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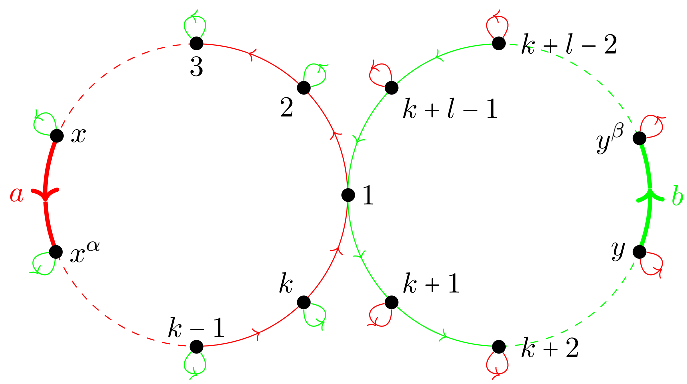
Faculty of Science

Iterated monodromy groups of critically fixed (anti-)rational maps



Master's Thesis

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Mathematical Sciences

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Abstract

Iterated monodromy groups (in short, IMGs) are self-similar groups naturally associated to iterations of (anti-)rational maps on the Riemann sphere. In this thesis, we study the properties of the IMGs of critically fixed (anti-)rational maps; critically fixed maps being those maps whose critical points are also fixed points. More specifically, we prove that the IMGs of critically fixed (anti-)polynomials are regular branch on the subgroup of group elements with even permutational part. In the case of polynomials, we make use of the one-to-one correspondence between the conformal conjugacy classes of critically fixed polynomials and the isomorphism classes of connected planar embedded graphs. Similarly, in the case of anti-polynomials, we use that there is a one-to-one correspondence between the conformal conjugacy classes of critically fixed anti-rational maps and the isomorphism classes of unobstructed topological Tischler graphs. Not being able to prove a similar statement in the general case of (anti-)rational maps, we discuss some motivating examples and explain some of the difficulties.

Contents

Chapter 1

Introduction

In this thesis we study *iterated monodromy groups* (in short, IMGs) of (anti-)rational maps on the Riemann sphere; these maps we denote by $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$. An iterated monodromy group is a group which acts on the preimages of a base point by a monodromy action. IMGs can be used to study dynamical systems, as they encode combinatorial information about the dynamics of the corresponding maps. Furthermore, IMGs are interesting objects to study from a group theoretic perspective. They are self-similar, meaning that for an iterated monodromy group G there is a canonical homomorphism from G to $G^d \times \mathcal{S}_d$. Additionally, they provide examples of groups of intermediate growth.

We focus on studying the algebraic structure of the iterated monodromy groups of *critically fixed* (anti-)rational maps on the Riemann sphere; critically fixed maps being those maps whose critical points are also fixed points. In particular, we study the branchness properties of the iterated monodromy groups of these maps. For a group G to be regular branch on a subgroup H means that H^d embeds into H and that H has finite index in G.

One motivation to study the branchness properties of groups is a result by Bartholdi. He proved that finitely generated, contracting, semi-fractal, regular branch groups are finitely L-pesented, but are not finitely presented [Bar03]. This is particularly interesting for iterated monodromy groups, as IMGs are groups that are finitely generated, contracting and semi-fractal.

Branchness properties of iterated monodromy groups of postcritically finite quadratic polynomials have been studied by Bartholdi and Nekrashevych in [BN08]. Šunić (and his collaborators) studied the branchness properties of iterated monodromy groups of critically fixed polynomials and gave a talk on their results in May 2021 at Banff International Research Station [Šun21]. In this thesis we reprove Šunić's results, as well as extend them to the case of iterated monodromy groups of critically fixed anti-polynomials; an anti-polynomial being a complex conjugate of a polynomial map.

To study the IMGs of critically fixed rational maps, we use that there is a one-to-one correspondence between the conformal conjugacy classes of critically fixed rational maps (of degree at least two) and the isomorphism classes of connected planar embedded graphs (with at least one edge and no loops) [Hlu19]. We use these graphs to describe the generators of the IMGs in terms of their wreath recursion.

For the polynomial case, we reprove the statements of Šunić. These statements can be summarized in the following theorem.

Theorem 1.0.1. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed polynomial with $|C_f| \geq 3$. Let $G \coloneqq \mathrm{IMG}(f)$ and* *let $E < G$ be the subgroup of all elements of G with even permutational part. Then G is regular branch on E.*

We prove this theorem by induction, starting with a base case in which f has two critical points, meaning that G has two generators.

We use a similar strategy to study the IMGs of anti-rational maps. We namely use that there is a one-to-one correspondence between conformal conjugacy classes of critically fixed anti-rational maps (of degree at least 2) and isomorphism classes of unobstructed topological Tischler graphs. We again use these graphs to describe the generators of the IMGs in terms of their wreath recursion.

We can then use the lemmas we have proven for the polynomial case to prove the following statement.

Theorem 1.0.2. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed anti-polynomial with $|C_f| \geq 3$. Let $G \coloneqq \mathrm{IMG}(f)$ and let $E < G$ be the subgroup of all elements of G with even permutational part. Then G is regular branch on E.*

We have not been able to prove similar results in the more general case of (anti-)rational maps. We will however discuss specific examples of rational maps, and look at some of the difficulties regarding proving a general statement in the case of rational maps.

Acknowledgements

I want to thank my daily supervisor Mikhail Hlushchanka for proposing the interesting subject that is the topic of my thesis, for his patience and willingness to discuss and answer questions on the material of my thesis. Furthermore, I would like to thank Gunther Cornelissen and Valentijn Karemaker for being my UU project supervisor and second examinor. Lastly, I want to thank all the people around me for their support this past year, which has definitely not been the easiest.

Notation

Before starting on the background, we introduce some notation that we will use throughout this thesis.

$\mathbb{N}$	the set of natural numbers, excluding 0
$\mathbb{C}$	the set of complex numbers
n	an element of $\mathbb{N}$
$\mathbb{N}_{\geq n}$	the set of natural numbers greater than or equal to n
$\widehat{\mathbb{C}}$	the Riemann sphere
$[n]$	the set $\{1, 2, \ldots, n\}$
$\mathcal{A}_n$	the alternating group on $[n]$
$\mathcal{S}_n$	the symmetric group on $[n]$
X	a (finite) alphabet
$\mathcal{A}(\mathsf{X})$	the alternating group on X
$\mathcal{S}(\mathsf{X})$	the symmetric group on X
X^*	the set $\{x_1 x_2 \ldots x_n \,\vert\, x_i \in \mathsf{X},\, n \geq 0\}$ of words in the alphabet X
T_X	the tree of words in the alphabet X
$\mathrm{Aut}(\mathsf{T}_\mathsf{X})$	the automorphism group of T_X
$\pi_1(\mathcal{M}, t)$	the fundamental group of a topological space $\mathcal{M}$ at a basepoint t
$f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$	an (anti-)rational map on $\widehat{\mathbb{C}}$
f^n	the n-th iterate of f
C_f	the set of critical points of f
P_f	the postcritical set of f
$\mathrm{Fix}(f)$	the set of fixed points of f
$\mathrm{IMG}(f)$	the iterated monodromy group of f
$\mathrm{Tisch}(f)$	the Tischler graph of f (when f is critically fixed)
$\mathrm{Charge}(f)$	the charge graph of f (when f is critically fixed)

Chapter 2

Background

The background in this chapter is based on [Nek05], [BGN03], [Hlu19] and [Gey22].

2.1 Tree of words T_X

Let X be a finite set, called an *alphabet.* We denote by

$$\mathsf{X}^* = \{x_1x_2\dots x_n \,|\, x_i \in \mathsf{X},\, n \in \mathbb{N}\cup\{0\}\}$$

the set of all finite words over the alphabet X, including the empty word $\varnothing$. We have $\mathsf{X}^* = \bigcup_{n\geq 0}\mathsf{X}^n$. For $v = x_1x_2\dots x_n \in \mathsf{X}^n$ we say that the *length* of the word v is n; this length is denoted $|v|$.

The set X^* naturally is a vertex set of the *tree of words* T_X. The root of this tree is the empty word $\varnothing$ and two words are connected by an edge if and only if they are of the form v and vx for some $v \in \mathsf{X}^*$ and $x \in \mathsf{X}$. The set X^n is called the *n-th level* of the tree T_X.

Let $w \in \mathsf{X}^*$ be arbitrary. We denote by T_w the subtree of T_X rooted at w, which has the vertex set $w\mathsf{X}^* = \{wu \,|\, u \in \mathsf{X}^*\}$. Note that the subtree T_w is isomorphic to T_X via the shift $\mathsf{s}_w : \mathsf{T}_w \to \mathsf{T}_\mathsf{X}$ defined by $wu \mapsto u$ for $u \in \mathsf{X}^*$.

2.2 Automorphism group of T_X

Let X be a finite alphabet. An *automorphism* of the tree of words T_X is a bijective map $f : \mathsf{X}^* \to \mathsf{X}^*$ that respects adjacency of the vertices, that is, for any two adjacent vertices $v, vx \in \mathsf{X}^*$ the vertices $f(v)$ and $f(vx)$ are also adjacent, i.e., there exist $u \in \mathsf{X}^*$ and $y \in \mathsf{X}$ such that $f(v) = u$ and $f(vx) = uy$. The set of all automorphisms of the rooted tree T_X forms a group under the operation of composition of maps; this group is called the *automorphism group* of T_X and is denoted $\mathrm{Aut}(\mathsf{T}_\mathsf{X})$. We always consider right actions, meaning that for $g_1, g_2 \in \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ in the product g_1g_2 the element g_1 acts first. The *n-th level stabilizer* $\mathsf{St}_{\mathrm{Aut}(\mathsf{T}_\mathsf{X})}(n)$ is the subgroup of all elements of $\mathrm{Aut}(\mathsf{T}_\mathsf{X})$ that fix all vertices of the n-th level of T_X.

For every $g \in \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ and $w \in \mathsf{X}^*$, we define the automorphism $g|_w : \mathsf{T}_\mathsf{X} \to \mathsf{T}_\mathsf{X}$ by

$$g|_w \coloneqq \mathsf{s}_{w^g} \circ g \circ \mathsf{s}_w^{-1}.$$

This automorphism is called the *restriction of g to the subtree* T_w or the *restriction of g at the vertex w.*

In the following, we assume that $\mathsf{X} = \{1, 2, \ldots, d\}$ to simplify the discussion. We also recall that $\mathcal{S}(\mathsf{X})$ denotes the symmetric group on the set X.

We have a natural isomorphism

$$\psi : \mathsf{St}_{\mathrm{Aut}(\mathsf{T}_\mathsf{X})}(1) \to \mathrm{Aut}(\mathsf{T}_\mathsf{X})^d, \quad g \mapsto \langle\!\langle g|_1, g|_2, \ldots, g|_d \rangle\!\rangle := (g|_1, g|_2, \ldots, g|_d).$$

In general, every element $g \in \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ can be written as

$$g = \langle\!\langle g|_1, g|_2, \ldots, g|_d \rangle\!\rangle \sigma_g, \tag{2.1}$$

where $\langle\!\langle g|_1, g|_2, \ldots, g|_d \rangle\!\rangle \in \mathsf{St}_{\mathrm{Aut}(\mathsf{T}_\mathsf{X})}(1)$ and $\sigma_g \in \mathcal{S}(\mathsf{X})$ is the permutation defining the action of g on the first level X^1. We call σ_g the *permutational part* of g. The expression (2.1) is called the *wreath recursion* of g.

Formally, the isomorphism ψ above extends to a canonical isomorphism

$$\psi : \mathrm{Aut}(\mathsf{T}_\mathsf{X}) \to \mathrm{Aut}(\mathsf{T}_\mathsf{X})^d \rtimes \mathcal{S}(\mathsf{X}),$$

where the semidirect product is taken with respect to the natural action of $\mathcal{S}(\mathsf{X})$ on every factor of $\mathrm{Aut}(\mathsf{T}_\mathsf{X})^d$. Hence, the automorphism group $\mathrm{Aut}(\mathsf{T}_\mathsf{X})$ is isomorphic to the *permutational wreath product* $\mathrm{Aut}(\mathsf{T}_\mathsf{X}) \wr \mathcal{S}(\mathsf{X})$. That is, wreath recursions of two elements $g, h \in \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ are multiplied in the following way

$$\langle\!\langle g|_1, g|_2, \ldots, g|_d \rangle\!\rangle \sigma_g \cdot \langle\!\langle h|_1, h|_2, \ldots, h|_d \rangle\!\rangle \sigma_h = \langle\!\langle g|_1 h|_{1^{\sigma_g}}, g|_2 h|_{2^{\sigma_g}}, \ldots, g|_d h|_{d^{\sigma_g}} \rangle\!\rangle \sigma_g \sigma_h.$$

There are the following standard subgroups of a group acting on a rooted tree.

Definition 2.2.1. Let $G \leq \mathrm{Aut}(\mathsf{T}_\mathsf{X})$.

1. For a vertex $v \in \mathsf{X}^*$, the *vertex stabilizer* is the subgroup $G_v = \{g \in G \mid v^g = v\}$.
2. The *n-th level stabilizer* $\mathsf{St}_G(n)$ is the subgroup of all elements of G that fix all vertices of the n-th level X^n of the tree T_X; i.e., $\mathsf{St}_G(n) = \bigcap_{v \in \mathsf{X}^n} G_v$.
3. The *rigid stabilizer* $\mathsf{RiSt}_G(v)$ of a vertex $v \in \mathsf{X}^*$ is the group of all elements of G that may only act non-trivially on the vertices of the form vu, for $u \in \mathsf{X}^*$. That is,
$$\mathsf{RiSt}_G(v) = \{g \in G \mid w^g = w \text{ for all } w \notin v\mathsf{X}^*\}.$$
4. The *n-th level rigid stabilizer* $\mathsf{RiSt}_G(n)$ is the subgroup generated by the rigid stabilizers of all vertices of the n-th level; i.e., $\mathsf{RiSt}_G(n) = \langle \mathsf{RiSt}_G(v) \mid v \in \mathsf{X}^n \rangle$.

Definition 2.2.2. A group $G \leq \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ is called *level-transitive* if it acts transitively on every level X^n of the tree T_X.

Given $H, H_1, \ldots, H_d < \mathrm{Aut}(\mathsf{T}_\mathsf{X})$, the notation $H_1 \times \cdots \times H_d \subset H$ means that H contains the preimage $\psi^{-1}(H_1 \times \cdots \times H_d)$, which is the subgroup of $\mathsf{St}_{\mathrm{Aut}(\mathsf{T}_\mathsf{X})}(1)$ that acts on the subtree $x\mathsf{T}_\mathsf{X}$ by an element from H_x for each $x \in \mathsf{X}$.

Definition 2.2.3. Let $G \leq \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ be level-transitive. If all of the rigid stabilizers $\mathsf{RiSt}_G(n), n \in \mathbb{N}$, are non-trivial, then G is said to be *weakly branch*. The group G is *branch* if all $\mathsf{RiSt}_G(n)$ have finite index in G.

The group G is *regular weakly branch on* $H \leq G$ if H is non-trivial and $\underbrace{H \times \cdots \times H}_{d \text{ factors}} \subset H$. If in addition H has finite index in G, we say that G is *regular branch on* H.

2.2.1 Self-similar groups

Definition 2.2.4. An action of a group G on the tree T_X by automorphisms is said to be *self-similar* if for every $g \in G$ and every $x \in \mathsf{X}$ there exist $h \in G$ and $y \in \mathsf{X}$ such that

$$(wx)^g = yw^h$$

for all $w \in \mathsf{X}^*$. In particular, $G < \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ is called self-similar if its action on T_X is self-similar. In other words, G is self-similar if the restriction $g|_w$ is an element of G for all $g \in G$ and $w \in \mathsf{X}^*$.

Definition 2.2.5. Let $G \leq \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ and let $S \subseteq G$. We define the *labeled Schreier graph* for G with respect to S. This is a directed graph with vertex set X' consisting of all $x \in \mathsf{X}$ for which it holds that there exists at least one $s \in S$ such that s acts non-trivially on x and directed edge set $\mathsf{X}' \times S$, where the edge (x, s) starts at x and ends at x^s (and is labeled $s|_x$, for every $x \in \mathsf{X}'$ and $s \in S$). For every $s \in S$, we choose a distinct color and every edge of the form (x, s) is colored in the color corresponding to s. Furthermore, the edge (x, s) is drawn as a thick line if the restriction $s|_x$ is non-trivial.

2.3 Iterated monodromy groups

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an (anti-)rational map. The *critical points* of f are those points $c \in \widehat{\mathbb{C}}$ at which f is not locally injective. The set of critical points of f is denoted by C_f. The set $P_f \coloneqq \bigcup_{n=1}^{\infty} f^n(C_f)$ is then called the *postcritical set* of f. The map f is said to be *postcritically finite* if the cardinality $|P_f|$ of the postcritical set is finite. Lastly, we have the set of *fixed points* of f, which is denoted by $\mathrm{Fix}(f)$.

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a postcritically finite (anti-)rational map of degree $d \geq 2$. Then f induces a d-to-1 covering

$$f : \mathcal{M}_1 \coloneqq \widehat{\mathbb{C}} \setminus f^{-1}(P_f) \to \mathcal{M} \coloneqq \widehat{\mathbb{C}} \setminus P_f. \tag{2.2}$$

We use this covering to construct the so-called *tree of preimages* T_f. Choose a basepoint $t \in \mathcal{M}$; this will be the root of the tree. For each $n \in \mathbb{N}$, the n-th level of the tree of preimages T_f is the set $f^{-n}(t)$ of preimages of t under the n-th iteration of f. We connect each vertex $z \in f^{-n}(t)$ with $f(z) \in f^{-(n-1)}(t)$. As the map f in (2.2) is a degree d covering, every vertex in the level $f^{-(n-1)}(t)$ is connected to exactly d vertices in the n-th level, which means that T_f is a d-regular rooted tree.

Let $\pi_1(\mathcal{M}, t)$ be the fundamental group of $\mathcal{M}$ with respect to the basepoint t. We denote by $[\gamma]$ the homotopy class of a closed loop γ, based at t. For each such loop $\gamma \in \pi_1(\mathcal{M}, t)$ we have that for every $n \geq 0$ and $z \in f^{-n}(t)$ there is exactly one lift γ_z of γ under f^n starting at z. We denote by z^γ the endpoint of the path γ_z. By the homotopy lifting property, z^γ depends only on the homotopy class $[\gamma]$, and not on the specific choice of the representative γ.

The map

$$z \mapsto z^\gamma$$

then defines a permutation on the set $f^{-n}(t)$. This action of $\pi_1(\mathcal{M}, t)$ on $f^{-n}(t)$ is called the *monodromy action.*

The permutation $z \mapsto z^\gamma$ defines an automorphism of the tree T_f. Indeed, let $z \in f^{-n}(t)$ and $f(z) \in f^{-(n-1)}(t)$ be two adjacent vertices, and let γ_z be the f^n-lift of γ starting in z and ending in z^γ. Then $f(\gamma_z)$ is the f^{n-1}-lift of γ starting in $f(z)$ and ending in $f(z^\gamma)$. It follows that

$f(z)^\gamma = f(z^\gamma)$, and thus z^γ and $f(z)^\gamma$ are adjacent as well. Hence, the map $z \mapsto z^\gamma$ defines an automorphism of T_f.

The discussion above implies that the fundamental group $\pi_1(\mathcal{M}, t)$ acts on the tree of preimages T_f by automorphisms. This action is called the *iterated monodromy action* of $\pi_1(\mathcal{M}, t)$ on T_f. It follows that we have a group homomorphism:

$$\phi_f : \pi_1(\mathcal{M}, t) \to \mathrm{Aut}(T_f)$$

from the fundamental group of $\mathcal{M} = \widehat{\mathbb{C}} \setminus P_f$ to the automorphism group $\mathrm{Aut}(T_f)$ of the rooted tree of preimages T_f.

Definition 2.3.1. The *iterated monodromy group* of f, denoted $\mathrm{IMG}(f)$, is the quotient of the fundamental group $\pi_1(\mathcal{M}, t)$ by the kernel of the iterated monodromy action. That is,

$$\mathrm{IMG}(f) = \pi_1(\mathcal{M}, t)/\mathrm{Ker}(\phi_f) \simeq \phi_f(\pi_1(\mathcal{M}, t)).$$

One can easily check that, up to conjugacy, the iterated monodromy group does not depend on the choice of the basepoint t.

2.3.1 Standard action

As noted before, the tree of preimages T_f is a d-regular rooted tree, and is thus isomorphic to the tree of words T_X for an alphabet X of d letters. There does not exist a canonical isomorphism, but there is a class of natural isomorphisms, which is constructed as follows.

Fix an alphabet X of d letters and a bijection $\Lambda : \mathsf{X} \to f^{-1}(t)$ between the vertices of the first level of the tree of words T_X and the first level of the tree of preimages T_f. For every $x \in \mathsf{X}$, choose a path ℓ_x starting in t and ending in $\Lambda(x)$. We now extend the map Λ to X^* inductively by the rule that $\Lambda(xv)$ for $x \in \mathsf{X}$ and $v \in \mathsf{X}^n$ is the endpoint of the f^n-lift of ℓ_x starting in $\Lambda(v) \in f^{-n}(t)$. The constructed *labelling map* $\Lambda : \mathsf{T}_\mathsf{X} \to T_f$ is an isomorphism of rooted trees.

The *standard action* of $\pi_1(\mathcal{M}, t)$ (or $\mathrm{IMG}(f)$) on the tree T_X is the action obtained by conjugating the iterated monodromy action on T_f by a labelling map $\Lambda : T_X \to T_f$.

Proposition 2.3.2 ([BGN03, Proposition 5.4])**.** *The standard action is self-similar and is given by*

$$(xv)^\gamma = y\left(v^{\ell_x \gamma_x \ell_y^{-1}}\right), \tag{2.3}$$

where γ_x is the f-lift of the loop γ starting at $\Lambda(x)$, and $y \in \mathsf{X}$ is such that $\Lambda(y)$ is the endpoint of γ_x.

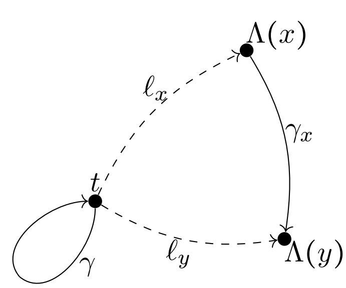


Figure 2.1: A visualisation of the recurrent formula (2.3) for the standard action.

2.4 Critically fixed rational maps

In this section, we introduce critically fixed rational maps and discuss a way to combinatorially classify them using planar embedded graphs.

Definition 2.4.1. A rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is *critically fixed* if $C_f \subset \operatorname{Fix}(f)$, that is, if f fixes each of its critical points.

2.4.1 Tischler graphs

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed rational map and let $c \in C_f$ be a critical point of f. The *basin of attraction of* c is the set

$$B_c := \{z \in \widehat{\mathbb{C}} \mid \lim_{n\to\infty} f^n(z) = c\}.$$

The *immediate basin of* c, denoted by Ω_c, is the connected component of B_c that contains c. It is a standard fact in complex dynamics that Ω_c is a simply connected open set. Furthermore, there exists a conformal map $\tau_c : \mathbb{D} \to \Omega_c$ and a natural number $d_c \in \mathbb{N}_{>2}$ such that

$$(\tau_c \circ f \circ \tau_c^{-1})(z) = z^{d_c}$$

for all $z \in \mathbb{D} := \{z \in \mathbb{C} \mid |z| < 1\}$. This conformal map extends to a continuous and surjective map $\tau_c : \overline{\mathbb{D}} \to \overline{\Omega}_c$ between closures.

An *internal ray of angle* $\theta \in [0,1)$ in the immediate basin Ω_c is the image of the radial arc $r(\theta) := \{te^{2\pi i\theta} \mid t \in [0,1]\}$ under the map τ_c. Every internal ray of angle θ has a *landing point*, which is defined to be the point $\tau_c(e^{2\pi i\theta}) \in \partial\Omega_c$. Note that the internal ray of angle θ is fixed (set-wise) under f if and only if $\theta = \frac{j}{d_c-1}$ for some $j = 0,\dots,d_c-2$. We denote by R_f the set of all landing points of all fixed internal rays. Note that $R_f \subset \operatorname{Fix}(f)$.

Definition 2.4.2. The *Tischler graph* of a critically fixed rational map f is the planar embedded graph $\operatorname{Tisch}(f)$ whose edge set consists of the fixed internal rays in the immediate basins of all critical points of f and whose vertex set consists of the endpoints of these rays.

Let now $T = \operatorname{Tisch}(f)$ be the Tischler graph of a critically fixed rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\deg(f) \geq 2$. We denote by V_T, E_T and F_T the vertex, edge and face sets respectively. Then, the following statements hold [Hlu19]:

- $V_T = C_f \cup R_f = \operatorname{Fix}(f)$;
- $|V_T| = \deg(f) + 1$, $|E_T| = 2\deg(f) - 2$ and $|F_T| = \deg(f) - 1$;
- T is bipartite and connected;
- Each face of T is a (possibly degenerate) quadrilateral and has exactly two critical points on the boundary.

2.4.2 Charge graphs

Let again $T = \operatorname{Tisch}(f)$ be the Tischler graph of a critically fixed rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\deg(f) \geq 2$. Now, for every face Q of T, we choose a Jordan arc e_Q that joins the two critical points of f on the boundary of Q, so that $\operatorname{int}(e_Q) \subset Q$.

Definition 2.4.3. A *charge graph* of f is the graph with vertex set C_f and edge set $\{e_Q \,|\, Q \in F_T\}$; it is denoted by Charge(f).

We now briefly describe the *blow-up operation*, which is a way to define a (topological) branched covering map $f_G : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ from a (connected) planar embedded graph G in $\widehat{\mathbb{C}}$. (Here, we assume that G may have multiple edges, but no loops.) Informally, *blowing up* the graph G means that we cut the sphere $\widehat{\mathbb{C}}$ open along the interior of each edge e of G and glue in a closed Jordan region patch D_e inside each cut along the boundary. To define the map f_G, we first send the boundary ∂D_e of each patch in a 2-to-1 fashion onto the respective edge e, so that the endpoints of e stay fixed. Then we continuously extend f_G to the whole sphere in the following way. The map f_G homeomorphically maps each complementary component of the union of the patches onto the respective face of G. Simultaneously, the interior of each Jordan region patch D_e is mapped by f_G homeomorphically onto the complement of the corresponding edge e. For a formal construction, see [PL98] and [Hlu19, Section 5.1].

Using the blow-up operation, we can state the following results on charge graphs.

Theorem 2.4.4 ([Hlu19, Proposition 8 and Theorem 2])**.** *Every critically fixed rational map f with* $\deg(f) \geq 2$ *can be obtained from a charge graph of f by blowing up its edges.*

Furthermore, this induces a canonical bijection between the conformal conjugacy classes of critically fixed rational maps of degree at least 2 and the isomorphism classes of connected planar embedded graphs with at least one edge and no loops.

We will now explain more formally the first part of the theorem above. Let f be a critically fixed rational map and let $G \coloneqq$ Charge(f) be its charge graph. Recall that every edge e_Q of G corresponds to a face Q of $T =$ Tisch(f). We have that under the map f, every edge e_Q will have exactly two lifts e_Q^+ and e_Q^- within the closure of the face Q; both of these arcs are isotopic to e_Q relative to $V_T \supset C_f$. In particular, the arcs e_Q, e_Q^+ and e_Q^- share their endpoints. The edges e_Q^+ and e_Q^- form the boundary of the Jordan region patch D_{e_Q}, such that $e_Q \subset \overline{D_{e_Q}} \subset \overline{Q}$. It follows that f maps the interior $\mathrm{int}(D_{e_Q})$ homeomorphically onto $\widehat{\mathbb{C}} \setminus e_Q$ for every edge e_Q of G. Furthermore, every complementary component W of $\bigcup_{Q \in F_T} D_{e_Q}$ is contained in a unique face U_W of Charge(f), and f maps W onto U_W by an orientation preserving homeomorphism so that $f(\partial W) = \partial U_W$. We define $G^{\pm}$ to be the graph with the vertex set C_f and the edge set $\{e_Q^+, e_Q^- \,|\, Q \in F_T\}$, G^+ to be the graph with vertex set C_f and edge set $\{e_Q^+ \,|\, Q \in F_T\}$ and G^- to be the graph with vertex set C_f and edge set $\{e_Q^- \,|\, Q \in F_T\}$.

2.4.3 Polynomial case

In the special case when $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ is a critically fixed polynomial map, we can say even more about the properties of $T =$ Tisch(f) and $G =$ Charge(f). For example, in T, every face is connected to ∞. To see this, we first note that the degree of the vertex ∞ in T equals $\deg(f) - 1$. When we start at one edge adjacent to ∞ and move clockwise to the next edge adjacent to ∞, we see that there is a unique face through which we move. This holds for every edge adjacent to the vertex corresponding to ∞, meaning that in this way we already have all $\deg(f) - 1$ faces of T. We conclude that ∞ is on the boundary of every face of the Tischler graph T. Consequently, in Charge(f), we see that every edge is connected to the vertex corresponding to ∞.

Because every face is a quadrilateral and ∞ is on the boundary of every face, we have that when you remove the vertex corresponding to ∞ from the graph T along with the edges adjacent to ∞, the remaining graph has only one face. This means that the remaining graph is acyclic. Furthermore, we have that the number of vertices of this remaining graph is $\deg(f)$ and the number of edges is $\deg(f) - 1$. Using Lemma A.1, we conclude that this graph must be connected, hence is a tree.

2.5 Critically fixed anti-rational maps

In this section, we talk about critically fixed anti-rational maps and how to classify them using certain planar embedded graphs.

Definition 2.5.1. An *anti-rational map* is the complex conjugate of a rational map. An *anti-polynomial* is the complex conjugate of a polynomial.

Like in the case of rational maps, an anti-rational map f is called *critically fixed* if all critical points of f are fixed by f. Furthermore, again following the set-up in the case of rational maps, the *Tischler graph* of a critically fixed anti-rational map is the graph whose edges are the fixed internal rays of f and whose vertices are the endpoints of these rays.

Let now $T = \mathrm{Tisch}(f)$ be the Tischler graph of a critically fixed anti-rational map $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ with $\deg(f) \geq 2$. We denote by V_T and F_T the vertex set and face set respectively. Then, the following statements hold [Gey22]:

- $V_T = C_f \cup R_f = \mathrm{Fix}(f)$;
- $|F_T| = \deg(f) + 1$;
- Each vertex corresponding to a critical point of multiplicity m has degree $m + 2 \geq 3$ and each vertex corresponding to a repelling fixed point has degree 2;
- Every face A of T is a Jordan domain and f maps A anti-conformally onto $\widehat{\mathbb{C}} \setminus \bar{A}$.

Definition 2.5.2. A *topological Tischler graph* is a connected planar embedded graph $T \subset S^2$ where every vertex has degree ≥ 3, and every face is a Jordan domain.

The topological Tischler graph for a critically fixed anti-rational map is obtained from its Tischler graph by forgetting all vertices of degree 2, which are exactly the vertices corresponding to repelling fixed points.

We now define a way to obtain a map, the so-called associated Schottky map, from a topological Tischler graph.

Definition 2.5.3. For a Jordan domain $U \subset S^2$, an associated *topological reflection (in U)* is an orientation-reversing homeomorphism $f_U : S^2 \to S^2$ such that $f_U = \mathrm{id}$ on ∂U, and $f_U^2 = \mathrm{id}$ on S^2. For a topological Tischler graph $T \subset S^2$, an associated *Schottky map* f_T is a map such that the restriction of f_T to each face U of T is a topological reflection associated to U.

Definition 2.5.4. A topological Tischler graph G is called *obstructed* if there is a pair of distinct faces A and B of G sharing two distinct edges $a, b \subset \partial A \cap \partial B$. G is called *unobstructed* if this is not the case.

Using these definitions, we can state the following classification of anti-rational maps.

Theorem 2.5.5 ([Gey22, Theorem 5.6, Corollary 5.12 and Theorem 5.13]). *Every critically fixed anti-rational map f with* $\deg(f) \geq 2$ *is isotopic relative to* C_f *to the Schottky map associated to the Tischler graph of* f.

Furthermore, this induces a canonical bijection between conformal conjugacy classes of critically fixed anti-rational maps of degree at least 2 and isomorphism classes of unobstructed topological Tischler graphs.

2.5.1 Anti-polynomial case

In the more specific case of anti-polynomials, the topological Tischler graph has certain special properties, which we will discuss in this section. To do so, we introduce the following definition.

Definition 2.5.6. A subset $T \subset \widehat{\mathbb{C}} \setminus \{\infty\}$ is an *unbounded planar embedded tree* if T is closed in $\mathbb{R}^2$ and homeomorphic to a planar embedded tree without vertices of degree 2, and with all its leaves removed. A *topological Tischler tree* is an unbounded planar embedded tree with ≥ 3 unbounded edges.

Let now $\hat{T}$ be the topological Tischler graph of a critically fixed anti-polynomial. Then the following statements hold:

- Every face has ∞ on its boundary;
- The vertex ∞ has degree $d-1$, where d is the degree of the anti-polynomial;
- The topological Tischler graph is unobstructed;
- $T = \hat{T} \cap \mathbb{R}^2$ is a topological Tischler tree.

Chapter 3

Polynomial case

In this chapter, we prove Theorem 1.0.1; our main result on critically fixed polynomials. The proof of this theorem is based on induction. Before providing the proof, we introduce two lemmas that allow us to execute the induction basis and the induction step in this proof. However, we start this chapter by considering a specific example of a critically fixed polynomial to get a feeling of the tools and ideas that we are going to utilise in the proofs.

3.1 Example

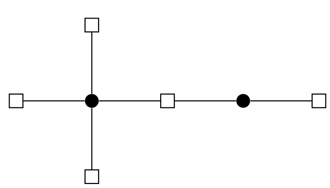
Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the polynomial of degree 7 given by

$$f(z) = \frac{-15z^7 + 35z^6 + 21z^5}{41}, \tag{3.1}$$

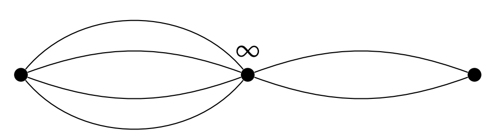
and denote by G the iterated monodromy group $\mathrm{IMG}(f)$. In this subsection, we prove that G is regular branch on itself.

The polynomial f is critically fixed, and the set of critical points of f is given by $C_f = \{0, 1, \infty\}$. The Tischler graph $\mathrm{Tisch}(f)$ of this polynomial is depicted in Figure 3.1a. Here, circles represent points in C_f and squares represent points in $\mathrm{Fix}(f) \setminus C_f$. To simplify the picture, we have left out infinity, which is connected to all vertices in $\mathrm{Fix}(f) \setminus C_f$. The corresponding charge graph $\mathrm{Charge}(f)$ of f is depicted in Figure 3.1b.

We choose a spanning subtree T of $\mathrm{Charge}(f)$ as shown in thick black edges in Figure 3.2a. Furthermore, we choose t to be the fixed point in the outer face of the charge graph. The fundamental group $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$ is then generated by the loops a and b, depicted in red and green respectively in Figure 3.2a. Note that each of them intersects exactly one edge of the chosen tree T. With a



(a)



(b)

Figure 3.1: The Tischler graph (a) and the charge graph (b) for the map f from (3.1).

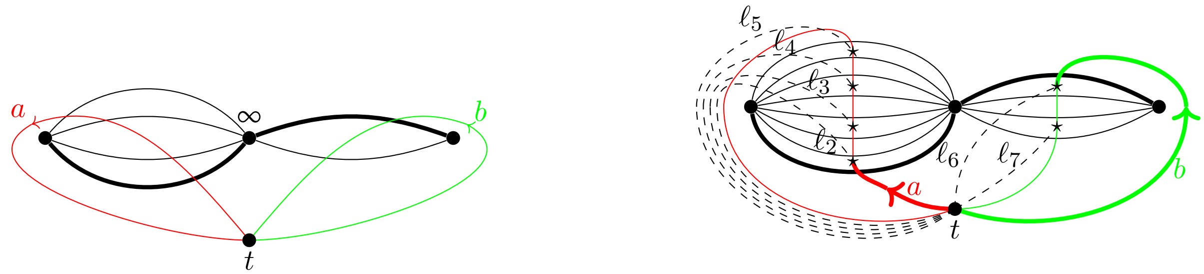


Figure 3.2: Computing the action of IMG(f): (a) The spanning subtree T of Charge(f) and the corresponding generators a and b; (b) The preimage of Charge(f) under f.

slight abuse of notation, we denote by a and b the corresponding elements in $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$ and also in G. Then a and b generate G as well.

In Figure 3.2b we see the preimage of 3.2a under f. For clarity, we have left out all preimages of a and b that have the form of a loop. We choose a spanning subtree T' of the preimage of Charge(f) under f, that is a subgraph of the preimage of T under f. This tree is shown in thick black edges in Figure 3.2b. The fixed point t has 7 preimages in total; the preimages not equal to t are depicted as $\star$. For each of these preimages, we choose a connecting path ℓ_i which connects t to the preimage and which does not cross any of the edges of the spanning subtree T'. Here, ℓ_1 is the constant path at the fixed point t. Furthermore, the edges corresponding to preimages of the generators which cross T' are marked. Using Figure 3.2b and Proposition 2.3.2, we can now write down the wreath recursions corresponding to the generators a and b. Namely,

$$
\begin{aligned}
a &= \langle\!\langle a, 1, 1, 1, 1, 1, 1 \rangle\!\rangle (1\,2\,3\,4\,5),\\
b &= \langle\!\langle b, 1, 1, 1, 1, 1, 1 \rangle\!\rangle (1\,6\,7).
\end{aligned}
$$

Note that the permutational parts $\sigma_a = (1\,2\,3\,4\,5)$ and $\sigma_b = (1\,6\,7)$ of a and b are even, and hence are elements of $\mathcal{A}_7$. Using these wreath recursions, we obtain the labeled Schreier graph of G on the first level of the 7-regular rooted tree. Here we use a coloured edge to describe the action of the corresponding generator; red and green for a and b respectively.

Using Figure 3.3, we try to come up with non-trivial elements in G that have only trivial restrictions. We use commutators of powers of a and b to produce such elements. The elements we

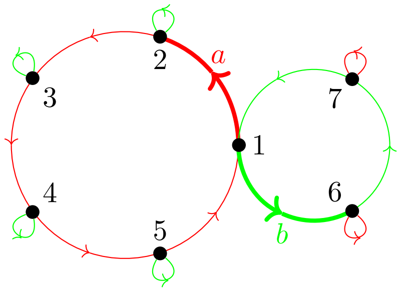


Figure 3.3: The labeled Schreier graph of G with respect to $\{a, b\}$.

construct are listed below.

$$
\begin{aligned}
[a^{-1}, b^{-1}] &= aba^{-1}b^{-1} &&= \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\,7\,5)\\
[a^{-2}, b^{-1}] &= a^2ba^{-2}b^{-1} &&= \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\,7\,4)\\
[a^{-3}, b^{-1}] &= a^3ba^{-3}b^{-1} &&= \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\,7\,3)\\
[a^{-4}, b^{-1}] &= a^4ba^{-4}b^{-1} &&= \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\,7\,2)\\
[a^{-1}, b^{-2}] &= ab^2a^{-1}b^{-2} &&= \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\,6\,5)\\
[a^{-2}, b^{-2}] &= a^2b^2a^{-2}b^{-2} &&= \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\,6\,4)\\
[a^{-3}, b^{-2}] &= a^3b^2a^{-3}b^{-2} &&= \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\,6\,3)\\
[a^{-4}, b^{-2}] &= a^4b^2a^{-4}b^{-2} &&= \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\,6\,2)
\end{aligned}
$$

We can also write this as

$$[a^{-i}, b^{-j}] = a^ib^ja^{-i}b^{-j} = \langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle(1\ 8-j\ 6-i)$$

for $1 \le i \le 4$ and $1 \le j \le 2$. Lemma A.3 then tells us that these constructed elements in fact generate $\langle\!\langle 1,1,1,1,1,1,1\rangle\!\rangle\mathcal{A}_7$. It follows that $\langle\!\langle a,1,1,1,1,1,1\rangle\!\rangle$ and $\langle\!\langle b,1,1,1,1,1,1\rangle\!\rangle$ are in G. More specifically, we have the following decompositions:

$$
\begin{aligned}
a\cdot[a^{-1},b^{-1}]\cdot[a^{-2},b^{-1}]^{-1}\cdot[a^{-1},b^{-1}]^{-1}\cdot[a^{-3},b^{-1}]\cdot[a^{-4},b^{-1}]^{-1}\cdot[a^{-3},b^{-1}]^{-1} &= \langle\!\langle a,1,1,1,1,1,1\rangle\!\rangle\\
b\cdot[a^{-1},b^{-1}]^{-1}\cdot[a^{-1},b^{-2}]\cdot[a^{-1},b^{-1}] &= \langle\!\langle b,1,1,1,1,1,1\rangle\!\rangle.
\end{aligned}
$$

To see this, note that

$$
\begin{aligned}
(1\,7\,5)(1\,7\,4)^{-1}(1\,7\,5)^{-1}(1\,7\,3)(1\,7\,2)^{-1}(1\,7\,3)^{-1} &= (1\,7\,5)(1\,4\,7)(1\,5\,7)(1\,7\,3)(1\,2\,7)(1\,3\,7)\\
&= (1\,5\,4)(1\,3\,2)\\
&= (1\,5\,4\,3\,2)\\
&= (1\,2\,3\,4\,5)^{-1}
\end{aligned}
$$

and

$$
\begin{aligned}
(1\,7\,5)^{-1}(1\,6\,5)(1\,7\,5) &= (1\,5\,7)(1\,6\,5)(1\,7\,5)\\
&= (1\,7\,6)\\
&= (1\,6\,7)^{-1}.
\end{aligned}
$$

We conclude that $G\times 1\times 1\times 1\times 1\times 1\times 1 \subset G$, and thus that $G\times G\times G\times G\times G\times G\times G \subset G$. That is, we proved the statement below.

Proposition 3.1.1. *Let f be the polynomial from* (3.1) *and G be the iterated monodromy group* IMG(f). *Then G is regular branch on itself.*

3.2 The base case lemma

In the following, we generalise the example from Section 3.1 to a more general setup which includes the IMGs of all critically fixed polynomials with exactly two finite critical points.

Lemma 3.2.1. *Let $G = \langle a, b\rangle$ be a subgroup of $\mathrm{Aut}(\mathsf{T}_{[k+l-1]})$ generated by two elements a and b, where $k, l \in \mathbb{N}_{\geq 2}$. Furthermore, suppose there are elements $\bar{a}, \bar{b} \in G$ such that their wreath recursions have the following form:*

$$\bar{a} = \langle\!\langle 1, \ldots, 1, \underbrace{a}_{x}, 1, \ldots, 1\rangle\!\rangle\alpha = \langle\!\langle 1, \ldots, 1, \underbrace{a}_{x}, 1, \ldots, 1\rangle\!\rangle(1\,2\,\ldots\,k),$$
$$\bar{b} = \langle\!\langle 1, \ldots, 1, \underbrace{b}_{y}, 1, \ldots, 1\rangle\!\rangle\beta = \langle\!\langle 1, \ldots, 1, \underbrace{b}_{y}, 1, \ldots, 1\rangle\!\rangle(1\ k+1\ \ldots\ k+l-1),$$

where $x \in [k]$ and $y \in \{1, k+1, \ldots, k+l-1\}$, and where α and β have the same parity as the permutational part of a and b respectively. Let $E < G$ be the subgroup of all elements of G with even permutational part. Then the following two statements hold:

(i) $\langle\!\langle 1, \ldots, 1\rangle\!\rangle\mathcal{A}_{k+l-1} \subset E$;

(ii) $\underbrace{E \times \cdots \times E}_{k+l-1 \text{ times}} \subset E$.

Proof. The labeled Schreier graph for the action of $\bar{a}$ and $\bar{b}$ on $[k+l-1]$ is depicted in Figure 3.4. We use this graph to create non-trivial elements that have only trivial restrictions. Like in Section 3.1, the constructed elements are special commutators. Namely, for $x \in [k]$ and $y = 1$, they have the following form:

$$[\bar{a}^{-i}, \bar{b}^{-j}] = \langle\!\langle 1, \ldots, 1\rangle\!\rangle(1\ k+l-j\ k-i+1) \text{ for } 1 \leq i \leq k-x \qquad \text{and } 1 \leq j \leq l-1$$
$$[\bar{a}^{i-k}, \bar{b}^{-j}] = \langle\!\langle 1, \ldots, 1\rangle\!\rangle(1\ k+l-j\ k-i+1) \text{ for } k-x+1 \leq i \leq k-1 \text{ and } 1 \leq j \leq l-1.$$

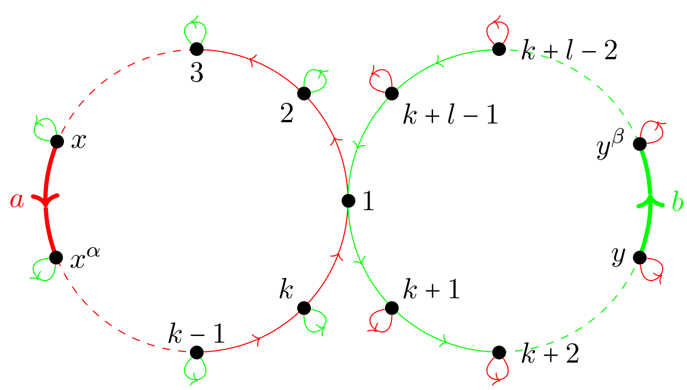


Figure 3.4: The labeled Schreier graph of $\bar{a}$ and $\bar{b}$ acting on $[k+l-1]$.

Furthermore, for $x \in [k]$ and $y \in \{k+1, \dots, k+l-1\}$, they have the following form:

$$\begin{aligned}
[\bar{a}^{-i}, \bar{b}^{-j}] &= \langle\!\langle 1, \dots, 1 \rangle\!\rangle (1\ k+l-j\ k-i+1) \text{ for } 1 \le i \le k-x && \text{and } 1 \le j \le k+l-1-y \\
[\bar{a}^{i-k}, \bar{b}^{-j}] &= \langle\!\langle 1, \dots, 1 \rangle\!\rangle (1\ k+l-j\ k-i+1) \text{ for } k-x+1 \le i \le k-1 && \text{and } 1 \le j \le k+l-1-y \\
[\bar{a}^{-i}, \bar{b}^{l-j}] &= \langle\!\langle 1, \dots, 1 \rangle\!\rangle (1\ k+l-j\ k-i+1) \text{ for } 1 \le i \le k-x && \text{and } k+l-y \le j \le l-1 \\
[\bar{a}^{i-k}, \bar{b}^{l-j}] &= \langle\!\langle 1, \dots, 1 \rangle\!\rangle (1\ k+l-j\ k-i+1) \text{ for } k-x+1 \le i \le k-1 && \text{and } k+l-y \le j \le l-1.
\end{aligned}$$

Using these elements and Lemma A.3, we see that we have $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}_{k+l-1} \subset \langle \bar{a}, \bar{b} \rangle \subseteq G$. The set $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}_{k+l-1}$ consists solely of elements with even permutational part, so by definition, we have $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}_{k+l-1} \subset E$, proving the first statement. To prove the second statement, we distinguish three different cases, depending on the parity of the permutational parts of the generators a and b.

Case 1: both a and b have even permutational part. In this case, both α and β are even permutations and the subgroup E is in fact the whole group G. Combining the fact that $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}_{k+l-1} \subset E$ with the fact that α and β are even permutations, we see that we can express the elements $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \alpha^{-1}$ and $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \beta^{-1}$ in terms of a and b. It follows that

$$\bar{a} \cdot \langle\!\langle 1, \dots, 1 \rangle\!\rangle \alpha^{-1} = \langle\!\langle 1, \dots, 1, \underbrace{a}_{x}, 1, \dots, 1 \rangle\!\rangle \alpha \cdot \langle\!\langle 1, \dots, 1 \rangle\!\rangle \alpha^{-1} = \langle\!\langle 1, \dots, 1, \underbrace{a}_{x}, 1, \dots, 1 \rangle\!\rangle$$

and

$$\bar{b} \cdot \langle\!\langle 1, \dots, 1 \rangle\!\rangle \beta^{-1} = \langle\!\langle 1, \dots, 1, \underbrace{b}_{y}, 1, \dots, 1 \rangle\!\rangle \beta \cdot \langle\!\langle 1, \dots, 1 \rangle\!\rangle \beta^{-1} = \langle\!\langle 1, \dots, 1, \underbrace{b}_{y}, 1, \dots, 1 \rangle\!\rangle$$

are both elements of $\langle a, b \rangle = E$. Combining this with $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}_{k+l-1} \subset E$, we conclude that $\underbrace{E \times \cdots \times E}_{k+l-1 \text{ times}} \subset E$, proving the second statement.

Case 2: both a and b have odd permutational part. In this case, both α and β are odd permutations. Furthermore, the subgroup E of all elements of G with even permutational part is generated by the set $\{a^2, b^2, ab\}$; see Lemma A.4. We prove that $\langle\!\langle 1, \dots, 1, \underbrace{a^2}_{x}, 1, \dots, 1 \rangle\!\rangle$, $\langle\!\langle 1, \dots, 1, \underbrace{b^2}_{y}, 1, \dots, 1 \rangle\!\rangle$ and $\langle\!\langle 1, \dots, 1, \underbrace{ab}_{x}, 1, \dots, 1 \rangle\!\rangle$ are all elements of the subgroup E.

We have that $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}_{k+l-1} \subset E$, which in particular means that $\langle\!\langle 1, 1, \dots, 1 \rangle\!\rangle (k+1\, x^\alpha\, x)$ is an element of E. We now look at the following products:

$$\begin{aligned}
\langle\!\langle 1, \dots, 1 \rangle\!\rangle (k+1\, x^\alpha\, x) \cdot \bar{a} &= \langle\!\langle 1, \dots, 1 \rangle\!\rangle (k+1\, x^\alpha\, x) \cdot \langle\!\langle 1, \dots, 1, \underbrace{a}_{x}, 1, \dots, 1 \rangle\!\rangle \alpha \\
&= \langle\!\langle 1, \dots, 1, \underbrace{a}_{x^\alpha}, 1, \dots, 1 \rangle\!\rangle (k+1\, x^\alpha\, x) \alpha,
\end{aligned}$$

$$\begin{aligned}
\bar{a} \cdot \langle\!\langle 1, \dots, 1, \underbrace{a}_{x^\alpha}, 1, \dots, 1 \rangle\!\rangle (k+1\, x^\alpha\, x) \alpha &= \langle\!\langle 1, \dots, 1, \underbrace{a}_{x}, 1, \dots, 1 \rangle\!\rangle \alpha \cdot \langle\!\langle 1, \dots, 1, \underbrace{a}_{x^\alpha}, 1, \dots, 1 \rangle\!\rangle (k+1\, x^\alpha\, x) \alpha \\
&= \langle\!\langle 1, \dots, 1, \underbrace{a^2}_{x}, 1, \dots, 1 \rangle\!\rangle \alpha (k+1\, x^\alpha\, x) \alpha.
\end{aligned}$$

Note that $\alpha(k+1\, x^\alpha\, x)\alpha$ is an even permutation, hence its inverse is also even. Again using that $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}_{k+l-1} \subset E$, it follows that $\langle\!\langle 1, \dots, 1, \underbrace{a^2}_{x}, 1, \dots, 1 \rangle\!\rangle \in E$.

We can use the same kind of argument to prove that $\langle\!\langle 1,\dots,1,\underbrace{b^2}_{y},1,\dots,1\rangle\!\rangle \in E$.

Lastly, we show that $\langle\!\langle 1,\dots,1,\underbrace{ab}_{x},1,\dots,1\rangle\!\rangle \in E$. For this, let $z \in [k+l-1] \setminus \{x^\alpha, y\}$ and note that $\langle\!\langle 1,\dots,1\rangle\!\rangle(z\,x^\alpha\,y) \in E$. We then look at the following products:

$$\begin{aligned}\langle\!\langle 1,\dots,1\rangle\!\rangle(z\,x^\alpha\,y)\cdot\bar{b} &= \langle\!\langle 1,\dots,1\rangle\!\rangle(z\,x^\alpha\,y)\cdot\langle\!\langle 1,\dots,1,\underbrace{b}_{y},1,\dots,1\rangle\!\rangle\beta \\ &= \langle\!\langle 1,\dots,1,\underbrace{b}_{x^\alpha},1,\dots,1\rangle\!\rangle(z\,x^\alpha\,y)\beta,\end{aligned}$$

$$\begin{aligned}\bar{a}\cdot\langle\!\langle 1,\dots,1,\underbrace{b}_{x^\alpha},1,\dots,1\rangle\!\rangle(z\,x^\alpha\,y)\beta &= \langle\!\langle 1,\dots,1,\underbrace{a}_{x},1,\dots,1\rangle\!\rangle\alpha\cdot\langle\!\langle 1,\dots,1,\underbrace{b}_{x^\alpha},1,\dots,1\rangle\!\rangle(z\,x^\alpha\,y)\beta \\ &= \langle\!\langle 1,\dots,1,\underbrace{ab}_{x},1,\dots,1\rangle\!\rangle\alpha(z\,x^\alpha\,y)\beta.\end{aligned}$$

The permutation $\alpha(z\,x^\alpha\,y)\beta$ again is an even permutation, from which we can conclude that $\langle\!\langle 1,\dots,1,\underbrace{ab}_{x},1,\dots,1\rangle\!\rangle \in E$. Combining the fact that $\langle\!\langle 1,\dots,1,\underbrace{a^2}_{x},1,\dots,1\rangle\!\rangle$, $\langle\!\langle 1,\dots,1,\underbrace{b^2}_{y},1,\dots,1\rangle\!\rangle$ and $\langle\!\langle 1,\dots,1,\underbrace{ab}_{x},1,\dots,1\rangle\!\rangle$ are all elements of E with $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}_{k+l-1} \subset E$, we conclude that in this case it also holds that $\underbrace{E\times\cdots\times E}_{k+l-1\text{ times}} \subset E$.

Case 3: one of the generators a, b has even permutational part and the other generator has odd permutational part. Without loss of generality, we assume that a has even permutational part and b has odd permutational part, so α is an even permutation and β is an odd permutation. In this case the subgroup E of all elements of G with even permutational part is generated by the set $\{a, b^2, bab^{-1}\}$; see Lemma A.4. We prove that $\langle\!\langle 1,\dots,1,\underbrace{a}_{x},1,\dots,1\rangle\!\rangle$, $\langle\!\langle 1,\dots,1,\underbrace{b^2}_{y},1,\dots,1\rangle\!\rangle$, $\langle\!\langle 1,\dots,1,\underbrace{bab^{-1}}_{y},1,\dots,1\rangle\!\rangle \in E$.

As in the case of two even permutations, from the fact that $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}_{k+l-1} \subset E$ it follows that $\langle\!\langle 1,\dots,1,\underbrace{a}_{x},1,\dots,1\rangle\!\rangle \in E$. Using the same reasoning as in the case of two odd permutations, we can state that $\langle\!\langle 1,\dots,1,\underbrace{b^2}_{y},1,\dots,1\rangle\!\rangle \in E$. To show that $\langle\!\langle 1,\dots,1,\underbrace{bab^{-1}}_{y},1,\dots,1\rangle\!\rangle \in E$, we let

$z \in [k+l-1] \setminus \{y^\beta, x\}$ and note that $\langle\!\langle 1,\dots,1\rangle\!\rangle(z\,y^\beta\,x) \in E$. We then look at the following equations:

$$\begin{aligned}
\langle\!\langle 1,\dots,1\rangle\!\rangle(z\,y^\beta\,x)\cdot\bar{a} &= \langle\!\langle 1,\dots,1\rangle\!\rangle(z\,y^\beta\,x)\cdot\langle\!\langle 1,\dots,1,\underbrace{a}_{x},1,\dots,1\rangle\!\rangle \\
&= \langle\!\langle 1,\dots,1,\underbrace{a}_{y^\beta},1,\dots,1\rangle\!\rangle(z\,y^\beta\,x) \\
\bar{b}\cdot\langle\!\langle 1,\dots,1,\underbrace{a}_{y^\beta},1,\dots,1\rangle\!\rangle(z\,y^\beta\,x) &= \langle\!\langle 1,\dots,1,\underbrace{b}_{y},1,\dots,1\rangle\!\rangle\beta\cdot\langle\!\langle 1,\dots,1,\underbrace{a}_{y^\beta},1,\dots,1\rangle\!\rangle(z\,y^\beta\,x) \\
&= \langle\!\langle 1,\dots,1,\underbrace{ba}_{y},1,\dots,1\rangle\!\rangle\beta(z\,y^\beta\,x) \\
\bar{b}^{-1} &= \langle\!\langle 1,\dots,1,\underbrace{b^{-1}}_{y^\beta},1,\dots,1\rangle\!\rangle\beta^{-1} \\
\langle\!\langle 1,\dots,1,\underbrace{ba}_{y},1,\dots,1\rangle\!\rangle\beta(z\,y^\beta\,x)\cdot\bar{b}^{-1} &= \langle\!\langle 1,\dots,1,\underbrace{ba}_{y},1,\dots,1\rangle\!\rangle\beta(z\,y^\beta\,x)\cdot\langle\!\langle 1,\dots,1,\underbrace{b^{-1}}_{y^\beta},1,\dots,1\rangle\!\rangle\beta^{-1} \\
&= \langle\!\langle 1,\dots,1,\underbrace{bab^{-1}}_{y},1,\dots,1\rangle\!\rangle\beta(z\,y^\beta\,x)\beta^{-1}.
\end{aligned}$$

Again, by noting that $\beta(z\,y^\beta\,x)\beta^{-1}$ is an even permutation, we can conclude that it holds that $\langle\!\langle 1,\dots,1,\underbrace{bab^{-1}}_{y},1,\dots,1\rangle\!\rangle \in E$. One last time combining $\langle\!\langle 1,\dots,1,\underbrace{a}_{x},1,\dots,1\rangle\!\rangle$, $\langle\!\langle 1,\dots,1,\underbrace{b^2}_{y},1,\dots,1\rangle\!\rangle$, $\langle\!\langle 1,\dots,1,\underbrace{bab^{-1}}_{y},1,\dots,1\rangle\!\rangle \in E$ with the fact that $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}_{k+l-1} \subset E$, we conclude that in the third case it also holds that $\underbrace{E\times\cdots\times E}_{k+l-1 \text{ times}} \subset E$. In conclusion, we have proven that in all three cases both statements hold, proving the lemma. $\square$

Lemma 3.2.1 leads to a more general corollary, but to phrase this corollary, we first need to introduce a useful notation.

Definition 3.2.2. Let $G < \mathrm{Aut}(\mathsf{T}_\mathsf{X})$, $H < G$ and $S \subseteq \mathsf{X}$. Then we define

$$H[S] := \{g \in \mathrm{Aut}(\mathsf{T}_\mathsf{X}) \mid g|_x \in H \text{ for } x \in S,\ g|_x = 1 \text{ for } x \notin S\}.$$

Recall that $\mathcal{A}(\mathsf{X})$ denotes the alternating group on the set X. We note that for $S \subseteq \mathsf{X}$, we may naturally view the group $\mathcal{A}(S)$ as a subgroup of $\mathcal{A}(\mathsf{X})$.

Corollary 3.2.3. *Let* X *be a finite set, let* $a_1, a_2 \in \mathrm{Aut}(\mathsf{T}_\mathsf{X})$ *and let* $G = \langle a_1, a_2\rangle$. *Furthermore, let* $\bar{a}_1, \bar{a}_2 \in G$ *and suppose that for* $i \in \{1,2\}$ *the following conditions hold:*

- *the permutational part of* $\bar{a}_i$ *consists of a single cycle* σ_i *on a set* $\mathsf{X}_i \subset \mathsf{X}$ *with* $|\mathsf{X}_i| \geq 2$*;*
- *the parity of* σ_i *coincides with the parity of the permutational part of* a_i*;*
- $\bar{a}_i$ *restricts to* a_i *at* $x_i \in \mathsf{X}_i$*; all other restrictions are trivial.*

Furthermore, suppose that $|\mathsf{X}_1 \cap \mathsf{X}_2| = 1$. *Let* $E < G$ *be the subgroup of all elements of* G *with even permutational part. Then the following holds:*

(i) $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{X}_1 \cup \mathsf{X}_2) \subset E$*;*

(ii) $E[\mathsf{X}_1 \cup \mathsf{X}_2] \subset E$.

3.3 The induction step lemma

In this section, we prove the lemma below, which we will later utilize in the induction step of the proof of our first main result (Theorem 3.4.1).

Lemma 3.3.1. *Let $m \in \mathbb{N}_{\geq 3}$ and $G = \langle a_1, a_2, \dots, a_m \rangle$ be a subgroup of $\mathrm{Aut}(\mathsf{T}_{[M]})$ for some $M \geq 4$. Suppose that for every $1 \leq i \leq m$ there exists an element $\bar{a}_i \in G$ such that:*

- *the permutational part of $\bar{a}_i$ consists of a single cycle σ_i on a set $\mathsf{X}_i \subset [M]$ of size $k_i \geq 2$;*
- *the parity of σ_i coincides with the parity of the permutational part of a_i;*
- *$\bar{a}_i$ restricts to a_i at some $x_i \in \mathsf{X}_i$, while all the other restrictions are trivial.*

For every $1 \leq i \leq m$, let

$$\mathsf{Y}_i \coloneqq \bigcup_{j=1}^{i} \mathsf{X}_j$$

and let $E_i < \langle a_1, \dots, a_i \rangle < G$ be the subgroup of all elements of $\langle a_1, \dots, a_i \rangle$ with even permutational part. Let $3 \leq n \leq m$ and suppose that E_{n-1} satisfies the following two conditions:

(i) $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}(\mathsf{Y}_{n-1}) \subset E_{n-1}$;

(ii) $E_{n-1}[\mathsf{Y}_{n-1}] \subset E_{n-1}$.

Next, assume that $|\mathsf{Y}_{n-1} \cap \mathsf{X}_n| \geq 1$ and $|\mathsf{Y}_{n-1} \smallsetminus \mathsf{X}_n| \geq 2$. Then E_n also satisfies conditions (i) *and* (ii).

Proof. Let $x \in \mathsf{Y}_{n-1} \cap X_n$ and let $y_1, y_2 \in \mathsf{Y}_{n-1} \smallsetminus \mathsf{X}_n$. According to the statement, we have

$$\bar{a}_n = \langle\!\langle 1, \dots, 1, \underbrace{a_n}_{x_n}, 1, \dots, 1 \rangle\!\rangle \sigma_n,$$

where $x_n \in \mathsf{X}_n$. Since E_{n-1} satisfies conditions (i) and (ii), for every $g \in E_{n-1}$ there is an element $\bar{g} \in E_{n-1}$ of the form

$$\bar{g} = \langle\!\langle 1, \dots, 1, \underbrace{g}_{x}, 1, \dots, 1 \rangle\!\rangle (x \, y_1 \, y_2).$$

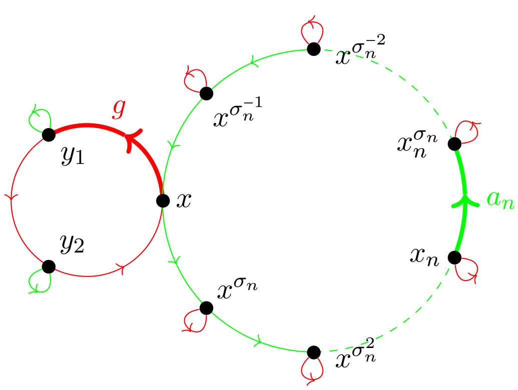


Figure 3.5: The labeled Schreier graph depicting $\bar{a}_n$ and $\bar{g}$.

Figure 3.5 illustrates the labeled Schreier graph for the action of $\bar{a}_n$ and $\bar{g}$. Here, we only depicted the "non-trivial part", that is, we do not show the vertices where both $\bar{a}_n$ and $\bar{g}$ act trivially.

We first prove that E_n satisfies the first condition, meaning that $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{Y}_n) \subset E_n$. Applying Corollary 3.2.3 to $\bar{a}_n$ and $\bar{g}$, we see that $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{X}_n \cup \{y_1,y_2\}) < E_n$. In particular, this means that $\langle\!\langle 1,\dots,1\rangle\!\rangle(x\ y_1\ m) \in E_n$ for all $m \in \mathsf{X}_n \setminus \{x\}$. We assumed that $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{Y}_{n-1}) \subset E_{n-1} \subset E_n$, from which it follows that $\langle\!\langle 1,\dots,1\rangle\!\rangle(x\ y_1\ m) \in E_n$ for all $m \in \mathsf{Y}_{n-1} \setminus \{x, y_1\}$. Combining these two statements, we see that we can use Lemma A.2 to conclude that $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{Y}_n) \subset E_n$, proving that E_n satisfies condition (i).

We now go on to proving the second condition, namely that $E_n[\mathsf{Y}_n] \subset E_n$. We do this by showing that $\langle\!\langle 1,\dots,1,\underbrace{s}_{x},1,\dots,1\rangle\!\rangle \in E_n$ for all generators s of E_n. To show this, we distinguish two different cases.

Case 1: k_n is odd, so σ_n is an even permutation. In this case, both $\bar{a}_n$ and a_n are elements of E_n. By Lemma A.4 and since $E_{n-1}[\mathsf{Y}_{n-1}] \subset E_{n-1} \subset E_n$, it is sufficient to show that the following statements hold:

- $\langle\!\langle 1,\dots,1,\underbrace{a_n}_{x},1,\dots,1\rangle\!\rangle \in E_n$;
- $\langle\!\langle 1,\dots,1,\underbrace{a_i a_n a_i^{-1}}_{x},1,\dots,1\rangle\!\rangle \in E_n$ for all $a_i \in \{a_1,\dots,a_{n-1}\}$ with odd permutational part.

Using that $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{Y}_n) \subset E_n$, we can state that $\bar{a}_n \cdot \langle\!\langle 1,\dots,1\rangle\!\rangle\sigma_n^{-1} = \langle\!\langle 1,\dots,1,\underbrace{a_n}_{x_n},1,\dots,1\rangle\!\rangle \in E_n$, and even that $\langle\!\langle 1,\dots,1,\underbrace{a_n}_{x},1,\dots,1\rangle\!\rangle \in E_n$.

Now fix $a_i \in \{a_1,\dots,a_{n-1}\}$ with odd permutational part, so σ_i is an odd permutation. Conjugating $\bar{a}_i$, $\bar{a}_n$ and $\bar{a}_i^{-1}$ by certain elements of $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{Y}_n)$, we can construct elements of the form

$$\langle\!\langle 1,\dots,1,\underbrace{a_i}_{x},1,\dots,1\rangle\!\rangle o_1, \langle\!\langle 1,\dots,1,\underbrace{a_n}_{x^{o_1}},1,\dots,1\rangle\!\rangle e, \langle\!\langle 1,\dots,1,\underbrace{a_i^{-1}}_{x^{o_1 e}},1,\dots,1\rangle\!\rangle o_2 \in \langle a_1,a_2,\dots,a_n\rangle,$$

where o_1, o_2 are odd permutations on Y_n and e is an even permutation on Y_n. Multiplying these three elements, we get

$$\begin{aligned}\langle\!\langle 1,\dots,1,\underbrace{a_i}_{x},1,\dots,1\rangle\!\rangle o_1 \cdot \langle\!\langle 1,\dots,1,\underbrace{a_n}_{x^{o_1}},1,\dots,1\rangle\!\rangle e \cdot \langle\!\langle 1,\dots,1,\underbrace{a_i^{-1}}_{x^{o_1 e}},1,\dots,1\rangle\!\rangle o_2 \\ = \langle\!\langle 1,\dots,1,\underbrace{a_i a_n a_i^{-1}}_{x},1,\dots,1\rangle\!\rangle o_1 e o_2 \in \langle a_1,\dots,a_n\rangle.\end{aligned}$$

Since $o_1 e o_2$ is an even permutation, we can now conclude that $\langle\!\langle 1,\dots,1,\underbrace{a_i a_n a_i^{-1}}_{x},1,\dots,1\rangle\!\rangle \in E_n$.

Case 2: k_n is even, so σ_n is an odd permutation. By Lemma A.4, it is sufficient to show that the following statements hold:

- $\langle\!\langle 1,\dots,1,\underbrace{a_n^2}_{x},1,\dots,1\rangle\!\rangle \in E_n$;

- $\langle\!\langle 1,\dots,1,\underbrace{a_i a_n}_{x},1,\dots,1\rangle\!\rangle \in E_n$ and $\langle\!\langle 1,\dots,1,\underbrace{a_n a_i}_{x},1,\dots,1\rangle\!\rangle \in E_n$ for all $a_i \in \{a_1,\dots,a_{n-1}\}$ with odd permutational part;
- $\langle\!\langle 1,\dots,1,\underbrace{a_n a_i a_n^{-1}}_{x},1,\dots,1\rangle\!\rangle \in E_n$ for all $a_i \in \{a_1,\dots,a_{n-1}\}$ with even permutational part.

Conjugating the element a_n by elements of $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{Y}_n)$, we may construct elements of the form $\langle\!\langle 1,\dots,1,\underbrace{a_n}_{x},1,\dots,1\rangle\!\rangle o_1 \in \langle a_1,\dots,a_n\rangle$ and $\langle\!\langle 1,\dots,1,\underbrace{a_n}_{x^{o_1}},1,\dots,1\rangle\!\rangle o_2 \in \langle a_1,\dots,a_n\rangle$, where o_1 and o_2 are odd permutations on Y_n. Multiplying these two elements, we get

$$\langle\!\langle 1,\dots,1,\underbrace{a_n}_{x},1,\dots,1\rangle\!\rangle o_1 \cdot \langle\!\langle 1,\dots,1,\underbrace{a_n}_{x^{o_1}},1,\dots,1\rangle\!\rangle o_2 = \langle\!\langle 1,\dots,1,\underbrace{a_n^2}_{x},1,\dots,1\rangle\!\rangle o_1 o_2 \in \langle a_1,\dots,a_n\rangle.$$

The permutation $o_1 o_2$ is an even permutation, because o_1 and o_2 are both odd permutations. From this, we can conclude that $\langle\!\langle 1,\dots,1,\underbrace{a_n^2}_{x},1,\dots,1\rangle\!\rangle \in E_n$.

Next, fix $a_i \in \{a_1,\dots,a_{n-1}\}$ with odd permutational part, which means that σ_i is an odd permutation. Conjugating $\bar{a}_i$ and $\bar{a}_n$ by certain elements of $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{Y}_n)$, we can construct elements $\langle\!\langle 1,\dots,1,\underbrace{a_i}_{x},1,\dots,1\rangle\!\rangle o_1 \in \langle a_1,\dots,a_n\rangle$ and $\langle\!\langle 1,\dots,1,\underbrace{a_n}_{x^{o_1}},1,\dots,1\rangle\!\rangle o_2 \in \langle a_1,\dots,a_n\rangle$, where o_1 and o_2 are odd permutations on Y_n. Multiplying these elements, we get

$$\langle\!\langle 1,\dots,1,\underbrace{a_i}_{x},1,\dots,1\rangle\!\rangle o_1 \cdot \langle\!\langle 1,\dots,1,\underbrace{a_n}_{x^{o_1}},1,\dots,1\rangle\!\rangle o_2 = \langle\!\langle 1,\dots,1,\underbrace{a_i a_n}_{x},1,\dots,1\rangle\!\rangle o_1 o_2 \in \langle a_1,\dots,a_n\rangle.$$

As $o_1 o_2$ is an even permutation, this also means that $\langle\!\langle 1,\dots,1,\underbrace{a_i a_n}_{x},1,\dots,1\rangle\!\rangle \in E_n$. Using the same kind of argument we can also conclude that $\langle\!\langle 1,\dots,1,\underbrace{a_n a_i}_{x},1,\dots,1\rangle\!\rangle \in E_n$.

Now fix $a_i \in \{a_1,\dots,a_{n-1}\}$ with even permutational part, so σ_i is an even permutation. Conjugating $\bar{a}_n$, $\bar{a}_i$ and $\bar{a}_n^{-1}$ by elements of $\langle\!\langle 1,\dots,1\rangle\!\rangle\mathcal{A}(\mathsf{Y}_n)$, we can construct elements of the form

$$\langle\!\langle 1,\dots,1,\underbrace{a_n}_{x},1,\dots,1\rangle\!\rangle o_1, \langle\!\langle 1,\dots,1,\underbrace{a_i}_{x^{o_1}},1,\dots,1\rangle\!\rangle e, \langle\!\langle 1,\dots,1,\underbrace{a_n^{-1}}_{x^{o_1 e}},1,\dots,1\rangle\!\rangle o_2 \in \langle a_1,\dots,a_n\rangle,$$

where o_1 and o_2 are odd permutations on Y_n and e is an even permutation on Y_n. Multiplying these three elements, we get

$$\begin{aligned}\langle\!\langle 1,\dots,1,\underbrace{a_n}_{x},1,\dots,1\rangle\!\rangle o_1 \cdot \langle\!\langle 1,\dots,1,\underbrace{a_i}_{x^{o_1}},1,\dots,1\rangle\!\rangle e \cdot \langle\!\langle 1,\dots,1,\underbrace{a_n^{-1}}_{x^{o_1 e}},1,\dots,1\rangle\!\rangle o_2 \\ = \langle\!\langle 1,\dots,1,\underbrace{a_n a_i a_n^{-1}}_{x},1,\dots,1\rangle\!\rangle o_1 e o_2 \in \langle a_1,\dots,a_n\rangle.\end{aligned}$$

The permutational part of this element is an even permutation, which means that we can conclude that $\langle\!\langle a_i a_n a_i^{-1},1,\dots,1\rangle\!\rangle$ is an element of E_n.

We have now proven that $\langle\!\langle 1,\dots,1,\underbrace{s}_{x},1,\dots,1\rangle\!\rangle \in E_n$ for all generators s of E_n, which implies that $E_n[\mathsf{Y}_n] \subset E_n$ and finishes the proof of the lemma. $\square$

3.4 Proof of branchness

Theorem 3.4.1. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed polynomial with $|C_f| \geq 3$. Let $G \coloneqq \mathrm{IMG}(f)$ and let $E < G$ be the subgroup of all elements of G with even permutational part. Then G is regular branch on E.*

Proof. Let d be the degree of f and let $C_f = \{c_1, c_2, \dots, c_n, \infty\}$ be the set of critical points of f. We start this proof by looking at the charge graph Charge(f). As shown in Section 2.4.3, in this graph, every edge is connected to the critical point ∞. Let now t be the fixed point in one of the faces of the charge graph and choose a spanning subtree T of Charge(f). Then, for every $i \in [n]$, we define a loop a_i based at t. We do this by connecting t to a small loop around c_i by a path that does not cross T. Note that for every $i \in [n]$, the loop a_i crosses exactly one edge of T exactly once, namely, the edge of T that connects ∞ and c_i. The fundamental group $\pi_1(\widehat{\mathbb{C}} \smallsetminus C_f, t)$ is then generated by these loops a_i. With a slight abuse of notation, we denote by a_i not only the element in $\pi_1(\widehat{\mathbb{C}} \smallsetminus C_f, t)$, but also in $G = \mathrm{IMG}(f)$. Then the set $\{a_1, \dots, a_n\}$ also generates the iterated monodromy group G.

Next, we look at the graph $G^{\pm}$ as defined in Section 2.4.2 and the lifts of the loops a_i under f. Denote by T^+ the graph consisting of the edges e^+ of G^+ such that $f(e^+)$ is an edge of T. Note that T^+ is a spanning subtree of $G^{\pm}$ which is isotopic to T relative to C_f. Furthermore, since f sends e_i^+ homeomorphically onto e_i, we have that $|f^{-1}(a_i) \cap e_i^+| = |a_i \cap e_i| = 1$. This means that exactly one lift of a_i crosses an edge of the tree T^+, and that this edge is exactly the edge e_i^+. Then, for every preimage t_j of t (t_1 being t itself), we choose a path ℓ_j connecting t to t_j which does not cross any edge of T^+. The path ℓ_1 is the constant path at t.

Using Proposition 2.3.2, we can now say things about the wreath recursions of the generators a_i. We can state the following properties of a_i:

- The permutational part of a_i consists of a single cycle σ_i on a set $\mathsf{X}_i \subset [d]$ of size $k_i \geq 2$. Here, the number k_i is given by the local degree of the critical point c_i. Furthermore, the set X_i corresponds exactly to the preimages of t in the blow-up of the edges which are crossed by the loop a_i exactly once.

- The generator a_i restricts to a_i at some $x_i \in \mathsf{X}_i$, while all other restrictions are trivial. The reason for this is that exactly one lift of a_i crosses an edge of T^+ and that none of the paths ℓ_j crosses an edge of T^+.

We use [Nek05, Proposition 6.8.2] to conclude that the reduced labeled Schreier graph of G with respect to $\{a_1, \dots, a_n\}$ is a tree of n cycles. This means that for every distinct $i, j \in [n]$ it holds that $|\mathsf{X}_i \cap \mathsf{X}_j| \in \{0, 1\}$ and for every $i \in [n]$ there exists at least one $j \in [n]$ such that $|\mathsf{X}_i \cap \mathsf{X}_j| = 1$.

We now want to use induction on the number of cycles of the reduced labeled Schreier graph to prove that G is regular branch on E. To do so, we need to have an order through which we are going to walk through the cycles. We make this order in the following way. Fix $a_i \in \{a_1, \dots, a_n\}$ such that the respective cycle corresponds to a leaf in the tree of cycles, i.e., for X_i it holds that there exists exactly one $j \in [n]$ such that $|\mathsf{X}_i \cap \mathsf{X}_j| = 1$. We define $g_n \coloneqq a_i$. We now remove the cycle corresponding to a_i from the tree of cycles. In the remaining tree, we again choose a cycle corresponding to a leaf; we define this to be g_{n-1}. We repeat this process until only one cycle remains. This cycle will be g_1.

Note that for each $i \in [n]$ the following conditions hold:

- the permutational part of g_i consists of a single cycle on a set $\mathsf{Z}_i \subset \mathsf{X}$ with $|\mathsf{Z}_i| \geq 2$;
- g_i restricts to g_i at $z_i \in \mathsf{Z}_i$; all other restrictions are trivial.

Define

$$\mathsf{Y}_i \coloneqq \bigcup_{j=1}^{i} \mathsf{Z}_j$$

and let $E_i < \langle g_1, \ldots, g_i \rangle < G$ be the subgroup of all elements of $\langle g_1, \ldots, g_i \rangle$ with even permutational part.

Note that we have that $|\mathsf{Z}_1 \cap \mathsf{Z}_2| = 1$. This means we can now use Corollary 3.2.3 to state the following:

(i) $\langle\!\langle 1, \ldots, 1 \rangle\!\rangle \mathcal{A}(\mathsf{Y}_2) \subset E_2$;

(ii) $E_2[\mathsf{Y}_2] \subset E_2$.

This will be our induction basis.

Now assume that condition (i) and condition (ii) hold for E_{i-1} with $i \in \{3, 4, \ldots, n\}$. Because of how we have chosen the order of cycles, we have that $|\mathsf{Y}_{i-1} \cap \mathsf{Z}_i| = 1$, from which it follows that $|\mathsf{Y}_{i-1} \setminus \mathsf{Z}_i| \geq 2$. This means that we can use Lemma 3.3.1 to state that

(i) $\langle\!\langle 1, \ldots, 1 \rangle\!\rangle \mathcal{A}(\mathsf{Y}_i) \subset E_i$;

(ii) $E_i[\mathsf{Y}_i] \subset E_i$.

Using induction, we can then also deduce that

(i) $\langle\!\langle 1, \ldots, 1 \rangle\!\rangle \mathcal{A}(\mathsf{Y}_n) \subset E_n$;

(ii) $E_n[\mathsf{Y}_n] \subset E_n$.

Note that $E_n = E$ and $\mathsf{Y}_n = [d]$. This then implies that G is regular branch on E, which concludes the argument. □

Chapter 4

Rational case

In this chapter, we look at two examples corresponding to two critically fixed rational maps that are not polynomials. In the first example, we see that we can apply the tools we have used to prove our statement for polynomials to prove a similar result in the case of this specific rational map. However, in the second example, we see that it is not possible to apply these tools and we prove a slightly different result.

4.1 Square example

Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be the rational map given by

$$f(z) = \frac{3z^5 - 20z}{5z^4 - 12}, \tag{4.1}$$

and denote by G the iterated monodromy group $\mathrm{IMG}(f)$. In this section, we will prove that G is regular branch on itself.

The rational map f is a critically fixed map, and the critical points of f are given by the set $C_f = \{-1 - i, -1 + i, 1 - i, 1 + i\}$. Furthermore, the fixed points of f are given by the set $\mathrm{Fix}(f) = \{-1 - i, -1 + i, 1 - i, 1 + i, 0, \infty\}$. The Tischler graph $\mathrm{Tisch}(f)$ is depicted in Figure 4.1a. Here, circles represent critical points and squares represent points in $\mathrm{Fix}(f) \setminus C_f$. The corresponding charge graph $\mathrm{Charge}(f)$ is depicted in Figure 4.1b.

We choose a spanning subtree T of the critical set C_f in $\mathrm{Tisch}(f)$ as shown in thick black edges in Figure 4.2a. Furthermore, we choose a fixed point t, as shown in Figure 4.2a. The fundamental

(a) (b)

Figure 4.1: The Tischler graph (a) and the charge graph (b) for the map f from (4.1).

(a) (b)

Figure 4.2: Computing the action of $\mathrm{IMG}(f)$: (a) The spanning subtree T of $\mathrm{Tisch}(f)$ and the corresponding generators a, b, c and d; (b) The preimage of $\mathrm{Charge}(f)$ under f.

group $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$ is then generated by the loops a, b, c and d, depicted in orange, red, green and blue respectively in Figure 4.2a. Note that each of them intersects exactly one edge of the chosen tree T. With a slight abuse of notation, we denote by a, b, c and d the corresponding elements in $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$ and also in G. Then a, b, c and d not only generate $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$, but also generate the iterated monodromy group G.

In Figure 4.2b we see the lifts of the four loops under f. To simplify the picture, we have left out all lifts that have the form of a loop. Using Figure 4.2b, we can write down the wreath recursions corresponding to the generators of G. Namely,

$$\begin{aligned}
a &= \langle\langle 1, 1, 1, a, 1 \rangle\rangle (1\,4\,5),\\
b &= \langle\langle 1, 1, 1, 1, b \rangle\rangle (1\,5\,2),\\
c &= \langle\langle 1, c, 1, 1, 1 \rangle\rangle (1\,2\,3),\\
d &= \langle\langle 1, 1, d, 1, 1 \rangle\rangle (1\,3\,4).
\end{aligned}$$

Next, we apply Corollary 3.2.3 to the generators a and c and conclude that it holds that $\langle\langle 1, 1, 1, 1, 1 \rangle\rangle \mathcal{A}_5 \subset G$. As all four generators have even permutational part, it follows that $\langle\langle 1, 1, 1, a, 1 \rangle\rangle$, $\langle\langle 1, 1, 1, 1, b \rangle\rangle$, $\langle\langle 1, c, 1, 1, 1 \rangle\rangle$ and $\langle\langle 1, 1, d, 1, 1 \rangle\rangle$ are elements of G. We can use this to conclude that $G \times G \times G \times G \times G \subset G$, which means that we have proven the statement below.

Proposition 4.1.1. *Let f be the polynomial from* (4.1) *and G be the iterated monodromy group* $\mathrm{IMG}(f)$*. Then G is regular branch on itself.*

Lastly, we note that we can apply a modified version of the proof above to the more general case of a critically fixed rational maps whose charge graph is a square with multiple edges.

4.2 Triangle example

In this section, we study the iterated monodromy group G associated to the charge graph from in Figure 4.3b. In Figure 4.3a, we see the Tischler graph corresponding to this charge graph depicted in red.

We choose a spanning subtree T of the Tischler graph; this tree is depicted in thick red lines in Figure 4.4a. Furthermore, we choose a basepoint t, as shown in Figure 4.4a. The iterated

Figure 4.3: The Tischler graph (a) and the charge graph (b).

monodromy group G is then generated by the loops a and b, depicted in blue and green respectively in Figure 4.4a. Note that each of them intersects exactly one edge of the chosen tree T.

In Figure 4.4b we see the lifts of the two loops. To simplify the picture, we have left out all lifts that have the form of a loop. Using Figure 4.2b, we can write down the wreath recursions corresponding to the generators of G. Namely,

$$\begin{aligned} a &= \langle\langle 1,1,1,a\rangle\rangle(1\,2\,4),\\ b &= \langle\langle 1,1,b,1\rangle\rangle(2\,3\,4). \end{aligned}$$

Let $G = \langle a, b\rangle$ be the group generated by a and b. Furthermore, let $K < G$ be the subgroup generated by the following elements:

$$\begin{aligned} g_1 &= a^2b &&= \langle\langle 1,a,b,a\rangle\rangle(1\,2)(3\,4),\\ g_2 &= aba &&= \langle\langle 1,1,ba,a\rangle\rangle(1\,3)(2\,4),\\ g_3 &= a^{-1}b &&= \langle\langle a^{-1},1,b,1\rangle\rangle(1\,2)(3\,4),\\ g_4 &= ba^{-1} &&= \langle\langle a^{-1},1,b,1\rangle\rangle(1\,4)(2\,3). \end{aligned}$$

We prove that G is regular branch on K. For this, we first consider the following commutators:

$$\begin{aligned} g_3^{-1}g_4 &= [b,a^{-1}] &&= \langle\langle 1,1,1,1\rangle\rangle(1\,3)(2\,4),\\ g_4g_2g_3^{-1}g_1^{-1} &= [b^{-1},a^{-1}b^{-1}] &&= \langle\langle 1,1,1,1\rangle\rangle(1\,4)(3\,4). \end{aligned}$$

Using these two elements, we see that we have $\langle\langle 1,1,1,1\rangle\rangle K_4 \subseteq K$, where K_4 denotes the Klein four-group. This means that we also have that $\langle\langle 1,a,b,a\rangle\rangle, \langle\langle 1,1,ba,a\rangle\rangle, \langle\langle a^{-1},1,b,1\rangle\rangle, \langle\langle a^{-1},1,b,1\rangle\rangle \in K$, and hence that $\langle\langle 1,a^{-1},b^{-1},a^{-1}\rangle\rangle, \langle\langle 1,1,a^{-1}b^{-1},a^{-1}\rangle\rangle, \langle\langle a,1,b^{-1},1\rangle\rangle, \langle\langle a,1,b^{-1},1\rangle\rangle \in K$.

Figure 4.4: Computing the action of $\mathrm{IMG}(f)$: (a) The spanning subtree T of the Tischler graph and the corresponding generators a and b; (b) The preimage of the charge graph.

We now use these elements to prove that $\langle\!\langle g_1, 1, 1, 1\rangle\!\rangle = \langle\!\langle a^2b, 1, 1, 1\rangle\!\rangle \in K$. For this, we look at the following products:

$$\begin{aligned}
\langle\!\langle a, 1, b^{-1}, 1\rangle\!\rangle \cdot \langle\!\langle 1, 1, ba, a\rangle\!\rangle &= \langle\!\langle a, 1, a, a\rangle\!\rangle, \\
\langle\!\langle 1, 1, 1, 1\rangle\!\rangle(1\,4)(2\,3) \cdot \langle\!\langle a, 1, a, a\rangle\!\rangle \cdot \langle\!\langle 1, 1, 1, 1\rangle\!\rangle(1\,4)(2\,3) &= \langle\!\langle a, a, 1, a\rangle\!\rangle, \\
\langle\!\langle 1, a^{-1}, b^{-1}, a^{-1}\rangle\!\rangle \cdot \langle\!\langle 1, 1, ba, a\rangle\!\rangle &= \langle\!\langle 1, a^{-1}, a, 1\rangle\!\rangle, \\
\langle\!\langle 1, 1, 1, 1\rangle\!\rangle(1\,2)(3\,4) \cdot \langle\!\langle 1, a^{-1}, a, 1\rangle\!\rangle \cdot \langle\!\langle 1, 1, 1, 1\rangle\!\rangle(1\,2)(3\,4) &= \langle\!\langle a^{-1}, 1, 1, a\rangle\!\rangle, \\
\langle\!\langle a, a, 1, a\rangle\!\rangle \cdot \langle\!\langle a^{-1}, 1, 1, a\rangle\!\rangle &= \langle\!\langle 1, a, 1, a^2\rangle\!\rangle, \\
\langle\!\langle 1, 1, 1, 1\rangle\!\rangle(1\,2)(3\,4) \cdot \langle\!\langle 1, a, 1, a^2\rangle\!\rangle \cdot \langle\!\langle 1, 1, 1, 1\rangle\!\rangle(1\,2)(3\,4) &= \langle\!\langle a, 1, a^2, 1\rangle\!\rangle, \\
\langle\!\langle a, 1, a^2, 1\rangle\!\rangle \cdot \langle\!\langle a^{-1}, 1, b, 1\rangle\!\rangle &= \langle\!\langle 1, 1, a^2b, 1\rangle\!\rangle.
\end{aligned}$$

Combining the fact that $\langle\!\langle 1, 1, a^2b, 1\rangle\!\rangle \in K$ with the fact that $\langle\!\langle 1, 1, 1, 1\rangle\!\rangle K_4 \subseteq K$, we conclude that $\langle\!\langle g_1, 1, 1, 1\rangle\!\rangle = \langle\!\langle a^2b, 1, 1, 1\rangle\!\rangle \in K$. We can use similar arguments to show that $\langle\!\langle g_i, 1, 1, 1\rangle\!\rangle \in K$ for all $i \in [4]$. Again using the fact that $\langle\!\langle 1, 1, 1, 1\rangle\!\rangle K_4 \subseteq K$, this means we can conclude that G is regular branch on K.

Chapter 5

Anti-polynomial case

In this chapter, we prove our main result for anti-polynomial maps. For this proof we make use of the lemmas we have proven in the chapter on polynomial maps.

Theorem 5.0.1. *Let $f : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a critically fixed anti-polynomial with $|C_f| \geq 3$. Let $G \coloneqq \mathrm{IMG}(f)$ and let $E < G$ be the subgroup of all elements of G with even permutational part. Then G is regular branch on E.*

Proof. Let $C_f = \{c_1, c_2, \dots, c_n, \infty\}$ be the set of critical points of f and let d be the degree of f. We start this proof by looking at the topological Tischler graph of the polynomial f. We denote this graph by $\hat{T}$. We now distinguish two different cases, depending on the structure of the faces of the graph $\hat{T}$.

Case 1: every face of $\hat{T}$ has at most two critical points on its boundary. From this it follows that every critical point is adjacent to ∞ in $\hat{T}$. To see this, suppose that there is a critical point which is not adjacent to ∞. Because of the structure of $\hat{T}$, this means that this critical point is adjacent to at least 3 other critical points. However, this would also mean that we have a face with more than 2 points on its boundary, namely the critical point we started it and 2 of the adjacent points. This contradicts our assumption, meaning that we indeed have that every critical point is adjacent to ∞ in $\hat{T}$.

We then choose a spanning tree T' of $\hat{T}$ such that all the critical points in $C_f \setminus \{\infty\}$ are leaves of this tree. Next, we define T to be the tree obtained by removing ∞ and all adjacent edges from $\hat{T}$. We choose a basepoint t in a face F_t which has two critical points on its boundary; one of which being a leaf of T. For every $i \in [n]$, we now define a loop a_i based at t around the critical point c_i which crosses exactly one edge of the spanning tree T' exactly once. We do this by connecting t to a small loop around c_i by a path that does not cross T'. The fundamental group $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$ is then generated by these loops. With a slight abuse of notation, we denote by a_i not only the element in $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$, but also in $\mathrm{IMG}(f)$. Then the set $\{a_1, \dots, a_n\}$ also generates $\mathrm{IMG}(f)$.

We use the Schottky map associated to $\hat{T}$ to study the lifts of the loops $\{a_1, \dots, a_n\}$. We have that the basepoint t has exactly d preimages under f; one in every face, except in F_t. Furthermore, we have that for every $i \in [n]$, exactly one lift of the loop a_i crosses an edge of the spanning tree T', namely the edge in T' connecting c_i to ∞. This means that a_i has exactly one non-trivial restriction; here it restricts to a_i^{-1}. Furthermore, the non-trivial lifts of every loop a_i close up to a cycle on the preimages of t in the faces of which c_i is a point on the boundary. This means that we have that the permutational part of a_i is a cycle for every $i \in [n]$.

We can use the structure of the wreath recursion of a_i to describe the structure of a_i^{-1}. Using the statements above, we see that for every $i \in [n]$ we have that:

- the permutational part of a_i^{-1} consists of a single cycle on a set X_i of size $k_i \geq 2$;
- the parity of the permutational part of a_i^{-1} coincides with the parity of the permutational part of a_i;
- a_i^{-1} restricts to a_i at some $x_i \in \mathsf{X}_i$; while all other restrictions are trivial.

For our proof based on induction, we fix an order through which to walk through the cycles corresponding to the inverses of our generators. We define g_1 to be the inverse of the generator corresponding to the critical point on the boundary of F_t; g_1 being that critical point which is a leaf of T. We let g_2 be the inverse of the generator corresponding to the second critical point on the boundary of F_t. Next, we let g_3 be the inverse of the generator corresponding to an arbitrary cycle adjacent to g_2. We continue this way, by letting g_{j+1} be the inverse of the generator corresponding to an arbitrary cycle adjacent to at least one of the cycles corresponding to the set $\{g_i \mid 1 \leq i \leq j\}$. Before continuing our proof, we first define the following things. For every $i \in [n]$, we define σ_i to be the permutational part of g_i, and X_i to be the set on which σ_i acts non-trivially. We define

$$\mathsf{Y}_i \coloneqq \bigcup_{j=1}^{i} \mathsf{X}_i$$

and let $E_i < \langle g_1^{-1}, \dots, g_i^{-1} \rangle < G$ be the subgroup of all elements of $\langle g_1^{-1}, \dots, g_i^{-1} \rangle$ with even permutational part.

Note that we have that $|\mathsf{X}_1 \cap \mathsf{X}_2| = 1$. This means we can now use Corollary 3.2.3 to state our induction basis:

(i) $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}(\mathsf{Y}_2) \subset E_2$;

(ii) $E_2[\mathsf{Y}_2] \subset E_2$.

Now assume that condition (i) and condition (ii) hold for E_{i-1} with $i \in \{3, 4, \dots, n\}$. For every $i \in \{3, 4, \dots, n\}$, we have that $\mathsf{X}_1 \cap \mathsf{X}_i = \varnothing$, meaning that $|\mathsf{Y}_{i-1} \setminus \mathsf{X}_i| \geq 2$, as $|\mathsf{X}_1| \geq 2$. Furthermore, because of how we have chosen the order of cycles, we have that $|\mathsf{Y}_{i-1} \cap \mathsf{X}_i| \geq 1$. This means that we can use Lemma 3.3.1 to state that

(i) $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}(\mathsf{Y}_i) \subset E_i$;

(ii) $E_i[\mathsf{Y}_i] \subset E_i$.

Using induction, we can then also state that

(i) $\langle\!\langle 1, \dots, 1 \rangle\!\rangle \mathcal{A}(\mathsf{Y}_n) \subset E_n$;

(ii) $E_n[\mathsf{Y}_n] \subset E_n$.

Note that $E_n = E$ and $\mathsf{Y}_n = [d]$. This then implies that G is regular branch on E, which concludes the argument of the first case.

Case 2: there is at least one face of $\hat{T}$ which has at least three critical points on its boundary. We start by choosing a basepoint t in a face with at least three critical points on its boundary. We will denote this face by F_t. The next thing we do is choose a spanning tree T' of $\hat{T}$ such that all critical points on the boundary of F_t are leaves of this tree. We do this in the following way:

1. We look at the graph obtained by removing all points on the boundary of F_t and the edges adjacent to them, except for the point corresponding to ∞. The graph that remains is a connected graph. The reason for this is that every face of $\hat{T}$ is a Jordan domain connected to ∞ and from the boundary of every face we only remove a set of edges adjacent to each other, of which at most one is an edge adjacent to ∞. This means we can then choose a spanning tree of this graph.

2. For every point on the boundary of F_t, we then add one edge to this spanning tree which connects the point to the tree.

For every $i \in [n]$, we define a loop a_i based at t. For every critical point c_i which is a leaf of T', we define a_i by connecting t to a small loop around c_i by a path that does not cross T'. This loop a_i crosses exactly one edge of the spanning tree T' exactly once. We then define a loop a_i for every critical point c_i which is a leaf of the tree T'' obtained by removing every leaf of T' as well as the edge adjacent to this leaf. We define a_i to be the loop based at t which crosses the edge of T'' adjacent to c_i and crosses no other edges of T' such that the faces which a_i crosses is minimal. We continue this process until we have defined a loop a_i for every $i \in [n]$. Note that all loops a_i cross exactly one edge of T' exactly once. The fundamental group $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$ is then generated by these loops. With a slight abuse of notation, we denote by a_i not only the element in $\pi_1(\widehat{\mathbb{C}} \setminus C_f, t)$, but also in $\mathrm{IMG}(f)$. Then the set $\{a_1, \dots, a_n\}$ also generates $\mathrm{IMG}(f)$.

Like in the first case, we use the Schottky map associated to $\hat{T}$ to lift the loops $\{a_1, \dots, a_n\}$. For every $i \in [n]$ for which c_i is a leaf of T', we have that a_i has the same structure as in the first case. For the critical points c_i which are not leaves of T', we have that the non-trivial lifts of a_i will close up to a cycle on the preimages of t in the faces of $\hat{T}$ which are crossed by a_i. This means that the permutational part of a_i is a cycle for every $i \in [n]$. Furthermore, exactly one of these lifts will cross an edge of T' exactly once, namely, the edge of T' that is crossed by a_i. This means that a_i has exactly one non-trivial restriction, where it restricts to a_i^{-1}.

We can use the structure of the wreath recursion of a_i to describe the structure of a_i^{-1}. Using the statements above, we see that for every $i \in [n]$ we have that:

- the permutational part of a_i^{-1} consists of a single cycle on a set X_i of size $k_i \geq 2$;
- the parity of the permutational part of a_i^{-1} coincides with the parity of the permutational part of a_i;
- a_i^{-1} restricts to a_i at some $x_i \in \mathsf{X}_i$; while all other restrictions are trivial.

We then choose an order through which to walk through the cycles corresponding to the inverses of our generators. Let m be the number of points on the boundary of the face F_t. We now discuss the structure of the cycles corresponding to these m critical points. For each of these critical points c_i, we have that the cycle corresponding to this point is a cycle on the preimages of t in the faces of which c_i is a point on the boundary, with the exception of F_t. This cycle will exactly intersect with the cycles corresponding to neighbouring points on the boundary of F_t, and with these cycles it will intersect exactly once. This means that the cycles corresponding to the m critical points on the boundary of F_t form a tree of m cycles.

This means we can use the tools in the proof of Theorem 3.4.1 to choose an order through which to walk through these cycles; we define g_1 up to g_m to be the inverses of the generators defined in this order. For $i \in [m]$, let σ_i denote the permutational part of g_i and let X_i be the set of points on which σ_i acts non-trivially. Furthermore, let $\mathsf{Y}_m = \bigcup_{i=1}^m \mathsf{X}_i$ and let $E_m < \langle g_1^{-1}, \dots, g_m^{-1} \rangle < G$ be the subgroup of all elements of $\langle g_1^{-1}, \dots, g_m^{-1} \rangle$ with even permutational part. We again use the proof of Theorem 3.4.1 to conclude that:

(i) $\langle\langle 1, \dots, 1 \rangle\rangle \mathcal{A}(\mathsf{Y}_m) \subset E_m$;

(ii) $E_m[\mathsf{Y}_m] \subset E_m$.

For every $m+1 \leq j \leq n$, we let g_{j+1} be the inverse of the generator corresponding to an arbitrary cycle adjacent to at least one of the cycles corresponding to the set $\{g_i \,|\, 1 \leq i \leq j\}$. We now denote for every $i \in [n]$ the permutational part of g_i by σ_i and the set on which σ_i acts non-trivially by X_i. We then have that for every $i \in [n]$ the following conditions hold:

- σ_i is a single cycle on the set X_i with $|X_i| \geq 2$;
- g_i restricts to g_i at some $x_i \in \mathsf{X}_i$; all other restrictions are trivial.

We define

$$\mathsf{Y}_i \coloneqq \bigcup_{j=1}^{i} \mathsf{X}_i$$

and let $E_i < \langle g_1^{-1}, \dots, g_i^{-1} \rangle < G$ be the subgroup of all elements of $\langle g_1^{-1}, \dots, g_i^{-1} \rangle$ with even permutational part.

Before continuing our proof, we make an important observation. We have that the boundary of the face F_t consists of at least 4 edges. Furthermore, every loop a_i crosses at most 2 edges of these boundary edges. This means that there are at least 2 faces adjacent to F_t through which our loop a_i does not pass. We can conclude that for every $m+1 \leq i \leq n$ we have that $|\mathsf{Y}_m \setminus \mathsf{X}_i| \geq 2$, and thus that $|\mathsf{Y}_{i-1} \setminus \mathsf{X}_i| \geq 2$. We also note that, because of how we chose the order of our generators, we have that $|\mathsf{Y}_{i-1} \cap \mathsf{X}_i| \geq 1$.

Now assume that condition (i) and condition (ii) hold for E_{i-1} with $i \in \{m+1, m+2, \dots, n\}$. Using the observations above, we see that we can use Lemma 3.3.1 to state that

(i) $\langle\langle 1, \dots, 1 \rangle\rangle \mathcal{A}(\mathsf{Y}_i) \subset E_i$;

(ii) $E_i[\mathsf{Y}_i] \subset E_i$.

Using induction, we can then also state that

(i) $\langle\langle 1, \dots, 1 \rangle\rangle \mathcal{A}(\mathsf{Y}_n) \subset E_n$;

(ii) $E_n[\mathsf{Y}_n] \subset E_n$.

Note that $E_n = E$ and $\mathsf{Y}_n = [d]$. This then implies that G is regular branch on E, which concludes the argument of the second case, also concluding the proof.

□

Chapter 6

Further research

In this thesis, we have proven results for (anti-)polynomials. However, the tools that we used to prove these results are more generally applicable. This raises the question whether it would be possible to prove similar results for (anti-)rational maps using these same tools. In the example in Section 4.1, we have for example seen that in that case we could indeed use Corollary 3.2.3 to prove the regular branchness of the iterated monodromy group. In the example in Section 4.2 however, we see that we can not apply these tools, because the structure of the generators does not fit the requirements of our lemmas.

This is exactly what makes it hard to generalise our statements. In the polynomial case, we have that the cycles corresponding to the generators form a tree of cycles. In the more general case of rational maps, we have that a pair of cycles can intersect at more than one point, which means that we can not always use Corollary 3.2.3 to prove a base case for our induction. This means that we might need to pose extra conditions on the Tischler graph to be able to prove a similar statements in the case of rational maps.

Furthermore, because the cycles can have more than one intersection point, it is not clear how we should choose the order in which to add new cycles to our base case. We can namely have that the cycle we want to add intersects so much with the already added cycles, that there are less than two points in the already added cycles which are not in the cycle we want to add. This would mean that we can not use Lemma 3.3.1 to do the induction step of the proof.

Another problem we run into when looking at rational maps, is that if we make use of a spanning tree of the charge graph corresponding to the map to calculate the wreath recursions of the generators, we can have that these wreath recursions have multiple non-trivial restrictions. This would for example be the case in the example in Section 4.1. However, we can solve this problem by using a spanning tree of the Tischler graph instead of the charge graph.

All in all, we believe that it would be possible to use our tools to prove similar statements in the more general case of (anti-)rational maps, but it might be necessary to include extra conditions in these statements.

Appendix A

Auxiliary lemmas

Here, we state and prove multiple lemmas that we use in the proofs in this thesis.

Lemma A.1. *Let $G = (V, E)$ be an acyclic graph for which it holds that $|E| = |V| - 1$. Then G is connected.*

Proof. Define $G_1, G_2, \dots, G_n$ to be the connected components of G, and write $G_i = (V_i, E_i)$. As G is acyclic, we know for every $i \in [n]$ that G_i is acyclic as well as connected, and hence is a tree. Furthermore, it is a well known fact that the number of edges of a tree equals the number of vertices minus one. This means that $|E_i| = |V_i| - 1$ for every $i \in [n]$. When we look at the following equation:

$$|E| = \sum_{i=1}^{n} |E_i| = \sum_{i=1}^{n} (|V_i| - 1) = \sum_{i=1}^{n} |V_i| - n = |V| - n,$$

we see that we can conclude that $n = 1$, hence that G is connected. □

The three lemmas below are technical lemmas about the generating sets of the subgroups $\mathcal{A}_n$ and $E(G)$ of even elements in the symmetric group $\mathcal{S}_n$ and a group $G < \mathrm{Aut}(\mathsf{T}_\mathsf{X})$, respectively.

Lemma A.2. *Let $n \in \mathbb{N}_{\geq 3}$. For distinct $i, j \in \mathbb{N}$ with $1 \leq i, j \leq n$, the set of 3-cycles*

$$\{(i\ j\ k) \mid 1 \leq k \leq n,\ k \neq i, j\}$$

generates the alternating group $\mathcal{A}_n$.

Proof. It is a standard fact that for $n \geq 3$ all 3-cycles in $\mathcal{S}_n$ generate the alternating subgroup $\mathcal{A}_n$; see, for example, [Arm88, Theorem 6.5]. We use this to prove the lemma by showing that with our set of 3-cycles we can generate all 3-cycles. For this, we first note that $(i\ j\ k)(i\ j\ k) = (i\ k\ j)$ for all $1 \leq k \leq n$, $k \neq i, j$. Next, by looking at the product

$$(i\ j\ b)(i\ j\ a)(i\ b\ j) = (i\ a\ b),$$

we see that we can make all 3-cycles of the form $(i\ a\ b)$ with $1 \leq a, b \leq n$, $a \neq b$, and $a, b \neq i, j$. We already showed that we can make the 3-cycles $(i\ j\ k)$ and $(i\ k\ j)$ for all $1 \leq k \leq n$, $k \neq i, j$. This means that we can in fact make any 3-cycle of the form $(i\ a\ b)$ with $1 \leq a, b \leq n$, $a \neq b$ and $a, b \neq i$. Let now $(a\ b\ c)$ be an arbitrary 3-cycle in $\mathcal{A}_n$. If a, b, or c equals i, we have already shown that we can make the cycle $(a\ b\ c)$, so suppose that $a, b, c \neq i$. In this case, we can make the 3-cycles $(i\ a\ b)$ and $(i\ c\ a)$, which means we can also make

$$(i\ a\ b)(i\ c\ a) = (a\ b\ c).$$

It follows that the set $\{(i\ j\ k) \mid 1 \leq k \leq n,\ k \neq i, j\}$ generates the set of all 3-cycles in $\mathcal{S}_n$, and hence generates $\mathcal{A}_n$. $\square$

Lemma A.3. *Let $k, l \in \mathbb{N}_{\geq 2}$. Then the set of 3-cycles*

$$\{(1\ k+l-j\ k-i+1) \mid 1 \leq i \leq k-1,\ 1 \leq j \leq l-1\}$$

generates the alternating group $\mathcal{A}_{k+l-1}$.

Proof. Our goal is to first show that we can generate the 3-cycle $(1\ k+l-1\ m)$ for all $2 \leq m \leq k+l-2$, after which we can use Lemma A.2 to prove the statement. Note that we already have $(1\ k+l-1\ m)$ for all $2 \leq m \leq k$; these elements we get by plugging in $1 \leq i \leq k-1$ and $j = 1$. In particular, by taking $i = k-1$ and $j = 1$, we get the element $(1\ k+l-1\ 2)$. Multiplying this 3-cycle with itself, we get $(1\ 2\ k+l-1)$. When we take $i = k-1$ and $2 \leq j \leq l-1$, we obtain the 3-cycles $(1\ m\ 2)$ for $k+1 \leq m \leq k+l-2$. We can use these 3-cycles to get

$$(1\ 2\ k+l-1)(1\ m\ 2)(1\ k+l-1\ 2) = (1\ k+l-1\ m)$$

for all $k+1 \leq m \leq k+l-2$, which means we have now generated the 3-cycles $(1\ k+l-1\ m)$ for all $2 \leq m \leq k+l-2$. Using Lemma A.2 we conclude that our set of 3-cycles generates the alternating group $\mathcal{A}_{k+l-1}$. $\square$

Lemma A.4. *Let $E < G$ be the subgroup of all elements of G with even permutational part and denote by $\mathrm{gen}(G)$ the set of generators of G. Furthermore, we define the following sets:*

$X_1 \coloneqq \{e \mid e \in \mathrm{gen}(G) \text{ has even permutational part}\}$,
$X_2 \coloneqq \{o_1 o_2 \mid o_1, o_2 \in \mathrm{gen}(G) \text{ have odd permutational part}\}$,
$X_3 \coloneqq \{oeo^{-1} \mid o \in \mathrm{gen}(G) \text{ has odd permutational part, } e \in \mathrm{gen}(G) \text{ has even permutational part}\}$.

Then E is generated by the following set of elements:

$$X \coloneqq X_1 \cup X_2 \cup X_3.$$

Proof. We will prove the statement by induction on the length of words in $\mathrm{gen}(G)$.

Let x be a word in $\mathrm{gen}(G)$ of length 1 with even permutational part. Then $x = e$ for some $e \in \mathrm{gen}(G)$ with even permutational part, which means that $x \in X_1$ and $x \in \langle X \rangle$.

Let x be a word in $\mathrm{gen}(G)$ of length 2 with even permutational part. We write $x = x_1 x_2$ for the decomposition of x into generators $x_1, x_2 \in \mathrm{gen}(G)$. We now distinguish two different cases:

1. First suppose that x_1 has even permutational part, so $x_1 \in X_1$. For the word $x = x_1 x_2$ also to have even permutational part, it must be that x_2 has even permutational part as well. Hence, $x_1, x_2 \in X_1$ and $x \in \langle X \rangle$.

2. Secondly, we suppose that x_1 has odd permutational part. It must then hold that x_2 too has odd permutational part, otherwise their product can never have even permutational part. This means that $x = x_1 x_2 \in X_2$ and hence that $x \in \langle X \rangle$.

Now suppose that for every word x in $\text{gen}(G)$ with even permutational part and of length smaller or equal to $n \geq 2$ it holds that it is generated by the set X. Let then $x \in E$ be an arbitrary word in $\text{gen}(G)$ of length $n+1$ which has even permutational part. We write $x = x_1 x_2 \dots x_{n+1}$ for the decomposition of x into generators. We again distinguish different cases:

1. The first case is the case that x_1 has even permutational part, so $x_1 \in X_1$. This means that $x_2 \dots x_{n+1}$ also has even permutational part. This is a word of length n, which means that we can use our induction hypothesis to state that it is generated by the set X. As $x_1 \in X_1$, we can then conclude that $x = x_1 x_2 \dots x_{n+1}$ also is generated by the set X.

2. The second case is the case that x_1 has odd permutational part and x_2 has even permutational part, so $x_1 x_2 x_1^{-1}, x_1 x_2^{-1} x_1^{-1} \in X_3$. The word $x_1 x_2^{-1} x_1^{-1} x = x_1 x_3 \dots x_{n+1}$ then has even permutational part, as both $x_1 x_2^{-1} x_1^{-1}$ and x have even permutational part. This is a word of length n, so we can use the induction hypothesis and state that it is generated by X. As $x_1 x_2 x_1^{-1} \in X_3$, we see that $x_1 x_2 x_1^{-1} x_1 x_3 \dots x_{n+1} = x$ is also generated by X.

3. The third and last case is the case that both x_1 and x_2 have odd permutational part, so $x_1 x_2 \in X_2$. In this case $x_3 x_4 \dots x_{n+1}$ is a word of length $n-1$ which has even permutational part, which means we can use the induction hypothesis to say that it is generated by the set X. As $x_1 x_2 \in X_2$, we conclude that $x_1 x_2 x_3 \dots x_{n+1} = x$ is also generated by X.

Using induction, we can now conclude that every word in $\text{gen}(G)$ with even permutational part can be generated by X, hence that the subgroup E is generated by the set X. $\square$

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