String theory with spontaneously broken supersymmetry and vacuum instability

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Abstract

We study the vacuum stability of orbifold compactifications of type IIB string theory. We focus in orbifolds of the shape $\left(S^1 \times T^4\right)/\mathbb{Z}_p$ that spontaneously break all supersymmetry. We compute the one-loop partition function for these models and integrate it to obtain the vacuum energy density. We then analyze the resulting vacuum and how instabilities appear depending on the S^1 radius.

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Chapter 1

Introduction

Our current theoretical frameworks for describing the most fundamental interactions and pieces of nature, general relativity, and the standard model of particle physics have successfully described a wide range of phenomena within their respective domains: the former governs the large-scale structures and the macroscopic nature of space-time, while the latter explains the interactions of subatomic particles through the electromagnetic, weak, and strong nuclear forces. However, these theories remain incompatible with each other, especially when attempting to describe conditions where both gravitational and quantum effects are significant, such as in black holes and the early universe.[26]

String theory stands out as the most promising candidate for a unified theory of quantum gravity. Unlike point particles in traditional quantum field theories, string theory postulates that the fundamental constituents of the universe are one-dimensional strings, while among the theory objects with all kind of dimensionalities, called branes, are present. The fundamental strings vibrate at different frequencies, with each vibration mode giving rise to different fields. Remarkably, string theory incorporates the graviton, the hypothetical quantum of gravity, thus providing a nice framework to explore a quantum description of gravity.

One of the key motivations for this work is the connection between string theory and the cosmological constant[25]. Quantum field theories predict a vacuum energy density, while general relativity links this value with the so-called cosmological constant Λ, a parameter that codifies the geometry of the space-time in absence of other content apart from vacuum fluctuations. String theory, with its intricate structure and higher-dimensional framework, offers different perspectives on this parameter. Exploring which models give different vacua, within the whole landscape that string theory offers, may result in possible constraints of this theory in order to match the experimental data.

One of the main historical arguments from critics of this theory was its requirement of a symmetry known as supersymmetry (SUSY) [28]. This symmetry reflects in a connection between bosons and fermions, and is a crucial element of string theory that arises when adding fermions to the theory, but until the date it has not been observed in nature. This aspect also implies the vanishing of the cosmological constant, forcing the expected geometry of the vacuum to be flat.

Another non compelling aspect of string theory is its requirement for extra spatial dimensions. Our observable universe seems to be four-dimensional, but string theory requires the existence of additional dimensions that are compactified, periodical or "curled-up", in a way that makes them invisible at our energy scale. Compactification procedures not only hide these extra dimensions but also significantly influence the resulting lower-dimensional physical theory. This thesis will focus on a particular method of compactification based on orbifolds, which involves modifying the compact dimensions to break SUSY spontaneously. This allows the theory to be consistent and keeps fermions in the spectrum, while explaining why SUSY has not been found in nature and prescribing a procedure that may give a non-vanishing vacuum amplitude.

Orbifolds provide a versatile tool for achieving SUSY breaking.[13] We aim to gain deeper insights into the quantum properties of the vacuum by studying the one-loop vacuum amplitude in Type IIB string theory with SUSY broken through the orbifold. For example, we will investigate if any kind of instabilities (tachyonic states) may appear in the theory by performing this method and if those instabilities may be erased by requiring more restrictive constraints. The one-loop vacuum amplitude is particularly significant as it encapsulates the first order quantum correction and provides information about the stability and dynamics of the vacuum state, as we will see.

1.1 Outline & Objective

The specific goal of this thesis involves giving a prescription to compute the one-loop vacuum amplitude for Type IIB string theory after spontaneous SUSY breaking by applying orbifold methods. This involves calculating the partition function of the theory and integrating it over the fundamental domain of the torus. By doing so, we can explore how quantum instabilities manifest in this setting and how the interplay between SUSY breaking and compactification shapes the physical properties of the vacuum.The outline to achieve this purpose will be as follows:

Since the theory exhibits conformal symmetry, we will study the conformal field theory (CFT) living in the world-sheet. To make this thesis accessible to people without String Theory or CFT backgrounds, we will start by briefly stating the basics of CFT's focusing on the results we will need in Section 2.

Using the CFT framework we will proceed to compute the partition function in Section 3. There we will introduce the action of the theory, the quantization scheme and how to get the different contributions to the partition function.

In Section 4, we will introduce the concept of orbifolds, specify the orbifold action on the coordinates and states and quickly proceed with the computation of the partition function in the case in which the group used for the quotient is a general symmetric freely acting $\mathbb{Z}_p$.

Once we have this partition function, in Section 5, we will study a standard method, sometimes known as the orbit method, to solve the integral of the partition function and, particularly, the extension of this method to the theory with orbifolds. We will end analytically computing some examples of one-loop vacuum amplitude for these models and study their properties.

Chapter 2

Conformal field theory

Conformal field theories occupy a central position in modern theoretical physics due to their wide range of applications. In string theory, CFT's describe the dynamics of strings through the perspective of the two-dimensional world-sheet that these objects define while they travel through the space-time[8]. The consistency of string theory heavily relies on the properties of these world-sheet CFT's and the study of its mathematical structures is a key part of its simple description. Moreover, the AdS/CFT correspondence, connects the CFT defined on the boundary of an Anti-de Sitter (AdS) space with the gravity theory in its bulk [23]. This duality has provided profound insights into both quantum gravity and strongly coupled quantum field theories, offering a non-perturbative formulation of string theory and enabling the study of black holes and quantum chromodynamics among other systems.

CFT's also have a major relevance in condensed matter physics, primary due to their ability to describe critical phenomena. At critical points, systems undergo phase transitions characterized by scale invariance. CFT's, with their inherent conformal symmetry, provide a natural framework for modeling these phase transitions and allow the exact determination of critical exponents and scaling functions, which are essential for understanding the behavior of materials near critical points.

CFT's are quantum field theories that are invariant under conformal transformations, which preserve angles but not necessarily distances. These symmetries simplify many calculations, allowing for solutions in certain cases where other methods fail.

2.1 Basics of 2D CFT's

Let us start this brief description of CFT's by defining what is a conformal transformation. Given two manifolds (M, g) and (N, h), where the first component represents a topological space and the second a metric over it, a local conformal transformation ϕ is defined to act as:[17]

$$\phi \circ g = \Lambda h \ . \tag{2.1}$$

The main idea of this kind of transformations is that they are angle preserving. For the purpose of this thesis we will delve into the study of conformal transformations over two-dimensional flat manifolds (M, η); the endomaps then lead to transformations of the shape:[1]

$$\phi \circ \eta = \Lambda \eta \ \longrightarrow \ \eta_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^\mu} \frac{\partial \phi^\beta}{\partial x^\nu} = \Lambda \eta_{\mu\nu} \ . \tag{2.2}$$

Where $\mu \in \{1, 2\}$. If we choose a flat metric $\eta_{\alpha\beta} = \delta_{\alpha\beta}$ then we get the following restrictions under the map ϕ:

$$\partial_1 \phi^1 = \partial_2 \phi^2 \quad \partial_2 \phi^1 = -\partial_1 \phi^2 \ . \tag{2.3}$$

Which is equivalent to the Cauchy-Riemann equations, and so these conformal maps are equivalent to biholomorphic maps. Following this idea, we can perform a change of variables $z = x^1 + ix^2$ $\overline{z} = x^1 - ix^2$ to change into the complex plane and better understand how the coordinates change under conformal maps,

$$ds^2 = dzd\overline{z} \xrightarrow{\phi} \left| \frac{\partial \phi}{\partial z} \right|^2 dzd\overline{z} = \Delta(z, \overline{z}) \ dzd\overline{z} \ . \tag{2.4}$$

We call Δ the scaling dimension. Following this perspective we can apply an infinitesimal transformation $z \to \tilde{z} = z + \epsilon(z)$ and obtain the conformal algebra; expanding for both the holomorphic and antiholomorphic coordinates we get that the generators of the algebra, and its conmutation relations, are:

$$\text{Holomorphic: } l_n = -z^{n+1} \partial_z, \ [l_n, l_m] = (n - m) \, l_{n+m} \ . \tag{2.5}$$

$$\text{Anti-Holomorphic: } \overline{l}_n = -\overline{z}^{n+1} \partial_{\overline{z}}, \ \left[\overline{l}_n, \overline{l}_m\right] = (n - m) \, \overline{l}_{n+m} \ . \tag{2.6}$$

Each of this algebras are called Witt Algebra. Strictly speaking, due to the independent treatment of the coordinates z and $\overline{z}$, the whole algebra is the direct sum of two Witt algebras $A \oplus \overline{A}$ together with the commutator relation $\left[l_n \overline{l}_m\right] = 0$. This is reduced when imposing a physical condition $\overline{z} = (z)^*$. Applying this condition we restrict our algebra to the one generated by the operators $l_n + \overline{l}_n$ and $l_n - \overline{l}_n$ for all $n \in \mathbb{Z}$.

In order to get the global conformal transformations we can upgrade this transformations onto the Riemann sphere $S^1 \equiv \mathbb{C} \cup \{\infty\}$ and require them to be well-defined. We then get four different types of transformations.[2]

- **Translations**:
  - Generated by $\left(l_{-1} + \overline{l}_{-1}\right)$ and $i \left(l_{-1} + \overline{l}_{-1}\right)$
  - Resulting in $x^\mu \to x^\mu + a^\mu$ with $a^\mu \in \mathbb{R}^2$
- **Rotations**:
  - Generated by $i \left(l_0 - \overline{l}_0\right)$
  - Resulting in $x^\mu \to M^\mu_\nu x^\nu$ $M^\mu_\nu \in SO(1, 1)$

[1] We are assuming that that the maps are orientation preserving, which is equivalent to $\left| \frac{\partial \phi(x^1, x^2)}{\partial (x^1, x^2)} \right| > 0$

[2] We give the description of this transformations on real coordinates since it is simpler to see their physical meaning.

- **Dilations**:
  - Generated by $\left(l_0 + \bar{l}_0\right)$
  - Resulting in $x^\mu \to \Lambda x^\mu$
- **Special Conformal Transformations**:
  - Generated by $\left(l_1 - \bar{l}_1\right)$ and $i\left(l_1 - \bar{l}_1\right)$
  - Resulting in $x^\mu \to \frac{x^\mu - b^\mu |x|^2}{1 - 2b_\mu x^\mu + |b|^2|x|^2}$

The first three transformations have a clear physical meaning; a theory which is conformal invariant will be invariant under the Poincaré group, as expected for a good physical theory, while the invariance under dilation transformations indicates that the system is invariant under changes of scale and the invariance under special conformal transformations indicates invariance under angle-preserving maps.

We can then relate the group of finite globally defined conformal transformations to the group of Möbius transformations over S^1, $SL(2,\mathbb{C})/\mathbb{Z}_2$:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \; a,b,c,d \in \mathbb{C}, \; ad - bc = 1 \implies z \to \frac{az+b}{c\tau + d} \tag{2.7}$$

When describing a physical system, each state can be understood as eigenstates of operators such the hamiltonian and momentum operators. In CFT's, physical states are then defined by the eigenvalues of the dilation and rotation operators. Since they are described in terms of $l_0, \bar{l}_0$ for these theories, we can focus on states that are simultaneously eigenstates for both operatos:

$$l_0|\psi\rangle = h|\psi\rangle \quad \bar{l}_0|\psi\rangle = \bar{h}|\psi\rangle \implies |\psi\rangle = |h,\bar{h}\rangle \;, \tag{2.8}$$

$$\left(l_0 + \bar{l}_0\right)|h,\bar{h}\rangle = \left(h + \bar{h}\right)|h,\bar{h}\rangle \quad \left(l_0 - \bar{l}_0\right)|h,\bar{h}\rangle = \left(h - \bar{h}\right)|h,\bar{h}\rangle \;. \tag{2.9}$$

The eigenvalue of the dilation operator is then the scaling dimension previously defined $\Delta = h + \bar{h}$, while the eigenvalue of the rotation operator is the spin $s = h - \bar{h}$.

2.2 The punctured plane and radial quantization

In the context of String Theory we will study the CFT living on the so-called world-sheet; which is the manifold generated by the trace of a string moving in the space-time. In this thesis we will focus in Type IIB String Theory, and we will focus on closed strings; this implies that the world-sheet to be homeomorphic to a cylinder:

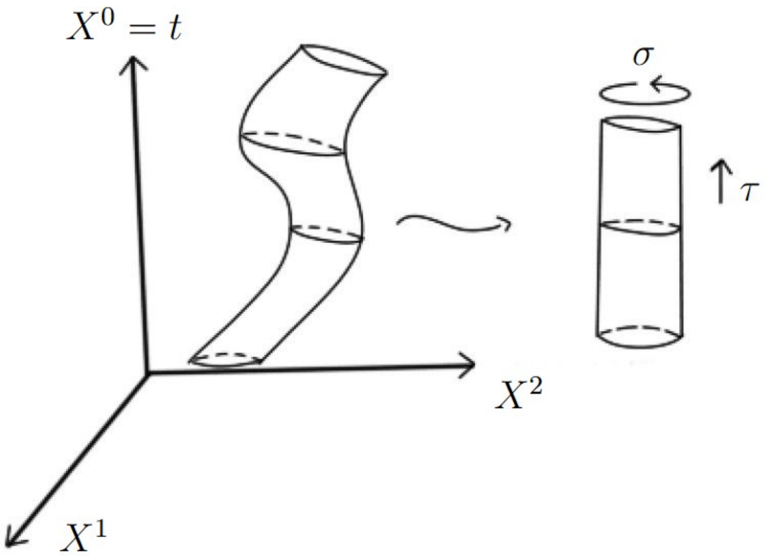

Figure 2.1: Representation of a closed string propagating through a space-time with coordinates (X^0, X^1, X^2). The surface that codifies its trajectory is known as world-sheet and can be reparametrized be a cylinder with coordinates (τ, σ).

We can use the symmetries of the theory to define a conformally flat metric, so we end up having a Minkowski space in D=2 with coordinates $\tau \in \mathbb{R}$ and $\sigma \in \mathbb{R}/L\mathbb{Z} \equiv S^1$, where L is the period of the cyclic coordinate.

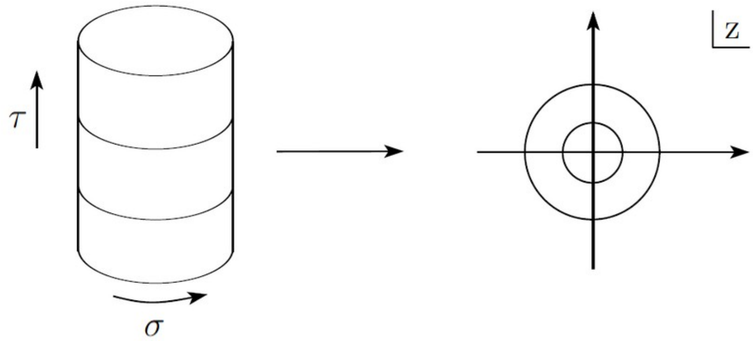

Figure 2.2: Map between the cylinder (world-sheet) and the punctured plane. Source:[22]

Then we perform a change of variables to $\zeta = \tau + i\sigma$ $\overline{\zeta} = \tau - i\sigma$, in which follows that the periodicity of σ forces $\zeta \approx \zeta + iL$. We can then relate the cylindrical world-sheet with the complex plane without including the origin, which is called punctured plane. This relation is given by a conformal map:

$$z = f(\zeta) = e^{\frac{2\pi\zeta}{L}} \tag{2.10}$$

This is going to be the picture in which we will develop the quantization scheme for this theory. While in a usual 1D quantization scheme we have a Hilbert space for each fixed time and then study the time evolution by introducing propagators between time slices, in the picture of the punctured plane we will have this Hilbert space defined in circles around the origin, for which the propagation will be given by increasing the radial direction.

Following this perspective, the hamiltonian, which is related to the time evolution, will correspond to the dilation operator from the conformal algebra, while the momentum operator, related to space translations, will correspond to the rotation operator.

Recall from QFT that we have to take special care when computing N-point correlators with the time ordering between them; this idea is implemented in radial quantization by imposing a radial ordering:

$$\mathcal{R}\left[\phi_1(z,\overline{z})\phi_2(w,\overline{w})\right] = \begin{cases} \phi_1(z,\overline{z})\phi_2(w,\overline{w}) & \text{if } |z| > |w| \ , \\ \pm\phi_2(w,\overline{w})\phi_1(z,\overline{z}) & \text{if } |z| < |w| \ . \end{cases} \tag{2.11}$$

Where the sign depends on the statistics of the field, $+$ for bosons and $-$ for fermions.

In Lagrangian mechanics one of the most interesting objects is the energy-momentum tensor $T_{\mu\nu}$. It is an invariant quantity for any Lorentz-invariant theory that arises as a conserved current under space-time translations and whose associated conserved charges are the momentum operators P^μ.

Returning to the perspective of the punctured plane, in complex coordinates, translational and scale invariance reduce the energy-momentum tensor to only two independent components: the chiral T_{zz} and antichiral $T_{\overline{z}\overline{z}}$ currents, holomorphic and antiholomorphic respectively. From here on, we will describe the procedures for the holomorphic part only, since the antiholomorphic one is equivalent; we will only mention them when it is necessary.

In the context of CFT's the energy momentum tensor has also another really interesting perspective. For a given theory with an action $S[\phi]$ depending on a field ϕ, we can compute the energy-momentum tensor by applying an infinitesimal transformation $x^\alpha \to x^\alpha + \epsilon^\alpha$:

$$T_{\mu\nu}(x^\alpha) = \frac{\partial \mathcal{L}}{\partial(\partial^\mu \phi)}\partial_\nu \phi - \eta_{\mu\nu}\mathcal{L}, \qquad \text{Quantization condition} : \langle 0|\ T\ |0\rangle = 0 \ . \tag{2.12}$$

In order to get the Hilbert space, we consider the mode expansion of the T tensor, as in the latter case to get the $l_n, \overline{l}_m$ we express in complex coordinates:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}L_n \implies L_n = \oint_{C_0} z^{n+1}T(z)\frac{dz}{2\pi i} \ . \tag{2.13}$$

Where C_0 is a closed contour containing 0. The L_n $n \in \mathbb{Z}$ are the generators of the so-called Virasoro algebra, which is the central extension of the already discussed Witt algebra. Nevertheless, we still need the commutation relation of these operators in order to define the algebra. To do so, we need to investigate the fields in the theory, since as shown in (2.13), the $T(z)$ does depend inherently in the theory.

In CFT's the fields are characterised by how they transform under conformal maps. In this context, we define two kind of fields, the primary (or conformal) fields and the secondary fields. The primary fields are defined to transform under a finite conformal transformation φ as:[17]

$$\phi(z,\overline{z}) \xrightarrow{z\to\tilde{z}=\varphi(z)} \left(\partial_z\tilde{z}\right)^h \left(\partial_{\overline{z}}\overline{\tilde{z}}\right)^{\overline{h}} \phi\left(\tilde{z},\overline{\tilde{z}}\right) \ . \tag{2.14}$$

And so they are called primary fields of weight $(h,\overline{h})$, while secondary fields are any other kind. Performing an infinitesimal transformation $z\to z+\epsilon(z)$, a primary field it changes as:

$$\delta\phi(w,\overline{w}) = h\partial\epsilon(w)\ \phi(w,\overline{w}) + \epsilon(w)\partial_w\phi(w,\overline{w}) \ . \tag{2.15}$$

Using the relation between the energy momentum tensor and infinitesimal transformations discussed before, we can compute how the field ϕ transforms under an infinitesimal transformation $\delta z = \epsilon(z)$:

$$\delta\phi(w,\overline{w}) = \frac{1}{2\pi i}\oint \epsilon(z)\left[T(z),\phi(\overline{w})\right] dz \ . \tag{2.16}$$

Which can be computed substracting the paths of integration as

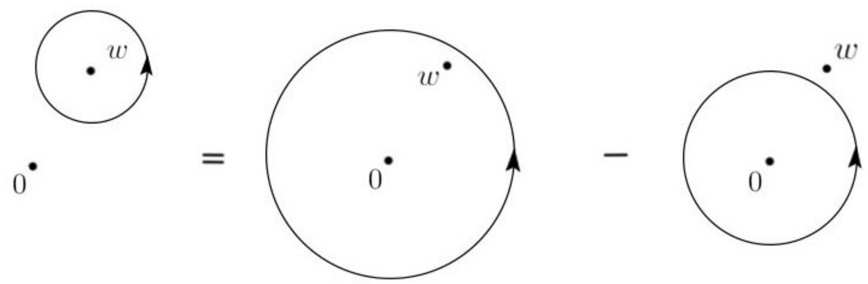

Figure 2.3: Substraction of the integration contours to simplify the commutator inside the integral. The direction of the path is given by the radial ordering. Source:[22]

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Through the residue theorem we know that we only need to expand the term $T(z)\phi(w)$ and check the singular points of this expansion when $z\to w$ in order to obtain the result of the integral; this kind of expansion is usually referred as OPE (Operator product expansion). In the case of primary fields (2.15), we already know the result of the integral, leading to the OPE:

$$\mathcal{R}\left[T(z)\phi(w,\overline{w})\right] = \frac{h}{(z-w)^2}\phi(w,\overline{w}) + \frac{1}{z-w}\partial_z\phi(w,\overline{w}) + \text{Non-singular terms} \ . \tag{2.17}$$

Then we can obtain the conmutators of the L_n Virasoro generators by computing the OPE of T with itself. In general, in CFT's this expansion leads to:

$$\mathcal{R}\left[T(z)T(z)\right] \sim \frac{c}{2\,(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} \ . \tag{2.18}$$

Which leads to the commutation relation, when expanding the $T(z)$ as in (2.12):

$$[L_n, L_m] = (n-m)\, L_{n+m} + \frac{c}{12} n \left(n^2 - 1\right) \delta_{n+m,0} \ . \tag{2.19}$$

The algebra characterised by this generators and commutation relation is known as Virasoro algebra. Repeating this process for the antichiral part we get a similar algebra for the $\overline{L}_n$ antichiral generators; we also get an extra relation $\left[L_n, \overline{L}_m\right] = 0$, which reveals that our full algebra is $Vir \oplus \overline{Vir}$.

2.2.1 Central Charge

Clearly the energy-momentum tensor is not a primary field, since its expansion has an extra term for theories with $c \neq 0$. This c is known as central charge, and depends on the lagrangian defining the theory. Before we analyze this central charge, Let us quickly show the particular form of the lagrangians that we are going to follow during the computations of the partition function in this thesis:

2.2.1.1 Example: Free Boson

Let us take the action of a free real scalar field:

$$S[\phi] \propto \frac{1}{2} \int \partial_\mu \phi \partial^\mu \phi d^2 x \ . \tag{2.20}$$

It will not be proven that this action is conformal invariant, which can be found in [27], but it will be motivated. Computing the energy-momentum tensor of the classical theory by applying an infinitesimal translation:

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial \left(\partial^\mu \phi\right)} \partial_\nu \phi - \eta_{\mu\nu} \mathcal{L} = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} \eta_{\mu\nu} \partial_\rho \phi \partial^\rho \phi \ . \tag{2.21}$$

We realise that this tensor is symmetric and traceless, the vanishing of the trace of the energy-momentum tensor motivates the scale invariance of the theory. On the other hand, we can obtain the tensor for the quantum theory, representing the energy-momentum tensor in complex coordinates:

$$\langle 0 | T(z) | 0 \rangle = 0 \implies T(z) = -2\pi : \partial_z \phi \partial_z \phi : \tag{2.22}$$

Where the : . : represents the normal ordering. We can then compute the OPE's related to this energy-momentum tensor:

$$\mathcal{R}\left[T(z) \partial_w \phi(w, \overline{w})\right] \sim \frac{\partial_z \phi}{(z-w)^2} + \frac{\partial_z^2 \phi}{(z-w)} \ , \tag{2.23}$$

$$\mathcal{R}\left[T(z) T(w)\right] \sim \frac{1}{2\,(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \ . \tag{2.24}$$

Comparing (2.23) and (2.17) shows that $\partial_z \phi$ is a conformal field of conformal weight $h = 1$, while equation (2.24) results in $c = 1$. The description also applies for the antiholomorphic contribution resulting in $\overline{h} = 1$, $\overline{c} = 1$. Summarising, the algebra of states will be determined by the central charge following (2.19). Since for the scalar real field $c = \overline{c} = 1$, and so we say bosonic fields have $c = 1$.

2.2.1.2 Example: Free Fermion

Recall the action for a free real Majorana fermionic field:

$$S[\psi] \propto \int \left(\psi\partial_z\psi + \overline{\psi}\partial_z\overline{\psi}\right) \ . \tag{2.25}$$

As in the latter case this theory is invariant under the full conformal group[27]. This time chirality is directly related with each of the fields; from the equations of motion one get that $\psi(z)$ is holomorphic and so chiral while $\overline{\psi}(\overline{z})$ is antiholomorphic and so antichiral. We compute the energy-momentum tensor:

$$T(z) = -\pi : \psi\partial_z\psi : \quad \overline{T}(\overline{z}) = -\pi : \overline{\psi}\partial_{\overline{z}}\overline{\psi} : \tag{2.26}$$

Which allows us to compute the OPE's

$$\mathcal{R}\left[T(z)\partial_w\psi(w,\overline{w})\right] \sim \frac{\frac{1}{2}\partial_z\psi}{(z-w)^2} + \frac{\partial_z^2\psi}{(z-w)} \ , \tag{2.27}$$

$$\mathcal{R}\left[T(z)T(w)\right] \sim \frac{\frac{1}{2}}{2\,(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{(z-w)} \ . \tag{2.28}$$

Getting a similar result for the atiholomorphic part. This fixes the theory of a free Majorana fermion as containing fields of conformal weight $h = \overline{h} = \frac{1}{2}$ and it fits in an algebra determined by $c = \overline{c} = \frac{1}{2}$.

As a side note, one important property of the central charge is that it is additive when considering decoupled CFT's. If we have n decoupled CFT's, the energy-momentum tensor satisfies:

$$T(z) = \sum_{i=1}^{n} T_i(z) \implies c = \sum_{i=1}^{n} c_i \ . \tag{2.29}$$

This property will be exploited in several points of our description. For example, it allows us to compute different bosonic contributions for a theory separately. Other feature of this property is that, when considering the ghost formalism, it restricts the number of dimensions in critical superstring theory to $D = 10$.[20].

2.2.1.3 Casimir Energy

To illustrate the physical meaning of the central charge, we can relate the energy-momentum tensor computed for the CFT in the plane again into the cylinder by the map (2.10). As we have already seen, $T_{\mu\nu}$ is not a primary field and so its transformation under a conformal map is not trivially obtained through equation (2.15), instead we have to compute the inverse of ζ and compute the transformation by brute force, doing so we obtain:

$$T(z) \xrightarrow{\zeta^{-1}} \frac{2\pi}{L}\left(z^2 T(z) - \frac{c}{12}\right) \implies L_0^{\text{Cylinder}} = L_0 - \frac{c}{24} \ . \tag{2.30}$$

Then we can obtain the expected value of the energy-momentum tensor for the cylinder. Requiring again the quantization condition:

$$\langle T(z)\rangle = \langle \overline{T}(\overline{z})\rangle = 0 \implies \langle T_{00}\rangle = -\frac{1}{2\pi}\left(\langle T_{\text{Cylinder}}(z)\rangle + \langle \overline{T}_{\text{Cylinder}}(\overline{z})\rangle\right) = \frac{\pi}{12L^2}\left(c+\overline{c}\right)\ . \tag{2.31}$$

From this term we can obtain the energy of the cylinder:

$$E_{\text{Cylinder}} = -\int_0^L \langle T_{00}\rangle dx = -\frac{\pi}{12L}\left(c+\overline{c}\right)\ . \tag{2.32}$$

Which is exactly the Casimir energy of a system, for example, if we choose to have a scalar field, then $c = \overline{c} = 1$ and so $E_{\text{Cylinder}} = -\frac{\pi}{6L}$, same result that we would obtain for the ground state of a sum of harmonic oscillators after performing a UV regularisation.

2.2.2 Hilbert spaces

Central charge allows us to mathematically characterize the representations of interest in our theory. In other words, it allows us to describe the different possible ground states that may appear in our system as well as the whole Hilbert space that can be built over it. To see how this happens, we need to define how we build up the Hilbert space for CFT's.

In order to have a well defined vacuum $|0\rangle$ for the CFT we must require the regularity of $T(z)|0\rangle$ when $z \to 0$[3], which can be attained by asking the state to verify:

$$L_n|0\rangle = \overline{L}_n|0\rangle = 0 \quad \forall n \geq -2\ . \tag{2.33}$$

Which implies invariance of the vacuum under transformations generated by $L_0, L_{\pm 1}$. Focusing on the holomorphic part, this means that the $|0\rangle$ vacuum must be $SL(2,\mathbb{R})$-invariant[4], invariant under global conformal transformations, as one would expect.

We can define a state $|h\rangle = \lim_{z\to 0}\phi(z)|0\rangle$ as created by acting with a primary field $\phi(z)$ of conformal weight h over the $SL(2,\mathbb{R})$-invariant vacuum; these states satisfy:

$$L_0|h\rangle = h|h\rangle \quad L_n|h\rangle = 0\ n \in \mathbb{N}^*\ . \tag{2.34}$$

This is the definition of a highest-weight state in algebra. Then we have a clear correspondence between primary fields and highest-weight states. From these highest-weight states we can obtain new states by acting with negative Virasoro generators L_{-n} $n > 0$, those are called descendant states. The intuition behind this construction is that highest-weight states represent the ground state for some specific Hilbert space while the negative Virasoro operators act as the creation operators.

[3]This is equivalent to the infinite past in the cylinder perspective.

[4]The $\mathbb{R}$ is due to choosing the holomorphic part for the discussion, the general framework involves a $|0\rangle_{\mathbb{C}} = |0\rangle \otimes \overline{|0\rangle}$ and is invariant under $SL(2,\mathbb{C})/\mathbb{Z}_2$.

The Hilbert space arising from the $|h\rangle$ is called Verma module, this name is also used for the space arising from $|h, \overline{h}\rangle$ and it forms a representation of the Virasoro algebra.

Since we can link a state $|h\rangle$ with a primary field, we can investigate this connection to check how to relate the descendants of a highest-weight state with fields. Laurent expanding the ϕ:

$$\phi(z) = \sum_{n\in\mathbb{Z}} \phi_n z^{-n-h} \implies \phi_n = \frac{1}{2\pi i}\oint \phi(z) z^{n+h} \ . \tag{2.35}$$

The modes ϕ_n are raised to operators. The effect of acting with L_{-n} over $|h\rangle$ is equivalent to applying one of these ϕ_n over it. This relation can be obtained by developing the commutation relations of the L_n and ϕ_n, this leads to a definition of the states as the modes of the primary field:

$$\phi_n|0\rangle = 0 \ \forall n > -h \qquad \phi_{-h}|0\rangle = |h\rangle \ . \tag{2.36}$$

The set of the primary field $\phi(z)$ and its descendant fields, usually denoted as $[\phi]$ is called the conformal family of ϕ, and it is simple to see that the Hilbert state built through these fields is equivalent to the one obtained by using the highest-weight state, this results in the so-called state-operator correspondence, which means that the set of Verma modules and the set of conformal families are bijective.

In order to have a well defined physical theory we would ask for the representation to be unitary. We can then find constraints over h and c to represent a physical theory. For example, if we compute the the norm of a descendant state:

$$0 \leq \langle h|L^{\dagger}_{-n}L_{-n}|h\rangle = \langle h|\left[L_n, L_{-n}\right]|h\rangle = \left[2nh + \frac{c}{12}\left(n^3 - n\right)\right]\langle h|h\rangle \ . \tag{2.37}$$

In general we can inspect the different options for h and c that lead to unitary representations, this method is done through the so-called Kac determinant; as this is not intended to be a fully formal description, we will just state the interesting result for our context, the rest can be found in [10].

For $c = 1$, the case of bosonic fields, we find a continuous infinite number of representations characterized by h whilst the ones satisfying $h = \frac{n^2}{4}$ $n \in \mathbb{Z}$ contain an extra null vector in their Hilbert space. For $c = \frac{1}{2}$, fermionic fields, we find three different unitary irreducible representations, characterized by $h \in \{0, \frac{1}{2}, \frac{1}{16}\}$.

2.2.3 Characters of the Virasoro Algebra

In the quest for computing the partition function we can think about its most classical description as sum over all the particles/states of our theory weighted by their energy/temperature . For CFT's we will see how the explicit connection with the algebra and the partition function is performed in Chapter 3, but for our purposes, it is a nice shortkey using the language of characters.

The character of an irreducible representation given by a highest-weight state $|h\rangle$, generating a Hilbert space $\mathcal{H}_h$ is defined to be:

$$\chi_i(\tau) = \mathrm{Tr}_{\mathcal{H}_h} q^{L_0 - \frac{c}{24}} = \sum_{n\in\mathbb{N}} dim(n+h) q^{n+h-\frac{c}{24}} \ . \tag{2.38}$$

Where $q = e^{2\pi i\tau}$ is defined in order to weight the states correctly as they will appear in the partition functions in the next sections, the τ parameter will be introduced later as the skew parameter of the torus. The $dim(n+h)$ factor just measures the number of linearly independent states in the Hilbert space $\mathcal{H}_h$ at the different levels. The most relevant theories for the objective of this thesis are $c = 1$ and $c = \frac{1}{2}$ as we have already discussed about.

In the $c = 1$ we have an infinite number of possible highest-weight representations whose characters are:[14]

$$h = 0 \implies \chi_{1,0}(\tau) = \frac{1}{\eta(\tau)} \ , \tag{2.39}$$

$$h = \frac{n^2}{4} \implies \chi_{1,\frac{n^2}{4}}(\tau) = \frac{1}{\eta(\tau)} q^{n^2/4}\left(1 - q^{n+1}\right) \ , \tag{2.40}$$

$$h \neq 0 \implies \chi_{1,h}(\tau) = \frac{1}{\eta(\tau)} q^h \ . \tag{2.41}$$

Where we followed the notation $\chi_{c,h}$, and $\eta(\tau)$ is Dedekind's η function defined as:

$$\eta(\tau) = q^{\frac{1}{24}} \prod_{n=1}^{\infty} \frac{1}{1-q^n} \ . \tag{2.42}$$

For the $c = \frac{1}{2}$ theory, we find three irreducible representations and their characters are:[14]

$$h = 0 \implies \chi_0(\tau) = \frac{1}{2\sqrt{\eta}}\left(\sqrt{\theta_3} + \sqrt{\theta_4}\right) \ , \tag{2.43}$$

$$h = \frac{1}{2} \implies \chi_{\frac{1}{2}}(\tau) = \frac{1}{2\sqrt{\eta}}\left(\sqrt{\theta_3} - \sqrt{\theta_4}\right) \ , \tag{2.44}$$

$$h = \frac{1}{16} \implies \chi_{\frac{1}{16}}(\tau) = \frac{1}{\sqrt{2\eta}}\sqrt{\theta_2} \ . \tag{2.45}$$

Where ϑ's are the Jacobi's ϑ functions defined as:

$$\vartheta\begin{bmatrix}\alpha\\ \beta\end{bmatrix}(\tau) = \eta(\tau) e^{2\pi i\alpha\beta} q^{\frac{1}{2}\alpha^2 - \frac{1}{24}} \prod_{n=1}^{\infty}\left(1 + q^{n+\alpha-\frac{1}{2}} e^{2\pi i\beta}\right)\left(1 + q^{n-\alpha-\frac{1}{2}} e^{-2\pi i\beta}\right) \ . \tag{2.46}$$

Using the notation:

$$\vartheta_3 = \vartheta\begin{bmatrix}0\\ 0\end{bmatrix}, \quad \vartheta_4 = \vartheta\begin{bmatrix}0\\ \frac{1}{2}\end{bmatrix}, \quad \vartheta_2 = \vartheta\begin{bmatrix}\frac{1}{2}\\ 0\end{bmatrix}, \quad \vartheta_1 = \vartheta\begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\end{bmatrix} = 0 \ . \tag{2.47}$$

The purpose of these characters is twofold. Firstly, it simplifies the notation while discussing the partition function. Secondly, they have transform in a particular way under modular transformations that we will use when computing the partition function.

For a more formal and extensive description about characters, we refer the reader to [8].

Chapter 3

One-loop Partition Function

The main objective of this thesis is computing the one-loop cosmological constant for a class of vacua determined by compactification on orbifolds.

For this objective, in this chapter we start by computing the one-loop partition function[1]. To do so, we will discuss the spectrum of the Type IIB theory and compute all different contributions to the partition function separately. We will end by discussing some general thermodynamical properties of this partition function and how to read the spectrum from it.

For historical reasons, we will focus in a spacetime $\mathbb{R}^{1,4} \times S^1 \times T^4$. The choice of having a T^n as compactified part can be seen as just simplicity, while the fact that we are considering 5 non-compact directions has mainly historical reasons linked to the study of the $D1-D5$ brane system. [31]

3.1 CFT Partition Function

To start constructing the partition function we have to first understand what is the geometry underneath the one-loop diagrams in Type IIB String Theory.

Figure 3.1: Expansion of vacuum-to-vacuum amplitude in terms of world-sheet geometries.

We are focusing on closed strings, the first intuition about one-loop amplitudes is that their related diagrams are determined by a genus 1 surface. This idea leads to the determination that one-loop vacuum diagrams are homeomorphic to the torus. The discussion and simplifications to obtain the partition function goes through understanding how the

[1]We will refer to it as just partition function from now on.

algebra of the CFT is related with the directions of the torus and the connection of torus with lattices.

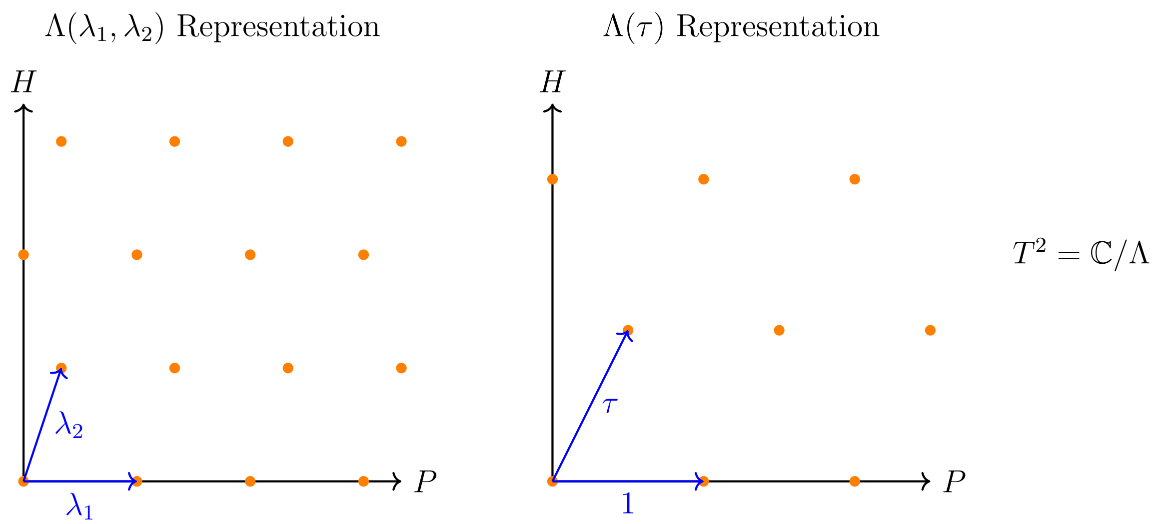


Figure 3.2: Representation of the lattice that generates the torus. The basis of the lattice is given by vectors λ_2 and λ_1 . After a reparametrization we represent it with the parameter τ.

We can link each specific possible torus shape with a lattice determined by two vectors as shown in Figure 3.2. λ_1, describes the cycle in which translations are generated by the $\hat{P}$ operator (the momentum operator) and λ_2 describes the $\hat{H}$ one (the hamiltonian operator). We can freely reparameterize the coordinates so that $\lambda_1 = 1$ and $\lambda_2 = \tau_1 + i\tau_2 = \tau \in \mathbb{H}^+ \equiv \{z \in \mathbb{C} \mid Im(z) > 0\}$.

Following this picture, we can construct the partition function through the generating function over euclidean periodic time as a typical QFT procedure. In order to do so, we need to identify one of the directions with the one related to the $\hat{H}$ operator, and so we will choose $\tau = \tau_2$. This assumption establishes one specific torus between the whole number of possible tori determined by $\tau \in \mathbb{H}^+$. In doing so τ_2 represents the period of euclidean time (also interpreted as the inverse of the temperature), and so the partition function reads as

$$Z(T) = \mathrm{Tr}\left(e^{-\beta\hat{H}}\right) \implies Z(0,\tau_2) = Tr\left[e^{-2\pi\tau_2\hat{H}}\right] = \mathrm{Tr}\left[e^{-2\pi\tau_2\left(L_0+\overline{L}_0-\frac{c+\bar{c}}{24}\right)}\right] \quad . \tag{3.1}$$

Where the last equality is obtained from the algebra of the CFT, L_0,$\overline{L}_0$ represent the Virasoro generators for right and left movers and c,$\bar{c}$ represent their central charges. In order to get a general torus that allows us to describe any possible shape of the lattice we can skew the previous example. To do so we perform a traslation over the horizontal axis of Figure 3.2); this traslation is implemented by $\hat{P} = i\left(L_0 - \overline{L}_0\right)$ from the CFT. For a generic $\tau \in \mathbb{C}$ we end up with the expression:

$$Z(\tau_1,\tau_2) = \mathrm{Tr}\left[e^{-2\pi\tau_2\left(L_0+\overline{L}_0-\frac{c}{24}-\frac{\bar{c}}{24}\right)}e^{-2\pi\tau_1 i\left(L_0-\overline{L}_0\right)}\right] = \mathrm{Tr}\left[q^{L_0-\frac{c}{24}}\bar{q}^{\overline{L}_0-\frac{\bar{c}}{24}}\right] \quad . \tag{3.2}$$

Where we have defined $q = e^{2\pi i\tau}$. This expression shows explicitly the factor related to the right-movers (involving q) and the factor related to the left-movers (involving $\overline{q}$). Now, we can expand L_0 as a sum over the Virasoro generators in the different directions. Using the properties of the trace, and knowing from the second chapter that the energy-momentum tensor splits as the sum of the different contributions, we can then expand the partition function in different pieces. Working in a $\mathbb{R}^{1,4} \times T^4 \times S^1$ spacetime and using (2.29) and (3.2):

$$Z(\tau, \overline{\tau}) = Z_{\mathbb{R}^{1,4}}(\tau, \overline{\tau})\ Z_{T^4}(\tau, \overline{\tau})\ Z_{S^1}(\tau, \overline{\tau})\ Z_F(\tau, \overline{\tau})\ . \tag{3.3}$$

Where the first three contributions in the right-hand side refer to the bosonic contribution in the directions given by the sub-index and the last contribution refers to the fermionic contributions. From the CFT perspective we are separating the total central charge in the contributions from separated bosons and fermions of $c = 1$ and $c = \frac{1}{2}$ respectively.

The details of each contribution can be obtained using standard techniques from quantum mechanics, nevertheless we will review briefly how to obtain the contributions from the perspective of the CFT and the characters of Virasoro algebras as well as drafting the basics on how to obtain the states in String Theory.

3.1.1 Modular transformations

Before computing the partition function, there is an important remark, the one-loop partition function must be modular invariant. As already stated, the partition function corresponds to the study of different fields over a toric geometry. Following this, the function must be invariant under the disconnected diffeomorphisms of the torus. These are not directly taken into account in the CFT computation of the partition function. These transformations form the so-called modular group $PSL(2, \mathbb{Z})$.

An intuitive way to construct the modular group is simply focusing on the lattice that generates the torus. In this context, the transformations of $\tau \in \mathbb{H}^+ \equiv \{z \in \mathbb{C} \mid Im(z) \geq 0\}$ that encode the periodicity over both cycles of the torus. They are generated by taking slices of the coordinate in $\hat{H}$ and making a twist in 2π and cutting a slice in the $\hat{P}$ direction and making a twist in 2π. By doing so, we get two different transformations:

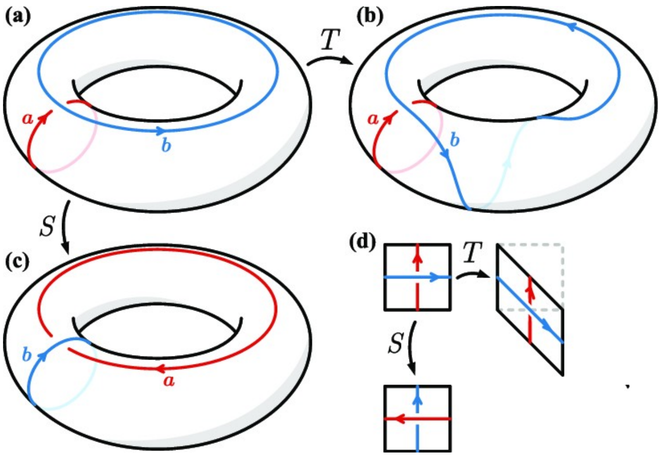


Figure 3.3: Intuition under the modular transformations S and T through their effect on the torus cycles and lattices. Source:[30]

- T **Transformation:** The T transformation is related with the P-cycle, and so it is defined as the traslation $T(\tau) = \tau + 1$.
- S **Transformation:** The twisting on the H direction generates the transformation $U(\tau) = \frac{\tau}{\tau+1}$, but the usual presentation of the group, due to the nice properties when discussing boundary conditions, is usually given on terms of $S = U^{T^{-1}}$ which leads to the transformation $S(\tau) = \frac{-1}{\tau}$

The modular group is then defined as the group generated by this transformations with the usual composition $PSL(2, \mathbb{Z}) \equiv \langle S, T \mid S^2 = (ST)^3 = Id \rangle$.

The domain of the τ parameter is then restricted by the fact that modular transformations do not generate different theories. In order to avoid over-counting, we require $\tau \in \mathbb{H}^+ / PSL(2, \mathbb{Z}) = \mathcal{F}$. We call $\mathcal{F}$ the fundamental domain of the torus, which can be explicitly written as $\mathcal{F} \equiv \left\{ z \in \mathbb{C} \mid |z| > 1 \ , \ Re(z) \in [-\frac{1}{2}, \frac{1}{2}] \ , \ Im(z) > 0 \right\}$.

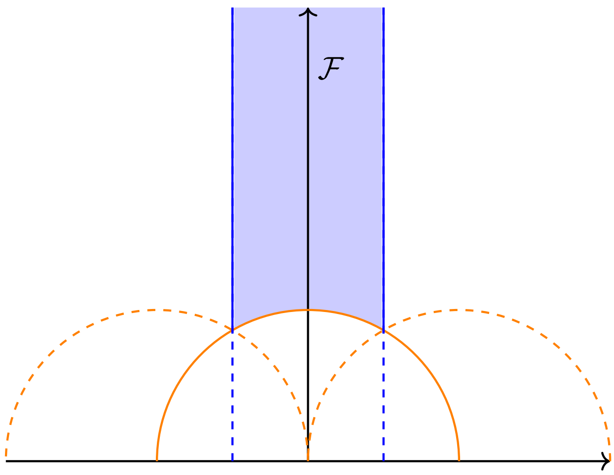


Figure 3.4: Fundamental domain, region filled in blue.

The region below the fundamental domain can be obtained with S transformations while the lateral regions are obtained by applying T transformations.

3.2 Bosonic contributions to the partition function

We start by computing here the bosonic contributions to the partiton function, we could directly do it following the description of Virasoro characters, but we prefer linking them with the direct computation of the trace in order to have both, an intuition on how the modular functions will appear here, but also giving a small description on how the bosonic fields arise in String Theory; this formalism will be required when introducing the orbifold compactification. So we start by presenting the action of the bosonic contribution to String Theory:

$$S = -\frac{1}{4\pi\alpha'}\int \partial_\alpha X^\mu \partial^\alpha X_\mu \; d\tau d\sigma \ . \tag{3.4}$$

In this action, the fields X^μ work as an embedding from the world-sheet to the space-time. They map the world-sheet, a smooth surface parametrized by (τ, σ), with the space-time, serving as coordinates/bosonic fields in it. The equations of motion arising from this action may be solved after requiring periodical boundary conditions[2] to get the expansion of the fields:

$$X^\mu(\tau,\sigma) = X^\mu_L(\tau+\sigma) + X^\mu_R(\tau-\sigma) \ , \tag{3.5}$$

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu\,(\tau-\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\alpha^\mu_n e^{-2\pi i n(\tau-\sigma)} \ , \tag{3.6}$$

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}\alpha' p^\mu\,(\tau+\sigma) + \sqrt{\frac{\alpha'}{2}}i\sum_{n\in\mathbb{Z}^*}\frac{1}{n}\overline{\alpha}^\mu_n e^{-2\pi i n(\tau+\sigma)} \ . \tag{3.7}$$

Then we can quantize the theory, promoting the modes α^μ_n to operators. The commutators for these operators read as:

$$[\alpha^\mu_n, \alpha^\nu_m] = [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\delta_{m+n,0}\eta^{\mu\nu}, \qquad [\alpha^\mu_n, \tilde{\alpha}^\nu_m] = 0 \ . \tag{3.8}$$

So they form two copies of the same algebra. We can extend the definition of the α^μ_n to $n = 0$ by just inspecting equations (4.5) and (4.6), and so defining $\alpha^\mu_0 = p^\mu$.

The so called L_0 operator, obtained from the energy-momentum tensor expansion and expression the hamiltonian of the system, is then obtained as total number of oscillators operator plus the contribution of the momenta, summarised:

$$L_0 = \frac{1}{2}\sum_{n\in\mathbb{Z}} :\alpha_{-n}\alpha_n: = \frac{1}{2}\alpha_0 + \sum_{n=0}^{\infty}\alpha_{-n}\alpha_n \ . \tag{3.9}$$

We are working on closed strings, we allow left and right movers.The vacuum will be the tensor product of the left and right vacua ($|0\rangle \otimes \overline{|0\rangle}$) and we can act over it with the left α^μ_n or right $\overline{\alpha}^\mu_n$ oscillators.

The action is conformal invariant[27] so we may study this theory as a CFT, allowing us to use the already developed partition function formula.

[2] This is not a required thing in general String Theory, but since we are working in Type IIB, the strings are closed and so we can impose this condition, avoiding open strings in our spectrum.

3.2.1 Non-compact contributions to the partition function

The first bosonic contribution comes from the free (non-compact) directions.

Each free direction will contribute with a boson for each right and left movers. In this context, if we have D free directions, since we are working on light-cone gauge this will translate to $D-2$ terms referred to the transverse directions.[3]. To compute each contribution, we can just perform the trace for one of the contributions.

$$\mathrm{Tr}\left(q^{L_0-\frac{c}{24}}\overline{q}^{L_0-\frac{\overline{c}}{24}}\right) = q^{-\frac{1}{24}}\overline{q}^{-\frac{1}{24}}\mathrm{Tr}\left[q^{\frac{1}{2}\alpha_0}\overline{q}^{\frac{1}{2}\alpha_0}\right]\mathrm{Tr}\left[q^{\sum_{n=1}^{\infty}\alpha_{-n}\alpha_n}\overline{q}^{\sum_{n=1}^{\infty}\overline{\alpha}_{-n}\overline{\alpha}_n}\right] \tag{3.10}$$

$$= \int_{\mathbb{R}} \sqrt{2}|q|^{\alpha_0^2}d\alpha_0 \left| q^{-\frac{1}{24}} \prod_{n=1}^{\infty}\sum_{k=0}^{\infty}(q^n)^k \right|^2 = \frac{1}{\sqrt{\tau_2}}\frac{1}{|\eta|^2} \ . \tag{3.11}$$

Working in light-cone gauge (In order to avoid a treatment of ghost in the theory) we only have 3 oscillators contributing to the whole partition function, leading to:

$$Z_{\mathbb{R}^{1,4}} = \left(\frac{1}{\sqrt{\tau_2}\eta\overline{\eta}}\right)^3 \ . \tag{3.12}$$

Remark that this result can also be obtained by using the Virasoro characters (2.42). In general the momentum of free bosons may be any continuous value $p \in \mathbb{R}$, which allows us to write $h = \frac{1}{2}p^2$. Examining the expansion in terms of the modes, we got the level-matching condition $(\alpha_0 - \overline{\alpha}_0)\,|\phi\rangle = 0$, which constraints the highest-weight states of our theory to $|h, \overline{h}\rangle,\ h = \overline{h}$, and so we can express these contributions to the partition function as:

$$Z(\tau_1, \tau_2) = \int_{\mathbb{R}} \chi_{c=1,h}\overline{\chi_{c=1,h}}dp = \frac{1}{\sqrt{\tau_2}}\frac{1}{|\eta|^2} \ . \tag{3.13}$$

We have two remarks from this relation, first of all is that both right and left movers are treated exactly in the same manner thanks to the condition $\left(L_0 - \overline{L_0}\right)|\phi\rangle = 0$. As we will see in the next sections, if this relation is not satisfied then the contributions are not conjugate. Secondly, we expect each contribution to be modular invariant, as they define a proper $c = 1$ theory by themselves. This can be easily checked following the modular properties of the η function:

$$\eta(\tau) \xrightarrow{S} \sqrt{-i\tau}\eta(\tau), \qquad \eta(\tau) \xrightarrow{T} e^{\frac{i\pi}{12}}\eta(\tau) \ . \tag{3.14}$$

And so each contribution transforms as:

$$\sqrt{Im(\tau)}|\eta(\tau)|^2 \xrightarrow{S} \frac{\sqrt{Im(\tau)}}{|\tau|}|\sqrt{-i\tau}\eta(\tau)|^2 = \sqrt{Im(\tau)}|\eta(\tau)|^2 \ , \tag{3.15}$$

$$\sqrt{Im(\tau)}|\eta(\tau)|^2 \xrightarrow{T} \sqrt{Im(\tau)}|e^{\frac{\pi i}{12}}\eta(\tau)|^2 = \sqrt{Im(\tau)}|\eta(\tau)|^2 \ . \tag{3.16}$$

Showing explicitly the modular invariance of this piece of the partition function.

[3] This result can also be justified using the formalism of ghosts, where there are bosonic ghost that will contribute with -2 to the central charge.[20]

3.2.2 Compact contributions

For the compact contributions, we will perform the computation of the S^1 and give the explicit version of the resulting lattice and how it transforms under modular maps; then we will consider the contribution of the T^4 and check which constraints must be realized in the lattice in order to ensure modular invariance.

We start with the S^1 contribution. In this context the bosonic coordinates are periodic, following:

$$X(\tau,\sigma) \sim Z(\tau,\sigma) + 2\pi R n \quad n \in \mathbb{Z} \ . \tag{3.17}$$

Where X represents the coordinate over the S^1. Following the mode expansion of the coordinates (3.6), (3.7), we can now have a shift between right and left momentum:

$$(\alpha_0 - \overline{\alpha_0})\,|\phi\rangle = Rn|\phi\rangle \ . \tag{3.18}$$

From here we follow [8]. We can choose a generic factor $\frac{1}{2}\Delta^2$, to be the conformal weight of the holomorphic part $h = \frac{1}{2}\Delta^2$, which constrains $\overline{h} = \frac{1}{2}(\Delta - Rn)^2$.Writing in terms of these two quantities the partition function:

$$Z_{S^1} = \frac{1}{|\eta|^2}\sum_{h,\overline{h}} \chi_{c=1,h}\overline{\chi_{c=1,\overline{h}}} = \frac{1}{|\eta|^2}\sum_{\Delta,n} q^{\frac{1}{2}\Delta^2}\overline{q}^{\frac{1}{2}(\Delta-Rn)^2} \ . \tag{3.19}$$

We can then restrict the possible values of the scaling factor by requiring this partition function to be modular invariant. Imposing invariance under T transformations requires:

$$q^{\frac{1}{2}\Delta^2}\overline{q}^{\frac{1}{2}(\Delta-Rn)^2} \xrightarrow{T:\ \tau\to\tau+1} q^{\frac{1}{2}\Delta^2}\overline{q}^{\frac{1}{2}(\Delta-Rn)^2} e^{2\pi i n\left(\Delta R - \frac{1}{2}R^2 n\right)} \implies \Delta R - \frac{1}{2}R^2 n = m \in \mathbb{Z} \ . \tag{3.20}$$

Solving for Δ, we present the usual version of the partition function for a compact boson on an S^1

$$Z_{S^1} = \frac{1}{|\eta|^2}\sum_{n,m\in\mathbb{Z}} q^{\frac{1}{4}\left(\frac{n}{R}+mR\right)^2}\overline{q}^{\frac{1}{4}\left(\frac{n}{R}-mR\right)^2} \ . \tag{3.21}$$

Through this discussion we used $\alpha' = 1$, which reflects in the exponent of $q\ \overline{q}$ being a linear combination of R and $\frac{1}{R}$. If we write it for a generic α' the easiest solution is shifting $R \to \frac{R}{\sqrt{\alpha'}}$, which fixes any dimensional inconvenience.

We can analyze more deeply this expression. First of all, if we follow the path integral perspective of the partition function, we would find that the partition function can be written as:[20]

$$Z_{S^1} = \frac{1}{|\eta|^2}\sum_{P_L,P_R} q^{\frac{\alpha'}{4}P_L^2}\overline{q}^{\frac{\alpha'}{4}P_R^2} \implies P_{L,R} = \frac{n}{R} \pm \frac{mR}{\alpha'} \ . \tag{3.22}$$

The last equality is given by comparison with (3.21). In the string theory language, states with $n \neq 0$ are called Kaluza-Klein modes, since the contribution $\frac{n}{R}$ can be related with the center of mass contribution to the momenta $P_{L/R}$. This result is typically obtained by compactifying a dimension by the Kaluza-Klein method. On the other hand, the m is called the winding number since it appears as a contribution to the squared momenta whose origin is the winding of the strings over the S^1 direction. In some sense, the number of loops of the string over this direction contributes to the momenta with a factor proportional to the radius of the S^1.

Now we can think that, in this contribution, right and left states are not treated in the same manner. Their momentum are different, this appears to violate level-matching since we could have different "masses" on both sides by having $p_R \neq p_L$. We will, for now, just state that this is naturally expected when applying condition (3.18), but we will exploit this property later on while discussing about the states after the orbifolding, and we will check explicitly how the level-matching is ensured when computing the one-loop amplitude.

As a last remark, we still need to check the invariance of this partition function under S and T transformations. This invariance is already trivially satisfied, but can be quite annoying trying to check it by brute force in this shape. To check the S invariance we then perform a Poisson resummation on the sum arising in (3.22)

$$Z_{S^1} = \frac{1}{|\eta|^2} \sum_{n,m\in\mathbb{Z}} q^{P_R^2} \overline{q}^{P_L^2} = \frac{R}{\sqrt{\alpha'}} \frac{1}{\sqrt{\tau_2}|\eta|^2} \sum_{n,m\in\mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2 \alpha'}|n+m\tau|^2} . \tag{3.23}$$

Then the prefactor is modular invariant as it is the contribution of a free direction. The sum is trivially modular invariant. The T transformation can be undone by shifting $n \to n-1$ which has no effect on the sum, as we would expect since the definition of the $p_{L/R}$ is given to ensure T invariance. Meanwhile the S transformation is easily undone by the change $m \to -m$, which leaves the sum invariant.

3.2.2.1 Momentum lattices

As we have shown, the S^1 contribution can be summarised as a prefactor $\frac{1}{|\eta|^2}$ and then a sum over the possible right and left momenta. This result can also be obtained in a slightly more general level .

If we consider a compact coordinate, we can represent it as a quotient over an infinite discrete group as:

$$X \sim X + 2\pi R m \implies X \in \mathbb{R}/2\pi R\mathbb{Z} . \tag{3.24}$$

This relation leads to an expansion of the bosonic coordinates, in terms of the world-sheet variables (τ, σ):

$$X(\tau, \sigma) = x + \alpha' p\tau + mR\sigma + \text{oscillators } (n > 0 \text{ terms}) . \tag{3.25}$$

On the other hand, we can split the fields in terms of left and right movers $X_R(\tau - \sigma), X_L(\tau + \sigma)$:

$$X_{L/R} = x_{L/R} + \frac{1}{2}\alpha' p_{L/R} (\tau \pm \sigma) + \text{oscillators } (n > 0 \text{ terms}) . \tag{3.26}$$

Where these $P_{L/R}$ are precisely the ones in (3.22) while computing the partition function of the S^1, justifying why we call them this way. On the other hand, when we quantize the theory, the operator $\hat{P}$ generates translations over x, then in order to have a well defined wave function e^{ipx}, we must require $p \in \frac{\mathbb{Z}}{R}$.

This leads to the idea that the identification in (3.24) may be generalised with the concept of a lattice. Formally, a lattice Γ is a set of points of $\mathbb{R}^n$ such that we can write any point as an integer linear combination of elements of its basis $B = \{e_i\}_{i\in I}$:

$$x \in \Gamma \iff x = \sum_{i\in I} m_i e_i \ , \ m_i \in \mathbb{Z} \ \forall i \in I \equiv \{1, 2, 3, ...\} \ . \tag{3.27}$$

Lattices may have also an associated inner product. When it is given by a metric whose signature is $diag\left(\underbrace{-1-, 1, ..., -1}_{r}, \underbrace{1, 1, ..., 1}_{s}\right)$ then is called Lorentzian of signature (r, s).

Given a lattice Γ with an inner product denoted by $\langle \cdot, \cdot \rangle$, we can define its dual lattice $\Gamma*$ as:

$$\Gamma^* \equiv \left\{ x \in \mathbb{R}^n \ \middle| \ \langle x, v \rangle \in \mathbb{Z} \ \forall \ v \in \Gamma \right\} \ . \tag{3.28}$$

Moreover, a lattice is called even if the norm of all vectors is an even number. If the lattice Γ verifies $\Gamma = \Gamma^*$, then it is called self-dual.

We can also give a notion of size to the lattice. This is through the idea of the volume of the unit cell, which is expressed as the determinant of the matrix $G_i^j = e_i^j \implies vol\,(\Gamma) = |G|$. Given the relation between the lattice and its dual, it is easy to see that $vol\,(\Gamma^*) = \frac{1}{vol(\Gamma)}$.

In our example of the S^1 compactification, we can identify $\Gamma = R\mathbb{Z}$, which leads to the restriction that the momenta p lives in Γ^*. Summarising this perspective, the compactification of the bosonic coordinates equivalent to the quotient over a lattice. Then, the total centre of mass momenta will live in the dual lattice.

To recover the left and right momenta, which we already obtained requiring modular invariance in the partition function, we can express the side momenta as:

$$p_{R/L} = p \pm L \quad L \in \Gamma, p \in \Gamma^* \ . \tag{3.29}$$

Recall that in the partition function the main quantities are $p_{L/R}$. Following this description, we can recognize that (p_L, p_R) form a Lorentzian lattice with metric $diag(-1, 1)$. We can then define the momentum lattice as:

$$\Gamma^* = span_{\mathbb{Z}} \left\langle \frac{1}{\sqrt{2}} \left(\frac{1}{\frac{R}{\sqrt{\alpha'}}}, \frac{1}{\frac{R}{\sqrt{\alpha'}}} \right), \frac{1}{\sqrt{2}} \left(\frac{R}{\sqrt{\alpha'}}, -\frac{R}{\sqrt{\alpha'}} \right) \right\rangle \ . \tag{3.30}$$

Following this description, the product of two lattice vectors will satisfy:

$$P \cdot P' = \frac{1}{2} \left(\frac{n}{\frac{R}{\sqrt{\alpha'}}} + mR \right) \left(\frac{n'}{R} + m'R \right) - \frac{1}{2} \left(\frac{n}{R} - mR \right) \left(\frac{n'}{R} - m'R \right) = nm' + n'm \ . \tag{3.31}$$

Which ensures than this lattice is even, as $P^2 \in 2\mathbb{Z}$; while its volume is $vol\,(\Gamma) = 1$. Those two conditions are sufficient to say that the lattice is self-dual.[29]

3.2.2.2 T^D contribution

Following the idea of the lattices, we can generalize this discussion for a general T^D. In this context, we will have D chiral bosons and D antichiral bosons. As already mentioned, the contributions from different coordinates are multiplicatives for the partition function, so the partition function of the T^D will be:

$$Z_{T^D}(\tau,\overline{\tau}) = \frac{1}{|\eta|^{2D}} \sum_{(p_L,p_R)\in\Gamma} q^{\frac{1}{2}p_L^2}\overline{q}^{\frac{1}{2}p_R^2} \ . \tag{3.32}$$

For a general D-dimensional lattice, given a general metric G induced on the compact direction, we can write down:[4][7]

$$p_{L/R}^2 = \frac{\alpha'}{2}\vec{m}^t G^{-1}\vec{m} + \frac{1}{2\alpha'}\vec{n}^t G\vec{n} \pm \vec{n}^t\vec{m} \quad \vec{n},\vec{m} \in \mathbb{Z}^D \ . \tag{3.33}$$

Further constraints to the lattice may be done in order to ensure modular invariance. Again, requiring a Lorentzian signature we find that Γ is an even lattice as $p_L^2 - p_R^2 = 2\vec{n}^t\vec{m} \in 2\mathbb{Z}$. This constraints leads, as in the case of the S^1, to T invariance of the partition function.

$$\frac{1}{|\eta|^{2D}} \sum_{(p_L,p_R)\in\Gamma} q^{\frac{1}{2}p_L^2}\overline{q}^{\frac{1}{2}p_R^2} = \frac{1}{|\eta|^{2D}} \sum_{(p_L,p_R)\in\Gamma} e^{-\pi\tau_2\left(p_L^2+p_R^2\right)}e^{\pi i\tau_1\left(p_L^2-p_R^2\right)} \tag{3.34}$$

$$\xrightarrow{T} \frac{1}{|\eta|^{2D}} \sum_{(p_L,p_R)\in\Gamma} e^{-\pi\tau_2\left(p_L^2+p_R^2\right)}e^{\pi i\tau_1\left(p_L^2-p_R^2\right)}e^{\pi i\left(p_L^2-p_R^2\right)} \ . \tag{3.35}$$

To check the S invariance, we perform a Poisson resummation as before. In this case we need the general formula, leading to:

$$\sum_{p\in\Gamma} f(p) = Vol(\Gamma^*) \sum_{b\in\Gamma^*} \tilde{f}(b), \qquad \tilde{f}(b) = \hat{F}\left[f(p)\right](b) \ . \tag{3.36}$$

To simplify this description, we can first apply an S transformation and then compute the resummation, This leads to

$$q^{\frac{1}{2}p_L^2}\overline{q}^{\frac{1}{2}p_R^2} \xrightarrow{S:\tau\to\frac{-1}{\tau}} e^{-\pi i\frac{p_L^2}{\tau}}e^{\pi i\frac{p_R^2}{\overline{\tau}}} \ . \tag{3.37}$$

Now performing the Poisson resummation we get:

$$\sum_{(p_L,p_R)\in\Gamma} q^{\frac{1}{2}p_L^2}\overline{q}^{\frac{1}{2}p_R^2} \longrightarrow \frac{vol\,(\Gamma^*)}{\sqrt{\tau_2}^D} \sum_{(b_L,b_R)\in\Gamma^*} q^{\frac{1}{2}b_L^2}\overline{q}^{\frac{1}{2}b_R^2} \ . \tag{3.38}$$

[4]We are setting $B = 0$.

Now, if we require the full partition function $Z_{T^D}(\tau)$ to be modular invariant we find that $vol(\Gamma^*) = 1$, which imposes that the lattice must be self-dual.

Through this results, we can express the contribution of the T^4 to the partition function as:

$$Z_{T^4}(\tau, \overline{\tau}) = \frac{1}{|\eta|^8} \sum_{(p_L, p_R) \in \Gamma_{4,4}} q^{\frac{1}{2}p_L^2} \overline{q}^{\frac{1}{2}p_R^2} \ . \tag{3.39}$$

Where the $\Gamma_{4,4}$ is an even self-dual Lorentzian lattice. This kind of lattices are usually called Narain lattices.

With this result we have already computed all the bosonic contributions and checked its modular invariance.

3.3 Fermionic contribution to the partition function

The fermionic part can be built in several ways depending on the approach. Since modular invariance is of major importance for this thesis, the main perspective would be a geometrical approach which will also has a deeper meaning when implementing the action of the orbifold. We will also briefly compare different techniques, such as the CFT direct approach or taking into account the different sectors in order to impose the GSO projection. In any case, we have to properly define the fermions in our theory; we will follow the RNS formalism for these contributions. The notation and thread of this discussion will follow [7].

In order to add fermions, we pair the bosonic fields $X^\mu(\tau, \sigma)$ $\mu \in [0, 1, ..., D]$ with fermionic partners $\psi^\mu(\tau, \sigma)$[6], anticommuting fields which are spinors on the world-sheet[5] and vectors on the space-time. The immediate change to the action is adding the standard Dirac term for D free massless fermions:

$$S_F = \frac{-1}{4\pi} \int \overline{\psi}^\mu \gamma^\alpha \partial_\alpha \psi_\mu d\tau d\sigma \ . \tag{3.40}$$

Where the ρ^α are the two-dimensional Dirac matrices obeying the usual Clifford algebra.

The addition of this term enhances the symmetries of the lagrangian that now presents the so-called supersymmetry, which can be summarised as:

$$\begin{aligned} \delta X^\mu &= \overline{\epsilon} \psi^\mu \ , \\ \delta \psi^\mu &= \rho^\alpha \partial_\alpha X^\mu \epsilon \ . \end{aligned} \tag{3.41}$$

Where ϵ is a constant infinitesimal Majorana spinor that consists of anticommuting Grassman numbers. In this case one can choose two possible periodicity conditions on each spinorial component $\psi_\pm$ for the boundary term to vanish. For the closed bosonic string it was taken to be the periodicity condition $X^\mu(\tau, \sigma + 2\pi) = X^\mu(\tau, \sigma)$, but due to the fermionic nature of these new fields they can also be antiperiodical, giving two possible expansions of the modes:

[5] The spinorial index $A = \pm$ is, as usual, dropped.

- **Neveu-Schwarz** conditions: Choosing antiperiodicity for thhe fermionc fields:

$$\psi^\mu_\pm(\tau,\sigma+2\pi) = -\psi^\mu_\pm(\tau,\sigma) \ . \tag{3.42}$$

  Then the mode decomposition of the fields will be:

$$\psi^\mu_-(\tau,\sigma) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} b^\mu_n e^{-in(\tau-\sigma)} \ , \tag{3.43}$$

$$\psi^\mu_+(\tau,\sigma) = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \bar{b}^\mu_n e^{-in(\tau+\sigma)} \ . \tag{3.44}$$

  The fermionic fields resulting from this choice are said to lay in the **NS sector**.

- **Ramond** conditions: Choosing periodicity for thhe fermionc fields:

$$\psi^\mu_\pm(\tau,\sigma+2\pi) = \psi_\pm(\tau,\sigma) \ . \tag{3.45}$$

  Then the mode decomposition of the fields will be:

$$\psi^\mu_-(\tau,\sigma) = \sum_{n\in\mathbb{Z}} b^\mu_n e^{-in(\tau-\sigma)} \ , \tag{3.46}$$

$$\psi^\mu_+(\tau,\sigma) = \sum_{n\in\mathbb{Z}} \bar{b}^\mu_n e^{-in(\tau+\sigma)} \ . \tag{3.47}$$

  The fermionic fields resulting from this choice are said to lay in the **R sector**.

The quantized theory is pretty straightforward. We will have inside each sector, two types of operators, $\{\alpha_n\}_{n\in\mathbb{Z}}$ the bosonic ones, exactly the same as in the bosonic theory, and the fermionic ones $\{b_n\}_{n\in I}$ where $I\equiv\mathbb{Z}$ for the R sector and $I\equiv\mathbb{Z}+\frac{1}{2}$ for the NS sector. The algebra for the operators are:

$$[\alpha^\mu_n,\alpha^\nu_m] = n\delta_{n+m,0}\eta^{\mu\nu}, \qquad \{b^\mu_n,b^\nu_m\} = \delta_{n+m,0}\eta^{\mu\nu} \ . \tag{3.48}$$

While the operator L_0 changes by adding a fermionic part:

$$L^f_0 = \frac{1}{2}\sum_{\{n\in I\}} n : b_{-n}\cdot b_n := \sum_{\{n\in I|n>0\}} nb_{-n}\cdot b_n \ . \tag{3.49}$$

The Hilbert space is determined by acting with the negative index operators over the ground states of both sectors, those ground states are defined as:

$$\alpha^\mu_n|0\rangle_{\text{NS}} = b^\mu_m|0\rangle_{\text{NS}} = 0, \qquad n\in\mathbb{N},\ m\in\mathbb{N}+\frac{1}{2}, \tag{3.50}$$

$$\alpha^\mu_n|a\rangle_{\text{R}} = b^\mu_m|a\rangle_{\text{R}} = 0, \qquad n\in\mathbb{N},\ m\in\mathbb{N}^* \ . \tag{3.51}$$

Where the a label reflects the degeneracy of the Ramond sector vacuum due to the existence of the b^μ_0 operators.These do not appear in the L_0 formula and so do not change the mass of the state, but indeed they rotate the state; as follow from equation (3.49), they satisfy the Clifford algebra, and so the ground state is non-unique and it is a spinor.

It is illustrative to remark that the normal ordering in the definition of L_0 arises a constant term. Using ζ-Riemann regularisation we obtain:[6]

$$\text{NS: } L_0^f = \sum_{n\in\mathbb{Z}+\frac{1}{2}} nb_{-n}\cdot b_n - \frac{1}{48}, \qquad \text{R: } L_0^f = \sum_{n\in\mathbb{Z}} nb_{-n}\cdot b_n + \frac{1}{24}\ . \tag{3.52}$$

Adding the bosonic extra term $(+\frac{1}{24})$ shift the vacuum energy in order to relate later the spaces descending from the ground state throught these operators with the highest-weight representations on CFT:

$$\text{NS: } L_0^f = \sum_{n\in\mathbb{Z}+\frac{1}{2}} nb_{-n}\cdot b_n, \qquad \text{R: } L_0^f = \sum_{n\in\mathbb{Z}} nb_{-n}\cdot b_n + \frac{1}{16}\ . \tag{3.53}$$

NS L_0 eigenvalue	State	**R** L_0 eigenvalue	State
0	$\|0\rangle$	$\frac{1}{16}$	$\left\|\frac{1}{16}\right\rangle$
$\frac{1}{2}$	$b_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{16}+1$	$b_{-1}\left\|\frac{1}{16}\right\rangle$
$\frac{3}{2}$	$b_{-\frac{3}{2}}\|0\rangle$	$\frac{1}{16}+2$	$b_{-2}\left\|\frac{1}{16}\right\rangle$
2	$b_{-\frac{3}{2}}b_{-\frac{1}{2}}\|0\rangle$	$\frac{1}{16}+3$	$b_{-3}\left\|\frac{1}{16}\right\rangle,\quad b_{-2}b_{-1}\left\|\frac{1}{16}\right\rangle$
$\frac{5}{2}$	$b_{-\frac{5}{2}}\|0\rangle$		
3	$b_{-\frac{5}{2}}b_{-\frac{1}{2}}\|0\rangle$		
$\frac{7}{2}$	$b_{-\frac{7}{2}}\|0\rangle$		
4	$b_{-\frac{7}{2}}b_{-\frac{1}{2}}\|0\rangle, b_{-\frac{5}{2}}b_{-\frac{3}{2}}\|0\rangle$		
...	...	...	...

Table 3.1: Hilbert space arising in the NS and R sectors.

Since we are working on closed string, we allow left and right movers, and so we will have different vacua obtained by tensoring the previous ones, this leads to 4 possible sectors on Type IIB, NS-NS, NS-R, R-NS, R-R. It is easy to see that the space-time fields arising from the NS-NS and R-R sectors will be bosons while NS-R R-NS will generate the fermions.

In superstrings this is not the whole story; we also need to impose the so-called GSO projection in order to reach a consistent theory. This operator can be justified in terms of representation theory and modular invariance as we will see later. In this case, we will just justify its implementation by requiring the spectrum of the theory to be supersymmetric. In the NS sector, it is represented by a projection $\pi_{NS} = \frac{1}{2}\left[1+(-1)^F\right]$ where the operator $F = \sum_{n\in\mathbb{N}+\frac{1}{2}} b_{-n}b_n - 1$ is the fermion number; while in the R sector the GSO holds in imposing one chirality to the ground field. In the case of closed strings, we have to impose this GSO for both left and right movers; the choice of the chirality of the R ground states lead to two possible consistent theories:

[6]This is the description of the Hilbert spaces on both sectors for one single contribution. Recall that, working in light-cone gauge, we will have $D-2$ contributions.

- **Type IIA**: $(-1)^F = 1 = -(-1)^{\overline{F}}$.
  This choice generates a nonchiral theory.
- **Type IIB**: $(-1)^F = 1 = (-1)^{\overline{F}}$.
  This choice generates a chiral theory.

In this thesis we will work on Type IIB, and so our theory will involve closed strings and chiral fields.

3.3.1 Spin structures and geometry

The fermionic nature of the ψ fields allows to choose $\pm$ periodical conditions for each of the cycles in the torus, so we may have time (anti)periodicity and space (anti)periodicity;; as we will see, this is closely related to modular invairance. The previous description chose antiperiodicity in the spatial direction. Choosing a specific combination is called as spin structure, following [14], we will denote a given spin structure (x, y) as $\mathrm{x}\underset{y}{\square}$, where the square represents a $\tau = \tau_2$ unit cell describing the torus, x representing the periodicity on the space direction and y the periodicity on the time direction.

While the spatial periodicity is directly codified in the mode expansion in (3.43)-(3.47) establishing the Hilbert spaces of the modes; the time (anti)periodicity involves the insertion of an operator reflecting the fermionic or bosonic behaviour of the states; the discussion of this operator is not trivial, but for the purposes of this thesis we will only highlight that such an operator must anticommute with the fermionic fields. The proper choice is $(-1)^F$, which correctly flips the boundary condition in time. Inserting this four possible conditions we get four different contributions to the trace, for the holomorphic part this summarises as:

$$-\underset{-}{\square} : \mathrm{Tr}_{NS}\left[q^{L_0-\frac{c}{24}}\right] , \qquad -\underset{+}{\square} : \mathrm{Tr}_{R}\left[q^{L_0-\frac{c}{24}}\right] , \tag{3.54}$$

$$+\underset{-}{\square} : \mathrm{Tr}_{NS}\left[(-1)^F q^{L_0-\frac{c}{24}}\right] , \qquad +\underset{+}{\square} : \mathrm{Tr}_{R}\left[(-1)^F q^{L_0-\frac{c}{24}}\right] . \tag{3.55}$$

Then, the fermionic part of the partition function must be a combination of these traces, the insertion of the left movers involve taking the absolute value square of the whole term. In order to perform this combination in a consistent way, we require this combination to be modular invariant.

Those terms can be expressed in terms of modular functions as:

$$-\underset{-}{\square} : \prod_{j=1}^{8} q^{-\frac{1}{48}} \prod_{r\in\mathbb{Z}+\frac{1}{2}} (1+q^n) = \frac{\vartheta_3^4}{\eta^4} , \qquad -\underset{+}{\square} : \prod_{j=1}^{8} q^{-\frac{1}{48}} \prod_{r\in\mathbb{Z}} (1+q^n) = \frac{\vartheta_2^4}{\eta^4} , \tag{3.56}$$

$$+\underset{-}{\square} : \prod_{j=1}^{8} q^{-\frac{1}{48}} \prod_{r\in\mathbb{Z}+\frac{1}{2}} (1-q^n) = \frac{\vartheta_4^4}{\eta^4} , \qquad +\underset{+}{\square} : \prod_{j=1}^{8} q^{-\frac{1}{48}} \prod_{r\in\mathbb{Z}} (1-q^n) = -\frac{\vartheta_1^4}{\eta^4} = 0 , \tag{3.57}$$

$$\tag{3.58}$$

where we have already introduced that we have 8 contributions since we have 10 directions. The index j moves along the contributions, as they are all equivalent, the result of this product is just powering one contribution to the 8th.

Applying S and T transformations results in interchanging the periodicity conditions, schematically:

$$\text{Fixed point under PSl(2,}\mathbb{Z}\text{): } {}^{+}\square_{+} \tag{3.59}$$

$$ {}^{-}\square_{+} \overset{S}{\longleftrightarrow} {}^{+}\square_{-}, \qquad {}^{-}\square_{-} \overset{T}{\longleftrightarrow} {}^{+}\square_{-} \; . \tag{3.60}$$

This results can be analytically computed through the modular properties of the Jacobi ϑ functions, but it can also be obtained from the geometrical perspective given on Section 3.1.1, where we already discussed how the S and T transformations affect the cycles of the torus.

Since those are all the possible pieces, we expect the partition function to be a linear combination of these elements, following the previous rules and adding the phases that arise during these transformations, the partition function must be proportional to:

$$Z_F \propto \left({}^{-}\square_{-}\right) - \left({}^{+}\square_{-}\right) - \left({}^{-}\square_{+}\right) \pm \left({}^{+}\square_{+}\right) \; . \tag{3.61}$$

Where, even if the last phase is not fixed, since that term vanishes there is no degeneracy; this is actually a key difference between Type IIA and Type IIB, this term does not vanish in Type IIA .
Seeing explicitly how modular invariance fixes what we introduced as GSO projection, to avoid over-counting we set the overall constant to be $\frac{1}{2}$, which also agrees with the overall factor of the GSO projection: $\pi_{GSO} = \frac{1}{2}\left(1 - (-1)^F\right)$.

Then the fermionic contribution is:

$$Z_F = \frac{1}{2}\left|\frac{\vartheta_3^4}{\eta^4} - \frac{\vartheta_4^4}{\eta^4} - \frac{\vartheta_2^4}{\eta^4}\right|^2 = 0 \; . \tag{3.62}$$

By virtue of the so-called Riemann identity for Jacobi ϑ functions this contribution vanishes, so the whole partition function of the system vanishes.

3.3.1.1 CFT perspective

Through the CFT perspective, we already know that there exist 3 different irreducible unitary representation for the $c = \frac{1}{2}$ theory. Comparing these representations with the algebras that arise for each sector 3.1, we can easily identify:

$$\text{NS} \leftrightarrow [0] \oplus \left[\frac{1}{2}\right], \qquad \text{R} \leftrightarrow \left[\frac{1}{16}\right] \; . \tag{3.63}$$

Imposing the GSO projection disentangles both representations in the NS sector, while the R sector only has one contribution $|\frac{1}{16}\rangle_+$ or $|\frac{1}{16}\rangle_-$.

From the perspective of characters, there is also another way to obtain this constraints based on fusion algebras and the Verlinde formula. Since those characters are related to the algebras verifying $c = \frac{1}{2}$, there should not appear new ones when considering modular transformations. This allows us to rewrite the modular transformations S and T in terms of matrices in the space of the characters, in such a way that the modular invariance is imposed in a more general way, for a disussion of this topic we suggest [8].

To summarize the notation for the computation of cosmological constant, we will express the fermionic contribution in terms of the characters for the $[0]$, $\left[\frac{1}{2}\right]$ and $\left[\frac{1}{16}\right]$:

$$-\underset{-}{\square} : \chi_0 + \chi_{\frac{1}{2}} , \qquad\qquad -\underset{+}{\square} : \chi_0 - \chi_{\frac{1}{2}} \ , \tag{3.64}$$

$$+\underset{-}{\square} : \sqrt{2}\chi_{\frac{1}{16}} \qquad\qquad +\underset{+}{\square} : 0 \ . \tag{3.65}$$

3.4 Connection with Thermodynamics & String States

In previous sections we have computed the full partition function for Type IIB string theory in a $\mathbb{R}^{1,4} \times T^4 \times S^1$ space-time, which can be expressed as:

$$Z(\tau, \overline{\tau}) = \frac{1}{2\sqrt{\tau_2}^3} \frac{1}{(\eta\overline{\eta})^{12}} \left(\sum_{(P_L, P_R) \in \Gamma_{S^1}} q^{\frac{\alpha'}{4} p_L^2} \overline{q}^{\frac{\alpha'}{4} p_R^2} \right) \left(\sum_{(p_L, p_R) \in \Gamma_{4,4}} q^{\frac{\alpha'}{4} P_L^2} \overline{q}^{\frac{\alpha'}{4} P_R^2} \right) \left| \vartheta_3^4 - \vartheta_4^4 - \vartheta_2^4 \right|^2 \ . \tag{3.66}$$

Where the Γ_{S^1} represents the lattice of the S^1 specified in (3.22). As it has been already discussed, the partition function has a natural intuition related to statistical physics.

If one sets $\tau_1 = 0$, then the expansion of the partition function is exactly the usual formula with a temperature given by the factor $\beta = 2\pi\tau_2$, this gives an insight about the meaning of τ_2 parameter. The limit when $\tau_2 \to 0$ is equivalent on the thermodynamical perspective to $T \to \infty$, and so we will call it the UV regime. On the other hand, the limits $\tau_2 \to \infty$ and $T \to 0$ are also equivalent, and so we will call this limit IR or massless regime.

On the other hand we can understand the partition function as a sum over all the states, in which particles contribute with $+$ for bosons and $-$ for fermions, where its sign is included in the partition function by the GSO projection [11].

$$Z = \mathrm{Tr}\left[(-1)^F q^{L_0 - \frac{c}{24}} \overline{q}^{\overline{L_0} - \frac{\overline{c}}{24}} \right] \approx \frac{1}{\tau_2^{1-D'/2}} \sum_{bosons} q^n \overline{q}^m - \frac{1}{\tau_2^{1-D'/2}} \sum_{fermions} q^n \overline{q}^m \tag{3.67}$$

$$= \frac{1}{\tau_2^{1-D'/2}} \sum_{n,m} \left[N_B(n,m) - N_F(n,m) \right] q^n \overline{q}^m \ . \tag{3.68}$$

Where n, m are the L_0 and $\overline{L_0}$ eigenvalues respectively and D' is the number of non-compact directions. As we will argue later, the level-matching $n - m = 0$ is not trivially satisfied and will arise naturally while performing the integral to get the vacuum energy density.

The supersymmetry, introduced in (3.41), connects fermionic and bosonic fields. When taken into space-time particles, this results on the fermionic and bosonic degrees of freedom being exactly the same for each mass level. Following (3.62), this symmetry implies that the partition function vanishes as we already computed. Moreover, we can track this vanishing effect back to the fermionic contribution, which is the contribution related to adding SUSY to the theory. But we can get more information from the partition function apart from ensuring that the spectrum is supersymmetric.

Focusing on states with no momenta or windings, we can then obtain the masses of the lowest energy states by expanding the characters, since the overall contribution is 0 in this case (Due to SUSY) we will split the partition function in the different sectors, and study their fields.

To perform the expansion, the easiest way is split the contributions from the Jacobi ϑ functions, which can be directly expressed as infinite sums, and the contributions from the η's, that will be expanded when $q, \overline{q} \to 0$. Using the sum expansion of the ϑ functions:[9]

$$\vartheta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (\tau) = \sum_{n \in \mathbb{Z}} q^{\frac{1}{2}(n+\alpha)^2} e^{2\pi i (n+\alpha)\beta} \ . \tag{3.69}$$

The NS sector results in ϑ functions with $\alpha = 0$, while the R sector exhibits $\alpha = \frac{1}{2}$. On the other hand, the η contributions can be Taylor expanded as:

$$\frac{1}{(\eta\overline{\eta})^{12}} = (q\overline{q})^{-\frac{1}{2}} \prod_{n=1}^{\infty} \frac{1}{(1-q^n)(1-\overline{q}^n)} = (q\overline{q})^{-\frac{1}{2}} \left[1 - 12q - 12\overline{q} + 144 q\overline{q} + \mathcal{O}\left(q^2, \overline{q}^2\right)\right] \ . \tag{3.70}$$

And so the lower energy contributions to the partition function on each sector read:

$$Z = \frac{1}{\tau_2^{3/2}} \times \text{Sum over } T^4 \times S^1 \times (q\overline{q})^{-\frac{1}{2}} \sum_{r,\tilde{r}} q^{\frac{1}{2}r^2} \overline{q}^{\frac{1}{2}\tilde{r}^2} \left[1 + \mathcal{O}(q, \overline{q})\right] \ . \tag{3.71}$$

Where $r, \tilde{r} \in \left(\mathbb{Z} + \frac{1}{2}\right)^4$ for the NS- or -NS sectors respectively and $r, \tilde{r} \in \mathbb{Z}^4$ for the R- and -R sectors.

The modular invariance is apparently broken in this expression since we are in the low energy approximation, as well as the level-matching is not trivially satisfied at this level, so when considering the possible states in each sector, we have to take into account both effects. While level matching implies that the exponent for q and $\overline{q}$ are the same, modular invariance may be recovered by only accepting states allowed by the GSO projection,resulting in $\sum_i s_i \in 2\mathbb{Z}$.

Inspecting the spectrum we get:

$$\begin{array}{|c|c|}
\hline
\text{NS} & \text{R} \\
\hline
(\pm 1,0,0,0) & \left(\pm\frac{1}{2},\pm\frac{1}{2},\frac{1}{2},\frac{1}{2}\right) \\
(0,\pm 1,0,0) & \left(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) \\
(0,0,\pm 1,0) & \left(\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) \\
(0,0,0,\pm 1) & \left(-\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}\right) \\
 & \left(\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) \\
 & \left(-\frac{1}{2},\frac{1}{2},-\frac{1}{2},\frac{1}{2}\right) \\
\hline
\end{array} \tag{3.72}$$

Multiplying R-R or NS-NS states will create space-time bosons, while products of R-NS and NS-R are space-time fermions. We clearly see the intuition under (3.62), since the terms with the $-$ in the partition function expansion are exactly the ones referred to the fermionic sectors when we compute the absolute value squared.

From the partition function we can also obtain the masses of such states. The power of the $q, \overline{q}$ is $\frac{1}{2}\alpha' p^2 = m^2$, and so we can read the mass squared for each state. In this case as the square of all the vectors $r, \tilde{r}$ is 1 and the powers of $q, \overline{q}$ are the same in the fermionic contribution, without the S^1 contribution the level matching is trivially satisfied. When considering the S^1, level-matching imposes constraints on the allowed values of momenta and winding according to (3.18).

Computing the masses for this states we check that for all the possible products, the masses of the states are 0. As the contribution from the S^1 is always positive, we infer that the spectrum has no tachyonic states, and so we will not expect having IR divergences. In the next section we will also study what happens when the level-matching is not trivially satisfied.

There is also another important remark about the GSO projection and the stability of the vacuum. As we discussed, there is no tachyon in this theory, but if we take naively all the possible particles that may arise in the theory without imposing modular invariance, then tachyonic states will appear. A simple example comes from taking the state $(0,0,0,0) \times (0,0,0,0)$, which implies an overall factor of $(q\overline{q})^{-\frac{1}{2}}$, which translates into a negative square mass state, a tachyon. Then, the requirement of modular invariance / GSO projection erases this state from the spectrum.

Chapter 4

Orbifolds

In physics we usually work on geometrical spaces codified through the concept of manifold (M, g), topological spaces over which we defined some charts (U, ϕ) that give us a local representation of the points in the topological space as geometrical objects in $\mathbb{R}^n$ or other usual set of numbers, such as $\mathbb{C}$.

On the other hand, from the algebraic perspective, given a group G we can define an action over a set X; the intuition under this idea can vary depending on the group and the set, from permutations over a set of elements, rotations in polygons, transformations of geometrical figures, we will revisit the most vital concepts associated with group actions while talking about the mathematical structure in this chapter. The idea of group actions is also common in physics, for instance when describing the spin of the electron with the $SU(2)$ group or in crystallography when describing the crystal lattice points through just the nodes of the unit cell.

Mixing these two concepts results in the idea of orbifolds. By exploiting a symmetry coming from a group action in a physical system, we can start with the manifold in which the theory lives in, "mod out" the symmetry and reach into a new system in which this symmetry is (spontaneously or totally) broken by only performing geometrical and algebraic procedures in the formalism. This idea is linked with String Theory as a nice way of break SUSY and recover a theory without explicit supersymmetry by performing changes in the compactified dimensions as we will see in the next chapters.

4.1 Mathematical Structure

We will start the formal discussion about orbifolds by refreshing the concept of group actions. Then we will define what is an orbifold and give some examples. This is not intended as a totally formal introduction, but as a way of formally introducing this topic and give an intuition.

Given a group G and a set X, an action ϕ of an element $g \in G$ over an element $x \in X$ is a map which is compatible with the group structure, i.e. :

$$\begin{aligned} \phi : G \times X &\longrightarrow X \\ (g, x) &\longrightarrow g \circ x = x' \quad , \end{aligned} \tag{4.1}$$

satisfying $\phi[g, \phi(h,x)] = \phi(gh,x)$. This action is said effective when $Ker(\phi) = \{e\}$, with $e \in G$ the identity element, it is also called free if it does not have non-trivial fixed points, i.e. $G_x \equiv \{x \in X \mid g \circ x = x \; \forall g \in G\} = \{e\} \; \forall x \in X$.

Once we have defined an action, we can define an equivalence relation $x, y \in X \; x \sim y \iff x = g \circ y, \; g \in G$. This allows use to have a well defined coset description X/G.

Some of the most intuitive examples of group actions are the continuous rotations of the plane and the discrete rotations.

- **SO(2) rotations**: Consider $G = SO(2)$ and the set $X = \mathbb{R}^2$. We can define an action of G over X acting as rotations over the plane. We can represent the group through a parameter θ which represent the angle of rotation. Doing this procedure, we have that the action of G sends one point $(x,y) \in \mathbb{R}^2$ to other point following Figure 4.2. This allows us to build the quotient $\mathbb{R}^2/SO(2)$ in which we identify points $(x,y) \sim g \circ (x,y) \forall g \in SO(2)$. The quotient results in the positive real line.

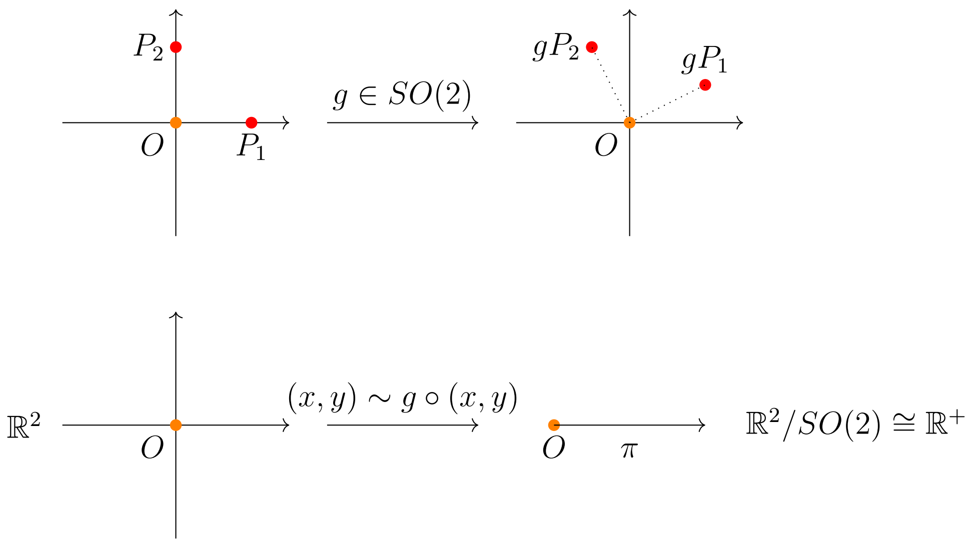


Figure 4.1: Example of $SO(2)$ action over $\mathbb{R}^2$ and its related coset construction.

- **Discrete rotations and cones**: Consider $G = \mathbb{Z}_p \equiv \{1, g, g^2, g^3, ..., g^{n-1}\} \; p \geq 2$. We can define an action of G over $\mathbb{R}^2$ as a rotation of $\frac{2\pi n}{p}$ around the origin. In this case we find a well defined quotient $\mathbb{R}^2/\mathbb{Z}_p$ which generates a cone of angle $\frac{2\pi}{p}$.

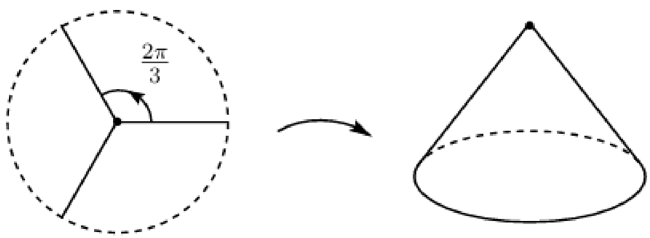


Figure 4.2: Example of quotient of a plane under $G = \mathbb{Z}_3$. Source:[18]

Those two examples are obviously non-freely acting; in both cases the origin $(0,0)$ is invariant under the action of the whole group and so is a fixed point under the action. In terms of the quotient $\mathbb{R}^2/\mathbb{Z}_p$ we can related this fixed point with the singularity that arises at the end of the cone.

There can also be groups acting on sets which are not related to rotations, one example of this kind of action is the torus T^n. Taking $G = \mathbb{Z}^2$, we can define an action such that it performs an entire translation on $\mathbb{R}^2$: $g = (p,q) \in \mathbb{Z}^2$, $a = (x,y) \in \mathbb{R}^2$ $g \circ x = (x+p, y+q) \in \mathbb{R}^2$. Taking the quotient will lead to an equivalence relation $a_1 \sim a_2 \iff x_1 - x_2, y_1 - y_2 \in \mathbb{Z}$. From the geometrical perspective, this quotient is equivalent to a 2-dimensional torus $T^2 \equiv \mathbb{R}^2/\mathbb{Z}^2$.

After recalling the concepts of manifold and group actions, we can properly define an orbifold, it requires first some definitions linked to the previous mentioned.

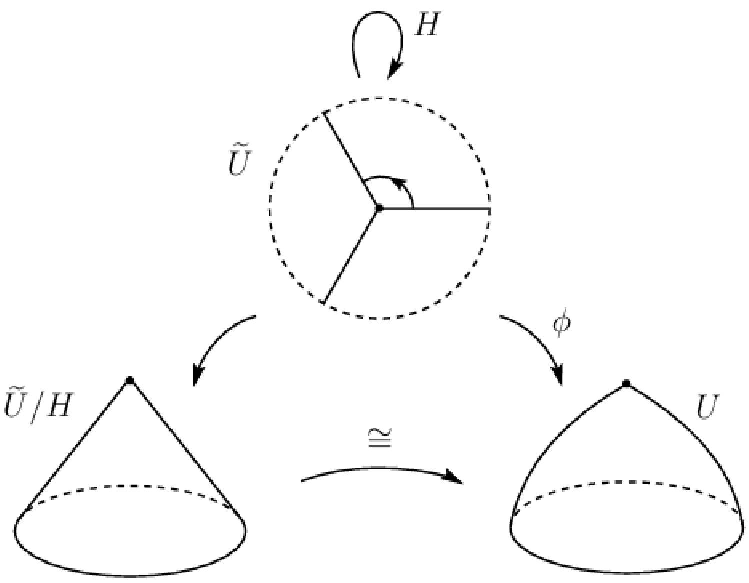


Figure 4.3: Idea of orbifold chart. The lower-right picture represents an open set U. The upper figure is the image resulting on $\mathbb{R}^2$ as in the manifold case, the map ϕ induces an homeomorphism between the initial open set and the lower-left image, the geometrical figure in which the quotient over the symmetry group H is present. Source:[18]

Let $U \subset X$ an open set of a topological space X, $\tilde{U} \subset \mathbb{R}^n$ an open set and G a finite group acting smoothly and effectively over X. Then we can define an orbifold chart as the collection $\left(\tilde{U}, G, \phi\right)$, where $\phi : \tilde{U} \to U$ is a continuous H-invariant map that induces an homeomorphism $\tilde{U}/H \cong U$.

Following this definition, we can proceed with requiring compatibility between the orbifold charts, defining an orbifold atlas and then defining an orbifold as a manifold M equipped with an orbifold atlas exactly as in the manifold case. The usual manifold charts connect locally topological spaces with euclidean spaces, while the orbifold charts locally connect this space to quotients of euclidean spaces over groups.

For our purposes we attend to a relation stating that the quotient of a manifold over a group action has a natural orbifold structure, and so we will just treat orbifolds as quotients of manifolds over groups.[18]

With this picture we can easily understand what is the main difference between an orbifold and a usual manifold. Clearly manifolds are specific cases of orbifolds in which the groups for each orbifold chart are trivial $H_i = \{e\}$, then we recover the usual definition of manifold. However, orbifolds are extensions of manifolds. An example of an orbifold which cannot be described as a manifold is $\mathbb{R}^3/\mathbb{Z}_2$ where the $\mathbb{Z}_2$ acts as the reflection $(x, y, z) \to (-x, -y, -z)$. This orbifold is an open cone in the real projective space $\mathbb{RP}^2$, and so is a contractible space, so its homotopy group is trivial. However, extracting the singular point at the vertex modifies the homotopy group to be the same as for $\mathbb{RP}^2$ which is $\mathbb{Z}/2\mathbb{Z}$. In smooth manifolds, extracting a point does not change the homotopy, and so we can determine that this orbifold is not a manifold.

As already occurs with group actions, if we think about a D=2 smooth manifold, a smooth surface; from the orbifold picture, choosing the group to act as in the conical action, we are allowing local conical singularities in our orbifold, and so we are in some sense allowing "spikes" or singularities which are determined by the fixed points of the group action. In this thesis we are going to avoid confronting singularities by defining only freely acting orbifolds.

4.2 Orbifolds in Type IIB String Theory

As we already checked from the partition function, the spectrum of our theory is supersymmetric. When we added fermionic fields to the world-sheet, we had to impose a consistency condition known as the GSO projection, also motivated by modular invariance. This projection, crops part of the spectrum, erasing tachyons that may appear otherwise, but also resulting in a supersymmetric spectrum in which the fermionic and bosonic degrees of freedom are the same at all mass levels. This result has two inconveniences, the first one is that experimentally we have not detected supersymmetry, which may be because this one is broken or hidden in some sense; but also the equivalence between fermionic and bosonic degrees of freedom makes the partition function vanish, which as we will argue later, makes the cosmological constant be exactly zero.

The main use of orbifolds in this thesis will be using some symmetry of the theory to mod out a group acting on the T^5. Following this idea, as we will check explicitly, we will spontaneously break SUSY in the theory, giving mass to all the fermions and resulting in a non-trivial vacuum whose stability and values depend on particularities of the theory.

The theory in which we are working is Type IIB compactified on $\mathbb{R}^{1,4} \times T^5$. Since we want to keep the non-compact directions intact, we will only apply the orbifold over the T^5. To do so, we need to define suitable coordinates that allows us to describe the group action.

In this thesis we will study cyclic groups $\mathbb{Z}_p \equiv \{1, g, g^2, g^3, ..., g^{p-1}\}$, $p \in \mathbb{N}$; but as already seen in the previous examples, if we take a random action of this group over T^n, the resulting orbifold, may have fixed points; so we need to take care of avoiding this in order to link this description with Schwerk-Schwarz reductions that spontaneously break SUSY [31].

4.2.1 Specifications about the coordinates

We will only investigate the X^μ affected by the group action, so we are only interested in the T^5. To avoid fixed points we will split $T^5 \equiv T^4 \times S^1$ and require the group to act differently in the torus and the circle. We will represent the circle coordinate by Z and the torus coordinates as $(Y^i)_{\imath\in\{1,2,3,4\}}$.

Due to the orbifold action, we prefer using complex coordinates in the torus, and so we will use $W^j = \frac{1}{\sqrt{2}}(Y^{2j} + iY^{2j-1})\ \ j \in \{1,2\}$. Those coordinates are then split between left and right movers as done before:

$$W^j(\tau,\sigma) = W^j_L(\tau+\sigma) + W^j_R(\tau-\sigma)\ . \tag{4.2}$$

We then choose the group action so it performs, over the bosonic coordinates, rotation on the torus and a translation in the circle, ensuring no fixed points. Following [15], we will choose an orbifold whose effect is codified by four mass parameters $\{m_i\}_{i\in[1,4]}$, $m_i = \pm\frac{2\pi k}{n}\ k \in \mathbb{Z}$ represents the mass that the states acquire after the orbifold. Following this formalism, we will break all SUSY whenever all the mass parameters are turned on. We can think about this as equivalent to the Higgs mechanism. As we spontaneously break the symmetry, we give masses to the fields. In the case of SUSY, we start having $\mathcal{N}=8$ and as we turn on the mass parameters we break it till $\mathcal{N}=0$.
We can represent the action of the group over the coordinates as:

Torus coordinates	Circle coordinate
$W^1_L \xrightarrow{g} e^{i(m_1+m_3)}W^1_L$	$Z \xrightarrow{g} Z + \frac{2\pi R}{p}$
$W^2_L \xrightarrow{g} e^{i(m_1-m_3)}W^2_L$	
$W^1_R \xrightarrow{g} e^{i(m_2+m_4)}W^1_R$	
$W^2_R \xrightarrow{g} e^{i(m_2-m_4)}W^2_R$	

The mass parameters are usually codified in the so called twist vectors:

$$u = (0,0,\frac{m_1+m_3}{2\pi},\frac{m_1-m_3}{2\pi})\ , \tag{4.3}$$

$$\tilde{u} = (0,0,\frac{m_2+m_4}{2\pi},\frac{m_2-m_4}{2\pi})\ . \tag{4.4}$$

When $u=\tilde{u}$ the orbifold is said symmetric, since it acts in the same manner over right and left movers, and asymmetric if $u \neq \tilde{u}$
We can translate this action to the bosonic states:

- Circle coordinates: Acting with the g element performs a shift on the coordinate of $+2\pi R\frac{1}{p}$, this can be translated to the momenta space as:

$$g \circ (|n,m\rangle) = e^{2\pi i n\frac{1}{p}}|n,m\rangle\ . \tag{4.5}$$

- Torus coordinates: We directly use that the action perform a rotation on the coordinates to get the action over the non-zero modes:

$$g \circ \left(\alpha^j_{-n}|0\rangle\right) = e^{2\pi i u_j} g \circ (|0\rangle) = e^{2\pi i u_j}|0\rangle \ . \tag{4.6}$$

Where the j runs over the torus coordinates. We discuss later what occurs with the momentum contribution.

Meanwhile, the identification $X \sim g \circ X$ introduces new options for the boundary conditions. These new options will result in new sectors of the Hilbert space of states, the so-called twisted sectors $\mathcal{H}_k$ (Tagged by the parameter $k \in \{0, 1, 2, ..., p\}$). For the closed states we have:[1]

$$Z(\tau, \sigma + 2\pi) = Z(\tau, \sigma) + 2\pi R(m + \frac{k}{p}), \tag{4.7}$$

$$W^1_L(\tau, \sigma + 2\pi) = e^{2\pi i \tilde{u}_3 k} W^1_L(\tau, \sigma), \qquad W^1_R(\tau, \sigma + 2\pi) = e^{2\pi i u_3 k} W^1_R(\tau, \sigma), \tag{4.8}$$

$$W^2_L(\tau, \sigma + 2\pi) = e^{2\pi i \tilde{u}_4 k} W^2_L(\tau, \sigma), \qquad W^2_R(\tau, \sigma + 2\pi) = e^{2\pi i u_4 k} W^2_R(\tau, \sigma). \tag{4.9}$$

The discussion about fermions is equivalent. For the NS sector the vacuum is a scalar and so it will be a fixed point of the group action as in the case for the bosons $g \circ |0\rangle_{\rm NS} = |0\rangle_{\rm NS}$. For the R sector, the vacua are spinors as seen in Section 3.3, and so we will act as rotation $g \circ |0\rangle_{\rm R} = e^{2\pi i \vec{u}\vec{s}}|0\rangle_{\rm R}$ where $vecs$ represents the spin. As we will work with 8 contributions, we have 4 spin numbers.

[1] Recall, $u_{3/4} = \frac{m_1 \pm m_3}{2\pi}$; $\tilde{u}_{3/4} = \frac{m_2 \pm m_4}{2\pi}$

4.3 One-loop Partition Function: Orbifolded Theory

We have already discussed how to compute the full partition function in Section 3 and in this chapter we introduced the concept of orbifold. In terms of the states in the theory, the group action by which we are modding has two effects: give mass to some states in our Hilbert space and add some extra states in the new twisted sectors.

The introduction of these modifications is quite direct.

- To represent the projection arising from the orbifold over a group[2] G we can just define a projection operator $\pi_G = \frac{1}{|G|}\sum_{g\in G} g$.
  For the case of the cyclic groups $\mathbb{Z}_p$ we are going to work with, we can represent it as

$$\pi_G = \frac{1}{p}\sum_{i=0}^{n-1} g^i = \frac{1}{p}\left[1 + g + g^2 + ... + g^{p-1}\right] \ . \tag{4.10}$$

- To introduce the extra sectors we can easily express the total Hilbert space as a direct sum of the spaces in the different sectors $\mathcal{H} = \oplus_{g\in G}\mathcal{H}_g$, applying the properties of the trace $\mathrm{Tr}_{\mathcal{H}}\left(A\right) = \sum\limits_{g\in G} \mathrm{Tr}_{\mathcal{H}_g}\left(A\right)$

Combining both descriptions we can represent the partition function of the orbifolded theory over a $\mathbb{Z}_p$ as

$$Z = \frac{1}{p}\sum_{g,h\in G} \mathrm{Tr}_{\mathcal{H}_h}\left(g q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right) = \frac{1}{p}\sum_{i,j=0}^{p-1} \mathrm{Tr}_{\mathcal{H}_{g^k}}\left(g^l q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right) = \frac{1}{p}\sum_{k,l=0}^{p-1} Z[k,l] \ . \tag{4.11}$$

Following this idea, we split the partition function of the theory over an orbifold in different blocks $Z[k,l]$ in which the first number refers to the twisted sector and the second one to the order of the projection over the group. After this discussion, our main objective now is computing the different blocks:

$$Z[k,l] = \mathrm{Tr}_{\mathcal{H}_{g^k}}\left(g^l q^{L_0-\frac{c}{24}}\overline{q}^{\overline{L}_0-\frac{\overline{c}}{24}}\right) \ . \tag{4.12}$$

Let us start this computation by focusing on the untwisted sector, i.e., the blocks of the shape $Z[0,l]$. The block $Z[0,0] = 0$, which correspond to the theory compactified on $\mathbb{R}^{1,4}\times T^5$ was already computed, so we only care about the terms with $1 \leq l \leq p-1$.

4.3.1 $S^1/\mathbb{Z}_p$ contribution

Starting with the bosonic contributions, the action of the group over the S^1 is taken to perform an extra shift on the circle depending on the order of the group and the order of

[2]For this definitions we are assuming the group is abelian

the element, following (4.7):

$$Z_{S^1}[0,l] = \text{Tr}\left[g^l q^{L_0-\frac{1}{24}}\overline{q}^{\overline{L}_0-\frac{1}{24}}\right] = \frac{1}{\eta\overline{\eta}}\sum_{n,m\in\mathbb{Z}}\langle n,m|g^l q^{L_0-\frac{1}{24}}\overline{q}^{\overline{L}_0-\frac{1}{24}}|n,m\rangle$$
$$= \frac{1}{\eta\overline{\eta}}\sum_{n,m} e^{2\pi i n\frac{l}{p}} q^{\frac{\alpha'}{4}P_L(n,m)^2}\overline{q}^{\frac{\alpha'}{4}P_R(n,m)^2} \ . \qquad (4.13)$$

Then we compute the contribution from the twisted sectors. These sectors arise from new possible boundary conditions, as it was specified in (4.7)-(4.9). In this formalism we see clearly that the natural extension to the k-th sector is shifting $m \to m+\frac{k}{p}$. Combining the result of the projection and this shifting we get the general formula for the S^1 contribution:

$$Z[k,l] = \frac{1}{\eta\overline{\eta}}\sum_{n,m\in\mathbb{Z}} e^{2\pi i n\frac{l}{p}} q^{\frac{\alpha'}{4}P_L\left(n,m+\frac{k}{p}\right)^2}\overline{q}^{\frac{\alpha'}{4}P_R\left(n,m+\frac{k}{p}\right)^2} \ . \qquad (4.14)$$

We could ask ourselves if there could be some extra phase or terms arising when going into a non-trivial projection in the twisted sectors and try to sanity check that modular invariance still present, but as we will see later, modular invariance of the partition function blocks is not as trivial as it is for the whole partition function, so we will discuss it in the next case.

Examining this contribution, we can see both phenomena that we expected in the orbifold. Examining an example of the $\mathbb{Z}_2$ we have four blocks building up the partition function:

$$Z[k,l] = \frac{1}{|\eta|^2}\sum_{n,m\in\mathbb{Z}} (-1)^{ln}\, q^{\frac{\alpha'}{4}p_L\left(n,m+\frac{k}{2}\right)^2}\overline{q}^{\frac{\alpha'}{4}p_R\left(n,m+\frac{k}{2}\right)^2} \ . \qquad (4.15)$$

First of all, we have the twisted sectors in which the momentum is shifted by a value. It is easy to check that in this case the lowest momentum is shifted by $\Delta p_{L/R} = \pm\frac{R}{\sqrt{8\alpha'}}$, showing that the energy difference between the twisted sectors will depend on the radius of the S^1. This will have a consequence for the least energy states of the theory, leading to possible instabilities as we will discuss later. Following this idea, we explicitly see that there are new fields in our spectrum. On the other hand, if we fix a sector, we get that the partition function is a sum of two terms:

$$Z[k] = \frac{1}{2}\left(Z[k,0]+Z[k,1]\right) = \frac{1}{2|\eta|^2}\sum_{n,m\in\mathbb{Z}}\left[1+(-1)^n\right] q^{\frac{\alpha'}{4}p_L\left(n,m+\frac{k}{2}\right)^2}\overline{q}^{\frac{\alpha'}{4}p_R\left(n,m+\frac{k}{2}\right)^2} \ . \quad (4.16)$$

This expansion is just considering the orbifold projection applied to the k-th sector, but here it is easy to see that all the states with $n \in 2\mathbb{Z}+1$ acquire an extra term that, when considering the full partition function, will result in extra mass for those states.

4.3.2 $T^4/\mathbb{Z}_p$ contribution

In the case of the T^4 the rotation projects out some of the possible momentum over the lattice. Intuitively speaking, the rotation moves some momentum states in the trace to other direction, resulting in the product $\langle p|g|p\rangle = 0$. After the orbifold, we must specify which sublattice of a $\Gamma_{4,4}$ survives the projection. The possible sublattices depend on the specific group of symmetry we are modding out, we can follow [21] for a general description and [5] for a description about the symmetries of lattices. In general for symmetric orbifolds we just have to care about choosing an appropriate lattice that presents the symmetry $\mathbb{Z}_p$ that we need in order to correctly define the orbifold and have that contribution not destroyed by the projection. Examples of those lattices are $A_1 \oplus A_1 \oplus A_1 \oplus A_1 \subset \Gamma_{4,4}$ or a $A_1 \oplus A_1$ for a $\mathbb{Z}_4$ quotient[15]. If we choose an asymmetric orbifold that leaves no direction on the right movers unrotated, then the sublattices may be chosen to be any chiral lattice of the shape $\Gamma_{4,0}$.

As we discussed in Section 3.2.2, the lattices related to the T^4 are self-dual and so they have volume 1. When applying the orbifold, the sublattice that survives the projection may fail to be self-dual. Under modular transformation (particularly, under S) a new factor arises. To ensure modular invariance we will need to divide over the volume of the sublattice.

Proceeding with the contribution related to the oscillators, we can easily obtain it from the action of the group on the coordinates of T^4 (4.8)-(4.9), which is summarized for the right-movers as:

$$Z_{T^4}[0,l] = \prod_{j=3}^{4} q^{-\frac{1}{24}} \mathrm{Tr}\left(g^l q^{L_0^j}\right) = \prod_{j=3}^{4} \left[q^{\frac{1}{24}} \prod_{n\in\mathbb{N}^*} \left(1 - q^n e^{2\pi i l u_j}\right)\left(1 - q^n e^{-2\pi i l u_j}\right)\right]^{-1} \tag{4.17}$$

$$= \prod_{j=3}^{4} 2\sin\left(\pi l u_j\right) \frac{\eta(\tau)}{\vartheta[{}^{\frac{1}{2}}_{-\frac{1}{2}+lu_j}](\tau)} \ . \tag{4.18}$$

with a similar shape for the left-movers replacing $u \to \tilde{u}$ and taking the complex conjugate. To compute this term we have assumed that $u_j \notin \mathbb{Z}$, if this occurs, the direction is left unrotated from the bosonic perspective, which means that the contribution from that T^2 is the usual one.

The prefactor corresponds to the number of fixed points by the action over the T^4 coordinates[7], recall that the full action over $S^1 \times T^4$ has not fixed points due to the shifting on the S^1, but the rotation of the T^4 may have them.

4.3.3 Fermionic contribution

For the fermions the computations are quite similar as in the bosonic case. We rely again on the rotating nature of the action over the T^4 which just involves a shifting on the second components of the ϑ functions. Directly addressing the traces we obtain, for the right movers:

$$Z_{NS}[0,l] = \frac{1}{2}\left\{ \left(\frac{\vartheta_3}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[{}^{0}_{-lu_j}]}{\eta} - \left(\frac{\vartheta_4}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[{}^{0}_{-lu_j-\frac{1}{2}}]}{\eta} \right\} \ . \tag{4.19}$$

$$Z_R[0,l] = \frac{1}{2}\left\{\left(\frac{\vartheta_2}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[^{1/2}_{-lu_j}]}{\eta} - \left(\frac{\vartheta_1}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[^{1/2}_{-lu_j-\frac{1}{2}}]}{\eta}\right\} . \tag{4.20}$$

Where the last term is important for considering modular invariance but vanishes as $\vartheta_1 = 0$.

The whole fermionic contribution, introducing both the holomorphic and anti-holomorphic parts, reads as:

$$Z_F[0,l] = \frac{1}{2}\left|\left(\frac{\vartheta_3}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[^{0}_{-lu_j}]}{\eta} - \left(\frac{\vartheta_4}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[^{0}_{-lu_j-\frac{1}{2}}]}{\eta} - \left(\frac{\vartheta_2}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[^{1/2}_{-lu_j}]}{\eta}\right|^2 . \tag{4.21}$$

4.3.4 Modular invariance and projection over the orbifold

Now, instead of analysing the states in the twisted sectors $\mathcal{H}_{g^i}$, which will be tedious since we will have to compute several different Hilbert spaces for the new boundary conditions of the fields introduced by the group action, we can just recall how the time and space boundary conditions were interchanged under modular transformations in (3.60).

Equivalent to the computations performed for the fermionic states, we can understand the insertion of the operators g^l in the trace as a "time" boundary condition on the time cycle of the torus, this is explicitly justified with the current description of the orbifold action over the states (4.12). On the other hand, the Hilbert spaces of the twisted sectors arise in a similar manner as the "space" boundary conditions, with a proper justification of this perspective on (4.7)-(4.9), this leads to represent the different blocks of the partition function as

$$l\underset{k}{\square} .$$

As we performed modular transformations in order to modify the spin structures, we can do the same thing with these blocks. Under a general modular transformation these blocks transform as:[14]

$$1\underset{k}{\square} \overset{\tau\to\frac{a\tau+b}{c\tau+d}}{\longrightarrow} \text{la+kb} \underset{lc+kd}{\square} . \tag{4.22}$$

Particularly focusing on the S transformation, it related the untwisted sector with a g^l insertion with the unprojected l-th sector. The S transformation corresponds with $a=0, b=-1, c=1, d=0$, which then results in the transformation:

$$1\underset{0}{\square} \overset{\tau\to-\frac{1}{\tau}}{\longrightarrow} 0\underset{l}{\square} . \tag{4.23}$$

This prescription allows us to directly compute any contribution $Z[k,l]$ from the different blocks related to all the possible group actions $Z[0,l]$. This idea is sustained in the fact that those transformations connect all the blocks of the partition function, sometimes referred as having a complete symmetry.

Using carefully these properties, we get the general contributions:[3]

$$Z_{T^4}[k,l] = \left| \prod_{i=3}^{4} 2\sin\left[\pi \mathrm{gcd}(k,l) u_i\right] \frac{\eta}{\vartheta \begin{bmatrix} \frac{1}{2} - ku_i \\ -\frac{1}{2} + lu_i \end{bmatrix}} \right|^2 . \tag{4.24}$$

While the fermionic components $Z_F[k,l] = |Z_{NS} - Z_R|^2$ read as:

$$Z_{NS}[0,l] = \frac{1}{2}\left\{ \left(\frac{\vartheta_3}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[{}^{ku_j}_{-lu_j}]}{\eta} - \left(\frac{\vartheta_4}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[{}^{ku_j}_{-lu_j - \frac{1}{2}}]}{\eta} \right\} , \tag{4.25}$$

$$Z_R[0,l] = \frac{1}{2}\left\{ \left(\frac{\vartheta_2}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[{}^{1/2+ku_j}_{-lu_j}]}{\eta} - \left(\frac{\vartheta_1}{\eta}\right)^2 \prod_{j=3}^{4} \frac{\vartheta[{}^{1/2+ku_j}_{-lu_j - \frac{1}{2}}]}{\eta} \right\} . \tag{4.26}$$

Where, as stated before, the last term will be 0 as $\vartheta_1 = 0$.

4.3.5 Example of a $\mathbb{Z}_2$ with $u = \tilde{u} = (0, 0, 1, 0)$

In order to explore a specific example, we will evaluate the previous description for a $\mathbb{Z}_2$ orbifold and understand its low energy spectrum. There are two inequivalent twist vectors that result on a $\mathbb{Z}_2$ symmetric orbifold with group action following the already discussed one. One of the orbifolds breaks all SUSY and the other does not do it. We can check which will be the outcome by referring to the mass parameters.If we turn on all the mass parameters setting $m_i = \pi$, we get the twist vectors $u = \tilde{u} = (0, 0, 1, 0)$, there are also other options but their resulting partition function is essentially the same.

As we have already discussed, the spectrum of closed bosonic string theory presents tachyonic states, understood as instabilities in the associated quantum theory. When adding fermions we had to impose modular invariance (a consistency condition) by which we crop part of the spectrum and get a supersymmetric spectrum, which makes the partition function vanish. In this framework the problem of the tachyonic states is solved by this GSO projection that erases the negative mass states avoiding the divergences of $Z(\tau, \overline{\tau})$ that arise when $q, \overline{q} \to 0$, the IR regime.

In this chapter we used the orbifold formalism, which erases some states but also added some others. As the orbifold projection over the untwisted sector results in adding extra mass to the states we do not have to worry about possible tachyonic states here. As we have seen in the previous sections, the twisted sectors present a significant difference in energy from the untwisted sector; when this difference is negative, possible tachyons will appear.

[3] gcd stands for great common divisor.

We can check this explicitly by studying the partition function, as was already discussed in Section 4.4, we can associate the exponent of $q, \overline{q}$ with the mass squared of the states, so if we have an overall negative term there, tachyons will be present in the spectrum.

We know the close form for all terms except for the sum over the $T^4/\mathbb{Z}_2$ lattice. In this example with $u = \tilde{u} = (0, 0, 1, 0)$ the action of the group over the bosonic contributions for the T^4, following (4.2.1), is trivial, since $u_j \in \mathbb{Z} \forall j$. This means that the lattice does not change after the orbifold compactification. The bosonic part is summarised as:

$$Z_B[k, l](\tau, \overline{\tau}) = \frac{1}{|\eta|^{16}} \frac{1}{\sqrt{\tau_2}^3} \left[\sum_{(p_L, p_R) \in \Gamma_{4,4}} q^{\frac{\alpha'}{4} p_L^2} \overline{q}^{\frac{\alpha'}{4} p_R^2} \right] \left[\sum_{n,m \in \mathbb{Z}} (-1)^{nl} \, q^{\frac{\alpha'}{2} p_L (n, m + \frac{k}{2})^2} \overline{q}^{\frac{\alpha'}{2} p_R (n, m + \frac{k}{2})^2} \right] . \tag{4.27}$$

The fermionic contribution is directly computed after applying the obtained formulas (4.25)-(4.26), the result reads:

$$Z_F[k, l] = \frac{1}{4} \left| \left(\frac{\vartheta_3}{\eta} \right)^3 \frac{\vartheta \begin{bmatrix} k \\ -l \end{bmatrix}}{\eta} - e^{\pi i k} \left(\frac{\vartheta_4}{\eta} \right)^3 \frac{\vartheta \begin{bmatrix} k \\ -\frac{1}{2} - l \end{bmatrix}}{\eta} - \left(\frac{\vartheta_2}{\eta} \right)^3 \frac{\vartheta \begin{bmatrix} \frac{1}{2} + k \\ -l \end{bmatrix}}{\eta} \right|^2 . \tag{4.28}$$

To analyze the spectrum in search of tachyonic states we can expand the full partition function as discussed in Section 4.4 . Following that description, we describe the partition function in terms of $q, \overline{q}$ ($|q| << 1$, equivalent to approach the IR spectrum), impose the level matching and study the masses of the lowest energy states. As we argued before, the block $Z[0, 0]$ is the untwisted unprojected block, and so it matches the partition function of a general Type IIB compactified in a T^5 already described. We will repeat the analysis, this time for the whole untwisted sector $Z[0, l]$.

The series expansion of the ϑ function with the shift implied by the orbifold reads as:

$$Z_F[k, l] \sim e^{\pi i \sum_{i=3}^4 kl \left(u_i^2 - \tilde{u}_i^2 \right)} \sum_{r, \tilde{r}} q^{\frac{1}{2}(r + ku)^2} \overline{q}^{\frac{1}{2}(\tilde{r} + k\tilde{u})^2} e^{-2\pi i l [(r + ku)u - (\tilde{r} + k\tilde{u})\tilde{u}]} .$$

Where, as in the other case, the range of $r, \tilde{r}$ depend on the choice of the NS or R sector for left and right movers. The overall phase vanishes for symmetric orbifolds. Neglecting the contribution from the T^4 momenta, we can expand the different blocks of the partition function for this $\mathbb{Z}_2$ case as:

$$Z[k, l] = \frac{1}{\sqrt{\tau_2}^3} (q\overline{q})^{-\frac{1}{2}} \sum_{n, w \in \mathbb{Z}} e^{\frac{2\pi i n}{p} l} q^{\frac{\alpha^4}{4} P_R^2(k)} (\overline{q})^{\frac{\alpha'}{4} P_L^2(k)} \sum_{r, \tilde{r}} q^{\frac{1}{2}(r + ku)^2} \overline{q}^{\frac{1}{2}(\tilde{r} + k\tilde{u})^2} e^{-2\pi i l [(r + ku)u - (\tilde{r} + k\tilde{u})\tilde{u}]} . \tag{4.29}$$

Remark that this expansion is not general for all orbifolds, for a big amount of them we should introduce the overall contribution of modular function from the T^4 as (4.20) instead of just adding the η functions as in this case.

4.3.5.1 Untwisted Sector spectrum

For the untwisted sector we have already shown the possible weight vectors: As mentioned, the untwisted sector is only constrained by the GSO projection and allows the states:

Sector	$\tilde{r}, r$
NS	$(\pm\underline{1, 0}, 0, 0)$
	$(0,0,\pm 1, 0)$
	$(0,0,0,\pm 1)$

Sector	$\tilde{r}, r$
R	$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$
	$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$
	$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2}\right)$
	$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right)$

(4.30)

Where the underline means that the written state and the state obtained after exchanging the underlined components are valid.

Now we have to check which states are not projected out and the contributions to the masses. The orbifold charge will be trivial when $r_3 - \tilde{r}_3 \in \mathbb{Z}$, so for any possible tensor product in the NS-NS and R-R sectors will remain in the spectrum while the R-NS and NS-R will need the contribution from the circle to remain in the spectrum, missing level-matching and we can get it again as done before, as a summary:

- R-R & NS-NS Sectors: The possible states will be the tensor product between 2 of all the possible weight vectors in the NS or R respectively. Those states always verify $r^2 = 1$ and so having no winding from the orbifold, $n = 0$, and cancelling the $q^{\frac{1}{2}r^2 - \frac{1}{2}}$ contribution in any case, this means that those states are all massless.
- R-NS & NS-R Sectors: As in the case of the twisted sector the contribution of the orbifold charge must be cancelled by requiring $n \in 2\mathbb{Z} + 1$, but in this case the level matching is trivially confirmed, and so all the possible fermions are also in the spectrum but they all become massive due to the $n = \pm 1$ condition, their mass will be $m = \left|\frac{1}{R}\right|$

4.3.5.2 Twisted sector spectrum

To build the states of the closed string we have to do the tensor product of right and left movers. For the twisted sector we fix $k = 1$:

- NS-NS Sector: The lowest energy state in the NS-NS sector is given by:

  $$(0, 0, -1, 0) \otimes (0, 0, -1, 0) \ .$$

  Its mass will be given by a winding appearing from k=1 and since it is the lowest mass state n=0. It is easy to check where this extra contribution to the mass is arising, as we can see in (4.14), the twisted sectors provide a fractional extra winding contribution to the momenta.

  We have to take into account also that the term $q^{\frac{1}{2}(r-u)^2} = 1$ and so we have an overall factor $q^{-\frac{1}{2}}$ contributing to the momenta in the circle, so the term will read as $q^{\frac{1}{4}\left[\alpha' P_R(0,1/2)^2 - 2\right]}$, which gives a mass square term of:

  $$\alpha' m^2 = \frac{R^2}{4\alpha'} - 2 \ . \tag{4.31}$$

So this state will be tachyonic if $R < 2\sqrt{2\alpha'}$, meaning that instabilities will appear in this theory when the radius of the S^1 is lower than a critical value R_*.

- R-R Sector: First we must check whether those states are projected out due to the orbifolding; in the NS-NS this is not relevant since they have trivial charge under the orbifold, but in this case since $r, \tilde{r} \in \mathbb{Z} + \frac{1}{2}$ this charge may be not trivial. The charge under the orbifold will be given by $e^{-2\pi i l(r_3 - \tilde{r}_3)}$ and so the condition for those states to not be unprotected is $r_3 - \tilde{r}_3 \in \mathbb{Z}$, but in any case this is also trivially verified. On the other hand, looking at the S^1 contribution, states with $r - \tilde{r} \in 2\mathbb{Z}$ will remain massless, while the rest will have extra momentum. The first massive states will be given by:

$$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right) \qquad \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right)$$

$$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \qquad \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

$$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right) \qquad \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{1}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{1}, -\frac{3}{2}, -\frac{1}{2}\right)$$

$$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{3}{2}, -\frac{1}{2}\right) \qquad \left(\pm\frac{1}{2}, \pm\frac{1}{1}, -\frac{1}{2}, -\frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{3}{2}, -\frac{1}{2}\right)$$

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right) \qquad \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{3}{2}, -\frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{1}}, -\frac{1}{2}, \frac{1}{2}\right)$$

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \qquad \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{3}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right) \qquad \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{3}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right)$$

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{3}{2}, -\frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{1}}, -\frac{3}{2}, -\frac{1}{2}\right) \qquad \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{3}{2}, -\frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{1}}, -\frac{3}{2}, -\frac{1}{2}\right)$$

  For the masses we check that the factor $(r-u)^2 = 1$ for all those states both for left and right movers, so this power of $q\bar{q}$ cancells the contribution from the η functions and have no effect in the mass, and so the mass is given by:

$$\alpha' m^2 = \left(\frac{R}{2\sqrt{\alpha'}}\right)^2 \implies m = \left|\frac{R}{2\alpha'}\right| \, . \tag{4.32}$$

  We link this masses with the fractional winding that appears in the twisted sector, as from the S^1 we can read that, going to the twisted sector shifts $m \to m + \frac{1}{2}$ in the momenta of the S^1.

- NS-R & R-NS sectors: In principle, we could have different states coming from the tensor product of the different R sector states times the only one on the NS sector in both sides, but we have to check again if those states get a charge under the orbifold and get projected out.

To check this, we return to the expansions. The term related to the orbifold charge will read as:

$$e^{-2\pi il(r_3-\tilde{r}_3)} = e^{-2\pi il\left(\pm\frac{1}{2}\right)} = e^{\pm\pi il} \ . \tag{4.33}$$

So this states are going to be projected out since they get a non-trivial charge under the orbifold. In the $n=0$ momentum state this occurs, but when $n \neq 0$ this effect can be mixed with the momenta sum over the circle, since as we have already seen there exist a phase in the sum which is exactly $e^{\pi inl}$, so if $n \in 2\mathbb{Z}+1$ those states of the R-NS and NS-R sector are not projected out; so in order to get the lightest states we ask for states with $n=1$ or $n=-1$, the specific value between $+1$ and -1 will be determined may be determined by the specific phases but this case is quite special since both contributions always vanish for any odd n, but we have another constraint, the level matching. By comparing the mass from the left or right movers we have another constraint over n coming from the algebra, by assuring level-matching we just have the following options:

– NS-R Sector:

$$\begin{aligned}
[n= \ -1] \quad & (0,0,-1,0)\otimes\left(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right) \\
[n= \ -1] \quad & (0,0,-1,0)\otimes\left(\underline{\frac{1}{2},-\frac{1}{2}},-\frac{1}{2},\frac{1}{2}\right)
\end{aligned}$$

– R-NS Sector:

$$\begin{aligned}
[n=1] \quad & \left(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\otimes(0,0,-1,0) \\
[n=1] \quad & \left(\underline{\frac{1}{2},-\frac{1}{2}},-\frac{1}{2},\frac{1}{2}\right)\otimes(0,0,-1,0)
\end{aligned}$$

We can compute the mass on these sectors. For all these states the mass is the same, but Let us compute it in one specific combination first to check everything is alright. For example we can take the right movers in the first of the given states, the momentum on the circle reads initially as:

$$P^2_{L/R}(n,k=1;w=0) = \frac{n^2}{R^2}+\frac{R^2}{4\alpha'^2}\pm\frac{n}{\alpha} \ . \tag{4.34}$$

But checking the partition function it receives a correction:

$$\alpha' m_L^2 = \frac{\alpha' n^2}{R^2}+\frac{R^2}{4\alpha'}+n \ +2\left(r+u\right)^2-2 \ , \tag{4.35}$$

$$\alpha' m_R^2 = \frac{\alpha' n^2}{R^2}+\frac{R^2}{4\alpha'}-n \ +2\left(\tilde{r}+u\right)^2-2 \ . \tag{4.36}$$

In the first case $(r+u)^2=1$ and $(\tilde{r}+u)^2=0$ so we get:

$$\alpha' m_L^2 = \frac{\alpha' n^2}{R^2}+\frac{R^2}{4\alpha'}+n \ , \tag{4.37}$$

$$\alpha' m_R^2 = \frac{\alpha' n^2}{R^2}+\frac{R^2}{4\alpha'}-n \ -2 \ . \tag{4.38}$$

So choosing $n = -1$ the states satisfy level-matching so the mass of those states are:

$$\alpha' m^2 = \frac{\alpha'}{R^2} + \frac{R^2}{4\alpha'} - 1 \implies m = \left| \frac{1}{R} - \frac{R}{2\alpha'} \right| \ . \tag{4.39}$$

These states will be massless if $R = \sqrt{2\alpha'} < 2\sqrt{2\alpha'}$, and so the tachyonic bound prevents this states to be massless.

For a general $\mathbb{Z}_p$ symmetric orbifold acting as described, the highest contribution to the energy as it occurs in this example between the twisted sectors arise in the $k = 1$ sector. This allows to establish a bound for the radius of the S^1 to avoid tachyons in our theory which reads as [15]:

$$R \geq \sqrt{2np\alpha'} \qquad n = ||u_3| - |u_4|| \ . \tag{4.40}$$

4.3.6 Example of a Z_2 with $u = \tilde{u} = (0,0,1/2,1/2)$

This is, as performed before, a symmetric orbifold with twist vectors $u = \tilde{u} = (0,0,1/2,1/2)$.

Simplifying a bit the expression and factorising out the η's since we will expand it at the end, following the same procedure as before, we get:[4]

$$Z[k,l] = (\sqrt{\tau_2})^{-3} \frac{1}{(\eta\overline{\eta})^6} \left| \frac{\chi_j[k]}{\vartheta\left[\begin{smallmatrix} \frac{1}{2}-\frac{k}{2} \\ -\frac{1}{2}+\frac{l}{2} \end{smallmatrix}\right]} \right|^4 \times \text{ Sum over momenta on } S^1 \text{ and } T^4 \times$$
$$\times \frac{1}{4} \left| \left(\vartheta_3^2 \vartheta^2 \begin{bmatrix} \frac{k}{2} \\ -\frac{l}{2} \end{bmatrix} - e^{\pi i \sum_{i=3}^4 \frac{k}{2}} \vartheta_4^2 \vartheta^2 \begin{bmatrix} \frac{k}{2} \\ -\frac{1}{2}-\frac{l}{2} \end{bmatrix} \right) - \left(\vartheta_2^2 \vartheta^2 \begin{bmatrix} \frac{1}{2}+\frac{k}{2} \\ -\frac{l}{2} \end{bmatrix} + e^{\pi i \sum_{i=3}^4 k u_i} \vartheta_1^2 \vartheta^2 \begin{bmatrix} \frac{1}{2}+\frac{k}{2} \\ -\frac{1}{2}-\frac{l}{2} \end{bmatrix} \right) \right|^2 \tag{4.41}$$

First let us expand the first factor with the η functions to check the $q\overline{q}$ terms out of the fermionic part. The contribution is given by (In the twisted sector):

$$\frac{1}{4} (\eta\overline{\eta})^{-6} \left| \prod_{j=3}^4 \frac{\chi_j[1]}{\vartheta\left[\begin{smallmatrix} \frac{1}{2}-u_j \\ -\frac{1}{2}+lu_j \end{smallmatrix}\right]} \right|^2 = 4 (\eta\overline{\eta})^{-6} \left(\vartheta\left[\begin{smallmatrix} 0 \\ \frac{l-1}{2} \end{smallmatrix}\right]^{-2} \right) \left(\overline{\vartheta}\left[\begin{smallmatrix} 0 \\ \frac{l-1}{2} \end{smallmatrix}\right]^{-2} \right) . \tag{4.42}$$

Now Let us expand the holomorphic contribution first for simplicity (the anti-holomorphic part is equivalent with the complex conjugate):

$$\begin{aligned} \eta^{-6} \left(\vartheta\left[\begin{smallmatrix} 0 \\ \frac{l-1}{2} \end{smallmatrix}\right]^{-2} \right) &= \eta^{-8} q^{\frac{1}{12}} \prod_{n=1}^{\infty} \left(1 + q^{n-1/2}(-1)^{l-1}\right)^{-4} \\ &= q^{-\frac{1}{4}} \prod_{n=1}^{\infty} (1-q^n)^{-8} \left(1 + q^{n-1/2}(-1)^{l-1}\right)^{-4} \\ &\approx q^{-\frac{1}{4}} \left[1 + (-1)^{(l+1)} 4\sqrt{q} + 18q + (-1)^{(l+1)} 56 q^{3/2} + \mathcal{O}(q^2\right] . \end{aligned} \tag{4.43}$$

So the overall factors for the lightest states will be a $(q\overline{q})^{\frac{1}{4}}$. Indeed we can compute the overall contribution for the $k-$th twisted sector as $E_k - \frac{1}{2}$ with $E_k = \sum_{i \in I} \frac{1}{2} k\{u_i\} (1 - \{u_i\})$.

Now we must expand as a series, as done before, the fermionic contribution:

$$Z_{NS}[k,l] \propto \sum_{r,\tilde{r} \in \mathbb{Z}^4} q^{\frac{1}{2}(r+ku)^2} \overline{q}^{\frac{1}{2}(\tilde{r}+k\tilde{u})^2} e^{-\pi i l \sum_{i=3}^4 (r_i - \tilde{r}_i)} . \tag{4.44}$$

And so in the the twisted sector we have:

$$Z_{NS}[1,l] \propto \sum_{r,\tilde{r} \in \mathbb{Z}^4} q^{\frac{1}{2}(r+u)^2} \overline{q}^{\frac{1}{2}(\tilde{r}+\tilde{u})^2} e^{-\pi i l \sum_{i=3}^4 (r_i - \tilde{r}_i)} . \tag{4.45}$$

So the weight vectors for the lightest states will be given by $r, \tilde{r} \in \{(0,0,-1,0), (0,0,0,-1)\}$ or following the notation for the permutations $(0,0,\underline{-1,0})$.
In the R sector the lightest states will have only two possible vector: $r, \tilde{r} = \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$

[4]Recall $\chi_j[k] = 2\sin(\pi k u_j)$ when $k \neq 0$ and the whole quotient vanish when $k = 0$ in this case.

4.3.6.1 Untwisted Spectrum

As in the non-supersymmetric case we only have to check the orbifold charge and the level-matching. The condition for this orbifold is that the state won't be projected out whenever $\sum_{i=3}^{4}(r_i - \tilde{r}_i) \equiv 0 \ mod \ 2$. This condition is not satisfied for all the possible tensor products in any possible sector, but whenever this condition is not satisfied we can do the same procedure as before for the $R - NS$ and $NS - R$ states, on the other hand since all the possible vectors satisfy $r^2 = 1$ we will have no mismatch between m_R and m_L and there are going also to be no external contributions to the momentum since $E_0 = \overline{E}_0 = 0$, so the overall factor will be $(q\overline{q})^{\frac{1}{2}}$ which cancels; this change of the factor comes from the fact that the term $\propto \frac{\eta}{\vartheta}$ becomes 1 for the $k = 0$ sector.

- NS-NS Sector: First we start with the states given by tensor products with trivial orbifold charge:

$$(\underline{\pm 1, 0}, 0, 0) \otimes (\underline{\pm 1, 0}, 0, 0)$$
$$(0, 0, \pm 1, 0) \otimes (0, 0, \pm 1, 0)$$
$$(0, 0, \pm 1, 0) \otimes (0, 0, 0, \pm 1)$$
$$(0, 0, 0, \pm 1) \otimes (0, 0, \pm 1, 0)$$
$$(0, 0, 0, \pm 1) \otimes (0, 0, 0, \pm 1)$$

  All these states have $n = 0$ and no winding nor external factors so **they are massless**. Then we have massive states coming from the products, setting $n = \pm 1$:

$$(\underline{\pm 1, 0}, 0, 0) \otimes (0, 0, \pm 1, 0)$$
$$(\underline{\pm 1, 0}, 0, 0) \otimes (0, 0, 0, \pm 1)$$
$$(0, 0, \pm 1, 0) \otimes (\underline{\pm 1, 0}, 0, 0)$$
$$(0, 0, 0, \pm 1) \otimes (\underline{\pm 1, 0}, 0, 0)$$

  With masses $m = \left|\frac{1}{R}\right|$

- R-R Sector: We do the same procedure, I am not writing all of them here because they are many different combinations from the given table, but basically we will have massless bosons whenever the weight vectors verify $|r_3 + r_4| = |\tilde{r}_3 + \tilde{r}_4|$ and $m = \left|\frac{1}{R}\right|$ in other case, to give some examples:
  Massless:

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$
$$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2}\right) \otimes \left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right)$$

  Massive:

$$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, \frac{1}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

- R-NS & NS-R Sectors: It is exactly the same as in the latter cases, we will have massless and massive fermions of mass $m = \left|\frac{1}{R}\right|$ depending on if the condition $|r_3 + r_4| = |\tilde{r}_3 + \tilde{r}_4|$ is satisfied or not, I am not listing all of them since we have so many possibilities but here are 4 examples:
Masless:
$$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right) \otimes \left(\underline{\pm 1, 0}, 0, 0\right)$$
Massive:
$$\left(\underline{\frac{1}{2}, -\frac{1}{2}}, -\frac{1}{2}, \frac{1}{2}\right) \otimes (0, 0, 0, \pm 1)$$

4.3.6.2 Twisted Sector spectrum

So we can now construct the possible lowest energy states:

- NS-NS Sector: We have 4 options, summarized as:

$$(0, 0, \underline{-1, 0}) \otimes (0, 0, \underline{-1, 0})$$

The level matching is trivially confirmed since we are in a symmetric orbifold, but we have to check the orbifold charge, in this case for any combination $\sum_{i=3}^{4} (r_i - \tilde{r}_i) = 0$ and so they have trivial charge, so we have no problems with those states, in general this conditions reads as $\sum_{i=3}^{4} (r_i - \tilde{r}_i) \equiv 0 \; mod \; 2$.

The mass of this states is given by:

$$\alpha' m^2 = \frac{R^2}{4\alpha'} \; . \tag{4.46}$$

- R-R Sector: We have four options summarised as:

$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$
$$\left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right) \otimes \left(\pm\frac{1}{2}, \pm\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\right)$$

Again the level-matching is trivially confirmed and the orbifold charge is trivial since the sum gives out 2. The mass of those states is also:

$$\alpha' m^2 = \frac{R^2}{4\alpha'} \; . \tag{4.47}$$

- R-NS & NS-R Sectors: First we check the orbifold charge, this is rather easy since the contribution from the R sector is always going to be -1 and the NS contribution -1 so those states are also not projected out. Then we have to check the level-matching, the contribution to the exponent of the q's are the same, since at the end $r + u = (0, 0, -\frac{1}{2}, \frac{1}{2})[NS]$ or $(\pm\frac{1}{2}, \pm\frac{1}{2}, 0, 0)[R]$ and so its square is the same, giving that the level matching is trivially successful in these states and we do not have to add $n \neq 0$ contributions. So the possible states are:

– R-NS Sector:

$$\left(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)\otimes(0,0,\underline{-1,0})$$

– NS-R Sector:

$$(0,0,\underline{-1,0})\otimes\left(\pm\frac{1}{2},\pm\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)$$

All of them have the same mass as the states on the other sectors.

Chapter 5

Cosmological Constant

5.1 EFT vs. String Perspective

Let us start by refreshing the idea of the vacuum energy from the field theory point of view by taking the effective action action of a free massive scalar field. After performing a Wick rotation the path integral defines the vacuum energy Γ_E:[7]

$$e^{-\Gamma_E} = \int e^{-S_E[\phi]} \mathcal{D}\phi$$

We can evaluate the integral on the right side as a determinant getting to the expression:

$$\Gamma_E = -ln\left[\sqrt{|-\Box + m^2|}\right]$$

With the log-trace formula we can evaluate the terms inside the logarithm. Using momentum representation, we get:

$$\Gamma_E = -\frac{V}{2}\int_{R^D} ln\left(p^2 + m^2\right)\frac{d^D p}{\left(2\pi\right)^D} \ . \tag{5.1}$$

$$\tag{5.2}$$

Where V stands for the volume of he non-compact directions and D runs through the directions of the space-time. Adding several fields introduces a sum over the different particles:

$$\Gamma_E = -\frac{V}{2}\int_{R^D}\sum_{i\in I}\left(-1\right)^{F_i} ln\left(p^2 + m_i^2\right)\frac{d^D p}{\left(2\pi\right)^D} \tag{5.3}$$

$$= \frac{-V}{2}\sum_{i\in I}\int_{R^D}\int_{\mathbb{R}+}\left(-1\right)^{F_i} e^{-2\pi t\left(p^2+m_i^2\right)}\frac{dt}{t}\frac{d^D p}{\left(2\pi\right)^D} \ . \tag{5.4}$$

Where I enumerates all the particles, the F_i corrects the statistics for the fermionic fields with a $-$ sign and the t integral from the second line comes from using the formal identity [20]:

$$ln(A) = -\int_{\mathbb{R}+} e^{-2\pi t A}\frac{dt}{t} \ . \tag{5.5}$$

V_D stands for the volume of the space-time, if we take it into the left hand side we get the vacuum energy density, and so the cosmological constant Λ. We can then transform the sum over the particles as the trace by recalling the result from the algebra:

$$L_0 + \overline{L}_0 = H = \frac{\alpha'}{2}\left(p^2 + m^2\right) \ . \tag{5.6}$$

And introducing the particles as the base $|i>$, and so:

$$\Lambda = \frac{-1}{2}\int_{R^D}\int_{\mathbb{R}+} Tr'\left((-1)^{F_i}\, e^{-\frac{4\pi}{\alpha'}t\left(L_0+\overline{L}_0\right)}\right)\frac{dt}{t}\frac{d^Dp}{(2\pi)^D} \ .$$

Where the trace is denoted by T' since the zero modes are introduced by the p integral. From the algebra we must impose the level-matching condition. We can do so in the integral by adding a delta:

$$\delta\left(L_0 - \overline{L}_0\right) = \int_{\mathbb{R}} e^{2\pi i\theta\left(L_0-\overline{L}_0\right)}d\theta \ . \tag{5.7}$$

And so the one-loop vacuum amplitude reads as:

$$\Lambda = \frac{-1}{2}\int_{R^D}\int_{\mathbb{R}+}\int_{\mathbb{R}} Tr'\left(e^{-\frac{4\pi}{\alpha'}t\left(L_0+\overline{L}_0\right)}\right)e^{2\pi i\theta\left(L_0-\overline{L}_0\right)}\frac{dt}{t}\frac{d^Dp}{(2\pi)^D}d\theta \ . \tag{5.8}$$

Now performing changes of variables we can rewrite this in a similar shape as the partition function on the torus:

$$t = \frac{\alpha'\tau_2}{2}, \qquad \theta = \tau_1 \ . \tag{5.9}$$

$$\Lambda = \frac{-1}{4}\int_{R^D}\int_{\mathbb{R}+}\int_{\mathbb{R}} Tr'\left(e^{2\pi i\left[(i\tau_1-\tau_2)L_0-(i\tau_1+\tau_2)\overline{L}_0\right]}\right)\frac{d\tau_1 d\tau_2}{\tau_2}\frac{d^Dp}{(2\pi)^D} \ . \tag{5.10}$$

And so defining $\tau = \tau_1 + i\tau_2$, $q = e^{2\pi i\tau}$ and denoting $\mathbb{C}^+$ as the upper part of the plane $\{(x, iy)\,|y \geq 0\}$ the result reads as:

$$\Lambda = \frac{-1}{4}\int_{R^D}\int_{\mathbb{C}^+} Tr'\left(q^{L_0}\overline{q}^{\overline{L}_0}\right)\frac{d\tau_1 d\tau_2}{\tau_2^2}\frac{d^Dp}{(2\pi)^D} \ . \tag{5.11}$$

The integral over the momenta will give the 0 mode contribution and so it will result in the partition function inside the integral, so in general:

$$\Lambda = \frac{-1}{4\left(2\pi\sqrt{\alpha'}\right)^D}\int_{\mathbb{C}^+} Z(\tau_1,\tau_2)\frac{d\tau_1 d\tau_2}{\tau_2^2} \ .$$

But this integral will have divergences depending on the particular cases:

- **UV Divergences:** This kind will appear since each bosonic contribution, at least for the free bosons, will read $\propto \left(\sqrt{\tau_2}\eta\overline{\eta}\right)^{-1}$, and so $\tau_2 \to 0 \in \mathbb{C}$ we will have a divergence. This corresponds to the $T \to \infty$ regime and so to a UV divergence.

- **IR Divergences:** In the case of the bosonic string or the presence of tachyons in the IR spectrum lead to divergences when $\tau_2 \to \infty$ will also be present.

For avoiding the UV divergences we can impose a cutoff in a natural way. When we defined the skew parameter τ we dicussed that it belongs to the fundamental domain $\mathcal{F} \equiv \mathbb{C}^+/PSL(2,\mathbb{Z})$ to avoid over-counting; this also agrees with the idea that the whole integral must be modular invariant, since the partition function is modular invariant and the measure $\frac{d\tau}{Im(\tau)^2}$ also. And so the one-loop cosmological constant reads:

$$\Lambda = \frac{-1}{4\left(2\pi\sqrt{\alpha'}\right)^D} \int_{\mathcal{F}} Z(\tau_1,\tau_2)\frac{d\tau_1 d\tau_2}{\tau_2^2} \ . \tag{5.12}$$

Where $\mathcal{F} \equiv \mathbb{C}^+/PSL(2\mathbb{Z}) \equiv \{\tau = \tau_1 + i\tau_2 \mid \tau_1 \in [-1/2, 1/2],\ \tau_2 > 0,\ \tau_1^2 + \tau_2^2 \geq 1\}$ We will denote the integral over the fundamental domain, and so the amplitude of the torus, as Ω:

$$\Omega = \int_{\mathcal{F}} Z(\tau_1,\tau_2)\frac{d\tau_1 d\tau_2}{\tau_2^2} \ . \tag{5.13}$$

5.2 Unfolding Procedure

In order to perform integrals of the shape of (5.13) we have a major problem. The fundamental domain defines a region in which the boundaries of one of the variables is entangled with the other, for example:

$$\Omega = \int_{\mathcal{F}} Z d\mu = \int_{-1/2}^{1/2}\int_{\sqrt{1-\tau_1^2}}^{\infty} \frac{Z(\tau_1,\tau_2)}{\tau_2^2} d\tau_2 d\tau_1 \ . \tag{5.14}$$

Where $d\mu = \frac{d\tau_1 d\tau_2}{\tau_2^2}$ is the invariant under $PSL(2,\mathbb{Z})$ measure. This expression is clearly difficult to approach analytically, and so we would prefer to modify the domain in such a way that we recover an expression close to the result from the field theory in which the domain of integration does not mix both variables. The main problems associated with this are the existence of IR divergences in the integrand ($\tau_2 \to \infty$) and the already mentioned difficult domain of integration.

The solution to the second issue is straightforward; if we manage to mod out part of the symmetry of the integrand by algebraic procedures we may change the domain of integration in order to reduce the $PSL(2,\mathbb{Z})$ into other group G such that the quotient $\mathbb{C}^+/G$ is a domain that does not entangle the variables. In order to do this, we will first focus on the integral of the partition function before performing the orbifold compactification and then we will generalize to the case of orbifolds.

Let us start by focusing on the Ω integral (5.8). The partition function inside the integral is built as a product of modular invariant pieces, one of them is the sum over the momenta lattice $\Lambda_{S^1}(\tau)$, contribution which is expanded as an infinite sum:

$$\Lambda_{S^1}(x) = \sum_{n,m\in\mathbb{Z}} q^{P_R^2}\overline{q}^{P_L^2} = \frac{R}{\sqrt{\alpha'}}\frac{1}{\sqrt{\tau_2}}\sum_{n,m\in\mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2\alpha'}|n+m\tau|^2} \ . \tag{5.15}$$

Now we recall that the $PSL(2,\mathbb{Z})$ group can be represented as:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad a_i \in \mathbb{Z}, \quad ad - bc = 1 \ . \tag{5.16}$$

Focusing on the sum over the lattice we can rewrite, extracting the great common divisor from n, m:

$$\sum_{n,m\in\mathbb{Z}} e^{-\alpha|n+m\tau|^2} = \sum_{p\in\mathbb{Z}} \sum_{c,d\in\mathbb{N}} e^{-\alpha\frac{p^2}{\tau_2}|c+d\tau|^2} \ . \tag{5.17}$$

We can relate each exponential through one specific T transform using transformations $g \in PSL(2,\mathbb{Z})/\mathcal{T}$:[24]

$$\begin{pmatrix} a_1 & a_2 \\ c & d \end{pmatrix} \quad : \quad e^{-\alpha\frac{p^2}{\tau_2}} \xrightarrow{g} e^{-\alpha\frac{p^2}{\tau_2}|c+d\tau|^2} \ . \tag{5.18}$$

And so we can recover the S^1 sum over the lattice as the orbit of this element over $PSL(2,\mathbb{Z})/\mathcal{T}$:

$$\sum_{n,m\in\mathbb{Z}} e^{-\frac{\pi R^2}{\tau_2\alpha'}|n+m\tau|^2} = \sum_{p\in\mathbb{Z}} \sum_{g\in PSL(2,\mathbb{Z})/\mathcal{T}} e^{-\frac{\pi R^2}{\tau_2\alpha'}p^2} = \sum_{g\in PSL(2,\mathbb{Z})/\mathcal{T}} g \circ \tilde{\Lambda}_{S^1} \ . \tag{5.19}$$

Now we directly realize that we have expressed the lattice in such a way that we have a $\mathcal{T}$ invariant term and the sum over the whole orbit over $PSL(2,\mathbb{Z})/\mathcal{T}$. In order to simplify the discussion, we will add it to the Ω integral. To do so we expand:

$$1 = \frac{\Lambda_{S^1}}{\Lambda_{S^1}} = \sum_{g\in PSL(2,\mathbb{Z})} \frac{g \circ \tilde{\Lambda}_{S^1}}{\Lambda_{S^1}} = \sum_{g\in PSL(2,\mathbb{Z})} g \circ \left(\frac{\tilde{\Lambda}_{S^1}}{\Lambda_{S^1}}\right) \ . \tag{5.20}$$

Where in the last equality we used that the sum over the lattice Λ_{S^1} is modular invariant. Inserting this in the Ω equation:

$$\Omega = \int_{\mathcal{F}} Z d\mu = \int_{\mathcal{F}} \sum_{g\in PSL(2,\mathbb{Z})} g \circ \left(\frac{\tilde{\Lambda}_{S^1}}{\Lambda_{S^1}}\right) Z d\mu \ . \tag{5.21}$$

Now, using that both the partition function and the measure are modular invariant, we can act with g^{-1} and they remain the same, and so:

$$\Omega = \int_{\mathcal{F}} \sum_{g\in PSL(2,\mathbb{Z})} g \circ \left(\frac{\tilde{\Lambda}_{S^1}}{\Lambda_{S^1}} Z d\mu\right) = \sum_{g\in PSL(2,\mathbb{Z})} \int_{g^{-1}\circ\mathcal{F}} \frac{\tilde{\Lambda}_{S^1}}{\Lambda_{S^1}} Z d\mu = \int_{\mathbb{C}^+/\mathcal{T}} \tilde{\Lambda}_{S^1} \frac{Z}{\Lambda_{S^1}} d\mu \ . \tag{5.22}$$

A quick inspection gives $\mathbb{C}^+/\mathcal{T} \equiv \{(\tau_1,\tau_2) \in \mathbb{C} \mid \tau_1 \in [-1/2, 1/2],\ \tau_2 > 0\}$, so we have correctly modded out part of the modular invariance and just left an integrand invariant under T transformation, unfolding the domain. Studying the properties of the integrand we realize that it is no longer UV divergence due to the presence of the term $e^{-\alpha p^2\frac{1}{\tau_2}}$; this procedure gives also the correct prescription in order to avoid the UV divergences, the Poisson resummation does not alter the function but ensures the convergence of the integral when $\tau_2 \to 0$. On the other hand, the τ_1 can now be interpreted as the imposition of the level-matching.

To give an example, consider a theory with the contributions coming from two free bosons and a compact one on an S^1. The partition function will read as:

$$Z(\tau_1,\tau_2) = \frac{R}{\sqrt{\alpha'}}\frac{1}{\tau_2^3}\frac{1}{(\eta\overline{\eta})^3}\Lambda_{S^1}(\tau)\,, \qquad \frac{1}{\eta^3} = q^{-\frac{1}{8}}\sum_{n\in\mathbb{N}} a_n q^n,\ a_n = \partial_q \prod_{n=1}^{\infty}(1-q^n)^{-3}\,|_{q=0}\ . \tag{5.23}$$

With the expansion of η functions we can rewrite:

$$Z(\tau_1,\tau_2) = \frac{R}{\sqrt{\alpha'}}\sum_{n,m\in\mathbb{N}} a_n\overline{a}_m \frac{q^{-\frac{1}{8}}\overline{q}^{-\frac{1}{8}}}{\tau_2^3} q^n q^m \Lambda_{S^1}\left(\pi\frac{R^2}{\alpha'}\right)\ . \tag{5.24}$$

Expanding the q's:

$$Z(\tau_1,\tau_2) = \frac{R}{\sqrt{\alpha'}}\frac{1}{\tau_2^3}\sum_{n,m\in\mathbb{N}} a_n\overline{a}_m q^{-\frac{1}{8}} e^{2\pi\tau_2\left(n+m-\frac{1}{4}\right)} e^{2\pi i\tau_1(n-m)}\Lambda_{S^1}\left(\pi\frac{R^2}{\alpha'}\right)\ . \tag{5.25}$$

Now we can perform the integral over the fundamental domain. Following the prescription we developed we just change $\Lambda_{S^1} \to \tilde{\Lambda}_{S^1}$:

$$\Omega = \frac{R}{\sqrt{\alpha'}}\sum_{p\in\mathbb{Z}}\sum_{n,m\in\mathbb{N}} a_n\overline{a}_m \int_0^{\infty}\frac{1}{\tau_2^3}e^{-2\pi\tau_2\left(n+m-\frac{1}{4}\right)}e^{-\pi\frac{R^2}{\tau_2\alpha'}p^2}d\tau_2\int_{-1/2}^{1/2}e^{2\pi i\tau_1(n-m)}d\tau_1\ . \tag{5.26}$$

Now we check explicitly that the τ_1 integral imposes the level matching, in fact, the result of its integral is a delta $\delta_{n,m}$, while the τ_2 can be represented by a Basset function (5.43):

$$\Omega = \frac{\sqrt{\alpha'}}{R}\sum_{p\in\mathbb{Z}}\sum_{n\in\mathbb{N}} a_n^2\frac{8n-1}{p^2}\mathcal{K}_2\left[\sqrt{2}\pi|p|\frac{R}{\sqrt{\alpha'}}\sqrt{8n-1}\right]\ . \tag{5.27}$$

The objective of this example is just to show how to use this unfolding method to compute those kind of amplitudes. We will perform the analysis of the orbifolded theory in the next section. For this case, it can be shown that this amplitude diverges, indeed, a closer inspection of the integral (5.26) shows that in the IR ($\tau_2 \to \infty$), the contributions from $n = 0$ generate an exponential divergence. this is reflected on the final result inside the $\mathcal{K}_2$ function, in which the argument turns imaginary. We link this divergence with the presence of tachyons in the bosonic string spectrum; which leads to vacuum instabilities. We will deeply analyze this problem when discussing the theories on orbifolds.

5.2.1 Extension to orbifolds

As already discussed the partition function of a theory compactified on orbifolds splits in different blocks. The resulting partition function is given by:

$$Z = \frac{1}{p}\sum_{k,l\in[0,p-1]} Z[k,l]\ . \tag{5.28}$$

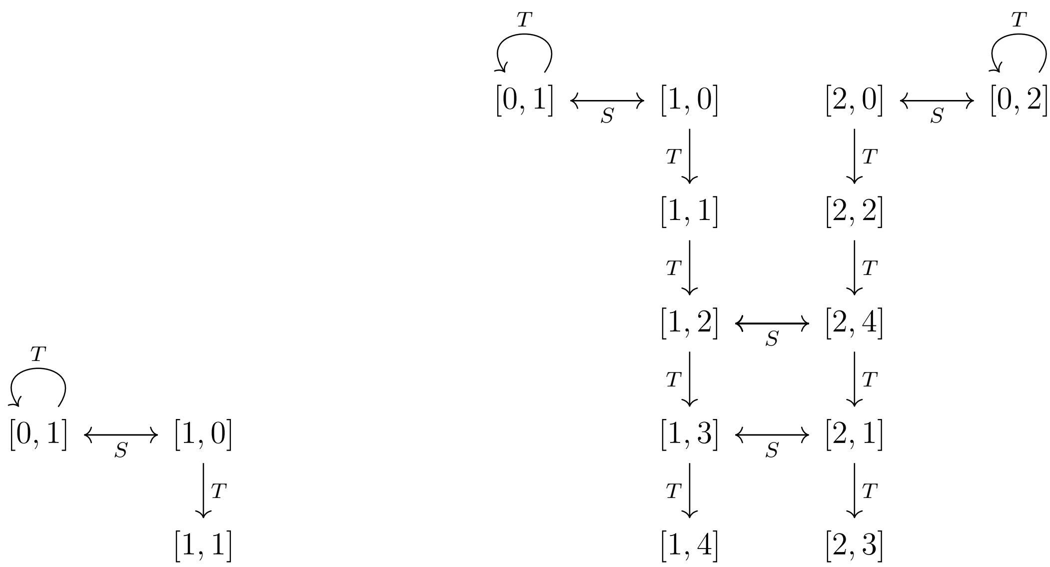
As the orbifold is taken to act over the S^1 direction the sum over the related lattice is modified and so the previous unfolding does not fail to generate an unfolded integral, since the only requirement is the modular invariance of Z, but the introduction of $\tilde{\Lambda}$ only cancells the contribution in $Z[0,0]$, while on the other terms we get impossible to solve (analytically) integrals. The first intuitive movement would be trying to upgrade the proccess in order to use $\tilde{\Lambda}_{S^1}[k,l]$ that cancels each contribution differently, but each $Z[k,l]$ block is not, in general, modular invariant by itself as we have already seen.

The natural extension is looking for blocks that are T invariant and trying to apply a similar method to them and lately relate all the terms to these ones through modular transformations.

In order to give a prescription, Let us assume that the orbifold is a $\mathbb{Z}_p$ with p prime. Our starting point will be expressing the whole partition function as the orbit of an element[1] over a subgroup $G \subset PSL(2,\mathbb{Z})$. We know that this is indeed possible due to the relation (4.24) between the $[k,l]$ blocks under modular transformations, i.e.:

$$Z = \frac{1}{p}\sum_{k,l=1}^{p} Z[k,l] = \frac{1}{p}\sum_{g\in G_{k_0,l_0}\subset PSL(2,\mathbb{Z})} g\circ\left(Z[k_0,l_0]\right) \ . \qquad (5.29)$$

Obviously this set of transformations is determined by the block $[k_0,l_0]$ chosen to represent the sum. Figure 5.1 shows the "descendent" nets from T-invariant terms. The connections between all the terms $[k,l]$ with fixed k through S and T transformation starting from a T-invariant term $[0,k]$.



(a) Net for a $\mathbb{Z}_2$ orbifold. (b) Example of net for a $\mathbb{Z}_5$ orbifold.

Figure 5.1: Examples of descendent nets obtained from modular transformations from $[0,1]$ and $[0,2]$. The connection between the $[0,1]$ and $[0,2]$ descendent nets through S transformations is also showed.

[1]The term [0,0] vanishes so it's implicitly taken out of the sum.

Following the spirit of the latter development, we intend to end with a domain only modded by the $\mathcal{T}$ group, this means that the natural choices for $[k_0, l_0]$ are the ones that are T-invariant. From the examples above, we see that the states $[0, i]$ are natural choices in this context, but any T-invariant term is accepted for this procedure.

Given Z_0, a T-invariant block, we have a well defined G set such that we can recover the full partition function by acting with its elements over Z_0. In this context, there is a uniquely defined group Γ such that[32]:

$$\forall \gamma \in \Gamma, \gamma \circ Z_0 = Z_0 \ , \qquad \sum_{g \in G} g \circ \Gamma = PSL(2, \mathbb{Z}) \ . \tag{5.30}$$

For simplicity, we will focus on the $[0, 1]$ term. Following the discussion for the case without the group quotient, we investigate now the S^1 lattice associated to this term. After the Poisson resummation, reads as:

$$\Lambda_1^p = \sum_{n,m \in \mathbb{Z}} e^{-\frac{\pi R^2}{\alpha' p^2 \tau_2} |pn+1+pm\tau|^2} \ . \tag{5.31}$$

This lattice is invariant under the the so-called $\Gamma_0^1[p]$ subgroup of $PSL(2, \mathbb{Z})$, given by matrices of the modular group with the shape:

$$\Gamma_0^1[p] \equiv \left\langle \begin{pmatrix} pa+1 & b \\ pc & pd+1 \end{pmatrix} \mid ad - bc = 1 \right\rangle \subset PSL(2, \mathbb{Z}) \ . \tag{5.32}$$

As a side remark, as done before, a modular transformation may be represented as:

$$\tau \to \frac{a\tau + b}{c\tau + d}, \quad ad - bc \neq 0, \ a, b, c, d \in \mathbb{Z} \ . \tag{5.33}$$

And so if we represent them as matrices:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \sim \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix} \quad ad - bc \neq 0, \ a, b, c, d \in \mathbb{Z} \ . \tag{5.34}$$

. From this symmetry it follows that the group $\Gamma_0^1[p]$ can be also represented as:

$$\Gamma_0^1[p] \equiv \left\langle \begin{pmatrix} pa-1 & b \\ pc & pd-1 \end{pmatrix} \mid ad - bc = 1 \right\rangle \subset PSL(2, \mathbb{Z}) \ . \tag{5.35}$$

The next step is representing the S^1 sum over the lattice Λ_1^p as the orbit of an element through the group $\Gamma_0^1[p]/\mathcal{T}$. The main problem is that the decomposition is not so straightforward in this case.

For $p = 2, 3$, we can rewrite the lattice as an orbit following:

$$\Lambda_1^p(x) = \sum_{m \in \mathbb{Z}} \sum_{\{(c,d) \in \mathbb{Z}^2 | (pc+1, pd) = 1\}} e^{-x(pm+1)^2 \frac{1}{p^2 \tau_2} |pc+1+pd\tau|^2} = \sum_{j \in \mathbb{Z}} \sum_{\gamma \in \Gamma_0^1[n]/\mathcal{T}} \gamma \circ \left(e^{-x \frac{(pj+1)}{p^2}} \right) \ . \tag{5.36}$$

Indeed for these cases $\sum_{g\in G} g\circ(\Gamma_0^1[p]) = PSL(2,\mathbb{Z})/\mathcal{T}$, which is why we can express the sum in this way. If this does not hold, as we will see later with the $p=5$ example, the expansion of the lattice is slightly different.

For $p>3$ we divide the sum in several terms which are related to the other T invariant terms in the net of $\mathbb{Z}_p$ apart from $Z[0,1]$. In order to illustrate the description we will just do the example for $p=5$. When expanding the lattice $\Lambda_1^5(x)$ we cannot do it directly in such a way that the subgroup Γ that stabilizes the T-invariant term [0,1] verifies $\sum_{g\in G} g\circ(\Gamma) = PSL(2,\mathbb{Z})/\mathcal{T}$. the group that stabilizes this $[0,1]$ element is the one already described $\Gamma_0^1[5]$; but an expansion of the lattice $\Lambda_1^5(x)$ requires two terms, if we want to express the contribution as the orbit of an element through a group as in the latter case:

$$\Lambda_1^{(5)} = \sum_{\substack{m,c,d\in\mathbb{Z}\\(5c+1,5d)=1}} e^{-(5m+1)^2\frac{|5c+1+5d\tau|^2}{25\tau_2}} + \sum_{\substack{m,c,d\in\mathbb{Z}\\(5c+3,5d)=1}} e^{-(5m+2)^2\frac{|5c+3+5d\tau|^2}{25\tau_2}} \,. \tag{5.37}$$

The expansion in two terms is also required algebraically because of the 5 multiplying the c,d in the exponent. The first term is treated in a similar way as in the latter case, by expressing it as the orbit of an element under $\Gamma_0^1[5]$; this idea can also be performed in the second term, but modifying the matrices that we are multiplying to the main element.

One can easily find that the subgroup we are looking for is the one given by matrices of the shape:

$$\begin{pmatrix} 5a+3 & b \\ 5c & 5d+2 \end{pmatrix} \sim \begin{pmatrix} 5a-3 & b \\ 5c & 5d-2 \end{pmatrix} \,. \tag{5.38}$$

The matrices of this shape form a subgroup of $PSL(2,\mathbb{Z})$, usually denoted as $\Gamma_0^2[5]$. Its generalization to $p\neq 5$ is trivial, just substituing $5\to p$.

Having this setup, we can think about how does choosing a $T-$invariant block different than the $[0,1]$ affects this procedure. In principle, we have no reason to take one particular T-invariant block, the result should be equivalent for all of them. With this idea, we can expect to have different contributions from the different T-invariant terms of the net. In the net of $\mathbb{Z}_5$ we have two suitable elements for this description, the first is the basis of the previous development $[0,1]$ while the second is $[0,2]$; checking the net diagram, these elements are related by a transformation $h = ST^2ST^3S \in PSL(2,\mathbb{Z})$; $h\circ Z[0,1] = Z[0,2]$. This specific transformation can be linked with the $\Gamma_0^1[5]$ and $\Gamma_0^2[5]$ groups:

$$ST^2ST^3S\circ\Gamma_0^1[5]/\mathcal{T} = \Gamma_0^2[5] \quad \left[ST^2ST^3S, \Gamma_0^1[5]\right] = 0 \,. \tag{5.39}$$

Following this, we can repeat the expansion as in the other cases expressing the two terms as:

$$\tilde{\Lambda}_i^{(n)}(x) = \sum_{p\in\mathbb{Z}} e^{-\frac{x}{\tau_2}\left(p+\frac{i}{n}\right)^2} \implies \Lambda_1^{(5)} = \sum_{\gamma\in\Gamma_0^1[5]/\mathcal{T}} \gamma\circ\left[\tilde{\Lambda}_1^{(5)}(x) + h\circ\tilde{\Lambda}_2^{(5)}(x)\right] \,. \tag{5.40}$$

But if we think about all the possible blocks $Z[k,l]$ that may appear in a $\mathbb{Z}_5$ we may also think that we would expect similar terms arising for $[0,3]$ and $[0,4]$, since they are also

T-invariant, but as we already mentioned, the Γ groups for $[0,3]$ and $[0,4]$ are exactly the same as for $[0,2]$ and $[0,1]$ respectively as seen in (5.38) and (5.35), which means that we are already taking them into account.

Returning into the Ω integral, we can express the quotient $h \circ \left(\frac{Z[0,1]}{\Lambda_1^{(5)}}\right) = \frac{Z[0,2]}{\Lambda_2^{(5)}}$, and follow the unfolding procedure:

$$
\begin{aligned}
5\Omega &= \int_{\mathcal{F}} \sum_{g\in G} g \circ (Z[0,1])\, d\mu = \int_{\mathcal{F}} \sum_{g\in G} g \circ \left(\frac{\Lambda_1^{(5)}}{\Lambda_1^{(5)}} Z[0,1]\right) d\mu \\
&= \sum_{g\in G,\ \gamma\in\Gamma_0^1[5]} \int_{\mathcal{F}} g\gamma \circ \left\{ \frac{Z[0,1]}{\Lambda_1^{(5)}} \left[\tilde{\Lambda}_1^{(5)} + h \circ \left(\tilde{\Lambda}_2^{(5)}\right)\right] \right\} d\mu \\
&= \sum_{g\in G,\ \gamma\in\Gamma_0^1[5]} \int_{\mathcal{F}} g\gamma \circ \left\{ \frac{Z[0,1]}{\Lambda_1^{(5)}} \tilde{\Lambda}_1^{(5)} + h \circ \left(\frac{Z[0,2]}{\Lambda_2^{(5)}} \tilde{\Lambda}_2^{(5)}\right) \right\} d\mu \\
&= \sum_{g\in G,\ \gamma\in\Gamma_0^1[5]} \int_{\mathcal{F}} g\gamma \circ \left(\frac{Z[0,1]}{\Lambda_1^{(5)}} \tilde{\Lambda}_1^{(5)}\right) + g\gamma h \circ \left(\frac{Z[0,2]}{\Lambda_2^{(5)}} \tilde{\Lambda}_2^{(5)}\right) d\mu \\
&= \sum_{g\in G,\ \gamma\in\Gamma_0^1[5]} \int_{\mathcal{F}} g\gamma \circ \left(\sum_{i=1}^{2} \frac{Z[0,i]}{\Lambda_i^{(5)}} \tilde{\Lambda}_i^{(5)}\right) d\mu = \int_{\mathbb{H}^+} \sum_{i=1}^{2} \frac{Z[0,i]}{\Lambda_i^{(5)}} \tilde{\Lambda}_i^{(5)} d\mu \ .
\end{aligned}
\tag{5.41}
$$

Where the last line is obtained by using $[\gamma, h] = 0$ as already stated, and $h \in G$, since $[0,1] \xrightarrow{h} [0,2]$, so it is contained in this set by definition. And so we can express in the integral over the fundamental domain as:

$$
\Omega = \frac{1}{5} \sum_{i=1}^{2} \int_{\mathbb{H}^+} \frac{Z[0,i]}{\Lambda_i^{(5)}} \tilde{\Lambda}_i^{(5)} d\mu \ . \tag{5.42}
$$

For any orbifold over the group $\mathbb{Z}_5$, obviously this method is intended to simplify the partition functions arising in our calculations and that is why the lattice that we are dividing by is exactly the one from the contributions in the terms $Z[0,i]$.

Generalizing the previous discussion to $p \in \mathbb{N}^*, p > 2$ is quite intuitive. We focus on those descending chains of blocks like the one arising from $Z[0,1]$ or $Z[0,2]$ in the previous example; since we know it should split the sum as in (5.27), and so we focus on the $Z[0,l]$ such that l is coprime with p and the rest of the T-invariant terms already considered, and then we repeat exactly the same procedure. By doing so we get the following formula:[32]

$$
\Omega = \int_{\mathcal{F}} \frac{1}{n} \sum_{i,j=0}^{p} Z[i,j] d\mu = \int_{\mathbb{H}^+} \frac{1}{n} \sum_{i\in I} \frac{Z[0,i]}{\Lambda_i^{(n)}} \tilde{\Lambda}_i^{(n)} \frac{d\tau_1 d\tau_2}{\tau_2^2} \ . \tag{5.43}
$$

Where $I \equiv \left\{ i \in \mathbb{N}^* \mid gcd(i,n) = 1 \ \wedge \ i < \frac{1}{2}(n-1) \right\}$.

The most interesting consequence of the formula is that, not only the integral is simplified avoiding contours difficult to compute analytically; but it also shows an interesting property. The only blocks of the partition function that we need in order to compute the vacuum-amplitude are in the untwisted sector, and within these, we only need the ones coming from projections whose orders are coprime with the order of the group and less than its half. This reduces substantially the number of terms one should compute to get this magnitude. For example, for the case of a $\mathbb{Z}_9$, we would potentially have 80 different blocks we should introduce in the initial integral. After the unfolding, we only need to compute 2 terms, $Z[0,1]$ and $Z[0,2]$.

On the other hand, it also simplifies the treatment of the lattices while computing this integral. In some cases the initial Narain lattice $\Gamma_{4,4}$ that appears in the untwisted sector is broken in the twisted sectors by the action of the orbifold; but applying the unfolding procedure we do not need to use the twisted sector blocks, and so we will not have to study those lattices in order to compute the vacuum amplitude.

5.3 Example for $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$

Let us start by a simple example. Consider a Type IIB string theory compactified in $\mathbb{R}^{1,9} \times S^1/\mathbb{Z}_2$. Without the orbifold, we just have 7 bosonic free contributions and 1 compact contribution on a S^1 equivalent to the discussed in Section 3.2.2;. The $Z[0,0]$ block vanishes due to the presence of supersymmetry. To breaK SUSY then we act over the S^1 with a translation orbifold and over the fermionic coordinates with a fermion index $(-1)^F$ operator. In practice, we already know how to compute its partition function, since the action of the group is equivalent to the torus action over fermions with the $\mathbb{Z}_2$.

Having this framework, we already know that the partition function splits in different blocks and how to obtain the cosmological constant:

$$Z(\tau_1,\tau_2) = \frac{1}{2}\sum_{k,l=0}^{1} Z[k,l](\tau_1,\tau_2), \qquad \Lambda = -\frac{1}{4\left(2\pi\sqrt{\alpha'}\right)^9}\int_{\mathcal{F}} Z(\tau_1,\tau_2)d\mu \; . \tag{5.44}$$

Then, in order to get the cosmological constant arising for this model, we are interested in the integral; given the unfolding procedure we can rewrite:

$$\Omega = \int_{\mathcal{F}} Z(\tau_1,\tau_2)d\mu = \frac{1}{2}\int_{\mathbb{H}^+} \frac{Z[0,1]}{\Lambda_1^{(2)}\left(\frac{\pi R^2}{\alpha'}\right)} \sum_{j\in\mathbb{Z}} e^{-\frac{\pi R^2}{\alpha'\tau_2}\left(n+\frac{1}{2}\right)^2} \; . \tag{5.45}$$

Recalling the result for the orbifold, we get that the [0,1] blovk of the partition function is expressed as:

$$Z[0,1] = Z_{\mathbb{R}^{1,8}} \times Z_{S^1}[0,1] \times Z_F[0,1] \; . \tag{5.46}$$

$$Z_{\mathbb{R}^{1,8}} = \left(\sqrt{\tau_2}\,|\eta|^2\right)^{-7}; \; Z_{S^1} = \frac{1}{|\eta|^2}\frac{R}{\sqrt{\alpha'\tau_2}}\Lambda_1^{(2)}\left(\frac{\pi R^2}{\alpha'}\right) \; ; \; Z_F = \left|\chi_0 + \chi_{\frac{1}{16}}\right|^2 \; . \tag{5.47}$$

While the fermionic part and the η contributions can be expanded in terms of Taylor series over q:[2]

$$\chi_0+\chi_{\frac{1}{16}}=\frac{1}{2}\left[\left(\frac{\vartheta_3}{\eta}\right)^4-\left(\frac{\vartheta_4}{\eta}\right)^4+\left(\frac{\vartheta_2}{\eta}\right)^4\right]=\left(\frac{\vartheta_2}{\eta}\right)^4, \qquad \left|\frac{\chi_0+\chi_{\frac{1}{16}}}{\eta^8}\right|^2=\left|\frac{\vartheta_2}{\eta^3}\right|^8 . \tag{5.48}$$

$$\frac{\vartheta_2}{\eta^3}=\frac{2\eta q^{1/12}\prod_{n=1}^{\infty}(1+q^n)^2}{\eta q^{1/12}\prod_{n=1}^{\infty}(1-q^n)^2}=2\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^2 \implies \left(\frac{\vartheta_2}{\eta^3}\right)^4=2^4\sum_{n=0}^{\infty}A_nq^n . \tag{5.49}$$

And so the expansion leads to:

$$\left|\frac{\chi_0+\chi_{\frac{1}{16}}}{\eta^8}\right|^2=2^8\sum_{n,m\in\mathbb{N}}A_n\overline{A}_mq^n\overline{q}^m, \qquad A_n=\partial_q^n\Big|_{q=0}\prod_{n=1}^{\infty}\left(\frac{1+q^n}{1-q^n}\right)^8 . \tag{5.50}$$

Returning to the Ω integral, then we have an expanded version:

$$\Omega=2^7\frac{R}{\sqrt{\alpha'}}\sum_{j\in\mathbb{Z}}\sum_{n,m\in\mathbb{N}}A_n\overline{A}_m\int_{\mathbb{H}^+}\frac{1}{\tau_2^6}e^{-2\pi\tau_2(n+m)+2\pi i\tau_1(n-m)}e^{-\frac{\pi R^2}{\alpha'\tau_2}\left(j+\frac{1}{2}\right)^2}d\tau_1d\tau_2 . \tag{5.51}$$

Where the first exponential term arises from the $q^n\overline{q}^m$ term from the fermionic expansion. Now we can perform the τ_1 and τ_2 integrals separately because of the $\mathbb{H}^+$ domain. The τ_1 integral involves just one term:

$$\int_{-\frac{1}{2}}^{\frac{1}{2}}e^{2\pi i(n-m)}d\tau_1\propto sin\left[\pi\left(n-m\right)\right] \implies \int_{-\frac{1}{2}}^{\frac{1}{2}}e^{2\pi i(n-m)}d\tau_1=\delta_{n,m} . \tag{5.52}$$

On the other hand the τ_2 integral is split in two parts, the first one refers to the term $n=m=0$ and the second one compiles all the other terms and gives out a modified Bessel function of the second kind, also called Basset function, defined as:

$$K_\nu(z)=\frac{\pi^{\frac{1}{2}}\left(\frac{1}{2}z\right)^\nu}{\Gamma\left(v+\frac{1}{2}\right)}\int_1^\infty \mathrm{e}^{-zt}\left(t^2-1\right)^{\nu-\frac{1}{2}}\,\mathrm{d}t=\frac{1}{2}\left(\frac{1}{2}z\right)^\nu\int_0^\infty\exp\left(-t-\frac{z^2}{4t}\right)\frac{\mathrm{d}t}{t^{\nu+1}} . \tag{5.53}$$

Introducing the $\delta_{n,m}$ in the sum and the τ_2 integral we get the final shape of this integral:

$$\Omega=\frac{2^{11}\,31}{3^3\;7\;5}\pi^5\left(\frac{\sqrt{\alpha'}}{R}\right)^9+\frac{2^{18}}{\pi^{5/2}}\left(\frac{\sqrt{\alpha'}}{R}\right)^4\sum_{\substack{M\in\mathbb{N}^*\\ j\in 2\mathbb{Z}+1}}|A_M|^2\left(\frac{\sqrt{\pi M}}{|j|}\right)^5 K_5\left[2\pi\frac{R}{\sqrt{\alpha'}}|j|\sqrt{M}\right] . \tag{5.54}$$

The first term, that goes as $\frac{1}{R^9}$ is related to the low-energy theory, while the rest of the contributions are contained in the infinite sum of the second term. The index of the sum are M, which runs over the different mass levels, and j which arised from the momenta contribution of the S^1. This function is definely positive, as any of the terms in the sum can be negative and the first term is always non-zero except for the limit $R\to 0$, in

[2]We have used the Riemann identity for the first equality

which we decompactify the S^1 recovering a supersymmetric spectrum. The $\mathcal{K}_5$ is closely linked with the number of non-compact directions, as the $5 = 1 + \frac{D-1}{2}$ where $D = 9$ is the number of non-compact directions; this comes from the prefactor $\frac{1}{\tau_2^6} = \frac{1}{\tau_2^{2+\frac{D-1}{2}}}$ in the integral.

The A_M correspond to the number of string states in the M mass level, as it arises from the expansion of $\frac{\chi_0+\chi_{\frac{1}{16}}}{\eta^3}$, which is the fermionic contribution; so it reflects the number that arises when expanding the partition function with the term q^M.

5.3.1 Hagedorn phase transition

From the analysis of the partition function in Section 4.4 we already know that the lowest energy states in the $k = 1$ twisted sector may be tachyonic depending on the radius of the S^1 compactification, particularly:

$$R < \sqrt{8\alpha'} \implies \text{Tachyonic} \qquad R = \sqrt{8\alpha'} \implies \text{Massless} \qquad R > \sqrt{8\alpha'} \implies \text{Massive}$$

So we expect that in the case of $R < \sqrt{8\alpha'}$ the integral (5.13) should diverge due to the presence of tachyons in the spectrum. From the previous analysis we realize that the term $M = 0$ does not diverge for any value of the radius, so the divergence should appear in the contribution given by $M > 0$ terms.

We then focus on the second term in (5.54):

$$\sum_{M\in\mathbb{N}^*}\sum_{j\in\mathbb{Z}} a_{j,M}, \qquad a_{j,M} = |A_M|^2\left(\frac{\sqrt{M}}{|2j+1|}\right)^5 K_5\left[2\pi|2j+1|\frac{R}{\sqrt{\alpha'}}\sqrt{M}\right] . \tag{5.55}$$

It can be easily seen that, for a fixed $M \in \mathbb{N}^*$, $\lim_{n\to\infty} a_{j,M} = 0$, so the divergences are not probably coming from this sum, on the other hand, in the M limit there are two possible sources of divergence, the coefficients A_M and the $\sqrt{M}^5$ term.

So we know that the expected divergence is secretly hidden in the M sum. Now we can try to get this behaviour analytically. To do so, we can expand the K_ν modified Bessel function of the second kind:

$$K_5(x) = \frac{\sqrt{\pi}}{\Gamma\left(\frac{11}{2}\right)}\left(\frac{x}{2}\right)^5\int_1^\infty e^{-xt}\sqrt{t^2-1}^9 . \tag{5.56}$$

So in our case the function is expanded as:

$$K_5\left[2\pi|2n+1|\frac{R}{\sqrt{\alpha'}}\sqrt{\pi M}\right] = \frac{\pi^5\sqrt{\pi}}{\Gamma(11/2)}\left(|2n+1|\frac{R}{\sqrt{\alpha'}}\right)^5 M^{5/2}\int_1^\infty e^{-t\left(2\pi|2n+1|\frac{R}{\sqrt{\alpha'}}\sqrt{M}\right)}\sqrt{t^2-1}^9 dt . \tag{5.57}$$

From this perspective the convergence of the M sum will be determined by comparing the divergence coming from the A_M coefficients and the terms with M on the last equation:

$$a_{n,M} = |A_M|^2\frac{\pi^5\sqrt{\pi}}{\Gamma(11/2)}\left(\frac{R}{\sqrt{\alpha'}}\right)^5 M^5\int_1^\infty e^{-t\left(2\pi|2n+1|\frac{R}{\sqrt{\alpha'}}\sqrt{M}\right)}\sqrt{t^2-1}^9 dt . \tag{5.58}$$

We can approximate the behaviour of the Taylor coefficients A_M when M is large, so in order to check if those terms diverge we can specify:[3]

$$A_M \approx \frac{1}{M^{11/4}} e^{\pi\sqrt{8M}} \ . \tag{5.59}$$

This means that the number of string states grows exponentially with the energy. For large M we get:

$$a_{j,M} = \frac{\pi^5\sqrt{\pi}}{\Gamma(11/2)\sqrt{M}} \left(\frac{R}{\sqrt{\alpha'}}\right)^5 \int_1^\infty e^{2\pi\sqrt{8M} - t\left(2\pi|2j+1|\frac{R}{\sqrt{\alpha'}}\sqrt{M}\right)} \sqrt{t^2-1}^9 dt \ . \tag{5.60}$$

In the limit $M \to \infty$ the convergence is given by the negativeness of the power of the exponential:

$$2\pi\sqrt{8M} - t\left(2\pi|2j+1|\frac{R}{\sqrt{\alpha'}}\sqrt{M}\right) \leq 0 \ . \tag{5.61}$$

This must hold for any j, which is verified if it is for $j = 0$. On the other hand $t > 1$ in the whole domain of the integral, and so the most restrictive case is $t = 1$. Following this ideas, the inequality that ensures the convergence is:

$$\sqrt{8} \leq \frac{R}{\sqrt{\alpha'}} \implies R \geq 2\sqrt{2\alpha'} \ . \tag{5.62}$$

Recovering the tachyonic bound that we already got from the study of the spectrum, showing that there are not IR divergencies over this bound.

The underlying theormodynamics here hide a so-called Hagedorn phase transition [16].

To give a brief picture, we started with the idea of fundamental vibrating strings which may have an instability but, performing the series expansion and the Poisson resummation, we sent this divergences to the high-energy regime. We are now describing closed strings in which the instability, appears now when heating up the system. At some critical temperature given by the point of this critical radius value, strings are copiously produced leading to a divergence in the number of states and so in terms of the energy. When the radius is small enough, the density of states decays after some value so the system presents no divergence.
rature from the typical partition function perspective.

5.4 Example for $\left(T^4 \times S^1\right)/\mathbb{Z}_2$

The procedure is exactly as the latter case, we only have to focus on the $Z[0,1]$ in order to perform the integral Ω. In this case the T^4 compactification gives out a sum over the momentum lattice, so this block reads as:

$$Z[0,1] = \frac{R}{\sqrt{\alpha'}}\frac{1}{\tau_2^2} \sum_{(p_L,p_R)\in\Gamma_{4,4}} q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2} \times \Lambda_1^{(2)}\left(\frac{\pi R^2}{\alpha'}\right) \times \left|\frac{\chi_0 + \chi_{\frac{1}{16}}}{\eta^8}\right|^2 \ . \tag{5.63}$$

[3]This relation is stated in [2], but also is known as the Cardy's Formula.

In order to perform the integral we need to give an explicit expression for the lattice $\Gamma_{4,4}$. In this case, we have a similar case as the last one, with the difference that we have less bosonic free contributions and we have the sum over the torus lattice. Performing the same unfolding as before we start with the expression:

$$\Omega = \frac{1}{2}\int_S \frac{Z[0,1]}{\Lambda_1^{(2)}}\tilde{\Lambda}_1^{(2)} d\mu \ . \tag{5.64}$$

Again we can expand the terms:

$$Z[0,1] = Z_{\text{free}} \times Z_{T^4} \times Z_{S^1}[0,1] \times |\chi_0 + \chi_{\frac{1}{16}}|^2 \tag{5.65}$$

$$= \frac{1}{(\sqrt{\tau_2}|\eta|^2)^3} \frac{\sum_{(p_L,p_R)\in\Gamma_{4,4}} q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2}}{|\eta^4|^2} \frac{R}{\sqrt{\alpha'\tau_2}|\eta|^2}\Lambda_1^{(2)}(\frac{\pi R^2}{\alpha'})|\chi_0 + \chi_{\frac{1}{16}}|^2 \tag{5.66}$$

$$= \frac{R}{\sqrt{\alpha'}}\frac{1}{\tau_2^2} \times \sum_{(p_L,p_R)\in\Gamma_{4,4}} q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2} \times \Lambda_1^{(2)}(\frac{\pi R^2}{\alpha'}) \times \left|\frac{\chi_0 + \chi_{\frac{1}{16}}}{\eta^8}\right|^2 \ . \tag{5.67}$$

And so the Ω reads as:

$$\Omega = \frac{R}{2\sqrt{\alpha'}}\int_S \frac{1}{\tau_2^4} \times \sum_{(p_L,p_R)\in\Gamma_{4,4}} q^{\frac{\alpha'}{4}p_L^2}\bar{q}^{\frac{\alpha'}{4}p_R^2} \times \tilde{\Lambda}_1^{(2)}\left(\frac{\pi R^2}{\alpha'}\right) \times \left|\frac{\chi_0 + \chi_{\frac{1}{16}}}{\eta^8}\right|^2 d\tau_2 d\tau_1 \ . \tag{5.68}$$

The unfolding is defined exactly as in the latter case, there is no difference in the process from the algebraic point of view since the extra factor Z_{T^4} is itself modular invariant as the lattice is required to be Lorentzian even self-dual. Now, the contributions to τ_1 arise also from the sum over the torus lattice, so the integral over τ_1 must include this contribution.

From the study of the algebra over lattices, we know that the geometry of the lattice is defined by its background field G. With respect to them we can write a closed form for generic $p_{L/R}^2$:[7]

$$p_{L/R}^2 = \vec{m}^t G^{-1}\vec{m} + \frac{1}{\alpha'^2}\vec{n}^t G\vec{n} \pm \frac{2}{\alpha'}\vec{n}^t\vec{m} \quad \vec{n},\vec{m}\in\mathbb{Z}^4 \ . \tag{5.69}$$

To perform this calculation we will fix a lattice Γ_0, in which the metric is diagonal. In general a general $\Gamma_{4,4}$ can be related with this Γ_0 lattice by a roation in $SO(4,4)$. The theory we are describing is symmetric under $SO(4)\times SO(4)$ (rotations in the left and right lattices independently), and so selecting a point in the space $SO(4,4)/SO(4)\times SO(4)$ is choosing a specific point in this moduli space. For dimensional reasons we will choose $G = \alpha' Id$, in this context the square of the chiral momentum can be explicitly computed:

$$p_{L/R}^2 = \frac{1}{\alpha'}\vec{m}^2 + \frac{1}{\alpha'}\vec{n}^2 \pm \frac{2}{\alpha'}\vec{n}^t\vec{m} \ . \tag{5.70}$$

And so:

$$\frac{\alpha'}{4}p_{L/R}^2 = \frac{1}{4}\left(\vec{m}^2 + \vec{n}^2\right) \pm \frac{1}{2}\vec{n}^t\vec{m} \quad \vec{n},\vec{m}\in\mathbb{Z}^4 \ . \tag{5.71}$$

The contribution from the lattice is then expanded as:

$$\sum_{(p_L,p_R)\in\Gamma_{4,4}} q^{\frac{\alpha'}{4}p_L^2}\overline{q}^{\frac{\alpha'}{4}p_R^2} = \sum_{(p_L,p_R)\in\Gamma_{4,4}} e^{-\pi\tau_2\left(\frac{\alpha'}{2}p_L^2+\frac{\alpha'}{2}p_R^2\right)}e^{-\pi i\tau_1\left(\frac{\alpha'}{2}p_L^2-\frac{\alpha'}{2}p_R^2\right)} = \sum_{\vec{n},\vec{m}\in\mathbb{Z}^4} e^{-\pi\tau_2\left(\vec{m}^2+\vec{n}^2\right)}e^{-2\pi i\tau_1\left(\vec{n}^t\vec{m}\right)} . \tag{5.72}$$

The contribution to the τ_1 integral then vanishes, since $\vec{n}^t\vec{m}\in\mathbb{Z}$ the contribution to the τ_1 integral is neglected, since after the integral the factor becomes a factor $e^{\pm 2\pi z}$ $z\in\mathbb{Z}$, so we only have to change the τ_2 contribution.

Explicitly the Ω integral is expressed as:

$$\Omega = \frac{R}{2\sqrt{\alpha'}}\int_S \frac{1}{\tau_2^4}\times \sum_{(p_L,p_R)\in\Gamma_{4,4}} q^{\frac{\alpha'}{4}p_L^2}\overline{q}^{\frac{\alpha'}{4}p_R^2} \times\tilde{\Lambda}_1^{(2)}\left(\frac{\pi R^2}{\alpha'}\right)\times\left|\frac{\chi_0+\chi_{\frac{1}{16}}}{\eta^8}\right|^2 d\tau_2 d\tau_1 \tag{5.73}$$

$$= \frac{R}{2\sqrt{\alpha'}}\int_S \frac{1}{\tau_2^4}\times \sum_{\vec{n},\vec{m}\in\mathbb{Z}^4} e^{-\pi\tau_2\left(\vec{m}^2+\vec{n}^2\right)}e^{-2\pi i\tau_1\left(\vec{n}^t\vec{m}\right)} \times\tilde{\Lambda}_1^{(2)}\left(\frac{\pi R^2}{\alpha'}\right)\times\left|\frac{\chi_0+\chi_{\frac{1}{16}}}{\eta^8}\right|^2 d\tau_2 d\tau_1 . \tag{5.74}$$

As computed before:

$$\left|\frac{\chi_0+\chi_{\frac{1}{16}}}{\eta^8}\right|^2 = \sum_{N,M\in\mathbb{N}} A_N\overline{A}_M q^N\overline{q}^M = \sum_{N,M\in\mathbb{N}} A_N\overline{A}_M e^{-2\pi\tau_2(N+M)}e^{2\pi i\tau_1(N-M)} . \tag{5.75}$$

And so we can expand the integral as: .

$$\Omega = \frac{R}{2\sqrt{\alpha'}}\sum_{\substack{\vec{n},\vec{m}\in\mathbb{Z}^4\\ N,M\in\mathbb{N}}} A_N\overline{A}_M\int_S\frac{1}{\tau_2^4}e^{-\pi\tau_2\left(\vec{m}^2+\vec{n}^2+2N+2M\right)}e^{-2\pi i\tau_1\left(N-M+\vec{n}^t\vec{m}\right)} \times\tilde{\Lambda}_1^{(2)}\left(\frac{\pi R^2}{\alpha'}\right) d\tau_2 d\tau_1 . \tag{5.76}$$

The level-matching integral now gives out a different delta $\delta_M^{N+\vec{n}^t\vec{m}}$, introducing it in the sum:

$$\Omega = \frac{R}{2\sqrt{\alpha'}}\sum_{\substack{\vec{n},\vec{m}\in\mathbb{Z}^4\\ N\in\mathbb{N}\\ j\in\mathbb{Z}}} A_N\overline{A}_{N+\vec{n}^t\vec{m}}\int_0^\infty\frac{1}{\tau_2^4}e^{-\pi\tau_2\left[(\vec{n}+\vec{m})^2+4N\right]}\ e^{\frac{-\pi R^2}{\alpha'\tau_2}(j+1/2)^2}d\tau_2 . \tag{5.77}$$

The latter expression can be directly integrated. We must distinguish between the term $[(\vec{n}+\vec{m})^2+4N]$ vanishing or not, this leads to:

$$\frac{2\sqrt{\alpha'}}{R}\Omega = \sum_{\substack{\vec{n},\vec{m}\in\mathbb{Z}^4\\ N\in\mathbb{N}\\ j\in\mathbb{Z}}} A_N\overline{A}_{N+\vec{n}^t\vec{m}}\int_0^\infty\frac{1}{\tau_2^4}e^{-\pi\tau_2\left[(\vec{n}+\vec{m})^2+4N\right]}\ e^{\frac{-\pi R^2}{\alpha'\tau_2}(j+1/2)^2}d\tau_2 \tag{5.78}$$

$$= \frac{2^7}{\pi^3}\left(\frac{\sqrt{\alpha'}}{R}\right)^6\sum_{j\in\mathbb{Z}}(1+2j)^{-6} \tag{5.79}$$

$$+\,2^4\pi^{3/2}\left(\frac{\sqrt{\alpha'}}{R}\right)^3{\sum_{\substack{\vec{n},\vec{m}\in\mathbb{Z}^4\\ N\in\mathbb{N}\\ j\in 2\mathbb{Z}+1}}}' A_N\overline{A}_{N+\vec{n}^t\vec{m}}\left[\frac{\sqrt{(\vec{n}+\vec{m})^2+4N}}{|j|}\right]^3\mathcal{K}_3\left[\pi\frac{R}{\sqrt{\alpha'}}|j|\sqrt{(\vec{n}+\vec{m})^2+4N}\right] .$$

The first sum can be computed exactly:

$$\frac{2\sqrt{\alpha'}}{R}\Omega = \frac{2^2}{3\cdot 5}\,\pi^3\left(\frac{\sqrt{\alpha'}}{R}\right)^6 \tag{5.80}$$
$$+\,2^4\pi^{3/2}\left(\frac{\sqrt{\alpha'}}{R}\right)^3 \sum_{j\in 2\mathbb{Z}+1}\ \sideset{}{'}\sum_{\substack{\vec{n},\vec{m}\in\mathbb{Z}^4\\ N\in\mathbb{N}}} A_N\overline{A}_{N+\vec{n}^t\vec{m}}\left[\frac{1}{j}\sqrt{(\vec{n}+\vec{m})^2+4N}\right]^3\,\mathcal{K}_3\left[\pi j\frac{R}{\sqrt{\alpha'}}\sqrt{(\vec{n}+\vec{m})^2+4N}\right]\,.$$

Where the $'$ denotes that the term $\vec{n}=\vec{m}=\vec{0}$ $N=0$ is not considered on the sum. Leading to the result:

$$\Omega = \frac{2}{3\cdot 5}\,\pi^3\left(\frac{\sqrt{\alpha'}}{R}\right)^5 \tag{5.81}$$
$$+\,2^3\pi^{3/2}\left(\frac{\sqrt{\alpha'}}{R}\right)^2 \sum_{j\in 2\mathbb{Z}+1}\ \sideset{}{'}\sum_{\substack{\vec{n},\vec{m}\in\mathbb{Z}^4\\ N\in\mathbb{N}}} A_N A_{N+\vec{n}^t\vec{m}}\left[\frac{1}{j}\sqrt{(\vec{n}+\vec{m})^2+4N}\right]^3\,\mathcal{K}_3\left[\pi j\frac{R}{\sqrt{\alpha'}}\sqrt{(\vec{n}+\vec{m})^2+4N}\right]\,.$$

5.4.1 Numerical approximation of Ω

Computing this sum analytically is almost impossible, while numerically we will have 10 different sum parameters with unbounded domains, so it is also quite difficult. We can try to make an approximation by turning off the lattice momenta $\vec{n}=\vec{m}=0$ (Since the behaviour is also partially codified with the N parameter). Doing so we plot how this amplitude behaves when increasing the radius:

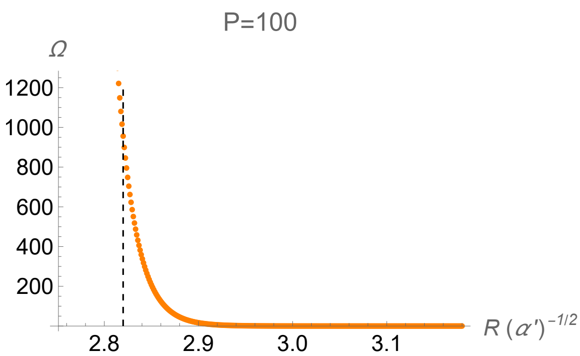


Figure 5.2: Ω for various $R/\sqrt{\alpha'}$ values. $N\in[1,100]$

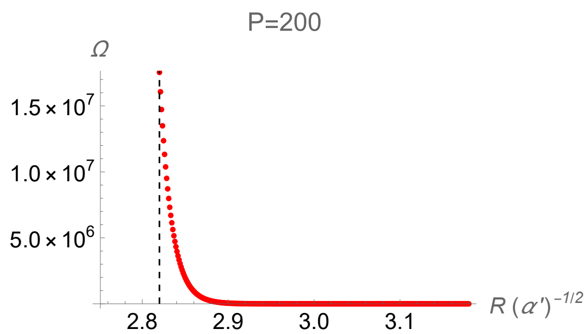


Figure 5.3: Ω for various $R/\sqrt{\alpha'}$ values. $N\in[1,200]$

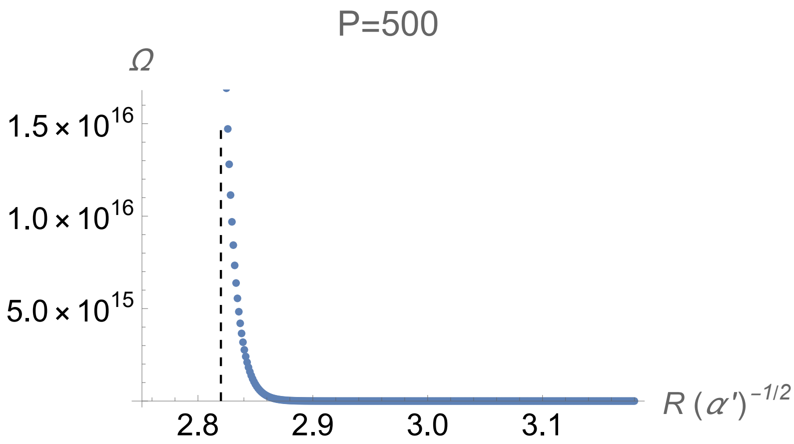


Figure 5.4: Ω for various $R/\sqrt{\alpha'}$ values. $N \in [1, 500]$

We can see that, as the radius tends to infinity, the amplitude tends to zero, while as the radius is small, it explodes. This divergence is encoded in the asymptote that occurs at $\frac{R}{\alpha'} = \sqrt{8}$. Theoretically, for this model, we argued that there should be a divergence arising for $R < \sqrt{8\alpha'}$. The existence of this critical radius is common to orbifold models of this kind, it is a symptom of tachyons (states of negative mass square as discussed in Section 4.3.5) appearing in the spectrum, and so it leads to vacuum instabilities. This issue only appears when adding "high energy" states, and so when we consider $N \to \infty$, as we discussed with the Hagedorn phase transition. This was not doable numerically, but we can clearly see that as we increase the number of terms P in the sum of Ω, we get the asymptotic behaviour, with the limit tending to $\Omega(\sqrt{8}) \to \infty$ as we expected analytically.

5.4.2 Decompactification limit

This expression was obtained after choosing a suitable metric for the $\Gamma_{4,4}$ $g = \alpha' Id$, since we can always Lorentz rotate from any general lattice to the one characterised by this choice of metric (sometimes called Γ_0). In order to decompactify, the easiest way goes through explicitly showing the radius of each S^1 in T^4 and taking its limit $R_i \to \infty$. To simplify the computations, we will choose the same radius to all the cycles in the torus and perform directly the limit. We expect to recover the result from the theory on $\mathbb{R}^{1,8} \times S^1/\mathbb{Z}_2$.

To do this,let us specify some notation and the metric we will choose for the T^4. First of all, we already have a radius (related to the $S^1/\mathbb{Z}_2$), all the terms related to this radius R_{S^1} go as $\left(\frac{R_{S^1}}{\sqrt{\alpha'}}\right)^a$ $a \in \mathbb{Z}$; on the other hand we choose $g = \frac{\alpha'}{R^2_{T^4}}\, Id$ to be the metric on the T^4, choosing it to be diagonal and having all the same entries basically we take the same idea as in the last case in which we separate $T^4 = (S^1)^4$ but showing explicitly the radius. So in order to simplify notation we define in this case:[4]

$$\overline{R} = \frac{R_{S^1}}{\sqrt{\alpha'}} \quad ; \quad R = R_{T^4} \implies g = \frac{\alpha'}{R^2}\, Id\ . \tag{5.82}$$

[4]The expression is symmetric in $\vec{n}$ and $\vec{m}$ before introducing the radius in the metric, and so we don't really care about adding $\frac{1}{R}$ or R. On the other hand, we use its square to relate with the already known result of the circle.

Using this as input for equation (5.69) we realize that the result is exactly the same expression after rescaling:

$$\vec{m} \to \vec{m}R \quad ; \quad \vec{n} \to \frac{\vec{n}}{R} \; . \tag{5.83}$$

Instead of trying to take the limit where $R \to 0$ directly from the close expression we obtained for Ω in (5.81), for the sake of simplicity, we will perform the limit in the τ_2-integral expression for Ω:

$$\frac{2}{\overline{R}}\Omega_{T^2\times S^1} = 2^8 \sum_{\substack{\vec{n},\vec{m}\in\mathbb{Z}^4 \\ N\in\mathbb{N} \\ j\in\mathbb{Z}}} A_N \overline{A}_{N+\vec{n}^t\vec{m}} \int_0^\infty \frac{1}{\tau_2^4} e^{-\pi\tau_2\left[\left(\frac{\vec{n}}{R}+\vec{m}R\right)^2+4N\right]} e^{\frac{-\pi\overline{R}^2}{\tau_2}\left(j+\frac{1}{2}\right)^2} d\tau_2 \; . \tag{5.84}$$

When $R \to \infty$ all the terms with $\vec{m} \neq 0$ vanish since they get exponentially suppressed, and so we can directly rewrite:

$$\frac{2}{\overline{R}}\Omega_{T^2\times S^1} = 2^8 \sum_{N\in\mathbb{N}} \sum_{j\in\mathbb{Z}} A_N^2 \int_0^\infty \frac{1}{\tau_2^4} e^{-4\pi\tau_2 N} \sum_{\vec{n}\in\mathbb{Z}^4} e^{-\pi\tau_2\frac{\vec{n}^2}{R^2}} \; e^{\frac{-\pi\tilde{r}^2}{\tau_2}(j+1/2)^2} d\tau_2 \; . \tag{5.85}$$

Upgrading the sum into an integral by saying $\frac{n}{R} \to x$:

$$\sum_{\vec{n}\in\mathbb{Z}^4} e^{-\pi\tau_2\frac{\vec{n}^2}{R^2}} = \left[\sum_{n\in\mathbb{Z}} e^{-\pi\tau_2\frac{n^2}{R^2}}\right]^4 \xrightarrow[R\to\infty]{} \left[\int_{\mathbb{R}} e^{-\pi\tau_2 x^2} dx\right]^4 = \frac{1}{\tau_2^2} \; . \tag{5.86}$$

Probes directly that:

$$\Omega_{T^4\times S^1}\left(\overline{R}, R\right) \xrightarrow[R\to\infty]{} \Omega_{S^1}\left(\overline{R}\right) \; . \tag{5.87}$$

5.4.3 Cosmological Constant analysis

As we developed in (5.13), for our model the one-loop contribution for the cosmological constant is given in terms of the Ω integral as:

$$\Lambda = -\frac{1}{4}\frac{\Omega}{\left(2\pi\sqrt{\alpha'}\right)^5} \; . \tag{5.88}$$

The first insight that we can get from here is that, for all these class of vacua, the one-loop cosmological constant is always negative This is independent of the radius of the the S^1 direction, as $\Omega\left(\frac{R}{\sqrt{\alpha'}}\right)$ is positive defined. In case this one-loop contribution is relevant enough, the overall vacuum would have AdS geometry.

Including the prefactor from (5.88) in (5.81), the cosmological constant reads as:

$$\Omega = -\frac{1}{2^6\cdot 3\cdot 5\pi^2}\frac{1}{R^5} \tag{5.89}$$
$$- \frac{1}{2^4\pi^{\frac{7}{2}}\sqrt{\alpha'}^3 R^2} \sum_{j\in 2\mathbb{Z}+1} \sideset{}{'}\sum_{\substack{\vec{n},\vec{m}\in\mathbb{Z}^4 \\ N\in\mathbb{N}}} A_N A_{N+\vec{n}^t\vec{m}} \left[\frac{1}{|j|}\sqrt{\left(\vec{n}+\vec{m}\right)^2+4N}\right]^3 \mathcal{K}_3\left[\pi|j|\frac{R}{\sqrt{\alpha'}}\sqrt{\left(\vec{n}+\vec{m}\right)^2+4N}\right] \; .$$

In the no string limit, in which the scale of the string is taken to be zero $\alpha' \to 0$, the second sum vanishes, this can be easily checked by using the asymptotic expansion for large argument of the Basset function:

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} \sum_{k=0}^{\infty} \frac{a_k(\nu)}{z^k} \ . \tag{5.90}$$

In the limit $\alpha' \to 0$, the leading term in terms of alpha will be $\sqrt{\alpha'}^{-3} e^{-A\frac{1}{\sqrt{\alpha'}}} \to 0$ vanishing the second contribution.

The first term does correctly not have any dependence on α' and so it is related with t low energy theory, so Type IIB supergravity. This calculation was also performed in supergravity [15], in which $\Lambda \propto \frac{1}{R^5}$, obviously this result was also expected from dimensional analysis. The exact prefactor to this $\frac{1}{R^5}$ should not be compared directly with the supergravity calculation, unfortunately the precise formula that links both computations, in string theory and supergravity, does not work for the case $D = 5$.[12]

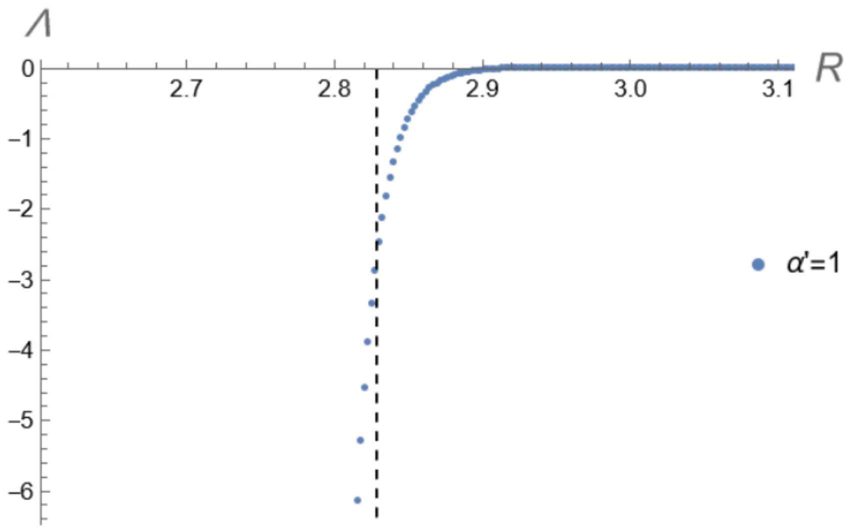


Figure 5.5: Cosmological constant over the radius of the S^1 direction, R. Fixed $\alpha' = 1$. The dashed line represents thhe critical value $R = \sqrt{8\alpha'}$.

Thus, taking only into consideration the first term results in a cosmological constant which is unstable since it will tend to roll down and the radius will tend to zero leading to a vacuum instability that cannot be fixed. On the other hand, the pure stringy term presents also this instability but at a critical value of the radius $R_0 > 0$, allowing a region $R \in (0, R_0)$ whose behaviour is completely out of the former analysis. If we try to extend naively region of validity of this constant Λ to $R < R_0$, then we will just find tachyons whose mass increases in absolute value, so the vacuum would be unstable.

Moreover, the integral Ω is always positive, since the first term is always positive the solution of the Basset function and its prefactor is always positive and the A_N are also positive. This forces the cosmological constant to be strictly negative, and so we expect that, prior to further loop corrections, the vacuum of this model exhibits Anti-de-Sitter geometry.

Chapter 6

Conclusions & Outlook

In this thesis, we studied orbifold compactifications of Type IIB String Theory focusing on the computation of one-loop partition function for these theories.

We performed the calculations of the one-loop vacuum amplitude arising from Type IIB string theory with a $\mathbb{R}^{1,4} \times (S^1 \times T^4)/\mathbb{Z}_p$ spacetime. Particularly for $\mathbb{Z}_p$ symmetric freely-acting orbifolds, using standard techniques such as the orbits method to unfold the integral of the partition function over the fundamental domain.

We then focused on an $\mathbb{Z}_2$ orbifold example and explicitly showed how to express the result of this integral as a series expansion. We checked how the expected tachyonic states in the spectrum lead to divergencences at low R and they are linked to well-known phenomena such as Hagedorn phase transitions. We also proved some limit regimes as the decompactification limit and obtained the cosmological constat at one-loop from the String Theory perspective.

We compared how this cosmological constant has the same tendency in the $\alpha' \to 0$ limit ($\Lambda \sim \frac{1}{R^5}$) as in the supergravity computation. We also showed that, for this kind of model, the one-loop contribution to the vacuum energy is always negative.

Future research may imply several directions.One of the main ideas would be to explicitly compare this result with the one arising from supergravity [1][12]. For doing so, there could be modifications on the formulae already existing for D even.

On the other hand, we expect a natural roll-down of the vacuum energy over R_{S^1} till the phase transition in which it became unstable; further investigation will be needed to actually confirm what may happen in the regime R_{S^1} under the tachyonic bound. There is some ongoing research on these topics, as trying to formally get some arguments inspired by T-duality intuition to investigate that region.[11]

Moreover, some modifications during the integral of the unfolding may lead to easier functions in the amplitude sum, such as performing a Poisson resummation on the T^4 contribution. This would help some analysis, such as showing explicitly how to drop the dependence on the T^4 modes for the A_N coefficients. Moreover, extensions to orbifolds in which the sum over the S^1 is not present may be also done analytically by using other kind of methods .[4][19][3]

Bibliography

[1] Steven Abel, Keith R. Dienes, and Eirini Mavroudi. "Towards a nonsupersymmetric string phenomenology". In: *Physical Review D* 91.12 (June 2015). ISSN: 1550-2368. DOI: 10.1103/physrevd.91.126014. URL: http://dx.doi.org/10.1103/PhysRevD.91.126014.

[2] Carlo Angelantonj, Matteo Cardella, and Nikos Irges. "An alternative for moduli stabilisation". In: *Physics Letters B* 641 (Sept. 2006), pp. 474–480. DOI: 10.1016/j.physletb.2006.08.072.

[3] Carlo Angelantonj, Ioannis Florakis, and Boris Pioline. "A new look at one-loop integrals in string theory". In: *Commun. Num. Theor. Phys.* 6 (2012), pp. 159–201. DOI: 10.4310/CNTP.2012.v6.n1.a4. arXiv: 1110.5318 [hep-th].

[4] Carlo Angelantonj, Ioannis Florakis, and Boris Pioline. "Rankin-Selberg methods for closed strings on orbifolds". In: *Journal of High Energy Physics* 2013.7 (July 2013). ISSN: 1029-8479. DOI: 10.1007/jhep07(2013)181. URL: http://dx.doi.org/10.1007/JHEP07(2013)181.

[5] Michael Baake. "A Guide to Mathematical Quasicrystals". In: *Quasicrystals: An Introduction to Structure, Physical Properties and Applications*. Ed. by Jens-Boie Suck, Michael Schreiber, and Peter Häussler. Berlin, Heidelberg: Springer Berlin Heidelberg, 2002, pp. 17–48. ISBN: 978-3-662-05028-6. DOI: 10.1007/978-3-662-05028-6_2. URL: https://doi.org/10.1007/978-3-662-05028-6_2.

[6] Katrin Becker, Melanie Becker, and John H. Schwarz. *String Theory and M-Theory: A Modern Introduction*. Cambridge University Press, 2006.

[7] Ralph Blumenhagen, Dieter Lüst, and Stefan Theisen. "Toroidal Compactifications: 10-Dimensional Heterotic String". In: *Basic Concepts of String Theory*. Berlin, Heidelberg: Springer Berlin Heidelberg, 2013, pp. 263–320. ISBN: 978-3-642-29497-6. DOI: 10.1007/978-3-642-29497-6_10. URL: https://doi.org/10.1007/978-3-642-29497-6_10.

[8] Ralph Blumenhagen and Erik Plauschinn. *Introduction to Conformal Field Theory: With Applications to String Theory*. Vol. 779. Jan. 2009. ISBN: 978-3-642-00449-0. DOI: 10.1007/978-3-642-00450-6.

[9] Håvard Damm-Johnsen. *Theta functions and their applications*. https://users.ox.ac.uk/~quee4127/theta.pdf. [Accessed 04-07-2024]. 2019.

[10] Philippe Di Francesco, Pierre Mathieu, and David Senechal. "Minimal Models I". In: *Conformal Field Theory*. New York, NY: Springer New York, 1997, pp. 200–238.

[11] Keith R. Dienes, Moshe Moshe, and Robert C. Myers. “String Theory, Misaligned Supersymmetry, and the Supertrace Constraints”. In: *Physical Review Letters* 74.24 (June 1995), pp. 4767–4770. ISSN: 1079-7114. DOI: 10.1103/physrevlett.74.4767. URL: http://dx.doi.org/10.1103/PhysRevLett.74.4767.

[12] Keith R. Dienes, Moshe Moshe, and Robert C. Myers. “String Theory, Misaligned Supersymmetry, and the Supertrace Constraints”. In: *Physical Review Letters* 74.24 (June 1995), pp. 4767–4770. ISSN: 1079-7114. DOI: 10.1103/physrevlett.74.4767. URL: http://dx.doi.org/10.1103/PhysRevLett.74.4767.

[13] G. v. Gersdorff and M. Quirós. “Supersymmetry breaking on orbifolds from Wilson lines”. In: *Physical Review D* 65.6 (Feb. 2002). ISSN: 1089-4918. DOI: 10.1103/physrevd.65.064016. URL: http://dx.doi.org/10.1103/PhysRevD.65.064016.

[14] Paul Ginsparg. *Applied Conformal Field Theory*. 1988. arXiv: hep-th/9108028.

[15] George Gkountoumis et al. “Freely acting orbifolds of type IIB string theory on T^5”. In: *JHEP* 08 (2023), p. 089. DOI: 10.1007/JHEP08(2023)089. arXiv: 2302.09112 [hep-th].

[16] Rolf Hagedorn. “Statistical thermodynamics of strong interactions at high energies”. In: *Nuovo Cimento, Suppl.* 3 (1965), pp. 147–186. URL: https://cds.cern.ch/record/346206.

[17] Dr. Hans Jockers. *Conformal Field Theory*. https://www.math.uni-hamburg.de/home/stern/Notes/CFT/Notes_CFT.pdf. [Accessed 04-07-2024].

[18] Francisco C. Caramello Jr. *Introduction to orbifolds*. 2022. arXiv: 1909.08699 [math.DG].

[19] Paul Kiefer. “Orthogonal Eisenstein series and theta lifts”. In: *International Journal of Number Theory* 19.06 (Feb. 2023), pp. 1305–1335. ISSN: 1793-7310. DOI: 10.1142/s1793042123500641. URL: http://dx.doi.org/10.1142/S1793042123500641.

[20] Elias Kiritsis. *String Theory in a Nutshell: Second Edition*. USA: Princeton University Press, Apr. 2019. ISBN: 978-0-691-15579-1, 978-0-691-18896-6.

[21] W. Lerchie, A.N. Schellekens, and N.P. Warner. “Lattices and strings”. In: *Physics Reports* 177.1 (1989), pp. 1–140. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(89)90077-X. URL: https://www.sciencedirect.com/science/article/pii/037015738990077X.

[22] Joaquin Liniado. *Two Dimensional Conformal Field Theory and a Primer to Chiral Algebras*. Oct. 2021.

[23] Juan Maldacena. In: *International Journal of Theoretical Physics* 38.4 (1999), pp. 1113–1133. ISSN: 0020-7748. DOI: 10.1023/a:1026654312961. URL: http://dx.doi.org/10.1023/A:1026654312961.

[24] K. H. O’Brien and C.-I Tan. “Modular invariance of the thermo-partition function and global phase structure of the heterotic string”. In: *Phys. Rev. D* 36 (4 Aug. 1987), pp. 1184–1192. DOI: 10.1103/PhysRevD.36.1184. URL: https://link.aps.org/doi/10.1103/PhysRevD.36.1184.

[25] Cormac O'Raifeartaigh et al. "One hundred years of the cosmological constant: from "superfluous stunt" to dark energy". In: *The European Physical Journal H* 43.1 (Mar. 2018), pp. 73–117. ISSN: 2102-6467. DOI: 10.1140/epjh/e2017-80061-7. URL: http://dx.doi.org/10.1140/epjh/e2017-80061-7.

[26] T Padmanabhan. "Combining general relativity and quantum theory: points of conflict and contact". In: *Classical and Quantum Gravity* 19.13 (June 2002), pp. 3551–3566. ISSN: 0264-9381. DOI: 10.1088/0264-9381/19/13/312. URL: http://dx.doi.org/10.1088/0264-9381/19/13/312.

[27] A.N. Schellekens. *Conformal Field Theory*. https://www.nikhef.nl/7Et58/CFT.pdf. [Accessed 04-07-2024].

[28] John H. Schwarz. *The Early History of String Theory and Supersymmetry*. 2012. arXiv: 1201.0981 [physics.hist-ph]. URL: https://arxiv.org/abs/1201.0981.

[29] Marjorie Senechal. "Introduction to Lattice Geometry". In: *From Number Theory to Physics*. Ed. by Michel Waldschmidt et al. Berlin, Heidelberg: Springer Berlin Heidelberg, 1992, pp. 476–495. ISBN: 978-3-662-02838-4. DOI: 10.1007/978-3-662-02838-4_10. URL: https://doi.org/10.1007/978-3-662-02838-4_10.

[30] Adam Smith, Omri Golan, and Zohar Ringel. "Intrinsic sign problems in topological quantum field theories". In: *Physical Review Research* 2 (Sept. 2020). DOI: 10.1103/PhysRevResearch.2.033515.

[31] Koen Christiaan Stemerdink. "Black Holes from Branes: Various string theoretical constructions". English. Doctoral thesis 1 (Research UU / Graduation UU). Universiteit Utrecht, Sept. 2022. ISBN: 978-94-6458-428-8. DOI: 10.33540/1379.

[32] Michele Trapletti. "On the unfolding of the fundamental region in integrals of modular invariant amplitudes". In: *Journal of High Energy Physics* 2003.02 (Feb. 2003), pp. 012–012. ISSN: 1029-8479. DOI: 10.1088/1126-6708/2003/02/012. URL: http://dx.doi.org/10.1088/1126-6708/2003/02/012.