamma = \mathcal{L}_{[\rho(\alpha),\rho(\beta)]} f\gamma$$

which implies indeed

$$\rho([\alpha,\beta]_A) = [\rho(\alpha), \rho(\beta)].$$

□

We also have a natural notion of integrable algebroids. Morphisms of Lie algebroid will be discussed later, but the idea of this definition should be clear.

Definition 2.31. A Lie algebroid $A \to M$ is called **integrable** if there exists some groupoid $\mathcal{G} \rightrightarrows M$ such that the Lie algebroid of $\mathcal{G}$ is isomorphic to A.

In Section 2.6 we will give an overview of the theory on integrability of Lie algebroids, developed in detail in [CF11].

We have already seen a few examples of integrable algebroids in the previous section, as those examples all came from groupoids. We will now shortly describe an example coming from Poisson geometry. The details of this example are not important for this thesis. The example is included to show a situation where the algebroid first shows up, and the corresponding groupoid is then sougth after. This example was already mentioned in the introduction: the search for symplectic realizations lead to the search for symplectic groupoids integrating the algebroid defined below. A thourough introduction into Poisson geometry can be found in [CFM21], where this relation to algebroids and groupoids is also extensively discussed.

Example 2.32. Let (M, π) be a Poisson manifold. That is, M is a manifold with a Poisson bracket $\{\cdot,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$. In order to study this additional structure, the bracket is often encoded in a bivector denoted π:

$$\pi : \Omega^1(M) \times \Omega^1(M) \to \Omega^1(M) \text{ such that } \{f, g\} = \pi(df, dg).$$

Associated to π, there is an induced map $\pi^\#$:

$$\pi^\# : T^*M \to TM, \ \alpha \mapsto i_\alpha(\pi).$$

And using these two maps, we find a Lie bracket on T^*M as follows:

$$[\alpha, \beta]_\pi = \mathcal{L}_{\pi^\#(\alpha)}(\beta) - \mathcal{L}_{\pi^\#(\beta)}(\alpha) - d(\pi(\alpha, \beta)).$$

These objects all come from the original Poisson bracket on $C^\infty(M)$, but in this formulation, it can be shown that T^*M is a Lie algebroid with Lie bracket $[\cdot,\cdot]_\pi$ and $\pi^\#$ as anchor. This is called the Poisson algebroid or cotangent algebroid.

As mentioned before, finding symplectic realisations of the Poisson structure now corresponds to finding a symplectic groupoid (for the definition of a symplectic groupoid, see [CFM21]) integrating the Poisson algebroid.

Moving back to general algebroids, we study the anchor map $\rho : A \to TM$ in more detail. Specifically, consider for $x \in M$ the kernel $\mathrm{Ker}(\rho_x) \subset A_x$.

Lemma 2.33. *The kernel $Ker(\rho_x) \subset A_x$ has a well-defined Lie bracket induced from $[\cdot, \cdot]_A$.*

Proof. Let α, α', β, β' in $\Gamma(A)$ such that $\alpha(x) = \alpha'(x) \in \mathrm{Ker}(\rho_x)$ and $\beta(x) = \beta'(x) \in \mathrm{Ker}(\rho_x)$. We want to prove that for the bracket $[\cdot, \cdot]_A$ on $\Gamma(A)$,

$$[\alpha, \beta]_A(x) = [\alpha', \beta']_A(x).$$

It suffices to compute the brackets in a neighbourhood U around x. Choose a local frame $\{e^1, \dots, e^r\}$ of $A|_U$. Write the sections in terms of this frame, for $y \in U$:

$$\alpha(y) = \sum_{i=1}^{r} a_i(y) e^i(y),$$

and similarly α', β and β' are determined by coordinate functions a_i', b_i and b_i'.

Then we compute the term $[\alpha', \beta']_A(x)$ in these coordinates. We will apply the Leibniz rule many times, resulting in terms of the form $\mathcal{L}_{\rho(\alpha)}(f)$; however, since all sections α, α', β, β' are in $\mathrm{Ker}(\rho_x)$, after evaluating at x all these extra terms will vanish. So, we can ignore these terms in the computation and we find

$$\begin{aligned}
[\alpha', \beta']_A(x) &= \left(\sum_{i=1}^{r} b_i' [\alpha', e^i] \right)(x) \\
&= \left(\sum_{i=1}^{r} \sum_{j=1}^{r} b_i' a_j' [e^j, e^i] \right)(x) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} b_i'(x) a_j'(x) [e^j, e^i](x) \\
&= \sum_{i=1}^{r} \sum_{j=1}^{r} b_i(x) a_j(x) [e^j, e^i](x) \\
&= [\alpha, \beta]_A(x).
\end{aligned}$$

Here we used that $\alpha'(x) = \alpha(x)$ and $\beta'(x) = \beta(x)$, which implies their coordinate functions also agree at the point x.

We see that the bracket $[\cdot, \cdot]_A$ induces a well-defined bracket on the kernel $\mathrm{Ker}(\rho_x)$. □

Definition 2.34. The **isotropy Lie algebra** at $x \in M$ is $\mathfrak{g}_x(A) := \mathrm{Ker}(\rho_x)$ with the Lie bracket from the previous lemma.

The name suggests that this Lie algebra should be related to the isotropy groups in the integrable case, which we see in the following lemma.

Lemma 2.35. *Let $\mathcal{G}$ be a groupoid with connected source fibers and algebroid A. Then for each $x \in M$, the isotropy Lie algebra $\mathfrak{g}_x(A)$ is the Lie algebra of the isotropy Lie group $\mathcal{G}_x$.*

Proof. The isotropy group is $\mathcal{G}_x = s^{-1}(x) \cap t^{-1}(x)$. The identity element is 1_x, so its Lie algebra is given by $T_{1_x}(s^{-1}(x) \cap t^{-1}(x))$. The isotropy Lie algebra is the kernel of the map

$$\rho_x : A_x \to T_x M, \text{ where } A_x = T_{1_x} s^{-1}(x).$$

Since the anchor is $\rho_x = d_{1_x} t$, this is precisely the Lie algebra of $\mathcal{G}_x$. □

Besides the kernel of the anchor, we are also interested in the image of the anchor. The results mentioned here will not be proven in detail; we refer to [CF11] for a more extensive discussion. At each point $x \in M$, we have $\mathrm{Im}(\rho_x) \subset T_x M$, and together these form a distribution on M. If the dimension of these subspaces is constant, the algebroid is called regular.

This distribution induced by the anchor map also defines a foliation of M, where the leaves are called the **orbits of the algebroid**. A leaf $\mathcal{O}$ of this foliation is a maximal immersed submanifold of M, such that $T_x\mathcal{O} = \mathrm{Im}(\rho_x)$ for all points $x \in \mathcal{O}$. The intepretation of these leaves, is that a leaf contains points that you can reach if you were to move by directions provided by the anchor map. A Lie algebroid is called **transitive** if the anchor map is surjective. In that case, the algebroid has one orbit.

If the algebroid is integrable, its orbits are related to those of the groupoid:

Proposition 2.36. *Let $\mathcal{G}$ be a groupoid with connected source fibers, and let A be its algebroid. Then the orbits of $\mathcal{G}$ coincide with the orbits of A.*

For a proof, see for example [CF11].

Corollary 2.37. *Let $\mathcal{G}$ be a groupoid with connected source fibers. If $\mathcal{G}$ is transitive, then its algebroid is transitive as well.*

Proof. If $\mathcal{G}$ is transitive, it has precisely one orbit. With source-connected fibers, this orbit will be the entire base M. By the previous proposition, its algebroid $A \to M$ will have one orbit, M, as well. Then it follows that for any $x \in M$, we have $T_x M = \mathrm{Im}(\rho_x)$, so indeed ρ is surjective and A is transitive. □

2.4 Morphisms of Lie algebroids

Morphisms of Lie algebroids are suprisingly difficult to formulate. Intuitively, they should be vector bundle maps that preserve the anchor map and the Lie bracket. However, vector bundle maps do not necessarily induce a map of sections, while the Lie brackets are defined on these sections. This makes it difficult to formulate the last condition. Of course, there is an 'easy' case where we do find an induced map of sections, if we consider two Lie algebroids over the same base manifold.

Definition 2.38. A **Lie algebroid morphism** between two Lie algebroids $A \to M$ and $B \to M$ over the same base manifold, is a vector bundle map

$$\Phi : A \to B$$

such that $\rho_B \circ \Phi = \rho_A$ and $\Phi([\alpha, \beta]_A) = [\Phi(\alpha), \Phi(\beta)]_B$ for all α, $\beta \in \Gamma(A)$.

In this case, Φ induces a map of sections simply by composition. Compatibility with the anchor is expressed nicely in the following diagram, which should be commuatitive:

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ & {\scriptstyle\rho_A}\searrow \quad \swarrow{\scriptstyle\rho_B} & \\ & TM & \end{array}$$

When we consider two general algebroids $A \to M$ and $B \to N$, there are several approaches to define morphisms. We discuss here two of them. The first one gives a seemingly unnatural condition, but does not require additional definitions. The second one requires additional theory, namely defining Lie algebroid cohomology, but the definition of Lie algebroid morphisms becomes quite natural.

For yet another description of Lie algebroid morphisms, we refer to [Mei17], where a vector bundle map $\Phi : A \to B$ is a Lie algebroid morphism if its graph is a sub Lie algebroid of the product $A \times B$ (which of course first requires one to prove that $A \times B \to M \times N$ is an algebroid itself).

Approach one: pullback vector bundle.

This approach is taken for example in [CF11]. First we define morphisms in general, then we specify what compatibility means in this sense.

Definition 2.39. A **Lie algebroid morphism** between Lie algebroids $A \to M$ and $B \to N$ is a vector bundle map $\Phi : A \to B$ covering a map $\phi : M \to N$ that is compatible with the anchor, so

$$d\phi \circ \rho_A = \rho_B \circ \Phi,$$

and compatible with the Lie brackets in the sense explained below.

Again, compatibility with the anchor is expressed as the following diagram being commutative.

$$\begin{array}{ccc} A & \xrightarrow{\Phi} & B \\ \downarrow{\scriptstyle\rho_A} & & \downarrow{\scriptstyle\rho_B} \\ TM & \xrightarrow{d\phi} & TN \end{array}$$

To define compatibility with the brackets, consider the pullback vector bundle $\phi^* B \to M$. Then for any section $\alpha \in \Gamma(A)$, the map $\Phi(\alpha) := \Phi \circ \alpha$ is a section of this pullback bundle $\Phi^* B$. And for any section $\alpha' \in \Gamma(B)$, the map $\phi^*(\alpha') := \alpha' \circ \phi$ is again a section of the pullback bundle $\Phi^* B$.

We can now express for any $\alpha_i \in \Gamma(A)$, the induced section $\Phi(\alpha_i)$ as a finite linear combination

$$\Phi(\alpha_j) = \sum_{i=1}^{n} f_j^i \beta_i^j \circ \phi$$

for $f_j^i \in C^\infty(M)$ and $\beta_i^j \in \Gamma(B)$. Then the compatibility of Φ with the Lie brackets is expressed as the condition

$$\Phi \circ [\alpha_1, \alpha_2] = \sum_{i=1}^{n} \mathcal{L}_{\rho_A(\alpha_1)}(f_2^i)\beta_i^2 \circ \phi - \sum_{i=1}^{n} \mathcal{L}_{\rho_A(\alpha_2)}(f_1^i)\beta_i^1 \circ \phi + \sum_{i,j=1}^{n} f_1^i f_2^j [\beta_i^1, \beta_j^2] \circ \phi.$$

Approach two: cohomology.

This second approach is discussed in [CFM21]. Here it is developed specifically for the algebroid of a Poisson structure, but in general this approach takes the following form. We first define Lie algebroid cohomology. Let $A \to M$ a Lie algebroid, and define the space of k-forms on A by

$$\Omega^k(A) = \Gamma(\Lambda^k A^*) = \{\, \omega : \Gamma(A) \times \cdots \times \Gamma(A) \to C^\infty(M) \mid \omega \text{ alternating and } C^\infty(M)\text{-linear} \,\}.$$

This comes with a differential $d_A : \Omega^k(A) \to \Omega^{k+1}(A)$, defined by the usual Koszul-type formula:

$$(d_A\omega)(a_0, .., a_k) = \sum_{i=0}^{k} (-1)^i \mathcal{L}_{\rho(a_i)}(\omega(a_0, .., \hat{a}_i, .., a_k)) + \sum_{i<j} (-1)^{i+j} \omega([a_i, a_j]_A, a_0, .., \hat{a}_i, .., \hat{a}_j, .., a_k)$$

for $\omega \in \Omega^k(A)$ and $a_i \in \Gamma(A)$. The square of this map is 0, which can be proven by a straightforward computation. With these definitions, one can define Lie algebroid cohomology classes in the usual way.

The complex and differential defined above can also be used to define algebroid morphisms. Note that a vector bundle map $\Phi : A \to B$ induces a map of forms $\Phi^* : \Omega^\bullet(B) \to \Omega^\bullet(A)$ by

$$(\Phi^*\omega)(a_1, \ldots, a_k) = \omega_{\phi(x)}(\Phi(a_1), \ldots, \Phi(a_k)).$$

Using this, we can formulate an alternative (but equivalent) definition of morphisms of Lie algebroids:

Definition 2.40. A **Lie algebroid morphism** between Lie algebroids $A \to M$ and $B \to N$ is a vector bundle map $\Phi : A \to B$ covering a map $\phi : M \to N$ such that the induced map on forms $\Phi^* : \Omega^\bullet(B) \to \Omega^\bullet(A)$ is a cochain map, i.e., it satisfies $d_A \Phi^* = \Phi^* d_B$.

Note that this definition also implies compatibility with the anchor as before.

Remark 2.41. If the two algebroids in question are the tangent bundle, and we want to define a Lie algebroid morphism between $TM \to M$ and $TN \to N$, compatibility with the anchor implies

$$d\phi \circ \rho_A = \rho_B \circ \Phi \implies d\phi = \Phi$$

as the anchor of the tangent bundle is the identity. We see that in this specific case, a Lie algebroid morphism is forced to be the differential of some map $\phi : M \to N$.

Finally, groupoid morphisms induce morphisms of their Lie algebroids (Theorem 13.38 in [CFM21]):

Lemma 2.42. *Let $\Psi : \mathcal{G} \to \mathcal{F}$ be a morphism of groupoids. Then this induces a morphism $\Phi : \mathcal{L}ie(\mathcal{G}) \to \mathcal{L}ie(\mathcal{F})$ of their algebroids, where Φ is the differential of Ψ restricted to $\mathcal{L}ie(\mathcal{G})$.*

2.5 Path homotopy in Lie algebroids

In this section we discuss homotopies of paths in Lie algebroids. This will be an important ingredient when looking for the integration of an algebroid in the next section. The discussion here closely follows [CF11].

Recall that a path homotopy in a manifold M between two paths γ_0 and γ_1, is a map $\gamma_\epsilon(t) : [0,1] \times [0,1] \to M$ from γ_0 to γ_1 such that $\gamma_\epsilon(0)$ and $\gamma_\epsilon(1)$ are constant. We now try to mimic this definition for an algebroid A to define A-path homotopies. We first define general paths in an algebroid $A \to M$.

Definition 2.43. An A-path above a path γ in M is a map $a : [0,1] \to A$ such that

$$\rho \circ a(t) = \frac{d\gamma}{dt}(t).$$

A homotopy between two A-paths should be some variation $a_\epsilon(t)$ of A-paths, with a condition at time $t = 0$ and $t = 1$. However, since $a(t)$ represents (via the anchor map) the time-t derivative of the base-path $\gamma_\epsilon(t)$, we should not require that $a_\epsilon(0)$ or $a_\epsilon(1)$ is constant; we would like to impose that the ϵ-derivative of $\gamma_\epsilon(t)$ is constant. We thus introduce an additional map, the variation in the ϵ-direction:

$$Var(\epsilon, t) : [0,1] \times [0,1] \to A, \quad \text{such that } \rho \circ Var(\epsilon, t) = \frac{d\gamma_\epsilon(t)}{d\epsilon}.$$

The requirement on path-homotopies in M that it is constant at time 0 and 1, translates for A-path homotopies to the condition that $Var(\epsilon, 0) = Var(\epsilon, 1) = 0$. There are several ways to define/compute this variation, all of which are equivalent.

For the first approach, we choose a connection on A. Ignoring the algebroid structure of A for a second, and simply viewing it as a vector bundle over M, we choose a connection

$$\nabla : \mathfrak{X}(M) \times \Gamma(A) \to \Gamma(A),$$

which is $C^\infty(M)$-linear in the first component and satisfies the Leibniz rule in the second.

Associated to this connection, we define the torsion.

$$T_\nabla : \Gamma(A) \times \Gamma(A) \to \Gamma(A),\ (a,b) \mapsto \nabla_{\rho(a)}(b) - \nabla_{\rho(b)}(a) - [a,b]_A.$$

Fixing a connection, the following differential equation admits a solution. We call this the homotopy equation:

$$\begin{cases} \nabla_{\frac{d\gamma}{d\epsilon}}(a_\epsilon(t)) - \nabla_{\frac{d\gamma}{dt}}(b_\epsilon(t)) = T_\nabla(a_\epsilon(t), b_\epsilon(t)) \\ b_\epsilon(0) = 0. \end{cases} \tag{2}$$

Then we define the variation $V(\epsilon,t)$ to be the solution $b(\epsilon,t)$ of the homotopy equation. It is independent of the chosen connection. The differential equation requires already $Var(\epsilon,0) = 0$, so the only remaining condition for $a_\epsilon(t)$ to be an A-path homotopy is that $b_\epsilon(t)$ is 0 at time $t = 1$.

For shorter notation, we will sometimes denote terms like $\nabla_{\frac{d\gamma}{dt}}(a)$ by $\frac{d}{dt}a$.

Definition 2.44. An A-**path homotopy** (or A-homotopy) between two A-paths $a_0(t)$ and $a_1(t)$ is a variation of A-paths $a_\epsilon(t) : [0,1]\times[0,1] \to A$ above a homotopy of the base paths $\gamma_\epsilon(t)$, such that $b(\epsilon,1) = 0$, where $b(\epsilon,t)$ is the solution of the **homotopy equation** (Equation (2)).

The second approach to defining the variation $Var(\epsilon,t)$ is by time-dependent sections. Choose a section of A, depending on ϵ and t, denoted $\xi_\epsilon(t) : M \to A$, with the property

$$\xi_\epsilon(t)(\gamma_\epsilon(t)) = a_\epsilon(t).$$

Then, the following differential equation has a solution

$$\begin{cases} \frac{d}{d\epsilon}\xi_\epsilon(t) - \frac{d}{dt}\beta_\epsilon(t) = [\xi_\epsilon(t), \beta_\epsilon(t)]_A \\ \beta_\epsilon(0) = 0 \end{cases}$$

and this solution corresponds to $b(\epsilon,t)$ found above (Proposition 3.15 in [CF11]). Thus, $a_\epsilon(t)$ is an A-path homotopy if and only if $\beta_\epsilon(1) = 0$ for all ϵ. This approach will not be used explictily in the rest of this thesis.

Remark 2.45. An A-path can equivalently be defined as a Lie algebroid morphism $\Phi : TI \to A$ (where I is the interval on which the A-path is defined). Such a morphism can always be written as $\Phi = a(t)dt$ for $a(t) : [0,1] \to A$ an A-path as defined above. With this expression we see that this is equivalent to the first definition of A-paths. From this perspective, an A-path homotopy is equivalent to a (regular) homotopy of Lie algebroid morphisms

$$a_\epsilon(t)dt + b_\epsilon(t)d\epsilon : T(I \times I) \to A,$$

where a and b correspond to the variations defined above. For this perspective, we refer to [CFM21], where this is discussed in Chapter 10 for the algebroid of a Poisson structure. This construction however generalizes to all algebroids.

Lemma 2.46. *Let $a_\epsilon(t)$ a variation of A-paths. Then $b_\epsilon(t)$ is a solution of the homotopy equation (Equation (2)) if and only if the following holds for all $\xi \in \Gamma(A^*)$:*

$$\frac{d}{d\epsilon}(\xi(a)) - \frac{d}{dt}(\xi(b)) = d_A\xi(a,b). \tag{3}$$

Proof. We prove this lemma by showing that the equations are locally equivalent, so in a neighbourhood of any point $(\widetilde{\epsilon}, \widetilde{t})$. To write these equations in local coordinates, consider $x \in M$ and a neighbourhood $U \subset M$ around x with local coordinates $x_1, \ldots, x_m$. Choose a local frame of $A|_U$, denoted by

$$\{e^1, \ldots, e^k\}.$$

Then in some neighbourhood V around $(\widetilde{\epsilon}, \widetilde{t})$ (specifically, a neighbourhood such that $\gamma_\epsilon(t) \in U$) we write

$$a_\epsilon(t) = \sum_i \alpha_i(\epsilon, t)e^i$$
$$b_\epsilon(t) = \sum_i \beta_i(\epsilon, t)e^i$$

for coordinate functions $\alpha_i(\epsilon, t), \beta_i(\epsilon, t) \in C^\infty(V)$.

To make the equations more readable, we will introduce the following notation. Denote by $E \in \mathfrak{X}(M)$ the vector field on M which has as integral curves $\left(\frac{d\gamma}{d\epsilon}\right)$. Denote by $T \in \mathfrak{X}(M)$ the vector field with integral curves $\left(\frac{d\gamma}{dt}\right)$.

Consider now a connection ∇ on A. Locally, a connection can be described by a connection matrix $(\omega_i^j)_i^j$ of 1-forms on M, such that for $\alpha = \sum_i \alpha_i e^i \in \Gamma(A|_U)$:

$$\nabla_X(\alpha) = \sum_i \alpha_i \sum_j \omega_i^j(X)e^i + \sum_i \mathcal{L}_X \alpha_i e^i.$$

Now consider the homotopy equation. Recall that the torsion is defined as

$$T_\nabla(a,b) = \nabla_{\rho(a)}b - \nabla_{\rho(b)}a - [a,b]_A.$$

The first two terms in this expression correspond to the terms on the left hand side, as $\rho(b) = d\gamma/d\epsilon = E_{\gamma_\epsilon(t)}$ and $\rho(a) = d\gamma/dt = T_{\gamma_\epsilon(t)}$. It will not be necessary to compute the local coordinate functions of $[a,b]_A$, so we will write this bracket locally as

$$[a,b]_A = \sum_i C_i e^i.$$

We can now rewrite the homotopy equation in its local version:

$$\sum_i \alpha_i(\epsilon,t) \sum_j \omega_i^j(E)e^i + \sum_i \mathcal{L}_E(\alpha_i(\epsilon,t))e^i - \sum_i \beta(\epsilon,t) \sum_j \omega_i^j(T)e^i - \sum_i \mathcal{L}_T(\beta_i(\epsilon,t))e^i$$
$$= -\frac{1}{2}\sum_i C_i e^i.$$

Since the collection $\{e^i\}_i$ form a local frame, this equation holds if and only if it holds for every component e^i, i.e., if and only if for every $i \in \{1, \ldots, k\}$ we have

$$\alpha_i(\epsilon, t) \sum_j \omega_i^j(E) + \mathcal{L}_E(\alpha_i(\epsilon, t)) - \beta(\epsilon, t) \sum_j \omega_i^j(T) - \mathcal{L}_T(\beta_i(\epsilon, t)) = -\frac{1}{2} C_i. \tag{4}$$

Furthermore, as the solution $b_\epsilon(t)$ was independent of the chosen connection, this holds for all connections, i.e., for all 1-forms $\omega_i^j \in \Omega^1(M)$.

Now consider the equation introduced in the lemma. To give a local form, we consider sections $\xi \in \Gamma(A^*)$. The local frame on A induces a dual local frame on A^*, denoted

$$\{ e_1, \ldots, e_k \}$$

and in this frame we can write

$$\xi = \sum_i \xi^i e_i, \ \xi^i \in C^\infty(U).$$

Then for a section $\alpha \in \Gamma(A)$, which is locally written as $\alpha = \sum_i \alpha_i e_i$, we find

$$\xi(\alpha) = \sum_i \xi^i \sum_j \alpha_j \delta_i^j = \sum_i \xi^i \alpha_i.$$

Now consider the term $d_A \xi(a, b)$. Using the definition of d_A (a Koszul-type formula), we see

$$d_A \xi(a, b) = \mathcal{L}_{\rho(a)}(\xi(b)) - \mathcal{L}_{\rho(b)}(\xi(a)) - \xi([a, b]_A).$$

Since $\xi(a)$ and $\xi(b)$ are just smooth functions on V, their derivatives are given by the Lie derivative:

$$\frac{d}{d\epsilon}(\xi(a)) = \mathcal{L}_E(\xi(a)).$$

Again, since $\rho(a) = T$ and $\rho(b) = E$, the first two terms of $d_A \xi(a, b)$ correspond to the terms on the left hand side of Equation (3), and we can rewrite that equation to

$$\mathcal{L}_E\left(\sum_i \xi^i \alpha_i\right) - \mathcal{L}_T\left(\sum_i \xi^i \beta_i\right) = -\frac{1}{2} \sum_i \xi^i e_i(\sum_j C_j e^j).$$

On the left hand side, we can apply the Leibniz rule. On the right hand side, we use the fact that e^i and e_i are dual bases to find that this is equivalent to:

$$\sum_i \left(\xi^i \mathcal{L}_E(\alpha_i) + \mathcal{L}_E(\xi^i) \alpha_i\right) - \sum_i \left(\xi^i \mathcal{L}_T(\beta_i) + \mathcal{L}_T(\xi^i) \beta_i\right) = -\frac{1}{2} \sum_i \xi^i C_i.$$

Again, this should hold for any section ξ, so for any set of coordinate functions $\{\xi^i\}_i$, from which we find Equation (3) holds if and only if for all $i \in \{1, \ldots, k\}$, $\xi_i \in C^\infty(U)$:

$$\xi^i \mathcal{L}_E(\alpha_i) + \mathcal{L}_E(\xi^i)\alpha_i - \xi^i \mathcal{L}_T(\beta_i) + \mathcal{L}_T(\xi^i)\beta_i = -\frac{1}{2}\xi^i C_i.$$

What we should prove now is the following. Assuming that the usual homotopy equation holds for all connections, consider some section $\xi \in \Gamma(A^*)$. We have to show that Equation (3) holds for ξ, by choosing a connection matrix that reduces Equation (4) to the equation we want to prove.

In the other direction, assuming that Equation (3) holds for all sections $\xi \in \Gamma(A^*)$, consider a connection ∇ on A. Then we have to prove that by the right choice of ξ, Equation (3) induces the homotopy equation w.r.t. ∇.

Comparing the local expressions of either equation, we see that they already look quite alike. The only terms that are left to show correspond, are $\mathcal{L}_E(\xi^i)\alpha_i$ on one hand, and $\alpha_i \sum_j \omega_i^j(E)$ on the other hand. By writing both the Lie derivative and the 1-forms in local coordinates on M, it follows that the free choice of ξ^i and ω_i^j allows one to always find a correspondence described above, proving the lemma. □

Lemma 2.46 will be useful in Section 2.7.

We now discuss a few examples of A-homotopies, by considering cases where $a_\epsilon(t)$ and $b_\epsilon(t)$ described above are co-linear in some sense. In that case, the torsion T_∇ will be zero, and the homotopy equation (Equation (2)) is simpler.

Example 2.47. Let $a(t)$ be an A-path above $\gamma : [0,1] \to M$. Consider now any reparametrization function of the form:

$$\chi : [0,1] \times [0,1] \to [0,1] \text{ such that } \chi(\epsilon, 0) = 0,\ \chi(\epsilon, 1) = 1.$$

Then $\gamma(\chi(\epsilon, t))$ gives a homotopy from the path $\gamma(\chi(0,t))$ to the path $\gamma(\chi(1,t))$. We can now define a variation of A-paths above this homotopy, denoted $a_\epsilon(t)$. This should satisfy

$$\rho \circ a_\epsilon(t) = \frac{d\gamma(\chi(\epsilon, t))}{dt} = \frac{\partial \chi(\epsilon, t)}{\partial t}\frac{d\gamma(t)}{dt}.$$

We see that the choice

$$a_\epsilon(t) = \frac{\partial \chi(\epsilon, t)}{\partial t} a(\chi(\epsilon, t))$$

satisfies this relation. Furthermore, we can now quickly find a solution of Equation (2) by choosing

$$b_\epsilon(t) = \frac{\partial \chi(\epsilon, t)}{\partial \epsilon} a(\chi(\epsilon, t)).$$

We see that, since partial derivatives of $\chi(\epsilon, t)$ commute, both sides of Equation (2) are zero. We conclude that $a_\epsilon(t)$ is an A-path homotopy if and only if $b_\epsilon(1) = 0$.

An important consequence of this example is the reparametrization of A-paths. Consider a reparametrization function $\tau : [0,1] \to [0,1]$, by which we mean an isomorphism such that $\tau(0) = 0$ and $\tau(1) = 1$. Assume that τ has derivatives vanishing at the endpoints. Then the reparametrization of an A-path $a : [0,1] \to A$ is $a^\tau(t) := \tau'(t)a(\tau(t))$. This reparametrization procedure will be important to concatenate A-paths in a smooth manner later on, for which it is important that the original path is homotopic to the reparametrized path.

Corollary 2.48. *With the above notation, $a(t)$ and $a^\tau(t)$ are A-path homotopic.*

Proof. This is a specific instance of the example above, where $\chi(\epsilon, t) = (1-\epsilon)t + \epsilon\,\tau(t)$. Choosing $a_\epsilon(t)$ and $b(\epsilon, t)$ as in the example above, we find the variation of A-paths given by

$$a_\epsilon(t) = (1 - \epsilon + \epsilon\,\tau'(t))\,a((1-\epsilon)t + \epsilon\,\tau(t)),$$

$$b_\epsilon(t) = (-t + \tau(t))a((1-\epsilon)t + \epsilon\tau(t)).$$

Indeed we have $a_0(t) = a(t)$ and $a_1(t) = a^\tau(t)$, and this is a solution of Equation (2) satisfying $b_\epsilon(1) = 0$. This shows that $a_\epsilon(t)$ is an A-path homotopy from $a(t)$ to $a^\tau(t)$. □

2.6 Lie algebroid integrability

As we have already mentioned, not all algebroids are integrable. We refer to [CF11] for some examples of non-integrable Lie algebroids. Furthermore, [CF11] gives the first extensive discussion on the integrability of general Lie algebroids, introducing the monodromy groups that control integrability. We give here an overview of the steps in this construction, leaving out the proofs. Specifically, we construct the Weinstein groupoid, which is the candidate for the groupoid integrating a given algebroid. This candidate groupoid can always be constructed, but the integrability problem considers whether this groupoid has a smooth structure. We state the theorem that determines when this smooth structure exists, which is proven in Chapter 4 of [CF11].

In order to integrate Lie algebroids, it is natural to first consider the case of integration of Lie algebras, and then to try and extend this procedure. In the case of Lie algebras, we have the famous third Lie theorem:

Theorem 2.49. *Every finite-dimensional Lie algebra is isomorphic to the Lie algebra of a Lie group.*

There are several proofs, and in our case it is interesting to consider the proof by Duistermaat and Kolk [DK00], also discussed in [CF11]. This proof explicitly constructs the integrating group G. First we observe the following about the Lie algebra $\mathfrak{g}$ of a Lie group G. Consider the space of paths in G starting at the identity:

$$P(G) := \{\, g : [0,1] \to G \mid g \text{ is smooth, } g(0) = e \,\}.$$

Define an equivalence relation on this set by $g \sim g'$ if they are homotopic paths (where the homotopy preserves the endpoints). Furthermore $P(G)$ admits a group structure; multiplication is given by path concatenation, where we first apply some smoothening procedure on the endpoints of paths (which ensure the result is again smooth). Now it is true that

$$\widetilde{G} := P(G)/\sim$$

is the simply-connected Lie group integrating $\mathfrak{g}$.

However, one can recover this group solely from information in $\mathfrak{g}$. Consider the space of paths in $\mathfrak{g}$:

$$P(\mathfrak{g}) = \{\, g : [0,1] \to \mathfrak{g} \mid g \text{ is smooth} \,\}.$$

Any path in $P(G)$ induces also an element in $P(\mathfrak{g})$, and this allows us to give $P(\mathfrak{g})$ an equivalence relation as well, even independently from $P(G)$ (again see [CF11] for the details). Then it follows that $G(\mathfrak{g}) := P(\mathfrak{g})/\sim$ is a topological group which, if $\mathfrak{g}$ is integrable, is precisely the simply-connected Lie group integrating $\mathfrak{g}$. The point is that, as groups,

$$\widetilde{G} = \frac{\{G\text{-paths}\}}{G\text{-path homotopy}} \cong \frac{\{\mathfrak{g}\text{-paths}\}}{\mathfrak{g}\text{-path homotopy}} = G(\mathfrak{g}),$$

and it can be shown that the latter group always has a smooth structure. This is the outline of the proof that every Lie algebra is integrable to a Lie group $G(\mathfrak{g})$. In the case of Lie algebroids, we can follow an analogous procedure to construct a groupoid associated to an algebroid. However, it is not true that this groupoid will always have a smooth structure.

To extend the procedure to algebroids, again we first consider the case where $\mathcal{G}$ is a groupoid with associated algebroid A. A $\mathcal{G}$-path is a map $g : [0,1] \to \mathcal{G}$ such that for a certain point $x \in M$, $g(0) = 1_x$ and for all $t \in [0,1]$, $s(g(t)) = x$. In words, it is a path that starts at a unit and stays inside one source fiber. Denote by $P(\mathcal{G})$ the space of $\mathcal{G}$-paths. Denote by $\sim$ the equivalence relation on $P(\mathcal{G})$ that is induced by smooth homotopies with fixed endpoints. It holds that

$$\widetilde{\mathcal{G}} := P(\mathcal{G})/\sim$$

is a Lie groupoid with 1-connected source fibers integrating A. The structure maps of this groupoid follow from those of $\mathcal{G}$: the source of a class of $\mathcal{G}$-paths $[g(t)]$ is the source of the arrow $g(0)$, and the target is the target of the arrow $g(1)$. The multiplication in this groupoid is induced by path concatenation: if $g_1(t)$ and $g_2(t)$ are composable, their concatenation is

$$g_2 \cdot g_1(t) = \begin{cases} g_1(t) & 0 \le t \le \frac{1}{2} \\ g_2(t) & \frac{1}{2} \le t \le 1 \end{cases}.$$

This concatenation is not necessarily smooth, but it holds that any $\mathcal{G}$-path is homotopic to one with vanishing derivatives with the endpoints, and choosing such representatives does give a smooth concatenation. Furthermore, this groupoid always has a smooth structure.

Similarly to the case of Lie algebras, we now consider how this descends to paths in the algebroid. It follows (with some computations, see [CF11]) that $\mathcal{G}$-paths 'descend' to A-paths, and $\mathcal{G}$-path homotopies 'descend' to A-path homotopies.

Furthermore, for A-paths, we find a concatenation procedure as well. Let $a_1(t)$, $a_2(t)$ A-paths such that $\pi(a_1(1)) = \pi(a_2(0))$ (with $\pi : A \to M$). We call these paths composable, however it is not true that $a_1(1) = a_2(0)$. In order to make their composition smooth, we consider a reparametrization $\tau : [0,1] \to [0,1]$ with derivatives vanishing at the endpoints. Define then $a_i^\tau(t) = \tau'(t)a_i(\tau(t))$. By Corollary 2.48, the paths a_i^τ and a_i are A-homotopic, and we can choose these reparametrized representatives to define:

$$a_2 \cdot a_1(t) = a_2^\tau \cdot a_1^\tau(t) = \begin{cases} 2a_1^\tau(2t) & 0 \leq t \leq \frac{1}{2} \\ 2a_2^\tau(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}. \tag{5}$$

We form the following quotient:

$$\mathcal{G}(A) := \frac{\{A\text{-paths}\}}{A\text{-path homotopy}},$$

and with this concatenation this has the structure of a groupoid. The other structure maps are clear. Now it follows that, as topological groupoids, we have again

$$\widetilde{\mathcal{G}} = \frac{\{\mathcal{G}\text{-paths}\}}{\mathcal{G}\text{-path homotopy}} \cong \frac{\{A\text{-paths}\}}{A\text{-path homotopy}} = \mathcal{G}(A),$$

and $\mathcal{G}(A)$ is *the* candidate for the source-connected groupoid integrating A. This groupoid is called the **Weinstein groupoid**. We see that the problem of integrating A reduces to finding a smooth structure on the Weinstein groupoid. If it admits a smooth structure, then it is unique in the following sense (Theorem 2.16 from [CF11]):

Theorem 2.50. *If an algebroid A is integrable, there exists a unique groupoid integrating A with 1-connected source fibers.*

It follows that this groupoid has to be the Weinstein groupoid.

We will now discuss the condition for integrability of A. This condition is expressed using monodromy groups. To define monodromy groups, we consider first the case of a Lie groupoid $\mathcal{G}$ and its Lie algebroid A. Assume $\mathcal{G}$ has 1-connected source fibers, so by the previous theorem we may assume $\mathcal{G} = \mathcal{G}(A)$. Consider now at $x \in M$ the identity componenent of the isotropy group, $\mathcal{G}_x^0(A)$. This is a connected Lie group integrating the isotropy Lie algebra $\mathfrak{g}_x(A)$. On the other hand, we can construct $G_x := G(\mathfrak{g}_x(A))$, the simply-connected integration of this Lie algebra (by the usual Lie group-Lie algebra correspondence). There are now two Lie groups integrating the same Lie algebra. Then there exists a discrete subgroup $\widetilde{\mathcal{N}}_x \subset G_x$ such that

$$\mathcal{G}_x^0(A) \cong G_x/\widetilde{\mathcal{N}}_x.$$

This group, $\widetilde{\mathcal{N}}_x$, is called the **monodromy group** at x.

Now we wish to define the monodromy groups in the case where $\mathcal{G}(A)$ is not known to have a smooth structure. In that case, $\mathcal{G}(A)^0_x$ is not necessarily a Lie group, so the above procedure does not work. We still consider the isotropy groups of $\mathcal{G}(A)$. We can describe the isotropy group at x by

$$\mathcal{G}(A)_x = \frac{\{\, A\text{-paths covering a loop in } \mathcal{O}_x \text{ based at } x \,\}}{A\text{-path homotopies}}.$$

The identity component $\mathcal{G}(A)^0_x$ of this group is found by considering only A-loops above contractible loops in the orbit $\mathcal{O}_x$ based at x.

We can now construct two short exact sequences. The first one uses a map $\mathcal{G}_x(A) \to \pi_1(\mathcal{O}_x)$ sending a class of A-homotopies to the class of its base-path homotopy in the fundamental group $\pi_1(\mathcal{O}_x)$, giving the sequence

$$0 \to \mathcal{G}_x(A)^0 \hookrightarrow \mathcal{G}_x(A) \to \pi_1(\mathcal{O}_x) \to 0.$$

Second, using the anchor map ρ restricted to $A|_{\mathcal{O}_x}$ we construct

$$0 \to \mathfrak{g}_{\mathcal{O}_x} \to A|_{\mathcal{O}_x} \to T\mathcal{O}_x \to 0.$$

From these sequences, it is possible to find a map $\partial_x : \pi_2(\mathcal{O}_x) \to G(\mathfrak{g}_x)$ such that the following sequence is exact:

$$\cdots \to \pi_2(\mathcal{O}_x) \xrightarrow{\partial_x} G(\mathfrak{g}_x) \to \mathcal{G}_x(A) \to \pi_1(\mathcal{O}_x).$$

This map $\partial_x : \pi_2(\mathcal{O}_x) \to G(\mathfrak{g}_x)$ is called the **monodromy morphism** at x, and its image

$$\mathrm{Im}(\partial_x) = \{\, [a] \in \mathcal{G}(\mathfrak{g}_x) \mid a \text{ is } A\text{-homotopic to } 0_x \,\}$$

is the **monodromy group** at x, denoted $\widetilde{\mathcal{N}}_x(A)$. The fact that the image is defined by this set of elements in $G(\mathfrak{g}_x)$ follows from the definition of ∂_x, see [CF11]. It then follows that $\mathcal{G}_x(A)^0 \cong G(\mathfrak{g}_x)/\widetilde{\mathcal{N}}_x(A)$. From intersecting $\widetilde{\mathcal{N}}_x(A)$ with the identity component of the center of $G(\mathfrak{g}_x)$, we find a smaller group $\mathcal{N}_x(A)$, and with this group we have the following result on integrability of Lie algebroids (Theorem 4.1 in [CF11]).

Theorem 2.51. *A Lie algebroid A is integrable $\iff$ the Weinstein groupoid $\mathcal{G}(A)$ has a smooth structure $\iff$ all $\mathcal{N}_x(A)$ are locally uniformly discrete.*

We see that the image of this monodromy map is the crucial ingredient in determining whether an algebroid is integrable. In the case that A is a transitive algebroid, the condition for integrability reduces to:

Theorem 2.52. *A transitive Lie algebroid A is integrable if and only if all Im(∂_x) is discrete.*

Furthermore, if A is transitive and integrable, the Weinstein groupoid $\mathcal{G}(A)$ is also transitive (as the orbits of A and $\mathcal{G}(A)$ coincide).

2.7 Actions of groupoids and algebroids

In this section we discuss groupoid and algebroid actions, and the relationship between the two. In Part III, we will consider a free action of a transitive algebroid. For this reason, this section focuses on the transitive case as well.

Let $\mathcal{G}$ be a groupoid over a manifold M.

Definition 2.53. A **(left) $\mathcal{G}$-space** is a surjective submersion $\mathrm{pr} : \mathcal{N} \to M$ together with a map

$$\mathcal{G} \times_M \mathcal{N} = \{ (g, p) \in \mathcal{G} \times \mathcal{N} \mid s(g) = \mathrm{pr}(p) \} \to \mathcal{N}$$

denoted $(g, p) \mapsto gp$ such that

1. $\mathrm{pr}(gp) = t(g)$
2. $g(hp) = (gh)p$
3. $1_{\mathrm{pr}(p)}p = p$.

An **action** of $\mathcal{G}$ on a surjective submersion $pr : \mathcal{N} \to M$ is a map satisfying these properties and such that, additionally, each $g \in \mathcal{G}$ induces a diffeomorphism $\mathcal{N}_{s(g)} \to \mathcal{N}_{t(g)}$.

Definition 2.54. An **action** of a Lie algebroid $A \to M$ on a surjective submersion $\mathrm{pr} : \mathcal{N} \to M$ is a Lie algebra morphism $a : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$ such that

1. $a(f\alpha) = \mathrm{pr}^*(f)a(\alpha), \ \forall f \in C^\infty(M), \alpha \in \Gamma(A)$
2. $d_p\mathrm{pr}(a(\alpha)_p) = \rho(\alpha)_{pr(p)}, \ \forall p \in \mathcal{N}$.

Note that the first condition is equivalent to a being a vector bundle map $a : \mathrm{pr}^*A \to T\mathcal{N}$. The second condition can be summarized as a being compatible with the anchor map. An algebroid action is called **free** if the map a is fiberwise injective. This means that for each $p \in \mathcal{N}$, the following map is injective:

$$a_p : A_{\mathrm{pr}(p)} \to T_p\mathcal{N}.$$

Lemma 2.55. *An action of a groupoid $\mathcal{G} \rightrightarrows M$ on $pr : \mathcal{N} \to M$ induces an algebroid action of $A := \mathcal{L}ie(\mathcal{G})$ on $\mathcal{N}$.*

Proof. Let $p \in \mathcal{N}$, $x = \mathrm{pr}(p)$. Consider right translation by p on the groupoid, arising from the groupoid action:

$$R_p : s^{-1}(x) \to \mathcal{N}, \ g \mapsto gp.$$

Consider the differential of this map, at the unit arrow 1_x:

$$d_{1_x}R_p : T_{1_x}s^{-1}(x) = A_x \to T_p\mathcal{N}.$$

This defines an algebroid action $\mathscr{a} : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$ where $\alpha \in \Gamma(A)$ is mapped to the vector field $X \in \mathfrak{X}(\mathcal{N})$, such that for $p \in \mathcal{N}_x$, $X_p = d_{1_x} R_p(\alpha(x))$.

Since this is defined by the differential of a smooth map, it is automatically a vector bundle morphism (satisfying condition (1)). To see that $\mathscr{a}$ is compatible with the anchor map, note that by the first condition on groupoid actions, we have $\mathrm{pr} \circ R_p = t$. Then we find the second condition for algebroid actions:

$$d_p\mathrm{pr}(\mathscr{a}(\alpha)_p) = d_p\mathrm{pr}(d_{1_x} R_p(\alpha(x))) = d_{1_x}(\mathrm{pr} \circ R_p)(\alpha(x)) = d_{1_x} t(\alpha(x)) = \rho(\alpha(x)).$$

□

Conversely, starting with an algebroid action, we can construct an action of its Weinstein groupoid. This groupoid action can only be smooth when the algebroid is integrable. But even without integrability, the induced action of the Weinstein groupoid will always be a well-defined set-theoretical action. We will use this set-theoretical action later on. In the integrable case, we have the following lemma, which holds only for complete actions. In the proof we will explain what this completeness condition entails.

Lemma 2.56. *For an integrable algebroid A, with $\mathcal{G}$ the integration with 1-connected source fibers, and a complete (in the sense defined below) algebroid action $\mathscr{a} : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$, there exists a unique groupoid action of $\mathcal{G}$ on $\mathcal{N}$ that induces $\mathscr{a}$.*

Proof. First note that we may assume $\mathcal{G} = \mathcal{G}(A)$, the Weinstein groupoid, by Theorem 2.50. To define the action of $\mathcal{G}$, we consider $g \in \mathcal{G}$, with representative A-path $g = [a]$. Denote the base-path of $a : [0,1] \to A$ by $\gamma : [0,1] \to M$. Note $\gamma(0) = s(g)$ and $\gamma(1) = t(g)$ (which follows from the construction of the Weinstein groupoid). We now look for an appropriate isomorphism $\mathcal{N}_{\gamma(0)} \to \mathcal{N}_{\gamma(1)}$ induced by $[a]$. This map is defined with the following procedure, that we also call "A-parallel transport":

$$\begin{aligned} &Act_g : \mathcal{N}_{\gamma(0)} \to \mathcal{N}_{\gamma(1)}, \\ &Act_g(u_0) = u(1), \text{ where } u(t) \text{ is the solution of} \\ &\qquad \begin{cases} \frac{d}{dt}u(t) = \mathscr{a}_{u(t)}(a(t)), \\ u(0) = u_0. \end{cases} \end{aligned}$$

We will also denote this action by $[a] \cdot u_0 = u(1)$. We call an algebroid action complete if this transporting procedure is defined, i.e., if the solution $u(t)$ exists for $t \in [0,1]$. Under this assumption, this defines an action of $\mathcal{G}$ on $\mathcal{N}$. It satisfies the first condition of groupoid actions as

$$\mathrm{pr}([a] \cdot u_0) = \mathrm{pr}(u_1) = \gamma(1) = t(g).$$

The second condition holds because transporting along a concatenation of paths is the same as first transporting along the first path, and then along the second path. For the third condition, note that a unit $1_{\mathrm{pr}(p)}$ acts by transporting along a constant path, which induces the identity map. Finally, each map Act_g is an isomorphism with the action of $[a^{-1}]$ as its inverse. Smoothness of these actions follows by a general theorem on the solutions of such ODE's, see Appendix 1 in [KN96].

It remains to show that this action is well-defined, i.e., A-homotopic paths induce the same A-parallel transport. So, assume a_0 and a_1 are A-homotopic paths, connected by the A-homotopy $a_\epsilon(t)$. We use the characterisation of A-homotopies of Lemma 2.46. Then for the solution $b_\epsilon(t)$ of the homotopy equation and for any $\xi \in \Gamma(A^*)$, the following holds:

$$\frac{d}{d\epsilon}(\xi(a)) - \frac{d}{dt}(\xi(b)) = d_A\xi(a, b).$$

Now consider any one-form $\theta \in \Omega^1(\mathcal{N})$, and consider the pullback form $\not{a}^*\theta := \theta \circ \not{a}$. This is a section in $\Gamma(A^*)$, so we know that the following holds:

$$\frac{d}{d\epsilon}(\not{a}^*\theta(a)) - \frac{d}{dt}(\not{a}^*\theta(b)) = d_A\not{a}^*\theta(a, b).$$

We can rewrite this identity to

$$\frac{d}{d\epsilon}(\theta(\not{a}(a))) - \frac{d}{dt}(\theta(\not{a}(b))) = d\theta(\not{a}(a), \not{a}(b)).$$

Now note that we have by construction of $u(t)$: $\frac{du(t)}{dt} = \not{a}(a(t))$. Finally, the following identity holds in general for one-forms θ and paths $u(t)$ on $\mathcal{N}$:

$$\frac{d}{d\epsilon}\left(\theta\left(\frac{du(t)}{dt}\right)\right) - \frac{d}{dt}\left(\theta\left(\frac{du(t)}{d\epsilon}\right)\right) = d\theta\left(\frac{du(t)}{dt}, \frac{du(t)}{d\epsilon}\right).$$

From this we conclude that we must have $\not{a}(b) = \frac{du(t)}{d\epsilon}$. Since $b_\epsilon(1) = 0$, we find that $\frac{du_\epsilon}{d\epsilon}(1) = 0$, so $u_\epsilon(1)$ is constant. Since $[a_\epsilon]$ acts by mapping to $u_\epsilon(1)$, we see all $a_\epsilon(t)$ induce the same action on $\mathcal{N}$. □

As an extension of this lemma, we find another characterisation of A-homotopies if additionally the algebroid action is free.

Lemma 2.57. *Let $A \to M$ an algebroid with a free, complete algebroid action $\not{a} : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$. Then a variation $a_\epsilon(t)$ between two A-paths is an A-homotopy if and only if their induced A-parallel transport maps on $\mathcal{N}$ agree.*

Proof. Consider the proof of the previous lemma again. We have already proven that A-homotopic paths induce the same A-parallel transport. Assume now that we have a variation of A-paths $a_\epsilon(t)$ such that for all ϵ, $a_\epsilon(t)$ induce the same parallel transport. Denote by $b_\epsilon(t)$ the solution to the homotopy equation (Equation (2)). By Lemma 2.46 combined with the proof above, we see that the following holds:

$$\mathcal{a}(b_\epsilon(t)) = \frac{du_\epsilon(t)}{dt}.$$

Since all parallel transports agree, all $u_\epsilon(1)$ are equal. Since $\mathcal{a}$ is injective, we see that $b_\epsilon(1) = 0$. Indeed, the variation $a_\epsilon(t)$ is an A-homotopy. □

The groupoid action defined in Lemma 2.56 is not necessarily free. The following corollary gives an instance where this does hold. We assume here already the result of Proposition 2.61, where we will show that the image of an algebroid action $\mathrm{Im}(\mathcal{a}) \subset T\mathcal{N}$ gives a regular foliation of $\mathcal{N}$. This implies that points in the same leaf of this foliation can be connected via the action of some A-path.

Corollary 2.58. *Let $A \to M$ an algebroid and $\mathcal{N} \to M$ a surjective submersion. Consider a free, complete algebroid action $\mathcal{a} : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$ and the induced foliation of $\mathcal{N}$. If the leaves of this foliation are simply connected, then the groupoid action of the Weinstein groupoid defined in Lemma 2.56 is also free.*

Proof. Consider two arrows $g, h \in \mathcal{G}(A)$ that act identically. Take two representative A-paths $a_0(t)$ and $a_1(t)$ of g and h respectively. Let $\gamma_0(t)$ be the basepath of $a_0(t)$. For any point v in the fiber $\mathcal{N}_{\gamma_0(t)}$, the A-paths $a_0(t)$ and $a_1(t)$ induce by parallel transport the paths $u_0(t)$ and $u_1(t)$ in $\mathcal{N}$, with initial point v and final point $[a_0] \cdot v = [a_1] \cdot v$. These paths lie inside a single leaf of the foliation on $\mathcal{N}$ induced by $\mathcal{a}$. Since this leaf is simply-connected, we can find a leafwise homotopy between $u_0(t)$ and $u_1(t)$ in $\mathcal{N}$. From this leafwise homotopy we can find a variation of A-paths between $a_0(t)$ and $a_1(t)$. By the previous lemma, this variation is an A-homotopy, as $a_0(t)$ and $a_1(t)$ act identically on $\mathcal{N}$. □

Restricting our attention to actions of transitive groupoids, we find that these are equivalent to certain Lie group actions on manifolds. Recall that by Proposition 2.18, any transitive Lie groupoid $\mathcal{G}$ is isomorphic to a gauge groupoid induced by a principal bundle. The structure group of this principal bundle is the isotropy group of $\mathcal{G}$, denoted G, which is independent of basepoint. Groupoid actions of $\mathcal{G}$ are characterised by group actions of G by the following lemma.

Lemma 2.59. *If $\mathcal{G}$ is transitive, with isotropy groups G, then there is a 1-1 correspondence between manifolds $pr : \mathcal{N} \to M$ with a $\mathcal{G}$-action, and manifolds F with a G-action.*

Proof. Since $\mathcal{G}$ is transitive, it is a gauge groupoid induced by the principal G-bundle $s^{-1}(x) \to M$, where $G = \mathcal{G}_x$, independent of $x \in M$.

First, consider any surjective submersion $\mathrm{pr} : \mathcal{N} \to M$ with a $\mathcal{G}$-action. Let $x \in M$, and define $F = \mathcal{N}_x$. Since G is the isotropy group $\mathcal{G}_x$, it consists of arrows $g : x \to x$. Since we have a groupoid action, each $g \in G_x$ induces an isomorphism $\mathcal{N}_x \to \mathcal{N}_x$. This is the G-action on F.

Consider now a manifold F with a G-action. The desired manifold $\mathcal{N}$ is defined by

$$\mathcal{N} = P \times_G F, \text{ where } (p, f) \sim (pg, fg) \text{ for } p \in P,\ f \in F,\ g \in G.$$

Recall that the projection of P onto M is the target map; the projection $\mathrm{pr} : \mathcal{N} \to M$ is defined as $\mathrm{pr}([(p, f)]) = t(p)$.

Note that each fiber of $\mathcal{N}$ is isomorphic to F. To see this, fix for any $y \in M$ a point in its fiber $p_y \in P_y$ (this will be some arrow from x to y). Any other point in the fiber P_y can now be written as $p' = p_y g$ for some $g \in G$. Then the fiber of $\mathcal{N}$ above y can be rewritten as

$$\mathcal{N}_y = \{\, [p_y g, f] \mid g \in G,\ f \in F \,\}.$$

Note that $[p_y g, f] \sim [p_y, fg^{-1}]$, and by the G-action on F, fg^{-1} is equal to some $f' \in F$. We find

$$N_y = \{\, [p_y, f] \mid f \in F \,\} \cong F,$$

indeed all fibers of $\mathcal{N}$ are isomorphic to F.

We now define the $\mathcal{G}$-action on $\mathcal{N}$. Let $g \in \mathcal{G}$, denoted by $g = [(p, q)]$ where $p \in P_y$ and $q \in P_z$ (so g is an arrow from y to z). Using the fixed point $p_y \in P_y$, there exists some $h \in G$ such that $g = [(p_y, qh^{-1})]$. Using the fixed point $p_z \in P_z$, there exists also some $k \in G$ such that $qh^{-1} = p_z k$. Now we define the action of g, which should be a mapping $\mathcal{N}_y \to \mathcal{N}_z$; since all fibers are isomorphic to F, we should define an isomorphism $g : F \to F$. Let $f \in F \cong \mathcal{N}_y$, and define $g \cdot f = fk$.

Remark that this definition could also be written more explicitly in the following way. Let $g \in \mathcal{G}$ as above, let $[(p_y, f)]$ be an element of $\mathcal{N}_y$ as above. The action can be written explicitly as $g \cdot [(p_y, f)] = [(p_z, fk)]$ where $k \in G$ such that $p_z k = qh^{-1}$. From this description it is clear that we have defined a map from $\mathcal{N}_y$ to $\mathcal{N}_z$.

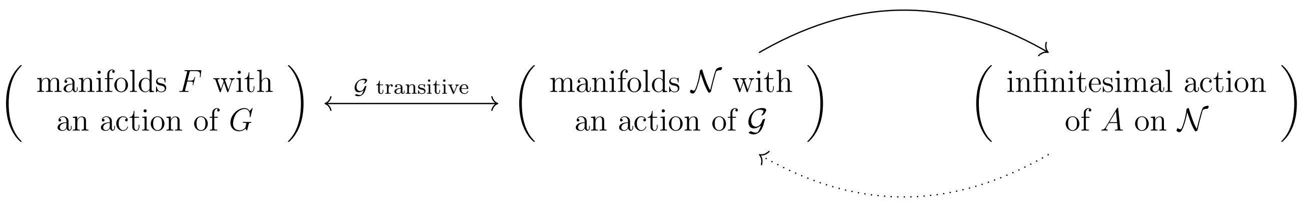
We now show that this satisfies the groupoid action axioms. The first and third requirements are immediate by the construction. Associativity can be proven by using the fixed element p_i in each fiber to explicitly write down the resulting action; but from our initial description it should be clear that the groupoid action is induced entirely from the group action of G on F, from which it also inherits these required properties. Finally, this also shows that each $g \in \mathcal{G}$ induces an isomorphism of fibers (as the group action on F is already an isomorphism).

It remains to show that this correspondence is 1-1. It is clear that starting with a manifold F, then defining $\mathcal{N}$ as above, and taking the fiber $\mathcal{N}_x$ with induced action returns the original manifold F with its G-action. We focus on the other direction; starting with $\mathrm{pr} : \mathcal{N} \to M$ with a $\mathcal{G}$-action, we find $F = \mathcal{N}_x$ with a G-action, and construct $\mathcal{N}' = P \times_G \mathcal{N}_x$. Then the following mappings give an isomorphism between $\mathcal{N}$ and $\mathcal{N}'$:

$$\mathcal{N}' \to \mathcal{N},\ [(g, p)] \mapsto gp,$$
$$\mathcal{N} \to \mathcal{N}',\ p \mapsto [(h, h^{-1}p)] \text{ where } h \in \mathcal{G} \text{ is an arrow from } x \text{ to } \mathrm{pr}(p).$$

After fixing a point in every fiber of P, it is a tedious but straightforward check that these maps are well-defined, commute with the $\mathcal{G}$-action and are inverses. This proves that the correspondence given above is indeed 1-1. $\square$

These last three lemmas can be summarized in the diagram below. A groupoid action of $\mathcal{G}$ always induces an action of its algebroid A. The reverse direction, pictured as a dotted arrow, holds if A is integrable with $\mathcal{G}$ the integration with 1-connected source fibers. Finally if $\mathcal{G}$ is transitive, groupoid actions are equivalent to group actions of its isotropy groups.



Finally from this diagram and the proof of Lemma 2.59 we find the following strong result.

Corollary 2.60. *Let A be an integrable transitive Lie algebroid and $pr : \mathcal{N} \to M$ a surjective submersion. Let $\dashv : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$ be a free and complete algebroid action. Then there exists an isomorphism*

$$\mathcal{N} \cong P \times_G \mathcal{N}_x$$

where G is the isotropy group $\mathcal{G}_x(A)$ of the Weinstein groupoid of A and $P = s^{-1}(x)$ a principal G-bundle.

Proof. By the dotted arrow in the diagram, or equivalently by Lemma 2.56, there exists a groupoid action of the Weinstein groupoid $\mathcal{G}(A)$ on $\text{pr} : \mathcal{N} \to M$ that induces $\dashv$. Since A is transitive, the Weinstein groupoid is transitive as well. Then the groupoid action is equivalent to a group action of its isotropy group $G := \mathcal{G}_x(A)$ on the fiber $\mathcal{N}_x$ by Lemma 2.59. This correspondence is 1-1, and the proof of this lemma gives the desired isomorphism

$$\mathcal{N} \cong P \times_G \mathcal{N}_x,$$

where $P = s^{-1}(x)$ is a principal G-bundle. $\square$

2.8 Free actions and integrability

In this section we show that the existence of a free action of a transitive algebroid forces it to be integrable. To prove this result, we first introduce the foliation algebroid induced by a free algebroid action. Then we consider the monodromy groupoid, which integrates this foliation algebroid. Using this monodromy groupoid we show that the original algebroid is integrable as well.

Consider a free algebroid action $\dashv : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$. The image of this action lies in $T\mathcal{N}$, and gives a foliation on $\mathcal{N}$, called the orbit foliation of the action.

Proposition 2.61. *If the action* $\mathscr{a} : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$ *is free, the orbit foliation* $\mathcal{F}_{\mathscr{a}} = Im(\mathscr{a}) \subset T\mathcal{N}$ *is a regular foliation.*

Proof. Clearly $\mathrm{Im}(\mathscr{a}) \subset T\mathcal{N}$ defines a distribution, which is smooth as $\mathscr{a}$ is smooth. We prove that this distribution is involutive, i.e., closed under the Lie bracket. In order to show this, consider $\mathscr{a} : \mathrm{pr}^*A \to T\mathcal{N}$ as a vector bundle map (of vector bundles over $\mathcal{N}$), which is by assumption injective. So, restricting the codomain to $\mathcal{F}_{\mathscr{a}}$ we find an isomorphism

$$\mathscr{a} : \mathrm{pr}^*A \xrightarrow{\sim} \mathcal{F}_{\mathscr{a}}.$$

This also induces an isomorphism on the space of sections

$$\mathscr{a} : \Gamma(\mathrm{pr}^*A) \xrightarrow{\sim} \Gamma(\mathcal{F}_{\mathscr{a}}).$$

Any section $\alpha \in \Gamma(A)$ induces a section $\mathrm{pr}^*\alpha := \alpha \circ \mathrm{pr} \in \Gamma(\mathrm{pr}^*A)$. For just these sections it is easy to prove they are closed under the Lie bracket, but these do not give all sections in $\Gamma(\mathrm{pr}^*A)$. An arbitrary section $\alpha \in \Gamma(\mathrm{pr}^*A)$ can be written as

$$\alpha = \sum_{i=1}^{k} f_i \mathrm{pr}^*(\alpha_i), \;\; f_i \in C^\infty(\mathcal{N}), \;\; \alpha_i \in \Gamma(A).$$

A section β of $\mathcal{F}_{\mathscr{a}}$ is then by the isomorphism above generally written as

$$\beta = \sum_i f_i \mathscr{a}(\alpha_i), \;\; f_i \in C^\infty(\mathcal{N}), \;\; \alpha_i \in \Gamma(A).$$

where $\mathscr{a}$ is interpreted again as a map $\mathscr{a} : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$.

Now using this general description of sections, we can check that $\mathcal{F}_{\mathscr{a}}$ is involutive. Using the Leibniz rule for Lie brackets of vector fields, we compute the Lie bracket of two general sections of $\mathcal{F}_{\mathscr{a}}$:

$$\begin{aligned}
\left[\sum_i f_i \mathscr{a}(\alpha_i), \sum_j g_j \mathscr{a}(\beta_j)\right] &= \sum_{i,j} [f_i \mathscr{a}(\alpha_i), g_j \mathscr{a}(\beta_j)] \\
&= \sum_{i,j} \big(g_j [f_i \mathscr{a}(\alpha_i), \mathscr{a}(\beta_j)] + \mathcal{L}_{f_i \mathscr{a}(\alpha_i)}(g_j) \mathscr{a}(\beta_j)\big) \\
&= \sum_{i,j} \big(f_i g_j [\mathscr{a}(\alpha_i), \mathscr{a}(\beta_j)] + \mathcal{L}_{f_i \mathscr{a}(\alpha_i)}(g_j) \mathscr{a}(\beta_j) - g_j \mathcal{L}_{\mathscr{a}(\beta_j)}(f_i) \mathscr{a}(\alpha_i)\big) \\
&= \sum_{i,j} f_i g_j \mathscr{a}([\alpha_i, \beta_j]) + \sum_{i,j} \mathcal{L}_{f_i \mathscr{a}(\alpha_i)}(g_j) \mathscr{a}(\beta_j) - \sum_{i,j} g_j \mathcal{L}_{\mathscr{a}(\beta_j)}(f_i) \mathscr{a}(\alpha_i).
\end{aligned}$$

In the last line, we used the fact that $\mathscr{a}$ is a Lie algebra morphism. We find that this is again a section of $\mathcal{F}_{\mathscr{a}}$, as it is a linear combination of terms $f_i \mathscr{a}(\alpha_i)$, so indeed the image of $\mathscr{a}$ is an involutive distribution.

Then by Frobenius' theorem, this corresponds to a foliation of $\mathcal{N}$. The leaves of this foliation are determined by their tangent spaces, which in turn are determined by $\mathcal{F}_{a}$: for L a leaf in the foliation, and $p \in L$, we have

$$T_pL = \{\, a(\alpha)(p) \in T_p\mathcal{N} \mid \alpha \in \Gamma(A) \,\}.$$

□

In general, a regular foliation of $\mathcal{N}$ can be seen as a sub-algebroid of the tangent bundle $T\mathcal{N}$. By Frobenius' theorem, a regular foliation is equivalent to an involutive distribution $\mathcal{F} \subset T\mathcal{N}$. Involutivity implies $\mathcal{F}$ is closed under the Lie bracket of vector fields on $T\mathcal{N}$, which allows us to endow $\mathcal{F}$ with a Lie bracket. The anchor map $\rho : \mathcal{F} \to T\mathcal{N}$ is simply the inclusion map. The previous lemma shows that a free algebroid action $a : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$ gives rise to a regular foliation of $\mathcal{N}$, which can be seen as an algebroid $\mathcal{F}_{a} \to \mathcal{N}$.

Consider now any foliation $\mathcal{F}$ of $\mathcal{N}$, viewed as an algebroid, and consider the Weinstein groupoid integrating it. This groupoid is defined by equivalence classes of $\mathcal{F}$-paths. In this case, the Weinstein groupoid can be described by considering leafwise paths on $\mathcal{N}$, up to leafwise homotopies, as the leaves of $\mathcal{N}$ are precisely the orbits of the algebroid $\mathcal{F}$. From this viewpoint, the resulting groupoid is also called the **monodromy groupoid** over $\mathcal{N}$, defined by

$$\mathcal{M}on(\mathcal{N}, \mathcal{F}) = \frac{\text{leafwise paths}}{\text{leafwise homotopy}} \rightrightarrows \mathcal{N}.$$

For a class $[u(t)] \in \mathcal{M}on(\mathcal{N}, \mathcal{F})$ the source and target maps are given by $s([u]) = u(0)$, $t([u]) = u(1)$. The unit map sends p to the class of the constant path at p; the inversion sends $[u(t)]$ to $[u(t)^{-1}]$ and the multiplication is determined by concatenation of paths.

Theorem 2.62. *Let $\mathcal{F} \to \mathcal{N}$ be an algebroid arising from a foliation of $\mathcal{N}$. Then the groupoid $\mathcal{M}on(\mathcal{N}, \mathcal{F})$ is smooth, hence all algebroids of this type are integrable.*

For the proof of this theorem, and a more detailed discussion of integrations of foliation algebroids, we refer to [Phi87]. Combining the last theorem with Proposition 2.61, we find an additional integrability theorem for transitive algebroids with free actions.

Theorem 2.63. *If A is a transitive algebroid with a free and complete algebroid action on a surjective submersion $\mathcal{N} \to M$, then A is integrable.*

Proof. Following Section 2.6, we construct the Weinstein groupoid $\mathcal{G}(A)$ as the candidate for the integration, and we look for a smooth structure on this groupoid.

By Proposition 2.61, we know that the free action induces a foliation on $\mathcal{N}$, which can be integrated to the monodromy groupoid

$$\mathcal{M}on(\mathcal{N}, \mathcal{F}_{a}) \rightrightarrows \mathcal{N},$$

and it is known that this is a smooth groupoid.

On the other hand, the algebroid action induces a groupoid action of $\mathcal{G}(A)$ on $\mathcal{N} \to M$. We do not know whether $\mathcal{G}(A)$ is smooth, but the construction in Lemma 2.56 does always provide a set-theoretical action. With this we can construct the set-theoretical action groupoid

$$\mathcal{G}(A) \times \mathcal{N} \rightrightarrows \mathcal{N}$$

which is smooth if and only if $\mathcal{G}(A)$ is smooth.

We now prove that this action groupoid is isomorphic to the monodromy groupoid $\mathcal{M}on(\mathcal{N}, \mathcal{F}_{\mathscr{a}})$. This provides a smooth structure on the action groupoid, which in turn implies that the Weinstein groupoid $\mathcal{G}(A)$ is smooth.

To define the isomorphism, recall that an arrow in the action groupoid is a pair (g, p) where $g \in \mathcal{G}(A)$ and $p \in \mathcal{N}$ such that $s(g) = \mathrm{pr}(p)$. The groupoid action of $\mathcal{G}(A)$ on $\mathcal{N} \to M$ was defined in Lemma 2.56 by so-called A-parallel transport, and evaluating this at time $t = 1$. Here, we consider the A-parallel transport itself. Let $a(t)$ be a representative of g, so an A-path above $\gamma : [0, 1] \to M$. Then we denote by $\tau_a(t) : \mathcal{N}_{\gamma(0)} \to \mathcal{N}_{\gamma(t)}$ the path on $\mathcal{N}$ found by solving the differential equation

$$\tau_a(t)(u_0) = u(t), \text{ where } u(t) \text{ is the solution of}$$
$$\begin{cases} \frac{d}{dt}u(t) = \mathscr{a}_{u(t)}(a(t)) \\ u(0) = u_0. \end{cases}$$

Now define the map

$$\Phi : \mathcal{G}(A) \times \mathcal{N} \to \mathcal{M}on(\mathcal{N}, \mathcal{F}_{\mathscr{a}})$$
$$\Phi([a], p) = [\tau_a(t)(p)].$$

By construction this map commutes with all structure maps. Note that it covers the map $\mathrm{Id} : \mathcal{N} \to \mathcal{N}$.

In the other direction, consider a class $[u(t)] \in \mathcal{M}on(\mathcal{N}, \mathcal{F}_{\mathscr{a}})$. Thus $u(t)$ is a path in $\mathcal{N}$, such that for all time t, we have

$$\frac{d}{dt}u(t) \in \mathcal{F}_{\mathscr{a}}(u(t)) = \mathrm{Im}(\mathscr{a})_{u(t)}.$$

Now just as in the proof of Proposition 2.61, we can identify $\mathrm{Im}(\mathscr{a})$ with the pullback bundle pr^*A over $\mathcal{N}$. Then at any time t, we have

$$\frac{d}{dt}u(t) \in (\mathrm{pr}^*A)_{u(t)} \cong A_{\mathrm{pr}(u(t))}.$$

Now define the map

$$\Psi : \mathcal{M}on(\mathcal{N}, \mathcal{F}_{\mathscr{a}}) \to \mathcal{G}(A) \times \mathcal{N}$$
$$\Psi([u(t)]) = \left(\left[\frac{d}{dt}u(t) \right], u(0) \right).$$

By the identification that we made above, this maps indeed into $\mathcal{G}(A) \times \mathcal{N}$, and again it is an easy check to see it commutes with the structure maps.

We still have to show these maps are well-defined. First note that Φ and Ψ are inverses to eachother, which is easily checked. These maps represent the motto

"$\mathcal{G}(A)$ gives speeds of paths in $\mathcal{N}$, and $\mathcal{N}$ has the initial points.
Then $\mathcal{M}on(\mathcal{N}, \mathcal{F}_{a})$ contains the entire paths in $\mathcal{N}$."

The algebroid action ensures that on both sides, we find paths that stay inside the leaves of $\mathcal{F}_{a}$. Showing that Ψ and Φ are well-defined now boils down to showing that leafwise homotopic paths on $\mathcal{N}$ correspond to A-homotopic paths via a. We see that this is basically the content of Lemma 2.57, which requires the assumption that the algebroid action is free and complete.

Now that we have shown the isomorphism

$$\mathcal{M}on(\mathcal{N}, \mathcal{F}_{a}) \cong \mathcal{G}(A) \times \mathcal{N},$$

we see that the smooth structure on the monodromy groupoid induces a smooth structure on the Weinstein groupoid $\mathcal{G}(A)$, and indeed $A \to M$ is an integrable algebroid. □

3 The general linear groupoid and representations

In this section we discuss one example of a groupoid, namely the general linear groupoid of a vector bundle. This example is closely related to the theory of groupoid and algebroid representations. We will first define the general linear groupoid. Then we discuss groupoid representations, and show that a representation is the same thing as a groupoid morphism to the general linear groupoid.

We will then define algebroid representations. We want to discuss how algebroid representations are then the same thing as an algebroid morphism to the algebroid of the general linear groupoid. In order to prove this, we first have to compute the algebroid of the general linear groupoid. This section is devoted completely to this example. It is not explicitly used in the rest of the thesis, but it is a nice introduction to the theory in Section 4.

Definition 3.1. Let M be a manifold, and $E \to M$ a vector bundle over M. The **general linear groupoid** of E is the groupoid $GL(E) \rightrightarrows M$, with base space M and arrow space

$$GL(E) = \{ (y, A, x) \mid A : E_x \to E_y \text{ linear isomorphism}, x, y \in M \}$$

with the following structure maps:

- source map $s(y, A, x) = x$,
- target map $t(y, A, x) = y$,
- unit map $u(x) = (x, \mathrm{Id}, x)$ with Id the identity map on E_x,

- inverse map $i(y, A, x) = (x, A^{-1}, y)$,
- composition $m((z, B, y), (y, A, x)) = (z, BA, x)$ with BA composition of maps.

It is straightforward to check that this definition satisfies the groupoid axioms. This groupoid is transitive, as any two fibers admit an isomorphism between them. Furthermore, the isotropy group at $x \in M$ is given by

$$GL(E)_x = \{ (x, A, x) \mid A : E_x \to E_x \text{ is a linear isomorphism} \} \cong Aut(E_x).$$

3.1 Groupoid and algebroid representations

The general linear groupoid can be seen as the standard example of a representation of a groupoid.

Definition 3.2. A **representation of a groupoid** $\mathcal{G} \rightrightarrows M$ is a vector bundle $\pi : F \to M$ together with an action map

$$\mu : \mathcal{G} \times_M F \to F$$

where $\mathcal{G} \times_M F = \{(g, e) \in \mathcal{G} \times F \mid s(g) = \pi(e)\}$, such that the following hold:

(i) $\pi(\mu(g, e)) = t(g)$,

(ii) $\mu(g, \mu(h, e)) = \mu(gh, e)$,

(iii) $\mu(g, -) : F_{s(g)} \to F_{t(g)}$ is a linear isomorphism.

The collection of representations is denoted $\text{Rep}(\mathcal{G})$.

Example 3.3. On the general linear groupoid $GL(E)$, one finds a very straightforward example of a representation by taking the vector bundle $E \to M$ and defining the action map by $\mu((y, A, x), v) = A(v)$ where $v \in E_x$. The axioms of the representation encode precisely the fact that A is a linear isomorphism between fibers.

Since the structure of a groupoid representation and the general linear groupoid are so similar, we can view this as the 'standard example' of a groupoid representation, which motivates the following lemma.

Lemma 3.4. *For a groupoid $\mathcal{G} \rightrightarrows M$, a representation $E \in Rep(\mathcal{G})$ is the same thing as a groupoid morphism $\mathcal{G} \to GL(E)$.*

Proof. Given a representation $E \in \text{Rep}(\mathcal{G})$ with action map μ, we define a groupoid morphism $\phi : \mathcal{G} \to GL(E)$ by $\phi(g) = (t(g), \mu(g, -), s(g))$. From the axioms in Definition 3.2 it follows that this is a morphism to $GL(E)$, commuting with the structure maps. Given a groupoid morphism $\phi : \mathcal{G} \to GL(E)$ we make E into a representation by defining the map $\mu : \mathcal{G} \times_M E \to E$, $\mu(g, v) = A(v)$ where $\phi(g) = (t(g), A, s(g))$. Again the properties of groupoid morphisms make this into an action map. Clearly this is a 1-1 correspondence. □

Example 3.5. For the general linear groupoid $GL(E)$, the identity map $GL(E) \to GL(E)$ induces the standard representation introduced in Example 3.3.

Remark 3.6. A groupoid representation can be seen simply as a groupoid action on a vector bundle. From Lemma 2.55, we know that this induces an algebroid action on the vector bundle as well. On the other hand, we will now introduce algebroid representations. It is not immediately clear how these are related to the infinitesimal versions of groupoid actions that we find from Lemma 2.55. This relationship will be discussed in Section 3.3.

Definition 3.7. Let $A \to M$ be a Lie algebroid. A **representation of a Lie algebroid** A is a vector bundle $E \to M$ with a bilinear map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$, $(\alpha, s) \mapsto \nabla_\alpha(s)$, such that for all $f \in C^\infty(M)$, $\alpha, \beta \in \Gamma(A)$ and $s \in \Gamma(E)$,

(i) $\nabla_{f\alpha}(s) = f\nabla_\alpha(s)$,

(ii) $\nabla_\alpha(fs) = f\nabla_\alpha(s) + \mathcal{L}_{\rho(\alpha)}(f)(s)$,

(iii) $\nabla_{[\alpha,\beta]}(s) = \nabla_\alpha\nabla_\beta(s) - \nabla_\beta\nabla_\alpha(s)$.

A map ∇ satisfying item (i) and (ii) is called an A-connection on $E \to M$. An A-connection satisfying item (iii) is called flat. With this terminology, an algebroid representation is a vector bundle with a flat A-connection. For $A = TM$ we find the usual notion of connections on vector bundles.

The obvious question is now how a groupoid representation might induce a representation of its algebroid. Consider $E \in \mathrm{Rep}(\mathcal{G})$ and $A = \mathcal{L}ie(\mathcal{G})$. Consider a section $\alpha \in \Gamma(A)$, which we can identify with the right-invariant vector field α^R on $\mathcal{G}$. Corresponding to this vector field, there is a flow on $\mathcal{G}$. We define the map $\Phi^\alpha_x(t) : M \times \mathbb{R} \to \mathcal{G}$ by, for $x \in M$, flowing along this vector field starting at the unit 1_x:

$$\Phi^\alpha_x(t) := \phi^t_{\alpha^R}(1_x).$$

Lemma 3.8. *For any t at which the flow of α^R is defined, $\Phi^\alpha_x(t) \in \mathcal{G}$ is an arrow from x to $\phi^t_{\rho(\alpha)}(x)$ (i.e., the flow of α^R covers the flow of $\rho(\alpha)$).*

Proof. It is clear that $\Phi^\alpha_x(t)$ is an arrow in $\mathcal{G}$. We want to compute its source and target. From the definition of α^R, we have an explicit expression for the speed of $\Phi^\alpha_x(t)$:

$$\frac{d}{dt}\Phi^\alpha_x(t) = \alpha^R_{\Phi^\alpha_x(t)} = d_{1_{t(\Phi^\alpha_x(t))}} R_{\Phi^\alpha_x(t)} \left(\alpha_{t(\Phi^\alpha_x(t))}\right).$$

This is a map into $T\mathcal{G}$, giving the speed of $\Phi^\alpha_x(t)$. We want to consider the target map $t : \mathcal{G} \to M$. The differential is a map $dt : T\mathcal{G} \to TM$. Applying this differential to the speed of $\Phi^\alpha_x(t)$ gives, using the chain rule:

$$\begin{aligned} d_{\Phi^\alpha_x(t)} t \left(d_{1_{t(\Phi^\alpha_x(t))}} R_{\Phi^\alpha_x(t)} \left(\alpha_{t(\Phi^\alpha_x(t))}\right)\right) &= d_{1_{t(\Phi^\alpha_x(t))}} \left(t \circ R_{\Phi^\alpha_x(t)}\right) \left(\alpha_{t(\Phi^\alpha_x(t))}\right) \\ &= d_{1_{t(\Phi^\alpha_x(t))}} (t) \left(\alpha_{t(\Phi^\alpha_x(t))}\right) \text{ since } t \circ R_g = t, \\ &= \rho_{t(\Phi^\alpha_x(t))} \left(\alpha_{t(\Phi^\alpha_x(t))}\right) \end{aligned}$$

where the last step follows as the differential dt restricted to A is precisely the anchor map ρ.

Now we have found that the composition $t \circ \Phi_x^\alpha(t)$, giving the target of $\Phi_x^\alpha(t)$, is a map into M with differential $\rho(\alpha)$. This implies indeed that

$$t \circ \Phi_x^\alpha(t) = \phi_{\rho(\alpha)}^t(x).$$

Now for the source of $\Phi_x^\alpha(t)$, we can do a similar computation, applying ds to the speed of $\Phi_x^\alpha(t)$. We use the identification $s = t \circ i$. After applying the chain rule, we use that $t \circ i \circ R_g = s(g)$, which is a constant map for fixed t:

$$\begin{aligned}
d_{\Phi_x^\alpha(t)}(t \circ i)\left(d_{1_{t(\Phi_x^\alpha(t))}} R_{\Phi_x^\alpha(t)}\left(\alpha_{t(\Phi_x^\alpha(t))}\right)\right) &= d_{1_{t(\Phi_x^\alpha(t))}}\left(t \circ i \circ R_{\Phi_x^\alpha(t)}\right)\left(\alpha_{t(\Phi_x^\alpha(t))}\right) \\
&= d_{1_{t(\Phi_x^\alpha(t))}} s(\Phi_x^\alpha(t))\left(\alpha_{t(\Phi_x^\alpha(t))}\right) \text{ since } t \circ R_g = t, \\
&= 0.
\end{aligned}$$

This implies that $s \circ \Phi_x^\alpha(t)$ is constant, and since at $t = 0$, $\Phi_x^\alpha(0) = 1_x$, we have for all t that $s \circ \Phi_x^\alpha(t) = x$ indeed. □

Now consider the inverse arrow $(\Phi_x^\alpha(t))^{-1}$. This is an arrow from $y := \phi_{\rho(\alpha)}^t(x)$ to x, meaning it can act on vectors in E that are in the fiber E_y. Finally note that for any section $s \in \Gamma(E)$, $s(y)$ lies in this fiber, and applying the arrow $(\Phi_x^\alpha(t))^{-1}$ gives a vector in E_x. Thus, it makes sense to define the following map ([CF11], Corollary 2.34):

Lemma 3.9. *Let $E \in Rep(\mathcal{G})$ with action map μ and $A = \mathcal{L}ie(\mathcal{G})$. Then E is also a representation of A, with the A-connection $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ defined by*

$$\nabla_\alpha(s)(x) = \left.\frac{d}{dt}\right|_{t=0} \mu((\Phi_x^\alpha(t))^{-1}, s(\phi_{\rho(\alpha)}^t(x))).$$

We have already argued that this is a well-defined map of sections. It remains to show that this satisfies the three properties in Definition 3.7. However, from this expression it is quite difficult to prove property (iii). What we will do instead is apply this construction to the groupoid $GL(E)$. We then compute the algebroid of $GL(E)$ using this map. After that, it will follow naturally that this construction works for any groupoid.

3.2 The algebroid of the general linear groupoid

Consider the general linear groupoid $GL(E)$. Its algebroid will be denoted $\mathfrak{gl}(E)$. It is quite difficult to compute this algebroid directly. Using the map ∇ defined above, we will prove that its space of sections $\Gamma(\mathfrak{gl}(E))$ is isomorphic to the space of derivations on E. First we will define derivations, and see that they actually admit the structure of a space of sections of a Lie algebroid.

Definition 3.10. For a vector bundle $E \to M$, a **derivation** is a pair (D, V) where $D : \Gamma(E) \to \Gamma(E)$ is a linear map of sections and $V \in \mathfrak{X}(M)$ is a vector field, satisfying the following Leibniz rule:

$$D(fs) = fD(s) + \mathcal{L}_V(f)(s) \text{ for all } f \in C^\infty(M),\ s \in \Gamma(E).$$

V is called the **symbol** of D. The space of derivations is denoted $\mathcal{D}er(E)$.

Lemma 3.11. *Assume E is not the zero vector bundle. Then any derivation (D, V) is uniquely determined by the linear map $D : \Gamma(E) \to \Gamma(E)$.*

Proof. Suppose vector fields V and V' are both symbols satisfying the Leibniz rule for D. Then we find

$$\mathcal{L}_{V'}(f)(s) = \mathcal{L}_V(f)(s),\ \forall f \in C^\infty(M),\ s \in \Gamma(E).$$

Consider this expression for a neighbourhood around some point $x \in M$. We can always find a section s of E such that $s(x)$ is nonzero. Then the above equation shows that V and V' agree at the point x. Since this can be done around any point, we find that V and V' are the same vector field. Each linear map $D : \Gamma(E) \to \Gamma(E)$ admits a unique symbol making it into a derivation. □

As a result of this lemma, we will often refer to derivations simply by the linear map D. We describe the associated symbol of D by the **symbol map** $\sigma : \mathcal{D}er(E) \to \mathfrak{X}(M)$, $\sigma(D) = V$.

Lemma 3.12. *The space $\mathcal{D}er(E)$ has a Lie bracket given by the commutator of maps, with the symbol induced by the usual Lie bracket of vector fields, i.e.:*

$$[(D_1, V_1), (D_2, V_2)] := (D_1 \circ D_2 - D_2 \circ D_1, [V_1, V_2]).$$

Proof. Clearly the commutator is again a linear map of sections. We check that this commutator satisfies the Leibniz rule. Let $f \in C^\infty(M)$, $s \in \Gamma(E)$.

$$\begin{aligned}(D_1 \circ D_2 - D_2 \circ D_1)(fs) =& D_1(fD_2(s) + \mathcal{L}_{V_2}(f)(s)) - D_2(fD_1(s) + \mathcal{L}_{V_1}(f)(s))\\ =& fD_1 \circ D_2(s) + \mathcal{L}_{V_1}(f)D_2(s) + \mathcal{L}_{V_2}D_1(s) + \mathcal{L}_{V_1} \circ \mathcal{L}_{V_2}(f)(s)\\ &- fD_2 \circ D_2(s) - \mathcal{L}_{V_2}(f)D_1(s) - \mathcal{L}_{V_1}(f)D_2(s) - \mathcal{L}_{V_2} \circ \mathcal{L}_{V_1}(f)(s)\\ =& f(D_1 \circ D_2 - D_2 \circ D_2)(s) + (\mathcal{L}_{V_1} \circ \mathcal{L}_{V_2} - \mathcal{L}_{V_2} \circ \mathcal{L}_{V_1})(f)(s)\\ =& f(D_1 \circ D_2 - D_2 \circ D_2)(s) + \mathcal{L}_{[V_1,V_2]}(f)(s)\end{aligned}$$

This shows that indeed the commutator is again in $\mathcal{D}er(E)$, and that the Lie bracket of vector fields $[V_1, V_2]$ is the symbol of this commutator. □

This lemma also shows that the symbol map $\sigma : \mathcal{D}er(E) \to \mathfrak{X}(M)$ is a Lie algebra morphism. We see that $\mathcal{D}er(E)$, equipped with this Lie bracket and the anchor map σ, has the structure of the space of sections of a Lie algebroid. We will prove that this is isomorphic to the sections of $\mathfrak{gl}(E)$.

Theorem 3.13. *Let $E \to M$ a vector bundle. Consider the groupoid $GL(E)$ and its algebroid $\mathfrak{gl}(E)$. Then $\Gamma(\mathfrak{gl}(E)) \cong \mathcal{D}er(E)$ as Lie algebras, and the anchor ρ of $\mathfrak{gl}(E)$ corresponds to the symbol map σ.*

The proof follows from a sequence of lemmas. First we prove that the map defined in Lemma 3.9 is a Lie algebra morphism for $A = \mathfrak{gl}(E)$, and that this map commutes with the symbol and anchor maps. Then, we will construct a commutative diagram of short exact sequences involving $\mathfrak{gl}(E)$ and $\mathcal{D}er(E)$. Finally, we use this diagram to prove the theorem.

Lemma 3.14. *Consider the groupoid $GL(E)$ and its algebroid $\mathfrak{gl}(E)$. Then the construction of Lemma 3.9 defines a Lie algebra morphism*

$$\nabla : \Gamma(\mathfrak{gl}(E)) \to \mathcal{D}er(E)$$

that commutes with the anchor map ρ and the symbol map σ.

Proof. Applied to the groupoid $GL(E)$, Lemma 3.9 defines a map $\nabla : \Gamma(\mathfrak{gl}(E)) \times \Gamma(E) \to \Gamma(E)$ by

$$\nabla_\alpha(s)(x) = \left.\frac{d}{dt}\right|_{t=0} \mu((\Phi_x^\alpha(t))^{-1}, s(\phi^t_{\rho(\alpha)}(x))).$$

Here μ is the action map as in Example 3.3, the standard action of $GL(E)$ on E. We can interpret ∇ as a map $\nabla : \Gamma(\mathfrak{gl}(E)) \to \mathcal{D}er(E)$. We have to prove a few things. First of all, to be a Lie algebra morphism, this has to be a linear map. This property corresponds to item (i) in Definition 3.7. We consider the following expression:

$$\nabla_{f\alpha}(s)(x) = \left.\frac{d}{dt}\right|_{t=0} \mu((\Phi_x^{f\alpha}(t))^{-1}, s(\phi^t_{\rho(f\alpha)}(x))).$$

First note that we have $\rho(f\alpha) = f\rho(\alpha)$ for $f \in C^\infty(M)$. Then it is a general property of the flow that $\phi^t_{f\rho(\alpha)}(x) = \phi^{f(x)t}_{\rho(\alpha)}$.

For the term $\Phi_x^{f\alpha}(t)$, recall that $(f\alpha)^R = (f \circ t)\alpha^R$ (with t the target map). From the same general flow property we find $\phi^t_{(f\circ t)\alpha^R}(1_x) = \phi^{(f\circ t)(1_x)t}_{\alpha^R}(1_x) = \phi^{f(x)t}_{\alpha^R}(1_x)$.

Now we apply a change of coordinates $t \to t/f(x)$, so $d/dt \to f(x)d/dt$, to find indeed $\nabla_{f\alpha}(s) = f\nabla_\alpha(s)$.

Second, for $\alpha \in \Gamma(\mathfrak{gl}(E))$, ∇_α should satisfy the Leibniz rule. Note this is equivalent to property (ii) of algebroid representations (Definition 3.7). We prove this by the following computation, where we use the product rule in line 3, and then the fact that $\mu(1_x, s(x)) = s(x)$.

$$
\begin{aligned}
\nabla_\alpha(fs) &= \frac{d}{dt}\bigg|_{t=0} \mu((\Phi_x^\alpha(t))^{-1}, (fs)(\phi^t_{\rho(\alpha)}(x))) \\
&= \frac{d}{dt}\bigg|_{t=0} f(\phi^t_{\rho(\alpha)}(x))\mu((\Phi_x^\alpha(t))^{-1}, s(\phi^t_{\rho(\alpha)}(x))) \\
&= \left(\frac{d}{dt}\bigg|_{t=0} f(\phi^t_{\rho(\alpha)}(x))\right)\mu(1_x, s(x)) + f(x)\,\frac{d}{dt}\bigg|_{t=0} \mu((\Phi_x^\alpha(t))^{-1}, s(\phi^t_{\rho(\alpha)}(x))) \\
&= \mathcal{L}_{\rho(\alpha)}(f)(s)(x) + f\nabla_\alpha(s)(x).
\end{aligned}
$$

From this computation, we also see that indeed $\sigma \circ \nabla = \rho$.

Finally, to be a Lie algebra morphism, this map has to commute with the brackets. Since this is a local propety, this follows with a proof in local coordinates. □

Lemma 3.15. *We have the following commutative diagram of short exact sequences, where σ is the symbol map, and ρ is the anchor of A.*

$$
\begin{array}{ccccc}
\Gamma(End(E)) & \xrightarrow{\ i\ } & \mathcal{D}er(E) & \xrightarrow{\ \sigma\ } & \mathfrak{X}(M) \\
 & \searrow^{j} & \uparrow{\scriptstyle\nabla} & \nearrow^{\rho} & \\
 & & \Gamma(\mathfrak{gl}(E)) & &
\end{array}
$$

Proof. We have to prove that both rows are exact, and that all maps commute. We have already proven commutativity on the right side, $\sigma \circ \nabla = \rho$.

We start with exactness in the top row. Consider the space $\mathcal{D}er(E)$. Choose any vector bundle connection ∇^E on E. Then any derivation can be written as $(\nabla^E_X + A, X)$, where $X \in \mathfrak{X}(M)$ and A is a $C^\infty(M)$-linear map of sections. The Leibniz rule for vector bundle connections and the fact that A is linear implies that the symbol of this derivation is X. We can find such an derivation for any vector field X, showing that σ is surjective. On the other hand, the kernel of σ contains precisely those derivation that are $C^\infty(M)$-linear, i.e. have symbol 0. By viewing these derivation as maps sending $x \in M$ to a linear map $E_x \to E_x$, we see that these form the sections of the endomorphism bundle $End(E)$. By this construction the top row is a short exact sequence.

For the bottom row we follow a similar procedure. Since $GL(E)$ is transitive, $\mathfrak{gl}(E)$ is as well, so ρ is surjective. Now consider for any $x \in M$ the kernel $\mathrm{Ker}(\rho_x)$. This is the isotropy Lie algebra at x, which we know is the Lie algebra of the isotropy group at x. Recall that the isotropy group is $GL(E)_x \cong Aut(E_x)$. Then its Lie algebra is $\mathrm{Ker}(\rho_x) \cong End(E_x)$, and indeed we find on the level of sections $\Gamma(Ker(\rho)) \cong \Gamma(End(E))$. The map j is simply the composition of this isomorphism with the inclusion $\mathrm{Ker}(\rho) \hookrightarrow A$. By construction, this sequence is then exact.

It remains to show that the left triangle is commutative. Let $T \in \Gamma(End(E))$. So, T is a map $T : M \to End(E)$ such that at any point $x \in M$, $T_x : E_x \to E_x$. Now consider the inclusion $j(T) \in \Gamma(A)$. We aim to show $\nabla \circ j(T) = i(T)$. We prove this pointwise, using the definition of ∇. Let $s \in \Gamma(E)$ and $x \in M$.

$$\begin{aligned}\nabla_{j(T)}(s)(x) &= \frac{d}{dt}|_{t=0}\mu((\Phi_x^{j(T)}(t))^{-1}, s(\phi^t_{\rho(j(T))}(x))) \\ &= \frac{d}{dt}|_{t=0}\mu((\Phi_x^{j(T)}(t))^{-1}, s(x)).\end{aligned}$$

Here we used that $j(T)$ lies in the kernel of ρ. Since $\Phi_x^{j(T)}(T)$ covers the flow of $\phi^t_{j(T)}(x)$, and the latter is the constant path at x, we see that $\Phi_x^{j(T)}(t)$ will at any time t be an arrow from x to x. So, it is a path in the isotropy group $GL(E)_x$, which is isomorphic to $Aut(E_x)$. Note that $\Phi_x^{j(T)}(t)$ is determined by the flow of the vector field $j(T)^R$. We see that we only consider the values of $j(T)^R$ at arrows $g \in GL(E)_x \cong Aut(E_x)$. For such g, we find then

$$j(T)^R_g = d_{1_{t(g)}} R_g(j(T)(x)) = d_{1_x} R_g(j(T)(x))$$

and since 1_x in $Aut(E_x)$ is simply the identity morphism, we find the usual Lie group-Lie algebra correspondence here. Combined with the fact that for $GL(E)$, the action μ is simply application of the linear morphism, we find

$$\nabla_{j(T)}(s)(x) = T(x)(s(x))$$

and the left triangle commutes as well. □

Using this diagram, we can now prove that $\nabla : \Gamma(\mathfrak{gl}(E)) \to \mathcal{D}er(E)$ is an isomorphism.

Proof of Theorem 3.13. To show that ∇ is an isomorphism, it remains to prove it is bijective. We use commutativity of the diagram above to show this.

First it is injective; suppose s is a section of A that is in the kernel of ∇. Then $\sigma \circ \nabla(s) = \sigma(0) = 0$, and by commutativity we must have $\rho(s) = 0$. That means s has some preimage $a \in \Gamma(End(E))$ such that $j(a) = s$. By commutativity of the left triangle then $i(a) = \nabla(s) = 0$, and since i is injective, we find $a = 0$. Since $s = j(a)$, we also have $s = 0$. So, the map ∇ has kernel zero and it must be injective.

For surjectivity, consider some derivation D. Since the anchor map is surjective, there is some section s of A such that $\rho(s) = \sigma(D)$. Now consider $\nabla(s) = D'$, which is not necessarily equal to D, but by commutativity it does have the same symbol as D. So $D - D' \in \mathrm{Ker}(\sigma)$ and by exactness there is some $a \in \Gamma(End(E))$ such that $i(a) = D - D'$. Now by commutativity of the left triangle, $\nabla(j(a)) = D - D'$, and we conclude $\nabla(j(a) + s) = D$, i.e. we have found a preimage of D and thus ∇ is surjective. □

We now move to the proof of Lemma 3.9, which stated that any groupoid representation E on $\mathcal{G}$ induces a representation on its algebroid A. We have seen that it holds specifically for $GL(E)$ with the standard action map and its algebroid $\mathfrak{gl}(E)$. In general, we first have the following lemma.

Lemma 3.16. *A representation on a Lie algebroid $A \to M$ is equivalent to a Lie algebra morphism*

$$\nabla : \Gamma(A) \to \mathcal{D}er(E)$$

which commutes with the anchor and symbol map (so $\sigma \circ \nabla = \rho$).

Proof. Recall Definition 3.7: a Lie algebroid representation is a vector bundle $E \to M$ with a map

$$\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E).$$

We can also interpret ∇ as a map

$$\nabla : \Gamma(A) \to \mathcal{D}er(E).$$

Property (i) in Definition 3.7 is equivalent to this being a linear map. Property (ii) is equivalent to the Leibniz rule for derivations, where $\rho(\alpha)$ is the symbol of ∇_α. Property (iii) is equivalent to ∇ preserving the brackets, which combined with being a linear map, is equivalent to being a Lie algebra morphism. □

Proof of Lemma 3.9. Consider $E \in \mathrm{Rep}(\mathcal{G})$ with action map μ and Lie algebroid $A = \mathcal{L}ie(\mathcal{G})$. By Lemma 3.4, this representation on $\mathcal{G}$ is equivalent to a groupoid morphism $\mathcal{F} : \mathcal{G} \to GL(E)$. By Lemma 2.42, this induces an algebroid morphism $F : A \to \mathfrak{gl}(E)$. Then via the induced map on sections, $F : \Gamma(A) \to \Gamma(\mathfrak{gl}(E)) \cong \mathcal{D}er(E)$, we find by the previous lemma that this is equivalent to a representation of E on the algebroid A.

Denote by $\nabla^{st} : \mathfrak{gl}(E) \to \mathcal{D}er(E)$ the isomorphism defined in Lemma 3.14, and denote by μ^{st} the standard action map of $GL(E)$ on E. Then the representation induced on A by the procedure described above, is a map $\nabla : \Gamma(A) \times \Gamma(E) \to \Gamma(E)$ defined by

$$\begin{aligned}\nabla_\alpha(s)(x) &= (\nabla^{st} \circ F)(\alpha)(s)(x)\\ &= \nabla^{st}_{F\circ\alpha}(s)(x)\\ &= \left.\frac{d}{dt}\right|_{t=0} \mu^{st}\left(\left(\phi^t_{(F\circ\alpha)^R}(1_x)\right)^{-1}, s(\phi^t_{\rho(F\circ\alpha)}(x))\right).\end{aligned}$$

For the flow, note the following:

$$\phi_{(F\circ\alpha)^R} = \mathcal{F}(\phi_{\alpha^R}).$$

Now $\mathcal{F}$ corresponds to the original action μ of $\mathcal{G}$ on E. Furthermore, since the map $\Gamma(A) \to \mathcal{D}er(E)$ commutes with the anchor map, we find

$$\nabla_\alpha(s)(x) = \left.\frac{d}{dt}\right|_{t=0} \mu\left(\left(\phi^t_{\alpha^R}(1_x)\right)^{-1}, s(\phi^t_{\rho\circ\alpha}(x))\right),$$

which is precisely the connection constructed in Lemma 3.9. □

3.3 Relating algebroid representations to infinitesimal actions

Let $\mathcal{G}$ a groupoid with algebroid A. A groupoid representation of $\mathcal{G}$ is basically a groupoid action μ on a vector space E. This induces on one hand an algebroid representation in the form of a flat A-connection ∇ on E as seen in Lemma 3.9. On the other hand, by Lemma 2.55 this induces an algebroid action $\mathscr{a} : \Gamma(A) \to \mathfrak{X}(E)$. Here we shows the relationship between the induced connection ∇ and the induced action $\mathscr{a}$.

First of all, let's recall how the induced action $\mathscr{a}$ is defined. We basically revisit the proof of Lemma 2.56, writing it in the notation we use for representations. Let $E \in \mathrm{Rep}(\mathcal{G})$ with moment map μ. Then for any $p \in E$, μ_p acts on the source fiber $s^{-1}(\pi(p))$ by

$$\mu_p : s^{-1}(\pi(p)) \to E, \ \mu_p(g) = \mu(g, p).$$

We denote $\pi(p) = x$. Taking the differential of this map at the unit $1_x \in s^{-1}(x)$, we find a map on the fiber of A above x:

$$d_{1_x}\mu_p : T_{1_x}s^{-1}(x) = A_x \to T_pE.$$

Then on the level of sections, we find the induced infinitesimal action

$$\mathscr{a} : \Gamma(A) \to \mathfrak{X}(E), \ \mathscr{a}(\alpha)(p) = (d_{1_{\pi(p)}}\mu_p)(\alpha_{\pi(p)}).$$

We find an equivalent description by revisiting the map $\Phi^\alpha_x(t)$ defined before for $\alpha \in \Gamma(A)$:

$$\Phi^\alpha_x(t) = \phi^t_{\alpha^R}(1_x).$$

From Lemma 3.8 we know that at any time t, this gives an arrow in $\mathcal{G}$ from x to $\phi^t_{\rho(\alpha)}(x)$. This means that $\Phi^\alpha_x(t)$ is an arrow acting by μ on the fiber E_x. Furthermore this flow encodes precisely the infinitesimal action on E induced by α, so equivalently the action is described by

$$\mathscr{a} : \Gamma(A) \to \mathfrak{X}(E), \ \mathscr{a}(\alpha)(p) = \left.\frac{d}{dt}\right|_{t=0} \Phi^\alpha_{\pi(p)}(t) \cdot p. \tag{6}$$

From this formula we find a third description of the infinitesimal action: $\mathscr{a}(\alpha)$ is a vector field on E with flow $\phi^t_{\mathscr{a}(\alpha)}(p) = \Phi^\alpha_{\pi(p)}(t) \cdot p = \phi^t_{\alpha^R}(1_{\pi(p)}) \cdot p$.

In order to relate this infinitesimal action to the induced A-connection (the induced algebroid representation) ∇, we have to consider the dual vector bundle E^*. We first show that this dual is again a groupoid representation.

Lemma 3.17. *For any representation E of $\mathcal{G}$, the dual vector bundle E^* is also a representation of $\mathcal{G}$ in a natural way.*

Proof. Let μ denote the moment map of the representation E. Let $(g, \xi) \in \mathcal{G} \times_M E^*$ and let $v \in E_{t(g)}$. Then we define $\mu' : \mathcal{G} \times_M E^* \to E^*$ by

$$\mu'(g, \xi)(v) = \xi(\mu(g^{-1}, v)). \tag{7}$$

We just have to check that this satisfies the three properties in Definition 3.2.

First of all, $\mu(g^{-1}, \cdot)$ is a map that takes elements in $E_{t(g)}$ since μ is a moment map. Then we know $\mu(g^{-1}, \cdot) \in E_{s(g)}$ which is a valid argument for ξ. This proves that the map μ' is well-defined, and also shows that $\mu'(g, \xi) \in E^*_{t(g)}$, which is the first property of moment maps.

Using the fact that μ is a moment map, we can compute the second property:

$$\begin{aligned} \mu'(g, \mu'(h, \xi))(v) &= \mu'(h, \xi)(\mu(g^{-1}, v)) \\ &= \xi(\mu(h^{-1}, \mu(g^{-1}, v))) \\ &= \xi(\mu(h^{-1}g^{-1}, v)) \\ &= \mu'(gh, \xi)(v). \end{aligned}$$

Finally to show that $\mu'(g, \cdot)$ is an isomorphism we construct an explicit inverse. Define $\mu''(g, \cdot) : E^*_{t(g)} \to E^*_{s(g)}$ using the fact that $\mu(g^{-1}, \cdot)$ has an inverse, by

$$\mu''(g, \eta)(v) = \eta(\mu^{-1}(g^{-1}, v)).$$

Then we see indeed

$$\begin{aligned} \mu''(g, \mu'(g, \xi))(v) &= \mu'(g, \xi)(\mu^{-1}(g^{-1}, v)) \\ &= \xi(\mu(g, \mu^{-1}(g^{-1}, v))) = \xi(v) \\ &\text{and} \\ \mu(g, \mu''(g, \eta))(v) &= \mu''(g, \eta)(\mu(g^{-1}, v)) \\ &= \eta(\mu^{-1}(g^{-1}, \mu(g^{-1}, v))) = \eta(v). \end{aligned}$$

□

Because of this lemma, for any representation $E \in \text{Rep}(\mathcal{G})$, we can consider $E^* \in \text{Rep}(\mathcal{G})$ and the induced infinitesimal action $a : \Gamma(A) \to \mathfrak{X}(E^*)$. This is of course directly related to the infinitesimal action of E. In order to relate this to ∇, we introduce some notation.

Let $s \in \Gamma(E)$. We want to express that this induces a function on E^* by evaluating at s. To express this, we define the function $f_s \in C^\infty(E^*)$ by

$$f_s(\xi) = \xi(s \circ \pi(\xi)). \tag{8}$$

Using these functions, we get to the main statement of this section.

Lemma 3.18. *Let $E \in Rep(\mathcal{G})$, $A = \mathcal{L}ie(\mathcal{G})$. The infinitesimal action $a : \Gamma(A) \to \mathfrak{X}(E^*)$ is related to the A-connection (as from Lemma 3.9) by*

$$\mathcal{L}_{a(\alpha)}(f_s) = f_{\nabla_\alpha(s)}.$$

Proof. We just do the computation, using the infinitesimal action as in Equation (6) (which describes the flow of the vector field $\mathscr{a}(\alpha)$).

$$\begin{aligned}
\mathcal{L}_{\mathscr{a}(\alpha)}(f_s)(\xi) &= \frac{d}{dt}|_{t=0} f_s \circ \left(\phi^t_{\mathscr{a}(\alpha)}(\xi)\right) \\
&= \frac{d}{dt}|_{t=0} f_s \circ \left(\phi^t_{\alpha^R}(1_{\pi(\xi)}) \cdot \xi\right) \\
&\text{then using the definition of } f_s\text{, and since } \phi^t_{\alpha^R}(1_{\pi(\xi)}) \cdot \xi \in E^*_{\phi^t_{\rho(\alpha)}(\pi(\xi))}, \\
&= \frac{d}{dt}|_{t=0} \left(\phi^t_{\alpha^R}(1_{\pi(\xi)}) \cdot \xi\right) \left(s(\phi^t_{\rho(\alpha)}(\pi(\xi)))\right) \\
&\text{then using the moment map on } E^* \text{ as in Equation (7),} \\
&= \frac{d}{dt}|_{t=0} \xi \left(\left(\phi^t_{\alpha^R}(1_{\pi(\xi)})\right)^{-1} \cdot s(\phi^t_{\rho(\alpha)}(\pi(\xi))) \right) \\
&= \xi(\nabla_\alpha(s)(\pi(\xi))) \\
&= f_{\nabla_\alpha(s)}(\xi).
\end{aligned}$$

□

Remark 3.19 (Outlook for the coming chapter). In this chapter, we have discussed in detail the general linear groupoid $GL(E)$, alongside the theory of groupoid and algebroid representations. This is similarly to the theory of Lie groups and Lie algebroids, where the general linear group and group representations are related in a similar manner.

However for the next chapter, we will focus on a different aspect of the theory that we have already developed here. In order to characterise the algebroid of $GL(E)$, we have constructed an isomorphism $\Gamma(\mathfrak{gl}(E)) \cong \mathcal{D}er(E)$. To construct this, we made use of the following sequence (the bottom part of the diagram in Lemma 3.15):

$$\Gamma(End(E)) \xrightarrow{j} \Gamma(\mathfrak{gl}(E)) \xrightarrow{\rho} \mathfrak{X}(M).$$

We saw that $\Gamma(End(E)) \cong \Gamma(Ker(\rho))$, and this is a short exact sequence because ρ is surjective. In this case, $\Gamma(End(E))$ and $\mathfrak{X}(M)$ also fitted into a short exact sequence around $\mathcal{D}er(E)$, which gave a nice characterisation of $\mathfrak{gl}(E)$. But even in general, any transitive algebroid A can fit into such a sequence:

$$\mathrm{Ker}(\rho) \hookrightarrow A \xrightarrow{\rho} TM.$$

This is called an Atiyah sequence, and the next chapter will be devoted entirely to such sequences associated with transitive algebroids.

4 Transitive and Atiyah algebroids

In this section we will study transitive algebroids. The study of general linear groupoids and algebroids in the previous section gave rise to a notion of representations of groupoids and algebroids. Similarly, the study of transitive algebroids will give rise to a natural notion of connections on algebroids.

We will see that integrable transitive algebroids always come from principal bundles. Connections on principal bundles will induce a natural notion of connections on these integrable algebroids. This will be discussed in Section 4.2. For non-integrable transitive algebroids, there is no associated principal bundle, but the notion of connections and curvature can be extended in a natural way. This general notion of connections will be introduced in Section 4.1. In Section 4.3 we will revisit how this forms a natural extension of the principal bundle case.

Transitive algebroids naturally come with a short exact sequence, discussed already in Remark 3.19. These sequences are called Atiyah sequences. As a preparation for Part II and Part III, we will also discuss Atiyah sequences arising from fibrations in Section 4.4.

4.1 Abstract Atiyah algebroids

Consider a transitive Lie algebroid $A \to M$ with anchor ρ. Since ρ is surjective, A is regular, and the isotropy Lie algebras $\mathrm{Ker}(\rho_x)$ form a Lie algebra bundle over M, see for example Section 10.2 in [Mei17]. This Lie algebra bundle will be denoted $\mathfrak{g}(A)$. Then since ρ is surjective, we can form the following short exact sequence.

Definition 4.1. Given a transitive Lie algebroid $A \to M$ with anchor ρ, its **abstract Atiyah sequence** is the following short exact sequence:

$$0 \longrightarrow \mathfrak{g}(A) \hookrightarrow A \xrightarrow{\rho} TM \longrightarrow 0 \ .$$

Using this sequence, we can introduce a notion of connections on transitive Lie algebroids.

Definition 4.2. Given a transitive Lie algebroid $A \to M$, a **connection** is a right splitting of the Atiyah sequence, i.e., a map $\sigma : TM \to A$ such that $\rho \circ \sigma = \mathrm{id}_{TM}$.

The **curvature** of a connection σ is the 2-form $\Omega_\sigma \in \Omega(M; \mathfrak{g}(A))$ defined by

$$\Omega_\sigma(X, Y) = [\sigma(X), \sigma(Y)]_A - \sigma([X, Y]).$$

Lemma 4.3. *Ω_σ is indeed a $\mathfrak{g}(A)$-valued form.*

Proof. We have to check that ρ composed with this two-form is zero. We use the fact that ρ preserves Lie brackets, and that $\rho \circ \sigma = \mathrm{id}_{TM}$ to find for any X, $Y \in \mathfrak{X}(M)$:

$$\begin{aligned} \rho(\Omega_\sigma(X, Y)) &= \rho([\sigma(X), \sigma(Y)]_A) - \rho \circ \sigma([X, Y]) \\ &= [\rho \circ \sigma(X), \rho \circ \sigma(Y)] - [X, Y] \\ &= [X, Y] - [X, Y] = 0. \end{aligned}$$

□

In words, the curvature measures the failure of σ to preserve Lie brackets. This corresponds to the interpretation of curvature in the case of principal connections. We will now discuss transitive algebroids that are integrable, and we will see that connections in this case really come from principal bundle connections.

4.2 Principal bundles and classical Atiyah sequences

Let pr : $P \to M$ be a principal G-bundle. For principal bundles we know that quotienting by G gives the base manifold: $P/G \cong M$. We expect that the quotient TP/G is then a vector bundle over M. This is actually a specific instance of a more general result:

Lemma 4.4. *Given a principal G-bundle $pr : P \to M$ and a G-equivariant vector bundle $\pi_E : E \to P$, the quotient E/G is a vector bundle over M.*

Proof. Let $[p, v] \in E/G$, so $p \in P$ and $v \in E_p$. We define the projection map $\pi_E : E/G \to M$ as $\pi_E([p, v]) = \mathrm{pr}(p)$. This is well-defined as $[p, v] \sim [q, w]$ implies $q = pg$ for some $g \in G$, and $\mathrm{pr}(pg) = \mathrm{pr}(p)$. Since pr is a projection map, it follows that π_E is smooth and surjective.

The fact that E/G is smooth follows from the G-action. The action of G on P is free and proper. The same holds for the action on E, making E/G into a smooth manifold.

Local triviality follows from combining local sections of $P \to M$ with local frames of $E \to P$; the composition gives a local frame of the vector bundle $E/G \to M$.

Finally consider the fibers; for $x \in M$, choosing a point $p_0 \in P_x$ allows us to write

$$(E/G)_x = \pi_E^{-1}(x) = \{[p_0, v] | v \in T_{p_0}E\} \simeq T_{p_0}E$$

which shows that the fibers are linear vector spaces, and we conclude that E/G is a vector bundle over M. □

Corollary 4.5. *$A = TP/G \to M$ is a vector bundle over M with projection $\pi([p, v]) = pr(p)$.*

As the notation suggests, we want to make this vector bundle into an algebroid over M. This requires us to define a bracket on its space of sections and endow it with an anchor map. To define the bracket, we consider the G-invariant vector fields on P, denoted $\mathfrak{X}(P)^G$:

$$\mathfrak{X}(P)^G = \{ X \in \mathfrak{X}(P) \mid X(pg) = d_pR_g(X(p)) \,\forall p \in P,\, g \in G \}.$$

Note that these vector fields are automatically projectable vector fields, because pushforwards are only defined for projectable vector fields. Since the Lie bracket commutes with pushforwards of projectable vector fields, this space is closed under the Lie bracket. Now this induces a Lie bracket on the space of sections of the algebroid $A = TP/G$ by the following lemma:

Lemma 4.6. *$\Gamma(A) \cong \mathfrak{X}(P)^G$.*

Proof. We prove this lemma by explicitly constructing the map between these two spaces. First let $s : M \to TP/G$ be a section of A. Consider the pullback bundle $\mathrm{pr}^*(TP/G)$ over P, with fibers $\mathrm{pr}^*(TP/G)_p = (TP/G)_{\pi(p)}$ for $p \in P$. As we saw before, this implies that the fiber $\mathrm{pr}^*(TP/G)_p$ is isomorphic to T_pP. In words, this pullback bundle first quotients TP by G, but then by taking the pullback we remove this quotient again; we see that the pullback bundle is isomorphic to $TP \to P$.

Now a smooth section $s : M \to TP/G$ induces a section $s' : P \to \mathrm{pr}^*(TP/G) \simeq TP$, i.e. a vector field on P. Explicitly, this is defined as $s'(p) = (p, s(\mathrm{pr}(p)))$. The corresponding vector field is defined as $p \mapsto v_p$, where v_p is a representative such that $(p, v_p) \in s(\mathrm{pr}(p))$ as a class in TP/G. Since $\mathrm{pr}(p) = \mathrm{pr}(pg)$, we have that $(p, v_p) \sim (pg, v_{pg})$, i.e.,

$$X(pg) = v_{pg} = d_p R_g(v_p) = d_p R_g(X(p)).$$

We see indeed that this vector field on P is G-invariant.

For the inverse map, let $X \in \mathfrak{X}(P)^G$. Let $x \in M$ and fix $p_0 \in P_x$. Define $s : M \to TP/G$ by $s(x) = [(p_0, X(p_0))]$. Since $X(p_0) \in T_{p_0}P$, this represents a class in TP/G, and since $\mathrm{pr}(p_0) = x$ it lies in the fiber above x. To show this is well-defined, consider a different point in the fiber $q_0 \in P_x$. Then for some $g \in G$, $q_0 = p_0 \cdot g$. Now since X is a G-invariant vector field on P, we have that $X(q_0) = d_{p_0} R_g(X(p_0))$, so $(p_0, X(p_0)) \sim (q_0, X(q_0))$, and this section is well-defined.

Finally it is obvious that these operations are inverse to each other and we have an isomorphism $\phi : \Gamma(TP/G) \xrightarrow{\sim} \mathfrak{X}(P)^G$. □

Next we want to endow $A \to M$ with an anchor map $\rho : A \to TM$. The projection $\mathrm{pr} : P \to M$ induces the differential $d\mathrm{pr} : TP \to TM$. This induces a well-defined vector bundle map $d\mathrm{pr} : TP/G \to TM$. To see that this is well-defined, let $v \in T_pP$ and $w \in T_qP$ such that $v \sim w$. That is, there exists an element $g \in G$ such that $q = pg$. Then using the fact that $\mathrm{pr} \circ R_g = \mathrm{pr}$ we have that

$$d_q\mathrm{pr}(w) = d_q\mathrm{pr}(d_p R_g(v)) = d_p(\mathrm{pr} \circ R_g)(v) = d_p\mathrm{pr}(v).$$

Hence $d\mathrm{pr}$ is well-defined on the quotient TP/G. This will be the anchor map of A:

$$\rho := d\mathrm{pr} : A \to TM.$$

Definition 4.7. The **Atiyah algebroid** associated to a principal G-bundle $P \xrightarrow{\mathrm{pr}} M$ is the vector bundle $A = TP/G \to M$ with bracket induced by $\Gamma(A) \cong \mathfrak{X}(P)^G$ and anchor map $d\mathrm{pr}$.

Since pr is a submersion, $d\mathrm{pr}$ is surjective, hence this algebroid is transitive. This allows us to construct the following short exact sequence.

Definition 4.8. The **classical Atiyah sequence** corresponding to the Atiyah algebroid of a principal G-bundle $P \xrightarrow{\mathrm{pr}} M$ is the short exact sequence

$$0 \longrightarrow \mathrm{Ker}(d\mathrm{pr}) \hookrightarrow TP/G \xrightarrow{d\mathrm{pr}} TM \longrightarrow 0 \ . \tag{9}$$

Similar to Section 4.1, we have the following notion of connections on Atiyah algebroids.

Definition 4.9. A **connection** on an Atiyah algebroid is a splitting of the classical Atiyah sequence, i.e., a map $\sigma : TM \to A$ such that $d\pi \circ \sigma = \mathrm{id}_{TM}$.

In the case of Atiyah algebroids, connections on the algebroid are completely determined by principal bundle connections. This correspondence is described in the following lemma due to Atiyah.

Lemma 4.10. *Principal bundle connections on a principal G-bundle $P \to M$ are in 1-1 correspondence with connections on the Atiyah algebroid $TP/G \to M$.*

This result becomes immediate after a basic discussion of principal bundle connections.

Connections on principal bundles

Principal bundle connections are often described in several equivalent ways. We will show here that these descriptions all arise from a short exact sequence associated to the principal bundle. First we construct this sequence.

Consider the G-action on P, $A : P \times G \to P$. Pointwise (for $p \in P$)this action can be written as a map $A_p : G \to P$. Since the fibers of P are G-torsors, A_p maps back into $P_{\mathrm{pr}(p)}$ and is a diffeomorphism of this fiber. Associated to the action A, there is an induced infinitesimal action $\mathscr{a} : \mathfrak{g} \to \mathfrak{X}(P)$. Pointwise, the infinitesimal action is simply the differential of A_p, and is also an isomorphism:

$$\mathscr{a}_p = d_e A_p : \mathfrak{g} \xrightarrow{\sim} T_p P_{\mathrm{pr}(p)} \subset T_p P.$$

Consider now $d\mathrm{pr} : TP \to TM$. The kernel of this map is formed by the collection of vectors that are tangent to the fibers of pr. This space is denoted

$$T_p^v P := T_p P_{\mathrm{pr}(p)} = \mathrm{Ker}(d_p \mathrm{pr}).$$

Combining this with the isomorphism $\mathscr{a}_p : \mathfrak{g} \to T_p P_{\mathrm{pr}(p)}$, we can construct a pointwise short exact sequence:

$$0 \to \mathfrak{g} \xrightarrow{\mathscr{a}_p} T_p P \xrightarrow{d_p \mathrm{pr}} T_{\mathrm{pr}(p)} M \to 0. \tag{10}$$

By attaching $\mathfrak{g}$ to every point in P, we construct $P \times \mathfrak{g}$, fitting into a short exact sequence of bundles over P:

$$0 \to P \times \mathfrak{g} \xrightarrow{\mathscr{a}} TP \xrightarrow{d\mathrm{pr}} \mathrm{pr}^* TM \to 0.$$

Short exact sequences can admit *splittings.* We describe splittings of sequences in three equivalent ways, given in the lemma below. See [Hat02] (p. 147) for a more detailed discussion on splittings of short exact sequences.

Lemma 4.11 (Splitting lemma.)**.** *For a short exact sequence*

$$0 \longrightarrow A \stackrel{i}{\longrightarrow} B \stackrel{p}{\longrightarrow} C \longrightarrow 0$$

the following are equivalent:

1. *A right splitting, i.e., a map $q : C \to B$ such that $p \circ q = id_C$;*

2. A left splitting, i.e., a map $j : B \to A$ such that $j \circ i = id_A$;

3. An isomorphism $\phi : B \to A \oplus C$ such that $\phi \circ i$ is the natural inclusion of A into $A \oplus C$ and $\phi^{-1} \circ p$ is the natural projection of $A \oplus C$ onto C.

We now describe the usual notions of principal bundle connections, and show that they are simply splittings of Equation (10) with a G-invariance condition.

First, a principal connection is often described as a vector subbundle $\mathcal{H} \subset TP$ that is horizontal, so such that $\forall p \in P$, $T_pP = \mathcal{H}_p \oplus T_p^vP$, and that is G-invariant, so such that $\mathcal{H}_{pg} = d_pR_g(\mathcal{H}_p)$ for $p \in P$ and $g \in G$. So a connection is a choice that allows us to talk about horizontal vectors.

Second, a principal connection can be described as a $\mathfrak{g}$-valued 1-form on P, denoted $\omega \in \Omega^1(P; \mathfrak{g})$. This should satisfy $\omega(\mathscr{a}(v)) = v$, and be G-invariant. Pointwise, this acts is a map $\omega_p : T_pP \to \mathfrak{g}$.

Third, a principal connection is given by a horizontal lifting map, denoted $\mathscr{h} : \mathfrak{X}(M) \to \mathfrak{X}(P)$. This should satisfy $d\pi \circ \mathscr{h} = \mathrm{id}$ (so pointwise, $d_p\pi(\mathscr{h}(X)_p) = X_{\pi(p)}$), and should be G-invariant in the sense that $\mathrm{Im}(\mathscr{h}) \subset \mathfrak{X}(P)^G$.

These three descriptions are related by $\mathcal{H} = \mathrm{Im}(\mathscr{h}) = \mathrm{Ker}(\omega)$. We see that they correspond to the three descriptions of a splitting of a short exact sequence, where we additionally have a G-invariance condition. The splitting lemma stated above already claims that these descriptions are equivalent, but for our purposes it is nice to see explicitly how a horizontal subbundle $\mathcal{H} \subset TP$ induces a horizontal lifting map $\mathscr{h} : \mathfrak{X}(M) \to \mathfrak{X}(P)^G$.

Lemma 4.12. *A horizontal subbundle $\mathcal{H} \subset TP$ induces a map $\mathscr{h} : \mathfrak{X}(M) \to \mathfrak{X}(P)^G$ such that $d\pi \circ \mathscr{h} = id$ and $Im(\mathscr{h}) = \mathcal{H}$.*

Proof. Let $X \in \mathfrak{X}(M)$. We aim to define $\mathscr{h}(X) \in \mathfrak{X}(P)^G$, hence we want to assign for any $p \in P$ a vector $\mathscr{h}(X)_p \in T_pP$. Denote $x = \mathrm{pr}(p)$. Note that $d_p\mathrm{pr}|_{\mathcal{H}_p} : \mathcal{H}_p \to T_xM$ is an isomorphism by the splitting lemma above. We define

$$\mathscr{h}(X)_p := \left(d_p\mathrm{pr}|_{\mathcal{H}_p}\right)^{-1}(X_x) \in \mathcal{H}_p \subset T_pP.$$

From the G-invariance of $\mathcal{H}$, it follows that the resulting vector field $\mathscr{h}(X)$ is also G-invariant. The constructed mapping $\mathscr{h}$ maps into $\mathfrak{X}(P)^G$, and has as image $\mathcal{H}$.

□

With this lemma, we have a 1-1 corresondence between subbundles $\mathcal{H}$ and lifting maps $\mathscr{h}$, the two descriptions of connections that we use most in this thesis. We now move to the proof of Lemma 4.10.

Proof of Lemma 4.10. Recall that a connection on the Atiyah algebroid is a splitting of the Atiyah sequence, which is a map $\mathscr{h} : \mathfrak{X}(M) \to \Gamma(TP/G)$ such that $d\pi \circ \mathscr{h} = \mathrm{id}$. Recall that we identified $\Gamma(TP/G) \simeq \mathfrak{X}(P)^G$. Then by the discussion above, we see that this is equivalent to a principal bundle connection on P. □

4.3 Integrable and non-integrable transitive algebroids

In Section 4.2, we have seen that principal bundles induce transitive algebroids, and principal bundle connections induce a natural notion of connections on these transitive algebroids. These connections took the form of a splitting of the classical Atiyah sequence. This notion of connection can be naturally extended to general transitive algebroids that do not come from principal bundles, which was introduced in Section 4.1. In this section, we discuss the integrability of transitive algebroids. We show that integrable transitive algebroids always come from principal bundles. Then using this fact, we can argue that principal connections are the only sort of connection that can induce some notion of connection on algebroids, further endorsing the definitions in Section 4.1. These statements break down into a few lemma's.

Lemma 4.13. *The Atiyah algebroid associated to a principal G-bundle $pr : P \to M$ is precisely the algebroid of the gauge groupoid $(P \times P)/G \rightrightarrows M$.*

Proof. To prove this, we compute the algebroid associated to the gauge groupoid and see that we find the Atiyah algebroid $TP/G \to M$. To find the vector bundle $A \to M$ associated to $\mathcal{G} = (P \times P)/G$, we determine its fibers

$$A_x = d_{1_x} s^{-1}(x).$$

Fix for any $x \in M$ one point in its fiber $p_x \in P_x$. We will use these basepoints to work with homotopy classes. We can now compute the fiber of A above x:

$$\begin{aligned} A_x &= T_{[p_x,p_x]} \left(\{ [(q, p_x)] \mid q \in P \}\right) \\ &\cong (T_{p_x} P \times T_{p_x}\{p_x\}) \,/ \sim \end{aligned}$$

where $\sim$ is the equivalence relation induced by the G-action. We see indeed $A = TP/G$. With a similar computation using fixed basepoints in each fiber, it follows that

$$\mathfrak{X}^s_{inv}\left((P \times P)/G\right) \cong \mathfrak{X}(P)^G,$$

so the Lie bracket on the Atiyah algebroid agrees with the bracket on the algebroid of the gauge groupoid. Finally, for both algebroids the anchor is defined as the differential of the projection map. We see that the Atiyah algebroid induced by a principal bundle $P \to M$ is the algebroid of the gauge groupoid of $P \to M$. □

We now also have an alternative proof for Corollary 2.37:

Corollary. *Let $\mathcal{G}$ be a groupoid with connected source fibers. If $\mathcal{G}$ is transitive, then its algebroid is transitive as well.*

Proof. We know by Proposition 2.18 that a transitive groupoid $\mathcal{G}$ is isomorphic to a gauge groupoid for some principal bundle $P \to M$. Then the algebroid of $\mathcal{G}$ is isomorphic to the Atiyah algebroid $TP/G \to M$, which transitive as its anchor $\rho = d\text{pr}$ is surjective. □

Corollary 4.14. *Any integrable transitive algebroid is the Atiyah algebroid induced by a principal bundle.*

Proof. Let A be an integrable transitive algebroid. Then its Weinstein groupoid $\mathcal{G}(A)$ is transitive as well, and integrates A. Since $\mathcal{G}(A)$ is transitive, it is isomorphic to the gauge groupoid of some principal bundle $P \to M$. Then A must be isomorphic to the induced Atiyah algebroid $TP/G \to M$ by Lemma 4.13. □

The point of this section is to show the following. Any transitive algebroid that is also integrable, is actually an Atiyah algebroid by this corollary, and connections on the algebroid are naturally induced by principal bundle connections by Lemma 4.10. These integrable transitive algebroids sit inside a larger class of transitive algebroids. For general transitive algebroids, the notions of connection can be extended naturally as we have discussed. Moreover, since the only groupoids integrating transitive algebroids are gauge groupoids, there is no other natural notion of connection around on our groupoids than principal bundle connections.

Example 4.15. As an application of this theory, we consider again the case introduced in Section 3. Consider a vector bundle $E \to M$, and the induced general linear groupoid $GL(E) \rightrightarrows M$ consisting of tuples (y, A, x) where $x, y \in M$ and $A : E_x \to E_y$ a linear isomorphism. We have considered its Lie algebroid $\mathfrak{gl}(E)$, and proven that its space of sections is isomorphic to the space of derivations on E: $\Gamma(\mathfrak{gl}(E)) \cong \mathcal{DO}(E)$.

In order to prove this, we used our first example of a classical Atiyah sequence:

$$\Gamma(End(E)) \to \Gamma(\mathfrak{gl}(E)) \to \mathfrak{X}(M).$$

Since any two points in M admit an isomorphism between their fibers in E, the groupoid $GL(E)$ is transitive. This also implies its algebroid $\mathfrak{gl}(E)$ is transitive, and this sequence is indeed exact. Now there is some principal bundle over M, for which the induced gauge groupoid is $GL(E)$, and for which the space of sections of the induced Atiyah algebroid is $\mathcal{D}er(E)$. Of course this principal bundle is the frame bundle of E. We now make this more explicit.

The principal bundle associated to a transitive groupoid is found by considering the source fiber at some fixed base point $x \in M$. We have

$$s^{-1}(x) = \{\, (y, A, x) \mid y \in M,\ A : E_x \to E_y \text{ a linear isomorphism} \,\},$$

and its fiber at y is precisely

$$s^{-1}(x)_y = \{\, (y, A, x) \mid A : E_x \to E_y \text{ a linear isomorphism} \,\}.$$

Its structure group is the isotropy group at x:

$$\mathcal{G}_x = \{\, (x, A, x) \mid A : E_x \to E_x \text{ a linear isomorphism} \,\},$$

which acts by arrow composition. This boils down to composition of the linear isomorphisms. Then we have an isomorphism of groupoids:

$$GL(E) \cong s^{-1}(x) \times s^{-1}(x)/\mathcal{G}_x.$$

On the other hand, associated to the vector bundle $E \to M$, there is a principal bundle $Fr(E)$, the frame bundle. Its fiber at y is

$$Fr(E)_y = \{ A \mid A : \mathbb{R}^k \to E_y \text{ a linear isomorphism} \},$$

and its structure group is

$$GL_k(\mathbb{R}) = \{ M \mid M : \mathbb{R}^k \to \mathbb{R}^k \},$$

where k is the rank of the vector bundle E. This acts on the frame bundle by matrix multiplication.

Now since E is a vector bundle of rank k, we can choose an isomorphism $\phi : E_x \to \mathbb{R}^k$. With this identification, we find that the principal $\mathcal{G}_x$-bundle $s^{-1}(x)$ is isomorphic to the frame bundle $Fr(E)$.

This also gives another description of the algebroid $\mathfrak{gl}(E)$; we see that this is the Atiyah algebroid

$$\mathfrak{gl}(E) = (T\, Fr(E))/GL_k(\mathbb{R}).$$

We see that Section 3 was an example of the theory developed in this chapter, but the additional vector bundle structure allowed us to develop the theory of groupoid and algebroid representations as well.

Considering connections, recall that vector bundle connections on E are in 1-1 corresondence with principal connections on $Fr(E)$. This implies that a connection on the Atiyah algebroid $\mathfrak{gl}(E)$ is the same thing as a vector bundle connection on E. Note that these connections are really different objects than the A-connections ∇ that we associate to an algebroid representation of A, even though we use the same symbol.

4.4 Atiyah sequence of a proper fibration

In Part II and Part III we will consider proper fibrations, and principal bundles will form the 'base example' of the developed theory. We first define what we understand by proper fibrations, and then we define a sequence for these spaces that behaves like the Atiyah sequence associated to principal bundles.

Definition 4.16. A **proper fibration** is a surjective submersion $\text{pr} : \mathcal{N} \to M$ that is additionally a proper map.

Remark 4.17.

- These conditions imply that a proper fibration is always a locally trivial fibration by Ehresmann's lemma.
- We assume properness to force completeness of certain vector fields, which we will make more precise below.

Definition 4.18. Given a proper fibration $\text{pr} : \mathcal{N} \to M$, an **Ehresmann connection** is one of two equivalent descriptions:

- a horizontal subbundle $\mathcal{H} \subset T\mathcal{N}$;
- a horizontal lifting map $\hslash : \mathfrak{X}(M) \to \mathfrak{X}(\mathcal{N})$.

These descriptions are related by $\mathrm{Im}(\hslash) = \mathcal{H}$.

By properness of pr, vector fields in the image $\hslash(\mathfrak{X}(M))$ are complete, which will be a crucial property in Section 5.

Recall the classical Atiyah sequence (Equation (9)) associated to a principal bundle. We now construct a similar sequence associated to a proper fibration. At any point $p \in \mathcal{N}$, we can construct a short exact sequence of vector spaces:

$$0 \to T_p^v\mathcal{N} \to T_p\mathcal{N} \to T_{\mathrm{pr}(p)}M \to 0,$$

where $T_p^v\mathcal{N}$ denotes the vectors at p tangent to the fibers of pr. On the level of sections, we find a sequence of vector bundles over $\mathcal{N}$:

$$0 \to \mathfrak{X}^v(\mathcal{N}) \to \mathfrak{X}^{proj}(\mathcal{N}) \to \mathfrak{X}(M) \to 0,$$

where $\mathfrak{X}^{proj}(\mathcal{N})$ denotes the projectable vector fields. These are vector fields $X \in \mathfrak{X}(\mathcal{N})$ for which there exists $X' \in \mathfrak{X}(M)$ such that $d_p\mathrm{pr}(X_p) = X'_{\mathrm{pr}(p)}$ for all $p \in \mathcal{N}$.

We now want to interpret this last sequence as a sequence of vector bundles over M, in order to have any similarity to the classical Atiyah sequences. This forces us to deal with the following infinite-dimensional vector bundles:

- $\mathfrak{X}^v \to M$ with fibers $\mathfrak{X}^v_x := \mathfrak{X}(\mathcal{N}_x)$, vector fields on the fiber $\mathcal{N}_x$ tangent to the fiber. Sections of this vector bundle are precisely vertical vector fields on $\mathcal{N}$:

$$\Gamma(\mathfrak{X}^v) = \mathfrak{X}^v(\mathcal{N}).$$

- $\mathfrak{X}^{proj} \to M$ with fibers $\mathfrak{X}^{proj}_x := \mathfrak{X}^{proj}(\mathcal{N}_x)$, vector fields on the fiber $\mathcal{N}_x$ which are projectable. Sections of this vector bundle are projectable vector fields on $\mathcal{N}$:

$$\Gamma(\mathfrak{X}^{proj}) = \mathfrak{X}^{proj}(\mathcal{N}).$$

These give rise to a short exact sequence of vector bundles over M:

$$0 \to \mathfrak{X}^v \to \mathfrak{X}^{proj} \to TM \to 0.$$

On the level of sections, we find a sequence of infinite-dimensional vector bundles over M, which we call the **Atiyah sequence associated to the fibration**:

$$0 \to \mathfrak{X}^v(\mathcal{N}) \to \mathfrak{X}^{proj}(\mathcal{N}) \to \mathfrak{X}(M) \to 0.$$

In this description, an element $V \in \mathfrak{X}^v(\mathcal{N})$ is a section of the bundle $\mathfrak{X}^v \to M$, and for any $x \in M$, $V_x \in \mathfrak{X}(\mathcal{N}_x)$ is a vector field such that

$$V_x(p) := V_p \in T_p\mathcal{N}_x, \ \forall p \in \mathcal{N}_x.$$

Comparing this sequence to the Atiyah sequence of a principal bundle, we observe that in the Atiyah sequence, the middle term is actually an algebroid over M. In the case of a fibration, it is not clear whether the same is true. In Proposition 7.1, we will see that (under additional assumptions) there is a finite-dimensional subsequence which does have an algebroid over M as the middle term.

Part II
Holonomy and the Ambrose-Singer theorem

In this part, we introduce holonomy groups and state the Ambrose-Singer theorem. We will first introduce holonomy for principal bundles, and discuss several properties of these groups. Then we generalize to proper fibrations. In both cases, we fix a connection, which allows us to lift paths from the base manifold to paths in fibers. Elements of the holonomy group are then found by lifting along loops in the base. These groups will also have a smooth structure. The Lie algebra of the holonomy groups will be determined entirely by the curvature of the chosen connection. This is the statement of the Ambrose-Singer theorem. This theorem was originally stated for principal bundles, but can be generalized to surjective submersions. For the classical proof of this theorem, we refer to [AS53] or [KN96]. In this thesis we will not focus on this classical proof; instead, we will take a different viewpoint using algebroids in Part III, from which we try to prove the theorem and gain other insights.

5 Holonomy groups

In this chapter, we follow mostly the description in the book of Kobayashi and Nomizu ([KN96]).

5.1 Holonomy group of a principal bundle

Consider a principal G-bundle $\mathrm{pr} : P \to M$ (with a right G-action on P) with a connection $\hbar : \mathfrak{X}(M) \to \mathfrak{X}(P)^G$, corresponding to a horizontal subbundle $\mathcal{H} \subset TP$. A **horizontal curve** is a curve $u(t)$ in P, such that for all times t its speed is a horizontal vector, i.e., $\dot{u}(t) \in \mathcal{H}_{u(t)}$. The path $\gamma(t) := \mathrm{pr} \circ u(t)$ on M is called the base path of u.

Starting from a path γ in M, we can lift γ to a horizontal path u in P using the connection $\hbar$. For any initial point u_0 in the fiber at $\gamma(0)$, the horizontal path is determined by the ODE:

$$\begin{cases} \frac{du}{dt}(t) = \hbar_{u(t)}\left(\frac{d\gamma}{dt}(t)\right) \\ u(0) = u_0. \end{cases} \tag{11}$$

It is well-known that this admits a unique solution, see for example Proposition 3.1 in Chapter 2 in [KN96]. We say that γ admits a unique lift to a horizontal curve u in P. Now we can define parallel transport on principal bundles.

Definition 5.1. Parallel transport on P along γ with respect to the connection $\hbar$ is the mapping $\tau_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ that sends $u_0 \in P_{\gamma(0)}$ to $u(1)$, where $u(t)$ is the unique horizontal lift of γ (i.e., the solution to Equation (11)).

When discussing holonomy groups, we will use the following properties of parallel transport maps.

Lemma 5.2. *The parallel transport map commutes with the G-action: for $p \in P_x$, $g \in G$ and γ a path from x to y, we have that $\tau_\gamma(pg) = \tau_\gamma(p)g$.*

Proof. Denote by $u_p(t)$ the solution of Equation (11) for $u(0) = p$, and by $u_{pg}(t)$ the solution for $u(0) = pg$. Then we see that at time $t = 0$, $u_{pg}(0) = u_p(0)g$. Consider now at any time t the differential of $u_p(t)g = R_g(u_p(t))$, which is by the chain rule:

$$d_{u_p(t)}R_g\left(\frac{du_p(t)}{dt}\right) = d_{u_p(t)}R_g\left(\hbar_{u_p(t)}\left(\frac{d\gamma}{dt}\right)\right) = \hbar_{u_p(t)g}\left(\frac{d\gamma}{dt}\right).$$

Here we used the fact that $\hbar$ maps to right-invariant vector fields. We see that this is equal to the speed of the path $u_{pg}(t)$, thus by uniqueness of ODE's those paths agree at any time t. □

Lemma 5.3. *The parallel transport map $\tau_\gamma : P_{\gamma(0)} \to P_{\gamma(1)}$ is a diffeomorphism of G-torsors.*

Proof. It clearly has an inverse given by $\tau_{\gamma^{-1}}$. By the last lemma, it commutes with the G-action on both fibers. Smoothness follows from a general theorem on ODE's, stating that equations of the form of Equation (11) have a smooth solution, see Appendix 1 in [KN96]. □

Note that when γ is a loop based at $x \in M$, the parallel transport map is a diffeomorphism of P_x. Now we come to the definition of holonomy groups. To compose elements in these groups, we have to work with piecewise differentiable loops γ in the base. Note that for piecewise smooth loops, Equation (11) admits a piecewise smooth solution, and Lemma 5.2 and Lemma 5.3 still apply (see also Chapter 2, Section 3 in [KN96]).

Definition 5.4. The holonomy group based at x is the collection of parallel transport maps τ_γ where γ is a piecewise differentiable loop based at x:

$$\mathrm{Hol}_x^{\hbar} = \{\, \tau_\gamma : P_x \to P_x \mid \gamma \text{ a loop based at } x \,\}.$$

The restricted holonomy group is defined by restricition our attention to contractible loops:

$$\mathrm{Hol}_x^{\hbar,0} = \{\, \tau_\gamma : P_x \to P_x \mid \gamma \text{ a contractible loop based at } x \,\}.$$

Lemma 5.5. *$Hol_x^{\hbar}$ is a group, and $Hol_x^{\hbar,0}$ is a subgroup of $Hol_x^{\hbar}$.*

Proof. The constant loop at x induces a constant parallel transport map, which is the identity element and is in both groups. The group operation is concatenation of paths downstairs, which is again piecewise smooth. Inverse elements are found by transporting along the inverse of the base path. Since concatenation and inverses of contractible loops remain contractible, $\mathrm{Hol}_x^{\hbar,0}$ is a subgroup. □

We now study the dependence of $\mathrm{Hol}_x^{\hbar}$ on the basepoint x. First of all, we remark that for principal bundles, holonomy groups are often introduced with basepoints in P, for example in [KN96]. We will now introduce this viewpoint as well. Choosing a basepoint $p \in P_x$ in P, we use the fact that P_x is a G-torsor. For any element $\tau_\gamma \in \mathrm{Hol}_x^{\hbar}$, we have $\tau_\gamma(p) \in P_x$. Then there exists an element $g \in G$ such that $\tau_\gamma(p) = pg$. This allows us to define holonomy groups with basepoints in P as follows:

Definition 5.6. The holonomy group based at $p \in P$ is defined as

$$\mathrm{Hol}_p^{\hbar} = \{\, g \in G \mid \tau_\gamma(p) = pg \text{ for some piecewise smooth loop } \gamma \text{ based at } \mathrm{pr}(p) \,\}.$$

The restricted holonomy group based at $p \in P$ is defined as

$$\mathrm{Hol}_p^{\hbar,0} = \{\, g \in G \mid \tau_\gamma(p) = pg \text{ for some contractible piecewise smooth loop } \gamma \text{ based at } \mathrm{pr}(p) \,\}.$$

Lemma 5.7. *These are subgroups of G: $Hol_p^{\hbar,0} \leq Hol_p^{\hbar} \leq G$.*

Proof. Both groups contain the identity element e, obtained from taking the parallel transport along the constant path at $\mathrm{pr}(p)$. If $g, h \in \mathrm{Hol}_p^{\hbar}$, then they are 'induced' by loops γ_g and γ_h, and it is easy to check that the concatenation $\gamma_g \circ \gamma_h$ induces $hg \in \mathrm{Hol}_p^{\hbar}$, and the inverse path γ_g^{-1} induces $g^{-1} \in \mathrm{Hol}_p^{\hbar}$. The same holds for $\mathrm{Hol}_p^{\hbar,0}$, and by the same argument as above this is closed under these operations (and thus a subgroup itself). □

Now the following two statements hold for these groups. We assume here that M is path-connected, and otherwise this holds for any path component of M.

- For all $p \in P_x$, there is an isomorphism $\mathrm{Hol}_x^{\hbar(,0)} \cong \mathrm{Hol}_p^{\hbar(,0)}$.
- For any $p, q \in P$, there is an element $g \in G$ such that $\mathrm{Hol}_p^{\hbar(,0)} = g^{-1}\mathrm{Hol}_q^{\hbar(,0)} g$.

Together, these statements imply that holonomy groups are independent of basepoints (in M or P), up to isomorphisms and conjugation. We now prove these two statements.

Lemma 5.8. *For any $p \in P_x$, the (restricted) holonomy groups based at x and p are isomorphic.*

Proof. To prove this lemma, we first rewrite the holonomy group $\mathrm{Hol}_x^{\hbar}$ based at x in the following way. We know that the maps $\tau_\gamma : P_x \to P_x$ are diffeomorphisms of G-torsors, and $\tau_\gamma(p) = pg$ for some $g \in G$. We can rewrite the parallel transport map as a map $\tau'_\gamma : P_x \to G$ mapping p to $g \in G$ such that $\tau_\gamma(p) = pg$, and the holonomy group based at x is equivalent to

$$\mathrm{Hol}_x^{\hbar} \cong \{\, \tau'_\gamma : P_x \to G \mid p \cdot \tau'_\gamma(p) = \tau_\gamma(p) \ \forall p \in P_x \,\}.$$

Now it is clear by construction that the group $\mathrm{Hol}_p^{\hbar}$ is determined by evaluating all maps τ'_γ at the point p, giving an explicit mapping from $\mathrm{Hol}_x^{\hbar}$ to $\mathrm{Hol}_p^{\hbar}$.

In the other direction, suppose the group $\mathrm{Hol}^{\not{h}}_p$ is known. This means that for any piecewise differentiable loop γ, we know one value of the map τ'_γ, namely the value at p. Then by Lemma 5.2, the value at any other point $q = pg \in P_x$ is determined as well by $\tau'_\gamma(q) = \tau'_\gamma(p)g$. So we can recover $\mathrm{Hol}^{\not{h}}_x$ entirely from $\mathrm{Hol}^{\not{h}}_p$ as well, and we have found a bijection between these two groups.

For the restricted holonomy groups we find a bijection in precisely the same way. Finally it should be clear from the discussion above (for example the proof Lemma 5.7) that the group structures agree, and we conclude that these bijections are group isomorphisms. □

In order to prove the second statement, we look at yet another description of holonomy groups. This description is used in, for example, [AS53]. We define an equivalence relation on P by saying $p \sim q$ if p and q can be joined by a horizontal curve (a piecewise differentiable curve in P with speeds in $\mathcal{H}$). Then the holonomy group based at $p \in P$ is defined as:

$$\mathrm{Hol}^{\not{h}}_p = \{ g \in G \mid p \sim pg \}.$$

It is straightforward to check that this is precisely the holonomy group at p defined above. The advantage of this description is that we can now easily compare holonomy groups with basepoints in different fibers. We assume here that M is path-connected, and otherwise, the following holds for each path component.

Let $p, q \in P$ and consider the points $\mathrm{pr}(p)$ and $\mathrm{pr}(q)$ in M. Since M is path-connected there is a path γ from $\mathrm{pr}(p)$ to $\mathrm{pr}(q)$. Parallel transport along this path maps p to some point in the fiber of $\mathrm{pr}(q)$, so for some $g \in G$, $p \sim qg$.

Lemma 5.9. *With this element g, $Hol^{\not{h}}_p = g^{-1} Hol^{\not{h}}_q g$.*

Proof. Suppose $h \in \mathrm{Hol}^{\not{h}}_q$, so $q \sim qh$. We need to prove $g^{-1}hg \in \mathrm{Hol}^{\not{h}}_p$. Using Lemma 5.2, we find the following sequence of equivalences:

$$pg^{-1}hg \sim qhg \sim qg \sim p$$

where we used the equivalences $q \sim pg^{-1}$, $qh \sim q$ and $qg \sim p$ in that order. This shows that $g^{-1}hg \in \mathrm{Hol}^{\not{h}}_p$. The other direction follows with a similar computation. □

The same lemma holds for the reduced holonomy groups. Combining this lemma with Lemma 5.8, we find the promised result:

Corollary 5.10. *When M is path-connected, the (reduced) holonomy group is independent of basepoint in M or P, up to isomorphisms and/or conjugation.*

Remark 5.11. In the literature, holonomy groups are most often defined with basepoints in P. This approach has the big advantage that one can use the inherited group structure of G. In our case, we want to generalise to proper fibrations. For proper fibrations, there is no natural group structure and the description of holonomy groups with basepoints in the fibers is not insightful at all. For this reason, we also focus on basepoints in M when describing holonomy groups of principal bundles.

5.2 Holonomy groups of proper fibrations

Consider now a proper fibration $\mathrm{pr} : \mathcal{N} \to M$ (see Definition 4.16), and fix an Ehresmann connection given by a horizontal subbundle $\mathcal{H} \subset T\mathcal{N}$. Recall that this can also be seen as a horizontal lifting map $\hbar : \mathfrak{X}(M) \to \mathfrak{X}(\mathcal{N})$, such that $d\mathrm{pr} \circ \hbar$ is the identity map. Clearly $\hbar$ maps into projectable vector fields on $\mathcal{N}$. Pointwise, $\hbar_p : T_{\mathrm{pr}(p)}M \to T_p\mathcal{N}$ maps v to the unique vector in $\mathcal{H}_p$ such that $d_p\mathrm{pr}(\hbar_p(v)) = v$. Using this lifting map we can define parallel transport for proper fibrations similarly to the principal bundle case. Throughout we will assume curves to be piecewise differentiable.

Definition 5.12. Let γ be a piecewise differentiable curve in M. Parallel transport along γ with respect to the connection $\hbar$ is the map $\tau_\gamma : \mathcal{N}_{\gamma(0)} \to \mathcal{N}_{\gamma(1)}$, mapping $u_0 \in \mathcal{N}_{\gamma(0)}$ to $u(1)$ where $u(t)$ is the unique solution of the differential equation

$$\begin{cases} \frac{du}{dt}(t) = \hbar\left(\frac{d\gamma}{dt}(t)\right), \\ u(0) = u_0. \end{cases}$$

Remark 5.13. For this definition, it is important that pr is a proper map. We can view $\gamma(t)$ as an integral curve of some vector field X on the base. Then the horizontal lift $u(t)$ can be interpreted as an integral curve of the lifted vector field $\hbar(X)$. We say the connection is complete if this solution exists on the entire interval $[0, 1]$, i.e., if the lifted vector field $\hbar(X)$ is complete. Choosing X compactly supported, we see that properness of the map pr ensures completeness of the connection, and the horizontal lift is well-defined on the interval $[0, 1]$.

Lemma 5.14. *For any curve γ on M, the parallel transport map $\tau_\gamma : \mathcal{N}_{\gamma(0)} \to \mathcal{N}_{\gamma(1)}$ is a diffeomorphism.*

Proof. The proof is similar to that of Lemma 5.3. By smoothness of solutions of such ODE's, proven in Appendix 1 of [KN96], it follows that all such maps are smooth, and the inverse of τ_γ is given by parallel transport along the inverse path $\tau_{\gamma^{-1}}$. □

Definition 5.15. The holonomy group based at x is the collection of parallel transport maps τ_γ where γ is a piecewise differentiable loop based at x:

$$\mathrm{Hol}_x^{\hbar} = \{\, \tau_\gamma : P_x \to P_x \mid \gamma \text{ a loop based at } x \,\}.$$

The restricted holonomy group is defined by restricition our attention to contractible loops:

$$\mathrm{Hol}_x^{\hbar,0} = \{\, \tau_\gamma : P_x \to P_x \mid \gamma \text{ a contractible loop based at } x \,\}.$$

The holonomy group has a natural group structure, and the restricted holonomy group is a subgroup of it, with the same proof as for Lemma 5.5.

Lemma 5.16. *For any two points x and y in a path component of M, the holonomy groups $Hol_x^{\hbar}$ and $Hol_y^{\hbar}$ are isomorphic.*

Proof. Consider a path γ in M such that $\gamma(0) = x$ and $\gamma(1) = y$. Note that the parallel transport $\tau_\gamma : \mathcal{N}_x \to \mathcal{N}_y$ induces a map by conjugation

$$\tau_\gamma^C : \mathrm{Diff}(\mathcal{N}_y) \to \mathrm{Diff}(\mathcal{N}_x),\ \phi \mapsto \tau_\gamma^{-1} \circ \phi \circ \tau_\gamma.$$

This is clearly smooth, and conjugation with the inverse γ^{-1} gives the inverse map, so this is an isomorphism. This maps the holonomy group based at x to the holonomy group based at y, from which we conclude these are isomorphic. □

A big difference with the previous section is that it is not very interesting to talk about holonomy groups based at points in $\mathcal{N}$; these do not inherit a natural group structure from $\mathcal{N}$, so this does not add new insights. We do have in both cases the important property that the holonomy groups are independent of basepoints (in M). This suggests that, once we assume they are Lie groups, their Lie algebras should also be in some sense independent of base points. This motivates some choices that we make in the next chapter, where we introduce the candidate for this Lie algebra.

6 The classical Ambrose-Singer theorem

Finally in this chapter we discuss the classical Ambrose-Singer theorem. We have already mentioned that this theorem is about the Lie algebras of the holonomy groups. We first introduce the candidate for this Lie algebra, which is the holonomy Lie algebra. We do this immediately for the general case of surjective submersions, but of couse this covers principal bundles as well.

6.1 The holonomy Lie algebra

Recall that the holonomy group is defined for a given connection $\hbar : \mathfrak{X}(M) \to \mathfrak{X}(\mathcal{N})$. We can associate to this connection the notion of curvature, which measures the failure of $\hbar$ to preserve Lie brackets and is defined as follows:

Definition 6.1. The **curvature** associated to a connection $\hbar : \mathfrak{X}(M) \to \mathfrak{X}(\mathcal{N})$ is the map

$$\Omega : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}^v(\mathcal{N}),\ (X, Y) \mapsto \hbar([X, Y]) - [\hbar(X), \hbar(Y)].$$

Lemma 6.2. *The curvature Ω maps to projectable vector fields on $\mathcal{N}$.*

Proof. Let $X, Y \in \mathfrak{X}(M)$. Applying $d\mathrm{pr}$ to $\Omega(X, Y)$ gives

$$d\mathrm{pr}\,(\Omega(X, Y)) = d\mathrm{pr} \circ \hbar([X, Y]) - d\mathrm{pr}[\hbar(X), \hbar(Y)].$$

Since $d\mathrm{pr} \circ \hbar = \mathrm{id}$ and the pushforward of pr commutes with Lie brackets of projectable vector fields, we find

$$d\mathrm{pr}\,(\Omega(X, Y)) = [X, Y] - [X, Y] = 0.$$

□

Pointwise, the curvature gives a map

$$\Omega_x : T_xM \times T_xM \to \mathfrak{X}(\mathcal{N}_x).$$

We are now interested in the image of this map $\mathrm{Im}(\Omega_x) \subset \mathfrak{X}(\mathcal{N}_x)$. The moral of the Ambrose-Singer theorem is that this image tells us what happens infinitesimally when doing parallel transport along loops based at x, i.e., this image is the Lie algebra of $\mathrm{Hol}_x^{\hbar}$.

However, simply defining these Lie algebras as $\mathrm{Im}(\Omega_x)$ gives rise to a problem. Recall Lemma 5.16 (or, in the case of principal bundles, Corollary 5.10), where we saw that holonomy groups at different basepoints are isomorphic under the conjugation map τ_γ^C. This suggests that the corresponding Lie algebras at different basepoints should also be isomorphic under parallel transport maps. Taking this into account, we can define the holonomy Lie algebras.

Let γ be any path in M starting at x. Then τ_γ induces a map of vector fields, by 'pulling back' vector fields on the fiber at $\gamma(1)$ to the fiber at x:

$$\tau_\gamma^* : \mathfrak{X}(\mathcal{N}_{\gamma(1)}) \to \mathfrak{X}(\mathcal{N}_{\gamma(0)}),$$
$$\tau_\gamma^*(X)(p) = d_{\tau_\gamma(p)}\tau_\gamma^{-1}(X(\tau_\gamma(p))).$$

Using this map, we can 'pull back' the image of the curvature at any point y back to x (as $\mathrm{Im}(\Omega_y) \subset \mathfrak{X}(\mathcal{N}_y)$). This defines the holonomy Lie algebra at x:

Definition 6.3. The **holonomy Lie algebra** at a point x, denoted $\mathfrak{hol}_x^{\hbar}$, is defined by the span of all transported images $\tau_\gamma^*(\mathrm{Im}(\Omega_{\gamma(1)}))$ where γ is a path in M starting at x.

We see that in any path-connected neighbourhood $U \subset M$, for any points x, $y \in U$ and γ a path from x to y it follows that

$$\tau_\gamma^* : \mathfrak{hol}_x^{\hbar} \to \mathfrak{hol}_y^{\hbar}$$

is an isomorphism. This has resolved the issue we had before, where holonomy Lie algebras at different basepoints could be different from eachother. Additionally, these isomorphisms give in some sense a local trivialization

$$\mathfrak{hol}^{\hbar}|_U \cong U \times \mathfrak{hol}_x^{\hbar}$$

where we consider $\mathfrak{hol}^{\hbar}$ as a bundle over M, with fiber $\mathfrak{hol}_x^{\hbar}$ at x.

Recall the description of $\mathfrak{X}^v(\mathcal{N})$ as the space of sections of the infinite-dimensional vector bundle $\mathfrak{X}^v \to M$ introduced in Section 4.4. Assuming that each $\mathfrak{hol}_x^{\hbar}$ is finite-dimensional, we can define a space of sections

$$\Gamma(\mathfrak{hol}_x^{\hbar}) := \{\, s \in \mathfrak{X}^v(\mathcal{N}) \mid s(x) \in \mathfrak{hol}_x^{\hbar}\ \forall x \in M \,\}.$$

This is clearly a $C^\infty(M)$-module. Furthermore, by covering M by a finite number of connected charts it can be shown this is a finitely generated projective module, which by Swan's theorem implies we get a vector bundle $\mathfrak{hol}^{\hbar} \to M$. Together with the local trivializations and the following lemma, it is a locally trivial bundle of Lie algebras.

Lemma 6.4. *For any point x, the holonomy Lie algebra $\mathfrak{hol}_x^{\hbar}$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{N}_x)$.*

It is possible to prove this lemma directly, but since the proof will arise naturally inside the proof of 7.1, we postpone it until then.

6.2 The Ambrose-Singer theorem

Finally we can discuss the statement of the Ambrose-Singer theorem. In [AS53], this theorem is proven for principal bundles, and we start by stating the version for principal bundles.

Theorem 6.5. *Let $P \to M$ a principal G-bundle endowed with a connection $\hbar$. Then*

1. *Each group $Hol_x^{\hbar}$ is a Lie group with identity component $Hol_x^{\hbar,0}$;*
2. *The Lie algebra of $Hol_x^{\hbar}$ is $\mathfrak{hol}_x^{\hbar}$.*

In this thesis, we focus on proper fibrations, for which we can state a more general version of this theorem. In our discussion, we will discuss holonomy from the viewpoint of algebroids. In order to this, we need an additional assumption on the connection.

Definition 6.6. A **holonomic connection** on a proper fibration $\mathrm{pr} : \mathcal{N} \to M$ is a connection $\hbar$ such that each holonomy Lie algebra $\mathfrak{hol}_x^{\hbar} \subset \mathfrak{X}(\mathcal{N}_x)$ is finite-dimensional, and consists of vector fields that are either nowhere vanishing or trivial.

Remark 6.7. Let $P \to M$ be a principal G-bundle with a connection $\hbar$. Then $\hbar$ is G-invariant, from which it immediately follows that elements of $\mathfrak{hol}_x^{\hbar}$ are either nowhere vanishing or trivial. We see that principal bundle connections automatically satisfy this condition.

In Proposition 7.1 it will become apparant why we need this condition. We can now state the generalised version of the Ambrose-Singer theorem.

Theorem 6.8. *Let $\hbar$ be a holonomic connection on a fibration $pr : \mathcal{N} \to M$. Then*

1. *Each group $Hol_x^{\hbar}$ is a Lie group, with identity component $Hol_x^{\hbar,0}$,*
2. *The Lie algebra of $Hol_x^{\hbar}$ is $\mathfrak{hol}_x^{\hbar}$,*
3. *There exists a principal $Hol_x^{\hbar}$-bundle $P \to M$ such that there is an isomorphism between $\mathcal{N} \to M$ and $P \times_{Hol_x^{\hbar}} \mathcal{N}_x \to M$,*
4. *There exists a principal connection on P that induces the original connection $\hbar$.*

Recall that in Section 4, we constructed the Atiyah sequences of princpal bundles and of fibrations. We already hinted at the fact that the sequence of a fibration will contain a subsequence that is due to some algebroid (which we will prove in Proposition 7.1). With this subsequence, the situation looks a lot like the principal bundle case, which is further endorsed by items 3 and 4 in this theorem. These items suggest as well that the situation for proper fibrations with holonomic connections is very closely related to the principal bundle case.

We now explain *why* we will not discuss the classical proof here, but take a different approach. We have shown that (if M is path-connected) all holonomy groups are isomorphic, independent of basepoint. In the classical discussion of the Ambrose-Singer theorem this is neglected. Mackenzie in [Mac05] has initiated the idea that holonomy should be discussed in the context of transitive groupoids; the holonomy groups are simply the isotropy groups of a transitive groupoid (recall that in a transitive groupoid, indeed all isotropy groups are isomorphic). Proving the Ambrose-Singer theorem then comes down to computing the algebroid of this groupoid.

In Mackenzie's approach, it is proven that the holonomy groups are Lie groups, and then a holonomy groupoid is constructed with these groups as isotropy groups. Then he proves that the algebroid of this groupoid has as isotropy algebra the holonomy Lie algebra. We will revisit this approach in Section 8.2. In the coming chapter, we take a different approach. Starting with the holonomy Lie algebras, we will see that an algebroid arises in a very natural way. We will call this the Ambrose-Singer algebroid. Then we consider the Weinstein groupoid of this algebroid, and attempt to compute its isotropy groups. The goal is to prove that the isotropy groups are isomorphic to the holonomy groups.

Part III
Relating the Ambrose-Singer theorem to algebroids

In this final part, we combine the theory of holonomy with the theory of groupoids and algebroids. This 'new perspective' is due to Mackenzie in [Mac05], and describes holonomy groups as the isotropy groups of a transitive groupoid. As we have mentioned, we will take a different approach than that of Mackenzie.

First, we will show that a holonomic connection $\hbar$ on a proper fibration $\mathcal{N} \to M$ induces an algebroid, which we will call the Ambrose-Singer algebroid. This algebroid will have the bundle $\mathfrak{hol}^{\hbar}$ as its isotropy Lie algebra. Furthermore, this will be a transitive algebroid with a free action, forcing it to be integrable. We consider its integration, and attempt to prove that its isotropy groups are isomorphic to the holonomy groups, which will prove the Ambrose-Singer theorem.

Besides discussing this classical result, we will be able to prove item 3 and 4 of the generalised Ambrose-Singer theorem (Theorem 6.8), showing that $\mathcal{N}$ is isomorphic to the fibered product of a principal bundle.

7 Re-interpreting the holonomy Lie algebra in the theory of algebroids

We consider, as before, a proper fibration $\mathrm{pr} : \mathcal{N} \to M$ endowed with a connection $\hbar$. Recall that the curvature of this connection gives rise to a Lie algebra $\mathfrak{hol}^{\hbar}_x$ at every point in M. We first show that if the connection is holonomic, we can construct a transitive Lie algebroid which is integrable and acts on $\mathrm{pr} : \mathcal{N} \to M$. Then we consider path homotopies for this algebroid, which we use to analyse its Weinstein groupoid with induced groupoid action.

7.1 The Ambrose-Singer algebroid $A^{\hbar}$

Proposition 7.1. *Let $\hbar$ be a connection on a proper fibration $\mathcal{N} \to M$. The following are equivalent:*

- *$\hbar$ is a holonomic connection;*
- *there exists a transitive Lie algebroid $A \to M$ with a free action $a : \Gamma(A) \to \mathfrak{X}(N)$, such that the image $\hbar(\mathfrak{X}(M))$ lies inside Im(a).*

Moreover, in this case there exists a smallest such algebroid denoted $A^{\hbar}$, and the isotropy bundle of this algebroid is precisely $\mathfrak{hol}^{\hbar}$. This is the ***Ambrose-Singer algebroid****.*

Proof. First, suppose such an algebroid A with free action a exists. The freeness of this action has a few consequences. First of all, since a is fiberwise injective, it is injective, hence we can view it as an inclusion $a : \Gamma(A) \hookrightarrow \mathfrak{X}^{proj}(\mathcal{N})$.

Next, for any $x \in M$ we consider the isotropy Lie algebra $\mathfrak{g}_x = \mathrm{Ker}(\rho_x) \subset A_x$. The action of A restricts to

$$\begin{aligned} a_x : \mathfrak{g}_x &\to \mathfrak{X}(\mathcal{N}_x), \\ v &\mapsto X \\ \text{where } X_p &= a_p(v) \ \ (\text{for } p \in \mathcal{N}_x)\,. \end{aligned}$$

From $d\mathrm{pr} \circ a = \rho$ it follows that this does map $\mathfrak{g}_x$ to vector fields tangent to the fiber at x. We now claim that this map is an inclusion as well. The kernel of this map consists of $v \in \mathfrak{g}_x$ such that for all $p \in \mathcal{N}_x$, $a_p(v) = 0$. Since the action is assumed to be fiberwise injective, this implies $v = 0$. Hence the restriction $a_x : \mathfrak{g}_x \to \mathfrak{X}(\mathcal{N}_x)$ is injective and can be seen as an inclusion.

Lastly, from the discussion above we see that if $a_x(v)$ is zero at any point $p \in \mathcal{N}_x$, it already follows from injectivity of a_p that v is 0. In other words, the action a sends $\mathfrak{g}_x$ to vector fields that are either nowhere vanishing, or the 0-vector field.

We can now use these inclusions to construct the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \Gamma(\mathfrak{g}) & \hookrightarrow & \Gamma(A) & \overset{\rho}{\underset{\overset{\hbar}{\dashleftarrow}}{\longrightarrow}} & \mathfrak{X}(M) & \longrightarrow & 0 \\ & & \downarrow a & & \downarrow a & & \downarrow = & & \\ 0 & \longrightarrow & \mathfrak{X}^v(\mathcal{N}) & \hookrightarrow & \mathfrak{X}^{proj}(\mathcal{N}) & \overset{d\mathrm{pr}}{\underset{\underset{\hbar}{\dashleftarrow}}{\longrightarrow}} & \mathfrak{X}(M) & \longrightarrow & 0 \end{array} \tag{12}$$

Note that since $\hbar(\mathfrak{X}(M)) \subset \mathrm{Im}(a)$, we can draw the map $\hbar$ in the upper sequence as well. Furthermore, the diagram commutes; in the left square, we just have commutativity of inclusions, and in the right square we find that the Lie algebroid action property $d\mathrm{pr} \circ a = \rho$ induces commutativity.

Now we consider the inclusion $\mathfrak{g}_x \hookrightarrow \mathfrak{X}(\mathcal{N}_x)$ and recall that the holonomy group $\mathfrak{hol}_x^{\hbar}$ also lives in $\mathfrak{X}(\mathcal{N}_x)$. We will show that $\mathfrak{hol}_x^{\hbar} \subset \mathfrak{g}_x$, and since the latter is finite-dimensional, we then conclude $\mathfrak{hol}_x^{\hbar}$ is also finite-dimensional. To prove this inclusion, recall that $\mathfrak{hol}_x^{\hbar}$ is spanned by all $\hbar$-parallel transported images of the curvature (see Definition 6.3), so we aim to show that these lie in $\mathfrak{g}_x$.

First of all, consider the curvature itself:

$$\Omega(X,Y) = \hbar([X,Y]) - [\hbar(X), \hbar(Y)], \text{ where } X, Y \in \mathfrak{X}(M).$$

Since $\hbar$ maps into $\Gamma(A)$ and $\Gamma(A)$ is closed under the bracket, $\Omega(X,Y)$ lives in $\Gamma(A)$; furthermore, since ρ is a Lie algebra morphism and $\rho \circ \hbar = \mathrm{id}$, we find $\Omega_x(X_x, Y_x) \in \mathfrak{g}_x$. To generate other elements in $\mathfrak{hol}_x^{\hbar}$, we need to transport these terms with $\hbar$-parallel transport. We claim that $\hbar$-parallel transport restricted to terms in $\mathfrak{g}$ corresponds with the parallel transport of a certain connection on the vector bundle $\mathfrak{g} \to M$. First, we define this connection as follows, using the notation $[\cdot,\cdot]_A$ for the Lie bracket on sections of A:

$$\nabla^{\hbar} : \mathfrak{X}(M) \times \Gamma(\mathfrak{g}) \to \Gamma(\mathfrak{g}),\ \nabla_X^{\hbar}(\xi) = [\hbar(X), \xi]_A \text{ for } X \in \mathfrak{X}(M),\ \xi \in \mathfrak{g}.$$

It is straightforward to check that this defines a connection using the fact that $\rho([\hbar(X), \xi]) = [\rho \circ \hbar(X), \rho(\xi)] = [X, 0] = 0$ (so it indeed maps back into $\mathfrak{g}$). Now we have the following lemma, the proof of which we postpone.

Lemma 7.2. *The $\hbar$-parallel transport on the fibration $pr : \mathcal{N} \to M$ restricted to vectors coming from the inclusions $\mathfrak{g}_x \subset \mathfrak{X}(\mathcal{N}_x)$, is equivalent to the $\nabla^{\hbar}$-parallel transport on the vector bundle $\mathfrak{g} \to M$.*

By this lemma, it is immediately clear that the $\hbar$-transported images of any $\Omega_x(X_x, Y_x)$ stay inside $\mathfrak{g}$. This generates all elements of the holonomy Lie algebra, so we conclude $\mathfrak{hol}_x^{\hbar} \subset \mathfrak{g}_x$, and thus the holonomy Lie algebra is finite dimensional. Since we saw above that $\mathfrak{g}_x$ is included in $\mathfrak{X}(\mathcal{N}_x)$ by nowhere vanishing vector fields only, the same holds for $\mathfrak{hol}_x^{\hbar}$, and it follows that $\hbar$ is a holonomic connection.

In the other direction, suppose that $\hbar$ is a holonomic connection, i.e., each $\mathfrak{hol}_x^{\hbar}$ is finite-dimensional and consists of nowhere vanishing vector fields. We construct an algebroid A with a free action on $\mathcal{N}$. There are two requirements already; the image of the action should contain the image of $\hbar$, and its isotropy Lie algebra must contain $\mathfrak{hol}^{\hbar}$. We construct immediately the smallest possible algebroid satisfying all conditions, which will be the Ambrose-Singer algebroid $A^{\hbar}$. From the two requirements just mentioned, the smallest possible algebroid is defined by the condition

$$\Gamma(A^{\hbar}) = \Gamma(\mathfrak{hol}^{\hbar}) \oplus \hbar(\mathfrak{X}(M)).$$

This is a direct sum, as the holonomy Lie algebras are generated by the curvature and sit in the space of vertical vector fields on $\mathcal{N}$. This also defines $A^{\hbar}$ as a vector bundle, by

$$A^h = \mathfrak{hol}^{\hbar} \oplus TM.$$

Note that $\hbar(\mathfrak{X}(M))$ is identified with $\mathfrak{X}(M)$, as $\hbar$ is injective. The anchor map is projection onto the second coordinate:

$$\rho : A^{\hbar} \to TM,\ \rho(\xi, X) = X.$$

We see that the isotropy Lie algebra of $A^{\hbar}$ is indeed $\mathfrak{g} = \mathfrak{hol}^{\hbar}$, showing that $A^{\hbar}$ is the smallest possible algebroid satisfying all conditions.

It remains to define a Lie bracket on the space of sections and a free algebroid action. We first define the action, as it gives an additional condition on the Lie bracket. There is a straightforward choice for this action:

$$\begin{aligned} a : \Gamma(A^{\hbar}) = \Gamma(\mathfrak{hol}^{\hbar}) \oplus \hbar(\mathfrak{X}(M)) &\to \mathfrak{X}^{proj}(\mathcal{N}) \\ (\xi, \hbar(X)) &\mapsto \xi + \hbar(X) \end{aligned}$$

where the vector field on the right is determined pointwise by

$$(\xi + \hbar(X))_p = (\xi \circ \mathrm{pr}(p))_p + \hbar(X)_p \in T_p\mathcal{N}_{\mathrm{pr}(p)} \oplus \mathcal{H}_p.$$

This action satisfies $d\mathrm{pr} \circ a = \rho$ and its image contains $\hbar(\mathfrak{X}(M))$. To show that it is fiberwise injective, take $p \in \mathcal{N}_x$ and consider a_p. If $a_p(\xi, \hbar(X)) = 0$, then both $\hbar(X)_p$ and $(\xi(\mathrm{pr}(p)))_p$ must be 0. By injectivity of $\hbar$, we see $X = 0$. By assumption, ξ is either nowhere vanishing or the zero-vector field, so it must be the latter: $\xi = 0$. It follows that this action is fiberwise injective.

Finally this action is required to be a Lie algebra morphism, which induces a condition on the Lie bracket on $\Gamma(A^{\hbar})$. This condition is written in the equation below. Note that the brackets on the right hand side are the usual brackets of vector fields on $\mathcal{N}$ or M, and on the left hand side we have the bracket on $A^{\hbar}$ that is to be defined.

$$\begin{aligned} a\left([(\xi, \hbar(X)), (\eta, \hbar(Y))]\right) &= [a(\xi, \hbar(X)), a(\eta, \hbar(Y))] \\ &= [\xi + \hbar(X), \eta + \hbar(Y)] \\ &= [\xi, \eta] + [\xi, \hbar(Y)] + [\hbar(X), \eta] + [\hbar(X), h(Y)] \\ &= [\xi, \eta] + \nabla^{\hbar}_X(\eta) - \nabla^{\hbar}_Y(\xi) - \Omega(X, Y) + \hbar([X, Y]) \end{aligned}$$

Here we use again the notation from the discussion above: $\nabla^{\hbar}_X(\xi) = [\hbar(X), \xi]$ (and later we will see that this still defines a connection, endorsing this notation). The last term of this expression clearly lives in $\hbar(\mathfrak{X}(M))$. We define the bracket on $\Gamma(A^{\hbar})$ then as follows:

$$\begin{aligned} &[\cdot, \cdot] : \Gamma(A^{\hbar}) \times \Gamma(A^{\hbar}) \to \Gamma(A^{\hbar}), \\ &[(\xi, \hbar(X)), (\eta, \hbar(Y))] = \left([\xi, \eta] + \nabla^{\hbar}_X(\eta) - \nabla^{\hbar}_Y(\xi) - \Omega(X, Y),\ \hbar([X, Y])\right). \end{aligned} \tag{13}$$

With this definition the action a is immediately a Lie algebroid morphism. It remains to show that this is a well-defined Lie bracket, first of all by showing that the first component indeed lies in $\Gamma(\mathfrak{hol}^{\hbar})$ and then proving that it is a Lie bracket. We show now that the first four terms of the expression above all lie in $\Gamma(\mathfrak{hol}^{\hbar})$.

- First of all, the terms $\Omega(X, Y)$ lie in $\Gamma(\mathfrak{hol}^{\hbar})$ by construction.
- Next, consider the terms $\nabla^{\hbar}_X(\eta)$ and $\nabla^{\hbar}_Y(\xi)$. Recall that the Lie bracket can be described by

$$\nabla^{\hbar}_X(\eta) := [\hbar(X), \xi] = \left.\frac{d}{dt}\right|_{t=0} (d\phi^t_{\hbar(X)})^{-1}(\xi),$$

where $\phi^t_{\hbar(X)}$ is the flow on $\mathcal{N}$ along the vector field $\hbar(X)$. This flow is precisely the map giving $\hbar$-parallel transport along integral curves of X. Recall that the holonomy Lie algebra is invariant under parallel transport: this implies that for $\xi \in \Gamma(\mathfrak{hol}^{\hbar})$, we have $(d\phi^t_{\hbar(X)})^{-1}(\xi) \in \Gamma(\mathfrak{hol}^{\hbar})$, which then must hold for its differential as well. So, for $\xi \in \Gamma(\mathfrak{hol}^{\hbar})$, all expressions of the form $[\hbar(X), \xi] = \nabla^{\hbar}_X(\xi)$ are again in $\Gamma(\mathfrak{hol}^{\hbar})$.

- Finally, consider the term $[\xi, \eta]$. By applying the Jacobi identity to the previous result, it follows that $\mathfrak{hol}^{\hbar}$ is invariant under the operations $[[\hbar(X), \hbar(Y)], \xi]$, and then also under the operations $[\Omega(X,Y), \xi]$. Since these terms generate the holonomy Lie algebra, we have now shown that the holonomy Lie algebra is indeed closed under the Lie bracket, so a Lie subalgebra of $\mathfrak{X}(\mathcal{N}_x)$.

This proves that the map $[\cdot,\cdot]$ given above is well-defined (maps back into $\Gamma(A^{\hbar}) = \Gamma(\mathfrak{hol}^{\hbar}) \oplus \hbar(\mathfrak{X}(M))$). Additionally, the last point in this list also proves Lemma 6.4:

Lemma. *For any $x \in M$, the holonomy Lie algebra $\mathfrak{hol}^{\hbar}_x$ is a Lie subalgebra of $\mathfrak{X}(\mathcal{N}_x)$.*

Next, we prove the usual properties of the Lie bracket. Bilinearity, antisymmetry and the Jacobi identity are all inherited from the Lie brackets that we use to define it, which can be proven with some straightforward computations.

The final property it needs to satisfy, and which is a bit more difficult to check, is the Leibniz rule with respect to the anchor map. Let $f \in C^\infty(M)$. Under the identification $\Gamma(A) \cong a(\Gamma(A))$, we consider the induced map $\tilde{f} := f \circ \mathrm{pr} \in C^\infty(\mathcal{N})$. Then we consider the expression

$$[(\xi, \hbar(X)), \tilde{f}(\eta, \hbar(Y))]$$

using Equation (13). This gives five terms, which we discuss seperately. Note that all these terms are defined using regular Lie brackets of vector fields, allowing us to apply the regular Leibniz rule.

- The first term is $[\xi, \tilde{f}\eta]$. We note that $\tilde{f}$ is constant on the fibers of $\mathcal{N}$. This implies $\mathcal{L}_\xi(\tilde{f}) = 0$, because ξ is a vector field tangent to the fibers (as for each $x \in M$, $\xi(x) \in \mathfrak{X}(\mathcal{N}_x)$). By the usual Leibniz rule, we find $[\xi, \tilde{f}\eta] = \tilde{f}[\xi, \eta]$.
- For the second term, we see $\nabla^{\hbar}_X(\tilde{f}\eta) = [\hbar(X), \tilde{f}\eta] = \tilde{f}[\hbar(X), \eta] + \mathcal{L}_{\hbar(X)}(\tilde{f})\eta$.
- For the third term, we see $-\nabla^{\hbar}_{fY}(\xi) = -[\tilde{f}\hbar(Y), \xi] = -\tilde{f}[\hbar(Y), \xi] + \mathcal{L}_\xi(\tilde{f})$. We have seen $\mathcal{L}_\xi(\tilde{f}) = 0$, so we find $-\nabla^{\hbar}_{fY}(\xi) = -\tilde{f}[\hbar(Y), \xi]$.
- For the fourth term, using the definition of the curvature form we find

  $$-\Omega(X, fY) = -\tilde{f}\Omega(X,Y) - \hbar(\mathcal{L}_X(f)Y) + \mathcal{L}_{\hbar(X)}(\tilde{f})\hbar(Y).$$

  Of these three terms, the last two are clearly horizontal, so they should end up in the second component. All other terms that we have found so far are by our discussion above still in $\Gamma(\mathfrak{hol}^{\hbar})$.

- For the final term we find $\hbar([X, fY]) = \tilde{f}\hbar([X,Y]) + \hbar(\mathcal{L}_X(f)Y)$. Combining everything we find

$$\begin{aligned}[(\xi, \hbar(X)), \tilde{f}(\eta, \hbar(Y))] =& (\tilde{f}[\xi,\eta] + \tilde{f}[\hbar(X),\eta] + \mathcal{L}_{\hbar(X)}(\tilde{f})\eta - \tilde{f}[\hbar(Y),\xi] - \tilde{f}\Omega(X,Y), \\ & - \hbar(\mathcal{L}_X(f)Y) + \mathcal{L}_{\hbar(X)}(\tilde{f})\hbar(Y) + \tilde{f}\hbar([X,Y]) + \hbar(\mathcal{L}_X(f)Y)) \\ =& \tilde{f}[(\xi,\hbar(X)),(\eta,\hbar(Y))] + (\mathcal{L}_{\hbar(X)}(\tilde{f})\eta, \mathcal{L}_{\hbar(X)}(\tilde{f})\hbar(Y)) \\ =& \tilde{f}[(\xi,\hbar(X)),(\eta,\hbar(Y))] + \mathcal{L}_{\rho(\xi,\hbar(X))}(\tilde{f})(\eta,\hbar(Y))\end{aligned}$$

  Recall that the anchor map ρ was projection onto the second coordinate.

Taken together, these computations prove that the Leibniz rule holds for our bracket, finishing the proof. Items 2 and 3 of the above list also prove that $\nabla^{\hbar}$ is in fact a vector bundle connection:

Lemma 7.3. *On the vector bundle $\mathfrak{hol}^{\hbar} \to M$, there is a connection denoted $\nabla^{\hbar}$ defined by*

$$\nabla^{\hbar} : \mathfrak{X}(M) \times \Gamma(\mathfrak{hol}^{\hbar}) \to \Gamma(\mathfrak{hol}^{\hbar}),\ \nabla^{\hbar}_X(\xi) = [\hbar(X), \xi]. \tag{14}$$

□

This finishes the proof of Proposition 7.1. We now prove Lemma 7.2.

Proof of Lemma 7.2. Consider the $\hbar$-parallel transport on $\mathcal{N}$ along a curve γ. We choose a compactly supported vector field X on M that has γ as an integral curve. Note that X being compactly supported implies it is complete. The parallel transport along γ is precisely the flow $\phi^t_{\hbar(X)}$ on $\mathcal{N}$. The induced parallel transport of tangent vectors to $\mathcal{N}$ is the differential of this flow, so for $\xi \in \Gamma(\mathfrak{g})$, $\hbar$-parallel transport is given by $d\phi^t_{\hbar(X)}(\xi)$.

Now consider the introduced connection $\nabla^{\hbar}_X(\xi) = [\hbar(X), \xi]_A$ on the vector bundle $\mathfrak{g} \to M$. Since a is a Lie algebra morphism, this is equivalent to the usual Lie bracket $[\hbar(X), \xi]$ on $\mathfrak{X}(\mathcal{N})$. Recall that this bracket measures the change of ξ along the flow of $\hbar(X)$, which can be described as

$$[\hbar(X), \xi] = \frac{d}{dt}|_{t=0}(d\phi^t_{\hbar(X)})^{-1}(\xi).$$

Now we claim that the map $d\phi^t_{\hbar(X)}$ is precisely the map giving $\nabla^{\hbar}$-parallel transport on $\mathfrak{g} \to M$. We need to prove that this map gives a path in $\Gamma(\mathfrak{g})$, that it covers γ and that it is $\nabla^{\hbar}$-parallel.

First of all we show that for $\xi \in \Gamma(\mathfrak{g}) \subset \mathfrak{X}^v(\mathcal{N})$, $d\phi^t_{\hbar(X)}(\xi)$ remains in $\Gamma(\mathfrak{g})$. We know that the bracket $[\hbar(X), \xi]$ does map back into $\Gamma(\mathfrak{g})$, and from the expression above, we see that this bracket gives the infinitesimal version of the path $d\phi^t_{\hbar(X)}(\xi)$. From this, we can conclude that this path itself also maps back into $\Gamma(\mathfrak{g})$.

Consider now its base path; on $\mathcal{N}$, $d\phi^t_{\hbar(X)}(\xi)$ covers the path $\phi^t_{\hbar(X)}$, and from $\mathrm{pr} \circ \phi^t_{\hbar(X)} = \phi_X \circ \mathrm{pr}$ we see that it covers γ on M (since γ is an integral curve of X).

Finally the path needs to be $\nabla^{\hbar}$-parallel. To see that this is true, consider $v \in \mathfrak{g}$, which corresponds to $a(v)$ on $\mathcal{N}$. We compute the $\nabla^{\hbar}$-differential of the flow $d\phi^t_{\hbar(X)}(a(v))$:

$$\begin{aligned}\nabla^{\hbar}_{\dot\gamma(t)}(d\phi^t_{\hbar(X)}(a(v)))(\gamma(t)) &= [\hbar(X), d\phi^t_{\hbar(X)}(a(v))]\gamma(t)\\ &= \frac{d}{dt}|_{t=0}(d\phi^t_{\hbar(X)})^{-1}(d\phi^t_{\hbar(X)})(a(v))(\gamma(t))\\ &= 0.\end{aligned}$$

Indeed this path $d\phi^t_{\hbar(X)}$ is $\nabla^{\hbar}$-parallel, showing it is precisely the $\nabla^{\hbar}$-parallel transport on $\mathfrak{g}$, which is thus the same as $\hbar$-parallel transport on $\mathcal{N}$. □

Corollary 7.4. *The Ambrose-Singer algebroid $A^{\hbar}$ is integrable.*

Proof. This follows immediately from Theorem 2.63, as we have an algebroid with a free algebroid action. □

From now on, we will assume that the connection $\hbar$ is a holonomic connection, allowing us to talk about the Ambrose-Singer algebroid. Since this algebroid is integrable and is endowed with a free algebroid action, the Weinstein groupoid $\mathcal{G}(A^{\hbar})$ admits an action on $\mathrm{pr} : \mathcal{N} \to M$ as well by Lemma 2.56.

We now shortly recall the construction of this induced groupoid action. The action of an arrow $g \in \mathcal{G}(A^{\hbar})$ is determined by first taking a representative $A^{\hbar}$-path a (so $g = [a]$). Then $a(a(t))$ lies in $T\mathcal{N}$ and gives the speed of a certain curve $u(t)$ in $\mathcal{N}$. The groupoid action induced by g now maps $u_0 \in \mathcal{N}$ to $u(1)$, where $u(t)$ is the curve in $\mathcal{N}$ with speed $a(a(t))$ and initial point $u(0) = u_0$. The following proposition shows that, in the case of the action of $A^{\hbar}$, the groupoid action is related to parallel transport w.r.t. the connection $\hbar$.

Proposition 7.5. *Given any curve $\gamma : [0, 1] \to M$, its parallel transport $\tau_\gamma : \mathcal{N}_{\gamma(0)} \to \mathcal{N}_{\gamma(1)}$ is equivalent to the groupoid action induced by the class/arrow $\left[\hbar\left(\frac{d\gamma}{dt}\right)\right] \in \mathcal{G}(A^{\hbar})$.*

Proof. The proof follows by comparing the differential equations that are involved in either parallel transport or the groupoid action. Consider a curve $\gamma : [0, 1] \to M$. Let $u_0 \in \mathcal{N}_{\gamma(0)}$. Then $\hbar$-parallel transport along γ, applied to u_0, is determined by solving

$$\begin{cases}\frac{d}{dt}u(t) = \hbar\left(\frac{d\gamma}{dt}\right), \\ u(0) = u_0\end{cases}$$

and parallel transport maps u_0 to $u(1)$. Here, the algebroid has not come into play, and $\hbar$ is simply a map $\hbar : \mathfrak{X}(M) \to \mathfrak{X}(\mathcal{N})$.

On the other hand, since $A^{\hbar} = \mathfrak{hol}^{\hbar} \oplus \hbar(TM)$, $\hbar\left(\frac{d\gamma}{dt}\right)$ defines an $A^{\hbar}$-path above γ. Here we interpret $\hbar$ as a map into $\Gamma(A)$, by Diagram 12. This $A^{\hbar}$-path in turn represents a class in the Weinstein groupoid $\mathcal{G}(A^{\hbar})$. The action of this groupoid is determined by solving, for $u_0 \in \mathcal{N}_{\gamma(0)}$,

$$\begin{cases} \frac{d}{dt}u(t) = a\left(\hbar\left(\frac{d\gamma}{dt}\right)\right), \\ u(0) = u_0 \end{cases}$$

and u_0 is mapped to $u(1)$.

Now by commutativity of Diagram 12 it follows that the two differential equations defined above on $\mathcal{N}$ are precisely the same, and indeed parallel transport along γ agrees with the groupoid action of $\left[\hbar\left(\frac{d\gamma}{dt}\right)\right]$. □

In order to relate this groupoid action to the holonomy group, we restrict our attention to the isotropy group $\mathcal{G}_x(A^{\hbar})$. This restriction induces a group homomorphism denoted

$$\Phi : \mathcal{G}_x(A^{\hbar}) \to \mathrm{Diff}(\mathcal{N}_x).$$

Claim 7.6. *The image $\Phi(\mathcal{G}_x(A^{\hbar})) \subset Diff(\mathcal{N}_x)$ is precisely the holonomy group $Hol_x^{\hbar}$.*

The fact that the image contains the holonomy group is immediately clear by the previous proposition. To prove the other inclusion, we have to show that any element of the isotropy group $\mathcal{G}_x(A^{\hbar})$ is mapped to an element of $\mathrm{Hol}_x^{\hbar}$. It suffices to show that any $A^{\hbar}$-path above a loop in the base is $A^{\hbar}$-homotopic to some $A^{\hbar}$-path that is mapped to an element of $\mathrm{Hol}_x^{\hbar}$ (i.e., that is acting by $\hbar$-parallel transport along some loop). In order to do this, we need to discuss $A^{\hbar}$-homotopies in more detail.

Remark 7.7. Claim 7.6 is not proven yet in this thesis. In the next section, we work towards a possible geometric construction to prove this, but it is incomplete. The difficulty can be seen as the essence of the Ambrose-Singer theorem; showing that elements of the holonomy Lie algebra $\mathfrak{hol}_x^{\hbar}$ generate elements of the holonomy groups $\mathrm{Hol}_x^{\hbar}$.

In Section 8, we give a few ideas on how to further solve this problem. But even without proving the Ambrose-Singer theorem from this approach, we will see in Section 7.3 how our approach gives more insight than the classical discussions of holonomy concerning the structure of the fibration $\mathcal{N} \to M$. These results endorse Mackenzie's ideas that holonomy should be viewed in the context of groupoids, as groupoid theory can encompass their (global) properties in a natural way that the classical discussion does not.

7.2 $A^{\hbar}$-homotopies

In order to understand the Weinstein groupoid of $A^{\hbar}$, defined as

$$\mathcal{G}(A^{\hbar}) = \frac{A^{\hbar}\text{-paths}}{A^{\hbar}\text{-path homotopies}},$$

we aim to understand $A^{\hbar}$-homotopies in more detail. We have already discussed general algebroid homotopies in Section 2.5, and we now develop this theory further for the Ambrose-Singer algebroid:

$$A^{\hbar} = \mathfrak{hol}^{\hbar} \oplus TM.$$

We can decompose elements (and paths) in $A^{\hbar}$ into a 'vertical' part lying in $\mathfrak{hol}^{\hbar}$ and a 'horizontal' part lying in TM. For an $A^{\hbar}$-path $a(t)$ above a basepath γ in M we will write

$$a(t) = (a^v(t), a^h(t)).$$

Recall that the anchor of $A^{\hbar}$ is simply projection onto the second coordinate. Combined with the condition that $\rho \circ a(t)$ should give the speed of the base path γ, we find the following condition:

$$a^h(t) = \frac{d\gamma}{dt}(t).$$

Hence, in general an $A^{\hbar}$-path can always be written as

$$a(t) = \left(V(t), \frac{d\gamma}{dt}(t)\right),$$

where $\gamma(t)$ is the base path and $V(t)$ is a path in $\mathfrak{hol}^{\hbar}$ such that $V(t) \in \mathfrak{hol}^{\hbar}_{\gamma(t)}$.

Next we consider a variation of $A^{\hbar}$ paths,

$$a_\epsilon(t) = \left(V_\epsilon(t), \frac{d\gamma_\epsilon}{dt}(t)\right),$$

above a path-homotopy $\gamma_\epsilon(t)$ in M. In order for this to be an $A^{\hbar}$-path homotopy, we have to compute the variation $Var_\epsilon(t)$ defined in Section 2.5. We consider the solution to Equation (2), which will be of the form

$$b_\epsilon(t) = \left(W_\epsilon(t), \frac{d\gamma_\epsilon}{d\epsilon}(t)\right),$$

where $W_\epsilon(t)$ is again a path in $\mathfrak{hol}^{\hbar}$, such that $W_\epsilon(t) \in \mathfrak{hol}^{\hbar}_{\gamma_\epsilon(t)}$. Here $b_\epsilon(t)$ is found by solving the following differential equation, with ∇ any TM-connection on $A^{\hbar}$:

$$\begin{cases} \nabla_{\frac{d\gamma}{d\epsilon}} a_\epsilon(t) - \nabla_{\frac{d\gamma}{dt}} b_\epsilon(t) = T_\nabla(a_\epsilon(t), b_\epsilon(t)) \\ b_\epsilon(0) = 0. \end{cases} \tag{15}$$

By definition, $a_\epsilon(t)$ is an $A^{\hbar}$-homotopy if and only if $b_\epsilon(1) = 0$. However, this condition may be reduced to one depending only on V, W and γ.

Proposition 7.8. *The variation $a_\epsilon(t)$ is an $A^{\hbar}$-homotopy if and only if for the solution $W_\epsilon(t)$ of*

$$\begin{cases} \nabla^{\hbar}_{\frac{d\gamma}{d\epsilon}} V_\epsilon(t) - \nabla^{\hbar}_{\frac{d\gamma}{dt}} W_\epsilon(t) = -\left[V_\epsilon(t), W_\epsilon(t)\right] - \Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) \\ W_\epsilon(0) = 0 \end{cases} \tag{16}$$

it holds that $W_\epsilon(1) = 0$.

Proof. We consider Equation (15), and reduce it to Equation (16). First consider the boundary conditions $b_\epsilon(0) = 0$ and $b_\epsilon(1) = 0$. The horizontal part of $b_\epsilon(t)$ (i.e., the TM-part) is known to be $\frac{d\gamma}{d\epsilon}$. Since $\gamma_\epsilon(t)$ is a path homotopy in M, $\gamma_\epsilon(0)$ and $\gamma_\epsilon(1)$ are constant, and the horizontal parts of $b_\epsilon(0)$ and $b_\epsilon(1)$ are thus forced to be 0. The two boundary conditions reduce to $W_\epsilon(0) = 0$ and $W_\epsilon(1) = 0$.

Next, recall that the solution $b_\epsilon(t)$ to Equation (15) is independent of the connection ∇ on $A^\hbar$. We can choose explicitly a connection on $A^\hbar$ to simplify the equation. By Lemma 7.3, we have a connection $\nabla^\hbar$ on $\mathfrak{hol}^\hbar$, given by

$$\nabla^\hbar : \mathfrak{X}(M) \times \Gamma(\mathfrak{hol}^\hbar) \to \Gamma(\mathfrak{hol}^\hbar),\ \nabla_X(\xi) = [\hbar(X), \xi].$$

On TM, we choose any Levi-Civita connection, denoted $\nabla^{LC} : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$. Combined, this gives a connection ∇ on $A^\hbar$, defined by:

$$\begin{aligned} \nabla : \mathfrak{X}(M) \times \Gamma(A^\hbar) &\to \Gamma(A^\hbar) \\ \nabla_Y(\xi, X) &= (\nabla^\hbar_Y(\xi), \nabla^{LC}_Y(X)). \end{aligned}$$

Consider now two general sections $a = (\xi, X)$, $b = (\eta, Y) \in \Gamma(A^\hbar)$. Using the definition of the bracket on $\Gamma(A^\hbar)$ in Equation (13), we compute the torsion:

$$\begin{aligned} T_\nabla(a, b) &= \nabla_{\rho(a)}(b) - \nabla_{\rho(b)}(a) - [a, b] \\ &= \nabla_X(\eta, Y) - \nabla_Y(\xi, X) - [(\xi, X), (\eta, Y)] \\ &= (\nabla^\hbar_X(\eta) - \nabla^\hbar_Y(\xi) - [\xi, \eta] - \nabla^\hbar_X(\eta) + \nabla^\hbar_Y(\xi) - \Omega(X, Y),\ \nabla^{LC}_X(Y) - \nabla^{LC}_Y(X) - [X, Y]) \\ &= (-[\xi, \eta] - \Omega(X, Y),\ 0). \end{aligned}$$

Note that the horizontal part vanishes, because any Levi-Civita connection is torsion-free. Moving back to $A^\hbar$-homotopies, we can now simplify Equation (15). We compute the left hand side for the chosen connection:

$$\nabla_{\frac{d\gamma}{d\epsilon}} a_\epsilon(t) - \nabla_{\frac{d\gamma}{dt}} b_\epsilon(t) = \left(\nabla^\hbar_{\frac{d\gamma}{d\epsilon}} V_\epsilon(t) - \nabla^\hbar_{\frac{d\gamma}{dt}} W_\epsilon(t),\ \nabla^{LC}_{\frac{d\gamma}{d\epsilon}} \left(\frac{d\gamma}{dt} \right) - \nabla^{LC}_{\frac{d\gamma}{dt}} \left(\frac{d\gamma}{d\epsilon} \right) \right).$$

The horizontal part is the Lie bracket $\left[\frac{d\gamma}{d\epsilon}, \frac{d\gamma}{dt}\right]$ (again because the Levi-Civita connection is torsion free), and since these vector fields commute, the horizontal part is 0. Combining the above expression with the expression we found for the torsion, we see that $a_\epsilon(t)$ is an $A^\hbar$-homotopy if and only if for the solution $W_\epsilon(t)$ of

$$\begin{cases} \nabla^\hbar_{\frac{d\gamma}{d\epsilon}} V_\epsilon(t) - \nabla^\hbar_{\frac{d\gamma}{dt}} W_\epsilon(t) = -[V_\epsilon(t), W_\epsilon(t)] - \Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) \\ W_\epsilon(0) = 0 \end{cases} \tag{17}$$

it holds that $W_\epsilon(1) = 0$. □

We now consider how, starting from a homotopy $\gamma_\epsilon(t)$ in M, we may construct an $A^{\hbar}$-homotopy covering it. The naive guess turns out to be too restrictive, as shown in the example below.

Example 7.9. Consider any path homotopy $\gamma_\epsilon(t)$ in M. Then we can find a variation of $A^{\hbar}$-paths covering this homotopy by

$$a_\epsilon(t) = \left(0, \frac{d\gamma_\epsilon(t)}{dt}\right).$$

To check whether this is an $A^{\hbar}$-homotopy, we have to solve Equation (16) for $W_\epsilon(t)$ with $V_\epsilon(t) = 0$. We find:

$$\begin{aligned} \nabla^{\hbar}_{\frac{d\gamma}{dt}} W_\epsilon(t) &= \Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) \\ \left[\hbar\left(\frac{d\gamma}{dt}\right), W_\epsilon(t)\right] &= \left[\hbar\left(\frac{d\gamma}{dt}\right), \hbar\left(\frac{d\gamma}{d\epsilon}\right)\right] - \hbar\left(\left[\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right]\right) \\ &= \left[\hbar\left(\frac{d\gamma}{dt}\right), \hbar\left(\frac{d\gamma}{d\epsilon}\right)\right]. \end{aligned}$$

This suggests the solution $W_\epsilon(t) = \hbar\left(\frac{d\gamma}{d\epsilon}\right)$, but $W_\epsilon(t)$ should be a 'vertical' path in $\mathfrak{hol}^{\hbar}$, which this is certainly not. We see that in general, this variation need not be an $A^{\hbar}$-homotopy.

In order to find an $A^{\hbar}$-homotopy, we have to add a vertical component to the variation.

Lemma 7.10. *Given a homotopy $\gamma_\epsilon(t)$ in M, the following variation of $A^{\hbar}$-paths is an $A^{\hbar}$-homotopy covering the base homotopy:*

$$\left(V_\epsilon(t), \frac{d\gamma_\epsilon(t)}{dt}\right)$$

where

$$V_\epsilon(t) = -\tau_\gamma^{(0,\epsilon)} \int_0^\epsilon \tau_\gamma^{(\tilde{\epsilon},0)} \left(\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\tilde{\epsilon}}\right)\right) d\tilde{\epsilon}.$$

Proof. Consider a general variation of $A^{\hbar}$-paths given by $\left(V_\epsilon(t), \frac{d\gamma_\epsilon(t)}{dt}\right)$. We assume that $V_\epsilon(0) = 0$. We aim to choose this vertical component $V_\epsilon(t)$ in such a way that $W_\epsilon(t) = 0$ solves Equation (16). Then the boundary conditions are automatically satisfied.

In order to find the component $V_\epsilon(t)$ satisfying this, we solve Equation (16) for $W_\epsilon(t) = 0$. This comes down to solving

$$\frac{d}{d\epsilon} V_\epsilon(t) = -\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right). \tag{18}$$

Recall that $V_\epsilon(t)$ will be a path in $\mathfrak{hol}^{\hbar}$, which is invariant under parallel transport. This implies that there exists for all (ϵ, t) a term

$$V_{\gamma_0(t)}(\epsilon, t) \in \mathfrak{hol}^{\hbar}_{\gamma_0(t)}$$

such that

$$V_\epsilon(t) = \tau_\gamma^{(0,\epsilon)}\left(V_{\gamma_0(t)}\right).$$

Note that we use here parallel transport along the coordinate ϵ, so along the path from $\gamma_0(t)$ to $\gamma_\epsilon(t)$ in M for constant t. We use the notation τ_γ both for the parallel transport on $\mathcal{N}$, and for the induced transport on $T\mathcal{N}$; it should be clear from context which one is meant.

With this description, consider the left hand side of the equation to solve:

$$\begin{aligned}
\frac{d}{d\epsilon}V_\epsilon(t) &= \nabla^{\hbar}_{\frac{d\gamma}{d\epsilon}} V_\epsilon(t) \\
&= \left[\hbar\left(\frac{d\gamma}{d\epsilon}\right), \tau_\gamma^{(0,\epsilon)}\left(V_{\gamma_0(t)}(\epsilon,t)\right)\right] \\
&= \frac{d}{d\tilde{t}}\bigg|_{\tilde{t}=0}\left(d\phi^{\tilde{t}}_{\hbar\left(\frac{d\gamma}{d\epsilon}\right)}\right)^{-1}\left(d\phi^{t}_{\hbar\left(\frac{d\gamma}{d\epsilon}\right)}\right)\left(V_{\gamma_0(t)}(\epsilon,t)\right) \\
&= \tau_\gamma^{(0,\epsilon)}\left[\hbar\left(\frac{d\gamma}{d\epsilon}\right), V_{\gamma_0(t)}(\epsilon,t)\right] \\
&= \tau_\gamma^{(0,\epsilon)}\left(\frac{d}{d\epsilon}V_{\gamma_0(t)}(\epsilon,t)\right).
\end{aligned}$$

Here we used the fact that parallel transport along $\gamma_\epsilon(t)$ with respect to ϵ is equivalent to the flow of $\hbar\left(\frac{d\gamma}{d\epsilon}\right)$. Since this vector field commutes with itself, the flows commute and we can take the term $\tau_\gamma^{(0,\epsilon)}$ out in front.

Now solving Equation (18) for $V_\epsilon(t)$ we find:

$$\begin{aligned}
& \frac{d}{d\epsilon}V_\epsilon(t) = -\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) \\
\iff & \tau_\gamma^{(0,\epsilon)}\left(\frac{d}{d\epsilon}V_{\gamma_0(t)}(\epsilon,t)\right) = -\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) \\
\iff & \frac{d}{d\epsilon}V_{\gamma_0(t)}(\epsilon,t) = -\tau_\gamma^{(\epsilon,0)}\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right) \\
\iff & V_{\gamma_0(t)}(\epsilon,t) = -\int_0^\epsilon \tau_\gamma^{(\tilde{\epsilon},0)}\left(\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\tilde{\epsilon}}\right)\right) d\tilde{\epsilon} \\
\iff & V_\epsilon(t) = -\tau_\gamma^{(0,\epsilon)}\int_0^\epsilon \tau_\gamma^{(\tilde{\epsilon},0)}\left(\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\tilde{\epsilon}}\right)\right) d\tilde{\epsilon}.
\end{aligned}$$

□

The following result is now immediate, by evaluating at $\epsilon = 0$ and $\epsilon = 1$:

Corollary 7.11. *Given a path-homotopy $\gamma_\epsilon(t)$ in M, the following $A^\hbar$-paths are $A^\hbar$-homotopic:*

$$\left(0, \frac{d\gamma_0(t)}{dt}\right) \sim \left(V_1(t), \frac{d\gamma_1(t)}{dt}\right)$$

where

$$V_1(t) = -\tau_\gamma^{(0,1)} \int_0^1 \tau_\gamma^{(\epsilon,0)} \left(\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right)\right) d\epsilon$$
$$= -\int_0^1 \tau_\gamma^{(\epsilon,1)} \left(\Omega\left(\frac{d\gamma}{dt}, \frac{d\gamma}{d\epsilon}\right)\right) d\epsilon.$$

If the base homotopy is the contraction of a loop, we have an even stronger result.

Corollary 7.12. *Let $\gamma_\epsilon(t)$ be a contraction of a contractible loop $\gamma_0(t)$ based at x_0, where $\gamma_1(t) = const_{x_0}$. Then the following paths are $A^\hbar$-homotopic, with $V_1(t) \in \mathfrak{hol}^\hbar_{x_0}$ as before:*

$$\left(0, \frac{d\gamma_0(t)}{dt}\right) \sim (V_1(t), 0_{x_0}).$$

In this discussion we have now encountered two specific types of $A^\hbar$-paths; paths that are purely horizontal or purely vertical. Note that any path that is purely vertical, will have as basepath a single point in M, and will thus be a path within one fiber of $\mathfrak{hol}^\hbar$. We will denote these paths as follows:

1. For $\gamma : [0,1] \to M$, we set $g_\gamma := \left(0, \frac{d\gamma(t)}{dt}\right)$
2. For $v : [0,1] \to \mathfrak{hol}^\hbar_{x_0}$, we set $h_v := (v, 0_{x_0})$.

With this notation, the statement of Corollary 7.12 can be rephrased as $g_{\gamma_0} = h_{V_1}$, with V_1 as above.

Lemma 7.13. *With the same notation, the following holds for concatenation of $A^\hbar$-paths, where v is a path in $\mathfrak{hol}^\hbar_{x_0}$ and γ a path in M starting at x_0:*

$$g_\gamma \cdot h_v = \left[\left(\tau_\gamma^{0,t}(v^\tau(t)), \frac{d\gamma}{dt}\right)\right].$$

Proof. Denote by $a_0(t)$ the left hand side of this equation, and by $a_1(t)$ the right hand side. Then using Equation (5) to make the concatenation of $A^\hbar$-paths explicit, it is straightforward to check that the following is a variation of $A^\hbar$-paths:

$$a(\epsilon, t) = \begin{cases} ((2-\epsilon)v^\tau(2t), 0) & \text{if } t \leq \frac{1-\epsilon}{2} \\ \left((2-\epsilon)\tau^t_{\gamma\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)}(v^\tau(\frac{2\epsilon}{1+\epsilon}(t-1)+1)), \frac{2}{\epsilon+1}\frac{d\gamma}{dt}\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)\right) & \text{if } t \geq \frac{1-\epsilon}{2} \end{cases}.$$

The solution to the differential equation in Equation (16) is

$$W(\epsilon, t) = \begin{cases} \frac{\epsilon^2+3\epsilon}{1-\epsilon} t(2-\epsilon)v^\tau(2t) & \text{if } t \leq \frac{1-\epsilon}{2} \\ \left(\frac{\epsilon+1}{2} - \frac{t}{1+\epsilon}\right)(2-\epsilon)\tau^t_{\gamma\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)}(v^\tau(\frac{2\epsilon}{1+\epsilon}(t-1)+1)) & \text{if } t \geq \frac{1-\epsilon}{2} \end{cases}.$$

which is also straightforward to check. Furthermore, $W(\epsilon, 1) = 0$, which proves that $a_\epsilon(t)$ is an $A^{\hbar}$-homotopy. In Appendix A we give a more detailed proof, where we show in detail how this homotopy is constructed and how Equation (16) can be solved. □

Corollary 7.14. *Any $A^{\hbar}$-path $a(t)$ can be decomposed as $a(t) \sim g_\gamma \cdot h_v$, for γ the base-path of $a(t)$ and $v(t)$ some path in $\mathfrak{hol}^{\hbar}_{\gamma(0)}$.*

Proof. Consider an $A^{\hbar}$-path above a path γ, denoted

$$a(t) = \left(a^v(t), \frac{d\gamma}{dt}\right).$$

Define the path $v(t) \in \mathfrak{hol}^{\hbar}_{\gamma(0)}$ such that

$$a^v(t) = \tau_\gamma^{(0,t)}(v(t)),$$

which is possible as the holonomy Lie algebras are invariant under parallel transport. Now by Lemma 7.13 we have indeed

$$a(t) \sim \left(\tau_\gamma^{0,t}(v(t)), \frac{d\gamma}{dt}\right) \sim g_\gamma \cdot h_v.$$

□

Now we turn to Claim 7.6. Recall that we found a group homomorphism $\Phi : \mathcal{G}_x(A^{\hbar}) \to \mathrm{Diff}(\mathcal{N}_x)$ induced by the groupoid action of $\mathcal{G}(A^{\hbar})$. The claim is that the image of this group homomorphism is precisely $\mathrm{Hol}^{\hbar}_x$. In order to prove this, it suffices to show that any $A^{\hbar}$-path above a loop is $A^{\hbar}$-path homotopic to an $A^{\hbar}$-path that acts like parallel transport along a loop in the base.

Consider an $A^{\hbar}$-path $a(t) : [0,1] \to A^{\hbar}$ above a loop γ based at x. By Corollary 7.14, this can be decomposed as $a(t) \sim g_\gamma \cdot h_v$. The first part, g_γ, acts on $\mathcal{N}_x$ as $\hbar$-parallel transport along γ by Proposition 7.5. It remains to show that h_v for some $v(t) : [0,1] \to \mathfrak{hol}^{\hbar}_x$ also acts by parallel transport along some loop λ in M. We see that the proof of Claim 7.6 has been reduced to proving the following claim:

Claim 7.15. *For any path $v(t) : [0,1] \to \mathfrak{hol}^{\hbar}_x$, there is a loop λ in M such that any of the following equivalent statements hold:*

- $\Phi([v(t)]) = \tau_\lambda$;
- *$A^{\hbar}$-parallel transport along $v(t)$ is equivalent to $\hbar$-parallel transport along λ;*
- $h_v \sim g_\lambda$.

From Corollary 7.12, we find one possible way to prove this last claim. It suffices to find a contractible loop λ based at x, with contraction $\lambda_\epsilon(t)$, such that

$$v(t) \sim V_1(t) = \int_0^1 \tau_\lambda^{(\epsilon,1)} \left(\Omega \left(\frac{d\lambda}{dt}, \frac{d\lambda}{d\epsilon} \right) \right) d\epsilon. \tag{19}$$

By Corollary 7.12, this implies $h_v \sim g_\lambda$ as desired.

The challenge is to find this loop λ for a given path $v(t) : [0,1] \to \mathfrak{hol}_x^{\hbar}$. It is unclear whether this loop even admits an explicit expression, or can only be found as the limit of a sequence of approaching loops. We will discuss this problem further in Section 8.1.

7.3 Proper fibrations as fibered products of principal bundles

From the groupoid action of $\mathcal{G}(A^{\hbar})$ on $\mathrm{pr} : \mathcal{N} \to M$, we can deduce that this proper fibration actually comes from a principal bundle. In general, principal bundles give rise to fibrations in the following way.

Lemma 7.16. *Let $P \to M$ be a principal G-bundle, and let F be a smooth manifold with a G-action. Then*

$$P \times_G F \to M$$

is a locally trivial fibration with fibers F.

Furthermore, if $\hbar$ is a principal bundle connection on P, then this induces also an Ehresmann connection on the fibration $P \times_G F$.

In this section, we prove a result in the other direction, related to items 3 and 4 of the Ambrose-Singer theorem as stated in Theorem 6.8. The precise statement of that theorem is discussed in Section 7.4, and the proof of that statement depends on Claim 7.6. But even without assuming Claim 7.6, we can prove the following:

Proposition 7.17. *Let $\mathrm{pr} : \mathcal{N} \to M$ be a proper fibration endowed with a holonomic connection $\hbar$. Then for any $x \in M$, there exists a Lie group G and a principal G-bundle $P \to M$ such that*

$$\mathcal{N} \to M \cong P \times_G \mathcal{N}_x \to M,$$

and P admits a principal bundle connection inducing $\hbar$.

Proof. The result follows by collecting a few lemmas from the discussion of groupoid actions and the discussion of connections on Atiyah algebroids.

First of all, from Proposition 7.1 we find the integrable, transitive Ambrose-Singer algebroid $A^{\hbar}$ with an algebroid action on $\mathcal{N}$. This induces a groupoid action of $\mathcal{G}(A^{\hbar})$ on $\mathrm{pr} : \mathcal{N} \to M$. By Lemma 2.59, this is equivalent to a $\mathcal{G}_x(A^{\hbar})$-action on $\mathcal{N}_x$, independent of the point $x \in M$. Recall that the Weinstein groupoid of a transitive algebroid is also transitive, hence $\mathcal{G}(A^{\hbar})$ is isomorphic to a gauge groupoid. Denote now by G the isotropy group $\mathcal{G}_x(A^{\hbar})$, and denote by $P \to M$ the principal G-bundle $s^{-1}(x) \to M$. In Lemma 2.59 we have proven the isomorphism

$$\mathcal{N} \cong (P \times \mathcal{N}_x)/G,$$

which proves the first part of our proposition.

Furthermore, we know that $A^{\hbar}$ is transitive and integrable, so it must be the Atiyah algebroid associated to $P \to M$. We have seen in Diagram 12 that the connection $\hbar$ on $\mathcal{N} \to M$ can also be seen as a connection on the algebroid $A^{\hbar}$. By Lemma 4.10, this is equivalent to a principal bundle connection on $P \to M$. □

We remark that this proof does not depend on the structure of $A^{\hbar}$ at all. We can write a more general version, for which the same proof applies:

Lemma 7.18. *Let $P \to M$ be a principal G-bundle and $\mathcal{N} \to M$ a surjective submersion with a connection $\hbar$. Assume the gauge groupoid associated to P is endowed with a groupoid action on $pr : \mathcal{N} \to M$, inducing an infinitesimal action $a : TP/G \to \mathfrak{X}(\mathcal{N})$. If $Im(a) \subset \hbar(\mathfrak{X}(M))$, then there is an isomorphism*

$$\mathcal{N} \cong (P \times \mathcal{N}_x)/G,$$

and the connection $\hbar$ is induced by a principal bundle connection.

7.4 The holonomy groupoid

In this section we discuss a second groupoid integrating $A^{\hbar}$ besides the Weinstein groupoid. The reason is that even if Claim 7.6 is proven, it is not clear yet how this proves the Ambrose-Singer theorem. The action of $\mathcal{G}(A^{\hbar})$ on $\mathcal{N}$ is not necessarily free, so the isotropy group $\mathcal{G}_x(A^{\hbar})$ will not be isomorphic to $\mathrm{Hol}_x^{\hbar}$. We now consider a different groupoid that also integrates $A^{\hbar}$, but has precisely the holonomy groups $\mathrm{Hol}_x^{\hbar}$ as its isotropy groups.

To find this other groupoid, we consider again the algebroid action $a : \Gamma(A^{\hbar}) \to \mathfrak{X}(\mathcal{N})$ and the induced foliation of $\mathcal{N}$, determined by the distribution $\mathcal{F}_a = \mathrm{Im}(a) \subset T\mathcal{N}$. We have seen that the Weinstein groupoid $\mathcal{G}(A^{\hbar})$ is smooth because of the isomorphism

$$\mathcal{G}(A^{\hbar}) \times \mathcal{N} \cong \mathcal{M}on(\mathcal{N}, \mathcal{F}_a).$$

We can look for other groupoids for which a similar isomorphism exists. We can search in two directions;

- On the left hand side, we look for quotients $\mathcal{H} \rightrightarrows M$ of $\mathcal{G}(A^{\hbar}) \rightrightarrows M$ that still act on $\mathcal{N}$, allowing us to construct the action groupoid $\mathcal{H} \times \mathcal{N} \rightrightarrows \mathcal{N}$.
- On the right hand side, we look for groupoids integrating the foliation algebroid $\mathcal{F}_a$.

The latter, groupoids integrating a regular foliation, are discussed in detail in [Phi87] and [MM03]. The paper by Phillips describes the 'holonomic imperative'; the monodromy groupoid is the largest smooth groupoid integrating a foliation, and the holonomy groupoid is the smallest. The importance of the holonomy groupoid explains the name 'holonomic imperative', but a second reason for the naming convention is that the smooth structure on the monodromy groupoid can be constructed with germs of holonomy diffeomorphisms. These same germs are needed to construct the holonomy groupoid, and we will now make more precise how this groupoid is constructed.

We follow the description in [MM03]. First recall the notion of *germs* of locally defined diffeomorphisms. Let K and R be manifolds, and $x \in K$ and $y \in R$ any points. A germ of maps from x to y is an equivalence class of maps from opens around x to opens around y, such that x is mapped to y. Two such maps $f : U \to V$ and $f' : U' \to V'$ are equivalent if there is an open $W \subset U \cap U'$ around x such that $f|_W = f'|_W$.

Let U be an open around x and V an open around y, then it follows that any map $f : U \to V$ such that $f(x) = y$ determines a germ, which we will denote by $\mathrm{germ}_x(f)$. We will focus here on germs of diffeomorphisms, i.e., germs of maps $f : U \to V$ such that f is a diffeomorphism. These germs form a group under composition of maps. Using the notion of germs, we consider paths in our foliated manifold, and assign to them a notion of 'holonomy':

Definition 7.19. For any leafwise path $u(t) : p \to q$ in a foliation manifold $(\mathcal{N}, \mathcal{F})$, let T and S be transversals to the foliation at p and q respectively. The **holonomy** of u is a germ of a diffeomorphism denoted

$$\mathrm{hol}(u) = \mathrm{hol}^{T,S}(u) : (T, p) \to (S, q),$$

defined by the construction below.

Consider a leafwise path $u : p \to q$ and transversals T at p and S at q, i.e., submanifolds of $\mathcal{N}$ that are transversal to the leaves of $\mathcal{N}$. We can interpret the holonomy of u as a map from T to S which draws leafwise paths in $\mathcal{N}$, 'horizontal' to u. To define this, we distinguish two cases.

1. If there is a foliation chart U of $\mathcal{F}$ such that the path $u(t)$ lies entirely in U, we define the holonomy as follows. Find a neighbourhood V of p such that $V \subset U$ where we can define a function $f : V \to S$, such that

   - $f(p) = q$
   - for any $p' \in V$, $f(p')$ and p' lie in the same plaque of U.

   Choose V small enough to ensure that f is a diffeomorphism onto its image, and define

   $$\mathrm{hol}^{S,T}(u) = \mathrm{germ}_x(f).$$

   This definition is independent of choice of U and f. Note that if $v(t)$ is another leafwise path between p and q lying entirely in U, then $\mathrm{hol}^{S,T}(u) = \mathrm{hol}^{S,T}(v)$.

2. If there is no chart U of $\mathcal{F}$ containing the entire path $u(t)$, we proceed as follows. Choose a sequence of foliation charts $U_1, \dots, U_k$ which together cover $u(t)$, by which we mean

$$u_i := u\left(\left[\frac{i-1}{k}, \frac{i}{k}\right]\right) \subset U_i, \text{ for all } i \in \{1, \dots, k\}.$$

Choose a sequence of transversals $T = T_0, T_1, \dots, T_{k-1}, T_k = S$ at points in the intersection of these charts (i.e., T_i is a transversal at some point $p_i \in U_i \cap U_{i+1}$). Then within each chart U_i, we can do the procedure described above with respect to T_{i-1} and T_i. Composing the holonomies from all charts, we define

$$\mathrm{hol}^{S,T}(u) = \mathrm{hol}^{T_k, T_{k-1}}(u_k) \circ \dots \circ \mathrm{hol}^{T_1, T_0}(u_1).$$

Again, this definition is independent of the sequence U_i.

The above procedure assigns a notion of holonomy to any leafwise path in $\mathcal{N}$. This has the following properties (see [MM03] for more details):

- It is associative; for $u : p \to q$ and $v : q \to r$ with T, S, R transversals at p, q and r, we have
$$\mathrm{hol}^{T,R}(v \circ u) = \mathrm{hol}^{S,R}(v) \circ \mathrm{hol}^{T,S}(u).$$
- Leafwise homotopic paths have the same holonomy.
- Holonomy defines an equivalence relation on the space of leafwise paths in $\mathcal{N}$, where paths $u : p \to q$ and $v : p \to q$ have the same holonomy if for a transversal T at p,
$$\mathrm{hol}^{T,T}(v^{-1}u) = \mathrm{Id}.$$
- The above equivalence relation is also well-defined on the space of homotopy classes of leafwise paths in $\mathcal{N}$, denoted $\sim_{\mathrm{hol}}$.

The last item is an important conclusion, as it allows us to finally define the holonomy groupoid:

Definition 7.20. The **holonomy groupoid** of a foliated manifold $(\mathcal{N}, \mathcal{F})$ is the quotient

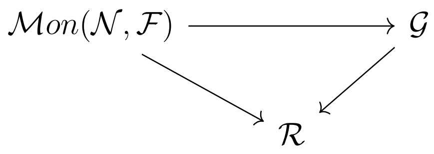
$$\mathcal{H}ol(\mathcal{N}, \mathcal{F}) = \mathcal{M}on(\mathcal{N}, \mathcal{F}) / \sim_{\mathrm{hol}} .$$

This groupoid is always smooth. In both [MM03] and [Phi87] the reader can find a construction of the smooth structure on the holonomy groupoid and the monodromy groupoid. Both smooth structures are constructed using the holonomies of paths defined above, which is one part of the 'holonomic imperative'.

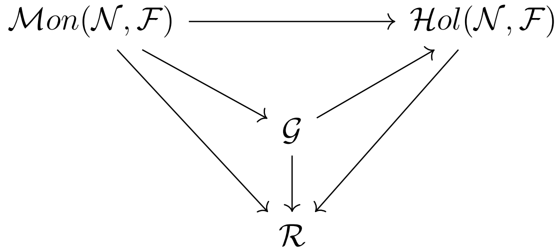
The other part of this concept is that the holonomy groupoid is the smallest possible smooth groupoid integrating a foliation, while the monodromy groupoid is the largest one. Furthermore, any other smooth integration will 'sit between' the monodromy and holonomy groupoids. This is made more precise in the following statement, which is the main theorem of [Phi87]:

Theorem 7.21. *Let $(\mathcal{N}, \mathcal{F})$ be a foliated manifold, and denote by $\mathcal{R}$ the equivalence relation on $\mathcal{N}$ where $(p, q) \in \mathcal{R}$ if and only if p and q lie in the same leaf. Let $\mathcal{G} \rightrightarrows \mathcal{N}$ be a Lie groupoid such that the map $(s, t) : \mathcal{G} \to \mathcal{R}$ is surjective.*

If there is a surjective groupoid morphism $\mathcal{M}on(\mathcal{N}, \mathcal{F}) \to \mathcal{G}$ such that the following diagram commutes



then there is a unique surjective groupoid morphism $\mathcal{G} \to \mathcal{H}ol(\mathcal{N}, \mathcal{F})$ such that the following diagram commutes:



Moreover, if $(s, t) : \mathcal{G} \to \mathcal{R}$ is an immersion, the map $\mathcal{G} \to \mathcal{H}ol(\mathcal{N}, \mathcal{F})$ is an isomorphism.

We now move back to our goal in this chapter. We are looking for a groupoid integrating $A^{\hbar}$, whose isotropy groups are precisely $\mathrm{Hol}_x^{\hbar}$, and which still acts on $\mathcal{N}$. We have seen that these groupoids are closely related to groupoids integrating the foliation $\mathcal{F}_a$, for which the theory above applies. To find our desired groupoid, it makes sense to simply take a quotient of the Weinstein groupoid $\mathcal{G}(A^{\hbar})$ where we quotient out classes that have the same action on $\mathcal{N}$. Clearly, the resulting groupoid will still act on $\mathcal{N}$.

Definition 7.22. Let $A^{\hbar}$ be the Ambrose-Singer algebroid, and denote by τ_a the $A^{\hbar}$-parallel transport on $\mathcal{N}$ induced by an $A^{\hbar}$-path $a(t)$. The **holonomy groupoid of $A^{\hbar}$ relative to $\mathcal{N}$** is defined by

$$\mathcal{H}ol_{\mathcal{N}}(A^{\hbar}) = \frac{A^{\hbar}\text{-paths } a : [0,1] \to A^{\hbar}}{a_0 \sim a_1 \text{ iff } \tau_{a_0} = \tau_{a_1}}.$$

With this definition we get to the main statement of this section, proving finally the Ambrose-Singer theorem under the assumption that Claim 7.6 holds;

Theorem 7.23. *Let $pr : \mathcal{N} \to M$ be a proper fibration with a holonomic connection $\hbar$. Then the holonomy groupoid of $A^{\hbar}$ relative to $\mathcal{N}$, $\mathcal{H}ol_{\mathcal{N}}(A^{\hbar})$, has the following properties:*

- *It is a smooth groupoid over M, integrating $A^{\hbar}$.*
- *Its isotropy groups are the holonomy groups $Hol_x^{\hbar}$.*

- *The s-fiber at x is a principal $Hol_x^{\hbar}$-bundle, denoted P_x.*

From these properties, we conclude that:

- *The Lie algebra of $Hol_x^{\hbar}$ is $\mathfrak{hol}_x^{\hbar}$, the isotropy Lie algebra of $A^{\hbar}$ at x.*
- *There is an isomorphism*

$$\mathcal{N} \to M \cong P_x \times_{Hol_x^{\hbar}} \mathcal{N}_x \to M,$$

and the connection $\hbar$ is equivalent to a principal bundle connection on $P \to M$.

Proof sketch. For the first part, we look at the correspondence of groupoids above. Since $\mathcal{H}ol_{\mathcal{N}}(A^{\hbar})$ still acts by $A^{\hbar}$-parallel transport (the action is even free), we can construct the action groupoid $\mathcal{H}ol_{\mathcal{N}}(A^{\hbar}) \times \mathcal{N} \rightrightarrows \mathcal{N}$. The first part of the theorem then follows from the lemma:

Lemma 7.24. *The groupoid $\mathcal{H}ol_{\mathcal{N}}(A^{\hbar}) \times \mathcal{N} \rightrightarrows \mathcal{N}$ is smooth.*

The fact that this groupoid is smooth should follow with the same proofs for the holonomy and monodromy groupois over $\mathcal{N}$, using the holonomy of leafwise paths. However, the details of this proof are beyond the scope of this thesis.

The second part follows if Claim 7.6 is assumed. By that claim, the image of the action of $\mathcal{G}_x(A^{\hbar})$ is the holonomy group. Since $\mathcal{H}ol_{\mathcal{N}}(A^{\hbar})_x$ is constructed by quotienting out elements with the same action, it follows that it is precisely the holonomy group.

The third item follows from the second, as $\mathcal{H}ol_{\mathcal{N}}(A^{\hbar})$ is transitive.

The fourth part of this theorem is the statement of the Ambrose-Singer theorem. The isotropy groups of $\mathcal{H}ol_{\mathcal{N}}(A^{\hbar})$ are the holonomy groups $\mathrm{Hol}_x^{\hbar}$, and they integrate the isotropy Lie algebras which are precisely $\mathfrak{hol}_x^{\hbar}$.

Finally, the last part follows from simply applying Lemma 7.18. Since $\mathcal{H}ol_{\mathcal{N}}(A^{\hbar})$ still acts on $\mathcal{N}$ and its isotropy groups are the holonomy groups, we find the desired isomorphism. □

8 Outlook

In this chapter we discuss possible future directions of this project. Of course, the main open question at this point is how to prove Claim 7.6 or Claim 7.15. Since the Ambrose-Singer is true by the 'classical' proof, we know they hold, but proving this from the groupoid perspective has been difficult. One possibility is applying the 'classical' proof techniques to these claims, which we discuss in Section 8.1. Another approach could be to follow the construction of Mackenzie, who also proved the Ambrose-Singer theorem using groupoid theory. We discuss this in Section 8.2.

Next, in Section 8.3 we look back to the proof that the Ambrose-Singer algebroid is integrable. This required a strong result on the integrability of foliation algebroids (in other words, the fact that monodromy groupoids are smooth). We show how one could prove the integrability of $A^{\hbar}$ without using this result, additionally giving more insight into the groupoid action of $\mathcal{G}(A^{\hbar})$. Finally, in Section 8.4 we discuss how the current framework can be applied more generally in the discussion of algebroid connections.

8.1 Applying classical methods

We return here to Claim 7.6, the claim that the image of the action $\Phi : \mathcal{G}_x(A^{\hbar}) \to \mathrm{Diff}(\mathcal{N}_x)$ is precisely the holonomy group $\mathrm{Hol}_x^{\hbar}$. First we explain why we believe this claim to be true, then we give a possible method of proving this claim via a proof of Claim 7.15.

We are using this claim to prove the Ambrose-Singer theorem from a new perspective, but of course this theorem is already proven in [AS53]. Acknowledging that this theorem holds, we find that the claim follows for the identity components of the groups involved. We will now make this more precise.

We focus on the identity component $\mathcal{G}_x(A^{\hbar})^0$ of the isotropy group. The action Φ reduces to an action on this subgroup. On the other hand, recall the identification

$$\mathcal{G}_x(A^{\hbar})^0 = G(\mathfrak{hol}_x^{\hbar})/\mathrm{Im}(\partial_x).$$

Hence there is a surjective map from $G(\mathfrak{hol}_x^{\hbar})$ to $\mathcal{G}_x(A^{\hbar})^0$. At the same time, $G(\mathfrak{hol}_x^{\hbar})$ is the simply connected Lie group integrating $\mathfrak{hol}_x^{\hbar}$, while $\mathrm{Hol}_x^{\hbar}$ is another Lie group integration. Then there is also a surjection from $G(\mathfrak{hol}_x^{\hbar})$ to $\mathrm{Hol}_x^{\hbar,0}$. All together, we find a commutative diagram:

$$\begin{array}{ccc} \mathcal{G}_x(A^{\hbar})^0 & \xrightarrow{\Phi} & \mathrm{Diff}(\mathcal{N}_x) \\ \uparrow & & \uparrow \\ G(\mathfrak{hol}_x^{\hbar}) & \longrightarrow & \mathrm{Hol}_x^{\hbar,0} \end{array}$$

from which we conclude that $\Phi(\mathcal{G}_x(A^{\hbar})^0) = \mathrm{Hol}_x^{\hbar,0}$.

Now that the idea of why this claim should hold is clear, we move on to possible proofs. Recall that the claim is proven if we find a proof for Claim 7.15. Given any path $v(t) : [0,1] \to \mathfrak{hol}_x^{\hbar}$, we want to find a loop λ in M inducing the same parallel transport as $v(t)$. As mentioned before, it is possible that this loop exists, but admits no explicit expression. To prove that at least this loop exists, we can look for a contractible loop satisfying Equation (19). We search for a way to write $v(t)$ as a path through $\mathrm{Im}(\Omega)$ with varying basepoints. Then at each point, we describe λ as a sequence of lasso's. These lasso's are small rectangular loops that lie in one coordinate chart. In that chart, we can do computations and prove that Equation (19) holds locally. By taking the limit of this lasso-approximation, we find the desired loop λ. This technique is similar to the techniques used in the classical proof of the Ambrose-Singer theorem, which makes it likely that this would work.

8.2 Mackenzie's approach

In [Mac05], Mackenzie gives a detailed discussion of transitive groupoids and path liftings, resulting in an alternative proof to the Ambrose-Singer theorem in his Theorem 6.4.20 (which is even stronger than Ambrose-Singer). In this section, we give an outline of this discussion, mostly of Chapters 6.3 and 6.4 in [Mac05]. We will translate the notation that Mackenzie uses to the notation used in this thesis. We first give a short summary of the entire procedure.

Starting with a Lie groupoid $\mathcal{G} \rightrightarrows M$ and a connection on its algebroid, one can define a holonomy subgroupoid of $\mathcal{G}$ using 'parallel transport' in $\mathcal{G}$ along the algebroid connection. The algebroid of this holonomy groupoid is generated by the curvature of the connection.

Starting with a principal G-bundle $P \to M$ with a connection, we can apply this procedure to the gauge groupoid $\mathcal{G}$ associated to P. It follows that in this case, the isotropy groups of the holonomy groupoid are precisely the holonomy groups. By computing the algebroid of $\mathcal{G}$, the classical Ambrose-Singer theorem is recovered.

In order to apply this approach in the general case of proper fibrations, one has to find a relationship between the isotropy groups of the holonomy groupoid and the holonomy groups. This relationship is more difficult to find, and requires a correct choice of 'ambient groupoid'. By ambient groupoid, we mean the groupoid of which the holonomy groupoid forms a subgroupoid.

We will now make this a bit more precise. Consider a Lie groupoid $\mathcal{G} \rightrightarrows M$. Denote by $\mathcal{P}(M)$ the space of paths on M, and by $\mathcal{P}(\mathcal{G})$ the space of $\mathcal{G}$-paths. Recall that a $\mathcal{G}$-path is a path $g(t) : [0,1] \to \mathcal{G}$ such that for some $x \in M$, $g(0) = 1_x$ and for all $t \in [0,1]$, $g(t) \in s^{-1}(x)$. The following is Definition 6.3.1 in [Mac05].

Definition 8.1. A **path connection** on $\mathcal{G} \rightrightarrows M$ is a lift $\Sigma : \mathcal{P}(M) \to \mathcal{P}(\mathcal{G})$ such that:

- $\Sigma(\gamma)(0) = 1_{\gamma(0)}$;
- $\Sigma(\gamma)(t) \in t^{-1}(\gamma(t))$;
- Lifting commutes with reparametrizations;
- If γ is smooth at time t, then $\Sigma(\gamma)$ is smooth at time t;
- If γ, $\gamma' \in \mathcal{P}(M)$ such that
$$\frac{d\gamma}{dt}(t') = \frac{d\gamma'}{dt}(t')$$
for some time t', then also
$$\frac{d\Sigma(\gamma)}{dt}(t') = \frac{d\Sigma(\gamma')}{dt}(t').$$

Path connections correspond to algebroid connections defined in Definition 4.2 by the following lemma (Theorem 6.3.5 in [Mac05]):

Lemma 8.2. *Path connections $\Sigma : \mathcal{P}(M) \to \mathcal{P}(\mathcal{G})$ on a groupoid $\mathcal{G} \rightrightarrows M$ are in 1-1 correspondence with connections $\sigma : TM \to A$ on $A = \mathcal{L}ie(\mathcal{G})$.*

This correspondence is determined by

$$\frac{d}{dt}\Sigma(\gamma)(t_0) = dR_{\Sigma(\gamma)(t_0)}\left(\sigma\left(\frac{d}{dt}\gamma(t_0)\right)\right),\ \gamma \in \mathcal{P}(M).$$

The curvature of an algebroid connection σ is a 2-form measuring the failure of σ to preserve Lie brackets. The corresponding notion for path connections is called holonomy.

Definition 8.3. The **holonomy of a path** $\gamma \in \mathcal{P}(M)$ is the arrow $\hat{\gamma} := \Sigma(\gamma)(1) \in \mathcal{G}$.

Using these holonomies, we construct a subgroupoid of $\mathcal{G}$.

Definition 8.4. The **holonomy subgroupoid** associated to Σ is defined by

$$\Theta = \Theta(\Sigma) := \{\, \hat{\gamma} \mid \gamma \in \mathcal{P}(M) \,\} \subset \mathcal{G}.$$

For any $x \in M$, the **holonomy group** at x is defined as the isotropy group of the holonomy groupoid:

$$\Theta_x = \{\, \hat{\gamma} \mid \gamma \text{ a loop at } x \,\}.$$

Note that any lift $\Sigma(\gamma)(t)$ lies entirely in Θ by choosing reparametrizations. By Theorem 6.3.19 in [Mac05], Θ is actually a Lie subgroupoid of $\mathcal{G}$. The main step in the proof of this theorem is showing that the isotropy groups of Θ are Lie groups.

The next step is to compute the Lie algebroid of Θ. Let A be the Lie algebroid of the ambient groupoid $\mathcal{G}$. Denote by $\mathfrak{g}(A)$ the isotropy Lie algebra bundle of A. Define $\mathfrak{g}(A)^\sigma$ to be the least sub-Lie algebra bundle of $\mathfrak{g}(A)$ such that

- $R_\sigma(X, Y) \in \mathfrak{g}(A)^\sigma$ for all $X, Y \in TM$, and
- $(ad \circ \sigma)(\Gamma(\mathfrak{g}(A)^\sigma)) \subset \Gamma(\mathfrak{g}(A)^\sigma)$.

Here, $R_\sigma : TM \times TM \to \mathfrak{g}(A)$ is the curvature

$$R_\sigma(X, Y) = \sigma([X, Y]) - [\sigma(X), \sigma(Y)],$$

and $ad : A \to \mathcal{D}er(\mathfrak{g}(A))$ is the adjoint representation (see Definition 5.2.16 in [Mac05]) determined by

$$ad(X)(V) = [X, V],\ X \in \Gamma(A),\ V \in \Gamma(\mathfrak{g}(A)).$$

After defining $\mathfrak{g}(A)^\sigma$, we can find a subalgebroid of A denoted A^σ which has $\mathfrak{g}(A)^\sigma$ as its isotropy Lie algebra, and which contains the image $\sigma(TM)$. This algebroid A^σ is defined by its space of sections:

$$\Gamma(A^\sigma) = \{\, X \in \Gamma(A) \mid X - \sigma \circ \rho(X) \in \Gamma(\mathfrak{g}(A)^\sigma) \,\}.$$

Theorem 6.4.20 in [Mac05] states:

Theorem 8.5. *The algebroid of the holonomy Lie groupoid Θ is precisely the algebroid A^σ.*

This concludes our summary of the Mackenzie's theory. We will now discuss how this relates to the classical Ambrose-Singer theorem, and to our Ambrose-Singer algebroid. Mackenzie also discusses how the Ambrose-Singer theorem for principal bundles can be recovered in 6.4.21. In our discussion we start with the algebroid A^σ of Θ, which we claim corresponds to the Ambrose-Singer algebroid $A^{\hbar}$. We give here an idea of the relationship between our work and Mackenzie's, but the proof details have to be worked out.

Lemma 8.6. *Let $\hbar$ be a holonomic connection on a proper fibration. Denote by σ the induced connection on the Ambrose-Singer algebroid $A^{\hbar}$ by Equation* (12). *The isotropy Lie algebra bundle $\mathfrak{g}(A)^\sigma$ defined above is precisely the holonomy Lie algebra bundle $\mathfrak{hol}^{\hbar}_x$.*

Proof sketch. The first condition, that $R_\sigma(X,Y)$ lies in $\mathfrak{g}(A)^\sigma$, corresponds to the construction of $\mathfrak{hol}^{\hbar}$ as the collection of images of the curvature. The second condition is precisely what is stated in Lemma 7.3. □

Lemma 8.7. *The algebroid A^σ is precisely the Ambrose-Singer algebroid $A^{\hbar}$.*

Proof sketch. Since A^σ is defined as having isotropy Lie algebra $\mathfrak{hol}^{\hbar}$ and containing $\sigma(TM) \cong TM$, we see that this corresponds to the Ambrose-Singer algebroid $A^{\hbar} = \mathfrak{hol}^{\hbar} \oplus TM$. □

In order to prove the Ambrose-Singer theorem, it remains to show that the isotropy groups of the holonomy groupoid Θ are precisely the holonomy groups $\mathrm{Hol}^{\hbar}$. In the case of principal bundles, this can be done using the gauge groupoid.

Let $P \to M$ a principal G-bundle with a connection $\hbar$. Consider the gauge groupoid $(P \times P)/G$ and the Atiyah algebroid TP/G. The connection $\hbar$ induces a connection on the Atiyah algebroid, which we denote by σ. We remark that for principal bundles, the Ambrose-Singer algebroid $A^{\hbar} = \mathfrak{hol}^{\hbar} \oplus TM$ will lie inside the Atiyah algebroid. This is due to G-invariance of the principal bundle connection $\hbar$, by which we find that $\mathfrak{X}(M)$ is identified with $\hbar(\mathfrak{X}(M)) \subset \mathfrak{X}(P)^{\mathcal{G}}$.

The connection σ on TP/G is equivalent to a path connection Σ on the gauge groupoid. Consider any loop γ in M based at a point x. We want to relate the following notions:

- $\hbar$-parallel transport along γ: $\tau_\gamma : P_x \to P_x$,
- The holonomy element $\hat{\gamma} := \Sigma(\gamma)(1) \in (P \times P)/G$.

Lemma 8.8. *Denote $\hat{\gamma} = [(p,q)]$ for some $p, q \in P_x$. Then the two notions given above are related by*

$$\tau_\gamma(p) = q.$$

Proof sketch. Recall that the holonomy element $\hat{\gamma}$ is determined via the path lifting Σ, which is determined by the differential equation

$$\frac{d}{dt}\Sigma(\gamma)(t) = dR_{\Sigma(\gamma)(t)}\left(\sigma_{\gamma(t)}\left(\frac{d}{dt}\gamma(t)\right)\right).$$

On the other hand, $\hbar$-parallel transport along γ is determined by solving the differential equation

$$\frac{d}{dt}u(t) = \hbar_{\gamma(t)}\left(\frac{d}{dt}\gamma(t)\right).$$

The connections $\hbar$ and σ are the same maps (see their relationship in Lemma 4.10). Furthermore, both of these notions are G-invariant, from which we see they are equivalent and $\tau_\gamma(p) = q$ if $\hat{\gamma} = [(p, q)]$. □

Corollary 8.9. *The isotropy groups of* Θ *are* $\Theta_x = Hol_x^{\hbar}$.

Proof. The isotropy groups Θ_x are determined by holonomy (in the sense of Definition 8.3) of loops in the base, which by the previous lemma corresponds to the $\hbar$-parallel transport along that loop. We see that the holonomy groups $\mathrm{Hol}_x^{\hbar}$ are recovered. □

By this corollary and Lemma 8.6, we find that Mackenzie's approach provides a proof for the Ambrose-Singer theorem for principal bundles as we now find

$$\mathcal{L}ie(\mathrm{Hol}_x^{\hbar}) = \mathcal{L}ie(\Theta_x) = \mathfrak{g}_x(A^\sigma) = \mathfrak{g}_x(A^{\hbar}) = \mathfrak{hol}_x^{\hbar}.$$

Of course, our discussion here hides a lot of details around the theorems of Mackenzie, including the proof that the isotropy groups Θ_x are Lie groups.

Unfortunately it is not as straightforward to generalise this discussion to proper fibrations. The algebroid A^σ will still be the Ambrose-Singer algebroid, but it becomes difficult to prove a result similar to Corollary 8.9 for proper fibrations. For principal bundles, we chose the gauge groupoid as ambient groupoid, which made it easy to relate elements of that groupoid to holonomy. For proper fibrations, we would like to take the Weinstein groupoid $\mathcal{G}(A^{\hbar})$ as ambient groupoid, and find holonomy subgroupoid $\Theta = \mathrm{Hol}_{\mathcal{N}}(A^{\hbar})$. However, in this discussion the same issue comes up that we couldn't solve in Claim 7.6: proving that the isotropy groups of $\mathrm{Hol}_{\mathcal{N}}(A^{\hbar})$ are the holonomy groups $\mathrm{Hol}_x^{\hbar}$.

In conclusion, this alternative approach does not immediately resolve the problems raised in Section 7, but further examination of these techniques might give more insight into a geometric proof of the Ambrose-Singer theorem. Besides that, other authors have build upon the framework of Mackenzie, and their techniques might prove useful as well. We will see an example of this in Section 8.4.

8.3 A different approach to integrability of the Ambrose-Singer algebroid

In this section, we consider again the result of Corollary 7.4; the algebroid $A^{\hbar}$ is integrable. We have proven this using the free action of $A^{\hbar}$ on $\mathcal{N} \to M$. This action induced on one hand an action groupoid $\mathcal{G}(A^{\hbar}) \times \mathcal{N}$, and on the other hand a monodromy groupoid $\mathcal{M}on(\mathcal{N}, \mathrm{Im}(a))$. An isomorphism between these groupoids forced $\mathcal{G}(A^{\hbar})$ to be smooth. This proof relies on the fact that monodromy groupoids over foliated manifolds are smooth. In this section, we discuss the integrability of $A^{\hbar}$ without using this strong result.

Consider the algebroid action $a : \Gamma(A) \to \mathfrak{X}(\mathcal{N})$ and the induced action of the Weinstein groupoid $\mathcal{G}(A^{\hbar})$. Restricting this action to the identity component of any isotropy group, we find an action of $\mathcal{G}_x(A^{\hbar})^0$ on the fiber $\mathcal{N}_x$. Now recall that

$$\mathcal{G}_x(A^{\hbar})^0 \cong G(\mathfrak{hol}_x^{\hbar})/\mathrm{Im}(\partial_x).$$

It holds that the groupoid action of $\mathcal{G}_x(A^{\hbar})^0$ is equivalent to a group action of $G(\mathfrak{hol}_x^{\hbar})$ which is trivial on $\mathrm{Im}(\partial_x)$. The following lemma defines this group action.

Lemma 8.10. *Interpret a path $a(t) : [0,1] \to \mathfrak{hol}_x^{\hbar} \subset \mathfrak{X}(\mathcal{N}_x)$ as a time-dependent vector field on $\mathcal{N}_x$. Consider the induced isotopy ϕ_a^t. Then the following defines a well-defined group homomorphism, which is trivial on $Im(\partial_x)$:*

$$\varphi : G(\mathfrak{hol}_x^{\hbar}) \to \mathit{Diff}(\mathcal{N}_x), \ \varphi([a]) = \phi_a^1.$$

Proof sketch. First, we recall what we mean by isotopy of a time-dependent section, described in Chapter 6 in [CdS01]. An isotopy of $\mathcal{N}_x$ is a map $\phi^t(p) : \mathcal{N}_x \times \mathbb{R} \to \mathcal{N}_x$ such that $\phi^0 = \mathrm{Id}_{\mathcal{N}_x}$ and each $\phi^t : \mathcal{N}_x \to \mathcal{N}_x$ is a diffeomorphism. A time-dependent vector field v_t on $\mathcal{N}_x$ induces an isotopy by

$$\frac{d\phi^t}{dt} = v_t \circ \phi^t.$$

We discuss first how to show this group homomorphism is well-defined. Consider two $\mathfrak{hol}_x^{\hbar}$-paths $a(t)$ and $b(t)$ which are $\mathfrak{hol}_x^{\hbar}$-homotopic. This means that there exist variations $a_\epsilon(t)$ and $b_\epsilon(t)$ of paths in $\mathfrak{hol}_x^{\hbar}$ such that $a_0(t) = a(t)$, $a_1(t) = b(t)$, and

$$\frac{da}{d\epsilon} - \frac{db}{dt} = [a,b] \text{ and } b_\epsilon(0) = b_\epsilon(1) = 0.$$

Interpreting these variations as time-dependent vector fields again, and computing their isotopies, it follows that $\phi_a^1 = \phi_b^1$. Furthermore, concatenation of $\mathfrak{hol}_x^{\hbar}$-paths commutes with composition of flows (isotopies), from which we conclude φ is a well-defined group homomorphism.

To prove that this is trivial on $\mathrm{Im}(\partial_x)$, recall that this image is precisely the collection of classes $[a] \in G(\mathfrak{hol}_x^{\hbar})$ such that a is $A^{\hbar}$-homotopic to 0_x. The proof that the action is well-defined also holds for $A^{\hbar}$-homotopies, and we see that $\phi_a^1 = \phi_{0_x}^1 = \mathrm{Id}$. □

Lemma 8.11. *The kernel of the action φ is a discrete subgroup of $G(\mathfrak{hol}_x^{\hbar})$.*

This lemma has to be proven, hiding the difficult details of the desired result:

Corollary 8.12. *The Ambrose-Singer algebroid $A^{\hbar}$ is integrable.*

Proof. By Theorem 2.52, the (transitive) algebroid $A^{\hbar}$ is integrable if and only if each $\mathrm{Im}(\partial_x)$ is discrete. By Lemma 8.10, $\mathrm{Im}(\partial_x)$ lies in $\mathrm{Ker}(\varphi)$, which is assumed to be discrete. □

Furthermore, we have (even without assuming Lemma 8.11) a relationship between this action and the holonomy group:

Lemma 8.13. *The image of $\varphi : G(\mathfrak{hol}_x^{\hbar}) \to \mathit{Diff}(\mathcal{N}_x)$ contains the reduced holonomy group $Hol_x^{\hbar,0}$.*

Proof sketch. Consider any element $\tau \in \mathrm{Hol}_x^{\hbar,0}$. We may assume τ arises from parallel transport along a contractible loop γ_0. Then the following defines a path in $G(\mathfrak{hol}_x^{\hbar})$:

$$a(\epsilon)(\tau_{\gamma_\epsilon}(p)) = \left(\frac{d}{d\epsilon}\tau_{\gamma_\epsilon}\right)(p),$$

which satisfies $\phi_a^1 = \tau$. To show that this is indeed a path in $\mathfrak{hol}_x^{\hbar}$, one can prove the following formula:

$$\frac{d}{d\epsilon}\tau_{\gamma_\epsilon}(p) = (\tau_{\gamma_\epsilon})_* \left(\int_0^1 (\tau_{\gamma_\epsilon}^{t,0})_* \left(\Omega\left(\frac{d\gamma}{d\epsilon}, \frac{d\gamma}{dt}\right)\right)(p)dt\right).$$

The image of the curvature form lies in the holonomy Lie algebra, which is invariant under parallel transport. Thus this expression stays inside the holonomy Lie algebra. □

Finally, we show that this group action is indeed related to the groupoid action of the Weinstein groupoid.

Lemma 8.14. *The following two actions of $\mathcal{G}_x(A^{\hbar})^0$ on $\mathcal{N}_x$ are equivalent:*

- *The action induced by $\varphi : G(\mathfrak{hol}_x^{\hbar}) \to \mathit{Diff}(\mathcal{N}_x)$*
- *The restriction of $\Phi : \mathcal{G}_x(A^{\hbar}) \to \mathit{Diff}(\mathcal{N}_x)$*

Proof. Let $g \in \mathcal{G}_x(A^{\hbar})^0$ with a representative A-path $a(t) : [0,1] \to \mathfrak{hol}_x^{\hbar}$. $a(t)$ induces two actions on $\mathcal{N}$. Both actions are defined by construction of a path in $\mathcal{N}$ and evaluating at $t = 1$. The first path is denoted ϕ^t and satisfies

$$\frac{d\phi^t}{dt} = a(t) \circ \phi^t.$$

The second path is denoted $u(t)$ and satisfies

$$\frac{du(t)}{dt} = a_u(t)(a(t)) = a(t) \circ u(t).$$

Here we used the definition of the action of the Ambrose-Singer algebroid in Proposition 7.1. Since these differential equations are the same, the two actions of $a(t)$ agree. □

With this lemma, the proof of Lemma 8.13 follows immediately from Claim 7.6. But on the other hand, we can now view the theory developed in Section 7 from a different perspective. Starting with this group action of $G(\mathfrak{hol}_x^{\hbar})$ on $\mathcal{N}$, we have shown that this action is well-defined on $\mathcal{G}_x(A^{\hbar})^0$. If one can prove that the image of this induced action is precisely $\mathrm{Hol}_x^{\hbar,0}$, this forms (part of) the proof of Claim 7.6. Then, this group action can be generalised to the groupoid action introduced in Section 7.

8.4 Application to algebroid connections

In this section, we consider a generalisation of the notion of connections, and see whether this generalisation can fit inside our framework as well. We focus on a specific generalisation, namely algebroid connections, introduced in this form by Fernandes in [Fer02]. So far, we have considered horizontal connections on proper fibrations $\mathrm{pr} : \mathcal{N} \to M$. We defined a connection as a lifting map, lifting tangent vectors on M to tangent vectors on $\mathcal{N}$, or on the level of vector fields:

$$\hbar : \mathfrak{X}(M) \to \mathfrak{X}^{proj}(\mathcal{N}).$$

We will call this situation the 'covariant case'. Now let $A \to M$ be an algebroid, and recall the definition of an algebroid action (Definition 2.54). This is a vector bundle map $a : A \to T\mathcal{N}$, which is a Lie algebra morphism on the level of sections:

$$a : \Gamma(A) \to \mathfrak{X}^{proj}(\mathcal{N}).$$

Interpreting A as an alternative tangent bundle, we can interpret an algebroid action as a generalisation of a connection. We remove the assumption that $a : \Gamma(A) \to \mathfrak{X}^{proj}(\mathcal{N})$ is a Lie algebra morphism; after all, connections $\hbar : \mathfrak{X}(M) \to \mathfrak{X}^{proj}(\mathcal{N})$ in general do not commute with the Lie brackets. This leads to the following definition, which is Definition 2.1 from [Fer02] generalised to proper fibrations.

Definition 8.15. Let $A \to M$ be a Lie algebroid and $\mathrm{pr} : \mathcal{N} \to M$ a proper fibration. Consider the pullback vector bundle $\mathrm{pr}^*A \to \mathcal{N}$. An A-**connection** is a vector bundle map

$$\hbar^A : \mathrm{pr}^*A \to T\mathcal{N}$$

which commutes with the anchor, i.e.,

$$d\mathrm{pr} \circ \hbar^A = \rho.$$

In the case of principal G-bundles, we additionally require $\hbar^A$ to be G-invariant.

Remark 8.16. This definition can be related to different concepts defined in this thesis:

- The covariant case can be recovered by considering the algebroid $TM \to M$. That is, throughout this thesis we have discussed holonomy induced by TM-connections on $\mathrm{pr} : \mathcal{N} \to M$.

- If an A-connection is additionally a Lie algebra morphism, we call it flat. Flat A-connections are precisely the algebroid actions defined in Definition 2.54.
- If $\mathcal{N} \to M$ is a vector bundle with a flat A-connection, we recover the notion of representations introduced in Definition 3.7.

To any A-connection we can associate a curvature form. Similar to the covariant case, this measures the failure of the connection to preserve Lie brackets.

Definition 8.17. Let $\hbar^A$ an algebroid connection. The associated curvature is

$$\Omega : \Gamma(A) \times \Gamma(A) \to \mathfrak{X}^v(\mathcal{N}),$$

$$\Omega(\alpha, \beta) = \hbar^A([\alpha, \beta]_A) - [\hbar^A(\alpha), \hbar^A(\beta)].$$

For any $x \in M$, this induces a pointwise map

$$\Omega_x : A_x \times A_x \to \mathfrak{X}^v(\mathcal{N}_x).$$

In order to generalise holonomy and the Ambrose-Singer theorem, we define parallel transport with respect to an A-connection.

Definition 8.18. Given an A-path $a(t)$, parallel transport along $a(t)$ with respect to the A-connection $\hbar^A$ is the map

$$\tau_a : \mathcal{N}_{\gamma(0)} \to \mathcal{N}_{\gamma(1)}$$

transporting u_0 to $u(1)$, where $u(t)$ is the curve in $\mathcal{N}$ and the solution of the ODE

$$\begin{cases} \frac{d}{dt}u(t) = \hbar^A_{u(t)}(a(t)), \\ u(0) = u_0. \end{cases}$$

Remark 8.19. If the connection is flat, this parallel transport is precisely the transport that we used to define an action of the Weinstein groupoid $\mathcal{G}(A)$ on $\mathcal{N}$.

Using parallel transport, we can define holonomy induced by an A-connection:

Definition 8.20. The A-**holonomy group** at $x \in M$ is defined as

$$\mathrm{Hol}_x^{\hbar^A} = \{\, \tau_a \mid a(t) \text{ an } A\text{-path covering a loop} \,\}.$$

The **reduced** A-**holonomy group** is defined by considering only contractible loops in the base.

Note that loops in the base must be contained in a leaf of the algebroid foliation.

Finally we can generalise the Ambrose-Singer theorem. This is Theorem 2.1 from [Fer02]. Here we state a version for proper fibrations:

Theorem 8.21 (Ambrose-Singer-Fernandes theorem.)**.** *Let $\hslash^A$ an A-connection on a proper fibration $\mathcal{N} \to M$. The Lie algebra of the holonomy group $Hol_x^{\hslash^A}$ is the linear span of*

- $\tau_a^*(Im(\Omega_{\gamma(1)}))$ *for any A-path $a(t)$ covering $\gamma(t)$ on M such that $\gamma(0) = x$, and*
- $\tau_a^*(\hslash^A(\alpha))$ *for $\alpha \in Ker(\rho_{\gamma(1)})$.*

Compared to the classical case, the holonomy Lie algebras pick up an extra term. In words, we find a sum of the 'classical' holonomy Lie algebras and the isotropy Lie algebra of A. In the case of the tangent Lie algebroid $TM \to M$, we see that the isotropy Lie algebra is trivial, hence the classical Ambrose-Singer theorem is recovered for TM-holonomy.

Fitting this theory into our framework, we turn to Proposition 7.1, and construct the Ambrose-Singer algebroid for A-connections. The Ambrose-Singer algebroid in the covariant case was defined as the direct sum of $\mathfrak{hol}^{\hslash}$ and TM, where TM was identified with the image $\hslash(TM)$ of the connection. In the case of A-connections, the image of the connection $\hslash^A$ is identified with the algebroid A, and the Ambrose-Singer algebroid for the connection $\hslash^A$, denoted AS, is:

$$AS = \mathfrak{hol}^{\hslash^A} \oplus A$$

with anchor map $\rho_{AS}(v, a) = \rho_A(a)$. The isotropy Lie algebra of this algebroid is clearly

$$\mathfrak{g}(AS) = \mathfrak{hol}^{\hslash^A} \oplus \mathfrak{g}(A)$$

which agrees with the Ambrose-Singer-Fernandes theorem stated above.

Finally, one can wonder if this can be generalised to other connections. An example of a result in this direction is a paper by Crampin and Saunders ([CS12]). This paper discusses holonomy of nonlinear connections, specifically for Landsberg spaces in Finsler geometry. Using the groupoid approach of Mackenzie, they prove a version of the Ambrose-Singer theorem for those spaces.

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A Homotopy for concatenation of $A^{\hbar}$-paths

In this appendix, we give the full proof of Lemma 7.13:

Lemma. *Let $\gamma : [0,1] \to M$ with $\gamma(0) = x_0$, and let $v : [0,1] \to \mathfrak{hol}^{\hbar}_{x_0}$. Denote induced $A^{\hbar}$-paths by*

$$g_\gamma := \left(0, \frac{d\gamma}{dt}\right), \; h_v := (v, 0_{x_0}).$$

Then for their concatenation, the following holds:

$$g_\gamma \cdot h_v = \left[\left(\tau_\gamma^{0,t}(v^\tau(t)), \frac{d\gamma}{dt}\right)\right].$$

Proof. This lemma is of course easily proven by checking that the provided variation of $A^{\hbar}$-paths satisfies all conditions. In this appendix however we choose to show how to construct such a homotopy.

We construct the homotopy $a_\epsilon(t)$ such that $a_0(t)$ is the left hand side of the equation and $a_1(t)$ is the right hand side. First we have to compute in detail what concatenation of $A^{\hbar}$-paths is, using Equation (5), and compute the base-paths of either side.

$$a_0(t) = \begin{cases} (2v^\tau(2t), 0_{x_0}) & \text{if } t \leq \frac{1}{2} \\ (0, 2\frac{d\gamma}{dt}(2t-1)) & \text{if } t \geq \frac{1}{2} \end{cases}$$

with basepath

$$\gamma_0(t) = \begin{cases} x_0 & \text{if } t \leq \frac{1}{2} \\ \gamma(2t-1) & \text{if } t \geq \frac{1}{2} \end{cases}$$

and

$$a_1(t) = \left(\tau_\gamma^{0,t}(v^\tau(t)), \frac{d\gamma}{dt}\right)$$

with basepath

$$\gamma_1(t) = \gamma(t)$$

Now to find an $A^{\hbar}$-homotopy, we take three steps; we first define a homotopy of their base-paths, then we define a variation above this homotopy, and then we prove that the variation is actually a homotopy. For the homotopy of base-paths, there is a clear candidate, which we denote by $\tilde{\gamma}(\epsilon, t)$ to easily distinguish it from the original base path γ.

$$\tilde{\gamma}(\epsilon, t) = \begin{cases} x_0 & \text{if } 0 \leq t \leq \frac{1-\epsilon}{2} \\ \gamma\left(\frac{2t}{\epsilon+1} + \epsilon - 1\right) & \text{if } \frac{1-\epsilon}{2} \leq t \leq 1 \end{cases}$$

It is easy to check that this is indeed a homotopy between $\gamma_0(t)$ and $\gamma_1(t)$.

We now move on to define the variation of $A^{\hbar}$-paths $a(\epsilon, t)$. Its horizontal part has to be the speed of $\tilde{\gamma}$. This splits into two cases, just like $\tilde{\gamma}$. The vertical part can be chosen any way we want, as long as it maps to the correct fibers. We define $a(\epsilon, t)$ here as general as possible, and will explore then the conditions that suggest what the vertical terms should be.

$$a(\epsilon, t) = \begin{cases} (V(\epsilon, t), 0) & \text{if } 0 \leq t \leq \frac{1-\epsilon}{2} \\ \left(U(\epsilon, t), \frac{2}{\epsilon+1} \frac{d\gamma}{dt} \left(\frac{2t}{\epsilon+1} + \epsilon - 1\right)\right) & \text{if } \frac{1-\epsilon}{2} \leq t \leq 1 \end{cases}$$

This comes with the following conditions on V and U:

$$V(\epsilon, t) \in \mathfrak{hol}^{\hbar}_{x_0}, \; U(\epsilon, t) \in \mathfrak{hol}^{\hbar}_{\gamma\left(\frac{2t}{\epsilon+1} + \epsilon - 1\right)}.$$

Furthermore, from the condition $a(0, t) = a_0(t)$ we find that we should have

$$U(0, t) = 0, \; V(0, t) = 2v^{\tau}(2t)$$

and from the condition $a(1, t) = a_1(t)$ we find

$$U(1, t) = \tau^{0,t}_{\gamma}(v^{\tau}(t)).$$

The other suggested condition for $V(1, 0)$ is actually a specific case of the following; at the 'middle point' $t = (1 - \epsilon)/2$, the terms U and V should agree, and we find the more general condition

$$V\left(\epsilon, \frac{1-\epsilon}{2}\right) = U\left(\epsilon, \frac{1-\epsilon}{2}\right).$$

This condition ensures that the resulting path is continuous. Since any continuous A-path is homotopic to a reparametrized smooth version, it is enough to impose continuity.

Any functions $U(\epsilon, t)$ and $V(\epsilon, t)$ satisfying these conditions would make $a(\epsilon, t)$ into a variation. We now consider when it is an $A^{\hbar}$-homotopy, which will add further conditions on $V(\epsilon, t)$ and $U(\epsilon, t)$.

In order to make this variation into a homotopy, we have to solve the following equations for $W(\epsilon, t)$, where $W(\epsilon, t)$ is a path above $\tilde{\gamma}$:

$$\begin{cases} \nabla^{\hbar}_{\frac{d\tilde{\gamma}}{d\epsilon}} V(\epsilon, t) - \nabla^{\hbar}_{\frac{d\tilde{\gamma}}{dt}} W(\epsilon, t) = -\left[V(\epsilon, t), W(\epsilon, t)\right] - \Omega\left(\frac{d\tilde{\gamma}}{dt}, \frac{d\tilde{\gamma}}{d\epsilon}\right) & \text{if } 0 \leq t \leq \frac{1-\epsilon}{2} \\ \nabla^{\hbar}_{\frac{d\tilde{\gamma}}{d\epsilon}} U(\epsilon, t) - \nabla^{\hbar}_{\frac{d\tilde{\gamma}}{dt}} W(\epsilon, t) = -\left[U(\epsilon, t), W(\epsilon, t)\right] - \Omega\left(\frac{d\tilde{\gamma}}{dt}, \frac{d\tilde{\gamma}}{d\epsilon}\right) & \text{if } \frac{1-\epsilon}{2} \leq t \leq 1 \\ W(\epsilon, 0) = 0 \end{cases} \tag{20}$$

After solving this, the variation is a homotopy if and only if $W(\epsilon, 1) = 0$. In this computation, we can take this as an additional condition on the system that we are solving.

These equations can already be simplified quite much when we consider what the base homotopy $\tilde{\gamma}$ actually is. It is a reparametrization of the path γ, where we stay at the intial point x_0 for a certain amount of time and then speed through the rest of the path. This means that the derivatives w.r.t. t and ϵ actually point in the same direction. More explicitly, we can just compute them. For $0 \leq t \leq (1-\epsilon)/2$, both derivatives are 0. For $\frac{1-\epsilon}{2} \leq t \leq 1$, we find:

$$\frac{d\tilde{\gamma}}{dt}(\epsilon, t) = \frac{2}{\epsilon+1}\frac{d\gamma}{dt}\left(\frac{2t}{1+\epsilon} + \epsilon - 1\right)$$
$$\frac{d\tilde{\gamma}}{d\epsilon}(\epsilon, t) = \left(\frac{-2t}{(1+\epsilon)^2} + 1\right)\frac{d\gamma}{dt}\left(\frac{2t}{\epsilon+1} + \epsilon - 1\right).$$

This means that the curvature forms in Equation (20) will be 0. Furthermore, as both derivatives are zero for $0 \leq t \leq (1-\epsilon)/2$, the connection terms in the first equation vanish as well, leaving for the first equation only:

$$[V(\epsilon, t), W(\epsilon, t)] = 0 \text{ for } 0 \leq t \leq \frac{1-\epsilon}{2}. \tag{21}$$

For the second equation, we can plug in the computed derivatives and the definition of the connection $\nabla^{\hbar}$ to find for $\frac{1-\epsilon}{2} \leq t \leq 1$:

$$\begin{aligned} -[U(\epsilon, t), W(\epsilon, t)] = \left(1 - \frac{2t}{(1+\epsilon)^2}\right)\left[\hbar\left(\frac{d\gamma}{dt}\left(\frac{2t}{\epsilon+1} + \epsilon - 1\right)\right), U(\epsilon, t)\right] - \\ \frac{2}{\epsilon+1}\left[\hbar\left(\frac{d\gamma}{dt}\left(\frac{2t}{\epsilon+1} + \epsilon - 1\right)\right), W(\epsilon, t)\right]. \end{aligned} \tag{22}$$

We are now in the following situation: we want to find paths $V(\epsilon, t)$, $U(\epsilon, t)$ and $W(\epsilon, t)$ that satisfy all the above conditions. Then the variation $a(\epsilon, t)$ is indeed a homotopy. Now that we have gathered all conditions, we start solving the system.

We start by choosing a solution for $W(\epsilon, t)$. Recall that this should be above $\tilde{\gamma}$ as well, so a decomposition into the cases around $t = (1-\epsilon)/2$ makes sense. Note that Equation (21) suggests that for low t, W should be colinear with V. For high t, Equation (22) is simplified if W is colinear with U. This motivates the following choice for $W(\epsilon, t)$, where $f, g \in \mathbb{C}^{\infty}(I \times I)$ are to be determined:

$$W(\epsilon, t) = \begin{cases} f(\epsilon, t)V(\epsilon, t) & \text{if } 0 \leq t \leq \frac{1-\epsilon}{2} \\ g(\epsilon, t)U(\epsilon, t) & \text{if } \frac{1-\epsilon}{2} \leq t \leq 1 \end{cases}.$$

From the condition $W(\epsilon, 0) = 0$ we find that either $f(\epsilon, 0) = 0$ or $V(\epsilon, 0) = 0$. From the condition $W(\epsilon, 1) = 0$ we find that either $g(\epsilon, 1) = 0$ or $U(\epsilon, 1) = 0$. Besides, we require that the two cases agree at $t = (1-\epsilon)/2$. Since we imposed that U and V agree at this point, we find that we should have

$$f\left(\epsilon, \frac{1-\epsilon}{2}\right) = g\left(\epsilon, \frac{1-\epsilon}{2}\right).$$

Now clearly Equation (21) is satisfied with this definition. For Equation (22), we see that the bracket on the left hand side is 0, and the remaining condition is

$$\left(\left(1-\frac{2t}{(1+\epsilon)^2}\right)-\frac{2}{\epsilon+1}g(\epsilon,t)\right)\left[\hbar\left(\frac{d\gamma}{dt}\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)\right),U(\epsilon,t)\right]=0.$$

This gives the following solution for the function $g(\epsilon, t)$:

$$g(\epsilon,t)=\frac{\epsilon+1}{2}-\frac{t}{1+\epsilon}.$$

Note that it is not possible to satisfy the condition $g(\epsilon, 1) = 0$. It follows that we must have $U(\epsilon, 1) = 0$ in order to satisfy $W(\epsilon, 1) = 0$.

To solve for $f(\epsilon, t)$, note that we have two known points: we would like to satisfy the condition $f(\epsilon, 0) = 0$, and we have that f and g should agree on the midpoint $t = (1-\epsilon)/2$. From this it is easy to find f as a linear function of t, and we have for W:

$$W(\epsilon,t)=\begin{cases}\left(\frac{\epsilon^2+3\epsilon}{1-\epsilon}t\right)V(\epsilon,t) & \text{if } t\le\frac{1-\epsilon}{2}\\ \left(\frac{\epsilon+1}{2}-\frac{t}{1+\epsilon}\right)U(\epsilon,t) & \text{if } t\ge\frac{1-\epsilon}{2}\end{cases}.$$

Since we have satisfied $W(\epsilon, 0) = 0$ with this choice of $f(\epsilon, t)$ we do not need to require $V(\epsilon, 0) = 0$.

It remains to solve for $U(\epsilon, t)$ and $V(\epsilon, t)$ with the following conditions:

1. $U(\epsilon,t)\in\mathfrak{hol}^{\hbar}_{\gamma\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)}$
2. $U(0,t)=0$
3. $U(1,t)=\tau_\gamma^{(0,t)}(v^\tau(t))$
4. $U(\epsilon,1)=0$
5. $V(\epsilon,t)\in\mathfrak{hol}^{\hbar}_{x_0}$
6. $V(0,t)=2v^\tau(2t)$
7. $V\left(\epsilon,\frac{1-\epsilon}{2}\right)=U\left(\epsilon,\frac{1-\epsilon}{2}\right)$.

We remark that condition 4 is only needed to make the solution an $A^\hbar$-homotopy, and the other conditions are imposed to construct a well-defined variation of $A^\hbar$-paths.

These requirements suggest the following solutions, where $f, h \in C^\infty(I \times I)$ and g, k reparemetrizations of (ϵ, t) are to be determined:

$$V(\epsilon,t)=f(\epsilon,t)v^\tau(g(\epsilon,t))$$

$$U(\epsilon,t) = h(\epsilon,t)\tau^t_{\gamma\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)}(v^\tau(k(\epsilon,t))).$$

In this way, conditions 1 and 5 are automatically satisfied. Note that between $t = 0$ and $t = (1-\epsilon)/2$, the path $\gamma(\epsilon,t)$ is constantly x_0, and the parallel transport map is simply the identity map in this time interval. The remaining conditions now all induce conditions on these functions f, g, h, k:

- Condition 2 implies that either $h(0,t) = 0$, or $k(0,t)$ is 0 or 1. Note that v^τ is the reparametrized path, which is 0 at its endpoints.
- Condition 3 implies that $h(1,t) = 1$ and $k(1,t) = t$.
- Condition 4 implies that $h(\epsilon,1) = 0$ or $k(\epsilon,1)$ is 0 or 1, similar to condition 2.
- Condition 6 implies that $f(0,t) = 2$ and $g(0,t) = 2t$.
- Condition 7 implies that at $t = (1-\epsilon)/2$, the functions f and h should agree, and the functions g and k should agree.

First of all, we let $f(\epsilon,t) = h(\epsilon,t) = 2-\epsilon$. This satisfies $f(0,t) = 2$, $h(1,t) = 1$, and makes f and h agree at $t = (1-\epsilon)/2$. Then we define $g(\epsilon,t) = 2t$, which satisfies the only explicit condition on g: $g(0,t) = 2t$. What remains is to find $k(\epsilon,t)$ with the conditions

- $k(0,t), k(\epsilon,1) \in \{0,1\}$,
- $k(1,t) = t$,
- $k(\epsilon,(1-\epsilon)/2) = 1-\epsilon$

The last condition comes from the requirement $g(\epsilon,(1-\epsilon)/2) = 1-\epsilon$. The solution is given by

$$k(\epsilon,t) = \frac{2\epsilon}{1+\epsilon}(t-1)+1.$$

This finishes the proof. To summarize, the final $A^{\hbar}$-homotopy is given by

$$a(\epsilon,t) = \begin{cases} ((2-\epsilon)v^\tau(2t),0) & \text{if } 0 \le t \le \frac{1-\epsilon}{2} \\ \left((2-\epsilon)\tau^t_{\gamma\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)}(v^\tau(\frac{2\epsilon}{1+\epsilon}(t-1)+1)), \frac{2}{\epsilon+1}\frac{d\gamma}{dt}\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)\right) & \text{if } \frac{1-\epsilon}{2} \le t \le 1 \end{cases}.$$

The solution to the differential equation in Equation (20) is

$$W(\epsilon,t) = \begin{cases} \frac{\epsilon^2+3\epsilon}{1-\epsilon}t(2-\epsilon)v^\tau(2t) & \text{if } 0 \le t \le \frac{1-\epsilon}{2} \\ \left(\frac{\epsilon+1}{2} - \frac{t}{1+\epsilon}\right)(2-\epsilon)\tau^t_{\gamma\left(\frac{2t}{\epsilon+1}+\epsilon-1\right)}(v^\tau(\frac{2\epsilon}{1+\epsilon}(t-1)+1)) & \text{if } \frac{1-\epsilon}{2} \le t \le 1 \end{cases}$$

and since this satisfies $W(\epsilon,1) = 0$, we have a homotopy between $a_0(t)$ and $a_1(t)$, which shows

$$g_\gamma \cdot h_v = \left[\left(\tau^{0,t}_\gamma(v^\tau(t)), \frac{d\gamma}{dt}\right)\right].$$

□