Universiteit Utrecht

Graduate School of Natural Sciences

The Functorial Nature of Enriched Data Types

Master's Thesis

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July 8, 2024

Acknowledgements

Many people have directly and indirectly contributed to the coming to be of this thesis, and I express my gratitude towards them all.

In particular, I would like to thank Maximilien Péroux, who has been encouraging and inspirational. Thanks should go to Wouter Swierstra, who has generously donated his time and energy.

Most of all I want to express my gratitude to Paige North, who has guided me both within my thesis and outside of it, for saying exactly the right things at the right time.

Abstract

In computer science, many popular data types can be modeled in a categorical framework. More specifically, they can be modeled as an initial algebra in the category of F-algebras for some endofunctor $F : \mathbf{C} \to \mathbf{C}$. Examples include the booleans, the natural numbers, lists, trees and more generally algebraic data types, W-types and inductive data types. These initial algebras have the special property that algebra morphisms out of it to any other algebra always exist and are unique. This gives us tools to easily reason about functions defined on these data types. The downside to this approach is that we lose a lot of flexibly when defining maps, since they must respect the algebra structures completely.

In a recent paper it has been shown that the category of F-algebras is enriched over the category of F-coalgebras, if the category $\mathbf{C}$ and functor F involved are sufficiently well-behaved [1]. This gave a structured way to define so called measurings, which respect the algebra structure up to a certain point, granting more flexibility when defining morphisms. They also introduced the notion of a C-initial algebra for some F-coalgebra C, which shares the desirable properties of an initial algebra while accommodating the idea of a measuring.

So far, the above has been applied to the example of natural numbers in [1]. First off, we will consider more examples, including the previously mentioned booleans, lists and trees. We will also expand on the existing theory, showing the construction of the enriched category of F-algebras is functorial in F. That is, given a natural transformation $\mu : F \to G$, we will construct enriched functors from the category of F-algebras to the category of G-algebras and vice versa. This gives rise to functors $\mathbf{Endo}(\mathbf{C}) \to \mathbf{EnrCat}$, where $\mathbf{Endo}(\mathbf{C})$ is the category of (sufficiently well-behaved) endomorphisms on $\mathbf{C}$ and natural transformations. The category $\mathbf{EnrCat}$ is the category of enriched categories and enriched functors, and will be defined in this thesis. Throughout the thesis, we will continue to pay attention to C-initial algebras, providing many examples and showing in which cases they are preserved between categories of algebras.

Contents

1 Introduction

Every computer program manipulates data in some way, shape or form. In order to properly reason about a program, this data needs to be made accessible in a structured way. To this end, an abstract description of this data is of great use, where we need not concern ourselves with exactly how data is stored, and instead interface with the data through abstract, predefined procedures. A first step is to define different data types, or structured ways of storing data. An example is the boolean data type, which contains only `True` and `False`, or the list type `List a` which contains lists with elements of another type `a`. We wish not to concern ourselves with exactly how a program stores and retrieves data, and instead would prefer to give a purely algebraic specification of the data type. An example of this is the implementation of the list type in Haskell. Instead of having to deal with the inner mechanics of the list type, we simply define the list type containing elements of type `a` using two constructors

```
List a = Nil | Cons a (List a).
```

To construct a list we have one of two choices, specified by each the constructors. Either we construct the empty list using `Nil`, or we add a new element of type `a` to an existing list containing elements of type `a` using `Cons a (List a)`.

In general, a way to give such an algebraic specification of a data type is by defining a data type as an algebraic data type. An algebraic data type is a (possibly recursive) sum type of product types. This is quite abstract, but can be made more concrete if we think of the product type as a way of taking two types `a` and `b` and combining them into a single type `(a,b)` which has values consisting of a value of type `a` and a value of type `b`. On the other hand, a sum type takes two types `a` and `b` and combines them into a single type `a | b` which has values consisting of a value of type `a` or a value of type `b`. This is parallel to taking the product of two sets A and B to obtain $A \times B = \{(a, b) \mid a \in A, b \in B\}$ and the disjoint union (or sum) of two sets A and B to obtain $A + B = \{x \mid x \in A \vee x \in B\}$. The list type is definitely an example of this. We have the constructor `Nil`, which construct the type with a single value which we interpret as the empty list. We also have the constructor `Cons`, which takes something of type `a` and something of type `List a` and combines them as a product type. Finally, to construct the type `List a` we take the sum of the types constructed by `Nil` and `Cons a (List a)`.

A big advantage of algebraic data types is that it is very easy to define functions on them. It suffices to give function definitions for the constructors of the type, instead of defining how the function should act on each specific instance of that type. Viewing an algebraic data type as a sum type (of product types), a function definition for each constructor tells us what to do on every component of said sum of types. We revisit our list example. Classically, to obtain the length of a list one would count the amount of elements in the list, keeping a running total while doing so. Having defined it as an algebraic data type, we simply need to say what our length function `len` should do on each constructor. We define `len(Nil) = 0`, since the length of the empty list is 0, and define `len(Cons x xs) = 1 + len(xs)`, where `x` is of type `a` and `xs` is of type `List a`.

Notice how we gave an inductive definition of the length function in the previous example. What facilitates this is the fact that `List a` is an inductive type; a type which may be defined in terms of itself. There are many inductive types. All algebraic data types are inductive types, and more general all W-types (or containers) are inductive types [2]. Examples include the booleans, natural numbers, lists, binary trees and other tree-like structures. These guiding examples will accompany us throughout this thesis. When reasoning about a program, inductive types have two major advantages which are closely related. In order define functions on them, we only have to provide a definition on a number of base cases and on the inductive steps. To check if two functions are equal, we simply verify they agree on the base cases and do the same thing on the inductive steps [3]. This makes defining functions on inductive types and proving properties of these functions easy.

1.1 A mathematical framework

To make this more precise, we turn to mathematics, and in particular category theory. Inspecting our definition of `List a`, we can translate it to mathematics as follows. Let A be a set, and let A^* denote the set of all lists with elements in A. We can reflect `Nil` as an element $[\,] \in A^*$, or equivalently as a function $1 \to A^*, * \mapsto [\,]$, where $1 \cong \{*\}$ is the set containing only one element. The constructor `Cons` can be reflected as a function $A \times A^* \to A^*, (a_0, [a_1, a_2, \dots, a_n]) \mapsto [a_0, a_1, a_2, \dots, a_n]$. We can pack this together to obtain the function $\alpha : 1 + A \times A^* \to A^*$. Observe that the domain of α is given by $1 + A \times A^*$, which reflects our definition of `List a` as a recursive sum of product types `Nil | Cons a (List a)`. In order to define our length function $\text{len} : A^* \to \mathbb{N}$, we can again proceed as before by defining

$$\begin{aligned} \text{len} : A^* &\to \mathbb{N} \\ [\,] &\mapsto 0 \\ [a_0, a_1, a_2, \dots, a_k] &\mapsto 1 + \text{len}([a_1, a_2, \dots, a_k]). \end{aligned}$$

Now we make an important observation. When defining len, we defined it on $\alpha(*) = [\,]$ and on $\alpha(a_0, [a_1, a_2, \dots, a_k]) = [a_0, a_1, a_2, \dots, a_k]$. What of instead of defining the function len directly on A^*, we define it using $\alpha : 1 + A \times A^* \to A^*$. We can do this by defining a function $\beta : 1 + A \times \mathbb{N} \to \mathbb{N}$ where $\beta(*) = 0$ and $\beta(a_0, k) = k + 1$, and asking that $\text{len}(\alpha(*)) = \beta(*) = 0$, and $\text{len}(\alpha(a_0, [a_1, a_2, \dots, a_k])) = \beta(a_0, \text{len}([a_1, a_2, \dots, a_k]))$. The function β is precisely reflecting the inductive definition of len; for the base case we return 0, and for the inductive step we add 1 to the result of the function acting on the rest of the list. We can summarize this in the commutative diagram

$$\begin{array}{ccc} 1 + A \times A^* & \xrightarrow{\text{id}_1 + \text{id}_A \times \text{len}} & 1 + A \times \mathbb{N} \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A^* & \xrightarrow{\text{len}} & \mathbb{N}. \end{array}$$

It turns out that defining a function $A^* \to B$ in an inductive manner corresponds exactly to defining $\beta : 1 + A \times B \to B$. We see $1 \to B$ corresponds to handling the base case, and $A \times B \to B$ corresponds to the inductive step. These two combined then give completely determine a function $f : A^* \to B$. This pattern is commonly known as a `fold`, or catamorphism. Indeed, the type signature of `fold` is given by

```
fold : b -> (a -> b -> b) -> [a] -> b.
```

Given an value of type `b` and a function `a -> b -> b`, we obtain a function `[a] -> b`. This corresponds to defining the functions $1 \to B$ and $A \times B \to B$, which together constitute $\beta : 1 + A \times B \to B$, in order to obtain a function $A^* \to B$.

We would like to explicitly point out that given a function $\beta : 1 + A \times B \to B$, there always exists a unique function $f : A^* \to B$ such that $f(\alpha(*)) = \beta(*)$ and $f(\alpha(a_0, [a_1, a_2, \dots, a_k])) = \beta(a_0, f([a_1, a_2, \dots, a_k]))$. Indeed, we simply define f using the above equations. What makes this work is the fact that every element of A^* can be uniquely written as either $\alpha(*) = []$ or $\alpha(a_0, [a_1, a_2, \dots, a_k])$. In other words, $\alpha : 1 + A \times A^* \to A$ is a bijection. Comparing functions $f : A^* \to B$ defined this way to each other is easy. We simply need to check the functions $\beta : 1 + A \times B \to B$ coincide, which is a lot easier than checking two functions $A^* \to B$ coincide. Many functions can be defined this way, and we provide examples in Section 3.5.2.

We have promised category theory and more precision, but seem to have seen none so far. But, both are around the corner. If we define a category which as objects has $\beta : 1 + A \times B \to B$, and as morphisms $f : B \to B'$ such that the diagram

$$\begin{array}{ccc} 1 + A \times B & \xrightarrow{\text{id}_1 + \text{id}_A \times f} & 1 + A \times B' \\ \downarrow{\scriptstyle \beta} & & \downarrow{\scriptstyle \beta'} \\ B & \xrightarrow{f} & B' \end{array}$$

commutes, we see all of our previous work fits perfectly into this framework. Indeed, $\alpha : 1 + A \times A^* \to A$ and $\beta : 1 + A \times \mathbb{N} \to \mathbb{N}$ are both objects of this category, and $\text{len} : A^* \to \mathbb{N}$ is a morphism in this category. We have already remarked there exists a unique function $f : A^* \to B$ for all $\beta : 1 + A \times B \to B$. This is precisely the universal property of an initial object. We see we have specified A^* as the initial object in this category.

Generalizing this, we can consider a functor $F : \textbf{Set} \to \textbf{Set}$ and objects $\beta : F(B) \to B$. Together with morphisms $f : B \to B'$ such that the square

$$\begin{array}{ccc} F(B) & \xrightarrow{F(f)} & F(B') \\ \downarrow{\alpha} & & \downarrow{\beta} \\ B & \xrightarrow{f} & B' \end{array}$$

commutes, this forms the category of F-algebras. We will denote the category of F-algebras by $\mathbf{Alg}^F$, or simply **Alg** if F is understood. In the case of lists, the functor F was given by $F : B \mapsto 1 + A \times B$ for some fixed set A. An initial object in this category is called an initial algebra. We have constrained ourselves to functors $F : \mathbf{Set} \to \mathbf{Set}$, but this all immediately generalizes to $F : \mathbf{C} \to \mathbf{C}$ for any category $\mathbf{C}$. An inductive type is then any type whose interpretation is given by an initial algebra for some functor $F : \mathbf{C} \to \mathbf{C}$.

But why go through so much effort? As previously hinted at, the universal property of the initial algebra makes it so powerful. In order to define a function from an initial algebra one only needs to define the target, and any two functions from an initial algebra are equal if and only if their targets are equal. A word of warning though; two objects in the category of F-algebras B and B' are only equal if the functions $\beta : F(B) \to B$ and $\beta' : F(B') \to B'$ are equal. It does not suffice to check if B and B' are equal as objects in the category $\mathbf{C}$.

To illustrate this we provide an example. Consider the set of lists of length at most n, denoted A^*_n. We can give A^*_n an algebra structure by

$$\begin{aligned} \alpha_n : 1 + A \times A^*_n &\to A^*_n \\ * &\mapsto [\,] \\ (a_0, [a_1, a_2, \dots a_k]) &\mapsto \begin{cases} [a_0, a_1, a_2, \dots a_{k-1}] & \text{if } k = n - 1 \\ [a_0, a_1, a_2, \dots a_k] & \text{otherwise.} \end{cases} \end{aligned}$$

The algebra structure on A^*_n adds an element $a \in A$ to the front of the list, and discards the last element of the list if its length would exceed n. Note how we could also have chosen to discard the element we are adding to the list instead of the last element of the list whenever the length would exceed n. This would result in a different algebra, with different behavior. For example, in the case of discarding the last element of the list, the unique function $A^* \to A^*_n$ sends a list to its first n elements, while in the other case a list is sent to its last n elements.

We would like to provide an example where we verify two functions are equal using the fact that $\alpha : 1 + A \times A^* \to A^*$ is an initial algebra, taken from [3]. The aim is to prove the function which duplicates each element in a list, given by

$$\begin{aligned} d : A^* &\to A^* \\ [a_0, a_1, \dots, a_k] &\mapsto [a_0, a_0, a_1, a_1, \dots a_k, a_k] \end{aligned}$$

doubles the length of the list. To use initiality of A^*, we must make d an algebra morphism. We must also somehow capture the idea of doubling a number in this framework. We will not go into full detail in this example, and leave some things for the reader to verify.

In order for d to be an algebra morphism, we must specify an algebra structure on the target A^*, which will be distinct from the algebra structure on A^* we have already seen. The algebra structure we desire is given by

$$\begin{aligned} \alpha_d : 1 + A \times A^* &\to A^* \\ * &\mapsto [\,] \\ (a, [a_0, a_1, \dots, a_k]) &\mapsto [a, a, a_0, a_1, \dots, a_k]. \end{aligned}$$

Notice how α_d adds an element $a \in A$ to the front of the list twice instead of once. The function $d : A^* \to A^*$ now coincides with the unique function $(A^*, \alpha) \to (A^*, \alpha_d)$.

The idea of doubling a number can be captured in a similar fashion. We define a different algebra structure β_t on $\mathbb{N}$ as follows

$$\begin{aligned} \beta_t : 1 + A \times \mathbb{N} &\to \mathbb{N} \\ * &\mapsto 0 \\ (a, k) &\mapsto k + 2. \end{aligned}$$

With this algebra structure, the function $t : k \mapsto 2 * k$ is an algebra morphism $t : (\mathbb{N}, \beta) \to (\mathbb{N}, \beta_t)$.

We can also define the function $\mathrm{len}_2 : (A^*, \alpha_d) \to (\mathbb{N}, \beta_t), [a_0, a_1, \ldots, a_{n-1}] \mapsto n$, which is an algebra morphism. Notice how the underlying function of sets $\mathrm{len}_2 : A^* \to \mathbb{N}$ coincides with the function $\mathrm{len} : A^* \to \mathbb{N}$.

With all this in place, we can now show the desired result. The algebra morphisms $t \circ \mathrm{len}$ and $\mathrm{len}_2 \circ d$ are both morphisms from (A^*, α) to $(\mathbb{N}, \beta_t)$, so they must coincide by (A^*, α) being an initial algebra. This means the functions $t \circ \mathrm{len}$ and $\mathrm{len}_2 \circ d$ also coincide as functions between the sets A^* and $\mathbb{N}$. The former takes the length of the list and then multiplies it by two, where the latter doubles the list and then computes its length. This shows the function $d : A^* \to A^*$ doubles the length of the list.

1.2 Introducing measurings

This framework does come at a disadvantage. For example, there does not exist a morphism $A_n^* \to A^*$ in the category of F-algebras, since $f([a_0, a_1, \ldots, a_{n-1}])$ would be undefined. To see why, consider the diagram that must commute for f to be a morphism

$$\begin{array}{ccc} 1 + A \times A_n^* & \xrightarrow{\mathrm{id}_1 + \mathrm{id}_A \times f} & 1 + A \times A^* \\ \downarrow{\scriptstyle \alpha_n} & & \downarrow{\scriptstyle \alpha} \\ A_n^* & \xrightarrow{\quad f \quad} & A^*. \end{array}$$

We can write $f([a_0, a_1, \ldots, a_{n-1}]) = f(\alpha_n(a_0, [a_1, \ldots, a_{n-1}, a_n]))$ for any $a_n \in A$, since α_n discards a_n anyway. However, by f being an algebra morphism, it must satisfy

$$f([a_0, a_1, \ldots, a_{n-1}]) = f(\alpha_n(a_0, [a_1, \ldots, a_{n-1}, a_n])) = \alpha(a_0, f([a_1, \ldots, a_{n-1}, a_n]))$$

for all $a_n \in A$, which gives a contradiction.

Even the most straightforward idea of embedding $A_n^* \hookrightarrow A^*$ does not yield an algebra morphism. This can severely limit the types of computations this framework allows. One could notice the embedding $A_n^* \hookrightarrow A^*$ does respect the algebra structures up to lists of length $n - 1$. In a sense, we do have a partial morphism.

This is precisely what motivated the idea of a measuring in [1]. A measuring can be thought of as a family of morphisms $\varphi_c : B \to B'$ indexed by $c \in C$. The set of all measurings from B to B' indexed by C will be denoted $\mathfrak{m}_C(B, B')$. This indexing gives us more freedom than a regular morphisms $B \to B'$ would provide, but it does still have some constraints. We will not give all details here, but save them for Section 2.5. One thing we would like to point out is that a measuring by 1 gives us just a single morphism, namely $\varphi_* : B \to B'$. Due to the constraints placed on a measuring, this morphism is actually a morphism in the category of F-algebras. Hence we have the isomorphism $\mathfrak{m}_1(B, B') \cong \mathbf{Alg}(B, B')$.

We remark that C is no ordinary set, but instead an F-coalgebra, which gives us the extra structure necessary to define measurings. Where we defined an F-algebra as $\beta : F(B) \to B$, we define an F-coalgebra as $\chi : C \to F(C)$. Dual to the construction of $\mathbf{Alg}^F$, we can construct the category of F-coalgebras which we will denote by $\mathbf{CoAlg}^F$, or $\mathbf{CoAlg}$ if F is understood. For instance, we can give A_n^* a coalgebra structure by defining

$$\begin{aligned} \chi : A_n^* &\to 1 + A \times A_n^* \\ [\,] &\mapsto * \\ [a_0, a_1, \ldots, a_k] &\mapsto (a_0, [a_1, \ldots, a_k]). \end{aligned}$$

As it turns out, there does exist a measuring $\varphi_c : A_n^* \hookrightarrow A^*$ for $c \in A_n^*$, which reflects the fact that the embedding $A_n^* \hookrightarrow A^*$ respects algebra structures up to lists of length $n - 1$.

The concept of a measuring extends even further, yielding an enrichment of the category of F-algebras in the category of F-coalgebras if $F : \mathbf{C} \to \mathbf{C}$ is sufficiently well-behaved. This entails that measurings really do generalize morphisms in the category of F-algebras in a well-behaved manner. A hint of this could already be seen when we remarked a measuring by 1 is equivalent to a morphism in the category of F-algebras. The enrichment is shown by defining an object $\underline{\mathbf{Alg}}(A, B)$, which can be thought of as the object containing all measurings from A to B by any F-coalgebra C. In particular it also contains all morphisms in the category of F-algebras. A measuring is then equivalent to a coalgebra morphism $C \to \underline{\mathbf{Alg}}(A, B)$. It turns out even more structure is present, since we can define objects $C \triangleright A$ and $[C, B]$ and show there exist isomorphisms

$$\mathfrak{m}_C(A, B) \cong \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A, B)) \cong \mathbf{Alg}(C \triangleright A, B) \cong \mathbf{Alg}(A, [C, B]).$$

In the case of $\mathbf{Set}$, we can use the above to see $\mathbf{Alg}(A, B) \cong \mathfrak{m}_1(A, B) \cong \mathbf{CoAlg}(1, \underline{\mathbf{Alg}}(A, B))$, showing that a coalgebra morphism $1 \to \underline{\mathbf{Alg}}(A, B)$ really picks out an F-algebras morphism $A \to B$.

Measurings also allow us to define a more general notion of initial algebra, namely that of C-initial algebra. We call an F-algebra $\beta : F(B) \to B$ C-initial if there exists a unique measuring $\varphi_c : B \to B'$ for all other F-algebras $\beta' : F(B') \to B'$. This gives a C-initial algebra the same advantages as an initial algebra; both definitions ask for existence and uniqueness of measurings and morphisms respectively. Since we can vary the F-coalgebra C, this gives a large collection of objects with the same advantages as the initial algebra. Moreover, while an initial algebra is unique (up to isomorphism), C-initial algebras are not, allowing us to find even more objects with the desirable properties of an initial algebra.

1.3 Overview of the thesis

In this thesis, our first goal is to provide many examples of measurings, for many different endofunctors F. So far the only worked out example is the example of natural numbers in [1]. We aim to provide examples in the case of booleans, lists and trees among others. In these examples we will pay special attention to so called terminal C-initial algebras, which are unique (up to isomorphism).

Our second goal is to see how these examples interact. So far, we have considered F-algebras and left the functor F fixed. What happens if we compare F-algebras to G-algebras? Are they related in some way? Can we say something about (terminal) C-initial algebras for different functors? All these questions will be answered in this thesis.

In Section 2 we start off by providing all necessary preliminaries for the rest of the thesis. We will assume knowledge about basic category theory and functional programming, but aim to make the material as accessible as possible. To see the enrichment in action, we will provide many examples in Section 3. In Section 4, we will work towards and present the main results of this thesis, showing the functorial nature of the enrichment of the category of algebras. Along with the main results we will provide relevant examples that build on the examples in Section 3. Finally, we focus our attention on (terminal) C-initial algebras, providing some general theory and applying said theory to familiar examples.

2 Preliminaries

In this section we give an overview of all the concepts underlying the content of this thesis. The first few subsections will concern themselves with concepts not taught in a standard graduate level category theory course. Topics include algebras and coalgebras, monoidal and enriched categories as well as the notion of a 2-category. After that, we have all the tools needed to dive into the material found in [1], which forms the starting point for the rest of the thesis.

2.1 The categories of algebras and coalgebras

First things first, we need to introduce the idea of an algebra and a coalgebra, since these will be the primary objects we will be studying. There are many examples of algebras and coalgebras, and we will see plenty of them in Section 3. In this section, we aim to give the abstract definition, as well as provide a few results which are immediately obtainable. Almost all of the material in this section can be found in [4][Sec. 10.5]. It might be helpful to keep an example from the next section at hand in order to see the theory come to life. Throughout this section, $\mathbf{C}$ will be a category and $F : \mathbf{C} \to \mathbf{C}$ will be an endofunctor.

Our very first definition is that of an *algebra*.

Definition 2.1.1. Given an endofunctor $F : \mathbf{C} \to \mathbf{C}$, an *$F$-algebra* is a morphism $\alpha : F(A) \to A$ in $\mathbf{C}$, denoted (A, α). The *category of F-algebras* has F-algebras as objects, and as morphisms $f : (A, \alpha) \to (B, \beta)$ morphisms $f : A \to B \in \mathbf{C}$ such that

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

Given $(A, \alpha) \in \mathbf{Alg}^F$, we will often call A the *carrier of the algebra* and α the *algebra structure*. If the endofunctor F is understood we will write $\mathbf{Alg}$ instead of $\mathbf{Alg}^F$ for the category of F-algebras.

Dually, we have a notion of *coalgebra*.

Definition 2.1.2. Given an endofunctor $F : \mathbf{C} \to \mathbf{C}$, the *category of F-coalgebras* $\mathbf{CoAlg}^F$ has as objects *F-coalgebras* $\chi : C \to F(C)$, denoted (C, χ), and as morphisms $f : (C, \chi) \to (D, \delta)$ morphisms $f : C \to D \in \mathbf{C}$ such that

$$\begin{array}{ccc} F(C) & \xrightarrow{F(f)} & F(D) \\ {\scriptstyle\alpha}\uparrow & & {\scriptstyle\beta}\uparrow \\ C & \xrightarrow{f} & D \end{array}$$

commutes.

Given $(C, \chi) \in \mathbf{CoAlg}^F$, we will often call C the *carrier of the coalgebra* and χ the *coalgebra structure*. If the endofunctor F is understood we will write $\mathbf{CoAlg}$ instead of $\mathbf{CoAlg}^F$ for the category of F-coalgebras.

We can construct the category of F-algebras and the category of F-coalgebras for any endofunctor $F : \mathbf{C} \to \mathbf{C}$. However, we are often interested in endofunctors F such that the resulting categories $\mathbf{Alg}^F$ and $\mathbf{CoAlg}^F$ have some desirable properties. In particular, categories which are complete and cocomplete are of interest to us. To this end, we state the following theorems found in [5][Def. 1.17, Def. 2.17, Cor. 2.75 & Ex. 2.j]

Theorem 2.1.3. *Let $\mathbf{C}$ be a locally presentable category and et $F : \mathbf{C} \to \mathbf{C}$ be a accessible functor. Then $\mathbf{Alg}$ and $\mathbf{CoAlg}$ are also locally presentable.*

Corollary 2.1.4. *Let $\mathbf{C}$ be a locally presentable category and let $F : \mathbf{C} \to \mathbf{C}$ be a accessible functor. Then $\mathbf{Alg}$ and $\mathbf{CoAlg}$, as locally presentable categories, are both complete and cocomplete.*

In this thesis, we will almost always assume the category $\mathbf{C}$ to be locally presentable and $F : \mathbf{C} \to \mathbf{C}$ to be accesible. These conditions are necessary to develop the theory underlying this thesis, which will be exhibited in Section 2.5.

As noted in Section 1, the category of algebras is of interest to us since its initial object provides semantics for a broad class of data types. We will call the initial object in $\mathbf{Alg}^F$ the *initial algebra*. Dually, the terminal object in $\mathbf{CoAlg}^F$ is called the *terminal coalgebra*. Both the initial algebra and the terminal coalgebra have a special property, which will be the contents of the following lemmas.

Lemma 2.1.5. *If the initial object (ι, I) in the category of algebras exists, $\iota : F(I) \to I$ is an isomorphism.*

We will call the initial object in the category of algebras the *initial algebra*.

Proof. Consider the algebra $F(\iota) : F(F(I)) \to F(I)$. Since (I, ι) is the initial algebra, there exists a unique morphism

$$\begin{array}{ccc} F(I) & \xrightarrow{F(f)} & F(F(I)) \\ \downarrow{\scriptstyle \iota} & & \downarrow{\scriptstyle F(\iota)} \\ I & \xrightarrow{f} & F(I). \end{array}$$

We also observe the algebra structure $\iota : F(I) \to I$ is also an algebra morphism $\iota : (F(I), F(\iota)) \to (I, \iota)$. By initiality of I we can state $\iota \circ f = \mathrm{id}_I$. This also means $F(\iota) \circ F(f) = \mathrm{id}_{F(I)}$, and by f being an algebra morphism we can deduce

$$\mathrm{id}_{F(I)} = F(\iota) \circ F(f) = f \circ \iota,$$

hence f is the inverse of ι. We conclude $\iota : F(I) \to I$ is an isomorphism. $\square$

Corollary 2.1.6. *The initial algebra $\iota : F(I) \to I$ is also an coalgebra $\iota^{-1} : I \to F(I)$.*

Dually, we can state the following lemma.

Lemma 2.1.7. *If the terminal object (τ, T) in the category of coalgebras exists, $\tau : T \to F(T)$ is an isomorphism.*

Proof. The proof is completely dual to Lemma 2.1.5 $\square$

Corollary 2.1.8. *The terminal coalgebra $\tau : T \to F(T)$ is also an algebra $\tau^{-1} : F(T) \to T$.*

A useful tool to compute the initial algebra and the terminal coalgebra is the following.

Proposition 2.1.9. *Let $\mathbf{C}$ be a category with initial object $0 \in \mathbf{C}$ and let $F : \mathbf{C} \to \mathbf{C}$ be an endofunctor. Consider the chain*

$$0 \xrightarrow{\mathsf{i}} F(0) \xrightarrow{F(\mathsf{i})} \dots F^n(0) \xrightarrow{F^n(\mathsf{i})} F^{n+1}(0) \to \dots$$

and suppose $\mathbf{C}$ contains the colimit of this chain, denoted I. If F preserves the colimit I, then I is the initial algebra.

As maybe expected, we also have the dualized statement for coalgebras.

Proposition 2.1.10. *Let $\mathbf{C}$ be a category with terminal object $1 \in \mathbf{C}$ and let $F : \mathbf{C} \to \mathbf{C}$ be an endofunctor. Consider the chain*

$$\dots \to F^{n+1}(1) \xrightarrow{F^n(!)} F^n(1) \dots \xrightarrow{F(!)} F(1) \xrightarrow{!} 1$$

and suppose $\mathbf{C}$ contains the limit of this chain, denoted T. If F preserves the limit T, then T is the terminal coalgebra.

One might wonder about terminal algebras and initial coalgebras. These are much easier to compute, since they are respectively given by the terminal and initial object in $\mathbf{C}$.

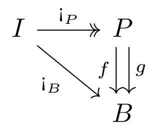
A class of algebras which we will frequently revisit is that of *preinitial algebras*.

Definition 2.1.11. Let I be the initial algebra. An algebra P is called *preinitial* if the unique morphism $\mathsf{i}_P : I \to P$ is an epimorphism.

A nice characterization of preinitial algebras is the following.

Lemma 2.1.12. *An algebra P is preinitial if and only if their exists at most one morphism $P \to B$ for any algebra B.*

Proof. Suppose P is preinitial. Let $f, g : P \to B$ be two morphisms out of P. We can draw the diagram



which must commute since I is the initial object. Since $f \circ \mathsf{i}_P = \mathsf{i}_B = g \circ \mathsf{i}_P$ and i_P is epic, we conclude $f = g$.

For the converse, suppose there exists at most one morphism $P \to B$ for any algebra B. Assume there exists morphisms f, g such that $f \circ \mathsf{i}_P = g \circ \mathsf{i}_P$. Since there exists at most one morphism out of P, the morphisms f and g must coincide. We see i_P is epic, hence that P is preinitial. □

Dually, we have a similar story for *subterminal coalgebras.*

Definition 2.1.13. Let T be the terminal coalgebra. A coalgebra S is called *subterminal* if the unique morphism $S \to T$ is a monomorphism.

Lemma 2.1.14. *A coalgebra S is subterminal if and only if their exists at most one morphism $C \to S$ for any coalgebra C.*

Proof. The proof is completely dual to the proof of Lemma 2.1.12. □

Given an algebra (A, α), we can "forget" the algebra structure and only consider the underlying object, or carrier, of the algebra. We can do the same for coalgebras, and this underlies a functor.

Definition 2.1.15. Given a category of F-algebras $\mathbf{Alg}^F$ we denote the canonical forgetful functor as

$$\begin{aligned} U_{\mathbf{Alg}} : \mathbf{Alg}^F &\to \mathbf{C} \\ (A, \alpha) &\mapsto A \\ (f : (A, \alpha) \to (B, \beta)) &\mapsto (f : A \to B). \end{aligned}$$

Similarly, the canonical forgetful functor $\mathbf{CoAlg}^F$ is denoted

$$\begin{aligned} U_{\mathbf{CoAlg}} : \mathbf{CoAlg}^F &\to \mathbf{C} \\ (C, \chi) &\mapsto C \\ (f : (C, \chi) \to (D, \delta)) &\mapsto (f : C \to D). \end{aligned}$$

We will often omit the subscript and write $U : \mathbf{Alg}^F \to \mathbf{C}$ if the domain of the functor is clear.

Using the forgetful functor, we can now say more about the preinitial algebras and subterminal coalgebras. For that, we need the following tool, namely [1, Lemma 38].

Lemma 2.1.16. *If the forgetful functor preserves epimorphisms, it underlies an isomorphism between the set of quotient algebras of (A, α) and the set of quotient objects Q of A for which α restricts to a morphism $\alpha_Q : F(Q) \to Q$.*

Now we obtain a simpler characterization of preinitial algebras.

Corollary 2.1.17. *Assuming the forgetful functor U preserves epimorphisms, preinitial algebras P correspond to quotient objects of the carrier of initial algebra I in $\mathbf{C}$ for which the algebra structure on I restricts to a morphism $F(P) \to P$.*

For the dual, we have a similar tool, namely [1, Lemma 42], but it does come with more assumptions.

Lemma 2.1.18. *If F is* well-equipped *[1, Def. 41] there is an isomorphism between the set of subobjects of (C, χ) and the set of subobjects of C for which χ restricts to a morphism $\chi_S : F(S) \to S$.*

This in turn gives us the following characterization of subterminal coalgebras.

Corollary 2.1.19. *If F is well-equipped, subterminal coalgebras correspond to subobjects of the carrier of terminal coalgebra T in $\mathbf{C}$ for which the coalgebra structure on I restricts to a morphism $F(S) \to S$.*

A functor going in the opposite direction $\mathbf{C} \to \mathbf{Alg}$ to the forgetful functor would be a way to construct an algebra out of any object in $\mathbf{C}$. This is a common theme in abstract algebra and is often referenced as a “free” object. In our case such a functor exists under reasonable assumptions by [5][Def. 1.17, Def. 2.17, Cor. 2.75 & Ex. 2.j].

Proposition 2.1.20. *Given a locally presentable category $\mathbf{C}$ and an accessible endofunctor $F : \mathbf{C} \to \mathbf{C}$, the forgetful functor $U : \mathbf{Alg}^F \to \mathbf{C}$ has a left adjoint. We will call this left adjoint the free functor $\mathrm{Fr} : \mathbf{C} \to \mathbf{Alg}^F$.*

The dual notion is that of a cofree functor.

Proposition 2.1.21. *Given a locally presentable category $\mathbf{C}$ and an accessible endofunctor $F : \mathbf{C} \to \mathbf{C}$, the forgetful functor $U : \mathbf{CoAlg}^F \to \mathbf{C}$ has a right adjoint. We will call this right adjoint the cofree functor $\mathrm{Cof} : \mathbf{C} \to \mathbf{CoAlg}^F$.*

These functors lie at the heart of many of the constructions found in this thesis.

Now that we have the main objects of this thesis introduced, it is time to exhibit some more general category theory.

2.2 Monoidal categories

The first type of category we will be concerning ourselves with are *monoidal categories*. Why we are interested in them will become apparent in Section 2.3 about enriched categories, where they will form the basis of the enrichment. The material in this section can also be found in [6]. A lot of categories endowed with some kind of product, such as the cartesian product, cocartesian product or the tensor product, are considered monoidal. Keeping this in mind, we state the following definition.

Definition 2.2.1. A *monoidal category* is a category $\mathbf{V}$ equipped with

1. a functor $\otimes : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$,
2. an object $\mathbb{I} \in \mathbf{V}$ called the *unit*,
3. a natural isomorphism $a_{A,B,C} : (X \otimes Y) \otimes Z \xrightarrow{\cong} X \otimes (Y \otimes Z)$ called the *associator*,
4. a natural isomorphism $\lambda_A : \mathbb{I} \otimes A \xrightarrow{\cong} A$ called the *left unitor*,
5. a natural isomorphism $\rho_A : A \otimes \mathbb{I} \xrightarrow{\cong} A$ called the *right unitor*,

such that the triangle identity

$$\begin{array}{ccc} (A \otimes \mathbb{I}) \otimes B & \xrightarrow{a_{A,\mathbb{I},B}} & A \otimes (\mathbb{I} \otimes B) \\ & {\scriptstyle \rho_A \otimes \mathrm{id}_B} \searrow \quad \swarrow {\scriptstyle \mathrm{id}_A \otimes \lambda_B} & \\ & A \otimes B & \end{array}$$

and the pentagon identity

$$\begin{array}{ccccc} & (A \otimes (B \otimes C)) \otimes D & \xrightarrow{a_{A,B\otimes C,D}} & A \otimes ((B \otimes C) \otimes D) & \\ {\scriptstyle a_{A,B,C} \otimes \mathrm{id}_D} \nearrow & & & & \searrow {\scriptstyle \mathrm{id}_A \otimes a_{B,C,D}} \\ ((A \otimes B) \otimes C) \otimes D & & & & A \otimes (B \otimes (C \otimes D)) \\ & {\scriptstyle a_{A\otimes B,C,D}} \searrow & (A \otimes B) \otimes (C \otimes D) & \nearrow {\scriptstyle a_{A,B,C\otimes D}} & \end{array}$$

commute.

The triangle and pentagon identity together assure us that the monoidal structure is associative and unital.

There are some desireable properties a monoidal category V can have.

Definition 2.2.2. A monoidal category $(\mathbf{V}, \otimes, \mathbb{I})$ is called *strict* if a, λ and ρ are all identities. In this case the triangle and pentagon identity hold trivially.

Definition 2.2.3. A monoidal category $(\mathbf{V}, \otimes, \mathbb{I})$ is called *symmetric* if there exists a natural isomorphism $b_{A,B} : A \otimes B \xrightarrow{\cong} B \otimes A$ such that $b_{B,A} \circ b_{A,B} = \mathrm{id}_{A,B}$ and such that it coheres with the monoidal unit and associator in the expected way.

Another way of putting it, is to say the monoidal structure is commutative.

Having defined what a monoidal category is, we supply some examples to keep in mind when thinking of monoidal categories.

Example 2.2.4. The canonical example is the symmetric monoidal category $(\mathbf{Set}, \times, 1)$. △

Example 2.2.5. Another symmetric monoidal structure on **Set** is $(\mathbf{Set}, +, \emptyset)$. △

Example 2.2.6. Any monoid $(X, \bullet, e)$ can be viewed as a category with objects given by elements of X and only identity morphisms. Then $\bullet$ gives this category a monoidal structure with e as unit. △

Example 2.2.7. The category of pointed sets $\mathbf{Set}_*$ has elements (A, a_0) where $a_0 \in A$, and morphisms $f : (A, a_0) \to (B, b_0)$ such that $f(a_0) = b_0$. It can be given a monoidal structure using the *smash product*, defined as

$$\begin{aligned} \vee : \mathbf{Set}_* \times \mathbf{Set}_* &\to \mathbf{Set}_* \\ ((A, a_0), (B, b_0)) &\mapsto A \times B / (a_0, b) \sim (a, b_0) \quad \text{for all } a, b \in A \times B, \end{aligned}$$

where the basepoint of $(A, a_0) \vee (B, b_0)$ is $[a_0, b_0]$. The unit is $(1, *) \in \mathbf{Set}_*$. △

Example 2.2.8. The category of $\mathbb{K}$ vector spaces $\mathbb{K}-\mathbf{Vect}$ equipped with the tensor product of vector spaces $\otimes$ and unit the trivial vector space $\{0\}$. △

Example 2.2.9. Given a category $\mathbf{C}$, we can define the category $\mathbf{Endo}(C)$, the category of endofunctors on $\mathbf{C}$. It has elements $F : \mathbf{C} \to \mathbf{C}$, and its morphisms are given by natural transformations $\mu : F \to G$. We can give it a non-symmetric monoidal structure using composition of functors and the identity as unit. This gives the monoidal category $(\mathbf{Endo}(C), \circ, \mathrm{id})$. △

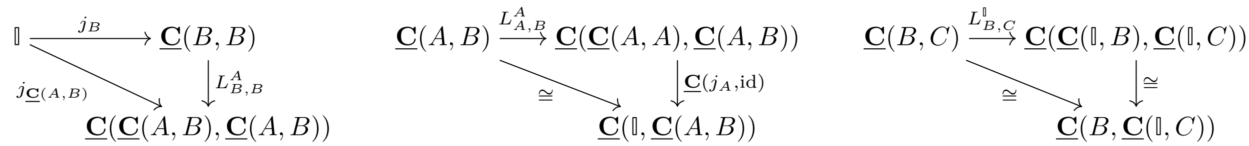
There is another bit of additional structure we can endow on a (symmetric) monoidal category, namely that of being *closed*. This is a closed category such that the closed structure is compatible with the monoidal structure. We will start with the definition of a closed category, which informally states that the set of all morphisms $\mathbf{C}(A, B)$ is an element of $\mathbf{C}$ itself.

Definition 2.2.10. A category $\mathbf{C}$ is called *closed* if there exist

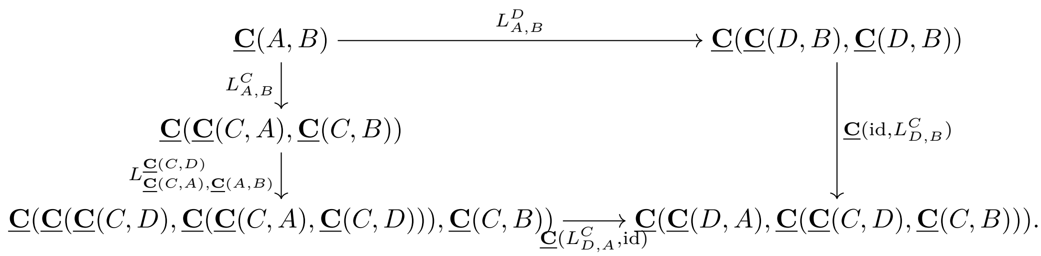
1. a functor $\underline{\mathbf{C}} : \mathbf{C}^{\mathbf{op}} \times \mathbf{C} \to \mathbf{C}$ called the *internal hom*,
2. an object $\mathbb{I} \in \mathbf{C}$ called the *unit object*,
3. a natural isomorphism $\mathrm{id}_{\mathbf{C}} \cong \underline{\mathbf{C}}(\mathbb{I}, \_)$,

4. a extranatural transformation $j_A : \mathbb{I} \to \underline{\mathbf{C}}(A, A)$,
5. a extranatural transformation $L^A_{B,C} : \underline{\mathbf{C}}(A, B) \to \underline{\mathbf{C}}(\underline{\mathbf{C}}(C, A), \underline{\mathbf{C}}(C, B))$,

such that a number of coherence conditions is satisfied. The first three are given by the diagrams



and the last one by the diagram



Finally, we require that the function $\mathbf{C}(A, B) \to \mathbf{C}(\mathbb{I}, \underline{\mathbf{C}}(A, B)), f \mapsto \underline{\mathbf{C}}(A, f)(j_A)$ to be a bijection.

This definition is quite lengthy and a bit opaque, but starts to make sense when $\underline{\mathbf{C}}(A, B)$ is thought of as the collection of morphisms from A to B in $\mathbf{C}$. A morphism $\mathbb{I} \to \underline{\mathbf{C}}(A, B)$ is picking out a single morphism $A \to B$, $j_A : \mathbb{I} \to \underline{C}(A, A)$ picks out the identity morphism and $L^A_{B,C}$ encapsules postcomposing a morphism $C \to A$ with a morphism $A \to B$ to obtain a morphism $C \to B$. The coherence conditions state composition and identities behave as expected. Now we are ready to see what kind of interplay there is between a closed structure and a monoidal structure.

Definition 2.2.11. A symmetric monoidal category $(\mathbf{V}, \otimes, \mathbb{I})$ is called *closed* if there exist a natural isomorphism

$$\mathbf{V}(C \otimes A, B) \cong \mathbf{V}(C, \underline{\mathbf{V}}(A, B)).$$

A different way of putting it is to say the internal hom functor $\underline{\mathbf{V}}(A, \_)$ is right adjoint to $\_ \otimes A$. As expected, every closed monoidal category is a closed category.

Lemma 2.2.12. *A closed symmetric monoidal category* $(\mathbf{V}, \otimes, \mathbb{I})$ *is closed.*

Proof sketch. The monoidal unit $\mathbb{I}$ and the unit object coincide and the internal hom of $\mathbf{V}$ is given by $\underline{\mathbf{V}}(\_, \_)$. The extranatural transformation $j_A : \mathbb{I} \to \underline{\mathbf{V}}(A, A)$ is given by the transpose of the left unitor $\lambda : \mathbb{I} \otimes A \to A$. Finally, the extranatural transformation $L^A_{B,C}$ is given by the transpose

$$\mathbb{I} \to \underline{\mathbf{V}}(\underline{\mathbf{V}}(A, B), \underline{\mathbf{V}}(\underline{\mathbf{V}}(C, A), \underline{\mathbf{V}}(C, B)))$$

where this transpose is constructed using the isomorphism $\mathbf{V}(A, B) \cong \mathbf{V}(\mathbb{I}, \underline{\mathbf{V}}(A, B))$ and regular composition in $\mathbf{V}$. □

Remark 2.2.13. If the monoidal category $(\mathbf{V}, \otimes, \mathbb{I})$ is not necessarily symmetric, we still have a definition of being *closed*. In this case, there is no guarantee the functors $A \otimes \_$ and $\_ \otimes A$ are isomorphic, and either one or both may have an adjoint. Often times one speaks of a *left closed*, *right closed* and *biclosed* category. We will not concern ourselves with this situation in this thesis, but it is good to keep in mind one could drop the requirement of symmetry if one really wanted to.

Having defined monoidal categories, a natural next question would be if there is a corresponding notion for functors. This is the case, and we can even extend this to natural transformations.

Definition 2.2.14. Given two monoidal categories $(\mathbf{V}, \otimes, \mathbb{I})$ and $(\mathbf{V}', \otimes', \mathbb{I}')$ a *lax monoidal functor* $(F, \nabla, \eta) : (\mathbf{V}, \otimes, \mathbb{I}) \to (\mathbf{V}', \otimes', \mathbb{I}')$ consists of a functor $F : \mathbf{V} \to \mathbf{V}'$, a natural transformation $\nabla_{A,B} : F(A) \otimes' F(B) \to F(A \otimes B)$ and a morphism $\eta : \mathbb{I}' \to F(\mathbb{I})$. The lax monoidal functor (F, ∇, η) must respect the associative and unital structure.

Definition 2.2.15. Given two monoidal functors $F, G : (\mathbf{V}, \otimes, \mathbb{I}) \to (\mathbf{V}', \otimes', \mathbb{I}')$ a *monoidal natural transformation* $\mu : F \to G$ is a natural transformation $\mu_A : F(A) \to G(A)$ such that

$$
\begin{array}{ccc}
F(A) \otimes' F(B) & \xrightarrow{\mu_A \otimes' \mu_B} & G(A) \otimes' G(B) \\
\downarrow{\scriptstyle \nabla^F} & & \downarrow{\scriptstyle \nabla^G} \\
F(A \otimes B) & \xrightarrow[\mu_{A \otimes B}]{} & G(A \otimes B)
\end{array}
\qquad
\begin{array}{ccc}
\mathbb{I}' & & \\
\downarrow{\scriptstyle \eta^F} & \searrow{\scriptstyle \eta^G} & \\
F(\mathbb{I}) & \xrightarrow{\mu_{\mathbb{I}}} & G(\mathbb{I})
\end{array}
$$

commute.

We see that we ask lax monoidal functors and natural transformations to respect the monoidal structure present.

In the next section we can see what all this work has been for when consider enriched categories. In an enriched category, the collection of morphisms between two objects is made to be an element of a monoidal category. Some notion of unit and product is needed in order to make sense of an enrichment. The unit will play a similar role similar to the role of the unit in the definition of a closed category, and the product is relied on heavily when coming up with a suitable notion of composition.

2.3 Enriched categories

With the definition of a closed category we made ourselves familiar with the idea of the collection of morphisms from A to B to not be a set, but rather an object in a category. We can take this idea even further, arriving at the idea of *enrichment*. There are many notions of enrichment, but we will contain ourselves to the idea of enrichment in a monoidal category. The main intuition is that the collection of morphisms $\mathbf{C}(A, B)$ is not a set, but rather an element of a monoidal category. This material is taken from [7].

Definition 2.3.1. Let $(\mathbf{V}, \otimes, \mathbb{I})$ be a monoidal category as in Definition 2.2.1. A *category enriched in* $\mathbf{V}$, or a $\mathbf{V}$-*enriched category* $\mathbf{C}$ consists of

- a set of objects $\mathbf{C}_0$,
- for each pair $(A, B) \in \mathbf{C}_0 \times \mathbf{C}_0$ an object $\mathbf{C}(A, B) \in \mathbf{V}_0$ called the *object of morphisms*,
- for each triple $(A, B, C) \in \mathbf{C}_0 \times \mathbf{C}_0 \times \mathbf{C}_0$ a morphism $\circ_{A,B,C} : \mathbf{C}(B, C) \otimes \mathbf{C}(A, B) \to \mathbf{C}(A, C)$ called the *composition morphism*,
- for each object $A \in \mathbf{C}_0$ a morphism $j_A : \mathbb{I} \to \mathbf{C}(A, A)$ called the *identity element*,

such that the diagrams

$$
\begin{array}{ccccc}
(\mathbf{C}(C, D) \otimes \mathbf{C}(B, C)) \otimes \mathbf{C}(A, B) & & \xrightarrow{\quad a \quad} & & \mathbf{C}(C, D) \otimes (\mathbf{C}(B, C) \otimes \mathbf{C}(A, B)) \\
\downarrow{\scriptstyle \circ_{B,C,D} \otimes \mathrm{id}_{\mathbf{C}(A,B)}} & & & & \downarrow{\scriptstyle \mathrm{id}_{\mathbf{C}(C,D)} \otimes \circ_{A,B,C}} \\
\mathbf{C}(B, D) \otimes \mathbf{C}(A, B) & \xrightarrow[\circ_{A,B,D}]{} & \mathbf{C}(A, D) & \xleftarrow[\circ_{A,C,D}]{} & \mathbf{C}(C, D) \otimes \mathbf{C}(A, C)
\end{array}
$$

and

$$
\begin{array}{ccccc}
\mathbf{C}(B, B) \otimes \mathbf{C}(A, B) & \xrightarrow{\circ_{A,B,B}} & \mathbf{C}(A, B) & \xleftarrow{\circ_{A,A,B}} & \mathbf{C}(A, B) \otimes \mathbf{C}(A, A) \\
\uparrow{\scriptstyle j_B \otimes \mathrm{id}_{\mathbf{C}(A,B)}} & \nearrow{\scriptstyle \ell} & & \nwarrow{\scriptstyle r} & \uparrow{\scriptstyle \mathrm{id}_{\mathbf{C}(A,B)} \otimes j_A} \\
\mathbb{I} \otimes \mathbf{C}(A, B) & & & & \mathbf{C}(A, B) \otimes \mathbb{I}
\end{array}
$$

commute.

As with the definition of a closed category, the diagrams we ask to commute may seem a bit daunting. Unpacking them, we see they state that composition is associative and respects units. Another thing to keep in mind is that a morphism $\mathbb{I} \to \mathbf{C}(A, B)$ can be thought of as picking out a morphism $A \to B$ in $\mathbf{C}$.

Example 2.3.2. Every locally small category $\mathbf{C}$ is enriched in $(\mathbf{Set}, \times, 1)$ by definition. The morphisms $j_A : 1 \to \mathbf{C}(A, A)$ are given by sending $* \in 1$ to the identity morphism $\mathrm{id}_C \in \mathbf{C}(A, A)$. △

Example 2.3.3. The category of vector spaces and linear maps **Vect** is enriched over itself, since the set $\mathbf{Vect}(A, B)$ itself is a vector space. △

Example 2.3.4. Consider the monoidal category $(\{\bot, \top\}, \wedge, \top)$ with a single non-identity morphism $\bot \to \top$. Viewing a preordered set as a category, this category is enriched in $(\{\bot, \top\}, \wedge, \top)$. △

Remark 2.3.5. There is a subtle difference between being a closed monoidal category and a category enriched over itself. Every closed monoidal category is enriched over itself in the expected way. However, not every self enriched monoidal category is closed. This is because in a closed monoidal category we require the adjunction

$$\mathbf{C}(A \otimes B, C) \cong \mathbf{C}(A, \underline{\mathbf{C}}(B, C)),$$

where in a self enriched category we do not.

A concrete example of this is the monoidal category **Top** of topological spaces and continuous maps. Its monoidal structure is given by the cartesian product $\times : \mathbf{Top} \times \mathbf{Top} \to \mathbf{Top}$ and unit the one point space $1 \in \mathbf{Top}$. We can endow the sets $\mathbf{Top}(A, B)$ with a topology by giving it the trivial topology. However, in general the product $\_ \times B$ is not left adjoint since it does not preserve colimits. Hence, **Top** can not be a closed category, but can be self enriched.

Here, we also have a corresponding notion of functor and natural transformation, similar to the case of closed categories.

Definition 2.3.6. Given two $\mathbf{V}$-enriched categories $\mathbf{C}$ and $\mathbf{D}$, a $\mathbf{V}$-enriched functor $F : \mathbf{C} \to \mathbf{D}$ consists of a function $F_0 : \mathbf{C}_0 \to \mathbf{D}_0$ and a collection of morphisms in $\mathbf{V}$

$$F_{A,B} : \mathbf{C}(A, B) \to \mathbf{D}(F_0(A), F_0(B))$$

such that F respects composition, that is the diagram

$$\begin{array}{ccc} \mathbf{C}(B, C) \otimes \mathbf{C}(A, B) & \xrightarrow{\circ_{A,B,C}} & \mathbf{C}(A, C) \\ \Big\downarrow{\scriptstyle F_{B,C} \otimes F_{A,B}} & & \Big\downarrow{\scriptstyle F_{A,C}} \\ \mathbf{D}(F_0(B), F_0(C)) \otimes \mathbf{D}(F_0(B), F_0(A)) & \xrightarrow[\circ_{F_0(A),F_0(B),F_0(C)}]{} & \mathbf{D}(F_0(C), F_0(A) \end{array}$$

commutes in $\mathbf{V}$, and such that F respects identities, that is the diagram

$$\begin{array}{ccc} & \mathbb{I} & \\ {\scriptstyle j_A} \swarrow & & \searrow {\scriptstyle j_{F_0(A)}} \\ \mathbf{C}(A, A) & \xrightarrow[F_{A.A}]{} & \mathbf{D}(F_0(A), F_0(A)) \end{array}$$

commutes in $\mathbf{V}$.

Comparing this to the definition of a ordinary functor, we see we still assign objects of $\mathbf{C}$ to objects of $\mathbf{D}$, and similarly for morphisms. The diagrams assure the functor respects composition and identities.

Definition 2.3.7. We define the category **V-Cat** to have $\mathbf{V}$-enriched categories as objects and $\mathbf{V}$-enriched functors as morphisms.

We also have a suitable notion of $\mathbf{V}$-enriched natural transformations.

Definition 2.3.8. Let $\mathbf{C}, \mathbf{D} \in \mathbf{V}\text{-}\mathbf{Cat}$ be two $\mathbf{V}$ enriched categories and let $F, G : \mathbf{C} \to \mathbf{D}$ be a pair of $\mathbf{V}$-enriched functors. A $\mathbf{V}$-*enriched natural transformation* $\mu : F \to G$ is a family of morphisms in $\mathbf{V}$

$$\mu_A : \mathbb{I} \to \mathbf{D}(F(A), G(A))$$

such that for any pair $A, B \in \mathbf{C}_0$ the diagram

$$\begin{array}{ccccc}
 & \mathbf{C}(A,B) \otimes \mathbb{I} \xrightarrow{G_{A,B} \otimes \mu_A} \mathbf{D}(G(A), G(B)) \otimes \mathbf{D}(F(A), G(A)) & & & \\
\overset{r^{-1}}{\nearrow} & & & \searrow^{\circ_{F(A),G(A),G(B)}} & \\
\mathbf{C}(A,B) & & & & \mathbf{D}(F(A), G(B)) \\
\underset{\ell^{-1}}{\searrow} & & & \nearrow_{\circ_{F(A),F(B),G(B)}} & \\
 & \mathbb{I} \otimes \mathbf{C}(A,B) \xrightarrow[\mu_B \otimes F_{A,B}]{} \mathbf{D}(F(B), G(B)) \otimes \mathbf{D}(F(A), F(B)) & & &
\end{array}$$

commutes.

Using the idea that morphisms $\mathbb{I} \to \mathbf{D}(F(A), G(A))$ correspond to morphisms $F(A) \to G(A)$, we recover the usual definition of a natural transformation as an family of morphisms indexed by $\mathbf{C}$. The commuting diagram is enriched version of the naturality square

$$\begin{array}{ccccc}
A & \qquad & F(A) & \xrightarrow{\mu_A} & G(A) \\
{\scriptstyle f}\big\downarrow & & {\scriptstyle F(f)}\big\downarrow & & \big\downarrow{\scriptstyle G(f)} \\
B & & F(B) & \xrightarrow[\mu(B)]{} & G(B).
\end{array}$$

One might wonder if it is possible to change the enriching category. This is indeed the case whenever it can be done in a structured fashion. The structure is provided by a lax monoidal functor between enriching categories, and is exhibited in the following definition.

Proposition 2.3.9. *Given two monoidal categories* $\mathbf{V}$ *and* $\mathbf{V}'$*, a lax monoidal functor* $F : \mathbf{V} \to \mathbf{V}'$ *induces a functor*

$$F_* : \mathbf{V}\text{-}\mathbf{Cat} \to \mathbf{V}'\text{-}\mathbf{Cat}.$$

Given a $\mathbf{V}$*-enriched category* $\mathbf{C}$ *we can construct a* $\mathbf{V}'$*-enriched category* $F_*(C)$ *such that*

- $F_*(\mathbf{C})$ *has the same objects as* $\mathbf{C}$
- $F_*(\mathbf{C})(A, B) = F(\mathbf{C}(A, B))$
- *The composition, unit, associator and unitor morphisms in* $F_*(\mathbf{C})$ *are the images of those of* $\mathbf{C}$ *composed with the structure morphisms of the lax monoidal functor* (F, ∇, η).

It might have stood out that a lot of the extra structures introduced so far on categories extend not only to functors, but also to natural transformations. This is a general pattern, where we not only consider morphisms between categories (functors), but also morphisms between morphisms of categories (natural transformations). In the upcoming section, we will provide a general framework within which this fits.

2.4 2-Categories

In this section we take away the most relevant material from [8]. The main idea is that of a 2-category, which has objects, morphisms between objects called 1-morphisms, and morphisms between morphisms, called 2-morphisms. A prime examples is that of the category **Cat**, the category which has categories as objects, functors as 1-morphisms and natural transformations as 2-morphisms. Many of the 2-categories considered in this paper are akin to **Cat**, albeit that we restrict ourselves to categories with extra structure. Having an idea, we now present a concrete definition.

Definition 2.4.1. A *strict 2-category* has a collection of objects, such that for any two objects A and B there exists a category $\mathrm{Hom}(A, B)$ of morphisms from A to B.

Remark 2.4.2. Another way of putting it is that a 2-category is a category enriched in **Cat**.

One may notice the word *strict* in the definition above. There are weaker notions of 2-categories, where we relax some conditions on composition of morphisms. We will not be concerning ourselves with those however, and only focus on the strict case.

Many of the regular notions from category theory such as functors, limits and adjunctions generalize to the context of 2-categories. We will only give the definition of a functor here, but more can be found in [8]

Definition 2.4.3. A *strict 2-functor* between 2-categories is a **Cat**-enriched functor.

Of course, this itself again forms a category.

Definition 2.4.4. We will denote the category of strict 2-categories and strict 2-functors by $2 - \mathbf{Cat}$.

Without knowing it, we have already encountered many examples of 2-categories.

Example 2.4.5. As suggested earlier, the category **Cat** of categories, functors and natural transformations is a 2-category. △

Example 2.4.6. Given a monoidal category V, **V-Cat** is a 2-category consisting of V-enriched categories, functors and natural transformations. △

Example 2.4.7. The category of monoidal categories, monoidal functors and monoidal natural transformations **MonCat** is a 2-category. △

Example 2.4.8. Although we have not defined all of the following concepts, we would like to remark the category of closed categories, closed functors and closed natural transformations forms a 2-category. △

A nice example which ties everything together is the following.

Example 2.4.9. In the previous section, we have seen a change of enriching category $F_* : \mathbf{V\text{-}Cat} \to \mathbf{V'\text{-}Cat}$ induced by a monoidal functor $F : V \to V'$. It turns out this functor also respects monoidal natural transformations. Using our newly developed language of 2-categories, we can tie this together nicely. Consider the functor sending a monoidal category to the category of V-enriched categories.

$$\begin{aligned} \mathbf{Enr} : \mathbf{MonCat} &\to 2\text{-}\mathbf{Cat} \\ V &\mapsto \mathbf{V\text{-}Cat} \\ (F : V \to V') &\mapsto (F_* : \mathbf{V\text{-}Cat} \to \mathbf{V'\text{-}Cat}). \end{aligned}$$

We see F_* is an example of a 2-functor. △

Of course, there is nothing stopping us from defining a 3-category, where we ask there to be objects, 1-morphisms between objects, 2-morphisms between 1-morphisms and 3-morphisms between 2-morphisms. Extending this pattern, we arrive at n-categories and even ∞-categories. For this thesis, we will only be needing 2-categories, but it is good to know this idea can be extended.

2.5 Enrichment of the category of algebras

This section is an exposition of the material found in [1][Sec. 3], and forms the starting point for the rest of the thesis. The punchline of this section is that for any sufficiently well behaved endofunctor $F : \mathbf{C} \to \mathbf{C}$ the category of F-algebras is enriched in the category of F-coalgebras. We will be stating many results without proof, for which we defer the reader to [1]. Throughout this section we will fix a monoidal category $(\mathbf{C}, \otimes, \mathbb{I})$ and a lax monoidal endofunctor $(F, \nabla, \eta) : (\mathbf{C}, \otimes, \mathbb{I}) \to (\mathbf{C}, \otimes, \mathbb{I})$. Later in this section, we will place some extra condition on $\mathbf{C}$ and F. We will denote the category of F-algebras by $\mathbf{Alg}$ and the category of F-coalgebras by $\mathbf{CoAlg}$, and we will often denote algebras algebras by $(A, \alpha), (B, \beta) \in \mathbf{Alg}$ and coalgebras by $(C, \chi), (D, \delta) \in \mathbf{CoAlg}$.

Let us start with what we will think of as the enriched functions. There are many different ways to view these functions; as functions indexed by a coalgebra, as partial functions or as measurings.

Definition 2.5.1. Let $(A, \alpha), (B, \beta) \in \mathbf{Alg}$ and let $(C, \chi) \in \mathbf{CoAlg}$. We call a morphism $\varphi : C \otimes A \to B$ in $\mathbf{C}$ a *measuring from A to B by C* if it makes the diagram

$$\begin{array}{ccccc}
 & F(C) \otimes F(A) & \xrightarrow{\nabla_{C,A}} & F(C \otimes A) & \xrightarrow{F(\varphi)} & F(B) \\
 {}^{\chi \otimes \mathrm{id}_{F(A)}}\nearrow & & & & & \downarrow \beta \\
C \otimes F(A) & & & & & \\
 {}_{\mathrm{id}_C \otimes \alpha}\searrow & & & & & \\
 & C \otimes A & & \xrightarrow{\quad\quad\varphi\quad\quad} & & B
\end{array}$$

commute. The set of all measurings from A to B by C is denoted $\mathfrak{m}_C(A, B)$.

Precomposing a measuring with a coalgebra or algebra morphism or postcomposing with an algebra morphism will again result in a measuring. Hence, we have a functor

$$\mathfrak{m} : \mathbf{CoAlg}^{\mathbf{op}} \times \mathbf{Alg}^{\mathbf{op}} \times \mathbf{Alg} \to \mathbf{Set}.$$

If we place some conditions on $\mathbf{C}$ and $F : \mathbf{C} \to \mathbf{C}$, we will see this functor is representable in each of its three arguments.

Remark 2.5.2. The monoidal unit $\mathbb{I}$ carries a coalgebra structure through $\eta : \mathbb{I} \to F(\mathbb{I})$. We claim a measuring from A to B by $\mathbb{I}$ and an algebra morphism $A \to B$ are equivalent. This can be seen by drawing the diagram

$$\begin{array}{ccccccc}
 & F(I) \otimes F(A) & \xrightarrow{\nabla_{C,A}} & F(I \otimes A) & \xrightarrow{\cong} & F(A) & \longrightarrow & F(B) \\
 {}^{\eta \otimes \mathrm{id}_{F(A)}}\nearrow & & & {}^{\cong}\nearrow & & & & \\
\mathbb{I} \otimes F(A) & & & & & \downarrow \alpha & & \downarrow \beta \\
 {}_{\mathrm{id}_{\mathbb{I}} \otimes \alpha}\searrow & & & & & & & \\
 & \mathbb{I} \otimes A & & \xrightarrow{\quad\cong\quad} & & A & \longrightarrow & B
\end{array}$$

where the isomorphisms are supplied by the lax monoidal structure of F. We conclude $\mathfrak{m}_{\mathbb{I}}(A, B) \cong \mathbf{Alg}(A, B)$.

Given two measurings $\varphi : C \otimes A \to A'$ and $\psi : D \otimes A' \to A''$ we would like to compose them to obtain a measuring from A to A''. It would make sense this would be a measuring by $D \otimes C$. However, $D \otimes C$ does not yet have a coalgebra structure. We can give it a coalgebra structure by the composition

$$D \otimes C \xrightarrow{\delta \otimes \chi} F(D) \otimes F(C) \xrightarrow{\nabla_{D,C}} F(D \otimes C).$$

The composite of measurings φ and ψ is given by

$$D \otimes C \otimes A \xrightarrow{\mathrm{id}_D \otimes \varphi} D \otimes A' \xrightarrow{\psi} A''.$$

One can verify this composite is indeed a measuring by verifying $\psi \circ (\mathrm{id}_D \otimes \varphi)$ makes the diagram in Definition 2.5.1 commute. We will denote this composition of measurings by

$$\circ_{\mathfrak{m}} : m_D(B, T) \times m_C(A, B) \to m_{D \otimes C}(A, T).$$

Definition 2.5.3. Let $A, B \in \mathbf{Alg}$. The *category of measurings from A to B* has as objects pairs $(C, \varphi) \in \mathbf{CoAlg} \times \mathfrak{m}_C(A, B)$ and morphisms $f : (C, \varphi) \to (D, \psi)$ are morphisms $f \in \mathbf{CoAlg}(C, D)$ such that

$$\begin{array}{ccc} C \times A & \xrightarrow{\varphi} & B \\ {\scriptstyle f\times\mathrm{id}}\downarrow & \nearrow_{\psi} & \\ D \times A & & \end{array}$$

commutes.

Definition 2.5.4. Let $A, B \in \mathbf{Alg}$. The *universal measuring* is the terminal object in the category of measurings from A to B, if it exists. It is denoted $(\underline{\mathbf{Alg}}(A, B), \mathrm{ev})$

Proposition 2.5.5. *If the universal measuring exists, it is a representing object of* $\mathfrak{m}\_(A, B)$.

Proof sketch. We aim to show $\mathfrak{m}_C(A, B) \cong \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A, B))$. Given $\varphi : C \otimes A \to B$, we know there exists a unique coalgebra morphism

$$\begin{array}{ccc} C \times A & \xrightarrow{\varphi} & B \\ {\scriptstyle f\times\mathrm{id}}\downarrow & \nearrow_{\mathrm{ev}} & \\ \underline{\mathbf{Alg}}(A, B) \times A & & \end{array}$$

since $\underline{\mathbf{Alg}}(A, B)$ is the terminal object in the category of measurings from A to B. Conversely, given a coalgebra morphism $f \in \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A, B))$ we obtain the measuring

$$\mathrm{ev} \circ (f \otimes \mathrm{id}_A) : C \otimes A \xrightarrow{f \otimes \mathrm{id}_A} \underline{\mathbf{Alg}}(A, B) \otimes A \xrightarrow{\mathrm{ev}} B.$$

This gives a natural bijection $\mathfrak{m}_C(A, B) \cong \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A, B))$. □

With this in hand, we can define a new type of initial object, called a C-initial algebra.

Definition 2.5.6. Given a coalgebra $C \in \mathbf{CoAlg}$, we call an algebra $A \in \mathbf{Alg}$ a *C-initial algebra* if for all $B \in \mathbf{Alg}$ there exists a unique measuring

$$\varphi : C \otimes A \to B.$$

The *terminal C-initial algebra* is the terminal object in the full subcategory of $\mathbf{Alg}$ spanned by C-initial algebras, if it exists.

Remark 2.5.7. There are many equivalent formulations of this definition. For now, we point out being C-initial is equivalent to asking $\mathfrak{m}_C(A, B) \cong 1$ for all $B \in \mathbf{Alg}$. The definition of the initial algebra I is that $\mathbf{Alg}(I, B) \cong 1$ for all $B \in \mathbf{Alg}$. Using that a measuring by $\mathbb{I}$ is equivalent to an algebra morphism, we can rephrase this as

$$\mathbf{Alg}(I, B) \cong m_{\mathbb{I}}(I, B) \cong 1.$$

From this perspective, being a C-initial algebra is a generalization of being an initial algebra, giving us a parameter C we can vary.

In order to develop the theory further, we assume $\mathbf{C}$ and $F : \mathbf{C} \to \mathbf{C}$ satisfy some extra conditions. From now on, we ask $\mathbf{C}$ to be a symmetric and closed monoidal category, and that it is *locally presentable*. Furthermore we ask F to *accessible*. The motivation for these assumptions is that by Proposition 2.1.20 and Proposition 2.1.21, the free and cofree functors $\mathrm{Fr} : \mathbf{C} \to \mathbf{Alg}^F$ and $\mathrm{Cof} : \mathbf{C} \to \mathbf{CoAlg}^F$ exist. We will make use of these functors when constructing the representing objects of $\mathfrak{m}$. Moreover, as remarked earlier the categories $\mathbf{Alg}^F$ and $\mathbf{CoAlg}^F$ are also locally presentable whenever $\mathbf{C}$ is, hence complete and cocomplete.

Another consequence is that F is not only lax monoidal, but also *lax closed*, respecting the closed structure of $\mathbf{C}$. In particular, this means we have a transformation

$$\tilde{\nabla}_{A,B} : F(\underline{\mathbf{C}}(A,B)) \to \underline{\mathbf{C}}(F(A),F(B))$$

natural in both arguments.

We have already seen the first representing object of $\mathfrak{m}$, namely $\underline{\mathbf{Alg}}(A,B)$, the terminal object in the category of measurings from A to B. However, up until now there was guarantee of the existence of this object. Using the cofree functor, we can construct $\underline{\mathbf{Alg}}(A,B)$ as an equalizer.

Proposition 2.5.8. *Given $A, B \in \mathbf{Alg}$, consider the following composite*

$$f : \mathrm{Cof}(\underline{\mathbf{C}}(A,B)) \xrightarrow{\chi_{\mathrm{Cof}}} F(\mathrm{Cof}(\underline{\mathbf{C}}(A,B))) \xrightarrow{F(\varepsilon_{\mathrm{Cof}})} F(\underline{\mathbf{C}}(A,B)) \xrightarrow{\tilde{\nabla}_{A,B}} \underline{\mathbf{C}}(F(A),F(B)) \xrightarrow{\beta_*} \underline{\mathbf{C}}(F(A),B)$$

where $\varepsilon_{\mathrm{Cof}}$ is the counit of the cofree-forgetful adjunction. The universal measuring $\underline{\mathbf{Alg}}(A,B)$ is given by the following equalizer in $\mathbf{CoAlg}$

$$\underline{\mathbf{Alg}}(A,B) \overset{\mathrm{eq}}{\dashrightarrow} \mathrm{Cof}(\underline{\mathbf{C}}(A,B)) \underset{\tilde{f}}{\overset{\mathrm{Cof}(\underline{\mathbf{C}}(\alpha,B))}{\rightrightarrows}} \mathrm{Cof}(\underline{\mathbf{C}}(F(A),B)),$$

where $\tilde{f}$ is the transpose of f. The evaluation map $\mathrm{ev} : \underline{\mathbf{Alg}}(A,B) \otimes A \to B$ *is obtained as the transpose of the composite*

$$\tilde{\mathrm{ev}} : \underline{\mathbf{Alg}}(A,B) \xrightarrow{\mathrm{eq}} \mathrm{Cof}(\underline{\mathbf{C}}(A,B)) \xrightarrow{\varepsilon_{\mathrm{Cof}}} \underline{\mathbf{C}}(A,B).$$

The next representing object is constructed in a similar fashion, this time making use of the free functor.

Proposition 2.5.9. *Given $A \in \mathbf{Alg}$ and $C \in \mathbf{CoAlg}$, consider the composite*

$$f : C \otimes F(A) \xrightarrow{\chi \otimes \mathrm{id}} F(C) \otimes F(A) \xrightarrow{\nabla_{C,A}} F(C \otimes A) \xrightarrow{F(\eta_{\mathrm{Fr}})} F(\mathrm{Fr}(C \otimes A)) \xrightarrow{\alpha_{\mathrm{Fr}}} \mathrm{Fr}(C \otimes A),$$

where η_{Fr} is the unit of the free-forgetful adjunction. The representing object of $\mu_C(A, \_)$ exists, is denoted $C \triangleright A$ and is given by the coequalizer

$$\mathrm{Fr}(C \otimes F(A)) \underset{\tilde{f}}{\overset{\mathrm{Fr}(\mathrm{id}_C \otimes \alpha)}{\rightrightarrows}} \mathrm{Fr}(C \otimes A) \overset{\mathrm{coeq}}{\dashrightarrow} C \triangleright A$$

where $\tilde{f}$ is the transpose of f.

To construct our final representing object we make use of the fact that $\mathbf{C}$ is closed.

Proposition 2.5.10. *Given $B \in \mathbf{Alg}$ and $C \in \mathbf{CoAlg}$, we can endow the set $\underline{\mathbf{C}}(C,B)$ with the coalgebra structure*

$$F(\underline{\mathbf{C}}(C,B)) \xrightarrow{\tilde{\nabla}_{C,B}} \underline{\mathbf{C}}(F(C),F(B)) \xrightarrow{\beta_* \chi^*} \underline{\mathbf{C}}(A,B).$$

We denote this coalgebra by $[C,B]$, and it is a representing object of $\mathfrak{m}_C(\_, B)$.

Summarizing the above, we have that for $A, B \in \mathbf{Alg}$ and $C \in \mathbf{CoAlg}$ we have

$$\mathfrak{m}_C(A,B) \cong \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A,B)) \cong \mathbf{Alg}(A,[C,B]) \cong \mathbf{Alg}(C \triangleright A, B).$$

All these natural isomorphisms will be very useful down the road. Being able to effectively utilize all the structure present is one of the challenges, and one where examples can be of great help.

Using the representing objects, we have different ways to take an algebra and construct a coalgebra from it. Two of these we give special attention, and are defined below.

Definition 2.5.11. Let $I \in \mathbf{Alg}$ denote the initial algebra. We define the functor

$$\begin{aligned}(\_)^* : \mathbf{CoAlg^{op}} &\to \mathbf{Alg} \\ C &\mapsto [C, I]\end{aligned}$$

and call C^* the *dual algebra of* C. We also define the functor

$$\begin{aligned}(\_)^\circ : \mathbf{Alg^{op}} &\to \mathbf{CoAlg} \\ A &\mapsto \underline{\mathbf{Alg}}(A, I)\end{aligned}$$

and call A° the *dual coalgebra of* A

Due to the natural isomorphisms above, we have a special relation between these duals.

Lemma 2.5.12. *The functors* $(\_)^\circ$ *and* $(\_)^*$ *form a dual adjunction*

$$\mathbf{Alg}(A, C^*) \cong \mathbf{CoAlg}(C, A^\circ).$$

The punchline is that $\underline{\mathbf{Alg}}(\_, \_)$ gives the category of algebras an enrichment in the category of coalgebras. For an enrichment to make sense, the enriching category must be a monoidal category. We have already seen a glimpse of the monoidal structure on **CoAlg** and we give more details here.

Proposition 2.5.13. *The category* **CoAlg** *has a symmetric monoidal structure for which the forgetful functor* $U : \mathbf{CoAlg} \to \mathbf{C}$ *is strong symmetric monoidal.*

Proof sketch. Given two coalgebras $C, D \in \mathbf{CoAlg}$, the product $C \otimes D$ is given by

$$C \otimes D \xrightarrow{\chi \otimes \delta} F(C) \otimes F(D) \xrightarrow{\nabla_{C,D}} F(C \otimes D).$$

The unit is given by $\mathbb{I}$, with coalgebra structure $\eta : \mathbb{I} \to F(\mathbb{I})$. □

Theorem 2.5.14. *The category* **Alg** *is enriched, copowered, and powered over the symmetric monoidal category* **CoAlg** *respectively via*

$$\mathbf{Alg}^{\mathrm{op}} \times \mathbf{Alg} \xrightarrow{\underline{\mathbf{Alg}}(\_,\_)} \mathbf{CoAlg}, \quad \mathbf{CoAlg} \times \mathbf{Alg} \xrightarrow{\_\triangleright\_} \mathbf{Alg}, \quad \mathbf{CoAlg}^{\mathrm{op}} \times \mathbf{Alg} \xrightarrow{[\_,\_]} \mathbf{Alg}.$$

Remark 2.5.15. We have already seen how to compose measurings. As it turns out, this precisely corresponds to the enriched composition. Enriched composition is given by a coalgebra morphism

$$\circ : \underline{\mathbf{Alg}}(A', A'') \otimes \underline{\mathbf{Alg}}(A, A') \to \underline{\mathbf{Alg}}(A, A'').$$

This is induced by the composition of measurings

$$(\underline{\mathbf{Alg}}(A', A'') \otimes \underline{\mathbf{Alg}}(A, A')) \otimes A \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_{A,A'}} \underline{\mathbf{Alg}}(A', A'') \otimes A' \xrightarrow{\mathrm{ev}_{A',A''}} A''.$$

Now that we have the preliminaries covered, we will continue to showcase many examples. The aim is that each new example adds a new layer of complexity in order to get an increasingly better feel for this enrichment.

3 Examples

In the previous section we have built a general theory showing that the category of algebras of an endofunctor $F : \mathbf{C} \to \mathbf{C}$ is enriched in the category of coalgebras of that same endofunctor. We also saw several objects representing the measuring functor $\mathfrak{m} : \mathbf{CoAlg}^{\mathbf{op}} \times \mathbf{Alg}^{\mathbf{op}} \times \mathbf{Alg} \to \mathbf{Set}$ in each of its arguments. Lastly, we assigned significance to C-initial algebras, which are algebras that have the same desirable properties as initial algebras, but are more abundant. We did have to impose some conditions on F and $\mathbf{C}$ though. We asked that $\mathbf{C}$ is a monoidal category which is locally presentable and closed and that F is an accessible monoidal endofunctor.

In this section, we would like to bring the theory to life by studying some examples. Fixing an endofunctor F in each example, we will consider its initial algebra and terminal coalgebra, the representing objects of $\mathfrak{m}$ and C-initial algebras. Starting with some straightforward examples to get a feel, we will work towards those which are important in computer science. For most of the examples we will only give the results and the necessary ingredients to obtain them. However, there are two selected examples where we will go into more detail, showing intermediary steps and confirming the results. The first of the two selected examples is the monoid type, which is the first example where we see the monoidal structure in action. The second is the list type, which is the first example which has semantics commonly used in computer science. To get a better feel for C-initial algebras, we will be providing a new constructive proof regarding some specific C-initial algebras already mentioned in [1] in the section on the natural numbers type. However, we will leave most of our results regarding C-initial algebras to Section 5.

In all our examples we consider the monoidal category $(\mathbf{Set}, \times, 1)$ which is locally presentable and closed. The endofunctors $F : \mathbf{Set} \to \mathbf{Set}$ in the examples will all be accessible and we will provide the monoidal structure of the endofunctor explicitly in each example. Once an endofunctor is fixed we will denote the category of F-algebras by $\mathbf{Alg}$ and the category of F-coalgebras by $\mathbf{CoAlg}$. Recall that due to $\mathbf{C}$ being locally presentable and F being accessible, the left adjoint of the forgetful functor $U : \mathbf{Alg} \to \mathbf{Set}$ exists. For the same reasons the right adjoint to forgetful functor $U : \mathbf{CoAlg} \to \mathbf{Set}$. The free and cofree functors will be denoted by Fr and Cof respectively, and will be used throughout this section. Finally, we will usually denote F-algebras by $(A, \alpha), (B, \beta) \in \mathbf{Alg}$ and F-coalgebras by $(C, \chi), (D, \delta) \in \mathbf{CoAlg}$.

3.1 The unit type

First off, we wish to start with a simple example. We will see the initial algebra will have $1 \in \mathbf{Set}$ as initial algebra. Hence, this example corresponds to the unit type in functional programming, the type with only one value.

Let $F : \mathbf{Set} \to \mathbf{Set}$ be the functor given by

$$\begin{aligned} F : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1. \end{aligned}$$

Its lax monoidal structure is given by the identity $\mathrm{id}_1 : 1 \to 1$ and the isomorphism that $1 \times 1 \xrightarrow{\cong} 1$.

The category of F-algebras, denoted $\mathbf{Alg}$ has elements $\alpha : 1 \to A, * \mapsto a_0$ denoted (A, a_0) and morphisms $f : (A, a_0) \to (B, b_0)$ satisfying $f(a_0) = b_0$. It can be regarded as the category of pointed sets, or as the under category $1/\mathbf{Set}$. If it is clear we are considering a pointed set, we will sometimes simply write $A \in \mathbf{Alg}$.

The category of F-coalgebras, denoted $\mathbf{CoAlg}$ has elements $\chi : C \to 1$ and morphisms $f : C \to D$, Which is isomorphic to the category of sets.

3.1.1 Initial and terminal objects

The initial algebra is given by $\mathrm{id} : (1, *) \to (1, *)$, since for every algebra (A, a_0) there exists exactly one pointed morphism out of $\mathrm{id} : 1 \to 1$. For any pre-initial algebra (I, i_0) the unique morphism $(1, *) \to (I, i_0)$ must be epic. This can only be the case whenever the underlying function $1 \to I$ is surjective. The only set which satisfies this condition is 1, so we conclude there are no pre-initial algebras besides the initial algebra.

The terminal coalgebra is given by 1 since the $\mathbf{CoAlg}$ is isomorphic to $\mathbf{Set}$. A subterminal coalgebra is any set A such that $A \rightarrowtail 1$ is injective. The only sets satisfying this are $\emptyset$ and 1, so we conclude these are the only subterminal coalgebras.

3.1.2 Measurings

Definition 3.1.1. Let $(A, a_0), (B, b_0) \in \mathbf{Alg}$ and $(C, \chi) \in \mathbf{CoAlg}$. A *measuring from A to B by C* is a function $\varphi : C \times A \to B$, such that $\varphi(c, a_0) = b_0$ for all $c \in C$.

We find that in this case, a measuring $\varphi : C \times A \to B$ is nothing but a family of algebra morphisms $A \to B$ indexed by C. In an upcoming section we will see the functor $\mathfrak{m}$ is representable in each of its three arguments by providing the representing objects.

The *category of measurings from A to B* has as elements measurings $\varphi : C \times A \to B$, and as morphisms $f : \varphi \to \psi$ functions $f : C \to D$ such that

$$\begin{array}{ccc} C \times A & \xrightarrow{\varphi} & B \\ {\scriptstyle f \times \mathrm{id}} \downarrow & \nearrow_{\psi} & \\ D \times A & & \end{array}$$

commutes. Another way of putting it is that the category of measurings from A to B is the over category $\mathbf{Set}/\mathbf{Alg}(A, B)$.

3.1.3 Free and cofree functors

In order to compute the representing objects, we wish to make use of the constructions provided in of [1, Sec. 3]. These constructions make use of the free and cofree functor.

The free functor is given by

$$\begin{aligned} \mathrm{Fr} : \mathbf{Set} &\to \mathbf{Alg} \\ A &\mapsto (A + 1, \mathrm{inr}) \\ f &\mapsto f + \mathrm{id}_1 \, . \end{aligned}$$

The unit and counit are given by $\eta_A^{\mathrm{Fr}} : A \to A + 1, a \mapsto a$ and $\varepsilon_A^{\mathrm{Fr}} : A + 1 \to A, * \mapsto \alpha(*), a \mapsto a$. One can easily check these satisfy the triangle identities and hence form an adjunction.

Since $\mathbf{CoAlg} \cong \mathbf{Set}$, the forgetful functor U is the identity. Hence the cofree functor is the identity as well.

3.1.4 Representing objects

Now we have everything in place to compute the representing objects of $\mathfrak{m}$. We will not provide the explicit computation, but using the free and cofree functor defined above one should be able to repeat the constructions in Section 3 of [1] to obtain the same results. In a later example we do provide full details.

The first representing object is the terminal object in the category of measurings, denoted $\underline{\mathbf{Alg}}(A, B)$. Its underlying set is given by $\mathbf{Set}(A \setminus \{a_0\}, B)$ with the coalgebra structure

$$\begin{aligned} \mathrm{ev} : \mathbf{Set}(A \setminus \{a_0\}, B) \times A &\to B \\ (f, a) &\mapsto \begin{cases} f(a) & \text{if } a \neq a_0 \\ b_0 & \text{if } a = a_0, \end{cases} \end{aligned}$$

Indeed, given a measuring $\varphi : C \times A \to B$, there is exactly one coalgebra morphism $f : C \to \underline{\mathbf{Alg}}(A, B)$ which results in a morphism of measurings, namely

$$\begin{aligned} f : C &\to \underline{\mathbf{Alg}}(A, B) \\ c &\mapsto \varphi(c)(\_)|_{A \setminus \{a_0\}}. \end{aligned}$$

Conversely, given a coalgebra morphism $f : C \to \underline{\mathbf{Alg}}(A, B)$, composing with ev yields a measuring $\mathrm{ev} \circ (f \times \mathrm{id}) : C \times A \to B$. So, we see there is a bijective correspondence between maps $f : C \to \mathbf{Set}(A \setminus \{a_0\}, B)$ and measurings $\varphi : C \times A \to B$. Moreover, this correspondence is natural in C, A and B, resulting in an isomorphism

$$\mathfrak{m}_C(A, B) \cong \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A, B)),$$

from which we can conclude $\mathfrak{m}\_(A, B)$ is represented by $\underline{\mathbf{Alg}}(A, B)$. Moreover, a measuring by 1 results in an algebra morphism $\varphi : 1 \times A \to B$. So, $\mathbf{Alg}(A, B) \cong \mathfrak{m}_1(A, B) \cong \mathbf{CoAlg}(1, \underline{\mathbf{Alg}}(A, B))$.

Next, we want to represent $\mathfrak{m}_C(A, \_)$. The representing object is denoted $C \triangleright A$, and has as underlying set

$$C \times A/(c, a_0) \sim (c', a_0),$$

for all $c, c' \in C$, with $[c, a_0]$ giving the algebra structure. We have the natural bijection

$$\mathfrak{m}_C(A, B) \cong \mathbf{Alg}(C \triangleright A, B),$$

sending a measuring φ to its unique factorization through $C \triangleright A$. A measuring φ always uniquely factorizes since $\varphi(c, a_0) = \varphi(c', a_0) = b_0$ for all $c, c' \in C$. We can now state $\mathfrak{m}_C(A, \_)$ is represented by $C \triangleright A$.

Lastly, given $C \in \mathbf{CoAlg}, B \in \mathbf{Alg}$, we can turn $\mathbf{Set}(C, B)$ into an algebra by taking the constant function $\text{const}_{b_0} : C \to B, c \mapsto b_0$ to be the preferred point in $\mathbf{Set}(C, B)$. We call this the *convolution algebra* and denote it $[C, B]$. We claim an algebra morphism $(A, a_0) \to ([C, B], \text{const}_{b_0})$ corresponds to a measuring $\varphi : C \times A \to B$. Indeed, given a morphism $\tilde{\varphi} : (A, a_0) \to ([C, B], \text{const}_{b_0})$, we know it satisfies $\tilde{\varphi}(a_0)(c) = b_0$ for all $c \in C$, which coincides with the condition on a measuring $\varphi : C \times A \to B$. Moreover, this correspondence is natural in A, B and C, so

$$\mathfrak{m}_C(A, B) \cong \mathbf{Alg}(A, [C, B]),$$

and we see $\mathfrak{m}_C(\_, B)$ is represented by $[C, B]$.

3.1.5 C-initial algebras

Given a coalgebra C, we say $A \in \mathbf{Alg}$ is a *C-initial algebra* if for all algebras $X \in \mathbf{Alg}$ there exists a unique coalgebra morphism

$$C \to \underline{\mathbf{Alg}}(A, X).$$

Our first remark is that if $C = \emptyset$ every algebra is C-initial, since $\emptyset$ is the initial object in $\mathbf{Set}$. For $C \neq \emptyset$, being C-initial implies the underlying set of $\underline{\mathbf{Alg}}(A, X)$ given by $\mathbf{Set}(A \setminus \{a_0\}, X)$ contains only a single element. This is because for any other $1 \neq D \in \mathbf{CoAlg}$, there exists more than one function $C \to D$. For $\mathbf{Set}(A \setminus \{a_0\}, X)$ to contain only one function for all X, it must be that $A \setminus \{a_0\} = \emptyset$, hence $A \cong 1$. The only algebra satisfying this is the initial algebra $\text{id} : 1 \to 1$. We conclude for $C \neq \emptyset$, the only C-initial algebra is $\text{id} : 1 \to 1$.

3.2 The empty type

Perhaps an even more fundamental example is that of the empty type. The empty type is the type with no values, which means we wish to consider a functor which has the empty set as initial algebra.

Let $F : \mathbf{Set} \to \mathbf{Set}$ be the functor given by

$$\begin{aligned} F : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto A. \end{aligned}$$

Its lax monoidal structure is given given by identities.

The category of F-algebras, denoted $\mathbf{Alg}$ has elements $\alpha : A \to A$ denoted (A, α). Morphisms $f : (A, \alpha) \to (B, \beta)$ are given by functions $f : A \to B$ which make the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{f} & B \end{array}$$

The category of F-coalgebras, denoted $\mathbf{CoAlg}$ has elements $\chi : C \to C$ denoted (C, χ). Morphisms $f : (C, \chi) \to (D, \delta)$ are given by functions $f : C \to D$ which make the following diagram commute:

$$\begin{array}{ccc} C & \xrightarrow{f} & D \\ \uparrow{\scriptstyle \chi} & & \uparrow{\scriptstyle \delta} \\ C & \xrightarrow{f} & D \end{array}$$

We observe the category **CoAlg** is isomorphic to the category of **Alg** by definition. Nevertheless, we will distinguish between these categories to see which role they play in the upcoming constructions.

3.2.1 Initial and terminal objects

The initial algebra is given by $\mathrm{id} : \emptyset \to \emptyset$, since for every algebra (A, α) there exists exactly morphism out of $\mathrm{id} : \emptyset \to \emptyset$. The only preinitial algebra is given by the initial algebra itself, since there exist no epimorphisms out of the empty set other than the identity.

The terminal coalgebra is given by $\mathrm{id} : 1 \to 1$ since there exists exactly one morphism into $\mathrm{id} : 1 \to 1$ for every coalgebra (C, χ). A subterminal coalgebra is then any coalgebra (C, χ) such that $(C, \chi) \rightarrowtail (1, \mathrm{id})$ is injective. The only coalgebras satisfying this are $(\emptyset, \mathrm{id})$ and $(1, \mathrm{id})$ and we conclude $\emptyset$ is the only subterminal coalgebra besides the terminal algebra.

3.2.2 Measurings

Definition 3.2.1. Let $(A, \alpha), (B, \beta) \in$ **Alg** and $(C, \chi) \in$ **CoAlg**. A *measuring from A to B by C* is a function $\varphi : C \times A \to B$, such that $\varphi(c, \alpha(a)) = \beta(\varphi(\chi(c), a))$ for all $(c, a) \in C \times A$.

Another way to state the condition on measurings is to say the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi(\chi(c))(\_)} & B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\varphi(c)(\_)} & B. \end{array}$$

must commute for all $c \in C$.

As a first, we consider a toy example.

Example 3.2.2. Consider the algebras $(\mathbb{N}, (\cdot 4)), (\mathbb{N}, (\cdot 8))$ and the coalgebra $(\mathbb{N}, (\cdot 2))$. A measuring from $(\mathbb{N}, (\cdot 4))$ to $(\mathbb{N}, (\cdot 8))$ by $(\mathbb{N}, (\cdot 2))$ is given by

$$\begin{aligned} \varphi : \mathbb{N} \times \mathbb{N} &\to \mathbb{N} \\ (i, j) &\mapsto i \cdot j^2. \end{aligned}$$

To verify this is a measuring, we must check $\varphi(i, 4j) = 8 \cdot \varphi(2i, j)$, which is the case since

$$\varphi(i, 4j) = i \cdot (4j)^2 = 16ij^2 = 8 \cdot 2i \cdot j^2 = 8 \cdot \varphi(2 \cdot i, j).$$

△

The *category of measurings from A to B* has as elements measurings $\varphi : C \times A \to B$, and as morphisms $f : \varphi \to \psi$ functions $f : C \to D$ such that

$$\begin{array}{ccc} C \times A & \xrightarrow{\varphi} & B \\ {\scriptstyle f \times \mathrm{id}}\downarrow & \nearrow{\scriptstyle \psi} & \\ D \times A & & \end{array}$$

3.2.3 Free and cofree functors

In order to compute the representing objects in the upcoming section, we need the left and right adjoint of the forgetful functors. They are the free and cofree functor and are respectively given by

$$\begin{aligned} \mathrm{Fr} : \mathbf{Set} &\to \mathbf{Alg} \\ A &\mapsto \begin{pmatrix} \coprod_{i \in \mathbb{N}} A_i & \xrightarrow{\mathrm{Sh}} \coprod_{i \in \mathbb{N}} A_i \\ A_i \ni a & \mapsto a \in A_{i+1} \end{pmatrix}, \end{aligned}$$

where $A_i = A$, and

$$\begin{aligned} \mathrm{Cof} : \mathbf{Set} &\to \mathbf{CoAlg} \\ C &\mapsto \begin{pmatrix} \prod_{i\in\mathbb{N}} C_i & \xrightarrow{\mathrm{CoSh}} \prod_{i\in\mathbb{N}} C_i \\ (c_0, c_1, c_2, \dots) & \mapsto (c_1, c_2, \dots) \end{pmatrix}, \end{aligned}$$

where $C_i = C$. Their behavior on functions is in both cases given by applying the function component wise.

To verify these are adjoints, one must verify $\mathbf{Set}(A, B) \cong \mathbf{Alg}(\mathrm{Fr}(A), B)$ and $\mathbf{Set}(C, D) \cong \mathbf{Alg}(C, \mathrm{Cof}(D))$. This can be done by constructing explicit natural bijections. The first bijection is given by $\mathbf{Set}(A, B) \to \mathbf{Alg}(\mathrm{Fr}(A), B), f \mapsto \coprod_{i\in\mathbb{N}} \beta^i \circ f$ with inverse $\mathbf{Alg}(\mathrm{Fr}(A), B) \to \mathbf{Set}(A, B), \coprod_{i\in\mathbb{N}} f_i \mapsto f_0$. The second bijection is given by $\mathbf{Set}(C, D) \to \mathbf{CoAlg}(C, \mathrm{Cof}(D)), f \mapsto \langle f \circ \chi^i \rangle_{i\in\mathbb{N}}$ with inverse $\mathbf{CoAlg}(C, \mathrm{Cof}(D)) \to \mathbf{Set}(C, D), \langle f_i \rangle_{i\in\mathbb{N}} \mapsto f_0$. Checking naturality is straightforward and left to the reader.

3.2.4 Representing objects

We wish to define the terminal object in the category of measurings from A to B, which is also a representation of $\mathfrak{m}\_(A, B)$. To define the terminal object in the category of measurings, we define the coalgebra $\underline{\mathbf{Alg}}(A, B)$, which has as underlying set

$$\{(f_0, f_1, \dots) \in \prod_{i\in\mathbb{N}} \mid \beta \circ f_{i+1} = f_i \circ \alpha\}.$$

Its coalgebra structure is given by shifting the entries to the left, or more precisely by the restriction of CoSh to $\underline{\mathbf{Alg}}(A, B)$.

The terminal object in the category of measurings from A to B is then given by

$$\begin{aligned} \mathrm{ev} : \underline{\mathbf{Alg}}(A, B) \times A &\to B \\ ((f_i)_{i\in\mathbb{N}}, a) &\mapsto f_0(a). \end{aligned}$$

Indeed, given a measuring $\varphi : C \times A \to B$, there is exactly one coalgebra morphism $f : C \to \underline{\mathbf{Alg}}(A, B)$ which results in a morphism of measurings, namely $f : C \to \underline{\mathbf{Alg}}(A, B), c \mapsto (\varphi(\chi^i(c))(\_))_{i\in\mathbb{N}}$. Conversely, given a morphism $f : C \to \underline{\mathbf{Alg}}(A, B)$, composing with ev yields a measuring $\mathrm{ev} \circ (f \times \mathrm{id}) : C \times A \to B$. This yields a bijective correspondence between morphisms $f : C \to \underline{\mathbf{Alg}}(A, B)$ and measurings $\varphi : C \times A \to B$. This correspondence is natural in C, A and B, resulting in an isomorphism

$$\mathfrak{m}_C(A, B) \cong \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A, B)),$$

from which we can conclude $\mathfrak{m}\_(A, B)$ is represented by $\underline{\mathbf{Alg}}(A, B)$.

We also wish to represent $\mathfrak{m}_C(A, \_)$. The representing object is denoted $C \triangleright A$, and has as underlying set

$$\coprod_{i\in\mathbb{N}} (C \times A)_i / \sim,$$

where the equivalence relation $\sim$ is the equivalence relation generated by $(\chi(c), a)_{i+1} \sim (c, \alpha(a))_i$. Its algebra structure is given by shifting to the right, namely by the function Sh. We have the natural bijection

$$\mathfrak{m}_C(A, B) \cong \mathbf{Alg}(C \triangleright A, B),$$

which sends a measuring φ to the algebra morphism $\tilde{\varphi} : C \triangleright A \to B, (c, a)_i \mapsto \varphi(\chi^i(c))(a)$ and which sends the algebra morphism $\tilde{\varphi} : C \triangleright A \to B$ to the measuring $\varphi = \tilde{\varphi}|_{(C\times A)_0}$. We can then state $\mathfrak{m}_C(A, \_)$ is represented by $C \triangleright A$.

Finally, given $C \in \mathbf{CoAlg}, B \in \mathbf{Alg}$, we define the convolution algebra $[C, B]$ to have $\mathbf{Set}(C, B)$ as underlying set with algebra structure given by

$$\begin{aligned} \beta_* \chi^* : \mathbf{Set}(C, B) &\to \mathbf{Set}(C, B) \\ f &\mapsto \beta \circ f \circ \chi. \end{aligned}$$

In doing so, an algebra morphism $(A, \alpha) \to ([C, B], \beta_* \chi^*)$ corresponds to a measuring $\varphi : C \times A \to B$. This can be seen by simply computing the condition for an algebra morphism $\tilde{\varphi} : (A, \alpha) \to ([C, B], \beta_* \chi^*)$, which is $\beta \circ \tilde{\varphi}(a) \circ \chi = \tilde{\varphi} \circ \alpha(a)$ for all $a \in A$. This corresponds to the condition on a measuring from A to B by C, which is $\beta(\varphi(\chi(c), a)) = \varphi(c, \alpha(a))$, for all $(c, a) \in C \times A$. Moreover, this correspondence is natural in A, B and C, so

$$\mathfrak{m}_C(A, B) \cong \mathbf{Alg}(A, [C, B]),$$

and we see $\mathfrak{m}_C(\_, B)$ is represented by $[C, B]$.

3.2.5 C-initial algebras

Given a coalgebra C, we say $A \in \mathbf{Alg}$ is a *C-initial algebra* if for all algebras $X \in \mathbf{Alg}$ there exists a unique coalgebra morphism

$$C \to \underline{\mathbf{Alg}}(A, X).$$

Our first remark is that if $C = \emptyset$ every algebra is C-initial, with $(1, \mathrm{id})$ being the terminal $\emptyset$-initial algebra. For $C \neq \emptyset$ the only C-initial algebra is $(\emptyset, \mathrm{id})$, since every other algebra would give too much freedom in constructing maps from C to $\underline{\mathbf{Alg}}(A, X)$ for there to be a unique morphism. This means $(\emptyset, \mathrm{id})$ is also the terminal C-initial algebra for all $C \in \mathbf{CoAlg}$.

3.3 The monoid type

In this example we will see begin to see the flexibility measurings provide. This is also the motivation for discussing more details regarding the representing objects than previously done. We will be considering the type which takes values in a fixed monoid $(X, \bullet, e)$. This type is not frequently used in computer science, but it does give us insight into the general theory. To study this type we must find a functor which has it as an initial algebra. This functor exists and is the next thing we will be defining.

Let $F : \mathbf{Set} \to \mathbf{Set}$ be the functor given by

$$\begin{aligned} F : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto X \\ f &\mapsto \mathrm{id}_X \end{aligned}$$

where $(X, \bullet, e)$ is some fixed monoid in $\mathbf{Set}$. The lax monoidal structure is given by monoid structure on X, namely $\nabla_{A,B} : X \times X \to X, (x', x) \mapsto x' \bullet x$ and $\eta : 1 \to X, * \mapsto e$. Since we will need it later, we also remark the lax closed structure is given by $\tilde{\nabla}_{A,B} : X \to \mathbf{Set}(X, X), x \mapsto r_x$ where $r_x : X \to X, x' \mapsto x' \bullet x$ is the function which multiplies an element of X by $x \in X$ from the right.

The category $\mathbf{Alg}$ has elements $\alpha : X \to A$, with morphisms $f : (A, \alpha) \to (B, \beta)$ given by commuting squares (or triangles if you prefer)

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\beta} \\ A & \xrightarrow{f} & B. \end{array}$$

The category $\mathbf{CoAlg}$ has elements $\chi : C \to X$, with morphisms $f : (C, \chi) \to (D, \delta)$ given by commuting squares

$$\begin{array}{ccc} X & \xrightarrow{\mathrm{id}_X} & X \\ {\scriptstyle\chi}\uparrow & & \uparrow{\scriptstyle\delta} \\ C & \xrightarrow{f} & D. \end{array}$$

Notice how these categories correspond to under category $X/\mathbf{Set}$ and the over category $\mathbf{Set}/X$ respectively.

3.3.1 Initial and terminal objects

The initial algebra and the terminal coalgebra are both given by

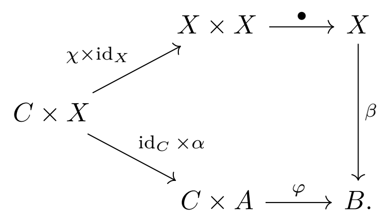
$$\mathrm{id}_X : X \to X.$$

Any algebra for which the function $\alpha : X \to A$ is surjective gives a preinitial algebra. Similarly, any coalgebra for which the function $\chi : C \to X$ is injective will give a subterminal coalgebra.

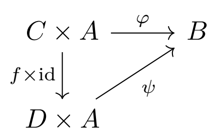
3.3.2 Measuring

Definition 3.3.1. Let $(A, \alpha), (B, \beta) \in \mathbf{Alg}$ and $(C, \chi) \in \mathbf{CoAlg}$. A *measuring from A to B by C* is a function $\varphi : C \times A \to B$, such that $\varphi(c, \alpha(x)) = \beta(\chi(c) \bullet x)$ for all $c \in C$ and $x \in X$.

Comparing this to the condition on a regular algebra morphism $f : A \to B$, given by $f(\alpha(x)) = \beta(x)$, we see a coalgebra C introduces some sort of twisting. The above condition on φ is equivalent to asking the following diagram commutes



The *category of measurings from A to B* has as elements measurings $\varphi : C \times A \to B$, and as morphisms $f : \varphi \to \psi$ functions $f : C \to D$ such that



commutes.

Here we provide a few examples of measurings to get a feel for them.

Example 3.3.2. Consider the algebra (X, id_X) and the coalgebra (X, id_X). A trivial example of a measuring is the monoid multiplication $\bullet : X \times X \to X$. △

Example 3.3.3. Let $A \in \mathbf{Set}$ be arbitrary and consider the algebra

$$\begin{aligned} \alpha : X &\to \mathbf{Set}(A, X) \\ x &\mapsto \mathrm{const}_m \end{aligned}$$

and the coalgebra (X, id). A measuring from $(\mathbf{Set}(A, X), \alpha)$ to itself by (X, id) is given by

$$\begin{aligned} \varphi : X \times \mathbf{Set}(A, X) &\to \mathbf{Set}(A, X) \\ (x, f) &\mapsto (a \mapsto f(a) \bullet x). \end{aligned}$$

Indeed, we can verify $\varphi(x', \mathrm{const}_m) = \mathrm{const}_{x' \bullet x} = \alpha(x' \bullet x)$. △

Example 3.3.4. Let X^* denote the set of finite lists with elements in X. This is far from a precise definition, but if the reader is interested in one, see Section 3.5 We can give it an algebra structure by $\alpha : X \to X^*, x \mapsto [x]$, and a coalgebra structure by $\chi : X^* \to X, [\,] \mapsto e, x : xs \mapsto x : \chi(xs)$. If acquainted with functional programming, one might recognize the definition of fold in the coalgebra structure.

A measuring from (X^*, α) to itself by (X^*, χ) is given by

$$\begin{aligned} \varphi : X^* \times X^* &\to X^* \\ (xs', [\,]) &\mapsto [\,] \\ (xs', x : xs) &\mapsto (\chi(xs') \bullet x) : xs, \end{aligned}$$

collapsing the entire list xs' onto the first element of the list $x : xs$ using the monoidal structure on X. To verify this is a measuring we check $\varphi(xs', \alpha(x)) = \varphi(xs', [x]) = [\chi(xs') \bullet x] = \alpha(\chi(xs') \bullet x)$.

Note this is not the only measuring we would define. Another measuring is given by

$$\begin{aligned} \varphi' : X^* \times X^* &\to X^* \\ xs', [\,] &\mapsto [\,] \\ xs', x : xs &\mapsto [\chi(xs') \bullet x], \end{aligned}$$

simply ignoring the last part of the list $x : xs$. Again we can verify this is a measuring, using the exact same equation as before to find $\varphi(xs', \alpha(x)) = \varphi(xs', [x]) = [\chi(xs') \bullet x] = \alpha(\chi(xs') \bullet x)$. △

3.3.3 Free and cofree functors

Again, we wish to make use of the left and right adjoint of the forgetful functors. The left adjoint free functor is given by

$$\begin{aligned} \mathrm{Fr} : \mathbf{Set} &\to \mathbf{Alg} \\ A &\mapsto (A + X, \alpha_{\mathrm{Fr}}) \\ f &\mapsto f + \mathrm{id}_X\,. \end{aligned}$$

where $\alpha_{\mathrm{Fr}} = \mathrm{inr}$. The unit and counit are given by $\eta^{\mathrm{Fr}}_A : A \to A + X, a \mapsto a$ and $\varepsilon^{\mathrm{Fr}}_A : A + X \to A, x \mapsto \alpha(x), a \mapsto a$. One can easily check these satisfy the triangle identities and hence form an adjunction.

The right adjoint cofree functor is given by

$$\begin{aligned} \mathrm{Cof} : \mathbf{Set} &\to \mathbf{CoAlg} \\ C &\mapsto (C \times X, \chi_{\mathrm{Cof}}) \\ f &\mapsto (f \times \mathrm{id}_X). \end{aligned}$$

where $\chi_{\mathrm{Cof}} = \mathrm{pr}_X$. The unit and counit are given by $\eta^{\mathrm{Cof}}_C : C \to C \times X, c \mapsto (c, \chi(c))$ and $\varepsilon^{\mathrm{Cof}}_C : C \times X \to C, (c, x) \mapsto c$. Again, one can easily check the triangle identities to verify the adjunction.

3.3.4 Representing objects

In this section we compute the representing objects of $\mathfrak{m}$. We will employ the constructions stated in Section 3 of [1] and verify the constructed objects have the desired properties afterwards.

The representing object $\underline{\mathbf{Alg}}(A, B)$ is given by the following equalizer

$$\underline{\mathbf{Alg}}(A, B) \overset{\mathrm{eq}}{\dashrightarrow} \mathbf{Set}(A, B) \times X \underset{\Phi}{\overset{\mathbf{Set}(\alpha, B) \times \mathrm{id}_X}{\rightrightarrows}} \mathbf{Set}(X, B) \times X$$

where Φ is the transpose of

$$\tilde{\Phi} : \mathbf{Set}(A, B) \times X \xrightarrow{\chi_{\mathrm{Cof}}} X \xrightarrow{F(\varepsilon^{\mathrm{Cof}})} X \xrightarrow{\tilde{\nabla}_{A,B}} \mathbf{Set}(X, X) \xrightarrow{\beta_*} \mathbf{Set}(X, B).$$

If we compute the composition we find $\tilde{\Phi} : \mathbf{Set}(A, B) \times X \to \mathbf{Set}(X, B), (f, x) \mapsto \beta \circ r_x$, where we recall $r_x : X \to X, x' \mapsto x' \bullet x$. Taking its transpose $\Phi = \mathrm{Cof}(\tilde{\Phi}) \circ \eta$ we find the explicit formula for Φ to be

$$\begin{aligned} \Phi : \mathbf{Set}(A, B) \times X &\to \mathbf{Set}(X, B) \times X \\ (f, x) &\mapsto (x' \mapsto \beta(x' \bullet x), x). \end{aligned}$$

Now that we know explicitly which maps we want to equalize, we can compute

$$\underline{\mathbf{Alg}}(A,B) = \{(f,x) \in \mathbf{Set}(A,B) \times X \mid (f \circ \alpha)(x') = \beta(x \bullet x')\}.$$

with coalgebra structure given by $\mathrm{pr}_X : \underline{\mathbf{Alg}}(A,B) \to X, (f,x) \mapsto x$. We would like to verify $\underline{\mathbf{Alg}}(A,B)$ is indeed a representing object. In other words, we would like to construct a natural bijection

$$\Psi : \mathfrak{m}_C(A,B) \xrightarrow{\cong} \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A,B)).$$

This bijection is given by sending a measuring $\varphi \in \mathfrak{m}_C(A,B)$ to

$$\Phi(\varphi)(c) = (a \mapsto \varphi(c,a), \chi(c)).$$

This is a well-defined coalgebra morphism since $\chi(c) = \mathrm{pr}_X(\Phi(\varphi)(c))$. Its inverse Ψ^{-1} sends a coalgebra morphism $f : C \to \underline{\mathbf{Alg}}(A,B)$ to the measuring

$$\begin{aligned} \Psi^{-1}(f) : C \times A &\to B \\ (c,a) &\mapsto (\mathrm{pr}_{\mathbf{Set}(A,B)} \circ f)(c)(a). \end{aligned}$$

We must verify this is a measuring, which we can do using that f is a coalgebra morphism. Observe $(\mathrm{pr}_{\mathbf{Set}(A,B)} \circ f)(c)(\alpha(x)) = \beta((\mathrm{pr}_X \circ f)(c) \bullet x) = \beta(\chi(c), x)$. It is easy to verify the Ψ and Ψ^{-1} are inverses of each other and that the bijection is natural.

The object $C \triangleright A$ is given by the coequalizer

$$(C \times X) + X \underset{\Phi}{\overset{\mathrm{id}_C \times \alpha + \mathrm{id}_X}{\rightrightarrows}} (C \times A) + X \overset{\mathrm{coeq}}{\dashrightarrow} C \triangleright A,$$

where Φ is the transpose of

$$\tilde{\Phi} : C \times X \xrightarrow{\chi \times \mathrm{id}_X} X \times X \xrightarrow{\nabla_{A,B}} X \xrightarrow{F(\eta^{\mathrm{Fr}})} X \xrightarrow{\alpha_{\mathrm{Fr}}} (C \times A) + X.$$

If we compute the composition we find $\tilde{\Phi} : C \times X \to (C \times A) + X, (c,x) \mapsto \chi(c) \bullet x$. Taking its transpose $\Phi = \varepsilon \circ \mathrm{Fr}(\tilde{\Phi})$ we find the explicit formula for Φ to be

$$\begin{aligned} \Phi : (C \times X) + X &\to (C \times A) + X \\ (c,x) &\mapsto \chi(c) \bullet x \\ x' &\mapsto x'. \end{aligned}$$

Now that we know explicitly which maps we want to coequalize, we can compute

$$C \triangleright A \cong \big((C \times A) + X\big)/(c, \alpha(x)) \sim (\chi(c) \bullet x).$$

with algebra structure given by $X \to C \triangleright A, x \mapsto [x]$. A slightly more intuitive may be to write $C \triangleright A$ as the pushout

$$\begin{array}{ccc} C \times X & \xrightarrow{f} & X \\ {\scriptstyle \mathrm{id}_C \times \alpha}\downarrow & \ulcorner & \downarrow \\ C \times A & \longrightarrow & C \triangleright A \end{array}$$

where $f : C \times X \to X, (c,x) \mapsto \chi(c) \bullet x$. To verify $C \triangleright A$ is a representing object, we construct a natural bijection

$$\Psi : \mathfrak{m}_C(A,B) \xrightarrow{\cong} \mathbf{Alg}(C \triangleright A, B).$$

This bijection is given by sending a measuring $\varphi \in \mathfrak{m}_C(A,B)$ to

$$\begin{aligned} \Psi(\varphi) : C \triangleright A &\to B \\ [c,a] &\mapsto \varphi(c,a) \\ [x] &\mapsto \beta(x). \end{aligned}$$

We must check this is well-defined, which is the case since $\Psi(\varphi)([c, \alpha(x)]) = \varphi(c, \alpha(x)) = \beta(\chi(c) \bullet x) = \Psi(\varphi)([\chi(c) \bullet x])$ and the equivalence relation defining $C \triangleright A$ is generated by $(c, \alpha(x)) \sim \chi(c) \bullet x$. It is a well-defined algebra morphism since $[x] = \beta(x)$ by definition. Its inverse Ψ^{-1} sends an algebra morphism $f : C \triangleright A \to B$ to the measuring

$$\begin{aligned}\Psi^{-1}(f) : C \times A &\to B \\ (c, a) &\mapsto f([c, a]).\end{aligned}$$

We must verify this is a measuring, which we can do using that f is an algebra morphism. Observe $f([c, \alpha(x)]) = f([\chi(c) \bullet x]) = \beta([\chi(c) \bullet x])$, which shows $\Psi^{-1}(f)$ is a measuring. It is easy to verify the Ψ and Ψ^{-1} are inverses of each other and that the bijection is natural.

Lastly, the representing object $[C, B]$ has as underlying set $\mathbf{Set}(C, B)$ with algebra structure given by the composition

$$X \xrightarrow{\tilde{\nabla}_{C,B}} \mathbf{Set}(X, X) \xrightarrow{\beta_* \circ \chi^*} \mathbf{Set}(C, B).$$

If we compute the composition we find the algebra structure on $\mathbf{Set}(C, B)$ is given by $X \to \mathbf{Set}(C, B), x \mapsto (c \mapsto \beta(\chi(c) \bullet x))$. To verify $[C, B]$ is a representing object, we construct a natural bijection

$$\Psi : \mathfrak{m}_C(A, B) \xrightarrow{\cong} \mathbf{Alg}(A, [C, B]).$$

This bijection is given by sending a measuring $\varphi \in \mathfrak{m}_C(A, B)$ to

$$\Psi(\varphi)(a) = (c \mapsto \varphi(c, a)).$$

We must check this is an algebra morphism, which is the case since $\Psi(\varphi)(\alpha(x)) = (c \mapsto \varphi(c, \alpha(x))) = (c \mapsto \beta(\chi(c) \bullet x))$. Its inverse Ψ^{-1} sends an algebra morphism $f : A \to [C, B]$ to the measuring

$$\begin{aligned}\Psi^{-1}(f) : C \times A &\to B \\ (c, a) &\mapsto f(a)(c).\end{aligned}$$

We must verify this is a measuring, which we can do using that f is an algebra morphism. Observe $f(\alpha(x))(c) = \beta(\chi(c) \bullet x)$ which shows $\Psi^{-1}(f)$ is a measuring. It is easy to verify the Ψ and Ψ^{-1} are inverses of each other and that the bijection is natural.

3.3.5 C-initial algebras

One of the characterizations of a C-initial algebra A is that $C \triangleright A$ is isomorphic to the initial object. In our case this would mean that

$$X \cong \big((C \times A) + X\big)/(c, \alpha(x)) \sim (\chi(c) \bullet x),$$

which is the case if and only if $\alpha : X \to A$ is surjective. We conclude any preinitial algebra $\alpha : X \twoheadrightarrow A$ is also C-initial for any coalgebra C. This also implies the terminal algebra $X \to 1$ is the terminal C-initial algebra for all coalgebras C.

3.4 The natural numbers type

In [1], the main example provided was the example of the natural numbers type. In this section we will repeat and elaborate on that example. Finally, we will add a new constructive proof regarding C-initial algebras. To study this type we must find a functor which has it as an initial algebra.

Consider the functor

$$\begin{aligned}F : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1 + A \\ f &\mapsto \mathrm{id}_1 + f\end{aligned}$$

The lax monoidal structure is given by

$$\begin{aligned} \nabla_{A,B} : 1 + A \times 1 + B &\to 1 + (A \times B) \\ (*, b) &\mapsto * \\ (a, *) &\mapsto * \\ (a, b) &\mapsto (a, b) \end{aligned}$$

and $\mathrm{inr} : 1 \to 1 + 1$.

The category of F-algebras has elements $\alpha : 1 + A \to A$ denoted (A, α). Sometimes we will denote $\alpha(*)$ by 0_A, and more generally $\alpha^n(0_A)$ will be denoted as n_A. Morphisms $f : (A, \alpha) \to (B, \beta)$ are given by functions $f : A \to B$ which make the following diagram commute:

$$\begin{array}{ccc} 1 + A & \xrightarrow{\mathrm{id} + f} & 1 + B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{f} & B \end{array}$$

The category of F-coalgebras has elements $\chi : C \to 1 + C$, denoted (C, χ). Morphisms $f : (C, \chi) \to (D, \delta)$ are given by functions $f : C \to D$ which make the following diagram commute:

$$\begin{array}{ccc} 1 + C & \xrightarrow{\mathrm{id}_1 + f} & 1 + D \\ {\scriptstyle \chi}\uparrow & & {\scriptstyle \delta}\uparrow \\ C & \xrightarrow{f} & D \end{array}$$

3.4.1 Initial and terminal objects

The initial algebra is the given by the set of natural numbers $\mathbb{N}$. Its algebra structure is given by succession, resembling the Peano axioms of the natural numbers. To be more precise, it is given by

$$\begin{aligned} 1 + \mathbb{N} &\to \mathbb{N} \\ * &\mapsto 0 \\ n &\mapsto n + 1. \end{aligned}$$

To verify $\mathbb{N}$ is indeed the initial algebra, let (A, α) be an arbitrary algebra. The unique algebra morphism $\mathbb{N} \to A$, denoted $\mathsf{i}_A : \mathbb{N} \to A$ is forced by the commutative square

$$\begin{array}{ccc} 1 + \mathbb{N} & \xrightarrow{\mathrm{id} + \mathsf{i}_A} & 1 + A \\ \downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathbb{N} & \xrightarrow{\mathsf{i}_A} & A, \end{array}$$

and defined as $\mathsf{i}_A : \mathbb{N} \to A$
$i \mapsto \alpha^{i+1}(*)$. The strict preinitial algebras are all of the form $\mathbb{n} := \{0, 1, \ldots, n\}$, with algebra structure given by

$$\begin{aligned} 1 + \mathbb{n} &\to \mathbb{n} \\ * &\mapsto 0 \\ i &\mapsto \min(i, n). \end{aligned}$$

The terminal coalgebra is given by $\mathbb{N}_\infty := \mathbb{N} + \{\infty\}$. Its coalgebra structure is given by

$$\begin{aligned} \mathbb{N}_\infty &\to 1 + \mathbb{N}_\infty \\ i &\mapsto \begin{cases} * & \text{if } i = 0 \\ i - 1 & \text{otherwise} \end{cases} \\ \infty &\mapsto \infty. \end{aligned}$$

To verify this is indeed the terminal coalgebra, let (C, χ) be an arbitrary coalgebra. The unique coalgebra morphism $C \to \mathbb{N}_\infty$, denoted $!_C : C \to \mathbb{N}_\infty$' is forced by the commutative square

$$\begin{array}{ccc} 1 + C & \xrightarrow{\text{id} + !_C} & 1 + \mathbb{N}_\infty \\ {\scriptstyle\chi}\big\uparrow & & \big\uparrow \\ C & \xrightarrow{\quad !_C \quad} & \mathbb{N}_\infty, \end{array}$$

and defined as $!_C : C \to \mathbb{N}_\infty, c \mapsto \begin{cases} 0 \text{ if } \chi(c) = * \\ 1 + !_C(c') \text{ if } \chi(c) = c' \end{cases}$. Subterminal coalgebras are given by subsets of $\mathbb{N}_\infty$ which are closed under the coalgebra structure. Some of the subterminal coalgebras share the same underlying set as preinitial algebras. In order to distinguish between the two, we will use $(\_)^\dagger$ to denote the coalgebra whenever it might be ambiguous. They are given by $\emptyset, \mathbb{0}^\dagger, \mathbb{1}^\dagger, \ldots, \mathbb{N}^\dagger$ as well as the union of $\{\infty\}$ with any of the these sets. The coalgebra structure is simply a restriction of the coalgebra structure of $\mathbb{N}_\infty$. Finally, we would like to point out $\mathbb{n}^\dagger$ has universal property. Since $\mathbb{n}^\dagger$ is subterminal, there exists at most one morphism $C \to \mathbb{n}^\dagger$ into it for every $C \in \mathbf{CoAlg}$. This morphism exists if and only if $!_C(c) \leq n$ for all $c \in C$, which is the universal property of $\mathbb{n}^\dagger$.

3.4.2 Measuring

Definition 3.4.1. Let $(A, \alpha), (B, \beta) \in \mathbf{Alg}$ and $(C, \chi) \in \mathbf{CoAlg}$. A *measuring from A to B by C* is a function $\varphi : C \times A \to B$ satisfying

1. $\varphi(c)(0_A) = 0_B$ for all $c \in C$
2. $\varphi(c)(\alpha(a)) = 0_B$ if $\chi(c) = *$
3. $\varphi(c)(\alpha(a)) = \beta(\varphi(c')(a))$ if $\chi(c) = c'$.

Intuitively, a measuring is a partial algebra morphism from A to B, where the coalgebra C determines up to which point we are computing the algebra morphism.

In the examples below, we wish to explore the new possibilities measurings have opened up for us, as well as show the limitations of measurings. The first example is taken from Section 2 of [1].

Example 3.4.2. Notice that the algebra structures involved can be quite limiting. We claim there does not exist a total algebra morphism $f : \mathbb{n} \to \mathbb{N}$. Denoting the algebra structure of $\mathbb{n}$ by $\alpha_n : 1 + \mathbb{n} \to \mathbb{n}$, we can create a contradiction by noting that for $n \in \mathbb{n}$, we must have $f(n) = f(\alpha_n(n)) = 1 + f(n)$ which can not be the case. However, we are able to define a measuring from $\mathbb{n}$ to $\mathbb{N}$. We define the measuring

$$\begin{aligned} \varphi : \mathbb{n}^\dagger \times \mathbb{n} &\to \mathbb{N} \\ (i, j) &\mapsto \min(i, j) \end{aligned}$$

We note that checking this is a measuring can be done by comparing the definition of measuring and that of φ. Moreover, comparing with the definition also reveals this morphism is unique. △

We would also like to provide a non-example to see a situation in which the measuring does not exist.

Example 3.4.3. Say we would like to define a measuring $\varphi : \mathbb{n}^\dagger \times (\mathbb{n} - 1) \to \mathbb{N}$. By definition φ must satisfy

$$\begin{aligned} \varphi(0, j) &= 0 && \text{for all } j \in \mathbb{n} - 1 \\ \varphi(i, 0) &= 0 && \text{for all } i \in \mathbb{n}^\dagger \\ \varphi(i, \alpha_{n-1}(j)) &= 1 + \varphi(i - 1, j) && \text{for all } i \neq 0, j \in \mathbb{n} - 1. \end{aligned}$$

This forces the definition of φ to be $\varphi(i, j) = \min(i, j)$ for all $0 \leq i, j \leq n - 1$. However, we run in to trouble when computing $\varphi(n, n - 1)$. Since $\alpha_{n-1}(n - 2) = \alpha_{n-1}(n - 1) = n - 1$, we must have that

$$\varphi(n, n - 1) = \varphi(n, \alpha_{n-1}(n - 2)) = 1 + \varphi(n - 1, n - 2) = 1 + \min(n - 1, n - 2) = n - 1,$$

but also

$$\varphi(n, n-1) = \varphi(n, \alpha_{n-1}(n-1)) = 1 + \varphi(n-1, n-1) = 1 + \min(n-1, n-1) = n.$$

We see there does not exist a measuring $\varphi : \mathbb{n}^\dagger \times (\mathbb{n} - 1) \to \mathbb{N}$. This is due to the fact that we tried to take the inductive definition of φ one step to far. △

The *category of measurings from* A *to* B has as elements measurings $\varphi : C \times A \to B$, and as morphisms $f : \varphi \to \psi$ functions $f : C \to D$ such that

$$\begin{array}{ccc} C \times A & \xrightarrow{\varphi} & B \\ {\scriptstyle f \times \mathrm{id}} \downarrow & \nearrow_{\psi} & \\ D \times A & & \end{array}$$

commutes.

3.4.3 Free and cofree functors

Again, we aim to use the free and cofree functors to construct the representing objects. The free functor is defined by mapping $A \in \mathbf{Set}$ to the algebra

$$\begin{aligned} \alpha : 1 + \mathbb{N} \times (A + \{0_A\}) &\to \mathbb{N} \times (A + \{0_A\}) \\ * &\mapsto (0, 0_A) \\ (i, 0_A) &\mapsto (i+1, 0_A) \\ (i, a) &\mapsto (i+1, a) \end{aligned}$$

The behavior of Fr on functions $f : A \to B$ is forced and given by

$$\begin{aligned} \mathrm{Fr}(f) : \mathbb{N} \times (A + \{0_A\}) &\to \mathbb{N} \times (B + \{0_B\}) \\ (i, a) &\mapsto (i, f(a)) \\ (i, 0_A) &\mapsto (i, 0_B). \end{aligned}$$

Checking this is a left adjoint to the forgetful functor can be done by constructing the natural bijection

$$\mathbf{Set}(A, B) \cong \mathbf{Alg}(\mathrm{Fr}(A), B).$$

The bijection is given by $\mathbf{Set}(A, B) \to \mathbf{Alg}(\mathrm{Fr}(A), B)$, $f \mapsto \tilde{f}$, where

$$\begin{aligned} \tilde{f} : \mathbb{N} \times (A + \{0_A\}) &\to B \\ (0, e_A) &\mapsto e_B \\ (0, a) &\mapsto f(a) \\ (i, e_A) &\mapsto \beta(\tilde{f}(i-1, e_A)) \\ (i, a) &\mapsto \beta(\tilde{f}(i-1, a)). \end{aligned}$$

The inverse of the bijection $\mathbf{Set}(A, B) \to \mathbf{Alg}(\mathrm{Fr}(A), B)$ is given by restriction to A. A quick check will also show this bijection is natural.

The cofree functor is defined by mapping $D \in \mathbf{Set}$ to the coalgebra

$$\begin{aligned} \delta : D \times D^*_\infty &\to 1 + D \times (D)^*_\infty \\ (d_0, [\,]) &\mapsto * \\ (d_0, d : ds) &\mapsto (d, ds) \end{aligned}$$

The behavior of Cof on functions $f : C \to D$ is forced and given by applying f to all elements of C in any element of $C \times C^*_\infty$. Checking this is a right adjoint to the forgetful functor can be done by constructing the natural bijection

$$\mathbf{Set}(C, D) \cong \mathbf{CoAlg}(C, \mathrm{Cof}(D)),$$

The bijection is given by $\mathbf{Set}(C, D) \to \mathbf{CoAlg}(C, \mathrm{Cof}(D)), f \mapsto \tilde{f}$, where

$$\begin{aligned}\tilde{f} = \langle \tilde{f}_0, \tilde{f}_1 \rangle : C &\to D \times D^*_\infty \\ c &\mapsto \begin{cases} (f(c), f(c') : \tilde{f}_1(c')) \text{ if } \chi(c) = c' \\ (f(c), [\,]) \text{ if } \chi(c) = *. \end{cases}\end{aligned}$$

Notice the function $\tilde{f}_0 = f$, which also gives the inverse for the bijection. Verifying $\tilde{f}$ is a coalgebra morphism is left to the reader.

3.4.4 Representing objects

In this section we wish to compute the representing objects of $\mathfrak{m} : \mathbf{CoAlg}^{\mathbf{op}} \times \mathbf{Alg}^{\mathbf{op}} \times \mathbf{Alg} \to \mathbf{Set}$. Here, we will be fairly brief. For more details, we refer the reader to the upcoming section about lists, since it is a direct generalization of what we are considering here.

First we wish to construct the terminal object in the category of measurings, $\underline{\mathbf{Alg}}(A, B)$. The underlying set of $\underline{\mathbf{Alg}}(A, B)$ is a subset of $\mathbf{Set}(A, B) \times \mathbf{Set}(A, B)^*_\infty$, given by

$$\{(f_0, f_{i+1})_{i \in I} \mid f_i(0_A) = 0_B, f_i(\alpha(a)) = \beta(f_{i+1}(a)), f_{\max}(\alpha(a)) = \beta(*) \text{ for all } a \in A, x \in X\},$$

where by $f_{\max}$ we mean the last element of the stream $(f_{i+1})_{i \in I}$, if it exists. Unpacking the above, we find elements of $\underline{\mathbf{Alg}}(A, B)$ are non empty streams $(f_0, f_{i+1})_{i \in I}$ such that $f_i(\alpha(a)) = \beta(f_{i+1}(a))$, and the last element of the stream (if it exists) should be the function mapping all elements of the form $\alpha(a)$ to the element $0_B \in B$. The coalgebra structure of $\underline{\mathbf{Alg}}(A, B)$ is given by shifting to the left, $(f_0, (f_{i+1})_i) \mapsto (f_1, (f_{i+2})_i), (f_0, [\,]) \mapsto *$. Lastly, we give the evaluation map

$$\begin{aligned}\mathrm{ev} : \underline{\mathbf{Alg}}(A, B) \times A &\to B \\ ((f_0, (f_{i+1})_{i \in I}), a) &\mapsto f_0(a).\end{aligned}$$

The evaluation map is measuring from A to B by $\underline{\mathbf{Alg}}(A, B)$ by definition, making it an element of the category of measurings from A to B. To verify $\mathfrak{m}_C(A, B) \cong \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A, B))$, we refer the reader to the next section.

Next we would like to compute $C \triangleright A$. By Theorem 22 from [1], $\mathfrak{m}_C(A, B) \cong \mathbf{Alg}(C \triangleright A, B)$, where $C \triangleright A$ has as underlying set

$$\mathrm{Fr}(C \times A)/\sim \; = \mathbb{N} \times ((C \times A) + \{e_{C \times A}\})/\sim .$$

Here the equivalence relation $\sim$ is generated by

$$\begin{aligned}(i, c, 0_A) &\sim (i, e_{C \times A}) \\ (i, c, \alpha(a)) &\sim (i, e_{C \times A}) \text{ if } \chi(c) = * \\ (i, c, \alpha(a)) &\sim (i + 1, c', a) \text{ if } \chi(c) = c'.\end{aligned}$$

The algebra structure is given by the function

$$\begin{aligned}1 + X \times \mathrm{Fr}(C \times A) &\to \mathrm{Fr}(C \times A) \\ * &\mapsto ([\,], e_{C \times A}) \\ (x, (xs, c, a)) &\mapsto (x : xs, c, a)\end{aligned}$$

composed with the quotient map $\mathrm{Fr}(C \times A) \to \mathrm{Fr}(C \times A)/\sim$. To verify $\mathfrak{m}_C(A, B) \cong \mathbf{Alg}(C \triangleright A, B)$ we again refer the reader to the upcoming section.

Finally, we define the convolution algebra $[C, B]$ to have underlying set $\mathbf{Set}(C, B)$ and algebra structure given by

$$\begin{aligned}1 + \mathbf{Set}(C, B) &\to \mathbf{Set}(C, B) \\ * &\mapsto (\_ \mapsto 0_B) \\ f &\mapsto \left(c \mapsto \begin{cases} \beta(f(c')) \text{ if } \chi(c) = c' \\ 0_A \text{ if } \chi(c) = * \end{cases} \right).\end{aligned}$$

Verifying this is indeed the convolution algebra is left to the next section.

We would like to give an example of a representing object to get a feel for them. For now, we will only state an example and leave most of the computations leading up to it to the next section where we will see a more general case. For the curious reader, a full computation of this example can be found in Section 2 of [1].

We would like to calculate $\underline{\mathbf{Alg}}(\mathbb{n}, B)$ for arbitrary $B \in \mathbf{Alg}$. Our aim is to leverage the isomorphism

$$\mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(\mathbb{n}, B)) \cong \mathfrak{m}_C(\mathbb{n}, B) \cong \mathbf{Alg}(\mathbb{n}, [C, B]).$$

To do so, we make an observation about $\mathbf{Alg}(\mathbb{n}, Z)$ for an arbitrary algebra $(Z.\zeta) \in \mathbf{Alg}$. Since $\mathbb{n}$ is preinitial, any morphism out of $\mathbb{n}$ is unique. So, $\mathbf{Alg}(\mathbb{n}, Z)$ has at most one element. The question now becomes if there is a condition on Z for $\mathbf{Alg}(\mathbb{n}, Z)$ to be inhabited. We claim this condition is

$$\zeta(\mathsf{i}_Z(i)) = \zeta(\mathsf{i}_Z(\min(n-1, i))) \text{ for all } i \in \mathbb{N}. \tag{3.1}$$

Now that we have an indication on whether $\mathbf{Alg}(\mathbb{n}, Z)$ is inhabited for arbitrary $Z \in \mathbf{Alg}$, we can focus on $Z = [C, B]$. Observing the algebra structure on $[C, B]$, we see the morphism $\mathsf{i}_{[C,B]}$ is given by

$$\begin{aligned} \mathsf{i}_{[C,B]} : \mathbb{N} &\to [C, B] \\ 0 &\mapsto (c \mapsto 0_B) \\ i+1 &\mapsto \left(c \mapsto \begin{cases} \beta(\mathsf{i}_{[C,B]}(i)(c')) \text{ if } \chi(c) = c' \\ 0_B \text{ if } \chi(c) = * \end{cases} \right). \end{aligned}$$

Denoting the algebra structure on $[C, B]$ by α, we would like to know under which conditions on C and B $\alpha(\mathsf{i}_{[C,B]}(i)) = \alpha(\mathsf{i}_{[C,B]}(\min(n-1)(i)))$. The above condition can be satisfied if $!_C(c) \leq n$ for all $c \in C$ or if Eq. (3.1) holds for B. So, we have the following:

$$\mathbf{Alg}(\mathbb{n}, [C, B]) \cong \begin{cases} \{*\} \text{ if } \beta(\mathsf{i}_B(i)) = \beta(\mathsf{i}_B(\min(n-1)(i))) \text{ for all } i \in \mathbb{N} \\ \{*\} \text{ if } !_C(c) \leq n \text{ for all } c \in C \\ \emptyset \text{ otherwise.} \end{cases}$$

Now we can make use of the isomorphism $\mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(\mathbb{n}, B)) \cong \mathbf{Alg}(\mathbb{n}, [C, B])$ We observe that in the case of $\beta(\mathsf{i}_B(i)) = \beta(\mathsf{i}_B(\min(n-1)(i)))$, $\underline{\mathbf{Alg}}(\mathbb{n}, B)$ has the universal property of the terminal coalgebra $\mathbb{N}_\infty$. Whenever this is not the case, $\underline{\mathbf{Alg}}(\mathbb{n}, B)$ has the universal property of $\mathbb{n}^\dagger \in \mathbf{CoAlg}$. We conclude

$$\underline{\mathbf{Alg}}(\mathbb{n}, B) = \begin{cases} \mathbb{N}_\infty \text{ if } \beta(\mathsf{i}_B(i)) = \beta(\mathsf{i}_B(\min(n-1)(i))) \text{ for all } i \in \mathbb{N} \\ \mathbb{n}^\dagger \text{ otherwise.} \end{cases}$$

This also shows that $\mathbb{n}^\circ = \underline{\mathbf{Alg}}(\mathbb{n}, \mathbb{N}) \cong \mathbb{n}^\dagger$. We will write $\mathbb{n}^\circ$ instead of $n^\dagger$ from now on. In the next example we will see it is not always the case that P and P° have the same underlying set for some preinitial algebra P.

3.4.5 C-initial algebras

In this section we would like to give a proof of the fact that $\mathbb{n}$ is the terminal $\mathbb{n}^\circ$-initial algebra. The proof is different from the proof presented in Section 2.5 of [1], with as main advantage that it does not use the law of excluded middle. In Section 5, we will give a more general version of this proof, regarding unbounded trees. Using techniques developed in the rest of the thesis, we will then be able to prove statements about C-initial algebras for other examples as well. A final remark is where in the previous examples we could make a complete classification of C-initial algebras, here we can not. The algebra structures involved are more complex, and hence we will restrict ourselves to the most useful cases.

Recall an algebra A is called C-initial if there exists a unique measuring $\varphi : C \times A \to X$ for all $X \in \mathbf{Alg}$. The terminal C-initial algebra is the terminal object, if it exists, in the subcategory of $\mathbf{Alg}$ spanned by the C-initial algebras. In other words, for any C-initial algebra A there must exist a unique algebra morphism from A to the terminal C-initial algebra.

We aim to show $\mathbb{n}$ is the terminal $\mathbb{n}^\circ$-initial algebra. In order to do so, we first show there exists a morphism $A \to \mathbb{n}$ for any $\mathbb{n}^\circ$-initial algebra A using induction. After that, we will show the morphism is unique using a similar argument as in Example 15 of [1].

We start with a lemma which allows us to use induction later on.

Lemma 3.4.4. *Let A be a $\mathbb{n}^\circ$-initial algebra. Then A is also $\mathbb{n}-1^\circ$ initial.*

Proof. Consider the coalgebra morphism $m : \mathbb{n}-1^\circ \to \mathbb{n}^\circ, i \mapsto i$. This induces a morphism

$$\begin{aligned} m \triangleright A : \mathbb{n}-1^\circ \triangleright A &\to \mathbb{n}^\circ \triangleright A \\ [n,i,a] &\mapsto [n,i,a]. \end{aligned}$$

This morphism is monomorphic by definition. Since A is $\mathbb{n}^\circ$-initial, we know $\mathbb{n} \triangleright A$ is an initial object. This means $m \triangleright A$ is a monomorphism into the initial object, hence an isomorphism. □

As an immediate consequence we have the following corollary.

Corollary 3.4.5. *Let A be a $\mathbb{n}^\circ$-initial algebra. Then A is also $\mathbb{k}^\circ$ initial for all $k \leq n$.*

The next lemma is a technical lemma which we will be able to leverage during the induction step.

Lemma 3.4.6. *Let A be a $\mathbb{n}^\circ$-initial algebra and let $\varphi : \mathbb{n}^\circ \times A \to \mathbb{N}$ be the unique measuring from A to $\mathbb{N}$ by $\mathbb{n}^\circ$. Then for all $i \in \mathbb{n}^\circ$ and $0 \leq j \leq n-i$ we have*

$$\varphi(i,a) = \varphi(i, \mathfrak{i}_A \circ \varphi(i+j,a)).$$

Proof. Let $k \leq n$. Define the coalgebra morphism

$$\begin{aligned} p_j : \mathbb{k}^\circ &\to \mathbb{n}^\circ \times \mathbb{n}^\circ \\ i &\mapsto (i, i+j) \end{aligned}$$

for all $0 \leq j \leq n-k$. Consider the composition of measurings $\varphi \circ (\mathrm{id} \times \mathfrak{i}_A \circ \varphi) \in \mathfrak{m}_{\mathbb{n}^\circ \times \mathbb{n}^\circ}(A, \mathbb{N})$. We can precompose this measuring with the coalgebra morphism p_j to obtain a measuring

$$\mathbb{k}^\circ \times A \xrightarrow{p_j \times \mathrm{id}_A} \mathbb{n}^\circ \times \mathbb{n}^\circ \times A \xrightarrow{\varphi \circ (\mathrm{id} \times \mathfrak{i}_A \circ \varphi)} \mathbb{N}.$$

We also have the coalgebra morphism $f : \mathbb{k}^\circ \to \mathbb{n}^\circ, i \mapsto i$ which we can precompose with φ. This gives us a measuring

$$\mathbb{k}^\circ \times A \xrightarrow{f \times \mathrm{id}_A} \mathbb{n}^circ \times A \xrightarrow{\varphi} \mathbb{N}.$$

Since A is also $\mathbb{k}^\circ$-initial, we know these measurings must coincide. Hence we can state $\varphi(i,a) = \varphi(i, \mathfrak{i}_A \circ \varphi(i+j,a))$ for all $i \in \mathbb{k}^\circ$ and $0 \leq j \leq n-k$. The only restriction placed on k was that $k \leq n$. Iterating over all $0 \leq k \leq n$, we arrive at the desired result

$$\varphi(i,a) = \varphi(i, \mathfrak{i}_A \circ \varphi(i+j,a))$$

for all $0 \leq i \leq n$ and $0 \leq j \leq n-i$. □

Corollary 3.4.7. *Let A be $\mathbb{n}^\circ$-initial and let φ be the unique measuring to $\mathbb{N}$. For all $i \in \mathbb{n}^\circ$ and $a \in A$, $\varphi(i,a) \leq i$.*

Proof. Since $\varphi(i,a) = \varphi(i, \mathfrak{i}_A(\varphi(i,a)))$ by the previous lemma, we know $[0,i,a] = [0,i,\mathfrak{i}_A(\varphi(i,a))]$ in $\mathbb{n}^\circ \triangleright A$. We also have $[\varphi(i,a), z] = [0,i,a]$, and by definition of the equivalence relation $\sim$, we have $[0,i,\mathfrak{i}_A(\varphi(i,a))] = [\min(i, \varphi(i,a)), z]$. From this we conclude

$$\varphi(i,a) = \min(i, \varphi(i,a)),$$

hence $\varphi(i,a) \leq i$ for all $i \in \mathbb{n}^\circ$. □

Now we are ready to define the a family of functions which will culminate in an algebra morphism $A \to \mathbb{n}$ for any $\mathbb{n}^\circ$ initial algebra A.

Definition 3.4.8. Let A be $\mathbb{n}^\circ$-initial, let φ be the unique measuring to $\mathbb{N}$ and let $0 \leq k \leq n$. Define the functions

$$\begin{aligned} \varphi_k : A &\to \mathbb{k} \\ a &\mapsto \varphi(k, a), \end{aligned}$$

which are well-defined by the previous corollary.

Proposition 3.4.9. *The functions $\varphi_k : A \to \mathbb{k}$ from Definition 3.4.8 are algebra morphisms for all $0 \leq k \leq n$.*

Proof. We proceed by induction over k. For $k = 0$, we have that $\mathbb{k} \cong 1$, the terminal object in the category of algebras. Hence $\varphi_0 : A \to 1$ is an algebra morphism. For the inductive step, assume $\varphi_{k-1} : A \to \mathbb{k} - 1$ is an algebra morphism. We wish to show $\alpha_k(\varphi_k(a)) = \varphi_k(\alpha(a))$ for all $a \in A$. If $\varphi_k(a) = 0$, we know

$$\varphi(k, \alpha(a)) = 1 + \varphi(k-1, a) = 1 + \varphi(k-1, \mathsf{i}_A(\varphi(k, a))) = 1 + \varphi(k-1, 0_A) = 1 = \alpha_k(0) = \alpha_k(\varphi(k, a)).$$

If $\varphi_k(a) \geq 0$ we can make the following deduction

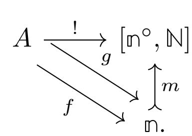
$$\begin{aligned} \alpha_k(\varphi_k(a)) &= \alpha_k(\varphi(k, a)) \\ &= 1 + \alpha_{k-1}(\varphi(k, a) - 1) \\ &= 1 + \alpha_{k-1}(\varphi(k, \mathsf{i}_A \circ \varphi(k, a)) - 1) \\ &= 1 + \alpha_{k-1}(\varphi(k-1, \mathsf{i}_A(\varphi(k, a) - 1))) \\ &= 1 + \varphi(k-1, \mathsf{i}_A \circ \varphi(k, a)) \\ &= 1 + \varphi(k-1, a) \\ &= \varphi(k, \alpha(a)). \end{aligned}$$

□

In particular, this lemma shows $\varphi_n : A \to \mathbb{n}$ is an algebra morphism. It still remains to show this algebra morphism is unique.

Lemma 3.4.10. *For any $\mathbb{n}^\circ$-initial algebra A, there exist at most one algebra morphism $A \to \mathbb{n}$.*

Proof. By Proposition 36 in [1], there exists a unique morphism $A \to [\mathbb{n}^\circ, \mathbb{N}]$ for any $\mathbb{n}^\circ$-initial algebra A. Since $\mathbb{n}$ is $\mathbb{n}^\circ$-initial we know there exists a unique morphism $m : \mathbb{n} \rightarrowtail [\mathbb{n}^\circ, \mathbb{N}]$. Upon closer inspection m turns out to be a monomorphism. Given any two morphisms $f, g : A \to \mathbb{n}$, we can draw the following diagram



Since the morphism $A \to [\mathbb{n}^\circ, \mathbb{N}]$ is unique, we know the composites $m \circ f = m \circ g$, and by m being mono we conclude $f = g$. Hence, there can be at most one algebra morphism from a $\mathbb{n}^\circ$-initial algebra A to $\mathbb{n}$. □

Putting all the above together, we arrive at the following result.

Theorem 3.4.11. *The algebra $\mathbb{n}$ is the terminal $\mathbb{n}^\circ$-initial algebra.*

Proof. Given an $\mathbb{n}^\circ$ initial algebra A, by Proposition 3.4.9 we obtain an algebra morphism $\varphi_n : A \to \mathbb{n}$. By Lemma 3.4.10 it is unique. We conclude $\mathbb{n}$ is the terminal $\mathbb{n}^\circ$-initial algebra. □

3.5 The list type

In Section 3.3 we have seen how introducing a monoidal structure allows measurings to introduce some "twist". On the other hand in Section 3.4 we have seen how measurings are able to control "shape" in a sense. In this section we are combining the two, which comes together nicely in the list type. Given a monoid $(X, \bullet, e)$, we can consider the type which contains all possible finite lists with elements in X. This example is used ubiquitously throughout computer science and exhibits the flexibility gained when considering not just algebra morphisms, but measurings as well. To study the list type we must find a functor which has it as an initial algebra. This functor exists and is the next thing we will be defining.

Let $(X, \bullet, e)$ be a monoid in **Set** and consider the functor

$$\begin{aligned} F : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1 + A \times X \\ f &\mapsto \mathrm{id}_1 + f \times \mathrm{id}_X \,. \end{aligned}$$

The lax monoidal structure is given by

$$\begin{aligned} \nabla_{A,B} : (1 + X \times A) \times (1 + X \times B) &\to 1 + X \times A \times B \\ (*, (x, b)) &\mapsto * \\ ((x, a), *) &\mapsto * \\ ((x, a), (x', b)) &\mapsto (x \bullet x', a, b) \end{aligned}$$

and $\eta : 1 \to 1 + X, * \mapsto e$. Since we will use it explicitly later, we remark the lax closed structure is given by

$$\begin{aligned} \tilde{\nabla}_{A,B} : 1 + X \times \mathbf{Set}(A, B) &\to \mathbf{Set}(1 + X \times A, 1 + X \times B) \\ * &\mapsto \mathrm{const}_* \\ (x, f) &\mapsto \tilde{\nabla}_{A,B}(x, f), \end{aligned}$$

where

$$\begin{aligned} \tilde{\nabla}_{A,B}(x, f) : 1 + X \times A &\to 1 + X \times B \\ * &\mapsto * \\ (x', a) &\mapsto x \bullet x', f(a). \end{aligned}$$

The category of F-algebras has elements $\alpha : 1 + X \times A \to A$ denoted (A, α). We will write $\alpha(*) = e_A$, thinking of $\alpha(*)$ as an "empty element". Sometimes when considering an algebra we will speak of a "list-like algebra". Morphisms $f : (A, \alpha) \to (B, \beta)$ are given by functions $f : A \to B$ which make the following diagram commute:

$$\begin{array}{ccc} 1 + X \times A & \xrightarrow{F(f)} & 1 + X \times B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

The category of F-coalgebras has elements $\chi : C \to 1 + X \times C$, denoted (C, χ). Sometimes when considering a coalgebra we will speak of a "stream-like coalgebra". Morphisms $f : (C, \chi) \to (D, \delta)$ are given by functions $f : C \to D$ which make the following diagram commute:

$$\begin{array}{ccc} 1 + X \times C & \xrightarrow{F(f)} & 1 + X \times D \\ {\scriptstyle \chi}\uparrow & & {\scriptstyle \delta}\uparrow \\ C & \xrightarrow{\quad f \quad} & D \end{array}$$

3.5.1 Initial and terminal objects

We denote the set of lists of length n with elements in X by X^n. The underlying set of the initial algebra is given by

$$X^* := \coprod_{i \in \mathbb{N}} X^i.$$

Again borrowing notation from functional programming, we will write $[\,] \in X^0$ for the empty list and given a list $xs \in X^*$ and an element $x \in X$, appending x to the list xs is denoted by $x : xs$. Moreover, we will call the element which was appended to the list most recently the first element, and conversely the element which was appended to the list earliest the last element. The algebra structure of X^* is given by

$$\begin{aligned} 1 + X \times X^* &\to X^* \\ * &\mapsto [\,] \\ (x, xs) &\mapsto x : xs. \end{aligned}$$

This also explains the term "list-like algebra", since every algebra $\alpha : 1 + X \times A \to A$ has an empty element $e_A = \alpha(*)$ and a way of appending an element $x \in X$ to a "list-like" $a \in A$. To verify X^* is indeed the initial algebra, let (A, α) be an arbitrary algebra. The unique algebra morphism $X^* \to A$, denoted $¡_A : X^* \to A$ is forced by the commutative square

$$\begin{array}{ccc} 1 + X \times X^* & \xrightarrow{F(f)} & 1 + X \times A \\ \big\downarrow & & \big\downarrow \alpha \\ X^* & \xrightarrow{\quad ¡_A \quad} & A, \end{array}$$

and defined as

$$\begin{aligned} ¡_A : X^* &\to A \\ [\,] &\mapsto e_A \\ x : xs &\mapsto \alpha(x, f(xs)). \end{aligned}$$

Preinitial algebras correspond to equivalence relations $\sim$ on X^* which satisfy $xs \sim xs' \Rightarrow x : xs \sim x : xs'$ for all $x \in X$ by [1, Lemma 38]. An example is the equivalence relation which identifies all lists with length greater than n with the list of their last n elements. This results in a preinitial algebra which we will denote by $X^*_n := \coprod_{0 \leq i \leq n} X^i$. Its algebra structure is given by

$$\begin{aligned} 1 + X \times X^*_n &\to X^*_n \\ * &\mapsto [\,] \\ (x, xs) &\mapsto \text{take}(n)(x : xs), \end{aligned}$$

where take takes the first n elements of a list. Another example is the equivalence relation generated by

$$x : xs \sim xs \text{ if } x \notin X',$$

where $X' \subseteq X$. This results in the preinitial algebra $(X')^*$ for any $X' \subseteq X$. The unique algebra morphism $X^* \to (X')^*$ corresponds to filtering the elements of a list.

One can check that in the above cases the unique morphisms $X^* \to X^*_n$ and $X^* \to (X')^*$ are epic. The above two can be combined to yield filtered lists of length at most n, leading to even more preinitial algebras. However, keep in mind this list is not exhaustive, and that there are even more preinitial algebras.

The terminal coalgebra is given by lists in X of finite and infinite length and is denoted X^*_∞. Formally, it is given by $X^*_\infty = X^* + (\prod_{n \in \mathbb{N}})X$. To verify this is indeed the terminal coalgebra, let (C, χ) be an arbitrary coalgebra. The unique algebra morphism $C \to X^*_\infty$, denoted $!_C : C \to X^*_\infty$, is forced by the commutative square

$$\begin{array}{ccc} 1 + X \times C & \xrightarrow{F(f)} & 1 + X \times X^*_\infty \\ \chi \big\uparrow & & \big\uparrow \\ C & \xrightarrow{\quad !_C \quad} & X^*_\infty, \end{array}$$

and defined as

$$\begin{aligned} !_C : C &\to X^*_\infty \\ c &\mapsto \begin{cases} [\,] \text{ if } \chi(c) = * \\ x : f(c') \text{ if } \chi(c) = (x, c') \end{cases} . \end{aligned}$$

Subterminal coalgebras correspond to subsets $A \subseteq X^*_\infty$ satisfying $x : xs \in A \Rightarrow xs \in A$ by Lemma 42 in [1]. As with with the case of the natural numbers, some of the subterminal coalgebras share the same underlying set as preinitial algebras. We will use $(\_)^\dagger$ to denote the coalgebra whenever it might be ambiguous. Some examples are $\emptyset, X_0^{*\dagger} \cong 1, X_n^{*\dagger}, X^{*\dagger}, \prod_{n\in\mathbb{N}} X$ and $(X')^*_\infty, X' \subseteq X$, with their coalgebra structure inherited from the coalgebra structure on X^*_∞. One can verify the all maps from subterminal coalgebra into X^*_∞ are injective.

3.5.2 Functions as algebra morphisms

In this section we would like to demonstrate the power of the algebraic approach to datatypes. We wish to give the definitions of some basic functions frequently seen in computer science as algebra morphisms. Parts of this section are based on [9].

Example 3.5.1. The first function is the length function len, which takes a list and returns its length. We want to word it in the language of algebras, so we first define an algebra

$$\begin{aligned} 1 + X \times \mathbb{N} &\to \mathbb{N} \\ * &\mapsto 0 \\ (x, n) &\mapsto n + 1. \end{aligned}$$

The function len on lists $xs \in X^*$ is then given by the unique algebra morphism $X^* \to \mathbb{N}$. △

Example 3.5.2. The second function considered takes two lists and concatenates them. Classically, this would be denoted as a function $X^* \times X^* \to X^*$, but to frame it as an algebra morphism, we consider it as a function $X^* \to \mathbf{Set}(X^*, X^*)$. To this end, consider the the algebra

$$\begin{aligned} \alpha : 1 + X \times \mathbf{Set}(X^*, X^*) &\to \mathbf{Set}(X^*, X^*) \\ * &\mapsto \mathrm{id} \\ (x, f) &\mapsto (xs \mapsto x : f(xs)). \end{aligned}$$

The concatenation function is denoted by $(++)$ and is defined uniquely by the following diagram

$$\begin{CD} 1 + X \times X^* @>{F(++)}>> 1 + X \times \mathbf{Set}(X^*, X^*) \\ @VVV @VV{\alpha}V \\ X^* @>{(++)}>> \mathbf{Set}(X^*, X^*), \end{CD}$$

where uniqueness follows from the fact X^* is the initial algebra. Given $xs \in X^*$, the function $(++)(xs) \in \mathbf{Set}(X^*, X^*)$ is the function which takes a list xs' and concatenates it with xs, resulting in the list $(++)(xs)(xs') = xs ++ xs'$. △

Example 3.5.3. The next function is head, which is a partially defined function which attempts to extract the first element of a list-like. We define the algebra

$$\begin{aligned} \alpha : 1 + X \times (X + \{\bot\}) &\to X + \{\bot\} \\ * &\mapsto \bot \\ (x, xs) &\mapsto x. \end{aligned}$$

The function head is then defined as the following algebra morphism head : $X^* \to (X + \{\bot\}, \beta), [\,] \mapsto \bot, x : xs \mapsto x$ △

Example 3.5.4. The next function is take(n), which is a function that takes the first $n \in \mathbb{N}_\infty$ elements of a list. We have already defined this function "by hand" when defining the algebra structure on X_n^*, but we would like view it from our algebraic perspective. As maybe suspected, it is given by the unique function

$$\begin{aligned} \mathrm{take}(n) : X^* &\to X_n^* \\ xs &\mapsto \mathrm{take}(n)(xs). \end{aligned}$$

△

Example 3.5.5. The filter function is a function which takes a predicate $p \in P_X := \mathbf{Set}(X, \{\top, \bot\})$ and a list $xs \in X^*$ and returns only the elements in xs which satisfy p. Classically, this would be written as $\text{filter} : X^* \times P_X \to X^*$, but in our case we want to consider it as a function $\text{filter} : X^* \to \mathbf{Set}(P_X, X^*)$. First, we define an algebra structure on $\mathbf{Set}(P_X, X^*)$ by

$$\begin{aligned}\beta : 1 + X \times \mathbf{Set}(P_X, X^*) &\to \mathbf{Set}(P_X, X^*)\\ * &\mapsto \text{const}_{[\,]}\\ (x, f) &\mapsto \left(p \mapsto \begin{cases} x : f(p) \text{ if } p(x) \\ f(p) \text{ otherwise} \end{cases}\right).\end{aligned}$$

The function filter is then given by the unique algebra morphism $X^* \to \mathbf{Set}(P_X, X^*)$. Another way to construct the filter function using a predicate $p \in P_X$ is to compute the set $X_p = \{x \in X \mid p(x) = \top\}$. Since $X_p \subset X$, X_p^* has the algebra structure corresponding to filtering. The function $\text{filter}(p)$ is the given by the unique function $\text{filter}(p) : X^* \to X_p^*$. △

Notice that all functions above can be described as folds. This is no coincidence. Constructing the target algebra (A, α) as done above corresponds to finding the accumulator and the binary operator of the fold.

3.5.3 Measuring

Definition 3.5.6. Let $(A, \alpha), (B, \beta) \in \mathbf{Alg}$ and $(C, \chi) \in \mathbf{CoAlg}$. A *measuring from A to B by C* is a function $\varphi : C \times A \to B$ satisfying

1. $\varphi(c)(e_A) = e_B$ for all $c \in C$
2. $\varphi(c)(\alpha(x, a)) = e_B$ if $\chi(c) = *$
3. $\varphi(c)(\alpha(x, a)) = \beta(x' \bullet x, \varphi(c')(a))$ if $\chi(c) = (x', c')$.

Intuitively, a measuring is a function that takes a stream-like coalgebra and a list-like algebra and combines their elements using the monoidal structure on X, then wraps them up in a new list-like. We see the coalgebra introduces some element wise "twist", as well as limits the "shape" of the list-like.

Example 3.5.7. First, we start with an example inspired by Section 2 from [1]. Let $(X, e, \bullet)$ be a commutative monoid. We remark there does not exist a total algebra morphism $f : X_n^* \to X^*$, since for a list $xs \in X_n^*$ of length n, we must have that

$$x : f(\text{take}(n-1)(xs)) = f(x : \text{take}(n-1)(xs)) = f(x : xs) = x : f(xs),$$

which can not be the case. Note that in the above computation we have used $\text{take}(n)(x : xs) = x : \text{take}(n-1)(xs)$. The problem here is that we are disregarding the last element of the list, where an algebra morphism into X^* does need that information.

However, we are able to define a measuring from X_n^* to X^*. We define the measuring

$$\begin{aligned}\varphi : X_n^{*\dagger} \times X_n^* &\to X^*\\ ([\,], xs) &\mapsto [\,]\\ (xs', [\,]) &\mapsto [\,]\\ (x' : xs', x : xs) &\mapsto (x' \bullet x) : \varphi(xs', xs).\end{aligned}$$

We note that checking this is a measuring can be done by comparing the definition of measuring and that of φ. This example is valid because the coalgebra $X_n^{*\dagger}$ limits up to which point we are considering elements of the algebra X_n^*. The problem we had earlier, caused by disregarding the last element of a list due to the maximum length of the list, is mitigated by the introduction of the coalgebra $X_n^{*\dagger}$. Since the elements of $X_n^{*\dagger}$ all have length at most n, we can truly disregard the last element of a list whenever its length exceeds n. △

Example 3.5.8. Next, we investigate another interesting preinitial algebra given by $(X')^*$ for some subset $X' \subseteq X$. We would like to find a measuring $\varphi : C \times (X')^* \to B$ for some non-empty coalgebra $\emptyset \neq C \in \mathbf{CoAlg}$ and some algebra $B \in \mathbf{Alg}$. This measuring must satisfy

$$\varphi(c)(x : xs) = \begin{cases} \beta((x \bullet x'), \varphi(c')(xs)) \text{ if } \chi(c) = (x', c') \\ e_B \text{ otherwise.} \end{cases}$$

By the coalgebra structure on $(X')^*$, we have that for any $x \notin X'$, $x : xs = xs$. In the case that $\chi(c) = (x', c')$ and $x \notin X'$, the above condition becomes $\varphi(c)(xs) = \varphi(c)(x : xs) = \beta((x \bullet x'), \varphi(c')(xs))$, giving us a contradiction since we need $\varphi(c)(xs) = \beta((x \bullet x'), \varphi(c')(xs))$ for all $x \notin X'$. We conclude a measuring $\varphi : C \times (X')^* \to B$ does not exist. The problem here is a loss of information. This loss stems from the fact that $x : x : \cdots : x : xs = xs$ whenever $x \notin X'$, whereas the measuring does need to take into account the added elements x. △

Example 3.5.9. Using measurings, we are also able to define a more flexible version of the map function. Traditionally, map takes a function f and a list and applies f to all elements of the list element wise. Using measurings, we can actually specify a list of functions which gets applied to elements of a list in a pairwise fashion. For each $x \in X$, define the function $r_X : X \to X, x' \mapsto x' \bullet x$ which multiplies an element of X by x from the right. We define the algebra

$$\begin{aligned} 1 + X \times \mathbf{Set}(X, X)^*_\infty &\to \mathbf{Set}(X, X)^*_\infty \\ * &\mapsto r_e \\ (x, fs) &\mapsto r_x : fs. \end{aligned}$$

We can now define the measuring

$$\begin{aligned} \varphi : X^{*\dagger} \times \mathbf{Set}(X, X)^*_\infty &\to X^* \\ ([\,], fs) &\mapsto [\,] \\ (xs', [\,]) &\mapsto [\,] \\ (x' : xs', f : fs) &\mapsto f(x') : \varphi(xs', fs). \end{aligned}$$

We verify this is a measuring by checking $\varphi(x' : xs', r_x : fs) = r_x(x') : \varphi(xs, fs) = (x' \bullet x) : \varphi(xs, fs)$. Note that we really need to define an algebra structure on $\mathbf{Set}(X, X)^*_\infty$ instead of a coalgebra structure, since there is no obvious way to extract $x \in X$ from a function $f : X \to X$ in such a way that the conditions on a measuring are satisfied. △

The *category of measurings from A to B* has as elements measurings $\varphi : C \times A \to B$, and as morphisms $f : \varphi \to \psi$ functions $f : C \to D$ such that

$$\begin{array}{ccc} C \times A & \xrightarrow{\varphi} & B \\ {\scriptstyle f \times \mathrm{id}} \big\downarrow & \nearrow_{\psi} & \\ D \times A & & \end{array}$$

commutes.

3.5.4 Free and cofree functors

Again, we aim to use the free and cofree functors to construct the representing objects. The free functor is defined by mapping $A \in \mathbf{Set}$ to

$$\begin{aligned} \alpha_{\mathrm{Fr}} : 1 + X \times X^* \times (A + \{e_A\}) &\to X^* \times (A + \{e_A\}) \\ * &\mapsto ([\,], e_A) \\ (x, xs, e_A) &\mapsto (x : xs, e_A) \\ (x, xs, a) &\mapsto (x : xs, a) \end{aligned}$$

The behavior of Fr on functions $f : A \to B$ is forced and given by

$$\begin{aligned}\mathrm{Fr}(f) : X^* \times (A + \{e_A\}) &\to X^* \times (B + \{e_B\}) \\ (xs, a) &\mapsto (xs, f(a)) \\ (xs, e_A) &\mapsto (xs, e_B).\end{aligned}$$

Checking this is a left adjoint to the forgetful functor can be done by constructing the natural bijection

$$\mathbf{Set}(A, B) \cong \mathbf{Alg}(\mathrm{Fr}(A), B).$$

The bijection is given by $\mathbf{Set}(A, B) \to \mathbf{Alg}(\mathrm{Fr}(A), B), f \mapsto \tilde{f}$, where

$$\begin{aligned}\tilde{f} : X^* \times (A + \{e_A\}) &\to B \\ ([\,], e_A) &\mapsto e_B \\ ([\,], a) &\mapsto f(a) \\ (x : xs, e_A) &\mapsto \beta(x, \tilde{f}(xs, e_A)) \\ (x : xs, a) &\mapsto \beta(x, \tilde{f}(xs, a)).\end{aligned}$$

The inverse of the bijection $\mathbf{Set}(A, B) \to \mathbf{Alg}(\mathrm{Fr}(A), B)$ is given by restriction to A. Informally, $\tilde{f}$ takes a list $xs \in X^*$ and an element $a \in A$ and appends xs to $f(a)$ in B. A quick check will also show this bijection is natural. The unit and counit are given by $\eta_A^{\mathrm{Fr}} : A \to X^* \times (A + \{e_A\}), a \mapsto ([\,], a)$ and

$$\begin{aligned}\varepsilon_A^{\mathrm{Fr}} : X^* \times (A + \{e_A\}) &\to A \\ ([\,], e_A) &\mapsto \alpha(*) \\ ([\,], a) &\mapsto a \\ (x : xs, a) &\mapsto \alpha(x, \varepsilon_A^{\mathrm{Fr}}(xs, a)).\end{aligned}$$

The cofree functor is defined by mapping $D \in \mathbf{Set}$ to

$$\begin{aligned}\delta_{\mathrm{Cof}} : D \times (X \times D)^*_\infty &\to 1 + X \times D \times (X \times D)^*_\infty \\ (d_0, [\,]) &\mapsto * \\ (d_0, (x, d) : xds) &\mapsto (x, d, xds)\end{aligned}$$

The behavior of Cof on functions $f : C \to D$ is forced and given by applying f to all elements of C in any element of $C \times (X \times C)^*_\infty$. Checking this is a right adjoint to the forgetful functor can be done by constructing the natural bijection

$$\mathbf{Set}(C, D) \cong \mathbf{CoAlg}(C, \mathrm{Cof}(D)),$$

The bijection is given by $\mathbf{Set}(C, D) \to \mathbf{CoAlg}(C, \mathrm{Cof}(D)), f \mapsto \tilde{f}$, where

$$\begin{aligned}\tilde{f} = \langle \tilde{f}_0, \tilde{f}_1 \rangle : C &\to D \times (X \times D)^*_\infty \\ c &\mapsto \begin{cases} (f(c), (x, f(c')) : \tilde{f}_1(c')) \text{ if } \chi(c) = (x, c') \\ (f(c), [\,]) \text{ if } \chi(c) = *. \end{cases}\end{aligned}$$

Notice the function $\tilde{f}_0 = f$, and hence this also gives the inverse for the bijection. Verifying $\tilde{f}$ is an coalgebra morphism is left to the reader. Informally, $\tilde{f}$ takes a stream-like $c \in C$ and applies f to all its elements. The unit and counit are given by $\eta_D^{\mathrm{Cof}} : D \to D \times (X \times D)^*_\infty, d \mapsto (d, [])$ and $\varepsilon_D^{\mathrm{Cof}} : D \times (X \times D)^*_\infty \to D, (d, xds) \mapsto d$.

3.5.5 Representing objects

Here, we will give full details when constructing the representing objects. We will finally see which roles the free and cofree functor play in this construction.

First we wish to construct the terminal object in the category of measurings, $\underline{\mathbf{Alg}}(A,B)$. The representing object $\underline{\mathbf{Alg}}(A,B)$ is given by the following equalizer

$$\underline{\mathbf{Alg}}(A,B) \overset{\text{eq}}{\dashrightarrow} \mathbf{Set}(A,B)\times(X\times\mathbf{Set}(A,B))^*_\infty \underset{\Phi}{\overset{\mathrm{Cof}(\alpha^*)}{\rightrightarrows}} \mathbf{Set}(1+X\times A,B)\times(X\times\mathbf{Set}(1+X\times A,B))^*_\infty$$

where Φ is the transpose of

$$\tilde{\Phi}:\mathbf{Set}(A,B)\times(X\times\mathbf{Set}(A,B))^*_\infty \xrightarrow{\chi_{\mathrm{Cof}}} 1+X\times\mathbf{Set}(A,B)\times(X\times\mathbf{Set}(A,B))^*_\infty \xrightarrow{F(\varepsilon^{\mathrm{Cof}})}$$
$$1+X\times\mathbf{Set}(A,B)\xrightarrow{\tilde{\nabla}_{A,B}}\mathbf{Set}(1+X\times A,1+X\times B)\xrightarrow{\beta_*}\mathbf{Set}(1+X\times A,B).$$

If we compute the composition we find

$$\begin{aligned}\tilde{\Phi}:\mathbf{Set}(A,B)\times(X\times\mathbf{Set}(A,B))^*_\infty &\to \mathbf{Set}(1+X\times A,B)\\ (f',[\,]) &\mapsto \mathrm{const}_{e_B}\\ (f',(x,f):xfs) &\mapsto \begin{cases} * & \mapsto e_B\\ (x',a) & \mapsto \beta(x\bullet x',f(a)).\end{cases}\end{aligned}$$

We see $\tilde{\Phi}$ only depends on the first element of the list xfs, so for brevity we may write $\tilde{\Phi}((f',(x,f):xfs)) = \tilde{\Phi}((x,f))$. Taking its transpose $\Phi = \mathrm{Cof}(\tilde{\Phi})\circ\eta$ we find the explicit formula for Φ to be

$$\begin{aligned}\Phi:\mathbf{Set}(A,B)\times(\mathbf{Set}(A,B)\times X)^*_\infty &\mapsto \mathbf{Set}(1+X\times A,B)\times(\mathbf{Set}(1+X\times A,B)\times X)^*_\infty\\ (f',[\,]) &\mapsto (\mathrm{const}_{e_B},[\,])\\ (f',(x,f):xfs) &\mapsto (\tilde{\Phi}((x,f)),[\,]).\end{aligned}$$

Taking the equalizer, we find the underlying set of $\underline{\mathbf{Alg}}(A,B)$ to be a subset of $\mathbf{Set}(A,B)\times(X\times\mathbf{Set}(A,B))^\infty$, given by

$$\{(f_0,(x_i,f_{i+1})_{i\in I}) \mid f_i(e_A)=e_B, (f_i\circ\alpha)(x,a)=\beta((x_i\bullet x),f_{i+1}(a)), f_{\max}(\alpha(x,a))=e_B \text{ for all } a\in A, x\in X\},$$

where $f_{\max}$ is the last function in the stream $(f_i)_{i\in I}$ if the stream is finite. Unpacking the above, we find $\underline{\mathbf{Alg}}(A,B)$ consists of two streams $(x_i)_{i\in I}\in X^*_\infty$ and $(f_i)_{i\in I+1}\in\mathbf{Set}(A,B)^*_\infty$, with the condition $f_i(\alpha(x,a)) = \beta((x_i\bullet x),f_{i+1}(a))$, and the last element of the stream (f_i) should be the function mapping all elements of the form $\alpha(x,a)$ to the element $e_B\in B$. The coalgebra structure of $\underline{\mathbf{Alg}}(A,B)$ is given by shifting to the left, $(f_0,(x_i,f_{i+1})_i)\mapsto(x_0,(f_1,(x_{i+1},f_{i+2})_i)), (f_0,[\,])\mapsto *$. Lastly, we give the evaluation map

$$\begin{aligned}\mathrm{ev}:\underline{\mathbf{Alg}}(A,B)\times A&\to B\\ ((f_0,(x_i,f_{i+1})_{i\in I}),a)&\mapsto f_0(a).\end{aligned}$$

The evaluation map is measuring from A to B by $\underline{\mathbf{Alg}}(A,B)$ by definition, making it an element of the category of measurings from A to B.

To verify $\mathfrak{m}_C(A,B)\cong\mathbf{CoAlg}(C,\underline{\mathbf{Alg}}(A,B))$, we explicitly construct the bijection. It is given by $\Psi:\mathfrak{m}_C(A,B)\to\mathbf{CoAlg}(C,\underline{\mathbf{Alg}}(A,B)), \varphi\mapsto\Psi(\varphi)$, where

$$\begin{aligned}\Psi(\varphi):C&\to\underline{\mathbf{Alg}}(A,B)\\ c&\mapsto\left(\varphi_c,\left(\chi_0^i(c),\varphi_{\chi_1^i(c)}\right)_{1\le i\le\mathrm{len}(c)}\right),\end{aligned}$$

where we write $\chi=\langle\chi_0,\chi_1\rangle : C\to 1+X\times C$ and $\mathrm{len}(c)\in\mathbb{N}_\infty$ is defined as the unique number such that $\chi^{\mathrm{len}(c)}(c)=*$. First, we verify $\Psi(\varphi)$ is well-defined. This is the case since $\varphi(\chi_1^{\mathrm{len}(c)}(c))(\alpha(x,a))=e_B$ by definition of a measuring and $\mathrm{len}(c)$ and $\varphi_{\chi_1^i(c)}(\alpha(x,a))=\beta(x\bullet\chi_0^i(c),\varphi_{\chi_1^{i+1}(c)}(a))$ by definition of a measuring. Second, we verify $\Psi(\varphi)$ is a morphism of coalgebras. To this end, we note $\Psi(\varphi)(\chi(c)) = (\varphi(\chi(c))(\_),\left(\chi_0^{i+1}(c),\varphi_{\chi_1^{i+1}(c)}\right)_{2\le i\le\mathrm{len}(c)})$, which agrees with the coalgebra structure on $\underline{\mathbf{Alg}}(A,B)$.

The inverse of the bijection $\Psi : \mathfrak{m}_C(A,B) \to \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A,B))$ is given by sending a morphism $\tilde{\varphi} : C \to \underline{\mathbf{Alg}}(A,B)$ to the measuring $\Psi^{-1}(\tilde{(\varphi)}) : (c,a) \mapsto (\mathrm{pr}_0 \circ \tilde{\varphi}(c))(a)$. This is well-defined by the properties on $\underline{\mathbf{Alg}}(A,B)$. The bijection $\Psi : \mathfrak{m}_C(A,B) \to \mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(A,B))$ is natural, which can be checked by a straightforward calculation. We conclude $\underline{\mathbf{Alg}}(A,B)$ represents $\mathfrak{m}\_(A,B)$. By the above the evaluation map is also the terminal object in the category of measurings, since any measuring $\varphi : C \times A \to B$ factors uniquely through $\mathrm{ev} : \underline{\mathbf{Alg}}(A,B) \times A \to B$.

Next we would like to compute $C \triangleright A$. The object $C \triangleright A$ is given by the coequalizer

$$X^* \times (C \times (1 + X \times A) + \{e_{C\times F(A)}\}) \underset{\Phi}{\overset{\mathrm{id}_{X^*} \times (\mathrm{id}_C \times \alpha + \mathrm{id}_{\{e\}})}{\rightrightarrows}} X^* \times (C \times A + \{e_{C\times A}\}) \overset{\mathrm{coeq}}{\dashrightarrow} C \triangleright A,$$

where Φ is the transpose of

$$\tilde{\Phi} : C \times 1 + X \times A \xrightarrow{\chi \times \mathrm{id}_{F(A)}} 1 + X \times C \times 1 + X \times A \xrightarrow{\nabla_{A,B}} 1 + X \times (C \times A) \xrightarrow{F(\eta^{\mathrm{Fr}})}$$
$$1 + X \times (X^* \times (C \times A + \{e_{C\times A}\})) \xrightarrow{\alpha_{\mathrm{Fr}}} X^* \times (C \times A + \{e_{C\times A}\}).$$

If we compute the composition we find

$$\begin{aligned} \tilde{\Phi} : C \times 1 + X \times A &\to X^* \times (C \times A + \{e_{C\times A}\}) \\ (c, *) &\mapsto \chi(c) \bullet ([\,], e_{C\times A}) \\ (c,(x,a)) &\mapsto \begin{cases} ([\,], e_{C\times A}) & \text{if } \chi(c) = * \\ ([x' \bullet x], (c', a)) & \text{if } \chi(c) = (x', c'). \end{cases} \end{aligned}$$

Taking its transpose $\Phi = \varepsilon \circ \mathrm{Fr}(\tilde{\Phi})$ we find the explicit formula for Φ to be

$$\begin{aligned} \Phi : X^* \times (C \times (1 + X \times A) + \{e_{C\times F(A)}\}) &\to X^* \times (C \times A + \{e_{C\times A}\}) \\ (xs, e_{C\times F(A)}) &\mapsto (xs, e_{C\times A}) \\ (xs, (c, *)) &\mapsto (xs, e_{C\times A}) \\ (xs, (c,(x,a))) &\mapsto \begin{cases} (xs, e_{C\times A}) & \text{if } \chi(c) = * \\ (xs {+}{+} [x' \bullet x], (c', a)) & \text{if } \chi(c) = (x', c'). \end{cases} \end{aligned}$$

Now that we know explicitly which maps we want to coequalize, we can compute

$$\mathrm{Fr}(C \times A)/\sim\; = X^* \times ((C \times A) + \{e_{C\times A}\})/\sim .$$

Here the equivalence relation $\sim$ is generated by

$$\begin{aligned} (xs, c, e_A) &\sim (xs, e_{C\times A}) \\ (xs, c, \alpha(x,a)) &\sim (xs, e_{C\times A}) \text{ if } \chi(c) = * \\ (xs, c, \alpha(x,a)) &\sim (xs {+}{+} [x \bullet x'], c', a) \text{ if } \chi(c) = (x', c'). \end{aligned}$$

Its algebra structure is given by

$$\begin{aligned} 1 + X \times C \triangleright A &\to C \triangleright A \\ * &\mapsto [[\,], e_{C\times A}] \\ (x, [xs, (c,a)]) &\mapsto [x : xs, (c,a)]. \end{aligned}$$

Intuitively, given an element (xs, c, a) the equivalence relation transfers the first elements of the stream-like c and the list-like a to the beginning of the list xs by combining them using the monoid structure on X.

To verify $\mathfrak{m}_C(A,B) \cong \mathbf{Alg}(C \triangleright A, B)$, we explicitly construct the bijection. It is given by

$$\begin{aligned} \Psi : \mathfrak{m}_C(A,B) &\to \mathbf{Alg}(C \triangleright A, B) \\ \varphi &\mapsto \Psi(\varphi), \end{aligned}$$

where $\Psi(\varphi)$ is given by

$$\begin{aligned}\Psi(\varphi) : C \triangleright A &\to B \\ [([\,], e_{C\times A})] &\mapsto e_B \\ [(x : xs, e_{C\times A})] &\mapsto \beta(x, \Psi(\varphi)([(xs, e_{C\times A})])) \\ [([\,], (c, a))] &\mapsto \varphi(c, a) \\ [(x : xs, (c, a))] &\mapsto \beta(x, \Psi(\varphi)([(xs, (c, a))])).\end{aligned}$$

To check $\Psi(\varphi)$ is well-defined, one needs to perform a straightforward check using the fact that φ is a measuring. To see $\Psi(\varphi)$ is a morphism of algebras, we note this is the case by definition of $C \triangleright A$ and the definition of $\Psi(\varphi)$.

The inverse of the bijection $\Psi : \mathfrak{m}_C(A, B) \to \mathbf{Alg}(C \triangleright A, B)$ is given by sending a morphism $\tilde{\varphi} : C \triangleright A \to B$ to the measuring

$$\Psi^{-1}(\tilde{\varphi}) : (c, a) \mapsto \tilde{\varphi}([([\,], (c, a))]),$$

which is well-defined by the properties on $C \triangleright A$. The bijection $\Psi : \mathfrak{m}_C(A, B) \to \mathbf{Alg}(C \triangleright A, B)$ is natural, which can be checked by a straightforward calculation. We conclude $C \triangleright A$ represents $\mathfrak{m}_C(A, \_)$.

Finally, we define the convolution algebra $[C, B]$ to have underlying set $\mathbf{Set}(C, B)$ and algebra structure given by

$$\begin{aligned}1 + X \times \mathbf{Set}(C, B) &\to \mathbf{Set}(C, B) \\ * &\mapsto (\_ \mapsto e_B) \\ (x, f) &\mapsto \left(c \mapsto \begin{cases} \beta(x \bullet x', f(c')) \text{ if } \chi(c) = (x', c') \\ e_A \text{ if } \chi(c) = * \end{cases} \right).\end{aligned}$$

By Section 3.4 in [1], we have the natural bijection $\mathfrak{m}_C(A, B) \cong \mathbf{Alg}(A, [C, B])$, which we verify by constructing it. It is given by $\Psi : \mathfrak{m}_C(A, B) \to \mathbf{Alg}(A, [C, B]), \varphi \mapsto \Psi(\varphi)$, where $\Psi(\varphi)$ is given by $\Psi(\varphi) : A \to [C, B], a \mapsto \varphi(\_)(a)$. To verify this is an algebra morphism, we check

$$\Psi(\varphi)(\alpha(x, a))(c) = \varphi(c)(\alpha(x, a)) = \begin{cases} \beta(x \bullet x', \varphi(c')(a) \text{ if } \chi(c) = (x', c')) \\ e_B \text{ if } \chi(c) = * \end{cases},$$

which corresponds to the algebra structure on $[C, B]$.

The inverse is given by sending $\tilde{\varphi} \in \mathbf{Alg}(A, [C, B])$ to the measuring $\Psi^{-1}(\tilde{\varphi}) : C \times A \to B, (c, a) \mapsto \tilde{\varphi}(a)(c)$. This is a measuring by definition of the algebra structure on $[C, B]$.

The bijection $\Psi : \mathfrak{m}_C(A, B) \to \mathbf{Alg}(A, [C, B])$ is natural, which can be checked by a straightforward calculation. We conclude $[C, B]$ represents $\mathfrak{m}_C(\_, B)$.

Example 3.5.10. We would like to calculate $\underline{\mathbf{Alg}}(X_n^*, B)$ for arbitrary $B \in \mathbf{Alg}$. Our aim is to leverage the isomorphism

$$\mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(X_n^*, B)) \cong \mathfrak{m}_C(X_n^*, B) \cong \mathbf{Alg}(X_n^*, [C, B]).$$

To do so, we make an observation about $\mathbf{Alg}(X_n^*, Z)$ for an arbitrary algebra $(Z.\zeta) \in \mathbf{Alg}$. Since X_n^* is preinitial, any morphism out of X_n^* is unique. So, $\mathbf{Alg}(X_n^*, Z)$ has at most one element. The question now becomes if there is a condition on Z for $\mathbf{Alg}(X_n^*, Z)$ to be inhabited. We claim this condition is

$$\zeta(x, \mathsf{i}_Z(xs)) = \zeta(x, \mathsf{i}_Z(\mathrm{take}(n-1)(xs))) \text{ for all } (x, xs) \in X \times X_n^*. \tag{3.2}$$

Notice that $\mathrm{take}(n-1)(xs) = xs$ for all lists of length less than n, so the only non-trivial case is when $\mathrm{len}(xs) = n$. If Eq. (3.2) is satisfied, we claim the unique algebra morphism $X_n^* \to Z$ is given by

$$\mathsf{i}_Z \circ m : X_n^* \hookrightarrow X^* \to Z,$$

where $m \in \mathbf{Set}(X_n^*, X^*)$ is an inclusion of sets, not an algebra morphism. For brevity, we will write $\mathsf{i}_Z \circ m = \mathsf{i}_Z$ if the domain is understood. To verify this is an algebra morphism, we must check it commutes with the algebra structures. If we denote the algebra structure on X_n^* by α_n, we need to check $\mathsf{i}_Z(\alpha_n(x, xs)) = \zeta(x, \mathsf{i}_Z(xs))$, since the case of the empty list is trivial. Using Eq. (3.2) and that $\mathsf{i}_Z : X^* \to Z$ is an algebra morphism we can make the following deduction:

$$\begin{aligned}(\mathsf{i}_Z \circ m)(\alpha_n(x, xs)) &= (\mathsf{i}_Z \circ m)(x : \mathrm{take}(n-1)(xs)) \\ &= \mathsf{i}_Z(x : \mathrm{take}(n-1)(xs)) \\ &= \zeta(x, \mathsf{i}_Z(\mathrm{take}(n-1)(xs))) \\ &= \zeta(x, \mathsf{i}_Z(xs)) \\ &= \zeta(x, (\mathsf{i}_Z \circ m)(xs)).\end{aligned}$$

Conversely, if there exists $xs \in X_n^*$ such that $\zeta(x, \mathsf{i}_Z(xs)) \neq \zeta(x, \mathsf{i}_Z(\mathrm{take}(n-1)(xs)))$ it is impossible to construct an algebra morphism $f : X_n^* \to Z$. This can be seen by trying to construct such a morphism f and observing $f(\alpha_n(x, xs)) \neq \zeta(x, f(xs))$ in this case.

Now that we have an indication on whether $\mathbf{Alg}(X_n^*, Z)$ is inhabited for arbitrary $Z \in \mathbf{Alg}$, we can focus in $Z = [C, B]$. Observing the algebra structure on $[C, B]$, we see the morphism $\mathsf{i}_{[C,B]}$ is given by

$$\begin{aligned}\mathsf{i}_{[C,B]} : X^* &\to [C, B] \\ [\,] &\mapsto (c \mapsto e_B) \\ x : xs &\mapsto \left(c \mapsto \begin{cases}\beta(x \bullet x', \mathsf{i}_{[C,B]}(xs)(c')) \text{ if } \chi(c) = (x', c') \\ e_B \text{ if } \chi(c) = *.\end{cases}\right)\end{aligned}$$

Denoting the algebra structure on $[C, B]$ by α, we would like to know under which condition on C and B $\alpha(x, \mathsf{i}_{[C,B]}(xs)) = \alpha(x, \mathsf{i}_{[C,B]}(\mathrm{take}(n-1)(xs)))$. Unpacking the above, we find this is the case if

$$\beta(x \bullet x', \mathsf{i}_{[C,B]}(xs)(c')) = \beta(x \bullet x', \mathsf{i}_{[C,B]}(\mathrm{take}(n-1)(xs))(c')) \text{ for all } (x', c') \in \mathrm{im}(\chi).$$

By definition of $\mathsf{i}_{[C,B]}$, the above condition can be satisfied if $\mathrm{len}(c) \leq n$ for all $c \in C$ or if Eq. (3.2) holds for B. So, we have the following:

$$\mathbf{Alg}(X_n^*, [C, B]) \cong \begin{cases}\{*\} \text{ if } \beta(x, \mathsf{i}_B(xs)) = \beta(x, \mathsf{i}_B(\mathrm{take}(n-1)(xs))) \text{ for all } (x, xs) \in X \times X_n^* \\ \{*\} \text{ if } \mathrm{len}(c) \leq n \text{ for all } c \in C \\ \emptyset \text{ otherwise.}\end{cases}$$

Now we can make use of the isomorphism $\mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(X_n^*, B)) \cong \mathbf{Alg}(X_n^*, [C, B])$ We observe that in the case of $\beta(x, \mathsf{i}_B(xs)) = \beta(x, \mathsf{i}_B(\mathrm{take}(n-1)(xs)))$, $\underline{\mathbf{Alg}}(X_n^*, B)$ has the universal property of the terminal coalgebra X_∞^*. Whenever this is not the case, $\underline{\mathbf{Alg}}(X_n^*, B)$ has the universal property of $X_n^* \in \mathbf{CoAlg}$. We conclude

$$\underline{\mathbf{Alg}}(X_n^*, B) = \begin{cases}X_\infty^* \text{ if } \beta(x, \mathsf{i}_B(xs)) = \beta(x, \mathsf{i}_B(\mathrm{take}(n-1)(xs))) \text{ for all } (x, xs) \in X \times X_n^* \\ X_n^* \text{ otherwise.}\end{cases}$$

This also shows that $X_n^{*\circ} = \underline{\mathbf{Alg}}(\mathbb{n}, X^*) \cong X_n^{*\dagger}$. We will write $X_n^{*\circ}$ instead of $X_n^{*\dagger}$ from now on. △

3.5.6 C-initial algebras

We have already seen that $\mathbb{n}$ is the terminal $\mathbb{n}^\dagger\mathbb{n}^\circ$-initial algebra. An educated guess is that X_n^* is the terminal $X_n^{*\dagger} \cong X_n^{*\circ}$-initial algebra as well. This is indeed the case! Moreover, for any $k \leq n$, X_n^* is a $X_k^{*\circ}$-initial algebra, just as in the case of natural numbers. We have already seen a proof for the natural numbers, which in this context corresponds to the trivial monoid $X \cong 1$. One could adapt that proof to hold for any monoid X, but we wish to take a slightly different approach. We will defer the proof to Section 5.3 where we aim to kill many birds with one stone.

One could also wonder about the preinitial algebra $(X')^*$. An educated guess might be that $(X')^*$ is a $(X')^{*\dagger}$-initial algebra. Alas, we have already seen at the start of this section that there does not exist a measuring $\varphi : C \times (X')^* \to B$, for any $B \in \mathbf{Alg}$. This immediately tells us that $(X')^{*\circ} = \underline{\mathbf{Alg}}((X')^*, X^*) = \emptyset$, and hence that $(X')^{*\circ} \ncong (X')^{*\dagger}$. We also see that $(X')^*$ can not be an $(X')^{*\dagger}$-initial algebra.

But what then can we say about coalgebras stemming from a subset $X' \subseteq X$, such as $(X')^{*\dagger}$ and $(X')_n^{*\dagger}$? As it turns out, they inherit their initial algebras from their counterparts stemming from the entire monoid X. This is due to the fact that in this case, the monomorphisms $m_n : (X')_n^{*\dagger} \rightarrowtail X_n^{*\circ} \in \mathbf{CoAlg}$ induce monomorphisms

$$\begin{aligned} m_n \triangleright \mathrm{id}_A : (X')_n^{*\dagger} \triangleright A &\to X_n^{*\circ} \triangleright A \\ [xs, xs', a] &\mapsto [xs, xs', a]. \end{aligned}$$

An algebra A is $X_n^{*\circ}$-initial if and only if $X_n^{*\circ} \triangleright A \cong I$, where $I \cong X^*$ is the initial algebra. A monomorphism into an initial object is an isomorphism, so using $m_n \triangleright \mathrm{id}_A$ we see $(X')_n^{*\dagger} \triangleright A \cong I$. This implies any $X_n^{*\circ}$-initial algebra A is also $(X')_n^{*\dagger}$-initial. Repeating the above argument for the monomorphisms $m_{k,n} : (X')_k^{*\dagger} \rightarrowtail X_n^{*\circ}$ for $k \leq n$, we can generalize the above statement. We see any $X_n^{*\circ}$-initial algebra is also $(X')_k^{*\dagger}$-initial.

3.6 The binary tree type

Having seen the theory applied to the case of lists, we can continue to explore datatypes frequently used in computer science. One of them is the binary tree type. This datatype adds to the complexity by allowing a branching structure. The branching structure is very predictable however, which makes it a nice stepping stone to the last section where we allow all branching structures. To study the binary tree type we must find a functor which has it as an initial algebra. This functor exists and is the next thing we will be defining.

Let F be the functor given by

$$\begin{aligned} F : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1 + X \times A \times A, \end{aligned}$$

where $(X, \bullet, e)$ is a commutative monoid. We claim F is lax monoidal by $\eta : 1 \to 1 + X$
$* \mapsto e$ and

$$\begin{aligned} \nabla_{A,B} : (1 + X \times A \times A) \times (1 + X \times B \times B) &\to 1 + X \times A \times B \times A \times B \\ (*, \_) &\mapsto * \\ (\_, *) &\mapsto * \\ ((x, a, a'), (x', b, b')) &\mapsto (x \bullet x', a, b, a', b'). \end{aligned}$$

The category of F-algebras has elements $\alpha : 1 + X \times A \times A \to A$ denoted (A, α). Morphisms $f : (A, \alpha) \to (B, \beta)$ are given by functions $f : A \to B$ which make the following diagram commute:

$$\begin{array}{ccc} 1 + X \times A \times A & \xrightarrow{F(f)} & 1 + X \times B \times B \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

The category of F-coalgebras has elements $\chi : C \to 1 + X \times C \times C$, denoted (C, χ). Morphisms $f : (C, \chi) \to (D, \delta)$ are given by functions $f : C \to D$ which make the following diagram commute:

$$\begin{array}{ccc} 1 + X \times C \times C & \xrightarrow{F(f)} & 1 + X \times D \times D \\ \uparrow{\scriptstyle \chi} & & \uparrow{\scriptstyle \delta} \\ C & \xrightarrow{\quad f \quad} & D \end{array}$$

3.6.1 Initial and terminal objects

The initial algebra is given by the collection of all finite binary trees. In order to define the underlying set, we define the sets $T_{X,n}$, which are the binary trees with values in X of depth at most n. Let $T_{X,0} = \{e_T\} \cong 1$, then we can write

$$T_{X,n} = X \times T_{X,n-1} \times T_{X,n-1} + T_{X,0}.$$

Note that we have a filtration $T_0 \subseteq T_1 \subseteq \cdots \subseteq T_{X,n-1} \subset T_{X,n} \subseteq T_{X,n+1} \subseteq \dots$. The underlying set of the initial algebra is given by $T_X := \bigcup_{n\in\mathbb{N}} T_{X,n}$. Its algebra structure is given by

$$\begin{aligned} 1 + X \times T_X \times T_X &\to T_X \\ * &\mapsto e_T \\ (x, \ell, r) &\mapsto (x, \ell, r). \end{aligned}$$

Given an element $(x, \ell, r) \in T_X$, we will call x the value stored in the node, ℓ the left child and r the right child. Some examples of preinitial algebras include $T_{X,n}$ for all $n \in \mathbb{N}$, $T_{X'}$ and $T_{X',n}$, where $X' \subseteq X$. A slightly less obvious example is X^*, with algebra structure

$$\begin{aligned} \alpha_\ell : 1 + X \times X^* \times X^* &\to X^* \\ * &\mapsto [\,] \\ (x, \ell, r) &\mapsto x : \ell. \end{aligned}$$

The unique algebra morphism $T_X \to (X^*, \alpha_\ell)$ returns the left most branch of the tree as a list. Altering the algebra structure on X^* does give different results. Consider the algebra structure

$$\begin{aligned} \alpha_{\max} : 1 + X \times X^* \times X^* &\to X^* \\ * &\mapsto [\,] \\ (x, \ell, r) &\mapsto \begin{cases} x : \ell & \text{if } \operatorname{len}(\ell) \geq \operatorname{len}(r) \\ x : r & \text{otherwise.} \end{cases} \end{aligned}$$

In this case, the unique algebra morphism $T \to (X^*, \alpha_{\max})$ gives the (leftmost) longest branch. Lastly, we remark there exists a morphism which can capture the shape of the binary tree while forgetting its contents. It is defined as the unique algebra morphism $\text{shape} : T_X \to T_1$.

In order te define the terminal coalgebra we need to define one more set which contains trees of infinite depth. This set can be defined recursively as $\tilde{T}_{X,\infty} = \{(x, \ell, r) \in X \times \tilde{T}_{X,\infty} \times \tilde{T}_{X,\infty}\}$. The terminal coalgebra has as underlying set $T_{X,\infty} = T_X + \tilde{T}_{X,\infty}$ with coalgebra structure given by

$$\begin{aligned} T_{X,\infty} &\to 1 + X \times T_{X,\infty} \times T_{X,\infty} \\ e_T &\mapsto * \\ (x, \ell, r) &\mapsto (x, \ell, r). \end{aligned}$$

Subterminal coalgebras are given by subsets of $T_{X,\infty}$ which respect the algebra structure. Again, we will use $(\_)^\dagger$ to distinguish algebras and coalgebras which have the same underlying set. Some examples are ${T_{X,n}}^\dagger, T_X^\dagger, \tilde{T}_{X,\infty}$, but we also have the example of

$$\begin{aligned} X^*_\infty &\to 1 + X \times X^*_\infty \times X^*_\infty \\ [\,] &\mapsto * \\ x : xs &\mapsto (x, xs, [\,]). \end{aligned}$$

Again, changing the coalgebra structure on X^*_∞ will give a plethora of different coalgebras, each with their own interpretation.

3.6.2 Measurings

Definition 3.6.1. Given algebras $A, B \in \mathbf{Alg}$ and a coalgebra $C \in \mathbf{CoAlg}$, a measuring is a function $\varphi : C \times A \to B$ satisfying

1. $\varphi(c)(e_A) = e_B$ for all $c \in C$
2. $\varphi(c)(\alpha(x, a_\ell, a_r)) = e_B$ if $\chi(c) = *$
3. $\varphi(c)(\alpha(x, a_\ell, a_r)) = \beta(x' \bullet x, \varphi(c_\ell)(a_\ell), \varphi(c_r)(a_r))$ if $\chi(c) = (x', c_\ell, c_r)$.

Comparing this with the definition of measuring found in Section 3.5, we see they are very similar. The only difference is that in this case, we are accommodating the branching structure of binary trees.

Example 3.6.2. For similar reasons as in Example 3.5.7, we remark there does not exist an algebra morphism $T_{X,n} \to T_X$. There does however exist a measuring from $T_{X,n}$ to T_X by $T^{\dagger}_{X,n}$. It is given by

$$\begin{aligned}
\varphi : T^{\dagger}_{X,n} \times T_{X,n} &\to T_X \\
(e_T, (x, \ell, r)) &\mapsto e_T \\
((x', \ell', r'), e_T) &\mapsto e_T \\
((x', \ell', r'), (x, \ell, r)) &\mapsto y(x' \bullet x, \varphi(\ell', \ell), \varphi(r', r)),
\end{aligned}$$

which is a measuring by definition. Again, for reasons similar to those seen in Example 3.5.7 a measuring to T_X does exist, where an algebra morphism does not. The coalgebra $T^{\dagger}_{X,n}$ gives us control over which parts of an element in $T_{X,n}$ we consider. △

Example 3.6.3. If we only care about modifying the shape of an element in T_X and don't want to introduce any twist, we can consider the measuring

$$\begin{aligned}
\varphi : T^{\dagger}_{\{e\}} \times T_X &\to T_X \\
(e_T, (x, \ell, r)) &\mapsto e_T \\
((e, \ell', r'), e_T) &\mapsto e_T \\
((e, \ell', r'), (x, \ell, r)) &\mapsto y(x, \varphi(\ell', \ell), \varphi(r', r)),
\end{aligned}$$

which is a measuring by definition. This measuring takes a tree containing only the unit of the monoid $t' \in T^{\dagger}_{\{e\}}$ and uses its shape to modify a tree $t \in T_X$ to fit within the shape of t'. △

Example 3.6.4. Again, we can generalize the map function using a measuring to obtain more fine grained control over which function gets applied to which elements. Recall we define the function $r_x : X \to X, x' \mapsto x' \bullet x$ for all $x \in X$. We define an algebra structure on $T_{\mathbf{Set}(X,X)}$ by

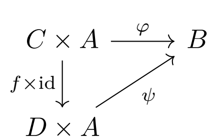
$$\begin{aligned}
1 + X \times T_{\mathbf{Set}(X,X)} \times T_{\mathbf{Set}(X,X)} &\to T_{\mathbf{Set}(X,X)} \\
* &\mapsto e_T \\
(x, \ell, r) &\mapsto (r_x, \ell, r).
\end{aligned}$$

Now we have a measuring given by

$$\begin{aligned}
\varphi : T^{\dagger}_{X} \times T_{\mathbf{Set}(X,X)} &\to T_X \\
(e_T, (f, \ell, r)) &\mapsto e_T \\
((x', \ell', r'), e_T) &\mapsto e_T \\
((x', \ell', r'), (f, \ell, r)) &\mapsto (f(x), \varphi(\ell', \ell), \varphi(r', r)).
\end{aligned}$$

To check this is a measuring, we verify $\varphi((x', \ell', r'), (r_x, \ell, r)) = (r_x(x'), \varphi(\ell', \ell), \varphi(r', r)) = (x' \bullet x, \varphi(\ell', \ell), \varphi(r', r))$. △

The *category of measurings from A to B* has as elements measurings $\varphi : C \times A \to B$, and as morphisms $f : \varphi \to \psi$ functions $f : C \to D$ such that



commutes.

3.6.3 Free and cofree functors

By now it should come as no surprise that we aim to use the free and cofree functors to construct the representing objects.

Given $A \in \mathbf{Set}$, the free functor has as underlying set

$$U(\mathrm{Fr}(A)) = \{(x, \ell, r) \in X \times U(\mathrm{Fr}(A)) \times U(\mathrm{Fr}(A))\} + A + \{e_{\mathrm{Fr}}\},$$

with algebra structure

$$\begin{aligned} 1 + X \times \mathrm{Fr}(A) \times \mathrm{Fr}(A) &\to \mathrm{Fr}(A) \\ * &\mapsto e_{\mathrm{Fr}} \\ (x, \ell, r) &\mapsto (x, \ell, r). \end{aligned}$$

One can think of $\mathrm{Fr}(A)$ as the set of all finite trees with elements in X, where we allow the leaves to contain elements of A. The behavior of Fr on functions $f : A \to B$ is forced and given by applying f to all elements of A in any element of $\mathrm{Fr}(A)$. Checking this is a left adjoint to the forgetful functor can be done by constructing the natural bijection

$$\mathbf{Set}(A, B) \cong \mathbf{Alg}(\mathrm{Fr}(A), B).$$

It is given by sending a function $f : A \to B$ to the morphism

$$\begin{aligned} \tilde{f} : \mathrm{Fr}(A) &\to B \\ (x, \ell, r) &\mapsto \beta(x, \tilde{f}(\ell), \tilde{f}(r)) \\ a &\mapsto f(a) \\ e_{\mathrm{Fr}} &\mapsto \beta(*). \end{aligned}$$

and sending a morphisms $\tilde{f} : \mathrm{Fr}(A) \to B$ to its restriction to $A \subseteq \mathrm{Fr}(A)$. The morphisms $\tilde{f}$ then sends the nodes to their embedding into B, and then appends the leaves with the trees $f(a) \in B$.

The cofree functor is defined by mapping $D \in \mathbf{Set}$ to

$$\begin{aligned} \delta : D \times T_{X \times D \times D, \infty} &\to 1 + X \times D \times T_{X \times D \times D, \infty} \times D \times T_{X \times D \times D, \infty} \\ (d_0, e_T) &\mapsto * \\ (d_0, (x, d_\ell, d_r, t_\ell, t_r)) &\mapsto (x, d_\ell, t_\ell, d_r, t_r) \end{aligned}$$

The behavior of Cof on functions $f : C \to D$ is forced and given by applying f to all elements of C in any element of $D \times T_{X \times D \times D, \infty}$. Checking this is a right adjoint to the forgetful functor can be done by constructing the natural bijection

$$\mathbf{Set}(C, D) \cong \mathbf{CoAlg}(C, \mathrm{Cof}(D)).$$

It is given by sending a function $f : C \to D$ to the morphism

$$\begin{aligned} \tilde{f} : C &\to D \times T^{\infty}_{X \times D \times D} \\ c &\mapsto \begin{cases} (f(c), e_T) \text{ if } \chi(c) = * \\ (f(c), (x, f(c_\ell), f(c_r)), \tilde{f}(c_\ell), \tilde{f}(c_r)) \text{ if } \chi(c) = (x, c_\ell, c_r). \end{cases} \end{aligned}$$

Conversely, a morphism $\tilde{f} : C \to \mathrm{Cof}(D)$ is sent to its composition with the projection on the first coordinate.

3.6.4 Representing objects

Here, we we will be very brief about the representing objects of $\mathfrak{m}$. Constructing them and verifying they do indeed represent $\mathfrak{m}$ is completely analogous to the constructions and verifications done in Section 3.5

The representing object $\underline{\mathbf{Alg}}(A, B)$ is a subset $\underline{\mathbf{Alg}}(A, B) \subseteq T^{\infty}_{X \times \mathbf{Set}(A,B)}$. We can view elements of $T^{\infty}_{X \times \mathbf{Set}(A,B)}$ as two trees with the same shape, one containing elements in X and the other functions $A \to B$.

We will write $(t_x, t_f) \in T^{\infty}_{X\times\mathbf{Set}(A,B)}$ for these two trees. An element $(t_x, t_f) \in T^{\infty}_{X\times\mathbf{Set}(A,B)}$ is included in $\underline{\mathbf{Alg}}(A, B)$ if and only if $f_i(e_A) = e_B$ for all $f_i \in t_f$ and $f_i(\alpha(x, a_\ell, a_r)) = \beta(x \bullet x_i, f_{i_\ell}(a_\ell), f_{i_r}(a_r))$, where f_{i_ℓ}, f_{i_r} are the children of f_i if they exist and are otherwise are taken to be the constant function const_{e_B}.

The representing object $C \triangleright A$ is given by

$$C \triangleright A = \mathrm{Fr}(C \times A)/\sim$$

where

$$\begin{aligned}(c, e_A) &\sim e_{\mathrm{Fr}}\\ (c, \alpha(x, a_\ell, a_r)) &\sim e_{\mathrm{Fr}} \text{ if } \chi(c) = *\\ (c, \alpha(x, a_\ell, a_r)) &\sim (x' \bullet x, (c_\ell, a_\ell), (c_r, a_r)) \text{ if } \chi(c) = (x', c_\ell, c_r)\end{aligned}$$

and we also apply the above relation recursively throughout the tree in $\mathrm{Fr}(C \times A)$. An element of $C \triangleright A$ can be viewed as a tree containing elements of X in its nodes and having elements of $C \times A$ at its leaves. Given a tuple $(c, a) \in C \times A$ at the leaf, we attempt te expand out its tree structure using α and χ, and replace it with an empty leaf if one of these two turns out to be empty.

Given an algebra B and a coalgebra C, the convolution algebra, denoted $[C, B]$, has as underlying set $\mathbf{Set}(C, B)$ and algebra structure

$$\begin{aligned}1 + X \times \mathbf{Set}(C, B) \times \mathbf{Set}(C, B) &\to \mathbf{Set}(C, B)\\ * &\mapsto (c \mapsto e_B)\\ (x, f_\ell, f_r) &\mapsto \left(c \mapsto \begin{cases} e_B \text{ if } \chi(c) = * \\ \beta(x' \bullet x, f_\ell(c_\ell), f_r(c_r)) \text{ if } \chi(c) = (x', c_\ell, c_r) \end{cases} \right).\end{aligned}$$

3.6.5 C-initial algebras

Many of our results from Section 3.5 carry over to this case. As before, one can compute that $T^{\dagger}_{X,n} \cong T^{\circ}_{X,n}$,and in line with our expectations, $T_{X,n}$ is indeed the terminal $T^{\circ}_{X,n}$-initial algebra. For a proof, we defer the reader to Section 5.3. We note the algebras $T_{X,n}$ are $T^{\dagger}_{X',k}$-initial for all $k \leq n$ and subsets $X' \subseteq X$, again similar to Section 3.5. We can see this by noting the monomorphisms $m : (X')^{*\dagger}_k \rightarrowtail X^{*\circ}_n \in \mathbf{CoAlg}$ induce monomorphisms $m \triangleright \mathrm{id}_A : {T^{*}_{X',n}}^{\dagger} \triangleright A \to {T_{X,n}}^{\circ} \triangleright A$ and through the same line of reasoning as in Section 3.5 we conclude any $T^{\circ}_{X,n}$-initial algebra is also $T^{\dagger}_{X',k}$-initial.

3.7 The unbounded tree type

For our final example, we explore what happens if we let loose control over the amount of branching. We want to study the unbounded tree type, which contains trees where each node can have any (possibly infinite) number of children. Again, we will see measurings introduce control over the shape of tree, as well as some element wise twisting.

Let F be the functor given by

$$\begin{aligned}F : \mathbf{Set} &\to \mathbf{Set}\\ A &\mapsto 1 + X \times A^{*}_{\infty}\\ f &\mapsto \mathrm{id}_! + \mathrm{id}_X \times \mathrm{map}(f)\end{aligned}$$

where $(X, \bullet, e)$ is a commutative monoid and map is a function frequently used in functional programming. It is defined as

$$\begin{aligned}\mathrm{map} : \mathbf{Set}(A, B) \times A^{*}_{\infty} &\to B^{*}_{\infty}\\ (f, [\,]) &\mapsto [\,]\\ (f, a : as) &\mapsto f(a) : \mathrm{map}(f)(as),\end{aligned}$$

hence takes a list $as \in A^*_\infty$ and a function $f : A \to B$ and applies f to every element of the list as, returning a list containing elements of B. We claim F is lax monoidal by $\eta : 1 \to 1 + X \times 1^*_\infty \cong 1 + X \times \mathbb{N}_\infty, * \mapsto (e, \infty)$ and

$$\begin{aligned} \nabla_{A,B} : (1 + X \times A^*_\infty) \times (1 + X \times B^*_\infty) &\to 1 + X \times (A \times B)^*_\infty \\ (*, \_) &\mapsto * \\ (\_, *) &\mapsto * \\ ((x, as), (x', bs)) &\mapsto (x \bullet x', \mathrm{zip}(as, bs)), \end{aligned}$$

where zip is a function borrowed from functional programming and defined as

$$\begin{aligned} \mathrm{zip} : A^*_\infty \times B^*_\infty &\to (A \times B)^*_\infty \\ ([\,], bs) &\mapsto [\,] \\ (as, [\,]) &\mapsto [\,] \\ (a : as, b : bs) &\mapsto (a, b) : \mathrm{zip}(as, bs). \end{aligned}$$

The category of F-algebras has elements $\alpha : 1 + X \times A^*_\infty \to A$ denoted (A, α). Morphisms $f : (A, \alpha) \to (B, \beta)$ are given by functions $f : A \to B$ which make the following diagram commute:

$$\begin{array}{ccc} 1 + X \times A^*_\infty & \xrightarrow{F(f)} & 1 + X \times B^*_\infty \\ \downarrow{\scriptstyle \alpha} & & \downarrow{\scriptstyle \beta} \\ A & \xrightarrow{\quad f \quad} & B \end{array}$$

The category of F-coalgebras has elements

$$\chi : C \to 1 + X \times C^*_\infty,$$

denoted (C, χ). Morphisms $f : (C, \chi) \to (D, \delta)$ are given by functions $f : C \to D$ which make the following diagram commute:

$$\begin{array}{ccc} 1 + X \times C^*_\infty & \xrightarrow{F(f)} & 1 + X \times D^*_\infty \\ \uparrow{\scriptstyle \chi} & & \uparrow{\scriptstyle \delta} \\ C & \xrightarrow{\quad f \quad} & D. \end{array}$$

3.7.1 Initial and terminal objects

The initial algebra is given by the collection of all finite unbounded trees. In order to define the underlying set, we define the sets $S_{X,n}$, which are the unbounded trees with values in X of depth at most n. Let $S_{X,0} = \{e_S\} \cong 1$, then we can write

$$S_{X,n} = X \times (S_{X,n-1})^*_\infty \times S_{X,0}.$$

Note that we have a filtration $S_0 \subseteq S_1 \subseteq \cdots \subseteq S_{X,n-1} \subseteq S_{X,n} \subseteq S_{X,n+1} \subseteq \ldots$. The underlying set of the initial algebra is given by $S_X := \bigcup_{n \in \mathbb{N}} S_{X,n}$. Its algebra structure is given by

$$\begin{aligned} 1 + X \times (S_X)^*_\infty &\to S_X \\ * &\mapsto e_S \\ (x, ss) &\mapsto (x, ss). \end{aligned}$$

For $(x, ss) \in S_X$ we will call x the value stored at the node and ss the list of children. Note that the empty tree e_S leads to some interesting behavior in combination with the varying number of children of each node. For example, what does one make of the difference between the trees $(x, [\,]) \in S_X$ and $(x, [e_S]) \in S_X$? In practice these differences are insignificant, but it does have an effect on the shape of a tree.

Some examples of preinitial algebras include $S_{X,n}$ for all $n \in \mathbb{N}$, $S_{X'}$ and $S_{X',n}$, where $X' \subseteq X$. A slightly less obvious example is X^*, with algebra structure

$$\begin{aligned} \alpha : 1 + X \times (X^*)^*_\infty &\to X^* \\ * &\mapsto [\,] \\ (x, [\,]) &\mapsto [x] \\ (x, xs : xss) &\mapsto x : xs. \end{aligned}$$

The unique algebra morphism $S_X \to X^*$ returns the left most branch of the tree as a list. Altering the algebra structure on X^* does give different results. Consider the algebra structure

$$\begin{aligned} \alpha_{\max} : 1 + X \times (X^*)^*_\infty &\to X^* \\ * &\mapsto [\,] \\ (x, [\,]) &\mapsto [x] \\ (x, xss) &\mapsto x : xs \text{ where } xs = \operatorname{argmax}_{xs' \in xss}(\operatorname{len}(xs')). \end{aligned}$$

In this case, the unique algebra morphism $T \to (X^*, \alpha_{\max})$ gives the longest branch. Lastly, we remark there exists a morphism which can capture the shape of the unbounded tree while forgetting its contents. It is defined as the unique algebra morphism $\text{shape} : S_X \to S_1$.

In order te define the terminal coalgebra we need to define one more set which contains trees of infinite depth. This set can be defined recursively as $\tilde{S}_{X,\infty} = \{(x, ss) \in X \times (\tilde{S}_{X,\infty})^*_\infty\}$. The terminal coalgebra has as underlying set $S_{X,\infty} = S_X + \tilde{S}_{X,\infty}$ with coalgebra structure given by

$$\begin{aligned} S_{X,\infty} &\to 1 + X \times (S_{X,\infty})^*_\infty \\ e_S &\mapsto * \\ (x, ss) &\mapsto (x, ss). \end{aligned}$$

Subterminal coalgebras are given by subsets of $S_{X,\infty}$ which respect the algebra structure. Again, we will use $(\_)^\dagger$ to distinguish algebras and coalgebras which have the same underlying set. Some examples are ${S_{X,n}}^\dagger, S_X^\dagger, \tilde{S}_{X,\infty}$, but we also have the example of

$$\begin{aligned} X^*_\infty &\to 1 + X \times (X^*_\infty)^*_\infty \\ [\,] &\mapsto * \\ x : xs &\mapsto (x, [xs]). \end{aligned}$$

Again, changing the coalgebra structure on X^*_∞ will give a plethora of different coalgebras, each with their own interpretation.

3.7.2 Measurings

Definition 3.7.1. Given algebras $(A, \alpha), (B, \beta) \in \mathbf{Alg}$ and a coalgebra $(C, \chi) \in \mathbf{CoAlg}$, a measuring is a function $\varphi : C \times A \to B$ satisfying

1. $\varphi(c)(e_A) = e_B$ for all $c \in C$
2. $\varphi(c)(\alpha(x, as)) = e_B$ if $\chi(c) = *$
3. $\varphi(c)(\alpha(x, as)) = \beta(x' \bullet x, \operatorname{map}(\varphi)(\operatorname{zip}(cs, as)))$ if $\chi(c) = (x', cs)$.

It is again very similar to previous definitions of measurings. The most notable difference is that we are zipping the lists of children c and $\alpha(x, as)$ have. This means a measuring does not only give us control over the depth of a tree, which would be analogous to lists, but gives us control over the branching structure of the tree as well.

Example 3.7.2. For similar reasons as in previous examples, we remark there does not exist an algebra morphism $S_{X,n} \to S_X$. There does however exist a measuring from $S_{X,n}$ to S_X by $S^\dagger_{X,n}$. It is given by

$$\begin{aligned} \varphi : S^\dagger_{X,n} \times S_{X,n} &\to S_X \\ (e_S, (x, ss)) &\mapsto e_S \\ ((x', ss'), e_T) &\mapsto e_T \\ ((x', ss'), (x, ss)) &\mapsto (x' \bullet x, \mathrm{map}(\varphi)(\mathrm{zip}(ss', ss))), \end{aligned}$$

which is a measuring by definition. Again, reasoning similar to that seen in Example 3.5.7 applies as to why a measuring to S_X does exist, where an algebra morphism does not. The coalgebra $S^\dagger_{X,n}$ gives us control over which parts of an element in $S_{X,n}$ we consider. △

Example 3.7.3. Consider the measuring

$$\begin{aligned} \varphi : S^\dagger_{\{e\}} \times S_X &\to S_X \\ (e_S, (x, ss)) &\mapsto e_S \\ ((e, ss'), e_T) &\mapsto e_T \\ ((e, ss'), (x, ss)) &\mapsto (x, \mathrm{map}(\varphi)(\mathrm{zip}(ss', ss))), \end{aligned}$$

which is a measuring by definition. This measuring takes a tree containing only the unit of the monoid $t' \in S^\dagger_1$ and uses its shape to modify a tree $t \in S_X$ to fit within the shape of t'. △

Example 3.7.4. Again, we can generalize the map function using a measuring to obtain more fine grained control over which function gets applied to which elements. Recall we define the function $r_x : X \to X, x' \mapsto x' \bullet x$ for all $x \in X$. We define an algebra structure on $S_{\mathbf{Set}(X,X)}$ by

$$\begin{aligned} 1 + X \times S_{\mathbf{Set}(X,X)} \times S_{\mathbf{Set}(X,X)} &\to S_{\mathbf{Set}(X,X)} \\ * &\mapsto e_T \\ (x, \ell, r) &\mapsto (r_x, \ell, r). \end{aligned}$$

Now we have a measuring given by

$$\begin{aligned} \varphi : S^\dagger_X \times S_{\mathbf{Set}(X,X)} &\to S_X \\ (e_S, (f, ss)) &\mapsto e_S \\ ((x', ss'), e_S) &\mapsto e_S \\ ((x', ss'), (f, ss)) &\mapsto (f(x'), \mathrm{map}(\varphi)(\mathrm{zip}(ss', ss))). \end{aligned}$$

To check this is a measuring, we verify $\varphi((x', ss'), (r_x, ss)) = (r_x(x'), \mathrm{map}(\varphi)(\mathrm{zip}(ss', ss))) = (x' \bullet x, \mathrm{map}(\varphi)(\mathrm{zip}(ss', ss)))$. △

The *category of measurings from A to B* has as elements measurings $\varphi : C \times A \to B$, and as morphisms $f : \varphi \to \psi$ functions $f : C \to D$ such that

$$\begin{array}{ccc} C \times A & \xrightarrow{\varphi} & B \\ {\scriptstyle f \times \mathrm{id}} \downarrow & \nearrow_{\psi} & \\ D \times A & & \end{array}$$

commutes.

3.7.3 Free and cofree functors

The final free and cofree functors are computed in this section. Given $A \in \mathbf{Set}$, the free functor has as underlying set

$$U(\mathrm{Fr}(A)) = \{(x, as) \in X \times U(\mathrm{Fr}(A))^*_\infty\} + A + \{e_{\mathrm{Fr}}\},$$

with algebra structure

$$\begin{aligned} 1 + X \times \mathrm{Fr}(A)^*_\infty &\to \mathrm{Fr}(A) \\ * &\mapsto e_{\mathrm{Fr}} \\ (x, as) &\mapsto (x, as). \end{aligned}$$

One can think of $\mathrm{Fr}(A)$ as the set of all finite unbounded trees with elements in X, where we allow the leaves to contain elements of A. The behavior of Fr on functions $f : A \to B$ is forced and given by applying f to all elements of A in any element of $\mathrm{Fr}(A)$. Checking this is a left adjoint to the forgetful functor can be done by constructing the natural bijection

$$\mathbf{Set}(A, B) \cong \mathbf{Alg}(\mathrm{Fr}(A), B).$$

It is given by sending a function $f : A \to B$ to the morphism

$$\begin{aligned} \tilde{f} : \mathrm{Fr}(A) &\to B \\ (x, as) &\mapsto \beta(x, \mathrm{map}(\tilde{f})(as)) \\ a &\mapsto f(a) \\ e_{\mathrm{Fr}} &\mapsto \beta(*). \end{aligned}$$

and sending a morphisms $\tilde{f} : \mathrm{Fr}(A) \to B$ to its restriction to $A \subseteq \mathrm{Fr}(A)$. The morphisms $\tilde{f}$ sends the nodes to their embedding into B, and then appends the leaves with the trees $f(a) \in B$.

The cofree functor is defined by mapping $D \in \mathbf{Set}$ to

$$\begin{aligned} \delta : D \times S_{\mathrm{Cof}(D)} &\to 1 + X \times (D \times S_{\mathrm{Cof}(D)})^*_\infty \\ (d_0, e_S) &\mapsto * \\ (d_0, (x, dts)) &\mapsto (x, dts) \end{aligned}$$

The behavior of Cof on functions $f : C \to D$ is forced and given by applying f to all elements of C in any part of $\mathrm{Cof}(D)$. Checking this is a right adjoint to the forgetful functor can be done by constructing the natural bijection

$$\mathbf{Set}(C, D) \cong \mathbf{CoAlg}(C, \mathrm{Cof}(D)).$$

It is given by sending a function $f : C \to D$ to the morphism

$$\begin{aligned} \tilde{f} : C &\to D \times S_{\mathrm{Cof}(D)} \\ c &\mapsto \begin{cases} (f(c), e_S) \text{ if } \chi(c) = * \\ (f(c), (x, \mathrm{map}(\tilde{f})(cs)) \text{ if } \chi(c) = (x, cs). \end{cases} \end{aligned}$$

Conversely, a morphism $\tilde{f} : C \to \mathrm{Cof}(D)$ is sent to its composition with the projection on the first coordinate.

3.7.4 Representing objects

Here, we we will be very brief about the representing objects of $\mathfrak{m}$. Constructing them and verifying they do indeed represent $\mathfrak{m}$ is analogous to the constructions and verifications done in Section 3.5

The representing object $\underline{\mathbf{Alg}}(A, B)$ is a subset $\underline{\mathbf{Alg}}(A, B) \subseteq S^\infty_{X \times \mathbf{Set}(A,B)}$. We can view elements of $S^\infty_{X \times \mathbf{Set}(A,B)}$ as two trees with the same shape, one containing elements in X and the other functions $A \to B$. We will write $(t_x, t_f) \in S^\infty_{X \times \mathbf{Set}(A,B)}$ for these two trees. An element $(t_x, t_f) \in \underline{\mathbf{Alg}}(A, B)$ if and only if

1. $f_i(e_A) = e_B$ for all $f_i \in t_f$

2. $f_i(\alpha(x, as)) = \beta(x \bullet x_i, \text{zipapply}(fs, as))$,

where here by $\text{zipapply}(fs, as)$ we mean applying the functions in fs to elements in as in a pair wise fashion.

The representing object $C \triangleright A$ is given by

$$C \triangleright A = \text{Fr}(C \times A)/\sim$$

where

$$\begin{aligned}(c, e_A) &\sim e_{\text{Fr}}\\ (c, \alpha(x, as)) &\sim e_{\text{Fr}} \text{ if } \chi(c) = *\\ (c, \alpha(x, as)) &\sim (x' \bullet x, \text{zip}(cs, as)) \text{ if } \chi(c) = (x', cs)\end{aligned}$$

and we also apply the above relation recursively throughout the tree in $\text{Fr}(C \times A)$. The object $C \triangleright A$ can be interpreted as being a tree containing elements of X in its nodes and having elements of $C \times A$ at its leaves. Given a tuple $(c, a) \in C \times A$ at the leaf, we attempt te expand out its tree structure using α and χ, and replace it with an empty leaf if one of the two turns out to be empty.

Given an algebra B and a coalgebra C, the convolution algebra, denoted $[C, B]$, has as underlying set $\mathbf{Set}(C, A)$ and algebra structure

$$\begin{aligned}1 + X \times \mathbf{Set}(C, B)^* &\to \mathbf{Set}(C, B)\\ * &\mapsto (\_ \mapsto e_B)\\ (x, fs) &\mapsto \left(c \mapsto \begin{cases} e_B \text{ if } \chi(c) = * \\ \beta(x' \bullet x, \text{zipapply}(fs, cs)) \text{ if } \chi(c) = (x', cs)\end{cases}\right).\end{aligned}$$

3.7.5 C-initial algebras

In this section we would like to give a proof of the fact that $S_{X,n}$ is the terminal $S^\dagger_{X,n}$-initial algebra. Similar to before, one can compute $S^\dagger_{X,n} \cong S^\circ_{X,n}$, and we will write $S^\circ_{X,n}$ from now on. We have already seen a proof of a similar result for the natural numbers, and this proof has the same outline.

Recall an algebra A is called C-initial if there exists a unique measuring $\varphi : C \times A \to X$ for all $X \in \mathbf{Alg}$. The terminal C-initial algebra is the terminal object, if it exists, in the subcategory of **Alg** spanned by the C-initial algebras. In other words, for any C-initial algebra A there must exist a unique algebra morphism from A to the terminal C-initial algebra.

We aim to show $S_{X,n}$ is the terminal $S^\circ_{X,n}$-initial algebra. In order to do so, we first show there exists a morphism $A \to S_{X,n}$ for any $S^\circ_{X,n}$-initial algebra A using induction. After that, we will show the morphism is unique using a similar argument as in [1][Ex. 15].

We start with a lemma which allows us to use induction later on.

Lemma 3.7.5. *Let A be a $S^\circ_{X,n}$-initial algebra. Then A is also $S^\circ_{X,n-1}$ initial.*

Proof. Consider the coalgebra morphism $m : S^\circ_{X,n-1} \to S^\circ_{X,n}, s \mapsto s$. This induces a morphism

$$\begin{aligned}m \triangleright A : S^\circ_{X,n-1} \triangleright A &\to S^\circ_{X,n} \triangleright A\\ [e_{\text{Fr}}] &\mapsto [e_{\text{Fr}}]\\ [s, a] &\mapsto [s, a]\\ [x, sas] &\mapsto [x, \text{map}(m \triangleright A)(sas)].\end{aligned}$$

This morphism is monomorphic by definition. Since A is $S^\circ_{X,n}$-initial, we know $S_{X,n} \triangleright A$ is an initial object. This means $m \triangleright A$ is a monomorphism into the initial object, hence an isomorphism. ☐

As an immediate consequence we have the following corollary.

Corollary 3.7.6. *Let A be a $S^\circ_{X,n}$-initial algebra. Then A is also $S^\circ_{X,k}$ initial for all $k \leq n$.*

The next lemma is a technical lemma which we will be able to leverage during the induction step.

Lemma 3.7.7. *Let A be a $S^\circ_{X,n}$-initial algebra and let $\varphi : S^\circ_{X,n} \times A \to S_X$ be the unique measuring from A to S_X by $S^\circ_{X,n}$. For all $0 \leq i \leq n$, let $s_i \in S^\circ_{X,n}$ be the largest tree of depth i in $S_{X,n}$ which contains only the unit $e \in X$. Then for all $0 \leq i \in n$ and $0 \leq j \leq n - i$ we have*

$$\varphi(s_i, a) = \varphi(s_i, \mathfrak{i}_A \circ \varphi(s_{i+j}, a)).$$

Proof. Let $k \leq n$. First, we define a function which takes two trees and appends the second tree to all the roots of the first tree. We define it as

$$\begin{aligned} (++) : S_{X,\infty} \times S_{X,\infty} &\to S_{X,\infty} \\ (e_S, s) &\mapsto s \\ ((x, ss), s) &\mapsto (x, \mathrm{map}((++)s)(ss)). \end{aligned}$$

For all $0 \leq j \leq n - k$, define the coalgebra morphism

$$\begin{aligned} p_j : S^\circ_{X,k} &\to S^\circ_{X,n} \times S^\circ_{X,n} \\ s &\mapsto (s, \tilde{s} + +s_j), \end{aligned}$$

where $\tilde{s}$ is the unique tree $\tilde{s} \in S_{\{e\},k}$ such that $\mathrm{shape}(\tilde{s}) = \mathrm{shape}(s)$. Consider the composition of measurings $\varphi \circ (\mathrm{id} \times \mathfrak{i}_A \circ \varphi) \in \mathfrak{m}_{S^\circ_{X,n} \times S^\circ_{X,n}}(A, S_X)$. We can precompose this measuring with the coalgebra morphism p_j to obtain a measuring

$$S^\circ_{X,k} \times A \xrightarrow{p_j \times \mathrm{id}_A} S^\circ_{X,n} \times S^\circ_{X,n} \times A \xrightarrow{\varphi \circ (\mathrm{id} \times \mathfrak{i}_A \circ \varphi)} S_X.$$

We also have the coalgebra morphism $m : S^\circ_{X,k} \to S^\circ_{X,n}, s \mapsto s$ which we can precompose with φ. This gives us a measuring

$$S^\circ_{X,k} \times A \xrightarrow{m \times \mathrm{id}_A} S^\circ_{X,n} \times A \xrightarrow{\varphi} S_X.$$

Since A is also $S^\circ_{X,k}$-initial, we know these measurings must coincide. Hence we can state $\varphi(s, a) = \varphi(s, \mathfrak{i}_A \circ \varphi(\tilde{s} + +s_j, a))$ for all $s \in S^\circ_{X,k}$ and $0 \leq j \leq n - k$. The only restriction placed on k was that $k \leq n$. Iterating over all $0 \leq k \leq n$, we arrive at the desired result

$$\varphi(s, a) = \varphi(s, \mathfrak{i}_A \circ \varphi(\tilde{s} + +s_j, a))$$

for all $0 \leq i \leq n$ and $0 \leq j \leq n - i$. In particular this holds for $s = s_i$, and noting $s_i + +s_j = s_{i+j}$ we obtain the desired result. □

Corollary 3.7.8. *Let A be $S^\circ_{X,n}$-initial and let φ be the unique measuring to S_X. For all $0 \leq i \leq n$ and $a \in A$, we have $\varphi(s_i, a) \in S_{X,i}$.*

Proof. We will proceed by induction over i. For the base case $i = 0$, we have $s_0 = e_S$, so $\varphi(s_0, a) = e_S \in S_{X,0}$. For the induction step, assume $\varphi(s_i, a) \in S_{X,i}$. Using $s_{i+1} = (e, (s_i)^{(\infty)})$ and the previous lemma we can write $\varphi(s_{i+1}, a) = \varphi((e, (s_i)^{(\infty)}), \mathfrak{i}_A(\varphi(s_{i+1}, a)))$. Let $\alpha : 1 + X \times (S_X)^*_\infty \to S_X$, then we can write $(x, ss) = \alpha^{-1}(\varphi(s_{i+1}, a))$. We then know by definition of a measuring that

$$\varphi(s_{i+1}, a) = \varphi((e, (s_i)^{(\infty)}), \mathfrak{i}_A(\varphi(s_{i+1}, a))) = (e \bullet x, \mathrm{map}(\varphi)(\mathrm{zip}((s_i)^{(\infty)}, as))),$$

where $as = \mathrm{map}(\mathfrak{i}_A)(ss)$. By the induction hypothesis, we know $\mathrm{map}(\varphi)(\mathrm{zip}((s_i)^{(\infty)}, as)) \in (S_{X,i})^*_\infty$, hence $\varphi(s_{i+1}, a) \in S_{X,i+1}$ by definition. This concludes the induction step and the proof. □

Now we are ready to define the a family of functions which will culminate in an algebra morphism $A \to S_{X,n}$ for any $S^\circ_{X,n}$ initial algebra A.

Definition 3.7.9. Let A be $S^{\circ}_{X,n}$-initial, let φ be the unique measuring to S_X and let $0 \leq k \leq n$. Define the functions

$$\begin{aligned} \varphi_k : A &\to S_{X,k} \\ a &\mapsto \varphi(s_k, a), \end{aligned}$$

which are well-defined by the previous corollary.

Remark 3.7.10. With this new definition, we can restate Lemma 3.7.7 as

$$\varphi_i = \varphi_i \circ \mathsf{i}_A \circ \varphi_{i+j}$$

for all $0 \leq i \leq n$ and $0 \leq j \leq n - i$.

Proposition 3.7.11. *The functions $\varphi_k : A \to S_{X,k}$ from Definition 3.7.9 are algebra morphisms for all $0 \leq k \leq n$.*

Proof. We proceed by induction over k. For $k = 0$, we have that $S_{X,k} \cong 1$, the terminal object in **Set**. Hence $\varphi_0 : A \to S_{X,0}$ is an algebra morphism. For the inductive step, assume $\varphi_{k-1} : A \to S_{X,k-1}$ is an algebra morphism. We wish to show φ_k is an algebra morphism. In more detail, we aim to show $\alpha_k(x, \mathrm{map}(\varphi_k)(as)) = \varphi_k(\alpha(x, as))$ for all $(x, as) \in X \times A^*_\infty$. First, we will write $as = (a_i)_{i=0}^{j}$, where $j \in \mathbb{N}_\infty$. We can now write

$$\begin{aligned} \alpha_k(x, \mathrm{map}(\varphi_k)(as)) &= \alpha_k(x, \mathrm{map}(\varphi_k)((a_i)_i)) \\ &= \alpha_k(x, (\varphi_k(a_i))_i) \\ &= \alpha_k(x, (\varphi_k \circ \mathsf{i}_A \circ \varphi_k(a_i))_i). \end{aligned}$$

Writing $\varphi_k \circ \mathsf{i}_A \circ \varphi_k(a_i) = t_i$, we can make the case distinction

$$t_i = \begin{cases} e_S & \text{if } \varphi_k(a_i) = e_S \in S_X \\ (x_i, \mathrm{map}(\varphi_{k-1} \circ \mathsf{i}_A)(ss_i)) & \text{if } \varphi_k(a_i) = (x_i, ss_i) \in S_X \end{cases}$$

by $\varphi : S^{\circ}_{X,n} \times A \to S_X$ being a measuring. This means $\alpha_k(x, \mathrm{map}(\varphi_k)(as)) = \alpha_k(x, (t_i)_i)$. By the algebra structure of $S_{X,k}$, we can write this as

$$\alpha_k(x, (t_i)_i) = (x, (r_i)_i),$$

where

$$r_i = \begin{cases} e_S & \text{if } \varphi_k(a_i) = e_S \in S_X \\ \alpha_{k-1}(x_i, \mathrm{map}(\varphi_{k-1} \circ \mathsf{i}_A)(ss_i)) & \text{if } \varphi_k(a_i) = (x_i, ss_i) \in S_X. \end{cases}$$

By the induction hypothesis $\alpha_{k-1}(x_i, \mathrm{map}(\varphi_{k-1} \circ \mathsf{i}_A)(ss_i)) = (\varphi_{k-1} \circ \mathsf{i}_A)(x_i, ss_i)$ and we can write

$$r_i = (\varphi_{k-1} \circ \mathsf{i}_A)(\varphi_k(a_i)) = (\varphi_{k-1} \circ \mathsf{i}_A \circ \varphi_{k-1})(a_i)$$

using Lemma 3.7.7 Retracing our steps, we now compute

$$\begin{aligned} \alpha_k(x, \mathrm{map}(\varphi_k)(as)) &= (x, (r_i)_i) \\ &= (x, (\varphi_{k-1} \circ \mathsf{i}_A \circ \varphi_{k-1})(a_i)_i) \\ &= (x, \mathrm{map}(\varphi_{k-1})((\mathsf{i}_A \circ \varphi_{k-1})(a_i))_i) \\ &= \varphi_k(\alpha(x, (\mathsf{i}_A \circ \varphi_{k-1})(a_i))_i)) \\ &= (\varphi_k \circ \mathsf{i}_A \circ \varphi_k)(\alpha(x, (a_i)_i)) \\ &= \varphi_k(\alpha(x, as)) \end{aligned}$$

using that φ is a measuring twice. We conclude $\varphi_k : A \to S_{X,k}$ is an algebra morphism for all $0 \leq k \leq n$. $\square$

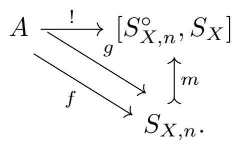
In particular, this lemma shows $\varphi_n : A \to S_{X,n}$ is an algebra morphism. It still remains to show this algebra morphism is unique.

Lemma 3.7.12. *For any $S^\circ_{X,n}$-initial algebra A, there exist at most one algebra morphism $A \to S_{X,n}$.*

Proof. By [1][Prop. 35], there exists a unique morphism $A \to [S^\circ_{X,n}, S_X]$ for any $S^\circ_{X,n}$-initial algebra A. Since $S_{X,n}$ is $S^\circ_{X,n}$-initial we know there exists a unique morphism $m : S_{X,n} \rightarrowtail [S^\circ_{X,n}, S_X]$. This map is given by

$$\begin{aligned} m : S_{X,n} &\to [S^\circ_{X,n}, S_X] \\ e_S &\mapsto \mathrm{const}_{e_S} \\ (x, ss) &\mapsto \left(s' \mapsto \begin{cases} e_S & \text{if } s' = e_S \\ (x' \bullet x, \mathrm{map}(m)(\mathrm{zip}(ss, ss'))) & \text{if } s' = (x', ss'). \end{cases} \right) \end{aligned}$$

We can easily conclude m is a monomorphism by observing that $m(s)(s_n) = s$ for all $s \in S_{X,n}$. Given any two morphisms $f, g : A \to S_{X,n}$, we can draw the following diagram



Since the morphism $A \to [S^\circ_{X,n}, S_X]$ is unique, we know the composites $m \circ f = m \circ g$, and by m being mono we conclude $f = g$. Hence, there can be at most one algebra morphism from a $S^\circ_{X,n}$-initial algebra A to $S_{X,n}$. ☐

Putting all the above together, we arrive at the following result.

Theorem 3.7.13. *The algebra $S_{X,n}$ is the terminal $S^\circ_{X,n}$-initial algebra.*

Proof. Given an $S^\circ_{X,n}$ initial algebra A, by Proposition 3.7.11 we obtain an algebra morphism $\varphi_n : A \to S_{X,n}$. By Lemma 3.7.12 it is unique. We conclude $S_{X,n}$ is the terminal $S^\circ_{X,n}$-initial algebra. ☐

Finally, we wish to point out a corollary which will play in important role when computing some terminal C-initial algebras for the list type and the binary tree type in Section 5. In the proof of Theorem 3.7.13 we have only made use of very specific elements of $S^\circ_{X,n}$, namely the elements $s_i \in S^\circ_{X,n}$ which are the largest trees of depth i containing only the monoidal unit $e \in X$. Using this observation, we can state the following corollary.

Corollary 3.7.14. *Consider the coalgebra*

$$\begin{aligned} \chi_n : \mathbb{n}^\circ &\to 1 + X \times (\mathbb{n}^\circ)^*_\infty \\ 0 &\mapsto * \\ i &\mapsto (e, [i-1, i-1, \dots]) \end{aligned}$$

where $\mathbb{n}^\circ \cong \{0, 1, \dots, n\}$. Let (C, χ) be a coalgebra that has $(\mathbb{n}, \chi_n)$ as a subobject and A be a C-initial algebra. Then there exist algebra morphisms $\varphi_k : A \to S_{X,k}$ for all $k \in \mathbb{n}$.

This corollary concludes this section.

By now, we have plenty of examples at our disposal. It might have stood out that in many of the examples the computations were very similar. One could wonder if these examples have a certain relationship to one and other. This is definitely the case and will be the topic of the next section, where we explore how a lax monoidal natural transformation $\mu : F \to G$ induces functors between the categories of algebras and coalgebras of F and G. This will culminate in a notion of enriched functors between the categories of F-algebras and G-algebras, which will preserve much of the structures seen in the examples.

4 Enriched functors between categories of algebras

We have already noted the repetitive nature of some of the examples in the previous section. This suggests there exists a relationship between the categories of algebras of different endofunctors, which also respects the enrichment. Indeed, this is the case and the punchline of this section, which can be found in Corollary 4.3.15.

In order to arrive at Corollary 4.3.15, we first define the category **EnrCat** in Section 4.1, which as objects has pairs $(\mathbf{C}, \mathbf{V})$, where $\mathbf{C}$ is a $\mathbf{V}$-enriched category. By Theorem 2.5.14, for well-behaved endofunctors $F, G : \mathbf{C} \to \mathbf{C}$ we know $(\mathbf{Alg}^F, \mathbf{CoAlg}^F)$ and $(\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ are elements of **EnrCat**. The main idea is that a consistent way to transform an F-measuring into a G-measuring is equivalent to a morphism $(\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ in **EnrCat**. With this idea in hand we show different ways in which natural transformations between F and G result in consistent transformations of measurings, and by extension morphisms in **EnrCat**, in Section 4.2 and Section 4.3. We wrap this all up by showing the functorial nature of these constructions, giving functors from the category of endofunctors to **EnrCat**. Finally, we will give a lot of examples showcasing the theory in Section 4.4

4.1 The category EnrCat

We start off by giving a definition for the category of enriched categories, **EnrCat**. We will be very explicit in our construction, but the main idea is to apply the Grothendieck construction to the functor $\mathbf{Enr} : \mathbf{V} \mapsto \mathbf{V}\text{-}\mathbf{Cat}$ which maps a monoidal category to the category of $\mathbf{V}$-enriched categories. This will yield a category opfibered over **MonCat**, the category of monoidal categories.

We will denote the 2-category of monoidal categories, lax monoidal functors and lax monoidal natural transformations by **MonCat**, and for any $\mathbf{V} \in \mathbf{MonCat}$ we will denote the 2-category of $\mathbf{V}$ enriched categories, $\mathbf{V}$-enriched functors and $\mathbf{V}$-enriched natural transformations by $\mathbf{V}\text{-}\mathbf{Cat}$.

Definition 4.1.1. Let **EnrCat** be the category which as elements has pairs $(\mathbf{C}, \mathbf{V})$, where $\mathbf{V}$ is a monoidal category and $\mathbf{C}$ is a $\mathbf{V}$-enriched category. A morphism $(\mathbf{C}, \mathbf{V}) \to (\mathbf{C}', \mathbf{V}')$ is given by a pair of functors

$$\begin{aligned} \pi : \mathbf{V} \to \mathbf{V}' &\in \mathbf{MonCat} \\ \rho : \pi_*(\mathbf{C}) \to \mathbf{C}' &\in \mathbf{V}'\text{-}\mathbf{Cat}, \end{aligned}$$

where $\pi_* : V\text{-}\mathbf{Cat} \to V'\text{-}\mathbf{Cat}$ is the functor induced by the lax monoidal functor π as in Proposition 2.3.9 and ρ is a $\mathbf{V}'$-enriched functor.

Given pairs of functors $(\rho, \pi), (\rho', \pi') \in \mathbf{EnrCat}$, its composition

$$(\mathbf{C}, \mathbf{V}) \xrightarrow{(\rho,\pi)} (\mathbf{C}', \mathbf{V}') \xrightarrow{(\rho',\pi')} (\mathbf{C}'', \mathbf{V}'')$$

is given $(\rho'', \pi'_* \circ \pi_*)$, where $\rho'' : (\pi'_* \circ \pi_*)(\mathbf{C}) \to \mathbf{C}''$ is given by the composition $\rho' \circ \rho$ on objects. On morphisms ρ'' is given by the composition

$$(\pi' \circ \pi)_*(\mathbf{C}(A, B)) \xrightarrow{\pi'(\rho_{A,B})} \pi'_*(\mathbf{C}'(\rho(A), \rho(B))) \xrightarrow{\rho'_{\rho(A),\rho(B)}} \mathbf{C}''((\rho' \circ \rho)(A), (\rho' \circ \rho)(B)).$$

Identities are inherited from the underlying categories.

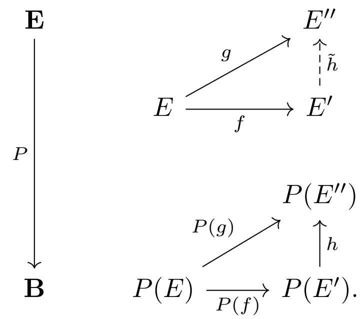
We can think of **EnrCat** as the category containing all enriched categories, indexed by their enrichment. There is an obvious functor

$$\begin{aligned} \mathbf{EnrCat} &\to \mathbf{MonCat} \\ (\mathbf{C}, \mathbf{V}) &\mapsto \mathbf{V} \\ (\rho, \pi) &\mapsto \pi \end{aligned}$$

which could remind one of a set fibered over another. We can make this more precise, but in order to do so we need the notion of a *cocartesian morphism.*

Definition 4.1.2. Given a functor $P : \mathbf{E} \to \mathbf{B}$, a morphism $f : E \to E'$ in $\mathbf{E}$ is called *cocartesian* if for any morphism $g : E \to E''$ in $\mathbf{E}$ and $h : P(E') \to P(E'')$ in $\mathbf{E}$ such that $h \circ P(f) = P(g)$ there exists a unique lift $\tilde{h} : E' \to E''$ such that $\tilde{h} \circ f = g$ and $P(\tilde{h}) = h$.

The definition can be summarized in the following diagram



Remark 4.1.3. The definition encountered more frequently in the literature is that of a *cartesian* morphism, which is dual to that of a cocartesian morphism. In our case we are dealing with cocartesian morphisms, which is why we state the definition above.

Now we can make precise what the fibrational structure entails.

Definition 4.1.4. A functor $P : \mathbf{E} \to \mathbf{B}$ is called an *opfibration* if for all objects $E \in \mathbf{E}$ and morphisms $f : P(E) \to B$ there exists a cocartesian morphism $\tilde{f} : E \to E'$ such that $P(\tilde{f}) = f$.

The category **EnrCat** can be viewed as a opfibered category over **MonCat**.

Lemma 4.1.5. *The functor*

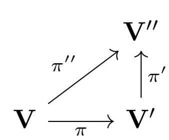
$$\begin{aligned} \mathbf{EnrCat} &\to \mathbf{MonCat} \\ (\mathbf{C}, \mathbf{V}) &\mapsto \mathbf{V} \\ (\rho, \pi) &\mapsto \pi \end{aligned}$$

is an opfibration.

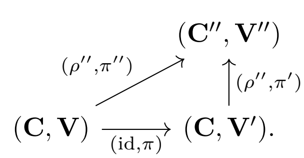
Proof. Let $(\mathbf{C}, \mathbf{V}) \in \mathbf{EnrCat}$ and let $\pi : \mathbf{V} \to \mathbf{V}' \in \mathbf{MonCat}$. The cocartesian lift of π is given by

$$(\mathrm{id}, \pi) : (\mathbf{C}, \mathbf{V}) \to (\pi_*(\mathbf{C}), \mathbf{V}').$$

Indeed, given a morphism $(\rho'', \pi'') : (\mathbf{C}, \mathbf{V}) \to (\mathbf{C}'', \mathbf{V}'')$ and a commutative diagram



in **MonCat**, the unique lift of π' is given by (ρ'', π'), as seen in the following diagram



□

We can even go a step further and make **EnrCat** into a 2-category, giving the notion of an *enriched natural transformation.*

Definition 4.1.6. Given a pair of morphisms $(\rho, \pi), (\rho', \pi') : (\mathbf{C}, \mathbf{V}) \to (\mathbf{C}', \mathbf{V}')$ in **EnrCat**, an *enriched natural transformation from* (ρ, π) *to* (ρ', π') is given by a pair

$$(\mu, \nu) : (\rho, \pi) \to (\rho', \pi'),$$

where $\mu : \rho \to \rho'$ is a $\mathbf{V}'$-enriched natural transformation and $\nu : \pi \to \pi'$ is a lax monoidal natural transformation, such that the following diagram in $\mathbf{C}'$ commutes

$$\begin{array}{ccccccc}
\pi(\mathbf{C}(A,B)) & \xrightarrow{\cong} & \mathbb{I} \otimes \pi(\mathbf{C}(A,B)) & \xrightarrow{\mu_B \otimes \rho'_{A,B}} & \mathbf{C}'(\rho(B), \rho'(B)) \otimes \mathbf{C}'(\rho(A), \rho(B)) & & \\
 & & & & & \searrow^{\circ^{\mathbf{C}'}} & \\
\downarrow{\scriptstyle \nu_{\mathbf{C}(A,B)}} & & & & & & \mathbf{C}'(\rho(A), \rho'(B)). \\
 & & & & & \nearrow_{\circ^{\mathbf{C}'}} & \\
\pi'(\mathbf{C}(A,B)) & \xrightarrow[\cong]{} & \pi'(\mathbf{C}(A,B)) \otimes \mathbb{I} & \xrightarrow[\rho'_{A,B} \otimes \mu_A]{} & \mathbf{C}'(\rho'(A), \rho'(B)) \otimes \mathbf{C}'(\rho(A), \rho'(A)) & &
\end{array}$$

Composition of enriched natural transformations is done by composing the components of the natural transformation. The identity natural transformation is given by the combining the identity natural transformations from **V-Cat** and **MonCat**. This makes **EnrCat** into a 2-category. We would like to remark the above definition is closely related to Definition 2.3.8, the definition of an enriched natural transformations. The only difference is that we add the vertical map $\nu_{\mathbf{C}(A,B)}$ to account for the possibly distinct hom-objects of $\mathbf{C}$ in $\mathbf{V}'$ under π and π'.

Remark 4.1.7. We would like to remark the above precisely corresponds to using the Grothendieck construction on the 2-functor

$$\begin{aligned}
\mathbf{MonCat} &\to \mathbf{Cat} \\
\mathbf{V} &\mapsto \mathbf{V}\text{-}\mathbf{Cat} \\
(\pi : \mathbf{V} \to \mathbf{V}') &\mapsto (\pi_* : \mathbf{V}\text{-}\mathbf{Cat} \to \mathbf{V}'\text{-}\mathbf{Cat}) \\
(\nu : \pi \to \pi') &\mapsto (\nu_* : \pi_* \to \pi'_*).
\end{aligned}$$

We have made the effort to define natural transformations in **EnrCat**, which gives us the tools to speak about adjunctions. However, we will not be utilizing much of the 2-categorical structure in this thesis. The reason for giving the 2-categorical structure of **EnrCat** is for completeness' sake, and so we may continue our research in the direction of adjunctions in the future.

Example 4.1.8. The motivating example for the construction of the above category is the enrichment of the category of algebras in the category of coalgebras. Given an accessible endofunctor $F : \mathbf{C} \to \mathbf{C}$ on a locally presentable closed monoidal category, the pair $(\mathbf{Alg}^F, \mathbf{CoAlg}^F)$ is an element of **EnrCat** △

So far, we have been speaking of a "consistent way of transforming F-measurings into G-measurings" without giving more details. Now that we have constructed **EnrCat**, and in particular have an idea of a morphism in **EnrCat**, we are in the right head space to give this definition.

Definition 4.1.9. Let $\rho : \mathbf{Alg}^F \to \mathbf{Alg}^G$ be a functor, $\pi : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ be a lax monoidal functor and let

$$\Phi_{A,B,C} : \mathfrak{m}^F_C(A, B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A), \rho(B))$$

constitute a natural transformation from $\mathfrak{m}^F$ to $\mathfrak{m}^G \circ (\pi \times \rho \times \rho)$. We say Φ *respects composition of measurings* if the diagram

$$\begin{array}{ccc}
\mathfrak{m}^F_D(B,T) \times \mathfrak{m}^F_C(A,B) & \xrightarrow{\Phi_{B,T,D} \times \Phi_{A,B,C}} & \mathfrak{m}^G_{\pi(D)}(\rho(B), \rho(T)) \times \mathfrak{m}^G_{\pi(C)}(\rho(A), \rho(B)) \\
 & & \downarrow{\scriptstyle \circ^G_{\mathfrak{m}}} \\
\downarrow{\scriptstyle \circ^F_{\mathfrak{m}}} & & \mathfrak{m}^G_{\pi(D) \otimes \pi(C)}(\rho(A), \rho(T)) \\
 & & \uparrow{\scriptstyle \mathfrak{m}^G_{(\nabla^\pi)}(\rho(B), \rho(T))} \\
\mathfrak{m}^F_{D \otimes C}(A,T) & \xrightarrow[\Phi_{A,T,D \otimes C}]{} & \mathfrak{m}^G_{\pi(D \otimes C)}(\rho(A), \rho(T))
\end{array}$$

commutes.

Note that we have packed three things in this definition. First, we assume there to exist functors $\rho : \mathbf{Alg}^F \to \mathbf{Alg}^G$ and $\pi : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$. Second, we assume these functors work well together to yield a natural transformation $\Phi_{A,B,C} : \mathfrak{m}^F_C(A,B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A), \rho(B))$. Third, we also ask this natural transformation to respect composition of measurings. Since we think of measurings as our generalized algebra morphisms, this should be enough to obtain a morphism in **EnrCat**. In order to obtain such morphism, we will concern ourselves with constructing a morphism

$$\rho_{A,B} : \pi(\underline{\mathbf{Alg}}^F(A,B)) \to \underline{\mathbf{Alg}}^G(\rho(A), \rho(B)).$$

The morphisms $\rho_{A,B}$ will be needed to obtain an enriched functor $\rho : \pi_*(\mathbf{Alg}^F) \to \mathbf{Alg}^G$. We will see that the constructed morphism $\rho_{A,B}$ respects identities, and show it also respects composition.

Lemma 4.1.10. *Let $\rho : \mathbf{Alg}^F \to \mathbf{Alg}^G$ be a functor and $\pi : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ be a lax monoidal functor. A natural transformation*

$$\Phi_{A,B,C} : \mathfrak{m}^F_C(A,B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A), \rho(B))$$

induces a morphism

$$\rho_{A,B} : \pi(\underline{\mathbf{Alg}}^F(A,B)) \to \underline{\mathbf{Alg}}^G(\rho(A), \rho(B)).$$

Note we do not yet ask the natural transformation Φ to respect composition.

Proof. For all algebras $A, B \in \mathbf{Alg}^F$ we have the function

$$\Phi_{A,B} : \mathfrak{m}^F_{\underline{\mathbf{Alg}}^F(A,B)}(A,B) \to \mathfrak{m}^G_{\pi(\underline{\mathbf{Alg}}^F(A,B))}(\rho(A), \rho(B))$$

and the natural isomorphisms

$$\mathfrak{m}^F_{\underline{\mathbf{Alg}}^F(A,B)}(A,B) \cong \mathbf{CoAlg}^F(\underline{\mathbf{Alg}}^F(A,B), \underline{\mathbf{Alg}}^F(A,B))$$
$$\mathfrak{m}^G_{\pi(\underline{\mathbf{Alg}}^F(A,B))}(\rho(A), \rho(B)) \cong \mathbf{CoAlg}^G(\pi(\underline{\mathbf{Alg}}^F(A,B)), \underline{\mathbf{Alg}}^G(\rho(A), \rho(B))).$$

Taking $\mathrm{id}_{\underline{\mathbf{Alg}}^F(A,B)} \in \mathbf{CoAlg}^F(\underline{\mathbf{Alg}}^F(A,B), \underline{\mathbf{Alg}}^F(A,B))$ gives us a measuring from A to B under the natural isomorphism. Composing this with $\Phi_{A,B}$ gives us a measuring from $\rho(A)$ to $\rho(B)$ by $\pi(\underline{\mathbf{Alg}}^F(A,B))$, which under the natural isomorphism gives the desired morphism

$$\rho_{A,B} : \pi(\underline{\mathbf{Alg}}^F(A,B)) \to \underline{\mathbf{Alg}}^G(\rho(A), \rho(B)). \quad \square$$

Remark 4.1.11. We can even be more explicit about the construction above. The image of $\mathrm{id}_{\underline{\mathbf{Alg}}^F(A,B)}$ under the natural isomorphism is precisely the evaluation map

$$\mathrm{ev}^F_{A,B} : \underline{\mathbf{Alg}}^F(A,B) \otimes A \to B.$$

Under Φ we obtain the G-measuring

$$\Phi(\mathrm{ev}^F_{A,B}) : \pi(\underline{\mathbf{Alg}}^F(A,B)) \otimes \rho(A) \to \rho(B).$$

Finally, $\rho_{A,B}$ is given as the unique map

$$\begin{array}{ccc} \pi(\underline{\mathbf{Alg}}^F(A,B)) \otimes \rho(A) & \xrightarrow{\Phi(\mathrm{ev}^F_{A,B})} & \rho(B) \\ {\scriptstyle \rho_{A,B} \otimes \mathrm{id}} \downarrow & \nearrow {\scriptstyle \mathrm{ev}^G_{\rho(A),\rho(B)}} & \\ \underline{\mathbf{Alg}}^F(\rho(A), \rho(B)) \otimes \rho(A) & & \end{array}$$

to the terminal object $\underline{\mathbf{Alg}}^F(\rho(A), \rho(B))$ in the category of measurings from $\rho(A)$ to $\rho(B)$.

Lemma 4.1.12. *Let $\rho : \mathbf{Alg}^F \to \mathbf{Alg}^G$ be a functor, $\pi : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ be a lax monoidal functor and let Φ be a natural transformation*

$$\Phi_{A,B,C} : \mathfrak{m}^F_C(A,B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A),\rho(B)).$$

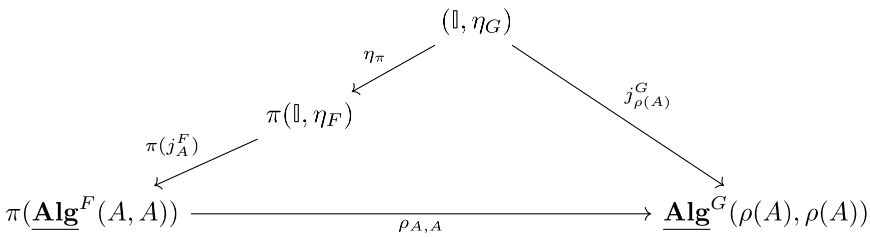
The induced morphism

$$\rho_{A,B} : \pi(\underline{\mathbf{Alg}}^F(A,B)) \to \underline{\mathbf{Alg}}^G(\rho(A),\rho(B))$$

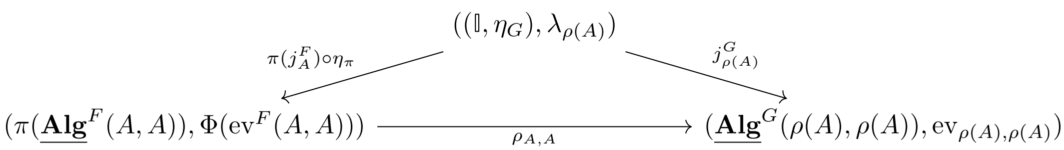
from Lemma 4.1.10 respects identities as in Definition 2.3.6

Again, we not yet ask Φ to respect composition.

Proof. In order to show $\rho_{A,B}$ respects identities we will use the characterization of $\mathrm{ev}_{A,B} : \underline{\mathbf{Alg}}(A,B)\otimes A \to B$ as the terminal object in the category of measurings from A to B. We must show the following diagram commutes



where $j_A : \mathbb{I} \to \underline{\mathbf{Alg}}(A,A)$ is the family of identity elements of the enriched category. We will turn to the category of measurings from $\rho(A)$ to $\rho(A)$ to show the above diagram commutes. In this category, we have the terminal object $(\underline{\mathbf{Alg}}^G(\rho(A),\rho(A)), \mathrm{ev}_{\rho(A),\rho(A)})$. Writing λ for the left unitor in the monoidal category $\mathbf{C}$, we also have the measurings $((\mathbb{I},\eta_G),\lambda_{\rho(A)})$ and $(\pi(\underline{\mathbf{Alg}}^F(A,A)), \Phi(\mathrm{ev}^F(A,A)))$. The above diagram is a diagram of coalgebra morphisms, which precisely corresponds to a diagram of measurings from $\rho(A)$ to itself



where $\rho_{A,A}$ and $j^G_{\rho(A)}$ are a morphism of measurings by definition, and $\pi(j^F_A)\circ\eta_\pi$ is morphism of measurings since π is a lax monoidal functor, hence respects the left unitor. The diagram commutes since $\underline{\mathbf{Alg}}^G(\rho(A),\rho(A))$ is the terminal object. We conclude

$$j^G_{\rho(A)} = \rho_{A,A} \circ \pi(j^F_A)$$

and hence that ρ respects identities. ☐

Lemma 4.1.13. *Let $\rho : \mathbf{Alg}^F \to \mathbf{Alg}^G$ be a functor, $\pi : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ be a lax monoidal functor and let Φ be a natural transformation*

$$\Phi_{A,B,C} : \mathfrak{m}^F_C(A,B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A),\rho(B))$$

which respects composition. The induced morphism

$$\rho_{A,B} : \pi(\underline{\mathbf{Alg}}^F(A,B)) \to \underline{\mathbf{Alg}}^G(\rho(A),\rho(B))$$

from Lemma 4.1.10 respects composition as in Definition 2.3.6

Proof. One can use a similar strategy as when showing $\rho_{A,B}$ respects identities for showing ρ respects composition. We must show the following diagram commutes

$$\begin{array}{ccc} \pi(\underline{\mathbf{Alg}}^F(B,T)) \otimes \pi(\underline{\mathbf{Alg}}^F(A,B)) & \xrightarrow{\pi(\circ^F)\circ\nabla^\pi} & \pi(\underline{\mathbf{Alg}}^F(A,T)) \\ \downarrow{\scriptstyle \rho_{B,T}\otimes\rho_{A,B}} & & \downarrow{\scriptstyle \rho_{A,T}} \\ \underline{\mathbf{Alg}}^G(\rho(B),\rho(T)) \otimes \underline{\mathbf{Alg}}^G(\rho(A),\rho(B)) & \xrightarrow{\circ^G} & \underline{\mathbf{Alg}}^G(\rho(A),\rho(T)). \end{array}$$

We turn to the category of measurings from $\rho(A)$ to $\rho(T)$, and recognize $(\underline{\mathbf{Alg}}^G(\rho(A),\rho(T)), \mathrm{ev}^G_{\rho(A),\rho(T)})$ is the terminal object in this category. Moreover, we have the following measurings from $\rho(A)$ to $\rho(T)$

$$\big(\pi(\underline{\mathbf{Alg}}^F(B,T)) \otimes \pi(\underline{\mathbf{Alg}}^F(A,B)),\, \Phi(\mathrm{ev}^F_{B,T}) \circ (\mathrm{id} \otimes \Phi(\mathrm{ev}^F_{A,B}))\big)$$
$$\big(\underline{\mathbf{Alg}}^G(\rho(B),\rho(T)) \otimes \underline{\mathbf{Alg}}^G(\rho(A),\rho(B)),\ \mathrm{ev}^G_{\rho(B),\rho(T)} \circ (\mathrm{id} \otimes \mathrm{ev}^G_{\rho(A),\rho(B)})\big)$$
$$\big(\pi(\underline{\mathbf{Alg}}^F(A,T)),\, \Phi(\mathrm{ev}^F_{A,T})\big).$$

Again, the above diagram of coalgebra morphisms corresponds exactly to a diagram of measurings in the category of measurings from $\rho(A)$ to $\rho(T)$. That all the vertical coalgebra morphisms are actually morphisms of measurings is by definition of maps of the form $\rho_{A,B}$. The bottom map $\circ^G$ is a morphism of measurings by definition as well. Finally, $\pi(\circ^F) \circ \nabla^\pi$ is a morphism of measurings since we asked Φ to respect composition of measurings. As a result, we know

$$\Phi(\mathrm{ev}^F_{B,T}) \circ (\mathrm{id} \otimes \Phi(\mathrm{ev}^F_{A,B})) = \Phi(\mathrm{ev}^F_{B,T} \circ (\mathrm{id} \otimes \mathrm{ev}^F_{A,B})) \circ \nabla^\pi,$$

which is together with the naturality of Φ implies $\pi(\circ^F) \circ \nabla^\pi$ is a morphism of measurings. Since in the above diagram the composites map into the terminal object in the category of measurings from $\rho(A)$ to $\rho(T)$, they must coincide. □

Inspecting the proof above, we can state the following corollary which reduces the amount of verification needed to check if a family of function Φ respects composition.

Corollary 4.1.14. *A family of functions*

$$\Phi_{A,B,C} : \mathfrak{m}^F_C(A,B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A),\rho(B))$$

respects composition of measurings if and only if

$$\Phi(\mathrm{ev}^F_{B,T}) \circ (\mathrm{id} \otimes \Phi(\mathrm{ev}^F_{A,B})) = \Phi(\mathrm{ev}^F_{B,T} \circ (\mathrm{id} \otimes \mathrm{ev}^F_{A,B})) \circ \nabla^\pi.$$

Having the above lemmas at hand, we can state the following result.

Proposition 4.1.15. *Let* $\rho : \mathbf{Alg}^F \to \mathbf{Alg}^G$ *be a functor and* $\pi : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ *be a lax monoidal functor. A natural transformation*

$$\Phi_{A,B,C} : \mathfrak{m}^F_C(A,B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A),\rho(B))$$

which respects composition induces a morphism

$$(\rho,\pi) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

in **EnrCat**.

Proof. Combining the previous three lemmas yields morphisms

$$\rho_{A,B} : \pi(\underline{\mathbf{Alg}}^F(A,B)) \to \underline{\mathbf{Alg}}^G(\rho(A),\rho(B))$$

which respect composition and identities, hence a morphism

$$(\rho,\pi) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

in **EnrCat**. □

Once might wonder if the converse is also true. This is the case, and the contents of the next lemma.

Lemma 4.1.16. *A morphism*

$$(\rho, \pi) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

in **EnrCat** *induces a natural transformation*

$$\Phi_{A,B,C} : \mathfrak{m}_C^F(A, B) \to \mathfrak{m}_{\pi(C)}^G(\rho(A), \rho(B))$$

which respects composition.

Proof. We define the natural transformation Φ as

$$\Phi : \mathfrak{m}_C^F(A, B) \cong \mathbf{CoAlg}^F(C, \underline{\mathbf{Alg}^F}(A, B)) \xrightarrow{\pi} \mathbf{CoAlg}^G(\pi(C), \pi(\underline{\mathbf{Alg}^F}(A, B))) \xrightarrow{(\rho_{A,B})_*} \mathbf{CoAlg}^G(\pi(C), \underline{\mathbf{Alg}^G}(\rho(A), \rho(B))) \cong \mathfrak{m}_{\pi(C)}^G(\rho(A), \rho(B)).$$

We remark the image of $\mathrm{ev}_{A,B}^F$ under Φ is given by

$$\Phi(\mathrm{ev}_{A,B}^F) = \mathrm{ev}_{\rho(A),\rho(B)}^G \circ (\rho_{A,B} \otimes \mathrm{id}_{\rho(A)}),$$

and the image of $\mathrm{ev}_{B,T}^F \circ (\mathrm{id} \otimes \mathrm{ev}_{A,B}^F)$ under Φ is given by

$$\mathrm{ev}_{\rho(A),\rho(T)} \circ (\rho_{A,T} \circ \mathrm{id}_{\rho(A)}) \circ (\pi(\circ^F) \otimes \mathrm{id}_{\rho(A)}).$$

By Corollary 4.1.14 it suffices to check

$$\Phi(\mathrm{ev}_{B,T}^F) \circ (\mathrm{id} \otimes \Phi(\mathrm{ev}_{A,B}^F)) = \Phi(\mathrm{ev}_{B,T}^F \circ (\mathrm{id} \otimes \mathrm{ev}_{A,B}^F)) \circ \nabla^\pi.$$

Since both are measurings from $\rho(A)$ to $\rho(T)$, we can consider their transposes in the category of measurings from $\rho(A)$ to $\rho(T)$. This yields the diagram

$$\begin{array}{ccccc}
\pi(\underline{\mathbf{Alg}^F}(B,T)) \otimes \pi(\underline{\mathbf{Alg}^F}(A,B)) & \xrightarrow{\nabla^\pi} & \pi(\underline{\mathbf{Alg}^F}(B,T) \otimes \underline{\mathbf{Alg}^F}(B,T)) & \xrightarrow{\pi(\circ^F)} & \pi(\underline{\mathbf{Alg}^F}(A,T)) \\
\downarrow{\scriptstyle \mathrm{id} \otimes \rho_{A,B}} & & & & \\
\pi(\underline{\mathbf{Alg}^F}(B,T)) \otimes \underline{\mathbf{Alg}^G}(\rho(A),\rho(B)) & & & & \downarrow{\scriptstyle \rho_{A,T}} \\
\downarrow{\scriptstyle \rho_{B,T} \otimes \mathrm{id}} & & & & \\
\pi(\underline{\mathbf{Alg}^F}(B,T)) \otimes \underline{\mathbf{Alg}^G}(\rho(A),\rho(B)) & & \xrightarrow[\circ^G]{} & & \underline{\mathbf{Alg}^G}(\rho(A),\rho(T))
\end{array}$$

which commutes since $\rho : (\mathbf{Alg}^F, \pi_*(\mathbf{CoAlg}^F)) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ is an enriched functor. We conclude Φ respects composition. $\square$

Now we see there is an equivalence between families of functions Φ which respect composition and morphisms in **EnrCat**.

Theorem 4.1.17. *Let* $\rho : \mathbf{Alg}^F \to \mathbf{Alg}^G$ *be a functor,* $\pi : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ *be a lax monoidal functor. A natural transformation*

$$\Phi_{A,B,C} : \mathfrak{m}_C^F(A, B) \to \mathfrak{m}_{\pi(C)}^G(\rho(A), \rho(B))$$

which respects composition of measurings is equivalent to a morphism

$$(\rho, \pi) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

in **EnrCat**.

Proof. The result follows from combining Lemma 4.1.16 and Proposition 4.1.15. □

This theorem confirms our intuition, and in the next sections we will see how we can apply this theorem in different cases. We conclude this section with a corollary which cements our intuition about composition of measurings.

Corollary 4.1.18. *If two families of functions*

$$\Phi_{A,B,C} : \mathfrak{m}^F_C(A,B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A),\rho(B))$$

and

$$\Psi_{A,B,C} : \mathfrak{m}^G_C(A,B) \to \mathfrak{m}^H_{\pi'(C)}(\rho'(A),\rho'(B))$$

respect composition, then their composite

$$(\Psi \circ \Phi)_{A,B,C} : \mathfrak{m}^F_C(A,B) \to \mathfrak{m}^G_{\pi(C)}(\rho(A),\rho(B)) \to \mathfrak{m}^H_{\pi'(\pi(C))}(\rho'(\rho(A)),\rho'(\rho(B)))$$

respects composition as well.

4.2 Embedding measurings

In this section we will see our first application of the theory presented in the previous section. The guiding intuition is that given an monomorphism $\mu : F \rightarrowtail G$, this should result in an embedding $\mathbf{Alg}^F \rightarrowtail \mathbf{Alg}^G$ which also respects the enrichment. We will see that we need to strengthen the hypothesis slightly, to $\mu : F \rightarrowtail G$ being a section rather that just a monomorphism. In Section 4.4.1, we will see this demonstrated in the case of natural numbers, lists and trees. This is in line with the idea that lists are a generalization of natural numbers, and that trees are a generalization of lists.

Before we get there, we first need to demonstrate what we can do given a natural transformation $\mu : F \to G$.

Definition 4.2.1. Given a natural transformation $\mu : F \to G$, the *pullback functor* is defined as

$$\mu^* : \mathbf{Alg}^G \to \mathbf{Alg}^F$$
$$(\alpha : G(A) \to A) \mapsto (\alpha \circ \mu_A : F(A) \to G(A) \to A)$$

As expected, there is a dual to this definition for coalgebras.

Definition 4.2.2. Given a natural transformation $\mu : F \to G$, the *pushforward functor* is defined as

$$\mu_* : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$$
$$(\chi : C \to F(C)) \mapsto (\mu_C \circ \chi : C \to F(C) \to G(C))$$

The pushforward functor has a special property, namely that of being *strict*.

Proposition 4.2.3. *Given a lax monoidal natural transformation $\mu : F \to G$, the pushforward functor $\mu_* : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ is a strict lax monoidal functor*

$$\mu_* : (\mathbf{CoAlg}^F, \otimes, (\mathbb{I}, \eta_F)) \to (\mathbf{CoAlg}^G, \otimes, (\mathbb{I}, \eta_G)),$$

Proof. The coherence maps are given by

$$\mathrm{id} : (\mathbb{I}, \eta_G) \to (\mathbb{I}, \mu_{\mathbb{I}} \circ \eta_F)$$

and given F-coalgebras (C,χ) and (D,δ)

$$\mathrm{id} : \mu_*(C) \otimes \mu_*(D) \to \mu_*(C \otimes D)$$

where these functions are coalgebra morphisms since $\eta_G = \mu_{\mathbb{I}} \circ \eta_F$ and $\nabla^G_{C,D} \circ (\mu_C \otimes \mu_D) = \mu_{C\otimes D} \circ \nabla^F_{C,D}$ by μ being a monoidal natural transformation. Checking the coherence maps satisfy associativity and unitality is trivial since all coherence maps are identities. Moreover, μ_* is a srict monoidal functor since all coherence maps are identities. □

We can also say more about the pullback functor μ^* whenever μ is epic.

Lemma 4.2.4. *If $\mu : F \to G$ is epic, μ^* is fully faithful.*

Proof. The bijection $\mathbf{Alg}^G(A, B) \cong \mathbf{Alg}^F(\mu^*(A), \mu^*(B))$ is given by

$$\begin{aligned} \mu^* : \mathbf{Alg}^G(A, B) &\to \mathbf{Alg}^F(\mu^*(A), \mu^*(B)) \\ f &\mapsto f \end{aligned}$$

and is well-defined since μ^* is a functor. Its inverse is well-defined since given a morphism $f \in \mathbf{Alg}^F(\mu^*(A), \mu^*(B))$ we can draw the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F(f)} & F(B) \\ {\scriptstyle \mu_A}\big\downarrow & & \big\downarrow{\scriptstyle \mu_B} \\ G(A) & \xrightarrow{G(f)} & G(B) \\ {\scriptstyle \alpha}\big\downarrow & & \big\downarrow{\scriptstyle \beta} \\ A & \xrightarrow[f]{} & B. \end{array}$$

We know the outer rectangle commutes since f is an algebra morphism and we know the upper square commutes since μ is a natural transformation. We hence see

$$f \circ \alpha \circ \mu_A = \beta \circ F(f) \circ \mu_A.$$

Since μ is epic, we know μ_A is epic as well and we conclude $f \circ \alpha = \beta \circ F(f)$, hence that the inverse is well-defined. □

Since we have already done a lot of work in the previous section relating transformations of measurings to morphisms in **EnrCat**, we already have everything in place to state the key result of this section.

Theorem 4.2.5. *Let (ν, μ) be a pair of lax monoidal natural transformations*

$$F \underset{\nu}{\overset{\mu}{\rightleftarrows}} G$$

such that $\nabla^F_{C,A}$ coequalizes $\mathrm{id}_F \otimes \nu$ and $(\nu \otimes \nu) \circ (\mu \otimes \mathrm{id}_G)$. Then

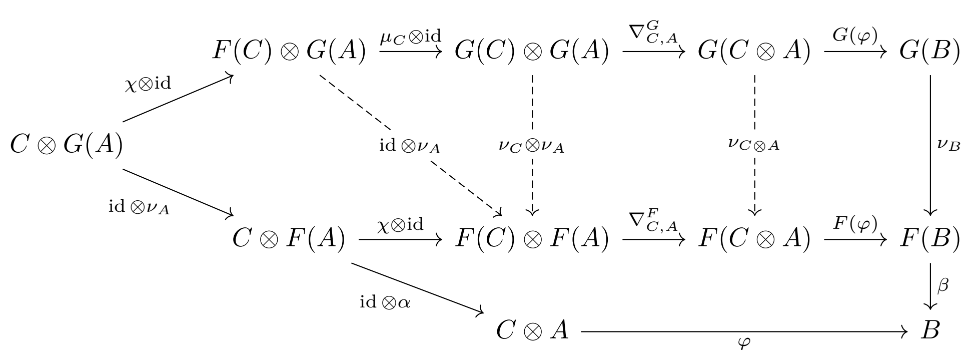
$$(\nu^*, \mu_*) : (\mathbf{Alg}^G, \mathbf{CoAlg}^G) \to (\mathbf{Alg}^F, \mathbf{CoAlg}^F)$$

is a morphism in **EnrCat**.

Proof. We define the natural transformation

$$\begin{aligned} \Phi_{A,B,C} : \mathfrak{m}_C(A, B) &\to \mathfrak{m}_{\mu_*(C)}(\nu^*(A), \nu^*(B)) \\ \varphi &\mapsto \varphi. \end{aligned}$$

This is well defined since given any F-measuring φ, the following diagram



commutes since ν is a lax monoidal natural transformation and $\nabla^F_{C,A}$ coequalizes $\mathrm{id} \otimes \nu_A$ and $(\nu_C \otimes \nu_A) \circ (\mu_C \otimes \mathrm{id})$. This shows any F-measuring $\varphi : C \otimes A \to B$ is also a G-measuring $\varphi : \mu_*(C) \otimes \nu^*(A) \to \nu^*(B)$. Observe Φ respects composition by Corollary 4.1.14 and the fact that

$$\Phi(\mathrm{ev}^F_{B,T}) \circ (\mathrm{id} \otimes \Phi(\mathrm{ev}^F_{A,B})) = \mathrm{ev}^F_{B,T} \circ \mathrm{id} \otimes \mathrm{ev}^F_{A,B} = \Phi(\mathrm{ev}^F_{B,T} \circ (\mathrm{id} \otimes \mathrm{ev}^F_{A,B})) \circ \nabla^{\mu_*}$$

since $\nabla^{\mu_*} = \mathrm{id}$ since μ^* is a strict monoidal natural transformation. By Theorem 4.1.17 we conclude

$$(\nu^*, \mu_*) : (\mathbf{Alg}^G, \mathbf{CoAlg}^G) \to (\mathbf{Alg}^F, \mathbf{CoAlg}^F)$$

is a morphism in **EnrCat**. □

Remark 4.2.6. The morphism $\nu^*_{A,B}$ is the unique map from $\mu_*(\underline{\mathbf{Alg}^F}(A, B))$ to $\underline{\mathbf{Alg}^G}(\nu^*(A), \nu^*(B))$ in the category of measurings from $\nu^*(A)$ to $\nu^*(B)$.

Returning to our intution of a section $\mu : F \to G$ resulting in an enriched embedding of categories, we state the following result.

Corollary 4.2.7. *Given a pair of lax monoidal natural transformations*

$$F \underset{\nu}{\overset{\mu}{\rightleftarrows}} G$$

such that $\nu \circ \mu = \mathrm{id}_F$, then

$$(\nu^*, \mu_*) : (\mathbf{Alg}^G, \mathbf{CoAlg}^G) \to (\mathbf{Alg}^F, \mathbf{CoAlg}^F)$$

is a morphism in **EnrCat**.

Proof. Since $\nu \circ \mu = \mathrm{id}_F$, any morphism coequalizes $\mathrm{id} \otimes \nu_A$ and $(\nu_C \otimes \nu_A) \circ (\mu_C \otimes \mathrm{id})$ since the morphisms coincide. By Theorem 4.2.5 $(\nu^*, \mu_*) : (\mathbf{Alg}^G, \mathbf{CoAlg}^G) \to (\mathbf{Alg}^F, \mathbf{CoAlg}^F)$ is a morphism in **EnrCat**. □

As promised, we wrap everything up in a functor from the category of endofunctors to **EnrCat**.

Corollary 4.2.8. *Let* $\mathbf{C}$ *be a locally presentable, closed symmetric monoidal category and let* $\mathbf{Endo}(\mathbf{C})_{retr.}$ *denote the category of accessible lax monoidal endofunctors on* $\mathbf{C}$*. Morphisms $G \to F$ in* $\mathbf{Endo}(\mathbf{C})_{retr.}$ *are given by pairs (ν, μ) of lax monoidal transformations*

$$F \underset{\nu}{\overset{\mu}{\rightleftarrows}} G$$

such that $\nu \circ \mu = \mathrm{id}_F$. There exists a functor

$$\begin{aligned} \mathbf{Endo}(\mathbf{C})_{retr.} &\to \mathbf{EnrCat} \\ F &\mapsto (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \\ (\nu, \mu) &\mapsto ((\nu^*, \mu_*) : (\mathbf{Alg}^G, \mathbf{CoAlg}^G) \to (\mathbf{Alg}^F, \mathbf{CoAlg}^F)) \end{aligned}$$

Proof. We have already defined the functor, and the only thing left to check is that it respects composition. Given two pairs of natural transformations $(\nu, \mu) : H \to G$ and $(\nu', \mu') : G \to F$, we obtain two morphisms (ν^*, μ_*) and (ν'^*, μ'_*). Composing the latter two morphisms in **EnrCat** yields the morphism $(\nu'^*, \mu'_*) \circ (\nu^*, \mu_*)$. By definition we already know $(\nu'^*, \mu'_*) \circ (\nu^*, \mu_*)$ and $((\nu' \circ \nu)^*, (\mu \circ \mu)_*)$ agree on objects. It remains to check the also agree on the enriched hom-objects. We need to verify the diagram

$$\begin{array}{ccc} \mu'_* \circ \mu_*(\underline{\mathbf{Alg}^F}(A, B)) & \xrightarrow{(\nu' \circ \nu)^*_{A,B}} & \underline{\mathbf{Alg}^H}(\nu'^* \circ \nu^*(A), \nu'^* \circ \nu^*(B)) \\ & \searrow^{\mu_*(\nu^*_{A,B})} \qquad \nearrow_{\nu'^*{}_{\nu^*(A),\nu^*(B)}} & \\ & \mu_* \underline{\mathbf{Alg}^G}(\nu^*(A), \nu^*(B)) & \end{array}$$

commutes. This is the case, since by definition all coalgebra morphisms involved are actually morphisms of measurings in the category of measurings from $\nu'^* \circ \nu^*(A)$ to $\nu'^* \circ \nu^*(B)$. Since $\underline{\mathbf{Alg}^H}(\nu'^* \circ \nu^*(A), \nu'^* \circ \nu^*(B))$ is the terminal object in this category, the morphisms must coincide. We conclude the functor respects composition. $\square$

4.3 Pushing forward and pulling back measurings

The previous section provides some nice results which confirm our intuition. However, we did require a lot, asking for a pair of lax monoidal natural transformations $F \xrightarrow{\mu} G \xrightarrow{\nu} F$ such that their composition is the identity on F. Nicer would be if we could simply consider *any* lax monoidal natural transformation $\mu : F \to G$, hopefully resulting in a morphism $(\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ in **EnrCat**. One of the drawbacks of having only a single natural transformation $\mu : F \to G$ is that we only get two functors between categories of algebras and coalgebras, namely the pullback functor $\mu^* : \mathbf{Alg}^G \to \mathbf{Alg}^F$ and the pushforward functor $\mu_* : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$. A morphism in **EnrCat** does ask for two functors but they have to be either both covariant or both contravariant. Of the pullback functor and the pushforward functor the former is contravariant where the latter is covariant, which poses a problem. To solve this issue, we turn to the left adjoint of the pullback functor $\mu_!$ and the right adjoint of pushforward functor $\mu_{¡}$.

In this section we aim to show that given a lax monoidal natural transformation $\mu : F \to G$, we obtain two morphisms

$$(\mu_!, \mu_*) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

and

$$(\mu^*, \mu_{¡}) : (\mathbf{Alg}^G, \mathbf{CoAlg}^G) \to (\mathbf{Alg}^F, \mathbf{CoAlg}^F)$$

in **EnrCat**.

Remark 4.3.1. One might wonder if instead of the left adjoint of μ^* we could also have considered the right adjoint. Sadly the right adjoint does not exist, since in general μ^* does not preserve colimits. In particular, it does not preserve the initial object. This can be easily seen when considering the categories $\mathbf{Alg}^F$ and $\mathbf{Alg}^G$ from Example 4.4.4. The initial objects are given by $\mathbb{N}$ and X^* respectively, and pulling back X^* using μ^* does not give the initial object $\mathbb{N}$ since their carriers are not the same. Similarly, the left adjoint to μ_* does not exist since it does not preserve limits. Again, in particular it does not preserve the terminal object. This does mean that we have a full understanding of what is possible when starting from a natural transformation $\mu : F \to G$.

Before we can get anything done, we must show the left adjoint to the pullback functor exists. This could be proven using some adjoint functor theorem, but we prefer to give an explicit construction.

Theorem 4.3.2. *Given a natural transformation $\mu : F \to G$, the pullback functor $\mu^* : \mathbf{Alg}^G \to \mathbf{Alg}^F$ has a left adjoint $\mu_! : \mathbf{Alg}^F \to \mathbf{Alg}^G$ given by the coequalizer in $\mathbf{Alg}^G$,*

$$\mathrm{Fr}^G(F(A)) \underset{\tilde{f}}{\overset{\mathrm{Fr}^G(\alpha)}{\rightrightarrows}} \mathrm{Fr}^G(A) \overset{coeq}{\dashrightarrow} \mu_!(A),$$

for any algebra $(A, \alpha) \in \mathbf{Alg}^F$. The morphism $\tilde{f}$ is obtained as adjunct under the free-forgetful adjunction of the composition

$$f : F(A) \xrightarrow{\mu_A} G(A) \xrightarrow{G(\eta_A)} G(\mathrm{Fr}^G(A)) \xrightarrow{\alpha_{\mathrm{Fr}^G}} \mathrm{Fr}^G(A)$$

with η being the unit of the free-forgetful adjunction.

Before we prove this theorem, we would like to exhibit the behavior of $\mu_!$ on morphisms. Given an algebra morphism $g : A \to B \in \mathbf{Alg}^F$, we can draw the following diagram where we abbreviate Fr^G to Fr:

$$\begin{array}{ccccc}
\mathrm{Fr}(F(A)) & \underset{\tilde{f}_A}{\overset{\mathrm{Fr}(\alpha)}{\rightrightarrows}} & \mathrm{Fr}(A) & \xrightarrow{q_A} & \mu_!(A) \\
\downarrow{\scriptstyle \mathrm{Fr}(F(g))} & & \downarrow{\scriptstyle \mathrm{Fr}(g)} & & \vdots{\scriptstyle \mu_!(g)} \\
\mathrm{Fr}(F(B)) & \underset{\tilde{f}_B}{\overset{\mathrm{Fr}(\beta)}{\rightrightarrows}} & \mathrm{Fr}(B) & \xrightarrow[q_B]{} & \mu_!(B).
\end{array}$$

We claim $\mu_!(g)$ is induced by the morphism $q_B \circ \mathrm{Fr}(g)$ coequalising $\mathrm{Fr}(\alpha)$ and $\tilde{f}_A$. To see this is the case, note q_B coequalizes $\mathrm{Fr}(\beta)$ and $\tilde{f}_B$ by definition. Moreover, $\mathrm{Fr}(g) \circ \mathrm{Fr}(\alpha) = \mathrm{Fr}(\beta) \circ \mathrm{Fr}(F(g))$ since $g \in \mathbf{Alg}^F$ and $\mathrm{Fr}(g) \circ \tilde{f}_A = \tilde{f}_B \circ \mathrm{Fr}(F(g))$ since $\mathrm{Fr}(g) \in \mathbf{Alg}^G$ and the other components of f_A and f_B are natural transformations. We can conclude $q_B \circ \mathrm{Fr}(g)$ coequalizes $\mathrm{Fr}(\alpha)$ and $\tilde{f}_A$ and hence there exists an induced map $\mu_!(g)$.

Proof. Throughout this proof, we will write $\mathrm{Fr}^G = \mathrm{Fr}$ since we are only considering the free functor $\mathbf{C} \to \mathbf{Alg}^G$. Also note that we will omit the forgetful functor $U^G : \mathbf{Alg}^G \to \mathbf{C}$ to avoid a notational mess.

Let $g : A \to B \in \mathbf{C}$. We claim $g \in \mathbf{Alg}^F(A, \mu^*(B))$ if and only if its transpose $\tilde{g} : \mathrm{Fr}(A) \to B$ coequalizes $\mathrm{Fr}(\alpha)$ and $\tilde{f}$. If this is the case, then there is a one to one correspondence between algebra morphisms $\mathbf{Alg}^F(A, \mu^*(B))$ and algebra morphisms $\mathbf{Alg}^G(\mu_!(A), B)$.

First, assume $g \in \mathbf{Alg}^F(A, \mu^*(B))$. Since $\tilde{g}$ is the transpose of g it is given by $\tilde{g} = \varepsilon_B \circ \mathrm{Fr}(g)$. Similarly, $\tilde{f} = \varepsilon_{\mathrm{Fr}(A)} \circ \mathrm{Fr}(f)$. We can now deduce

$$\begin{aligned} \tilde{g} \circ \mathrm{Fr}(\alpha) &= \varepsilon_B \circ \mathrm{Fr}(g \circ \alpha) \\ &= \varepsilon_B \circ \mathrm{Fr}(\beta \circ \mu_B \circ F(g)) \\ &= \varepsilon_B \circ \mathrm{Fr}(\beta \circ G(g) \circ \mu_A). \end{aligned}$$

We remark that by the triangle identities $\varepsilon_B \circ \eta_B = \mathrm{id}_B$ and that by $\tilde{g} \in \mathbf{Alg}^G$ we know

$$\tilde{g} \circ \alpha_{\mathrm{Fr}} = \beta \circ G(\tilde{g}).$$

Using this, we can continue our deduction by

$$\begin{aligned} \tilde{g} \circ \mathrm{Fr}(\alpha) &= \varepsilon_B \circ \mathrm{Fr}(\beta \circ G(g) \circ \mu_A) \\ &= \varepsilon_B \circ \mathrm{Fr}(\beta \circ G(\varepsilon_b \circ \eta_B \circ g) \circ \mu_A) \\ &= \varepsilon_B \circ \mathrm{Fr}(\beta \circ G(\varepsilon_b \circ \mathrm{Fr}(g) \circ \eta_A) \circ \mu_A) \\ &= \varepsilon_B \circ \mathrm{Fr}(\tilde{g} \circ \alpha_{\mathrm{Fr}} \circ G(\eta_A) \circ \mu_A) \\ &= \varepsilon_B \circ \mathrm{Fr}(\tilde{g} \circ f) \\ &= \tilde{g} \circ \varepsilon_{\mathrm{Fr}(A)} \circ \mathrm{Fr}(f) \\ &= \tilde{g} \circ \tilde{f} \end{aligned}$$

and conclude $\tilde{g}$ coequalizes $\mathrm{Fr}(\alpha)$ and $\tilde{f}$ whenever $g \in \mathbf{Alg}^F(A, \mu^*(B))$.

Second, assume $\tilde{g} : \mathrm{Fr}(A) \to B$ coequalizes $\mathrm{Fr}(\alpha)$ and $\tilde{f}$. Since $g : A \to B$ is the transpose of $\tilde{g}$, it is given by $g = \tilde{g} \circ \eta_A$. We can now make the following deduction:

$$\begin{aligned} g \circ \alpha &= \tilde{g} \circ \eta_A \circ \alpha \\ &= \tilde{g} \circ \mathrm{Fr}(\alpha) \circ \eta_{F(A)} \\ &= \tilde{g} \circ \tilde{f} \circ \eta_{F(A)} \\ &= \tilde{g} \circ \varepsilon_{\mathrm{Fr}(A)} \circ \mathrm{Fr}(f) \circ \eta_{F(A)} \\ &= \tilde{g} \circ \varepsilon_{\mathrm{Fr}(A)} \circ \eta_{\mathrm{Fr}(A)} \circ f, \end{aligned}$$

where on the first to last line we used the η is a natural transformation. Again, the triangle identities state $\varepsilon_{\mathrm{Fr}(A)} \circ \eta_{\mathrm{Fr}(A)} = \mathrm{id}_{\mathrm{Fr}(A)}$. Using this, we can continue our deduction by

$$\begin{aligned} g \circ \alpha &= \tilde{g} \circ \varepsilon_{\mathrm{Fr}(A)} \circ \eta_{\mathrm{Fr}(A)} \circ f \\ &= \tilde{g} \circ f \\ &= \tilde{g} \circ \alpha_{\mathrm{Fr}} \circ G(\eta_A) \circ \mu_A \\ &= \beta \circ G(\tilde{g}) \circ G(\eta_A) \circ \mu_A \\ &= \beta \circ G(\tilde{g} \circ \eta_A) \circ \mu_A \\ &= \beta \circ G(g) \circ \mu_A \\ &= \beta \circ \mu_B \circ F(g), \end{aligned}$$

from which we deduce $g \in \mathbf{Alg}^F(A, \mu^*(B))$.

We conclude the left adjoint of μ^* is given by the coequalizer of $\mathrm{Fr}(\alpha)$ and $\tilde{f}$. □

We would like to point out the relationship between $\mu_!$ and the free functors.

Lemma 4.3.3. *Given $\mu : F \to G$ and the free functors $\mathrm{Fr}^F : \mathbf{Set} \to \mathbf{Alg}^F$ and $\mathrm{Fr}^G : \mathbf{Set} \to \mathbf{Alg}^G$, we have*

$$\mu_! \circ \mathrm{Fr}^F \cong \mathrm{Fr}^G .$$

Proof. We have the natural isomorphisms

$$\mathbf{Alg}^G(\mu_! \circ \mathrm{Fr}^F(A), B) \cong \mathbf{Alg}^F(\mathrm{Fr}^F(A), \mu^*(B)) \cong \mathbf{C}(A, U(\mu^*(B))) = \mathbf{C}(A, U((B))),$$

hence by the universal property of the left adjoint we know $\mu_! \circ \mathrm{Fr}^F \cong \mathrm{Fr}^G$. □

We seen that whenever μ is epic the pullback functor μ^* is fully faithful in Lemma 4.2.4 Now that we have constructed the left adjoint, we can say even more.

Lemma 4.3.4. *If $\mu : F \to G$ is epic, then $\mu_! \circ \mu^* \cong \mathrm{id}_{\mathbf{Alg}^G}$.*

Proof. We have the following isomorphism

$$\mathbf{Alg}^G(\mu_! \circ \mu^*(A), B) \cong \mathbf{Alg}^F(\mu^*(A), \mu^*(B)) \cong \mathbf{Alg}^G(A, B).$$

by Lemma 4.2.4 and we conclude $\mu_! \circ \mu^* = \mathrm{id}_{\mathbf{Alg}^G}$ by the Yoneda embedding being full and faithful. □

Next is a slightly surprising result which will turn out to be absolutely key throughout this entire section.

Lemma 4.3.5. *Let $F, G : \mathbf{C} \to \mathbf{C}$ be lax monoidal endofunctors and let $\mu : F \to G$ be a lax monoidal natural transformation. Let $(B, \beta) \in \mathbf{Alg}^F$ and let $C \in \mathbf{CoAlg}^F$, then the F-algebras $[C, \mu^*(B)]$ and $\mu^*[\mu_*(C), B]$ are equal.*

Proof. Both $[C, \mu^*(B)]$ and $\mu^*[\mu_*(C), B]$ have $\underline{\mathbf{C}}(C, B)$ as underlying set. We claim $\mathrm{id}_{\underline{\mathbf{C}}(C,B)}$ is actually an algebra morphism. If this is the case, then $[C, \mu^*(B)]$ and $\mu^*[\mu_*(C), B]$ are equal. First, note μ is a lax monoidal natural transformation, hence a closed natural transformation [6]. By definition of a closed natural transformation

$$\underline{\mathbf{C}}(F(A), \mu_B) \circ \tilde{\nabla}^F = \underline{\mathbf{C}}(\mu_A, G(B)) \circ \tilde{\nabla}^G \circ \mu_{\underline{C}(A,B)}.$$

To see $\mathrm{id}_{\underline{\mathbf{C}}(C,B)}$ is an algebra morphism, we must check the following diagram in $\mathbf{C}$ commutes:

$$\begin{array}{ccc}
F(\underline{\mathbf{C}}(C,B)) & \xrightarrow{F(\mathrm{id})} & F(\underline{\mathbf{C}}(C,B)) \\
 & & \downarrow{\scriptstyle \mu_{\underline{\mathbf{C}}(C,B)}} \\
\downarrow{\scriptstyle \tilde{\nabla}^F} & & G(\underline{\mathbf{C}}(C,B)) \\
 & & \downarrow{\scriptstyle \tilde{\nabla}^G} \\
\underline{\mathbf{C}}(F(C), F(B)) & & \underline{\mathbf{C}}(G(C), G(B)) \\
\downarrow{\scriptstyle \underline{\mathbf{C}}(\chi, \beta \circ \mu_B)} & & \downarrow{\scriptstyle \underline{\mathbf{C}}(\mu_C \circ \chi, \beta)} \\
\underline{\mathbf{C}}(C,B) & \xrightarrow{\mathrm{id}} & \underline{\mathbf{C}}(C,B)
\end{array}$$

Since both vertical maps our equal by μ being a closed monoidal natural transformation, we see our diagram commutes. We conclude the algebras $[C, \mu^*(B)]$ and $\mu^*[\mu_*(C), B]$ are equal. □

Since we will be using this in the upcoming proof, we point out this corollary.

Corollary 4.3.6. *Let $F : \mathbf{C} \to \mathbf{C}$ be a monoidal endofunctor, $B \in \mathbf{Alg}^F$ and let $C \in \mathbf{CoAlg}^F$. Then the F-algebras $[C, \mu^* \circ \mu_!(B)]$ and $\mu^*[\mu_*(C), \mu_!(B)]$ are equal.*

Corollary 4.3.7. *There exists a natural isomorphism $\mathfrak{m}^F_C(A, \mu^*(B)) \cong \mathfrak{m}^G_{\mu_*(C)}(\mu_!(A), B)$.*

Proof. Let $B \in \mathbf{Alg}^G$, then

$$\begin{aligned}\mathbf{Alg}^G(\mu_!(A) \triangleright \mu_*(C), B) &\cong \mathbf{Alg}^G(\mu_!(A), [\mu_*(C), B]) \\ &\cong \mathbf{Alg}^F(A, \mu^*[\mu_*(C), B]) \\ &\cong \mathbf{Alg}^F(A, [C, \mu^*(B)]) \\ &\cong \mathbf{Alg}^F(C \triangleright A, \mu^*(B)).\end{aligned}$$

Since $\mathfrak{m}^F_C(A, \mu^*(B)) \cong \mathbf{Alg}^F(C \triangleright A, \mu^*(B)) \cong \mathbf{Alg}^G(\mu_!(A) \triangleright \mu_*(C), B) \cong \mathfrak{m}^G_{\mu_*(C)}(\mu_!(A), B)$ this gives the desired result. □

With Lemma 4.3.5 in hand, it is time to start leveraging it. Using the Yoneda embedding, we can carry over the result to another representing object.

Lemma 4.3.8. *For all coalgebras $C \in \mathbf{CoAlg}^F$ and algebras $A \in \mathbf{Alg}^F$ there exists a natural isomorphism $\mu_*(C) \triangleright \mu_!(A) \cong \mu_!(C \triangleright A)$.*

Proof. We aim to use the fact the the Yoneda embedding is full and faithful in combination with Corollary 4.3.6. For any algebra $B \in \mathbf{Alg}^G$, we have the following natural isomorphisms

$$\begin{aligned}\mathbf{Alg}^G(\mu_*(C) \triangleright \mu_!(A), B) &\cong \mathbf{Alg}^G(\mu_!(A), [\mu_*(C), B]) \\ &\cong \mathbf{Alg}^F(A, \mu^*([\mu_*(C), B])) \\ &= \mathbf{Alg}^F(A, [C, \mu_*(B)]) \\ &\cong \mathbf{Alg}^F(C \triangleright A, \mu^*(B)) \\ &\cong \mathbf{Alg}^G(\mu_!(C \triangleright A), B).\end{aligned}$$

Since the Yoneda embedding is full and faithful, we can conclude there is a natural isomorphism $\mu_*(C) \triangleright \mu_!(A) \cong \mu_!(C \triangleright A)$. □

We proceed with another technical lemma which we will leverage later.

Lemma 4.3.9. *For all coalgebras $C, D \in \mathbf{CoAlg}$ and algebras $B \in \mathbf{Alg}$ we have the natural isomorphism*

$$[C, [D, B]] \cong [D \otimes C, B].$$

Proof. The underlying objects of these algebras are given by $\underline{\mathbf{C}}(C, \underline{\mathbf{C}}(D, B))$ and $\underline{\mathbf{C}}(D \otimes C, B)$ respectively. Since $\mathbf{C}$ is a closed monoidal category, we know there is a natural isomorphism

$$\underline{\mathbf{C}}(C, \underline{\mathbf{C}}(D, B)) \cong \underline{\mathbf{C}}(D \otimes C, B).$$

We claim this isomorphism lifts to an isomorphism of algebras and we verify using the diagram

$$\begin{array}{ccc}
F(\underline{\mathbf{C}}(C, \underline{\mathbf{C}}(D, B))) & \xrightarrow{\cong} & F(\underline{\mathbf{C}}(D \otimes C, B)) \\
{\scriptstyle \tilde{\nabla}^F}\downarrow & & \downarrow{\scriptstyle \tilde{\nabla}^F} \\
\underline{\mathbf{C}}(F(C), F(\underline{\mathbf{C}}(D, B))) & & \underline{\mathbf{C}}(F(D \otimes C), F(B)) \\
{\scriptstyle \underline{\mathbf{C}}(F(C), \tilde{\nabla}^F)}\downarrow & & \downarrow{\scriptstyle (\nabla^F)^*} \\
\underline{\mathbf{C}}(F(C), \underline{\mathbf{C}}(F(D), F(B))) & & \underline{\mathbf{C}}(F(D \otimes C), F(B)) \\
{\scriptstyle \chi^* \circ \underline{\mathbf{C}}(F(C), \delta^* \circ \beta_*)}\downarrow & & \downarrow{\scriptstyle (\delta \otimes \chi)^* \circ \beta_*} \\
\underline{\mathbf{C}}(C, \underline{\mathbf{C}}(D, B)) & \xrightarrow{\cong} & \underline{\mathbf{C}}(D \otimes C, B)
\end{array}$$

which commutes by the closed monoidal structure on $\mathbf{C}$ and naturality of the isomorphism $\underline{\mathbf{C}}(C, \underline{\mathbf{C}}(D, B)) \cong \underline{\mathbf{C}}(D \otimes C, B)$. $\square$

Using Lemma 4.3.9, we can now show the representing object $\_ \triangleright \_$ respects the monoidal structure on the category of coalgebras.

Lemma 4.3.10. *For all coalgebras $C, D \in \mathbf{CoAlg}$ and algebras $A \in \mathbf{Alg}$ we have the natural isomorphism*

$$D \triangleright (C \triangleright A) \cong (D \otimes C) \triangleright A.$$

Proof. We again aim to use the fact the the Yoneda embedding is full and faithful in combination with Lemma 4.3.9. For any algebra $B \in \mathbf{Alg}$ we have the following natural isomorphisms

$$\begin{aligned} \mathbf{Alg}(D \triangleright (C \triangleright A), B) &\cong \mathbf{Alg}(C \triangleright A, [D, B]) \\ &\cong \mathbf{Alg}(A, [C, [D, B]]) \\ &\cong \mathbf{Alg}(A, [C \otimes D, B]) \\ &\cong \mathbf{Alg}((D \otimes C) \triangleright A, B) \end{aligned}$$

and by the Yoneda embedding we conclude our result. $\square$

Remark 4.3.11. The coalgebra $D \otimes C$ makes use of the lax monoidal structure of ∇ to obtain a coalgebra structure $D \otimes C \xrightarrow{\delta \otimes \chi} F(D) \otimes F(C) \xrightarrow{\nabla} F(D \otimes C)$. This implies the lax monoidal structure ∇ is also incorporated into the algebra structure of $D \triangleright (C \triangleright A)$.

Now we have enough tools at our disposal to start building towards the main result of this section. We aim to show a lax monoidal natural transformation $\mu : F \to G$ results in a morphism $(\mu_!, \mu_*) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ in $\mathbf{EnrCat}$. As a starting point, we will again make use of a transformation of measurings Φ.

Definition 4.3.12. We define

$$\Phi_{A,B,C} : \mathfrak{m}^F_C(A, B) \to \mathfrak{m}^G_{\mu_*(C)}(\mu_!(A), \mu_!(B))$$

as the composite

$$\mathfrak{m}^F_C(A, B) \cong \mathbf{Alg}^F(C \triangleright A, B) \xrightarrow{\mu_!} \mathbf{Alg}^G(\mu_!(C \triangleright A), \mu_!(B)) \cong \mathbf{Alg}^G(\mu_*(C) \triangleright \mu_!(A), \mu_!(B)) \cong \mathfrak{m}^G_{\mu_*(C)}(\mu_!(A), \mu_!(B)).$$

Having this transformation of measurings is nearly enough, but as seen in the previous sections we also need to ask it respects composition to prove our main result.

Lemma 4.3.13. *The family of functions defined in Definition 4.3.12 respects composition.*

Proof. We aim to show the diagram

$$\begin{array}{ccc} \mathfrak{m}^F_D(B, T) \times \mathfrak{m}^F_C(A, B) & \xrightarrow{\Phi_{B,T,D} \times \Phi_{A,B,C}} & \mathfrak{m}^G_{\mu_*(D)}(\mu_!(B), \mu_!(T)) \times \mathfrak{m}^G_{\mu_*(C)}(\mu_!(A), \mu_!(B)) \\ & & \downarrow \circ^G_{\mathfrak{m}} \\ \downarrow \circ^F_{\mathfrak{m}} & & \mathfrak{m}^G_{\mu_*(D) \otimes \mu_*(C)}(\mu_!(A), \mu_!(T)) \\ & & \uparrow \mathfrak{m}^G_{(\nabla \mu_*)}(\mu_!(B), \mu_!(T)) \\ \mathfrak{m}^F_{D \otimes C}(A, T) & \xrightarrow[\Phi_{A,T,D \otimes C}]{} & \mathfrak{m}^G_{\mu_*(D \otimes C)}(\mu_!(B), \mu_!(T)) \end{array}$$

commutes. To do so, we remark that composition of measurings

$$\begin{aligned} \circ_{\mathfrak{m}} : \mathfrak{m}_D(B, T) \times \mathfrak{m}_C(A, B) &\to \mathfrak{m}_{D \otimes C}(A, T) \\ (\psi, f) &\mapsto \psi \circ (\mathrm{id}_D \otimes f) \end{aligned}$$

under the natural identifications $\mathfrak{m}_C(A,B) \cong \mathbf{Alg}(C \triangleright A, B)$ corresponds to the the algebra morphism

$$\begin{aligned} \circ_{\mathfrak{m}} \mathbf{Alg}(D \triangleright B, T) \times \mathbf{Alg}(C \triangleright A, B) &\to \mathbf{Alg}(D \triangleright (C \triangleright A), T) \\ (\tilde{\psi}, \tilde{f}) &\mapsto \tilde{\psi} \circ (\mathrm{id}_D \triangleright \tilde{f}). \end{aligned}$$

Using Lemma 4.3.10, we can state that Φ respecting composition of measurings is equivalent to verifying

$$\begin{array}{ccc}
\mathbf{Alg}^F(D \triangleright B, T) \times \mathbf{Alg}^F(C \triangleright A, B) & \xrightarrow{\circ_{\mathfrak{m}}^F} & \mathbf{Alg}^F(D \triangleright (C \triangleright A), T) \\
\Big\downarrow {\scriptstyle \mu_! \times \mu_!} & & \Big\downarrow {\scriptstyle \mu_!} \\
\mathbf{Alg}^G(\mu_!(D \triangleright B), \mu_!(T)) \times \mathbf{Alg}^G(\mu_!(C \triangleright A), \mu_!(B)) & & \mathbf{Alg}^G(\mu_!(D \triangleright (C \triangleright A)), \mu_!(T)) \\
 & & \Big\downarrow \cong \\
\Big\downarrow \cong & & \mathbf{Alg}^G(\mu_*(D) \triangleright \mu_!(C \triangleright A), \mu_!(T)) \\
 & & \Big\downarrow \cong \\
\mathbf{Alg}^G(\mu_*(D) \triangleright \mu_!(B), \mu_!(T)) \times \mathbf{Alg}^G(\mu_*(C) \triangleright \mu_!(A), \mu_!(B)) & \xrightarrow{\circ_{\mathfrak{m}}^G} & \mathbf{Alg}^G(\mu_*(D) \triangleright (\mu_*(C) \triangleright \mu_!(A)), \mu_!(T))
\end{array}$$

commutes. Given a pair $(\tilde{\psi}, \tilde{f}) \in \mathbf{Alg}^F(D \triangleright B, T) \times \mathbf{Alg}^F(C \triangleright A, B)$, we need to verify

$$\mu_!(\tilde{\psi} \circ (\mathrm{id}_D \triangleright \tilde{f})) = \mu_!(\tilde{\psi}) \circ \mu_!(\mathrm{id}_D \triangleright \tilde{f})$$

corresponds to

$$\mu_!(\tilde{\psi}) \circ (\mathrm{id}_{\mu_*(D)} \triangleright \mu_!(\tilde{f}))$$

under the isomorphism $\mathbf{Alg}^G(\mu_!(D \triangleright (C \triangleright A)), \mu_!(T)) \cong \mathbf{Alg}^G(\mu_*(D) \triangleright \mu_!(C \triangleright A), \mu_!(T))$. This is indeed the case by naturality of the isomorphism $\mu_!(D \triangleright (C \triangleright A)) \cong (\mu_*(D) \triangleright \mu_!(C \triangleright A)$. □

Summarizing the above, we state the following theorem.

Theorem 4.3.14. *Given a lax monoidal natural transformation $\mu : F \to G$, we obtain a morphism*

$$(\mu_!, \mu_*) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

in **EnrCat**.

Proof. By Lemma 4.3.13, the family of function defined in Definition 4.3.12 using $\mu : F \to G$ respects composition. Using Theorem 4.1.17, we conclude we obtain a morphism

$$(\mu_!, \mu_*) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

in **EnrCat**. □

We wish to wrap this up nicely, and do so by proving the following functor is well-defined.

Corollary 4.3.15. *Let* **C** *be a locally presentable, closed symmetric monoidal category and let* **Endo**(**C**) *denote the category of accessible lax monoidal endofunctors on* **C** *and lax monoidal natural transformations. There exists a functor*

$$\begin{aligned} \mathbf{Endo}(\mathbf{C}) &\to \mathbf{EnrCat} \\ F &\mapsto (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \\ \mu &\mapsto (\mu_!, \mu_*) \end{aligned}$$

Proof. We have already defined the functor, and the only thing left to check is that it respects composition. Given two lax monoidal natural transformations $\mu : F \to G$ and $\nu : G \to H$, we obtain two morphisms $(\mu_!, \mu_*)$ and $(\nu_!, \nu_*)$. Composing the latter two morphisms in **EnrCat** yields the morphism $(\nu_!, \nu_*) \circ (\mu_!, \mu_*)$. By uniqueness of adjoints, we already know $(\nu_!, \nu_*) \circ (\mu_!, \mu_*)$ and $((\nu \circ \mu)_!, (\nu \circ \mu)_*)$ agree on objects. It remains to check the also agree on the enriched hom-objects. We need to verify the diagram

$$\begin{array}{ccc} \nu_* \circ \mu_*(\underline{\mathbf{Alg}^F}(A,B)) & \xrightarrow{(\nu\circ\mu)_{!A,B}} & \underline{\mathbf{Alg}^H}(\nu_! \circ \mu_!(A), \nu_! \circ \mu_!(B)) \\ & {\scriptstyle \nu_*(\mu_{!A,B})} \searrow \qquad \nearrow {\scriptstyle \nu_{!\mu_!(A),\mu_!(B)}} & \\ & \nu_*(\underline{\mathbf{Alg}^G}(\mu_!(A), \mu_!(B))) & \end{array}$$

commutes. This is the case, since by definition all coalgebra morphisms involved are actually morphisms of measurings in the category of measurings from $\nu_! \circ \mu_!(A)$ to $\nu_! \circ \mu_!(B)$. Since $\underline{\mathbf{Alg}^H}(\nu_! \circ \mu_!(A), \nu_! \circ \mu_!(B))$ is the terminal object in this category, the morphisms must coincide. We conclude the functor respects composition. □

We now continue on to the dual story, where the right adjoint to the pullback functor plays a central role. There are some subtle differences which we point out, and these subtle differences are also the reason we can not immediately dualize the statements above. We start out by giving an explicit construction of the right adjoint $\mu_{¡}$.

Theorem 4.3.16. *Given a natural transformation $\mu : F \to G$, the pushforward functor $\mu_* : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ has a right adjoint $\mu_{¡} : \mathbf{CoAlg}^G \to \mathbf{CoAlg}^F$ given by the equalizer in $\mathbf{CoAlg}^F$*

$$\mu_{¡}(C) \overset{\mathrm{eq}}{\dashrightarrow} \mathrm{Cof}^F(C) \underset{\tilde{f}}{\overset{\mathrm{Cof}(\chi)}{\rightrightarrows}} \mathrm{Cof}^F(G(C))$$

for any coalgebra $(C, \chi) \in \mathbf{CoAlg}^G$, where $\tilde{f}$ is obtained as adjunct under the free-forgetful adjunction of the composition

$$\mathrm{Cof}^F(C) \xrightarrow{\chi_{\mathrm{Cof}}} F(\mathrm{Cof}^F(C)) \xrightarrow{F(\varepsilon)} F(C) \xrightarrow{\mu_C} G(C)$$

with ε being the counit of the free-forgetful adjunction.

Proof. The proof is completely dual to that of Theorem 4.3.2 □

The functor $\mu_{¡}$ is lax monoidal if and only if μ_* is strong monoidal by [10]. Since μ_* is a strict monoidal functor by Lemma 4.2.3, it is in particular a strong monoidal functor. We conclude its right adjoint $\mu_!$ is a lax monoidal functor.

We have a statement dual to Lemma 4.3.3 regarding the cofree functor.

Lemma 4.3.17. *Given $\mu : F \to G$ and the cofree functors $\mathrm{Cof}^F : \mathbf{Set} \to \mathbf{CoAlg}^F$ and $\mathrm{Cof}^G : \mathbf{Set} \to \mathbf{CoAlg}^G$, we have*

$$\mathrm{Cof}^F \cong \mu_{¡} \circ \mathrm{Cof}^G .$$

Proof. We have the natural isomorphisms

$$\mathbf{CoAlg}^F(C, \mu_{¡} \circ \mathrm{Cof}^G(D)) \cong \mathbf{CoAlg}^G(\mu_*(C), \mathrm{Cof}^G(D)) \cong \mathbf{CoAlg}^G(U(\mu_*(C)), D) = \mathbf{CoAlg}^G(U(C), D)$$

hence by the universal property of the right adjoint we know $\mathrm{Cof}^F \cong \mu_{¡} \circ \mathrm{Cof}^G$. □

This is where the subtle difference manifests itself. When defining the family of function resulting in the morphism $(\mu_!, \mu_*) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ in **EnrCat**, we saw the family of functions was defined as applying $\mu_!$, combined with a bunch of natural isomorphisms. Here, we apply the pullback functor μ^*, as well as a natural transformation.

Definition 4.3.18. We define

$$\Phi_{A,B,C} : \mathfrak{m}^G_C(A,B) \to \mathfrak{m}^F_{\mu_{\mathsf{i}}(C)}(\mu^*(A), \mu^*(B))$$

as the composite

$$\Phi_{A,B,C} : \mathfrak{m}^G_C(A,B) \cong \mathbf{Alg}^G(A, [C,B]) \xrightarrow{\mathbf{Alg}^G(A,[\varepsilon,B])} \mathbf{Alg}^G(A, [\mu_* \circ \mu_{\mathsf{i}}(C), B]) \xrightarrow{\mu^*}$$
$$\mathbf{Alg}^F(\mu^*(A), \mu^*([\mu_* \circ \mu_{\mathsf{i}}(C), B])) = \mathbf{Alg}^F(\mu^*(A), ([\mu_{\mathsf{i}}(C), \mu^*(B)])) \cong \mathfrak{m}^F_{\mu_{\mathsf{i}}(C)}(\mu^*(A), \mu^*(B)),$$

where ε is the counit of the adjunction $\mu_* \dashv \mu_{\mathsf{i}}$.

Defining Φ in this way ensures it is well-defined, but does make for an opaque definition. Making it explicit at the level of the underlying category $\mathbf{C}$, Φ sends an G-measuring $f : C \otimes A \to B$ to the F-measuring

$$f \circ (\varepsilon_C \otimes \mathrm{id}_A) : \mu_{\mathsf{i}}(C) \otimes A \to B,$$

where we ignore the pushforward functor μ_* and pullback functor μ^* since they do not change the carrier of the coalgebra or algebra.

Having this explicit description in hand, we can go on to prove the following.

Lemma 4.3.19. *The family of functions defined in Definition 4.3.18 Φ respects composition.*

Proof. We know $\Phi(f) = f \circ (\varepsilon_C \otimes \mathrm{id}_A)$, so we explicitly check composition of measurings is respected. Given G-meausurings $f : C \otimes A \to B$ and $\psi : D \otimes B \to T$, it is necessary to check the diagram

$$\begin{array}{ccccc}
\mu_{\mathsf{i}}(D) \otimes \mu_{\mathsf{i}}(C) \otimes A & \xrightarrow{\mathrm{id}_{\mu_{\mathsf{i}}(D)} \otimes (f \circ (\varepsilon_C \otimes \mathrm{id}_A))} & \mu_{\mathsf{i}}(D) \otimes B & \xrightarrow{\psi \circ (\varepsilon_D \otimes \mathrm{id}_B)} & T \\
\downarrow{\scriptstyle \nabla^{\mu_{\mathsf{i}}}} & & & \nearrow{\scriptstyle (\psi \circ (\mathrm{id}_D \otimes f)) \circ (\varepsilon_{D \otimes C} \otimes \mathrm{id}_A)} & \\
\mu_{\mathsf{i}}(D \otimes C) \otimes A & & & &
\end{array}$$

commutes. This is the case since $\varepsilon : \mu_* \circ \mu_{\mathsf{i}} \to \mathrm{id}$ is a monoidal natural transformation, hence as morphisms in $\mathbf{C}$ we have

$$\varepsilon_D \otimes \varepsilon_C = \varepsilon_{C \otimes D} \circ \nabla^{\mu_{\mathsf{i}}}.$$

□

Similar to before, we obtain the following result.

Theorem 4.3.20. *Given a lax monoidal natural transformation $\mu : F \to G$, we obtain a morphism*

$$(\mu^*, \mu_{\mathsf{i}}) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

in $\mathbf{EnrCat}$.

Proof. By Lemma 4.3.19, the family of function defined in Definition 4.3.18 using $\mu : F \to G$ respects composition. Using Theorem 4.1.17, we conclude we obtain a morphism

$$(\mu_!, \mu_*) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

in $\mathbf{EnrCat}$. □

We could then again wrap up this section in the following clean statement.

Corollary 4.3.21. *Let $\mathbf{C}$ be a locally presentable, closed symmetric monoidal category and let $\mathbf{Endo}(\mathbf{C})$ denote the category of accessible lax monoidal endofunctors on $\mathbf{C}$ and lax monoidal natural transformations. There exists a functor*

$$\mathbf{Endo}(\mathbf{C})^{\mathbf{op}} \to \mathbf{EnrCat}$$
$$F \mapsto (\mathbf{Alg}^F, \mathbf{CoAlg}^F)$$
$$\mu \mapsto (\mu^*, \mu_{\mathsf{i}})$$

Proof. We have already defined the functor, and the only thing left to check is that it respects composition. Given two natural transformations $\mu : F \to G$ and $\nu : G \to H$, we obtain two morphisms $(\mu^*, \mu_{\mathsf{i}})$ and $(\nu^*, \nu_{\mathsf{i}})$. Composing the latter two morphisms in **EnrCat** yields the morphism $(\mu^*, \mu_{\mathsf{i}}) \circ (\nu^*, \nu_{\mathsf{i}})$. we already know $(\mu^*, \mu_{\mathsf{i}}) \circ (\nu^*, \nu_{\mathsf{i}})$ and $((\nu \circ \mu)^*, (\nu \circ \mu)_{\mathsf{i}})$ agree on objects by definition. It remains to check the also agree on the enriched hom-objects. First, we remark $(\mu_{\mathsf{i}} \circ \nu_{\mathsf{i}}) \cong (\mu \circ \nu)_{\mathsf{i}}$ by uniqueness of adjoints. We need to verify the diagram

$$\begin{array}{ccc} \mu_{\mathsf{i}} \circ \nu_{\mathsf{i}}(\underline{\mathbf{Alg}^F}(A,B)) & \xrightarrow{(\mu\circ\nu)^*{}_{A,B}} & \underline{\mathbf{Alg}^H}(\mu^* \circ \nu^*(A), \mu^* \circ \nu^*(B)) \\ & {\scriptstyle \mu_{\mathsf{i}}(\nu^*{}_{A,B})} \searrow \quad \nearrow {\scriptstyle \mu^*{}_{\nu^*(A),\nu^*(B)}} & \\ & \mu_{\mathsf{i}}(\underline{\mathbf{Alg}^G}(\nu^*(A), \nu^*(B))) & \end{array}$$

commutes. This is the case, since by definition all coalgebra morphisms involved are actually morphisms of measurings in the category of measurings from $\mu^* \circ \nu^*(A)$ to $\mu^* \circ \nu^*(B)$. Since $\underline{\mathbf{Alg}^H}(\mu^* \circ \nu^*(A), \mu^* \circ \nu^*(B))$ is the terminal object in this category, the morphisms must coincide. We conclude the functor respects composition. □

4.4 Examples of enriched functors

Now that we have all the general theory down, it is time to consider some examples to see it come to life. As in Section 3, we will be more elaborate in some examples than others, but will always provide the necessary ingredients to go into more detail using the general theory if the reader wishes to do so. Throughout this section we will only be considering lax monoidal endofunctors on the (closed, symmetric) monoidal category $(\mathbf{Set}, \times, 1)$, all of which we have seen in Section 3. All the natural transformations in this section are lax monoidal as well, even though will not be giving the lax monoidal structures. We will be consistent with notation between examples and denote the functor corresponding to the natural numbers type by

$$\begin{aligned} F : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1 + A, \end{aligned}$$

the functor corresponding to the list type by

$$\begin{aligned} G : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1 + X \times A, \end{aligned}$$

the functor corresponding to the binary tree type by

$$\begin{aligned} H : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1 + X \times X \times A \end{aligned}$$

and the functor corresponding to the unbounded tree type by

$$\begin{aligned} J : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1 + X \times A^{\infty}, \end{aligned}$$

where $(X, \bullet, e)$ is a fixed monoid. In general, we will denote monomorphic natural transformations by μ, and epimorphic natural transformations by ν.

4.4.1 Examples of embedding measurings

Our first example is the motivating example for the theory found in Section 4.2. We've already remarked the natural numbers can be viewed as a special case of lists when we only consider lists with one element.

Example 4.4.1. Take the functor corresponding to the natural numbers type $F : A \mapsto 1+A$ and the functor corresponding to the list type $G : A \mapsto 1 + X \times A$. Consider the natural transformations

$$\begin{aligned} \mu &: F \to G \\ \mu_A &: 1 + A \to 1 + X \times A \\ * &\mapsto * \\ a &\mapsto (e, a) \end{aligned}$$

and

$$\begin{aligned} \nu &: G \to F \\ \nu_A &: 1 + X \times A \to 1 + A \\ * &\mapsto * \\ (x, a) &\mapsto a. \end{aligned}$$

We verify check $\nu \circ \mu = \mathrm{id}_F$, and hence we can apply Theorem 4.2.5.

This gives us a family of functions

$$\Phi : \mathfrak{m}^F_C(A, B) \to \mathfrak{m}^G_{\mu_*(C)}(\nu^*(A), \nu^*(B))$$

and we would wish to explicitly check this is well-defined. Given an F-measuring $\varphi : C \times A \to B$, we know it must satisfy

1. $\varphi(c)(0_A) = 0_B$ for all $c \in C$
2. $\varphi(c)(\alpha(a)) = 0_B$ if $\chi(c) = *$
3. $\varphi(c)(\alpha(a)) = \beta(\varphi(c')(a))$ if $\chi(c) = c'$.

We check this is also a G-measuring $\varphi : \mu_*(C) \times \nu^*(A) \to \nu^*(B)$ by verifying

1. $\varphi(c)(e_A) = \varphi(c)(0_A) = 0_B = e_B$ for all $c \in C$
2. $\varphi(c)(\alpha \circ \nu_A(x, a)) = \varphi(c)(\alpha(a)) = 0_B = e_B$ if $\mu_C \circ \chi(c) = \chi(c) = *$
3. $\varphi(c)(\alpha \circ \nu_A(x, a)) = \varphi(c)(\alpha(a)) = \beta(\varphi(c')(a)) = \beta(x_0 \bullet x, \varphi(c')(a))$ if $\mu_C \circ \chi(c) = (x_0, c')$.

and are relieved to see our theorem hold in practice as well. △

Another observation is that lists can be viewed as trees where each node has at most one child. In the next example we explore this idea.

Example 4.4.2. Take the functor corresponding to the list type $G : A \mapsto 1 + X \times A$ and the functor corresponding to the (unbounded) tree type $J : A \mapsto 1 + X \times A^*_\infty$ and consider the monoidal natural transformations

$$\begin{aligned} \mu &: G \to J \\ \mu_A &: 1 + X \times A \to 1 + X \times A^*_\infty \\ * &\mapsto * \\ (x, a) &\mapsto (x, [a, a, \dots]) \end{aligned}$$

and

$$\begin{aligned} \nu &: J \to G \\ \nu_A &: 1 + X \times A^*_\infty \to 1 + X \times A \\ * &\mapsto * \\ (x, [\,]) &\mapsto * \\ (x, a : as) &\mapsto (x, a). \end{aligned}$$

We can check $\nu \circ \mu = \mathrm{id}_G$, so indeed we have a section μ.

To get an idea of what the pullback functors ν^* and μ^* do, we observe their behavior on the initial algebras, the set containing all lists X^* and the set containing all unbounded trees S_X respectively. The functor μ^* maps the initial algebra $\alpha : 1 + X \times {S_X}^*_\infty \to S_X$ to the algebra

$$\begin{aligned} \alpha \circ \mu_{S_X} : 1 + X \times S_X &\to S_X \\ * &\mapsto \emptyset \\ (x, t) &\mapsto (x, [t]). \end{aligned}$$

This also gives us a map from the initial object

$$\begin{aligned} \mathsf{i}_{S_X} : X^* &\to S_X \\ [\,] &\mapsto \emptyset \\ x : xs &\mapsto (x, \mathsf{i}_{X^*}(xs)). \end{aligned}$$

which takes a list and converts it into a tree with only one branch. The functor ν^* maps the initial algebra $\beta : 1 + X \times X^* \to X^*$ to the algebra

$$\begin{aligned} \beta \circ \nu_{X^*} : 1 + X \times X^*_{(\infty)} &\to X^* \\ * &\mapsto [\,] \\ (x, [\,]) &\mapsto [\,] \\ (x, l : ls) &\mapsto x : l. \end{aligned}$$

This also gives us a map from the initial object

$$\begin{aligned} \mathsf{i}_{X^*} : S_X &\to X^* \\ \emptyset &\mapsto [\,] \\ (x, [\,]) &\mapsto [\,] \\ (x, t : ts) &\mapsto x : \mathsf{i}_{S_X}(t). \end{aligned}$$

which takes the leftmost branch of a tree and converts it to a list. Notice the subtlety of the difference between the image of $(x, [\,])$ and $(x, [\emptyset])$. The former gets mapped to the empty list, discarding the information in x, where the latter gets mapped to the list containing only x. Also notice the choice of ν heavily influences how the tree gets converted into a list. If we chose ν to map elements of the form $(x, a : as)$ to the the pair (x, a'), where a' is the last element of the list $a : as$, we obtain the rightmost branch of the tree as a list. One could also write a recursive definition of ν to obtain the longest branch of the tree, or to obtain the entire tree flattened in a certain order. All these options would still be retractions of μ, so the theory would still hold.

By the above theory, we also have that G-measurings are also J-measurings when pulling back along ν. Notice however that the converse does not hold since $\nabla^J_{C,A}$ does not coequalize $\mathrm{id} \otimes \mu_A$ and $(\mu_C \otimes \mu_A) \circ (\nu_C \otimes \mathrm{id})$ However, if let $A^+ = A^* \setminus \{[\,]\}$ and modify J to

$$\begin{aligned} \tilde{J} : \mathbf{Set} &\to \mathbf{Set} \\ A &\mapsto 1 + X \times A^+ \end{aligned}$$

and ν to

$$\begin{aligned} \tilde{\nu}_A : 1 + X \times A^+ &\to 1 + X \times A \\ * &\mapsto * \\ (x, a : as) &\mapsto (x, a). \end{aligned}$$

does make $\nabla^{\tilde{J}}_{C,A}$ coequalize $\mathrm{id} \otimes \mu_A$ and $(\mu_C \otimes \mu_A) \circ (\tilde{\nu}_C \otimes \mathrm{id})$. So, every $\tilde{J}$-measuring can also be transferred to an G-measuring. The reason it does work for $\tilde{J}$ and not for J is due to what is discussed above about the subtle difference between the image of $(x, [\,])$ and $(x, [\emptyset])$. After modifying J to $\tilde{J}$, we see the case $(x, [\,])$ no longer occurs since we ask the list to always be non-empty. △

4.4.2 Examples of pushing forward measurings

In many of our examples some sort of monoidal structure is involved. For the first example of this section we would like to explore what happens when we compare different monoidal structures.

Example 4.4.3. Consider the lax monoidal functor $\mathrm{const}_X : \mathbf{Set} \to \mathbf{Set}, A \mapsto X$ for some fixed monoid $(X, \bullet, e)$. A lax monoidal natural transformation $\mu : \mathrm{const}_X \to \mathrm{const}_{X'}$ corresponds exactly to a monoid homomorphism $X \to X'$. The left adjoint to pullback functor, $\mu_! : \mathbf{Alg}^X \to \mathbf{Alg}^{X'}$, is given by

$$\mu_!(A) = A + X'/\alpha(x) \sim \mu(x).$$

The family of functions $\Phi : \mathfrak{m}_C^X(A, B) \to \mathfrak{m}_{\mu_*(C)}^{X'}(\mu_!(A), \mu_!(B))$ is given by sending a measuring $\varphi \in \mathfrak{m}_C^X(A, B)$ to

$$\Phi(\varphi)(c, [a]) = [\varphi(c, a)]$$
$$\Phi(\varphi)(c, [x']) = [\mu_C \circ \chi(C) \bullet x'].$$

△

For our next example we will again be comparing lists to trees. This is a nice motivating example and we will be quite elaborate.

Example 4.4.4. Consider endofunctors $F : A \mapsto 1 + A$ and $G : A \mapsto 1 + X \times A$. The category $\mathbf{Alg}^F$ has $\mathbb{N}$ as initial algebra, and $\mathbf{Alg}^G$ can be thought of as a category of "list-likes" and has X^* as initial algebra. We also have a monoidal natural transformation

$$\mu : F \to G$$
$$\mu_A : 1 + A \to 1 + X \times A$$
$$* \mapsto *$$
$$a \mapsto (e, a).$$

Notice μ is monoidal since we choose to map $a \in A$ to $(e, a) \in X \times A$. In general, mapping to any idempotent element of X will suffice.

The left adjoint of $\mu^* : \mathbf{Alg}^G \to \mathbf{Alg}^F$ is denoted as $\mu_! : \mathbf{Alg}^F \to \mathbf{Alg}^G$ and given by

$$\mu_!(A) = (X^* \times A)/\sim,$$

where the equivalence relation $\sim$ is generated by $(xs, \alpha(a)) \sim (xs +\!+ [e], a)$. Its algebra structure is given by

$$\mu_!(\alpha) : 1 + X \times (X^* \times A)/\sim \to (X^* \times A)/\sim$$
$$* \mapsto [[\,], \alpha(*)]$$
$$(x, [xs, a]) \mapsto [x : xs, a].$$

On morphisms $\mu_!$ acts by sending an algebra morphism $f : A \to A'$ to $(\mathrm{id}_{X^*} \times f)/\sim$. Given an algebra morphism $f : A \to \mu^*(B)$, its transpose is given by

$$\tilde{f}([\,], a) = f(a)$$
$$\tilde{f}(x : xs, a) = \beta(x, \tilde{f}(xs, a)).$$

Conversely, given $\tilde{f} : \mu_!(A) \to B$, its transpose is

$$f(a) = \tilde{f}([[\,], a]).$$

We need the unit and counit of the adjunction to compute Φ, and these are given by

$$\eta_A : A \to (\mu^* \circ \mu_!)(A)$$
$$a \mapsto [[\,], a]$$

and

$$\begin{aligned} \varepsilon_B : (\mu_! \circ \mu^*)(B) &\to B \\ [[\,], b] &\mapsto b \\ [x : xs, b] &\mapsto \beta(x, \varepsilon_B([xs, b])). \end{aligned}$$

Using the above, we can compute the functions $\Phi_{A,B,C} : \mathfrak{m}^F_C(A, B) \to \mathfrak{m}^G_{\mu_*(C)}(\mu_!(A), \mu_!(B))$ to be given by

$$\begin{aligned} \Phi_{A,B,C}(\varphi) : \mu_*(C) \times \mu_!(A) &\to \mu_!(B) \\ (c, [xs, a]) &\mapsto \begin{cases} [xs, \varphi(\chi^{\text{len}(xs)}(c), a)] & \text{if } [\![c]\!] > \text{len}(xs) \\ [\text{take}([\![c]\!])(xs), \beta(*)] & \text{otherwise} \end{cases} \end{aligned}$$

where $[\![c]\!]$ is the smallest integer $n \in \mathbb{N}$ such that $\chi^n(c) = *$, also called the *index* of $c \in C$.

Before we consider an example of what Φ does with a measuring, we first examine what $\mu_!$ does on (pre)initial objects. Since $\mu_!$ is a left adjoint, it preserves initial objects. Indeed, $((X^* \times \mathbb{N})/\sim) \cong X^*$ by the isomorphism

$$[xs, j] \mapsto xs + +e^j,$$

where by e^j we mean the list of length j containing only the unit e. All preinitial algebras in $\mathbf{Alg}^F$ are of the form $\mathbb{n}$, and under $\mu_!$ these are mapped to

$$(X^* \times \mathbb{n})/[xs, \alpha_n(i)] \sim [xs + + [e], i].$$

As a consequence we have that

$$[xs, n] = [xs + + e^j, n]$$

for all $j \in \mathbb{N}$. Hence, one can think of $\mu_!(\mathbb{n})$ as the lists in X^* which contain at most n consecutive copies of e at the beginning of the list. As an example of what happens to a measuring we consider

$$\begin{aligned} \min : \mathbb{n}^\circ \times \mathbb{n} &\to \mathbb{N} \\ (i, j) &\mapsto \min(i, j). \end{aligned}$$

Under Φ, this is mapped to

$$\begin{aligned} \Phi(\min) : \mathbb{n}^\circ \times (X^* \times \mathbb{n})/\sim &\to \mathbb{N} \\ (i, [xs, j]) &\mapsto \text{take}(i)(xs + + e^j). \end{aligned}$$

△

Of course, we could also consider a natural transformation going the other way, which will be the next example.

Example 4.4.5. Consider the functors F, G from the previous Example 4.4.4 and the natural transformation

$$\begin{aligned} \nu : G &\to F \\ \nu_A : 1 + X \times A &\to 1 + A \\ * &\mapsto * \\ (x, a) &\mapsto a. \end{aligned}$$

The left adjoint of $\nu^* : \mathbf{Alg}^F \to \mathbf{Alg}^G$ is denoted as $\nu_! : \mathbf{Alg}^G \to \mathbf{Alg}^F$ and given by

$$\nu_!(A) = A/\alpha(x, a) \sim \alpha(x', a)$$

for all $x, x' \in X$ and $a \in A$. Its algebra structure is given by

$$\begin{aligned} \nu_!(\alpha) : 1 + A/\sim &\to A/\sim \\ * &\mapsto [\alpha(*)] \\ [a] &\mapsto [\alpha(x, a)] \end{aligned}$$

for some $x \in X$, which is well-defined by definition. Intuitively, $\nu_!(A)$ simply forgets specifically which element $x \in X$ we are appending and is only concerned with the fact that we are appending in the first place. The unit and counit of the adjunction are given by

$$\begin{aligned} \eta_A : A &\to (\nu^* \circ \nu_!)(A) \\ a &\mapsto [a] \end{aligned}$$

and

$$\begin{aligned} \varepsilon_B : (\nu_! \circ \nu^*)(B) &\to B \\ [b] &\mapsto b \end{aligned}$$

respectively.

The maps $\Phi_{A,B,C} : \mathfrak{m}^F_C(A, B) \to \mathfrak{m}^G_{\nu_*(C)}(\nu_!(A), \nu_!(B))$ are given by

$$\begin{aligned} \Phi_{A,B,C}(\varphi) : \nu_*(C) \times \nu_!(A) &\to \nu_!(B) \\ (c, [a]) &\mapsto [\varphi(c, a)]. \end{aligned}$$

Verifying $\Phi(\varphi)$ is a measuring boils down to the observation that which element $x \in X$ we are adding is irrelevant.

We observe that in this example the behavior of $\nu_!$ and Φ reflects what the natural transformation ν is doing; stripping away any and all information regarding X. △

By now we have a good idea of the behavior of the monoidal structure under a transformation of measurings. For our next examples we will be considering what happens when going from lists to trees, adding a branching structure to the objects we are considering.

Example 4.4.6. Consider the functors $G : A \mapsto 1 + X \times A$ and $H : A \mapsto 1 + X \times A \times A$ corresponding to the list type and the binary tree type respectively. We define the lax monoidal natural transformation

$$\begin{aligned} \mu : G &\to H \\ \mu_A : 1 + X \times A &\to 1 + X \times A \times A \\ * &\mapsto * \\ (x, a) &\mapsto (x, a, a). \end{aligned}$$

The left adjoint to pullback functor, $\mu_! : \mathbf{Alg}^G \to \mathbf{Alg}^H$, is given by

$$\mu_!(A) = ((X \times \mu_!(A) \times \mu_!(A)) + A) \,/ \alpha(x, a) \sim (x, a, a).$$

The family of functions $\Phi : \mathfrak{m}^G_C(A, B) \to \mathfrak{m}^H_{\mu_*(C)}(\mu_!(A), \mu_!(B))$ sends a measuring $\varphi \in \mathfrak{m}^G_C(A, B)$ to the measuring

$$\begin{aligned} \Phi(\varphi)(c, [a]) &= \varphi(c, a) \\ \Phi(\varphi)(c, [x, \ell, r]) &= \begin{cases} [\beta(*)] & \text{if } \chi(c) = * \\ [x' \bullet x, \Phi(\varphi)(c', \ell), \Phi(\varphi)(c', r)] & \text{if } \chi(c) = (x', c'). \end{cases} \end{aligned}$$

△

Of course, we could also consider a natural transformation going the other way, which will be the next example.

Example 4.4.7. Again, consider the functors $G : A \mapsto 1 + X \times A$ and $H : A \mapsto 1 + X \times A \times A$ and define the lax monoidal natural transformation

$$\begin{aligned} \nu : H &\to G \\ \nu_A : 1 + X \times A \times A &\to 1 + X \times A \\ * &\mapsto * \\ (x, \ell, r) &\mapsto (x, \ell). \end{aligned}$$

The left adjoint to pullback functor, $\nu_! : \mathbf{Alg}^H \to \mathbf{Alg}^G$, is given by

$$\nu_!(A) = A/\alpha(x, \ell, r) \sim \alpha(x, \ell, r')$$

for all $r, r' \in A$. The family of function $\Phi : \mathfrak{m}^H_C(A, B) \to \mathfrak{m}^G_{\nu_*(C)}(\nu_!(A), \nu_!(B))$ is given by sending a measuring $\varphi \in \mathfrak{m}^H_C(A, B)$ to

$$\Phi(\varphi)(c, [a]) = [\varphi(c, a)].$$

△

Example 4.4.8. Consider the functors $G : A \mapsto 1 + X \times A$ and $J : A \mapsto 1 + X \times A^*_\infty$. We define the lax monoidal natural transformation

$$\begin{aligned} \mu : G &\to J \\ \mu_A : 1 + X \times A &\to 1 + X \times A^*_\infty \\ * &\mapsto * \\ (x, a) &\mapsto (x, a^\infty), \end{aligned}$$

where by $a^\infty = [a, a, \ldots]$ we mean an infinite list containing only copies of a. The left adjoint to pullback functor, $\mu_! : \mathbf{Alg}^G \to \mathbf{Alg}^J$, is given by

$$\mu_!(A) = (X \times \mu_!(A)^*_\infty) + A/\alpha(x, a) \sim (x, a^\infty).$$

The function $\Phi : \mathfrak{m}^G_C(A, B) \to \mathfrak{m}^J_{\mu_*(C)}(\mu_!(A), \mu_!(B))$ sends a measuring $\varphi \in \mathfrak{m}^G_C(A, B)$ to

$$\Phi(\varphi)(c, [a]) = \varphi(c, a)$$

$$\Phi(\varphi)(c, [x, as]) = \begin{cases} [\beta(*)] & \text{if } \chi(c) = * \\ [x' \bullet x, \mathrm{map}(\Phi(\varphi)(c', \_))(as)] & \text{if } \chi(c) = (x', c'). \end{cases}$$

△

We can toy a bit with the previous example, modifying it slightly to observe the differences.

Example 4.4.9. Consider the functors $G : A \mapsto 1 + X \times A$ and $J : A \mapsto 1 + X \times A^*_\infty$ from the previous example, and define the lax monoidal natural transformation

$$\begin{aligned} \mu : G &\to J \\ \mu_A : 1 + X \times A &\to 1 + X \times A^*_\infty \\ * &\mapsto * \\ (x, a) &\mapsto (x, a^n). \end{aligned}$$

for some $n \in \mathbb{N}$, where $a^n = [a, a, \ldots, a]$ we mean a list containing n copies of a. Note this example is similar to the previous one, but we allow ourselves a parameter $n \in \mathbb{N}$ controlling the length of list, instead of making it infinite. The left adjoint to pullback functor, $\mu_! : \mathbf{Alg}^G \to \mathbf{Alg}^J$, is given by

$$\mu_!(A) = ((X \times \mu_!(A)^*_\infty) + A)/\alpha(x, a) \sim (x, a^n).$$

Note the notation is a bit sloppy since we are disregarding the equivalence classes to an extent when noting $(x, a^n) \in \mu_!(A)^*_\infty$. The function $\Phi : \mathfrak{m}^G_C(A, B) \to \mathfrak{m}^J_{\mu_*(C)}(\mu_!(A), \mu_!(B))$ is given by sending a measuring $\varphi \in \mathfrak{m}^G_C(A, B)$ to

$$\Phi(\varphi)(c, [a]) = \varphi(c, a)$$

$$\Phi(\varphi)(c, [x, as]) = \begin{cases} [\beta(*)] & \text{if } \chi(c) = * \\ [x' \bullet x, \mathrm{map}(\Phi(\varphi)(c', \_))(\mathrm{take}(n)(as))] & \text{if } \chi(c) = (x', c'). \end{cases}$$

△

We would also like to consider a natural transformation going in the opposite direction.

Example 4.4.10. Again, consider the functors $G : A \mapsto 1 + X \times A$ and $J : A \mapsto 1 + X \times A^*_\infty$ from the previous examples. We define the natural transformation

$$\begin{aligned} \nu : J &\to G \\ \nu_A : 1 + X \times A^*_\infty &\to 1 + X \times A \\ * &\mapsto * \\ (x, [\,]) &\mapsto (x, \alpha(*)) \\ (x, a : as) &\mapsto (x, a). \end{aligned}$$

The left adjoint to pullback functor, $\nu_! : \mathbf{Alg}^J \to \mathbf{Alg}^G$, is given by

$$\nu_!(A) = A/\sim,$$

where $\sim$ is generated by

$$\begin{aligned} \alpha(x, a : as) &\sim \alpha(x, a : as') \\ \alpha(x, [\,]) &\sim \alpha(x, [\alpha(*)]) \end{aligned}$$

for all $x \in X$, $as, as' \in A^*_\infty$. The function $\Phi : \mathfrak{m}^J_C(A, B) \to \mathfrak{m}^G_{\nu_*(C)}(\nu_!(A), \nu_!(B))$ is given by

$$\Phi(\varphi)(c, [a]) = [\varphi(c, a)].$$

△

4.4.3 Examples of pulling back measurings

To conclude this section we give some examples concerning the final part of the general theory; namely that of pulling back measurings. We again start with an example concerning the monoidal structures involved.

Example 4.4.11. Consider the lax monoidal functor $\mathrm{const}_X : \mathbf{Set} \to \mathbf{Set}, A \mapsto X$ for some fixed monoid X. A lax monoidal natural transformation $\mu : \mathrm{const}_X \to \mathrm{const}_{X'}$ corresponds exactly to a monoid homomorphism $X \to X'$. The right adjoint to pushforward functor, $\mu_¡ : \mathbf{CoAlg}^{X'} \to \mathbf{CoAlg}^X$, is given by

$$\mu_¡(C) = \{(c, x) \in C \times X \mid \chi(c) = \mu(x)\}.$$

The family of function $\Phi : \mathfrak{m}^{X'}_C(A, B) \to \mathfrak{m}^{X'}_{\mu_¡(C)}(\mu^*(A), \mu^*(B))$ is given by sending a measuring $\varphi \in \mathfrak{m}^{X'}_C(A, B)$ to

$$\Phi(\varphi)(c, x, a) = \varphi(c, a).$$

△

Example 4.4.12. Again, consider $F : A \mapsto 1 + A$ and $G : A \mapsto 1 + X \times A$ and the natural transformation $\mu : F \to G$ from Example 4.4.4 defined by $\mu_A : a \mapsto (e, a)$. The right adjoint of $\mu_* : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$ is given by $\mu_¡ : \mathbf{CoAlg}^G \to \mathbf{CoAlg}^F$, where

$$\mu_¡(C) = \{c \in C \mid \chi(c) = * \text{ or } \chi(c) = (e, c'), c' \in \mu_¡(C)\}.$$

Note how $\mu_¡(C) \subseteq C$. The algebra structure is inherited from C. Transposing morphisms is done by restriction and inclusion. The family of functions Φ is also given by restriction, namely

$$\begin{aligned} \Phi_{A,B,C} : \mathfrak{m}^G_C(A, B) &\to \mathfrak{m}^F_{\mu_¡(C)}(\mu^*(A), \mu^*(B)) \\ \varphi &\mapsto \varphi|_{\mu_¡(C) \times A}. \end{aligned}$$

△

The previous example demonstrates that we can easily restrict a measuring under the right circumstances. For a slightly more involved example, consider the following.

Example 4.4.13. Consider $F : A \mapsto 1 + A$ and $G : A \mapsto 1 + X \times A$ and the natural transformation $\nu : G \to F$ from Example 4.4.5 defined by $\nu_A : (x, a) \mapsto a$. The right adjoint of $\nu_* : \mathbf{CoAlg}^G \to \mathbf{CoAlg}^F$ is given by $\nu_{\mathsf{i}} : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$, where

$$\nu_{\mathsf{i}}(C) = \{(c, xs) \in C \times X^\infty \mid [\![c]\!] = \mathrm{len}(xs)\}$$

with coalgebra structure

$$\begin{aligned} \nu_{\mathsf{i}}(\chi) : \nu_{\mathsf{i}}(C) &\to 1 + X \times \nu_{\mathsf{i}}(C) \\ (c, xs) &\mapsto \begin{cases} * & \text{if } [\![c]\!] = \mathrm{len}(xs) = 0 \\ (x, (\chi(c), xs')) & \text{if } xs = x : xs'. \end{cases} \end{aligned}$$

Recall $[\![c]\!]$ is the index of c as defined in Example 4.4.4. Also note $\nu_{\mathsf{i}}(C) \subseteq C \times X^\infty$.

The family of functions Φ is given by projection onto the first coordinate, namely

$$\begin{aligned} \Phi_{A,B,C} : \mathfrak{m}^F_C(A, B) &\to \mathfrak{m}^G_{\nu_{\mathsf{i}}(C)}(\nu^*(A), \nu^*(B)) \\ \varphi &\mapsto \varphi \circ (\mathrm{pr}_C \times \mathrm{id}_A). \end{aligned}$$

△

As a final example we consider what happens when we add a branching structure to our objects, by comparing the functor corresponding to the list type with the functor corresponding to the binary tree type.

Example 4.4.14. Consider $G : A \mapsto 1 + X \times A$ and $H : A \mapsto 1 + X \times A \times A$. We define the natural transformation

$$\begin{aligned} \mu : G &\to H \\ \mu_A : 1 + X \times A &\to 1 + X \times A \times A \\ * &\mapsto * \\ (x, a) &\mapsto (x, a, a). \end{aligned}$$

The right adjoint to pushforward functor, $\mu_{\mathsf{i}} : \mathbf{CoAlg}^H \to \mathbf{CoAlg}^G$, is given by

$$\mu_{\mathsf{i}}(C) = \{c \in C \mid \chi(c) = (x, c', c'), c' \in \mu_{\mathsf{i}}(C) \text{ or } \chi(c) = *\}.$$

Notice that $\mu_{\mathsf{i}}(C) \subseteq C$. The family of functions $\Phi : \mathfrak{m}^H_C(A, B) \to \mathfrak{m}^G_{\mu_{\mathsf{i}}(C)}(\mu^*(A), \mu^*(B))$ is given by restriction of $\varphi \in \mathfrak{m}^H_C(A, B)$ to $\mu_{\mathsf{i}}(C) \subseteq C$. △

All these examples help us build an understanding of the interplay between algebras of different endofunctors. One important part of the story that we have not been considering yet is the C-initial algebra. We have not seen how these objects fare under transformation of measurings, and that will be the contents of the next section.

5 C-initial algebras and enriched functors

In the previous section we have seen that given a lax monoidal natural transformation $\mu : F \to G$, we obtain two morphisms in **EnrCat**, namely $(\mu_!, \mu_*) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ and $(\mu^*, \mu_{¡}) : (\mathbf{Alg}^G, \mathbf{CoAlg}^G) \to (\mathbf{Alg}^F, \mathbf{CoAlg}^F)$. In this section we wish to observe their behavior when interacting with C-initial algebras. Since $\mu_!$ and μ_* are both left adjoint, one might expect $(\mu_!, \mu_*)$ to preserve C-initial algebras, and we will see this is indeed the case. Sadly, this does not hold for $(\mu^*, \mu_{¡})$.

In Section 5.1 we kick off by stating some insightful results regarding C-initial algebras. Not all of these results are immediately relevant, but they do help drive our intuition about C-initial algebras. Following that, we will see under which conditions a functor $(\rho, \pi) \in \mathbf{EnrCat}$ preserves C-initial algebras and apply that to $(\mu_!, \mu_*)$. Finally, in Section 5.3 we will provide some results on terminal C-initial algebras and use them to compute the terminal C-initial algebras promised in Section 3.

5.1 Some insightful results

In this section we will be giving some results regarding C-initial algebras in general. These results will not be used in this thesis, but might prove useful in future research. Some of the results have already been stated in [1] in a different form, but we would like to state them here for completeness. We start off with an observation made when computing C-initial algebras.

Remark 5.1.1. Given a C-initial algebra A and the initial algebra I, it suffices to compute $\varphi : C \otimes A \to I$ to obtain all relevant information. This is since every other measuring $\psi : C \otimes A \to B$ uniquely factors as

$$\begin{array}{ccc} C \otimes A & \xrightarrow{\varphi} & I \\ & \searrow^{\psi} & \downarrow{¡_B} \\ & & B. \end{array}$$

In a sense, φ totally captures the behavior of A as a C-initial algebra.

What follows are a series of lemmas regarding initial, preinitial, terminal and subterminal objects.

Lemma 5.1.2. *The initial object I is C-initial for all coalgebras C.*

Proof. The algebra I is C-initial whenever $m_C(I, B) \cong 1$ for all $B \in \mathbf{Alg}$. We note

$$\mathfrak{m}_C(I, B) \cong \mathbf{Alg}(I, [C, B]) \cong 1$$

since I is the initial object in the category of algebras and conclude our result. □

As an immediate consequence we have the following.

Corollary 5.1.3. *For all coalgebras C, we have that $C \triangleright I \cong I$.*

Using the above, we obtain the following lemma.

Lemma 5.1.4. *A any C-initial algebra A is also $D \otimes C$-initial.*

Proof. We aim to show $(D \otimes C) \triangleright A \cong I$, where I is the initial algebra. Since A is C-initial, we know $C \triangleright A$. Using Lemma 4.3.10 and the previous corollary we compute

$$(D \otimes C) \triangleright A \cong D \triangleright (C \triangleright A) \cong D \triangleright I \cong I,$$

and we conclude A is also $D \otimes C$-initial. □

Remark 5.1.5. Given a C-initial algebra A, we can compose the unique measuring $C \otimes A \to A'$ with any other measuring $D \otimes A' \to A''$ to yield a measuring

$$D \otimes C \otimes A \to A''.$$

By the previous result this measuring is unique.

Lemma 5.1.6. *Let I be the initial algebra. For all $B \in \mathbf{Alg}$, $\underline{\mathbf{Alg}}(I, B) \cong T$, where T is the terminal coalgebra.*

Proof. We verify $\underline{\mathbf{Alg}}(I, B)$ has the universal property of a terminal object by computing

$$\mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(I, B)) \cong \mathbf{Alg}(I, [C, B]) \cong 1,$$

and we conclude our result. □

Lemma 5.1.7. *Let $A, B \in \mathbf{Alg}$. Any measuring by the terminal coalgebra T from A to B gives a total algebra homomorphism from A to B.*

Proof. Recall a measuring by the monoidal unit $\mathbb{I}$ is equivalent to a total algebra morphism. Consider the unique map $!_{\mathbb{I}} : \mathbb{I} \to T$. This yields a function

$$\mathfrak{m}_T(A, B) \to \mathfrak{m}_{\mathbb{I}}(A, B) \cong \mathbf{Alg}(A, B),$$

which gives the desired result. □

In Section 3 we have paid a lot of attention to preinitial algebras when computing C-initial algebras. Using the isomorphisms regarding the representing objects of $\mathfrak{m}_C(A, B)$, we can give some general results regarding preinitial algebras and C-initial algebras as well.

Lemma 5.1.8. *Given algebras $P, B \in \mathbf{Alg}$ such that P is preinitial, the coalgebra $\underline{\mathbf{Alg}}(P, B)$ is subterminal.*

Proof. An algebra P is preinitial if and only if $\mathbf{Alg}(P, B)$ contains at most one element. Dually, a coalgebra S is subterminal if $\mathbf{CoAlg}(C, S)$ contains at most one element. Since we have the isomorphism

$$\mathbf{CoAlg}(C, \underline{\mathbf{Alg}}(P, B)) \cong \mathbf{Alg}(P, [C, B])$$

and P is preinitial, we conclude our result. □

Corollary 5.1.9. *Given a preinitial algebra $P \in \mathbf{Alg}$, its dual coalgebra P° is subterminal.*

Proof. The dual coalgebra P° is defined as $\underline{\mathbf{Alg}}(P, I)$, where I is the initial algebra. By Lemma 5.1.8 we conclude our result. □

Next up is a powerful general result which allows us to generate a large number of C-initial algebras for specific coalgebras C.

Proposition 5.1.10. *A preinitial algebra $P \in \mathbf{Alg}$ is P°-initial.*

Proof. The aim is to show $\mathfrak{m}_{P^\circ}(P, B) \cong 1$ for all $B \in \mathbf{Alg}$. Our first remark is that there exists at most one measuring from P to B by P°, since

$$\mathfrak{m}_{P^\circ}(P, B) \cong \mathbf{Alg}(P, [P^\circ, B])$$

and P is preinitial, hence has at most one morphism out of it. Second, since P° is subterminal by the previous lemma, $\mathbf{CoAlg}(P^\circ, P^\circ) \cong 1$. Moreover, we have the identification

$$1 \cong \mathbf{CoAlg}(P^\circ, P^\circ) \cong \mathbf{Alg}(P, (P^\circ)^*) = \mathbf{Alg}(P, [P^\circ, I]) \cong \mathfrak{m}_{P^\circ}(P, I),$$

where I is the initial algebra. Postcomposing the unique measuring $\varphi \in \mathfrak{m}_{P^\circ}(P, I)$ with the morphism $\mathfrak{i}_B : I \to B$ yields a measuring

$$\mathfrak{i}_B \circ \varphi \in \mathfrak{m}_{P^\circ}(P, B).$$

We have shown there exists a unique measuring from P to B by P° for all $B \in \mathbf{Alg}$ and conclude P is P° initial. □

As remarked before, the power of an initial algebra I is that for any other algebra B there exists a unique morphism $I \to B$. For any preinitial algebra P we know there exists at most one algebra morphism $P \to B$, but there are no guarantees of it existing. The above result tells us we can circumvent this disadvantage by not considering algebra morphisms $P \to B$, but instead measurings $P^\circ \otimes P \to B$. This gives the advantages of an initial algebra to a much broader class of algebras, namely all preinitial algebras. We can easily compute preinitial algebras using Corollary 2.1.17, which makes this a powerful and practical result.

We have seen cases where P is also the terminal P°-initial algebra, for example when taking $P = \mathbb{n}$ as seen in Theorem 3.4.11. One might wonder if this is the case in general.

Proposition 5.1.11. *In general, given a preinitial algebra $P \in \mathbf{Alg}$, P is not the terminal P°-initial algebra.*

Proof. We give a proof by counterexample. In Section 3.5.6 we have seen that the coalgebra $(X')^{*\circ} \cong \emptyset$. Every algebra A is $\emptyset$-initial since there exists a unique measuring $\varphi : \emptyset \times A \cong \emptyset \to B$ for all $B \in \mathbf{Alg}$. The terminal $\emptyset$-initial algebra is hence given by the terminal algebra $1 + X \to 1$, which is unequal to $(X')^*$. We conclude our result. □

5.2 C-initial algebra preserving functors

The main goal of this section is to show $(\mu_!, \mu_*)$ preserves C-initial algebras. In order to do so, we give a characterization of C-initial algebra preserving functors (ρ, π).

Proposition 5.2.1. *An enriched functor*

$$(\rho, \pi) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

preserves C-initial algebras if ρ preserves initial objects and for any C-initial algebra A we have $\pi(C) \triangleright \rho(A) \cong \rho(C \triangleright A)$. Moreover, if π is a strong monoidal functor the converse holds as well.

Proof. First we show that if ρ preserves initial objects and if for any C-initial algebra A we have $\pi(C) \triangleright \rho(A) \cong \rho(C \triangleright A)$ that then (ρ, π) preserves C-initial algebras. Let A be C-initial. We claim $\rho(A)$ is $\pi(C)$-initial. Since A is C-initial $C \triangleright A \cong I$. Since ρ preserves initial objects we know $\rho(C \triangleright A) \cong \pi(C) \triangleright \rho(A)$ is the initial object, hence $\rho(A)$ is $\pi(C)$-initial. We conclude (ρ, π) preserves C-initial algebras.

For the converse, let (ρ, π) be a C-initial algebra preserving functor and assume π is a strong monoidal functor. Recall that a measuring by the monoidal unit $\mathbb{I}$ is equivalent to an algebra morphism. Since the initial algebra I_F is $\mathbb{I}_F$-initial by definition and (ρ, π) preserves C-initial algebras, we have that $\rho(I_F)$ is $\pi(\mathbb{I}_F) \cong \mathbb{I}_G$-initial. Given a G-algebra B we can now compute

$$\mathbf{Alg}(\rho(I_F), B) \cong \mathfrak{m}_{\mathbb{I}_G}(\rho(I_F), B) \cong 1$$

and see $\rho(I_F) \cong I_G$, hence that ρ preserves initial objects. Given a C-initial F-algebra A, we know $\rho(C \triangleright A) \cong I_G$ since ρ preserves initial objects. We also have that

$$\mathbf{Alg}(\pi(C) \triangleright \rho(A), B) \cong \mathfrak{m}_{\pi(C)}(\rho(A), B) \cong 1,$$

for all G-algebras B since (ρ, π) preserve C-initial algebras. This shows $\pi(C) \triangleright \rho(A) \cong I_G \cong \rho(C \triangleright A)$ and we conclude our result. □

With Proposition 5.2.1 at hand, we can now prove the main result.

Theorem 5.2.2. *Given a lax monoidal natural transformation $\mu : F \to G$, the enriched functor $(\mu_!, \mu_*)$: $(\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ preserves C-initial algebras.*

Proof. Since $\mu_!$ is a left adjoint it preserves initial objects. By Lemma 4.3.8 we know $\mu_*(C) \triangleright \mu_!(A) \cong \mu_!(C \triangleright A)$ for all algebras A and coalgebras C. By Proposition 5.2.1 we conclude $(\mu_!, \mu_*)$ preserves C-initial algebras. □

In Section 4.2 we also constructed $(\mu^*, \mu_{¡})$. The next result shows its behavior regarding C-initial algebras and concludes this section.

Proposition 5.2.3. *The enriched functor* $(\mu^*, \mu_¡) : (\mathbf{Alg}^G, \mathbf{CoAlg}^G) \to (\mathbf{Alg}^F, \mathbf{CoAlg}^F)$ *does not preserve* C*-initial algebras.*

Proof. We will give a proof by counterexample. Consider the functor $F : \mathbf{Set} \to \mathbf{Set}, A \mapsto A$ from Section 3.2 and the functor $G : \mathbf{Set} \to \mathbf{Set}, A \mapsto 1$ from Section 3.1. We define the monoidal natural transformation $\mu : F \to G$ by

$$\begin{aligned} \mu_A : A &\to 1 \\ a &\mapsto 1. \end{aligned}$$

Previously we have noted that for F-coalgebras $C \neq \emptyset$ the only C-initial algebra is $(\emptyset, \mathrm{id})$. We have also seen that for G-coalgebras $C \neq \emptyset$ the only C-initial algebra is $(1, \mathrm{id})$. Pulling back the G-algebra $(1, \mathrm{id})$ to the F-algebra $\mu^*(1, \mathrm{id}) = (1, \mathrm{id})$ we see $(\mu^*, \mu_¡)$ does not preserve C-initial algebras. □

5.3 Regarding terminal C-initial algebras

So far we have had a promising start, observing $(\mu_!, \mu_*)$ always preserves C-initial algebras. We are also interested in seeing if the terminal C-initial algebras are also preserved. The conditions under which terminal C-initial algebras are preserved are more restricting, since we ask a functor to preserve something initial and something terminal.

Lemma 5.3.1. *An enriched functor*

$$(\rho, \pi) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$$

satisfying the conditions in Proposition 5.2.1 preserves terminal C*-initial algebras if* ρ *is full, essentially surjective on objects and reflects initial objects.*

Proof. Let $C \in \mathbf{CoAlg}^F$. Given a terminal C-initial algebra T_C, we claim $\rho(T_C)$ is the terminal $\pi(C)$-initial algebra. Let A' be a $\pi(C)$-initial algebra. This means $\pi(C) \triangleright A' \cong I_G$. Since ρ is essentially surjective, we know there exists $A \in \mathbf{Alg}^F$ such that $\rho(A) \cong A'$. This gives us the isomorphisms

$$\rho(C \triangleright A) \cong \pi(C) \triangleright \rho(A) \cong I_G.$$

Since ρ reflects initial objects, $C \triangleright A \cong I_F$, hence A is C-intitial. This means $\mathbf{Alg}^F(A, T_C) \cong 1$, and by ρ being full we have

$$1 \cong \mathbf{Alg}^F(A, T_C) \twoheadrightarrow \mathbf{Alg}^G(\rho(A), \rho(T_C)) \cong \mathbf{Alg}^G(A', \rho(T_C)) \cong 1,$$

hence there exists a unique morphism $A' \to \rho(T_C)$. We conclude $\rho(T_C)$ is the terminal $\pi(C)$-initial algebra. □

Remark 5.3.2. So far there have not been any examples of ρ satisfying these conditions, since none of the encountered functors reflect initial objects.

The above remark also leads us to the following lemma.

Lemma 5.3.3. *In general,* $(\mu_!, \mu_*) : (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \to (\mathbf{Alg}^G, \mathbf{CoAlg}^G)$ *does not preserve terminal* C*-initial algebras.*

Proof. We give a proof by counterexample. Consider the functors $F : A \mapsto 1 + A$ and $J : A \mapsto 1 + X \times A^*_\infty$, as seen in Section 3 and Section 4.4. We have the lax monoidal natural transformation $\mu : F \to J$ given by

$$\begin{aligned} \mu_A : 1 + A &\to 1 + X \times A^*_\infty \\ * &\mapsto * \\ a &\mapsto (e, a^\infty). \end{aligned}$$

The terminal $\mathbb{n}^\circ$-initial F-algebra is given by $\mathbb{n}$ by Theorem 3.4.11. We have also seen that $\S_{X,n}$ is the terminal $\mu_*(\mathbb{n}^\circ)$-initial J-algebra by Corollary 3.7.14. Computing $\mu_!(\mathbb{n})$ we find $\mu_!(\mathbb{n}) \cong (X \times \mu_!(\mathbb{n})) + (X^* \times \mathbb{n})/\sim$, where $\sim$ is generated by $(x : xs, i) \sim (x, (xs, i)^\infty)$ and $(xs, i+1) \sim (xs ++ [e], i)$. Clearly $\mu_!(\mathbb{n}) \ncong S_{X,n}$, and we conclude our result. □

We still wish to say something about some terminal C-initial algebras of endofunctors found in Section 3. In order to do so, we give the following result, which is quite specific but does give us the necessary tools.

Proposition 5.3.4. *Let (ν, μ) be a pair of lax monoidal natural transformations*

$$F \underset{\nu}{\overset{\mu}{\rightleftarrows}} G$$

such that $\nu \circ \mu = \mathrm{id}_F$. Let $T_{\mu_*(C)}$ be the terminal $\mu_*(C)$-initial algebra. Then $\nu_!(T_{\mu_*(C)})$ is the terminal C-initial algebra.

Proof. Let A be a C-initial algebra. Since $(\mu_!, \mu_*)$ preserve C-initial algebras by Theorem 5.2.2, we know $\mu_!(A)$ is $\mu_*(C)$-initial. Hence there exists a unique algebra morphism $\mu_!(A) \to T_{\mu_*(C)}$. Since $\nu \circ \mu = \mathrm{id}_F$, we also have $\nu_! \circ \mu_! = \mathrm{id}_{\mathbf{Alg}^F}$. This implies $\nu_!$ is full. Using this, we see

$$1 \cong \mathbf{Alg}^G(\mu_!(A), T_{\mu_*(C)}) \twoheadrightarrow \mathbf{Alg}^F(\nu_! \circ \mu_!(A), \nu_!(T_{\mu_*(C)})) \cong \mathbf{Alg}^F(A, \nu_!(T_{\mu_*(C)})) \cong 1,$$

and we conclude $\nu_!(T_{\mu_*(C)})$ is the terminal C-initial algebra. □

The next two results concern themselves with the functors $G : A \mapsto 1 + X \times A$ and $J : A \mapsto 1 + X \times A^*_\infty$, as seen in Section 3 and Section 4.4. The functor G corresponds to the list type, where the functor J corresponds to the unbounded tree type. We wish to use Corollary 3.7.14 together with Proposition 5.3.4 to show the G-algebra X^*_n is the terminal $X^{*\circ}_n$-initial algebra. To do so we first need the following lemma.

Lemma 5.3.5. *Consider the J-coalgebra*

$$\begin{aligned} \chi_{X,n} : X^{*\circ}_n &\to 1 + X \times (X^{*\circ}_n)^*_\infty \\ [\,] &\mapsto * \\ x : xs &\mapsto (x, [xs, xs, \ldots]). \end{aligned}$$

Then $S_{X,n}$ is the terminal $(X^{*\circ}_n, \chi_{X,n})$-initial algebra.

Proof. Consider the coalgebra

$$\begin{aligned} \chi_n : \mathbb{n}^\circ &\to 1 + X \times (\mathbb{n}^\circ)^*_\infty \\ 0 &\mapsto * \\ i &\mapsto (e, [i-1, i-1, \ldots]) \end{aligned}$$

where $\mathbb{n}^\circ \cong \{0, 1, \ldots, n\}$. We can construct a monomorphism

$$\begin{aligned} \mathbb{n}^\circ &\to X^\circ_n \\ i &\mapsto e^i \end{aligned}$$

and see $(\mathbb{n}^\circ, \chi_n)$ is a subobject of $(X^\circ_n, \chi_{X,n})$. By Corollary 3.7.14 we know there exists an algebra morphism $\varphi_n : A \to S_{X,n}$ for any $(X^{*\circ}_n, \chi_{X,n})$-initial algebra A. To conclude $S_{X,n}$ is the terminal $(X^{*\circ}_n, \chi_{X,n})$-initial algebra, we need to show this morphism is unique. We use the same approach as in Lemma 3.7.12 by constructing a monomorphism

$$\begin{aligned} m : S_{X,n} &\to [X^\circ_n, S_X] \\ e_S &\mapsto \mathrm{const}_{e_S} \\ (x, ss) &\mapsto \left(xs \mapsto \begin{cases} e_S & \text{if } xs = [\,] \\ (x' \bullet x, \mathrm{map}(m)(\mathrm{zip}(ss, [xs', xs', \ldots]))) & \text{if } xs = x' : xs'. \end{cases} \right) \end{aligned}$$

and through reasoning identical is in Lemma 3.7.12 conclude $\varphi_n : A \to S_{X,n}$ is unique. We conclude $S_{X,n}$ is the terminal $(X^\circ_n, \chi_{X,n})$-initial algebra □

Now we are ready to apply Proposition 5.3.4 to obtain the desired result.

Proposition 5.3.6. *The algebra X_n^* is the terminal $X_n^{*\circ}$-initial algebra.*

Proof. Consider the pair of lax monoidal natural transformations

$$G \underset{\nu}{\overset{\mu}{\rightleftarrows}} J$$

defined by

$$\begin{aligned} \mu_A : 1 + X \times A &\to 1 + X \times A^*_\infty \\ * &\mapsto * \\ (x, a) &\mapsto (x, a^\infty) \end{aligned}$$

and

$$\begin{aligned} \nu_A : 1 + X \times A^*_\infty &\to 1 + X \times A \\ * &\mapsto * \\ (x, [\,]) &\mapsto (x, \alpha(*)) \\ (x, a : as) &\mapsto (x, a) \end{aligned}$$

as seen in Section 4.4.2. We see $\nu \circ \mu = \mathrm{id}_G$ and hence can apply Proposition 5.3.4. The terminal $\mu_*(X_n^{*\circ})$-initial algebra is given by $S_{X,n}$ by Lemma 5.3.5. Hence, the terminal $X_n^{*\circ}$-initial algebra is given by $\nu_!(S_{X,n})$. By Section 4.4.2, $\nu_!(S_{X,n})$ is given by

$$\nu_!(S_{X,n}) = S_{X,n}/\sim,$$

where $\sim$ is generated by $(x, s : ss) \sim (x, s : ss')$ and $(x, [\,]) \sim (x, e_S)$ for all $x \in X$ and $ss, ss' \in (S_{X,n})^*_\infty$. To complete our proof, we construct the isomorphism

$$\begin{aligned} f : \nu_!(S_{X,n}) &\to X_n^* \\ [e_S] &\mapsto [\,] \\ [x, s : ss] &\mapsto x : f(s), \end{aligned}$$

and we conclude X_n^* is the terminal $X_n^{*\circ}$-initial algebra. □

For the next two results we will consider the functor $H : A \mapsto 1 + X \times A \times A$, the functor corresponding to the binary tree type. We will also be using $J : A \mapsto 1 + X \times A^*_\infty$, and both H and J are seen in Section 3 and Section 4.4. Using a similar approach as before we want to show the H-algebra $T_{X,n}$ is the terminal $T^\circ_{X,n}$-initial algebra.

Lemma 5.3.7. *Consider the J-coalgebra*

$$\begin{aligned} \chi_{T,n} : T^\circ_{X,n} &\to 1 + X \times (T^\circ_{X,n})^*_\infty \\ e_T &\mapsto * \\ (x, \ell, r) &\mapsto (x, [\ell, r, \ell, r, \dots]). \end{aligned}$$

Then $S_{X,n}$ is the terminal $(T^\circ_{X,n}, \chi_{T,n})$-initial algebra.

Proof. Again consider the coalgebra

$$\begin{aligned} \chi_n : \mathtt{n}^\circ &\to 1 + X \times (\mathtt{n}^\circ)^*_\infty \\ 0 &\mapsto * \\ i &\mapsto (e, [i-1, i-1, \dots]) \end{aligned}$$

where $\mathbb{n}^\circ \cong \{0, 1, \dots, n\}$. We can construct a monomorphism

$$\begin{aligned} m : \mathbb{n}^\circ &\to T^\circ_{X,n} \\ 0 &\mapsto e_T \\ i &\mapsto (e, m(i-1), m(i-1)) \end{aligned}$$

and see $(\mathbb{n}^\circ, \chi_n)$ is a subobject of $(T^\circ_{X,n}, \chi_{T,n})$. By Corollary 3.7.14 we know there exists an algebra morphism $\varphi_n : A \to S_{X,n}$ for any $(T^\circ_{X,n}, \chi_{T,n})$-initial algebra A. To conclude $S_{X,n}$ is the terminal $(T^\circ_{X,n}, \chi_{T,n})$-initial algebra, we need to show this morphism is unique. We use the same approach as in Lemma 3.7.12 by constructing a monomorphism

$$\begin{aligned} m' : S_{X,n} &\to [T^\circ_{X,n}, S_X] \\ e_S &\mapsto \mathrm{const}_{e_S} \\ (x, ss) &\mapsto \left(t \mapsto \begin{cases} e_S & \text{if } t = e_T \\ (x' \bullet x, \mathrm{map}(m')(\mathrm{zip}(ss, [\ell, r, \ell, r, \dots]))) & \text{if } t = (x, \ell, r). \end{cases} \right) \end{aligned}$$

and through reasoning identical is in Lemma 3.7.12 conclude $\varphi_n : A \to S_{X,n}$ is unique. We conclude $S_{X,n}$ is the terminal $(T^\circ_{X,n}, \chi_{T,n})$-initial algebra □

Now we can state and prove the following proposition, using Proposition 5.3.4.

Proposition 5.3.8. *The algebra $T_{X,n}$ is the terminal $T^\circ_{X,n}$-initial algebra.*

Proof. Consider the pair of lax monoidal natural transformations

$$H \underset{\nu}{\overset{\mu}{\rightleftarrows}} J$$

defined by

$$\begin{aligned} \mu_A : 1 + X \times A \times &\to 1 + X \times A^*_\infty \\ * &\mapsto * \\ (x, \ell, r) &\mapsto (x, [\ell, r, \ell, r, \dots]) \end{aligned}$$

and

$$\begin{aligned} \nu_A : 1 + X \times A^*_\infty &\to 1 + X \times A \times A \\ * &\mapsto * \\ (x, [\,]) &\mapsto (x, \alpha(*), \alpha(*)) \\ (x, [a]) &\mapsto (x, a, \alpha(*)) \\ (x, a : a' : as) &\mapsto (x, a, a'). \end{aligned}$$

We see $\nu \circ \mu = \mathrm{id}_H$ and hence can apply Proposition 5.3.4. The terminal $\mu_*(T^\circ_{X,n})$-initial algebra is given by $S_{X,n}$ by Lemma 5.3.7. Hence, the terminal $T^\circ_{X,n}$-initial algebra is given by $\nu_!(S_{X,n})$. We compute $\nu_!(S_{X,n})$ to be given by

$$\nu_!(S_{X,n}) = S_{X,n} / \sim,$$

where $\sim$ is generated by $(x, s : s'ss) \sim (x, s : s' : ss'), (x, [s]) \sim (x, [s, e_S])$ and $(x, [\,]) \sim (x, [e_S])$ for all $x \in X$ and $ss, ss' \in (S_{X,n})^*_\infty$. To complete our proof, we construct the isomorphism

$$\begin{aligned} f : \nu_!(S_{X,n}) &\to T_{X,n} \\ [e_S] &\mapsto [\,] \\ [x, s : s' : ss] &\mapsto (x, f(s), f(s')), \end{aligned}$$

and we conclude $T_{X,n}$ is the terminal $T^\circ_{X,n}$-initial algebra. □

To conclude this section, we conjecture the following.

Conjecture 5.3.9. *Let (ν, μ) be a pair of lax monoidal natural transformations*

$$F \underset{\nu}{\overset{\mu}{\rightleftarrows}} G$$

such that $\nu \circ \mu = \mathrm{id}_F$. Then $(\nu_!, \nu_*)$ preserves terminal C-initial algebras.

6 Conclusion & Outlook

In this thesis, we have shown the functorial nature of the enrichment of the category of algebras in the category of coalgebras. Given a closed symmetric monoidal category and an accesible lax monoidal endofunctor F acting on said category, we can construct the categories $\mathbf{Alg}^F$ and $\mathbf{CoAlg}^F$, the categories of F-algebras and F-coalgebras. By [1], $\mathbf{Alg}^F$ is enriched, tensored and powered in $\mathbf{CoAlg}^F$. This construction applies to a broad range of endofunctors, and in particular to W-types. In order to show the functorial nature of this enrichment, we constructed the 2-category $\mathbf{EnrCat}$ in Definition 4.1.1. The category $\mathbf{EnrCat}$ has as objects pairs $(\mathbf{C}, \mathbf{V})$, where $\mathbf{C}$ is a $\mathbf{V}$-enriched category and $\mathbf{V}$ is a monoidal category, and is fibered over $\mathbf{MonCat}$. By construction, $(\mathbf{Alg}^F, \mathbf{CoAlg}^F) \in \mathbf{EnrCat}$. We have shown this construction is functorial in Corollary 4.3.15 and Corollary 4.3.21 respectively, constructing two functors

$$
\begin{aligned}
\mathbf{Endo}(\mathbf{C}) &\to \mathbf{EnrCat} \\
F &\mapsto (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \\
\mu &\mapsto (\mu_!, \mu_*)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{Endo}(\mathbf{C})^{\mathbf{op}} &\to \mathbf{EnrCat} \\
F &\mapsto (\mathbf{Alg}^F, \mathbf{CoAlg}^F) \\
\mu &\mapsto (\mu^*, \mu_{¡})
\end{aligned}
$$

where $\mu_!$ is the left adjoint of the pullback functor $\mu^* : \mathbf{Alg}^G \to \mathbf{Alg}^F$ and $\mu_{¡}$ is the right adjoint of the pushforward functor $\mu_* : \mathbf{CoAlg}^F \to \mathbf{CoAlg}^G$. For all functors involved, explicit constructions have been provided.

In [1], special significance has been assigned to (terminal) C-initial algebras, which are a generalization of initial algebras and share their desirable properties. We have shown $(\mu_!, \mu_*)$ preserve C-initial algebras in Theorem 5.2.2, as well as given specific conditions under which terminal C-initial algebras are preserved in Proposition 5.3.4. We have also shown that any preinitial algebra P is also P°-initial in Proposition 5.1.10, providing a broad class of C-initial algebras. To accompany this theory, we have provided many examples. Besides computing how the enrichment plays out in these examples, we have also provided a constructive proof showing certain algebras are terminal C-initial algebras in the case of the unbounded tree type. Using techniques developed in this paper regarding the preservation of terminal C-initial algebras, we have used this result to compute C-initial algebras in the case of binary trees, lists and natural numbers as well.

There is still so much more to do! First off, in future work we would seek to prove or disprove Conjecture 5.3.9.

Also, it is conspicuous that we obtain two functors $\mathbf{Endo}(\mathbf{C}) \to \mathbf{EnrCat}$, one being covariant and the other being contravariant. The functors map to the same objects, and a morphism μ gets mapped to $(\mu_!, \mu_*)$ by one and to $(\mu^*, \mu_{¡})$ by the other. Even more conspicuous is that $\mu_!$ and μ_* are left adjoint to μ^* and $\mu_{¡}$ respectively. One might suspect $(\mu_!, \mu_*)$ is left adjoint to $(\mu^*, \mu_{¡})$, and we seek to prove this result in the future. We have already taken steps in that direction by defining $\mathbf{EnrCat}$ as a 2-category, which in turn gives a notion of adjunction. Notable is that this notion of adjunction is probably not strict. Taking inspiration from adjoint functors, given $\mu : F \to G$, one would expect an isomorphism of the form

$$\underline{\mathbf{Alg}}^G(\mu_!(A), B) \text{ “} \cong \text{ ” } \underline{\mathbf{Alg}}^F(A, \mu^*(B)).$$

Since these objects do not live in the same category, we need to use either μ_* or $\mu_{¡}$ to make sense of the isomorphism. This would yield isomorphisms

$$\mu_{¡}(\underline{\mathbf{Alg}}^G(\mu_!(A), B)) \cong \underline{\mathbf{Alg}}^F(A, \mu^*(B)) \tag{6.1}$$

$$\underline{\mathbf{Alg}}^G(\mu_!(A), B) \cong \mu_*(\underline{\mathbf{Alg}}^F(A, \mu^*(B))). \tag{6.2}$$

If we compute an example, taking $F : \mathbf{Set} \to \mathbf{Set}, A \mapsto 1$ and $G : \mathbf{Set} \to \mathbf{Set}, A \mapsto X$ for some fixed monoid X and $\mu_A : F(A) \to G(A), * \mapsto e$, we can compute

$$\underline{\mathbf{Alg}}^G(\mu_!(A), B) = \{(f, x) \in \mathbf{Set}(A + X/\alpha(*) \sim e, B) \times X \mid f[x'] = \beta(x \bullet x')\}.$$

and

$$\underline{\mathbf{Alg}}^F(A, \mu^*(B)) = \mathbf{Set}(A \setminus \{\alpha(*)\}, B).$$

Computing $\mu_{\mathsf{i}}(\underline{\mathbf{Alg}}^G(\mu_!(A), B))$ we find

$$\mu_{\mathsf{i}}(\underline{\mathbf{Alg}}^G(\mu_!(A), B)) \cong \mathbf{Set}(A \setminus \{\alpha(*)\})$$

which does align with Eq. (6.1). However, taking $A = 1$ we can compute

$$\underline{\mathbf{Alg}}^G(\mu_!(A), B) \cong X \ncong 1 \cong \mathbf{Set}(\emptyset, B) \cong \mu_*(\underline{\mathbf{Alg}}^F(A, \mu^*(B)))$$

which does not agree with Eq. (6.2). This computation suggests we need a weaker notion of adjunction than an isomorphism of enriched hom-objects.

Recently, the idea of an n-partial algebra homomorphism has been presented in [11, Remark 4.7.5]. In the future, we wish to study these n-partial algebra homomorphism more extensively, and using them give a broad class of terminal C-initial algebras.

What also warrants more research is the monoidal structure on $\mathbf{Endo}(\mathbf{C})$ given by composition. We wish to find out if we can respect the monoidal structure on $\mathbf{Endo}(\mathbf{C})$ when constructing functors $\mathbf{Endo}(\mathbf{C}) \to \mathbf{EnrCat}$.

Finally, throughout this thesis, we have left the ambient category $\mathbf{C}$ fixed. It would be interesting to see if the enrichment of the category of algebras in the category of coalgebras is also functorial in $\mathbf{C}$, not only just in the endofunctor $F : \mathbf{C} \to \mathbf{C}$. A possibility would be to construct the fibered category $\mathbf{Endo}$ which as elements has pairs $(\mathbf{C}, F : \mathbf{C} \to \mathbf{C})$, and then find a (fibered) functor $\mathbf{Endo} \to \mathbf{EnrCat}$. If done, this would give a complete picture of the enrichment of the category of algebras in the category of coalgebras.

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